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THE SCHMID-VILONEN CONJECTURE FOR VERMA MODULES OF HIGHEST ANTIDOMINANT WEIGHT AND DISCRETE SERIES REPRESENTATIONS

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BY
ANTHONY SANTIAGO CHAVES AGUILAR

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A mis padres y mis hermanos,

a mi comunidad Alajuelita,

y a la memoria de William Alvarado

"No pedagogy which is truly liberating can remain distant from the oppressed by treating them as unfortunates and by presenting for their emulation models from among the oppressors. The oppressed must be their own example in the struggle for their redemption "

—Paulo Freire, *Pedagogy of the Oppressed*

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ABSTRACT

In his study of the unitary dual of a real semisimple Lie group $G_{\mathbb{R}}$, Vogan and his co-workers introduced a hermitian form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$ on a Harish-Chandra module I that is invariant under the action of the maximal compact of the complexification G of $G_{\mathbb{R}}$. Understanding the signature of this form is related to the problem of determining if the Harish-Chandra module I is unitary or not.

In their paper [SV12], Vilonen and Schmid use the Beilinson-Bernstein localization theorem along with a twisted version of Saito's theory of mixed Hodge modules to extend the definition of this invariant form to a set of irreducible modules that are not necessarily in the Harish-Chandra category anymore. This set consists of modules that are "geometrically constructible", meaning they are induced from locally closed subvarieties of the flag variety using the Grothendieck functors for $\mathcal{D}$-modules. The corresponding form turns out to be the integral of the polarization.

From Saito's theory, these modules are endowed with natural Hodge filtrations. Schmid and Vilonen posted a conjecture that aims to study the signature of this form on the graded pieces of the Hodge filtration.

In this thesis we explain all the constructions of Schmid and Vilonen with particular emphasis in the case when the modules are induced from closed subvarities of the flag variety. We also provide geometric proofs of the conjecture in the cases when I is a Verma module of antidominant highest weight or a discrete series representation.

CHAPTER 1

INTRODUCTION

Let $G_{\mathbb{R}}$ be a real semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and maximal compact $K_{\mathbb{R}}$. We denote by G, $\mathfrak{g}$, and K their complexifications. Let $U_{\mathbb{R}}$ be a maximal compact subgroup inside G so that $U_{\mathbb{R}} \cap G_{\mathbb{R}} = K_{\mathbb{R}}$, and denote its Lie algebra by $\mathfrak{u}_{\mathbb{R}}$. A Harish-Chandra module is a $(\mathfrak{g}, K)$-module so that the K and $\mathfrak{g}$-actions are compatible and the K-action is locally finite.

Let H be the universal Cartan subgroup of G and denote by $\mathfrak{h}$ its Lie algebra. Let $\Lambda \subset h^*$ be the weight lattice. For any $\lambda \in \Lambda \subset h^*$, we denote by $\mathcal{U}\mathfrak{g}_\lambda$-mod the category of modules over the enveloping algebra $\mathcal{U}\mathfrak{g}$ with infinitesimal character χ_λ. Here χ_λ denotes the character of the center $Z\mathfrak{g}$ induced from λ via the Harish-Chandra isomorphism $Z\mathfrak{g} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$. We denote the category of Harish-Chandra modules with infinitesimal character χ_λ by HC_λ.

For any representation V of the group $G_{\mathbb{R}}$, we can construct a Harish-Chandra module $\tilde{V}$ by taking the smooth, K-finite vectors in V. As it turns out, this process yields a bijection between unitary irreducible representations of $G_{\mathbb{R}}$ and irreducible Harish-Chandra modules with a positive definite $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form. The former of these sets is known as the unitary dual of $G_{\mathbb{R}}$, and it has been a long lasting problem to understand its nature.

Assuming λ is a *real weight*, meaning $\lambda \in \mathfrak{h}^*_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$, Vogan constructs, using algebraic techniques, for any irreducible object $I \in \mathrm{HC}_\lambda$, a hermitian $U_{\mathbb{R}}$-invariant form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$ which is unique up to scaling. When the module I also admits a $\mathfrak{g}_{\mathbb{R}}$-invariant form, one can replace it by a scalar multiple and there exists a K-equivariant involution $T : I \to I$ so that

$$(u, v)_{\mathfrak{g}_{\mathbb{R}}} = (Tu, v)_{\mathfrak{u}_{\mathbb{R}}}$$

for any elements $u, v \in I$ (see [SV12], section 2). Since T is an involution, the space I splits as the direct sum of the ± 1-eigenspaces and each of those eigenspaces will be stable under the K-action. The form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$ has to be definite when restricted to the K-isotypic components.

Therefore the problem of understanding whether $(u, v)_{\mathfrak{g}_{\mathbb{R}}}$ is positive definite translates to understanding the signature of the form $(\cdot, \cdot)_{\mathfrak{u}_{\mathbb{R}}}$ on different K-isotypic components and the nature of the transformation T.

In their paper [SV12], Schmid and Vilonen use geometric techniques to construct a large class of irreducible $\mathcal{U}\mathfrak{g}$-modules, which are not necessarilly in the Harish-Chandra category anymore, and endow them with a $U_{\mathbb{R}}$-invariant hermitian form (again, assuming the weight λ is real). Let us briefly explain this construction. For any complex algebraic variety Y, we denote by $\mathcal{D}_Y$ its ring of differential operators, and by $\mathcal{D}_Y$-mod the category of $\mathcal{D}_Y$-modules. Let X be the flag variety of G. Let $\mathcal{D}_X^\lambda$ be the corresponding ring of twisted differential operators associated to the weight λ. It turns out that for any object $\mathcal{M} \in \mathcal{D}_X^\lambda$, its global sections $M = \Gamma(X, \mathcal{M})$ is an object in $\mathcal{U}\mathfrak{g}_\lambda$-mod. If λ is dominant and regular, the Beilinson-Bernstein localization theorem [BB81], gives an equivalence of categories:

$$\mathcal{U}\mathfrak{g}^\lambda\text{-mod} \underset{\Gamma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathcal{D}_X^\lambda\text{-mod}$$

Let $Q \overset{j}{\hookrightarrow} X$ be a locally closed subset of X. For any irreducible vector bundle with a flat connection $\mathcal{S}$ on Q, we can consider a twisted version of the minimal extension functor to produce an irreducible $\mathcal{D}_X^\lambda$-module $j_{!*}^\lambda \mathcal{S}$. This extension can be defined only for certain weights λ. Whenever it can be defined, the localization theorem ensures that $I = \Gamma(X, j_{!*}^\lambda \mathcal{S})$ is an irreducible object in $\mathcal{U}\mathfrak{g}_\lambda$. In addition, endowing $\mathcal{S}$ with the trivial Hodge and weight filtrations makes it an irreducible Hodge Module. By (a twisted version of) Saito's theory, the module I is also a Hodge module equipped with a canonical Hodge filtration $\{F_P I\}$.

If moreover the character λ is real, then the Hodge module is *polarized*. We will explain in more detail how the polarization is constructed in the next section. For now we point out it is a pairing $P : I \times \overline{I} \to \mathcal{C}^{-\infty}(X_{\mathbb{R}})$, where $\mathcal{C}^{-\infty}(X_{\mathbb{R}})$ denotes the space of distributions on the real manifold $X_{\mathbb{R}}$ that underlies X. From here Schmid and Vilonen give a geometric description of the $U_{\mathbb{R}}$-invariant hermitian form. This is

$$(u, v)_{\mathfrak{u}_{\mathbb{R}}} = \int_X P(u, \overline{v})\, dm, \tag{1.1}$$

where dm is the $U_{\mathbb{R}}$-invariant Haar measure on the flag variety. About the signature of this form, Schmid and Vilonen conjectured

Conjecture 1.1. *Let c be the minimum degree where the Hodge filtration is not trivial. Assume $v \in F_p I \cap (F_{p-1} I)^{\perp}$. Then*

$$(-1)^{p-c}(v, v) > 0.$$

The class of modules I constructed using the process we described includes the irreducible objects in category $\mathcal{O}$ (for some arbitrary choice of a Borel), and the irreducible Harish-Chandra modules. They appear respectively as minimal extensions of irreducible vector bundles on the Bruhat cells, and as minimal extensions of K-equivariant irreducible vector bundles on K-orbits on the flag variety. In this paper we focus to prove conjecture 1.1 in the most simple cases in each of those settings. Those are precisely when Q is the zero-dimensional Bruhat cell or when Q is a closed K-orbit, and K has the same rank as G. In both of those cases, since Q is closed, the minimal extension coincides with the usual pushforward.

When Q is a single point in the flag variety, the corresponding modules coincide with the Verma module of antidominant highest weight, whereas if Q is a closed K-orbit, the module is the (infinitesimal) discrete series representations. The main result in this paper is

Theorem 1.2. *Conjecture 1.1 holds true when I is either an irreducible Verma module of antidominant highest weight, or a discrete series representation.*

It turns out these results can be proven from purely algebraic considerations. Indeed, in the first case, the form $(\cdot, \cdot)_{\mathfrak{u}_{\mathbb{R}}}$ coincides with the Shapovalov form and it was studied by

Wallach [Wal84]. Schmid actually verified the conjecture holds true in that case by using the results of Wallach. On the other hand, it can be shown that the conjecture in the second case is equivalent to the well known fact that discrete series are unitary, and in particular it has to hold true. Even so, the nature of the conjecture (1.1) is geometric, in fact, both the filtration F_pI and the form $(\cdot,\cdot)_{\mathfrak{u}_\mathbb{R}}$ are constructed from purely geometric data. Therefore we were motivated by trying to provide geometric proofs of these "simple" cases. In particular, we do not use that the infinitesimal discrete series come from a unitary representation. Our proof of the conjecture for that case can be considered to be a geometric proof of the unitarity of the discrete series representations.

The strategy we follow in this paper is to study locally the integral in (1.1). It turns out that both the differential form dm and the polarization $P(\cdot,\cdot)$ admit explicit formulas on an open set isomorphic to a maximal unipotent subgroup $N \subset B$, for some Borel B of G. These formulas are given in terms of the function $a_{w_0}(n)$ which is defined to be the A-component in the Iwasawa decomposition $G = U_\mathbb{R}AN$ of the term $w_0^{-1}nw_0$, where $w_0 \in U_\mathbb{R}$ is a representative of the maximal element in the Weyl group. The study of the integral in (1.1) is closely related to behavior of the function $a_{w_0}^\lambda(n)$.

Lu introduced in [Lu99] a set of local coordinates for the flag variety that realizes the function $a_{w_0}^\lambda(n)$ in a very explicit form. The advantage of these coordinates is they allow us to give explicit formulas for the polarization. These formulas generalize the ones given in [SV15] when G is SL(2). However, since these coordinates are not holomorphic, it is difficult to describe polarization on the module I in those terms. This is the main obstacle we need to overcome in order to prove theorem 1.2.

This thesis is structured as follows. In section 2, we present the construction of Schmid and Vilonen, and explain how to come up with the right polarizations. In section 3 we show how the conjecture 1.1 is related to the unitarizability of irreducible $(\mathfrak{g}, K)$-modules. Section 4 is devoted to introduce Lu's coordinates, and sections 5 and 6 deal with the geometric proof of theorem 1.2 in the two settings we have presented.

CHAPTER 2

PRELIMINARIES

2.1 Pushforward of $\mathcal{D}$-modules

Let $f : Z \to Y$ be a morphism of smooth algebraic varieties. Given a $\mathcal{D}_Z$-module M, our goal is to construct a $\mathcal{D}_Y$-module in a functorial way. We follow the expositions in [HT07] and [Gin98] in this section.

First, define the *connector bimodule* $\mathcal{D}_{Z\to Y}$ by

$$\mathcal{D}_{Z\to Y} := f^*\mathcal{D}_Y = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{D}_Z.$$

This has the structure of a $f^{-1}\mathcal{D}_Z$ on the right, and of a $\mathcal{D}_Q$-module on the left as can be seen from the derivative map $\mathcal{T}_Z \to f^*\mathcal{T}_Y$, where $\mathcal{T}_Z$ and $\mathcal{T}_Y$ denote the tangent sheaves of Z and Y respectively and j^* is the restriction functor for $\mathcal{O}$-modules.

Let X be a smooth complex variety X and let Ω_X be the sheaf of top-degree differential forms on X. By means of the Lie derivative, this sheaf has the structure of a right $\mathcal{D}_X$-module. Let M be a left $\mathcal{D}_X$-module. One can see that for a vector field $\theta \in \mathcal{T}_X$, the formula

$$\theta(\omega \otimes m) = \omega \cdot \theta \otimes m - \omega \otimes \theta \cdot m.$$

defines a structure of a right $\mathcal{D}_X$-module on $\Omega_X \otimes_{\mathcal{O}_X} M$. Since the sheaf Ω_X is invertible, we obtain an equivalence of categories

$$\mathcal{D}_X\text{-mod} \leftrightarrow \mathcal{D}_X^{\mathrm{op}}\text{-mod}.$$

We call the corresponding functors *side changing functors.*

Define

$$\mathcal{D}_{Y\leftarrow Z} := \Omega_Z \otimes_{\mathcal{O}_Z} f^{-1}\Omega_Y^{-1} \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z\to Y}.$$

By the previous argument, we see $\mathcal{D}_{Y\leftarrow Z}$ is a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_Z)$-module.

Definition 2.1. Given a $\mathcal{D}_Z$-module M, we define the functor $f_* : D^b(\mathcal{D}_Z\text{-mod}) \to D^b(\mathcal{D}_Y\text{-mod})$ at the level of derived categories by

$$f_*M := Rf.(\mathcal{D}_{Y\leftarrow Z} \otimes^L_{\mathcal{D}_Z} M).$$

Now assume the morphism $i : Z \to Y$ is a closed embedding. Then one can check $\mathcal{D}_{Y\leftarrow Z}$ is locally free over $\mathcal{D}_Y$ and the functor $Ri.$ is exact. In particular i_* restricts to an exact functor $i_* : \mathcal{D}_Z\text{-mod} \to \mathcal{D}_Y\text{-mod}$.

Let $\mathcal{J}_Z$ be the defining sheaf of Z. Denote by $N(\mathcal{J}_Z\mathcal{D}_Y)$ the normalizer of $\mathcal{J}_Z$ in $\mathcal{D}_Y$. That is

$$N(\mathcal{J}_Z\mathcal{D}_Y) := \{\psi \in \mathcal{D}_Y : \psi\mathcal{J}_Z \subset \mathcal{J}_Z\mathcal{D}_Y\}.$$

The vector fields on Y that are tangent to Z are contained in $N(\mathcal{J}_Z\mathcal{D}_Y)$. Indeed, as a sheaf of algebras, $N(\mathcal{J}_Z\mathcal{D}_Y)$ is generated by these vector fields and the functions on Y. There is a canonical identification

$$N(\mathcal{J}_Z\mathcal{D}_Y)/\mathcal{J}_Z\mathcal{D}_Y = i.(\mathcal{D}_Q).$$

If we define $N(\mathcal{D}_Y\mathcal{J}_Z)$ in the analogous way, it turns out

Proposition 2.2. *The normalizers $N(\mathcal{J}_Z\mathcal{D}_Y) = N(\mathcal{D}_Y\mathcal{J}_Z)$ coincide. Moreover, there is a canonical identification*

$$N(\mathcal{D}_Y\mathcal{J}_Z)/\mathcal{D}_Y\mathcal{J}_Z = i.(\mathcal{N}_Z \otimes \mathcal{D}_Z \otimes \mathcal{N}_Z^*)$$

where $\mathcal{N}_Z$ and $\mathcal{N}_Z^*$ are the top exterior powers of the normal and conormal bundles of Z in Y respectively.

From this proposition, we get the following description of the pushforward functor of $\mathcal{D}$-modules for closed embeddings

Proposition 2.3. *Let M be a $\mathcal{D}_Z$-module. There is an isomorphism of $\mathcal{D}_Y$-modules:*

$$i_*M = \mathcal{D}_Y \otimes_{\mathcal{J}_Z \mathcal{D}_Y} i.(\mathcal{N}_Z \otimes_{\mathcal{O}_Z} M).$$

Where the $\mathcal{D}_Y$-module structure on the right hand side comes from left multiplication on $\mathcal{D}_Y$.

We remark that sections of $\mathcal{N}_Z$ can be thought of as distributions on Q, as such they can be differentiated with respect to vector fields on Q. In this sense, the module i_*M can be regarded as the extension of M by adding the formal derivatives in the normal directions of the corresponding "delta functions".

Let $\mathcal{D}_Y^Z$-mod denote the category of $\mathcal{D}_Y$-modules that are supported on Z. For any $N \in \mathcal{D}_Y$-mod, there is an action of $N(\mathcal{D}_Y \mathcal{J}_Z)/\mathcal{D}_Y \mathcal{J}_Z$ on $\ker(\mathcal{J}_Z, N)$, the sheaf of sections of N that are anihilated under the action of the ideal $\mathcal{J}_Z$. Therefore we can define a functor $i^! : \mathcal{D}_Y^Z\text{-mod} \to \mathcal{D}_Z\text{-mod}$ by

$$N \in \mathcal{D}_Y^Z\text{-mod} \mapsto i^! N := \mathcal{N}_Z^* \otimes_{\mathcal{O}_Z} i^{-1}\ker(\mathcal{J}_Z, N).$$

The functor $i^!$ is exact. The following result will be important later. See [Gin98], theorem 3.3.13.1 for details.

Theorem 2.4. *(Kashiwara equivalence theorem) Let $i : Z \hookrightarrow Y$ be a closed embedding. The pushforward functor $i_* : \mathcal{D}_Z\text{-mod} \to \mathcal{D}_Y^Z\text{-mod}$ from the category of $\mathcal{D}_Z$-modules to the category of $\mathcal{D}_Y$-modules supported on Z is an equivalence of categories. Its quasi-inverse is given by the functor $i^!$.*

2.2 Construction of irreducible modules

Let us define explicitly what are the class of objects Schmid and Vilonen consider in their paper. Let X be the flag variety of G. For any weight $\lambda \in \Lambda$, there is a unique G-equivariant line bundle $\mathcal{O}_\lambda$ on X characterized by the property that the Borel B that stabilizes a given point x acts on the geometric fiber via the character e^λ. We define the ring of twisted differential operators $\mathcal{D}_X^\lambda$ to be

$$\mathcal{D}_X^\lambda = \mathcal{O}_{\lambda-\rho} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\rho-\lambda},$$

where ρ is half the sum of the positive roots. One can make sense of $\mathcal{D}_X^\lambda$ even when $\lambda \in \mathfrak{h}^*$ is not an element in the weight lattice, in such case the line bundle $\mathcal{O}_{\lambda-\rho}$ can not be defined on X, but it can still be defined as a line bundle equipped with a $\mathfrak{g}$-action on different open sets of X. In this case, if B is the stabilizer of the point $x \in X$, we require $\mathfrak{b}$ to act on the fiber of $\mathcal{O}_{\lambda-\rho}$ via the character $\lambda - \rho$.

Denote by $J_\lambda \subset \mathbb{C}[\mathfrak{h}^*]$ the mazimal ideal of functions that vanish at λ. We denote by I_λ the pullback of this ideal to $Z\mathfrak{g}$. Beilinson and Bernstein [BB81] prove the following

Theorem 2.5. *The global section of $\mathcal{D}_X^\lambda$ are given by*

$$\Gamma(X, \mathcal{D}_X^\lambda) = \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g})I_\lambda$$

Therefore we have functors

$$\Gamma : \mathcal{D}_X^\lambda\text{-mod} \to \mathcal{U}\mathfrak{g}_\lambda\text{-mod}$$
$$\Delta : \mathcal{U}\mathfrak{g}_\lambda\text{-mod} \to \mathcal{D}_X^\lambda\text{-mod}$$

where Γ is the global sections functor and Δ is the localization functor given by $M \mapsto \mathcal{D}_X^\lambda \otimes_{\mathcal{U}\mathfrak{g}} M$. The functor Δ is left adjoint to Γ. Beilinson ans Bernstein also prove

Theorem 2.6. *If the weight λ is dominant, then*

1. *The functor Γ is exact.*

2. *The natural transformation $Id \to \Gamma \circ \Delta$ is an isomorphism.*

This implies (see [BG99], lemma 2.4 for example), whenever λ is dominant, there is an equivalence of categories

$$\mathcal{U}\mathfrak{g}_\lambda\text{-mod} \leftrightarrow \frac{\mathcal{D}_X^\lambda\text{-mod}}{\ker(\Gamma)}. \tag{2.1}$$

If λ is in addition regular, then the kernel of the functor Γ only contains the zero object. Indeed, in such case any $\mathcal{D}_X^\lambda$-module is generated by its global sections. This implies the version of the BB-correspondence we stated in the introduction. Notice formula (2.1) implies the space of global sections of an irreducible $\mathcal{D}_X^\lambda$ module is an irreducible object in $\mathcal{U}\mathfrak{g}_\lambda$-mod even if λ is not regular. We are interested in constructing irreducible $\mathcal{U}\mathfrak{g}$-modules, hence we can and will drop from now on the assumption that λ is regular.

Let $f : Q \hookrightarrow X$ be a locally closed smooth algebraic set. Factor f as the composition

$$Q \overset{i}{\hookrightarrow} X^o \overset{j}{\hookrightarrow} X.$$

Where $X^o := X \backslash \partial Q$. Notice i is a closed embedding and j is an open embedding. Motivated by the construction in the previous section, we write $\mathcal{J}_Q$ for the defining ideal of Q, and we define a ring of twisted differential operators on Q by

$$\mathcal{D}_Q^\lambda := i^{-1}\left(N(\mathcal{J}_Q\mathcal{D}_{X^o}^\lambda)/\mathcal{J}_Q\mathcal{D}_{X^o}^\lambda\right).$$

Proposition 2.2 carries over to this situation and we have

$$N(\mathcal{D}_{X^o}^\lambda\mathcal{J}_Q)/\mathcal{D}_{X^o}^\lambda\mathcal{J}_Q = i.\left(\mathcal{N}_Q \otimes \mathcal{D}_Q^\lambda \otimes \mathcal{N}_Q^*\right)$$

Hence we define the twisted pushforward $i_*^\lambda : \mathcal{D}_Q^\lambda\text{-mod} \to \mathcal{D}_{X^o}^\lambda\text{-mod}$ by

$$i_*^\lambda M := \mathcal{D}_{X^o}^\lambda \otimes_{N(\mathcal{J}_Q\mathcal{D}_{X^o}^\lambda)} i.(\mathcal{N}_Q \otimes M).$$

This functor turns out to be exact. For M' in the category of $\mathcal{D}^\lambda_{X^o}$-modules, we define $j^\lambda_* : D^b(\mathcal{D}^\lambda_{X^o}\text{-mod}) \to D^b(\mathcal{D}^\lambda_X\text{-mod})$ simply by $j_*M' = Rj.M'$, which has to be defined as a derived functor. Finally we set $f^\lambda_* = j^\lambda_* \circ i^\lambda_*$. In the case when the inclusion f is an affine morphism, the functor f^λ_* has no cohomology, and so if $M \in \mathcal{D}^\lambda_Q$-mod, then $f^\lambda_* M$ is an underived $\mathcal{D}^\lambda_X$-module.

Just like in the untwisted case, we can also define a functor $f^\lambda_! : D^b(\mathcal{D}^\lambda_Q\text{-mod}) \to D^b(\mathcal{D}^\lambda_X\text{-mod})$ in the derived categories (see for example [HT07], chapter 2 for the untwisted case). It turns out that for any $M \in D^b(\mathcal{D}^\lambda_Q\text{-mod})$, we have a morphism $f^\lambda_! M \to f^\lambda_* M$. Denote by $f^\lambda_{!*}$ its image. This functor is called *the minimal extension*. This name is justified since, for M irreducible, then $f^\lambda_{!*}M$ is the unique irreducible subobject of $f^\lambda_* M$.

Let $\mathcal{S}$ be an irreducible vector bundle on Q with a flat connection (hence, a $\mathcal{D}_Q$-module). For any λ so that $\mathcal{O}_{\lambda-\rho}$ can be defined in a neighborhood of Q, the sheaf $\mathcal{O}_{\lambda-\rho}|_Q \otimes_{\mathcal{O}_Q} \mathcal{S}$ is a $\mathcal{D}^\lambda_Q$ module. We denote

$$\mathcal{M}(Q,\lambda,\mathcal{S}) = R^0 f^\lambda_*(\mathcal{O}_{\lambda-\rho}|_Q \otimes_{\mathcal{O}_Q} \mathcal{S})$$
$$\mathcal{I}(Q,\lambda,\mathcal{S}) = f^\lambda_{!*}(\mathcal{O}_{\lambda-\rho}|_Q \otimes_{\mathcal{O}_Q} \mathcal{S})$$

Even though in general both functors f^λ_* and $f^\lambda_!$ are derived, the image of $f^\lambda_! \to f^\lambda_*$ is non-trivial only in degree zero, therefore $\mathcal{I}(Q,\lambda,\mathcal{S})$ is still the unique irreducible subobject of $\mathcal{M}(Q,\lambda,\mathcal{S})$.

For the close embedding $i : Q \hookrightarrow X^o$, one has $i^\lambda_* = i^\lambda_! = i^\lambda_{!*}$. Assume $U \subset X^o$ is the open set containing Q where the bundle $\mathcal{O}_{\lambda-\rho}$ can be defined. The category $\mathcal{D}^{\lambda,Q}_{X^o}$-mod of $\mathcal{D}^\lambda_{X^o}$-modules supported on Q is equivalent to the category $\mathcal{D}^{\lambda,Q}_U$-mod of $\mathcal{D}^\lambda_U$-modules supported on Q by Kashiwara equivalence, where $\mathcal{D}^\lambda_U$ is the restriction of $\mathcal{D}^\lambda_X$ to U. So we have functors

$$\begin{array}{ccc} \mathcal{D}_Q\text{-mod} & \xrightarrow{i_*} & \mathcal{D}_X^Q\text{-mod} \\ {\scriptstyle \mathcal{O}_{\lambda-\rho}|_Q\otimes\cdot}\downarrow & & \downarrow{\scriptstyle \mathcal{O}_{\lambda-\rho}\otimes\cdot} \\ \mathcal{D}_Q^\lambda\text{-mod} & \xrightarrow{i_*^\lambda} & \mathcal{D}_X^{\lambda,Q}\text{-mod} \end{array}$$

This diagram is commutative, from where we conclude

$$\mathcal{I}(Q,\lambda,\mathcal{S}) = j_{!*}^\lambda(\mathcal{O}_{\lambda-\rho} \otimes_{\mathcal{O}_U} i_*\mathcal{S})) \tag{2.2}$$

This is the description we will use in sections 5 and 6.

It is possible to use the localization theorem to give an explicit geometric characterization of the irreducible objects in HC_λ and the irreducible objects in category $\mathcal{O}$. Let now Q be a K-orbit in X, in particular Q is affinely embedded in X. Assume $\lambda \in \mathfrak{h}_\mathbb{R}$ is a weight so that $\mathcal{O}_{\lambda-\rho}$ makes sense in a neighborhood of Q. Let $\mathcal{S}$ be a K-equivariant line bundle on Q with a flat connection. It turns out (see [BB81]) that both $\mathcal{M}(Q,\lambda,\mathcal{S})$ and $\mathcal{I}(Q,\lambda,\mathcal{S})$ have locally finite K-actions which are compatible with the $\mathfrak{g}$-action. Therefore their global sections are Harish-Chandra modules. Conversely, any irreducible Harish-Chandra module is of the form $\Gamma(X, \mathcal{I}(Q,\lambda,\mathcal{S}))$, for some Q, $\mathcal{S}$ and λ. The latter statement is true even if λ is not regular.

For the case of category $\mathcal{O}$, it turns out that choosing Q to be a Bruhat cell (which are also affinely embedded) allows us to realize the dual Verma modules as pushforwards of trivial vector bundles. Then the irreducible objects all appear as the minimal extensions of such vector bundles (see [BG99], proposition 4.4 for example). Notice again we do not need to use that λ is regular

Schmid and Vilonen use a twisted version of Saito's theory of Mixed Hodge Modules [Sai90] to endow both $\mathcal{M}(Q,\lambda,\mathcal{S})$ and $\mathcal{I}(Q,\lambda,\mathcal{S})$ with canonical weight and Hodge filtrations which come from thinking of $\mathcal{S}$ as a Hodge module by endowing it with the trivial Hodge and weight filtrations. We won't define these filtrations in general, but we can give description of what the Hodge filtration looks like when $i : Z \hookrightarrow Y$ is a closed embedding of smooth algebraic varieties. Let M be a Hodge module on Z with Hodge filtration $F.M$. Let U be an open subset of Y which admits affine coordinates $x_1, \ldots, x_n$ so that $U \cap Z$ is cut out by the equations $x_1 = \cdots = x_k = 0$. In this case, we have

$$(i_*M)(U) = M(U \cap Z) \otimes \mathbb{C}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_k}](\delta_Z),$$

meaning we adjoint formal derivatives in the normal directions of the delta function to $M(U \cap Z)$, which we think of as a flat section of $\mathcal{N}_Z(U)$. Locally we have

$$F_p(i_*M)(U) = \sum_{|\alpha|+q+k \leq p} F_q M(U \cap Z) \partial_{x_1}^{\alpha_1} \ldots \partial_{x_k}^{\alpha_k}(\delta_Z). \tag{2.3}$$

Schmid and Vilonen explain that these filtrations still make sense in the twisted setting as long as $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. IF Q is a closed set, from (2.2), we can describe this filtration. It is:

$$F_p(\mathcal{O}_{\lambda-\rho} \otimes i_*\mathcal{S}) = \mathcal{O}_{\lambda-\rho} \otimes F_p i_*\mathcal{S}.$$

2.3 Polarization

For a smooth complex variety Y, denote by $Y_{\mathbb{R}}$ the underlying smooth real manifold. We denote by $\overline{Y}$ the same variety equipped with the conjugate complex structure. The map $Y \to \overline{Y}$ given by $y \mapsto \overline{y}$ is a diffeomorphism of real manifolds, and interchanges the holomorphic structure on Y with the antiholomorphic structure on $\overline{Y}$. The sheaf of differential operators $\overline{\mathcal{D}_Y}$ is defined in the natural way. For a $\mathcal{D}_Y$-module M, we define $\overline{M}$ to be the corresponding conjugate module with action given by $\overline{\psi} \cdot \overline{m} = \overline{\psi \cdot m}$, for any $\psi \in \mathcal{D}_Y$. A polarization on a Hodge Module M is a $\mathcal{D}_Y \times \overline{\mathcal{D}_Y}$-bilinear pairing

$$M \times \overline{M} \to C^{-\infty}(Y_{\mathbb{R}}).$$

Where $C^{-\infty}(Y_{\mathbb{R}})$ is the space of distributions on $Y_{\mathbb{R}}$. Let Y be an algebraic complex variety and Z a locally closed subset of Y, and $f : Z \hookrightarrow Y$ the inclusion. We will give a description of the polarization. Use the factorization

$$Z \overset{i}{\hookrightarrow} Y^o \overset{j}{\hookrightarrow} Y$$

where Y^o stands for $Y \backslash \partial Z$. Use the same coordinates given in (2.3) for the embedding i. Let M be an irreducible variation of Hodge structure (hence, a Hodge module) on Z endowed with a polarization $(\cdot, \cdot)_1 : M \times \overline{M} \to C^{\infty}(Z_{\mathbb{R}})$, where the target is the space of smooth functions on $Z_{\mathbb{R}}$. Then the Hodge module $i_* M$ is pure and admits a polarization. This polarization is locally given by

$$P(u\alpha(\delta_Z), \overline{v}\overline{\beta}(\delta_{\overline{Z}})) = (u, \overline{v})_1(\alpha\overline{\beta})(\delta_{Z_{\mathbb{R}}}). \tag{2.4}$$

Where $u, v \in M$, $\alpha, \beta \in \mathbb{C}[\partial_{k+1}, \ldots, \partial_n]$, and $\delta_{Z_{\mathbb{R}}}$ is the delta function of $Z_{\mathbb{R}}$, which we think as the section $\delta_Z \otimes \delta_{\overline{Z}}$ of the line bundle $\mathcal{N}_Z \otimes_{\mathbb{C}} \mathcal{N}_{\overline{Z}}$. It is characterized by $z_i \cdot \delta_{Z_{\mathbb{R}}} = \overline{z}_i \cdot \delta_{Z_{\mathbb{R}}} = 0$, for any $1 \leq i \leq k$, and $\partial_i \cdot \delta_{Z_{\mathbb{R}}} = \overline{\partial}_i \cdot \delta_{Z_{\mathbb{R}}} = 0$ for any $k+1 \leq i \leq n$.

The extension to Y is defined in the natural way. From the construction we see that

$$\begin{aligned} P(u, \overline{v}) \in & \mathcal{O}_{Y^o_{\mathbb{R}}} \otimes_{\mathcal{O}_Y \times \overline{\mathcal{O}}_Y} (f_*(\mathcal{O}_Z) \otimes \overline{f_*(\mathcal{O}_Z)}) \\ & = \mathcal{O}_{Y^o_{\mathbb{R}}} \otimes_{\mathcal{O}_Y \otimes \overline{\mathcal{O}}_Y} (\mathcal{D}_Y \otimes_{N(\mathcal{J}_Z \mathcal{D}_Y)} f.(\mathcal{N}_Z)) \otimes (\mathcal{D}_{\overline{Y}} \otimes_{N(\mathcal{J}_{\overline{Z}} \mathcal{D}_{\overline{Y}})} f.(\mathcal{N}_{\overline{Z}})) \\ & = \mathcal{D}_{Y^o_{\mathbb{R}}} \otimes_{N(\mathcal{J}_{Z_{\mathbb{R}}} \mathcal{D}_{Y^o_{\mathbb{R}}})} f.(\mathcal{N}_{Z_{\mathbb{R}}}) =: \Delta^Z_{\mathbb{R}}. \end{aligned}$$

Here the real subindices denote the smooth counterparts of the corresponding holomorphic objects. Just like in section 2.1, $\Delta^Z_{\mathbb{R}}$ can be regarded as the adjunction of the formal derivatives in the normal directions to the delta distributions (sections of the normal bundle) on $Z_{\mathbb{R}}$. We have a pairing

$$\langle \cdot, \cdot \rangle : f^{-1}\Omega_{Y_{\mathbb{R}}} \otimes_{\mathcal{D}_{Y_{\mathbb{R}}}} \Delta_{\mathbb{R}}^{Z} \to \Omega_{Z_{\mathbb{R}}},$$

defined by $\omega \otimes \phi \otimes \alpha \mapsto \langle \omega \cdot \phi, \alpha \rangle$, where $\langle \cdot, \cdot \rangle : \Omega_{Y_{\mathbb{R}}} \times \mathcal{N}_{Z_{\mathbb{R}}} \to \Omega_{Z_{\mathbb{R}}}$ is the contraction map. In particular it makes sense to define

$$\int_Y \delta\omega := \int_Z \langle \omega, \delta \rangle \tag{2.5}$$

for $\delta \in \Delta_{\mathbb{R}}^{Z}$ and $\omega \in \Omega_{Y_{\mathbb{R}}}$. The following is an immediate consequence of this formula

Proposition 2.7. *Assume Z is a closed set of Y. For any smooth measure ω, and any $\delta \in \Delta_{\mathbb{R}}$, the integral $\int_Y \delta\omega$ is absolutely convergent.*

Now we consider the case when $Y = X$ is the flag variety and $Z = Q$ is a locally closed set. Because we want to work in the twisted setting, we need an equivalent definition of polarization. Denote $\mathcal{M} = \mathcal{M}(Q, \lambda, \mathcal{S})$. Use the notation of (2.2) to write, $\mathcal{M} = j_*(\mathcal{O}_{\lambda-\rho} \otimes_{\mathcal{O}_U} i_*\mathcal{S})$. We consider a pairing $P : \mathcal{M} \times \overline{\mathcal{M}} \to C^{-\infty}(X_{\mathbb{R}})$ which can be described as follows:

1. Endow $\mathcal{S}$ with a flat hermitian metric $(\cdot, \cdot)$.

2. Extend to a polarization $P_1(\cdot, \cdot)$ on $i_*(\mathcal{S})$ by means of the formula (2.4).

3. This induces a pairing $P(\cdot, \cdot)$ on $\mathcal{O}_{\lambda-\rho} \otimes i_*(\mathcal{S})$ by using the trivial metric on $\mathcal{L}_{\lambda-\rho}$. Explicitely, this is given by

$$P(\sigma \otimes u, \overline{\tau} \otimes \overline{v}) = \sigma(\overline{\tau}) P_1(u, \overline{v}).$$

   This formula makes sense because, since λ is a real weight then $\overline{\mathcal{O}}_{\lambda-\rho} = \mathcal{O}_{\rho-\lambda}$.

4. Extend to $\mathcal{M}$ in the natural way.

From the definition, it is clear that $P(u,\overline{v}) \in \Delta_{\mathbb{R}}^{Q}$ for any $u, v \in \mathcal{M}$ and therefore can be integrated against measures on X. The main property of this construction is the following

Proposition 2.8. *The polarization P is $\mathfrak{u}_{\mathbb{R}}$-equivariant. This means*

$$\psi \cdot P(u,\overline{v}) = P(\psi \cdot u, \overline{v}) + P(u, \psi \cdot \overline{v})$$

for any $u, v \in \mathcal{M}(Q, \lambda, \mathcal{S})$ and $\psi \in \mathfrak{u}_{\mathbb{R}}$.

Proof. Assume the line bundle $\mathcal{O}_{\lambda-\rho}$ can be defined on an open set $U \subset X$ that contains Q. Then we have an identification $\overline{\mathcal{O}}_{\lambda-\rho} = \mathcal{O}_{\rho-\lambda}$ that comes from identifying X with $\overline{X}$. For any $u \in U_{\mathbb{R}}$ close enough to the identity, and $x \in U$, we have $u \cdot \overline{x} = \overline{u \cdot x}$. Therefore the pairing

$$\mathcal{O}_{\lambda-\rho} \times \overline{\mathcal{O}}_{\lambda-\rho} \to C^{\infty}(U)$$

is $\mathfrak{u}_{\mathbb{R}}$-equivariant.

By Saito's theory, the pairing P_1 on $i_*(\mathcal{S})$ is $\mathcal{D}_X \times \overline{\mathcal{D}}_{\overline{X}}$-bilinear, and in particular it is $\mathfrak{u}_{\mathbb{R}}$-equivariant. It follows that the pairing P on $\mathcal{M}$ is also $\mathfrak{u}_{\mathbb{R}}$-equivariant. □

As we defined in the introduction, let dm be the $U_{\mathbb{R}}$-invariant Haar measure on X. Schmid and Vilonen construct a hermitian form on I by

$$(u, v)_{\mathfrak{u}_{\mathbb{R}}} := \int_X P(u, \overline{v})\, dm.$$

In principle, it is not clear this integral will be convergent unless Q is a closed set. We will not discuss here exactly how to deal with the case when Q is not closed as our main interest is in the close embedding situation. Let us just say that it is possible to verify the integral $\int_X P(u,\overline{v})\omega$ absolutely convergent whenever $u, v \in F_0 I$ and ω is a smooth measure on X. Since the sheaf $\mathcal{I}$ is irreducible, one can also check that for any $u, v \in \mathcal{I}$, there exist $u_0, v_0 \in F_0\mathcal{I}$ and $\phi \in \mathcal{D}_{X_{\mathbb{R}}}$ so that $P(u,\overline{v}) = \phi \cdot P(u_0, \overline{v}_0)$. This is clear in the untwisted setting but is not obvious for twisted sheaves. The way to extend this integral then is by

$$\int_X P(u,\overline{v})\,dm := \int_X P(u_0,\overline{v}_0)(\,dm\cdot\phi)$$

The following is a result of Schmid and Vilonen (See [SV12], proposition 5.10

Theorem 2.9. *Let* dm *denote the* $U_{\mathbb{R}}$*-invariant measure on* X*. For any* $u,v\in\Gamma(X,\mathcal{I})$*, the hermitian form* $(u,v)_{\mathfrak{u}_{\mathbb{R}}}$ *defined by*

$$(u,v)_{\mathfrak{u}_{\mathbb{R}}} := \int_X P(u,\overline{v})\,dm.$$

is $\mathfrak{u}_{\mathbb{R}}$*-invariant.*

Proof. (Borrowed from [SV12]) Let $\psi\in\mathfrak{u}_{\mathbb{R}}$. We have

$$\begin{aligned}
(\psi\cdot u,v)_{\mathfrak{u}_{\mathbb{R}}}+(u,\psi\cdot v)_{\mathfrak{u}_{\mathbb{R}}} &= \int_X P(\psi\cdot u,\overline{v})+P(u,\overline{\psi\cdot v})\,dm\\
&= \int_X P(\psi\cdot u,\overline{v})+P(u,\psi\cdot\overline{v})\,dm\\
&= \int_X \psi\cdot P(u,\overline{v})\,dm\\
&= \int_X P(u,\overline{v})(dm\cdot\psi)\\
&= 0.
\end{aligned}$$

Where in the first line we used $\overline{\psi}=\psi$ since $\psi\in U_{\mathbb{R}}$ and in the last line we used that $dm\cdot\psi=0$ because dm is $U_{\mathbb{R}}$-invariant. □

CHAPTER 3

THE PROBLEM OF UNITARIZABILITY

In this section we explain how the Schmid-Vilonen conjecture can be used to prove unitarizability. Assume K is the group of fixed points of an involution $\theta : G \to G$ and Q is K-orbit on the flag variety. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition that corresponds to the split of $\mathfrak{g}$ in the 1 and -1 eigenspaces with respect to the involution θ. One has that $\mathfrak{k}$ is the Lie algebra of K.

Let $x \in Q$, and let $B \subset G$ be the Borel that fixes x. Then we have $Q = K/K \cap B$. Denote $B_K := K \cap B$. The group B_K is solvable and satisfies $B_K/[B_K, B_K] =: H'$ is a commutative subgroup of K. The inclusion $\mathfrak{b}_K \hookrightarrow \mathfrak{b}$ induces an inclusion $\mathfrak{h}' \hookrightarrow \mathfrak{h}$ on the corresponding Lie algebras. The condition that the bundle $\mathcal{O}_{\lambda-\rho}$ can be defined on a neighborhood of Q is equivalent to the condition that the pullback of $\lambda - \rho$ to $\mathfrak{h}'$ is an integer weight, meaning it can be exponentiated to a character of the group H' (see [HMSW87]). Choose λ so that this condition is satisfied. Just like before, denote by f the inclusion $f : Q \hookrightarrow X$. Let $\mathcal{I} = \mathcal{I}(Q, \lambda, \mathcal{S})$, and $I = \Gamma(X, \mathcal{I})$.

3.1 Generation by lowest piece of Hodge filtration

The Hodge filtration on I is said to be generated by its lowest piece if one has $\mathcal{U}^p\mathfrak{g} \cdot F_0 I = F_p I$. Assume I admits a $\mathfrak{g}_{\mathbb{R}}$-invariant form $(\cdot, \cdot)_{\mathfrak{g}_{\mathbb{R}}}$. As we said in the introduction (see [SV12], section 2), there exists a K-equivariant involution $T : V \to V$ so that

$$(u, v)_{\mathfrak{g}_{\mathbb{R}}} = (Tu, v)_{\mathfrak{u}_{\mathbb{R}}}.$$

It produces a decomposition $I = I^+ \oplus I^-$, associated with the 1 and -1 eigenspaces of T on I. Both spaces I^+ and I^- are K-stable and orthogonal to each other with respect to the form $(\cdot,\cdot)_{\mathfrak{u}_\mathbb{R}}$. Indeed, if $u \in I^+$ and $v \in I^-$, then

$$(u,v)_{\mathfrak{u}_\mathbb{R}} = (Tu,v)_{\mathfrak{u}_\mathbb{R}} = (u,v)_{\mathfrak{g}_\mathbb{R}} = \overline{(v,u)}_{\mathfrak{g}_\mathbb{R}} = \overline{(Tv,u)}_{\mathfrak{u}_\mathbb{R}} = -\overline{(v,u)}_{\mathfrak{u}_\mathbb{R}} = -(u,v)_{\mathfrak{u}_\mathbb{R}}.$$

We also have

$$\mathfrak{p}\cdot I^+ \subset I^-, \quad \mathfrak{p}\cdot I^- \subset I^+.$$

For the rest of this section, we reindex the Hodge filtration so that $F_0 I$ is the lowest non-trivial piece. From the local description (2.3), we see that $F_0 I$ is $F_c I$ in the previous indexing, where c is the codimension of Q in X.

Proposition 3.1. *The form $(\cdot,\cdot)_{\mathfrak{u}_\mathbb{R}}$ is positive definite on $F_0 I$*

Proof. It is possible to verify that $F_0\mathcal{I} \subset \mathcal{O}_{\lambda-\rho} \otimes f.(\mathcal{N}_Q \otimes \mathcal{S})$. Let U' be an affine open set of X that intersects Q. Since Q is affinely embedded in X, then $U = Q \cap U'$ is an open affine subset of Q. Let δ_Q be a section of $\mathcal{N}_Q$ in that neighborhood that is flat in U. The bundle $\mathcal{O}_{\lambda-\rho}|_Q \otimes \mathcal{N}_Q \otimes \mathcal{S}$ is trivializable on U, let $\sigma \otimes \delta_Q \otimes s$ be a section. The canonical polarization on $\mathcal{O}_{\lambda-\rho}$ is a hermitian $U_\mathbb{R}$-invariant metric, then one has from formula (2.5)

$$\int_{U'} P(\sigma\otimes\delta_Q\otimes s, \overline{\sigma}\otimes\overline{\delta}_Q\otimes\overline{s})\, dm = \int_U \|\sigma\|^2\|s\|^2\, d\mu$$

where $d\mu$ is the restriction to U of a $\mathfrak{k}$-invariant measure on Q. For a global section $u \in F_0 I$, the integral $\int_X P(u,\overline{u})\, dm$ is absolutely convergent and has positive integrand. Therefore $(u,u)_{\mathfrak{u}_\mathbb{R}} > 0$. □

The elements of $\mathfrak{k}$ form vector fields tangent to Q and therefore they do not increase the Hodge filtration. That is, for any $p \in \mathbb{N}$:

$$\mathfrak{k}\cdot F_p\mathcal{I} \subset F_p\mathcal{I}.$$

A consequence of this fact is the following

Theorem 3.2. *Assume the Hodge filtration on I is generated by the lowest piece. Then for any natural number p, $F_pI \cap (F_{p-1}I)^{\perp} \subset I^{\varepsilon}$, where $\varepsilon = sgn(-1)^p$.*

Proof. Since $\mathcal{U}^p\mathfrak{g} \cdot F_0I = F_pI$, we have

$$\begin{aligned} F_pI &= \sum_{i=0}^{p} \mathfrak{g}^i F_0 I \\ &= \sum_{i=0}^{p} (\mathfrak{k}+\mathfrak{p})^i F_0 I \\ &= \sum_{i=0}^{p} \mathfrak{p}^i F_0 I. \end{aligned}$$

By proposition 3.1, $F_0I \subset I^+$, hence $\mathfrak{p}^iF_0I \subset I^+$ if i is even, and $\mathfrak{p}^iF_0I \subset I^-$ if i is odd. Therefore

$$F_pI \cap I^+ = \sum_{i \le p,\ i \text{ even}} \mathfrak{p}^i F_0 I, \quad \text{and} \quad F_pI \cap I^- = \sum_{i \le p,\ i \text{ odd}} \mathfrak{p}^i F_0 I, \tag{3.1}$$

and these spaces are orthogonal to each other. Assume without loss of generality that p is even (the other case is proven in an analogous way). From (3.1), $F_pI \cap I^- \subset F_{p-1}I$. We conclude

$$F_pI \cap (F_{p-1}I)^{\perp} \subset F_pI \cap (F_pI \cap I^-)^{\perp} = F_pI \cap I^+ \subset I^+.$$

□

Therefore we obtain

Corollary 3.3. *The Schmid-Vilonen conjecture is true if and only if the form $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{R}}}$ is positive definite.*

Proof. Indeed, from proposition 3.2, it is clear that the Schmid-Vilonen conjecture in this case is true if and only if $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$ is positive definite on I^+ and negative definite on I^-. This is equivalent to $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{R}}}$ being positive definite. □

The conclusion of this section is the following

Theorem 3.4. *If $I = \Gamma(X, \mathcal{I}(Q, \lambda, \mathcal{S}))$ is an irreducible $(\mathfrak{g}, K)$-module that admits a $\mathfrak{g}_{\mathbb{R}}$-invariant form $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{R}}}$ and is generated by the lowest piece of the Hodge filtration, then $(\cdot,\cdot)_{\mathfrak{g}_{\mathbb{R}}}$ is positive definite (in other words, I comes from a unitary representation) if and only if the Schmid-Vilonen conjecture holds true for I.*

Conversely, Davis and Vilonen [DV22a] used the well known fact that tempered modules (which include the case of Q a closed K-orbit that we study in the next section) come from unitary representations to prove that the Hodge filtration is generated by the lowest piece in all those cases. The motivation of our work is in a sense, the converse result. We would like to prove the Schmid-Vilonen conjecture from a geometric perspective and then be able to determine if the corresponding module comes from a unitary representation or not. In section 6 we do this for *discrete series representations*, which occur when Q is a closed K-orbit and the rank of K coincides with the rank of G. In that case, proposition 3.6 in the next section ensures the Hodge filtration is generated by the lowest piece, and theorem 3.4 implies that discrete series representations are unitary.

3.2 Closed K-orbits

In this section we explain further how the theory works for closed embeddings. For the rest of this section, we assume Q is closed. Then B_K is solvable, but also parabolic, and so it is a Borel of K. In particular $Q = K/B_K$ is isomorphic to the flag variety of K.

Keeping in mind formula (2.2), the twisted Hodge filtration in this setting is given by

$$F_p\mathcal{I} = \mathcal{O}_{\lambda-\rho} \otimes F_p(i_*\mathcal{S}). \tag{3.2}$$

where the Hodge filtration on $i_*\mathcal{S}$ is the canonical one. The proof of theorem 3.1 is quite transparent in this case. Indeed, in this case we have $F_0\mathcal{I} = \mathcal{O}_{\lambda-\rho} \otimes i.(\mathcal{N}_Q \otimes \mathcal{S})$. Also, theorem 2.7 implies the integral defining $(u, \overline{v})_{\mathfrak{u}_{\mathbb{R}}}$ is always absolutely convergent, not only for $u, v \in F_0 I$.

Our goal is to prove that the modules I that are induced from Q satisfy their Hodge filtration is generated by their lowest piece.

Proposition 3.5. *For any $p > q \in \mathbb{N}$, and any $k > 0$, we have*

$$H^k(X, F_p\mathcal{I}) = H^k(X, F_p\mathcal{I}/F_q\mathcal{I}) = 0.$$

Proof. Since the sheaves $F_p\mathcal{I}$ are supported in Q, it is clear that

$$H^k(X, F_p\mathcal{I}) = H^k(Q, i^{-1}F_p\mathcal{I}), \text{ and } H^k(X, F_p\mathcal{I}/F_q\mathcal{I}) = H^k(Q, i^{-1}F_p\mathcal{I}/F_q\mathcal{I})$$

The sheaves $i^{-1}F_p\mathcal{I}$ and $i^{-1}F_p\mathcal{I}/F_q\mathcal{I}$ on Q are K-equivariant vector bundles, therefore they split as the direct sum $\oplus\mathcal{O}(\mu)$, for some weights μ in the weight lattice of K.

It is easy to see that in local coordinates, the vectors that span the normal bundle to Q are given by $\mathfrak{p}^+$, on which the Cartan of K acts by positive roots. Since the Cartan also acts by positive roots on $\mathcal{O}_{\lambda-\rho}$, we deduce the associated μ that appear in the decomposition are all positive. Therefore the result follows from the Borel-Weil-Bott theorem. □

From the local description (2.3), it is clear that in the close embedding case we have

$$\begin{aligned} F_{p+1}\mathcal{I} &= \mathcal{T}_X \cdot F_p\mathcal{I}. \\ &= \mathfrak{g} \cdot F_p\mathcal{I} \\ &= (\mathfrak{k} + \mathfrak{p}) \cdot F_p\mathcal{I} \\ &= F_p\mathcal{I} + \mathfrak{p} \cdot F_p\mathcal{I}. \end{aligned}$$

That is, we have a commutative diagram

$$\begin{array}{ccccccc} & & & & \mathfrak{p} \cdot F_p\mathcal{I} & & \\ & & & & \downarrow & \searrow & \\ 0 & \longrightarrow & F_p\mathcal{I} & \hookrightarrow & F_{p+1}\mathcal{I} & \twoheadrightarrow & 0 \end{array}$$

The following proposition provides a recursive description of the Hodge filtration on I

Proposition 3.6. *We have $F_{p+1}I = F_pI + \mathfrak{p} \cdot F_pI$.*

Proof. Proposition 3.5 implies that $\Gamma(X, F_p\mathcal{I}/F_q\mathcal{I}) = F_pI/F_qI$ for any $p > q$. Now, for the same argument of proposition 3.5, the kernel of the map $\mathfrak{p} \cdot F_p\mathcal{I} \to F_{p+1}\mathcal{I}/F_p\mathcal{I}$ has no higher cohomology, and therefore the map $\Gamma(X, \mathfrak{p} \cdot F_p\mathcal{I}) \to F_{p+1}I/F_pI$ is surjective.

Now we prove that $\Gamma(X, \mathfrak{p} \cdot F_p\mathcal{I}) = \mathfrak{p} \cdot F_pI$. Both $\Gamma(X, \mathfrak{p} \cdot F_p\mathcal{I}) = \Gamma(Q, i^{-1}\mathfrak{p} \cdot F_p\mathcal{I})$ and $\mathfrak{p} \cdot F_pI$ are finite-dimensional K-modules. By theorem 2.6 part 2, we only need to check their localizations on Q are isomorphic. Indeed, since $\mathfrak{k}$ normalizes $\mathfrak{p}$, we have

$$\begin{aligned} \mathcal{D}_Q \otimes_{\mathcal{U}\mathfrak{k}} \mathfrak{p} \cdot F_pI &= \mathfrak{p} \cdot (\mathcal{D}_Q \otimes_{\mathcal{U}\mathfrak{k}} F_pI) \\ &= \mathfrak{p} \cdot i^{-1}F_p\mathcal{I} \end{aligned}$$

We conclude that $\mathfrak{p} \cdot F_pI$ is a K-submodule of $F_{p+1}I$ that surjects onto $F_{p+1}I/F_{p+1}I$. Therefore $F_{p+1}I = F_pI + \mathfrak{p} \cdot F_pI$. □

We obtain as a corollary

Corollary 3.7. *The Hodge filtration in I is generated by its lowest piece.*

Proof. We proceed by induction. The base case is clear. Assume the statement holds true for p, then we have

$$\begin{aligned}
\mathcal{U}^{p+1}\mathfrak{g} \cdot F_0 I &= \mathfrak{g} \cdot (\mathcal{U}^p\mathfrak{g} \cdot F_0 I) \\
&= \mathfrak{g} \cdot F_p I \\
&= (\mathfrak{k} + \mathfrak{p}) \cdot F_p I \\
&= F_p I + \mathfrak{p} \cdot F_p I \\
&= F_{p+1} I.
\end{aligned}$$

where in the last line we used proposition 3.6. □

CHAPTER 4

LU'S COORDINATES.

In this section we explain some results from [Lu99]. Precisely, we describe a set of coordinates introduced by Lu for Bruhat cells in the flag variety and some applications.

Let G be a complex semisimple Lie group, with maximal Cartan H and associated Weyl group $W = N_G(H)/H$. As usual, we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Fix a Borel B containing H and let Φ be the associated root system where B determines a choice of positive roots. For each root $\alpha \in \Phi$, denote by $\mathfrak{n}_\alpha$ the eigenspace of $\mathfrak{g}$ associated to α. Let N be the unipotent radical of B and $\mathfrak{n} = \text{span}\{\mathfrak{n}_\alpha : \alpha \in \Phi^+\}$ its Lie algebra. Let B^- be the opposite Borel to B and define $\mathfrak{n}^-$, N^- in a similar way. For any $w \in W$, let N_w denote the unipotent subgroup of G given by

$$N_w = N \cap wN^-w^{-1}.$$

in other words, N_w is the group with Lie algebra

$$\mathfrak{n}_w = \{\mathfrak{n}_\alpha : \alpha \in \Phi_w^+\}$$

where $\Phi_w^+ = \{\alpha \in \Phi^+ : w^{-1}\alpha < 0\}$.

Fix a minimal length decomposition $w = w_{\gamma_1} \dots w_{\gamma_l}$, where the γ_i's are simple positive roots and w_{γ_i} are the corresponding reflections. For each γ_i, let $\dot{\gamma_i}$ be a representative of w_{γ_i} in $U_\mathbb{R}$ (which, as in the previous section, is a compact real form of G), and set $\dot{w} = \dot{\gamma_1} \dots \dot{\gamma_l}$. We abuse notation and denote $N_{s_{\gamma_i}}$ by N_{γ_i}.

Let $G = U_\mathbb{R}AN$ be the Iwasawa decomposition of G. Here A is an abelian subgroup of G with Lie algebra $\mathfrak{a}$ such that $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$, is the Cartan decomposition of $\mathfrak{h}$, with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{u}_\mathbb{R}$. There is an action of G on $U_\mathbb{R}$ given by $g \cdot u = u'$, where u' is the $U_\mathbb{R}$-component in the Iwasawa decomposition of gu. This action can be understood as a lift to $U_\mathbb{R}$ of the G-action on $G/B \cong U_{\mathbb{R}}/T$, where $T = U_{\mathbb{R}} \cap B$ is the maximal Torus in $U_{\mathbb{R}}$ with Lie algebra $\mathfrak{t}$.

We have the following result due to Lu:

Theorem 4.1. *There exists a diffeormorphism*

$$G_w : N_{\gamma_1} \times \cdots \times N_{\gamma_l} \to N_w$$

that is characterized by the condition

$$G_w(n_1, \ldots, n_l) \cdot \dot{w} = (n_1 \cdot \dot{\gamma}_1) \ldots (n_l \cdot \dot{\gamma}_l).$$

Identify the flag variety X of G with G/B. Let $\Sigma_w = BwB/B$ be the Bruhat cell associated to w. Denote by $C_{\gamma_i} = N_{\gamma_i} \cdot \dot{\gamma}_i$. For any $w \in W$, the stabilizer of $\dot{w}$ under the N_w-action is trivial, hence the action map gives a diffeomorphism from N_w to C_w. We have the diagram

$$\begin{array}{ccccc} N_{\gamma_1} \times \cdots \times N_{\gamma_l} & \xrightarrow{G_w} & N_w & & \\ \downarrow a & & \downarrow a & & \\ C_{\gamma_1} \times \cdots \times C_{\gamma_l} & \xrightarrow{m} & C_w & \xrightarrow{p} & \Sigma_w \end{array} \tag{4.1}$$

where m is the multiplication map, p is the projection from G to X, given by

$$g \mapsto g\dot{w}B = (g \cdot \dot{w})B,$$

and the maps a are the action maps. The content of theorem 4.1 is that m is a diffeomorphism, and G_w is the unique diffeomorphism making this diagram commute. Notice G_w and m are only diffeomorphisms of real manifolds, not biholomorphisms of complex varieties.

Lu uses theorem 4.1 to give coordinates for N_w. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$ and let $\gamma \in \Phi$ be any root. Choose vectors e_γ and f_γ in $\mathfrak{n}_\gamma, \mathfrak{n}_{-\gamma}$ respectively such that $\langle e_\gamma, f_\gamma \rangle = 1$.

Set $h_\gamma = [e_\gamma, f_\gamma]$. Let $\Psi_\gamma : \mathfrak{sl}_2 \to \mathfrak{g}$ be the injection of Lie algebras defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h_\gamma, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f_\gamma \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e_\gamma.$$

This homomorphism can be lifted to an injection of Lie groups $\Psi_\gamma : \mathrm{SL}(2,\mathbb{C}) \to G$.

In the case when $\gamma = \gamma_i$, the map

$$F_{\gamma_i} : \mathbb{C} \to N_{\gamma_i}, \quad z \mapsto \Psi_i \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

gives coordinates to N_{γ_i}. From here, we obtain coordinates for N_w simply by composition with G_w. Denote by $F_w : \mathbb{C}^l \to N_w$ the corresponding diffeomorphism. From now on we will refer to it as *Lu's coordinates*. Notice $F_{\gamma_i} = F_{w_{\gamma_i}}$ by definition.

The main importance for us of these coordinates is they provide explicit formulas for the $\mathfrak{u}_\mathbb{R}$-invariant differential forms on the Bruhat cells. Let dn be a N_w-invariant Haar measure on N_w. This measure is uniquely defined up to multiplication by scalars. Let $a_w : N_w \to A$ be the A-component of the Iwasawa decomposition of $w^{-1}nw$. That is $w^{-1}nw = ua_w(n)n_1$, where $u \in U_\mathbb{R}$, $a_w(n) \in A$ and $n_1 \in N$. The following is a theorem of Kostant (See [Kos63], or [Lu99]: Theorem 4.1). We add its proof because we will essentially repeat the same argument in the proof of Lemma 5.2

Theorem 4.2. *The differential form*

$$a_w(n)^{-2(\rho - w^{-1}\rho)}\, dn$$

is the pullback to N_w of the restriction to Σ_w of a $U_\mathbb{R}$-invariant differential l-form on X.

Proof. Let $x = wB \in X$. Let Ω_{N_w} and Ω_{Σ_w} be the sheaves of top holomorphic differential forms on N_w and Σ_w. As we mentioned, the projection map $p : N_w \to \Sigma_w$ given by $n \mapsto n\cdot x$

is an isomorphism and we use this isomorphism to identify Ω_{N_w} and Ω_{Σ_w}. We have

$$T_x\Sigma_w = \mathfrak{n}_w = \oplus_{\alpha\in\Phi_w^+}\mathfrak{n}_\alpha.$$

Using the Killing form, we get

$$T_x^*\Sigma_w = (\mathfrak{n}_w)^* \cong \oplus_{\alpha\in\Phi_w^+}\mathfrak{n}_{-\alpha}.$$

Hence $V = \wedge^l T_x^*\Sigma_w = \wedge_{\alpha\in\Phi_w^+}\mathfrak{n}_{-\alpha}$ is the geometric fiber of Ω_{Σ_w} at x. Set $v := \wedge_{\gamma\in\Phi_w^+} f_\gamma \in V$.

Let $\overline{\Omega}_w$ be the conjugate sheaf of antiholomorphic top differential forms. Let s be the unique N_w-invariant global form in $\Gamma(\Sigma_w, \Omega_{\Sigma_w})$ such that $s|_x = v$ and let $\overline{s} \in \Gamma(\Sigma_w, \overline{\Omega}_{\Sigma_w})$ be its conjugate. Then we have

$$s \wedge \overline{s} = dn_*$$

where dn_* is the pushforward of a Haar measure on N_w.

Notice V has a natural H-action coming from the H-action on N_w. This is given by

$$h \cdot v = h^{-\theta} v$$

where $\theta = \sum_{\gamma\in\Phi_w^+}\gamma$. In fact, since $\Phi_w^+ = \Phi^+ \cap w(\Phi^-)$, one has $\theta = \rho - w\rho$. By restricting this H-action to T, we can define Ω_w to be the $U_{\mathbb{R}}$-equivariant vector bundle on X with geometric fiber V at x. Clearly $\Omega_w|_{\Sigma_w} = \Omega_{\Sigma_w}$.

Let $n \in N_w$ and let $y = nx$. Let $\dot{w}^{-1}n\dot{w} = ua_w(n)m$ be the Iwasawa decomposition. We have $n = u'(\dot{w}a_w(n)\dot{w}^{-1})m'$, where $m' = \dot{w}m\dot{w}^{-1}$ is a unipotent element that fixes x and $u' = \dot{w}u\dot{w}^{-1}$ is an element in $U_{\mathbb{R}}$. Let μ be the restriction to Σ_w of the $U_{\mathbb{R}}$-invariant global section of $\Omega_w \wedge \overline{\Omega}_w$ that satisfies $\mu|_x = v \wedge \overline{v} =: v'$. We have

$$\begin{aligned} dn_*|_y &= n_*(v') \\ &= u'_*(\dot{w} a_w(n) \dot{w}^{-1})_* m'_*(v') \\ &= u'_*(\dot{w} a_w(n) \dot{w}^{-1})_*(u) \\ &= u'_*((\dot{w} a_w(n) \dot{w}^{-1})^{-2(\rho - w\rho)} v') \\ &= u'_*(a_w(n)^{-2w^{-1}(\rho - w\rho)} v') \\ &= a_w(n)^{2(\rho - w^{-1}\rho)} u'_*(v') \\ &= a_w(n)^{2(\rho - w^{-1}\rho)} \mu|_y \end{aligned}$$

From where we conclude

$$\mu = a_w(n)^{-2(\rho - w^{-1}\rho)}\, dn_*.$$

This implies the theorem. □

For any weight $\alpha \in \mathfrak{h}^*$, denote by H_α the coweight in $\mathfrak{h}$ associated to α. Let

$$\alpha_i := w_{\gamma_1} \dots w_{\gamma_{i-1}}(\gamma_i).$$

The set $\{\alpha_1, \dots, \alpha_l\}$ coincides with Φ_w^+. Let $\beta_i = -w^{-1}(\alpha_i)$. Lu proves in [Lu99]:

Theorem 4.3. *In terms of Lu's coordinates we have*

$$a_w(n) = \prod_{i=1}^{l} \exp\left(\log(1 + |z_i|^2) \frac{H_{\beta_i}}{\langle \beta_i, \beta_i \rangle}\right).$$

In addition

$$dn = \prod_{i=1}^{l} (1 + |z_i|^2)^{2\frac{\langle \rho, \beta_i \rangle}{\langle \beta_i, \beta_i \rangle} - 1} dz_i d\bar{z}_i$$

defines a bi-invariant Haar measure on N_w.

As a collorary of theorems 4.3 and 4.2, we get

Theorem 4.4. *In terms of Lu's coordinates the differential form*

$$\omega = (1+|z_1|^2)^{-2\frac{\langle\rho,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}-1} \dots (1+|z_l|^2)^{-2\frac{\langle\rho,\alpha_l\rangle}{\langle\alpha_l,\alpha_l\rangle}-1} dz_1 d\bar{z}_1 \dots dz_l d\bar{z}_l$$

is the pullback to N_w of the restriction to the Bruhat cell Σ_w of a $U_{\mathbb{R}}$-invariant $2l$-form on the flag variety.

CHAPTER 5

VERMA MODULES OF ANTIDOMINANT HIGHEST WEIGHT.

In this section we prove

Theorem 5.1. *Conjecture 1.1 holds true if* $I = \Gamma(X, \mathcal{I}(Q, \lambda, \mathbb{C}_x)$*, where* $Q = \{x\}$ *is a single point in the flag variety,* $\mathbb{C}_x$ *is the trivial bundle, and* $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ *is any dominant weight.*

Since Q is closed, we have $\mathcal{I} = \mathcal{I}(Q, \lambda, \mathcal{S}) = \mathcal{M}(Q, \lambda, \mathcal{S})$. Then, $I = \Gamma(X, \mathcal{I})$ is an irreducible $\mathcal{U}\mathfrak{g}_\lambda$-module. We remark if B is the Borel subgroup of G associated to x, then I coincides with the Verma module of highest weight $w_0(\lambda)$ in the block $\mathcal{O}_\lambda$ of the category $\mathcal{O}$ associated to B (see [BG99], section 4), where w_0 is the longest element in the Weyl group. This Verma module is irreducible because $w_0(\lambda)$ is an antidominant weight.

Fix a Borel B and without loss of generality assume $x = B^{\mathrm{op}} = w_0 B$. Then $x \in \Sigma_{w_0}$ (the open Bruhat cell centered at x) and the morphism $N \to \Sigma_{w_0}$ given by $n \mapsto n \cdot x$ is an isomorphism of complex algebraic varieties. For any λ, We can make sense of the $\mathfrak{g}$-equivariant bundle $\mathcal{O}_{\lambda-\rho}$ on Σ_{w_0} and it will be moreover N-equivariant. Let σ be an N-invariant section on Σ_{w_0} (this is unique up to scaling). Let $\langle \cdot, \cdot \rangle$ be the hermitian form on $\mathcal{O}_{\lambda-\rho}$ that is $\mathfrak{u}_{\mathbb{R}}$-invariant and satisfies $\langle \sigma, \sigma \rangle(x) = 1$.

Lemma 5.2. *We have a formula for* $\|\sigma\|^2 = \langle \sigma, \sigma \rangle$*:*

$$\|\sigma\|^2 (n \cdot x) = a_{w_0}^{2w_0(\lambda-\rho)}(n).$$

Notice the right hand side on this formula is a real number precisely because $\lambda \in \mathfrak{h}_{\mathbb{R}}$*.*

Proof. Let $w_0^{-1} n w_0 = u a_w(n) n_1$ be the Iwasawa decomposition of $w_0^{-1} n w_0$. Then

$$\begin{aligned} n &= (w_0 u w_0^{-1})(w_0 a_{w_0}(n) w_0^{-1})(w_0 n_1 w_0^{-1}) \\ &= u'(w_0 a_{w_0}(n) w_0^{-1}) n'. \end{aligned}$$

Where $u' \in U_{\mathbb{R}}$ and $n' \in N^{\mathrm{op}}$. Write $L_{\lambda-\rho}$ for the corresponding geometric line bundle associated to $\mathcal{O}_{\lambda-\rho}$. Let $\sigma(x) = v \in (L_{\lambda-\rho})_x$, we have

$$\begin{aligned}
\langle \sigma, \sigma \rangle (n \cdot x) &= \langle n_* v, n_* v \rangle \\
&= \langle (u')_*^{-1} n_* v, (u')_*^{-1} n_* v \rangle \\
&= \langle ((w_0 a_{w_0}(n) w_0^{-1}) n')_* v, ((w_0 a_{w_0}(n) w_0^{-1}) n')_* v \rangle \\
&= \langle (w_0 a_{w_0}(n) w_0^{-1})_* v, (w_0 a_{w_0}(n) w_0^{-1})_* v \rangle \\
&= \langle a_{w_0}^{w_0^{-1}(\lambda-\rho)}(n) v, a_{w_0}^{w_0^{-1}(\lambda-\rho)}(n) v \rangle \\
&= a_{w_0}^{2w_0^{-1}(\lambda-\rho)}(n)
\end{aligned}$$

Where we used that the action of n' on v is the identity because $\mathfrak{n}^-$ acts trivially on the fiber of $L_{\lambda-\rho}$ at x. Now the result follows from the fact that $w_0 = w_0^{-1}$. □

From (2.2) we can describe the twisted module I in terms of its untwisted version:

$$\begin{aligned}
I =& \Gamma(X, \mathcal{I}) \\
&= \Gamma(N, \mathcal{I}) \\
&= \Gamma(N, \mathcal{O}_{\lambda-\rho}) \otimes_{\Gamma(N, i^{-1}\mathcal{O}_N)} \Gamma(N, i_* \mathbb{C}_0) \\
&= \sigma \Gamma(N, i_* \mathbb{C}_0)
\end{aligned}$$

Denote by $I^0 := \Gamma(N, i_* \mathbb{C}_0)$. For elements $u \in I$, we write $u = \sigma u_0$, with $u_0 \in I^0$. The section σ can be understood as the twist that gives rise to the D_N^λ-action. Let P_0 be the polarization on I^0. From section 2.3, if $u, v \in I$ then the corresponding polarization P on $\mathcal{I}$ is given by

$$P(u, \overline{v}) = \|\sigma\|^2 P_0(u_0, \overline{v}_0).$$

From 4.2 and 5.2, we have

$$\begin{aligned}(u,v)_{\mathfrak{u}_{\mathbb{R}}} &= \int_X P(u,\overline{v})\,dm \\ &= \int_{\Sigma_w} P(u,\overline{v})\,dm \\ &= \int_N \|\sigma\|^2 P_0(u_0,\overline{v}_0) a_{w_0}^{-4\rho}(n)\,dn \\ &= \int_N P_0(u_0,\overline{v}_0) a_{w_0}^{2w_0\lambda-2\rho}(n)\,dn. \end{aligned} \tag{5.1}$$

5.1 Pushforward of differential operators

Our goal now is to use Lu's coordinates on N to rewrite the integral in (5.1). In order to do so, we need to understand how holomorphic differential operators pushforward under Lu's diffeomorphism. Before we go into the specific example of Lu's diffeomorphism, let's discuss a more general setting.

Let $\{z\} \overset{i}{\hookrightarrow} Y$ and $\{z'\} \overset{i'}{\hookrightarrow} Y'$ be smooth complex varieties. Assume there is a map $f : Y \to Y'$ that is a diffeomorphism at the level of smooth manifolds so that $f(z) = z'$. Remember that for a complex manifold Y, its complexification is given by $Y \times \overline{Y}$, and Y embeds on it via $\Delta_Y : Y \hookrightarrow Y \times \overline{Y}$, given by $\Delta_Y(y) = (y, \overline{y})$. If Y and Y' are replaced by small enough neighborhoods of $\{z\}$ and $\{z'\}$ respectively, we can assume the map f extends to a biholomorphism

$$f^{\mathbb{C}} : Y \times \overline{Y} \to W \subset Y' \times \overline{Y}'$$

so that $g^{\mathbb{C}}(z, \overline{z}) = (z', \overline{z}')$.

We have $\mathcal{D}_{Y\times\overline{Y}} = \mathcal{D}_Y \otimes \mathcal{D}_{\overline{Y}}$ and there is an isomorphism $f_*^{\mathbb{C}} : g_\cdot^{\mathbb{C}}(\mathcal{D}_Y \otimes \mathcal{D}_{\overline{Y}}) \to \mathcal{D}_W = \mathcal{O}_W \otimes_{\mathcal{O}_{Y'} \otimes \mathcal{O}_{\overline{Y}'}} \mathcal{D}_{Y'} \otimes \mathcal{D}_{\overline{Y}'}$ given by pushforward of differential operators. Denote by $\mathbb{C}_z$ and $\mathbb{C}_{z'}$ the trivial sheaves on z and z' respectively. Since $i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z}$ and $i'_*\mathbb{C}_{z'} \otimes \overline{i'_*\mathbb{C}_{z'}}$ are supported on $\{z\} \times \{\overline{z}\}$ and $\{z'\} \times \{\overline{z}'\}$ respectively, we also have an isomorphism

$$f_*^{\mathbb{C}} : f_\cdot^{\mathbb{C}}(i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z}) \to i'_*\mathbb{C}_{z'} \otimes \overline{i'_*\mathbb{C}_{z'}}$$

that exchanges the $f_\cdot^{\mathbb{C}}(\mathcal{D}_Y \otimes \mathcal{D}_{\overline{Y}})$-structure on the left with the $\mathcal{D}_W$-structure on the right. Let P be the polarization on $i_*\mathbb{C}_z$, which can be thought of as a function $P : i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z} \to \Delta_{\mathbb{R}}^Z$ (see the discussion leading to (2.5)). Let P' be the polarization on $i'_*\mathbb{C}_{z'}$. Then $f_*^{\mathbb{C}}$ interchanges these polarizations, and in particular

$$\int_Y P(\delta)\omega = \int_{Y'} P'(f_*^{\mathbb{C}}\delta)(f_*^{\mathbb{C}}\omega).$$

Denote by $i_1 : Y \to Y \times \overline{Y}$ the inclusion "in the first coordinate", meaning $i_1(y) = (y, \overline{z})$. Define similarly i_2, i'_1, and i'_2.

Lemma 5.3. *Let $f_1^{\mathbb{C}} = f^{\mathbb{C}} \circ i_1$ and $f_2^{\mathbb{C}} = f^{\mathbb{C}} \circ i_2$ be the restrictions of $f^{\mathbb{C}}$ of the first and second coordinates respectively. Assume the morphism $f^{\mathbb{C}}$ satisfies $f_1^{\mathbb{C}}(Y) \subset Y'$. Then it also satisfies $f_2^{\mathbb{C}}(\overline{Y}) \subset \overline{Y}'$. Moreover the natural isomorphisms*

$$f_{1*}^{\mathbb{C}} : f_{1\cdot}^{\mathbb{C}}(i_*\mathbb{C}_z) \to i'_*\mathbb{C}_{z'}$$
$$f_{2*}^{\mathbb{C}} : f_{2\cdot}^{\mathbb{C}}(\overline{i_*\mathbb{C}_z}) \to \overline{i'_*\mathbb{C}_{z'}}$$

satisfy $f_*^{\mathbb{C}} = f_{1*}^{\mathbb{C}} \otimes f_{2*}^{\mathbb{C}}$.

Proof. From the construction of $f^{\mathbb{C}}$, we have $f^{\mathbb{C}}(y_1, y_2) = \overline{f^{\mathbb{C}}(\overline{y}_2, \overline{y}_1)}$, where $\overline{(y'_1, \overline{y}'_2)} := (y'_2, \overline{y}'_1)$. Indeed, this identity is satisfied at all the real points $(y, \overline{y})$ because $f^{\mathbb{C}}$ extends f. Since $f^{\mathbb{C}}(Y \times \{\overline{z}\}) \subset Y' \times \{\overline{z}'\}$, then also

$$f^{\mathbb{C}}(\{z\} \times \overline{Y}) = \overline{f^{\mathbb{C}}(Y \times \{\overline{z}\})} \subset \overline{Y' \times \{\overline{z}'\}} = \{z\} \times \overline{Y}'.$$

Let $\mathcal{J}_Y$ the structure sheaf of $Y \times \{\overline{z}\}$ in $Y \times \overline{Y}$. We define similarly $\mathcal{J}_{Y'}$. Then $f_*^{\mathbb{C}}$ restricts to an isomorphism

$$f_*^{\mathbb{C}} : f_\cdot^{\mathbb{C}}(\ker(\mathcal{J}_Y, i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z})) \xrightarrow{\sim} \ker(\mathcal{J}_{Y'}, i'_*\mathbb{C}_{z'} \otimes \overline{i'_*\mathbb{C}_{z'}})$$

The normal bundle of Y in $Y \times \overline{Y}$ coincides with $\mathcal{O}_Y \otimes \mathcal{T}_{\overline{Y}}$. Hence it is globally trivializable and its top exterior power is the trivial bundle on Y. This means we have a canonical identification $i_1^!(i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z}) = \ker(\mathcal{J}_Y, i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z})$ (see section 2.1). We also have $i_{1*}(i_*\mathbb{C}_z) = i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z}$. By the equivalence of Kashiwara, this implies $i_1^!(i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z}) = i_*\mathbb{C}_z$. We have similar results substituting Y by Y', hence there is a commutative diagram

$$\begin{array}{ccc} i_*\mathbb{C}_z & \xrightarrow{\sim} & i'_*\mathbb{C}_{z'} \\ \big\downarrow & & \big\downarrow \\ i_*\mathbb{C}_z \otimes \overline{i_*\mathbb{C}_z} & \xrightarrow{\sim} & i'_*\mathbb{C}_{z'} \otimes \overline{i'_*\mathbb{C}_{z'}} \end{array}$$

substituting Y by $\overline{Y}$ yields a similar result, and we conclude $f_*^{\mathbb{C}} = f_{1*}^{\mathbb{C}} \otimes f_{2*}^{\mathbb{C}}$. □

5.2 The case of Lu's diffeomorphism

Now we prove Lu's diffeomorphism fits into the assumptions of lemma 5.3. Let $G = U_{\mathbb{R}}AN$ be the Iwasawa decomposition. For any real group H, denote by $H^{\mathbb{C}}$ its complexification. If we regard G as a real group, then $G^{\mathbb{C}} = G \times G$. The group G is embedded in $G \times G$ as a real form via $g \mapsto (g, g^{\zeta})$, where $g \mapsto g^{\zeta}$ is the involution of G whose fixed points are $U_{\mathbb{R}}$. The groups $U_{\mathbb{R}}^{\mathbb{C}}$, $A^{\mathbb{C}}$ and $N^{\mathbb{C}}$ are naturally subgroups of $G \times G$, and they're given by $U_{\mathbb{R}}^{\mathbb{C}} = \{(g, g), g \in G\}$, $A^{\mathbb{C}} = \{(h, h^{-1}), h \in H\}$, and $N^{\mathbb{C}} = N \times N^{\mathrm{op}}$.

There exists a dense open set $\Lambda \subset G \times G$ containing the identity element (see [SW02]) that can be decomposed as $\Lambda = U_{\mathbb{R}}^{\mathbb{C}}A^{\mathbb{C}}N^{\mathbb{C}}$. This decomposition extends the Iwasawa decomposition of G. Hence the G-action on $U_{\mathbb{R}}$ can be extended to a holomorphic map $(g, u) \mapsto g \cdot u \in U_{\mathbb{R}}^{\mathbb{C}}$, which is defined for $g \in G^{\mathbb{C}}$, $u \in U_{\mathbb{R}}^{\mathbb{C}}$, and $gu \in \Lambda$.

Lemma 5.4. *Let g_1, g_2 be elements in $G^{\mathbb{C}}$, and $u \in U_{\mathbb{R}}^{\mathbb{C}}$ so that $g_2 u \in \Lambda$ and $g_1 g_2 u \in \Lambda$, then $g_1(g_2 \cdot u) \in \Lambda$ and $g_1 \cdot (g_2 \cdot u) = (g_1 g_2) \cdot u$.*

Proof. By assumption

$$\begin{aligned} g_1(g_2 u) &= g_1(g_2 \cdot u) a_1 n_1, \quad a_1 \in A^{\mathbb{C}}, n_1 \in N^{\mathbb{C}} \\ g_1 g_2 u &= ((g_1 g_2) \cdot u) a_2 n_2, \quad a_2 \in A^{\mathbb{C}}, n_2 \in N^{\mathbb{C}} \end{aligned}$$

Therefore

$$\begin{aligned} g_1(g_2 \cdot u) &= ((g_1 g_2) \cdot u) a_2 n_2 n_1^{-1} a_1^{-1} \\ &= ((g_1 g_2) \cdot u) a_2 a_1^{-1} a_1 n_2 n_1^{-1} a_1^{-1} \\ &= ((g_1 g_2) \cdot u) a_3 n_3 \end{aligned}$$

where $a_3 = a_2 a_1^{-1} \in A^{\mathbb{C}}$, and $n_3 = a_1 n_2 n_1^{-1} a_1^{-1}$ is contained in $N^{\mathbb{C}}$. □

Let $w \in W$ be an element in the Weyl group of length l so that $w = w_{\gamma_1} \dots w_{\gamma_l}$ is its decomposition into simple reflections. Let $F_w : N_{\gamma_1} \times \cdots \times N_{\gamma_l} \to N_w$ be Lu's diffeomorphism. This map is not holomorphic, but it is a real-analytic diffeomorphism. It can be complexified and there exist open sets $e \in U_i \subset N_{\gamma_i}$ so that there is an open embedding holomorphism

$$\begin{aligned} &F_w^{\mathbb{C}} : (U_1 \times U_1^{\mathrm{op}}) \times \cdots \times (U_l \times U_l^{\mathrm{op}}) \to N_w \times N_w^{\mathrm{op}}, \\ &(x_1, y_1, \dots, x_n, y_n) = (x, y) \mapsto F_w^{\mathbb{C}}(x, y) \end{aligned}$$

that extends F_w. The open sets U_i can be taken small enough to ensure for each element $u_i \in U_i \times U_i^{\mathrm{op}}$, $u_i(\dot{\gamma}_i, \dot{\gamma}_i) \in \Lambda$, and that for each element u in the image of $F_w^{\mathbb{C}}$, the element $u(\dot{w}, \dot{w})$ also lands in Λ. Then the map $F_w^{\mathbb{C}}$ is characterized by the property

$$((x_1, y_1) \cdot (\dot{\gamma}_1, \dot{\gamma}_1)) \dots ((x_l, y_l) \cdot (\dot{\gamma}_l, \dot{\gamma}_l)) = F_w^{\mathbb{C}}(x, y) \cdot (\dot{w}, \dot{w}).$$

Theorem 5.5. *We have* $F_w^{\mathbb{C}}((U_1 \times \{e\}) \times \cdots \times (U_l \times \{e\})) \subset N_w \times \{e\}$.

Proof. Let $n \in N_w$, and let $u \in U_{\mathbb{R}}$ be an element such that $u^{-1}nu \in N_w^{\text{op}}$. We have

$$(n,e)(u,u) = (nu,nu)(e,e)(e,u^{-1}n^{-1}u).$$

which implies $(n,e)\cdot(u,u) = (nu,nu)$.

Notice that, $\dot{w}'^{-1}n\dot{w}' \in N_{w'}^{\text{op}}$ for any $w' \in W$ and $n \in N_{w'}$. Let $z = (x_1, e, \ldots, x_l, e)$. Then

$$\begin{aligned} F_w^{\mathbb{C}}(z)\cdot(\dot{w},\dot{w}) &= ((x_1,e)\cdot(\dot{\gamma}_1,\dot{\gamma}_1))\ldots((x_l,e)\cdot(\dot{\gamma}_l,\dot{\gamma}_l)) \\ &= (x_1\dot{\gamma}_1, x_1\dot{\gamma}_1)\ldots(x_l\dot{\gamma}_l, x_l\dot{\gamma}_l) \\ &= (x_1\dot{\gamma}_1\ldots x_l\dot{\gamma}_l, x_1\dot{\gamma}_1\ldots x_l\dot{\gamma}_l) \\ &= (n\dot{w}, n\dot{w}) \\ &= (n,e)\cdot(\dot{w},\dot{w}) \end{aligned}$$

where we used that $x_1\dot{\gamma}_1\ldots x_l\dot{\gamma}_l = n\dot{w}$ for some $n \in N_w$.

We claim that $F_w^{\mathbb{C}}(z) = (n,e)$. In order to prove that we just need to check the map $U' \subset (N_w \times N_w^{\text{op}}) \to U_{\mathbb{R}}^{\mathbb{C}}$ given by $(n,m) \mapsto (n,m)\cdot(\dot{w},\dot{w})$ is injective. Here U' denotes the domain where the action is well defined.

Assume $(n,m)\cdot(\dot{w},\dot{w}) = (n',m')\cdot(\dot{w},\dot{w})$. Then, from lemma 5.4

$$\begin{aligned} &((n')^{-1}n,(m')^{-1}m)\cdot(\dot{w},\dot{w}) = (\dot{w},\dot{w}) \\ &\Rightarrow ((n')^{-1}n,(m')^{-1}m)(\dot{w},\dot{w}) = (\dot{w}hn'', \dot{w}h^{-1}m''),\ h \in H, n'' \in N_w, m'' \in N_w^{\text{op}} \\ &\Rightarrow \dot{w}^{-1}(n')^{-1}n\dot{w} \in HN_w, \quad \text{and} \quad \dot{w}^{-1}(m')^{-1}m\dot{w} \in HN_w^{\text{op}}. \end{aligned}$$

Since we have $\dot{w}^{-1}(n')^{-1}n\dot{w} \in N_w^{\text{op}}$, and $\dot{w}^{-1}(m')^{-1}m\dot{w} \in N_w$, we conclude $(n')^{-1}n =$

$(m')^{-1}m = e$ which implies injectivity. We conclude $F^{\mathbb{C}}_{w}(z) = (n, e) \in N_w \times \{e\}$. $\square$

Write $Y = (U_1 \times \{e\}) \times \cdots \times (U_l \times \{e\})$ and $Y' = N_w \times \{0\}$. Then $\overline{Y} = (\{e\} \times U_1^{\text{op}}) \times \cdots \times (\{e\} \times U_l^{\text{op}})$ and $\overline{Y}' = \{0\} \times N_w^{\text{op}}$. Endow the set Y with coordinates as in Chapter 3 and denote by $i : \{0\} \hookrightarrow Y$ and $i' : \{e\} \hookrightarrow N_w$. The element 0 is taking the role of z in the previous section and e the one of z'. Theorem 5.5 and lemma 5.3 imply that there exists an isomorphism

$$F^{\mathbb{C}}_{w1*} : F^{\mathbb{C}}_{w1.}(i_*\mathbb{C}_0) \xrightarrow{\sim} (i'_*\mathbb{C}_e)$$

that interchanges the polarizations on both sides. We write $\Gamma(Y, i_*\mathbb{C}_0) = \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_l}]$. Taking global sections we obtain

Theorem 5.6. *Pushforward of differential operators under Lu's diffeomorphism induces an isomorphism of filtered vector spaces*

$$F_{w*} : \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_l}] \to I_w^0 := \Gamma(N_w, i'_*\mathbb{C}_e).$$

that respects the polarization on both sides. Meaning, if P_1 and P_2 are the polarizations on the space on the left and right respectively, then

$$P_1(\alpha, \overline{\beta}) = P_2(F_{w*}\alpha, \overline{F_{w*}\beta}).$$

5.3 Proof of theorem 5.1

We now let $w = w_0$ be the longest element in the Weyl group, so that $N = N_w$. Let n be the dimension of N. Using Lu's coordinates, and theorem 4.3, we obtain

$$\begin{aligned} a_{w_0}^{2w_0\lambda-2\rho}(n)\,dn &= \prod_{i=1}^{n}(1+|z_i|^2)^{2\frac{\langle w_0\lambda-\rho+\rho,\beta_i\rangle}{\langle\beta_i,\beta_i\rangle}-1}\,dz_i\,d\overline{z}_i \\ &= \prod_{i=1}^{n}(1+|z_i|^2)^{-2\frac{\langle\lambda,-w_0\beta_i\rangle}{\langle -w_0\beta_i,-w_0\beta_i\rangle}-1}\,dz_i\,d\overline{z}_i \\ &= \prod_{i=1}^{n}(1+|z_i|^2)^{-2\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}-1}\,dz_i\,d\overline{z}_i \end{aligned} \tag{5.2}$$

Write (5.2) by $f(z,\overline{z})\,dzd\overline{z}$. Let $\alpha,\beta\in\mathbb{C}[\partial_{z_1},\ldots,\partial_{z_n}]$. Let P_1 and P_2 be the polarizations on $i_*\mathbb{C}_0$ and $i'_*\mathbb{C}_e$ respectively. We conclude from (5.1) and theorem 5.6:

$$\begin{aligned} (\sigma F_{w*}\alpha,\sigma F_{w*}\beta)_{\mathfrak{u}_\mathbb{R}} &= \int_N P_2(F_{w*}\alpha,\overline{F_{w*}\beta})a_{w_0}^{2w_0\lambda-2\rho}(n)\,dn \\ &= \int_{\mathbb{C}^n} P_1(\alpha,\overline{\beta})f(z,\overline{z})\,dzd\overline{z} \\ &= (\alpha\overline{\beta})^t(f)|_{z=\overline{z}=0} =: B^\lambda(\alpha,\beta). \end{aligned} \tag{5.3}$$

Here $(\alpha\overline{\beta})^t$ denotes the adjoint of $\alpha\overline{\beta}$. At the level of monomials this adjoint is given by the formula

$$(\partial^\alpha)^t = (-1)^{\deg(\alpha)}\partial^\alpha.$$

From (5.2) we see that if $\alpha = \partial_{z_1}^{d_1}\ldots\partial_{z_n}^{d_n}$, $\beta = \partial_{z_1}^{c_1}\ldots\partial_{z_n}^{c_n}$, then $B^\lambda(\alpha,\beta) = 0$ whenever $(d_1,\ldots,d_n)\neq(c_1,\ldots,c_n)$. Hence, the monomials in $\mathbb{C}[\partial_{z_1},\ldots,\partial_{z_n}]$ provide an orthogonal basis with respect to the form B^λ. Since F_{w*} respects filtrations, theorem 5.6 ensures that in order to prove theorem 5.1, it is enough to prove B^λ has signature $(-1)^p$ on the homogeneous monomials of degree p. That is, we only to prove the following lemma

Lemma 5.7. *Let $\alpha = \partial_{z_1}^{d_1} \dots \partial_{z_n}^{d_n}$ with $d_1 + \cdots + d_n = p$. Then $(-1)^p B^\lambda(\alpha, \alpha) > 0$.*

Proof. We have

$$B^\lambda(\alpha, \alpha) = \prod_{i=1}^{n} \partial_{z_i}^{d_i} \partial_{\bar{z}_i}^{d_i} \left((1 + |z_i|^2)^{-2\frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} - 1} \right) |_{z_i = 0}$$

Because λ is a dominant weight, $\langle \lambda, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$ is a non-negative number. It is easy to check from the Taylor expansion that $\partial_z^d \partial_{\bar{z}}^d \left((1 + |z|^2)^{-t} \right) |_{z=0}$ has the same sign as $(-1)^d$ whenever t is a positive number. The lemma follows. □

Remark 5.8. Notice from formula (5.1), the $\mathfrak{u}_\mathbb{R}$-invariant form on I is encoded on the Taylor coefficients around the identity of the form $a_{w_0}^{2(w_0(\lambda) - \rho)}(n)\, dn$. The main importance of Lu's coordinates in this section is that it allowed us to provide an explicit orthogonal basis for I.

5.4 Example

Let's take $G = SL(3, \mathbb{C})$, and choose B to be the Borel of upper triangular matrices. Similarly, $U_\mathbb{R} = SU(3)$ is the set of hermitian matrices. In this case, N is endowed with holomorphic coordinates by

$$(a, b, c) \mapsto \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We choose as simple roots $\dot{\gamma}_1 = \dot{\gamma}_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\dot{\gamma}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$. Then we have

$$N_{\gamma_1} = \left\{ \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, N_{\gamma_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right\}, N_{\gamma_3} = \left\{ \begin{pmatrix} 1 & z_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Lu's change of coordinates is given in this case by (see [Lu99], example 2.9):

$$a = z_1, \quad b = \frac{\sqrt{1+|z_2|^2}z_3 - i\bar{z}_1 z_2}{\sqrt{1+|z_1|^2}}, \quad c = \frac{\sqrt{1+|z_2|^2}z_1 z_3 + iz_2}{\sqrt{1+|z_1|^2}}$$

Endow the complexifications with coordinates by

$$N^{\mathbb{C}} = \left\{ \left(\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2 b_2 - c_2 & -b_2 & 1 \end{pmatrix} \right) \right\},$$

$$N^{\mathbb{C}}_{\gamma_1} = \left\{ \left(\begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -y_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right\},$$

$$N^{\mathbb{C}}_{\gamma_2} = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -y_2 & 1 \end{pmatrix} \right) \right\},$$

$$N^{\mathbb{C}}_{\gamma_3} = \left\{ \left(\begin{pmatrix} 1 & x_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -y_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \right\}.$$

The conjugation map $N \to \overline{N}^{\text{op}}$ is given by $n \mapsto \overline{n}^{-T}$. Here, the second set of coordinates for $N^{\mathbb{C}}$, was chosen to ensure that this map is given in coordinates by $(a_1, b_1, c_1) \mapsto (\overline{a}_2, \overline{b}_2, \overline{c}_2)$. Therefore, using these coordinates, the complexification of Lu's diffeomorphism is given by

$$\begin{aligned} a_1 &= x_1, \quad b_1 = \frac{\sqrt{1+x_2y_2}x_3 - iy_1x_2}{\sqrt{1+x_1y_1}}, \quad c_1 = \frac{\sqrt{1+x_2y_2}x_1x_3 + ix_2}{\sqrt{1+x_1y_1}} \\ a_2 &= y_1, \quad b_2 = \frac{\sqrt{1+x_2y_2}y_3 + ix_1y_2}{\sqrt{1+x_1y_1}}, \quad c_2 = \frac{\sqrt{1+x_2y_2}y_1y_3 - iy_2}{\sqrt{1+x_1y_1}} \end{aligned}$$

This morphism is defined only in a neighborhood of the real embedding of $N_{\gamma_1} \times N_{\gamma_2} \times N_{\gamma_3}$ in its complexification. That is the set

$$S := \{y_1 = \overline{x}_1, y_2 = \overline{x}_2, y_3 = \overline{x}_3\}.$$

This is the morphism we denoted by $f^{\mathbb{C}}$ in section 5.1., where in this case $f = F_w$. Notice it induces the isomorphism $F_{w*}^{\mathbb{C}}$

$$\mathbb{C}[\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}] \to \mathbb{C}[\partial_{a_1}, \partial_{b_1}, \partial_{c_1}, \partial_{a_2}, \partial_{b_2}, \partial_{c_2}]$$

that interchanges the $D_{x,y}$-action on the left with the $D_{a,b,c}$-action on the right, where $D_{x,y}$ and $D_{a,b,c}$ denote the corresponding rings of differential operators.

The statement of theorem 5.5 is that, when $y_1 = y_2 = y_3 = 0$, we obtain $a_2 = b_2 = c_2 = 0$. When we restrict to the subvarieties cut out by those equations, we obtain an isomorphism

$$F_{w1}^{\mathbb{C}} : N_{\gamma_1} \times N_{\gamma_2} \times N_{\gamma_3} \to N$$

$$(x_1, x_2, x_3) \mapsto (a_1, b_1, c_1) = (x_1, x_3, ix_2 + x_1 x_3)$$

which is the morphism we denoted by $f_1^{\mathbb{C}}$ in section 5.1. This implies there exists an isomorphism of filtered vector spaces

$$F_{w1*}^{\mathbb{C}} : \mathbb{C}[\partial_{x_1}, \partial_{x_2}, \partial_{x_3}] \xrightarrow{\sim} \mathbb{C}[\partial_{a_1}, \partial_{b_1}, \partial_{c_1}]$$

that exchanges the $\mathbb{C}\langle x_1, x_2, x_3, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\rangle$-action on the left with the $\mathbb{C}\langle a_1, b_1, c_1, \partial_{a_1}, \partial_{b_1}, \partial_{c_1}\rangle$-action on the right that comes from pushforward of differential operators. Let's describe this map in lower degree terms. The pushforward of differential operators gives:

$$\partial_{x_1} = \partial_{a_1} + b\partial_{c_1} \quad \partial_{x_2} = i\partial_{c_1} \quad \partial_{x_3} = \partial_{b_1} + a\partial_{c_1}$$

From where the map $F^{\mathbb{C}}_{w1*}$ has the following form in degrees lower than 3:

$$\begin{array}{ccc}
\partial_{x_1} \mapsto \partial_{a_1} & \partial_{x_2} \mapsto i\partial_{c_1} & \partial_{x_3} \mapsto \partial_{b_1} \\
\partial^2_{x_1} \mapsto \partial^2_{a_1} & \partial^2_{x_2} \mapsto -\partial^2_{c_1} & \partial^2_{x_1} \mapsto \partial^2_{b_1} \\
\partial_{x_1}\partial_{x_2} \mapsto i\partial_{a_1}\partial_{c_1} & \partial_{x_1}\partial_{x_3} \mapsto \partial_{a_1}\partial_{b_1} - \partial_{c_1} & \partial_{x_2}\partial_{x_3} \mapsto i\partial_{b_1}\partial_{c_1}
\end{array}$$

The elements on the left are orthogonal with respect to the form B^{λ}, and so are the elements on the right. The commutative diagram in the proof of lemma 5.3 corresponds to the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[\partial_{x_1}, \partial_{x_2}, \partial_{x_3}] & \xrightarrow{F^{\mathbb{C}}_{w1*}} & \mathbb{C}[\partial_{a_1}, \partial_{b_1}, \partial_{c_1}] \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}[\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}] & \xrightarrow{F^{\mathbb{C}}_{w*}} & \mathbb{C}[\partial_{a_1}, \partial_{b_1}, \partial_{c_1}, \partial_{a_2}, \partial_{b_2}, \partial_{c_2}]
\end{array}$$

From where we see, as explained in section 5.3, that this provides an algorithm to find an orthogonal basis of $I = \sigma\mathbb{C}[\partial_a, \partial_b, \partial_c]$ with respect to the form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$.

CHAPTER 6

DISCRETE SERIES REPRESENTATIONS

In this chapter we use the notation from chapter 3. Assume from now on that Q is a closed K-orbit in X and that K has the same rank as G. In this case there is a discrete lattice of weights $\lambda \in \mathfrak{h}^*$ so that the line bundle $\mathcal{O}_{\lambda-\rho}$ can be defined around Q (see the discussion at the beginning of chapter 3). Since Q is the flag variety for K, it is simply connected and hence its only irreducible vector bundle with a flat connection is the trivial one $\mathcal{O}_Q$. The goal of this section is to provide a geometric proof of

Theorem 6.1. *Conjecture 1.1 holds true if $I = \Gamma(X, \mathcal{I}(Q, \lambda, \mathcal{O}_Q))$, where Q is a closed K-orbit embedded in the flag variety, and $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is a dominant weight so that $\mathcal{O}_{\lambda-\rho}$ makes sense in an open set of X containing Q.*

When K is the complexification of a maximal compact $K_{\mathbb{R}} \subset G_{\mathbb{R}}$, the correponding modules $I = \Gamma(X, \mathcal{I}(Q, \lambda, \mathcal{O}_Q))$ are the so-called (infinitesimal) *discrete series representaions* of $G_{\mathbb{R}}$.

6.1 Restricted Lu's coordinates

Let $x \in Q$ be any point. Unlike section 3, we assume here B^{op} is the Borel that fixes x. As we explained in section 3.2, we have B^{op}_K is a Borel subgroup of K. Therefore we have a natural isomorphism between the universal Cartan for G and the universal Cartan for K which yields a natural embedding of the roots for $\mathfrak{k}$ into the roots for $\mathfrak{g}$. We call the roots associated to $\mathfrak{k}$ *compact*, and the other *non-compact*. Denote by $\Lambda_\Phi \subset \mathfrak{h}^*$ the root lattice of G. Let $\Lambda^c_\Phi \subset \Lambda_\Phi$ be the elements in the root lattice for K, and $\Lambda^{nc}_\Phi = \Lambda_\Phi \backslash \Lambda^c_\Phi$. We call these the lattices of compact and non-compact roots respectively. It turns out that the sum of two compact or two non-compact elements is compact, whereas the sum of a compact and a non-compact element is non-compact.

The Borel B fixes a choice of simple roots. Let w be the longest element in the Weyl group with respect to this choice, and assume $w = w_{\gamma_1} \dots w_{\gamma_n}$, where the γ_i are simple roots and $n = \dim N^+$. Just like in section 4, write $\alpha_i = w_{\gamma_1} \dots w_{\gamma_{i-1}}(\gamma_i)$ and $\beta_i = -w^{-1}(\alpha_i)$. Both sets $\{\alpha_i, 1 \leq i \leq n\}$ and $\{\beta_i, 1 \leq i \leq n\}$ coincide with the set of all positive roots. If $\dim Q = m$, there are indexes $i_1, \dots, i_m$ so that $\{\beta_{i_j}, 1 \leq j \leq m\}$ is the set of positive compact roots.

The goal of this section is to prove Lu's coordinates can be restricted to an open set of Q. By identifying N with Σ_w, we can think of $Q' = Q \cap \Sigma_w$ as a closed subset of N. Define

$$K_{\mathbb{R}} := K \cap U_{\mathbb{R}}.$$

This is a compact real form for K. We have the following

Lemma 6.2. *For any $z \in N$, we have $z \in Q'$ if and only if $(z \cdot \dot{w})\dot{w}^{-1} \in K_{\mathbb{R}}$.*

Proof. For any $z \in N$, we have the Iwasawa decomposition

$$\begin{aligned} z\dot{w} &= (z \cdot \dot{w})an, \quad n \in N \\ \Rightarrow z &= (z \cdot \dot{w})\dot{w}^{-1}\dot{w}an\dot{w}^{-1} \\ \Rightarrow z &= (z \cdot \dot{w})\dot{w}^{-1}a'n', \quad a' \in A, n' \in N^{\text{op}}. \end{aligned} \tag{6.1}$$

From this we conclude that if $(z \cdot \dot{w})\dot{w}^{-1} \in K_{\mathbb{R}}$, then $zB^{\text{op}} \in K_{\mathbb{R}}B^{\text{op}}x = Q$, which means $z \in Q'$. Conversely, if $zB^{\text{op}} \in Q$, then (6.1) implies $(z \cdot \dot{w})\dot{w}^{-1} \in K_{\mathbb{R}}B^{\text{op}}$. Since $(z \cdot \dot{w})\dot{w}^{-1} \in U_{\mathbb{R}}$, and $U_{\mathbb{R}} \cap B^{\text{op}} \subset K_{\mathbb{R}}$, this implies $(z \cdot \dot{w})\dot{w}^{-1} \in K_{\mathbb{R}}$.

□

Now we prove that Lu's coordinates restrict to local coordinates for Q'

Theorem 6.3. *Let $F : N_{\gamma_1} \times \cdots \times N_{\gamma_n} \to N_w$ be the diffeomorphism from 4.1. Then F maps $N_{\gamma_{i_1}} \times \cdots \times N_{\gamma_{i_m}}$ diffeomorphically to Q'.*

Proof. Write $Z = N_{\gamma_{i_1}} \times \cdots \times N_{\gamma_{i_m}}$. We first prove the image of Z is contained in Q'. From lemma 6.2, all we need to prove is $(F(z) \cdot \dot{w})\dot{w}^{-1} \in K_{\mathbb{R}}$ for any $z \in Z$. Write $\dot{w}_i = \dot{\gamma}_1 \ldots \dot{\gamma}_i$. From theorem 4.1, we know that for some $n_i \in N_{\gamma_i}$

$$\begin{aligned}
(F(z) \cdot \dot{w})\dot{w}^{-1} &= \dot{\gamma}_1 \ldots (n_{i_1} \cdot \dot{\gamma}_{i_1}) \ldots (n_{i_m} \cdot \dot{\gamma}_{i_m})\dot{\gamma}_{i_m+1} \ldots \dot{\gamma}_n \dot{w}^{-1} \\
&= \dot{\gamma}_1 \ldots (n_{i_1} \cdot \dot{\gamma}_{i_1}) \ldots (n_{i_m} \cdot \dot{\gamma}_{i_m})\dot{\gamma}_{i_m}^{-1} \dot{w}_{i_m-1}^{-1} \\
&= \ldots \\
&= (\dot{w}_{i_1-1}(n_{i_1} \cdot \dot{\gamma}_{i_1})\dot{\gamma}_{i_1}^{-1}\dot{w}_{i_1-1}^{-1}) \ldots (\dot{w}_{i_m-1}(n_{i_m} \cdot \dot{\gamma}_{i_m})\dot{\gamma}_{i_m}^{-1}\dot{w}_{i_m-1}^{-1})
\end{aligned}$$

Hence, it is enough to prove that $\dot{w}_{i_l-1}(n_{i_l} \cdot \dot{\gamma}_{i_l})\dot{\gamma}_{i_l}^{-1}\dot{w}_{i_l-1}^{-1} \in K_{\mathbb{R}}$ for every $1 \leq l \leq m$.

Let G_α be the copy of $\mathrm{SL}(2, \mathbb{C})$ inside of G which arises from the morphism Ψ_α from section 2. Let $U_\alpha = G_\alpha \cap U$. From construction $(n_{i_l} \cdot \dot{\gamma}_{i_l})\dot{\gamma}_{i_l}^{-1} \in U_{\gamma_{i_l}}$. Conjugation by w_{i_l-1} gives an isomorphism from $G_{\gamma_{i_l}}$ to $G_{\beta_{i_l}}$ that sends $U_{\gamma_{i_l}}$ to $U_{\beta_{i_l}} \subset K_{\mathbb{R}}$. Therefore $\dot{w}_{i_l-1}(n_{i_l} \cdot \dot{\gamma}_{i_l})\dot{\gamma}_{i_l}^{-1}\dot{w}_{i_l-1}^{-1} \in K_{\mathbb{R}}$.

Now we prove that $F(Z) = Q'$. Since Q is a K-orbit, it is affinely embedded, and since Σ_w is an open affine subset of X, then $Q' = Q \cap \Sigma_w$ is an affine open subset of Q, in particular it is connected. The sets Z and Q' have the same dimension and the morphism F is a diffeomorphism from theorem 4.1, so $F(Z)$ must be an open set of Q'. Since Z is a closed subset of $N_{\gamma_1} \times \cdots \times N_{\gamma_n}$, $F(Z)$ is closed in N, in particular it is a closed subset of Q'. Since Q' is connected, we conclude $Q' = F(Z)$. ∎

6.2 Description of the polarized module

Recall we write $\mathcal{I} = \mathcal{I}(Q, \lambda, \mathcal{O}_Q)$ and $I = \Gamma(X, \mathcal{I})$. The fact that $\mathcal{O}_{\lambda-\rho}$ implies the weight $\lambda - \rho$ is a compact element in the root lattice. From (2.2) we have

$$\mathcal{I} = \mathcal{O}_{\lambda-\rho} \otimes i_* \mathcal{O}_Q.$$

Like in section 5, denote by σ an N-invariant section of $\mathcal{O}_{\lambda-\rho}(N)$. Then there is an inclusion

$$I \hookrightarrow \sigma(i_* \mathcal{O}_Q)(N).$$

Denote by S the set of indexes i such that α_i is a non-compact root. Endow N with Lu's coordinates. Theorem 6.3 implies $Q' = N \cap Q$ is cut out by the equations

$$Q' = \{z_i = \overline{z}_i = 0, i \in S\}.$$

Denote by $X(Q'_{\mathbb{R}})$ the algebra of complex-valued smooth functions on Q'. Also write $M := X(Q')[\partial_{z_i}, \partial_{\bar{z}_i}]$, where $i \in S$. From theorems 4.1 and 6.3, we obtain an inclusion $(i_* \mathcal{O}_Q)(N) \hookrightarrow M$. Therefore there is an injection

$$\iota : I \hookrightarrow \sigma M.$$

The space on the right is naturally filtered by degree of differential operators in the normal directions, and ι respects the filtrations on both sides. Moreover, M admits a polarization P' according to the construction of section 2.3. Let $u_0, v_0 \in M$ and $u = \sigma u_0, v = \sigma v_0$. Then

$$P(u, \overline{v}) = \|\sigma\|^2 P'(u_0, v_0)$$

The group H acts on N by conjugation (equivalently, on Σ_w). We have

Proposition 6.4. *The action of H on N is given in Lu's coordinates by*

$$t \cdot (z_1, \bar{z}_1 \ldots, z_n, \bar{z}_n) = (e^{\alpha_1}(t)z_1, e^{-\alpha_1}(t)z_1, \ldots, e^{\alpha_n}(t)z_n, e^{-\alpha_n}(t)z_n).$$

Proof. This is proposition 2.11 in [Lu99]. □

In particular H acts on $M\sigma$ in a semisimple way, and ι is H-equivariant. Theorem 6.4 implies the eigenvalues of the H-action on $X(Q')$ belong to Λ^c, and it acts on each $\partial_{z_i}, \partial_{\bar{z}_i}$ via weights in Λ^{nc}. Denote by M^i the subspace of M of homogeneous elements in degree i, and define

$$M^+ := \bigoplus_{i \text{ even}} M^i, \;\; M^- := \bigoplus_{i \text{ odd}} M^i. \tag{6.2}$$

Then M^+ and M^- are the eigenspaces where H acts by compact and non-compact weights in the root lattice respectively. We define similarly $I^+ := M^+\sigma \cap I$ and $I^- = M^-\sigma \cap I$. Since $\lambda - \rho$ is a compact weight, I^+ and I^- are respectively the spaces where H acts by compact and non-compact weights. We have $I = I^+ \oplus I^-$.

In chapter 3 we proved that the spaces I^+ and I^- were orthogonal to each other assuming there existed also a $\mathfrak{g}_{\mathbb{R}}$-invariant form. This is true in general:

Lemma 6.5. *Under the bilinear form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$, the spaces I^+ and I^- are orthogonal.*

Proof. Let $u, v \in I$ be two elements so that H acts on them with different elements in the root lattice. That is $h \cdot u = h^\alpha u$ and $h \cdot v = h^\beta v$, for all $h \in H$ and $\alpha \neq \beta$. The form $(\cdot,\cdot)_{\mathfrak{u}_{\mathbb{R}}}$ is K-invariant (because it is $\mathfrak{u}_{\mathbb{R}}$-invariant, in particular $\mathfrak{k}_{\mathbb{R}}$-invariant, and since the K-action is admissible, it is K-invariant), in particular H-invariant, therefore

$$(u, v)_{\mathfrak{u}_{\mathbb{R}}} = (h \cdot u, h \cdot v)_{\mathfrak{u}_{\mathbb{R}}} = (h^\alpha u, h^\beta v)_{\mathfrak{u}_{\mathbb{R}}} = h^{\alpha-\beta}(u, v)_{\mathfrak{u}_{\mathbb{R}}}.$$

From where $(u, v)_{\mathfrak{u}_{\mathbb{R}}} = 0$. All of the possible H-eigenspaces on I^+ and I^- have different weights, and this proves the lemma. $\square$

Each space F_pI is fixed under the H-action. Therefore $F_pI = F_pI^+ \oplus^\perp F_pI^-$. Moreover, from (6.2) we see that $F_pI^+ = F_{p-1}I^+$ if p is odd, and $F_pI^- = F_{p-1}I^-$ if p is even. Therefore

$$F^pI \cap (F^{p-1}I)^\perp \subset I^+, \text{ if } p \text{ is even,}$$
$$F^pI \cap (F^{p-1}I)^\perp \subset I^-, \text{ if } p \text{ is odd.}$$

We conclude

Theorem 6.6. *Theorem 6.1 is true if and only if the bilinear form $(\cdot,\cdot)_{\mathfrak{u}_\mathbb{R}}$ is positive definite when restricted to I^+, and negative definite when restricted to I^-.*

6.3 Proof of theorem 6.1

Just, like in section 4, we have in Lu's coordinates

$$\|\sigma\|^2\, dm = \|\sigma\|^2 a_{w_0}^{-4\rho}(n)\, dn = a_{w_0}^{2w_0\lambda-2\rho}(n)\, dn = \prod_{i=1}^{n}(1+|z_i|^2)^{-2\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}-1}\, dz_i\, d\overline{z}_i.$$

Write

$$h_i = (1+|z_i|^2)^{-2\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}-1},$$

and $h_c = \prod_{i \notin S} h_i$, $h_{nc} = \prod_{i \in S} h_i$. Let $u = \sigma(\sum_i f_i \partial^{\varepsilon_i})$, be an element of I with $f_i \in X(Q')$ and $\partial_i^{\varepsilon_i} \in \mathbb{C}[\partial_i, i \in S]$. Then

$$\begin{aligned}
(u,u)_{\mathfrak{u}_\mathbb{R}} &= \int_X P(u, \overline{u})\, dm \\
&= \int_N P(u, \overline{u}) a_{w_0}^{-4\rho}(n)\, dn \\
&= \int_{\mathbb{C}^n} \left(\sum_{i,j} f_i \overline{f}_j \partial^{\varepsilon_i} \overline{\partial}^{\varepsilon_j}(\delta) \right) h_c h_{nc}\, dz\, d\overline{z} \\
&= \int_{\mathbb{C}^n} \delta h_c \sum_{i,j} f_i \overline{f}_j (\partial^{\varepsilon_i} \overline{\partial}^{\varepsilon_j})^t (h_{nc})\, dz\, d\overline{z} \\
&= \int_{\mathbb{C}^m} h_c \sum_{i,j} f_i \overline{f}_j \left((\partial^{\varepsilon_i} \overline{\partial}^{\varepsilon_j})^t (h_{nc})|_0 \right)\, dz\, d\overline{z}
\end{aligned} \tag{6.3}$$

This last integral is absolutely convergent from 2.7. Therefore its value will be positive (or negative) if the integrand is a positive (respectively, negative) function everywhere on $\mathbb{C}^m$.

Let $A = \mathbb{C}[\partial_i, \overline{\partial}_i, i \in S]$. On A we define a hermitian form by $B(\alpha, \beta) = (a\overline{\beta})^t(h_{nc})|_0$. We know that h_c is always positive, hence in order to prove that $(\cdot,\cdot)_{\mathfrak{u}_\mathbb{R}}$ is positive definite on I^+ and negative definite on I^-, it is enough to prove

Proposition 6.7. *The form $B(\cdot,\cdot)$ is positive definite on A^+ and negative definite on A^-, where A^+ and A^- are the subspaces of A consisting of elements of purely even degree and purely odd degree respetively.*

The goal of the rest of this section is to prove this proposition. For a vector space V with a hermitian form B_V, we say V admits an *orthogonal signature decomposition* if $V = V^+ \oplus V^-$ with V^+ and V^- orthogonal with respect to B_V, and B_V is positive definite on V^+ and negative definite on V^-. We reduce the proof of proposition 6.7 to a "one-dimensional version" via the following lemma

Lemma 6.8. *Let (U, B_U) and (V, B_V) be two vector spaces with hermitian forms so that*

$$U = U^+ \oplus^\perp U^-, V = V^+ \oplus^\perp V^-$$

are orthogonal signature decompositions with respect to the corresponding forms. Then the space $U \otimes V$ endowed with the hermitian form B defined by

$$B(u_1 \otimes v_1, u_2 \otimes v_2) = B_U(u_1, u_2) B_V(v_1, v_2).$$

admits a perpendicular signature decomposition

$$U \otimes V = (U^+ \otimes V^+ \oplus U^- \otimes V^-) \oplus^\perp (U^+ \otimes V^- \oplus U^- \otimes V^+).$$

Because $\lambda \in \mathfrak{h}_\mathbb{R}$ is a dominant weight, the number $-2\frac{\langle\lambda,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle} - 1$ is strictly negative. From the previous lemma, we just need to prove

Proposition 6.9. *Let k be a positive number and let $D = \mathbb{C}[\partial_z, \partial_{\overline{z}}]$. Equip this space with the hermitian form*

$$B(\alpha, \beta) = (\alpha\overline{\beta})^t \left(\frac{1}{(1 + |z|^2)^k} \right) |_{z=0}.$$

Then D admits an orthogonal signature decomposition $D = D^+ \oplus^\perp D^-$, where D is the subspace of elements with pure even degree and D^- is the space of elements with pure odd degree.

Proof. The space D is graded by assigning the element ∂_z degree 1, and the element $\partial_{\overline{z}}$ degree -1. Denote by D^i the subspace of elements with pure degree i. Note that

$$D^i = \mathbb{C}[\Delta]\partial_z^i, \text{ for } i \geq 0, \text{ and } D^i = \mathbb{C}[\Delta]\partial_{\overline{z}}^i, \text{ for } i \leq 0.$$

Here Δ represents the operator $\partial_z \partial_{\overline{z}}$. It is clear that the spaces D^i are orthogonal to each other and therefore we just need to check B is positive definite on the pieces of even degree and negative definite on the pieces of odd degree.

Assume $i \geq 0$ (the other case is analogous), and choose the basis $\{\partial_z^i, \Delta\partial_z^i, \Delta^2\partial_z^i, \dots\}$ for the vector space D^i. Write $f = \frac{1}{(1+|z|^2)^k}$. This function is positive definite. In particular, by Bochner's theorem, we have

$$f(z) = \int_{\mathbb{C}} g(\xi) e^{-2\pi i \xi \cdot z} \, d\xi,$$

where g is a function that is positive everywhere on $\mathbb{C}$. Fix an arbitrary integer N and let A be the matrix given by $A_{kl} = B(\Delta^k \partial_z^i, \Delta^l \partial_z^i)$, where $1 \leq k, l \leq N$. Then we have

$$\begin{aligned} A &= [\Delta^{k+l+i}(f)(0)] \\ &= [\int_{\mathbb{C}} (-4\pi^2 |\xi|^2)^{k+l+i} g(\xi) \, d\xi] \\ &= \int_{\mathbb{C}} [(-(2\pi^2|\xi|)^2)^{k+l+i}] g(\xi) \, d\xi \end{aligned}$$

For any value of ξ, the matrix $[(-(2\pi|\xi|)^2)^{k+l+i}]$ is positive semidefinite if i is even, and negative semidefinite if i is odd. Indeed, let $A(\xi) = [(-r)^{k+l+i}], 1 \leq k, l \leq N$, where $r = 2\pi|\xi|$ is a positive number, and let $u = (x_1, \dots, x_N)$. We see that

$$u^t A(\xi) u = (-r)^i (x_1 - r x_2 + r^2 x_3 - \cdots + (-r)^N x_N)^2,$$

which is non-negative if i is even, and non-positive if i is odd. Clearly there is no vector u that makes $u^t A(\xi) u = 0$ for every value of ξ, hence A is a non-degenerate matrix which is the integral with respect to a positive measure of positive semidefinite matrices when i is even, and negative semidefinite matrices when i is odd. The conclusion is that A is positive definite in the first case, and negative definite in the latter situation. □

Remark 6.10. Just like in section 3, the signature of the form is encoded in properties of the function $a_w(n)$. In this case, the fact that this is a positive definite function. Lu's coordinates work particularly well to compute the signs of the integrands involved in the process.

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