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LYAPUNOV EXPONENTS AND RIGIDITY IN ELLIPTIC AND HYPERBOLIC SETTINGS

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To my grandparents:

Larry, Shirley, Jack, and Judy

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ABSTRACT

This thesis studies a pair of problems relating rigidity and Lyapunov exponents.

In Chapter 2, we study Anosov automorphisms of nilmanifolds. More precisely, we obtain necessary and sufficient conditions for an Anosov automorphism of a nilmanifold with simple Lyapunov spectrum to be locally Lyapunov spectrum rigid.

In Chapter 3, we study perturbations of random walks on isotropic manifolds. Our main result in this section is a necessary and sufficient criterion for this random walk to be isometric with respect to some metric. This criterion is a generalization of work of Dolgopyat and Krikorian [DK07]

CHAPTER 1

INTRODUCTION

Smooth dynamics investigates properties of diffeomorphisms of smooth manifolds that emerge when the diffeomorphisms are iterated. An important motivation for this study is classical mechanics. During the 20th century, it was discovered that some diffeomorphisms have strong rigidity properties: certain features of their dynamics persist even when small modifications are made to the diffeomorphisms. For example, periodic trajectories of some systems persist under perturbations. Some systems are "chaotic" in the sense that nearby trajectories quickly diverge, whereas others are isometric, in the sense that different trajectories remain a fixed distance apart as the system is iterated. In this work, we investigate rigidity phenomena for both types of systems related to numbers called Lyapunov exponents.

Lyapunov exponents describe the exponential infinitesimal convergence or divergence of trajectories in a dynamical system. Suppose that $A_1, A_2, \ldots$ is a sequence of linear transformations between d-dimensional inner product spaces such that the composition $A^n = A_n A_{n-1} \cdots A_1$ is defined. Consider the singular values of A^n, which we denote by $\sigma_1(n) \geq \sigma_2(n) \geq \cdots \geq \sigma_d(n)$. If $\lim_{n\to\infty} n^{-1} \log \sigma_j(n)$ exists, we denote this number by λ_j and call it a *Lyapunov exponent*.

Lyapunov exponents exist in many settings. For example, if the A_i are chosen i.i.d. from a finite set of matrices, then almost surely for each j the limit exists and is independent of the sequence of matrices. Lyapunov exponents are particularly important for studying dynamical systems: If $f\colon M \to M$ is a diffeomorphism of a closed Riemannian manifold M, by setting $A^n = D_x(f^n)$ we obtain Lyapunov exponents associated to the derivative of f; the Lyapunov exponents in this case are important information about the exponential rate of divergence of trajectories under the iteration of f. For example, if f is an isometry then all of its Lyapunov exponents are zero. In many situations, the existence of non-zero Lyapunov exponents implies that a system exhibits "chaotic" behavior.

Lyapunov Spectrum Rigidity of Anosov Automorphisms

The study of maps called Anosov diffeomorphisms is central in dynamical systems as these maps exhibit very strong "chaotic" behavior. One may construct such maps algebraically by the following method: If $L \in \mathrm{SL}(n, \mathbb{Z})$ and L has no eigenvalue of unit modulus, then the induced map on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is an Anosov diffeomorphism called an Anosov *automorphism*. If f is an Anosov diffeomorphism of $\mathbb{T}^n$, then there exists a unique Anosov automorphism L in the same homotopy class as f as well as a homeomorphism h such that $h^{-1}fh = L$. The map h is called a *conjugacy* and f and L are said to be conjugate. We think of L as the linear model of f. The conjugacy h is always Hölder, but need not be C^1 because this would imply that corresponding invariant measures of f and L have the same Lyapunov exponents, which need not be true.

Surprisingly, for some Anosov automorphisms L, if f is an volume-preserving Anosov diffeomorphism with L as its linear model and f has the same volume Lyapunov exponents as L, then f is automatically C^1 conjugate to L. Such an Anosov automorphism L is called *Lyapunov spectrum rigid*. Sometimes an automorphism is known only to be "locally" Lyapunov spectrum rigid, which means that the previous statement only holds when f is a sufficiently C^1 small perturbation of L.

Lyapunov spectrum rigidity is well studied in the case of the torus. Early work was done by de la Llave, Marco, and Moriyon [Dll87, Dll92] for Anosov diffeomorphisms of $\mathbb{T}^2$. More recently, Gogolev, Kalinin, and Sadovskaya [GKS18] showed rigidity under the assumption that no three eigenvalues of L have the same modulus and that L^4 has irreducible characteristic polynomial over $\mathbb{Q}$. Earlier results appeared in [Gog08] and [GKS11]. Recently, Saghin and Yang obtained several additional results on the torus [SY19]. The previously mentioned results are all for automorphisms of tori.

We say that an Anosov automorphism has simple spectrum if all of its Lyapunov exponents are distinct. In [Gog08], Gogolev characterizes local Lyapunov spectrum rigidity of Anosov automorphisms of tori with simple spectrum. The only manifolds known to support Anosov diffeomorphisms are those that are finitely covered by nilmanifolds. Based on a conjecture of Smale, these are the only manifolds supporting Anosov diffeomorphisms [Sma67], hence understanding Lyapunov spectrum rigidity requires studying nilmanifold automorphisms. The following theorem characterizes local Lyapunov spectrum rigidity in the case of nilmanifold automorphisms. Its proof introduces several new coarse geometric techniques needed for working on nilmanifolds.

Theorem 1.0.1. *Let L be an Anosov automorphism of a nilmanifold N/Γ with all Lyapunov exponents distinct. Then L is locally Lyapunov spectrum rigid if and only if L is irreducible and has sorted spectrum.*

Irreducibility is a generalization of the existing irreducibility criterion appearing in [GKS11]. The criterion of "sorted spectrum" requires a substantial development of nilmanifolds in order to state, and so will be explained in detail below. The above theorem first appeared in [DeW21].

Perturbations of Isometric Systems.

A dramatic example of a dichotomy involving Lyapunov exponents appears in the case of perturbations of some random isometric systems. Random dynamical systems are just like deterministic ones, except one chooses the map to apply in the following "random" way. If $(f_1, \ldots, f_m)$ is a tuple of diffeomorphisms of a manifold M, then choose a uniform i.i.d. sequence of numbers $\omega_i \in \{1, \ldots, m\}$. The random dynamics is then given by applying the maps $f_{\omega_1}, f_{\omega_2}, \ldots$ sequentially. We then speak of the *random dynamical system* associated to the tuple $(f_1, \ldots, f_m)$. The Lyapunov exponents of the random system are those associated to the random sequence of maps $D_x(f_{\omega_n}\cdots f_{\omega_1})$. A precise statement of when this limit exists requires the notion of an ergodic stationary measure, which we now define.

Stationary Measures. A probability measure μ on M is *stationary* for a random dynamical system if it satisfies $m^{-1}\sum_{i=1}^{m}(f_i)_*(\mu)=\mu$. Such a measure is called *ergodic* if it is not the non-trivial sum of two distinct stationary measures. Associated to an ergodic stationary measure μ are a list of numbers $\lambda_1(\mu)\geq\cdots\geq\lambda_d(\mu)$. For μ-a.e. $x\in M$ the Lyapunov exponents associated to the random sequence of maps $D_x(f_{\omega_n}\cdots f_{\omega_1})$ almost surely exist and are equal to $\lambda_1(\mu),\ldots,\lambda_d(\mu)$. In this way, the Lyapunov exponents depend only on the measure μ.

The following is a conceptual re-statement of Theorem 3.1.1 below and gives a converse to the statement that isometric systems have all Lyapunov exponents zero: a situation where the existence of a stationary measure with all Lyapunov exponents zero implies that a system is isometric.

Theorem 1.0.2. *Let M be a rank-1 symmetric space of compact type. Suppose that $(R_1,\ldots,R_m)$ is a tuple of isometries of M that generates a dense subset of the isometry group of M. If $(f_1,\ldots,f_m)$ is a small perturbation of $(R_1,\ldots,R_m)$, then either:*

1. *All Lyapunov exponents for the random dynamical system associated to the tuple $(f_1,\ldots,f_m)$ are zero and there exists a smooth diffeomorphism ψ such that for each i, $\psi f_i\psi^{-1}$ is an isometry, or*

2. *The random dynamical system associated to the tuple $(f_1,\ldots,f_m)$ has uniformly large top Lyapunov exponent, i.e. there exists $\epsilon>0$ such that for every ergodic stationary measure μ, $\lambda_1(\mu)>\epsilon$.*

Theorem 1.0.2 is a generalization of a result of Dolgopyat and Krikorian, who proved the same result when M is a sphere [DK07]. A novelty in the proof of Theorem 1.0.2 is the development of a framework for studying the nearness of diffeomorphisms to isometries by use of the strain tensor. This above theorem previously appeared in [DeW20].

CHAPTER 2

LOCAL LYAPUNOV SPECTRUM RIGIDITY OF NILMANIFOLD AUTOMORPHISMS

2.1 INTRODUCTION

Since their introduction, Anosov diffeomorphisms have been a central class of examples in the field of dynamical systems. A diffeomorphism f of a closed Riemannian manifold M is *Anosov* if the tangent bundle of M splits into the continuous direct sum of two Df-invariant subbundles $TM = E^u \oplus E^s$ such that Df uniformly expands the length of vectors in E^u and uniformly contracts the length of vectors in E^s (see Subsection 2.2.1 for a more precise description). We refer to E^u as the *unstable bundle* and E^s as the *stable bundle* associated to f. An important feature of Anosov diffeomorphisms is their structural stability. This means that there exists a C^1 neighborhood $\mathcal{U}$ of f such that if $g \in \mathcal{U}$ then there exists a homeomorphism h such that $hgh^{-1} = f$. The map h is called a conjugacy and f and g are said to be conjugate. See [KH97, Sec. 2.3] for more concerning structural stability.

In this paper, we study a rigidity phenomenon concerning conjugacies between two Anosov diffeomorphisms. It is well known that a conjugacy between two Anosov diffeomorphisms is necessarily Hölder continuous, and, in general, no more than Hölder continuity can be expected. If a conjugacy between two maps is C^1, then the maps are said to be C^1 conjugate. Maps that are C^1 conjugate have many common features and so there are natural reasons why two diffeomorphisms cannot be C^1 conjugate. In our work, we consider two obvious obstructions to the existence of a C^1 conjugacy. Our main result is to show, in a particular setting, that if there is not an obvious obstruction, then there is indeed a C^1

conjugacy between a map and its perturbation.

To begin, we will describe the most elementary obstruction to the existence of a conjugacy: the periodic data. Suppose that f and g are two diffeomorphisms that are C^1 conjugate by a conjugacy h, so that $f = hgh^{-1}$. If p is a periodic point of f of period n, then $h(p)$ is a periodic point of g of period n. By differentiating, we see that

$$D_p f^n = D_{h(p)} h D_{h(p)} g^n (D_{h(p)} h)^{-1}.$$

Consequently, we see that the differentials of f^n at p and g^n at $h(p)$ are conjugate. Given two diffeomorphisms f and g and a conjugacy h, we say that f and g have the same *periodic data with respect to* h if for each periodic point p, if p has period n, then $D_p f^n$ and $D_{h(p)} g^n$ are conjugate as linear maps. The previous discussion shows that for two diffeomorphisms to be C^1 conjugate, it is necessary for them to have the same periodic data.

We say that a map is C^{1+} when the map is C^1 and its derivative is θ-Hölder continuous for some $\theta > 0$. We write $\mathrm{Diff}^{1+}_{\mathrm{vol}}(M)$ for the set of volume-preserving diffeomorphisms of M that are C^{1+}. We are now able to introduce one of the two kinds of rigidity that we will study.

Definition 2.1.1. We say that an Anosov diffeomorphism f is *locally periodic data rigid* if there exists a C^1 neighborhood $\mathcal{U}$ of f inside of $\mathrm{Diff}^{1+}(M)$ such that if $g \in \mathcal{U}$ and f and g have the same periodic data with respect to a conjugacy h, then h is C^{1+}.

In this paper, we obtain local periodic data rigidity results for Anosov automorphisms of nilmanifolds. Before stating this result, we briefly explain this setting and why it is the appropriate generalization. A nilmanifold is a smooth manifold obtained by the following construction. One begins with a nilpotent Lie group N and a discrete subgroup Γ such that N/Γ is compact. The manifold N/Γ is then known as a nilmanifold. If L is an automorphism of N preserving Γ, then L descends to a map on the quotient N/Γ. Write $\mathfrak{n}$ for the Lie algebra of N. The automorphism L induces an automorphism of the Lie algebra $\mathfrak{n}$. If the induced map on $\mathfrak{n}$ has no eigenvalues of unit modulus, then the map on N/Γ is an Anosov diffeomorphism. We refer to the map on the quotient $L\colon N/\Gamma \to N/\Gamma$ as an *Anosov automorphism*. It is an open question, first raised by Smale [Sma67], whether if $f\colon M \to M$ is an Anosov diffeomorphism, then M is finitely covered by a nilmanifold. Consequently, studying Anosov automorphisms on nilmanifolds is quite natural. As far as the author is aware, there are no known examples of Anosov automorphisms exhibiting periodic data rigidity on a non-toral nilmanifold. In this paper we establish the first such example.

We say that an Anosov automorphism has *simple spectrum* if the magnitudes of all the eigenvalues of the induced map on $\mathfrak{n}$ are distinct. In this paper, we restrict our study to automorphisms with simple spectrum. However, even with this restriction not every automorphism exhibits local periodic data rigidity. In the toral case, a condition called *irreducibility* is necessary for periodic data rigidity. We generalize the definition of irreducibility to a nilmanifold automorphism. As it turns out, irreducibility alone is insufficient to ensure periodic data rigidity. We also introduce a condition on the spectrum that we call *sortedness*. Precise statements of these conditions are given later, as they rely on a more detailed development of the notion of an Anosov automorphism of a nilmanifold. For irreducibility, see Section 2.6. For sortedness, see Definition 2.2.3.

Theorem 2.1.2. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with simple spectrum. Then L is locally periodic data rigid if and only if L is irreducible and has sorted spectrum.*

The proof of sufficiency in this theorem relies on periodic approximation (Proposition 1), which reduces the sufficiency claim to that in Theorem 2.1.4. The necessity of the condition follows from Theorem 2.9.8.

As the title of this paper suggests, we also study Lyapunov spectrum rigidity. Before we can state our rigidity result in this direction, we briefly develop the necessary language. Suppose that f is a diffeomorphism of a manifold M of dimension n preserving an ergodic invariant measure μ. Then there exists a list of numbers $\lambda_1 \leq \cdots \leq \lambda_n$ such that for μ-a.e.

$x \in M$, and any non-zero $v \in T_xM$ there exists $1 \leq i \leq n$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda_i.$$

The numbers λ_i are referred to as the Lyapunov exponents of f with respect to μ. Note that some of the λ_i may be repeated. We refer to this list with multiplicity as the *Lyapunov spectrum* of f with respect to μ. In the case of an Anosov automorphism L, vol is an ergodic invariant measure. If the induced map on $\mathfrak{n}$ has eigenvalues $\lambda_1, ..., \lambda_n$, then the Lyapunov exponents of L with respect to volume are the numbers $\log|\lambda_i|$ for $1 \leq i \leq n$. Hence when we restrict to L with simple spectrum, the Lyapunov exponents of L with respect to volume are all distinct. For more information, see [KH97, Supplement].

There is a close relationship between Lyapunov exponents and periodic data. The work of Kalinin in [Kal11] contains a very useful result relating periodic data to Lyapunov exponents. We will not recapitulate this result in full, but instead state a conclusion that follows from it. Suppose that f is an Anosov diffeomorphism of a nilmanifold. Kalinin establishes the following for the Lyapunov exponents of measures invariant under f. Suppose that μ is an ergodic invariant measure and that $\chi_1 \leq \cdots \leq \chi_n$ are the Lyapunov exponents, listed with multiplicity, of f with respect to μ. For any periodic point p there is a natural ergodic invariant measure supported on the orbit of p, namely, the uniform measure. Write the Lyapunov exponents of this measure with multiplicity as $\chi_1^{(p)} \leq \cdots \leq \chi_n^{(p)}$. What Kalinin shows is that for every $\epsilon > 0$, there exists a point p, so that for $1 \leq i \leq n$, $\left|\chi_i - \chi_i^{(p)}\right| < \epsilon$ [Kal11, Thm. 1.4]. In this sense the Lyapunov exponents of μ are approximated by the Lyapunov exponents at a periodic point. If an Anosov diffeomorphism has the same periodic data as a linear example, then every periodic point has the same Lyapunov exponents. Consequently, we deduce the following proposition, which is immediate from Kalinin's work.

Proposition 1. *(Periodic Approximation) [Kal11] Suppose that f is an Anosov diffeomorphism with the same periodic data as an Anosov automorphism L. Then the Lyapunov* *exponents of every ergodic invariant measure of f coincide with those of L.*

We now introduce a notion of local rigidity that pertains to the volume Lyapunov spectrum.

Definition 2.1.3. Suppose that $L \in \mathrm{Diff}^{1+}_{\mathrm{vol}}(M)$ is an Anosov automorphism. We say that L is *locally Lyapunov spectrum rigid* if there exists a C^1 neighborhood $\mathcal{U}$ of L in $\mathrm{Diff}^{1+}_{\mathrm{vol}}(M)$ such that if $g \in \mathcal{U}$, and the Lyapunov spectrum of g with respect to volume is equal to the Lyapunov spectrum of L with respect to volume, then g is C^{1+} conjugate to L.

There are dynamical systems other than Anosov automorphisms that exhibit Lyapunov spectrum rigidity. For instance, Butler [But17] recently showed that closed locally symmetric spaces of negative curvature are characterized either in terms of the Lyapunov exponents of their geodesic flow or the periodic data of their geodesic flow. While Butler's result also concerns Lyapunov spectrum rigidity, his approach is quite different from the approach in this paper.

A C^{1+} Anosov diffeomorphism with the same periodic data as an Anosov automorphism preserves a C^{1+} volume [KH97, Thm 19.2.5]. Thus, by Proposition 1, we see that in order to show that an Anosov automorphism is locally periodic data rigid that it suffices to show that the automorphism is locally Lyapunov spectrum rigid. Obtaining local Lyapunov spectrum rigidity is our main result.

Theorem 2.1.4. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with simple Lyapunov spectrum. Then L is locally Lyapunov spectrum rigid if and only if L is irreducible and has sorted spectrum.*

This theorem follows from the combination of Theorem 2.8.1 and Theorem 2.9.8, each of which shows one direction of the equivalence. We show that the theorem is not vacuous by constructing explicit examples of such automorphisms in Section 2.10. We believe that this theorem establishes the first known instance of Lyapunov spectrum rigidity for an Anosov automorphism of a nilmanifold that is not a torus. For previous work on Lyapunov spectrum rigidity in the torus setting, see [SY19] as well as [GKS18].

In the nilmanifold case, there are several new complications that arise in the study of the local rigidity of Anosov automorphisms. One major complication is that certain weak foliations that exist in the toral case do not exist in the nilmanifold case because the distributions that define them are not integrable. Another difficulty is that the foliations that arise in our proof are not minimal, whereas in the work of Gogolev, Kalinin, and Sadovskaya, all foliations arising in their proof are minimal by assumption [GKS18, Thm. 1.1]. This work provides some of the first examples of rigidity in a setting without an abundance of minimal dynamical foliations. A final important difficulty is that in $\mathbb{R}^n$ a geodesic is a line, i.e. minimizes distance between its points. In a nilpotent Lie group, geodesics may take very inefficient paths at large scale. This difficulty is overcome by working directly within unstable manifolds and using their coarse geometry. This approach is one of the novelties of this work.

A more technical difference between our result and previous results is that we allow the perturbation to have slightly lower regularity. We assume that the perturbation is C^{1+} whereas in [GKS18] the perturbation is assumed to be C^2. This seems to be the lowest that the regularity of f can be lowered using the current approach. See Subsection 2.2.5 for a more detailed discussion.

2.1.1 Earlier Work

Periodic data rigidity is well studied in the case of Anosov automorphisms of tori. One of the first papers to study this problem was the paper of de La Llave [Dll87], which considered local periodic data rigidity on the two dimensional torus and showed that if a C^∞ Anosov diffeomorphism and a C^1 close C^∞ perturbation had the same periodic data then the two diffeomorphisms were C^∞ conjugate. De la Llave generalized this result to other regularities of diffeomorphism in [Dll92]. Later works considered generalizations of this problem to higher dimensional tori. For example, Gogolev generalized this result to the case of an arbitrary dimensional torus [Gog08] while retaining the assumption of simple spectrum. A substantial difficulty is caused when the Lyapunov spectrum is not simple. One of the first results not requiring simple spectrum was [GKS11] which showed local periodic data rigidity for toral automorphisms having at most two eigenvalues of equal modulus. Many of the aforementioned papers make use of the Livsic theorem in order to show that the derivative cocycles of a system and its perturbation are cohomologous. As equality of periodic data is a hypothesis required to apply the Livsic theorem, periodic data rigidity is quite natural to consider.

Recently, there has been interest in studying the questions of Lyapunov spectrum rigidity for toral automorphisms. For instance, Saghin and Yang [SY19] obtained several results on the torus. Most importantly, they showed local Lyapunov spectrum rigidity in the case of simple spectrum. A stronger version of this result is obtained in [GKS18, Thm. 1], which established local Lyapunov spectrum rigidity of an Anosov automorphism L of a torus under the assumptions that no three eigenvalues of L have the same modulus and that L and L^4 are both irreducible. In both cases, the authors study C^2 perturbations of an Anosov automorphism that are C^1 small and conclude that the conjugacy is C^{1+}. Hence our choice to study C^{1+} perturbations is reasonable.

In all the previous results, either an irreducibility assumption is explicitly assumed or is implied by another hypothesis such as the dimension of the manifold. In every case, considered above, the irreducibility condition assumed is either equivalent to or stronger than the irreducibility condition considered in this paper.

In either the case of Lyapunov spectrum or periodic data rigidity, arguments for regularity of the conjugacy typically show that that the derivative cocycle of the perturbation is quasi-conformal when restricted to summands in the splitting into Lyapunov subspaces. In the case that the stable and unstable distributions are one dimensional, this is immediate. However, in the case of non-simple Lyapunov spectrum more subtle arguments are required. For this reason, it is natural that we are considering the case of simple spectrum in this paper.

For more general background on rigidity theory related to the present context, the reader may find the notes on rigidity theory by Gogolev useful [Gog19].

2.1.2 Sketch of proof of Theorem 2.1.4

Our proof of the sufficiency of the condition in Theorem 2.1.4 follows the approach taken in [GKS18]. We construct a neighborhood $\mathcal{U}$ of L in the following way. By Theorem 2.2.2, we may choose the neighborhood $\mathcal{U}$ so that if $f \in \mathcal{U}$ then the unstable bundle $E^u = E^{u,f}$ for f splits into the direct sum of one-dimensional Hölder continuous Df-invariant subbundles $E_i^{u,f}$. We index these subbundles so that if $i < j$ then the expansion properties of Df acting on $E_i^{u,f}$ are weaker than those of Df acting on $E_j^{u,f}$. We call the E_i^u distribution the ith unstable distribution. For each $1 \leq j \leq \dim E^u$, the distribution $\oplus_{i\geq j} E_i^{u,f}$ uniquely integrates to a foliation, which we call the ith strong unstable foliation and denote by $\mathcal{S}_i^{u,f}$. We use this superscript notation analogously for other objects depending on the map f. Note that the indices i are arranged so that $\mathcal{S}_1^{u,f}$ is the full unstable foliation and the dimension of a leaf of $\mathcal{S}_i^{u,f}$ decreases as i increases. This construction is standard and recalled in Proposition 12. We say that a conjugacy h *intertwines* two foliations $\mathcal{F}$ and $\mathcal{G}$ if $h(\mathcal{F}(x)) = \mathcal{G}(h(x))$ for each x. Here and elsewhere, $\mathcal{F}(x)$ is the leaf of the foliation $\mathcal{F}$ through the point x.

The proof of the theorem is by an inductive argument. The core claim in the induction is that if h is a conjugacy and h intertwines $\mathcal{S}_i^{u,f}$ and $\mathcal{S}_i^{u,L}$, then h intertwines $\mathcal{S}_{i+1}^{u,f}$ and $\mathcal{S}_{i+1}^{u,L}$. To prove this claim, we construct a one-dimensional foliation $\mathcal{W}_i^{u,f}$ that restricted to an $\mathcal{S}_i^{u,f}$ leaf has a global product structure with the $\mathcal{S}_{i+1}^{u,f}$ foliation (see Subsection 2.2.4 for the definition of this term). The foliation $\mathcal{W}_i^{u,f}$ is tangent to the $E_i^{u,f}$ distribution, and so one thinks of the $\mathcal{W}_i^{u,f}$ foliation as the weakest part of the $\mathcal{S}_i^{u,f}$ foliation. Constructing the $\mathcal{W}_i^{u,f}$ foliation is involved and uses a detailed study of the coarse geometry of nilpotent Lie groups, which comprises Section 2.3. The proof also uses the result that h induces a quasi-isometry on leaves of the $\mathcal{S}_i^{u,f}$ foliation (see Corollary 2). In this argument, L having sorted spectrum plays a crucial role.

As a byproduct of the construction of $\mathcal{W}_i^{u,f}$ in Proposition 14, we also obtain that h intertwines the $\mathcal{W}_i^{u,f}$ foliation with an algebraically defined and analytic foliation $\mathcal{W}_i^{u,L}$. We pull back the $\mathcal{S}_{i+1}^{u,f}$ foliation by h to obtain a foliation $\mathcal{F}$. We then study the holonomies of $\mathcal{F}$ between leaves of the $\mathcal{W}_i^{u,L}$ foliation. Surprisingly, we prove in Lemma 2.8.2that these holonomies are isometries. Via an additional argument appearing in Section 2.7, we show that the foliation $\mathcal{F}$ is equal to the foliation $\mathcal{S}_{i+1}^{u,L}$. This additional argument also requires a detailed study of the geometry of nilmanifolds. In this argument, the assumption of irreducibility plays its most important role. One finally concludes that h is C^{1+} by studying the disintegration of volume along the $\mathcal{W}_i^{u,f}$ foliations; the key idea used is discussed in Subsection 2.2.5.

The proof of necessity of the condition in Theorem 2.1.4 is a systematic procedure for perturbing an automorphism L should either sortedness of the Lyapunov spectrum or irreducibility fail. The general idea is to shear a fast direction into a slower one. This argument appears in Section 2.9.

2.2 PRELIMINARIES

2.2.1 Anosov diffeomorphisms

We say that an automorphism of a finite dimensional real vector space is *hyperbolic* if it has no eigenvalues of modulus one. A hyperbolic linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ decomposes $\mathbb{R}^n$ into the direct sum of two subspaces: a stable subspace, E^s, on which A is a contraction, and an unstable subspace, E^u, on which A^{-1} is a contraction. We say that an eigenvector v of A is stable or unstable according to which subspace it lies in. In this paper, we study diffeomorphisms whose differentials satisfy an analogous property. A diffeomorphism f of a compact manifold M is *Anosov* if there exists a continuous splitting of TM into the direct sum of two Df-invariant subbundles $E^{s,f}$ and $E^{u,f}$, a Riemannian metric on M, and $\lambda > 1$

such that, for any $x \in M$,

$$\|D_x f\,|_{E^{s,f}}\,\| < \lambda^{-1} < 1 < \lambda < \|D_x f^{-1}\,|_{E^{u,f}}\,\|^{-1}.$$

We refer to the distributions $E^{s,f}$ and $E^{u,f}$ as the stable and unstable distributions of f, respectively.

A nilmanifold N/Γ is a smooth manifold obtained as the quotient of a nilpotent Lie group N by a cocompact lattice Γ. For information on nilmanifolds and nilpotent Lie groups, see [Rag72]. A right-invariant metric on N descends to a metric on N/Γ and makes the projection $\pi\colon N \to N/\Gamma$ a local isometry.

Definition 2.2.1. We say that a map $L\colon N/\Gamma \to N/\Gamma$ of a nilmanifold N/Γ is an *Anosov automorphism* if the natural lift of L to N is an automorphism of N and the differential of the lift at $e \in N$ is hyperbolic. In an abuse of notation, we may use L to refer to both the map on N and N/Γ.

An Anosov automorphism is an Anosov diffeomorphism. On a nilmanifold, every Anosov diffeomorphism is topologically conjugate to an Anosov automorphism by a result of Franks [Fra69] and Manning [Man74].

In this paper, we consider the regularity of a conjugacy h between an Anosov automorphism L and an Anosov diffeomorphism f. In general, there may be infinitely many such conjugacies. However, all conjugacies between f and L have the same regularity. For a discussion of all possible conjugacies, see [KKRH10].

Proposition 2. *Suppose that f is an Anosov diffeomorphism and L is an Anosov automorphism. If, for some $k \in \mathbb{N}$ and $\theta \in [0,1)$, there exists a $C^{k+\theta}$ conjugacy between f and L, then every other conjugacy between f and L is $C^{k+\theta}$.*

Proof. Conze and Marcuard [CM70, Thm. 1] proved that if γ is a homeomorphism and A is an Anosov automorphism such that $\gamma A \gamma^{-1} = A$, then γ is an affine transformation. Note that if $hf = Lh$ and $h'f = Lh'$, then $h'h^{-1}Lhh'^{-1} = L$ and $hh'^{-1}Lh'h^{-1} = L$. Applying the result of Conze and Marcuard to $h'h^{-1}$ and $h'^{-1}h$, we see that both are affine and hence C^∞. This implies that h' is as regular as h. □

Mather spectrum

We say that a diffeomorphism has *simple Mather spectrum* if there exists a continuous splitting of TM into one-dimensional Df-invariant subbundles E_i, i.e. $TM = E_1 \oplus \cdots \oplus E_{\dim M}$, and there exist constants $a_i < b_i$ such that $b_i < a_{i+1}$ and a Riemannian metric on M such that if $v \in E_i$,

$$a_i\|v\| \le \|D_x f v\| \le b_i\|v\|. \tag{2.1}$$

Note that this is different from the usual definition of Mather spectrum but is equivalent. See, for instance, the introduction to [JSdll95]. For a diffeomorphism f with simple Mather spectrum, we say that the spectrum is contained in $[a_i, b_i]$-rings in accordance with the constants a_i and b_i in equation (2.1).

We will use the following standard result when making perturbations. See, for example, [JSdll95, Thm. A].

Theorem 2.2.2. *Suppose that f has simple Mather spectrum contained in $[a_i, b_i]$-rings. For every $\epsilon > 0$, there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that the Mather spectrum of any diffeomorphism $g \in \mathcal{U}$ is contained in $[a_i', b_i']$ rings with $0 < a_i - a_i' < \epsilon$ and $0 < b_i' - b_i < \epsilon$, $1 \le i \le \dim M$, and the corresponding splitting $TM = \oplus_i E_i^g$ satisfies $\rho(E_i^f, E_i^g) \le \epsilon$, where ρ is a metric on the Grassmanian of 1-planes induced by the Riemannian metric on M.*

Note that simple Mather spectrum gives an estimate, equation (2.1), that is uniform over all of M. In the sequel, we work with the decomposition of TM into one-dimensional subbundles having the properties given in Theorem 2.2.2. In the case of simple Mather spectrum, there is a splitting of the unstable bundle into continuous one-dimensional Df-

invariant subbundles. In the notation of the theorem, we write:

$$E^{u,f} = \bigoplus_{i=1}^{\dim E^u} E_i^{u,f}, \tag{2.2}$$

and order the subspaces so that $i < j$ implies that the maximum expansion b_i of $E_i^{u,f}$ is smaller than the minimum expansion a_j of $E_j^{u,f}$. We refer to $E_i^{u,f}$ as the *ith unstable subspace.*

2.2.2 Dynamical foliations for Anosov automorphisms

We describe algebraically the stable and unstable manifolds of an Anosov automorphism. Let L be an automorphism of a nilpotent Lie group N. For $g \in N$, write R_g for right multiplication by g. By right translation, we identify TN with $N \times T_eN$.

As follows, this allows a precise study of the dynamics of an automorphism of N and of its induced map on a quotient N/Γ for an invariant lattice Γ.

Proposition 3. *(Dynamical foliations for nilmanifold automorphisms). Suppose that $L\colon N \to N$ is an automorphism of a nilpotent Lie group N such that $D_eL\colon \mathfrak{n} \to \mathfrak{n}$ has simple real spectrum with no eigenvalues of modulus 1. Index the eigenvalues of D_eL of modulus greater than one so that*

$$1 < |\lambda_1^u| < \cdots < |\lambda_k^u|,$$

and write $E_{\lambda_i^u}$ for the corresponding eigenspace.

1. *For $1 \leq i \leq k$, the "strong" linear subspace $\mathfrak{s}_i^u := \oplus_{j \geq i} E_{\lambda_j^u}$ is a subalgebra of $\mathfrak{n}$ tangent to a subgroup S_i^u. Similarly, the "weak" subspace $E_{\lambda_i^u}$ is tangent to a subgroup W_i^u. There exists a right-invariant foliation of N obtained from the right translates of these subgroups. The leaf through $x \in N$ of these foliations is equal to $S_i^u x$ and $W_i^u x$, respectively.*

2. *Suppose that Γ is a lattice such that $L(\Gamma) = \Gamma$. Then L descends to a map $N/\Gamma \to N/\Gamma$.*

Moreover, for each $1 \leq i \leq k$, the foliations of N with leaves $W_i^u x$ and $S_i^u x$ descend to L-invariant foliations of N/Γ. We refer to these foliations as the $\mathcal{W}_i^{u,L}$ and $\mathcal{S}_i^{u,L}$ foliations. For $x\Gamma \in N/\Gamma$, $\mathcal{W}_i^{u,L}(x\Gamma) = W_i^{u,L} x\Gamma$ and $\mathcal{S}_i^{u,L}(x\Gamma) = S_i^{u,L} x\Gamma$.

3. *The right-invariant distribution defined by $E_{\lambda_i^u}$ at the identity projects to a distribution on N/Γ. The leaves of the $\mathcal{W}_i^{u,L}$ foliation are characterized by their tangency to this distribution. Similarly, the leaves of the $\mathcal{S}_i^{u,L}$ foliation are characterized by their tangency to the projection of the right-invariant distribution arising from the subalgebra $\mathfrak{s}_i^u = \oplus_{j \geq i} E_{\lambda_j^u} \subseteq \mathfrak{n}$. In particular, these distributions are uniquely integrable.*

4. *Finally, any right-invariant framing of $E_{\lambda_i^u}$ on N projects to a framing of the corresponding distribution on N/Γ via the differential of $\pi\colon N \to N/\Gamma$. Fixing such a framing, we identify the action of the differential of L restricted to the $W_i^u x\Gamma$ foliation with the action of L on the right-invariant framing of $E_{\lambda_i^u}$ on N, which itself is identified with the action of $D_e L$ on $\mathfrak{n}$ via the trivialization. The action on $E_{\lambda_i^u} \subset \mathfrak{n}$ is multiplication by λ_i^u. So, with respect to this framing of the projection of $E_{\lambda_i^u}$ to N/Γ, the differential of L is multiplication by λ_i^u. Similar considerations apply to S_i^u.*

Remark 1. In the sequel we use the notation $\mathcal{W}_i^{u,L}$ and $\mathcal{S}_i^{u,L}$ to denote these "weak" and "strong" foliations on both N and N/Γ. Given the context in which this notation appears, this should cause no confusion.

We now outline the proof of the above proposition.

Proof of Proposition 3. Fixing a metric on T_eN, we obtain a right-invariant metric on N. The automorphism L has differential $D_eL\colon \mathfrak{n} \to \mathfrak{n}$ at e. With respect to the trivialization of the tangent bundle on the right, the map $DL\colon N \times T_eN \to N \times T_eN$ is

$$(g, v) \mapsto (L(g), (D_{L(g)} R_{L(g)^{-1}})(D_g L)(D_e R_g) v).$$

(Note that $L(g)$ is the image of g under the automorphism L and not left translation.) In the above expression, the composition of differentials applied to v is the differential of the map $h \mapsto L(hg)L(g)^{-1} = L(h)$ at e. As this map is equal to L, we see that the map being applied to v is D_eL. Consequently, with respect to the trivialization of TN by right translation, DL has the expression

$$D_g(g, v) = (L(g), (D_eL)v).$$

By assumption, $\mathfrak{n}$ splits into one-dimensional eigenspaces of DL, so that $\mathfrak{n} = \oplus_\lambda E_\lambda$. By pushing this splitting into eigenspaces through the trivialization, we obtain a right-invariant, DL-invariant splitting of TN. A right-invariant metric makes L Anosov on N. In particular, any estimate we have on the growth of norm of vectors in $E_\lambda \subset \mathfrak{n}$ now holds globally for the splitting on N.

1. Consider $D_eL \colon \mathfrak{n} \to \mathfrak{n}$ in this way. As we previously assumed, there is a splitting of $\mathfrak{n}$ into one-dimensional eigenspaces of D_eL that we write as $\mathfrak{n} = \oplus_\lambda E_\lambda$ where $\lambda \in \mathbb{R}$. Observe that if $v \in E_\lambda$ and $w \in E_{\lambda'}$ then $DL([v, w]) = [DLv, DLw] = \lambda\lambda'[v, w]$.

   As the bracket of eigenvectors is either an eigenvector or zero, for a fixed i, $\oplus_{|\lambda| \geq |\lambda_i^u|} E_\lambda$ is a subalgebra of $\mathfrak{n}$ and hence is tangent to an analytic subgroup of N. We write S_i^u for the analytic subgroup tangent to $\oplus_{|\lambda| \geq |\lambda_i^u|} E_\lambda$ at e. We also consider the one-dimensional subspaces E_λ, which are subalgebras as they are one-dimensional. We write W_i^u for the analytic subgroup tangent to $E_{\lambda_i^u}$ at e.

   We write $\mathfrak{n}^s$ for the subspace of $\mathfrak{n}$ spanned by eigenvectors with eigenvalue of modulus less than one and $\mathfrak{n}^u$ for the subspace of $\mathfrak{n}$ spanned by eigenvectors with eigenvalue of modulus greater than one. In this case we write N^u and N^s for the corresponding analytic subgroups. Note that $N^u = S_1^u$.

   Suppose that $E \subset \mathfrak{n}$ is a D_eL-invariant subspace. If E is a subalgebra of $\mathfrak{n}$, then E is tangent to an analytic subgroup of N, which we will call N_E. The right translates of N_E are tangent to the right translates of E. Hence any properties of the differential of L on E hold at every point of N_Ex for all $x \in N$, with respect to a right-invariant metric. Applying this reasoning to the subalgebras $\oplus_{|\lambda|\geq|\lambda_i^u|}E_\lambda$ and E_λ establishes the first part of the proposition.

2. Consider a lattice $\Gamma \subset N$ such that $L(\Gamma) = \Gamma$. Note that N/Γ is a compact manifold, see [Rag72, Thm. II.2.1], and, as the metric we chose on N is right-invariant, the quotient map $\pi\colon N \to N/\Gamma$ is a local isometry. Moreover, right-invariant structures on N descend to structures on N/Γ. As the foliations with leaves $W_i^u x$ and $S_i^u x$ are right-invariant, they descend to foliations on N/Γ via the projection. We refer to these foliations as $\mathcal{W}_i^{u,L}$ and $\mathcal{S}_i^{u,f}$ respectively.

3. The tangent distribution to the $W_i^u x$ foliation and $S_i^u x$ foliations are right-invariant, and hence descend to distributions tangent to the foliations $\mathcal{W}_i^{u,L}$ and $\mathcal{S}_i^{u,L}$. An integrable smooth distribution is uniquely integrable, so we see that the $\mathcal{W}_i^{u,L}$ and $\mathcal{S}_i^{u,L}$ distributions are characterized by their tangent distributions.

4. The final part is immediate. A fixed framing of the invariant splitting of $\mathfrak{n}$ extends to a right-invariant framing of N that projects to a framing of N/Γ. As $\pi\colon N \to N/\Gamma$ is a local isometry and respects the action of L, we see that the action on a frame may be computed by lifting the frame and then projecting back to N/Γ. Consequently, any estimates on the framing in N give estimates on the framing in N/Γ.

□

2.2.3 Automorphisms of nilmanifolds

In this section, we give an explicit description of the eigenspace decomposition associated to an automorphism of a real nilpotent Lie algebra with sorted simple spectrum.

We write $\mathbb{N} = \{1, 2, \ldots\}$. An $\mathbb{N}$-grading of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition into subspaces $\mathfrak{g} = \oplus_{i\in\mathbb{N}}\mathfrak{g}_{(i)}$ such that $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(j)}] = \mathfrak{g}_{(i+j)}$. We say that a $\mathbb{N}$-grading of a nilpotent Lie algebra is *Carnot* if $\mathfrak{g}_{(1)}$ generates $\mathfrak{g}$ as a Lie algebra, i.e. $\mathfrak{g}$ is the smallest subalgebra of itself containing $\mathfrak{g}_{(1)}$. For a nilpotent Lie algebra $\mathfrak{n}$, we write $\mathfrak{n}_i$ for the ith term in the lower central series of $\mathfrak{n}$. Recall that, by definition, $\mathfrak{n}_1 = \mathfrak{n}$ and $\mathfrak{n}_{i+1} = [\mathfrak{n}, \mathfrak{n}_i]$. If $\mathfrak{n}$ is a nilpotent Lie algebra equipped with an automorphism L that admits a real eigenbasis, we define the *L-grading* of $\mathfrak{n}$ in the following way: define $\mathfrak{n}_{(i)}$ to equal to the linear subspace generated by the eigenvectors of L acting on $\mathfrak{n}_i$ that do not lie in $\mathfrak{n}_{i+1}$. While the notation $\mathfrak{n}_{(i)}$ for the L-grading does not demonstrate the dependence on L, we only ever consider one automorphism at a time and this should not cause confusion.

We are now able to define sorted spectrum.

Definition 2.2.3. Suppose $L\colon \mathfrak{n} \to \mathfrak{n}$ is an automorphism of a nilpotent Lie algebra $\mathfrak{n}$. We say that L has *sorted unstable spectrum* if, for any $j > k$ and any two unstable eigenvectors v and w with eigenvalues λ_v and λ_w, if $v \in \mathfrak{n}_{(k)}$ and $w \in \mathfrak{n}_{(j)}$, then $|\lambda_w| > |\lambda_v|$. We say that L has *sorted stable spectrum* if L^{-1} has sorted unstable spectrum. We say that an Anosov automorphism of N/Γ has sorted spectrum if the induced map on $\mathfrak{n}$ has sorted stable and unstable spectrum.

Proposition 4. *Suppose that L is an automorphism of a real nilpotent Lie algebra $\mathfrak{n}$ with a real eigenbasis. Then the L-grading of $\mathfrak{n}$ is Carnot. Further, if $v \in \mathfrak{n}_{(i)}$ is an eigenvector, then there exist $v_1, \ldots, v_i \in \mathfrak{n}_{(1)}$ such that $v = [v_1, [v_2, \ldots, [v_{i-1}, v_i] \ldots]]$.*

Proof. We begin by showing inductively that for $i \geq 2$, $\mathfrak{n}_{(i)} = [\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}]$. So, suppose $i \geq 2$. Then a basis for $[\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}]$ is obtained as a subset of the brackets of a basis for $\mathfrak{n}_{(1)}$ with a basis for $\mathfrak{n}_{(i-1)}$. The bracket of eigenvectors of L is another eigenvector of L. Consequently, $[\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}]$ has a basis comprised of eigenvectors of L. Further, any bracket is in $\mathfrak{n}_i$. Thus $[\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}] \subseteq \mathfrak{n}_{(i)}$ by definition of $\mathfrak{n}_{(i)}$. Consequently, we are done once we show that $\mathfrak{n}_{(i)} \subseteq [\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}]$.

Suppose that $v \in \mathfrak{n}_{(i)}$. Then there exist $r_k \in \mathfrak{n}$ and $s_k \in \mathfrak{n}_{i-1}$ such that $v = \sum_k [r_k, s_k]$. Write $r_k = r_{k,1} + r_{k,2}$ where $r_{k,1} \in \mathfrak{n}_{(1)}$ and $r_{k,2} \in \mathfrak{n}_2$. Similarly, write $s = s_{k,1} + s_{k,2}$ where

$s_{k,1} \in \mathfrak{n}_{(i-1)}$ and $s_{k,2} \in \mathfrak{n}_i$. Then

$$v = \sum_k [r_k, s_k] = \sum_k [r_{k,1} + r_{k,2}, s_{k,1} + s_{k,2}]$$
$$= \sum_k [r_{k,1}, s_{k,1}] + [r_{k,1}, s_{k,2}] + [r_{k,2}, s_{k,1}] + [r_{k,2}, s_{k,2}].$$

The second and third terms inside the sum are in $\mathfrak{n}_{i+1}$, and the fourth term is in $\mathfrak{n}_{i+2}$. Thus as $v \in \mathfrak{n}_{(i)}$, $[r_{k,1}, s_{k,1}] \in \mathfrak{n}_{(i)}$, and $\mathfrak{n}_i = \mathfrak{n}_{(i)} \oplus \mathfrak{n}_{i+1}$, the final three terms are 0. So, $v = \sum_k [r_{k,1}, s_{k,1}]$, and so $\mathfrak{n}_{(i)} = [\mathfrak{n}_{(1)}, \mathfrak{n}_{(i-1)}]$.

We now show the second claim in the theorem: an eigenvector is the bracket of eigenvectors in $\mathfrak{n}_{(1)}$. The second claim immediately implies that the grading is Carnot. We give a proof by induction. Suppose that the claim holds for i. Let $V = \{v_j\}$ be an eigenbasis for $\mathfrak{n}_{(1)}$ and let $W = \{w_k\}$ be an eigenbasis for $\mathfrak{n}_{(i)}$. By hypothesis, each w_k is a repeated bracket of the vectors v_j. Observe that

$$\mathfrak{n}_{(i+1)} = [\mathfrak{n}_{(1)}, \mathfrak{n}_{(i)}] = \operatorname{span}\{[r, s] : r \in \mathfrak{n}_{(1)}, s \in \mathfrak{n}_{(i)}\}$$
$$= \operatorname{span}\{[v_j, w_k] : v_j \in V, w_k \in W\}.$$

Thus as $\mathfrak{n}_{(i+1)}$ is spanned by the vectors $[v_j, w_k]$, it has a basis consisting of vectors of the form $[v_j, w_k]$, each of which is an eigenvector. Thus every eigenvector of L in $\mathfrak{n}_{(i+1)}$ is of the form $[v_j, w_k]$, and hence is the repeated bracket of eigenvectors in $\mathfrak{n}_{(1)}$. $\square$

If an automorphism L of $\mathfrak{n}$ is hyperbolic, we write $\mathfrak{n}^s$ and $\mathfrak{n}^u$ for its stable and unstable subspaces. Suppose that L has simple spectrum. Then $\mathfrak{n}^s$ and $\mathfrak{n}^u$ are subalgebras of $\mathfrak{n}$ spanned by the stable and unstable eigenvectors of L acting on $\mathfrak{n}$, respectively. Consequently, these are each subalgebras of $\mathfrak{n}$ as eigenvectors of L bracket to other eigenvectors or to 0.

Proposition 5. *Suppose that L is an automorphism of a real nilpotent Lie algebra $\mathfrak{n}$ with sorted spectrum admitting an eigenbasis, and let $\mathfrak{n}^s$ and $\mathfrak{n}^u$ be the stable and unstable subalgebras. Then $[\mathfrak{n}^s, \mathfrak{n}^u] = 0$. Thus $\mathfrak{n} = \mathfrak{n}^s \oplus \mathfrak{n}^u$ as a Lie algebra.*

Proof. Suppose not. Then there exists a stable eigenvector $v \in \mathfrak{n}^s$ and an unstable eigenvector $w \in \mathfrak{n}^u$ such that $[v, w] \neq 0$. Suppose that $|\lambda_v|^{-1} < |\lambda_w|$; if the reverse inequality holds a similar argument applies. Suppose $w \in \mathfrak{n}_{(i)}$. Then $[v, w] \in \mathfrak{n}_{i+1}$ is an unstable eigenvector with eigenvalue smaller in magnitude than w, contradicting that L has sorted spectrum. □

The proof of the following is then almost immediate.

Proposition 6. *Suppose that L is an automorphism of a real nilpotent Lie algebra $\mathfrak{n}$ admitting a real eigenbasis and having sorted spectrum. Then the L-grading of the unstable algebra $\mathfrak{n}^u$ is Carnot.*

Proof. Proposition 5 implies that $\mathfrak{n}_i = \mathfrak{n}_i^s \oplus \mathfrak{n}_i^u$. This implies immediately that $\mathfrak{n}^u_{(i)} = \mathfrak{n}_{(i)} \cap \mathfrak{n}^u$ and that $\mathfrak{n}^s_{(i)} = \mathfrak{n}_{(i)} \cap \mathfrak{n}^s$. That the grading is Carnot now follows from Proposition 4. □

From this we easily deduce:

Proposition 7. *Suppose that L is an automorphism of a real nilpotent Lie algebra $\mathfrak{n}$ admitting a real eigenbasis and that L has sorted spectrum. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of the eigenvectors in $\mathfrak{n}^u_{(1)}$. Then if v is an eigenvector in $\mathfrak{n}^u_{(j)}$, then the eigenvalue of v is equal to $\lambda_{i_1} \cdots \lambda_{i_j}$ where each i_j satisfies $1 \leq i_j \leq k$.*

Proof. This is immediate because the L-grading of $\mathfrak{n}^u$ is Carnot; v is the bracket of j eigenvectors of L lying in $\mathfrak{n}^u_{(1)}$. □

The eigenvalues of L do not accurately reflect the divergence of points in the large scale geometry of a nilpotent Lie group. We say that an automorphism L of $\mathfrak{n}$ is *expanding* if $\mathfrak{n} = \mathfrak{n}^u$. Let L be an expanding automorphism of a nilpotent Lie algebra $\mathfrak{n}$. For an eigenvector $v \in \mathfrak{n}_{(i)}$, write λ_v for the eigenvalue of v. We define

$$\sigma_v = |\lambda_v|^{1/i}. \tag{2.3}$$

We refer to σ_v as the *escape speed of L in the direction v*.

Corollary 1. *Suppose that L is an expanding automorphism of a nilpotent Lie algebra $\mathfrak{n}$ and that L has simple real spectrum. Let v be an eigenvector associated to the smallest magnitude eigenvalue λ_1 of L. Then for any eigenvector w such that v and w are linearly independent, we have $\sigma_w > \sigma_v = |\lambda_1|$.*

Proof. By Proposition 4, the L-grading of $\mathfrak{n}$ is Carnot. So, for any other eigenvector $w \in \mathfrak{n}_{(i)}$, we may write w as the bracket of eigenvectors $v_1, \ldots, v_i \in \mathfrak{n}_{(1)}$, so that the eigenvalue of w is $\lambda_{v_1} \cdots \lambda_{v_i}$. By the assumption of simple spectrum, the modulus of one of the terms in this product is greater than $|\lambda_1|$ and the modulus of each other term is at least $|\lambda_1|$, so we obtain $\sigma_w = |\lambda_{v_1} \cdots \lambda_{v_i}|^{1/i} > |\lambda_1| = \sigma_v$. □

2.2.4 Foliations

We now recall some notions concerning foliations. For a more detailed discussion, see [PSW97]. Let $\mathcal{F}$ be a foliation of a closed manifold M. For $p \in M$, we may locally represent the leaf $\mathcal{F}(p)$ as a graph. For a normed space F, we write $F(\delta)$ for the closed disk of radius δ around 0. For small $\delta > 0$, there is a unique function $g(\cdot, y)\colon F_p(\delta) \to F_p^{\perp}$, where F_p is a subspace of T_pM, such that

$$\phi\colon (x, y) \mapsto \exp_p \circ (x, g(x, y))$$

is a foliation chart.

We say that a foliation has *uniformly $C^{k+\theta}$ leaves* if the map $x \mapsto g(x, y)$ is $C^{k+\theta}$ and its derivatives of order less than k with respect to x are continuous in (x, y) and the kth derivative varies Hölder continuously with exponent θ.

We say that a foliation has *uniformly $C^{k+\theta}$ holonomy* if the map $h\colon y \mapsto g(x, y)$ is $C^{k+\theta}$ and its derivatives of order less than k with respect to y depend continuously on (x, y) and the kth derivative varies Hölder continuously with exponent θ.

We say that two foliations $\mathcal{F}$ and $\mathcal{G}$ of a manifold M have *global product structure* if

$\dim \mathcal{F} + \dim \mathcal{G} = \dim M$ and for each distinct pair $x, y \in M$ the set $\mathcal{F}(x) \cap \mathcal{G}(y)$ consists of exactly one point. Note that given a global product structure, we may identify two leaves of the $\mathcal{F}$ foliation by "sliding" along the leaves of the $\mathcal{G}$ foliation. We say that two subfoliations $\mathcal{F}$ and $\mathcal{G}$ of a foliation $\mathcal{W}$ have *subordinate global product structure to* $\mathcal{W}$ if $\dim \mathcal{F} + \dim \mathcal{G} = \dim \mathcal{W}$ and the restrictions of the foliations $\mathcal{F}$ and $\mathcal{G}$ to any leaf of $\mathcal{W}$ give a global product structure on that leaf.

If $\mathcal{F}$ and $\mathcal{G}$ are two foliations of a manifold M, then we say that $\mathcal{F}$ is a subfoliation of $\mathcal{G}$ if the partition of M given by $\mathcal{F}$ is a refinement of the partition given by $\mathcal{G}$. Given two subfoliations $\mathcal{F}$ and $\mathcal{G}$ with subordinate global product structure to a foliation $\mathcal{W}$ and $a, b \in \mathcal{W}(c)$, define $H^{\mathcal{F}}_{a,b} \colon \mathcal{G}(a) \to \mathcal{G}(b)$ by $H^{\mathcal{F}}_{a,b}(x) = \mathcal{F}(x) \cap \mathcal{G}(b)$. We refer to $H^{\mathcal{F}}_{a,b}$ as the *$\mathcal{F}$-holonomy between the leaves $\mathcal{G}(a)$ and $\mathcal{G}(b)$*. Briefly we call it the *$\mathcal{F}$-holonomy.* Note that for any $a, b \in \mathcal{W}(c)$, $H^{\mathcal{F}}_{a,b}$ is continuous.

We will also take the opportunity to define some notation. Suppose $\mathcal{F}$ is a foliation with C^1 leaves of a Riemannian manifold M. The inclusion of a leaf of $\mathcal{F}$ into M is C^1, and so we may pullback the Riemannian metric on M to a metric on the leaf. We endow each leaf with this pullback metric. For two points x, y in the same leaf of $\mathcal{F}$, we define the distance $d_{\mathcal{F}}(x, y)$ to be the distance between x and y with respect to the pullback metric on $\mathcal{F}(x)$.

Later, in Lemma 2.8.2, we will obtain a foliation of N whose holonomies are isometries. For later use, we record an algebraic description of all such isometries.

Proposition 8. *Suppose that W is a one-parameter subgroup of nilpotent Lie group N and that $x, y \in N$. If $I \colon Wx \to Wy$ is an orientation-preserving isometry with respect to the induced Riemannian metric on Wx and Wy, then I is given by right multiplication on N restricted to Wx. In particular, if I takes x to y, then I is the restriction of*

$$z \mapsto z x^{-1} y.$$

Every other such isometry between Wx and Wy is of the form

$$z \mapsto zx^{-1}wy$$

for some $w \in W$.

This follows because right translation realizes every isometry of Wx.

2.2.5 A criterion for regularity

We summarize a result in Saghin and Yang [SY19, Thm. G] that we will use in the proof of Theorem 2.8.1 to establish the regularity of a conjugacy along the leaves of a foliation.

Suppose that $\mathcal{F}$ is a continuous foliation of a compact manifold M with uniformly $C^{r+\theta}$ leaves where $r \in \mathbb{N}$ and $\theta \in (0, 1]$. Assume that $\mathcal{F}$ is invariant under a $C^{r+\theta}$ diffeomorphism, $f\colon M \to M$, and that with respect to some Riemannian metric $\|\cdot\|$ on M there exists $\lambda > 1$ such that for all $x \in M$ $\|D_x f\mid_{T_x\mathcal{F}}\| \geq \lambda$. We say that such a foliation is a $C^{r+\theta}$ *expanding foliation* for f.

A particular type of absolutely continuous measure is used for detecting the regularity of a conjugacy intertwining expanding foliations. We now recall the notion of a disintegration of measures in a manner that is adapted to our context. For a detailed account, see [Rok52]. Let μ be a Borel measure on a manifold M and let $\mathcal{F}$ be a foliation on M. For a foliation chart $\phi\colon B^n_\tau \times B^{n-k}_\perp \to U \subset M$ the plaques of the chart form a partition $\mathcal{P}$ of the set U. The natural projection $\pi\colon U \to \mathcal{P}$ allows us to push foward the restriction of the measure μ to U to a measure $\overline{\mu}$ on $\mathcal{P}$. We then seek a system of measures $\{\mu_P\}_{P\in\mathcal{P}}$ where each μ_P is a Borel measure on the plaque P. By *system of measures* we mean that $\mu_P(P) = 1$ for $\overline{\mu}$-a.e. P, that for a fixed continuous f that the function $P \mapsto \int_P f\,d\mu_P$ is measurable, and further that we may express an integral over U against μ as an iterated integral:

$$\int_U f\,d\mu = \int_{\mathcal{P}} \int_P f\,d\mu_P\,d\overline{\mu}.$$

By the work of Rokhlin, there exists such a system of measures [Rok52]. Further, if we had two systems $\{\mu_P^\alpha\}_{P\in\mathcal{P}}$ and $\{\mu_P^\beta\}_{P\in\mathcal{P}}$, then $\mu_P^\alpha = \mu_P^\beta$ for $\overline{\mu}$-a.e. P. We refer to these systems of measures as the disintegration of the measure μ along the plaques of the chart ϕ. Given this notion of disintegration, we make the following slight modification of the definition of a Gibbs expanding state found in [SY19, Def. 2.2]. As before, we write C^{1+} for an object that is $C^{1+\theta}$ for some $\theta > 0$. Note that the notation C^{1+} does not make an implicit statement of uniformity: different maps that are C^{1+} may have different Hölder exponents.

Definition 2.2.4. Let $\mathcal{F}$ be an expanding foliation for a C^{1+} diffeomorphism f. An f-invariant measure μ is a *Gibbs expanding* state along $\mathcal{F}$ if, for any foliation chart of $\mathcal{F}$, the disintegration of μ along the plaques of the chart is equivalent to the Lebesgue measure on the plaque for μ-almost every plaque.

The following is an abridgement of a more general result, [SY19, Thm. 6], to the one-dimensional case. However, that result has an additional hypothesis that the dynamics be C^2 instead of C^{1+}. We state the sharpened version of this result and outline the proof, which is essentially contained in the implication (B1) implies (B5') in [SY19].

Lemma 2.2.5. *Let M be a smooth closed manifold, and let $f, g \in \mathrm{Diff}^{1+}(M)$. Let $\mathcal{F}$ be a one-dimensional expanding foliation for f, and let $\mathcal{G}$ be an expanding foliation for g such that $\mathcal{F}$ and $\mathcal{G}$ have uniformly C^{1+} leaves. Let μ be a Gibbs expanding state of f along $\mathcal{F}$. Suppose that f and g are topologically conjugate by a homeomorphism h and that h intertwines $\mathcal{F}$ and $\mathcal{G}$. Then the following two conditions are equivalent:*

1. *$\nu := h_*(\mu)$ is a Gibbs expanding state of g along the foliation $\mathcal{G}$.*

2. *h restricted to each $\mathcal{F}$ leaf within the support of μ is uniformly C^{1+}.*

Proof. Fix a foliation box B^f for $\mathcal{F}$. By this, we mean that B^f is the image of a foliation chart $\psi\colon D_k \times D_{n-k} \to M$, where D_k is a disk. As h intertwines $\mathcal{F}$ and $\mathcal{G}$, the set $h(B^f)$, which we call B^g, is a foliation box of $\mathcal{G}$. As μ and ν are expanding states, one may explicitly calculate their densities against volume on the leaves of $\mathcal{F}$. Write $B^f(x)$ for the plaque containing x. Specifically, for almost every $x \in B^f$ the disintegration of μ along the plaque containing x has density ρ_f against volume, where

$$\rho^f(z) = \frac{\Delta(x,z)}{\int \Delta(x,z)\, d\operatorname{vol}_{B^f(x)}}, \text{ and}$$
$$\Delta(x,z) = \lim_{n\to\infty} \frac{\|Df^n\,|_{T_z\mathcal{F}}\,\|}{\|Df^n\,|_{T_x\mathcal{F}\|}}.$$

One may check that this density is uniformly Hölder along almost every plaque of $\mathcal{F}$ [SY19, Prop. 2.3]. If we write μ_x for the disintegration of μ along the plaque $B^f(x)$, then we may show that $h_*(\mu_x) = \nu_{h(x)}$ for μ-a.e. $x \in B^f$; one may prove this by using the essential uniqueness of the disintegration. Continuity then forces $h_*(\mu_x) = \nu_{h(x)}$ for every $x \in B^f$. Now consider the restriction of h to a single plaque; we write $h_x \colon B^f(x) \to B^g(h(x))$ for the restricted map. We have already established that $h_*(\mu_x) = \nu_{h(x)}$. We also know that

$$\mu_x = \rho^f\, d\operatorname{vol}_{B^f(x)} \text{ and } \nu_{h(x)} = \rho^g\, d\operatorname{vol}_{B^g(h(x))},$$

where ρ^f and ρ^g are uniformly Hölder. This implies that if $y \in B^f(x)$ then

$$\int_x^y \rho^f\, d\operatorname{vol}_{B^f_x} = \int_{h(x)}^{h(y)} \rho^g\, d\operatorname{vol}_{B^g_{h(x)}}.$$

The implicit function theorem then implies that h_x is uniformly C^{1+}. By fixing a covering of M by finitely many foliation boxes, we obtain a uniform estimate over all of M. □

Remark 2. We remark that Lemma 2.2.5 is one of the major obstacles to lowering the regularity from C^{1+} in Theorem 2.1.4 to C^1. If the dynamics are only assumed to be C^1, then there is no a priori reason why the function $\Delta(x,z)$ that appears in the above proof would even be defined.

We now introduce one final result that will be of use. Suppose that $\mathcal{F}$ is an expanding foliation for a C^{1+} diffeomorphism f of a manifold M and that μ is an f-invariant measure. If $\mathcal{P}$ is a partition of a manifold M, then $\mathcal{P}$ is said to be a *measurable partition* with respect to a measure μ if there exists a sequence of Borel measurable subsets E_n of M such that

$$\mathcal{P} = \{E_1, M \setminus E_1\} \vee \{E_2, M \setminus E_2\} \vee \cdots \quad \text{mod } 0,$$

where the mod 0 refers to μ. Suppose that ξ is such a measurable partition of M. We say that ξ is *subordinate to* $\mathcal{F}$ *and* μ if

1. the common refinement $\vee_{i\leq 0} f^i\xi$ is the partition into points;
2. for all $x \in M$, $\xi(x)$ is contained in a single $\mathcal{F}$ leaf;
3. for μ-a.e. x, $\xi(x)$ is bounded and contains a neighborhood of x in $\mathcal{F}(x)$.

In addition, if the partition $f\xi$ is coarser than ξ we say that ξ is an *increasing* partition. The construction of increasing measurable partitions for expanding foliations is classical. In the C^{1+} setting, see, for instance, [Yan16, Sec. 3].

We now recall a useful result concerning the Pesin entropy formula. See [SY19, Sec. 2.4] for an explanation of this result in the present context. The argument there is an adaptation of the argument of Ledrappier presented in [LY85].

Lemma 2.2.6. *Let f be a C^{1+} diffeomorphism and let μ be an f-invariant measure. Suppose that $\mathcal{F}$ is a C^{1+} expanding foliation for f. Suppose that ξ is an increasing measurable partition subordinate to $\mathcal{F}$ and μ. Then the conditional measures of μ are absolutely continuous on the leaves of $\mathcal{F}$ if and only if*

$$H_\mu(f^{-1}\xi \mid \xi) = \int \log \|Df\mid_{T\mathcal{F}}\| \, d\mu,$$

where $H_\mu(f^{-1}\xi \mid \xi)$ is the conditional entropy of $f^{-1}\xi$ given ξ.

2.3 COARSE GEOMETRY OF RIEMANNIAN NILMANIFOLDS

This section studies the coarse geometry of Riemannian nilmanifolds. The important idea used in this section, namely the function ϕ defined below, is due to Guivarc'h and was defined in [Gui73]. For a recent use of these same estimates in a different context, see [Cor16].

Let G be a connected Lie group. Fix a compact symmetric neighborhood U of the identity of G and a right-invariant metric d_G on G. For $x, y \in G$, we define $d_U(x, y)$ to be the minimum n such that $yx^{-1} = u_1 u_2 \cdots u_n$ where $u_i \in U$. Note that d_U is right-invariant but not necessarily left-invariant. The following proposition is a special case of [Bre07, Prop. 4.4].

Proposition 9. *The metrics d_U and d_G are quasi-isometric: there exist $A \geq 1$,$B > 0$ such that for all $x, y \in G$*

$$\frac{1}{A} d_U(x, y) - B \leq d_G(x, y) \leq A d_U(x, y) + B.$$

Suppose that N is a nilpotent Lie group and write $\mathfrak{n}$ for the Lie algebra of N. As before, write $\mathfrak{n}_k$ for the kth term in the lower central series of $\mathfrak{n}$. A norm $\|\cdot\|$ on $\mathfrak{n}$ induces a norm on $\mathfrak{n}_k/\mathfrak{n}_{k+1}$. Choose vector space complements $\mathfrak{n}_{(k)}$ to $\mathfrak{n}_{k+1}$ inside of $\mathfrak{n}_k$. The norm restricts to a norm on these subspaces. Decompose an element $x \in \mathfrak{n}$ as $\sum_k x_k$ where each $x_k \in \mathfrak{n}_{(k)}$. Define the *Guivarc'h length* of an element $x \in \mathfrak{n}$ by

$$\phi(x) = \max_k \|x_k\|^{1/k}.$$

Note that if $\mathfrak{n}$ is not abelian then ϕ is not a norm. We will always use the Carnot grading for the choice of complements. The following theorem is implicit in the work of Guivarc'h [Gui73], though it does not seem to be explicitly stated. A thorough explication of Guivarc'h's result is given in [Bre07, Thm. 2.7].

Theorem 2.3.1. *Let N be a nilpotent Lie group endowed with a right-invariant Riemannian metric. Then there exist constants $A > 0, B \geq 0$ such that for any $x \in N$,*

$$\frac{1}{A}\phi(\log x) - B \leq d_N(e, x) \leq A\phi(\log x) + B.$$

Using this coarse estimate, we now return to the escape speeds defined in equation (2.3). The following proposition shows that points lying in the slowest subgroup of an expanding automorphism with simple real spectrum are characterized by their escape speed.

Proposition 10. *Suppose that N is a nilpotent Lie group and $L\colon N \to N$ is an expanding automorphism of N with simple sorted spectrum. Let λ be the smallest modulus eigenvalue of L and let v be an eigenvector of eigenvalue λ. Let Σ be an eigenbasis for L containing v. Let $\sigma = \min_{w\in\Sigma\setminus\{v\}} \sigma_w$, where σ_w is the escape rate in direction w as defined in equation (2.3). Then $\sigma > \sigma_v = |\lambda|$. Choose any η such that $\sigma_v < \eta < \sigma$. Then for any $x \in N$, if there exist C, D such that $d_N(L^n(x), e) \leq C\eta^n + D$ for all $n \geq 0$, then x lies in the subgroup tangent to v.*

Proof. That $\sigma > \sigma_v$ is the content of Corollary 1. Now suppose that $\sigma_v < \eta < \sigma$. Suppose that $x \in N$ and that there exist C, D such that $d_N(L^n(x), e) < C\eta^n + D$ for all $n \geq 0$. Write $\log(x) = \sum_k x_k$. Where $x_k \in \mathfrak{n}_{(k)}$. Then L acts on $\log(x)$ by scaling each of its components in the eigenspace decomposition. Specifically, write $\log(x) = \sum_{1\leq i\leq r} \sum_{1\leq j\leq \dim \mathfrak{n}_{(i)}} a_{ij} v_{i,j}$ where $v_{i,j}$ is is the jth slowest unstable eigenvector in $\mathfrak{n}_{(i)}$. Consequently, $dL^n(\log(x)) = \sum_i \sum_j \lambda^n_{v_{i,j}} a_{ij} v_{i,j}$. Now, $dL^n(\log x) = \log L^n(x)$. By Theorem 2.3.1, there exist A and B

such that $Ad_N(x,e) + B \geq \phi(\log(x))$. Thus, for sufficiently large n,

$$\begin{aligned} AC\eta^n + (B + AD) &\geq \phi(\log(L^n)) \\ &= \phi\left(\sum_i \sum_j \lambda_{v_{i,j}}^n a_{ij} v_{i,j}\right) \\ &= \max_i \|\sum_j \lambda_{v_{i,j}}^n a_{ij} v_{i,j}\|^{1/i} \\ &\geq (\sigma_{v_{i,j}})^n \, |a_{ij}|^{1/i} \, \|v_{i,j}\|^{1/i} \end{aligned}$$

But by choice $\eta < \sigma_{v_{i,j}}$ for all $v_{i,j}$ except $v_{1,1}$. Thus $a_{ij} = 0$ except possibly for $i = 1, j = 1$. Thus x lies in the subgroup tangent to $v_{1,1}$ as its logarithm is a multiple of $v_{1,1}$. □

Note that the proof of Proposition 10 provides detailed information about the distance between e and $L^n(x)$. However, we have only extracted the above statement concerning the slowest speed because it is all we need.

Theorem 2.3.2. *Suppose that N/Γ is a nilmanifold and L is an automorphism of N/Γ with simple sorted spectrum. Fix some i; then by Proposition 3, the ith strong foliation $\mathcal{S}_i^{u,L}$ exists. By the same proposition, the ith weak foliation $\mathcal{W}_i^{u,L}$ exists and subfoliates $\mathcal{S}_i^{u,L}$. Let Σ be an eigenbasis for the action of dL on $\mathfrak{s}_i^u$ and let v be the vector with smallest modulus eigenvalue in Σ. Let $\sigma = \min_{w\in\Sigma\setminus\{v\}} \sigma_w$ be the second slowest escape speed of the action of dL on $\mathfrak{s}_i^u$. Then $\sigma > \sigma_v$, where σ_v denotes the escape speed of v associated to the action of dL on $\mathfrak{s}_i^u$. Choose η such that $\sigma_v < \eta < \sigma$. Then for any $x \in N/\Gamma$ and $y \in \mathcal{S}_i^{u,L}(x)$, if there exist C, D such that $d_{\mathcal{S}_i^{u,L}}(L^n(x), L^n(y)) \leq C\eta^n + D$ for all $n \geq 0$, then $x \in \mathcal{W}_i^{u,L}(y)$.*

Proof. We reduce to Proposition 10. Suppose that that for some C, D,

$$d_{\mathcal{S}_i^{u,L}}(L^n(x), L^n(y)) < C\eta^n + D$$

for all $n \geq 0$. As distance along unstable leaves is the same in N/Γ or in the lifted foliation on N, it suffices to work in the universal cover. In the universal cover, the lifted foliation has leaf through x equal to $S_i^u x$, where S_i^u is the strong subgroup defined in Proposition 3. Consequently, we may write $y = mx$ for some $m \in S_i^u$. By assumption, $d_{\mathcal{S}_i^{u,L}}(L^n(x), L^n(y)) \leq C\eta^n + D$ for all $n \geq 0$. The strong unstable foliation is preserved by right multiplication. Right multiplication preserves the distance along leaves as the leaf metric is induced by a right-invariant metric on N. Thus,

$$\begin{aligned} d_{\mathcal{S}_i^{u,L}}(L^n(x), L^n(y)) = d_{\mathcal{S}_i^{u,L}}(e, L^n(y)(L^n(x))^{-1}) &= d_{\mathcal{S}_i^{u,L}}(e, L^n(yx^{-1})) \\ &= d_{\mathcal{S}_i^{u,L}}(e, L^n(m)) \end{aligned}$$

The restriction of L to S_i^u is an expanding automorphism with simple sorted spectrum, and so the previous proposition applies with the same choice of η. We conclude that m lies in the subgroup of S_i^u generated by v, as desired. □

2.4 COARSE GEOMETRY AND CONJUGACIES

In the first subsection, we show that a conjugacy intertwining the leaves of two sufficiently nice foliations induces a quasi-isometry between the leaves of those foliations. In the second section, we use this result to show that under suitable conditions the $E_i^{u,f}$ distribution is uniquely integrable. The proof of unique integrability is obtained by using that a quasi-isometry respects escape speeds.

Before we begin, we record a basic result showing that a conjugacy interwines stable manifolds. Recall that $\mathcal{S}_1^{u,f}$ is equal to the full unstable foliation.

Proposition 11. *Suppose that L is an Anosov automorphism, f is an Anosov diffeomorphism, and h is a conjugacy satisfying $h \circ f = L \circ h$. Then $h(\mathcal{S}_1^{u,f}) = \mathcal{S}_1^{u,L}$ and $h(\mathcal{S}_1^{s,f}) = \mathcal{S}_1^{s,L}$.*

Proof. Two points are in the same stable manifold if and only if they converge to each other under forward interation. If $d(f^n(x), f^n(y)) \to 0$, then

$$d(h(f^n(x)), h(f^n(y))) = d(L^n(h(x)), L^n(h(y))) \to 0$$

as h is uniformly continuous. So, if $x \in S_1^{s,f}(y)$, then $h(x) \in S_1^{s,L}(h(y))$. The proof is similar in the case of unstable manifolds. □

2.4.1 Bounded geometry

When working with non-compact Riemannian manifolds, it is easy to accidentally allow trivial counterexamples to seemingly reasonable analytic claims. However, there exists a natural class of manifolds that are suitable for analysis: those with bounded geometry. For further discussion and examples, see [Eld13, Ch. 2], which gives an extended discussion of bounded geometry in a dynamical setting.

Definition 2.4.1. We say that a smooth Riemannian manifold N equipped with a smooth metric, g, has *bounded geometry* if

1. The global injectivity radius of N is positive; i.e. there is a uniform lower bound on the injectivity radius of the exponential map over all points $n \in N$.

2. For each $k \geq 0$, there exists C_k such that pointwise

$$\|\nabla^k R\| \leq C_k,$$

   where $\|\cdot\|$ is the norm on tensors induced by g and R denotes the curvature of the Levi-Civita connection ∇ of g.

Consider the universal cover $\widetilde{N}$ of a compact Riemannian manifold N. Endowed with the pullback metric from N, $\widetilde{N}$ has bounded geometry. For an example of a manifold without bounded geometry, consider a Riemannian manifold with a two-dimensional cusp: for any

$\epsilon > 0$, there is a point x sufficiently deep in the cusp that the injectivity radius of the exponential map at x is less than ϵ.

We also introduce a class of submanifolds of bounded geometry manifolds that are also suitable for analysis.

Definition 2.4.2. [Eld13, Def. 2.21] Let $k \geq 1$ be an integer. Let N be a Riemannian manifold of bounded geometry and let $i\colon M \to N$ be a C^k immersion of a C^k manifold M into N. For $x \in M$, we write $M_{x,\delta}$ for the connected component of x in $i^{-1}(B_\delta(i(x)) \cap i(M))$, where $B_\delta(i(x))$ is the open ball of radius δ in N centered at $i(x)$. We say that M is a C^k *uniformly immersed submanifold of* N when there exists $\delta > 0$ such that for all $x \in M$, $M_{x,\delta}$ is represented in normal coordinates on N by the graph of a function $g_x\colon T_xM \to T_xM^\perp$. We also require that there is a uniform bound over all x on the C^k norm of g_x, where this norm is defined with respect to the natural Euclidean structure on T_xM and $T_xM^\perp$.

Note that our definition of uniformly immersed does not reference the pullback metric on M. The reason for this is that the definition of bounded geometry includes reference to the exponential map, which need not be defined if M or the pullback metric has regularity lower than C^2. However, in the case that i and M are both sufficiently regular being uniformly immersed is equivalent to M having low order bounded geometry [Eld13, Lem. 2.27]. One should also note that the definition is quite adapted to a dynamical setting. In the graph transform approach to constructing unstable manifolds, one essentially obtains the needed estimate in the course of constructing the unstable manifolds.

Proposition 12. *Suppose that f is a C^{1+} partially hyperbolic map of a smooth compact manifold M and that $\mathcal{S}^{u,f}$ is the strong unstable foliation of M defined by the partially hyperbolic splitting of f. Then $\mathcal{S}^{u,f}$ is a foliation with uniformly C^{1+} leaves. Moreover, each leaf of $\mathcal{S}^{u,f}$ is a C^1 uniformly immersed submanifold of M.*

Proof. This follows because the strong foliation $\mathcal{S}^{u,f}$ admits a C^1 plaquation, see [HPS77, Cor. 5.6]. Let $\operatorname{Emb}^1(D^k, M)$ be the space of C^1 embeddings of the closed k-dimensional unit disk in M endowed with the C^1 topology. Admitting such a plaquation implies that there is a subset $\mathcal{U} \subseteq \text{Emb}^1(D^k, M)$, where k is the dimension of a leaf of $\mathcal{S}^{u,f}$, such that each point in the foliation is in the image of such a disk. The precompactness of $\mathcal{U}$ immediately implies the uniformity estimate in the definition of uniformly immersed. □

The following proposition shows that for a uniformly immersed submanifold we may locally approximate distance along a leaf by the distance in the manifold.

Proposition 13. *[Eld13, Lem. 2.25] Let M be a C^1 uniformly immersed submanifold of a smooth Riemannian manifold N that has bounded geometry. Write d_M for the Riemannian distance on M induced by the pullback metric and d_N for the Riemannian distance on N. Then for any $C > 1$ there exists δ such that if $d_M(x, y) < \delta$, then*

$$d_N(x, y) \leq d_M(x, y) \leq C d_N(x, y).$$

For a full proof see [Eld13, Lem. 2.25]. However, we will describe briefly the idea. The assumption of bounded geometry allows us to locally approximate the geometry M by the geometry of a graph $g\colon B_\delta(0) \subseteq \mathbb{R}^m \to \mathbb{R}^k$ where m is the dimension of M, k is the codimension of M in N, and $D_0 g = 0$. Suppose $(p, g(p))$ and $(q, g(q))$ are two points on the graph of g. If δ is chosen to be sufficiently small, then the distance between $(p, g(p))$ and $(q, g(q))$ is approximately the same as the distance between p and q in $\mathbb{R}^m$. More needs to be said, particularly about uniformity, but this is essentially the idea.

Lemma 2.4.3. *Suppose that $h\colon N \to N$ is a continuous map of a compact smooth Riemannian manifold N. Suppose that M and M' are two C^1 uniformly injectively immersed submanifolds of M and that $h\colon M \to M'$ is a bijection. Then $h|_M$ is a quasi-isometry from M to M' in the induced metric.*

Proof. Let η and C be constants such that the conclusion of Proposition 13 holds for M and M'. Because h is a map of a compact manifold there exists δ, such that if $d_N(x, y) < \delta$, then

$d_N(h(x),h(y)) < \eta/2$. Fix a minimum length path γ between two points x and y in M such that $d_M(x,y) > \delta/C$. Divide γ into n segments of length δ/C and one segment, the last, of length less than δ/C. Write the endpoints of these segments as $x_1,\ldots,x_n = y$. We then have that

$$d_{M'}(h(x),h(y)) \le \left(\sum_{i=1}^{n-2} d_{M'}(h(x_i),h(x_{i+1}))\right) + d_{M'}(h(x_{n-1}),h(x_n)).$$

As $d_M(x_i,x_{i+1}) < \delta/C$, we see by the bilipschitz estimate from Proposition 13 that $d_N(x_i,x_{i+1}) < \delta$, and hence $d_N(h(x_i),h(x_{i+1})) < \eta/2$. By the bilipschitz estimate again, this time on M', $d_{M'}(h(x_i),h(x_{i+1}) \le C\eta/2$. So, we see that

$$d_{M'}(h(x),h(y)) \le n \cdot C\eta/2,$$

but $n = \lceil d_M(x,y)/(\delta/C)\rceil$, so

$$d_{M'}(h(x),h(y)) \le d_M(x,y)\frac{\eta}{\delta}C^2 + 1.$$

To obtain the lower bound, do the same argument using h^{-1}. This gives that there exist constants C', D' such that

$$d_M(h^{-1}(x),h^{-1}(y)) \le C' d_{M'}(x,y) + D$$

Thus by rearranging quasi-isometry follows. □

Suppose that $\mathcal{F}$ is a foliation with uniformly C^1 leaves and let $\mathcal{F}'$ be the space that is topologized and given the smooth structure of the disjoint union of the leaves of $\mathcal{F}$. The inclusion of $\mathcal{F}'$ into M is a C^1 uniform immersion, so the conclusion of Proposition 13 holds with a uniform constant over the inclusion of all leaves of $\mathcal{F}$ into M. Applying this observation to a map h intertwining two such foliations yields the following corollary.

Corollary 2. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two topological foliations with uniformly C^1 leaves of a smooth compact manifold M. If $h\colon M \to M$ is homeomorphism that intertwines the $\mathcal{F}$ and $\mathcal{G}$ foliations, then for all $x \in M$, h is a quasi-isometry from $\mathcal{F}(x)$ to $\mathcal{G}(h(x))$ and the constants of the quasi-isometry can be taken to be uniform over all leaves of $\mathcal{F}$.*

2.4.2 Quasi-isometry and unique integrability

Using Corollary 2, we show in this subsection that, under certain hypotheses, the $E_i^{u,f}$ distribution uniquely integrates to a 1-dimensional foliation.

Proposition 14. *Suppose that L is a Anosov automorphim of a nilmanifold N/Γ with simple sorted spectrum. Then there exists a C^1 neighborhood $\mathcal{U}$ of L in $\mathrm{Diff}^{1+}(M)$ with the following property. If $f \in \mathcal{U}$ and h_f is a conjugacy between f and L, and, if, for some i, $h_f(\mathcal{S}_i^{u,f}) = \mathcal{S}_i^{u,L}$, then the $E_i^{u,f}$ distribution is uniquely integrable and integrates to a foliation $\mathcal{W}_i^{u,f}$ with uniformly C^{1+} leaves. In addition,*

1. *For each $x \in N/\Gamma$, $h_f(\mathcal{W}_i^{u,f}(x)) = \mathcal{W}_i^{u,L}(h_f(x))$.*
2. *The $\mathcal{W}_i^{u,f}$ foliation and the $\mathcal{S}_{1+1}^{u,f}$ foliation have subordinate product structure to the $\mathcal{S}_i^{u,f}$ foliation.*
3. *The $\mathcal{S}_{i+1}^{u,f}$ holonomies between leaves of $\mathcal{W}_i^{u,f}$ are uniformly C^{1+}.*

Proof. It suffices to construct such a neighborhood for each i; intersecting these neighborhoods then gives the result.

Fix some $1 \le i \le \dim E^u$. We apply Theorem 2.3.2 on the manifold $\mathcal{S}_i^{u,L}$. Let Σ be an eigenbasis for the action of L on $\mathfrak{s}_i^u$ that contains v, a vector tangent to the subgroup W_i^u. Let $\sigma = \min_{w\in\Sigma\setminus\{v\}} \sigma_w > |\lambda_i^u|$. Note that the σ_w are the escape rates of the action on S_i^u. Choose η such that $\sigma_v < \eta < \sigma$. Then for any $x \in N/\Gamma$ and $y \in \mathcal{S}_i^{u,L}(x)$, if there exist C, D such that if $d_{\mathcal{S}_i^{u,L}}(L^n(x), L^n(y)) \le C\eta^n + D$ for all $n \ge 0$, then, by Theorem 2.3.2, x lies in the same $\mathcal{W}_i^{u,L}$ leaf as y.

Pick $\epsilon > 0$ such that $|\lambda_i^u|+\epsilon < \eta$. Then by Theorem 2.2.2, there exists a C^1 neighborhood $\mathcal{U}_\epsilon$ of L such that for any unit vector $w \in E_i^{u,f}$, we have $\|Dfw\| \leq (|\lambda_i^u| + \epsilon) < \eta$.

Suppose now that $f \in \mathcal{U}_\epsilon$ and that h is a conjugacy between f and L intertwining the ith strong foliations. Then all of the previously mentioned considerations hold for f. Suppose that $x \in N/\Gamma$ and let γ be a curve tangent to the $E_i^{u,f}$ distribution containing x. Suppose that y is another point on γ. Write $\ell(\gamma)$ for the length of γ. The inequality $\|Dfw\| \leq (|\lambda_i^u| + \epsilon) < \eta$ implies

$$\ell(f^n \circ \gamma) \leq (|\lambda_i^u| + \epsilon)^n \ell(\gamma),$$

and hence,

$$d_{\mathcal{S}_i^{u,f}}(f^n(x), f^n(y)) \leq (|\lambda_i^u| + \epsilon)^n \ell(\gamma).$$

By Proposition 12 the $\mathcal{S}_i^{u,f}$ foliation has uniformly C^1 leaves and so by Corollary 2 there exist constants C, D such that for all $n \geq 0$

$$d_{\mathcal{S}_i^{u,L}}(h(f^n(x)), h(f^n(y))) \leq C d_{\mathcal{S}_i^{u,f}}(f^n(x), f^n(y)) + D. \tag{2.4}$$

But as $h \circ f^n = L^n \circ h$, this implies that

$$d_{\mathcal{S}_i^{u,L}}(L^n(h(x)), L^n(h(y))) \leq C(|\lambda_i^u| + \epsilon)^n + D.$$

Consequently, as $|\lambda_i^u| + \epsilon < \eta$, we see that $h(x) \in \mathcal{W}_i^{u,L}(h(y))$. Thus $h(\gamma) \subseteq \mathcal{W}_i^{u,L}(h(x))$. This implies that $E_i^{u,f}$ is uniquely integrable and integrates to the $h^{-1}(\mathcal{W}_i^{u,L})$ foliation.

Finally, we need to show the claim about subordinate product structure. The dimensions of the foliations are correct, so we just need to show that a $\mathcal{W}_i^{u,f}$ leaf and a $\mathcal{S}_{i+1}^{u,f}$ leaf intersect at exactly one point if they lie in the same $\mathcal{S}_i^{u,f}$ leaf.

To see that there is at most one point of intersection, note that as $\mathcal{S}_{i+1}^{u,f}$ and $\mathcal{W}_i^{u,f}$ are uniformly transverse there is a uniform lower bound on the distance between points of

$\mathcal{S}^{u,f}_{i+1}(x) \cap \mathcal{W}^{u,f}_i(x)$ independent of x. If there were another point $y \in \mathcal{S}^{u,f}_{i+1}(x) \cap \mathcal{W}^{u,f}_i(x)$, then by iterating the dynamics backwards, we would obtain points $f^{-n}(x)$ and $f^{-n}(y)$ arbitrarily close to each other and with $f^{-n}(y) \in \mathcal{S}^{u,f}_{i+1}(x) \cap \mathcal{W}^{u,f}_i(x)$. This contradicts our previous observation about transversality.

Next, we show how to deduce that there exists a point of intersection. The argument is similar: the distributions that the $\mathcal{W}^{u,f}_i$ and $\mathcal{S}^{u,f}_{i+1}$ foliations are tangent to are uniformly transverse. Consequently, there exists $\epsilon > 0$ such that if there are two points $x, y \in \mathcal{S}^{u,f}_i(z)$ and $d_{\mathcal{S}^{u,f}_i}(x, y) < \epsilon$, then $\mathcal{W}^{u,f}_i(x) \cap \mathcal{S}^{u,f}_{i+1}(y)$ is non-empty. So, to see that for $y \in \mathcal{S}^{u,f}_i(x)$ that $\mathcal{W}^{u,f}_i(x) \cap \mathcal{S}^{u,f}_{i+1}(x)$ intersect, observe that for sufficiently large n, we have $d_{\mathcal{S}^{u,f}_i}(f^{-n}(x), f^{-n}(y)) < \epsilon$. This concludes the first two claims.

Finally, the C^{1+} uniformity of the holonomies follows from [Bro16, Sec. 2.2], which gives this result in the C^{1+} setting. □

2.5 WEAK AND STRONG DISTANCE ALONG FOLIATIONS

Suppose that two foliations $\mathcal{F}$ and $\mathcal{G}$ have subordinate product structure to a foliation $\mathcal{W}$ of a Riemannian manifold M and that all three foliations have uniformly C^1 leaves. Fix two points $x, y \in M$ such that $x \in \mathcal{W}(y)$. Then one can consider the distance between x and $\mathcal{F}(x) \cap \mathcal{G}(y)$ along the $\mathcal{F}$ foliation, which is denoted $d_{\mathcal{F}}(x, \mathcal{F}(x) \cap \mathcal{G}(y))$. While one might not expect such a quantity to be useful in general, in the algebraic setting it has a substantial application. As before, fix a nilmanifold N/Γ and an Anosov automorphism L with simple spectrum that is sorted and irreducible. In this section, we study this construction in the case of the foliations $\mathcal{W}^{u,L}_i$ and $\mathcal{S}^{u,L}_{i+1}$ subordinate to the foliation $\mathcal{S}^{u,L}_i$.

We begin by studying the analogous foliations on N. As we have seen in Proposition 3, the $\mathcal{S}^{u,L}_i$ foliation of N is subfoliated by the $\mathcal{S}^{u,L}_{i+1}$ foliation as well as the $\mathcal{W}^{u,L}_i$ foliation. We now formally introduce the quantity $d_{\mathcal{F}}(x, \mathcal{F}(x) \cap \mathcal{G}(y))$ for some foliations arising in our setting. Although we use the term "distance" in the definition below, note that the notion is not symmetric.

Definition 2.5.1. Suppose that L is an Anosov automorphism of the nilpotent group N endowed with a right-invariant metric. For $1 \leq i \leq \dim N^u$, we define the *ith weak* and *strong distances* on $\mathcal{S}_i^{u,L}(x) = S_i^u x$ as follows. For two points, $q, r \in \mathcal{S}_i^{u,L}(x)$, we define

$$d_{\mathcal{W}_i}(q,r) = d_{\mathcal{W}_i^{u,L}}(S_{i+1}^u q \cap W_i^u r, r),$$

$$d_{\mathcal{S}_i}(q,r) = d_{\mathcal{S}_{i+1}^{u,L}}(q, S_{i+1}^u q \cap W_i^u r).$$

In other words, we find the point $z = \mathcal{S}_{i+1}^{u,L}(q) \cap \mathcal{W}_i^{u,L}(r)$ and measure the distance from q to z along the $(i+1)$st strong unstable foliation to define the strong distance. One measures the distance from z to r along the $\mathcal{W}_i^{u,L}$ foliation to find the weak distance. Further, note that this notion depends on the automorphism L used to define the strong and weak foliations. We do not include the automorphism in the notation as we only ever consider one automorphism at a time.

The following is immediate due to the right-invariance of the dynamical foliations and the right-invariance of distance measured along these foliations.

Lemma 2.5.2. *The weak and strong distances are invariant under right multiplication by elements of N, i.e. for $n \in N$, and any $q, r \in S_i^u x$,*

$$d_{\mathcal{W}_i}(q,r) = d_{\mathcal{W}_i}(qn, rn),$$

$$d_{\mathcal{S}_i}(q,r) = d_{\mathcal{S}_i}(qn, rn).$$

We now record some lemmas concerning weak and strong distances that will be of use later.

Lemma 2.5.3. *Suppose that $w \in W_i^u$ and $s \in S_{i+1}^u$. Then*

$$d_{\mathcal{W}_i}(e, ws) = d_{\mathcal{W}_i}(e, w) \text{ and } d_{\mathcal{S}_i}(e, ws) = d_{\mathcal{S}_i}(e, s).$$

Proof. For the first claim, by definition, $d_{\mathcal{W}_i}(e, ws) = d_{\mathcal{W}_i^{u,L}}(S_{i+1}^u e \cap W_i^u ws, ws)$, but as $W_i^u ws = W_i^s s$, this is equal to $d_{\mathcal{W}_i^{u,L}}(s, ws)$, which by right invariance is equal to $d_{\mathcal{W}_i^{u,L}}(e, w)$, which is equal to $d_{\mathcal{W}_i^{u,L}}(S_{i+1}^u \cap W_i^u w, w)$. But this is equal to $d_{\mathcal{W}_i}(e, w)$ by definition. For the second claim, by definition, $d_{\mathcal{S}_i}(e, ws) = d_{\mathcal{S}_{i+1}^{u,L}}(e, S_{i+1}^u \cap W_i^u ws)$. Because $W_i^u ws = W_i^u s$, this is equal to $d_{\mathcal{S}_{i+1}^{u,L}}(e, S_{i+1}^u \cap W_i^u s)$. This is equal to $d_{\mathcal{S}_{i+1}^{u,L}}(e, s)$, which is equal to $d_{\mathcal{S}_i}(e, s)$ because $S_{i+1}^u e \cap W_i^u s = s$. □

Before the next proof, we record a useful observation about one-parameter subgroups.

Remark 3. Suppose that G is a Lie group endowed with a right-invariant metric. Fix an abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and let $H \coloneqq \exp(\mathfrak{h})$ be the analytic subgroup of G tangent to $\mathfrak{h}$ at $e \in G$. Then $\exp = \mathrm{Exp}$, where Exp is the Riemannian exponential map on H in the metric given by the restriction of the metric on G.

Proof. The right-invariant metric on G restricted to H is a bi-invariant metric on the abelian group H. The exponential map of any invariant metric on an abelian Lie group is the Lie exponential. Since H is abelian Exp and exp must coincide. □

We now study the change in weak and strong distance under an automorphism of N/Γ. As before, we consider an Anosov automorphism L of N/Γ such that L has simple sorted spectrum.

Claim 1. Suppose that L is an Anosov automorphism of a nilpotent Lie group N with simple sorted spectrum. Suppose that $x, y \in N$ with $y \in \mathcal{S}_i^{u,L}(x)$. Then for all $m \in \mathbb{Z}$,

$$d_{\mathcal{W}_i}(L^m(x), L^m(y)) = |\lambda_i^u|^m \, d_{\mathcal{W}_i}(x, y). \tag{2.5}$$

Proof. Note that $d_{\mathcal{W}_i}(L^m(x), L^m(y)) = d_{\mathcal{W}_i}(e, L^m(yx^{-1}))$ by right-invariance of weak distance. So, writing yx^{-1} as ws for some $w \in W_i^u$ and $s \in S_{i+1}^u$, we see that

$$d_{\mathcal{W}_i}(L^m(x), L^m(y)) = d_{\mathcal{W}_i}(e, L^m(w)L^m(s)) = d_{\mathcal{W}_i}(e, L^m(w)),$$

by Lemma 2.5.3.

Note that if $\exp(v) = w$, then $L^m(w) = \exp(dL_e^m v) = \exp((\lambda_i^u)^m v)$. By appealing to Remark 3, we see immediately that $d_{W_i^u}(e, \exp(tv)) = |t|\, d_{W_i^u}(e, w)$. Thus,

$$\begin{aligned} d_{\mathcal{W}_i}(e, L^m(w)) = |\lambda_i^u|^m\, d_{\mathcal{W}_i^u}(e, w) = |\lambda_i^u|^m\, d_{\mathcal{W}_i}(e, w) &= |\lambda_i^u|^m\, d_{\mathcal{W}_i}(e, ws) \\ &= |\lambda_i^u|^m\, d_{\mathcal{W}_i}(x, y). \end{aligned}$$

□

Like the previous claim, the following lemma follows from Lemma 2.5.3 and Remark 3 by estimating distance along one-parameter subgroups.

Lemma 2.5.4. *Let L be an Anosov automorphism of a nilpotent Lie group N with sorted simple spectrum. Suppose that $w \in W_i^u$ and $s \in S_{i+1}^u$. Then there exist $C, D > 0$ such that for all $m \in \mathbb{Z}$,*

$$d_{\mathcal{W}_i}(e, w^m s^m) = mD,$$

and

$$d_{\mathcal{S}_i}(e, w^m s^m) \leq mC.$$

2.6 IRREDUCIBILITY

In this section, we discuss irreducibility and show the equivalence between its algebraic and dynamical formulations.

2.6.1 The toral case

We begin by recalling the existing notion of irreducibility for toral automorphisms. An element $A \in \mathrm{SL}(n, \mathbb{Z})$ is said to be *irreducible* if the characteristic polynomial of A is irreducible over $\mathbb{Q}$. This notion appears in [GKS11] and [GKS18]. This is equivalent to each eigenspace of A being dense when projected to $\mathbb{T}^n$. If A is not irreducible, it is shown in [Gog08, Sec. 3], that A is not locally Lyapunov spectrum rigid. There is a small oversight in the argument of [Gog08], which occurs in the case of non-simple spectrum. We will not explain this here, however, as the argument below, which is based on [Gog08, Sec. 3], is self-contained.

2.6.2 Nilmanifold case

We now consider nilmanifolds. Fix an automorphism, L, of N/Γ. As in the preliminaries, we write $E^u = \oplus E_i^u$. If these subspaces are one-dimensional, then they are tangent to subgroups W_i^u at the identity. Moreover, the foliation by $W_i^u x$ leaves is invariant under left translation by elements of W_i^u. Now, descending to the quotient N/Γ, we are interested in the density of the $W_i^u x\Gamma$ leaves. As before, we write N_k for the kth term in the lower central series of N.

Definition 2.6.1. We say that an Anosov automorphism L of a nilmanifold N/Γ is *irreducible* if, for any eigenvector w of L such that $w \in \mathfrak{n}_{(k)}$, we have $\overline{W\Gamma} = N_k\Gamma$ in N/Γ where W is the one-parameter subgroup tangent to w. Here $\mathfrak{n}_{(k)}$ is the kth term in the L-grading as defined in Subsection 2.2.3.

Note that in the case that N/Γ is a torus that Definition 2.6.1 coincides with the definition of irreducibility recalled in the previous subsection. We think of the above definition as a dynamically defined irreducibility criterion because it concerns the density of leaves of a foliation defined using L. The following proposition gives an algebraic characterization of irreducibility. Recall that if N is a nilpotent Lie group admitting a lattice Γ, then $\exp^{-1}(\Gamma)$ is a discrete subset of $\mathfrak{n}$. Any basis of $\mathfrak{n}$ contained in the $\mathbb{Z}$-span of $\exp^{-1}(\Gamma)$ has rational structure constants. Such a choice of basis gives a $\mathbb{Q}$-structure on $\mathfrak{n}$ in the sense that it determines $\mathbb{Q}$-vector space V contained in $\mathfrak{n}$ of the same dimension as $\mathfrak{n}$. If $L\colon N \to N$ is an automorphism that preserves Γ, then with respect to a basis contained in $\exp^{-1}(\Gamma)$, dL is defined over $\mathbb{Q}$.

Proposition 15. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an automorphism of a nilmanifold with* *sorted simple spectrum. Then L is irreducible if and only if for each k, the induced action of L on $N_k/[N_k, N_k]$ is irreducible over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-structure given by Γ.*

Before we give the proof, we elaborate on what is meant by the statement of the proposition. If Γ is a lattice in N, then $\Gamma \cap N_k$ is a lattice in N_k, see [Rag72, Thm. 2.3 Cor. 1]. Consequently, if L is a map of N preserving Γ, then L restricts to a map on N_k preserving $\Gamma \cap N_k$. There is a quotient map $\pi\colon N_k \to N_k/[N_k, N_k]$. We may then further quotient by $\pi(\Gamma \cap N_k)$. The automorphism L induces an automorphism on the resulting torus $(N_k/[N_k, N_k])/\pi(\Gamma \cap N_k)$. The irreducibility in the theorem is equivalent to the irreducibility for each k of the automorphism on this torus in the sense mentioned in the previous subsection.

In the proof of the proposition, we use the following claim that is derived from the discussion of nilrotations in [EW13, Ch. 10].

Lemma 2.6.2. *Suppose that* $\exp(tv)$ *is a one-parameter subgroup of a nilpotent group N containing a lattice Γ. Then* $\exp(tv)$ *is dense in N/Γ if and only if* $\pi(\exp(tv))$ *is dense in* $(N/[N, N])/(\pi(\Gamma))$.

We now turn to the proof of the proposition.

Proof of Proposition 15. The equivalence is seen by using the claim stated in the previous paragraph. Suppose that W is a one-parameter subgroup tangent to w, which is an eigenvector of L in $\mathfrak{n}_{(k)}$. By Lemma 2.6.2, $W\Gamma$ is dense in $N_k/(\Gamma \cap N_k)$ if and only if $\pi(W)$ is dense in $(N_k/[N_k, N_k])/\pi(N_k \cap \Gamma)$. Note that $\pi(W)$ is a one-parameter subgroup tangent to $d\pi(w)$, which is an eigenvector tangent to an eigenspace of the induced automorphism on this quotient torus. Using the $\mathbb{Q}$-structure coming from the lattice, the induced map on the quotient torus $(N_k/[N_k, N_k])/\pi(N_k \cap \Gamma)$ may be identified with a matrix $A_k \in \mathrm{SL}_m(\mathbb{Z})$ for some m. Each eigenspace of A_k is dense in this torus if and only the induced automorphism A_k is irreducible in the sense that its characteristic polynomial does not factor over $\mathbb{Q}$. Thus we have obtained equivalence between irreducibility in the dynamical sense and in the algebraic sense. □

Remark 4. In the case of maps of the torus, one is able to obtain estimates on how many of the elements of $\mathrm{SL}_n(\mathbb{Z})$ define rigid automorphisms of the torus [GKS11, Prop. 3.1]. One is able to do this by stating reducibility as a Zariski closed condition on $\mathrm{SL}_n(\mathbb{R})$. However, in our case as the condition of sorted spectrum involves inequalities between eigenvalues it is unclear if this approach can be adapted.

2.7 FOLIATIONS WITH ISOMETRIC HOLONOMIES

In this section, we prove a rigidity result characterizing a particular type of topological foliation subordinate to the strong unstable foliation of an irreducible Anosov automorphism L on N/Γ with sorted simple spectrum.

Proposition 16. *Suppose that L is an irreducible automorphism of a nilmanifold N/Γ with sorted simple spectrum. Suppose that $\mathcal{F}$ is an L-invariant, continuous foliation subordinate to the $\mathcal{S}_i^{u,L}$ foliation. Moreover, suppose that $\mathcal{F}$ and $\mathcal{W}_i^{u,L}$ have subordinate product structure to the $\mathcal{S}_i^{u,L}$ foliation. Further, suppose that the $\mathcal{F}$ holonomy between the $\mathcal{W}_i^{u,L}$ leaves endowed with their right-invariant metric is an isometry. Then $\mathcal{F}$ coincides with $\mathcal{S}_{i+1}^{u,L}$.*

In the proof of this proposition we use the following lemma. Note that without the product structure the lemma is false. For example, consider the Reeb foliation.

Lemma 2.7.1. *Let L be a irreducible Anosov automorphism with sorted simple spectrum of a nilmanifold N/Γ. Assume that $\mathcal{F}$ is a continuous foliation that subfoliates the $\mathcal{S}_i^{u,L}$ foliation and has product structure with the $\mathcal{W}_i^{u,L}$ foliation subordinate to the $\mathcal{S}_i^{u,L}$ foliation. Write $\widetilde{\mathcal{F}}$ for the lift of this foliation to N. Suppose that for some $x_n, y_n \in N$ that $x_n \to x$ and $y_n \to y$ and $y_n \in \widetilde{\mathcal{F}}(x_n)$. Then $y \in \widetilde{\mathcal{F}}(x)$.*

Proof. We work in the universal cover. Suppose we have such a pair of sequences. Pick a transversal τ to the $\mathcal{S}_i^{u,L}$ foliation of N. Note that the algebraic structure of the $\mathcal{S}_i^{u,L}$

foliation allows us to ensure that any leaf of $\mathcal{S}_i^{u,L}$ intersects τ at exactly one point. We define a map $\Pi\colon \bigsqcup_{p\in\tau} \mathcal{W}_i^{u,L}(t) \to W_i^u$ as follows. First, using the subordinate product structure project along $\widetilde{\mathcal{F}}$ onto the $\mathcal{W}_i^{u,L}$ leaf containing p. Since $\mathcal{W}_i^{u,L}(p) = W_i^u p$, we then compose with the identification of $W_i^u p$ with W_i^u via $n \mapsto np^{-1}$. Note that Π is continuous due to the continuity of $\mathcal{F}$ and the subordinate product structure. Observe that if $x \in \mathcal{S}_i^{u,L}(y)$, then $\widetilde{\mathcal{F}}(x) = \widetilde{\mathcal{F}}(y)$ if and only if $\Pi(x) = \Pi(y)$ due to the product structure. To conclude, note that $\Pi(x_n) = \Pi(y_n)$, and so by continuity we have $\Pi(x) = \Pi(y)$. $\square$

The contradiction obtained in the following proof is the same contradiction as obtained in [GKS11, p. 851].

Proof of Proposition 16 . We proceed by induction and show that $\mathcal{F}$ is invariant under left translation by elements of the subgroups $S^u_{\dim E^u - j}$. We begin with $j = 0$, and increase j until we reach $j = \dim E^u - (i+1)$.

Suppose we know the result for j; we show it for $j+1$. By continuity of $\mathcal{F}$ and the density of periodic points of L, it suffices to prove that $\mathcal{F}(x) = \mathcal{S}_{i+1}^{u,L}(x)$ at each periodic point x. For any periodic point, we may pass to a power of the dynamics so that the point is fixed. If $p\Gamma$ is a fixed point of L then consider the cover $\pi \circ R_p\colon N \to N/\Gamma$ which sends $x \mapsto xp\Gamma$. Note that in this cover e is in the fiber over $p\Gamma$. As $L(p\Gamma) = p\Gamma$, $L\colon N \to N$ is a lift of $L\colon N/\Gamma \to N/\Gamma$ with respect to this cover. These two observations show that we have reduced to the case of the periodic point $p\Gamma$ being equal to the $e\Gamma$. Using the lift to this particular cover, it now suffices to show that $\widetilde{\mathcal{F}}(e)$ is equal to $\mathcal{S}_{i+1}^{u,L}(e)$.

Suppose that $s \in S^u_{\dim E^u - (j+1)}$. If $s \in \widetilde{\mathcal{F}}(e)$, then we are done. For the sake of contradiction, suppose not. By Proposition 8, which characterizes the isometries between leaves of the $\mathcal{W}_i^u$ foliation, there exists $w \in W_i^u$ such that

$$H^{\widetilde{\mathcal{F}}}_{e,s}(x) = xws. \tag{2.6}$$

By Proposition 15, the assumption of irreducibility implies that $S_i^u\Gamma \subseteq \overline{W_i^u\Gamma}$ in N/Γ.

Thus we may fix a sequence $b_i \in W_i^u$ and $\gamma_i \in \Gamma$ such that $b_i\gamma_i^{-1} \to ws$.

Now, we have that $b_i ws \in \widetilde{\mathcal{F}}(b_i)$ by equation (2.6). Note that by the definition of the commutator[1],

$$b_i ws = [b_i, ws] wsb_i,$$

and that $[b_i, ws] \in S^u_{\dim E^u - j}$, by the Carnot grading on N^u. Since $\widetilde{\mathcal{F}}$ is $S^u_{\dim E^u - j}$-invariant under left multiplication, we have

$$wsb_i \in \widetilde{\mathcal{F}}(b_i).$$

Consequently by right-invariance of $\widetilde{\mathcal{F}}$ under the action of Γ,

$$wsb_i\gamma_i^{-1} \in \widetilde{\mathcal{F}}(b_i\gamma_i^{-1}).$$

As $b_i\gamma_i^{-1} \to ws$, and left multiplication is continuous, we find that

$$wsb_i\gamma_i^{-1} \to (ws)^2.$$

So, by the Lemma 2.7.1, we see that

$$(ws)^2 \in \widetilde{\mathcal{F}}(ws) = \widetilde{\mathcal{F}}(e).$$

By once again using the Carnot structure and commutators we see that $wsws = s'w^2s^2$, for some $s' \in S^u_{\dim E^u - j}$. So, again using invariance under left multiplication, we see that $w^2s^2 \in \widetilde{\mathcal{F}}(e)$. By repeating the argument from the point where we chose the sequence b_i, we obtain that $w^{2^m}s^{2^m}$ is in $\widetilde{\mathcal{F}}(e)$ for all $m \geq 0$.

By considering the weak and strong distances, we show that this leads to a contradiction. By Lemma 2.5.4, there exist $C, D > 0$ such that

$$d_{\mathcal{S}_i}(e, w^{2^m}s^{2^m}) \leq 2^m C$$

[1] We define the commutator by $[g, h] = ghg^{-1}h^{-1}$.

and

$$d_{\mathcal{W}_i}(e, w^{2^m} s^{2^m}) = 2^m D.$$

We will obtain a contradiction by applying L^{-1}. Fix some small $\epsilon > 0$ and let $c_\epsilon(m)$ be the minimum $j \geq 0$ such that $\epsilon^2 < d_{\mathcal{S}_i}(e, L^{-j}(w^{2^m} s^{2^m})) < \epsilon$. We now estimate the weak and strong distances of the points $L^{-c_\epsilon(m)}(w^m s^m)$ from e. By Claim 1, we see that the weak distance is contracted by a multiple of $|\lambda_i^u|$. The strong distance is contracted by a rate of at least $\left|\lambda_{i+1}^u\right|$, as the norm of the differential of L restricted to S_{i+1}^u is bounded below by $\left|\lambda_{i+1}^u\right|$. Consequently, as $\epsilon^2 < d_{\mathcal{S}_i}(e, L^{-c_\epsilon(m)} w^m s^m)$ and $d_{\mathcal{S}_i}(e, L^{-c_\epsilon(m)} w^m s^m) \leq 2^m C / \left|\lambda_{i+1}^u\right|^{c_\epsilon(m)}$, we see that

$$\left|\lambda_{i+1}^u\right|^{c_\epsilon(m)} \leq 2^m C/\epsilon^2.$$

Consequently, for each m, by Claim 1 and the inequality,

$$\begin{aligned} d_{\mathcal{W}_i}(L^{-c_\epsilon(m)}(e), L^{-c_\epsilon(m)}(w^m s^m)) &= |\lambda_i^u|^{-c_\epsilon(m)} \, 2^m D \\ &\geq |\lambda_i^u|^{-c_\epsilon(m)} \left|\lambda_{i+1}^u\right|^{c_\epsilon(m)} \frac{D}{C} \epsilon^2. \end{aligned}$$

Observe that as $m \to \infty$, that $c_\epsilon(m) \to \infty$ as well, so this lower bound is going to ∞. This is impossible however: we will show that it violates the continuity of the $\widetilde{\mathcal{F}}(e)$ foliation.

We claim that the set

$$K = \{x \in \widetilde{\mathcal{F}}(e) \mid d_{\mathcal{S}_i}(e, x) \leq \epsilon\}$$

is compact. To see this, note that the $\widetilde{\mathcal{F}}(e)$ foliation and the $\mathcal{S}_{i+1}^{u,L}$ foliation both have subordinate global product structure with the $\mathcal{W}_i^{u,L}$ foliation to the $\mathcal{S}_i^{u,L}$ foliation. Consequently, there is a well-defined homeomorphism $\Pi\colon S_{i+1}^{u,L} \to \widetilde{\mathcal{F}}(e)$ given by $s \mapsto W_i^{u,L} s \cap \widetilde{\mathcal{F}}(e)$. Note that $\Pi(s) = ws$ for some $w \in W_i^u$, so, by Lemma 2.5.3, Π preserves strong distances from e. Consequently,

$$K = \Pi(\{x \in S_{i+1}^u \mid d_{\mathcal{S}_i}(e, x) \leq \epsilon\}),$$

is compact. Because weak distance varies continuously and K is compact, the function $x \mapsto d_{\mathcal{W}_i}(e, x)$ is bounded on K. But this contradicts the existence of the points $L^{-c(m)}(s^m w^m) \in \widetilde{\mathcal{F}}(e)$ that have strong distance from e of size at most ϵ, but for large m have unbounded weak distance from e. Having reached this contradiction, we see that $s \in \widetilde{\mathcal{F}}(e)$ and we are done. $\square$

2.8 PROOF OF MAIN RIGIDITY THEOREM

In this section, we prove the sufficiency of the condition in Theorem 2.1.4.

Theorem 2.8.1. *Suppose that L is an Anosov automorphism of a nilmanifold N/Γ that is irreducible and has sorted simple Lyapunov spectrum. Then there exists a C^1 neighborhood $\mathcal{U}$ of L in $\mathrm{Diff}^{1+}_{vol}(N/\Gamma)$ such that if $f \in \mathcal{U}$ and the Lyapunov spectrum of f with respect to volume coincides with that of L and h_f is a conjugacy between f and L, then h_f is C^{1+}.*

We begin with a lemma.

Lemma 2.8.2. *Let N/Γ and L be given as in Theorem 2.8.1. Suppose that $\mathcal{F}$ is a continuous, L-invariant foliation subordinate to the $\mathcal{S}^{u,L}_i$ foliation, that $\mathcal{F}$ and $\mathcal{W}^{u,L}_i$ have subordinate product structure, and that $\mathcal{F}$ has uniformly C^1 holonomies between the $\mathcal{W}^{u,L}_i$ leaves. Then the holonomies of $\mathcal{F}$ between the leaves of $\mathcal{W}^{u,L}_i$ are isometries in the induced metric on the $\mathcal{W}^{u,L}_i$ leaves.*

Proof. Write $H^{\mathcal{F}}_{a,b}$ for the holonomy of $\mathcal{F}$ between $\mathcal{W}^{u,L}_i(a)$ and $\mathcal{W}^{u,L}_i(b)$. As $\mathcal{F}$ is L-invariant, this holonomy satisfies

$$H^{\mathcal{F}}_{a,b} = L^n \mid_{\mathcal{W}^{u,L}_i(b_n)} \circ H^{\mathcal{F}}_{a_n,b_n} \circ L^{-n} \mid_{\mathcal{W}^{u,L}_i(a)},$$

where $a_n = L^{-n}a, b_n = L^{-n}b$.

It suffices to show that $\|DH^{\mathcal{F}}_{a,b}\| = 1$, as we are working with maps of 1-manifolds. The differentials are conformal, so the norm of a composition is the product of norms. In particular,

$$\|DH^{\mathcal{F}}_{a,b}\| = \|DL^n\|\|DH^{\mathcal{F}}_{a_n,b_n}\|\|DL^{-n}\|.$$

As we are regarding $\mathcal{W}^u_i$ with its right-invariant metric, the norm of DL is constant, so the first and third terms above multiply to 1. Thus we need only show that $\|DH^{\mathcal{F}}_{a_n,b_n}\| \to 1$ as $n \to \infty$. Pass to a subsequence so that $a_n, b_n \to c$ in N/Γ. Then as the holonomies are uniformly C^1, we see that $\|DH^{\mathcal{F}}_{a_n,b_n}\|$ converges to $\|DH^{\mathcal{F}}_{c,c}\| = 1$ because $H^{\mathcal{F}}_{c,c}$ is the identity. Thus $\|DH^{\mathcal{F}}_{a,b}\| = 1$. The result follows. □

Remark 5. If the foliation $\mathcal{W}^u_i$ had higher dimensional leaves and we assumed that DL is conformal on $\mathcal{W}^u_i$, then the proof of Lemma 2.8.2 still works and we get the same conclusion.

We now proceed to the proof of the theorem.

Proof of Theorem 2.8.1. By Proposition 14, there exists a C^1 neighborhood $\mathcal{U}$ of L in $\text{Diff}^{1+}_{\text{vol}}(M)$ such that if $f \in \mathcal{U}$ and h_f is a conjugacy between f and L, and if, for some i, $h_f(\mathcal{S}^{u,f}_i) = \mathcal{S}^{u,L}_i$, then there exists a continuous foliation $\mathcal{W}^{u,f}_i$ satisfying the properties mentioned in the conclusion of Proposition 14.

Suppose that $f \in \mathcal{U}$ and that the volume Lyapunov spectrum of f coincides with the volume Lyapunov spectrum of L. We will prove the claim about f by induction. However, before proceeding to the induction we make the following observation.

Lemma 2.8.3. *If $h_f(\mathcal{W}^{u,f}_i) = \mathcal{W}^{u,L}_i$, then h_f is uniformly C^{1+} along $\mathcal{W}^{u,f}_i$.*

Proof. If $h_f(\mathcal{W}^{u,f}_i) = \mathcal{W}^{u,L}_i$, then h_f intertwines the action of f and L on the $\mathcal{W}^{u,f}_i$ and $\mathcal{W}^{u,L}_i$ foliations. Both are expanding foliations, as elements of $\mathcal{U}$ have simple Mather spectrum. By the Pesin entropy formula, the volume entropy of f is equal to

$$\int_{N/\Gamma} \sum_{1\le i\le \dim E^u} \ln|\lambda^u_i| \, d\,\text{vol},$$

which coincides with the entropy of L against volume. Volume is the measure of maximal entropy for L. Consequently as f and L have the same volume entropy, vol is also the unique measure of maximal entropy for f. Thus $(h_f)_* \operatorname{vol} = \operatorname{vol}$ as vol is the unique measure of maximal entropy for f and L and a conjugacy intertwines the measures of maximal entropy. We next claim that the disintegration of volume along $\mathcal{W}_i^{u,L}$ and $\mathcal{W}_i^{u,f}$ leaves is absolutely continuous. The case of $\mathcal{W}_i^{u,L}$ is immediate by Fubini's theorem as $\mathcal{W}_i^{u,L}$ is analytic. We now explain why the disintegration along $\mathcal{W}_i^{u,f}$ is absolutely continuous. First, note that if ξ is an increasing measurable partition subordinate to the $\mathcal{W}_i^{u,L}$ foliation then $h_f^{-1}(\xi)$ is an increasing measurable partition subordinate to the $\mathcal{W}_i^{u,f}$ foliation as $(h_f)_*(\operatorname{vol}) = \operatorname{vol}$. Consequently, we have the following equality of conditional entropies:

$$H_{\operatorname{vol}}(f^{-1}(h_f^{-1}(\xi)) \mid h_f^{-1}(\xi)) = H_{\operatorname{vol}}(L^{-1}\xi \mid \xi) = \ln|\lambda_i^u|\,.$$

But as the volume spectrum of f is the same as the volume spectrum of L, we see that

$$\int \ln \|Df\,|_{\mathcal{W}_i^{u,f}}\,\|\,d\operatorname{vol} = \ln|\lambda_i^u|$$

as well. Consequently, the hypotheses of Lemma 2.2.6 are satisfied and so the disintegration of volume along the $\mathcal{W}_i^{u,f}$ foliation is absolutely continuous. Then, by Lemma 2.2.5, we conclude that h_f is uniformly C^{1+} along $\mathcal{W}_i^{u,f}$. □

We now proceed by induction to show that $h_f(\mathcal{S}_i^{f,u}) = \mathcal{S}_i^{u,L}$; i.e., that h_f carries strong foliations to strong foliations. We induct on $1 \le i \le \dim E^u$ beginning with $i = 1$. In the case that $i = 1$, this is the statement that a conjugacy carries unstable manifolds to unstable manifolds, which is verified in Proposition 11. Suppose now that the claim holds for i. Then as $f \in \mathcal{U}$ and the induction hypothesis, we see that there exists a foliation $\mathcal{W}_i^{u,f}$ such that $h_f(\mathcal{W}_i^{u,f}) = \mathcal{W}_i^{u,L}$ satisfying the conclusion of Proposition 14. By Lemma 2.8.3, h_f is uniformly C^{1+} along $\mathcal{W}_i^{u,f}$.

Let $\mathcal{F}$ denote the image of $\mathcal{S}_{i+1}^{u,f}$ by h_f. Then $\mathcal{F}$ is a subfoliation of $\mathcal{S}_i^{u,L}$, by the induction hypothesis. As $h_f(\mathcal{W}_i^{u,f}) = \mathcal{W}_i^{u,L}$, $\mathcal{F}$ and $\mathcal{W}_i^{u,L}$ have subordinate product structure to the $\mathcal{S}_i^{u,L}$ foliation. Further, we claim that the holonomy of $\mathcal{F}$ between $\mathcal{W}_i^{u,L}$ leaves is uniformly

C^{1+}. The holonomy $H^{\mathcal{F}}$ is the composition $h_f \circ H^{\mathcal{W}^{u,f}_{i+1}} \circ h_f^{-1}$. The conjugacy h_f restricted to $\mathcal{W}^{u,L}_i$ is uniformly C^{1+} by the previous discussion. The holonomies of the fast foliation $\mathcal{S}^{u,f}_{i+1}$ between leaves of the $\mathcal{W}^{u,f}_i$ foliation are uniformly C^{1+} by Proposition 14. Thus $\mathcal{F}$ satisfies the hypotheses of Lemma 2.8.2 and so $\mathcal{F}$ has isometric holonomies between $\mathcal{W}^{u,L}_i$ leaves. Consequently, $\mathcal{F}$ satisfies the hypotheses of Proposition 16, which implies $\mathcal{F} = \mathcal{S}^{u,L}_{i+1}$. Thus $h_f(\mathcal{S}^{u,f}_{i+1}) = \mathcal{S}^{u,L}_{i+1}$ and the induction holds.

Note that at each step in the induction that we concluded that h_f is uniformly C^{1+} along $\mathcal{W}^{u,f}_i$. This shows that for $f \in \mathcal{U}$ with the same volume spectrum as L that the map h_f is C^{1+} along $\mathcal{W}^{u,f}_i$ for each $1 \leq i \leq \dim E^u$. To conclude that h_f is C^{1+} on the full unstable manifold $\mathcal{S}^{u,f}_1$, we now appeal to Journé's lemma:

Lemma 2.8.4. *[Jou88] Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous transverse foliations with uniformly C^{1+} leaves. If f is uniformly C^{1+} along the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ then f is C^{1+}.*

We now apply Journé's lemma inductively. The foliations $\mathcal{S}^{u,f}_{i+1}$ and $\mathcal{W}^{u,f}_i$ are transverse subfoliations of $\mathcal{S}^{u,f}_i$. So, if h_f is C^{1+} along both, then, by the lemma, f is C^{1+} along $\mathcal{S}^{u,f}_i$. Proceeding inductively from strongest to weakest, we see that h_f is C^{1+} along the full unstable manifold. Repeating the argument for the stable manifold gives the full result. □

2.9 NECESSITY OF IRREDUCIBILITY AND SORTED SPECTRUM FOR LOCAL RIGIDITY

In this section, we establish through constructions a necessary condition for Lyapunov spectrum rigidity in the case of simple spectrum. We will frequently consider a nilmanifold N/Γ as well as the quotient nilmanifold $N/Z(N)/\pi(\Gamma)$, which we denote by $\underline{N}/\underline{\Gamma}$. As elsewhere, $Z(N)$ denotes the center of N. If L is an Anosov automorphism of N/Γ, we denote by $\underline{L}$ the induced map on $\underline{N}/\underline{\Gamma}$. To show necessity, we produce perturbations of L with the same periodic data as L that are not even Lipschitz conjugate to L. A volume-preserving map with the same periodic data as L has the same volume Lyapunov spectrum as L by Proposition 1.

This implies the necessity of the condition in Theorem 2.1.4 on volume Lyapunov spectrum rigidity. The proof of necessity proceeds by induction. The base case of the induction is the claim that if the induced automorphism $\underline{L}$ of $\underline{N}/\underline{\Gamma}$ is irreducible and has sorted spectrum but L does not, then L is not periodic data rigid. The induction step shows that if an automorphism is not rigid, then a central extension of this automorphism is also not rigid. By considering iterated central extensions, we reduce to the base case.

The organization of this section is as follows. First, we give explicit constructions in the base case depending on whether the automorphism is not reducible or fails to have sorted spectrum. The approach is an extension to nilmanifolds of the perturbative technique studied by Gogolev [Gog08] and de la Llave [Dll92] in the case of the torus. The general idea is to shear a fast unstable direction into a slower unstable direction. After giving the constructions in the base case, we give a separate construction for the induction step. In the final section, we conclude.

2.9.1 Non-sorted spectrum

In this section we show that if $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with unsorted simple spectrum and $\underline{L}\colon \underline{N}/\underline{\Gamma} \to \underline{N}/\underline{\Gamma}$ is sorted, then L is not rigid.

Proposition 17. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with simple spectrum of a nilmanifold N/Γ. Suppose that the induced action on $\mathfrak{n}$ has an unstable eigenvector $w \in \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$, and another unstable eigenvector $u \notin \mathfrak{z}$. Write λ_w and λ_u for the eigenvalues of u and w. If $|\lambda_u| > |\lambda_w|$, then L is not Lyapunov spectrum or periodic data rigid. Indeed, there exist arbitrarily C^∞-small perturbations of L with the same periodic data so that a conjugacy between L and the perturbation need not even be Lipschitz.*

Before we begin the proof, we outline the approach. We construct a family of perturbations of L by shearing the base dynamics into the slow direction, w, in the fiber. For each element in the family, we then obtain an explicit equation for a conjugacy between that element and L. By analyzing the resulting equation, we then obtain a necessary and sufficient condition for the conjugacy to be Lipschitz. We then show that this necessary condition for the conjugacy to be Lipschitz fails on arbitrarily small perturbations, which proves the proposition. The criterion in Lemma 2.9.3 first appeared in [Gog08, Prop. 1].

Proof of Proposition 17. The map L descends to a map $\underline{L}$ on $\underline{N}/\underline{\Gamma}$. If $x \in N/\Gamma$, we write $\underline{x}$ to denote the image of x in $\underline{N}/\underline{\Gamma}$. We will also use $\underline{x}$ to denote an element of $\underline{N}/\underline{\Gamma}$ even if we have not introduced an element x.

Recall that Z acts on N/Γ on the left and this action preserves the structure of the fibration $\mathbb{T}^n \to N/\Gamma \to (N/Z)/\pi(\Gamma)$. Consequently, for an element $z \in Z$ and a point $x \in N/\Gamma$, we may consider the point $x+z$. If $\phi \in C^\infty(\underline{N}/\underline{\Gamma}, \mathbb{R})$, we define a map $I_\phi \colon N/\Gamma \to N/\Gamma$ via $x \mapsto x + \phi(\underline{x})w$. We similarly define the perturbation $L_\phi := L(x) + \phi(\underline{x})w$. Finally, we observe that for $t \in \mathbb{R}$ and $x \in N/\Gamma$, that $L(x + tw) = L(x) + \lambda_w tw$.

Lemma 2.9.1. *The perturbation L_ϕ has the same periodic data as L.*

Proof. Consider the differential of L_ϕ when viewed in charts that trivialize the principal bundle structure of N/Γ. In such charts, L_ϕ is a map $B_1 \times \mathbb{T}^n \to B_2 \times \mathbb{T}^n$ where B_i is an open disk in $\mathbb{R}^k$ and $k = \dim \underline{N}/\underline{\Gamma}$. As L_ϕ is a bundle map, we may choose the chart so that the differential is of the form:

$$\begin{bmatrix} D\underline{L} & 0 \\ D\phi(\underline{x})w & L\,|_{Z(N)} \end{bmatrix}.$$

We see immediately from the block form of this matrix that L_ϕ has the same periodic data as L. □

Lemma 2.9.2. *If $\phi \in C^\infty(\underline{N}/\underline{\Gamma}, \mathbb{R})$ and $\psi \in C^0(\underline{N}/\underline{\Gamma}, \mathbb{R})$ satisfy the cohomological equation*

$$\phi(\underline{x}) + \psi(\underline{L}\,\underline{x}) = \lambda_w \psi(\underline{x}), \tag{2.7}$$

for all $x \in \underline{N}/\underline{\Gamma}$, then I_ψ is a conjugacy between L_ϕ and L.

Proof. We check by computation that $I_\psi \circ L_\phi = L \circ I_\psi$. Suppose that $x \in N/\Gamma$, then $I_\psi \circ L_\phi$ sends

$$x \mapsto L(x) + \phi(\underline{x})w \mapsto L(x) + \phi(\underline{x})w + \psi(\underline{L}\,\underline{x})w = I_\psi \circ L_\phi(x).$$

On the other hand, $L \circ I_\psi(x)$ is

$$L(x) + \lambda_w \psi(\underline{x})w.$$

Thus if the stated condition holds, then I_ψ is such a conjugacy. $\square$

In fact, we can write down an explicit function ψ satisfying equation (2.7). The form of ψ can be guessed by using the relation appearing in Lemma 2.9.2 as a recurrence. Define

$$\mathcal{J}(\phi) = \lambda_w^{-1} \sum_{k \geq 0} \lambda_w^{-k} \phi(\underline{L}^k\,\underline{x}), \tag{2.8}$$

so that $I_{\mathcal{J}(\phi)}$ provides a conjugacy between L and L_ϕ. Note that $\mathcal{J}(\phi)$ is a well-defined continuous function as this series is absolutely convergent.

Lemma 2.9.3. *The vector u defines a vector field U on $\underline{N}/\underline{\Gamma}$. Let $\mathcal{U}$ be the foliation of $\underline{N}/\underline{\Gamma}$ tangent to the integral curves of U. For $\phi \in C^\infty(\underline{N}/\underline{\Gamma}, \mathbb{R})$, $\mathcal{J}(\phi)$ is Lipschitz along $\mathcal{U}$ if and only if the following two sums converge and are equal in the sense of distributions,*

$$\sum_{k \geq 0} \lambda_w^{-k} \lambda_u^k \phi_u \circ \underline{L}^k = -\lambda_w^{-1} \sum_{k<0} \lambda_w^{-k} \lambda_u^k \phi_u \circ \underline{L}^k, \tag{2.9}$$

where $\phi_u = U(\phi)$ is the derivative of ϕ in the direction of U.

Proof. In the proof we write ψ for $\mathcal{J}(\phi)$ for both clarity and convenience. First, suppose that ψ is Lipschitz along $\mathcal{U}$. Then for Lebesgue-a.e. point p in $\underline{N}/\underline{\Gamma}$, ϕ_u exists at p, and hence by differentiating equation (2.7), there is an $\underline{L}$-invariant set of full volume such that

$$\phi_u + \lambda_u \psi_u \circ \underline{L} = \lambda_w \psi_u$$

on this set. This relation implies that almost everywhere

$$\psi_u = -\frac{1}{\lambda_u}\phi_u \circ \underline{L}^{-1} + \frac{\lambda_w}{\lambda_u}\psi_u \circ \underline{L}^{-1}.$$

By similarly using this relation as a recurrence and noting that $|\lambda_u| > |\lambda_w|$, we obtain that

$$\psi_u = -\lambda_u^{-1}\sum_{k<0}\left(\frac{\lambda_u}{\lambda_w}\right)^k \phi_u \circ \underline{L}^k, \tag{2.10}$$

almost everywhere. As $|\lambda_u| > |\lambda_w|$, the series above converges in C^0 sense. As ψ is Lipschitz along $\mathcal{U}$, it is absolutely continuous and is equal to the integral of its derivative by U. The foliation $\mathcal{U}$ is analytic, so we see that ψ is differentiable with derivative ψ_u, and ψ_u satisfies equation (2.10) along a.e. $\mathcal{U}$ leaf. In particular, this implies that the distributional derivative of ψ along almost every $\mathcal{U}$ leaf is given by pairing with ψ_u as in equation (2.10). This implies that the distributional derivative of ψ in the direction U is regular, and, in particular, is given by pairing with the expression in equation (2.10).

We may compute the distributional derivative of ψ in another way as well. In particular, by definition,

$$\psi = \sum_{k\geq 0}\lambda_w^{-k}\phi \circ \underline{L}^k.$$

To find the distributional derivative of ψ in the direction U, we differentiate term by term; hence, in the sense of distributions,

$$\psi_u = \sum_{k\geq 0}\lambda_w^{-k}\lambda_u^k\phi_u \circ \underline{L}^k.$$

Thus the claimed equality holds. This establishes the implication in the theorem.

Next, suppose that the stated sums converge in the sense of distributions and are equal.

The distribution given by pairing with

$$\psi = \sum_{k \geq 0} \lambda_w^{-k} \phi \circ \underline{L}^k,$$

has distributional derivative in direction U given by the sum of the distributions

$$\sum_{k \geq 0} \lambda_w^{-k} \lambda_u^k \phi_u \circ \underline{L}^k.$$

By assumption this is equal to the distribution

$$-\lambda_u^{-1} \sum_{k<0} \lambda_w^{-k} \lambda_u^k \phi_u \circ \underline{L}^k.$$

However, this distribution is regular as $|\lambda_u| > |\lambda_w|$, and is equal to pairing with some function $\omega \in C^0$. Hence the distributional derivative of ψ is given by pairing with ω, i.e. ψ is weakly differentiable along U with weak derivative ω. Note that ω is in C^0, as is ψ. Hence a standard argument implies that ψ is Lipschitz in the direction U with Lipschitz constant depending on $\|\omega\|_{C^0}$. □

The proof of Proposition 17 is then finished by the following lemma.

Lemma 2.9.4. *In any neighborhood of* 0 *in* $C^\infty(\underline{N}/\underline{\Gamma}, \mathbb{R})$*, there exists a function* ϕ *violating equation (2.9).*

Proof. It suffices to consider the case that the image of u is central in $\mathfrak{n}/\mathfrak{z}$. We may make this reduction by the following means. If N'/Γ' is a nilmanifold fibering over $\underline{N}'/\underline{\Gamma}'$, then we can pullback a function ϕ on $\underline{N}'/\underline{\Gamma}'$ to a function on N'/Γ'. Suppose that $\underline{\phi}$ is a function on $\underline{N}'/\underline{\Gamma}$ that fails to satisfy equation (2.9). Then there exists a function $\underline{\psi}$ on $\underline{N}'/\underline{\Gamma}'$ that pairs to different things with each side of equation (2.9). Denote by ϕ and ψ the pullbacks of these functions to N'/Γ'. We claim that ϕ does not satisfy equation (2.9). To see this note that if $\underline{\rho}$ and $\underline{\omega}$ are two functions on $\underline{N}'/\underline{\Gamma}'$ then the pairing of $\underline{\rho}$ with $\underline{\omega}$ is equal to the pairing of the pullbacks of ρ and ω.

We now assume that u is tangent to the central direction in $\underline{N}/\underline{\Gamma}$. As is standard, $L^2(\underline{N}/\underline{\Gamma},\mathbb{C})$ is a unitary representation of the group N/Z, in which $\exp(u)$ is central. Note that as the subgroup tangent to u is central in N/Z that $\exp(u)$ acts inside of irreducible representations by multiplication. Pick a non-trivial irreducible representation $V_\gamma\subseteq L^2(\underline{N}/\underline{\Gamma},\mathbb{C})$ as well as a C^∞ vector $\phi\in V_\gamma$ on which u acts nontrivially. Note that there exists such a vector as there are functions on $\underline{N}/\underline{\Gamma}$ that are not constant in the u direction.

Let U^t be the flow given by left translation by $\exp ut$. Observe that as u is central inside V_γ that U^t acts by multiplication by $e^{i\lambda_\gamma}$ for some $\lambda_\gamma\in\mathbb{R}\setminus\{0\}$. Suppose now that ϕ is a smooth function in V_γ. Then

$$\phi\circ U^t=e^{i\lambda_\gamma t}\phi.$$

Observe that

$$\phi\circ L\circ U^t=\phi\circ U^{\lambda_u t}\circ L.$$

Thus $\phi\circ L$ lies in a representation $V_{\gamma'}$ where u acts by $\lambda_{\gamma'}=\lambda_u\lambda_\gamma$. Note that $\phi_u=\lim_{t\to0}(\phi\circ U^t-\phi)/t$ lies within V_γ as ϕ is smooth. Similarly, by applying L^k, we obtain a function $\phi^{(k)}$ on which U^t acts by multiplication by $e^{i\lambda_\gamma\lambda_u^k t}$. For j and k such that $|j-k|$ is sufficiently large, $\phi^{(j)}$ and $\phi^{(k)}$ are orthogonal as

$$\langle\phi^{(j)},\phi^{(k)}\rangle=\langle U^t\phi^{(j)},U^t\phi^{(k)}\rangle=\langle e^{i\lambda_\gamma\lambda_u^j t}\phi,e^{i\lambda_\gamma\lambda_u^k t}\phi\rangle=e^{i\lambda_\gamma\lambda_u^j t}e^{-i\lambda_\gamma\lambda_u^k t}\langle\phi,\phi\rangle$$

must be constant in t.

Observe now that if we evaluated equation (2.9) on ϕ, that the two distributions in that equation are different. The functions $\phi\circ L^k$ for large positive and negative k are orthogonal by the discussion above. Consequently, in equation (2.9), when we pair with $\phi\circ L^k$ for sufficiently large k the left hand distribution gives a non-zero quantity, while the right hand gives a zero quantity. By working with the real and imaginary parts we may then obtain a function in $C^\infty(\underline{N}/\underline{\Gamma},\mathbb{R})$ with the same property. As the relation in equation (2.9) is linear, if it fails for ϕ it fails for $\epsilon\phi$ as well, and so the needed result holds in any neighborhood of 0 in $C^\infty(\underline{N}/\underline{\Gamma}, \mathbb{R})$. $\square$

This finishes the proof of Proposition 17, as we have now shown that there exist arbitrarily small ϕ such that L_ϕ has the same periodic data as L and is not Lipschitz conjugate to L. $\square$

2.9.2 Lack of irreducibility

In this section we show the following.

Proposition 18. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with simple spectrum. If the restricted map $L\mid_Z$, where Z is the center of Z, is reducible with respect to the $\mathbb{Q}$-structure given by Γ, then L is not periodic data rigid.*

This proposition follows because if the action of L is reducible, the map

$$\pi\colon N/\Gamma \to \underline{N}/\underline{\Gamma}$$

naturally decomposes into a sequence of two fiber bundles each with torus fiber. We then shear a fast unstable direction tangent to one of these bundles into a slower unstable direction tangent to the other just as we did in Proposition 17.

Proof of Proposition 18. If $L\mid_Z\colon Z \to Z$ is reducible with respect to the $\mathbb{Q}$-structure coming from $\Gamma \cap Z$ and $L\mid_Z$ has simple spectrum, then the characteristic polynomial of L splits over $\mathbb{Q}$ into two distinct factors p and q with no common roots. Let V_p be the subspace of $\mathfrak{z}$ tangent to the eigenvectors of $L\mid_Z$ with eigenvalues coming from p and V_q by the analogous subspace for q. Both of these subspaces are rational. Consequently, $V_q \cap \Gamma$ is a lattice in V_q.

Let $\overline{V_q}$ denote the image of V_q in $Z/(Z\cap\Gamma)$. In particular, observe that $\overline{V_q}$ is a torus. This torus acts freely and faithfully on N/Γ by translation. Note in addition that the map L commutes with the action of $\overline{V_q}$ up to an element of $\overline{V_q}$ because $\overline{V_q}$ is invariant under

L. Consequently, if we quotient by the action of $\overline{V_q}$, we obtain that N/Γ fibers over some manifold $(N/\overline{V_q})/(\pi(\Gamma))$ with torus fiber and that the map L descends to the quotient.

We may now repeat the proof of Proposition 17. Suppose that the action of L on V_q has a larger unstable eigenvalue λ than any eigenvalue of the action on V_p. Let w be a vector tangent to the subspace associated to λ. Using w we can shear $\overline{V_p}$ into $\overline{V_q}$ as in the proof of Proposition 17, and repeat the argument there to obtain that L is not Lyapunov rigid. □

2.9.3 Lack of rigidity persists in extensions

If one has a nilmanifold N/Γ and an automorphism L of N/Γ that is not Lyapunov spectrum rigid, then one may wonder if there exists an Anosov automorphism $L'\colon N'/\Gamma' \to N'/\Gamma'$ and a natural algebraic quotient map $\pi\colon N'/\Gamma' \to N/\Gamma$ such that $\pi \circ L' = L \circ \pi$ and L' is Lyapunov spectrum rigid. The content of the following proposition is that this cannot happen. The value of the proposition is that it allows for Lyapunov spectrum rigidity to be studied inductively.

The construction that follows is closely related to the homotopy theoretic approach developed by Gogolev, Otaneda, and Rodriguez Hertz in [GORH15]. For an automorphism A of a Lie group G, they study A-maps of principal G-bundles, which means that if $F\colon E \to F$ is a map of principal G-bundles, then $F(x.g) = F(x).A(g)$. Note that any Anosov automorphism $N/\Gamma \to N/\Gamma$ is an A-map of N/Γ when viewed as a map of principal torus bundles. In this case, the map A is the map restricted to the torus fiber through $e\Gamma$, which is $Z/(Z \cap \Gamma)$. The basic theory of such A-maps is developed clearly in [GORH15] and so we do not repeat the development here. We recall one result, Proposition 4.4., which gives that if $A, B \in \operatorname{Aut}(G)$, and f is an A map and g is a B map, then $f \circ g$ is an BA-map.

Proposition 19. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is a Anosov automorphism with simple spectrum that is not periodic data rigid. Then, if N' is a nilpotent group containing a lattice Γ' such that $N'/N'_k = N$ and $\Gamma'/(N'_k \cap \Gamma') = \Gamma$, and $L'\colon N'/\Gamma' \to N'/\Gamma'$ is an Anosov automorphism with simple spectrum inducing the map L on the quotient N/Γ, then L' is not* *periodic data rigid.*

Proof. By induction on the degree of nilpotency, it suffices to show the result when N' is a central extension of N so that $N'/\Gamma' \to N/\Gamma$ is a principal $\mathbb{T}^n$-bundle, where $n = \dim Z(N')$.

Lemma 2.9.5. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism and L' is a central extension of L to the torus bundle N'/Γ' that is an A-map, where A is the restriction of L' to the fiber through $e\Gamma'$. Then there exists a C^0 neighborhood $\mathcal{U}$ of L in $\mathrm{Diff}^\infty(N/\Gamma)$ such that if $f \in \mathcal{U}$ then there exists a smooth A-map F covering f such that $d_{C^k}(F, L') = O(d_{C^{k+1}}(f, L))$.*

In principle, the above lemma is immediate from Theorem 6.2 in [GORH15]. We will not recapitulate that theorem in full here as it requires developing the language of classifying spaces. Vaguely, the theorem says that the only obstruction to the existence of such a map is at the level of homotopy. As L' is an A-map covering L and f is homotopic to L, there is no obstruction to finding an A-map covering f. However, the theorem there does not assure us that F is near to L' nor that F is as smooth as L'.

Consequently, we will give a different more explicit proof which produces F that is C^∞ near to L'.

Proof. Choose $\mathcal{U}$ sufficiently C^0-small so that if $f \in \mathcal{U}$ then for any $x \in N/\Gamma$ there exists a unique length minimizing geodesic γ_x starting at $f(x)$ and ending at $L(x)$. We write E_x for the fiber of N'/Γ' over N/Γ.

As N'/Γ' is a principle $\mathbb{T}^n$-bundle over N/Γ, we may choose a $\mathbb{T}^n$-connection on this bundle, which gives an associated parallel transport. See, for instance, [KN63, Ch. 2 Sec. 3]. Consequently, parallel transport along γ_x gives a map $P_x\colon E_{f(x)} \to E_{L(x)}$, which in a trivialization, is a translation on $\mathbb{T}^n$. Observe that these maps P_x piece together to form a C^∞ map $\Psi\colon N'/\Gamma' \to N'/\Gamma'$, which is defined fiberwise and is an Id-map of $\mathbb{T}^n$-bundles covering $L \circ f^{-1}$. Note that $\|\Psi\|_{C^k} = O(\|L \circ f^{-1}\|_{C^{k+1}})$ due to the definition of Ψ via parallel transport. Consequently, $F = \Psi \circ L'$ is an A-map covering f such that $d_{C^k}(F, L') = O(d_{C^{k+1}}(f, L))$. $\square$

Lemma 2.9.6. *Suppose that F is an A-map of the bundle N'/Γ' covering a map $f\colon N/\Gamma \to$*

N/Γ. Suppose f has the same periodic data as L, then F has the same periodic data as L'. Further, if f is volume-preserving then so is F and, in this case, F has the same volume Lyapunov spectrum as L'.

Proof. To begin, we show that F and L' have the same periodic data. Note that every periodic point of F lies above a periodic point of f, i.e. if p is F periodic then $\pi(p)$ is f periodic. Suppose, for the moment, that x is a fixed point. Then consider the differential of F in a trivialization $U \times \mathbb{T}^n$. As F is an A-map, the trivialization looks like:

$$(u, z) \mapsto (f(u), Az + \phi(u)),$$

for some C^∞ function $\phi\colon U \to \mathbb{T}^n$, which gives the translational part of the map. Observe that the differential of the map is

$$\begin{bmatrix} Df & 0 \\ D\phi & A \end{bmatrix}. \tag{2.11}$$

Similar considerations apply at all periodic points. Consequently, the periodic data of F is the union of the periodic data of f with the periodic data of A. By Proposition 1, this implies that all of the Lyapunov exponents for every invariant measure of F coincide with those of L'.

It remains to show that F preserves a volume. Let ω be the volume form on N/Γ preserved by f. There is a well-defined n-form, η, on N'/Γ' coming from the principal torus bundle structure. Consider the form $(\pi^*\omega) \wedge \eta$, i.e. the pullback of volume on the base wedged with the volume on the fiber. That this form is preserved is immediate due to the block form of the differential of F obtained in equation (2.11). □

The following lemma shows that a conjugacy between L' and the perturbation F fibers over a map on N/Γ.

Lemma 2.9.7. *As above, suppose that $F\colon N/\Gamma \to N/\Gamma$ is an A-map and L is an Anosov automorphism with simple spectrum that is also an A-map. If H is a C^1 conjugacy between*

F and L then H perserves the fibers of the fibration $N/\Gamma \to \underline{N}/\underline{\Gamma}$.

Proof. By supposition as H is C^1, F has a number of dynamical features that it inherits from L. In particular, the stable and unstable subspaces are defined and have a continuous splitting into continuous one-dimensional subbundles. Let E_i be a subspace associated to an eigenvalue λ_i of A. We claim that H intertwines the corresponding distributions E_i associated to eigenvalues of the map A acting on the fiber. By inspecting equation (2.11), we see that for a periodic point of F that the subspace E_i is tangent to the fiber of the projection $N/\Gamma \to \underline{N}/\underline{\Gamma}$. This holds for each exponent arising from A as the distribution defined by the splitting into one-dimensional subbundles is continuous and periodic points are dense. Similarly, the one-dimensional stable and unstable subspaces of L arising from the action of L on the center of N are tangent to the fiber. Since we assumed simple spectrum, we see that DH carries $E_i^{*,F}$ to $E_i^{*,L}$ for $* \in \{s, u\}$, and consequently H preserves the fibers of the fibration. $\square$

We now show that the perturbation F of L' that we constructed cannot be C^1 conjugate to L'. Suppose, for the sake of contradiction, that it is so that there exists a C^1 conjugacy H satisfying $F \circ H = H \circ L'$. By the lemma, all three of these maps preserve the structure of the fibration $N'/\Gamma' \to N/\Gamma$. So, each descends to a map $N/\Gamma \to N/\Gamma$ and these quotient maps satisfy $\underline{F} \circ \underline{H} = \underline{H} \circ \underline{L'}$. By assumption, we already know what two of these maps are, so we have that $f \circ \underline{H} = \underline{H} \circ L$. But, as H was assumed to be a C^1 map, and we showed that it perserves the fibers of the bundle $N'/\Gamma' \to N/\Gamma$, we obtain that $\underline{H}$ is C^1. However, as h is not C^1, this contradicts Proposition 2, which shows that if one conjugacy between f and L is C^1 then all conjugacies are C^1. $\square$

Using the above we can prove the following theorem.

Theorem 2.9.8. *Suppose that $L\colon N/\Gamma \to N/\Gamma$ is an Anosov automorphism with simple spectrum such that either L is not irreducible or the exponents of L are not sorted. Then L is not locally Lyapunov spectrum rigid or periodic data rigid.*

Proof. There exists a term N_k in the lower central series such that the map induced by L on $(N/N_k)/\pi(\Gamma)$ satisfies the hypotheses of either Proposition 17 or Proposition 18. Consequently the induced map on the quotient is not periodic data rigid. By Proposition 19, we conclude that L is not periodic data rigid or Lyapunov spectrum rigid either. □

2.10 EXAMPLES, COUNTEREXAMPLES, AND CURIOSITIES

In this section we construct several examples of Anosov automorphisms of nilmanifolds illustrating novelties of the nilmanifold case. Most importantly, we construct an automorphism of a two-step nilmanifold that is irreducible and has sorted spectrum, which shows that the rigidity theorem in this paper is not vacuous. We also describe nilmanifolds that do not admit locally rigid automorphisms. We also give an example of an automorphism where the conjugacy is C^{1+} along the unstable foliation but not the stable foliation.

2.10.1 A locally rigid automorphism

In his seminal survey of smooth dynamics, [Sma67], Smale gave two examples of Anosov automorphisms on a particular two-step nilmanifold. However, Smale's examples are not rigid because on the quotient torus they are the direct sum of two automorphisms of $\mathbb{T}^2$. Consequently, they are reducible. Though Smale's particular examples are not rigid, it is possible to construct a rigid example on that nilmanifold.

Example 1 (Smale's nilmanifold)**.** Consider the Lie group given by the product of two copies of the Heisenberg group, H_1 and H_2. Write X_i, Y_i, Z_i, where $[X_i, Y_i] = Z_i$ for the usual basis of the Lie algebra $\mathfrak{h}_i$ of H_i. We then construct an automorphism of the Lie group and algebra by setting:

$$\begin{aligned}
X_1 &\mapsto (26+15\sqrt{3})X_1 + (8733+5042\sqrt{3})Y_1 \\
Y_1 &\mapsto (71+41\sqrt{3})X_1 + (28901+16686\sqrt{3})Y_1 \\
Z_1 &\mapsto (262087+151316\sqrt{3})Z_1 \\
X_2 &\mapsto (26-15\sqrt{3})X_2 + (8733-5042\sqrt{3})Y_2 \\
Y_2 &\mapsto (71-41\sqrt{3})X_2 + (28901-16686\sqrt{3})Y_2 \\
Z_2 &\mapsto (262087-151316\sqrt{3})Z_2.
\end{aligned}$$

Note that for X_2, Y_2, Z_2 we have just changed $+\sqrt{3}$ to $-\sqrt{3}$. This defines an automorphism of $H_1 \times H_2$. As in Smale's case, we define a lattice by first defining a lattice in $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. View this Lie algebra as matrices of the form

$$\begin{bmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{bmatrix}.$$

Now for the lattice, take the elements where the first matrix has entries in $\mathcal{O}(\mathbb{Q}(\sqrt{3}))$ and for the second factor take the conjugate entries to those chosen in the first factor. Explicitly, this lattice consists of matrices with entries in $\mathbb{Z} + \mathbb{Z}\sqrt{3}$ of the form

$$\begin{bmatrix} 0 & a+b\sqrt{3} & e+f\sqrt{3} \\ 0 & 0 & c+d\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & a-b\sqrt{3} & e-f\sqrt{3} \\ 0 & 0 & c-d\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

The automorphism specified preserves this lattice in $\mathfrak{h}$ because the automorphism is defined over $\mathbb{Z}[\sqrt{3}]$, and is invertible because its determinant is in $\mathbb{Z}[\sqrt{3}]^\times$. Thus it can be shown that the lattice, Γ, generated by the exponential image of this lattice in $H_1 \oplus H_2$ is preserved by the corresponding automorphism of $H_1 \oplus H_2$. Thus we obtain an automorphism of $(H_1 \oplus H_2)/\Gamma$.

We now check that this automorphism is sorted and totally irreducible. First we need irreducibility over $\mathbb{Q}$ of the action in the base and in the fiber. The map in the fiber is a map of a 2-torus and hence is irreducible as it is hyperbolic. The map in the fiber has eigenvalues $262087 + 151316\sqrt{3}$ and $262087 - 151316\sqrt{3}$.

In order to compute the map in the base, it suffices to study the automorphism we obtain of the lattice when we have quotiented out by the fiber. Represent an element of the basis for the quotient lattice by (a, b, c, d) corresponding to the matrix:

$$\begin{bmatrix} 0 & a + b\sqrt{3} & * \\ 0 & 0 & c + d\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & a - b\sqrt{3} & * \\ 0 & 0 & c - d\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

Written with respect to the coordinates (a, b, c, d), the automorphism is:

$$\begin{bmatrix} 26 & 45 & 71 & 123 \\ 15 & 26 & 41 & 71 \\ 8733 & 15126 & 28901 & 50058 \\ 5042 & 8733 & 16686 & 28901 \end{bmatrix}. \tag{2.12}$$

We then calculate the eigenvalues of this matrix using Mathematica. They are approximately:

```
57844.9, 9.06171, 0.0193643, 0.0000985198
```

Whereas the eigenvalues in the central fiber are approximately:

```
524174, .00000190779(=1.9*10e-6)
```

Thus we see that the stable and unstable spectrum are sorted. Finally, one may compute the characteristic polynomial of this matrix and check that it is irreducible over $\mathbb{Q}$. This automorphism is sorted and irreducible and so is locally Lyapunov spectrum rigid by the main theorem.

2.10.2 Non-rigid families

Using the necessity of sorted spectrum for local rigidity, we can exhibit nilmanifolds such that no Anosov automorphism with simple spectrum on that nilmanifold is Lyapunov spectrum rigid. For this construction we use a free nilpotent Lie group. Much additional information about automorphisms of free nilpotent groups is contained in [Pay09]. We consider the two-step free nilpotent group on 3 generators, $N_{3,2}$. Explicitly, define a Lie algebra $\mathfrak{n}$ via three generators x, y, z. These generators bracket to linearly independent elements $[x, y]$, $[x, z]$ and $[y, z]$, and we stipulate that $\mathfrak{n}_3 = 0$, so that this Lie algebra is two-step. In fact, we see that $\{x, y, z, [x, y], [x, z], [y, z]\}$ is a basis for $\mathfrak{n}$ with rational structure coefficients, and hence its $\mathbb{Z}$-span is a lattice in $\mathfrak{n}$. The exponential image of this lattice generates a lattice in N, which we call Λ. As $N_{3,2}$ is free, an automorphism L of $N_{3,2}$ is determined by the induced map $\underline{L}$ on $N_{3,2}/Z(N_{3,2})$, which we identify with $\mathbb{R}^3$. If $\underline{L}$ has eigenvalues $0 < |\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$, and preserves volume so that $|\lambda_1|\,|\lambda_2|\,|\lambda_3| = 1$, then we see that $L\mid_{Z(N_{2,3})}$ has as its exponents $|\lambda_i|^{-1}$, for $i \in \{1, 2, 3\}$. In particular, $1 > |\lambda_2|^{-1} > |\lambda_1|$, and so such an automorphism has its stable exponents out of order. Consequently, no such automorphism of $N_{3,2}/\Lambda$ is Lyapunov spectrum rigid.

As an example of this, consider the map induced by the companion matrix associated to the polynomial $x^3 + x^2 - 8x + 1$ acting on a basis that generate the Lie algebra to $N_{3,2}$. This polynomial defines a totally real extension of degree 3 over $\mathbb{R}$. Despite the resulting automorphism, L, not being rigid, it is still irreducible and its unstable exponents are sorted. Consequently, the argument in this paper shows that a conjugacy between L and a sufficiently small perturbation with the same spectrum is C^{1+} along the unstable foliation even though it might not be C^{1+} along the stable foliation.

2.10.3 Rigidity of non-linear examples

It is natural to ask whether there are non-linear Anosov diffeomorphisms that exhibit volume Lyapunov spectrum rigidity. Recent examples constructed by Erchenko show that even in the case of $\mathbb{T}^2$, two diffeomorphisms may have the same volume Lyapunov spectrum and not be C^1 conjugate. This follows from [Erc19, Thm. 1.1], which gives examples of diffeomorphisms with the same volume Lyapunov spectrum but different Lyapunov spectrum for the measure of maximal entropy. See Question 2 of [Erc19] for a more refined question relating rigidity to an associated pressure function.

CHAPTER 3

SIMULTANEOUS LINEARIZATION OF DIFFEOMORPHISMS OF ISOTROPIC MANIFOLDS

3.1 INTRODUCTION

A basic problem in dynamics is determining whether two dynamical systems are equivalent. A standard notion of equivalence is conjugacy: if f and g are two diffeomorphisms of a manifold M, then f and g are *conjugate* if there exists a homeomorphism h of M such that $hfh^{-1} = g$. Some classes of dynamical systems are distinguished up to conjugacy by a small amount of dynamical information. One of the most basic examples of this is Denjoy's theorem: a C^2 orientation preserving circle diffeomorphism with irrational rotation number is conjugate to a rotation [KH97, §12.1]. In the case of Denjoy's theorem, the rotation number is all the information needed to determine the topological equivalence class of the diffeomorphism under conjugacy.

Rigidity theory focuses on identifying dynamics that are distinguished up to conjugacy by particular kinds of dynamical information such as the rotation number. There are finer dynamical invariants than rotation number which require a finer notion of equivalence to study. For instance, one obtains a finer notion of equivalence if one insists that the conjugacy be a C^1 or even C^∞ diffeomorphism. A smoother conjugacy allows one to consider invariants such as Lyapunov exponents, which may not be preserved under conjugacy by homeomorphisms. For a single volume preserving Anosov diffeomorphism, the Lyapunov exponents with respect to volume are invariant under conjugation by C^1 volume preserving maps. Consequently, one is naturally led to ask, "If two volume preserving Anosov diffeomorphisms have the same

Lyapunov exponents are the two C^1 conjugate?" In some circumstances the answer is "yes". Such situations where knowledge about Lyapunov exponents implies systems are conjugate by a C^1 diffeomorphism are instances of a phenomenon called "Lyapunov spectrum rigidity". See [Gog19] for examples and discussion of this type of rigidity. For recent examples, see [But17], [DeW21],[GRH19],[GKS18], and [SY19].

In rigidity problems related to isometries, it is often natural to consider a family of isometries. A collection of isometries may have strong rigidity properties even if the individual elements of the collection do not. For example, Fayad and Khanin [FK09] proved that a collection of commuting diffeomorphisms of the circle whose rotation numbers satisfy a simultaneous Diophantine condition are smoothly simultaneously conjugated to rotations. Their result is a strengthening of an earlier result of Moser [Mos90]. A single diffeomorphism in such a collection might not satisfy the Diophantine condition on its own.

Although the two types of rigidity described above occur in the dissimilar hyperbolic and elliptic settings, a result of Dolgopyat and Krikorian combines the two. They introduce a notion of a Diophantine set of rotations of a sphere and use this notion to prove that certain random dynamical systems with all Lyapunov exponents zero are conjugated to isometric systems [DK07]. Our result is a generalization of this result to the setting of isotropic manifolds. We now develop the language to state both precisely.

Let $(f_1, ..., f_m)$ be a tuple of diffeomorphisms of a manifold M. Let $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with uniform distribution on $\{1, ..., m\}$. Given an initial point $x_0 \in M$, define $x_n = f_{\omega_n} x_{n-1}$. This defines a Markov process on M. We refer to this process as the random dynamical system associated to the tuple $(f_1, ..., f_m)$. Let f^n_ω be defined to equal $f_{\omega_n} \circ \cdots \circ f_{\omega_1}$. We say that a probability measure μ on M is a *stationary measure* for this process if $m^{-1}\sum_{i=1}^m (f_i)_*\mu = \mu$. A stationary measure is *ergodic* if it is not a non-trivial convex combination of two distinct stationary measures. Fix an ergodic stationary measure μ. For μ-almost every x, almost surely for any $v \in T_xM \setminus \{0\}$, the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x f_\omega^n v\| \tag{3.1}$$

and takes its value in a fixed finite list of numbers depending only on μ:

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_{\dim M}(\mu). \tag{3.2}$$

These are the *Lyapunov exponents* with respect to μ. In fact, for almost every ω and μ-a.e. x there exists a flag $V_1 \subset \cdots \subset V_j$ inside T_xM such that if $v \in V_i \setminus V_{i-1}$ then the limit in (3.2) is equal to $\lambda_{\dim M - \dim V_i}$. The number of times a particular exponent appears in (3.2) is given by $\dim V_i - \dim V_{i-1}$; this number is referred to as the multiplicity of the exponent. For more information, see [Kif86].

Our result holds for isotropic manifolds. By definition, an *isotropic manifold* is a Riemannian manifold whose isometry group acts transitively on its unit tangent bundle. The closed isotropic manifolds are S^n, $\mathbb{RP}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, and the Cayley projective plane. In the following we write G° for the identity component of a Lie group G.

Theorem 3.1.1. *Let M^d be a closed isotropic Riemannian manifold other than S^1. There exists k_0 such that if $(R_1, ..., R_m)$ is a tuple of isometries of M such that the subgroup of* $\mathrm{Isom}(M)$ *generated by this tuple contains* $\mathrm{Isom}(M)^\circ$*, then there exists $\epsilon_{k_0} > 0$ such that the following holds. Let $(f_1, ..., f_m)$ be a tuple of C^∞ diffeomorphisms satisfying* $\max_i d_{C^{k_0}}(f_i, R_i) < \epsilon_{k_0}$*. Suppose that there exists a sequence of ergodic stationary measures μ_n for the random dynamical system generated by $(f_1, ..., f_m)$ such that $|\lambda_d(\mu_n)| \to 0$, then there exists $\psi \in \mathrm{Diff}^\infty(M)$ such that for each i the map $\psi f_i \psi^{-1}$ is an isometry of M and lies in the subgroup of* $\mathrm{Isom}(M)$ *generated by* $(R_1, \ldots, R_m)$.

Dolgopyat and Krikorian proved Theorem 3.1.1 in the case of S^n [DK07, Thm. 1].

Dolgopyat and Krikorian also obtained a Taylor expansion of the Lyapunov exponents of the stationary measure of the perturbed system [DK07, Thm. 2]. Fix $(R_1, \ldots, R_m)$ generating $\mathrm{Isom}(S^n)^\circ$. Let $(f_1, \ldots, f_m)$ be a C^{k_0} small perturbation of $(R_1, \ldots, R_m)$ and μ

be any ergodic stationary measure for $(f_1, \ldots, f_m)$. Let $\Lambda_r = \lambda_1 + \cdots + \lambda_r$ denote the sum of the top r Lyapunov exponents. In [DK07, Thm. 2], it is shown that the Lyapunov exponents of μ satisfy

$$\lambda_r(\mu) = \frac{\Lambda_d}{d} + \frac{d-2r+1}{d-1}\left(\lambda_1 - \frac{\Lambda_d}{d}\right) + o(1)|\lambda_d(\mu)|, \tag{3.3}$$

where $o(1)$ goes to zero as $\max_i d_{C^{k_0}}(f_i, R_i) \to 0$. Using this formula Dolgopyat and Krikorian obtain an even stronger dichotomy for systems on even dimensional spheres: either $(f_1, \ldots, f_m)$ is simultaneously conjugated to isometries or the Lyapunov exponents of every ergodic stationary measure of the perturbation are uniformly bounded away from zero. By using this result they show if $(R_1, \ldots, R_m)$ generates $\mathrm{Isom}(S^{2n})^\circ$ and $(f_1, \ldots, f_m)$ is a C^{k_0} small perturbation such that each f_i preserves volume, then volume is an ergodic stationary measure for $(f_1, \ldots, f_m)$ [DK07, Cor. 2].

It is natural to ask if a similar Taylor expansion can be obtained in the setting of isotropic manifolds. Proposition 12 shows that Λ_r may be Taylor expanded assuming that $(R_1, ..., R_m)$ generates $\mathrm{Isom}(M)^\circ$ and the induced action of $\mathrm{Isom}(M)^\circ$ on $\mathrm{Gr}_r(M)$, the Grassmanian bundle of r-planes in TM, is transitive.

In Theorem 3.6.1, we give a Taylor expansion relating λ_1 and λ_d which holds for isotropic manifolds. However, we cannot Taylor expand every Lyapunov exponent as in equation (3.3) because if a manifold does not have constant curvature then its isometry group cannot act transitively on the two-planes in its tangent spaces. The argument of Dolgopyat and Krikorian requires that the isometry group act transitively on the space of k-planes in TM for $0 \leq k \leq d$.

It is natural to ask why the proof of Theorem 3.1.1 does not work in the case of S^1 even though S^1 is isotropic. As Proposition 6 shows, for a tuple $(R_1, \ldots, R_m)$ as in the theorem, uniformly small perturbations of $(R_1, \ldots, R_m)$ are uniformly Diophantine in a sense explained below. This uniformity is used crucially in the proof when we change the tuple of isometries that we are working with. The same uniformity of Diophantineness does not hold for tuples of isometries of S^1: a small perturbation may lose all Diophantine properties. The reason that the proof of Proposition 6 does not work for S^1 is that the isometry group of S^1 is not semi-simple.

There are not many other results like Theorem 3.1.1. In addition to the aforementioned result of Dolgopyat and Krikorian, there are some results of Malicet. In [Mal12], a similar linearization result is obtained that applies to a particular type of map of $\mathbb{T}^2$ that fibers over a rotation on S^1. In a recent work, Malicet obtained a Taylor expansion of the Lyapunov exponent for a perturbation of a Diophantine random dynamical system on the circle [Mal20].

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3.1.1 Outline

The proof of Theorem 3.1.1 follows the general argument of [DK07]. For readability, the argument in this paper is self-contained. While a number of the results below appear in [DK07], we have substantially reformulated many of them and in many places offer a different proof. Doing so is not merely a courtesy to the reader: the results in [DK07] are stated in too narrow a setting for us to use. Simply stating more general reformulations would unduly burden the reader's trust. In addition, as will be discussed below, there are some oversights in [DK07] which we explain in subsection 3.1.2 and that we have remedied in Section 3.5. We have also stated intermediate results and lemmas in more generality than is needed for the proof of Theorem 3.1.1 so that they may be used by others. Below we sketch the general argument of the paper and emphasize some differences with the approach in [DK07].

The proof of Theorem 3.1.1 is by an iterative KAM convergence scheme. Fix a closed isotropic manifold M. We start with a tuple of diffeomorphisms $(f_1, \ldots, f_m)$ nearby to a tuple of isometries $(R_1, \ldots, R_m)$. We must find some smooth diffeomorphism ψ such that $\widetilde{f}_i := \psi f_i \psi^{-1} \in \mathrm{Isom}(M)$. To do this we produce a conjugacy ψ that brings each f_i closer to being an isometry. To judge the distance from being an isometry, we define a strain tensor that vanishes precisely when a diffeomorphism is an isometry. By solving a particular coboundary equation and using that the Lyapunov exponents are zero, we can construct ψ so that $\widetilde{f}_i$ has small strain tensor. In our setting, a diffeomorphism with small strain is near to an isometry, so $(\widetilde{f}_1, \ldots, \widetilde{f}_m)$ is near to a tuple of isometries $(R_1', \ldots, R_m')$. We then repeat the procedure using these new tuples as our starting point. The results of performing a single step of this procedure comprise Lemma 3.6.1. Once Lemma 3.6.1 is proved, the rest of the proof of Theorem 3.1.1 is bookkeeping that checks that the procedure converges. Most of the paper is in service of the proof Lemma 3.6.1, which gives the result of a single step in the convergence scheme.

Proofs of technical and basic facts are relegated to a significant number of appendices. This has been done to focus the main exposition on the important ideas in the proof of Theorem 3.1.1 and not on the technical details. The appendices that might be most beneficial to look at before they are referenced in the text are appendices 3.7.1 and 3.7.2. These appendices concern C^k calculus and interpolation inequalities. Both contain estimates that are common in KAM arguments. The organization of the main body of the paper reflects the order of the steps in the proof of Lemma 3.6.1. There are several important results in the proof of Lemma 3.6.1, which we now describe.

The first part of the proof of Lemma 3.6.1 requires that a particular coboundary equation can be tamely solved. The solution to this equation is one of the main subjects of Section 3.2. The equation is solved in Proposition 7. This proposition is essential in the work of Dolgopyat and Krikorian [DK07] and its proof follows from the appendix to [Dol02]; it relies on a Diophantine property of the tuple of isometries $(R_1, \ldots, R_m)$. This property is formulated in subsection 3.2.2. The stability of this property under perturbations is crucial in the proof and an essential feature of our setting. In addition, the argument in Section

3.2 is different from Dolgopyat's earlier argument because we we use the Solovay-Kitaev algorithm (Theorem 3.2.1), which is more efficient than the procedure used in the appendix to [Dol02].

Section 3.3 considers stationary measures for perturbations of $(R_1, \ldots, R_m)$. Suppose M is a quotient of its isometry group, its isometry group is semisimple, and $(R_1, \ldots, R_m)$ is a Diophantine subset of $\mathrm{Isom}(M)$. Suppose $(f_1, \ldots, f_m)$ is a small smooth perturbation of $(R_1, \ldots, R_m)$. There is a relation between a stationary measure μ for the perturbed system and the Haar measure. Proposition 11 relates integration against μ with integration against the Haar measure. Lyapunov exponents are calculated by integrating the log Jacobian against a stationary measure of an extended dynamical system on a Grassmannian bundle over M. Consequently, this proposition relates stationary measures and their Lyapunov exponents to the volume on a Grassmannian bundle.

The relationship between Lyapunov exponents and stationary measures is explained in Section 3.4. Proposition 12 provides a Taylor expansion of the sum of the top r Lyapunov exponents of a stationary measure μ. Three terms appear in the Taylor expansion. The first two terms have a direct geometric meaning, which we interpret in terms of strain tensors introduced in subsection 3.4.2. The final term in the Taylor expansion depends on a quantity $\mathcal{U}(\psi)$. This quantity does not have a direct geometric interpretation. However, in the proof of Lemma 3.6.1, we show that by solving the coboundary equation from Proposition 7 the quantity $\mathcal{U}(\psi)$ can be made to vanish. Once $\mathcal{U}(\psi)$ vanishes, then we have an equation directly relating Lyapunov exponents to the strain. This equation then allows us to conclude that a diffeomorphism with small Lyapunov exponents also has small strain. We reformulate in a Riemannian geometric setting some arguments of [DK07] by using the strain tensor. This gives coordinate-free expressions that are easier to interpret.

Section 3.5 contains the most important connection between the strain tensor and isometries: diffeomorphisms of small strain on isotropic manifolds are near to isometries. The basic geometric fact proved in Section 3.5 is Theorem 3.5.1, which is true on any manifold.

Theorem 3.5.1 is then used to prove Proposition 13, which is a more technical result adapted for use in the KAM scheme. Proposition 13 then allows us to prove that our conjugated tuple is near to a new tuple of isometries, which allows us to repeat the process.

All of the previous sections combine in Section 3.6 to prove Lemma 3.6.1. We then obtain the main theorem, Theorem 3.1.1, and prove an additional theorem that relates the top and bottom Lyapunov exponents of a perturbation, Theorem 3.6.1.

3.1.2 An oversight and its remedy

Section 3.5 is entirely new and different from anything appearing in [DK07]. Consequently, the reader may wonder why it is needed. Section 3.5 provides a method of finding a tuple of isometries $(R_1', \ldots, R_m')$ near to the tuple $(\widetilde{f}_1, \ldots, \widetilde{f}_m)$ of diffeomorphisms. In [DK07], the new diffeomorphisms R_m are found in the following manner. As in equation (3.9), one may find vector fields Y_i such that

$$\exp_{R_i(x)} Y_i(x) = f_i(x).$$

If Z is a vector field on M, we define ψ_Z, as in equation (3.10) to be the map $x \mapsto \exp_x Z(x)$. There is a certain operator, the Casimir Laplacian, which acts on vector fields. This operator is defined and discussed in more detail in subsection 3.2.2. Dolgopyat and Krikorian then project the vector fields Y_i onto the kernel of the Casimir Laplacian, to obtain a vector field Y_i'. They then define R_i' to equal $\psi_{Y_i'} \circ R_i$. This happens in the line immediately below equation (19) in [DK07].

One difficulty is establishing that the maps $(R_1', \ldots, R_m')$ are close to the $(\widetilde{f}_i, \ldots, \widetilde{f_m})$. The argument for their nearness hinges on part (d) of Proposition 3 in [DK07], which essentially says that, up to a third order error, the magnitude of the smallest Lyapunov exponent is a bound on the distance. As written, the argument in [DK07] suggests that part (d) is an easy consequence of part (c) of [DK07, Prop. 3]. However, part (d) does not follow. Here is a simplification of the problem. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Pick a point

$x \in \mathbb{R}^n$ and write $D_x f = A+B+C$, where A is a multiple of the identity, B is symmetric with trace zero, and C is skew-symmetric. The results in part (c) imply that A and B are small, but they offer no information about C.[1] Concluding that the norm of Df is small requires that C be small as well. As C is skew-symmetric it is natural to think of it as the germ of an isometry. Our modification to the argument is designed to accommodate the term C by recognizing it as the "isometric" part of the differential. Pursuing this perspective leads to the strain tensor and our Proposition 13. Conversation with Dmitry Dolgopyat confirmed that there is a problem in the paper on this point and that part (d) of Proposition 3 does not follow from part (c).

3.2 A DIOPHANTINE PROPERTY AND SPECTRAL GAP

Fix a compact connected semisimple Lie group G and let $\mathfrak{g}$ denote its Lie algebra. Endow G with the bi-invariant metric arising from the negative of the Killing form on $\mathfrak{g}$. We denote this metric on G by d. We endow a subgroup H of G with the pullback of the Riemannian metric from G and denote the distance on H with respect to the pullback metric by d_H. We use the manifold topology on G unless explicitly stated otherwise. Consequently, whenever we say that a subset of G is dense, we mean this with respect to the manifold topology on G. We say that a subset S of G *generates* G if the smallest closed subgroup of G containing S is G. In other words, if $\langle S \rangle$ denotes the smallest subgroup of G containing S, then S generates if $\overline{\langle S \rangle} = G$.

Suppose that $S \subset G$ generates G. We begin this section by discussing how long a word in the elements of S is needed to approximate an element of G. Then using this approximation we obtain quantitative estimates for the spectral gap of certain operators associated to S. Finally, those spectral gap estimates allow us to obtain a "tameness" estimate for a particular operator that arises from S. This final estimate, Proposition 7, will be crucial in the KAM

[1] For those comparing with the original paper, A and B correspond to the terms q_1 and q_2, respectively, which appear in part (c) of [DK07, Prop. 3].

scheme that we use to prove Theorem 3.1.1.

The content of this section is broadly analogous to Appendix A in [Dol02]. However, our development follows a different approach and in some places we are able to obtain stronger estimates.

3.2.1 The Solovay-Kitaev algorithm

Suppose that S is a subset of G. We say that S is *symmetric* if $s \in S$ implies $s^{-1} \in S$. For a natural number n, let S^n denote the n-fold product of S with itself. Let S^{-1} be $\{s^{-1} : s \in S\}$. For $n < 0$, define S^n to equal $(S^{-1})^{-n}$. The following theorem says that any sufficiently dense symmetric subset S of a compact semisimple Lie group is a generating set. More importantly, it also gives an estimate on how long a word in the generating set S is needed to approximate an element of G to within error ϵ. If $w = s_1 \cdots s_n$ is a word in the elements of the set S, then we say that w is *balanced* if for each $s \in S$, s appears the same number of times in w as s^{-1} does.

Theorem 3.2.1. *[DN06, Thm. 1](Solovay-Kitaev Algorithm) Suppose that G is a compact semisimple Lie group. There exists $\epsilon_0(G) > 0$ and $\alpha > 0$ and $C > 0$ such that if S is any symmetric ϵ_0-dense subset of G then the following holds. For any $g \in G$ and any $\epsilon > 0$, there exists a natural number l_ϵ such that $d(g, S^{l_\epsilon}) < \epsilon$. Moreover, $l_\epsilon \leq C \log^\alpha(1/\epsilon)$. Further, there is a balanced word of length l_ϵ within distance ϵ of g.*

Later, we use a version of this result that does not require that the set S be symmetric. Using a non-symmetric generating set significantly increases the word length obtained in the conclusion of the theorem. It is unknown if there exists a version of the Solovay-Kitaev algorithm that does not require a symmetric generating set and keeps the $O(\log^\alpha(1/\epsilon))$ word length. See [BO18] for a partial result in this direction.

Proposition 1. *Suppose that G is a compact semisimple Lie group endowed with a bi-invariant metric. There exists $\epsilon_0(G) > 0$, $\alpha > 0$, and $C \geq 0$ such that if S is any ϵ_0-dense* *subset of G then the following holds. For any $g \in G$ and any $\epsilon > 0$, there exists a natural number l_ϵ such that $d(g, S^{l_\epsilon}) < \epsilon$. Moreover, $l_\epsilon \leq C\epsilon^{-\alpha}$.*

Our weakened version of the Solovay-Kitaev algorithm relies on the following lemma, which allows us to approximate the inverse of an element h by some positive power of h.

Lemma 3.2.1. *Suppose that G is a compact d-dimensional Lie group with a fixed bi-invariant metric. Then there exists a constant C such that for all $\epsilon > 0$ and any $h \in G$ there exists a natural number $n < C/\epsilon^d$ such that $d(h^{-1}, h^n) < \epsilon$.*

Proof. This follows from a straightforward pigeonhole argument. We cover G with sets of diameter ϵ. There exists a constant C so that we can cover G with at most $C\,\mathrm{vol}(G)/\epsilon^d$ such sets, where d is the dimension of G. Consider now the first $\lceil C\,\mathrm{vol}(G)/\epsilon^d \rceil$ iterates of h^2. By the pigeonhole principle, two of these must fall into the same set in the covering, and so there exist natural numbers n_i and n_j such that $0 < n_i < n_j < \lceil C\,\mathrm{vol}(G)/(\epsilon^d) \rceil$ and h^{2n_i} and h^{2n_j} lie in the same set in the covering. Thus $d(h^{2n_i}, h^{2n_j}) < \epsilon$. As h is an isometry it follows that $d(e, h^{2n_j - 2n_i}) < \epsilon$ and hence $d(h^{-1}, h^{2n_j - 2n_i - 1}) < \epsilon$ as well. This finishes the proof. □

We now prove the proposition.

Proof of Proposition 1. Let $\hat{S} = S \cup S^{-1}$. Note that as $\hat{S}$ is a symmetric generating set of G that by Theorem 3.2.1 for any $\epsilon > 0$, there exists a number $l_{\epsilon/2} = O(\log^\alpha(1/\epsilon))$ such that for any $g \in G$ there exists an element h in $\hat{S}^{l_{\epsilon/2}}$ such that $d(h, g) < \epsilon/2$. Further, by the statement of Theorem 3.2.1, we know that h is represented by a balanced word w in $\hat{S}^{l_{\epsilon/2}}$.

To finish the proof, we replace each element of w that is in S^{-1} by a word in S^j for some uniform $j > 0$. To do this we show that there exists a fixed j so that the elements of S^j approximate well the inverses of the elements of S. Write $S = \{s_1, \cdots, s_m\}$ and consider the element $(s_1, ..., s_m)$ in the group $G \times \cdots \times G$, where there are m terms in the product. By applying Lemma 3.2.1 to the group $G \times \cdots \times G$ and the element $(s_1, ..., s_m)$, we obtain that there exists a uniform constant C' and $j < C' 2^{dm} l_{\epsilon/2}^{dm}/\epsilon^{dm}$ such that any $s \in S^{-1}$ may be approximated to distance $\epsilon/(2l_{\epsilon/2})$ by an element in S^j.

We now replace each element of S^{-1} appearing in w with a word in S^j that is at distance $\epsilon/(2l_{\epsilon/2})$ away from it. Call this new word w'. Because w is balanced, we replace exactly half of the terms in w. Thus w' is a word of length $jl_{\epsilon/2}/2 + l_{\epsilon/2}/2$ as we have replaced half the entries of w, which has length $l_{\epsilon/2}$, with words of length j. Let h' be the element of G obtained by multiplying together the terms in w'.

Note that multiplication of any number of elements of G is 1-Lipschitz in each argument. Hence as we have modified the expression for h in exactly $l_{\epsilon/2}/2$ terms and each modification is of size $\epsilon/(2l_{\epsilon/2})$, h' is distance at most $\epsilon/2$ from h and hence at most distance ϵ from g. Thus $S^{jl_{\epsilon/2}/2+l_{\epsilon/2}/2}$ is ϵ dense in G and

$$jl_{\epsilon/2}/2 + l_{\epsilon/2}/2 < C'' l_{\epsilon/2}^{dm+1}/\epsilon^{dm} = O(\log^{(dm+1)\alpha}(1/\epsilon)\epsilon^{-dm}),$$

which establishes the proposition as m depends only on $|S|$. □

We record one final result that asserts that if $S \subseteq G$ generates, then the powers of S individually become dense in G.

Proposition 2. *Suppose that G is a compact connected Lie group. Suppose that $S \subseteq G$ generates G. Then for all $\epsilon > 0$ there exists a natural number n_ϵ such that S^{n_ϵ} is ϵ-dense in G.*

Proof. Let $\{g_1, ..., g_m\}$ be an $\epsilon/2$-dense subset of G. Because S generates, for each g_i there exists n_i and $w_i \in S^{n_i}$ such that $d(g_i, w_i) < \epsilon/2$. By a pigeonhole argument similar to the proof of Lemma 3.2.1, it holds that for all $\epsilon > 0$ there exists a natural number N such that for all $n \geq N$, $d(S^n, e) < \epsilon$. Thus there exists N such that for all $n \geq N$, S^n contains elements within distance $\epsilon/2$ of the identity. Thus $S^{N+\max_i\{n_i\}}$ is ϵ-dense in G. □

3.2.2 Diophantine Sets

We will now introduce a notion of a Diophantine subset of a compact connected semisimple Lie group G. Write $\mathfrak{g}$ for the Lie algebra of G. We recall the definition of the standard quadratic Casimir inside of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Write B for the Killing form on $\mathfrak{g}$ and let X_i be an orthonormal basis for $\mathfrak{g}$ with respect to B. We will also denote the inner product arising from the Killing form by $\langle \cdot, \cdot \rangle$. Then the *Casimir*, Ω, is the element of $U(\mathfrak{g})$ defined by

$$\Omega = \sum_i X_i^2.$$

The element Ω is well-defined and central in $U(\mathfrak{g})$. Elements of $U(\mathfrak{g})$ act on the smooth vectors of representations of G. Consequently, as Ω is central and every vector in an irreducible representation (π, V) is smooth, $\pi(\Omega)$ acts by a multiple of the identity. Given an irreducible unitary representation (π, V), Define $c(\pi)$ by

$$c(\pi)\,\mathrm{Id} = -\pi(\Omega). \tag{3.4}$$

The quantity $c(\pi)$ is positive in non-trivial representations. Further, as π ranges over all non-trivial representations, $c(\pi)$ is uniformly bounded away from 0. For further information see [Wal18, 5.6].

Definition 3.2.2. Let G be a compact, connected, semisimple Lie group. We say that a subset $S \subset G$ is (C, α)-*Diophantine* if the following holds for each non-trivial, irreducible, finite dimensional unitary representation (π, V) of G. For all non-zero $v \in V$ there exists $g \in S$ such that

$$\|v - \pi(g)v\| \geq Cc(\pi)^{-\alpha}\|v\|,$$

where $c(\pi)$ is defined in (3.4). We say that S is *Diophantine* if S is (C, α)-Diophantine for some $C, \alpha > 0$. If $(g_1, ..., g_m)$ is a tuple of elements of G, the we say that this tuple is (C, α)-Diophantine if the underlying set is (C, α)-Diophantine.

Our formulation of Diophantine is slightly different from the definition in [Dol02] as we refer directly to irreducible representations. We choose this formulation because it allows for a unified analysis of the action of Ω in diverse representations of G.

Proposition 3. *[Dol02, Thm. A.3] Suppose that S is a finite subset of a compact connected semisimple Lie group G. Then S is Diophantine if and only if $\overline{\langle S\rangle} = G$. Moreover, there exists $\epsilon_0(G)$ such that any ϵ_0-dense subset of G is Diophantine.*

Before proceeding to the proof we will show two preliminary results.

Lemma 3.2.3. *Suppose that G is a compact connected semisimple Lie group. Suppose that (π, V) is an irreducible unitary representation of G. Then for any $v \in V$ of unit length, any $X \in \mathfrak{g}$ of unit length, and $t \geq 0$,*

$$\|\pi(\exp(tX))v - v\| \leq t\sqrt{c(\pi)}.$$

Proof. A similar argument to the following appears in [Wal18, 5.7.13]. There exists an orthonormal basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ such that $X_1 = X$. Observe that

$$\pi(\exp(tX))v - v = td\pi(X)v + O(t^2).$$

The transformation $d\pi(X)$ is skew symmetric with respect to the inner product. Thus $d\pi(X)^2$ is positive semidefinite. Consequently:

$$\langle d\pi(X)v, d\pi(X)v\rangle = -\langle d\pi(X)^2 v, v\rangle \leq -\langle \pi(\Omega)v, v\rangle = c(\pi)\|v\|^2.$$

Hence

$$\|\pi(\exp(tX)v) - v\| \leq t\sqrt{c(\pi)} + O(t^2).$$

For $0 \leq i \leq n$, let $t_i = \frac{i}{n}t$. Then

$$\begin{aligned}\|\pi(\exp(tX))v - v\| &\leq \sum_{i=1}^{n} \|\pi(\exp(t_i X))v - \pi(\exp(t_{i-1}X))v\| \\ &\leq \sum_{i=1}^{n} \|\pi(\exp(tX/n))v - v\| \\ &\leq n\left(\frac{t}{n}\sqrt{c(\pi)} + O((t/n)^2)\right).\end{aligned}$$

Taking the liminf of the right hand side as $n \to \infty$ gives the result. □

The following lemma will be of use in the proof of Proposition 4.

Lemma 3.2.4. *Suppose that (π, V) is a non-trivial, irreducible, finite dimensional, unitary representation of a compact, connected, semisimple group G. Then for any $v \in V$, there exists g such that $\langle \pi(g)v, v\rangle = 0$.*

Proof. If such a g does not exist, then for all $g \in G$, $\pi(g)v$ lies in the same half-space as v. But then $\int_G \pi(g)v\, dg \neq 0$ and is a G invariant vector, which contradicts the irreducibility of π. □

Proposition 4. *Suppose that G is a compact connected semisimple Lie group. Then there exist $\epsilon_0, C, \alpha > 0$ such that any ϵ_0-dense subset of G is (C, α)-Diophantine. If S is a subset of G such that S^{n_0} is ϵ_0-dense in G, then S is $(C/n_0, \alpha)$ Diophantine.*

Proof. Let ϵ_0 equal the $\epsilon_0(G)$ in Theorem 3.2.1, the Solovay-Kitaev algorithm. In the case that S is already ϵ_0-dense, let $n_0 = 1$. By Theorem 3.2.1, there exist C and α such that for each ϵ there exists $l_\epsilon \leq C \log^{\alpha}(\epsilon^{-1})$ such that $S^{n_0 l_\epsilon}$ is ϵ-dense in G. Suppose that (π, V) is a non-trivial irreducible unitary representation of G and suppose that $v \in V$ is a unit vector. By Lemma 3.2.4 there exists $g \in G$ such that $\langle \pi(g)v, v\rangle = 0$. Now fix $\epsilon = 1/(100\sqrt{c(\pi)})$. Then there exists an element $w \in S^{n_0 l_\epsilon}$ such that $d(g, w) < \epsilon$. Thus by Lemma 3.2.3,

$$\|\pi(g)v - \pi(w)v\| \leq \epsilon\sqrt{c(\pi)} < \frac{1}{100}.$$

By the triangle inequality, this implies that

$$\|\pi(w)v - v\| \geq 1.$$

Write $w = g_1^{\sigma_1} \cdots g_{n_0 l_\epsilon}^{\sigma_{n_0 l_\epsilon}}$ where each $\sigma_i \in \{\pm 1\}$ and each $g_i \in S$. Let $w_i = g_1^{\sigma_1} \cdots g_i^{\sigma_i}$. Let $w_0 = e$. By applying the triangle inequality $n_0 l_\epsilon$ times, we see that

$$\sum_{i=0}^{n_0 l_\epsilon - 1} \|\pi(w_i)v - \pi(w_{i+1})v\| \geq \|v - \pi(w)v\| \geq 1.$$

Thus there exists some i such that

$$\|\pi(w_i)v - \pi(w_{i+1})v\| \geq \frac{1}{n_0 l_\epsilon}.$$

Applying $\pi(w_i^{-1})$ and noting by our choice of ϵ that $l_\epsilon \leq C \log^\alpha(c(\pi))$, we obtain that

$$\|v - \pi(g_i^{\sigma_i})v\| \geq \frac{1}{n_0 C' \log^\alpha(c(\pi))}. \tag{3.5}$$

Thus we are done as we have obtained an estimate that is stronger than the required lower bound of $C/c(\pi)^\alpha$. □

We now prove the equivalence of the Diophantine property appearing in Proposition 4 with that in Definition 3.2.2.

Proof of Proposition 3. To begin, suppose that S is Diophantine. For the sake of contradiction, suppose that $H := \overline{\langle S \rangle} \neq G$. Consider the action of G on $L^2(G/H)$ by left translation. Note that H acts trivially. However, $L^2(G/H)$ contains non-trivial representations of G. Thus $S \subset H$ cannot be Diophantine, which is a contradiction.

For the other direction, suppose that $\overline{\langle S \rangle} = G$. Then by Proposition 2 there exists n such that S^n is $\epsilon_0(G)$-dense and, hence S is Diophantine by Proposition 4. □

The stronger bound in equation (3.5) gives an equivalent characterization of Diophantineness.

Corollary 1. *Let G be a compact, connected, semisimple Lie group. A subset S of G is Diophantine if and only if there exist $C, \alpha > 0$ such that the following holds for each non-trivial, irreducible, finite dimensional, unitary representation (π, V) of G. For all $v \in V$ there exists $g \in S$ such that*

$$\|v - \pi(g)v\| \geq \frac{\|v\|}{C \log^{\alpha}(c(\pi))}.$$

Diophantine subsets of a group are typical in the following sense.

Proposition 5. *Suppose that G is a compact connected semisimple Lie group. Let $U \subset G\times G$ be the set of ordered pairs (u_1, u_2) such that $\{u_1, u_2\}$ is a Diophantine subset of G. Then U is Zariski open and hence open and dense in the manifold topology on $G \times G$.*

Proof. Let $U \subset G\times G$ be the set of points (u_1, u_2) such that $\{u_1, u_2\}$ generates a dense subset of G. Theorem 1.1 in [Fie99] gives that U is Zariski open and non-empty. By Proposition 3, this implies that $\{u_1, u_2\}$ is Diophantine. As U is non-empty, the final claim follows. □

3.2.3 Polylogarithmic spectral gap

In this subsection, we study spectral properties of an averaging operator associated to a tuple of elements of G. Consider a tuple $(g_1, ..., g_m)$ of elements of G. Let $\mathbb{R}[G]$ denote the group ring of G over $\mathbb{R}$. From this tuple we form $\mathcal{L} \coloneqq (g_1 + \cdots + g_m)/m \in \mathbb{R}[G]$. The element $\mathcal{L}$ acts in representations of G in the natural way. If (π, V) is a representation of G, then we write $\mathcal{L}_\pi$ for the action of $\mathcal{L}$ on V. The main result of this subsection is the following proposition, which gives some spectral properties of $\mathcal{L}_\pi$ under the assumption that $\{g_1, ..., g_m\}$ is Diophantine.

Proposition 6. *Suppose that G is a compact connected semisimple Lie group, $(g_1, ..., g_m)$ is a tuple of elements of G, and that $\{g_1, ..., g_m\}$ generates G. Then there exists a neighborhood*

N of $(g_1, ..., g_m)$ in $G \times \cdots \times G$ and constants $D_1, D_2, \alpha > 0$ such that if $(g_1', ..., g_m') \in N$, then $\{g_1', \ldots, g_m'\}$ is Diophantine and its associated averaging operator $\mathcal{L}$ satisfies

$$\|\mathcal{L}_\pi^n\| \leq D_1 \left(1 - \frac{1}{D_2 \log^\alpha(c(\pi))}\right)^n,$$

for each non-trivial irreducible unitary representation (π, V).

The proof of Proposition 6 uses the following lemma, which is a sharpening the triangle inequality for vectors that are not colinear.

Lemma 3.2.5. *Suppose that v, w are two vectors in an inner product space. Suppose that $\|v\| \leq \|w\|$ and let $\hat{v} = v/\|v\|$ and $\hat{w} = w/\|w\|$. If*

$$\|\hat{v} - \hat{w}\| \geq \epsilon,$$

then

$$\|v + w\| \leq (1 - \epsilon^2/10)\|v\| + \|w\|.$$

Proof. We begin by considering the following estimate for unit vectors.

Claim 2. Suppose that the angle between two unit vectors $\hat{v}$ and $\hat{w}$ is $\theta \in [0, \pi]$, then

$$\|\hat{v} + w\| \leq \|\hat{v}\| + (1 - \theta^2/10)\|\hat{w}\|.$$

Proof. It suffices to consider the two vectors $\hat{v} = (1, 0)$ and $\hat{w} = (\cos\theta, \sin\theta)$ in $\mathbb{R}^2$. It then suffices to show:

$$\|\hat{v} + \hat{w}\|^2 \leq \left(\|\hat{v}\| + \left(1 - \frac{\theta^2}{10}\right)\|\hat{w}\|\right)^2.$$

From the definitions,

$$\|\hat{v} + \hat{w}\|^2 = 2 + 2\cos\theta$$

and

$$\left(\|\hat{v}\| + \left(1 - \frac{\theta^2}{10}\right)\|\hat{w}\|\right)^2 = 4 - 4\frac{\theta^2}{10} + \frac{\theta^4}{100} \geq 4 - 4\frac{\theta^2}{10}.$$

Thus it suffices to show for $\theta \in [0, \pi]$ that

$$2 + 2\cos\theta \leq 4 - 4\frac{\theta^2}{10},$$

which follows because for $\theta \in [0, \pi]$ we have the estimate $\cos\theta \leq 1 - \theta^2/5$. □

We may prove the lemma once we have one more observation. Note that if $\hat{v}$ and $\hat{w}$ are two unit vectors, then $\|\hat{v} - \hat{w}\| = \epsilon$ is less than the angle θ between $\hat{v}$ and $\hat{w}$ because the distance between $\hat{v}$ and $\hat{w}$ along a unit circle they lie on is precisely θ. Thus we see that $\epsilon \leq \theta$ for $0 \leq \theta \leq \pi$.

We now compute. Note that without loss of generality we may assume that $\|w\| = 1$, which we do in the following. By the triangle inequality,

$$\|v + w\| \leq \|v\|\|\hat{v} + \hat{w}\| + (1 - \|v\|)\|\hat{w}\|.$$

By the claim it then follows that

$$\|v + w\| \leq \|v\|((1 - \theta^2)\|\hat{v}\| + \|\hat{w}\|) + (1 - \|v\|)\|\hat{w}\|.$$

Noting from before that $0 \leq \epsilon \leq \theta$ for $\theta \in [0, \pi]$, we then conclude:

$$\|v + w\| \leq \|v\|((1 - \epsilon^2/10)\|\hat{v}\| + \|\hat{w}\|) + (1 - \|v\|)\|\hat{w}\| = (1 - \epsilon^2/10)\|v\| + \|w\|.$$

□

Proof of Proposition 6. For convenience, let $W = (g_1, ..., g_m)$ and let $S = \{g_1, ..., g_m\}$. Let $\epsilon_0(G)$ be as in Proposition 4. By Proposition 2, because $\overline{\langle S \rangle} = G$ there exists some n_0 such that S^{n_0} is $\epsilon_0/2$-dense in G. Then let N be the neighborhood of $(g_1, ..., g_m)$ in $G \times \cdots \times G$ such that if $p = (g'_1, ..., g'_m) \in N$ then $\{g'_1, ..., g'_m\}^{n_0}$ is at least ϵ_0-dense in G. It now suffices to obtain the given estimate for the set $W = (g_1, ..., g_m)$ using only the assumption that S^{n_0} is ϵ_0-dense. Below, W^{n_0} is the tuple of the m^{n_0} words of length n_0 with entries in W.

By Proposition 4, there exist (C, α) such that any ϵ_0-dense set is (C, α)-Diophantine. As S^{n_0} is ϵ_0-dense, so is $S^{n_0}S^{-n_0}$, and hence $S^{n_0}S^{-n_0}$ is (C, α)-Diophantine.

Consider now a non-trivial irreducible finite dimensional unitary representation (π, V) of G. Since $S^{n_0}S^{-n_0}$ is (C, α)-Diophantine, Corollary 1 implies that for any unit length $v \in V$ there exist $w_1, w_2 \in S^{n_0}$ such that

$$\|v - \pi(w_1^{-1}w_2)v\| \geq \frac{1}{C \log^{\alpha}(c(\pi))},$$

and so

$$\|\pi(w_1)v - \pi(w_2)v\| \geq \frac{1}{C \log^{\alpha}(c(\pi))}.$$

Hence by Lemma 3.2.5, since π is unitary

$$\|\pi(w_1)v + \pi(w_2)v\| \leq \left(1 - \frac{1}{10C^2 \log^{2\alpha}(c(\pi))}\right) \|\pi(w_1)v\| + \|\pi(w_2)v\| \leq \left(2 - \frac{1}{10C^2 \log^{2\alpha}(c(\pi))}\right) \|v\|.$$

Then by the triangle inequality:

$$\begin{aligned}
\|\mathcal{L}_\pi^{n_0} v\| &= \left\| \frac{1}{|W|^{n_0}} \sum_{w \in W^{n_0}} \pi(w)v \right\| \\
&\leq \frac{1}{|W|^{n_0}} \left(\|\pi(w_1)v + \pi(w_2)v\| + \sum_{w \in W^{n_0} \setminus \{w_1, w_2\}} \|\pi(w)v\| \right) \\
&\leq \frac{1}{|W|^{n_0}} \left(2 - \frac{1}{10C^2 \log^{2\alpha}(c(\pi))}\right) \|v\| + \frac{|W|^{n_0} - 2}{|W|^{n_0}} \|v\| \\
&\leq \left(1 - \frac{1}{10C^2 |W^{n_0}| \log^{2\alpha}(c(\pi))}\right) \|v\|.
\end{aligned}$$

Interpolating gives that for all $n \geq 0$,

$$\|\mathcal{L}_\pi^n\| \leq \left(1 - \frac{1}{10C^2|W^{n_0}|\log^{2\alpha}(c(\pi))}\right)^{-1}\left(1 - \frac{1}{10C^2|W^{n_0}|\log^{2\alpha}(c(\pi))}\right)^{n/n_0}.$$

As (π, V) ranges over all non-trivial representations, $c(\pi)$ is uniformly bounded away from 0; see [Wal18, 5.6.7]. This implies that the first term above is uniformly bounded by some $D > 0$ independent of π. Applying the estimate $(1+x)^\epsilon \leq 1 + \epsilon x$ to the second term then gives the proposition. □

Notice that in Proposition 6 that we obtain an entire neighborhood of our initial set S on which we have the same estimates for $\mathcal{L}_\pi$. Consequently, because these estimates remain true under small perturbations, we think of them as being stable. We will use the term "stable" in the following precise sense.

Definition 3.2.6. Suppose that T is some property of a tuple $W = (g_1, ..., g_m)$ with elements in a Lie group G. We say that T is *stable* at $W = (g_1, ..., g_m)$ if there exists a neighborhood N of $(g_1, ..., g_m)$ in $G \times \cdots \times G$ such that if $(g_1', ..., g_m') \in N$ then T holds for $(g_1', ..., g_m')$. We will also say that T is *stable* without reference to a subset when the relevant tuples that T is stable on are evident.

A crucial aspect of the Diophantine property in compact semisimple Lie groups is that by Proposition 4 there is a stable lower bound on (C, α). This stability will be essential during the KAM scheme.

3.2.4 Diophantine sets and tameness

Consider a smooth vector bundle E over a closed manifold M. We may consider the space $C^\infty(M, E)$ of smooth sections of E. Consider a linear map $L\colon C^\infty(M,E) \to C^\infty(M,E)$. We say that L is *tame* if there exists α such that for all k there exists C_k, such that for all $s \in C^\infty(M, E)$,

$$\|Ls\|_{C^k} \leq C_k \|s\|_{C^{k+\alpha}}.$$

See [Ham82, II.2.1] for more about tameness. The main result of this section is to show such estimates for certain operators related to $\mathcal{L}$.

Though $\mathcal{L}$ acts in any representation of G, we are most interested in the action of G on the sections of certain vector bundles, which we now describe. Suppose that K is a closed subgroup of G and that E is a smooth vector bundle over G/K. We say that E is a *homogeneous vector bundle* over G/K if G acts on E by bundle maps and this action projects to the action of G on G/K by left translation. We now give an explicit description of all homogeneous vector bundles over G/K via the Borel construction. See [Wal18, Ch. 5] for more details about this topic and what follows. Suppose that (τ, E_0) is a finite dimensional unitary representation of K. Form the trivial bundle $G \times E_0$. Then K acts on this bundle by $(g, v) \mapsto (gk, \tau(k)^{-1}v)$. Then $(G \times E_0)/K$ is a vector bundle over G/K that we denote by $G \times_\tau E_0$. Note, for instance, that $C^\infty(G, \mathbb{R})$ is the space of sections of the homogeneous vector bundle obtained from the trivial representation of $\{e\} < G$. The left action of G on $G \times E_0$ descends to $G \times_\tau E_0$, and hence this is a homogeneous vector bundle.

In order to do analysis in a homogeneous vector bundle, we must introduce some additional structures. Suppose that $E = G \times_\tau E_0$ is a homogeneous vector bundle. The base G/K comes equipped with the projection of the Haar measure on G. As the action of K on $G \times E_0$ is isometric on fibers, the fibers of E are naturally endowed with an inner product. We may then consider the space $L^2(E)$, the space of all L^2 sections of E. In addition, we will write $C^\infty(E)$ for the space of all smooth sections of E. The action of G on E preserves $L^2(E)$ and $C^\infty(E)$.

We recall briefly how one may do harmonic analysis on sections of such bundles. As before, let Ω be the Casimir operator, which is an element of $U(\mathfrak{g})$. Then Ω acts on the C^∞ vectors of any representation of G. Denote by Δ the differential operator obtained by the action of $-\Omega$ on $C^\infty(E)$. Then Δ is a hypoelliptic differential operator on E. We then use the spectrum of Δ to define for any $s \geq 0$ the Sobolev norm H^s in the following manner. $L^2(E)$ may be decomposed as the Hilbert space direct sum of finite dimensional irreducible unitary representations V_π. Write $\phi = \sum_\pi \phi_\pi$ for the decomposition of an element $\phi \in L^2(E)$. Then the s-Sobolev norm is defined by

$$\|\phi\|_{H^s}^2 = \sum_\pi (1 + c(\pi))^s \|\phi_\pi\|_{L^2}^2.$$

We write $\|f\|_{C^s}$ for the usual C^s norm of a function or section of a vector bundle. It is not always necessary to work with the decomposition of $L^2(E)$ into irreducible subspaces, but instead use a coarser decomposition as follows. We let H_λ denote the subspace of $L^2(E)$ on which Δ acts by multiplication by $\lambda > 0$. There are countably many such subspaces H_λ and each is finite dimensional. In the sequel, those functions that are orthogonal to the trivial representations in $L^2(E)$ will be of particular importance. We denote by $L_0^2(E)$ the orthogonal complement of the trivial representations in $L^2(E)$, and $C_0^\infty(E)$ the subspace $L_0^2(E) \cap C^\infty(E)$.

We now consider the action of $\mathcal{L}$ on the sections of a homogeneous vector bundle.

Proposition 7. *[DK07, Prop 1.] (Tameness) Suppose that $(g_1, ..., g_m)$ is a Diophantine tuple with elements in a compact connected semisimple Lie group G. Suppose that E is a homogeneous vector bundle that G acts on. Then there exist constants $C_1, \alpha_1, \alpha_2 > 0$ such that for any $s \geq 0$ there exists C_s such that for any nonzero $\phi \in C_0^\infty(G/K, E)$ the following holds:*

$$\|(I - \mathcal{L})^{-1}\phi\|_{H^s} \leq C_1 \|\phi\|_{H^{s+\alpha_1}}$$

and

$$\|(I - \mathcal{L})^{-1}\phi\|_{C^s} \leq C_s \|\phi\|_{C^{s+\alpha_2}}.$$

Moreover, these estimates are stable.

Proof. As before, let H_λ be the λ-eigenspace of Δ acting on sections of E. Let $\mathcal{L}_\lambda$ denote the action of $\mathcal{L}$ on H_λ. From Proposition 6, we see that there exist D_1, D_2 and α_3 such that for all $\lambda > 0$, $\|\mathcal{L}_\lambda^n\|_{H^0} \leq D_1(1 - 1/(D_2 \log^{\alpha_3}(\lambda))^n$. Thus there exists C_3 such that

$\|(I-\mathcal{L}_\lambda)^{-1}\|_{H^0} \le C_3 \log^{\alpha_3}(\lambda)$. Now observe, that in the following sum that $\lambda \neq 0$ by our assumption that ϕ is orthogonal to the trivial representations contained in $L^2(E)$:

$$\begin{aligned}
\|(I-\mathcal{L})^{-1}\phi\|_{H^s}^2 &= \sum_{\lambda>0} (1+\lambda)^s \|(I-\mathcal{L}_\lambda)^{-1}\phi_\lambda\|_{L^2}^2 \\
&\le \sum_{\lambda>0} (1+\lambda)^s \|(I-\mathcal{L}_\lambda)^{-1}\|^2 \|\phi_\lambda\|_{L^2}^2 \\
&\le \sum_{\lambda>0} C_3^2 \log^{2\alpha_3}(\lambda)(1+\lambda)^s \|\phi_\lambda\|_{L^2}^2 \\
&\le \sum_{\lambda>0} C_4^2 (1+\lambda)^{s+\alpha_1} \|\phi_\lambda\|_{L^2}^2 \\
&\le C_4^2 \|\phi\|_{H^{s+\alpha_1}}^2,
\end{aligned}$$

for any $\alpha_1 > 0$ and sufficiently large C_4. The second estimate in the proposition then follows from the first by applying the Sobolev embedding theorem. □

3.2.5 Application to isotropic manifolds

We now introduce the class of isotropic manifolds, which are the subject of this paper and whose isometry groups may be studied along the above lines. We say that M is *isotropic* if $\operatorname{Isom}(M)$ acts transitively on the unit tangent bundle of M, T^1M. This is equivalent to $\operatorname{Isom}(M)^\circ$ acting transitively on T^1M. There are not many isotropic manifolds. In fact, all are globally symmetric spaces. See [Wol72, Thm. 8.12.2] for the full classification. The compact examples are:

1. $S^n = \operatorname{SO}(n+1)/\operatorname{SO}(n)$, sphere,
2. $\mathbb{RP}^n = \operatorname{SO}(n+1)/O(n)$, real projective space,
3. $\mathbb{C}\mathrm{P}^n = \operatorname{SU}(n+1)/U(n)$, complex projective space,
4. $\mathbb{H}\mathrm{P}^n = \operatorname{Sp}(n+1)/(\operatorname{Sp}(n)\times\operatorname{Sp}(1))$, quaternionic projective space,
5. $F_4/\operatorname{Spin}(9)$, Cayley projective plane.

Though S^1 is an isotropic manifold, we will exclude it in all future statements because its isometry group is not semisimple. The reason that we study isotropic manifolds is that if M is an isotropic manifold, $\mathrm{Isom}(M)$ is semisimple.

Lemma 3.2.7. *Suppose that M is a compact connected isotropic manifold other than S^1, then* $\mathrm{Isom}(M)$ *is semisimple. The same is true for* $\mathrm{Isom}^0(M)$*, the connected component of the identity.*

For a proof of this Lemma, see [Sha01], which computes the isometry groups for each of these spaces explicitly. In fact, these isometry groups all have simple Lie algebras.

One minor issue with applying what we have developed so far to isotropic manifolds is that $\mathrm{Isom}(M)$ need not be connected. Even in the case of S^2, $\mathrm{Isom}(M)$ is disconnected. In fact, Dolgopyat and Krikorian assume that the isometries in their theorem all lie in the identity component of $\mathrm{Isom}(M)$ and hence are rotations. Here, we consider the full isometry group. Hence Theorem 3.1.1 is a generalization even in the case of S^n. That said, the generalization is minor: the identity component is index 2 in the full isometry group.

Although connectedness of $\mathrm{Isom}(M)$ has not been the crux of previous arguments, if $\mathrm{Isom}(M) \neq \mathrm{Isom}(M)^\circ$, then there are "extra" representations of $\mathrm{Isom}(M)$ that appear in the definition of Diophantineness that would need to be dealt with slightly differently. For this reason we give the following definition, which is adapted to the case where $\mathrm{Isom}(M)$ is not connected.

Definition 3.2.8. We say that a tuple $(g_1, \ldots, g_m)$ with each $g_i \in \mathrm{Isom}(M)$ is *Diophantine* if there exists n such that if $S = \{g_1, \ldots, g_m\}$ then $S^n \cap \mathrm{Isom}(M)^\circ$ is (C, α)-Diophantine for some $C, \alpha > 0$. We say that such a tuple is (C, α, n)-Diophantine.

It follows from Proposition 3 that if a tuple is Diophantine, then there exists a neighborhood of that tuple such that the constants C, α, n may be taken to be uniform over that neighborhood. Thus Diophantineness in this more general sense is a stable property. The following analogue of Proposition 8 is then immediate.

Proposition 8. *Let M be a closed isotropic manifold of dimension at least 2 and S be a finite subset of $\mathrm{Isom}(M)$. The set S is Diophantine if and only if $\mathrm{Isom}(M)^\circ \subseteq \overline{\langle S \rangle}$. Moreover, there exists $\epsilon_0(M), C, \alpha, n > 0$ such that any subset of $\mathrm{Isom}(M)$ that is ϵ_0-dense in $\mathrm{Isom}(M)^\circ$ is stably (C, α, n)-Diophantine.*

We will show a tameness result in this setting. The important point is that $\mathrm{Isom}(M)^\circ$ is a semisimple connected Lie group and TM is a homogeneous vector bundle that $\mathrm{Isom}(M)^\circ$ acts on. Further, due to M being isotropic $L^2(M, TM)$ contains no trivial representations of $\mathrm{Isom}(M)^\circ$. Thus we are almost in a position where we can apply Proposition 7. There is one small issue: there may be representations of $\mathrm{Isom}(M)$ that are trivial on $\mathrm{Isom}(M)^\circ$ and hence the previous arguments do not apply directly to these representations. However, for the purpose of studying sections of TM, studying representations of $\mathrm{Isom}(M)^\circ$ suffices. The following Proposition explains how one may get around this issue to recover the appropriate analog of Proposition 6. It is important to note that there are many choices of a "Laplacian" acting on vector fields over a manifold, and they may not all be the same. In our case, we are choosing to work with the Casimir Laplacian, which arises from viewing TM as a homogeneous vector bundle.

Proposition 9. *Suppose that M is a closed isotropic manifold with $\dim M \geq 2$. Suppose that $(g_1, \ldots, g_m)$ is a Diophantine tuple with elements in $\mathrm{Isom}(M)$. There exists a neighborhood $\mathcal{N}$ of $(g_1, \ldots, g_m)$ in $\mathrm{Isom}(M) \times \cdots \times \mathrm{Isom}(M)$ and constants $D_1, D_2, \alpha > 0$ such that if $(g_1', \ldots, g_m') \in \mathcal{N}$, then $\{g_1', \ldots, g_m'\}$ is Diophantine. Let H_λ denote the λ-eigenspace of Δ acting on sections of TM. For any tuple in this neighborhood, the associated operator $\mathcal{L}$ acts on $L^2(M, TM)$ and preserves the H_λ-eigenspaces. In fact, writing $\mathcal{L}_\lambda$ for this induced action we have that:*

$$\|\mathcal{L}_\lambda^n\| \leq D_1 \left(1 - \frac{1}{D_2 \log^\alpha(\lambda)}\right)^n.$$

The same holds for the eigenspaces H_λ of Δ acting on other bundles over M assuming that $\mathrm{Isom}(M)$ acts isometrically on the space of sections of those bundles. In cases where there is *a trivial representation, we must also assume* $\lambda > 0$*. Examples of such bundles are* $L^2(M, \mathbb{R})$ *as well as* $L^2(\mathrm{Gr}_r(M), \mathbb{R})$ *in the case that* $\mathrm{Isom}(M)^\circ$ *acts transitively on the* r*-planes in* TM*.*

Proof. The key steps in the proof are substantially similar to those in Proposition 6, once we show that the elements of $\mathrm{Isom}(M)$ all preserve the spaces H_λ. Let Γ be a bundle as in the statement of the proposition that $\mathrm{Isom}(M)$ acts on isometrically.

Claim 3. Suppose that $V \subset \Gamma$ is an irrep of $\mathrm{Isom}(M)^\circ$ isomorphic to (π, W). Then for any $k \in \mathrm{Isom}(M)^\circ$, kV is an irrep of V isomorphic to $(\pi \circ \alpha, W)$ for some automorphism α of $\mathrm{Isom}(M)^\circ$. In particular, $c(\pi \circ \alpha) = c(\pi)$.

Proof. Let $g^k = k^{-1}gk$ as usual. We claim that for any $k \in \mathrm{Isom}(M)$ that kV is a representation of $\mathrm{Isom}(M)^\circ$. To see this note that for $v \in V$, that $gkv = kg^kv$, but $g^k \in \mathrm{Isom}(M)^\circ$, so $kg^kv \in kV$. Moreover, it is straightforward to see that the representation of $\mathrm{Isom}(M)^\circ$ on kV is isomorphic to the representation $(\pi \circ \alpha, W)$ where α is the automorphism $g \mapsto g^k$.

We now claim that $c(\pi \circ \alpha) = c(\pi)$. Because α is an automorphism, it preserves the Killing form, and hence we see that we can write the Casimir element as $\sum_i (d\alpha^{-1}(X_i))^2$. Now note that if one traces through the computation of what the value $c(\pi \circ \alpha)$ for the representation $\pi \circ \alpha$, that the α^{-1} we have introduced cancels with the α. Thus the computation reduces to the computation of $c(\pi)$ with the original expression $\sum_i X_i^2$. Hence $c(\pi \circ \alpha) = c(\pi)$. □

To conclude from this point, one does the same argument as in Proposition 6, except we start with the set S^{n_0} and only make use of the elements in $S^{n_0} \cap \mathrm{Isom}(M)^\circ$. No issues arise because any terms that do not lie in $\mathrm{Isom}(M)^\circ$ are isometries of H_λ as we have now shown. □

Having established the previous proposition the following is immediate and may be shown by repeating the argument of Proposition 7.

Proposition 10. *Suppose that* M *is a closed isotropic manifold with* $\dim M \geq 2$*. Suppose that* $(g_1, \ldots, g_m)$ *is a Diophantine tuple with elements in* $\mathrm{Isom}(M)$*. There exist constants*

$C_1, \alpha_1, \alpha_2 > 0$ such that for any $s \geq 0$ there exists C_s such that for any $\phi \in C^\infty(M, TM)$ the following holds:

$$\|(I - \mathcal{L})^{-1}\phi\|_{H^s} \leq C_1 \|\phi\|_{H^{s+\alpha_1}},$$

and

$$\|(I - \mathcal{L})^{-1}\phi\|_{C^s} \leq C_s \|\phi\|_{C^{s+\alpha_2}}.$$

Moreover these estimates are stable. The same holds for the action of $\mathcal{L}$ on any of the sections of any of the bundles that Proposition 9 applies to.

3.3 APPROXIMATION OF STATIONARY MEASURES

In this section, we introduce the notion of a stationary measure associated to a random dynamical system. We consider stationary measures of certain random dynamical systems associated to a Diophantine subset of a compact semisimple Lie group as well as perturbations of these systems. We begin by introducing these systems and some associated transfer operators. In Proposition 11, we give an asymptotic expansion of the stationary measures of a perturbation.

3.3.1 Random dynamical systems and their transfer operators

We now give some basic definitions concerning random dynamical systems. For general treatments of random dynamical systems and their basic properties, see [Kif86] or [Arn13]. If $(f_1, ..., f_m)$ is a tuple of maps of a standard Borel space M, then these maps generate a uniform Bernoulli random dynamical system on M. This dynamical system is given by choosing an index $1 \leq i \leq m$ uniformly at random and then applying the function f_i to M. To iterate the system further, one chooses additional independent uniformly distributed indices and repeats. We always use the words *random dynamical system* to mean uniform Bernoulli random dynamical system in the sense just described.

Associated to this random dynamical system are two operators. The first operator is called the *Koopman operator*. It acts on functions and is defined by

$$\mathcal{M}\phi \coloneqq \frac{1}{m}\sum_{i=1}^{m} \phi \circ f_i. \tag{3.6}$$

The second operator is called the *transfer operator*. It acts on measures and is defined by

$$\mathcal{M}^*\mu \coloneqq \frac{1}{m}\sum_{i=1}^{m} (f_i)_*\mu. \tag{3.7}$$

Depending on the space M, we may restrict the domains of these operators to a suitable subset of the spaces of functions and measures on M. We say that a measure is *stationary* if $\mathcal{M}^*\mu = \mu$. We assume that stationary measures have unit mass.

In this paper, we take M to be a compact homogeneous space G/K. If $g \in G$, then left translation by g gives an isometry of G/K that we also call g. As before, a tuple $(g_1, ..., g_m)$ with each $g_i \in G$ generates a random dynamical system on G/K. We will also consider perturbations of this random dynamical system. Consider a tuple $(f_1, ..., f_m)$ where each $f_i \in \text{Diff}^\infty(G/K)$. This collection also generates a random dynamical system on G/K. The indices $1, ..., m$ give a natural way to compare the two systems. We refer to the initial system as homogeneous or linear and to the latter system as non-homogeneous or non-linear.

We will simultaneously work with a homogeneous and non-homogeneous systems, so we now introduce notation to distinguish the transfer operators of each. We write $\mathcal{M}$ for the Koopman operator associated to the system generated by the tuple $(g_1, ..., g_m)$ and we write $\mathcal{M}_\epsilon$ for the Koopman operator associated to the tuple $(f_1, ..., f_m)$. Analogously we use the notation $\mathcal{M}^*$ and $\mathcal{M}_\epsilon^*$.

Later we will compare the homogeneous system given by a tuple $(g_1, ..., g_m)$ and a non-homogeneous perturbation $(f_1, ..., f_m)$. We thus introduce the notation

$$\varepsilon_k \coloneqq \max_i d_{C^k}(f_i, g_i), \tag{3.8}$$

for describing how large a perturbation is. In addition, it will be useful to have a linearization of the difference between f_i and g_i. The standard way to do this is via a chart on the Fréchet manifold $\mathrm{Diff}^\infty(G/K)$. If $d_{C^0}(f_i, g_i) <\hookrightarrow G/K$, then we associate f_i with the vector field Y_i defined at $g_i(x) \in G/K$ by

$$Y_i(g_i(x)) \coloneqq \exp_{g_i(x)}^{-1} f_i(x), \tag{3.9}$$

where we choose the minimum length preimage of $f_i(x)$ in $T_{g_i(x)}G/K$ under the map $\exp_{g_i(x)}^{-1}$. In addition, if Y is a vector field on M, then we define $\psi_Y \colon M \to M$ to be the map that sends

$$\psi_Y : x \mapsto \exp_x(Y(x)). \tag{3.10}$$

The following theorem asserts the existence of Lyapunov exponents for random dynamical systems.

Theorem 3.3.1. *[Kif86, Ch. 3, Thm. 1.1]. Suppose that E is measurable vector bundle over a Borel space M. Suppose that $F_1, F_2, \ldots$ is a sequence of independent and identically distributed bundle maps of E with common distribution ν and suppose that ν has finite support. Suppose that μ is an ergodic ν-stationary measure on M for the random dynamics on M induced by those on E.*

Then there exists a list of numbers, the Lyapunov exponents,

$$-\infty < \lambda^s < \lambda^{s-1} < \cdots < \lambda^1 < \infty,$$

such that for μ a.e. $x \in M$ and almost every realization of the sequence, there exists a filtration of linear subspaces

$$0 \subset V^s \subset \cdots \subset V^1 \subset E_x$$

such that, for that particular realization of the sequence, if $\xi \in V^{i+1} \setminus V^i$, where $V^i \equiv \{0\}$ for $i > s$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \|F^n \circ \cdots \circ F^1 \xi\| = \lambda^i.$$

3.3.2 Approximation of stationary measures

Let dm denote the push-forward of Haar measure to G/K. Note that Haar measure is stationary for the homogeneous random dynamical system given by $(g_1, ..., g_m)$. The following proposition compares the integral against a stationary measure μ for a perturbation $(f_1, ..., f_m)$ and the Haar measure. Up to higher order terms, the difference between integrating against Haar and against μ is given by the integral of a particular function $\mathcal{U}(\phi)$. We obtain an explicit expression for $\mathcal{U}(\phi)$, which is useful because we can tell when $\mathcal{U}(\phi)$ vanishes and thus when μ is near to Haar. Compare the following with [DK07, Prop. 2].

Proposition 11. *Suppose that $S = (g_1, ..., g_m)$ is a Diophantine tuple with elements in a compact connected semisimple group G or elements in $\operatorname{Isom}(M)$ for an isotropic manifold M with $\dim M \geq 2$. Let G/K be a quotient of G in the former case or a space $\operatorname{Isom}(M)^\circ$ acts transitively on in the latter. There exist constants k and C such that if $(f_1, ..., f_m)$ is a tuple with elements in $\operatorname{Diff}^\infty(G/K)$ with $\varepsilon_0 = \max_i d_{C^0}(f_i, g_i) <\hookrightarrow G/K$, then the following holds for each stationary measure μ for the uniform Bernoulli random dynamical system generated by the f_i. Let $Y_i = \exp^{-1}_{g_i(x)} f_i(x)$. Then for any $\phi \in C^\infty(G/K)$, we have*

$$\int_{G/K} \phi \, d\mu = \int_{G/K} \phi \, dm + \int_{G/K} \mathcal{U}(\phi) \, dm + O(\varepsilon_k^2 \|\phi\|_{C^k}), \tag{3.11}$$

where dm denotes the normalized push-forward of Haar measure to G/K and

$$\mathcal{U}(\phi) := \frac{1}{m} \sum_{i=1}^{m} \nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi \, dm). \tag{3.12}$$

Moreover,

$$\left| \int \mathcal{U}(\phi) \, dm \right| \leq C \|\phi\|_{C^k} \left\| \sum_{i=1}^{m} Y_i \right\|_{C^k}, \tag{3.13}$$

and the constants, including the constant in the big-O in equation (3.11), are stable in S.

Proof. The proof is similar to the proof of [Mal12, Prop. 4]. We write the proof for the connected group G; the proof for $\operatorname{Isom}(M)$ is identical with us using Proposition 10 instead of Proposition 7.

Note that a smooth real valued function defined on G/K is naturally viewed as a section of the trivial bundle over G/K. If we view the Koopman operator $\mathcal{M}$ associated to $(g_1, ..., g_m)$ as acting on the sections of the trivial bundle $G/K \times \mathbb{R}$, then $\mathcal{M}$ satisfies the hypotheses of Proposition 7. Thus there exists α and constants C_s such that for any $\phi \in C_0^\infty(G/K)$, the space of integral 0 smooth functions on G/K,

$$\|(I - \mathcal{M})^{-1}\phi\|_{C^s} \leq C_s\|\phi\|_{C^{s+\alpha}}. \tag{3.14}$$

Observe that for any i:

$$|\phi \circ f_i(x) - \phi \circ g_i(x)| \leq \varepsilon_0\|\phi\|_{C^1}.$$

Since μ is $\mathcal{M}_\epsilon^*$ invariant, this implies that

$$\left|\int \phi - \mathcal{M}\phi \, d\mu\right| = \left|\int \mathcal{M}_\varepsilon\phi - \mathcal{M}\phi \, d\mu\right| \leq \varepsilon_0\|\phi\|_{C^1}.$$

Substituting $(I-\mathcal{M})^{-1}(\phi-\int \phi \, dm)$ for the function ϕ in the previous line and using equation (3.14) yields a first order approximation:

$$\left|\int \phi \, d\mu - \int \phi \, dm\right| \leq \varepsilon_0 C_1\|\phi\|_{C^{1+\alpha}}. \tag{3.15}$$

We now use this first order approximation to obtain a better estimate. Note the Taylor expansion:

$$\phi \circ f_i(x) - \phi \circ g_i(x) = (\nabla_{Y_i}\phi)(g_i(x)) + O(\varepsilon_0^2\|\phi\|_{C^2}).$$

Integrating against μ yields

$$\int \phi - \mathcal{M}\phi \, d\mu = \int \mathcal{M}_\varepsilon\phi - \mathcal{M}\phi \, d\mu = \int \frac{1}{m}\sum_{i=1}^{m} \nabla_{Y_i}\phi(g_i(x)) \, d\mu + O(\varepsilon_0^2\|\phi\|_{C^2}).$$

We now plug in $(I - \mathcal{M})^{-1}(\phi - \int \phi \, dm)$ for ϕ in the previous line and use the estimate in equation (3.14) to obtain:

$$\int \phi\, d\mu - \int \phi\, dm = \int \frac{1}{m} \sum_{i=1}^{m} \left(\nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm) \right) (g_i(x))\, d\mu + O(\varepsilon_0^2 \|\phi\|_{C^{2+\alpha}}).$$

Using equation (3.15) on the first term on the right hand side above yields

$$\begin{aligned} \int \phi\, d\mu - \int \phi\, dm = & \int \frac{1}{m} \sum_{i=1}^{m} \left(\nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm) \right) (g_i(x))\, dm \\ & + O\left(\varepsilon_0 \left\| \sum_{i=1}^{m} \nabla_{Y_i} (I - \mathcal{M})^{-1} \phi \right\|_{C^{1+\alpha}} \right) + O(\varepsilon_0^2 \|\phi\|_{C^{2+\alpha}}). \end{aligned} \tag{3.16}$$

Note that

$$\left\| \sum_{i=1}^{m} \nabla_{Y_i} (I - \mathcal{M})^{-1} \phi \right\|_{C^{1+\alpha}} = O(\varepsilon_{2+\alpha} \|(I - \mathcal{M})^{-1} \phi\|_{C^{2+\alpha}}).$$

The application of equation (3.14) to $\|(I - \mathcal{M})^{-1}\phi\|_{C^{2+\alpha}}$ then gives that the first big O-term in (3.16) is $O(\varepsilon_0 \varepsilon_{2+\alpha} \|\phi\|_{C^{2+2\alpha}})$. Thus,

$$\int \phi\, d\mu - \int \phi\, dm = \int \frac{1}{m} \sum_{i=1}^{m} \left(\nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm) \right) (g_i(x))\, dm + O(\varepsilon_{2+\alpha}^2 \|\phi\|_{C^{2+2\alpha}}).$$

Now, by translation invariance of the Haar measure we may remove the g_i's:

$$\int \phi\, d\mu - \int \phi\, dm = \int \frac{1}{m} \sum_{i=1}^{m} \nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm)\, dm + O(\varepsilon_{2+\alpha}^2 \|\phi\|_{C^{2+2\alpha}}).$$

This proves everything except equation (3.13).

We now estimate the integral of

$$\begin{aligned} \mathcal{U}(\phi) &= \frac{1}{m} \sum_{i=1}^{m} \nabla_{Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm), \\ &= \nabla_{\frac{1}{m} \sum_{i=1}^{m} Y_i} (I - \mathcal{M})^{-1} (\phi - \textstyle\int \phi\, dm), \end{aligned}$$

against Haar. By equation (3.14) there exists C_1 such that

$$\left\|(I-\mathcal{M})^{-1}(\phi - \int \phi \, dm)\right\|_{C^1} \leq C_1 \|\phi\|_{C^{1+\alpha}},$$

which establishes equation (3.13) by a similar argument to the estimate of the big-O term occurring in the previous part of this proof. □

3.4 STRAIN AND LYAPUNOV EXPONENTS

In this section we study the Lyapunov exponents of perturbations of isometric systems. The main result is Proposition 12, which gives a Taylor expansion of the Lyapunov exponents of a perturbation. The terms appearing in the Taylor expansion have a particular geometric meaning. We explain this meaning in terms of two "strain" tensors associated to a diffeomorphism. These tensors measure how far a diffeomorphism is from being an isometry. After introducing these tensors, we prove Proposition 12. The Lyapunov exponents of a random dynamical system may be calculated by integrating against a stationary measure of a certain extension of the original system. By using Proposition 11, we are able to approximate such stationary measures by the Haar measure and thereby obtain a Taylor expansion.

3.4.1 Norms on Tensors

Throughout this paper we use the pointwise L^2 norm on tensors, which we now describe. For a more detailed discussion, see the discussion surrounding [Lee18, Prop. 2.40]. If V is an inner product space with orthonormal basis $[e_1, \dots, e_n]$, then $V^{\otimes k}$ has a basis of tensors of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_k}$$

where $1 \leq i_j \leq n$ for each $1 \leq j \leq k$. We declare the vectors of this basis to be orthonormal for the inner product on $V^{\otimes k}$. This norm is independent of the choice of orthonormal basis.

For a continuous tensor field T on a closed Riemannian manifold M, we write $\|T\|$ for $\max_{x \in M} \|T(x)\|$. If T is a tensor on a Riemannian manifold M, we then define its L^2 norm in the expected way by integrating the norm of $T(x)$ as a tensor on T_xM over all points $x \in M$, i.e.

$$\|T\|_{L^2} = \left(\int_M \|T(x)\|^2 \, d\,\mathrm{vol}(x) \right)^{1/2}.$$

3.4.2 Strain

If a diffeomorphism of a Riemannian manifold is an isometry, then it pulls back the metric tensor to itself. Consequently, if we are interested in how near a diffeomorphism is to being an isometry, it is natural to consider the difference between the metric tensor and the pullback of the metric tensor. This leads us to the following definition.

Definition 3.4.1. Suppose that f is a diffeomorphism of a Riemannian manifold (M, g). We define the *Lagrangian strain tensor* associated to f to be

$$E^f := \frac{1}{2} \left(f^*g - g \right).$$

This definition is consonant with the definition of the Lagrangian strain tensor that appears in continuum mechanics, c.f. [LRK09].

The strain tensor will be useful for two reasons. First, it naturally appears in the Taylor expansion in Proposition 12, which will allow us to conclude that a random dynamical system with small Lyapunov exponents has small strain. Secondly, we prove in Theorem 3.5.1 that for certain manifolds that a diffeomorphism with small strain is near to an isometry. The combination of these two things will be essential in the proof of our main linearization result, Theorem 3.1.1, which shows that perturbations with all Lyapunov exponents zero are conjugate to isometric systems.

We now introduce two refinements of the strain tensor that will appear in the Taylor expansion in Proposition 12. Note that E^f is a $(0, 2)$-tensor. Consequently, we may take its trace with respect to the ambient metric g.

Definition 3.4.2. Suppose that f is a diffeomorphism of a Riemannian manifold (M, g). We define the *conformal strain tensor* by

$$E_C^f := \frac{\mathrm{Tr}(f^*g - g)}{2d} g.$$

We define the *nonconformal strain tensor* by

$$E_{NC}^f := E^f - E_C^f = \frac{1}{2}\left(f^*g - g - \frac{\mathrm{Tr}(f^*g - g)}{d} g\right).$$

3.4.3 Taylor expansion of Lyapunov exponents

Suppose that M is a manifold and that f is a diffeomorphism of M. Let $\mathrm{Gr}_r(M)$ denote the Grassmannian bundle comprised of r-planes in TM. When working with $\mathrm{Gr}_r(M)$ we write a subspace of T_xM as E_x to emphasize the basepoint. Then f naturally induces a map $F\colon \mathrm{Gr}_r(M) \to \mathrm{Gr}_r(M)$ by sending a subspace $E_x \in \mathrm{Gr}_r(T_xM)$ to $D_xfE_x \in \mathrm{Gr}_r(T_{f(x)}M)$. If we have a random dynamical system on M, then by this construction we naturally obtain a random dynamical system on $\mathrm{Gr}_r(M)$. The following Proposition should be compared with [DK07, Prop. 3].

Proposition 12. *Suppose that M is a compact connected Riemannian manifold such that* $\mathrm{Isom}(M)$ *is semisimple and that* $\mathrm{Isom}(M)^\circ$ *acts transitively on* $\mathrm{Gr}_r(M)$. *Suppose that $S = (g_1, ..., g_m)$ is a Diophantine tuple of elements of* $\mathrm{Isom}(M)$. *Then there exists $\epsilon > 0$ and $k > 0$ such that if $(f_1, ..., f_m)$ is a tuple with elements in* $\mathrm{Diff}^\infty(M)$ *such that $d_{C^k}(f_i, g_i) < \epsilon$, then the following holds. Suppose that μ is an ergodic stationary measure for the random dynamical system obtained from the $(f_1, ..., f_m)$. Let Λ_r be the sum of the top r Lyapunov* *exponents of* μ*. Then*

$$\Lambda_r(\mu) = -\frac{r}{2dm}\sum_{i=1}^{m}\int_M \|E_C^{f_i}\|^2 \, d\,\mathrm{vol} + \frac{r(d-r)}{(d+2)(d-1)m}\sum_{i=1}^{m}\int_M \|E_{NC}^{f_i}\|^2 \, d\,\mathrm{vol} + \int_{\mathrm{Gr}_r(M)} \mathcal{U}(\psi)\, d\,\mathrm{vol} + O(\varepsilon_k^3). \tag{3.17}$$

where $\psi = \frac{1}{m}\sum_{i=1}^{m} \ln\det(Df_i \mid E_x)$*,* $\varepsilon_k = \max_i\{d_{C^k}(f_i, g_i)\}$*,* $\mathcal{U}$ *is defined as in Proposition 11, and* det *is defined in Appendix 3.7.4.*

Proof. Given the random dynamical system on M generated by the tuple $(f_1, ..., f_m)$, there is the induced random dynamical system on $\mathrm{Gr}_r(M)$ generated by the tuple $(F_1, ..., F_m)$. The Lyapunov exponents of the system on M may be obtained from the system on $\mathrm{Gr}_r(M)$ in the following way. By [Kif86, Ch. III, Thm 1.2], given an ergodic stationary measure μ on M, there exists a stationary measure $\overline{\mu}$ on $\mathrm{Gr}_r(M)$ such that

$$\Lambda_r(\mu) = \frac{1}{m}\sum_{i=1}^{m}\int_{\mathrm{Gr}_r(M)} \ln\det(Df_i \mid E_x)\, d\overline{\mu}(E_x).$$

Reversing the order of summation, this is equal to

$$\int_{\mathrm{Gr}_r(M)} \frac{1}{m}\sum_{i=1}^{m} \ln\det(Df_i \mid E_x)\, d\overline{\mu}(E_x). \tag{3.18}$$

As $\mathrm{Isom}(M)$ acts transitively on $\mathrm{Gr}_r(M)$, $\mathrm{Gr}_r(M)$ is a homogeneous space of $\mathrm{Isom}(M)$. Thus as $(g_1, ..., g_m)$ is Diophantine, we may apply Proposition 11 to approximate the integral in equation (3.18). Letting $\mathcal{U}$ be as in that proposition, there exists k such that

$$\Lambda_r(\mu) = \int_{\mathrm{Gr}_r(M)} \frac{1}{m}\sum_{i=1}^{m} \ln\det(Df_i \mid E_x)\, d\,\mathrm{vol}(E_x) + \int_{\mathrm{Gr}_r(M)} \mathcal{U}\left(\frac{1}{m}\sum_{i=1}^{m}\ln\det(Df_i \mid E_x)\right) d\,\mathrm{vol} + O\left((\max_i\{d_{C^k}(F_i, G_i)\})^2 \left\|\sum_{i=1}^{m}\ln\det(Df_i \mid E_x)\right\|_{C^k}\right) \tag{3.19}$$

We now estimate the error term. The following two estimates follow by working in a chart on $\mathrm{Gr}_r(M)$. If f, g are two maps of M and F, G are the induced maps on $\mathrm{Gr}_r(M)$, then $d_{C^k}(F,G) = O(d_{C^{k+1}}(f,g))$. In addition, by Lemma 3.7.13 we have that

$$\left\| \sum_{i=1}^{m} \ln\det(Df_i \mid E_x) \right\|_{C^k} = O(\varepsilon_{k+1}). \tag{3.20}$$

Thus the error term in (3.19) is small enough to conclude (3.17).

To finish, we apply the Taylor expansion in Proposition 18, which is in Appendix 3.7.5, to

$$\int_{\mathrm{Gr}_r(M)} \ln\det(Df_i \mid E_x)\, d\,\mathrm{vol}(E_x),$$

which gives precisely the first two terms on the right hand side of equation (3.17) and error that is $O(\varepsilon_1^3)$. □

3.5 DIFFEOMORPHISMS OF SMALL STRAIN: EXTRACTING AN ISOMETRY IN THE KAM SCHEME

In this section we prove Proposition 13, which gives that a diffeomorphism of small strain on an isotropic manifold is near to an isometry. In the KAM scheme, we will see that diffeomorphisms with small Lyapunov exponents are low strain and hence conclude by Proposition 13 that they are near to isometries. Proposition 13 follows from Theorem 3.5.1, which shows that certain diffeomorphisms with small strain of a closed Riemannian manifold are C^0 close to the identity.

Theorem 3.5.1. *Suppose that (M, g) is a closed Riemannian manifold. Then there exists $1 > r > 0$ and $C > 0$ such that if $f \in \mathrm{Diff}^2(M)$ and*

1. *there exists $x \in M$ such that $f(x) = x$ and $\|D_x f - \mathrm{Id}\,\| = \theta < r$,*

2. *$\|f^* g - g\| = \eta < r$, and*

3. $d_{C^2}(f, \mathrm{Id}) = \kappa < r$,

then for all $\gamma \in (0, r)$,

$$d_{C^0}(f, \mathrm{Id}) \leq C(\theta + \kappa\gamma + \eta\gamma^{-1}).$$

Theorem 3.5.1 is the main ingredient in the proof of our central technical result.

Proposition 13. *Suppose that* (M, g) *is a closed isotropic Riemannian manifold. Then for all* $\sigma > 0$ *and all integers* $\ell > 0$*, there exist* k *and* $C, r > 0$ *such that for every* $f \in \mathrm{Diff}^k(M)$*, if there exists an isometry* $I \in \mathrm{Isom}(M)$ *such that*

1. $d_{C^k}(I, f) < r$*, and*

2. $\|f^*g - g\|_{H^0} < r$,

then there exists an isometry $R \in \mathrm{Isom}(M)$ *such that*

$$d_{C^0}(R, I) < C(d_{C^2}(f, I) + \|f^*g - g\|_{H^0}^{1-\sigma}), \text{ and} \tag{3.21}$$

$$d_{C^\ell}(f, R) < C(\|f^*g - g\|_{H^0}^{1/2-\sigma} d_{C^2}(f, I)^{1/2-\sigma}). \tag{3.22}$$

Though the statement of Proposition 13 is technical, its use in the proof of Theorem 3.1.1 is fairly transparent: the proposition produces an isometry near to a diffeomorphism with small strain, which is the essence of iterative step in the KAM scheme. This remedies the gap in [DK07].

3.5.1 Low strain diffeomorphisms: Proof of Theorem 3.5.1

The main geometric idea in the proof of Theorem 3.5.1 is to study distances by intersecting spheres. In order to show that a diffeomorphism f is close to the identity, we must show that it does not move points far. As we shall show, a diffeomorphism of small strain distorts distances very little. Consequently, a diffeomorphism of small strain nearly carries spheres to spheres. If we have two points x and y that are fixed by f, then the unit spheres centered at

x and y are carried near to themselves by f. Consequently, the intersection of those spheres will be nearly fixed by f. By considering the intersection of spheres in this way, we may take a small set on which f nearly fixes points and enlarge that set until it fills the whole manifold.

Before the proof of the theorem we prove several lemmas.

Lemma 3.5.1. *Let M be a closed Riemannian manifold. There exists $C > 0$ such that the following holds. If $f \in \operatorname{Diff}^1(M)$ and $\|f^*g - g\| \le \eta$ then for all $x, y \in M$,*

$$(1 - C\eta)d(x, y) \le d(f(x), f(y)) \le (1 + C\eta)d(x, y).$$

Proof. If γ is a path between x and y parametrized by arc length, then $f \circ \gamma$ is a path between $f(x)$ and $f(y)$. The length of $f \circ \gamma$ is equal to

$$\begin{aligned}
\operatorname{len}(f \circ \gamma) &= \int_0^{\operatorname{len}(\gamma)} \sqrt{g(Df\dot{\gamma}, Df\dot{\gamma})}\, dt \\
&= \int_0^{\operatorname{len}(\gamma)} \sqrt{f^*g(\dot{\gamma}, \dot{\gamma})}\, dt \\
&= \int_0^{\operatorname{len}(\gamma)} \sqrt{g(\dot{\gamma}, \dot{\gamma}) + [f^*g - g](\dot{\gamma}, \dot{\gamma})}\, dt \\
&= \int_0^{\operatorname{len}(\gamma)} \sqrt{1 + [f^*g - g](\dot{\gamma}, \dot{\gamma})}\, dt.
\end{aligned}$$

By our assumption on the norm of $f^*g - g$, there exists C such that $|[f^*g - g](\dot{\gamma}, \dot{\gamma})| \le C\eta$. Then using that $\sqrt{1 + x} \le 1 + x$ for $x \ge 0$, we see that

$$\operatorname{len}(f \circ \gamma) \le \int_0^{\operatorname{len}(\gamma)} 1 + |[f^*g - g](\dot{\gamma}, \dot{\gamma})|\; dt \le \operatorname{len}(\gamma) + C\eta \operatorname{len}(\gamma).$$

The lower bound follows similarly by using that $1 + x \le \sqrt{1 + x}$ for $-1 \le x \le 0$. □

Lemma 3.5.2. *Let M be a closed Riemannian manifold. Then there exist $r, C > 0$ such that for all $f \in \operatorname{Diff}^2(M)$, if*

1. *there exists* $x \in M$ *such that* $f(x) = x$ *and* $\|D_x f - \mathrm{Id}\| = \theta < r$*, and*

2. $d_{C^2}(f, \mathrm{Id}) = \kappa < r$*,*

then for all $0 < \gamma < r$ *and* y *such that* $d(x, y) < \gamma$

$$d(y, f(y)) \leq C(\gamma\theta + \gamma^2\kappa).$$

Proof. Let $r = \hookrightarrow M/2$. We work in a fixed exponential chart centered at x, so that x is represented by 0 in the chart. Write

$$f(y) = 0 + D_0 f y + R(y) = y + (D_0 f - \mathrm{Id})y + R(y).$$

As the C^2 distance between f and the identity is at most κ, by Taylor's Theorem $R(y)$ is bounded in size by $C\kappa\,|y|^2$ for a uniform constant C. Thus

$$|f(y) - y| \leq \theta\,|y| + C\kappa\,|y|^2\,.$$

In particular, for all y such that $|y| \leq \gamma < r$,

$$|f(y) - y| \leq C'(\gamma\theta + \gamma^2\kappa).$$

But the distance in such a chart is uniformly bi-Lipschitz with respect to the metric on M, so the lemma follows. □

The following geometric lemma produces points on two spheres in a Riemannian manifold that are further apart than the centers of the spheres.

Lemma 3.5.3. *Let* M *be a closed Riemannian manifold. There exist* $C, r > 0$ *such that for all* $\beta \in (0, r)$*, if* $x, y \in M$ *satisfy* $\frac{\hookrightarrow M}{3} < d(x, y) < \frac{\hookrightarrow M}{2}$*, and there is a fixed* $p \in M$ *such that* $d(x, p) = d(y, x)$ *and* $d(p, y) < r$*, then there exists* $q \in M$ *depending on* p *such that:*

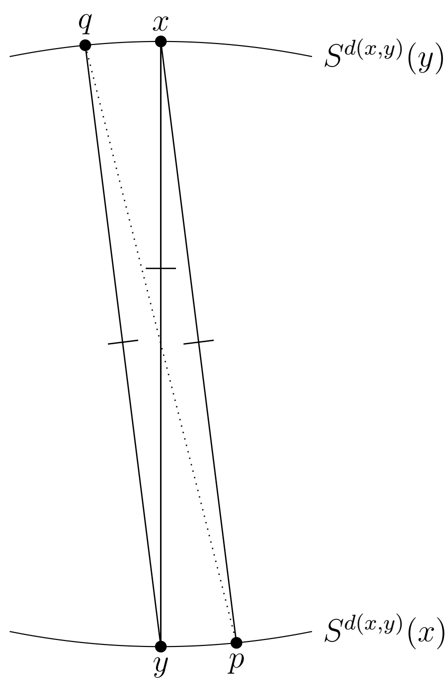

Figure 3.1: The four points x, y, p, q appearing in Lemma 3.5.3. Given x, y, p, the lemma produces the point q and gives an estimate on the length of the dotted line, which is longer than $d(x, y)$.

1. $d(q, y) = d(y, x)$,

2. $d(q, x) < \beta$, *and*

3. $d(q, p) \geq d(x, y) + Cd(y, p)\beta$.

In order to prove Lemma 3.5.3, we recall the following form of the second variation of length formula. For a proof of this and related discussion, see [CE75, Ch. 1,§6].

Lemma 3.5.4. *Let M be a Riemannian manifold and γ be a unit speed geodesic. Let $\gamma_{v,w}$ be a two parameter family of constant speed geodesics parametrized by $\gamma_{v,w}\colon [a, b] \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$ such that $\gamma_{0,0} = \gamma$. Suppose that $\frac{\partial \gamma_{v,w}}{\partial v} = V$ and $\frac{\partial \gamma_{v,w}}{\partial w} = W$ are both normal to $\dot{\gamma}_{0,0}$, which we denote by T. Then*

$$\frac{\partial^2 \operatorname{len}(\gamma_{v,w})}{\partial v \partial w} = \langle \nabla_W V, T\rangle|_a^b + \langle V, \nabla_T W\rangle|_a^b.$$

Proof of Lemma 3.5.3. We will give a geometric construction using the points x and y and then explain how this construction may be applied to the particular point p to produce a point q.

Let Q be a unit tangent vector based at y that is tangent to $S^{d(x,y)}(x)$, the sphere of radius $d(x, y)$ centered at x. Let $\gamma_t\colon [a, b] \to M$ be a one-parameter family of geodesics parametrized by arc length so that γ_0 is the unit speed geodesic from x to y, $\partial_t \gamma_t(b)|_{t=0} = Q$, $\gamma_t(b)$ is a path in $S^{d(x,y)}(x)$, and $\gamma_t(a) = x$ for all t. The variation γ_t gives rise to a Jacobi field Y. Note that $Y(a) = 0$, $Y(b) = Q$, and Y is a normal Jacobi field.

Next, let X be the Jacobi field along γ_0 defined by $X(b) = 0$ and $\nabla_T X|_b = Y(b)$, where T denotes $\dot{\gamma}_0$, i.e. the tangent to the curve γ_0. Such a field exists and has uniformly bounded norms because γ_0 is shorter than the injectivity radius. Let $\eta_t\colon [a, b] \to M$ be a one-parameter family of geodesics tangent to the field X such that $\eta_t(b) = y$, η_t is arc length parametrized, and $\eta_0 = \gamma_0$. Note that each η_t has length $d(x, y)$. Let T now denote $\dot{\gamma}_{s,t}$, which give the tangent direction to each curve $\gamma_{s,t}$ in the variation.

Define $\gamma_{s,t} \colon [a,b] \to M$ to be the arc length parametrized geodesic between $\eta_s(a)$ and $\gamma_t(b)$. The variation $\gamma_{s,t}$ is a two parameter variation satisfying the hypotheses of Lemma 3.5.4. Consequently, we see that

$$\frac{d^2 \operatorname{len}(\gamma_{s,t})}{dsdt} = \langle \nabla_X Y, T \rangle|_a^b + \langle Y, \nabla_T X \rangle|_a^b. \tag{3.23}$$

The first term may be rewritten as

$$\langle \nabla_X Y, T \rangle|_a^b = \nabla_X \langle Y, T \rangle|_a^b - \langle Y, \nabla_X T \rangle|_a^b. \tag{3.24}$$

As $Y(a) = 0$ and $X(b) = 0$, the second term in (3.24) is zero. Similarly $\nabla_X \langle Y, T \rangle|_b = 0$. We claim that $\nabla_X \langle Y, T \rangle|_a = 0$ as well. To see this we claim that $Y = \partial_t \gamma_{s,t}|_a = 0$ for all s. This is the case because $\gamma_{s,t}(a)$ is constant in t as $\gamma_{s,t}(a)$ depends only on s. Thus $\langle Y, T \rangle|_a = 0$. When we differentiate by X, we are differentiating along the path $\gamma_{s,0}(a)$. Thus $\nabla_X \langle Y, T \rangle|_a = 0$ as $\langle Y, T \rangle$ is 0 along this path. Thus $\langle \nabla_X Y, T \rangle|_a^b = 0$. Noting in addition that $Y(a) = 0$, equation (3.23) simplifies to

$$\frac{d^2 \operatorname{len}(\gamma_{s,t})}{dsdt} = \langle Y, \nabla_T X \rangle|_b.$$

Hence as we defined X so that $\nabla_T X|_b = Y(b)$,

$$\frac{d^2 \operatorname{len}(\gamma_{s,t})}{dsdt} = \langle Y(b), Y(b) \rangle = \|Q\| = 1.$$

Note next that $\frac{d^2}{ds^2} \operatorname{len}(\gamma_{s,t}) = 0$ because the geodesics $\gamma_{s,0}$ all have the same length. Similarly, $\frac{d^2}{dt^2} \operatorname{len}(\gamma_{s,t}) = 0$. Thus we have the Taylor expansion

$$\frac{d^2}{dsdt} \operatorname{len}(\gamma_{s,t}) = d(x,y) + st + O(s^3, t^3). \tag{3.25}$$

There exist $r_0 > 0$ and $C > 0$ such that for all $0 \le s, t < r_0$,

$$\text{len}(\gamma_{s,t}) \ge d(x, y) + Cst. \tag{3.26}$$

Consider now the pairs of points $\gamma_{s,0}(a)$ and $\gamma_{0,t}(b)$. We claim that if p is of the form $p = \gamma_{0,t}(b)$ for some small t then we may take $q = \gamma_{s,0}(a)$, where the choice of s will be dictated by β.

Note that

$$d(\gamma_{s,0}(a), x) = s\|X(a)\| + O(s^2) \text{ and } d(\gamma_{0,t}(b), y) = t\|Y(b)\| + O(t^2).$$

Hence there exists s_0 such that for $0 < s, t < s_0$,

$$d(\gamma_{s,0}(a), x) < 2s\|X(a)\| \text{ and } d(\gamma_{0,t}(b), y) < 2t\|Y(b)\|. \tag{3.27}$$

For any $\beta < \min\{2s_0\|X(a)\|, 2r_0\|X(a)\|\}$, by (3.26) taking $s = \beta/2\|X(a)\|$ we obtain

$$d(\gamma_{s,0}(a), \gamma_{0,t}(b)) \ge d(x, y) + t\beta C/2\|X(a)\|,$$

which by (3.27) implies

$$d(\gamma_{s,0}(a), \gamma_{0,t}(b)) \ge d(x, y) + \frac{C}{4\|X(a)\|\|Y(b)\|}\beta d(\gamma_{0,t}(b), y).$$

By (3.27) and our choice of s

$$d(\gamma_{s,0}(a), x) < \beta.$$

Finally, $d(\gamma_{s,0}(a), y) = d(x, y)$ by the construction of the variation. Thus the conclusion of the lemma holds for the points $p = \gamma_{0,t}(b)$ and $q = \gamma_{s,0}(a)$.

We claim that this gives the full result. First, note that for all pairs of points x and y

and choices of vectors Q in our construction that $\|X(a)\|$ and $\|Y(b)\|$ are bounded above and below. This is because the distance minimizing geodesic from X to Y does not cross the cut locus. Similarly, the constants C, r_0, and s_0 may be uniformly bounded below over all such choices of x and y by compactness. Thus as all these constants are uniformly bounded independent of x, y and Q, the above argument shows that for any pair x and y that there is a neighborhood N of y in $S^{d(x,y)}$ of uniformly bounded size, such that for any $p \in N$ there exists q satisfying the conclusion of the lemma. This gives the result as any p sufficiently close to y such that $d(x,p) = d(x,y)$ lies in such a neighborhood N. □

The following lemma shows that if a diffeomorphism with small strain nearly fixes a large region, then that diffeomorphism is close to the identity.

Lemma 3.5.5. *Let (M,g) be a closed Riemannian manifold. Then there exists $r_0 \in (0,1)$ such that for any $r', \beta \in (0, r_0)$, there exists $C > 0$ such that if $f \in \mathrm{Diff}^1(M)$ and*

1. $d_{C^0}(f, \mathrm{Id}) \le r_0$,

2. *there exists a point $x \in M$ such that all y with $d(x,y) < r'$ satisfy $d(y, f(y)) \le \beta \le r_0$, and*

3. $\|f^*g - g\| = \eta \le r_0$,

then

$$d_{C^0}(f, \mathrm{Id}) < C(\beta + \eta). \tag{3.28}$$

Proof. Let r_1, C_1 denote the r and C in Lemma 3.5.3. Let C_2 be the constant in Lemma 3.5.1. There exists a constant r_2 such that for any $x, y \in M$ with $\hookrightarrow (M)/3 < d(y,x) < \hookrightarrow (M)/2$ and any z such that $d(y,z) < r_2$, then $d(y, \hat{z}) < r_1$, where $\hat{z}$ is the radial projection of z onto $S^{d(x,y)}(x)$. Let $r_0 = \min\{r_1, r_2, \hookrightarrow (M)/24\}$.

Suppose that $x \in M$ has the property that $d(x,z) < r$ implies $d(z, f(z)) \le \beta$. Suppose that y is a point such that $\hookrightarrow (M)/3 < d(y,x) < \hookrightarrow (M)/2$. Let $\widehat{f(y)}$ be the radial projection of $f(y)$ onto $S^{d(x,y)}(x)$.

By choice of $r_0 \leq r_2$, $d(y, f(y)) < r_2$ and so $d(y, \widehat{f(y)}) \leq r_1$. Hence we may apply Lemma 3.5.3 with $\beta = r'$, $x = x$, $y = y$ and $p = \widehat{f(y)}$ to conclude that there exists a point $q \in M$ such that

$$d(q, y) = d(x, y), \tag{3.29}$$

$$d(q, x) < r', \tag{3.30}$$

$$d(q, \widehat{f(y)}) \geq d(x, y) + C_1 d(y, \widehat{f(y)}) r'. \tag{3.31}$$

Using the triangle inequality, we bound the left hand side of (3.31) to find

$$d(q, f(q)) + d(f(q), f(y)) + d(f(y), \widehat{f(y)}) \geq d(q, \widehat{f(y)}) \geq d(x, y) + C_1 d(y, \widehat{f(y)}) r'. \tag{3.32}$$

First, as $d(q, x) < r'$ and points within r' of x do not move more than β,

$$d(q, f(q)) \leq \beta.$$

Second, by Lemma 3.5.1, as the distance between q and y is bounded above by $\hookrightarrow (M)/2$, there exists C_3 such that

$$d(f(q), f(y)) \leq d(q, y)(1 + C_2 \eta) = d(x, y) + C_3 \eta.$$

Similarly, as $\hookrightarrow (M)/3 < d(x, y) <\hookrightarrow (M)/2$, Lemma 3.5.1 implies the following two bounds

$$d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y)) \leq \beta + d(x, y) + C_3 \eta \tag{3.33}$$

and similarly

$$d(x, f(y)) \geq d(x, y) - \beta - C_3 \eta. \tag{3.34}$$

For w sufficiently close to $S^{d(x,y)}(x)$ we claim that the radial projection $\hat{w}$ is the point in $S^{d(x,y)}(x)$ that minimizes the distance to w. To see this we use that below the injectivity radius geodesics are the unique distance minimizing path between two points. There are two cases: if $d(x,w) > d(x,y)$ and there is some other point $w' \in S^{d(x,y)}(x)$ with $d(w',w) \leq d(\hat{w},w)$, then the path from x to w' to w along geodesics must be strictly longer than the geodesic path from x directly to $\hat{w}$. If $d(x,w) < d(x,y)$ and $\hat{w} \neq w' \in S^{d(x,y)}(x)$, then one obtains two distance minimizing paths from x to $S^{d(x,y)}(x)$ passing through w: the first along a single geodesic and the second from x to w and then from w to w'. By the uniqueness of distance minimizing geodesics, the latter path must have length greater than $d(x,y)$ because it is not a geodesic. Thus $d(w,w') > d(w,\hat{w})$; a contradiction.

The estimates (3.33) and (3.34) imply that $|d(f(y),x) - d(x,y)| \leq \beta + C_3\eta$. Thus the distance from $f(y)$ to $S^{d(x,y)}(x)$ is at most $\beta + C_3\eta$. By the previous paragraph, $\widehat{f(y)}$ is the point in $S^{d(x,y)}(x)$ that minimizes distance to $f(y)$. Thus

$$d(f(y), \widehat{f(y)}) \leq \beta + C_3\eta. \tag{3.35}$$

Thus, we obtain from equation (3.32)

$$\beta + d(x,y) + C_3\eta + \beta + C_3\eta \geq d(x,y) + C_1 d(y, \widehat{f(y)})r'.$$

Thus

$$\frac{2\beta + 2C_3\eta}{C_1 r'} \geq d(y, \widehat{f(y)}).$$

Hence

$$d(y,f(y)) \leq d(f(y), \widehat{f(y)}) + d(y, \widehat{f(y)}) \leq \frac{2\beta + 2C_3\eta}{C_1 r'} + \beta + C_3\eta.$$

Thus by introducing a new constant $C_4 \geq 1$, we see that for any y satisfying $\hookrightarrow (M)/3 < d(y,x) < \hookrightarrow (M)/2$, that

$$d(y,f(y)) \leq C_4(\beta + \eta).$$

Note that the constant C_4 depends only on r' and (M, g).

Consider a point y where $(1/3+1/24) \hookrightarrow (M) < d(x, y) < (1/2-1/24) \hookrightarrow (M)$. Because $r' <\hookrightarrow (M)/24$ such a point y has a neighborhood of size r' on which points are moved at most distance $C_4(\beta + \eta)$ by f. Hence we may repeat the procedure taking y as the new basepoint. Let x be the given point in the statement of the lemma. Any point $q \in M$ may be connected to x via a finite sequence of points $x = x_0, \ldots, x_n = q$ such that each consecutive pair of points in the sequence are at a distance between $(1/3 + 1/24) \hookrightarrow (M)$ and $(1/2 - 1/24) \hookrightarrow (M)$ apart. As M is compact there is a uniform upper bound on the length of the shortest such sequence. If N is a uniform upper bound on the length of such a sequence, the above argument shows that for all $q \in M$

$$d(q, f(q)) \leq NC_4^N(\beta + \eta),$$

which gives the result. □

The proof of Theorem 3.5.1 consists of two steps. First a disk of uniform radius is produced on which f nearly fixes points. Then Lemma 3.5.5 is applied to this disk to conclude that f is near to the identity.

Proof of Theorem 3.5.1. Let r_1, C_1 be denote the r and C in Lemma 3.5.2, and let r_2, C_2 denote the r and c in Lemma 3.5.3. There will be a constant $r_3 > 0$ introduced later when it is needed. Let r_4 denote the constant r_0 appearing in Lemma 3.5.5. We let $r = \min\{1, r_1, r_2, r_3, r_4, \hookrightarrow (M)/24\}$. Let C_3 be the constant in Lemma 3.5.1. Let $\gamma \in (0, r)$ be given.

By Lemma 3.5.2, for all z such that $d(x, z) < \gamma$,

$$d(z, f(z)) \leq C_1(\theta\gamma + \gamma^2\kappa). \tag{3.36}$$

Suppose that y satisfies $\hookrightarrow (M)/3 < d(x, y) <\hookrightarrow (M)/2$. Let $\widehat{f(y)}$ be the radial projection of $f(y)$ onto the sphere $S^{d(x,y)}(x)$.

By Lemma 3.5.1,

$$d(x,y)(1 - C_3\eta) \leq d(f(x), f(y)) \leq d(x,y)(1 + C_3\eta).$$

As $f(x) = x$, this implies

$$d(x,y)(1 - C_3\eta) \leq d(x, f(y)) \leq d(x,y)(1 + C_3\eta).$$

Hence as $d(x,y)$ is uniformly bounded above and below, there exists C_4 such that

$$d(f(y), \widehat{f(y)}) < C_4\eta. \tag{3.37}$$

There exists $r_3 > 0$ such that if $\eta < r_3$, then $C_4\eta < r_2$. Hence by our choice of r, $d(y, \widehat{f(y)}) < r_2$ and we may apply Lemma 3.5.3 with $\beta = \gamma$, $x = x$, $y = y$, $p = \widehat{f(y)}$ to deduce that there exists q such that

$$d(q,y) = d(x,y), \tag{3.38}$$

$$d(q,x) < \gamma, \tag{3.39}$$

$$d(q, \widehat{f(y)}) \geq d(x,y) + C_2 d(y, \widehat{f(y)})\gamma. \tag{3.40}$$

By Lemma 3.5.1, and using that $d(x,y)$ is bounded by $\hookrightarrow (M)/2$, there exists C_5 such that

$$d(f(q), f(y)) \leq d(q,y)(1 + C_3\eta) \leq d(x,y) + C_5\eta. \tag{3.41}$$

By equation (3.36), as $d(q,x) < \gamma$,

$$d(q, f(q)) < C_1(\theta\gamma + \kappa\gamma^2). \tag{3.42}$$

Using the triangle inequality with (3.37), (3.41), (3.42), to bound the left hand side of equation (3.40), we obtain that

$$C_1(\theta\gamma+\kappa\gamma^2)+d(x,y)+C_5\eta+C_4\eta \geq d(q,f(q))+d(f(q),f(y))+d(f(y),\widehat{f(y)}) \geq d(x,y)+C_2 d(y,\widehat{f(y)})\gamma.$$

Moreover (3.37) gives the lower bound $d(y,\widehat{f(y)}) > d(y,f(y)) - C_4\eta$. We then obtain that

$$C_1(\theta\gamma+\kappa\gamma^2)+C_5\eta+C_4\eta \geq C_2 d(y,f(y))\gamma - C_2C_4\eta\gamma,$$

and so

$$\frac{C_1(\theta\gamma+\kappa\gamma^2)+C_5\eta+C_4\eta+C_2C_4\eta\gamma}{C_2\gamma} \geq d(y,f(y)).$$

The constants $C_1,\ldots,C_5$ are uniform over all y satisfying $\hookrightarrow(M)/3 < d(x,y) <\hookrightarrow(M)/2$. Thus there exists $C_6 > 0$ such that for all such y,

$$C_6(\eta\gamma^{-1}+\theta+\kappa\gamma) \geq d(y,f(y)). \tag{3.43}$$

Suppose that y is a point at distance $\frac{5}{12}\hookrightarrow(M)$ from x. The above argument shows if z satisfies $d(y,z) <\hookrightarrow(M)/12$ then (3.43) holds with y replaced by z, i.e.

$$C_6(\eta\gamma^{-1}+\theta+\kappa\gamma) \geq d(z,f(z)).$$

Define α by

$$\alpha = C_6(\eta\gamma^{-1}+\theta+\kappa\gamma). \tag{3.44}$$

Assuming that $\alpha < r_4$, z satisfies the second numbered hypothesis of Lemma 3.5.5 with $\beta = \alpha$ and any $r' \leq\hookrightarrow(M)/12$.

There are then two cases depending on whether $\alpha > r_4$ or $\alpha \leq r_4$. In the case that $\alpha \leq r_4$, we apply Lemma 3.5.5 with $x_0 = z$, $r' = r/2$, and $\beta = \alpha$. This gives that there exists a C_7 depending only on $r/2$ such that

$$d_{C^0}(f, \mathrm{Id}) \leq C_7(\eta\gamma^{-1} + \theta + \kappa\gamma).$$

If $\alpha > r_4$, then as $\kappa \leq r_4$,

$$d_{C^0}(f, \mathrm{Id}) \leq \kappa \leq r_4 \leq \alpha = C_6(\eta\gamma^{-1} + \theta + \kappa\gamma).$$

Thus letting $C_8 = \max\{C_6, C_7\}$, we have that

$$d_{C^0}(f, \mathrm{Id}) \leq C_8(\eta\gamma^{-1} + \theta + \kappa\gamma),$$

which gives the result. □

3.5.2 Application to isotropic spaces: proof of Proposition 13

We now prove Proposition 13, which is an application of Theorem 3.5.1 to isotropic spaces. The idea of the proof is geometric. We consider the diffeomorphism $I^{-1}f$. This diffeomorphism is small in C^0 norm, so there is an isometry R_1 that is close to the identity such that $R_1^{-1}I^{-1}f$ has a fixed point x. The differential of $R_1^{-1}I^{-1}f$ at x is very close to preserving both the metric tensor and curvature tensor at x. We then use the following lemma to obtain an isometry R_2 that is nearby to $R_1^{-1}I^{-1}f$.

Lemma 3.5.6. *[Hel01, Ch. IV Ex. A.6] Let M be a simply connected Riemannian globally symmetric space or $\mathbb{RP}^n$. Then if $x \in M$ and $L\colon T_xM \to T_xM$ is a linear map preserving both the metric tensor at x and the curvature tensor at x, then there exists $R \in \mathrm{Isom}(M)$ such that $R(x) = x$ and $D_xR = L$.*

We take the diffeomorphism in the conclusion of Proposition 13 to equal IR_1R_2. We then apply Theorem 3.5.1 to deduce that $R_2^{-1}R_1^{-1}I^{-1}f$ is near the identity diffeomorphism.

It follows that IR_1R_2 is near to f. Before beginning the proof, we state some additional lemmas.

Lemma 3.5.7. *Suppose that V_1 and V_2 are two subspaces of a finite dimensional inner product space W. Then there exists $C > 0$ such that if $x \in W$, then*

$$d(x, V_1 \cap V_2) < C(d(x, V_1) + d(x, V_2)).$$

Lemma 3.5.8. *Suppose that R is a tensor on $\mathbb{R}^n$. Let $\operatorname{stab}(R)$ be the subgroup of $\operatorname{GL}(\mathbb{R}^n)$ that stabilizes R under pullback. Then there exist $C, D > 0$ such that if $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map and $\|L - \operatorname{Id}\| < D$, then*

$$d_{\operatorname{GL}(\mathbb{R}^n)}(L, \operatorname{stab}(R)) \leq C\|L^*R - R\|.$$

Proof. Let $\mathfrak{s}$ be the Lie algebra to $\operatorname{stab}(R)$. Then consider the map ϕ from $\mathfrak{gl}$ to the tensor algebra on $\mathbb{R}^n$ given by

$$w \mapsto \exp(w)^*R - R.$$

We may write $w = v + v^{\perp}$, where $v \in \mathfrak{s}$ and $v \in \mathfrak{s}^{\perp}$. Because ϕ is smooth it has a Taylor expansion of the form

$$\phi(tv + tv^{\perp}) = 0 + tAv + tBv^{\perp} + O(t^2). \tag{3.45}$$

Note that A is zero because $v \in \mathfrak{s}$. We claim that B is injective. For the sake of contradiction, suppose $Bv^\perp = 0$ for some $v^\perp \in \mathfrak{s}^\perp$. Then $\exp(tv^\perp)^* R - R = O(t^2)$. But then

$$\begin{aligned}\exp(v^\perp)^* R - R &= \sum_{i=0}^{n-1} \exp((i+1)v^\perp/n)^* R - \exp(iv^\perp/n) R \\ &= \sum_{i=0}^{n-1} \exp(iv^\perp/n)^* (\exp(v^\perp/n)^* R - R) \\ &= O(1/n).\end{aligned}$$

And hence $\exp(v^\perp)^* R - R = 0$, which contradicts $v^\perp \notin \mathfrak{s}$. Thus B is an injection and hence by Taylor's theorem for small $v^\perp$ there exists C_1 such that

$$\|\exp(v^\perp)^* R - R\| \geq C_1 \|v^\perp\|. \tag{3.46}$$

By using the Taylor expansion (3.45) and noting that $A = 0$ there, we obtain from equation (3.46) that there exists $C_2 > 0$ such that

$$\|\exp(w)^* R - R\| \geq C_2 \|v^\perp\|. \tag{3.47}$$

It then follows there exists a neighborhood N of $\mathrm{Id} \in \mathrm{GL}(\mathbb{R}^n)$ such that $\mathrm{stab}(R) \cap N$ is the image of a disc $D \subset \mathfrak{s}$ under exp. Write $\mathfrak{gl} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ as a vector space. Thus as exp is bilipschitz in a neighborhood of $0 \in \mathfrak{gl}$ there exists C_3 such that if we write $w \in D$ as $w = v + v^\perp$, where $v \in \mathfrak{s}$ and $v^\perp \in \mathfrak{s}^\perp$, then

$$C_3^{-1} \|v^\perp\| \leq d_{\mathrm{GL}(\mathbb{R}^n)}(\exp(w), \exp(D)) \leq C_3 \|v^\perp\|. \tag{3.48}$$

As $\mathrm{stab}(R) \cap N = \exp(D)$, for all w in a smaller neighborhood $D' \subset D$, the middle term above is comparable to $d_{GL(\mathbb{R}^n)}(\exp(w), \mathrm{stab}(R))$.

Thus combining (3.48) with (3.47), we obtain

$$d_{\mathrm{GL}(\mathbb{R}^n)}(\exp(w), \mathrm{stab}(R)) \le C_2^{-1}C_3\|\exp(w)^*R - R\|.$$

This gives the result as exp is a surjection onto a neighborhood of $\mathrm{Id} \in \mathrm{GL}(\mathbb{R}^n)$. □

The following lemma is immediate from [Hel01, Thm. IV.3.3], which explicitly describes the isometries of globally symmetric spaces.

Lemma 3.5.9. *Suppose that M is a closed globally symmetric space. There exists $C > 0$ such that if $x, y \in M$, then there exists an isometry $I \in \mathrm{Isom}(M)^\circ$ such that $I(x) = y$ and $d_{C^0}(I, \mathrm{Id}) \le Cd(x, y)$. As $\mathrm{Isom}(M)^\circ$ is compact, it follows that for each k there exists a constant C_k such that one choose I with $d_{C^k}(I, \mathrm{Id}) \le C_k d(x, y)$.*

We also use the following lemma, which is the specialization of Lemma 3.5.8 to the metric tensor.

Lemma 3.5.10. *Suppose that V is a finite dimensional inner product space with metric g of dimension d. There exists a neighborhood U of $\mathrm{Id} \in \mathrm{GL}(V)$ and a constant C such that if $L \in U$ then*

$$d_{\mathrm{GL}(V)}(L, \mathrm{SO}(V)) \le C\|L^*g - g\|,$$

where $\mathrm{GL}(V)$ is endowed with the right-invariant Riemannian metric it inherits from the inner product space V.

We now prove the proposition.

Proof of Proposition 13. Pick $0 < \lambda < 1$ and a small τ such that

$$\frac{\lambda}{2} - \lambda\tau > \frac{1}{2} - \sigma \text{ and } \sigma > \tau > 0. \tag{3.49}$$

We also assume without loss of generality that $\ell \ge 3$. By Lemma 3.7.10 there exist k_0 and $\epsilon_0 > 0$ such that if s is a smooth section of the bundle of symmetric 2-tensors over M,

$\|s\|_{C^{k_0}} \leq 4$, and $\|s\|_{H^0} \leq \epsilon_0$, then $\|s\|_{C^\ell} \leq \|s\|_{H^0}^{1-\tau}$. Choose k such that

$$k > \max\{k_0, \frac{\ell}{1-\lambda}\}. \tag{3.50}$$

In addition, there are positive numbers $\epsilon_1, \ldots, \epsilon_7$ that will be introduced when needed in the proof below. We define

$$r = \min\{\epsilon_0, \epsilon_1^{1/(1-\tau)}, \epsilon_2, \ldots, \epsilon_7, 1\}.$$

Let $\epsilon_1 > 0$ be small enough that for any $x \in M$, if $L \colon T_xM \to T_xM$ is invertible and $\|L^*g - g\| \leq \epsilon_1$, then the conclusion of Lemma 3.5.10 holds for L.

Let $\eta = \|f^*g - g\|_{H^0}$ and $\varepsilon_2 = d_{C^2}(f, I)$. Consider the norm $\|f^*g - g\|_{C^{k_0}}$. As $d_{C^k}(I, f)$ is uniformly bounded, we see that $\|f^*g - g\|_{C^{k-1}}$ is uniformly bounded. In fact, there exists $\epsilon_2 > 0$ such that if $d_{C^k}(I, f) < \epsilon_2$, then $\|f^*g - g\|_{C^{k-1}} \leq 4$. As $r < \epsilon_0$, the discussion in the first paragraph of the proof implies that

$$\|f^*g - g\|_{C^3} \leq \eta^{1-\tau}. \tag{3.51}$$

Note that this is less than ϵ_1 by the choice of r.

For $x \in M$, we may consider the Lie group $\mathrm{GL}(T_xM)$ as well as its Lie algebra $\mathfrak{gl}$. There exists $\epsilon_3 > 0$ such that restricted to the ball of radius ϵ_3 about $0 \in \mathfrak{gl}$, the Lie exponential, which we denote by exp, is bilipschitz with constant 2.

Let $x \in M$ be a point that is moved the maximum distance by $I^{-1}f$. By Lemma 3.5.9, there exists a constant $D_k > 0$ independent of x and an isometry R_1 such that $R_1(x) = I^{-1}f(x)$ and $d_{C^k}(R_1, \mathrm{Id}) < D_k d(x, I^{-1}f(x))$. Let $h = R_1^{-1}I^{-1}f$ and note that h fixes x. Note that there exists $\epsilon_4 > 0$ such that if $d_{C^k}(f, I) < \epsilon_4$, then by the previous sentence R_1 can be chosen so that $d_{C^k}(R_1, \mathrm{Id})$ is small enough that

$$\|D_xh - \mathrm{Id}\,\| \leq C_0\varepsilon_2. \tag{3.52}$$

We claim that D_xh is near a linear map of T_xM that preserves both the metric tensor and the curvature tensor. Let $\mathrm{SO}(T_xM)$ be the group of linear maps preserving the metric tensor on T_xM and let G be the group of linear maps preserving the curvature tensor on T_xM. Both of these are subgroups of $\mathrm{GL}(T_xM)$. By the sentence after equation (3.51), D_xh pulls back the metric on T_xM to be within ϵ_1 of itself. Thus by Lemma 3.5.9, there exists a uniform constant C_1 such that D_xh is within distance $C_1\eta^{1-\tau}$ of $\mathrm{SO}(T_xM)$. Again by equation (3.51), we have that $\|h^*g - g\|_{C^3} \leq \eta^{1-\tau}$. In particular, as the curvature tensor is defined by the second derivatives of the metric, this implies by Lemma 3.5.8 that there exists a constant C_2 such that D_xh is within distance $C_2\eta^{1-\tau}$ of G.

The previous paragraph shows that there exists C_3 such that D_xh is within distance $C_3\eta^{1-\tau}$ of both $\mathrm{SO}(T_xM)$ and G. Consider now the exponential map of $\mathrm{GL}(T_xM)$. As before, let $\mathfrak{gl}$ denote the Lie algebra of $\mathrm{GL}(T_xM)$. Let $H = \exp^{-1}(D_xh) \in \mathrm{GL}(T_xM)$. Note that this preimage is defined as D_xh is near to the identity. Let $\mathfrak{so}$ be the Lie algebra to $\mathrm{SO}(T_xM)$ and let $\mathfrak{g}$ be the Lie algebra to G. As both $SO(T_xM)$ and G are closed subgroups and exp is bilipschitz we conclude that the distance both between H and each of $\mathfrak{so}$ and $\mathfrak{g}$ is bounded above by $2C_3\eta^{1-\tau}$. Thus by Lemma 3.5.7, there exists C_4 such that H is at most distance $C_4\eta^{1-\tau}$ from $\mathfrak{g} \cap \mathfrak{so}$. Let $X \in \mathrm{GL}(T_xM)$ be an element of $\mathfrak{g} \cap \mathfrak{so}$ minimizing the distance from H to $\mathfrak{g} \cap \mathfrak{so}$. There exists $\epsilon_5 > 0$ such that if $\eta \leq \epsilon_5$ then $C_4\eta^{1-\tau} < \epsilon_3$. Hence as $r < \epsilon_5$, the same bilipschitz estimate on the Lie exponential gives

$$d(\exp(X), D_xh) \leq 2C_4\eta^{1-\tau}. \tag{3.53}$$

Note that $\exp(X) \in SO(T_xM) \cap G$. By Lemma 3.5.6, there exists an isometry R_2 of M such that R_2 fixes x and $D_xR_2 = \exp(X)$. In fact, because of equation (3.52) and because X is within distance $C_4\eta^{1-\tau}$ of H, we may bound the norm of X and hence deduce that there exists C_5 such that

$$d_{C^k}(R_2, \mathrm{Id}) \leq C_5(\varepsilon_2 + \eta^{1-\tau}). \tag{3.54}$$

The map R in the conclusion of the proposition will be IR_1R_2. We must now check that $R = IR_1R_2$ satisfies estimates (3.21) and (3.22). The former is straightforward: (3.21) follows from (3.54) combined with knowing that R_1 was constructed so that $d(R_1, \mathrm{Id}) \leq D'\varepsilon_2$ for some uniform $D' > 0$.

Let $h_2 = R_2^{-1}h$. The map h_2 has x as a fixed point. There exists $C_6 > 0$ such that the following four estimates hold:

$$\|D_x h_2 - \mathrm{Id}\,\| \leq C_6 \eta^{1-\tau}, \tag{3.55}$$

$$\|h_2^* g - g\|_{C^3} \leq \eta^{1-\tau}, \tag{3.56}$$

$$d_{C^2}(h_2, \mathrm{Id}) \leq C_6(\varepsilon_2 + \eta^{1-\tau}), \tag{3.57}$$

$$d_{C^k}(h_2, \mathrm{Id}) \leq C_6(\eta^{1-\tau} + d_{C^k}(I, f)). \tag{3.58}$$

The first two estimates above are immediate from equations (3.53) and (3.51), respectively. The third and fourth follow from an estimate on C^k compositions, Lemma 3.7.5, and equation (3.54).

Let r_0 be the cutoff r appearing in Theorem 3.5.1. Note that there exists $\epsilon_6 > 0$ such that if $d_{C^k}(f, I) < \epsilon_6$ and $\eta < \epsilon_6$, then the right hand side of each of inequalities (3.55) through (3.58) is bounded above by r_0. Hence as $r < \epsilon_6$ we apply Theorem 3.5.1 to h_2 to conclude that there exists C_7 such that for all $0 < \gamma < r_0$,

$$d_{C^0}(\mathrm{Id}, h_2) < C_7(\eta^{1-\tau} + C_6(\varepsilon_2 + \eta^{1-\tau})\gamma + \eta^{1-\tau}\gamma^{-1}).$$

But $h_2 = R_2^{-1}R_1^{-1}I^{-1}f$, so

$$d_{C^0}(R, f) < C_8(\eta^{1-\tau} + C_6(\varepsilon_2 + \eta^{1-\tau})\gamma + \eta^{1-\tau}\gamma^{-1}). \tag{3.59}$$

We now obtain the high regularity estimate, equation (3.22), via interpolation. By similarly moving the isometries from one slot to the other, (3.58) gives that

$$d_{C^k}(R,f) < C_9(\eta^{1-\tau} + d_{C^k}(I,f)). \tag{3.60}$$

There exists $\epsilon_7 > 0$ such that if $d_{C^k}(I,f) < \epsilon_7$ and $\eta < \epsilon_7$, then the right hand side of equation (3.60) is at most 1.

We now apply the interpolation inequality in Lemma 3.7.7 and interpolate between the C^0 and C^k distance to estimate $d_{C^\ell}(R,f)$. Write $\ell = (1-\lambda')k$ for some λ' and note that $1 > \lambda' > \lambda$ by (3.50). We use the estimate in equation (3.59) to estimate the C^0 norm and use 1 to estimate the C^k norm, which we may do because $r < \epsilon_7$. Thus there exists C_{10} such that for $0 < \gamma < r_0$,

$$d_{C^\ell}(R,f) < C_{10}(\eta^{1-\tau}\gamma^{-1} + \varepsilon_2\gamma)^{\lambda'}. \tag{3.61}$$

Note that there exists $C_{11} > 0$ such that $\|f^*g - g\|_{H^0} \leq C_{11}\varepsilon_2$. Consequently, there exists a constant C_{13} such that $C_{12}\sqrt{\eta/\varepsilon_2}$ is less than the cutoff r_0. We take γ to equal $C_{12}\sqrt{\eta/\varepsilon_2}$ in equation (3.61), which gives

$$d_{C^\ell}(R,f) < C_{13}(\eta^{1/2-\tau}\varepsilon_2^{1/2} + \eta^{1/2}\varepsilon_2^{1/2})^{\lambda'} < C_{14}(\eta^{\lambda/2-\lambda\tau}\varepsilon_2^{\lambda/2} + \eta^{\lambda/2}\varepsilon_2^{\lambda/2}). \tag{3.62}$$

Hence by our choice of λ and τ in equation (3.49) and because $\eta < r < 1$,

$$d_{C^\ell}(R,f) < C_{15}\eta^{1/2-\sigma}\varepsilon_2^{1/2-\sigma}, \tag{3.63}$$

which establishes equation (3.22) and finishes the proof. □

3.6 KAM SCHEME

In this section we develop the KAM scheme and prove that it converges. A KAM scheme is an iterative approach to constructing a conjugacy between two systems in the C^∞ setting.

We begin by discussing the smoothing operators that will be used in the scheme. Then we state a lemma, Lemma 3.6.1, that summarizes the results of performing a step in the scheme. We then prove in Theorem 3.1.1 that by iterating the single KAM step that we obtain the convergence needed for this theorem. We conclude the section with a final corollary of the KAM scheme which gives an asymptotic relationship between the top exponent, the bottom exponent, and the sum of all the exponents.

3.6.1 One step in the KAM scheme

In the KAM scheme, we begin with a tuple of isometries $(R_1, ..., R_m)$ and a nearby tuple of diffeomorphisms $(f_1, ..., f_m)$. We want to find a diffeomorphism ϕ such that for all i, $\phi^{-1} f_i \phi = R_i$. However, such a ϕ may not exist.

We will then attempt construct a conjugacy, ϕ that has the following property. Let $\widetilde{f}_i$ equal $\phi^{-1} f_i \phi$. If we consider the tuple $(\widetilde{f}_1, ..., \widetilde{f}_m)$ and $(R_1, ..., R_m)$, we can arrange that the error term, $\mathcal{U}$, in Proposition 12, is small. Once we know that the error term is small, the estimate in Proposition 12 shows that small Lyapunov exponents imply that each $\widetilde{f}_i$ has small strain. Then using Proposition 13, small strain implies that there exist R_i' that each $\widetilde{f}_i$ is near to that R_i'. We then apply the same process to the tuples $(\widetilde{f}_1, ..., \widetilde{f}_m)$ and $(R_1', \ldots, R_m')$.

The previous paragraph contains the core idea of the KAM scheme. Following this scheme, one encounters a common technical difficulty inherent in KAM arguments: regularity. In our case, this problem is most crucial when we construct the conjugacy ϕ. There is not a single choice of ϕ, but rather a family depending on a parameter λ. The parameter λ controls how smooth ϕ is. Larger values of λ give less regular conjugacies. We refer to this as a *conjugation of cutoff* λ; the formal construction of the *conjugation of cutoff* λ appears in the proof in Lemma 3.6.1 which also gives estimates following from this construction. The nth time we iterate this procedure we will use a particular value λ_n as our cutoff. The proof of Theorem 3.1.1 shows how to pick the sequence λ_n so that the procedure converges.

We now introduce the smoothing operators. Suppose that M is a closed Riemannian manifold. As before, let Δ denote the Casimir Laplacian on M as in subsection 3.2.4. As Δ is self adjoint, it decomposes the space of L^2 vector fields into subspaces depending on the particular eigenvalue associated to that subspace. We call these subspaces H_λ. For a vector field X, we may write $X = \sum_\lambda X_\lambda$, where $X_\lambda \in H_\lambda$ is the projection of X onto the λ eigenspace of Δ. All of the eigenvalues of Δ are positive. By removing the components of X that lie in high eigenvalue subspaces, we are able to smooth X. Let $\mathcal{T}_\lambda X = \sum_{\lambda'<\lambda} X_{\lambda'}$ equal the projection onto the modes strictly less than λ in magnitude. Let $\mathcal{R}_\lambda X = \sum_{\lambda'\geq\lambda} X_{\lambda'}$ be the projection onto the modes of magnitude greater than or equal to λ. Then $X = \mathcal{T}_\lambda X + \mathcal{R}_\lambda X$.

We record two standard estimates which may be obtained by application of the Sobolev embedding theorem. For $s \geq 0$, there exists a constant $C_s > 0$ such that for any $\overline{s} \geq s$ and any C^∞ vector field X on M,

$$\|\mathcal{T}_\lambda X\|_{C^{\overline{s}}} \leq C_s \lambda^{k_3+(\overline{s}-s)/2} \|X\|_{C^s}, \tag{3.64}$$

$$\|\mathcal{R}_\lambda X\|_{C^s} \leq C_s \lambda^{k_3-(\overline{s}-s)/2} \|X\|_{C^{\overline{s}}}. \tag{3.65}$$

The smoothing operators and the above estimates on them are useful because without smoothing certain estimates appearing in the KAM scheme become unusable. One may see this by considering what happens in the proof of Lemma 3.6.1 if one removes the smoothing operator $\mathcal{T}_\lambda$ from equation (3.72).

The proof of the following lemma should be compared with [DK07, Sec. 3.4]

Lemma 3.6.1. *Suppose that (M^d, g) is a closed isotropic Riemannian manifold other than S^1. There exists a natural number l_0 such that for $\ell > l_0$ and any (C, α, n_0) the following holds. For any sufficiently small $\sigma > 0$, there exist a constant $r_\ell > 0$ and numbers k_0, k_1, k_2 such that for any $s > \ell$ and any m there exist constants $C_{s,\ell}, r_{s,\ell} > 0$ such that the following holds. Suppose that $(R_1, ..., R_m)$ is a (C, α, n_0)-Diophantine tuple with entries in* Isom(M) *and $(f_1, ..., f_m)$ is a collection of C^∞ diffeomorphisms of M. Suppose that the random dy-*

namical system generated by $(f_1, ..., f_m)$ has stationary measures with arbitrarily small in magnitude bottom exponent. Write ε_k for $\max_i d_{C^k}(f_i, R_i)$. If $\lambda \geq 1$ is a number such that

$$\lambda^{k_0}\varepsilon_{l_0} \leq r_\ell \tag{3.66}$$

and

$$\lambda^{k_1 - s/4}\varepsilon_s + \varepsilon_{l_0}^{3/2} < r_{s,\ell}, \tag{3.67}$$

then there exists a smooth diffeomorphism ϕ and a new tuple $(R_1', ..., R_m')$ of isometries of M such that for all i setting $\widetilde{f}_i = \phi f_i \phi^{-1}$, we have

$$d_{C^\ell}(\widetilde{f}_i, R_i') \leq C_{s,\ell}(\lambda^{k_1 - s/10}\varepsilon_s^{1-\sigma} + \varepsilon_{l_0}^{9/8}), \tag{3.68}$$

$$d_{C^0}(R_i, R_i') \leq C_{s,\ell}(\varepsilon_{l_0} + (\lambda^{k_1 - s/4}\varepsilon_s + \varepsilon_{l_0}^{3/2})^{1-\sigma}), \tag{3.69}$$

$$d_{C^s}(\widetilde{f}_i, R_i') \leq C_{s,\ell}\lambda^{k_2}\varepsilon_s, \text{ and} \tag{3.70}$$

$$d_{C^s}(\phi, \mathrm{Id}) \leq C_{s,\ell}\lambda^{k_2}\varepsilon_s. \tag{3.71}$$

The diffeomorphism ϕ is called a conjugation of cutoff λ.

Proof. As in equation (3.9), let Y_i be the smallest vector field on Y_i satisfying $\exp_{R(x)} Y_i(x) = f_i(x)$. Let $\mathcal{L}$ be the operator on vectors fields defined by $\mathcal{L}(Z) = m^{-1}\sum_{i=1}^m (R_i)_* Z$ as in Proposition 10. Let

$$V := -(1-\mathcal{L})^{-1}\left(\frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i\right) \tag{3.72}$$

and let $\widetilde{f}_i = \psi_V f_i \psi_V^{-1}$. Let $\widetilde{\varepsilon}_k = \max_i d_{C^k}(\widetilde{f}_i, R_i)$ and let $\widetilde{Y}_i$ be the pointwise smallest vector field such that $\exp_{R(x)} \widetilde{Y}_i(x) = \widetilde{f}_i(x)$. By Proposition 15, for a C^1 small vector field V,

$$\widetilde{Y}_i = Y_i + V - R_i V + Q(Y_i, V), \tag{3.73}$$

where Q is quadratic in the sense of Definition 3.7.1. By Proposition 7, we see that $\|V\|_{C^k} \leq C_k\varepsilon_{k+\alpha}$ for some fixed α. There exist β, D_1 such that $\|Q(Y_i, V)\|_{C^k} \leq D_k\varepsilon_{k+\beta}^2$. By estimating the terms in equation (3.73), it follows that for each $k > 0$ if $\varepsilon_{k+\alpha+\beta} < 1$ then there exists a constant $D_{2,k}$ such that

$$d_{C^k}(\widetilde{f}_i, R_i) < D_{2,k}\varepsilon_{k+\alpha+\beta}. \tag{3.74}$$

Let μ be an ergodic stationary measure on M for the tuple $(\widetilde{f}_1, ..., \widetilde{f}_m)$ as in the statement of the lemma. We now apply Proposition 12 with $r = d-1, d$ and recall why the hypotheses of that proposition are satisfied. First, by our assumption that M is isotropic, $\mathrm{Isom}(M)^\circ$ acts transitively on M and $\mathrm{Gr}_1(M)$. We have also assumed the tuple $(R_1, \dots, R_m)$ is Diophantine. The nearness of $(\widetilde{f}_1, \dots, \widetilde{f}_m)$ to $(R_1, \dots, R_m)$ is guaranteed by equation (3.74), a sufficiently small choice of r_ℓ, and sufficiently large choice of l_0 by equation (3.66) as $\lambda \geq 1$. Thus by applying Proposition 12 to the conjugated system, there exists k_1 such that, in the language of that proposition:

$$\Lambda_r(\mu) = \frac{-r}{2dm}\sum_{i=1}^m \int_M \|E_C^{\widetilde{f}_i}\|^2 + \frac{r(d-r)}{(d+2)(d-1)m}\sum_{i=1}^m \int_M \|E_{NC}^{\widetilde{f}_i}\|^2 \, d\,\mathrm{vol} + \int_{G_r(M)} \mathcal{U}(\psi_r) d\,\mathrm{vol} + O(\|\widetilde{Y}\|^3_{C^{k_1}}),$$

where $\psi_r(x) = \frac{1}{m}\sum_{i=1}^m \ln\det(D_x\widetilde{f}_i \mid E_x)$ and $\mathcal{U}$ is defined in Proposition 11.

Pick a sequence of ergodic stationary measures μ_n so that $|\lambda_d(\mu_n)| \to 0$. Subtracting the expression for $\Lambda_{d-1}(\mu_n)$ from the expression for $\Lambda_d(\mu_n)$, we obtain that

$$\begin{aligned}\lambda_d(\mu_n) = \Lambda_d(\mu_n) - \Lambda_{d-1}(\mu_n) = &\frac{-1}{2dm}\sum_{i=1}^m \int_M \|E_C^{\widetilde{f}_i}\|^2 \, d\,\mathrm{vol} + \frac{-(d-1)}{(d+2)(d-1)m}\sum_{i=1}^m \int_M \|E_{NC}^{\widetilde{f}_i}\|^2 \, d\,\mathrm{vol} \\ &- \int_{\mathrm{Gr}_{d-1}(M)} \mathcal{U}(\psi_{d-1}) \, d\,\mathrm{vol} + \int_{\mathrm{Gr}_d(M)} \mathcal{U}(\psi_d) \, d\,\mathrm{vol} + O(\|\widetilde{Y}\|^3_{C^{k_1}}).\end{aligned} \tag{3.75}$$

Write $\mathrm{Gr}_r(R)$ for the map on $\mathrm{Gr}_r(M)$ induced by R. Write $\mathbf{Y}_i$ for the shortest vector field on $\mathrm{Gr}_r(M)$ such that $\exp_{\mathrm{Gr}_r(R_i)(x)} \mathbf{Y}_i = \mathrm{Gr}_r(\widetilde{f}_i)(x)$. By Lemma 3.7.11, for each k there exists $C_{1,k}$ such that

$$\left\|\sum_{i=1}^m \mathbf{Y}_i\right\|_{C^k} \leq C_{1,k}\left(\left\|\sum_{i=1}^m \widetilde{Y}_i\right\|_{C^{k+1}} + \widetilde{\varepsilon}^2_{k+1}\right).$$

Hence by the above line and the final estimate in Proposition 11 there exists k_2 such that

$$\left|\int_{\mathrm{Gr}_r(M)} \mathcal{U}(\psi_r)\, d\,\mathrm{vol}\right| \leq C_2 \|\psi_r\|_{C^{k_2}} \left(\left\| \frac{1}{m}\sum_{i=1}^{m} \widetilde{Y}_i \right\|_{C^{k_2}} + \|\widetilde{Y}_i\|^2_{C^{k_2}} \right). \tag{3.76}$$

The term $\|\psi_r\|_{C^{k_2}}$ is bounded by a constant times $\widetilde{\varepsilon}_{k_2}$. By using equation (3.73) we may rewrite the second term appearing in the product in equation (3.76).

$$\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m}\widetilde{Y}_i &= \frac{1}{m}\sum_i Y_i + -(1-\mathcal{L})^{-1}(\frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i) - \frac{1}{m}\sum_i (R_i)_*(-(1-\mathcal{L})^{-1})(\mathcal{T}_\lambda Y_i) + \frac{1}{m}\sum_i Q(Y_i, V) \\
&= \frac{1}{m}\sum_i \mathcal{R}_\lambda Y_i + \frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i - (1-\mathcal{L})(1-\mathcal{L})^{-1}(\frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i) + \frac{1}{m}\sum_i Q(Y_i, V) \\
&= \frac{1}{m}\sum_i \mathcal{R}_\lambda Y_i + \frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i - \frac{1}{m}\sum_i \mathcal{T}_\lambda Y_i + \frac{1}{m}\sum_i Q(Y_i, V) \\
&= \frac{1}{m}\sum_i \mathcal{R}_\lambda Y_i + \frac{1}{m}\sum_i Q(Y_i, V)
\end{aligned}$$

By equation (3.65), there exists k_3 such that for all $s \geq 0$:

$$\|R_\lambda Y_i\|_{C^1} \leq C_{3,s}\lambda^{k_3 - s/2}\|Y_i\|_{C^s}.$$

As the Q term is quadratic, there exist ℓ_2, C_4 such that

$$\|Q(Y_i, V)\|_{C^{k_2}} \leq C_4\|Y_i\|_{C^{\ell_2}}\|V\|_{C^{\ell_2}} = C_4\|Y_i\|_{C^{\ell_2}}\|(1-\mathcal{L})^{-1}(\mathcal{T}_\lambda Y_i)\|_{C^{\ell_2}} \leq C_5\varepsilon^2_{\ell_3}$$

for some ℓ_3 by Proposition 10. Thus

$$\left\|\frac{1}{m}\sum_i \widetilde{Y}_i\right\|_{C^{k_2}} \leq C_{6,s}(\lambda^{k_3 - s/2}\varepsilon_s + \varepsilon^2_{\ell_3}).$$

Finally, by equation (3.74) we have that $\|\widetilde{Y}_i\|_{C^{k_2}} \leq C_7\varepsilon_{\ell_3}$ as before. Let $\ell_4 = \max\{\ell_3, k_2 +$

$\alpha + \beta\}$. Applying all of these estimates to (3.76) gives

$$\left| \int_{\mathrm{Gr}_r(M)} \mathcal{U}(\psi_r)\, d\,\mathrm{vol} \right| \leq C_{8,s} \varepsilon_{k_2} (\lambda^{k_3 - s/2} \varepsilon_s + \varepsilon_{\ell_4}^2). \tag{3.77}$$

By taking $\ell_5 > \max\{k_1 + \alpha + \beta, k_2, \ell_4\}$, using that $\lambda_d(\mu_n) \to 0$,[2] and combining equations (3.77) and (3.75) we obtain for $s \geq 0$ that there exists $C_{9,s}$ such that

$$C_{9,s}(\lambda^{k_3 - s/2} \varepsilon_s \varepsilon_{\ell_5} + \varepsilon_{\ell_5}^3) \geq \frac{1}{2dm} \sum_{i=1}^{m} \int_M \|E_C^{\widetilde{f}_i}\|^2 \, d\,\mathrm{vol} + \frac{(d-1)}{(d+2)(d-1)m} \sum_{i=1}^{m} \int_M \|E_{NC}^{\widetilde{f}_i}\|^2 \, d\,\mathrm{vol}\,. \tag{3.78}$$

Note that the coefficients on each of the strain terms are positive. If $s > \ell_5$, then by taking square roots, we see that there exist constants $C_{10,s}$ such that for each i

$$C_{10,s}(\lambda^{k_3/2 - s/4} \varepsilon_s + \varepsilon_{\ell_5}^{3/2}) \geq \|\widetilde{f}_i^* g - g\|_{H^0}. \tag{3.79}$$

We now give a naive estimate on the higher C^s norms under the assumption that ε_1 is bounded by a constant $\epsilon_1 > 0$. To begin, by combining equation (3.64) and Proposition 7 we see that there exists $\alpha > 0$ such that for each s there exists $D_{3,s}$ such that $\|V\|_{C^s} \leq D_{3,s} \lambda^\alpha \varepsilon_s$. Hence by Lemma 3.7.4, both $d_{C^s}(\psi_V, \mathrm{Id})$ and $d_{C^s}(\psi_V^{-1}, \mathrm{Id})$ are bounded by $D_{4,s} \lambda^\alpha \varepsilon_s$. This establishes equation (3.71).

Now applying the composition estimate from Lemma 3.7.5, we find that assuming $\lambda \geq 1$:

$$\begin{aligned} d_{C^s}(f \circ \psi_V^{-1}, R) &\leq C_{11,s}(d_{C^s}(f, R) + d_{C^s}(\psi_V^{-1}, \mathrm{Id})) \\ &\leq C_{12,s}(\varepsilon_s + \lambda^\alpha \varepsilon_s) \\ &\leq C_{13,s}(\lambda^\alpha \varepsilon_s). \end{aligned}$$

[2]Note that we did not need $\lambda(\mu_n) \to 0$ in order to conclude equation (3.78). It suffices to know that there μ such that $\lambda_d(\mu)$ is comparable to the right hand side of (3.77). This observation is the essence of the proof of Theorem 3.6.1.

We then apply the other estimate in Lemma 3.7.5, to find:

$$\begin{aligned} d_{C^s}(\psi_V \circ f \circ \psi_V^{-1}, R) &\le C_{11,s}(d_{C^s}(\psi_V, \mathrm{Id}) + d_{C^s}(f \circ \psi_V^{-1}, R)) \\ &\le C_{14,s}(\lambda^\alpha \varepsilon_s + \lambda^\alpha \varepsilon_s) \\ &\le C_{15,s}\lambda^\alpha \varepsilon_s. \end{aligned}$$

Hence under an assumption of the type in equation (3.66), namely $\varepsilon_1 < \epsilon_1$, we may conclude

$$d_{C^s}(\widetilde{f}_i, R) \le C_{15,s}\lambda^\alpha \varepsilon_s, \tag{3.80}$$

which establishes equation (3.70).

We now apply Proposition 13 to this system. Let k_σ and r_σ be the k and r in Proposition 13 for a given choice of σ and our fixed ℓ. In preparation for the application of the lemma, we record some basic estimates:

1. By combining equation (3.64) and Proposition 10 as before, we see that there exists ℓ_6 such that
$$d_{C^2}(\widetilde{f}_i, R_i) \le \varepsilon_{\ell_6}. \tag{3.81}$$

2. From the previous discussion we also have
$$\|\widetilde{f}_i^* g - g\|_{H^0} \le C_{10,s}(\lambda^{k_3/2 - s/4}\varepsilon_s + \varepsilon_{\ell_5}^{3/2}).$$

3. We also need the C^{k_σ} estimate
$$d_{C^{k_\sigma}}(\widetilde{f}_i, R) \le C_{15,k_\sigma}\lambda^\alpha \varepsilon_{k_\sigma}.$$

Hence if

$$C_{15,k_\sigma}\lambda^\alpha \varepsilon_{k_\sigma} < r_\sigma \tag{3.82}$$

and

$$C_{10,s}(\lambda^{k_3/2-s/4}\varepsilon_s + \varepsilon_{\ell_5}^{3/2}) \leq r_\sigma, \tag{3.83}$$

then by Proposition 13 and the previous estimates there exist C_6 and isometries R_i' such that

$$d_{C^\ell}(\widetilde{f}_i, R_i') \leq C_{16,s}(\lambda^{k_3/2-s/4}\varepsilon_s + \varepsilon_{\ell_5}^{3/2})^{1/2-\sigma}\varepsilon_{\ell_6}^{1/2-\sigma} \tag{3.84}$$

and

$$d_{C^0}(R_i', R_i) < C_{17,s}(\varepsilon_{\ell_6} + (\lambda^{k_3/2-s/4}\varepsilon_s + \varepsilon_{\ell_5}^{3/2})^{1-\sigma}). \tag{3.85}$$

Let $\ell_7 = \max\{\ell_5, \ell_6\}$. If $s > \ell_7$, then equation (3.84) implies

$$d_{C^\ell}(\widetilde{f}_i, R_i') \leq C_{16,s}(\lambda^{k_4-s/9}\varepsilon_s^{1-2\sigma} + \varepsilon_{\ell_7}^{5/4-(5/2)\sigma}),$$

which yields equation (3.68) under the assumption that $\sigma > 0$ is sufficiently small. Note that equation (3.85) establishes equation (3.69). Thus we are done as we have established these estimates assuming only bounds of the type appearing in equations (3.66) and (3.67). □

Remark 1. In the above lemma, we could instead have assumed that there exist stationary measures for which both the top exponent and the sum of all the exponents were arbitrarily small and concluded the same result. The reason being if we had considered $\Lambda_1 - \Lambda_d$ in equation (3.75), the coefficients of the strain terms would still have the same sign and so we could conclude the same result. By related modifications, one can produce many other formulations of the main result in [DK07] that require other hypotheses on the Lyapunov exponents.

3.6.2 Convergence of the KAM scheme

In this section we prove the main linearization theorem.

Theorem 3.1.1. *Let M^d be a closed isotropic Riemannian manifold other than S^1. There exists k_0 such that if $(R_1, ..., R_m)$ is a tuple of isometries of M such that the subgroup of*

$\text{Isom}(M)$ *generated by this tuple contains* $\text{Isom}(M)^\circ$*, then there exists* $\epsilon_{k_0} > 0$ *such that the following holds. Let* $(f_1, ..., f_m)$ *be a tuple of* C^∞ *diffeomorphisms satisfying* $\max_i d_{C^{k_0}}(f_i, R_i) < \epsilon_{k_0}$. *Suppose that there exists a sequence of ergodic stationary measures* μ_n *for the random dynamical system generated by* $(f_1, ..., f_m)$ *such that* $|\lambda_d(\mu_n)| \to 0$*, then there exists* $\psi \in \text{Diff}^\infty(M)$ *such that for each* i *the map* $\psi f_i \psi^{-1}$ *is an isometry of* M *and lies in the subgroup of* $\text{Isom}(M)$ *generated by* $(R_1, \ldots, R_m)$.

Before giving the proof, we sketch briefly the argument, which is typical of arguments establishing the convergence of a KAM scheme. In a KAM scheme where one wishes to show that some sequence of objects h_n converges there are often two parts. The first part of the proof is an inductive argument obtaining a sequence of estimates by the repeated application of the KAM step, which in our case is Lemma 3.6.1. The second half of the proof checks that the repeated application of the KAM step is valid by showing that we never leave the neighborhood of its validity and then checks that the procedure is converging in C^∞.

In the first part, one inductively produces a sequence of estimates by iterating a KAM step. The estimates produced usually come in two forms: a single good estimate in a low norm and bad estimates in high norms. The low regularity estimate probably looks like $\|h_n\|_{C^0} \le N^{-(1+\tau)^n}$ where $\tau > 0$, while for every s one has a high regularity estimate like $\|h_n\|_{C^s} \le N^{(1+\tau)^n}$. A priori, the h_n become superexponentially C^0 small, yet might be diverging in higher C^s norms. To remedy this situation one then interpolates between the low and high norms by using an equality derived from Lemma 3.7.7. In this case such an inequality for the objects h_n might assert something like

$$\|h_n\|_{C^{\lambda\cdot 0+(1-\lambda)s}} \le C_s \|f\|_{C^0}^{\lambda} \|f\|_{C^s}^{1-\lambda}.$$

If λ is sufficiently close to 1 and s is sufficiently large, a brief calculation then implies that the $C^{(1-\lambda)s}$ norm is also super exponentially small. By changing s and λ one then obtains convergence in C^∞.

Proof of Theorem 3.1.1. The proof is by a KAM convergence scheme. To begin we introduce the Diophantine condition we will use. By Proposition 8, $(R_1, ..., R_m)$ is (C', α', n')-Diophantine for some $C', \alpha' > 0$ and is stably so. By stability, there exist (C, α, n) and a C^0 neighborhood $\mathcal{U}$ of $(R_1, ..., R_m)$ such that any tuple in $\mathcal{U}$ is also (C, α, n)-Diophantine. Hence if $(R'_1, ..., R'_m) \in \mathcal{U}$, then the coefficients $C_{i,s}$ appearing in Lemma 3.6.1 are uniform over all of these tuples. Assuming we do not leave the set $\mathcal{U}$, the constants appearing in Lemma 3.6.1 will be uniform. We check this at the end of the proof in the discussion surrounding equation (3.90).

We now show that there exists a sequence of cutoffs λ_n so that if we repeatedly apply Lemma 3.6.1 with the cutoff λ_n on the nth time we apply the Lemma, then the resulting sequence of conjugates converges and the hypotheses of Lemma 3.6.1 remain satisfied. Given such a sequence λ_n the convergence scheme is run as follows. Let $(f_{1,1}, \dots, f_{m,1}) = (f_1, \dots, f_m)$ and let $(R_{1,1}, \dots, R_{m,1}) = (R_1, \dots, R_m)$. Given $(f_{1,n-1}, \dots, f_{m,n-1})$ and $(R_{1,n-1}, \dots, R_{m,n-1})$ we apply Lemma 3.6.1 with cutoff $\lambda = \lambda_n$ to produce a diffeomorphism ϕ_n and a tuple of isometries that we denote by $(R_{1,n}, \dots, R_{m,n})$. We set $f_{i,n} = \phi_n f_{i,n-1} \phi_n^{-1}$ to obtain a new tuple of diffeomorphisms $(f_{1,n}, \dots, f_{m,n})$. We write ψ_n for $\phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$, so that $f_{i,n} = \psi_n \circ f_i \circ \psi_n^{-1}$. Let $\varepsilon_{k,n} = \max_i d_{C^k}(f_{i,n}, R_{i,n})$.

We now show that such a sequence of cutoffs λ_n exist. Let σ be a small positive number and let l_0 and ϵ_{l_0} be as in Lemma 3.6.1. Let $k_0, k_1, k_2, r_\ell, C_{s,\ell}, r_{s,\ell}$ be as in Lemma 3.6.1 as well. To show that such a sequence of cutoffs λ_n exists we must also provide a fixed choice of s, ℓ for the application of Lemma 3.6.1. We will first show that the scheme converges in the C^{l_0} norm and then bootstrap to get C^∞ convergence. Fix some arbitrary $\ell > l_0$. The choice of ℓ does not matter in the sequel because we only will consider estimates on the l_0 norm. We will choose s such that

$$s > \ell. \tag{3.86}$$

Further, if s is sufficiently large and τ is sufficiently small, then we can pick α such that

$$\frac{2+\tau}{s/4-k_1} < \alpha < \min\{1/k_0, \tau/k_2\} \tag{3.87}$$

So, we increase s if needed and choose such a τ satisfying

$$1/8 > \tau > 0. \tag{3.88}$$

Pick s, α, τ so that each of equations (3.86), (3.87), (3.88) is satisfied.

Let $\lambda_n = N^{\alpha(1+\tau)^n}$ for some N we choose later. We will show that with this choice of cutoff at the nth step that the KAM scheme converges. In order to show this, we show the following two estimates hold inductively given a choice of sufficiently large N:

$$\varepsilon_{l_0,n} \leq N^{-(1+\tau)^n} \tag{H1}$$

$$\varepsilon_{s,n} \leq N^{(1+\tau)^n} \tag{H2}$$

$$\max_i\{d_{C^0}(R_{i,n}, R_{i,1})\} \leq \sum_{i=1}^{n} N^{-\frac{1}{2}(1+\tau)^i}. \tag{H3}$$

This involves two arguments. The first argument shows that there is a sufficiently large N such that if we have these estimates for n, then the hypotheses of Lemma 3.6.1 are satisfied. The second argument is the actual induction, which checks that if equations (H1) and (H2) hold for n then they also hold for $n+1$, i.e. we apply Lemma 3.6.1 and then deduce (H1) and (H2) for $n+1$ from this.

We begin by checking that for all sufficiently large $N > 0$ and any $n \in \mathrm{N}$ if (H1), (H2), and (H3) are satisfied, then the hypotheses of Lemma 3.6.1 are satisfied as well. To begin, as the summation in (H3) is summable, for all sufficiently large N, we are assured that $(R_{1,n}, \ldots, R_{m,n})$ lies in $\mathcal{U}$. The first numbered hypothesis of Lemma 3.6.1 is equation (3.66):

$$\lambda_n^{k_0}\varepsilon_{l_0,n} \leq r_\ell.$$

Given the choice of λ_n, if equations (H1) and (H2) hold it suffices to have

$$N^{\alpha k_0 (1+\tau)^n} N^{-(1+\tau)^n} < r_\ell,$$

which holds for N sufficiently large and all n by our choice of α. The other hypothesis of Lemma 3.6.1, equation (3.67), requires that

$$\lambda_n^{k_1 - s/4} \varepsilon_{s,n} + \varepsilon_{l_0,n}^{3/2} < r_{s,\ell}.$$

Given equations (H1) and (H2) and our choice of λ_n it suffices to have

$$N^{\alpha(k_1 - s/4)(1+\tau)^n} N^{(1+\tau)^n} + N^{-\frac{3}{2}(1+\tau)^n} < r_{s,\ell}.$$

Our choice of s and α implies that $\alpha(k_1 - s/4) < -1$, hence the above inequality holds for sufficiently large N. Thus the two hypotheses of Lemma 3.6.1 follow from equations (H1) and (H2). Thus we may apply Lemma 3.6.1 given (H1), (H2), (H3), and our choice of N.

We now proceed to the inductive argument. What we will show is that for all N sufficiently large, if we now require that our perturbation is small enough that (H1) and (H2) hold for $n = 1$ and our choice of N we check that we may continue applying Lemma 3.6.1 and that these estimates as well as (H3) continue to hold. Note that (H3) is trivial when $n = 1$. We must then check that equations (H1), (H2), and (H3) are satisfied for $n + 1$ given they hold for n. By the previous paragraph, we are free to apply the estimates from Lemma 3.6.1 as long as N is sufficiently large.

We now check that equation (H1) holds for $n + 1$. By equation (3.68), we obtain that

$$\varepsilon_{l_0,n+1} \leq C_{s,\ell}(\lambda_n^{k_1 - s/10} \varepsilon_{s,n}^{1-\sigma} + \varepsilon_{l_0,n}^{9/8}).$$

By applying equations (H1) and (H2) to each term on the right it suffices to show

$$C_{s,\ell}(N^{\alpha(k_1-s/10)(1+\tau)^n}N^{(1-\sigma)(1+\tau)^n}+N^{-9/8(1+\tau)^n})<N^{-(1+\tau)^{n+1}}. \tag{3.89}$$

By our choice of s, α, and τ, the lower bound in equation (3.87) implies that

$$\alpha(k_1-s/10)+(1-\sigma)<-(1+\tau).$$

In addition, by equation (3.88), $-9/8<-(1+\tau)$. Thus for sufficiently large N the left hand side of equation (3.89) is bounded above by $N^{-(1+\tau)^{n+1}}$.

Next we check equation (H2) holds for $n+1$. By equation (3.70),

$$\varepsilon_{s,n+1}\leq C_{s,\ell}\lambda_n^{k_2}\varepsilon_{s,n}.$$

Hence,

$$\varepsilon_s\leq C_{s,\ell}N^{k_2\alpha(1+\tau)^n}N^{(1+\tau)^n},$$

By equation (3.87), $1+k_2\alpha<1+\tau$ and hence, assuming N is sufficiently large, the right hand side is bounded by $N^{(1+\tau)^{n+1}}$, which shows equation (H2) is satisfied.

We now check that (H3). This follows easily by the application of equation (3.69), which gives

$$d_{C^0}(R_{i,n},R_{i,n+1})\leq C_{s,\ell}(\varepsilon_{l_0,n}+(\lambda_n^{k_1-s/4}\varepsilon_{s,n}+\varepsilon_{l_0,n}^{3/2})^{1-\sigma}) \tag{3.90}$$

Applying (H1) and (H2) and the definition of λ_n to estimate the right hand side of equation (3.90), we find that for the γ given in (H3) and N sufficiently large that

$$d_{C^0}(R_{i,n},R_{i,n+1})\leq N^{-\frac{1}{2}(1+\tau)^n}, \tag{3.91}$$

and (H3) holds for $n+1$.

We have now finished the induction but not the proof. We have shown that there exists a sequence λ_n and a choice s, α, ℓ, τ, N, so that if the initial conditions of the scheme are satisfied then we may iterate indefinitely and be assured of the estimates in equations (H1), (H2), (H3) at each step. We must now check that the conjugacies ψ_n are converging in C^∞ and that the tuples $(R_{1,n}, \dots, R_{m,n})$ are converging. The latter is immediate because by (3.91) this is a Cauchy sequence. In fact, we chose N large enough that we never leave $\mathcal{U}$, hence the limit is in $\mathcal{U}$. As the group of isometries of M is C^0 closed and the distance of the tuples $(f_{1,n}, \dots, f_{m,n})$ from a tuple of isometries is converging to 0, it follows that $(f_{1,n}, \dots, f_{m,n})$ is converging to a tuple of isometries. To show that the ψ_n converge in C^∞, we obtain for every s an estimate on $d_{C^s}(\phi_n, \mathrm{Id})$. By a similar induction to that just performed, the estimate (3.71) implies

$$d_{C^s}(\phi_n, \mathrm{Id}) \le C_s N^{(1+\tau)^n}.$$

Let $j > 0$ be an integer. By Lemma 3.7.8, interpolating with $\lambda = 1 - 1/10$ between the C^{l_0} distance and the C^{jl_0} distance of ϕ_n to the identity gives

$$d_{C^{.9l_0+(j/10)l_0}}(\phi_n, \mathrm{Id}) \le C_j N^{-.9(1+\tau)^n} N^{.1(1+\tau)^n} = C_j N^{-.8(1+\tau)^n}.$$

Thus by increasing j, we see that there exists $\tau' > 0$ such that for each C^s norm

$$d_{C^s}(\phi_n, \mathrm{Id}) < C'_s N^{-(1+\tau')^n}.$$

The previous line is summable in n. Hence we can apply Lemma 3.7.6 to obtain convergence of sequence of the $\psi_n = \phi_n \circ \cdots \circ \phi_1$ in the C^s norm for each s and thus C^∞ convergence.

Thus we see that we have simultaneously conjugated each f_i into $\mathrm{Isom}(M)$. In order to obtain the full theorem, we must be assured that $\psi^{-1} f_i \psi$ lies in the subgroup of $\mathrm{Isom}(M)$ generated by $(R_1, \dots, R_m)$. Note that $\mathrm{Isom}(M)/\mathrm{Isom}(M)^\circ$ is a finite group and that ψ is homotopic to the identity by construction. Thus we see that the image of the group generated by $(\psi^{-1}f_1\psi,\dots,\psi^{-1}f_m\psi)$ in $\operatorname{Isom}(M)/\operatorname{Isom}(M)^\circ$ is the same as the image of the group generated by $(R_1,\dots,R_m)$. By our choice of N, $(\psi^{-1}f_1\psi,\dots,\psi^{-1}f_m\psi)$ is in $\mathcal{U}$ and thus generates $\operatorname{Isom}(M)^\circ$. Thus the original tuple and the new one generate the same subgroup of $\operatorname{Isom}(M)$ and we are done. □

3.6.3 Taylor expansion of Lyapunov exponents

In order to recover Dolgopyat and Krikorian's Taylor expansion in the setting of isotropic manifolds, we would need to apply Proposition 12 for each $0 \le r \le \dim M$. However, one of the hypotheses of Proposition 12 is that $\operatorname{Isom}(M)^\circ$ acts transitively on $\operatorname{Gr}_r(M)$. In Proposition 14, we see that unless M is S^n or $\mathbb{RP}^n$, $\operatorname{Isom}(M)$ does not act transitively on $\operatorname{Gr}_r(M)$ for $r \neq 1$ or $d-1$. Despite Proposition 14, we are able to obtain a partial result: the greatest and least Lyapunov exponents are symmetric about the "average" Lyapunov exponent $\frac{1}{d}\Lambda_d(\mu)$.

Theorem 3.6.1. *Suppose that M^d is a closed isotropic manifold other than S^1 and that $(R_1,...,R_m)$ is a subset of $\operatorname{Isom}(M)$ that generates a subgroup of $\operatorname{Isom}(M)$ containing $\operatorname{Isom}(M)^\circ$. Suppose that $(f_1,...,f_n)$ is a collection of C^∞ diffeomorphisms of M. Then there exists k_0 such that if μ is an ergodic stationary measure of the random dynamical system generated by the $(f_1,...,f_m)$, then*

$$\left|\lambda_1(\mu) - \left(-\lambda_d(\mu) + \frac{2}{d}\Lambda_d(\mu)\right)\right| \le o(1)\left|\lambda_d(\mu)\right|. \tag{3.92}$$

where the $o(1)$ term goes to 0 as $\max_i d_{C^{k_0}}(f_i,R_i) \to 0$. The $o(1)$ term depends only on $(R_1,...,R_m)$.

Proof. By Theorem 3.1.1, there are two cases: either $(f_1,\dots,f_m)$ is conjugate to isometries or it is not. In the isometric case equation (3.92) is immediate, so we may assume that there there is an ergodic stationary measure μ with $\lambda_d(\mu)$ non-zero. The proof that follows is then essentially an observation about what happens when the KAM scheme is run on a system that has a measure with such a non-zero Lyapunov exponent. If we run the KAM scheme without assuming that $(f_1, ..., f_m)$ has a measure with zero exponents, we can keep running the scheme until the non-trivial exponents prevent us from continuing. At a certain point in the procedure, the non-trivial exponents cause a certain inequality fail. Using the failed inequality then gives the result.

We now give the details. Fix an ergodic stationary measure μ and consider equation (3.75) appearing in the KAM step:

$$\begin{aligned}\lambda_d(\mu) =& \frac{-1}{2dm}\sum_{i=1}^{m}\int_M \|E_C^{\widetilde{f}_i}\|^2\,d\,\mathrm{vol} + \frac{-(d-1)}{(d+2)(d-1)m}\sum_{i=1}^{m}\int_M \|E_{NC}^{\widetilde{f}_i}\|^2\,d\,\mathrm{vol} \\ &- \int_{\mathrm{Gr}_{d-1}(M)} \mathcal{U}(\psi_{d-1})\,d\,\mathrm{vol} + \int_{\mathrm{Gr}_d(M)} \mathcal{U}(\psi_d)\,d\,\mathrm{vol} + O(\|\widetilde{Y}\|^3_{C^{k_1}}).\end{aligned} \tag{3.93}$$

The above equation allows us to use that the exponent λ_d is small in magnitude. In the KAM step, we proceed from this estimate by estimating the $\|\widetilde{Y}\|^3_{C^{k_1}}$ term as well as the $\mathcal{U}$ terms. Equation (3.77) and the choice of ℓ_5 imply that these terms satisfy:

$$\left|\int_{\mathrm{Gr}_{d-1}(M)} \mathcal{U}(\psi_{d-1})\,d\,\mathrm{vol} - \int_{\mathrm{Gr}_d(M)} \mathcal{U}(\psi_d)\,d\,\mathrm{vol} + O(\|\widetilde{Y}\|^3_{C^{k_1}})\right| \le C_{8,s}\varepsilon_{\ell_5}(\lambda^{k_3-s/2}\varepsilon_s + \varepsilon_{\ell_5}^2). \tag{3.94}$$

Hence as long as

$$|\lambda_d(\mu)| < (C_{9,s} - C_{8,s})(\varepsilon_{\ell_5}(\lambda^{k_3-s/2}\varepsilon_s + \varepsilon_{\ell_5}^2)) \tag{3.95}$$

the proof of Lemma 3.6.1 may proceed to equation (3.78) even if there is not a sequence of measures μ_n such that $|\lambda_d(\mu_n)| \to 0$. Hence we may continue running the KAM scheme until equation (3.95) fails to hold.

Suppose that we iterate the KAM scheme until equation (3.95) fails. We consider the estimates available in the KAM scheme at the step of failure. By applying Proposition 12 with r equal to 1, d, and $d-1$, we obtain:

$$
\begin{aligned}
\Lambda_1(\mu) =& \frac{-1}{2dm}\sum_{i=1}^{m}\int_M \|E_C^{\widetilde{f}_i}\|^2\, d\,\mathrm{vol} + \frac{(d-1)}{(d+2)(d-1)m}\sum_{i=1}^{m}\int_M \|E_{NC}^{\widetilde{f}_i}\|^2\, d\,\mathrm{vol} + \int_{G_1(M)} \mathcal{U}(\psi_1)\, d\,\mathrm{vol} + \\
& O(\|\widetilde{Y}\|^3_{C^{k_1}}) \\
\Lambda_{d-1}(\mu) =& \frac{-(d-1)}{2dm}\sum_{i=1}^{m}\int_M \|E_C^{\widetilde{f}_i}\|^2\, d\,\mathrm{vol} + \frac{(d-1)}{(d+2)(d-1)m}\sum_{i=1}^{m}\int_M \|E_{NC}^{\widetilde{f}_i}\|^2\, d\,\mathrm{vol} + \int_{G_{d-1}(M)} \mathcal{U}(\psi_{d-1})\, d\,\mathrm{vol} + \\
& O(\|\widetilde{Y}\|^3_{C^{k_1}}) \\
\Lambda_d(\mu) =& \frac{-d}{2dm}\sum_{i=1}^{m}\int_M \|E_C^{\widetilde{f}_i}\|^2\, d\,\mathrm{vol} + \int_{G_d(M)} \mathcal{U}(\psi_d)\, d\,\mathrm{vol} + O(\|\widetilde{Y}\|^3_{C^{k_1}})
\end{aligned}
\tag{3.96}
$$

Write $\mathcal{U}_i$ as shorthand for the term $\int_{\mathrm{Gr}_i(M)} \mathcal{U}(\psi_i)\, d\,\mathrm{vol}$. Then,

$$
\lambda_1(\mu) - (-\lambda_d(\mu) + \frac{2}{d}\Lambda_d(\mu)) = \Lambda_1(\mu) - \Lambda_{d-1}(\mu) + \frac{(d-2)}{d}\Lambda_d(\mu) \tag{3.97}
$$

$$
= \mathcal{U}_1 + \mathcal{U}_{d-1} + \mathcal{U}_d + O(\|\widetilde{Y}\|^3_{C^{k_1}}). \tag{3.98}
$$

Using equations (3.77), (3.74), and that $\ell_5 > k_1 + \alpha$, we bound the right hand side of equation (3.98) to find

$$
\left|\lambda_1(\mu) - (-\lambda_d(\mu) + \frac{2}{d}\Lambda_d(\mu))\right| \le 4C_{8,s}(\lambda_n^{k_3 - s/2}\varepsilon_s\varepsilon_{\ell_5} + \varepsilon_{\ell_5}^3).
$$

But by the failure of estimate (3.95), we may bound the right hand side of the previous line to obtain:

$$
\left|\lambda_1(\mu) - (-\lambda_d(\mu) + \frac{2}{d}\Lambda_d(\mu))\right| \le \frac{4}{C_{9,s} - C_{8,s}}\left|\lambda_d(\mu)\right|. \tag{3.99}
$$

Note in the above equation that the larger $C_{9,s}$ is the smaller the left hand side of the equation is. We can take $C_{9,s}$ as large as we like and still run the KAM scheme. Running the KAM scheme while having a larger constant $C_{9,s}$ only requires that we assume our initial perturbation is closer to the original system of rotations in the C^{k_0} norm. Hence by assuming that the initial distance is arbitrarily small in the C^{k_0} norm, we may take $C_{9,s}$ as large as we like. Thus equation (3.92) follows from equation (3.99). $\square$

We now check the claim about isotropic manifolds.

Proposition 14. *Suppose that M is a closed isotropic manifold other than $\mathbb{RP}^n$ or S^n. Then* $\operatorname{Isom}(M)$ *does not act transitively on* $\operatorname{Gr}_k(M)$ *except if k equals* $= 0, 1, \dim M - 1$ *or* $\dim M$.

Proof. From subsection 3.2.5, we have a list of all the closed isotropic manifolds, so we may give an argument for each of the families, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, and $F_4/\operatorname{Spin}(9)$.

The isometry group of $\mathbb{C}\mathrm{P}^n$ is $\mathrm{PSU}(n+1)$. If we fix a point p in $\mathbb{C}\mathrm{P}^n$, then the isotropy group is naturally identified with $\mathrm{SU}(n)$. It is then immediate that the action of the isotropy group preserves complex subspaces of $\operatorname{Gr}_k(\mathbb{C}\mathrm{P}^n)$. Consequently $\operatorname{Isom}(\mathbb{C}\mathrm{P}^n)$ does not act transitively on $\operatorname{Gr}_k(\mathbb{C}\mathrm{P}^n)$ as $\mathbb{C}\mathrm{P}^n$ has subspaces that are not complex. In the case of $\mathbb{H}\mathrm{P}^n$, which is constructed similarly to $\mathbb{C}\mathrm{P}^n$, a similar argument works where we use instead that the isotropy group is $\mathrm{Sp}(k)$, the compact symplectic group.

We now turn to the Cayley plane, for which we give a dimension counting argument. The dimension of F_4 is 52 while $\dim F_4/\operatorname{Spin}(9) = 16$. Recall that if M is a manifold and $\dim M = d$ then $\dim \operatorname{Gr}_k(M) = (k+1)d + \frac{k(k+1)}{2}$. Hence $\dim \operatorname{Gr}_3(F_4/\operatorname{Spin}(9)) > 52$. If $\operatorname{Isom}(M)$ acts transitively on 2-planes then M must have constant sectional curvature and hence is a sphere. The Cayley plane does not have constant sectional curvature hence $k = 2$ is ruled out. Similarly, a dimension count excludes the possibility that F_4 acts transitively on $\operatorname{Gr}_k(F_4/\operatorname{Spin}(0))$ when $k \neq 0, 1, 15, 16$. $\square$

3.7 APPENDIX

3.7.1 C^k Estimates

In this subsection of the appendix, we collect some basic results concerning the calculus of C^k functions. Most of the estimates stated here are used to compare constructions coming from Riemannian geometry and constructions coming from a chart.

Most of the estimates we prove below involve the following definition, which is an appropriate form for a second order term in the C^k setting.

Definition 3.7.1. Suppose that X, Y, Z are all vector fields and that $Z = Z(X, Y)$ is a function of X and Y. We say that Z is *quadratic* in X and Y if there exists a fixed ℓ such that for each k there is a constant C_k depending only on Z such that:

$$\|Z\|_{C^k} \le C_k(\|X\|^2_{C^{k+\ell}} + \|Y\|^2_{C^{k+\ell}}). \tag{3.100}$$

In addition to quadratic, we may also refer to Z as being *second order* in X and Y. In the case that Z depends only on X the definition is analogous.

One thinks of equation (3.100) as a quadratic tameness estimate. Our main use of this notion is the following proposition, which allows us compose diffeomorphisms up to a quadratic error. As before, if Y is a vector field on M, we write ψ_Y for the map of M that sends $x \mapsto \exp_x(Y(x))$. To emphasize that ψ depends on a metric g, we may write ψ_Y^g.

The main result from this subsection is the following, which is used in the KAM scheme to see how the linearized error between f_i and R_i changes when f_i is conjugated by a diffeomorphism ψ.

Proposition 15. *[DK07, Eq. (8)] Suppose that (M, g) is a closed Riemannian manifold and that R is an isometry of M. Suppose that f is a diffeomorphism of M that is C^1 close to R. Let $Y(x) = \exp^{-1}_{R(x)} f(x)$. If C is a C^1 small vector field on M, then the error field*

$\exp_{R(x)}^{-1} \psi_C f \psi_C^{-1}$ *is equal to*

$$Y + C - R_* C + Q(C, Y),$$

where Q is quadratic in C and Y.

The proof of Proposition 15 is straightforward. It particularly relies on the following proposition, which simplifies working with diffeomorphisms of the form ψ_X.

Proposition 16. *Let M be a compact Riemannian manifold. If $X, Y \in \mathrm{Vect}^\infty(M)$ are sufficiently C^1 small and we define Z by*

$$\psi_Y \circ \psi_X = \psi_{X+Y+Z},$$

then there exists a fixed ℓ such that for each k there exists C_k such that

$$\|Z\|_{C^k} \leq C_k(\|X\|^2_{C^{k+\ell}} + \|Y\|^2_{C^{k+\ell}}),$$

i.e. Z is quadratic in X and Y.

The proof of Proposition 16 uses the following two lemmas concerning maps of $\mathbb{R}^n$.

Lemma 3.7.2. *[Hör76, Thm. A.7] Suppose that B is a compact convex domain in $\mathbb{R}^n$ with interior points. Then for $k \geq 0$, there exists C such if f, g are C^k maps from B to $\mathbb{R}$, then*

$$\|fg\|_{C^k} \leq C_k(\|f\|_{C^k}\|g\|_{C^0} + \|f\|_{C^0}\|g\|_{C^k}).$$

Lemma 3.7.3. *[Hör76, Thm. A.8] For $i \in \{1, 2, 3\}$, let B_i be a fixed compact convex domain in $\mathbb{R}^{n_i}$ with interior points. Let $k \geq 1$. There exists $C_k > 0$ such that if $f \colon B_1 \to B_2$ and $g \colon B_2 \to B_3$ are both C^k, then $f \circ g$ is C^k and*

$$\|f \circ g\|_{C^k} \leq C_k(\|f\|_{C^k}\|g\|^k_{C^1} + \|f\|_{C^1}\|g\|_{C^k} + \|f \circ g\|_{C^0}).$$

Using the previous two lemmas, we prove the following.

Proposition 17. *Suppose that g is a metric on $\mathbb{R}^n$. For a smooth vector field Y such that $\|Y\|_{C^1} < 1$, define*

$$Z(x) = \psi_Y^g(x) - Y(x) - x.$$

Let B be a compact convex domain in $\mathbb{R}^n$ with interior points. Then $Z|_B$ is quadratic in Y. In fact, for each k there exists C_k such that

$$\|Z|_B\|_{C^k} \leq C_k \|Y\|_{C^k}^2.$$

Proof. Let B be as in the statement of the proposition. Define $\gamma(Y(x), t)$ to be the map that sends $x \mapsto \exp_x tY(x) - x$, so that $\gamma(Y(x), 1) + x = \psi_Y^g$ and $\gamma(Y(x), 0) = 0$. We rewrite Z.

$$\begin{aligned} Z = \psi_Y^g(x) - x - Y(x) &= \gamma(Y(x), 1) - Y(x) \\ &= \int_0^1 \dot{\gamma}(Y(x), t) - Y(x)\, dt \\ &= \int_0^1 \dot{\gamma}(Y(x), t) - \dot{\gamma}(Y(x), 0)\, dt \\ &= \int_0^1 \int_0^1 t\ddot{\gamma}(Y(x), st)\, ds\, dt \\ &= \int_0^1 t \int_0^1 \ddot{\gamma}(Y(x), st)\, ds\, dt. \end{aligned}$$

By differentiating under the integral, we see that the nth derivatives of Z are controlled by the maximum of the nth derivatives of $\ddot{\gamma}(Y(x), t)$ for each fixed t. Hence it suffices to show for each $t \in [0, 1]$ that $\ddot{\gamma}(Y(x), t)$ is second order in Y.

Dropping the explicit dependence on x, we recall the coordinate expression of the geodesic equation. For a coordinate frame $[e_1, \ldots, e_n]$ and indices $1 \leq \mu, \nu, \lambda \leq n$, we define the Christoffel symbols $\Gamma_{\mu\nu}^{\lambda}$ by $\langle \nabla_{e_\mu} e_\nu, e_\lambda \rangle$. In addition, we write $\dot{\gamma}^\nu$ for $\langle \dot{\gamma}, e_\nu \rangle$ and similarly for

$\ddot{\gamma}$. The coordinate expression for the geodesic equation is then

$$\ddot{\gamma}^\lambda = -\Gamma^\lambda_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu.$$

We estimate the C^k norm of the right hand side. Write ϕ^t for the geodesic flow on TB. For fixed $r > 0$ in TB, let $TB(r)$ be the set of vectors $v \in TB$ such that $\|v\| < r$. Note that the restriction $\|\phi^t|_{TB(t)}\|_{C^k}$ is bounded. Let π be the projection from a tangent vector in $T\mathbb{R}^n$ to its basepoint in $\mathbb{R}^n$. Then

$$\gamma(x,t) = \pi \circ \phi^t \circ Y(x).$$

Hence, writing $\dot{\phi}$ for the geodesic spray,

$$\dot{\gamma}(x,t) = D\pi \circ \dot{\phi}\,|_{\phi^t(Y(x))}\,. \tag{3.101}$$

$D\pi \circ \dot{\phi}^t\,|_{TB(r)}$ has its C^k norm uniformly bounded in t by some D_k. By Lemma 3.7.3 because $\|Y\|_{C^1} < 1$ it follows that $\|\phi^t(Y(x),t)\|_{C^k} \leq C_k\|Y\|_{C^k}$.

Hence by applying Lemma 3.7.3 to (3.101), and similarly using that $\|Y\|_{C^1} < 1$ and $D\pi \circ \dot{\phi}$ is uniformly bounded we find

$$\|(D\pi \circ \dot{\phi}^t) \circ Y\|_{C^k} \leq C'_k(D_k\|Y\|_{C^1} + D_1\|Y\|_{C^k} + \|Y\|_{C^0}).$$

Hence

$$\|\dot{\gamma}(x,t)\|_{C^k} = \|D\pi \circ \dot{\phi}|_{\phi^t(Y(x))}\|_{C^k} \leq C_k\|Y\|_{C^k}.$$

The geodesic equation shows that at each point the coordinates of $\ddot{\gamma}$ are a quadratic polynomial in the coordinates of $\dot{\gamma}$. Hence by Lemma 3.7.2

$$\|\ddot{\gamma}(x,t)\|_{C^k} \leq C''_k\|Y\|^2_{C^k}$$

for all $t \in [0,1]$. Thus we obtain a uniform estimate on Z. $\square$

Proof of Proposition 16. As before, it suffices to prove the estimate in a chart. So, we are reduced to working in a neighborhood of $0 \in \mathbb{R}^n$. Fix some k, then by Proposition 17 we may write

$$\psi_Y(x) = x + Y(x) + Z_Y(x),$$

where $Z_Y(x)$ is quadratic in Y. Similarly define $Z_X(x)$ and $Z_{X+Y}(x)$. Then

$$\begin{aligned}
\psi_Y \circ \psi_X &= \psi_Y(x + X(x) + Z_X(x)) \\
&= x + X(x) + Z_X(x) + Y(x + X(x) + Z_X(x)) + Z_Y(x + X(x) + Z_X(x)).
\end{aligned}$$

To prove this proposition, we compare the previous line with

$$\psi_{X+Y} = x + X(x) + Y(x) + Z_{X+Y}(x).$$

The difference is

$$\psi_Y \circ \psi_X - \psi_{X+Y} = Z_X(x) - Z_{X+Y}(x) + Y(x + X(x) + Z_X(x)) - Y(x) + Z_Y(x + X(x) + Z_X(x))$$

The first and second terms satisfy the appropriate quadratic C^k estimate already. For the last term, we apply Lemma 3.7.3. Hence by assuming that $\|X\|_{C^1}$ is sufficiently small, we conclude that the Z_Y term is quadratic. We now turn to the Y terms:

$$Y(x + X(x) + Z_X(x)) - Y(x).$$

For this we apply the same trick as before. Write

$$Y(x + X(x) + Z_X(x)) - Y(x) = \int_0^1 Y'(x + t(X(x) + Z_X(x)))\|X(x) + Z_X(x)\|\, dt.$$

By differentiating under the integral, it suffices to show that the integrand is quadratic in X and Y. By Lemma 3.7.2, the integrand will be quadratic if there exists ℓ such that for each k there is a constant C_k such that both of $\|Y'(x+t(X(x)+Z_X(x)))\|_{C^k}$ and $\|X(x)+Z_X(x)\|_{C^k}$ are bounded by $C_k(\|X\|_{C^{k+\ell}} + \|Y\|_{C^{k+\ell}})$. This follows for both terms by the application of Lemma 3.7.3, so we are done. □

We now show another basic fact: near to the identity map a diffeomorphism and its inverse have comparable size.

Lemma 3.7.4. *Suppose that M is a closed Riemannian manifold. Then there exists $\epsilon > 0$ such that for all $k \geq 0$ then there exists C_k, such that if $f \in \operatorname{Diff}^k(M)$ and $d_{C^1}(f, \operatorname{Id}) < \epsilon$ then*

$$d_{C^k}(f^{-1}, \operatorname{Id}) \leq C_k d_{C^k}(f, \operatorname{Id}).$$

Proof. This proof follows the outline of the similar estimate in [Ham82, Lem. 2.3.6]. For convenience, write $g = f^{-1}$. In a chart, we write $f(x) = x + X(x)$ where the C^k norm of X is bounded by $d_{C^k}(f, \operatorname{Id})$. Similarly write $g(x) = x + Y(x)$. We now apply the chainrule to differentiate $g \circ f$. The case where $n = 1$ is immediate by differentiating $g \circ f = x + X(x) + Y(x + X(x))$, which gives that

$$DX + DY(\operatorname{Id} + DX) = 0.$$

Hence

$$DY = -DX(\operatorname{Id} + DX)^{-1},$$

which is uniformly comparable to $\|DX\|$ because $d_{C^1}(f, \operatorname{Id})$ is uniformly bounded.

For $k > 1$, we must estimate the higher order derivatives of Y. Note that for $k > 1$ that $D^k g = D^k Y$ and $D^k f = D^k X$.

Applying the chain rule to $f \circ g = \mathrm{Id}$ to calculate the kth derivative gives:

$$0 = \sum_{l=1}^{k} \sum_{j_1+\cdots+j_l=k} C_{l,j_1,\ldots,j_l} D^l_{g(x)} f\{D^{j_1}_x g, \ldots, D^{j_l}_x g\},$$

and hence

$$D^k_x g = -(D_{g(x)} f)^{-1} \sum_{l=2}^{k} \sum_{j_1+\cdots+j_l=k} C_{l,j_1,\ldots,j_l} D^l_{g(x)} f\{D^{j_1}_x g(x), \ldots, D^{j_l}_x g(x)\}. \tag{3.102}$$

As $(Df)^{-1}$ has uniformly bounded norm, it suffices to show that the each term in the sum has norm bounded by $\|X\|_{C^n}$.

We use the interpolation estimate in Lemma 3.7.7. If $j > 1$, then

$$\|D^j g\| = \|D^j Y\|,$$

By interpolation between the C^1 and C^{n-1} norms, for $1 \leq j \leq n-1$,

$$\|Y\|_{C^j} \leq C_{1,n-1} \|Y\|_{C^1}^{\frac{n-j-1}{n-2}} \|Y\|_{C^{n-1}}^{\frac{j-1}{n-2}}.$$

By interpolation between the C^1 and C^n norms, for $1 \leq j \leq n$,

$$\|X\|_{C^j} \leq C_{1,n} \|X\|_{C^1}^{\frac{n-j}{n-1}} \|X\|_{C^n}^{\frac{j-1}{n-1}}.$$

We now estimate the terms in the right hand side of equation (3.102). In the case that some $j_i = 1$, then $D^{j_i} g = \mathrm{Id} + DY$. Hence the right hand side of equation (3.102), may be rewritten as the sum of terms of the form

$$D^l_{g(x)} X\{A_1, ..., A_l\},$$

where each A_i is either equal to Id or $D^{j_i} Y$ and the sum of the j_i is less than or equal to

k. If $\|Y\|_{C^{k-1}} \leq 1$, then we are immediately done as the norm of this expression is at most $\|D^k f\|$. Otherwise, we may suppose that $\|Y\|_{C^{k-1}} \geq 1$. The C^1 norms of X and Y are uniformly bounded. Hence by interpolating between the C^1 and C^k norm to estimate the $D^l X$ term and the C^1 and the C^{k-1} norm to estimate the A_i terms, we find that

$$\|D^k_{g(x)} X\{A_1, ..., A_k\}\| \leq C' \|X\|_{C^k}^{\frac{l-1}{k-1}} \|Y\|_{C^{k-1}}^{\frac{k-r}{n-2}},$$

where $r \geq l$. But as $\|Y\|_{C^{k-1}} > 1$, this bounded above by

$$C' \|X\|_{C^k}^{\frac{l-1}{k-1}} \|Y\|_{C^{k-1}}^{\frac{k-l}{k-2}}.$$

Thus

$$\|D^k Y\|_{C^0} \leq C'' \sum_{l=2}^{k} \|X\|_{C^k}^{\frac{l-1}{k-1}} \|Y\|_{C^{k-1}}^{\frac{k-l}{k-2}}.$$

We may now proceed by induction on k. We already established the theorem for $k = 1$. Now, given that $\|Y\|_{C^{k-1}} \leq C_{k-1} \|X\|_{C^{k-1}}$, it follows that

$$\|D^k Y\|_{C^0} \leq C''' \sum_{l=2}^{k} \|X\|_{C^k}^{\frac{l-1}{k-1}} \|X\|_{C^{k-1}}^{\frac{k-l}{k-2}}.$$

By interpolation between the 1 and k norms, the uniform bound on the C^1 norm, we find that $\|X\|_{C^{k-1}} \leq D_k \|X\|_{C^k}^{\frac{k-2}{k-1}}$. This yields

$$\|D^k Y\|_{C^0} \leq D' \sum_{l=2}^{k} \|X\|_{C^k}^{\frac{l-1}{k-1}} \|X\|_{C^k}^{\frac{k-l}{k-1}} \leq D'' \|X\|_{C^k},$$

which is the desired result. □

We now obtain the following corollary.

Corollary 2. *Suppose that M is a closed Riemannian manifold. For smooth C^1 small vector* *fields X on M, we may write*

$$\psi_X^{-1} = \psi_{-X+Z},$$

where Z is quadratic in X.

Proof. To begin we know by Proposition 16 that

$$\psi_X \circ \psi_{-X} = \psi_Z,$$

where Z is quadratic in X. Note that $\psi_{-X} \circ \psi_Z^{-1} = \psi_X^{-1}$. By Lemma 3.7.4, $\psi_Z^{-1} = \psi_{Z'}$ where Z' is quadratic in X. Hence $\psi_X^{-1} = \psi_{-X} \circ \psi_{Z'}$. By Proposition 16, this gives that $\psi_X^{-1} = \psi_{-X+Z'+Q}$, where Q is quadratic in X and Z'. Hence as Z' is quadratic in X and the corollary follows. □

We can now complete the proof of the estimate on the error field of the conjugated system.

Proof of Prop. 15. To show this, we repeatedly apply Proposition 16 and Corollary 2. Writing Z for anything second order in C and Y, we find:

$$\begin{aligned}
\psi_C f \psi_C^{-1} &= \psi_C \psi_Y R \psi_C^{-1} \\
&= \psi_{C+Y+Z} R \psi_C^{-1} \\
&= \psi_{C+Y+Z} R \psi_{-C+Z} \\
&= \psi_{C+Y+Z+R_*(-C+Z)} R \\
&= \psi_{C+Y-R_*C+Z} R.
\end{aligned}$$

□

We now show two additional lemmas that we use in the KAM scheme.

Lemma 3.7.5. *Let M be a closed Riemannian manifold. Fix $k \geq 1$. There exist $C, \epsilon > 0$ such that if $R \in \operatorname{Isom}(M)$ and $f, g \in \operatorname{Diff}^k(M)$ satisfy $d_{C^1}(f, R) < \epsilon$, and $d_{C^1}(g, \mathrm{Id}) < \epsilon$,* *then*

$$d_{C^k}(f \circ g, R) \leq C_k(d_{C^k}(f, R) + d_{C^k}(g, \mathrm{Id})),$$

and

$$d_{C^k}(g \circ f, R) \leq C_k(d_{C^k}(f, R) + d_{C^k}(g, \mathrm{Id})).$$

Proof. We begin with a proof for the first inequality. In coordinates write $f(x) = R(x)+Y(x)$ and $g(x) = x + X(x)$. Then we just need to estimate

$$f \circ g(x) - R(x) = R(x + X(x)) - R(x) + Y(x + X(x)).$$

The last term is controlled by $d_{C^k}(f, R) + d_{C^k}(g, \mathrm{Id})$ by Lemma 3.7.3. So, it suffices to estimate the first term. The kth derivative of $R(x + X(x)) - R(x)$ is then

$$\sum_{l=1}^{k} \sum_{j_1+\cdots+j_l=k} C_{l,j_1,\ldots,j_l} D^l_{x+X(x)} R\{D^{j_1}_x g, \ldots, D^{j_l}_x g\} - D^l_x R.$$

For all the terms with $l < k$, the same interpolation approach as in Lemma 3.7.4 gives the appropriate estimate, i.e. they are bounded by

$$C \sum_{l=1}^{k-1} \|X\|_{C^k}^{\frac{l-1}{k-1}} \|X\|_{C^{k-1}}^{\frac{k-l}{k-2}}.$$

There are two remaining terms which are unaccounted for: $D^k R_{x+X(x)} - D^k R_x$. This is bounded by a constant time $\|X\|_{C^0}$ and the result follows.

We now consider the second inequality. As before we must estimate

$$g \circ f(x) - R(x) = X(x) + Y(R(x) + X(x)).$$

The important term is the second one. A similar argument to before then gives the result as all derviatives of R are uniformly bounded independent of R. □

Lemma 3.7.6. *Let M be a closed Riemannian manifold and $k \geq 0$. If $g_n \in \mathrm{Diff}^k(M)$ is a sequence of diffeomorphisms and $\sum_n d_{C^k}(g_n, \mathrm{Id}) < \infty$, then the sequence of compositions of diffeomorphisms $h_n = g_n g_{n-1} \cdots g_2 g_1$ converges in C^k to a diffeomorphism.*

Proof. As before, we check in charts. Having fixed a chart, write $g_n(x) = x + X_n(x)$. Write $h_n(x) = 1 + Y_n(x)$. Let $a_n = \|X_n\|_{C^k}$ and let $b_n = \|Y_n\|_{C^k}$. Note that

$$h_n(x) = x + Y_{n-1}(x) + X_n(x + Y_n(x)). \tag{3.103}$$

Suppose for the moment that $\|Y_{n-1}\|_{C^k} \leq 1$. Using Lemma 3.7.3 and that $\|Y_n\|_{C^k} \leq 1$,

$$\begin{aligned} \|X_n(x + Y_{n-1})\|_{C^k} &\leq C_k(\|X_n\|_{C^k}\|x + Y_{n-1}\|_{C^1}^k + \|X_n\|_{C^1}\|x + Y_{n-1}\|_{C^k} + \|X\|_{C^0}) && (3.104) \\ &\leq C_k'(a_n + a_n b_{n-1}) && (3.105) \end{aligned}$$

Hence it follows from equation (3.103) that there exists D_k such that if $b_{n-1} \leq 1$ then

$$b_n \leq b_{n-1} + D_k a_n(1 + b_{n-1}).$$

By induction, under the same assumption that $\|Y_j\|_{C^k} \leq 1$ for $j < n$, it follows that

$$b_n \leq -1 + \prod_{i=1}^{n}(1 + D_k a_i).$$

By noting that $\prod_{i=1}^{\infty}(1 + x_n) \leq \exp\left(\sum_{i=1}^{\infty} x_n\right)$ for $x_n \geq 0$, we can conclude that a tail of the sequence converges. This follows because as $\sum_n a_n$ converges we can inductively check that these inequalities hold starting the argument from an index N satisfying $\exp(\sum_{i=N}^{\infty} D_k a_i) - 1 < 1$. Hence as a tail of the infinite composition converges so does the whole composition.

□

3.7.2 Interpolation Inequalities

There is a basic C^k interpolation inequality, which may be found in the appendix of [Hör76, Thm A.5]. It states that:

Lemma 3.7.7. *Suppose that M is a closed Riemannian manifold. For $0 \leq a \leq b < \infty$ and $0 < \lambda < 1$ there exists a constant $C(a, b, \lambda)$ such that for any real valued C^b function f defined on M,*

$$\|f\|_{C^{\lambda a+(1-\lambda)b}} \leq C\|f\|_{C^a}^{\lambda}\|f\|_{C^b}^{1-\lambda}.$$

The following is an immediate consequence of Lemma 3.7.7.

Lemma 3.7.8. *Suppose that M is a closed Riemannian manifold. There exists $\epsilon > 0$ such that for $0 \leq a \leq b < \infty$ and $0 < \lambda < 1$ there exists a constant $C(a, b, \lambda)$ such that for any $f \in \operatorname{Diff}^{\infty}(M)$ such that $d_{C^0}(f, \operatorname{Id}) < \epsilon$, then*

$$d_{C^{\lambda a+(1-\lambda)b}}(f, \operatorname{Id}) \leq C d_{C^a}(f, \operatorname{Id})^{\lambda} d_{C^b}(f, \operatorname{Id})^{1-\lambda}.$$

Lemma 3.7.9. *Consider the space $C^{\infty}(M, N)$ where M and N are Riemannian manifolds and M and N are closed. For all $j, \sigma > 0$, there exists a natural number k and a number $\epsilon_0 > 0$ such that if $f, g \in C^{\infty}(M, N)$, $\|f - g\|_{H^j} < \epsilon_0 < 1$, and $\|f - g\|_{C^k} \leq 1/2$ then $\|f - g\|_{C^j} \leq \|f - g\|_{H^j}^{1-\sigma}$.*

Proof. The proof is a relatively straightforward application of the Sobolev embedding theorem and interpolation inequalities. First, we recall an interpolation inequality for Sobolev norms, see [BL76, Thm. 6.5.4]. For each $0 < \theta < 1$, s_0, s_1, there exists a constant C such that if we let $s = (1 - \theta)s_0 + \theta s_1$, then we have

$$\|f - g\|_{H^s} \leq C\|f - g\|_{H^{s_0}}^{1-\theta}\|f - g\|_{H^{s_1}}^{\theta}.$$

To begin the proof, note that it suffices to estimate $\|f - g\|_{C^{j+1}}$. Fix ℓ large enough that

H^ℓ embeds compactly in C^{j+1} by a Sobolev embedding. Then pick k large enough that

$$\|f-g\|_{H^\ell} \leq C_{\lambda,\ell}\|f-g\|_{H^j}^{1-\theta}\|f-g\|_{H^k}^{\theta},$$

where $0<\theta<\sigma$. The term $\|f-g\|_{H^k}^{\theta}$ is uniformly bounded by $C_k\|f-g\|_{C^k}^{\theta}$. Hence as H^ℓ compactly embeds in C^{j+1}, there exists $C'>0$ such that

$$\|f-g\|_{C^{j+1}} \leq C'\|f-g\|_{H^j}^{1-\theta} = C'\|f-g\|_{H^j}^{\sigma-\theta}\|f-g\|_{H^j}^{1-\sigma}.$$

If we choose ϵ_0 sufficiently small that $C'\|f-g\|_{H^j}^{\sigma-\theta}\leq 1$, then the result follows. □

A similar argument shows the following:

Lemma 3.7.10. *Suppose that E is a smooth Riemannian vector bundle over a closed Riemannian manifold M. For all choices $j,\ell,\sigma,D>0$ there exist k,ϵ_0 such that if f is a smooth section of E and $\|f\|_{H^j}\leq\epsilon_0<1$ and $\|f\|_{C^k}\leq D$ then $\|f\|_{C^\ell}\leq\|f\|_{H^j}^{1-\sigma}$.*

3.7.3 Estimate on Lifted Error Fields

The goal of this subsection is to prove a technical estimate on the error fields of a lifted system. The proof is a computation in charts.

Lemma 3.7.11. *Suppose that M is a closed Riemannian manifold. Fix numbers $m,k\geq 0$ and d such that $0\leq d\leq \dim M$. There exists a constant C such that the following holds. For any tuple $(f_1,...,f_m)$ of diffeomorphisms of M and $(r_1,...,r_m)$ a C^1 close tuple of isometries of M, let Y_i be the shortest vector field such that $\exp_{r_i(x)}Y_i(x)=f_i(x)$. Let F_i be the lift of f_i to $\operatorname{Gr}_d(M)$ and R_i be the lift of r_i to $\operatorname{Gr}_d(M)$. Let $\widetilde{Y}_i$ be the shortest vector field on $\operatorname{Gr}_d(M)$ such that $\exp_{R_i(x)}\widetilde{Y}_i(x)=F_i(x)$. If $\|\sum_i Y_i\|_{C^k}=\epsilon$ and $\max_i\|Y_i\|_{C^k}=\eta$, then*

$$\left\|\sum_{i=1}^m \widetilde{Y}_i\right\|_{C^{k-1}} \leq C(\epsilon+\eta^2).$$

Proof. The proof is straightforward but tedious. We give the proof in the case that each R_i is the identity. Removing this assumption both complicates the argument in purely technical ways and substantially obscures why the lemma is true. At the end of the argument, we indicate the modifications needed for the general proof.

For readability we redevelop some of the basic notions concerning Grassmannians. First we recall the charts on $\mathrm{Gr}_d(V)$, the Grassmannian of d-planes in a vector space V. Recall that given a vector space V and a pair of complementary subspaces P and Q of V that if $\dim P = d$ we obtain a chart on $\mathrm{Gr}_d(V)$ in the following manner. Let $L(P,Q)$ denote the space of linear maps from P to Q. For $A \in L(P,Q)$, we send A to the subspace $\{x + Ax \mid x \in P\} \in \mathrm{Gr}_d(V)$. This gives a smooth parametrization of a subset of $\mathrm{Gr}_d(V)$. Having fixed a complementary pair of subspaces P and Q, let π_P denote the projection to P along Q.

Suppose that U is a chart on M and let $\partial_1, ..., \partial_n$ denote the coordinate vector fields. We use the usual coordinate framing of TU to give coordinates on the Grassmannian bundle $\mathrm{Gr}_d(M)$. The tangent bundle to U naturally splits into sub-bundles spanned by $\{\partial_1, ..., \partial_d\}$ and $\{\partial_{d+1}, \ldots, \partial_n\}$. Call these sub-bundles P and Q, respectively. Let $\mathrm{End}(P,Q)$ denote the bundle of maps from P to Q. We obtain a coordinate chart via associating an element of $A \in \mathrm{End}(P,Q)$ and a point $x \in U$ with the graph of A in the tangent space over x.

As we have assumed that each r_i is the identity, in charts we write $f_i(x) = x + X_i(x)$. As the f_i are C^1 small, we work in a single chart. It now suffices to prove the corresponding estimate on the field X_i because X_i and Y_i are equal up to an error that is quadratic in the sense of Definition 3.7.1. We now calculate the action of f on $\mathrm{Gr}_d(U)$. Suppose that $A \in \mathrm{End}(P,Q)$. Then we have that $\{Df(v + Av)\}$ is a subspace of $T_{f(x)}M$. We must find the map A' whose graph gives the same subspace. Let I_A be the $n \times d$ matrix with top block I and bottom block A. Then the action of Df sends A to A' which is equal to

$$A' = DfI_A(\pi_P DfI_A)^{-1} - \mathrm{Id}\,.$$

To see that this is true, we must check that $A'V \subseteq Q$ and that $\{Dfv + DfAv \mid v \in V\}$ is the same as $\{v + A'v\}$. The second condition is evident from the definition of A'. If $v \in P$, then $(\pi_P DfI_A)^{-1}v = w$ is an element of P satisfying $\pi_P DfI_A w = v$. Thus $A'v = DfI_A(\pi_P DfI_A)^{-1}v - v \in Q$ and hence $A'V \subseteq Q$. Write F for the induced map on $\mathrm{Gr}_d(U)$. In coordinates F is the map that sends

$$(x, A) \mapsto (x, DfI_A(\pi_P DfI_A)^{-1} - \mathrm{Id}). \tag{3.106}$$

Write I_d for the d by d identity matrix. Let $\widehat{DX_i}$ be the matrix comprised of the first d rows of the matrix DX_i. In the estimates below, we will assume that the size of A is uniformly bounded. This does not restrict the generality as any subspace may be represented by such a uniformly bounded A. Then note that

$$\begin{aligned}(\pi_P D_f \begin{bmatrix} I_d \\ \hline A \end{bmatrix})^{-1} &= (I_d + \widehat{DX} \begin{bmatrix} I_d \\ \hline A \end{bmatrix})^{-1} \\ &= I_d - \widehat{DX} \begin{bmatrix} I_d \\ \hline A \end{bmatrix} + O(DX^2),\end{aligned}$$

where the $O(DX^2)$ is quadratic in the sense of Definition 3.7.1. Write X_A for the second term above.

We then have that

$$\begin{aligned} DfI_A(\pi_P DfI_A)^{-1} - \mathrm{Id} &= (\mathrm{Id} + DX) \begin{bmatrix} I_d \\ \hline A \end{bmatrix} (I_d - X_A) - \begin{bmatrix} I_d \\ \hline 0 \end{bmatrix} + O(DX^2). \\ &= \begin{bmatrix} I_d \\ \hline A \end{bmatrix} - \begin{bmatrix} I_d \\ \hline A \end{bmatrix} X_A + DX \begin{bmatrix} I_d \\ \hline A \end{bmatrix} + DX \begin{bmatrix} I_d \\ \hline A \end{bmatrix} X_A - \begin{bmatrix} I_d \\ \hline 0 \end{bmatrix} + O(DX^2). \\ &= \begin{bmatrix} 0 \\ \hline A \end{bmatrix} - \begin{bmatrix} I_d \\ \hline A \end{bmatrix} X_A + DX \begin{bmatrix} I_d \\ \hline A \end{bmatrix} + O(DX^2). \\ &= \begin{bmatrix} 0 \\ \hline A \end{bmatrix} + H(A, DX) + O(DX^2). \end{aligned}$$

where $H(A, DX)$ is the sum of the second and third terms two lines above. Note that H is linear in DX and that $\|H(A, DX) \leq C\|DX\|$ given our uniform boundedness assumption on A.

Thus we see that in this chart on $\mathrm{Gr}_d(U)$ that

$$F(x,A) - (x,A) = (f(x) - x, H(A, DX) + O(DX^2)). \tag{3.107}$$

In this chart, $\| \sum_i f_i(x) - x \|_{C^k} \leq \epsilon$. Hence writing $f_i(x) = x + X_i(x)$ as before, $\| \sum_i DX_i(x) \|_{C^{k-1}} \leq \epsilon$. Thus

$$\left\| \sum_i F_i(x,A) - (x,A) \right\|_{C^{k-1}} = \left\| \sum_i (f_i(x) - x, H(A, DX_i) + O(DX^2)) \right\|_{C^{k-1}} \leq C \left(\left\| \sum_i X_i \right\|_{C^k} + \max_i \|X_i\|$$

by the linearity of H. This completes the proof in the special case where $r_i = \mathrm{Id}$ for each i.

In the general setting one follows the same sequence of steps. One writes $f_i(x) = r_i(x) + X_i(r_i(x))$. One then does the same computation to determine the action on the Grassmannian bundle. This is complicated by additional terms related to R. Having finished this computation, one finds a natural analog of $H(A, DX)$, which now comprises eight terms instead of two, and also depends on r_i. Recognizing the cancellation is then somewhat complicated because of the dependence on r_i. However, this dependence does not cause an issue because the terms that would potentially cause trouble satisfy some useful relations. These relations emerge when one keeps in mind the base points, which is crucial when the isometries are non-trivial. □

3.7.4 Determinants

Suppose that V and W are finite dimensional inner product spaces. Consider a linear map $L\colon V \to W$. The determinant of the map L is defined as follows. If $\{v_i\}$ is an orthonormal basis for V, one may measure the size of the tensor $Lv_1 \wedge \cdots \wedge Lv_n$ with respect to the norm on tensors induced by the metric on W. If $\{v_1, ..., v_n\}$ is a basis for V, then we define

$$\det(L, g_1, g_2) \coloneqq \sqrt{\frac{\mathrm{Det}\left(\langle Lv_i, Lv_j\rangle_{g_2}\right)}{\mathrm{Det}\left(\langle v_i, v_j\rangle_{g_1}\right)}},$$

where Det is the usual determinant of a square matrix. Sometimes we have a map $L\colon V \to W$ and a subspace $E \subset V$. We then define

$$\det(L, g_1, g_2 \mid E) = \det(L|_E, g_1|_E, g_2). \tag{3.108}$$

When the spaces V and W are understood, we may write $\det(L \mid E)$.

There are some properties of det that we will record for later use.

Lemma 3.7.12. *Fix a basis and suppose that $V = W$. Working with respect to this basis, the determinant has the following properties:*

$$\det(L, g_1, g_2) = \det(\mathrm{Id}, g_1, L^* g_2), \tag{3.109}$$

$$\det(\mathrm{Id}, \mathrm{Id}, A) = \sqrt{\det(A, \mathrm{Id}, \mathrm{Id})} = \sqrt{|\mathrm{Det}(A)|}. \tag{3.110}$$

Proof. For the first equality, let $\{v_i\}$ be a basis of (V, g_1), then

$$\det(L, g_1, g_2) = \sqrt{\frac{\mathrm{Det}\langle Lv_i, Lv_j\rangle_{g_2}}{\mathrm{Det}\langle v_i, v_j\rangle_{g_1}}}$$

But, $\langle v_i, v_j\rangle_{L^* g_2} = \langle Lv_i, Lv_j\rangle_{g_2}$, so, this is equal to

$$\sqrt{\frac{\mathrm{Det}\langle v_i, v_j\rangle_{L^* g_2}}{\mathrm{Det}\langle v_i, v_j\rangle_{g_1}}},$$

which is the definition of $\det(\mathrm{Id}, g_1, L^* g_2)$.

For the second equality, fix an orthonormal basis $\{e_i\}$, then

$$\det(\mathrm{Id}, \mathrm{Id}, A) = \sqrt{\mathrm{Det}\langle e_i, e_j\rangle_A} = \sqrt{\mathrm{Det}\, A_{ij}}$$

whereas,

$$\det(A, \mathrm{Id}, \mathrm{Id}) = \sqrt{\mathrm{Det}\langle Ae_i, Ae_j\rangle_{\mathrm{Id}}} = \sqrt{\mathrm{Det}\, A^T A} = \sqrt{|\mathrm{Det}\, A|^2} = |\mathrm{Det}\, A|\,.$$

□

We record the following estimate which is used in the proof.

Lemma 3.7.13. *Let M be a closed manifold and let $0 \leq r \leq \dim M$. If g is an isometry of M, then $\ln\det(Df|E_x)$, which is defined on $\mathrm{Gr}_r(M)$, satisfies the following estimate:*

$$\|\ln\det(Df \mid E_x)\|_{C^k} = O(d_{C^{k+1}}(f, g)),$$

as $f \to g$ in C^{k+1}. The big-O is uniform over all isometries g.

Proof. It suffices to show that this estimate holds in charts. So, fix a pair of charts U and V on M such that $f(U)$ has compact closure inside of V. We define a map $H\colon \mathrm{Gr}_d(U)\times U\times V\times \mathbb{R}^{n^2} \to \mathbb{R}$ by sending the point (E, x, y, A) to the $\ln\det(A, g_x, g_y|E)$, where g_x and g_y denote the pullback metric from M. Using f we define a map $\widetilde{f}\colon \mathrm{Gr}_d \times U \to \mathrm{Gr}_d(U) \times U \times V \times \mathbb{R}^{n^2}$ by

$$(E, x) \mapsto (E, x, f(x), Df),$$

where we are using the coordinates to express Df as a matrix. Then the quantity we wish to estimate the C^k norm of is $H \circ \widetilde{f}$. If we analogously define $\widetilde{g}$, then note that $H \circ \widetilde{g} \equiv 0$ because g is an isometry. By writing out the derivatives using the chain rule and using that f is uniformly close to g, one sees that $\|H \circ \widetilde{g} - H \circ \widetilde{f}\|_{C^k} = O(d_{C^{k+1}}(f, g))$, and the result follows. □

3.7.5 Taylor Expansions

Taylor expansion of the log Jacobian

Proposition 18. *For C^1 small vector fields Y on a Riemannian manifold M, the following approximation holds*

$$\int_{\mathrm{Gr}_r(M)} \ln \det(D_x\psi_Y, \mathrm{Id}, g_{\psi_{Y(x)}} \mid E_x)\, d\,\mathrm{vol} = -\frac{r}{2d}\int_M \|E_C\|^2\, d\,\mathrm{vol} + \frac{r(d-r)}{(d+2)(d-1)}\int_M \|E_{NC}\|^2\, d\,\mathrm{vol} + O(\|Y\|^3_{C^1}),$$

where E_C and E_{NC} are the conformal and non-conformal strain tensors associated to ψ_Y as defined in subsection 3.4.2. In addition, det *is defined in Appendix 3.7.4 and ψ_Y is defined in equation* (3.10)*.*

The proof of this proposition is a lengthy computation with several subordinate lemmas.

Proof. In order to estimate the integral over M, we will first obtain a pointwise estimate on:

$$\int_{\mathrm{Gr}_r(T_xM)} \ln \det(D_x\psi_Y \mid E)\, dE.$$

To estimate this we work in an exponential chart on M centered at x. In this chart, x is 0 and $\psi_Y(0) = Y(0)$. Then

$$\int_{\mathrm{Gr}_r(T_xM)} \ln \det(D_x\psi_Y \mid E)\, dE = \int_{\mathrm{Gr}_r(T_xM)} \ln \det(D_0\psi_Y, \mathrm{Id}, g_{Y(0)} \mid E)\, dE.$$

We now rewrite the above line so that we can apply the Taylor approximation in Proposition 19.

Write the metric as $\mathrm{Id} + \hat{g}$. As we are in an exponential chart, $\|\hat{g}_{Y(0)}\| = O(\|Y\|^2_{C^0})$. Write $D\psi_Y = \mathrm{Id} + \hat{\psi}$. The integral we are calculating only involves $\hat{\psi}_0$ and $\hat{g}_{Y(0)}$, so below we drop the subscripts. Then

$$\int_{\mathrm{Gr}_r(T_xM)} \ln\det(D_x\psi_Y \mid E)\,dE = \int_{\mathrm{Gr}_r(T_xM)} \ln\det(\mathrm{Id}+\hat{\psi}, \mathrm{Id}, \mathrm{Id}+\hat{g} \mid E)\,dE.$$

Now applying the Taylor expansions in Propositions 19 and 20, we obtain the following expansion. For convenience let

$$K = (\hat{\psi} + \hat{\psi}^T)/2 - \frac{\mathrm{Tr}\,\hat{\psi}}{d}\,\mathrm{Id}\,. \tag{3.111}$$

Then

$$\int_{\mathrm{Gr}_r(T_xM)} \ln\det(D\psi_Y, \mathrm{Id}, g_{Y(0)} \mid E)\,dE = \tag{3.112}$$

$$\frac{r}{d}\,\mathrm{Tr}(\hat{\psi}) + \left[-\frac{r}{2d}\,\mathrm{Tr}(\hat{\psi}^2) + \frac{r(d-r)}{(d+2)(d-1)}\,\mathrm{Tr}(K^2)\right] + O(\|\hat{\psi}^3\|) + \frac{r}{2d}\,\mathrm{Tr}(\hat{g}) + O(\|\hat{g}\|^2) \tag{3.113}$$

Note that $\|\hat{\psi}\| = O(\|Y\|_{C^1})$ and $\|\hat{g}\| = O(\|Y\|^2_{C^0})$, hence the fourth and sixth terms in the above expression are each $O(\|Y\|^3_{C^1})$.

We now eliminate the two trace terms that are not quadratic in their arguments. For this, we use a Taylor expansion of the determinant.[3] Thus

$$\det(D\psi, \mathrm{Id}, g_{Y(0)}) = 1 + \mathrm{Tr}\,\hat{\psi} + \frac{(\mathrm{Tr}(\hat{\psi}))^2 - \mathrm{Tr}(\hat{\psi}^2)}{2} + \frac{\mathrm{Tr}(\hat{g})}{2} + O(\|Y\|^3_{C^1})$$

The integral of the Jacobian is 1, so integrating the previous line over M against volume

[3] Recall the usual Taylor expansion $\mathrm{Det}(\mathrm{Id}+A) = \mathrm{Id} + \mathrm{Tr}(A) + \frac{(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)}{2} + O(\|A\|^3)$. We combine this with the first order Taylor expansion

$$\det(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}+G) = \sqrt{\mathrm{Det}(1+G)} = \sqrt{1 + \mathrm{Tr}(G) + O(\|G\|^2)} = 1 + \frac{\mathrm{Tr}(G)}{2} + O(\|G\|^2).$$

we obtain

$$1 = 1 + \int_M \operatorname{Tr}\hat{\psi} + \frac{(\operatorname{Tr}(\hat{\psi}))^2 - \operatorname{Tr}(\hat{\psi}^2)}{2} + \frac{\operatorname{Tr}(\hat{g})}{2}\, d\operatorname{vol} + O(\|Y\|_{C^1}^3).$$

Thus

$$\int_M \operatorname{Tr}(\hat{\psi}) + \frac{\operatorname{Tr}(\hat{g})}{2} - \frac{\operatorname{Tr}(\hat{\psi}^2)}{2}\, d\operatorname{vol} = -\int_M \frac{(\operatorname{Tr}(\hat{\psi}))^2}{2}\, d\operatorname{vol} + O(\|Y\|_{C^1}^3).$$

Now, we integrate equation (3.112) over M and apply the previous line to eliminate the non-quadratic terms. This gives

$$\int_{\operatorname{Gr}_r(M)} \ln\det(D_x\psi_Y, \operatorname{Id}, g_{\psi_{Y(x)}} \mid E_x)\, dE_x = \int_M -\frac{r}{2d}(\operatorname{Tr}(\hat{\psi}_x))^2 + \frac{r(d-r)}{(d+2)(d-1)}\operatorname{Tr}(K_x^2)\, d\operatorname{vol} + O(\|Y\|_{C^1}^3), \tag{3.114}$$

where we have written $\hat{\psi}_x$ and K_x to emphasize the basepoint. The formula above is not yet very usable as both K_x and $\hat{\psi}_x$ are defined in terms of exponential charts. We now obtain an intrinsic expression for these terms. Recall that pointwise we use the L^2 norm on tensors. Below we suppress the x in $\|E_C(x)\|$ and $\hat{\psi}_x$.

Lemma 3.7.14. *Let E_C be the conformal strain tensor associated to ψ_Y. Then*

$$\int_M (\operatorname{Tr}(\hat{\psi}_x))^2\, d\operatorname{vol} = \int \|E_C\|^2\, d\operatorname{vol} + O(\|Y\|_{C^1}^3).$$

Proof. We use an exponential chart and compute a coordinate expression for $\|E_C\|^2$ in the center of this chart. As before, write $D\psi_Y = \operatorname{Id} + \hat{\psi}$, where $\hat{\psi} = O(\|Y\|_{C^1})$. Then working in exponential coordinates,

$$\begin{aligned}\operatorname{Tr}(\psi_Y^* g - g) &= \operatorname{Tr}((\operatorname{Id} + \hat{\psi})^T(\operatorname{Id} + O(\|Y\|_{C^0}^2))(\operatorname{Id} + \hat{\psi}) - \operatorname{Id}) \\ &= \operatorname{Tr}(\operatorname{Id} + \hat{\psi}^T + \hat{\psi} - \operatorname{Id} + O(\|Y\|_{C^1}^2) \\ &= 2\operatorname{Tr}(\hat{\psi}) + O(\|Y\|_{C^1}^2).\end{aligned}$$

Thus since $\hat{\psi} = O(\|Y\|_{C^1})$, by definition of E_C, we have

$$\begin{aligned}
\|E_C\|^2 &= \left\| \frac{\operatorname{Tr}(\psi_Y^* g - g)}{2d} \operatorname{Id} \right\|^2 \\
&= \left\| \frac{2 \operatorname{Tr}(\hat{\psi})}{2d} \operatorname{Id} \right\|^2 \\
&= \frac{\operatorname{Tr}(\hat{\psi})}{d} \| \operatorname{Id} \|^2 \\
&= \operatorname{Tr}(\hat{\psi}).
\end{aligned}$$

Integrating over M, we obtain the result. □

Lemma 3.7.15. *Let E_{NC} be the non-conformal strain tensor associated to ψ_Y and let K_x be as in equation* (3.111)*, then*

$$\int_M \operatorname{Tr}\left(K_x^2\right) \, d\operatorname{vol} = \int_M \|E_{NC}\|^2 \, d\operatorname{vol} + O(\|Y\|_{C^1}^3).$$

Proof. As before, we first compute a local expression for the integrand and check that this expression is comparable to the local expression for the non-conformal strain tensor. We compute at the center of an exponential chart. As before, write $D\psi_Y = \operatorname{Id} + \hat{\psi}$ where $\hat{\psi} = O(\|Y\|_{C^1})$. In this case

$$\psi_Y^* g = (\operatorname{Id} + \hat{\psi})^T (\operatorname{Id} + O(\|Y\|_{C^0}^2))(\operatorname{Id} + \hat{\psi}) = \operatorname{Id} + \hat{\psi}^T + \hat{\psi} + O(\|Y\|_{C^1}^2).$$

Using the above line and the definition of E_{NC} we then compute:

$$
\begin{aligned}
\|E_{NC}\|^2 &= \left\| \frac{1}{2} \left(\psi_Y^* g - g - \frac{\operatorname{Tr}(\psi_Y^* g - g)}{d} g \right) \right\|^2 \\
&= \frac{1}{4} \|(\operatorname{Id} + \hat{\psi})^T (\operatorname{Id} + O(\|Y\|_{C^0}^2)(\operatorname{Id} + \hat{\psi}) - \operatorname{Id} - 2\frac{\operatorname{Tr}\hat{\psi}}{d} \operatorname{Id} + O(\|Y\|_{C^1}^2)\|^2 \\
&= \frac{1}{4} \|\hat{\psi}^T + \hat{\psi} - 2\frac{\operatorname{Tr}\hat{\psi}}{d} \operatorname{Id} + O(\|Y\|_{C^1}^2)\|^2 \\
&= \frac{1}{4} \operatorname{Tr} \left(\left(\hat{\psi}^T + \hat{\psi} - 2\frac{\operatorname{Tr}\hat{\psi}}{d} \operatorname{Id} + O(\|Y\|_{C^1}^2) \right)^2 \right) \\
&= \operatorname{Tr} \left(\left(\frac{\hat{\psi}^T + \hat{\psi}}{2} - \frac{\operatorname{Tr}\hat{\psi}}{d} \operatorname{Id} \right)^2 \right) + O(\|Y\|_{C^1}^3) \\
&= \operatorname{Tr}(K^2) + O(\|Y\|_{C^1}^3).
\end{aligned}
$$

By integrating the above equality over M, the result follows. □

Finally, the proof of Proposition 18 follows by applying Lemma 3.7.14 and Lemma 3.7.15 to equation (3.114), which gives

$$
\begin{aligned}
\int_{\operatorname{Gr}_r(M)} \ln \det(D_x \psi_Y, \operatorname{Id}, g_{\psi_{Y(x)}} \mid E_x)\, dE_x = &- \frac{r}{2d} \int_M \|E_C\|^2 \, d\operatorname{vol} && (3.115) \\
&+ \frac{r(d-r)}{(d+2)(d-1)} \int_M \|E_{NC}\|^2 \, d\operatorname{vol} + O(\|Y\|_{C^1}^3). && (3.116)
\end{aligned}
$$

□

Approximation of integrals over Grassmanians

Let $\mathbb{G}_{r,d}$ be the Grassmanian of r-planes in $\mathbb{R}^d$. In this subsubsection, we prove the following simple estimate.

Proposition 19. *For $1 \le r \le d$, let $\Lambda_r \colon \operatorname{End}(\mathbb{R}^d) \to \mathbb{R}$ be defined by*

$$\Lambda_r(L) := \int_{\mathbb{G}_{r,d}} \ln \det(\operatorname{Id} + L, \operatorname{Id}, \operatorname{Id} \mid E)\, dE,$$

where dE denotes the Haar measure on $\mathbb{G}_{r,d}$. Then the second order Taylor approximation for Λ_r at 0 is

$$\Lambda_r(L) = \frac{r}{d} \operatorname{Tr} L + \left[-\frac{r}{2d} \operatorname{Tr}(L^2) + \frac{r(d-r)}{(d+2)(d-1)} \operatorname{Tr}(K^2) \right] + O(\|L\|^3),$$

where

$$K = \frac{L + L^T}{2} - \frac{\operatorname{Tr} L}{d} \operatorname{Id}.$$

Let $\lambda_r(L) = \Lambda_r(L) - \Lambda_{r-1}(L)$. Then the above expansion implies

$$\lambda_r(L) = \frac{1}{d} \operatorname{Tr} L + \left[-\frac{1}{2d} \operatorname{Tr}(L^2) + \frac{d-2r+1}{(d+2)(d-1)} \operatorname{Tr}(K^2) \right] + O(\|L\|^3).$$

Proof. Before beginning, note from the definition of Λ_r that if U is an orthogonal transformation, $\Lambda_r(U^T L U) = \Lambda_r(L)$. Consequently, if α_i is the ith term in the Taylor expansion of Λ_r, then α_i is invariant under conjugation by isometries.

The map Λ_r is smooth, so it admits a Taylor expansion:

$$\Lambda_r(L) = \alpha_1(L) + \alpha_2(L) + O(\|L\|^3),$$

where α_1 is linear in L and α_2 is quadratic in L. The rest of the proof is a calculation of α_1 and α_2. Before we begin this calculation we describe the approach. In each case, we reduce to the case of a symmetric matrix L. Then restricted to symmetric matrices, we diagonalize. There are few linear or quadratic maps from $\operatorname{End}(\mathbb{R}^n)$ to $\mathbb{R}$ that are invariant under conjugation by an orthogonal matrix. We then write α_i as a linear combination of such invariant maps from $\operatorname{End}(\mathbb{R}^n)$ to $\mathbb{R}$ and then solve for the coefficients of this linear combination.

We begin by calculating α_1.

Claim 4. With notation as above,

$$\alpha_1(L) = \frac{r}{d} \operatorname{Tr} L.$$

Proof. Let $\widetilde{\Lambda}_r(\operatorname{Id} + L) = \Lambda_r(L)$. Then from the definition, note that if U is an isometry then $\widetilde{\Lambda}_r(U(\operatorname{Id} + L)) = \widetilde{\Lambda}_r((\operatorname{Id} + L)U)) = \Lambda_r(L)$. Suppose that O_t is some path tangent to $O(n) \subset \operatorname{End}(\mathbb{R}^n)$ such that $O_0 = \operatorname{Id}$. Then $\widetilde{\Lambda}_r(O_t) = 0$. Write $O_t = \operatorname{Id} + tS + O(t^2)$ where S is skew symmetric. Then we see that

$$\widetilde{\Lambda}_r(\operatorname{Id} + tS + O(t^2)) = O(t^2),$$

So, $\Lambda_r(tS) = O(t^2)$. Hence α_1 vanishes on skew symmetric matrices.

Thus it suffices to evaluate α_1 restricted to symmetric matrices. Suppose that A is a symmetric matrix, then there exists an orthogonal matrix U so that $U^T A U$ is diagonal. Restricted to the space of diagonal matrices, which we identify with $\mathbb{R}^d$ in the natural way, observe that $\alpha_1 \colon \mathbb{R}^d \to \mathbb{R}$ is invariant under permutation of the coordinates in $\mathbb{R}^d$ because it is invariant under conjugation by isometries. There is a one dimensional space of maps having this property, and it is spanned by the trace, Tr. So, $\alpha_1(A) = \alpha_1(U^T A U) = a_1 \operatorname{Tr}(A)$ for some constant C. To compute the constant c it suffices to consider a specific matrix, e.g. $A = \operatorname{Id}$.

$$\begin{aligned}
\alpha_1(\operatorname{Id}) &= \frac{d}{d\epsilon} \int \ln \det(\operatorname{Id} + \epsilon \operatorname{Id} \mid E) \, dE \\
&= \frac{d}{d\epsilon} \int \ln(\operatorname{Id} + \epsilon)^r \, dE \\
&= \frac{d}{d\epsilon} r \ln(1 + \epsilon) \\
&= r.
\end{aligned}$$

So, $a_1 = r/d$. Thus for $L \in \mathrm{End}(\mathbb{R}^d)$, $\alpha_1(L) = \frac{r}{d}\operatorname{Tr}((L+L^T)/2) = \frac{r}{d}\operatorname{Tr}(L)$. □

We now compute α_2.

Claim 5. With notation as in the statement of Proposition 19,

$$\alpha_2(L) = -\frac{r}{2d}\operatorname{Tr}(L^2) + \frac{r(d-r)}{(d+2)(d-1)}\operatorname{Tr}(K^2).$$

Proof. Let $\widetilde{\Lambda}_r(\mathrm{Id}+L) = \Lambda_r(L)$. From the definition, note that for an isometry U, that $\widetilde{\Lambda}_r((\mathrm{Id}+L)U) = \widetilde{\Lambda}_r(U)$. Fix L and let $J = (L-L^T)/2$. Observe that

$$(\mathrm{Id}+L)e^{-J} = \mathrm{Id}+(L-J)+(J^2/2-LJ)+O(|L|^3).$$

Thus we see that

$$\begin{aligned}\Lambda_r(L) &= \widetilde{\Lambda}_r(\mathrm{Id}+L)\\ &= \widetilde{\Lambda}_r((L+\mathrm{Id})e^{-J})\\ &= \widetilde{\Lambda}_r(\mathrm{Id}+(L-J)+(J^2/2-LJ)+O(|L|^3))\\ &= \Lambda_r((L-J)+(J^2/2-LJ))+O(|L|^3).\end{aligned}$$

Now comparing the two Taylor expansions of $\widetilde{\Lambda}_r(\mathrm{Id}+L)$, we find:

$$\alpha_2(L) = \alpha_2(L-J)+\alpha_1(J^2/2-LJ).$$

Thus as we have already determined α_1:

$$\alpha_2(L) = \alpha_2((L+L^T)/2)+\frac{r}{d}\operatorname{Tr}(J^2/2-LJ).$$

So, we are again reduced to the case of a symmetric matrix S. In fact, by invariance of α_2 under conjugation by isometries, we are reduced to determining α_2 on the space of diagonal matrices. Identify $\mathbb{R}^d$ with diagonal matrices as before. We see that α_2 is a symmetric polynomial of degree 2 in d variables. The space of such polynomials is spanned by $\sum x_i^2$ and $\sum_{i,j} x_i x_j$. It is convenient to observe that for a diagonal matrix, D, $\operatorname{Tr}(D^2)$ and $\operatorname{Tr}(D)^2$ span this space as well. Hence

$$\alpha_2(S) = b_1 \operatorname{Tr}(S)^2 + b_2 \operatorname{Tr}(S^2)$$

Now in order to calculate b_1 and b_2 we will explicitly calculate $\alpha_2(\mathrm{Id})$ and $\alpha_2(P)$, where P is the orthogonal projection onto a coordinate axis.

In the first case,

$$2\alpha_2(\mathrm{Id}) = \frac{d}{d\epsilon_1}\frac{d}{d\epsilon_2}\int_{\mathbb{G}_{r,d}} \ln\det((\mathrm{Id}+\epsilon_1+\epsilon_2) \mid E)\, dE \,|_{\epsilon_1=0,\epsilon_2=0} = \frac{d^2}{d\epsilon}\ln(1+\epsilon)^r \,|_0 = -r.$$

So, $\alpha_2(\mathrm{Id}) = -r/2$.

Next suppose that P is projection onto a fixed vector e. Suppose that $\angle(e,E)=\theta$. We now compute $\ln\det(\mathrm{Id}+\epsilon P \mid E)$. We fix a useful basis of E. Let v be a unit vector making angle $\angle(e,E)$ with e. Then let $e_2, ..., e_r$ be unit vectors in E that are orthogonal to e and v. Then using the basis $v, e_2, ..., e_r$, we see that

$$\det(\mathrm{Id}+\epsilon P \mid E) = \frac{\|(\mathrm{Id}+\epsilon P)v \wedge (\mathrm{Id}+\epsilon P)e_2 \wedge \cdots \wedge (\mathrm{Id}+\epsilon P)e_r\|}{\|v \wedge \cdots \wedge e_r\|} = \sqrt{\langle (\mathrm{Id}+\epsilon P)v, (\mathrm{Id}+\epsilon P)v\rangle},$$

by considering the determinant defining the wedge product. But then as $Pv = \cos(\theta)e$,

$$\sqrt{\langle v+\epsilon\cos(\theta)e, v+\epsilon\cos(\theta)e\rangle} = \sqrt{\langle v,v\rangle + 2\epsilon\cos\theta\langle v,e\rangle + \epsilon^2\langle Pv,Pv\rangle} = \sqrt{1+2\epsilon\cos^2\theta+\epsilon^2\cos^2\theta}.$$

Now, the Taylor approximation for $\ln\sqrt{1+x}$ at $x=0$ is $x/2 - x^2/4 + O(x^3)$, so

$$\ln\det(\mathrm{Id} + \epsilon P \mid E) = \epsilon\cos\angle(E,e) + \epsilon^2\left[\frac{\cos\angle(E,e)}{2} - \cos^4\angle(E,e)\right] + O(\epsilon^3).$$

Hence, as this estimate is uniform over E, by integrating,

$$\int_{\mathbb{G}_{r,d}} \ln\det(\mathrm{Id} + \epsilon P \mid E)\, dE = \epsilon\int_{\mathbb{G}_{r,d}} \cos^2\angle(E,e)\, dE + \epsilon^2\int_{\mathbb{G}_{r,d}}\left[\frac{\cos^2\angle(E,e)}{2} - \cos^4\angle(E,e)\right] dE + O(\epsilon^3).$$

So, we are reduced to calculating the coefficient of ϵ^2 in the above expression. One may rewrite the above integrals in the following manner, by definition of the Haar measure as $\mathbb{G}_{r,d}$ is a homogeneous space of $\mathrm{SO}(d)$. Write $x_1, ..., x_d$ for the restriction of the Euclidean coordinates to the sphere. By fixing the coordinate plane $E_0 = \langle e_1, ..., e_r\rangle$, and letting $\theta = \angle((x_1, ..., x_d), E)$ we then have that $\cos(\theta) = \sqrt{\sum_{i=1}^r x_i^2}$. Thus

$$\begin{aligned}
\int_{\mathbb{G}_{r,d}} \cos^2\angle(E,e)\, dE &= \int_{\mathrm{SO}_d} \cos^2\angle(gE_0, e)\, dg \\
&= \int_{\mathrm{SO}_d} \cos^2\angle(E_0, ge)\, dg \\
&= \int_{S^{d-1}} \cos^2\angle(E_0, x)\, dx \\
&= \int_{S^{d-1}} \sum_{i=1}^r x_i^2\, dx,
\end{aligned}$$

Similarly, fixing the plane $E_0 = \langle e_1, ..., e_r\rangle$, we see that as $\cos^4\angle(E_0, x) = \left(\sum_{i=1}^r x_i^2\right)^2$

$$\int_{\mathbb{G}_{r,d}} \cos^4\angle(E,e) = \int_{S^{d-1}} \left(\sum_{i=1}^r x_i^2\right)^2 dx.$$

The evaluation of these integrals is immediate by using the following standard formulas:

$$\int_{S^{d-1}} x_1^2\, dx = \frac{1}{d}, \quad \int_{S^{d-1}} x_1^4\, dx = \frac{3}{d(d+2)}, \quad \int_{S^{d-1}} x_1^2 x_2^2\, dx = \frac{1}{d(d+2)}.$$

Thus we see that

$$\int_{\mathbb{G}_{r,d}} \frac{\cos^2 \angle(E,e)}{2} - \cos^4 \angle(E,e)\, dE = \frac{r}{2d} - \frac{r(r+2)}{d(d+2)}.$$

Thus

$$\alpha_2(P) = \frac{r}{2d} - \frac{r(r+2)}{d(d+2)}.$$

Returning to b_1, b_2, the coefficients of $(\mathrm{Tr}(S))^2$ and $\mathrm{Tr}(S^2)$, respectively, combining the cases of Id and P gives

$$-\frac{r}{2} = b_1 d^2 + b_2 d.$$

and

$$\frac{r}{2d} - \frac{r(r+2)}{d(d+2)} = b_1 + b_2.$$

We can now solve for b_1 and b_2 with respect to this basis of the space of conjugation invariant quadratic functionals. However, the computation will be more direct if instead we we use a different basis and write write $\alpha_2(S)$ as

$$b_1(\mathrm{Tr}(S))^2 + b_2 \,\mathrm{Tr}(\left(S - \frac{\mathrm{Tr}\, S}{d}\right)^2),$$

so that the second term is trace 0. Our computations from before now show that:

$$-\frac{r}{2} = b_1 d^2 + 0,$$

and

$$\frac{r}{2d} - \frac{r(r+2)}{d(d+2)} = b_1 + \frac{d-1}{d} b_2 \left(= b_1(\mathrm{Tr}(P))^2 + b_2 \,\mathrm{Tr}\left((P - \frac{\mathrm{Tr}\, P}{d}\,\mathrm{Id})^2\right)\right).$$

The first equation implies that

$$b_1 = -\frac{r}{2d^2},$$

The left hand side of the second equation of the pair is equal to

$$\frac{r(d-r)}{d(d+2)} - \frac{r}{2d}.$$

This gives

$$b_2 = \frac{r(d-r)}{(d-1)(d+2)} - \frac{r}{2d}.$$

So, for symmetric L, we have

$$\alpha_2(S) = \frac{-r}{2d^2}(\operatorname{Tr}(S))^2 + \left(\frac{r(d-r)}{(d-1)(d+2)} - \frac{r}{2d}\right)\operatorname{Tr}((S - \frac{\operatorname{Tr} S}{d}\operatorname{Id})^2). \tag{3.117}$$

Recall that we specialized to the case of a symmetric matrix, and that for a non-symmetric matrix there is another term. For $L \in \operatorname{End}\mathbb{R}^d$, setting $J = (L - L^T)/2$, as before,

$$\alpha_2(L) = \alpha_2\left(\frac{L + L^T}{2}\right) + \frac{r}{d}\operatorname{Tr}\left(\frac{J^2}{2} - LJ\right).$$

To simplify this we compute that:

$$\begin{aligned}\operatorname{Tr}\left(\frac{J^2}{2} - LJ\right) &= \operatorname{Tr}\left(\frac{L^2 - LL^T - L^TL + (L^T)^2}{8} - L\frac{L - L^T}{2}\right)\\ &= \operatorname{Tr}\left(\frac{LL^T - L^2}{4}\right).\end{aligned}$$

Write

$$S = \frac{L + L^T}{2}.$$

Observe that for an arbitrary matrix X, $\operatorname{Tr}((X - (\operatorname{Tr} X)/d\operatorname{Id})^2) = \operatorname{Tr}(X^2) - (\operatorname{Tr}(X))^2/d$.

Thus

$$\begin{aligned}
&- \frac{r}{2d^2}\,(\mathrm{Tr}(S))^2 - \frac{r}{2d}\,\mathrm{Tr}(\big(S-(\mathrm{Tr}\,S)/d\,\mathrm{Id})^2\big) + \frac{r}{d}\,\mathrm{Tr}(\frac{LL^T-L^2}{4}) \\
&= -\frac{r}{2d^2}\,(\mathrm{Tr}(S))^2 - \frac{r}{2d}\,\big(\mathrm{Tr}(S^2)\big) - \frac{-r}{2d^2}(\mathrm{Tr}(S))^2 + \frac{r}{d}\left(\mathrm{Tr}\left(\frac{LL^T-L^2}{4}\right)\right) \\
&= -\frac{r}{2d}\,\big(\mathrm{Tr}(S^2)\big) + \frac{r}{d}\left(\mathrm{Tr}\left(\frac{LL^T-L^2}{4}\right)\right) \\
&= \frac{r}{d}\left[\frac{-1}{2}\,\mathrm{Tr}(((L+L^T)/2)^2) + \mathrm{Tr}(\frac{LL^T-L^2}{4})\right] \\
&= \frac{r}{d}\left[\frac{-1}{2}(\mathrm{Tr}(\frac{L^2+(L^T)^2+2LL^T}{4})) + \mathrm{Tr}(\frac{LL^T-L^2}{4})\right] \\
&= -\frac{r}{2d}\,\mathrm{Tr}(L^2).
\end{aligned}$$

From before, we have that

$$\alpha_2(L) = -\frac{r}{2d^2}(\mathrm{Tr}(S))^2 + \left(\frac{r(d-r)}{(d-1)(d-2)} - \frac{r}{2d}\right)\mathrm{Tr}((S-\frac{\mathrm{Tr}\,S}{d}\,\mathrm{Id})^2) + \frac{r}{d}\,\mathrm{Tr}(\frac{LL^T-L^2}{4}).$$

So substituting the previous calculation we obtain:

$$\alpha_2(L) = -\frac{r}{2d}\,\mathrm{Tr}(L^2) + \left(\frac{r(d-r)}{(d-1)(d-2)}\right)\mathrm{Tr}\left(\left(\frac{L+L^T}{2} - \frac{\mathrm{Tr}\,L}{d}\,\mathrm{Id}\right)^2\right),$$

which is the desired formula. □

We have now calculated α_1 and α_2. This concludes the proof of Proposition 19. □

We will also use a first order Taylor expansion as well with respect to the metric.

Proposition 20. *Let $\Lambda_r(G)$ be defined for symmetric matrices G by*

$$\Lambda_r(G) \coloneqq \int_{\mathbb{G}_{r,d}} \ln\det(\mathrm{Id},\mathrm{Id},\mathrm{Id}+G \mid E)\,dE.$$

Then $\Lambda_r(G)$ admits the following Taylor development:

$$\Lambda_r(G) = \frac{r}{2d} \operatorname{Tr} G + O(\|G\|^2).$$

Proof. The proof of this proposition is substantially similar to that of the previous proposition. Let α_1 denote the first term in the Taylor expansion. Note that if U is an isometry that $\Lambda_r(U^T G U) = \Lambda_r(G)$. Thus α_1 is invariant under conjugation by isometries. Thus by conjugating by an orthogonal matrix, we are reduced to the case of G and diagonal matrix. As before, we see that $\alpha_1(D)$ is a multiple of $\operatorname{Tr}(D)$ as Tr spans the linear forms on $\mathbb{R}^d$ that are invariant under permutation of coordinates.

Thus it suffices to calculate the derivative in the case of $D = \mathrm{Id}$. So, we see that

$$\alpha_1(\mathrm{Id}) = \frac{d}{d\epsilon} \int_E \ln \det(\mathrm{Id}, \mathrm{Id}, \mathrm{Id} + \epsilon \, \mathrm{Id} \mid E) \, dE.$$

Thus the integral is equal to $\ln \sqrt{(1+\epsilon)^r}$ on every plane E. Thus the derivative is $r/2$ and so

$$\alpha_1(\mathrm{Id}) = \frac{r}{2} = \frac{r}{2d} \operatorname{Tr}(\mathrm{Id}).$$

And so the result follows. □

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