THE UNIVERSITY OF CHICAGO

SELF-SIMILARITY OF ZIGGURAT FRINGES AND RIGIDITY OF EXTREMAL FREE GROUP ACTIONS ON THE CIRCLE

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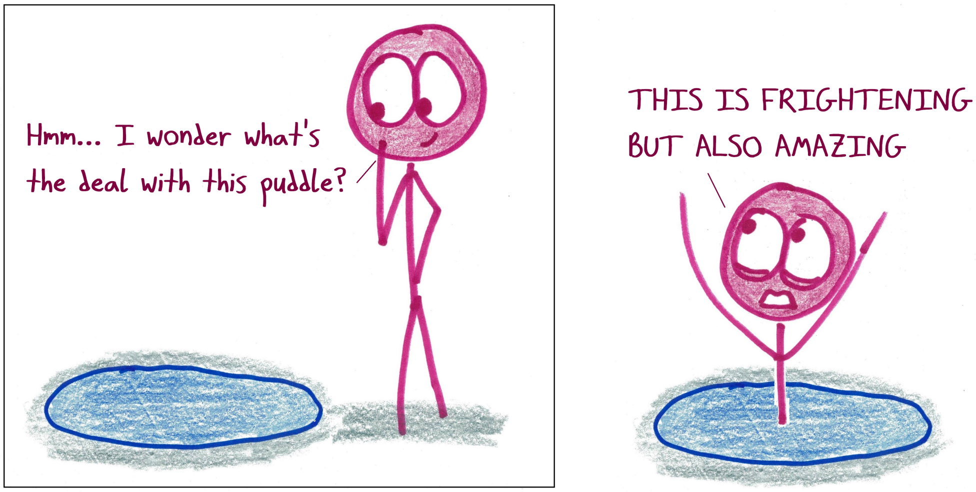
To my parents, who believed in me.



There are no puddles in math, just oceans in disguise.

- Ben Orlin

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT

We use combinatorial and number theoretic techniques to establish several new rigidity and rationality results in the area of nonabelian group actions on the circle. We show that Calegari-Walker *ziggurats* -- i.e. the graphs of extremal rotation numbers associated to positive words in free groups -- have projectively self-similar boundary and satisfy a power law for maximal regions of stability, by giving an explicit formula in a certain range. We give bounds on the complexity of the algorithm used to evaluate the formula and give other bounds characterizing the non-linearity of the extremal representations in some specific cases not at the boundary. Additionally, we establish certain sufficiency criteria for rationality of extremal rotation number in the general case of semi-positive and arbitrary words, using tools from one dimensional dynamics and theory of Diophantine approximations.

CHAPTER 1

INTRODUCTION

1.1 Statement of results

Let $G = \mathrm{Homeo}_+(S^1)$ denote the group of orientation preserving homeomorphisms of the circle, and let Γ be a finitely generated group. One of the fundamental question in geometry and topology is to study the structure of G by looking into the representation space $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))$ for different Γ. For a Lie group G, this can be done by analyzing the quotient space $\mathrm{Hom}(\Gamma, G)/G$ and studying *character varieties*. However, since $\mathrm{Homeo}_+(S^1)$ is not a Lie group, we need to find the appropriate analog for characters. To that end we use $\mathrm{rot}^\sim : \mathrm{Homeo}_+^\sim(S^1) \to \mathbb{R}$, Poincaré's (real-valued) rotation number, also known as translation number, which is semi-conjugacy invariant on representation classes $\rho : \Gamma \to G$ (see e.g. [6]).

We are particularly interested in the case when $\Gamma = F$, a free group on two generators a and b. Let w be a word in F. Following Calegari and Walker, we would like to examine the constraints satisfied by $\mathrm{rot}^\sim(\rho(w))$ over all representations ρ as above when we fix the rotation number of the image of the generators. When w is in the semigroup generated by a and b (such a w is said to be *positive*), it turns out that the maximum such $\mathrm{rot}^\sim$ is enough to provide a complete picture of the set of all possible rotation numbers for $\rho(w)$. In particular, consider the case when one of the $\mathrm{rot}^\sim(\rho(a))$ or $\mathrm{rot}^\sim(\rho(b))$ approaches 1 (from below), so that the action is almost conjugate to a translation, and it commutes with the action by the other generator. Let $h_a(w)$ and $h_b(w)$ be the number of a's and b's respectively in w.

Definition 1.1.1 (Fringe)**.** The *fringe* associated to w and a rational number $0 \leq p/q < 1$ is the set of $0 \leq t < 1$ for which there is a homomorphism ρ from F to $\mathrm{Homeo}_+^\sim(S^1)$ with $\mathrm{rot}^\sim(\rho(a)) = p/q$, $\mathrm{rot}^\sim(\rho(b)) = t$ and $\mathrm{rot}^\sim(\rho(w)) = h_a(w)p/q + h_b(w)$.

Technically, this should be called the *left fringe* and the *right fringe* is defined symmetrically by exchanging a and b.

Calegari and Walker show that there is some least rational number $s \in [0,1)$ so that the fringe associated to w and to p/q is equal to an interval of the form $[s,1)$. The *fringe length*, denoted $\mathrm{fr}_w(p/q)$, is defined to be equal to $1-s$. The purpose of this work is to study these fringes by giving an explicit formula for the *fringe length* and establishing its various self-similarity properties. Additionally, we prove some partial results that shed light on the structure of the graph of extremal rotation numbers when not near the fringes, and finally we try to discuss the problem in the case when w is not necessarily positive.

Our main result in chapter 4 is the following.

Fringe Formula 4.1.1. *If w is positive, and p/q is a reduced fraction, then*

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$$

where $\sigma_w(g)$ depends only on the word w and $g := \gcd(q, h_a(w))$. Moreover, $g \cdot \sigma(g)$ is an integer.

In chapter 5, we give specific bounds on the function $\sigma_w(g)$ and discuss its properties in some specific cases.

σ-inequality 5.1.1. *Suppose $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \dots a^{\alpha_n} b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b(w)}{h_a(w)} \leq \sigma_w(g) \leq \max_{1 \leq i \leq n} \beta_i$$

Moreover, $\sigma_w(g) = \dfrac{h_b(w)}{h_a(w)}$ when $h_a(w)$ divides q, and $\sigma_w(g) = \max \beta_i$ when q and $h_a(w)$ are coprime.

We also give some self-similarity and stability results for the fringes in the specific case of the example $w = abaab$ in chapter 5. In particular, we prove the following.

Theorem 5.3.4.

$$\mathrm{fr}_{abaab}(p/q) = \begin{cases} 1/q & \text{when } 3 \nmid q \\ 3/2q & \text{when } 3 \mid q \end{cases}$$

Furthermore, let

$$R(w;r,s) = \max\{\mathrm{rot}^{\sim}(\rho(w))\,|\,\mathrm{rot}^{\sim}(\rho(a)) = r, \mathrm{rot}^{\sim}(\rho(b)) = s\}.$$

If $(3,q) = 1$*, then*

$$R\left(abaab; t, 1 - \frac{1}{q}\right) \text{ is constant} \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q^2}\right).$$

If $(3,q) \neq 1$*, then*

$$R\left(abaab; t, 1 - \frac{3}{2q}\right) \text{ is constant} \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{2q^2}\right).$$

1.2 Known results and motivation

To give some context to this work, we explore some related areas of mathematics where similar results have been proved. Consider first the case of a Lie group G. Recovering a representation from a character is not always straightforward. Given a (finite) subset S of Γ, it becomes an interesting and subtle question to ask what conditions are satisfied by the values of a character on S. For example, the (multiplicative) Horn problem poses the problem of determining the possible values of the spectrum of the product AB of two unitary matrices given the spectra of A and B individually. It can be show that there is a map

$$\Lambda : SU(n) \times SU(n) \to \mathbb{R}^{3n}$$

taking A, B to the logarithms of the spectra of A, B and AB (suitably normalized). Agnihotri-Woodward [1] and Belkale [2] proved that the image of this map is a convex polytope, and explicitly described what it is.

When Γ is a surface group Γ_g and $G \subset \mathrm{Homeo}_+(S^1)$ is a transitive Lie group, the Milnor-Wood inequality and the work of W. Goldman gives a complete description of the structure of

$\mathrm{Hom}(\Gamma_g, G)$. In case when $G = \mathrm{Homeo}_+(S^1)$, K. Mann [10] gives a characterization of *geometric* representations in $\mathrm{Hom}(\Gamma_g, G)$ with rotation number $\frac{2g-2}{k}$ using k—fold central extensions of the group $PSL(2, \mathbb{Z})$.

Similarly, we would also like to be able to provide some sort of rigidity and stability results in the universal case $\Gamma = F$, a free group, by studying the values $x_i := \mathrm{rot}^{\sim}(\rho(w_i))$ for finitely many $w_i \in \Gamma$ on a common representation ρ. To that extent, we give explicit formula for certain *phase-locked* regions in the graphs of extremal rotation numbers associated to positive words in free groups, known as *ziggurats*. These formulae reveal (partial) integral projective self-similarity in ziggurat fringes, which are low-dimensional projections of characteristic polyhedra on the bounded cohomology of free groups.

1.3 Outline

We begin with some background material on rotation number and idea of ziggurats. The primary tool used in the proof of theorem 4.1.1 is an algorithm by Calegari-Walker that we explain in chapter 3. This chapter also outlines how the dynamical problem transforms to a combinatorial one using properties of rational rotation numbers.

The proof of theorem 4.1.1 is done in chapter 4. We start by observing that the Stairstep algorithm in § 3.2 reduces to a single linear programming problem due to the constraint that $\mathrm{rot}^{\sim}(b) \to 1^-$. In particular, we can find a unique solution to the optimization problem at the end of chapter 3, which using a result by Kaplan, and by some modular arithmetic produces the explicit formula.

In chapter 5, we elaborate on the formula 5.1.1 and give bounds on the constant σ. We also observe that by definition, a finite calculation giving a table of values of σ produces a complete list of fringe lengths for a fixed word. We prove the sharpness of the bounds by showing that equality occurs in specific cases, e.g. when h_a is a prime number. Finally in section 5.3, we analyze the special case of the word $abaab$ and besides the fringe length, we also try to estimate the size of the stability region in the other axis direction near the fringes.

The structure of fringe formula 4.1.1 automatically implies some sort of 'periodicity' in the fringes. In chapter 6, we give formula for the (partial) integral self-similarity in the fringes. As a related result, in section 6.2 we also give a bound on the height of the Ziggurat when not near the fringes, in a special case.

In chapter 7, we try to attempt to provide evidence towards a conjecture Calegari-Walker regarding the rationality properties of the maximal rotation number associated to arbitrary (i.e. not necessarily positive) words. In particular, we define a dynamical problem called the *interval game* related to the conjecture and try to find necessary and sufficient winning conditions for the same.

CHAPTER 2

BACKGROUND

2.1 Rotation number

Let $\mathrm{Homeo}_+(S^1)$ denote the group of orientation preserving homeomorphisms of the Circle. Consider the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_+^{\sim}(S^1) \to \mathrm{Homeo}_+(S^1) \to 0$$

whose center is generated by unit translation $z : p \to p + 1$.

Poincaré defined the *rotation number* $\mathrm{rot} : \mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$ as follows. First, define a function $\mathrm{rot}^{\sim} : \mathrm{Homeo}_+^{\sim}(S^1) \to \mathbb{R}$ by

$$rot^{\sim}(g) = \lim_{n\to\infty} \frac{g^n(x)}{n}$$

Note that this limit always exists, and is independent of the choice of the point $x \in S^1$. By definition, $\mathrm{rot}^{\sim}(gz^n) = \mathrm{rot}^{\sim}(g) + n$ for any integer n, so that $\mathrm{rot}^{\sim}$ descends to a well-defined function $\mathrm{rot} : \mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$.

We will use the following property of the rotation number function extensively in the later chapters. A homeomorphism $f \in \mathrm{Homeo}_+(S^1)$ has a periodic point of period n if and only if $\mathrm{rot}(f)$ is a rational number of the form m/n for some $m \in \mathbb{Z}$.

Our goal is to study the structure of $G = \mathrm{Homeo}_+^{\sim}(S^1)$ by looking into representations of a finitely generated group Γ into G. The function $\mathrm{rot}^{\sim}$ is semi-conjugacy invariant on G (which is not a Lie group) and can be thought of as an analog of a character in this context. Following Calegari-Walker [3] we would then like to understand what constraints are simultaneously satisfied by the value of $\mathrm{rot}^{\sim}$ on the image of a finite subset of Γ under a homomorphism to G. I.e. we study the values $x_i := \mathrm{rot}^{\sim}(\rho(w_i))$ for finitely many $w_i \in \Gamma$ on a common representation ρ.

2.2 Free groups, positive words, and ziggurats

The universal case to understand is that of a free group. Thus, let F be a free group with generators a and b, and for any element $w \in F$, let x_w be the function from conjugacy classes of representations $\rho : F \to \mathrm{Homeo}^{\sim}_{+}(S^1)$ to $\mathbb{R}$ which sends a representation ρ to $x_w(\rho) := \mathrm{rot}^{\sim}(\rho(w))$. The x_w are *coordinates* on the space of conjugacy classes of representations, and we study this space through its projections to finite dimensional spaces obtained from finitely many of these coordinates.

Definition 2.2.1. For any $w \in F$ and for any $r, s \in \mathbb{R}$ we define

$$X(w; r, s) = \{x_w(\rho) \mid x_a(\rho) = r, x_b(\rho) = s\}$$

The fact that set of representations with same rotation number is path-connected shows that $X(w; r, s)$ is a *compact* interval i.e. the extrema are achieved. By definition, it satisfies

$$X(w; r + m, s + n) = X(w; r, s) + mh_a(w) + nh_b(w)$$

where $h_a, h_b : F \to \mathbb{Z}$ count the signed number of copies of a and b respectively in each word.

If we define $R(w; r, s) = \max\{X(w; r, s)\}$ then $\min\{X(w; r, s)\} = -R(w; -r, -s)$. This is simply due to the fact that changing the orientation of the circle negates the rotation number. Thus all the information about $X(w; r, s)$ can be recovered from the function $R(w; \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$. In fact, by the observations made above, it suffices to restrict the domain of R to the unit square $[0, 1) \times [0, 1)$.

Definition 2.2.2. A word in F is *positive* if it is in the *semigroup* generated by a and b.

The theory developed in [3] is most useful when w is a positive word. In this case, $R(w; r, s)$ is lower semi-continuous, and monotone non-decreasing in both its arguments. Furthermore it is *locally constant* and takes *rational values* on an open and dense subset of $\mathbb{R}^2$. In fact,

Theorem 2.2.3 (Calegari-Walker [3] Thm. 3.4, 3.7). *Suppose w is positive (and not a power of a or b), and suppose r and s are rational. Then*

1. *$R(w; r, s)$ is rational with denominator no bigger than the smaller of the denominators of r and s; and*

2. *there is some $\epsilon(r, s) > 0$ so that $R(w; \cdot, \cdot)$ is constant on $[r, r + \epsilon) \times [s, s \times \epsilon)$.*

Furthermore, when r and s are rational and w is positive, Calegari-Walker give an explicit combinatorial algorithm to compute $R(w; r, s)$; it is the existence and properties of this algorithm that proves Theorem 2.2.3. Computer implementation of this algorithm allows one to draw pictures of the graph of R (restricted to $[0, 1) \times [0, 1)$) for certain short words w, producing a stairstep structure dubbed a *Ziggurat*; see Figure 2.1.

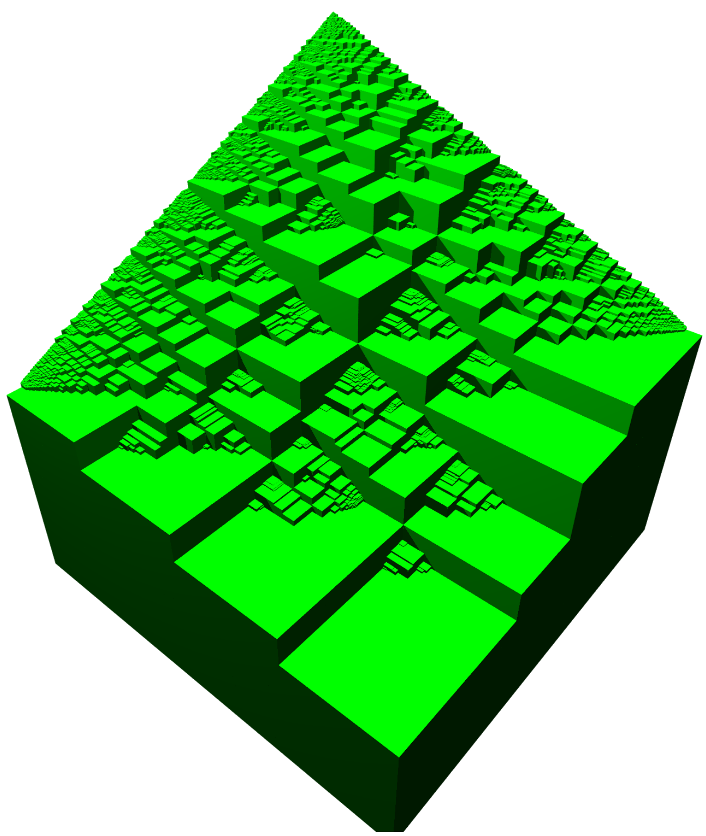

Figure 2.1: Graph of $R(abbbabaaaabbabb; \cdot, \cdot)$; colloquially, a *ziggurat*. Picture courtesy of Calegari-Walker.

In the special case of the word $w = ab$, a complete analysis can be made, and an explicit formula obtained for $R(ab;\cdot,\cdot)$. This case arose earlier in the context of the classification of taut foliations of Seifert fibered spaces, where the formula was conjectured by Jankins-Neumann [8] and proved by Naimi [11].

Theorem 2.2.4 (ab Theorem)**.** *For $0 \le r, s \le 1$, we have the formula*

$$R(ab;r,s) = \sup_{p_1/q \le r, p_2/q \le s} \frac{p_1 + p_2 + 1}{q}$$

But in *no other case* is any explicit formula known or even conjectured, and even the computation of $R(w;r,s)$ takes time which is an exponential function of the denominators of r and s.

2.3 Projective self-similarity and fringes

A Gordenko [7] gave a new analysis and interpretation of the ab formula, relating it to the Naimi formula in an unexpected way. Her formulation exhibits and explains an *integral projective self-similarity* of the ab-ziggurat, related to the theory of continued fractions, and the fact that the automorphism group of F_2 is $\mathrm{SL}(2,\mathbb{Z})$. Such global self-similarity is (unfortunately) not evident in ziggurats associated to other positive words; but there is a partial self-similarity (observed experimentally by Calegari-Walker and by Gordenko) in the *germ* of the ziggurats near the *fringes* where one of the coordinates r or s approaches 1 from below.

If we fix a positive word w and a rational number r, and (following [3]) we denote by $R(w;r,1-)$ the limit of $R(w;r,t)$ as $t \to 1$ from below, then the following can be proved:

Theorem 2.3.1 (Calegari-Walker [3], Prop. 3.15)**.** *If w is positive, and r is rational, there is a least rational number $s \in [0,1)$ so that $R(w;r,t)$ is constant on the interval $[s,1)$ and equal to $h_a(w)r + h_b(w)$.*

We refer to the number $1-s$ as in Theorem 2.3.1 (depending on the word w and the rational number r) as the *fringe length* of r, and denote it $\mathrm{fr}_w(r)$, or just by $\mathrm{fr}(r)$ if w is understood. In other words, $\mathrm{fr}_w(r)$ is the greatest number such that $R(w;r,1-\mathrm{fr}_w(r)) = h_a(w)r + h_b(w)$. More precisely, we should call this a "left fringe", where the right fringe should be the analog with the roles of the generators a and b interchanged.

To summarize, as $t \to 1$, the dynamics of F on S^1 is approximated better and better by a linear model. For t close to 1, the nonlinearity can be characterized by a perturbative model and fringes are the maximal regions where this perturbative model is valid. Our main theorem in chapter § 4 says that the fringe length, the size of this stability region follows a power law. This is a new example of (topological) nonlinear phase locking in 1-dimensional dynamics giving rise to a power law, of which the most famous example is the phenomenon of Arnol'd Tongues [5].

2.4 Arbitrary words and the interval game

Let us next consider the case of arbitrary (i.e. non-positive) words. The main problem that arises when we allowing w to contain a^{-1} or b^{-1} is that $R(w;r,s)$ is no longer a non-decreasing function. In fact Calegari and Walker show that there is a very strong restriction on the rotation number of a commutator.

Theorem 2.4.1 ([3], Example 4.9). *Let $w = aba^{-1}b^{-1}$.*

1. *If $r \notin \mathbb{Q}$ or $s \notin \mathbb{Q}$, then $\mathrm{rot}^\sim(aba^{-1}b^{-1}) = 0$ and hence $R(aba^{-1}b^{-1};r,s) = 0$.*

2. *If $r = p/q$, where $p/q \in \mathbb{Q}$ is in lowest terms, then $|\mathrm{rot}^\sim(aba^{-1}b^{-1})| \le 1/q$. If further $s = p'/q$ then $R(aba^{-1}b^{-1};p/q,p'/q) = 1/q$.*

These results show that we can't hope to get a ziggurat like picture in this case. Nonetheless, we can try to prove an analog of the Rationality theorem 2.2.3. In fact, the following is true for *semipositive* words i.e. words that either contain no a^{-1} or no b^{-1}.

Theorem 2.4.2 ([3]). *Let w be semipositive (without loss of generality, suppose it contains no a^{-1}). If r is rational, so is $R(w;r,s)$. Moreover, the denominator of R is no bigger than denominator of r.*

We would like to prove similar result in case of words $w \in [F,F]$. Calegari and Walker give the following strategy, in the form of a dynamical problem called *interval game*, to tackle the problem.

Definition 2.4.3. An *interval game* consists of a collection of elements from $\mathrm{Homeo}_+(S^1)$. We have one player ψ and a finite number of enemies $\phi_1, \phi_2, \ldots, \phi_m$. The goal is to find an *winning* interval $I \subset S^1$.

An interval $I \subset S^1$ *wins* if there exists some positive integer n such that

(i) $\psi^n(I_+)$ is in the interior of I. Here I_+ denotes the rightmost point of I.

(ii) $\psi^i(I)$ is disjoint from $\phi_j(I)$ for all $1 \le i \le n$, $1 \le j \le m$.

Assume (up to some conjugation) that w ends in a and let $w_1, w_2, \ldots, w_k$ be the suffixes of w that start with a^{-1}. Then taking $\phi_j = w_j^{-1}$ and $\psi = w$, and under the assumption that $\mathrm{rot}^\sim(\psi)$ is irrational, existence of a winning interval implies rationality of R (regardless of r and s). So we would ideally like to show that given arbitrary choice of $\psi, \phi_j \in \mathrm{Homeo}_+(S^1)$ with $\mathrm{rot}^\sim(w) \notin \mathbb{Q}$, we can always win in the interval game. In chapter § 7, we discuss necessary and sufficient conditions for existence of an winning interval in several scenarios.

CHAPTER 3

STAIRSTEP ALGORITHM

3.1 Dynamics using XY words

Consider a positive word W, and let $r = p_1/q_1$, $s = p_2/q_2$ are rational and expressed in reduced form. Theorem 2.2.3 says that $R(w; p_1/q_1, p_2/q_2)$ is rational, with denominator no bigger than $\min(q_1, q_2)$. Following [3], we present the Calegari-Walker algorithm to compute $R(w; p_1/q_1, p_2/q_2)$. The main idea is that since the rotation number essentially encodes the cyclic combinatorial order of the orbits in the circle, we can find $R(w; p_1/q_1, p_2/q_2)$ using purely combinatorial methods.

Definition 3.1.1 (XY-word)**.** An XY-word of *type* (q_1, q_2) is a cyclic word in the 2-letter alphabet X, Y of length $q_1 + q_2$, with a total of q_1 X's and q_2 Y's.

If W is an XY-word of type (q_1, q_2), we let W^∞ denote the bi-infinite string obtained by concatenating W infinitely many times, and think of this bi-infinite word as a function from $\mathbb{Z}$ to $\{X, Y\}$; we denote the image of $i \in \mathbb{Z}$ under this function by W_i, so that each W_i is an X or a Y, and $W_{i+q_1+q_2} = W_i$ for any i.

We define an action of the *semigroup* generated by a and b on $\mathbb{Z}$, associated to the word W (see Figure 3.1). The action is given as follows. For each integer i, we define $a(i) = j$ where j is the least index such that the sequence $W_i, W_{i+1}, \cdots, W_j$ contains exactly $p_1 + 1$ X's. Similarly, $b(i) = j$ where j is the least index such that the sequence $W_i, W_{i+1}, \cdots, W_j$ contains exactly $p_2 + 1$ Y's. Note that this means $W_{a(i)}$ is always an X and respectively $W_{b(i)}$ is always Y. We can then define

$$\mathrm{rot}_W^\sim(w) = \lim_{n\to\infty} \frac{w^n(1)}{n \cdot (q_1 + q_2)}.$$

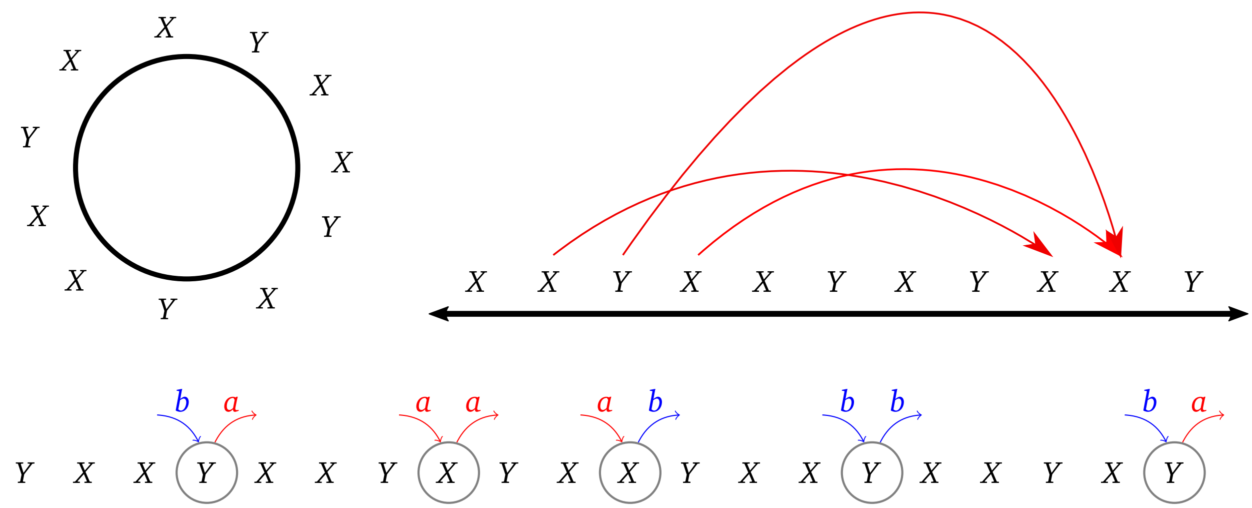

Figure 3.1: Action of a and b on W

Proposition 3.1.2 (Calegari-Walker formula)**.** *With notation as above, there is a formula*

$$R(w;p_1/q_1,p_2/q_2) = \max_W\{\mathrm{rot}^{\sim}_W(w)\}$$

where the maximum is taken over the finite *set of XY-words W of type (q_1,q_2).*

Evidently, each $\mathrm{rot}^{\sim}_W(w)$ is rational, with denominator less than or equal to $\min(q_1,q_2)$, proving the first part of Theorem 2.2.3. Though theoretically interesting, a serious practical drawback of this proposition is that the number of XY-words of type (q_1,q_2) grows exponentially in the q_i.

3.2 Stairstep theorem

Theorem 3.2.1 (Calegari-Walker [3], Thm. 3.11)**.** *Let w be a positive word, and suppose p/q and c/d are rational numbers so that c/d is a value of $R(w;p/q,\cdot)$. Then*

$$u := \inf\{t \;:\; R(w;p/q,t) = c/d\}$$

is rational, and $R(w;p/q,u) = c/d$.

The theorem is proved by giving an algorithm (the Stairstep Algorithm) to compute u and analyzing its properties. Note that the fringe length $\mathrm{fr}_w(p/q)$ is the value of $1-u$ where u is the output of the Stairstep Algorithm for $c/d = h_a(w)p/q + h_b(w)$. Observe that, whereas Theorem 2.3.1 proved the existence of a fringe length, this theorem proves that the length is in fact a rational number. We now explain this algorithm.

3.2.1 Reformulation using XY-words

Since R is monotone non-decreasing in both of its arguments, it suffices to prove that

$$\inf\{t \,:\, R(w; p/q, t) \geq c/d\} \tag{3.1}$$

is rational, and the infimum is achieved. Also, since R is locally constant from the right at rational points, it suffices to compute the infimum over rational t. So consider some $t = u/v$ (in lowest terms) such that $R(w; p/q, u/v) \geq c/d$. In fact, let W be a XY word of type (q, v) for which $R(w; p/q, u/v) = \mathrm{rot}^{\sim}_{W}(w)$. After some cyclic permutation, we can write

$$W = Y^{t_1} X Y^{t_2} X Y^{t_3} X \dots Y^{t_q} X$$

where $t_i \geq 0$ and $\sum_{i=1}^{q} t_i = v$. Our goal is then to minimize u/v over all such possible XY-words W.

After some circular permutation (which does not affect R), we may also assume without loss of generality that w is of the form

$$w = b^{\beta_n} a^{\alpha_n} \dots b^{\beta_2} a^{\alpha_2} b^{\beta_1} a^{\alpha_1}$$

where $\alpha_i, \beta_i > 0$. Also, assume that equality is achieved in (3.1) for u/v i.e. $R(w; p/q, t) = c/d$. Thus by construction, the action of w on W, defined via its action on $\mathbb{Z}$, is periodic with a period d, and a typical periodic orbit begins at $W_1 = Y$.

We fix some notations and try to analyze the action of each maximal string of a or b in w on W by inspecting its action on $\mathbb{Z}$. Note that, for

$$\tilde{s}_i = a^{\alpha_i} b^{\beta_{i-1}} a^{\alpha_{i-1}} \cdots b^{\beta_1} a^{\alpha_1}(1),$$

the $\tilde{s}_i$'th letter in W^∞ is always X. Let s_i be the index modulo q so that $W^\infty_{\tilde{s}_i}$ is the s_i'th X in W(cf. Figure 3.2). Thus for a periodic orbit starting at $W_1 = Y$, the string b^{β_i} is applied to the s_i'th X.

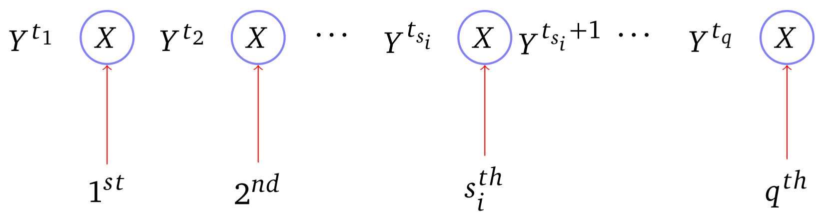

Figure 3.2: The XY word of type (q, ν).

Then by definition, $b^{\beta_i}(\tilde{s}_i)$ is the least number such that the sequence $W_{\tilde{s}_i}, W_{\tilde{s}_i+1}, \cdots, W_{b^{\beta_i}(\tilde{s}_i)}$ contains exactly $u\beta_i + 1$ Y's. Let l_i denote the number of X's in the sequence $W_{\tilde{s}_i}(=X), W_{\tilde{s}_i+1}, \cdots, W_{b^{\beta_i}(\tilde{s}_i)}(=Y)$(cf. Figure 3.3). Thus l_i is the smallest number such that

$$t_{s_i+1} + t_{s_i+2} + \ldots + t_{s_i+l_i+1} \geq u\beta_i + 1 \tag{3.2}$$

In other words, l_i is the biggest number such that

$$t_{s_i+1} + t_{s_i+2} + \ldots + t_{s_i+l_i} \leq u\beta_i \tag{3.3}$$

The purpose of rewriting this inequality was to make it homogeneous. Even if equality does not occur in (3.1), the inequality in (3.2) still holds true. The only difference is that l_i does not necessarily have to be the smallest number, however it does have to satisfy other constraints which we now describe.

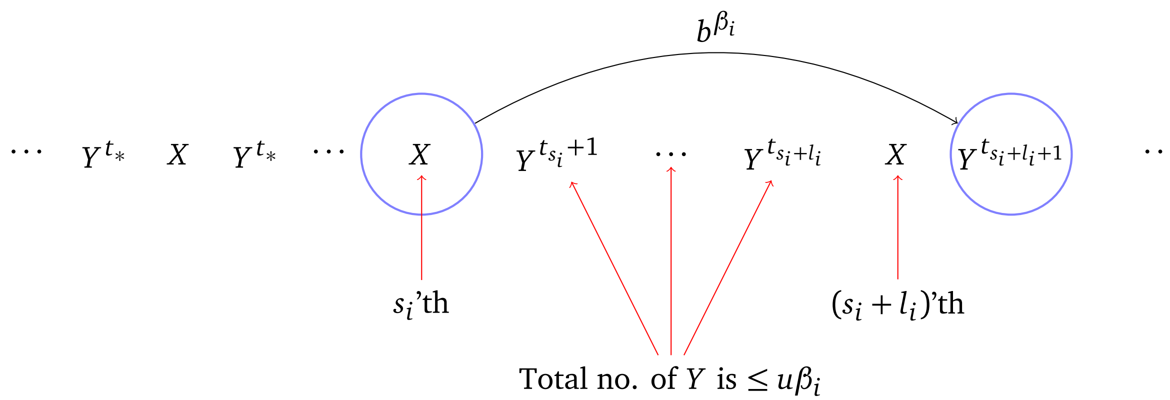

Figure 3.3: Action of b^{β_i}

We write w^d as

$$w^d = b^{\beta_k} a^{\alpha_k} b^{\beta_{k-1}} a^{\alpha_{k-1}} \cdots b^{\beta_1} a^{\alpha_1}$$

and instead of considering the action of w on W with a period d, assume that w^d acts on W^c by its action on $\mathbb{Z}$. Then the maximal a—strings and b—strings in w^d, all together cover exactly the total number of X's (and Y's) in W^c. For a similar reason, we know that intervals of the form of $\left(W_j, W_{a^{\alpha_i}(j)}\right)$ enclose precisely $p\alpha_i + 1$ X's. Thus we get the equality

$$\sum_{i=1}^{k} (l_i + (\alpha_i p + 1)) = cq.$$

Note that here α_i's are periodic as a function of i, with a period $k/d = n$, but in general, the l_i's are not periodic in i. We can also give a formula for s_i by counting the number of X's covered.

$$s_i = \sum_{j=1}^{i} (\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j.$$

Thus, we have formulated our minimization problem as a set of homogeneous linear integral equations subject to finitely many integral linear constraints. Because of homogeneity, it has a solution in integers if and only if it has a solution in rational numbers, and consequently, we can normalize the whole problem by rescaling to $v = 1$. The solution to this linear programming problem is necessarily rational and gives the minimal t such that $R(w; p/q, t) \geq c/d$. Also if equality is achieved then clearly $R(w; p/q, u) = c/d$, and thus the theorem is proved.

We summarize the whole algorithm in the next subsection.

3.2.2 Summary of the algorithm

Step 1. Replacing w by a cyclic permutation if necessary, write w^d in the form

$$w^d = b^{\beta_k} a^{\alpha_k} \dots b^{\beta_1} a^{\alpha_1}.$$

Step 2. Enumerate all non-negative integral solutions to

$$\sum_{i=1}^{k} l_i = cq - \sum_{i=1}^{k} (\alpha_i p + 1).$$

Note that we are counting each permutation of a certain solution set distinctly. This is important since the next step depends not only on the values of l_i but also their order.

Step 3. For each such solution set $(l_1, \dots, l_k)$, define

$$s_i = \sum_{j=1}^{i} (\alpha_j p + 1) + \sum_{j=1}^{i-1} l_j$$

Step 4. Find the smallest u which satisfies the system of inequalities

$$\begin{cases} \sum\limits_{i=1}^{q} t_i = 1, \\ t_i \geq 0 \ \ \forall\, i, \\ t_{s_i+1} + t_{s_i+2} + \ldots + t_{s_i+l_i} \leq u\beta_i \quad \forall\, 1 \leq i \leq k \,(\text{indices taken} \mod q) \end{cases}$$

Observe that this is a straightforward linear programming problem in the variables $t_1, t_2, \dots, t_q$, and u where, by construction, we can also impose the condition $0 \leq u \leq 1$.

Step 5. Find the smallest u over all solution sets $(l_1, \dots, l_k)$.

CHAPTER 4

A FORMULA FOR FRINGE LENGTH

In this chapter we will apply the Stairstep Algorithm to the computation of fringe lengths. The key idea is that in this special case, the equation

$$\sum_{i=1}^{k} l_i = cq - \sum_{i=1}^{k} (\alpha_i p + 1)$$

has a *unique* non-negative integral solution. This in turn reduces the last step of the algorithm to the solution of a *single* linear programming problem, rather than a system of (exponentially) many inequalities.

4.1 Statement of Fringe Formula

First let us state the Fringe Formula.

Theorem 4.1.1 (Fringe Formula). *If w is positive, and p/q is a reduced fraction, then*

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$$

where $\sigma_w(g)$ depends only on the word w and $g := \gcd(q, h_a(w))$; and $g \cdot \sigma_w(g)$ is an integer.

The formula for $\sigma_w(g)$ depends on both the α_i and the β_j in a complicated way, which we will explain in the sequel.

4.2 Proof of Fringe Formula 4.1.1

We now begin the proof of the fringe formula. This takes several steps, and requires a careful analysis of the Stairstep Algorithm. We therefore adhere to the notation in § 3.2. After cyclically permuting w if necessary we write w in the form

$$w = b^{\beta_n} a^{\alpha_n} \dots b^{\beta_1} a^{\alpha_1}.$$

4.2.1 Finding the optimal partition

First note that by Theorem (2.3.1), it is enough to find the minimum t such that

$$R(w; p/q, t) = \frac{h_a p + h_b q}{q}.$$

Thus to apply the stairstep algorithm (3.2.1), we are going to fix $c/d = (h_a p + h_b q)/q$ where c/d is the reduced form. Let us denote the gcd of h_a and q by g so that we have

$$c = \frac{h_a p + h_b q}{g}, \; d = \frac{q}{g}$$

since $(p, q) = 1$. Further writing $h_a = h' g$ and $q = q' g$, we rewrite the above equations as

$$c = h' p + h_b q', \; d = q'.$$

Thus step 1 of our algorithm becomes

$$w^{q'} = b^{\beta_{nq'}} a^{\alpha_{nq'}} \dots b^{\beta_1} a^{\alpha_1}$$

where clearly α_i, β_i are periodic as functions of i with period n. Similarly, step 2 of our algorithm transforms to

$$l_1 + \dots + l_{q'.n} = \underbrace{\frac{h_a.p + h_b.q}{g}}_{=c}.q - \underbrace{q'.h_a}_{=\sum_{i=1}^{nq'} \alpha_i}.p - q'.n$$

i.e.

$$l_1 + \ldots + l_{nq'} = h_b.qq' - nq' \tag{4.1}$$

and the equations in step 4 to find the minimum solution u, become

$$\sum_{i=1}^{q} t_i = 1 \tag{4.2}$$

$$t_i \geq 0 \qquad \forall\, i \tag{4.3}$$

$$t_{s_i+1} + t_{s_i+2} + \ldots + t_{s_i+l_i} \leq \beta_i u \qquad \forall\, 1 \leq i \leq nq' \tag{4.4}$$

where indices are taken (mod q). Now if any of the l_i is greater than or equal to $q\beta_i$, then the indices on the left hand side of equation (4.4) cycle through all of 1 through q a total of β_i times. Then using (4.2), we get that

$$\beta_i = \beta_i \sum_{1}^{q} t_i \leq t_{s_i+1} + t_{s_i+2} + \ldots + t_{s_i+l_i} \leq \beta_i u$$

implying $u \geq 1$, which is clearly not the optimal solution. Hence for the minimal solution u, we must have

$$l_i \leq q\beta_i - 1, \forall\, 1 \leq i \leq nq'.$$

Summing up all of these inequalities, we get that

$$\sum_{i=1}^{nq'} l_i \leq q \sum_{i=1}^{nq'} \beta_i - nq' = qq'h_b - nq'$$

But on the other hand, by step 2, equality is indeed achieved in the inequality above and hence

$$l_i = q\beta_i - 1, \forall\, 1 \leq i \leq nq' \tag{4.5}$$

is the *unique* non-negative integral solution to the partition problem in step 2. As mentioned before, this means we only need to deal with a single linear programming problem henceforth, formulated more precisely in the next section.

4.2.2 A linear programming problem

With the specific values of l_i found above, we can transform equations (4.2), (4.3) and (4.4) as follows. Note that for $l_i = q\beta_i - 1$, the set of indices $s_i + 1, s_i + 2, \cdots, s_i + l_i$ cycle through all of the values $1, 2, \cdots, q$ a total of β_i times, except one of them, namely $s_i \pmod q$, which appears $\beta_i - 1$ times. Then we can rewrite (4.4) as

$$\beta_i \left(\sum_{j=1}^{q} t_j \right) - t_{s_i} \leq \beta_i u \quad \forall\, 1 \leq i \leq nq'$$

i.e.

$$\frac{t_{s_i}}{\beta_i} \geq 1 - u \quad \forall\, 1 \leq i \leq nq'$$

Observe that in the above equation, β_i's are periodic with a period n whereas the s_i's are well defined modulo q (since t_i's have period q), which is usually much bigger than n. Then for the purpose of finding an u which satisfies the system of equations (4.2), (4.3) and (4.4), it will be enough to consider the indices i for which β_i is maximum for the same value of s_i.

To make the statement more precise, we introduce the following notation. Let the set of indices Λ be defined by

$$\Lambda = \left\{ i \;\middle|\; \beta_i = \max_{\substack{s_j = s_i \\ 1 \leq j \leq nq'}} \beta_j \right\}.$$

Then the first thing to note is that the set of numbers $\{s_i\}_{i \in \Lambda}$ are all distinct. Next recall that we are in fact trying to find the fringe length, which is $1 - t$, where t is the solution to the stairstep algorithm. So with a simple change of variable, our algorithm becomes the following *linear* *programming problem*:

$$\begin{aligned} \text{Find maximum of} \quad & \min_{i\in\Lambda}\left\{\frac{1}{\beta_i}t_{s_i}\right\} \\ \text{Subject to} \quad & \sum_{i\in\Lambda} t_{s_i} \le 1,\ t_{s_i} \ge 0\ \forall i \end{aligned}$$

But since we are trying to find the maximum, we may as well assume that $\sum_{i\in\Lambda} t_{s_i} = 1$ and $t_k = 0$ if $k \neq s_i$ for some $i \in \Lambda$. Then by a theorem of Kaplan [9], we get that the optimal solution occurs when for all $i \in \Lambda$, the number t_{s_i}/β_i equals some constant T independent of i. To find T, observe that

$$\frac{t_{s_i}}{\beta_i} = T \Rightarrow \sum_{i\in\Lambda}\beta_i T = 1 \Rightarrow T = \frac{1}{\sum_{i\in\Lambda}\beta_i}.$$

Thus the optimal solution to the linear programming problem, which is also the required fringe length is given by

$$\mathrm{fr}_w(p/q) = \frac{1}{\sum_{i\in\Lambda}\beta_i} \tag{4.6}$$

So all that remains is to figure out what the set of indices Λ looks like. In the rest of this section we try to characterize Λ and prove the fringe formula 4.1.1.

4.2.3 Reduction to combinatorics

It is clear from the definition that to figure out the set Λ, we need to find out exactly when two of the s_i's are equal as i ranges from 1 to nq'. Recall that the indices s_i's are taken modulo q. Using the optimal partition, we get that

$$s_i + l_i = \sum_{j=1}^{i}(p\alpha_j + 1 + q\beta_j - 1)$$

and hence

$$s_I = s_J \Leftrightarrow \sum_{j=1}^{I} \alpha_j \equiv \sum_{j=1}^{J} \alpha_j \pmod q$$

since $l_I \equiv l_J \pmod q$. Thus the elements of Λ are in bijective correspondence with the number of residue classes modulo q in the following set of numbers:

$$\begin{aligned}
A_1 &= \alpha_1 \\
A_2 &= \alpha_1 + \alpha_2 \\
A_3 &= \alpha_1 + \alpha_2 + \alpha_3 \\
A_4 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
&\vdots \\
A_{nq'} &= \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{nq'}
\end{aligned}$$

So we can rewrite the formula for the set Λ as

$$\Lambda = \left\{ i \,\middle|\, \beta_i = \max_{\substack{A_j \equiv A_i \pmod q \\ 1 \le j \le nq'}} \beta_j \right\}$$

Note that $A_n = h_a$ and α_i's are periodic with period n. So we have, $A_{n+i} = A_i + h_a$ or in other words, the collection of numbers $A_1, A_2, \ldots, A_{nq'}$ is nothing but a union of disjoint translates of the collection $(A_1, A_2, \ldots, A_n)$ by $0, h_a, 2h_a, \ldots, (q'-1)h_a$.

Let us refer to the n-tuple $(A_1, A_2, \ldots, A_n)$ as the first "n-block". Similarly the h_a-translate of the first n-block is referred to as the second n-block and so on. Note that $q'h_a = h'q$, so the $q'h_a$-translate of the first n-block is identical to itself modulo q. Hence we may think of translation by $(q'-1)h_a$ as translation by $-h_a$.

Next we claim that

Claim. *The numbers $0, h_a, 2h_a, \ldots, (q'-1)h_a$ are all distinct modulo q.*

Proof. If q divides the difference between any two such numbers, say mh_a, then $q' \mid mh' \Rightarrow q' \mid m \Rightarrow m \geq q'$, which is a contradiction. $\square$

In fact since h' is invertible modulo q, the set of numbers $\{0, h_a, \ldots, (q'-1)h_a\}$ is the same as $\{0, g, 2g, \ldots, (q'-1)g\}$ modulo q. Thus to determine the congruence classes in the collection $A_1, A_2, \ldots, A_{nq'}$, it is enough to find out which n-blocks overlap with the first n—block. Note that translating an n—block by $h_a (= h'g)$ takes it off itself entirely, so the *only* translates of an n-block that could overlap with itself are the translates by ig for $|i| < h'$ (See Figure 4.1).

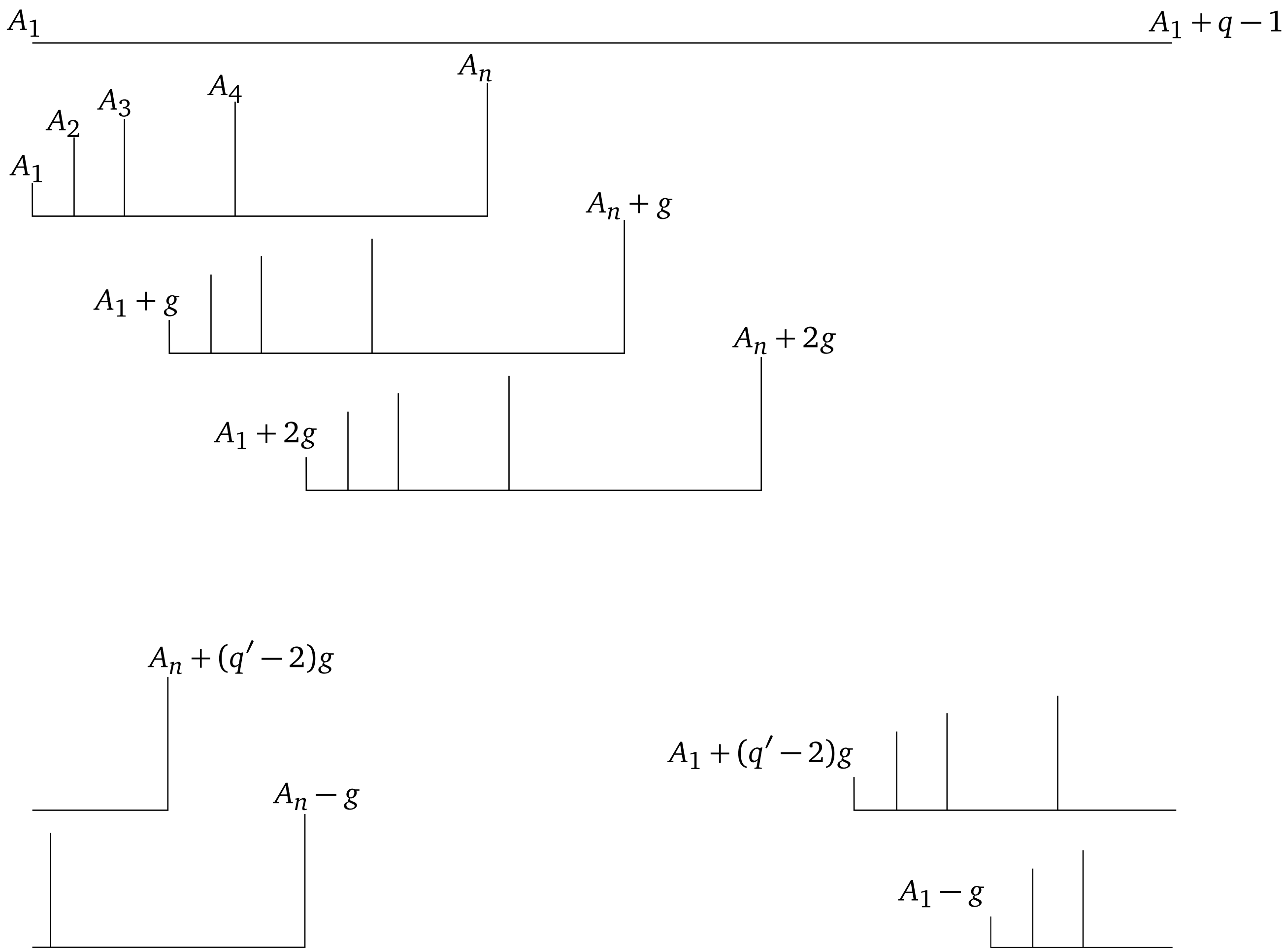

Figure 4.1: Translates of the first n-block

Finally observe that if we start with the the n-block given by $(A_1 + g, A_2 + g, \ldots, A_n + g)$ instead, we get overlaps at the same multiple of g as the first n-block; only translated by g. Thus starting from A_1, if we divide the residue class of q into a total of q' number of g—sized groups, then each β_i's appears the same number of times in each group and the overlaps appear at the same places translated by multiples of g. Hence to calculate the sum of $\max\{\beta_i\}$ over all residue classes, it is enough to calculate it for the residue classes which appear among $A_1, A_1+1, A_1+2, \ldots$ up to $A_1+(g-1)$ and then multiply the result by q'.

Let us summarize the results we have found so far in the form of an algorithm.

Step 1. Write down $A_1, A_2, \ldots, A_n$ where $A_i = \alpha_1 + \ldots + \alpha_i$.

Step 2. For each $0 \leq i \leq g-1$, let $\mathfrak{B}_i$ be defined as follows:

$$\mathfrak{B}_i = \max\left\{\beta_k \mid A_k + mg \equiv A_1 + i \pmod{q} \text{ where } -h' < m < h', 1 \leq k \leq n\right\}$$

Note that in case $q' < h'$, we replace h' with q' in above definition.

Step 3. Let S be the sum of $\mathfrak{B}_i$'s for $0 \leq i \leq g-1$. Then the fringe length is given by

$$\mathrm{fr}_w(p/q) = \frac{1}{q'S} \tag{4.7}$$

To finish the proof, define $\sigma_w(g) := S/g$ and note that by the structure of the algorithm, $\sigma_w(g)$ depends only on $g = gcd(q, h_a)$ and the word w. As a corollary, we also get the remarkable consequence that

Corollary 4.2.1. *The fringe length* does not *depend on p.*

i.e. the fringes are "periodic" on every scale. In section § 6.1 we elaborate on this phenomenon in a particular example, and discuss possible generalizations.

We finish this chapter by giving a picture of the Fringes corresponding to the Ziggurat in figure 4.2.

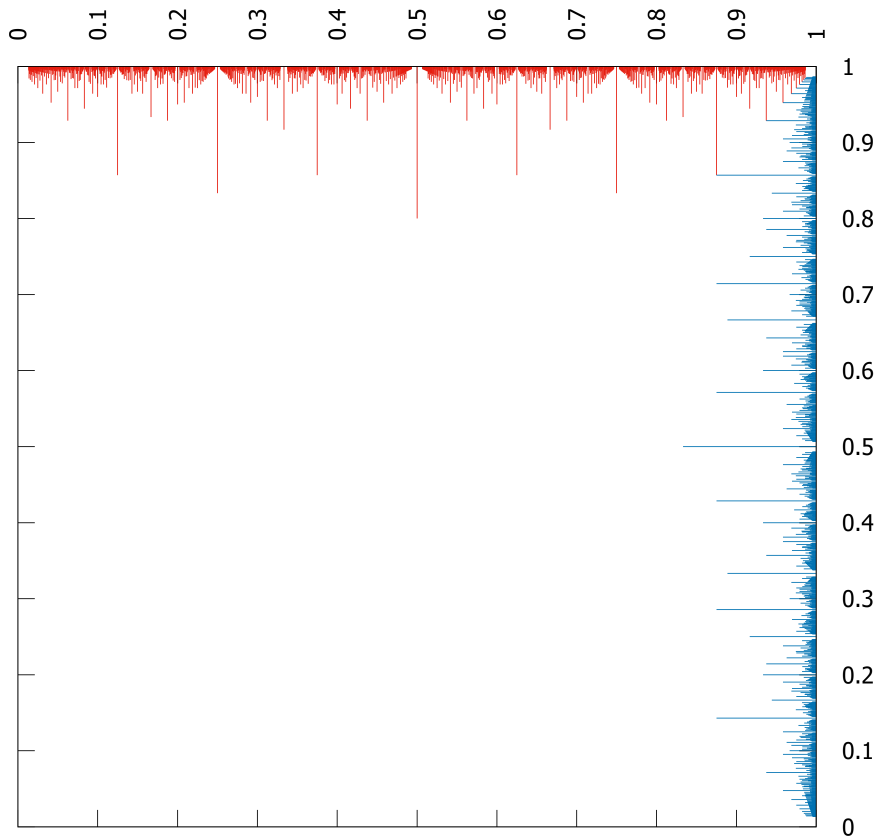

Figure 4.2: Plot of the fringes of $abbbabaaaabbabb$, $q = 1$ to 75

CHAPTER 5

BOUNDS ON σ AND SPECIAL CASES

In this chapter we give some examples to illustrate the complexity of the function σ. First we prove the following theorem and its corollary in the special case when $h_a(w)$ is prime.

5.1 Statement of σ-inequality

Theorem 5.1.1 (σ-inequality)**.** *Suppose $w = a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \dots a^{\alpha_n} b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b}{h_a} \leq \sigma_w(g) \leq \max_{1 \leq i \leq n} \beta_i$$

where the first equality is achieved in the case when h_a divides q and the second equality occurs when $(q, h_a) = 1$.

Corollary 5.1.2. *If h_a is a prime number then*

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{h_a}{q \cdot h_b}, & \text{if } h_a \mid q \\ \dfrac{1}{q \cdot \max\limits_{1 \leq i \leq n} \beta_i}, & \text{if } h_a \nmid q \end{cases}.$$

5.2 Proof of σ-inequality 5.1.1

For the first inequality, recall the numbers $A_1, A_2, \dots, A_{nq'}$ from last chapter. Note that the fact that $h_a \cdot q' = h' \cdot q$ tells us that there are at most h' elements in each residue class modulo q among $A_1, \dots, A_{nq'}$. Thus

$$\sum_{i=1}^{nq'} t_{s_i} \leq h' \cdot \sum_{i \in \Lambda} t_{s_i} \leq h' \cdot \sum_{i=1}^{q} t_i = h'$$

On the other hand, adding all the nq' inequalities in (4.4), and using $l_i = q\beta_i - 1$, we get that

$$u.\sum_{i=1}^{nq'} \beta_i \geq \sum_{i=1}^{nq'} \left(\beta_i \sum_{j=1}^{q} t_j - t_{s_i} \right) = \sum_{i=1}^{nq'} \beta_i - \sum_{i=1}^{nq'} t_{s_i} \geq \sum_{i=1}^{nq'} \beta_i - h'$$

$$u \geq 1 - \frac{h'}{h_b.q'} = 1 - \frac{h_a}{h_b q}$$

Hence, for the minimal u giving the fringe length we get that

$$\sigma_w(g) \geq \frac{h_b}{h_a}.$$

For the second inequality, observe that by definition,

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g)q} = \frac{1}{\sum_{i\in\Lambda} \beta_i} \geq \frac{1}{|\Lambda| \cdot \max_{i\in\Lambda} \beta_i} \geq \frac{1}{q \cdot \max_{i\in\Lambda} \beta_i}$$

since number of elements in Λ is at most the number of residue classes modulo q. Hence

$$\sigma_w(g) \leq \max_{i\in\Lambda} \beta_i \leq \max_{1\leq i\leq n} \beta_i.$$

We will finish the proof by showing that equality is indeed achieved in the following special cases:

Case 1 : $h_a \mid q$

In this case $h' = 1$. Hence all the s_i's are distinct.

Consider the specific example where $t_{s_i} = \beta_i/(h_b q')$ for all i and the rest of the t_i's are zero. Then we have

$$\beta_i.u \geq \sum_{j\neq i} \frac{\beta_j}{h_b q'}.\beta_i + \frac{\beta_i}{h_b q'}.(\beta_i - 1) = \beta_i.\frac{h_b q'}{h_b q'} - \frac{\beta_i}{h_b q'} \Rightarrow u \geq 1 - \frac{1}{h_b q'}.$$

Thus the minimum u_0 which gives a solution to (4.2), (4.3), (4.4) is $1-1/(q'h_b) = 1-h_a/(h_b q)$. Thus equality is achieved in the first part of Theorem 5.1.1.

We can give a second proof of this same fact using the algorithm developed in last section. Since $h_a \mid q$, the gcd of h_a and q is h_a. So any $g-$translate of the n-block is disjoint from itself. Hence $S = h_b$, giving the same formula as above.

Case 2: $gcd(h_a, q) = 1$

In this situation, $g = 1$. Hence $c = h_a.p + h_b.q$ and $d = q$ since $q = q'$.

Let $W = Y^{t_1}XY^{t_2}\ldots Y^{t_q}X$ as in the proof of Theorem 3.2.1. Since w now has a periodic orbit of period exactly q, we get that any $b-$string starting on adjacent X's must land in adjacent Y^* strings. Thus the constraints of the linear programming problem are invariant under permutation of the variable t_i, and by convexity, extrema is achieved when all t_i's are equal. But then we get

$$q.t_i = 1 \Rightarrow t_i = \frac{1}{q}$$

and

$$\beta_i u \geq l_i.t_i = \frac{(q\beta_i - 1)}{q} \Rightarrow u \geq 1 - \frac{1}{q\beta_i} \qquad \forall\, 1 \leq i \leq nq$$

Hence the minimum u which gives a solution to the system of equation is given by

$$u = 1 - \frac{1}{q.\max_{1\leq i\leq n}\{\beta_i\}}.$$

Observing that equality is indeed achieved in case of the word $\left(XY^{\max\{\beta_i\}}\right)^q$, we get equality in the second part of Theorem 5.1.1.

Again, we can give a much simpler proof of this result using the algorithm in the last chapter. In this case, we have $g = 1$ so that $q = q'$. So S is the maximum of all the β_i's which correspond to any A_i which is a translate of A_1 by one of $-h_a, -h_a + 1, \ldots, 0, \ldots, h_a - 1, h_a$; i.e. all of the A_i's. Thus $S = \sigma_w(g) = \max_{1\leq i\leq n}\{\beta_i\}$ since $g = 1$.

Remark 5.2.1. The function $\sigma_w(g)$ depends on $g = gcd(h_a, q)$ in a complicated way when h_a is not prime as we can see from the following table:

Word		$p/q = 1/5$	$p/q = 1/2$	$p/q = 1/3$	$p/q = 1/6$
$h_a = 6$	h_b	$g = 1$	$g = 2$	$g = 3$	$g = 6$
$aaabaaabbbb$	5	4	5/2	4/3	5/6
$abaabaaabbbb$	6	4	5/2	5/3	1
$abbaabaaabbbb$	7	4	3	2	7/6
$abbbaabaaabbbb$	8	4	7/2	4/3	7/3
$abbbababaaabbbb$	9	4	7/2	8/3	3/2
$abbbaabbaaabbbb$	9	4	7/3	7/3	3/2
$abbbababbaaabbbb$	10	4	7/2	8/3	5/3

Table 5.1: Values of $\sigma_w(g)$ for different w and g

Observe that when h_a is a prime number, above two cases are the only possibilities, and hence we easily get corollary 5.1.2.

5.3 One specific example $w = abaab$

Let us consider the case of the word $w = abaab$. Here $h_a = 3$ and $h_b = 2$, both prime numbers, and hence corollary 5.1.2 holds.

5.3.1 *Fringe Formula*

For $w = abaab$, the left fringe lengths are given by

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{3}{2q} & \text{when } 3 \mid q \\ \dfrac{1}{q} & \text{when } 3 \nmid q \end{cases}$$

and the right fringe lengths are given by

$$\mathrm{fr}_w(p/q) = \begin{cases} \dfrac{2}{3q} & \text{when } q \text{ is even.} \\ \dfrac{1}{2q} & \text{when } q \text{ is odd} \end{cases}.$$

The cases when $3 \nmid q$ and $2 \nmid q$ were also discussed in [3], p 18.

We give a fringe plot for both sides for the word $w = abaab$. Putting the origin at the point $(r = 0, s = 0)$, we have the following picture.

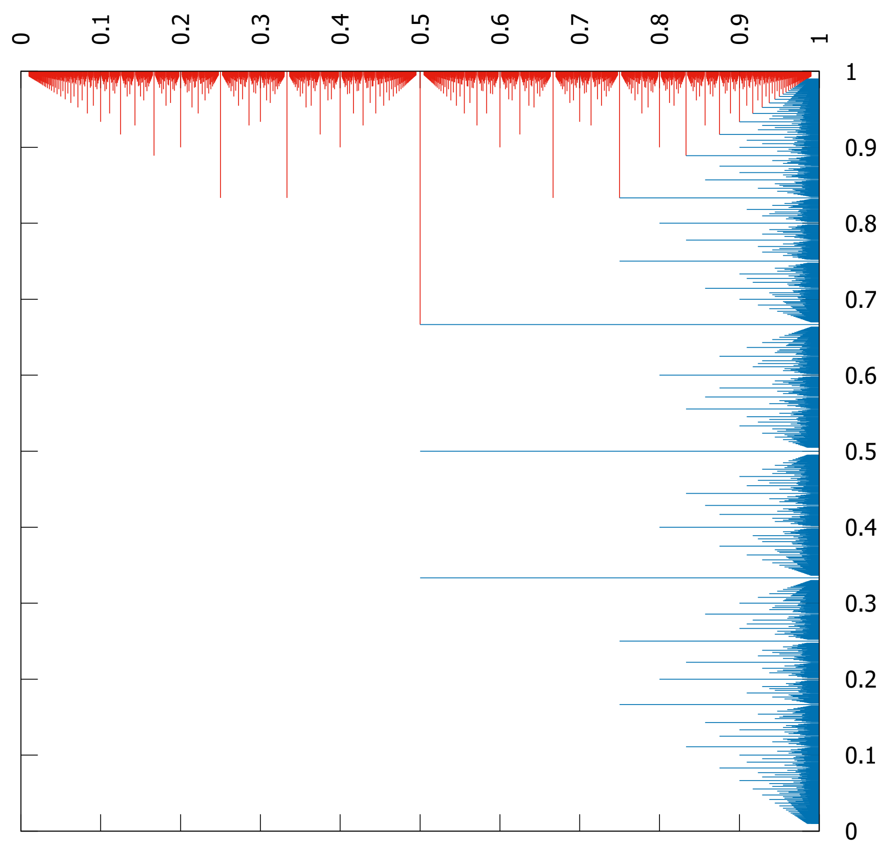

Figure 5.1: Plot of the fringes of $abaab$, $q = 1$ to 100

5.3.2 A lower bound on the size of stability region

In this section we will try to give a lower bound on the size of the region where R is locally constant when we move inwards from the fringes. Note that existence of such a region was guaranteed by theorem 2.2.3. However, the original theorem only gave an in optimal upper bound. So our goal is follows: given p/q, we wish to find the biggest ϵ such that $R(w; p/q+\epsilon, 1-\mathrm{fr}_w(p/q))$ is equal to $R(w; p/q, 1-)$.

We are going to stick to the word $w = abaab$. Assume first that $3 \nmid q$. From last subsection, we know that $\mathrm{fr}(p/q) = \frac{1}{q}$. We want to find a nontrivial lower bound on ϵ such that

$$R\left(abaab; t, \frac{q-1}{q}\right) = \frac{3p+2q}{q} \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q}+\epsilon\right)$$

We will be using the following notations:

Definition 5.3.1. Consider a generalization of the Farey sequence of order n, denoted $\mathscr{F}_n$, whose terms are all the positive reduced fractions with denominators not exceeding n, listed in order of their size, from 0 to $+\infty$.

If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in $\mathscr{F}_n$ with $\frac{a}{b} > \frac{c}{d}$ then we define the function ν_n by

$$\nu_n\left(\frac{a}{b}\right) = \frac{c}{d}$$

Clearly $\nu_n\left(\frac{a}{b}\right)$ is defined for all $a/b \in \mathbb{Q}^+$ and whenever $n \geq b$.

We make the following observation. Suppose there exists some $\frac{u}{v} > \frac{p}{q}$ such that $R(abaab; u/v, 1-) = \frac{3u+2v}{v}$ satisfies

$$\nu_n\left(\frac{3p+2q}{q}\right) = \frac{3u+2v}{v}$$

for some $n \geq q$. Assume $t \in \left[\frac{p}{q}, \frac{u}{v}\right)$. Since the denominator of $R\left(abaab; t, \frac{q-1}{q}\right)$ is at most q, we get that

$$R\left(abaab;t,\frac{q-1}{q}\right)\in\mathscr{F}_n,$$

which in turn implies that

$$R\left(abaab;t,\frac{q-1}{q}\right)=\frac{3p+2q}{q}$$

Note that $R\left(abaab;t,\frac{q-1}{q}\right)\neq\frac{3u+2v}{v}$ because

$$R\left(abaab;t,\frac{q-1}{q}\right)\leq R(abaab;t,1)\lneqq R\left(abaab;\frac{u}{v},1\right).$$

We have proved the following proposition:

Proposition 5.3.2. *Suppose there exists some $\frac{u}{v}>\frac{p}{q}$ such that*

$$\nu_n\left(\frac{3p+2q}{q}\right)=\frac{3u+2v}{v}$$

for some n. Then

$$R\left(abaab;t,\frac{q-1}{q}\right)=\frac{3p+2q}{q}$$

for all $t\in\left[\frac{p}{q},\frac{u}{v}\right)$.

So we would like to find out when the condition in the proposition holds. Note that,

$$\nu_n\left(\frac{3p+2q}{q}\right)=\frac{3u+2v}{v}\Longleftrightarrow\nu_n\left(\frac{3p}{q}\right)=\frac{3u}{v}$$

We prove the following property of these generalized Farey sequences.

Lemma 5.3.3. *If $\nu_{3n}\left(\frac{p}{q}\right)=\frac{u}{v}$ and $n\geq q$, then $\nu_n\left(\frac{3p}{q}\right)=\frac{3u}{v}$.*

Proof. Suppose not. Then either $n<q$ or there exists $\frac{c}{d}$, a reduced fraction, such that $\frac{3p}{q}<\frac{c}{d}<\frac{3u}{v}$ and $d\leq n$, in which case

$$\frac{p}{q}<\frac{c}{3d}<\frac{u}{v}$$

and the denominator of $\frac{c}{3d}$ is at most $3d \leq 3n$. Thus $\frac{c}{3d}$ is a term of $\mathscr{F}_{3n}$ in between $\frac{p}{q}$ and $\frac{u}{v}$. Contradiction! $\square$

Thus whenever we have $\frac{u}{v}$ and n such that $\nu_{3n}\left(\frac{p}{q}\right) = \frac{u}{v}$, and $n \geq q$, the requirement of the above proposition will be fulfilled. By properties of Farey sequence $\mathscr{F}_{3n}$, we know that the difference between consecutive $\frac{u}{v}$ and $\frac{p}{q}$ is maximum when the order of the sequence containing both is minimum. However, we are constrained to have $n \geq q$. Hence we get the following lower bound

$$\epsilon \geq \frac{1}{qv} \geq \frac{1}{3qn} = \frac{1}{3q^2}$$

It is easy to see that all the reasoning are similar in the case $3 \mid q$. The only difference is that we need $n \geq \frac{2q}{3}$. Thus we get the bound

$$\epsilon \geq \frac{1}{3n.q} = \frac{1}{2q^2}$$

We have shown:

Theorem 5.3.4. *If* $(3, q) = 1$*, then*

$$R\left(abaab; t, 1 - \frac{1}{q}\right) \text{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q^2}\right).$$

If $(3, q) \neq 1$*, then*

$$R\left(abaab; t, 1 - \frac{3}{2q}\right) \text{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{2q^2}\right).$$

This lower bound is not sharp, and in general the length depends on p since we can get a better bound on $\frac{1}{qv}$ if we use p. For a general word the bound depends on σ.

For the sake of fluency, we introduce the following term:

Definition 5.3.5. Given $w \in F_2$, and $a, b \in (0,1)$, we denote the set

$$\{(r,s)\,|R(w;r,s) = R(w;a,b), r \geq a, s \geq b\}$$

by $C(a,b)$, called the Cone of (a,b).

Using this new language, we will next try to find a nontrivial rectangle that we can fit in $C(p/q, 1-\mathrm{fr}(p/q))$ one of whose vertices is $(p/q, 1-\mathrm{fr}(p/q))$. We have already found one of the sides of this rectangle. We would now like to get a general idea of the shape of the cone.

Let us do the case $3 \nmid q$ first. Consider the point $\left(\frac{p}{q}, 1-\frac{1}{q+1}\right)$ in the case $3 \nmid q$. Again suppose there exists some $\frac{u}{v} > \frac{p}{q}$ such that

$$v_n\left(\frac{3p+2q}{q}\right) = \frac{3u+2v}{v}$$

for some n. Now by the same reasoning as above if we want $R\left(abaab; t, 1-\frac{1}{q+1}\right) = \frac{3p+2q}{q}$ for $t \in [p/q, u/v)$, then we need that $n \geq q+1$. After that it is exactly the same argument as above to see that

$$R\left(abaab; t, 1-\frac{1}{q+1}\right) \text{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q(q+1)}\right)$$

and in general,

$$R\left(abaab; t, 1-\frac{1}{q+i}\right) \text{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q(q+i)}\right).$$

In particular, by the monotonicity of R we find that

$$R\big(abaab; t, t'\big) \text{ is constant } \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q^2+3q}\right) \text{ and } \forall\, t' \in \left[1-\frac{1}{q}, 1-\frac{1}{q+1}\right).$$

Similarly in the case $3 \mid q$, we have

$$R\big(abaab; t, t'\big) \text{ is constant } \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{2q^2+3q}\right) \text{ and } \forall\, t' \in \left[1-\frac{3}{2q}, 1-\frac{3}{2q+3}\right).$$

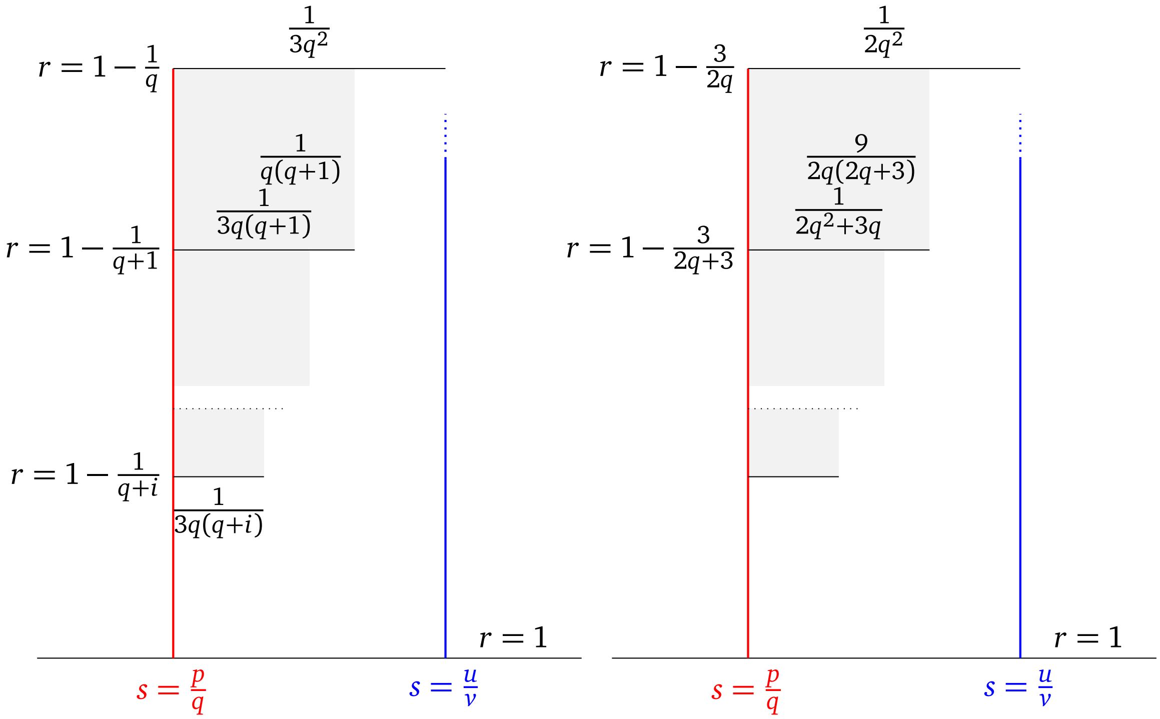

Figure 5.2: Parts of $C\left(\frac{p}{q}, 1 - \mathrm{fr}\left(\frac{p}{q}\right)\right)$

CHAPTER 6

FURTHER RESULTS

6.1 Projective self-similiarity

A. Gordenko shows in her paper [7] that the Ziggurat of the word $w = ab$ is self similar under two projective transformation (Theorem 4). In this section we show that similar transformations exist in case of the word $w = abaab$, which gives a different way to look at the Fringe formula.

Let us first look at the self-similarities of the left Fringe. Below (see figure 6.1) is a plot of the Fringe lengths where x-axis is the value of $\mathrm{rot}^{\sim}(a)$ and $y-$axis is value of $\mathrm{fr}_{abaab}(x)$. Thus for $x = p/q$ we have $\mathrm{fr}_{abaab}(x)$ defined as in § 5.3.1. We will drop the subscript $abaab$ for the next part.

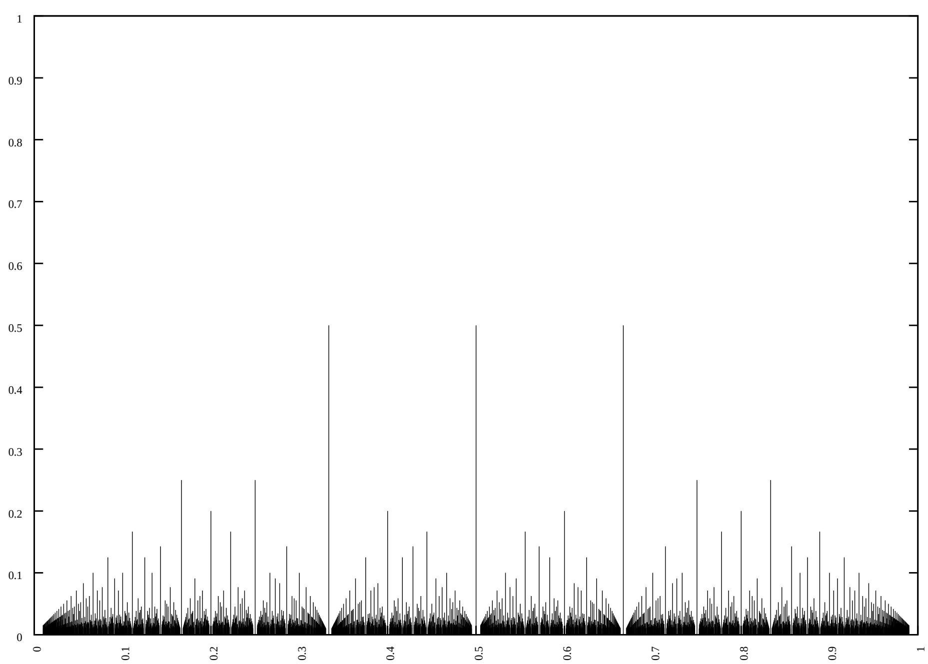

Figure 6.1: Plot of Left Fringe, $q = 1$ to 100

We prove that the unit interval can be decomposed into some finite number of intervals Δ_i such that there exist a further decomposition of each Δ_i into a disjoint union of subintervals $I_{i,j}$

such that the graph of $\mathrm{fr}(x)$ on each of $I_{i,j}$ is similar to that on some $\Delta_{k(i,j)}$ under projective linear transformations as follows:

Theorem 6.1.1. *Let* $\Delta_1 = (0,1/3), \Delta_2 = (1/3,1/2), \Delta_3 = (1/2,2/3)$ *and* $\Delta_4 = (2/3,1)$. *Then we have the following decomposition into* $I_{i,j}$ *and transformations* $T_{i,j}$*:*

$$
\begin{aligned}
I_{1,1} &= (0,1/4), & T_{1,1}(I_{1,1}) &= \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 = [0,1], \\
& & T_{1,1}(x,y) &= \left(\frac{x}{1-3x}, \frac{y}{1-3x}\right) \\
I_{1,2} &= (1/4,1/3), & T_{1,2}(I_{1,2}) &= \Delta_1, \\
& & T_{1,2}(x,y) &= \left(\frac{4x-1}{9x-2}, \frac{y}{9x-2}\right) \\
I_{2,1} &= (1/3,1/2), & T_{2,1}(I_{2,1}) &= \Delta_1, \\
& & T_{2,1}(x,y) &= \left(\frac{1-2x}{2-3x}, \frac{y}{2-3x}\right)
\end{aligned}
$$

Since the graph is clearly symmetric about $x = 1/2$, *similar decomposition exists for* Δ_3 *and* Δ_4 *(see figure 6.2).*

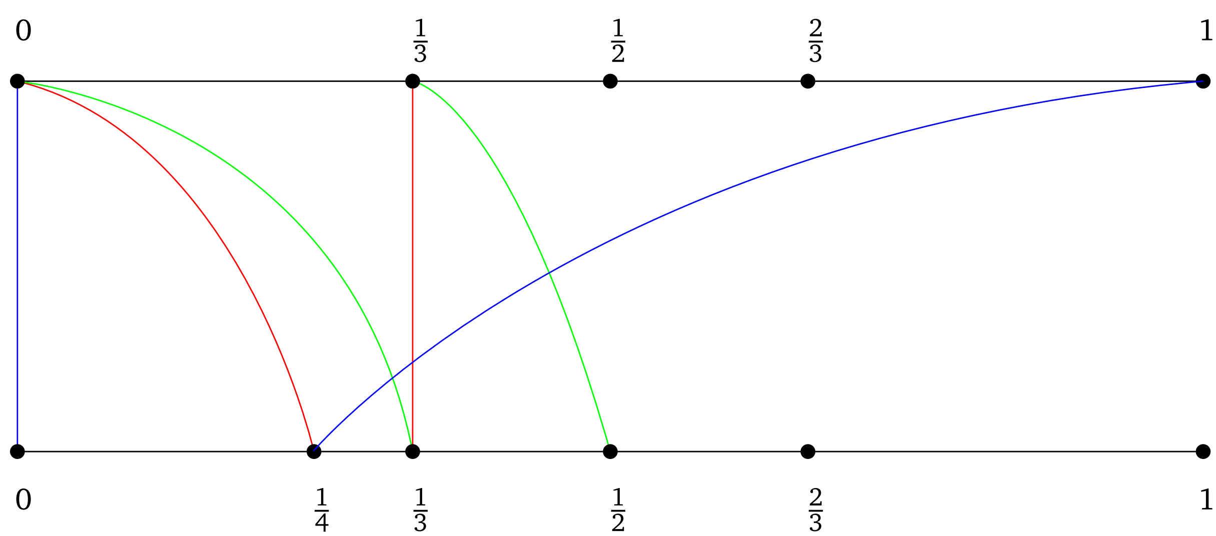

Figure 6.2: Intervals of projective self similarity in case of $w = abaab$

Proof. For each of the transformations note that the denominator of the image of p/q has the same gcd with h_a as q. Also, in each case, the numerator and denominator are coprime. The proof then follows easily by checking the length of images in each case. □

We thus note that in fact Δ_1 contains all the information necessary to determine the fringe dynamics. In fact, for h_a prime the following similarity result always holds:

Theorem 6.1.2. *Let $\Delta_1 = (0, 1/h_a)$ where h_a is a prime number. Then we can decompose Δ_1 into $I_{i,j}$ and find transformations $T_{i,j}$ as follows:*

$$\begin{aligned}
I_{1,1} &= (0, 1/(h_a+1)), & T_{1,1}(I_{1,1}) &= [0,1], \\
& & T_{1,1}(x,y) &= \left(\frac{x}{1-h_a x}, \frac{y}{1-h_a x}\right) \\
I_{1,2} &= (1/(h_a+1), 1/h_a), & T_{1,2}(I_{1,2}) &= \Delta_1, \\
& & T_{1,2}(x,y) &= \left(\frac{(h_a+1)x-1}{h_a^2 x-(h_a-1)}, \frac{y}{h_a^2 x-(h_a-1)}\right)
\end{aligned}$$

It is also easy to prove in the case of prime h_a that the plot on $\Delta = \left[\frac{(h_a-1)}{2h_a}, \frac{1}{2}\right]$ is similar to Δ_1 under the transformation

$$T(x,y) = \left(\frac{2-4x}{(h_a+1)-2h_a x}, \frac{2y}{(h_a+1)-2h_a x}\right)$$

Note that in case of $h_a = 3$, we have $(h_a-1)/2h_a = 1/h_a$, which explains Theorem 6.1.1.

6.2 Proof of Slippery conjecture in a specific case

In their paper [3], Calegari and Walker posed the following conjecture that gives a bound on R in terms of r, s, and the word w. The Slippery Conjecture states that

Conjecture 6.2.1 (Slippery Conjecture)**.** *For any positive w of the form $w = b^{\beta_n} a^{\alpha_n} \cdots b^{\beta_1} a^{\alpha_1}$, if $R(w; r, s) = c/d$ where c/d is reduced, then $|c/d - h_a(w)r - h_b(w)s| \le n/d$.*

We can easily see that $R(w; r, s) \ge h_a(w)r + h_b(w)s$ by considering the representation where a and b both act by rotation. Let $r = p/q$ and $s = u/v$. Now without loss of generality, we fix r and assume that the denominators of r and R are equal. In this specific case, if we want to maximize the difference in left hand side of above inequality, we need to minimize s. The Stairstep Algorithm (§ 3.2) does exactly that. Using the same notations from that section, we use the XY-word given by $W = Y^{t_1}XY^{t_2}XY^{t_3}X\dots Y^{t_q}X$. Since both a and w have periodic orbits of length q, we find that by convexity, extremal solution to the algorithm is obtained when all t_i's are equal. But then,

$$\sum_{i=1}^{q} t_i = v \Rightarrow t_i = \frac{v}{q}$$

and

$$t_{s_i+1} + \dots + t_{s_i+l_i} \le u\beta_i \Rightarrow \frac{v}{q} l_i \le u\beta_i \Rightarrow l_i \le \frac{u\beta_i q}{v}$$

Then

$$\begin{aligned}
q\left|\frac{p}{q} - h_a\frac{c}{q} - h_b\frac{u}{v}\right| &= \left(p - cqh_a - \frac{uqh_b}{v}\right) \\
&= \left(p - c\sum_{i=1}^{nq}\alpha_i - \frac{u}{v}\sum_{i=1}^{nq}\beta_i\right) \\
&= \sum_{i=1}^{nq}\left(\frac{l_i+1}{q} - \frac{u\beta_i}{v}\right) \\
&\le \sum_{i=1}^{nq}\left(\frac{\frac{u\beta_i q}{v}+1}{q} - \frac{u\beta_i}{v}\right) \\
&= \sum_{i=1}^{nq}\frac{1}{q} = n
\end{aligned}$$

Thus we have shown that the Slippery Conjecture is true in this specific case.

Proposition 6.2.2. *For any positive word w of the form $w = b^{\beta_n}a^{\alpha_n}\cdots b^{\beta_1}a^{\alpha_1}$, if $R(w; p/q, s) = c/q$ where c and p are coprime to q, then we have the inequality*

$$|c/q - h_a(w)p/q - h_b(w)s| \le n/q.$$

CHAPTER 7

ARBITRARY WORDS AND THE INTERVAL GAME

7.1 Known results

We recall the definition of *Interval game* here for convenience.

Definition 7.1.1. An *interval game* consists of a collection of elements from $\text{Homeo}_+(S^1)$. We have one player ψ and a finite number of enemies $\phi_1, \phi_2, \ldots, \phi_m$. The goal is to find an *winning* interval $I \subset S^1$.

An interval $I \subset S^1$ *wins* if there exists some positive integer n such that

(i) $\psi^n(I_+)$ is in the interior of I. Here I_+ denotes the rightmost point of I.

(ii) $\psi^i(I)$ is dijoint from $\phi_j(I)$ for all $1 \le i \le n$, $1 \le j \le m$.

Recall that we are only interested in finding a winning interval in the case $\text{rot}^{\sim}(\psi)$ is irrational. Within this restriction, it turns out, we can give an essentially complete description of winning criteria when we have only one enemy. If $\psi = R_\alpha$ and $\phi = R_\beta$ are rigid rotations (up to a semiconjugacy), we can recursively generate an open dense subset U of the unit square such that there is an winning interval if and only if $(\alpha, \beta) \in U$. Following ideas in [3], we give a correct picture of this set U in figure 7.1.

On the other hand if ϕ is not necessarily semiconjugate to a rigid rotation, we have the following sufficiency criteria for winning.

Theorem 7.1.2 ([3]). *consider the interval game with a single enemy ϕ and suppose* $\text{rot}(\psi)$ *is irrational and well-approximated.*[1] *Let μ be an invariant probability measure for ψ. If ϕ does not preserve μ, then an winning interval I exists.*

In fact, we can relax the criteria further. The main requirement for winning is existence of a point $r \in S^1$ such that there is slope 1 straight line that locally supports the graph of ϕ

1. *well-approximated* is defined at the beginning of next section.

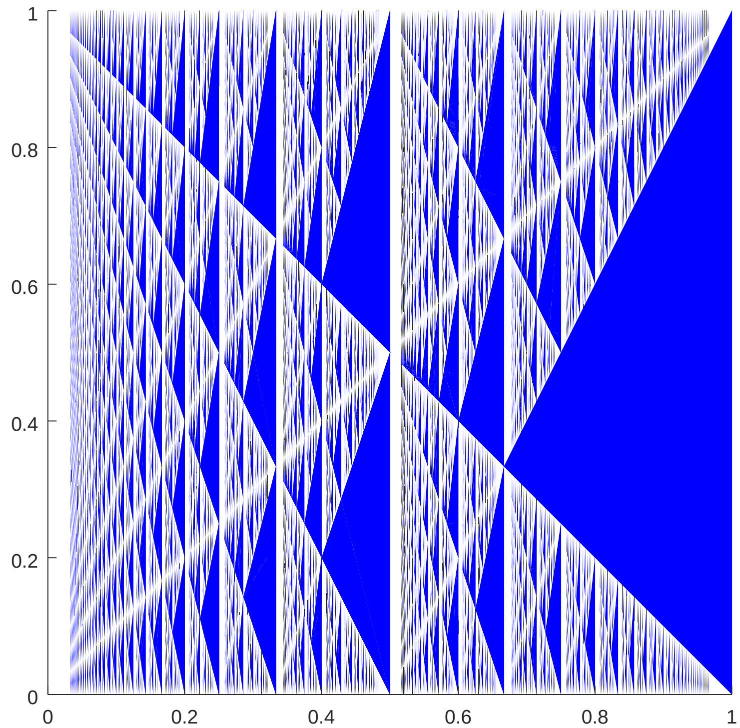

Figure 7.1: The set U containing points for which interval game can be won, 30 iterations

from above in a neighbourhood to the right of r. In other words, there exist $\epsilon \geq 0$, such that for $s > r$ and for $|s-r| \leq \epsilon$, we have $|\phi[r,s]| < |[r,s]|$. We say ϕ is *strongly contracting* to the right of r in this case. Our main result of this chapter is a generalization of this criteria based on a specific observation and gives a winning condition in the case of two or more enemies.

7.2 Proof of theorem 7.1.2

Before stating the main result of this chapter, we will give a short but careful analysis of the proof of theorem 7.1.2, since we will use some of the same techniques to prove the more genral case. Note that the proof of theorem 7.1.2 that appears in [3] is incorrect. We will use similar ideas but differ in the choice of winning interval. To start, we will need the following result from one-dimensional dynamics.

Lemma 7.2.1 ([4]). *Let α be an irrational number and let $r_i := R_\alpha^i(r)$ be the forward orbit of any point $r \in S^1$ under the rigid rotation by α for $i \geq 0$. Then there exists a sequence of* best rational approximations *for α of the form*

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \frac{p_6}{q_6} < \cdots < \frac{p_7}{q_7} < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

that converges to α and satisfies

(i) $|r_{q_n} - r| < |r_{q_{n-1}} - r|$ *and*

(ii) if $j > 0$ and $|r_j - r| \leq |r_{q_n} - r|$, then $j \geq q_n$.

Moreover, r_{q_n}'s converge to r alternately from left and right.

Note that the distance $\theta_n = |r_{q_n} - r|$ doesn't depend on r. We say α is *well-approximated* if $\theta_n/\theta_{n-1} \to 0$. Irrational numbers are generically well-approximated in the sense that they have full measure on the unit interval (see e.g. [4]).

Now first assume that ψ is conjugate to a rigid rotation and rescale coordinates so that $\psi = R_\alpha$, where $\alpha = \mathrm{rot}(\psi)$ is irrational. Then μ has full support and since ϕ does not preserve μ, the graph Γ of ϕ is monotone and does not have slope 1 everywhere.

Hence we can find a point $r \in S^1$ such that there is slope 1 straight line that locally supports the graph of ϕ from above in a neighbourhood to the right of r. In other words, ϕ is strongly contracting to the right of r and there exist $\epsilon \geq 0$, such that for $s > r$ and for $|s - r| \leq \epsilon$, we have $|\phi[r,s]| < |[r,s]|$.

In fact, due to the strictness of above inequality, there exist $\epsilon > 0$ such that for all $0 < \delta \leq \epsilon$, we can find a *minimum* s_δ such that $|\phi[r, s_\delta]| = |r, s_\delta| - \delta$. In fact, it is easy to see from figure 7.2 that ϕ is strongly contracting to the left of every such s_δ.

We will use the notations r_i as in lemma 7.2.1. Note that by the lemma, we can find arbitrarily large m such that $r_m < r$ and $|r_j - r| < |r_m - r|$ implies $j > m$. In particular for $0 \leq a, b \leq m$, we have

$$|a - b| < m \Rightarrow |r_a - r_b| = |r_{a-b} - r| > |r_m - r|$$

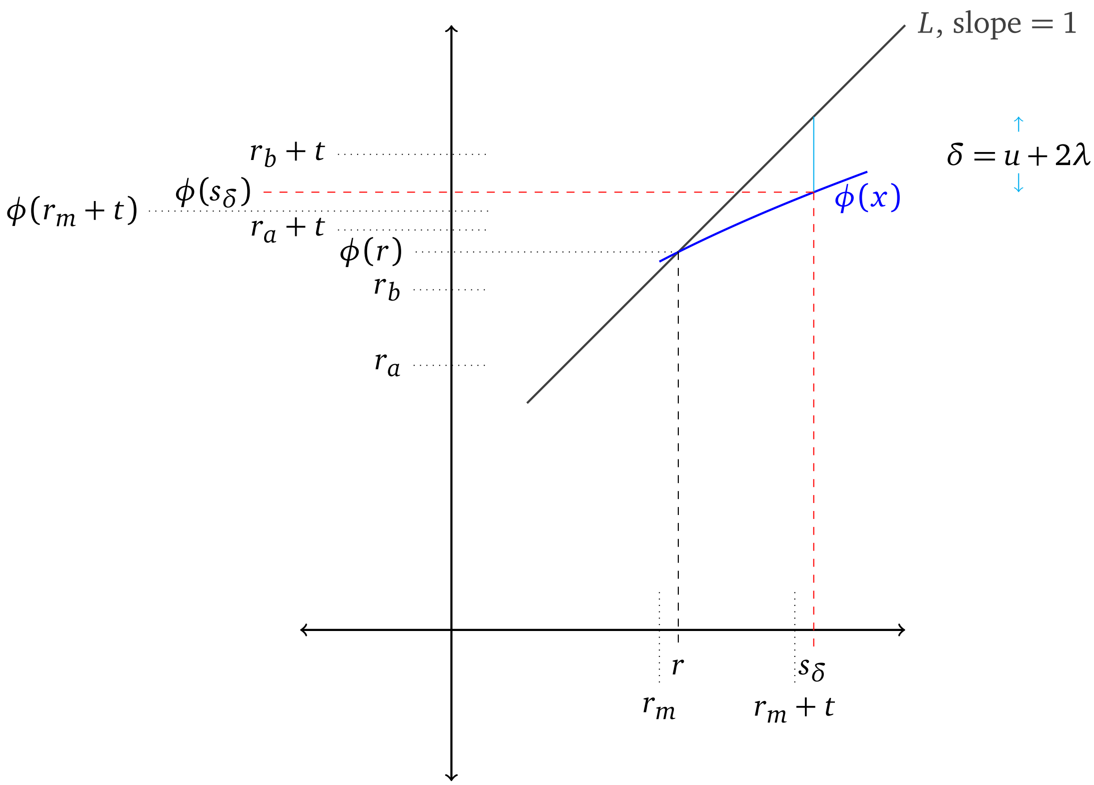

Figure 7.2: The case of one enemy

Let $\lambda = |r_m - r|$. Also let r_a and r_b be the closest two points to the left of $\phi(r)$ in the orbit of r upto r_m. Thus $0 \leq, a, b \leq m$ and

$$r_a < r_b \leq \phi(r)$$

Since α is well approximated, we can take m large enough to ensure

$$|r_b - r_a| = |r_{b-a} - r| > 3\lambda.$$

Let $u = |\phi(r) - r_b|$. Then we can make u as small as we want by making m large enough. In particular, we can assume $u + 2\lambda < \epsilon$, where ϵ is as above. Hence for $\delta = u + 2\lambda$ there exists a point $s_\delta > r$ such that

$$|\phi[r, s_\delta]| = |[r, s_\delta]| - \delta$$

It follows that,

$$\phi(s_\delta) = \phi(r) + (s_\delta - r) - \delta = (\phi(r) - u) - 2\lambda + (s_\delta - r) = r_b + (s_\delta - r) - 2\lambda$$

Let $t = s_\delta - r$. Since ϕ is strongly contracting to the left of $r + t = s_\delta$, we get that $|\phi(r_m + t, r + t)| < |r_m - r| = \lambda$ which implies

$$\begin{aligned} \phi(r_m + t) &> \phi(r + t) - \lambda \\ &= \phi(s_\delta) - \lambda \\ &= r_b - 3\lambda + t \\ &> r_a + t \end{aligned}$$

To summarize, we have the following chain of inequalities

$$r_a + t < \phi(r_m + t) < \phi(r + t) = r_b + t - 2\lambda < r_b + t - \lambda < r_b + t \tag{7.1}$$

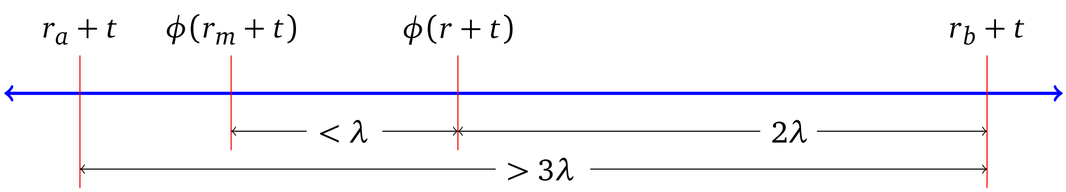

Figure 7.3: Relative Positions

Let's concentrate on the interval $I = [r_m + t, r + t]$. Note that $R_\alpha^b(I) = [r_b + t - \lambda, r_b + t]$. Hence according to the above inequalities, the image of I under ϕ is completely disjoint from its image under the rotation $\psi = R_\alpha$. Consequently, for a suitable choice of $\tau > 0$ to ensure 7.1.1.(ii), the interval $[r_m + t - \tau, r + t]$ is an winning interval.

When μ does not have full support, we replace Γ by the curve $\Gamma' := \left\{ \left(\int_0^r d\mu, \int_0^{\phi(r)} d\mu \right) \right\}$. This curve may contain horizontal and vertical segments. But still a point r as above exists and hence we can use the same argument in this case. $\square$

Remark 7.2.2. Note that s_δ is an increasing function of δ due to the monotonic nature of ϕ.

7.3 Further generalization and conjecture

First we make the following crucial observation in the proof from section § 7.2 that will help us generalize it to the case of multiple enemies. Note that, to satisfy the chain of inequalities 7.1, we don't need δ to be exactly equal to $u + 2\lambda$. In fact, as long as δ is chosen such that $|\phi(r+t) - (r_a + t)|$ and $|\phi(r+t) - (r_b + t)|$ are both more than λ, we will get disjointedness of $\phi(I)$ and $R_\alpha^j(I)$.

Now consider the interval game with two enemies ϕ_1 and ϕ_2 and suppose rotation number of ψ is a well-approximated irrational number. Following the same approach as before, take an invariant probability measure $\mu$for ψ and rescale so that ψ is conjugate to an irrational rotation. If either ϕ_1 or ϕ_2 is locally a rigid rotation at any point, we can simplify to the case of one enemy. So without loss of generality, we can assume that there is no point $r \in S^1$, where either ϕ_i locally preserves μ. We conjecture the following:

Conjecture 7.3.1. *Assume that there exists a point $r \in S^1$ such that ϕ_i's are either* strongly contracting *or* strongly expanding *to the right of r. Then a winning interval exists.*

Let's consider the case when both ϕ_1 and ϕ_2 are strongly contracting to the right of r. As in the proof of the case of one enemy, our goal will be to find a t such that $I = [r_m + t - \mu, r + t]$ will be the winning interval.

Following the same reasoning as last section, there exist $\epsilon_1, \epsilon_2 \geq 0$, such that for $s > r$ and for $|s - r| \leq \epsilon_i$, we have $|\phi_i[r,s]| < |[r,s]|$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

In particular, we can ensure that for all $0 < \delta \leq \epsilon$, we can find $s_{1,\delta}$ and $s_{2,\delta}$ such that

$$|\phi_i[r, s_{i,\delta}]| = |r, s_{i,\delta}| - \delta$$

and ϕ_i's are strongly contracting to the *left* of $s_{i,\delta}$.

For $i = 1, 2$, let r_{a_i} and r_{b_i} be the closest two points to the left of $\phi_i(r)$ in the orbit of r upto r_m where r_m is one of the closest approaches to r from the left. We define $\lambda = |r_m - r|$ and take m large enough to ensure both $|r_{b_i} - r_{a_i}|$ are bigger than 3λ.

Define $u_i = \left|\phi_i(r) - r_{b_i}\right|$ and choose m appropriately so that we can assume $u_i + 2\lambda \le \epsilon$ for both $i = 1, 2$. Here is where we differ from the proof in last section.

Choose δ_i and δ_i' appropriately using u_i's such that the corresponding s_i and s_i' satisfy

$$|\phi(s_i) - (r_{a_i} + s_i - r)| = \lambda \text{ and } |\phi(s_i') - (r_{b_i} + s_i' - r)| = \lambda$$

Then by our observation above and by remark 7.2.2, any point in the interval (s_i, s_i') can work as the right end point of a winning interval if we had only one enemy ϕ_i. In particular, if the intervals have overlap for $i = 1$ and 2, we are done.

Define the right difference quotient in a neighborhood to the right of r as $\liminf\limits_{t \to s^+} \frac{\phi(t) - \phi(s)}{t - s}$ for $s > r$. If the right difference quotients for both ϕ_1 and ϕ_2 are bounded away from 1 in a neighborhood of r, then we can apply a measure-theoretic argument by Calegari-Walker [3] to show that such an overlap exists. If the right difference quotient for ϕ_i is arbitrarily close to 1 at every point to the right of r in a neighborhood, then the structure of ϕ_i becomes very restricted. We hope to leverage these restrictions to finish the proof of conjecture 7.3.1 in future.

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