THE UNIVERSITY OF CHICAGO

RANDOM TOPOLOGICAL STRUCTURES

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BY
WEI CHAI

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ABSTRACT

Data science is a recent focus across many fields such as mathematics, statistics and computer sciences aiming at discovering and understand patterns from data. Topological data analysis (TDA) is a solution to discover the underlying topological structures.

The first part of this thesis is on the random coverage. We estimate the number of data points to cover an underlying manifold with possibly boundaries or singularities and with possibly non-uniform probability distribution. In general, the expected number to cover the space depends not only on the minimum of the probability measure but also on its decaying rates near the minimum point (characterized by two constants).

The second part is about the thresholds of correct Čech homology on a flat torus $\mathbb{T}^d$. The thresholds are estimated by the critical points of distance function and Morse theory. This thesis examines a special type of critical points and makes an improvement on the upper bound of the thresholds over the existing result.

The last part proposes a method to detect topological periodicity in a time series. Topological periodicity allows distortion of the function's domain. The proposed method is about encoding the topological information of a function by a tree. Comparison between functions counting re-parametrization can then be reduced to comparison between two corresponding trees.

CHAPTER 1

INTRODUCTION

Data science is a recent focus across many fields such as mathematics, statistics, and computer sciences aiming at discovering and understand patterns from data. Data points are discrete and random. We need to assume some sophisticated models that describe data and the relations among data points well. Traditional statistical analysis focuses on building appropriate probabilistic models. Data sources and noises are characterized by some adequate probability distribution. Then, the common tasks include purifying the data and making inferences based on the specific probability model.

However, discreteness is a commonly neglected aspect of data. People have to make assumptions about the association of the discrete data sets. The default treatment is to think the samples are from the real vector space $\mathbb{R}^d$, commonly with Euclidean structure, which is one of the simplest topological spaces and simplest example of a (Riemannian) manifold.

In general, topology captures the properties of a space under continuous transformations which is defined in a topological structure. Topology allows us to associate or to differentiate various spaces. For example, the Euclidean space or a solid ball can be continuously deformed to a single point while a circle cannot do so within itself. If distances can be defined between any two points in a space satisfying some conditions, the distance function automatically induces a topological structure on the space and the space is called a metric space. Riemannian manifold is in particular a metric space which is locally a resemblance of the Euclidean space.

The naïve embedding of data into the Euclidean space of $\mathbb{R}^d$ sometimes results in a very high dimension d which is difficult to process. For example, a study in [27] estimates that the face images may live in a space with dimension as high as 100. The data may actually lie on a lower dimensional subspace of $\mathbb{R}^d$, usually a submanifold. Dimensionality reduction and manifold learning methods such as Principal Component Analysis (PCA) are used in data science to discover such low-dimensionality relations.

T-distributed Stochastic Neighbor Embedding (t-SNE) is a popular nonlinear dimensionality reduction technique which is usually used in visualization. It is developed by [26]. The technique provides a map of high-dimensional data to low-dimensional. The distribution of the distances between the resulted lower dimensional data points resembles the distribution of the distances in the original data set. A weakness of such method is that the distances are changed and the topology is not preserved. Other methods like Isomap are developed to bring quasi-isometric maps. However, as shown in [15] , [34], when the data manifolds have complicated topology the algorithm of Isomap may fail to work. Indeed, the embedding of the data set to the lower dimensional Euclidean space is impossible for every situations without a compromise.

Instead of treating the data set as merely a Euclidean space, topological data analysis (TDA) is a solution to discover the underlying topological structures. From the nature of topology, we can expect this method gives results robust to noises since the topological structures generally tolerant perturbation to a certain degree. Some studies about the relation of topological methods and noisy data can be found [28] [2] [13]. For instance, in [28], Niyogi, Smale & Weinberger show that

Theorem (Niyogi, Smale & Weinberger). *Let M be a compact submanifold of $\mathbb{R}^d$ and let μ be a probability measure on $\mathbb{R}^d$ satisfying some conditions involving the submanifold M. Then, there is an algorithm that computes homology from points sampled with respect to μ such that the homology and the homotopy type is the same as M with high probability which can be estimated by the conditions of μ.*

An important object in topological data analysis is persistent homology because it can be used as a tool for measuring the impact of noise. Homology groups of a topological space X is a sequence of Abelian groups $H_0(X), H_1(X), \dots, H_d(X)$. In most cases of this thesis, we will consider homology with field coefficient and then homology groups are vector spaces. The rank of $H_k(X)$ is called the k-th Betti number β_k. The definition of homology can be found [22]. Intuitively, k-th homology groups and the Betti numbers count the number of k-dimensional cycles which are not boundaries of other objects, where cycles are objects with no boundaries. For instance, $H_0(X)$ corresponds to the connected components of X. The persistent homology is defined given a filtration of topological spaces $\{X_r : \text{ some } r\}$ such that

$$X_r \subset X_{r'}, \qquad \text{as long as } r \leq r'. \tag{1.1}$$

Then the inclusion $X_r \subset X_{r'}$ induces homomorphism between homology groups:

$$H_k(X_r) \rightarrow H_k(X_r'). \tag{1.2}$$

The family of homology groups of $H_k(X_r)$ is called the persistent homology. It is proved in [10] that under some condition, the persistent homology (as a family of vector spaces) is isomorphic to the direct sum of vector spaces generated by pairs (a, b) for $a < b$. Here the pair (a, b) can be interpreted as persistent elements living from a to b. For each persistent element given by (a, b), one map plot the point (a, b) on the plane with birth time and death time as coordinate. In this way, we get a persistence diagram that represents the persistent homology. One may also use bars starting from a ending at b to represent the element and get bar code representation for the persistent homology. For detailed references of persistent homology, see [10], [3] , [9].

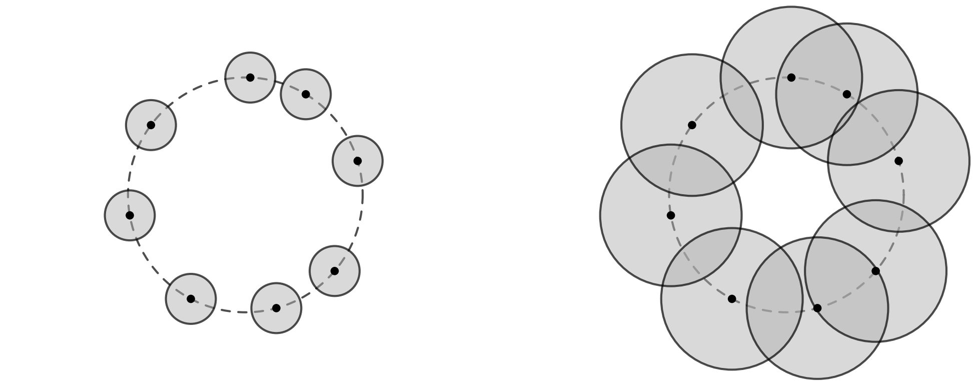
Persistent homology is a powerful tool to discover the topological structure behind the data sets. It already sees applications to a wide variety of fields. [11] and [10] discuss the shape of data sets of 3×3 patches from pictures. With the help of persistent homology, they discover that the data set forms a subspace in $\mathbb{R}^9$ as a Klein bottle. The results in turn have applications in image compression and texture recognition. [32] is a more recent study on social and biological contagions using persistent homology. It recognizes the topological structure of data to construct contagion maps which serve as dimensionality reduction.

The data points are discrete as we mentioned before. To apply topological data analysis such as persistent homology, it is still necessary to convert the discrete data to some meaningful topological structure. A natural idea is to use small balls and to consider their union rather than individual points. Here is an example of such attempt. Suppose we have a collection of data points on a plain shown as below. They are sampled on or near a circle.

In the first figure 1.1a, the union is broken and does not have the topological feature of a circle (i.e. the first homology group is 0.) We may increase the radius or obtain more sample points to get the correct topology like in Figure 1.1b. In such case the circle is covered by the samples. Since the circles live in an ambient plane, if the radius is even larger, we may once again lose the topology as seen in Figure 1.1c. The noises may also create non-contractible structures shown in the last figure 1.1d, having a negative impact on the topological inference.

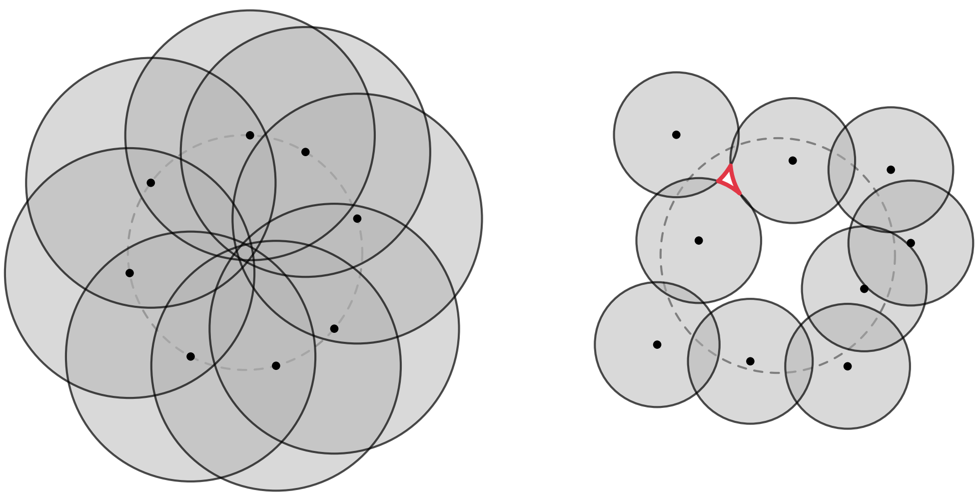
This thesis will mainly focus on the conditions of how phenomena like Figure 1.1a transfer to those like Figure 1.1b. We also only consider intrinsic distances instead of ambient distances (or chordal distances) to avoid the situation where the radii are exceedingly large or noises exist. The intrinsic distance between two data points is the one measured along the space where the data are sampled. We don't assume an ambient space. The balls on a circle are now just segments of arcs shown as below. For a fixed closed Riemannian manifold, if we have sufficiently large number of points and if the distances between points are small, then the chordal distance approximates the geodesic distance on the manifold.

The first part of this thesis will be on the random coverage. The number of data points to cover an underlying space will be estimated. Coverage is a sufficient condition to yield the entire topological structure. It is also useful in multiple applied sciences. [25], [31] are



(a) Radius is too small. The union is homotopy equivalent to the discrete set of points.

(b) Radius is neither too small nor too large. The union is homotopy equivalent to the circle.

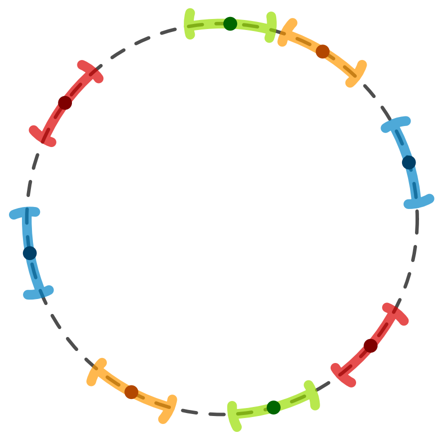


(c) Radius is too large. The union once again loses all the topological features and is homotopy equivalent to a single point.

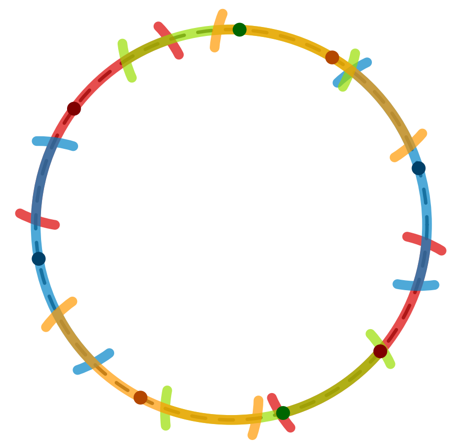
(d) Noises can create extra homology classes (marked as red above as an example) that are not from the underlying space.

Figure 1.1
Various cases of union of balls centered at data points on a circle.

examples of applying coverage problem to biology and military sciences. [18] is one of the pioneers to the problem. Suppose the data points are sampled from a Riemannian manifold with no boundary according to the uniform distribution. If N_{rm} is the number of points necessary to cover the manifold with geodesic balls with radius r by m times, they show that



(a) Radius is too small.



(b) Radius is large. The union covers and equals the circle.

Figure 1.2
Geodesic balls lying on the circle.

Theorem (Flatto & Newman)**.** *For some constant C,*

$$\mathbb{P}\left\{\left|\alpha N_{rm} - \left(\log\frac{1}{\alpha} + (d-m+1)\log\log\frac{1}{\alpha}\right)\right| > x\right\} \le e^{-Cx}. \tag{1.3}$$

Moreover, the expectation satisfies that

Theorem (Flatto & Newman)**.**

$$\mathbb{E}\left[N_{rm}\right] = \frac{1}{\alpha}\left[\log\frac{1}{\alpha} + (d-m+1)\log\log\frac{1}{\alpha} + O(1)\right], \ \text{as } r \to 0, \tag{1.4}$$

where $\alpha = \omega_d r^d$ is the Euclidean volume of a r-ball.

One of the limitations of the result of [18] is that it only deals with closed manifolds. In TDA applications, the manifold hypothesis is a nontrivial one, and it is important to study the effects of boundary or other more serious types of singularity on the coverage problem. On the other hand, the underlying probability measure is not necessarily uniform.

Let us first use a Riemannian manifold with boundary as an example. In a tubular neighborhood of the boundary, the volume of a ball will be significantly smaller than the volume of a ball in the interior. The expected number of the balls to cover a manifold with boundary increases over the number with no boundary. Moreover, the volume of the ball near the boundary changes with respect to the distance to the singularity, the boundary in this case. A Riemannian manifold that we consider here can have multiple types of singularities other than boundaries, for example, a manifold with corners. The probability measure can also be non-uniform. The different types of singularity and non-uniformity behave differently in affecting the coverage. In general, the expected number to cover a manifold depends not only on the minimum of the probability measure m but also on its decaying rates near the minimum point (characterized by constants a and b). It turns out that both the log term and the $\log\log$ term in the expected number need to be modified in both cases.

Theorem (Covering on a manifold with singularities)**.**

$$\mathbb{E}\left[N_r\right] = \frac{1}{m\alpha}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + O(1)\right], \quad \textit{as } r \to 0. \tag{1.5}$$

It should be noted that the expectation is not simply a multiple of the results from Flatto & Newman by a factor of the minimum m of the probability measure in general. Intuitively, the parameter a gives the co-dimension of the singularity and b gives the rate of growth of the measure of balls near the singularity.

In particular, when we have a manifold with a boundary, $m = \frac{1}{2}$, $a = 1$ and $b = 1$ as shown in Section 2.3.2, and hence we have

Theorem (Covering on a manifold with boundary)**.** *Suppose M is a d-dimensional Riemannian manifold with a non-empty boundary and $d \geq 2$. Then,*

$$\mathbb{E}\left[N_r\right] = \frac{2}{\alpha}\left[\frac{d-1}{d}\log\frac{1}{\alpha} + (d-1)\log\log\frac{1}{\alpha} + O(1)\right], \quad \textit{as } r \to 0. \tag{1.6}$$

Suppose we have a closed Riemannian manifold with no boundary is endowed with a probability density with a positive minimum $m > 0$ and the minimum point is non-degenerate. For instance, the density is a Morse function. We will show in Section 2.3.3 that $a = 0$ and $b = \frac{1}{2}$, and thus we have

Theorem (Covering on a manifold with Morse probability density)**.** *If the density function f has a positive and non-degenerate minimum $m > 0$. i.e. the Hessian matrix of f at the minimum point p_m is non-singular, then we have that*

$$\mathbb{E}\left[N_r\right] = \frac{1}{m\alpha}\left[\log\frac{1}{\alpha} + \frac{d}{2}\log\log\frac{1}{\alpha} + O(1)\right], \quad \text{as } r \to 0. \tag{1.7}$$

The proof first isolates different types of singularities by families of neighborhoods and characterizes the growth of the probability measure of balls in each family of sets. Since the general theorem covers a wide variety of problems, the section after the proof is dedicated for several examples. It shows how the general theorem applies to specific problems like Riemannian manifold with smooth boundary, non-uniform probability measure with a single positive minimum density, etc.

The second part of this thesis will be on the homology of Čech complexes derived from data points. There are cases where coverage is too strong a requirement. We may use examine the homology groups instead to get a more careful analysis of data. The union of small balls are usually hard to deal with. Some random complexes are constructed to represent or to approximate the union. [8] provides a good reference on the random structures and their applications in geometric and topological inference.

Čech complexes and Vietoris-Rips complexes are among the most common ones. Fixing radius r, the points $p_0, p_1, p_2, \ldots, p_k$ form a k-simplex of Čech complex if and only if

$$\bigcap_{i=0}^{k} B(p_i; r) \neq \emptyset \tag{1.8}$$

By nerve lemma, the Čech complex is homotopy equivalent to the union of balls. However, direct computing the entire Čech complex is very hard. Vietoris-Rips complex is a similar structure which is easier to compute. Fixing radius r, the points $p_0, p_1, p_2, \ldots, p_k$ form a k-simplex of Vietoris-Rips complex if and only if

$$d(p_i, p_j) \leq r \text{ for any } i, j \in \{0, \ldots, k\}. \tag{1.9}$$

One can prove that the Vietoris-Rips complex with radius r is contained in Čech complex with radius r which is in turn contained in the Vietoris-Rips complex with radius $2r$.

The question can be posed that what threshold of radius r should be to make the complexes have the correct homology groups as the underlying space. For such problems, the flat torus $\mathbb{T}^d$ is a first prototypical example of the underlying space since it has non-trivial homology groups unlike $\mathbb{R}^d$ and we will not be distracted by nuances brought by nonzero curvatures. [6] shows how to generalize the results to Riemannian manifold.

Let $\Lambda = n\omega_d r^d$ where n is the number of points, r is the radius, d is the dimension of the torus $\mathbb{T}^d$ and ω_d is the Euclidean volume of the unit ball in $\mathbb{R}^d$. For H_0 or equivalently the connectivity, [30] uses the Erdös-Rényi random graph model to get that the threshold for connectivity is $\Lambda = 2^{-d} \log n$.

For the connected closed manifold $\mathbb{T}^d$, the top homology H_d is correct if and only if the whole torus is covered. The threshold from the coverage problem [18] is $\Lambda = \log n + (d - 1) \log \log n$.

For the k-th homology, there are multiple regimes where the homology groups behave differently. [5] is a comprehensive survey on this subject. Some results can also be found in [23]. The k-homology first appears at $\Lambda = \Theta(n^{-\frac{1}{k+1}})$ for Čech complexes and at $\Lambda = \Theta(n^{-\frac{1}{2k+1}})$ for Vietoris-Rips complexes. When $\Lambda = \lambda \in (0, \infty)$, the Čech complex satisfies that $\mathbb{E}[\beta_k] \sim n$. For Vietoris-Rips complexes, we have that $\mathbb{E}[\beta_k] = O(n\Lambda^d e^{-c_d \Lambda})$.

The bound of $\mathbb{E}[\beta(n)]$ for Čech complex is more difficult to find. The recent studies [7] (on a flat torus), [6] (on a Riemannian manifold) utilize the critical points of distance function and Morse theory to give the following estimate of the threshold of homology being right.

Theorem (Bobrowski & Weinberger, Bobrowski & Oliveira)**.**

$$\lim_{n\to\infty} \mathbb{E}\left[\beta_k(r)\right] = \begin{cases} \infty & \Lambda = \log n + (k-2)\log\log n - w(n) \\ \beta_k & \Lambda = \log n + k\log\log n + w(n) \end{cases} \tag{1.10}$$

This thesis gives an improvement of the upper bound by a more careful investigation of a certain type of critical points.

Theorem.

$$\lim_{n\to\infty} \mathbb{E}\left[\beta_k(r)\right] = \begin{cases} \infty & \Lambda = \log n + (k-2)\log\log n - w(n) \\ \beta_k & \Lambda = \log n + (k-1)\log\log n + w(n) \end{cases} \tag{1.11}$$

An estimate for the upper bound of the probability of homology being correct is also included in this part. It can be proved that if $\Lambda = n\omega_d r^d$ is sufficiently large, the complement of the union of balls consists of small components.

Theorem (Complement of the union of balls)**.** *Suppose the radius $r < r_{max}/3$ where r_{max} is the injectivity radius of the torus. Then, there exists some constant $1/2 < \kappa < 1$ and for $\Lambda = n\omega_d r^d \geq \kappa \log n$, the probability of each component of the complement being contained in a ball with radius $2r$ tends to 1 as $n \to \infty$.*

Therefore, the global homology classes of the torus cannot exist in the complement by Mayer-Vietoris sequence. The remaining homology classes are all local to small holes (uncovered regions). This gives an estimation of the upper bound in terms of probability.

The last part of this thesis proposes a method to detect topological periodicity in a time series. Unlike the ordinary periodicity which requires that function values by a fixed offset are identical, topological periodicity allows distortion of the function's domain. In other words, if for some re-parametrization ϕ, $f \circ \phi$ is periodic in the common sense, then f is called topologically periodic. A traditional approach to the task is using Fourier transform. See [17]. However, Fourier transform does not work well when the domain is re-parametrized.

If we focus on H_0 rather than higher homology, then the topological periodicity should be preserved regardless of re-parametrization. In [16], the bottleneck distance between persistence diagrams that encodes persistent homology is defined. They show that

Theorem (Edelsbrunner & Harer)**.** *Let $\mathbb{X}$ be a topological space with two functions $f, g : \mathbb{X} \to \mathbb{R}$ satisfying some conditions. Then, for each dimension p, the bottleneck distance between the persistence homology is*

$$d_B(\mathrm{Dgm}_p(f), \mathrm{Dgm})_p(g)) \leq \|f - g\|_\infty = \sup_{x \in \mathbb{X}} |f(x) - g(x)| \,. \tag{1.12}$$

The proposed method in this thesis is about encoding the topological information of a function by a tree. The tree structure is inspired by [4] which describes a natural metric between phylogenetic trees. In particular, the methods of converting between an ordered tree with edge lengths and a function are provided in the last part. Once the distance between trees is defined, we can derive that the distance between two functions counting re-parametrization is bounded by the distance between the corresponding trees, which resembles the result in [16]:

Proposition. *If two continuous functions are defined on a compact interval has finitely many local extreme points, the distance between two functions is less than the distance of their corresponding persistent dendrograms.*

Remark. *Throughout this thesis, we use C, C', or any C_i to indicate unspecified constants. The constants that appear in different places may be different numbers.*

CHAPTER 2

RANDOM COVERING ON A MANIFOLD WITH BOUNDARIES AND SINGULARITIES

2.1 Main Results

2.1.1 Measure Characterization of Boundaries and Singularities

Let M be a compact d-dimensional Riemannian manifold possibly with singularities such as boundaries or corners. Suppose the manifold M is endowed with a probability measure ν. Denote the Riemannian volume of a set U on M by $\mathrm{vol}(U)$ or $\mathrm{vol}\, U$. Define $c(p;r) = \nu(B(p;r))/(\omega_d r^d)$ which is the ratio of the probability measure of the ball centered at $p \in M$ with radius r to the Euclidean volume of the ball with radius r, $\omega_d r^d$.

In order to isolate different levels of singularities, we divide the manifold into several region that "isolate" various singularities. For any radius $r > 0$, suppose $M = \bigcup_i U_i(r)$ for a finite number of regions $U_i = U_i(r)$.

In each of U_i, we have for $|w_i(r)| \leq Cr$ and for any $p \in U_i(r)$,

$$c(p;r) \geq m_i - w_i(r) \tag{2.1}$$

where $m_i > 0$ is a constant independent of r and the volume

$$\mathrm{vol}(U_i(r)) \leq Cr^{a_i} \tag{2.2}$$

for some constants a_i and C that depends only on the manifold M. Assume that $\{m_i\}$ is increasing in i.

The region which is the most difficult to cover is the one which maximizes $\frac{d-a_i}{m_i}$.

Furthermore, for each triple $(m_i, a_i, U_i(r))$ that maximizes $\frac{d-a_i}{m_i}$, we assume there exists a family of subsets $U_i(r;k) \subset U_i(r)$ for integers $k \geq 1$. Let $\epsilon = \frac{1}{|\log r|}$. The sets $U_i(r;k)$ satisfy:

$$\bigcup_{k\geq 1} U_i(r;k) = U_i(r), \tag{2.3}$$

$$C_1 r^{a_i}(k\epsilon)^{b_i} \leq \mathrm{vol}(U_i(r;k)) \leq C_2 r^{a_i}(k\epsilon)^{b_i}, \tag{2.4}$$

$$\text{for any } p \in U_i(r;k),\ c(p;r) \leq m_i(1 + C_3 k\epsilon), \tag{2.5}$$

$$\text{for any } p \in U_i(r) \setminus U_i(r;k),\ c(p;r) \geq m_i(1 + C_4(k+1)\epsilon), \tag{2.6}$$

for some constants b_i, C_1, C_2, C_3, C_4 which do not depend on r. In particular, if we set $U_i'(r;k) = U_i(r;k+1) \setminus U_i(r;k)$ and $U_i'(r;0) = U_i(r;1)$,

$$\mathrm{vol}(U_i'(r;k)) \leq C r^{a_i}\epsilon^{b_i} k^{b_i}, \tag{2.7}$$

$$\text{for any } p \in U_i'(r;k) \text{ and } k \geq 1,$$

$$m_i(1 + C_1 k\epsilon) \leq c(p;r) \leq m_i(1 + C_2(k+1)\epsilon), \tag{2.8}$$

$$\text{for any } p \in U_i'(r;0),$$

$$m_i(1 - w_i(r)) \leq c(p;r) \leq m_i(1 + C_2\epsilon) \tag{2.9}$$

Let $(m, a, U(r), U(r;k), b)$ be any of the tuples $(m_i, a_i, U_i(r), U_i(r;k), b_i)$ such that maximize $\frac{d-b_i}{m_i}$ among those that maximize $\frac{d-a_i}{m_i}$. We have that for any valid i, $\frac{d-a_i}{m_i} \leq \frac{d-a}{m}$ and $\frac{d-b_i}{m_i} \leq \frac{d-b}{m}$. Write $U'(r) = U'(r;0) = U(r;1)$ for short.

Here $U_i(r;k)$ and the parameter b characterize the rate of growth of probability measure within a range near the extreme value m.

Within the set $U'(r)$, we require that the union of two balls has a lower bound of the volume in terms of the distance between the centers. More precisely speaking, for any p, q

in $U'(r)$, we have that if $d(p,q) < 2r$, then

$$\nu(B(p;r) \cup B(q;r)) \geq m\omega_d r^d (1 + C_1 r^{-1} d(p,q) - C_2 \epsilon), \tag{2.10}$$

where $d(p,q)$ is the distance between two points and C_1, C_2 are some constants independent of p, q and r.

We say that a family of sets $U(k)$ on M admits a scale-sensitive ρ-net with respect to the Riemannian volume if there exists a discrete set of points with number bounded from above which covers the union $\bigcup_k U(k)$ by ρ-balls centered at the points and each $U(k)$ contains points with number bounded by the volume. That is, we have the following definition.

Definition 1. *A family of sets $U(k)$ (or just a single set where k has only one value) on the d-dimensional manifold M is said to admit a scale-sensitive ρ-net with respect to the Riemannian volume for any $\rho > 0$, if there exists a discrete set $S \subset \bigcup_k U(k)$ satisfying the following conditions:*

1. *the number of points $|S \cap U(k)| \leq C\rho^{-d}\mathrm{vol}(U(k))$, where C is a constant that only depend on the manifold M and is independent of k or ρ;*

2. *ρ-balls given by $S \cap U(k)$ cover the set $U(k)$, i.e.*

$$\sup_{p \in U(k)} d(p, S \cap U(k)) \leq \rho. \tag{2.11}$$

As an example, for a fixed submanifold P of M, the family of sets of the tubular neighborhood of P with variable radius r admits a scale-sensitive ρ-net with respect to the Riemannian volume when r is sufficiently small. This example will be discussed in Section 2.3.1.

Now return to our topic about singularities. In our settings, we require that all the sets $U_i(r)$ and the families of sets $U_i'(r;k)$ for every sufficiently small $r > 0$ admit scale-sensitive ρ-nets with respect to the Riemannian volume for sufficiently small $\rho > 0$.

For how these conditions fit to specific examples such as a manifold with boundaries or with a non-uniform probabilistic distribution, see Section 2.3.

2.1.2 General Result

Given the assumptions in the previous section, one has the following result:

Theorem 1. *Define the random variable N_r to be the number of balls necessary to cover M with respect to the given probability measure ν. Let $\alpha = \omega_d r^d$ Then,*

$$\mathbb{E}\left[N_r\right] = \frac{1}{m\alpha}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + R(\alpha)\right], \tag{2.12}$$

where $R(\alpha) = O(1)$ as $r \to 0$.

Comparing with the result for coverage on a manifold with no boundary in [18],

$$\mathbb{E}\left[N_r\right] = \frac{1}{\alpha}\left[\log\frac{1}{\alpha} + d\log\log\frac{1}{\alpha} + R(\alpha)\right], \tag{2.13}$$

where $R(\alpha) = O(1)$ as $r \to 0$, the parameter a that characterize the dimension of the singularity affects the log term, the parameter b that characterize the growth of the probability measure affects the $\log\log$ term, and the (usually minimal) value of the probability density makes an overall change to the quantity.

2.2 Proofs

Define a random variable X from N_r

$$N_r = \frac{1}{m\alpha}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + X\right]. \tag{2.14}$$

For each $x \in \mathbb{R}$, denote

$$N = \frac{1}{m\alpha}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + x\right]. \tag{2.15}$$

We will bound the probability by the following two lemmas.

Lemma 1. *For $x > 0$, the probability that N_r is large decays exponentially, i.e.*

$$\mathbb{P}\left\{N_r > N\right\} = \mathbb{P}\left\{X > x\right\} \leq Ce^{-\lambda x}, \tag{2.16}$$

where λ is a positive constant only depending on the manifold M and the probability measure ν on it.

This estimation restricts the probability of having too many points than expected, and thus gives the upper bound. The lemma will be proved in Section 2.2.2.

Lemma 2. *For $x \leq 0$, the probability that N_r is small decays exponentially, i.e.*

$$\mathbb{P}\left\{N_r \leq N\right\} = \mathbb{P}\left\{X \leq x\right\} \leq Ce^{x}. \tag{2.17}$$

This estimation restricts the probability of having too few points than expected, and thus gives the lower bound. The proof is in Section 2.2.3 and 2.2.4.

2.2.1 Proof of Theorem 1

The main theorem follows from the fact

Lemma 3. *Let A be any real random variable. Suppose that the tail distribution is bounded by exponential function, then $\mathbb{E}\left[A\right]$ is finite. That is, if $\mathbb{P}\left\{A > a\right\} \leq Ce^{-\lambda a}$ for some positive constants C and λ, then $\mathbb{E}\left[A\right] < \infty$.*

In fact, $\mathbb{E}[A] \leq \int_0^\infty a d\mathbb{P}\{A \leq a\} = -\lim_{R\to\infty} \int_0^R a d\mathbb{P}\{A > a\}$, and then using integration by parts,

$$\int_0^R x d\mathbb{P}\{A > a\} = R\mathbb{P}\{A > R\} - \int_0^R \mathbb{P}\{A > a\} da \tag{2.18}$$

$$\geq -\int_0^R \mathbb{P}\{A > a\} da \geq -\int_0^R Ce^{-\lambda a}. \tag{2.19}$$

Therefore, $\mathbb{E}[A] < \infty$.

Now take $A = |X|$. From Lemma 2 and Lemma 1, $\mathbb{P}\{A > a\} \leq Ce^{-\lambda a}$ for sufficiently small r and some positive constants C and λ independent on r. From the lemma above, we have that $R(\alpha) = \mathbb{E}[|X|] < \infty$, as $r \to \infty$; that is,

$$\mathbb{E}[N_r] = \frac{1}{m\alpha}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + R(\alpha)\right], \tag{2.20}$$

where $R(\alpha) = O(1)$ as $r \to 0$.

2.2.2 Estimation on Upper Bound

Define

$$\rho = \frac{r}{|\log r|} \quad \text{and } r' = r - \rho. \tag{2.21}$$

Since for all i, the sets $U_i(r;k)$ admit scale-sensitive ρ-nets with respect to k according to the assumptions, there exists sets $S_i \subset U_i(r) = \bigcup_k U_i(r;k)$ of at most $C\rho^{-d} \cdot \mathrm{vol}(U_i(r))$ points for all i. The sets S_i satisfy that the number of points

$$|S_i \cap U_i(r;k)| \leq C\rho^{-d} \cdot \mathrm{vol}(U_i(r;k)). \tag{2.22}$$

and that

$$\text{for any point } p \in U_i, \quad \sup_{p \in U_i} d(p, S_i) \leq \rho, \tag{2.23}$$

$$\text{for any point } p \in U_i(r; k), \quad \sup_{p \in U_i(r;k)} d(p, S_i \cup U_i(r; k)) \leq \rho. \tag{2.24}$$

Set $\mathbf{S} = \bigcup_i S_i$ to be the union of all points in S_i. Now we have that since $M = \bigcup_i U_i(r)$,

$$\sup_{p \in M} d(p, \mathbf{S}) \leq \rho. \tag{2.25}$$

Assume that the radius r is sufficiently small so that $r' = r - \rho \geq r(1 - C/|\log r|) > 0$. Let N' be the number of balls with radius r' necessary to cover $\mathbf{S}$. We have that the balls with radius r covers M if the balls with radius r' cover $\mathbf{S}$ because every point on M is within $\rho = r - r'$ away from some points in $\mathbf{S}$. That is,

$$\mathbb{P}\left\{N_r > N\right\} \leq \mathbb{P}\left\{N' > N\right\}. \tag{2.26}$$

Then, from $1 + t \leq \exp t$ for any $t \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{P}\left\{N_r > N\right\} \leq \mathbb{P}\left\{N' > N\right\} \\ \leq &\sum_{p \in \mathbf{S}} (1 - \nu(B(p; r')))^N \leq \sum_{p \in \mathbf{S}} \exp\left(-\nu(B(p_i; r'))N\right) \end{aligned} \tag{2.27}$$

$$\begin{aligned} = &\sum_{\substack{i, \\ \frac{d-a_i}{m_i} < \frac{d-a}{m}}} \sum_{p \in S_i} \exp\left(-\nu(B(p; r'))N\right) \\ + &\sum_{\substack{i, \\ \frac{d-a_i}{m_i} = \frac{d-a}{m}}} \sum_{p \in S_i} \exp\left(-\nu(B(p; r'))N\right) \end{aligned} \tag{2.28}$$

We first estimate the sums $\sum_{p \in S_i} \exp\left(-\nu(B(p;r'))N\right)$ for i such that $\frac{d-a_i}{m_i} < \frac{d-a}{m}$. For any point p from S_i, we have that $m_i - w_i(r') \leq c(p;r') < m_{i+1}$ and so

$$\nu(B(p;r')) \geq (m_i - w_i(r'))\omega_d r'^d \tag{2.29}$$

$$= (m_i - w_i(r'))(1-\epsilon)^d \omega_d r^d \tag{2.30}$$

Since $|w_i(r)| \leq Cr \leq C\epsilon$, the above is bounded by

$$(m_i - C\epsilon)\omega_d r^d, \tag{2.31}$$

for some constant C that does not depend on r. Then,

$$\sum_{p \in S_i} \exp\left(-\nu(B(p;r'))N\right)$$

$$\leq \sum_{p \in S_i} \exp\left(-m_i \alpha N\right) \exp\left(C\epsilon r^d N\right) \tag{2.32}$$

$$= |S_i| \exp\left(-m_i \alpha N\right) \exp\left(C\epsilon r^d N\right) \tag{2.33}$$

we have that

$$\epsilon r^d N$$

$$= \frac{1}{m\,|\log r|}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + x\right] \tag{2.34}$$

$$\leq C + C\epsilon x. \tag{2.35}$$

Set $\lambda = \frac{1}{2}\min_i \frac{m_i}{m}$. We have that for sufficiently small r,

$$\exp\left(C\epsilon r^d N\right) \leq Ce^{\lambda x}, \tag{2.36}$$

From the inequality (2.2), the number of points in S_i,

$$|S_i| \leq C\rho^{-d} \cdot \mathrm{vol}(U_i) \leq C\rho^{-d} r^{a_i} \leq Cr^{-(d-a_i)}\epsilon^{-d}. \tag{2.37}$$

Since we have $\frac{d-a_i}{m_i} \leq \frac{d-a}{m}$, then it follows that

$$\begin{aligned}
&\sum_{p \in S_i} \exp\left(-\nu(B(p;r'))N\right) \\
&\leq Ce^{\lambda x} \cdot r^{-(d-a_i)}\epsilon^{-d} \exp\left(-m_i \alpha N\right) && (2.38) \\
&\leq Ce^{\lambda x} \alpha^{-\frac{d-a_i}{d}} \left(\log \frac{1}{\alpha}\right)^d \\
&\cdot \exp\left\{-\frac{m_i}{m}\left[\frac{d-a}{d} \log \frac{1}{\alpha} + (d-b) \log\log \frac{1}{\alpha} + x\right]\right\} && (2.39) \\
&\leq C\alpha^{-\frac{m_i}{md}\left(\frac{d-a_i}{m_i} - \frac{d-a}{m}\right)} \left(\log \frac{1}{\alpha}\right)^{d - \frac{m_i}{m}(d-b)} \cdot e^{\lambda x} e^{-\frac{m_i}{m} x}. && (2.40)
\end{aligned}$$

Since $-\frac{m_i}{md}\left(\frac{d-a_i}{m_i} - \frac{d-a}{m}\right) < 0$, $\frac{1}{\alpha} \log \frac{1}{\alpha} \to 0$ as $r \to 0$, and $\lambda \leq \frac{m_i}{2m}$, then for sufficiently small radius r and small volume α, the formula above is bounded:

$$\sum_{p \in S_i} \exp\left(-m_i \alpha N\right) \leq Ce^{\lambda x} e^{-\frac{m_i}{m} x} \leq e^{-\lambda x}. \tag{2.41}$$

We will estimate the next sum $\sum\limits_{p \in S_i} \exp\left(-\nu(B(p;r'))N\right)$ for i that $\frac{d-a_i}{m_i} = \frac{d-a}{m}$. Note that $\frac{d-b_i}{m_i} \leq \frac{d-b}{m}$ for these i's. Unlike the previous estimate, we will further divide $U_i(r)$ into the union of $U_i'(r;k) = U_i(r;k) \setminus U_i(r;k-1)$.

For $U_i'(r;0)$, similar to the estimate in the previous estimate, we have that for any p in $U_i'(r;0)$,

$$\nu(B(p;r')) \geq (m_i - C_0\epsilon)\omega_d r^d, \tag{2.42}$$

In each of $U_i'(r;k)$ for $k \geq 1$ we have that for any $p \in U_i'(r;k)$ from (2.7),

$$c(p;r) \geq m_i(1 + Ck\epsilon). \tag{2.43}$$

Then, the volume

$$\begin{aligned} \nu(B(p;r')) &\geq m_i(1 + Ck\epsilon))(1-\epsilon)^d \omega_d r^d && (2.44) \\ &\geq m_i(1 - C_1\epsilon + C_2 k\epsilon)\omega_d r^d, && (2.45) \end{aligned}$$

for some constant C_1, C_2 that do not depend on k.

Note that the number of points

$$\begin{aligned} \left|S_i \cap U_i'(r;k)\right| &\leq C\rho^{-d} \cdot \mathrm{vol}(U_i'(r;k)) && (2.46) \\ &\leq C(r\epsilon)^{-d} r^{a_i} \epsilon^{b_i} k^{b_i} && (2.47) \\ &= C r^{-(d-a_i)} \epsilon^{-(d-b_i)} k^{b_i} && (2.48) \end{aligned}$$

The sum is now

$$\begin{aligned} &\sum_{p \in S_i} \exp\left(-\nu(B(p;r'))N\right) \\ &= \sum_{k \geq 1} \sum_{p \in S_i \cap U_i'(r;k)} \exp\left(-\nu(B(p;r'))N\right) && (2.49) \\ &\leq \sum_{k \geq 1} \left|S_i \cap U_i'(r;k)\right| \exp\left(-m_i(1 - C_1\epsilon + C_2 k\epsilon)\omega_d r^d\right) && (2.50) \\ &\leq \sum_{k \geq 1} C r^{-(d-a_i)} \epsilon^{-(d-b_i)} k^{b_i} \exp\left(-m_i \alpha N\right) \exp\left(C\epsilon r^d N\right) \exp\left(-Ck\epsilon r^d N\right) && (2.51) \\ &= C r^{-(d-a_i)} \epsilon^{-(d-b_i)} \exp\left(-m_i \alpha N\right) \cdot \exp\left(C\epsilon r^d N\right) \\ &\quad \cdot \sum_{k \geq 1} k^{b_i} \exp\left(-Ck\epsilon r^d N\right). && (2.52) \end{aligned}$$

We know from the estimate of the previous sum that if $\lambda = \frac{1}{2}\min_i \frac{m_i}{m}$, then for sufficiently small r,

$$\exp\left(C\epsilon r^d N\right) \le Ce^{\lambda x}. \tag{2.53}$$

Also since $N \ge \frac{1}{m\alpha}\log\frac{1}{\alpha} = \frac{C}{r^d \epsilon}$,

$$\sum_{k\ge 1} k^{b_i} \exp\left(-Ck\epsilon r^d N\right) \tag{2.54}$$

$$\le \sum_{k\ge 1} k^{b_i} \exp\left(-Ck\epsilon \cdot r^d \frac{C}{r^d \epsilon}\right) \tag{2.55}$$

$$\le \sum_{k\ge 1} k^{b_i} \exp\left(-Ck\right). \tag{2.56}$$

Since the constant C does not depend on k or b_i, the series above converges to some constant $< \infty$.

Thus, we have

$$\sum_{p\in S_i} \exp\left(-\nu(B(p;r'))N\right)$$

$$\le Cr^{-(d-a_i)}\epsilon^{-(d-b_i)} \exp\left(-m_i \alpha N\right) \cdot e^{\lambda x} \tag{2.57}$$

$$\le Cr^{-(d-a_i)}\epsilon^{-(d-b_i)} \cdot e^{\lambda x}$$
$$\cdot \exp\left\{-\frac{m_i}{m}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha} + x\right]\right\} \tag{2.58}$$

$$\le Cr^{-\frac{m_i}{m}\left(\frac{d-a_i}{m_i} - \frac{d-a}{m}\right)} \epsilon^{\frac{d-b_i}{m_i} - \frac{d-b}{m}} \cdot e^{\lambda x} e^{-\frac{m_i}{m}x} \tag{2.59}$$

$$\le Ce^{-\lambda x}. \tag{2.60}$$

The last inequality is from the fact that $\frac{d-a_i}{m_i} = \frac{d-a}{m}$ and $\frac{d-b_i}{m_i} \le \frac{d-b}{m}$.

It concludes from equations (2.60) and (2.41) that

$$\mathbb{P}\left\{X > x\right\} = \mathbb{P}\left\{N_r > N\right\} \leq Ce^{-\lambda x}. \tag{2.61}$$

2.2.3 Estimation on Lower Bound: the far end

Pick constant K such that $K > 4c(p;r)$ for any $p \in M$ and $r > 0$. We first proof the lower bound estimate when $N \leq \frac{1}{K\alpha}\log\frac{1}{\alpha}$; that is, we are going to estimate the probability of coverage when there are much fewer points on the manifold than the desired expectation.

Let $\mathcal{N}$ be a discrete subset of n points on the manifold M such that $n > C/\alpha$ and the pairwise distance between any two points p, q in $\mathcal{N}$ is greater than r. Then, the balls $B(p;r)$ for all $p \in \mathcal{N}$ are all disjoint pairwise. Since $\mathcal{N}$ is a subset, then if N random balls cover the manifold, they cover $\mathcal{N}$, i.e.

$$\mathbb{P}\left\{N_r \leq N\right\} \leq \mathbb{P}\left\{\mathcal{N} \text{ is covered}\right\} \tag{2.62}$$

$$= \mathbb{P}\left\{\begin{array}{c}\text{Every } B(p;r) \text{ for } p \in \mathcal{N} \\ \text{has at least one point from the } N \text{ random points}\end{array}\right\} \tag{2.63}$$

[18] give an estimate of the probability of such event in a multinomial distribution. It can be restated as following.

Lemma 4. *Suppose in a random experiment, E_i, $i = 1, \ldots, n$ are independent random events with probabilities $\mathbb{P}\left\{E_i\right\} = x_i$ and $\sum_{i=1}^n x_i \leq 1$. Now repeat the random experiment independently for N times. Then the probability that all events occur at least once is less than $\prod_{i=1}^n (1 - (1 - x_i)^N)$.*

In our setting, the event E_i is that for p_i in $\mathcal{N}$, $B(p_i;r)$ has at least one point from the

N random points and $\mathbb{P}\{E_i\} = c(p_i; r)\alpha$. Then, by the lemma above,

$$\mathbb{P}\{N_r \leq N\} \leq \prod_{i=1}^{n}(1-(1-c(p_i;r)\alpha)^N). \tag{2.64}$$

Because for sufficiently small t, $1-t > \exp(-2t)$, we have that

$$(1-c(p_i;r)\alpha)^N > \exp(-2c(p_i;r)\alpha N) \tag{2.65}$$

$$\geq \exp\left(-2c(p;r)\alpha \cdot \frac{1}{K\alpha}\log\frac{1}{\alpha}\right) \geq \alpha^{1/2}. \tag{2.66}$$

It follows that

$$\mathbb{P}\{N_r \leq N\} \leq (1-\alpha^{1/2})^n \leq \exp(-C\alpha^{-1/2}). \tag{2.67}$$

For sufficiently small radius r, we have $\alpha^{-1/2}$ is larger than any constant multiple of $\log\frac{1}{\alpha}$, in particular, bigger than x. Therefore, $\mathbb{P}\{N_r \leq N\} \leq \exp(-Cx)$.

2.2.4 Estimation on Lower Bound: near expectation

In this part, we assume $N > \frac{1}{K\alpha}\log\frac{1}{\alpha}$.

Let μ be the volume of the subset of $U'(r)$ where the points are not covered by the first N balls with radius of r. Let χ be the random variable such that $\chi = 0$ if the manifold M is covered by N balls and $\chi = 1$ otherwise.

We have that $\mu = \chi\mu$ and so $\mathbb{E}[\mu] = \mathbb{E}[\chi\mu]$. By Cauchy–Schwarz inequality,

$$\mathbb{E}[\mu]^2 \leq \mathbb{E}\left[\chi^2\right]\mathbb{E}\left[\mu^2\right] = \mathbb{P}\{\chi = 1\}\mathbb{E}\left[\mu^2\right] \leq \mathbb{P}\{N_r > N\}\mathbb{E}\left[\mu^2\right]. \tag{2.68}$$

As a result,

Lemma 5. $\mathbb{P}\{N_r > N\} \geq \mathbb{E}[\mu]^2 / \mathbb{E}[\mu^2]$.

Let $\phi(p)$ be a random variable such that $\phi(p) = 1$ if p is covered by the first N balls and $\phi(p) = 0$ otherwise. It follows that $\mu = \int_{U'(r)} \phi(p)dp$ which is an integral with respect to Riemannian volume. By Fubini's Theorem

$$\mathbb{E}[\mu] = \mathbb{E}\left[\int_{U'(r)} \phi(p)dp\right] = \int_{U'(r)} \mathbb{E}[\phi(p)]\, dp \tag{2.69}$$

$$= \int_{U'(r)} \mathbb{P}\{p \text{ is not covered}\}\, dp = \int_{U'(r)} (1 - c(p;r)\alpha)^N\, dp \tag{2.70}$$

Similarly,

$$\mathbb{E}\left[\mu^2\right] = \mathbb{E}\left[\int_{U'(r)} \int_{U'(r)} \phi(p)\phi(q)dpdq\right] \tag{2.71}$$

$$= \int_{U'(r)} \int_{U'(r)} \mathbb{P}\{\text{both } p \text{ and } q \text{ are not covered}\}\, dp\, dq \tag{2.72}$$

$$= \int_{U'(r)} \int_{U'(r)} (1 - \nu(B(p;r) \cup B(q;r)))^N\, dp\, dq \tag{2.73}$$

We may split the integral into two parts:

$$\mathbb{E}\left[\mu^2\right] = \iint_{d(p,q)\geq 2r} (1 - \nu(B(p;r) \cup B(q;r)))^N\, dp\, dq \tag{2.74}$$

$$+ \iint_{d(p,q)<2r} (1 - \nu(B(p;r) \cup B(q;r)))^N\, dp\, dq \tag{2.75}$$

$$=: \Omega_1 + \Omega_2. \tag{2.76}$$

In the first integral Ω_1, the two balls $B(p;r)$ and $B(q;r)$ centered at p and q respectively do not overlap, while in the second Ω_2, the two points are closer and the two balls will overlap.

We can isolate the effect of two balls $B(p;r)$ and $B(q;r)$ in the first integral Ω_1 since they are not overlapping. The estimate is given by the following result.

Lemma 6.

$$\Omega_1 \leq \mathbb{E}\left[\mu\right]^2 \tag{2.77}$$

This is true since

$$\Omega_1 = \iint_{d(p,q)\geq 2r} (1 - c(p;r)\alpha - c(q;r)\alpha)^N \, dp \, dq \tag{2.78}$$

$$\leq \int_{U'(r)} \int_{U'(r)} (1 - c(p;r)\alpha - c(q;r)\alpha + c(p;r)c(q;r)\alpha^2)^N \, dp \, dq \tag{2.79}$$

$$= \int_{U'(r)} \int_{U'(r)} (1 - c(p;r)\alpha)^N (1 - c(q;r)\alpha)^N \, dp \, dq \tag{2.80}$$

$$\leq \left[\int_{U'(r)} (1 - c(p;r)\alpha)^N \, dp\right] \left[\int_{U'(r)} (1 - c(q;r)\alpha)^N \, dq\right] = \mathbb{E}\left[\mu\right]^2 \tag{2.81}$$

The next step is about the second integral Ω_2. It can be proved to be bounded by the quantity below.

Lemma 7.

$$\Omega_2 \leq \frac{C}{\alpha^{d-1} N^d} \int_{U'(r)} (1 - c(p;r)\alpha)^N \, dp. \tag{2.82}$$

By the assumption (2.10), we know that the union satisfies

$$\nu(B(p;r) \cup B(q;r)) \geq m\omega_d r^d (1 + C_1 r^{-1} d(p,q) - C_2 \epsilon), \tag{2.83}$$

for $\epsilon = \frac{1}{|\log r|}$ and some constants C_1, C_2.

To get Lemma 7, we integrate with respect to one point q first and define the integral for point p,

$$
\begin{aligned}
I(p) &= \int_{d(p,q)<2r} (1-\nu(B(p;r)\cup B(q;r)))^N \, dq && (2.84)\\
&= \int_{d(p,q)<2r} (1-m\omega_d r^d(1+C_1 r^{-1} d(p,q) - C_2\epsilon))^N \, dq && (2.85)\\
&= \int_{d(p,q)<2r} (1-m\alpha - C_1 r^{d-1} d(p,q) + C_2\alpha\epsilon)^N \, dq && (2.86)
\end{aligned}
$$

We have that

$$
\begin{aligned}
&1-m\alpha - C_1 r^{d-1} d(p,q) + C_2\alpha\epsilon\\
&\le (1-m\alpha + C_2\alpha\epsilon)(1-C_1 r^{d-1} d(p,q)) && (2.87)
\end{aligned}
$$

for small r such that $m > C_2\epsilon = \frac{C_2}{|\log r|}$

Then the integral in (2.84) becomes

$$
\begin{aligned}
I(p) &= \int_{d(p,q)<2r} (1-m\alpha - C_1 r^{d-1} d(p,q) + C_2\alpha\epsilon)^N \, dq && (2.88)\\
&\le \int_{d(p,q)<2r} (1-m\alpha + C_2\alpha\epsilon)^N (1-C_1 r^{d-1} d(p,q))^N \, dq && (2.89)\\
&\le (1-m\alpha + C_2\alpha\epsilon)^N \int_{d(p,q)<2r} (1-C_1 r^{d-1} d(p,q))^N \, dq && (2.90)
\end{aligned}
$$

Now use the polar coordinates at point p. For $\delta \in \mathbb{R}$ and σ in a region of the sphere S^{d-1}, let $q = \exp_p(\delta\sigma)$ where exp is the exponential map on the Riemannian manifold M. Then, the Jacobian of the coordinate transform is uniformly bounded by $\delta^{d-1} J_0$ for $\delta = d(p,q) < 2r$, where $J_0 > 0$ is a constant independent on δ. Then,

$$\begin{aligned}
&\int_{d(p,q)<2r} (1 - C_1 r^{d-1} d(p,q))^N \, dq \\
&\leq \int_{S^{d-1}} \int_0^{2r} (1 - C_1 r^{d-1} \delta)^N \delta^{d-1} J_0 \, d\delta \, d\sigma \qquad (2.91) \\
&\leq C \int_0^{2r} (1 - C_1 r^{d-1} \delta)^N \delta^{d-1} \, d\delta. \qquad (2.92)
\end{aligned}$$

Set $t = C_1 r^{d-1} \delta$, then it follows

$$\begin{aligned}
&\int_{d(p,q)<2r} (1 - C_1 r^{d-1} d(p,q))^N \, dq \\
&\leq C \int_0^{C_1 r^d} (1-t)^N (C r^{-d+1} t)^{d-1} C r^{-d+1} \, dt \qquad (2.93) \\
&\leq C r^{-d(d-1)} \int_0^1 (1-t)^N t^{d-1} dt \qquad (2.94) \\
&= \frac{C}{\alpha^{d-1}} \cdot \frac{N!}{(N+d)!} \leq \frac{C}{\alpha^{d-1} N^d}. \qquad (2.95)
\end{aligned}$$

On the other hand, since $1 + t \leq e^t$ for any $t \in \mathbb{R}$,

$$\begin{aligned}
&(1 - m\alpha + C_2 \alpha \epsilon)^N \\
&= (1 - m\alpha)^N \left(1 + \frac{C_2 \alpha \epsilon}{1 - m\alpha}\right)^N \qquad (2.96) \\
&\leq (1 - m\alpha)^N \cdot \exp\left(\frac{C_2 \alpha N \epsilon}{1 - m\alpha}\right) \qquad (2.97) \\
&\leq (1 - m\alpha)^N \cdot \exp\left(\frac{C_2}{\log \frac{1}{\alpha}} \left[\frac{d-a}{d} \log \frac{1}{\alpha} + (d-b) \log\log \frac{1}{\alpha}\right]\right) \qquad (2.98) \\
&\leq C(1 - m\alpha)^N, \qquad (2.99)
\end{aligned}$$

for sufficiently small r (and α). We then have that

$$I(p) \leq \frac{C(1-m\alpha)^N}{\alpha^{d-1}N^d} \tag{2.100}$$

It concludes that

$$\Omega_2 = \int_{U'(r)} I(p)\, dp \tag{2.101}$$

$$\leq \frac{C\mathrm{vol}(U'(r))}{\alpha^{d-1}N^d}(1-m\alpha)^N \tag{2.102}$$

Now combining Lemma 6 and Lemma 7, we can prove Lemma 2.

$$\mathbb{P}\{N_r \leq N\} \leq 1 - \mathbb{E}[\mu]^2 / \mathbb{E}\left[\mu^2\right] \tag{2.103}$$

$$\leq 1 - \frac{\mathbb{E}[\mu]^2}{\Omega_2 + \mathbb{E}[\mu]^2} \leq \frac{\Omega_2}{\mathbb{E}[\mu]^2} \tag{2.104}$$

$$\leq \frac{\frac{C\mathrm{vol}(U'(r))}{\alpha^{d-1}N^d}(1-m\alpha)^N}{\left[\int_{U'(r)}(1-c(p;r)\alpha)^N dp\right]^2} \tag{2.105}$$

$$= \frac{C\mathrm{vol}(U'(r))(1-m\alpha)^N}{\alpha^{d-1}N^d\left[\int_{U'(r)}(1-c(p;r)\alpha)^N\, dp\right]^2}. \tag{2.106}$$

Since $U'(r)$ only contains points with $c(p;r) \leq m(1+C\epsilon)$. The integral in (2.106) can be further simplified to

$$\int_{U'(r)}(1-c(p;r)\alpha)^N dp$$

$$\geq \int_{U'(r)} \left[1 - m(1+C\epsilon)\alpha\right]^N\; dp \tag{2.107}$$

$$\geq \mathrm{vol}(U'(r))\left[1 - m(1+C\epsilon)\alpha\right]^N \tag{2.108}$$

Note that $1-t \geq e^{-2t}$ for $0 \leq t < 1/2$. Suppose r is sufficiently small so that $1-m\alpha > 1/2$ and $m\epsilon\alpha < 1/4$. Also note that $\epsilon = \frac{1}{|\log r|}$. It follows

$$[1 - m(1 + C\epsilon)\alpha]^N = (1 - m\alpha)^N \left(1 - \frac{Cm\epsilon\alpha}{1 - m\alpha}\right)^N \tag{2.109}$$

$$\geq (1 - m\alpha)^N (1 - 2Cm\epsilon\alpha)^N \geq (1 - m\alpha)^N \exp(-4Cm\epsilon\alpha N) \tag{2.110}$$

$$= (1 - m\alpha)^N \cdot \exp\left(-\frac{4C}{\log\frac{1}{\alpha}}\left[\frac{d-a}{d}\log\frac{1}{\alpha} + (d-b)\log\log\frac{1}{\alpha}\right]\right) \tag{2.111}$$

$$\geq C(1 - m\alpha)^N, \text{ for sufficiently small } \alpha \text{ (and } r\text{)}. \tag{2.112}$$

Then, the integral

$$\int_{U'(r)} (1 - c(p;r)\alpha)^N dp \tag{2.113}$$

$$\geq C\mathrm{vol}(U'(r))\,[1 - m(1 + C\epsilon)\alpha]^N \tag{2.114}$$

$$\geq C\mathrm{vol}(U'(r))\,(1 - m\alpha)^N \tag{2.115}$$

It follows that

$$\mathbb{P}\{N_r \leq N\} \leq \frac{C\mathrm{vol}(U'(r))(1 - m\alpha)^N}{\alpha^{d-1}N^d\left[\int_{U'(r)}(1 - c(p;r)\alpha)^N\,dp\right]^2} \tag{2.116}$$

$$\leq \frac{C\mathrm{vol}(U'(r))(1 - m\alpha)^N}{\alpha^{d-1}N^d\left[\mathrm{vol}(U'(r))(1 - m\alpha)^N\right]^2} \tag{2.117}$$

$$= \frac{C}{\alpha^{d-1}N^d\mathrm{vol}(U'(r))(1 - m\alpha)^N} \tag{2.118}$$

$$\leq \frac{C\left(\log\frac{1}{\alpha}\right)^b}{(\alpha N)^d\alpha^{a/d-1}(1 - m\alpha)^N} \tag{2.119}$$

Use $1 - t \geq e^{-t-t^2}$ for sufficiently small $t > 0$ and the fact that

$$\alpha^2 N \leq \frac{\alpha}{m} \left(\log \frac{1}{\alpha} + d \log \log \frac{1}{\alpha} \right) \tag{2.120}$$

is bounded above by a constant depending only on m and d. We can bound the quantity

$$\begin{aligned} (1 - m\alpha)^N &\geq \exp\left(-m\alpha N - m^2 \alpha^2 N \right) && (2.121) \\ &\geq C \exp(-m\alpha N). && (2.122) \end{aligned}$$

Note that we assume $N > \frac{1}{K\alpha} \log \frac{1}{\alpha}$ where K is a constant. Then, Equation (2.106) becomes

$$\mathbb{P}\left\{ N_r \leq N \right\} \leq \frac{C \left(\log \frac{1}{\alpha} \right)^b \cdot e^{m\alpha N}}{\alpha^{a/d-1} (\alpha N)^d} \tag{2.123}$$

$$\leq \frac{C \left(\log \frac{1}{\alpha} \right)^b \cdot \exp\left(\frac{d-a}{d} \log \frac{1}{\alpha} + (d-b) \log \log \frac{1}{\alpha} + x \right)}{\alpha^{a/d-1} \left[(\log \frac{1}{\alpha})/K \right]^d} \tag{2.124}$$

$$\leq \frac{C \left(\log \frac{1}{\alpha} \right)^b \cdot \alpha^{a/d-1} \left(\log \frac{1}{\alpha} \right)^{d-b}}{\alpha^{a/d-1} \cdot \left(\log \frac{1}{\alpha} \right)^d} e^x = Ce^x. \tag{2.125}$$

Here, the constant C only depend on the dimension d, the manifold M and the probability measure ν. This finishes the proof for Lemma 2.

2.3 Examples

The previous section gives the proof of the general result about random covering. One may be interested in particular cases like a manifold with a boundary or a manifold with no boundary but endowed with a non-uniform probabilistic distribution. We will discuss a few specific examples and show how the result before applies in those situations in this section.

According to Section 2.1.1, we need to divide the manifold M into several regions $U_i(r)$ and $M = \bigcup_i U_i(r)$, each with different properties of measures. The regions must satisfy requirements (2.1), (2.2), (2.4), (2.7), (2.10) and (2.11). In particular, the parameters m, a, b are from the following statements

1. In each $U_i(r)$, for $p \in U_i(r)$

$$c(p;r) \geq m_i - w_i(r). \tag{2.126}$$

2. The volume

$$\mathrm{vol}(U_i(r)) \leq C r^{a_i}. \tag{2.127}$$

3. For i that maximizes $\frac{d-a_i}{m_i}$, we further divide $U_i(r) = \bigcup_k U_i(r;k)$ and

$$C_1 r^{a_i}(k\epsilon)^{b_i} \leq \mathrm{vol}(U_i(r;k)) \leq C_2 r^{a_i}(k\epsilon)^{b_i}. \tag{2.128}$$

2.3.1 Tubular Neighborhood

In most of the following examples, the region may appear as a tubular neighborhood of a submanifold in M. Let P be a compact q-dimensional submanifold possibly with boundaries in M. Define the closed tubular neighborhood of radius $r \geq 0$ about P to be the set

$$\begin{aligned} T(P;r) = \{m \in M : m = \exp_p v \text{ for some } p \in P \\ \text{and } v \in N_pP \text{ with } \|v\| \leq r\}, \end{aligned} \tag{2.129}$$

where N_pP is the normal space of P at point p. That is, $T(P;r)$ is the set of points in m that has a geodesic with length less than r and meet P orthogonally.

The volume of a tubular neighborhood is studied in [35] and [21]. The estimation in [21] gives that if P is q-dimensional,

$$C_1 r^{d-q} \leq \mathrm{vol}(T(P;r)) \leq C_2 r^{d-q}, \tag{2.130}$$

for some constant C and sufficiently small r.

On the other hand, for some small r, the map exp. is a diffeomorphism from the disk bundle in NP to the tubular neighborhood $T(P;r)$. That means $T(P;r)$ is locally diffeomorphic to $B_{\mathbb{R}^{d-q}}(0;r) \times U$ for a neighborhood U on P and $B_{\mathbb{R}^{d-q}}(0;r)$ is the Euclidean ball with radius r in $\mathbb{R}^{d-q}$. Let ϕ be the diffeomorphism. For any $\rho > 0$, we have that there exists discrete sets $S_1 \subset B_{\mathbb{R}^{d-q}}(0;1)$ and $S_2 \subset U$ satisfying the following properties.

For $S_1 \subset B_{\mathbb{R}^{d-q}}(0;1)$, we require that for every $0 < r' \leq r$,

$$\sup_{p \in B_{\mathbb{R}^{d-q}}(0;r')} d(p, S_1 \cap B_{\mathbb{R}^{d-q}}(0;r')) \leq \rho, \tag{2.131}$$

and the number of points

$$\left|S_1 \cap B_{\mathbb{R}^{d-q}}(0;r')\right| \leq C r'^{d-q} \rho^{-(d-q)}. \tag{2.132}$$

For $S_2 \subset U$, we require that for any point $p \in U \subset P$,

$$\sup_{p \in U} d(p, S_2) \leq \rho, \tag{2.133}$$

and the number of points

$$|S_2| \leq C\rho^{-q}, \tag{2.134}$$

Now let $S = S_1 \times S_2 \in B_{\mathbb{R}^{d-q}}(0; r) \times U$, we have that the number of points for $0 < r' \leq r$ is

$$\left|S \cap \mathrm{vol}(T(P; r'))\right| \leq C r'^{d-q} \rho^{-(d-q)} \cdot C\rho^{-q} \tag{2.135}$$

$$\leq C\rho^{-d} r'^{d-q} \leq C\rho^{-d} \mathrm{vol}(T(P; r')). \tag{2.136}$$

Moreover, the distance from any point in $B_{\mathbb{R}^{d-q}}(0; r') \times U$ to S is bounded by some constant times ρ. Notice that $T(p; r)$ is compact, the union of all the sets $S' = \phi(S)$ under the diffeomorphism is a scale-sensitive ρ-net with respect to the Riemannian volume given by Definition 1.

As a summery, we have the following lemma

Lemma 8. *Let P be a compact q-dimensional submanifold possibly with boundaries in M. Then, the closed tubular neighborhood of radius $r \geq 0$ about P defined above, $T(P; r)$, satisfies the following properties:*

1. $C_1 r^{d-q} \leq \mathrm{vol}(T(P; r)) \leq C_2 r^{d-q}$
2. *For any $\rho > 0$, there exists a discrete set S in $T(P; r)$ and S is a scale-sensitive ρ-net; that is, for any $0 < r' \leq r$,*
   - (a) *the number of points $\left|S \cap T(P; r')\right| \leq C\rho^{-d} \mathrm{vol}(T(P; r'))$, where C is a constant that only depend on the manifold M and is independent of r' and ρ;*

(b) ρ-balls given by $S \cap T(P; r')$ cover the set $T(P; r')$, i.e.

$$\sup_{p \in T(P;r')} d(p, S \cap T(P; r')) \leq \rho.$$

2.3.2 Manifold with Boundaries

Here, we consider the random covering problem on a compact d-dimensional Riemannian manifold M with a boundary and with uniform probability distribution. We assume that the manifold is subset of a d-dimensional manifold with no boundary $\tilde{M}$ while the total volume of $M \subset \tilde{M}$ is still 1. Use $\tilde{B}(p; r)$ to denote the Riemannian ball on $\tilde{M}$ with center p and radius r. We have that $B(p; r) = M \cap \tilde{B}(p; r)$ for p in M. Let $\tilde{\exp}_p$ be the exponential map on $\tilde{M}$.

Partition of the Manifold

The small geodesic balls entirely lying in the interior of M has the same volume as in the $\tilde{M}$, the manifold without boundary. If the center is p and the radius is r, we have that $\text{vol}(B(p; r)) = \omega_d r^d + O(r^{d+1})$ and so

$$c(p; r) = 1 + O(r) \text{ as } r \to 0. \tag{2.137}$$

Define $U_0(r) = \{p \in M : d(p, \partial M) > r\}$. Then the volume of this set $\text{vol}(U_0(r)) \leq 1$. Since the volume does not depend on r, the ordinary ρ-net (or commonly known as ϵ-net) on the whole manifold gives a scale-sensitive ρ-net on U_0 which we need. Here, in $U_0(r)$, the parameters are

$$m_0 = 1,\ a_0 = 0. \tag{2.138}$$

For the rest part, $U_\partial(r) = \{p \in M : d(p, \partial M) < 2r\}$ is the set of all the points near the boundary. It is in a tubular neighborhood of the boundary ∂M in $\tilde{M}$. From Lemma 8,

$$\mathrm{vol}(U_\partial(r)) \leq C r^{d-1}, \tag{2.139}$$

and the scale-sensitive ρ-net necessary in the proof exists.

Volume of Balls Near the Boundary

Next, we need to estimate $c(p; r)$ for p in $U_\partial(r)$. Intuitively, the volume of balls near the boundary when the radius is small is about the half of the normal one. We may heuristically start from an Euclidean ball capped by a plane. In that case, when the plane is close to the center with distance δ, the Euclidean volume is greater than $1/2\omega_d r^d$ and is $1/2\omega_d r^d + O(r^{d-1}\delta)$, i.e. bounded by $1/2\omega_d r^d + C r^{d-1}\delta$ with different positive constants C when δ is small. The reason is that the ball capped by a plane is the union of a half ball and a region which can be bounded by two cylinders with height δ and the base as $d-1$ balls.

We will show that the similar formula holds for $c(p; r)$ in $U_\partial(r)$. More precisely, we claim that for p in $U_\partial(r)$,

$$c(p; r) \geq \frac{1}{2} + O(r),\ \text{ as } r \to 0, \tag{2.140}$$

and if we set $U'_\partial(r;k) = \left\{p \in M : \frac{kr}{|\log r|} \le d(p, \partial M) \le \frac{(k+1)r}{|\log r|}\right\}$ for $k \ge 0$ and

$$U_\partial(r;k) = \bigcup_{i<k} U'_\partial(r;i), \tag{2.141}$$

then,

$$\frac{1}{2}(1 + C_1 k\epsilon) \le \frac{1}{2} + C_1 r^{-1} d(p;\partial M) \tag{2.142}$$

$$\le c(p;r) \le \tag{2.143}$$

$$\frac{1}{2} + C_2 r^{-1} d(p;\partial M) \le \frac{1}{2}(1 + C_2(k+1)\epsilon), \tag{2.144}$$

$$\text{for } p \in U'_\partial(r;k) \tag{2.145}$$

and

$$\frac{1}{2} + O(r) \le c(p;r) \le \frac{1}{2} + C_2 r^{-1} d(p;\partial M) \le \frac{1}{2}(1 + C_2\epsilon). \tag{2.146}$$

Using results from Section 2.3.1 about tubular neighborhood,

$$\mathrm{vol}(U_\partial(r)) \le Cr \text{ and } \frac{C_1 kr}{|\log r|} \le \mathrm{vol}(U_\partial(r;k)) \le \frac{C_2 kr}{|\log r|}. \tag{2.147}$$

In fact, suppose $p \in U_\partial(r)$ and $q \in \partial M$ such that $d(p,q) = d(p,\partial M)$. Then the geodesic joining p and q meets ∂M orthogonally. Assume that the radius is small and $\tilde{\exp}_p$ gives a diffeomorphism from a ball with radius r and center at origin in Euclidean space T_pM, to the ball surrounding p. Let Q be the surface which is the intersection of $\tilde{B}(p;r)$ and the image under the diffeomorphism $\tilde{\exp}_p$ of the plane through the pre-image of q orthogonal to the line joining the pre-image of q and the origin. Let $P = \partial M \cap \tilde{B}(p;r)$ be the boundary within the ball at p. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference. We will estimate the volume between P and Q which is the difference of the ball on the boundary and the image of a Euclidean ball capped by a plane, $B(p;r)\,\Delta\,\tilde{\exp}_p$(Euclidean ball capped by plane).

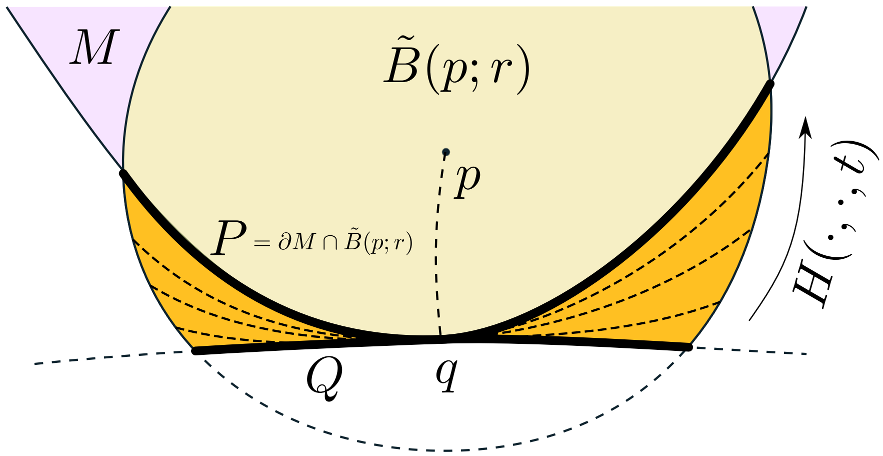


Figure 2.1
Variation H from surface Q to P.

We write S^{d-2} to denote a $(d-2)$-dimensional unit sphere of T_qM on the $(d-1)$-dimensional hyperplane orthogonal to the direction to p. Let ρ be the radius of the pre-image of Q under $\tilde{\exp}_p$ which is a Euclidean ball. Now we define a variation from Q to P in polar coordinates, $H : S^{d-2}\times[0,\rho]\times[0,1]\to\tilde{M}$, satisfying that the image of $H(\cdot,\cdot,0)$ is $Q\subset\partial M$, the image of $H(\cdot,\cdot,1)$ is P, $\frac{\partial H}{\partial t}(u,0,s)=u$ for any u and s, and the image of H contains the region between P and Q in the ball at p. We need the Jacobian of the variation H.

Define a vector field $X(t)$ along $H(u,t,s)$ such that

$$X(t)=\frac{\partial H}{\partial s}(u,t,s). \tag{2.148}$$

Similarly, for a certain frame $(u_1,u_2,\ldots,u_{d-2})$ on S^{d-2}, define vector fields $Y_i(t)$ along $H(u,t,s)$ such that $Y_i(t)=\frac{\partial H}{\partial u_i}(u,t,s)$. Then, since $H(u,0,s)=q$ is a constant map for any u and s, we have that

$$X(0)=0,\ \text{ and} \tag{2.149}$$

$$Y_i(0)=0 \quad \text{for all } i=1,\ldots d-2. \tag{2.150}$$

Moreover,

$$X'(t) = \nabla_{\frac{\partial}{\partial t}} \frac{\partial H}{\partial s} = \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial t} = 0, \tag{2.151}$$

because $\frac{\partial H}{\partial t}(u, 0, s) = u$ is constant for any s at $t = 0$.

By Taylor theorem, we have that as $t \to 0$,

$$Y_i(t) = tY_i'(0) + O(t^2) \quad \text{for all } i = 1, \ldots d-2, \tag{2.152}$$

$$X(t) = t^2 X''(0) + O(t^3). \tag{2.153}$$

It follows that the Jacobian J_H of the variation H at (u, t, s) is

$$|J_H(u, t, s)| \leq t^d J_0 \text{ as } t \to 0, \tag{2.154}$$

where J_0 is a certain constant. Therefore, we have that the volumes

$$\left|\mathrm{vol}(B(p;r)) - \mathrm{vol}(\tilde{\exp}_p(\text{Euclidean ball capped by plane}))\right| \tag{2.155}$$

$$\leq \mathrm{vol}[B(p;r) \Delta \tilde{\exp}_p(\text{Euclidean ball capped by plane})] \tag{2.156}$$

$$\leq \int_{\text{region between } P \text{ and } Q} 1 \tag{2.157}$$

$$\leq \int_{S^{d-1} \times [0,\rho] \times [0,1]} |J_H(u, t, s)| \, du \, dt \, ds \tag{2.158}$$

$$\leq C\rho^{d+1} \tag{2.159}$$

$$\leq Cr^{d+1}, \tag{2.160}$$

for sufficiently small r and thus ρ and $t \in [0, \rho]$ are small.

From [24], for an orthonormal basis $\{u_1, \ldots, u_{d-1}\}$ and $u = u_1$, the Jacobian of $\exp_p$ is

$$\exp_p^* \mathrm{vol}(ru) = \left[1 + \mathrm{Ric}(u,u)\frac{r^2}{6} + O(r^3)\right] \mathrm{vol}_e(ru), \tag{2.161}$$

where $\mathrm{vol}_e(ru)$ is the Euclidean volume form.

It concludes that firstly, as $r \to 0$,

$$\mathrm{vol}(B(p;r)) \geq \frac{1}{2}\omega_d r^d(1 + Cr^2) \tag{2.162}$$
$$\geq \frac{1}{2}\omega_d r^d + O(r^{d+2}) \tag{2.163}$$

for any $\delta = d(p, \partial M)$. If further we have $\frac{r}{|\log r|} \leq \delta = d(p, \partial M) \leq \frac{2r}{|\log r|}$, then $\delta \gg r^2$. Since

$$\left(\frac{1}{2}\omega_d r^d + O(r^{d-1}\delta)\right)(1 + O(r^2)) = \frac{1}{2}\omega_d r^d + O(r^{d-1}\delta), \tag{2.164}$$

then we have

$$\frac{1}{2}\omega_d r^d + C_1 r^{d-1}\delta \leq \mathrm{vol}(B(p;r)) \leq \frac{1}{2}\omega_d r^d + C_2 r^{d-1}\delta. \tag{2.165}$$

This finishes the proof for the claim.

We may derive from the claim that in $U_\partial(r)$,

$$m_\partial = \frac{1}{2}, \tag{2.166}$$
$$a_\partial = 1, \tag{2.167}$$
$$b_\partial = 1. \tag{2.168}$$

We have that $\frac{d-a_\partial}{m_\partial} = 2(d-1)$ and $\frac{d-a_0}{m_0} = d$. For $d \geq 2$, $\frac{d-a_\partial}{m_\partial} \geq \frac{d-a_0}{m_0}$ so $(m, a, b) = (\frac{1}{2}, 1, 1)$.

Volume of Two Balls Near the Boundary

We still need to estimate the volume of the union $\mathrm{vol}(B(p;r) \cup B(q;r))$ for p, q in $U_\partial(r)$ for p, q from $U'(r)$ and $d(p,q) < 2r$ and show that

$$\mathrm{vol}(B(p;r) \cup B(q;r)) \geq m\omega_d r^d (1 + C_1 r^{-1} d(p,q) - C_2 \epsilon). \tag{2.169}$$

In this part, we will frequently use the following simple facts in set theory:

Lemma 9. *For sets A, B, A', B', U, we have that*

$$(U \cap A)\Delta(U \cap B) \subset A\Delta B, \tag{2.170}$$

$$A\Delta B \subset (A\Delta B') \cup (B\Delta B'), \text{ and} \tag{2.171}$$

$$\begin{aligned} A \setminus B &\subset (A' \setminus B') \cup (A' \setminus A) \cup (B' \setminus B) \\ &\subset (A' \setminus B') \cup (A'\Delta A) \cup (B'\Delta B). \end{aligned} \tag{2.172}$$

Since $U'(r)$ is a tubular neighborhood of the boundary with small radius, the set $U'(r)$ is diffeomorphic to a disk bundle. Suppose $p_0 \in \partial M$ is the projection of the point p to the boundary given by the disk bundle. We have that $d(p, p_0) \leq \frac{Cr}{|\log r|} = Cr\epsilon$. Then, from [18], we have that

$$\begin{aligned} &\mathrm{vol}(B(p;r)\Delta B(p_0;r)) \\ &= \mathrm{vol}((M \cap \tilde{B}(p;r))\Delta(M \cap \tilde{B}(p_0;r))) && (2.173) \\ &\leq \mathrm{vol}(\tilde{B}(p;r)\Delta\tilde{B}(p_0;r)) && (2.174) \\ &\leq Cr^{d-1} d(p,p_0) && (2.175) \\ &\leq Cr^d\epsilon. && (2.176) \end{aligned}$$

Denote the Euclidean ball as $B_e(\cdot;\cdot)$. For sufficiently small r, $\tilde{\exp}_{p_0}$ is a diffeomorphism from $B_e(0;3r)$ to $\tilde{M}$. Let $P \subset T_{p_0}\tilde{M}$ be the tangent space of ∂M at p_0. The half space given by P towards the manifold M is denoted as $\mathbb{R}_+$. Using the same proof in the previous part, we have that

$$\mathrm{vol}(B(p_0;r)\Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))) \le Cr^{d+1}. \tag{2.177}$$

Since

$$\begin{aligned} &B(p;r)\Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r)) \\ &\subset (B(p;r)\Delta B(p_0;r)) \cup \left(B(p_0;r)\Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))\right), \end{aligned} \tag{2.178}$$

it follows that

$$\begin{aligned} &\mathrm{vol}(B(p;r)\Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))) \\ &\le \mathrm{vol}(B(p;r)\Delta B(p_0;r)) + \mathrm{vol}(B(p_0;r)\Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))) \qquad (2.179) \\ &\le Cr^d\epsilon. \qquad (2.180) \end{aligned}$$

Let $q_0 \in \tilde{\exp}_{p_0}(P)$ be the nearest point on $\tilde{\exp}_{p_0}(P)$ to the point q. Let $q_0 = \tilde{\exp}_{p_0}(v)$ for some $v \in P$. We have that

$$\mathrm{vol}((B(q_0;r)\Delta B(q;r)) \le Cr^d\epsilon, \tag{2.181}$$

$$\mathrm{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))\Delta B(q_0;r)) \le Cr^{d+1}. \tag{2.182}$$

Similar to the arguments above, it follows that

$$\mathrm{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))\Delta B(q;r)) \le Cr^d\epsilon. \tag{2.183}$$

The next thing we need is to estimate the difference of $(\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))$ and $\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))$ given that $q_0 = \tilde{\exp}_{p_0}(v)$. We claim that

$$\operatorname{vol}\left(\tilde{B}(q_0;r)\,\Delta\,\tilde{\exp}_{p_0}(B_e(v;r))\right) \leq Cr^{d+2}. \tag{2.184}$$

Then, we have that

$$\begin{aligned}
&\operatorname{vol}\left[(\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))\,\Delta\,\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))\right] \\
&= \operatorname{vol}\left[\left(\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r)\right)\Delta\left(\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap \tilde{\exp}_{p_0}(B_e(v;r))\right)\right] \qquad (2.185)\\
&\leq \operatorname{vol}\left(\tilde{B}(q_0;r)\,\Delta\,\tilde{\exp}_{p_0}(B_e(v;r))\right) \qquad (2.186)\\
&\leq Cr^{d+2}, \qquad (2.187)
\end{aligned}$$

and that

$$\begin{aligned}
&\operatorname{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r)))\Delta B(q;r)) \\
&\leq \operatorname{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))\Delta B(q;r)) \\
&\quad + \operatorname{vol}\left[(\tilde{\exp}_{p_0}(\mathbb{R}_+) \cap B(q_0;r))\,\Delta\,\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))\right] \qquad (2.188)\\
&\leq Cr^d\epsilon + Cr^{d+2} \qquad (2.189)\\
&\leq Cr^d\epsilon. \qquad (2.190)
\end{aligned}$$

Now that $B_e(v;r)$ and $B_e(0;r)$ are two Euclidean balls with centers on the plane P, we have that the difference has Euclidean measure

$$\begin{aligned}
&\operatorname{vol}_e\left[(\mathbb{R}_+ \cap B_e(0;r)) \setminus (\mathbb{R}_+ \cap B_e(v;r))\right] \\
&\leq Cr^{d-1}\left\|v\right\| \qquad (2.191)\\
&= Cr^{d-1}d(p_0,q_0). \qquad (2.192)
\end{aligned}$$

Since the Jacobian of $\tilde{\exp}_{p_0}$ is bounded by $1+O(r^2)$ from [24], it implies that for sufficiently small r.

$$\begin{aligned}
&\mathrm{vol}\left[\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r)) \setminus \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))\right] \\
&= \mathrm{vol}\left\{\tilde{\exp}_{p_0}[(\mathbb{R}_+ \cap B_e(0;r)) \setminus (\mathbb{R}_+ \cap B_e(v;r))]\right\} && (2.193) \\
&= Cr^{d-1} d(p_0, q_0)(1 + O(r^2)) && (2.194) \\
&\leq Cr^{d-1} d(p_0, q_0). && (2.195)
\end{aligned}$$

From the lemma about set theory,

$$\begin{aligned}
&\mathrm{vol}\left[\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r)) \setminus \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))\right] \\
&\leq \mathrm{vol}(B(p;r) \setminus B(q;r)) \\
&\quad + \mathrm{vol}(B(p;r) \Delta \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))) && (2.196) \\
&\quad + \mathrm{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))) \Delta B(q;r)).
\end{aligned}$$

Using the inequalities (2.180), (2.190), (2.195) and

$$\begin{aligned}
d(p_0, q_0) &\geq d(p,q) - d(p, p_0) - d(q, q_0) && (2.197) \\
&\geq d(p,q) - Cr\epsilon. && (2.198)
\end{aligned}$$

, we have that

$$
\begin{aligned}
&\mathrm{vol}(B(p;r) \setminus B(q;r)) \\
&\geq \mathrm{vol}\left[\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r)) \setminus \tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r))\right] \\
&\quad - \mathrm{vol}(B(p;r)\Delta\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(0;r))) \qquad (2.199) \\
&\quad - \mathrm{vol}((\tilde{\exp}_{p_0}(\mathbb{R}_+ \cap B_e(v;r)))\Delta B(q;r)) \\
&\geq C_1 r^{d-1} d(p_0, q_0) - Cr^d\epsilon - Cr^d\epsilon \qquad (2.200) \\
&\geq C_1 r^{d-1} d(p, q) - C_2 r^d \epsilon. \qquad (2.201)
\end{aligned}
$$

Remark. *In fact, one can show that*

$$
\begin{aligned}
&C_1 r^{d-1} d(p,q) - C_2 r^d \epsilon \\
&\leq \mathrm{vol}(B(p;r) \setminus B(q;r)) \qquad (2.202) \\
&\leq C_1 r^{d-1} d(p,q) + C_2 r^d \epsilon
\end{aligned}
$$

It concludes that the union of two nearby balls satisfies (2.10)

$$
\begin{aligned}
&\mathrm{vol}(B(p;r) \cup B(q;r)) \\
&= \mathrm{vol}(B(p;r)) + \mathrm{vol}(B(p;r) \setminus B(q;r)) \qquad (2.203) \\
&\geq m\omega_d r^d (1 + C_1 r^{d-1} d(p,q) - C_2 r^d \epsilon). \qquad (2.204)
\end{aligned}
$$

It remains to show that the claim (2.184),

$$
\mathrm{vol}\left(\tilde{B}(q_0;r)\,\Delta\,\tilde{\exp}_{p_0}(B_e(v;r))\right) \leq Cr^{d+2}. \qquad (2.205)
$$

For this, we need a lemma about the distance of two points under the exponential map on a compact Riemannian manifold.

Lemma 10. *Let p be a point on the compact Riemannian manifold M. Suppose* $\exp_p$ *is the exponential map at point p. Let σ, u in T_pM be two fixed vector. Then, the Riemannian distance satisfies*

$$\begin{aligned} & d(\exp_p(su), \exp_p(s(u+\sigma))) \\ &= \|s\sigma\| \cdot \left(1 - \frac{K}{6}s^2 + O(s^3)\right), \ \textit{where} \qquad (2.206) \\ & K = \frac{R(\sigma, u, \sigma, u)}{\|\sigma\|^2}, \qquad (2.207) \end{aligned}$$

as $s \to 0$.

Proof (Lemma 10). For $0 \leq s, t \leq 1$, define a variation $H(s,t)$ from $\exp_p(su)$ to $\exp_p(s(u+\sigma))$ such that

$$H(s,0) = \exp_p(su), \qquad (2.208)$$

$$H(s,1) = \exp_p(s(u+\sigma)), \qquad (2.209)$$

$$\begin{gathered} \text{For every } s,\ H(s,\cdot) \text{ is a geodesic} \\ \text{from } \exp_p(su) \text{ to } \exp_p(s(u+\sigma)). \end{gathered} \qquad (2.210)$$

Define vector fields X_t along $H(\cdot, t)$ and Y_s along $H(s, \cdot)$ satisfying

$$X_t(s) = \frac{\partial H}{\partial t}(s,t), \qquad (2.211)$$

$$Y_s(t) = \frac{\partial H}{\partial s}(s,t). \qquad (2.212)$$

We have that X_t is the tangent vector field along geodesics and Y_s is a Jacobi field for each $0 \leq s \leq 1$ and that

$$Y_s''(t) = -R(X_t(s), Y_s(t))X_t(s). \qquad (2.213)$$

Since $H(0,t)$ is constant, we have

$$X_t(0) = 0 \quad \text{for any } t. \tag{2.214}$$

As a result of (2.213), $Y_0''(t) = 0$ for any t. From (2.208) and (2.209), we have that $Y_0(0) = u$ and $Y_0(1) = u + \sigma$. We get

$$Y_0(t) = u + t\sigma. \tag{2.215}$$

Then, we have that for any t,

$$X_t'(0) = \nabla_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial t}(s,t)\bigg|_{s=0} \tag{2.216}$$

$$= \nabla_{\frac{\partial}{\partial t}} \frac{\partial H}{\partial s}(s,t)\bigg|_{s=0} = Y_0'(t) = \sigma. \tag{2.217}$$

Next,

$$X_t''(0) = \nabla^2_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial t}(s,t)\bigg|_{s=0} \tag{2.218}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} Y_s(t)\bigg|_{s=0} \tag{2.219}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + R(X_t(0), Y_0(t))Y_0(t) \tag{2.220}$$

$$= \left(\nabla_{\frac{\partial}{\partial s}} Y_s\bigg|_{s=0} \right)'(t), \tag{2.221}$$

as $X_t(0) = 0$.

Taking $\nabla_{\frac{\partial}{\partial s}}$ of the ODE (2.213), the left hand side will be

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial s}} Y_s''(t)\Big|_{s=0} \\
&= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(X_t(0), Y_0(t))Y_0'(t) && (2.222)\\
&= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} && (2.223)\\
&= \nabla^2_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + \nabla_{\frac{\partial}{\partial t}} \left(R(X_t(0), Y_0(t))Y_0(t)\right) && (2.224)\\
&= \left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)''(t) && (2.225)
\end{aligned}$$

as $X_t(0) = 0$ and $\nabla_{\frac{\partial}{\partial t}} X_t(0) = 0$; the right hand side will be

$$\nabla_{\frac{\partial}{\partial s}} \left(-R(X_t(s), Y_s(t))X_t(s)\right)\Big|_{s=0} = 0, \tag{2.226}$$

since there will be at least one $X_t(0)$ left in the tensor and it is 0. Since $H(0,t)$ and $H(1,t)$ are two geodesic given by exponential map, we have the boundary condition

$$\left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(0) = \left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(1) = 0. \tag{2.227}$$

The unique solution is $\left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(t) = 0$ for any $0 \leq t \leq 1$. Therefore,

$$X_t''(0) = \left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)'(t) = 0. \tag{2.228}$$

Now, we proceed to $X_t'''(t)$:

$$X_t'''(0) = \nabla^3_{\frac{\partial}{\partial s}} \frac{\partial H}{\partial t}(s,t)\bigg|_{s=0} \tag{2.229}$$

$$= \nabla^2_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} Y_s(t)\bigg|_{s=0} \tag{2.230}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + \nabla_{\frac{\partial}{\partial s}} \left(R(X_t(s), Y_s(t))Y_s(t)\right)\bigg|_{s=0} \tag{2.231}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + R(X_t'(0), Y_0(t))Y_0(t) \tag{2.232}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + R(\sigma, u+t\sigma)(u+t\sigma) \tag{2.233}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + R(\sigma, u)(u+t\sigma). \tag{2.234}$$

The first term is

$$\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla^2_{\frac{\partial}{\partial s}} Y_s(t)\bigg|_{s=0} + \left(R(X_t(s), Y_s(t))(\nabla_{\frac{\partial}{\partial s}} Y_s(t))\right)\bigg|_{s=0}, \tag{2.235}$$

$$= \left(\nabla^2_{\frac{\partial}{\partial s}} Y_s\bigg|_{s=0}\right)'(t). \tag{2.236}$$

Now take $\nabla^2_{\frac{\partial}{\partial s}}$ of the ODE (2.213). The left hand side will be

$$\nabla^2_{\frac{\partial}{\partial s}} Y_s''(t)\Big|_{s=0}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + \nabla_{\frac{\partial}{\partial s}} \left(R(X_t(s), Y_s(t)) Y_s'(t) \right)\Big|_{s=0} \tag{2.237}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(X_t'(0), Y_0(t)) Y_0'(t) \tag{2.238}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(\sigma, u + t\sigma)\sigma \tag{2.239}$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(\sigma, u)\sigma \tag{2.240}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla^2_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(X_t(0), Y_0(t)) \left(\nabla_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} \right) + R(\sigma, u)\sigma \tag{2.241}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla^2_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0} + R(\sigma, u)\sigma. \tag{2.242}$$

The first term

$$\nabla_{\frac{\partial}{\partial t}} \nabla^2_{\frac{\partial}{\partial s}} Y_s'(t)\Big|_{s=0}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \left(R(X_t(s), Y_s(t)) Y_s(t) \right)\Big|_{s=0} \tag{2.243}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + \nabla_{\frac{\partial}{\partial t}} \left(R(X_t'(s), Y_s(t)) Y_s(t) \right)\Big|_{s=0} \tag{2.244}$$

$$\text{(since } X_t(0) = 0 \text{ and } \left(\nabla_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0} \right)(t) = 0 \text{)} \tag{2.245}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + \frac{\partial}{\partial t} \left(R(X_t'(0), Y_0(t)) Y_0(t) \right) \tag{2.246}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + \frac{\partial}{\partial t} \left(R(\sigma, u + t\sigma)(u + t\sigma) \right) \tag{2.247}$$

$$= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} Y_s(t)\Big|_{s=0} + R(\sigma, u)\sigma. \tag{2.248}$$

Again, we have that

$$
\begin{aligned}
&\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}Y_s(t)\Big|_{s=0}\\
&= \nabla^2_{\frac{\partial}{\partial t}}\nabla^2_{\frac{\partial}{\partial s}}Y_s(t)\Big|_{s=0} + \nabla_{\frac{\partial}{\partial t}}\left(R(X_t(s),Y_s(t))\left(\nabla_{\frac{\partial}{\partial s}}Y_s(t)\right)\right)\Big|_{s=0} \qquad (2.249)\\
&= \left(\nabla^2_{\frac{\partial}{\partial s}}Y_s\Big|_{s=0}\right)''(t) \qquad (2.250)
\end{aligned}
$$

as $\nabla_{\frac{\partial}{\partial s}}Y_s(t)\Big|_{s=0} = 0$. It follows that the left hand side of (2.213) becomes

$$
\begin{aligned}
&\nabla^2_{\frac{\partial}{\partial s}}Y_s''(t)\Big|_{s=0}\\
&= \left(\nabla^2_{\frac{\partial}{\partial s}}Y_s\Big|_{s=0}\right)''(t) + 2R(\sigma,u)\sigma. \qquad (2.251)
\end{aligned}
$$

Taking $\nabla^2_{\frac{\partial}{\partial s}}$ of the right hand sid of (2.213), we will get

$$
\begin{aligned}
&\nabla^2_{\frac{\partial}{\partial s}}\left(-R(X_t(s),Y_s(t))X_t(s)\right)\Big|_{s=0}\\
&= -R(X_t'(0),Y_0(t))X_t'(0) \qquad (2.252)\\
&= -R(\sigma,u+t\sigma)\sigma = -R(\sigma,u)\sigma. \qquad (2.253)
\end{aligned}
$$

Therefore, we have a differential equation

$$
\left(\nabla^2_{\frac{\partial}{\partial s}}Y_s\Big|_{s=0}\right)''(t) + 2R(\sigma,u)\sigma = -R(\sigma,u)\sigma. \qquad (2.254)
$$

That is

$$
\left(\nabla^2_{\frac{\partial}{\partial s}}Y_s\Big|_{s=0}\right)''(t) = -3R(\sigma,u)\sigma. \qquad (2.255)
$$

We have the boundary conditions

$$\left(\nabla^2_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(0) = \left(\nabla^2_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(1) = 0. \tag{2.256}$$

The unique solution is that

$$\left(\nabla^2_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)(t) = -\frac{3}{2}t(t-1)R(\sigma,u)\sigma. \tag{2.257}$$

Finally,

$$X_t'''(0) = \left(\nabla^2_{\frac{\partial}{\partial s}} Y_s\Big|_{s=0}\right)'(t) + R(\sigma,u)(u+t\sigma) \tag{2.258}$$

$$= -(3t-\frac{3}{2})R(\sigma,u)\sigma + R(\sigma,u)(u+t\sigma) \tag{2.259}$$

$$= -(2t-\frac{3}{2})R(\sigma,u)\sigma + R(\sigma,u)u. \tag{2.260}$$

By Taylor theorem, we have that as $s \to 0$,

$$X_t(s) = X_t(0) + sX_t'(0) + \frac{s^2}{2}X_t''(0) + \frac{s^3}{6}X_t'''(0) + O(s^4) \tag{2.261}$$

$$= s\sigma + \frac{s^3}{6}\left[-(2t-\frac{3}{2})R(\sigma,u)\sigma + R(\sigma,u)u\right] + O(s^4). \tag{2.262}$$

Then, the norm squared is

$$\|X_t(s)\|^2 = s^2\|\sigma\|^2 - \frac{s^4}{6}\cdot(2R(\sigma,u,\sigma,u)) + O(s^5) \tag{2.263}$$

$$= \|s\sigma\|^2\left(1 - \frac{s^2R(\sigma,u,\sigma,u)}{3\|\sigma\|^2} + O(s^3)\right) \tag{2.264}$$

as $R(\sigma, u, \sigma, \sigma) = 0$. It follows that

$$\|X_t(s)\| = \|s\sigma\| \left(1 - \frac{s^2 R(\sigma, u, \sigma, u)}{6\|\sigma\|^2} + O(s^3)\right). \tag{2.265}$$

In conclusion,

$$\begin{aligned} d(\exp_p(su), \exp_p(s(u+\sigma))) &= \operatorname{length}(H(s,\cdot)) && (2.266) \\ &= \int_0^1 \|X_t(s)\| \, dt && (2.267) \\ &= \|s\sigma\| \left(1 - \frac{s^2 R(\sigma, u, \sigma, u)}{6\|\sigma\|^2} + O(s^3)\right) && (2.268) \\ &= \|s\sigma\| \left(1 - \frac{s^2 K}{6} + O(s^3)\right), && (2.269) \end{aligned}$$

where $K = \frac{R(\sigma,u,\sigma,u)}{\|\sigma\|^2}$. □

Remark. *Lemma 10 implies that if σ and u are bounded and manifold M is compact, for sufficiently small r, there exists constant C such that*

$$\left| d(\exp_p(rv), \exp_p(r(u+\sigma))) - \|ru\| \right| \leq Cr^3. \tag{2.270}$$

Now to prove the original claim (2.184), for each σ from the unit sphere S^{d-1} as a subset of $T_p\tilde{M}$, define $r'(\sigma) = d(\tilde{\exp}_p(ru), \tilde{\exp}_p(ru + r\sigma))$ where $u = v/r$. Note that $\tilde{\exp}_p(v) = q$. We have $r'(\sigma) = d(q, \tilde{\exp}_p(ru + r\sigma))$. By Lemma 10,

$$\begin{aligned} \left| r'(\sigma)^d - r^d \right| &= \left| d(\tilde{\exp}_p(ru), \tilde{\exp}_p(ru + r\sigma))^d - \|r\sigma\|^d \right| && (2.271) \\ &\leq Cr^d \cdot r^2 = Cr^{d+2}. && (2.272) \end{aligned}$$

In fact, the volume of $\tilde{\exp}_{p_0}(B_e(v; r))$ is the integral of $r'(\sigma)$ with respect to $\sigma \in S^{d-1}$ in spherical coordinates. Therefore, we have that if $\rho^{d-1} J_\sigma$ is the Jacobian of the spherical coordinate at σ with radius ρ, then

$$\begin{aligned}
&\operatorname{vol}\left(\tilde{B}(q_0;r)\,\Delta\,\tilde{\exp}_{p_0}(B_e(v;r))\right)\\
&= \int_{\sigma\in S^{d-1}} \left| \int_0^{r'(\sigma)} \rho^{d-1} d\rho - \int_0^r \rho^{d-1} d\rho \right| J_\sigma d\sigma && (2.273)\\
&\le C \int_{\sigma\in S^{d-1}} \left| r'(\sigma)^d - r^d \right| J_\sigma d\sigma && (2.274)\\
&\le C r^{d+2}. && (2.275)
\end{aligned}$$

This finishes the proof of the claim (2.184) and so the estimate (2.169) and (2.10).

Conclusion

In the previous parts, we have already checked all the conditions required in the main theorem. Note that we have

$$m = \frac{1}{2}, \tag{2.276}$$

$$a = 1, \tag{2.277}$$

$$b = 1. \tag{2.278}$$

By the main theorem (Theorem 1),

Corollary 1. *Suppose M is a d-dimensional Riemannian manifold with a non-empty boundary and $d \ge 2$. Let N_r be the number of balls with radii r necessary to cover M under the uniform distribution (i.e. the probability measure is the Riemannian measure if the total volume is 1). Then*

$$\mathbb{E}\left[N_r\right] = \frac{2}{\alpha}\left[\frac{d-1}{d}\log\frac{1}{\alpha} + (d-1)\log\log\frac{1}{\alpha} + O(1)\right], \quad \textit{as } r \to 0. \tag{2.279}$$

2.3.3 Probability Density with Single Minimum Point and Positive Minimum

In this section, we consider a d-dimensional Riemannian manifold M with no boundary and with total volume 1. Suppose ν is a probability measure on M with Radon–Nikodym derivative with respect to the Riemannian volume f. Suppose p_m on M is the unique minimum point of f and $f(p_m) = \min_M f = m > 0$ is strictly positive. Furthermore, we assume that there exists some constant ξ such that

$$m(1 + C_1 d(p_m, p)^{\xi}) \leq f(p) \leq m(1 + C_2 d(p_m, p)^{\xi}). \tag{2.280}$$

Following the similar scheme of the proof, we set $U(r) = M$. Then, for any $p \in U(r)$, $c(p; r) \geq m$. $\mathrm{vol}(U(r)) = 1$ and so

$$a = 0. \tag{2.281}$$

Let $U(r; k) = B(p_m; (k\epsilon)^{\frac{1}{\xi}})$ where $\epsilon = \frac{1}{|\log r|}$ For any $p \in U(r; k)$, since $\epsilon = \frac{1}{|\log r|} \gg r$ as $r \to 0$ and $d(p_m, p) \leq (k\epsilon)^{\frac{1}{\xi}}$,

$$\nu(B(p; r)) \leq m\mathrm{vol}(B(p; r))(1 + C((k\epsilon)^{\frac{1}{\xi}})^{\xi}) \tag{2.282}$$

$$\leq m\omega_d r^d (1 + Cr^2)(1 + C((k\epsilon)^{\frac{1}{\xi}})^{\xi}) \tag{2.283}$$

$$\leq m(1 + Ck\epsilon). \tag{2.284}$$

Therefore, we have that $c(p; r) \leq m(1 + Ck\epsilon)$.

For any $p \in U(r) \setminus U(r;k)$, $d(p, p_m) \geq (k\epsilon)^{\frac{1}{\xi}}$, then

$$\nu(B(p;r)) \geq m\mathrm{vol}(B(p;r))(1 + C((k\epsilon)^{\frac{1}{\xi}})^{\xi}) \tag{2.285}$$
$$\geq m\omega_d r^d(1 + Cr^2)(1 + C((k\epsilon)^{\frac{1}{\xi}})^{\xi}) \tag{2.286}$$
$$\geq m(1 + Ck\epsilon) \geq m(1 + C'(k+1)\epsilon). \tag{2.287}$$

Here, we have $c(p;r) \geq m(1 + C(k+1)\epsilon)$

Since $U(r;k)$ are balls on the manifold, the volume

$$C_1(k\epsilon)^{\frac{d}{\xi}} = C_1\left((k\epsilon)^{\frac{1}{\xi}}\right)^d \leq \tag{2.288}$$
$$\mathrm{vol}(U'(r)) = \mathrm{vol}(B(p_m; (k\epsilon)^{\frac{1}{\xi}})) \tag{2.289}$$
$$\leq C_2\left((k\epsilon)^{\frac{1}{\xi}}\right)^d = C_2(k\epsilon)^{\frac{d}{\xi}}. \tag{2.290}$$

It follows that

$$b = \frac{d}{\xi}. \tag{2.291}$$

We still need to verify (2.10)

$$\nu(B(p;r) \cup B(q;r)) \geq m\omega_d r^d(1 + C_1 r^{-1} d(p,q) - C_2\epsilon), \tag{2.292}$$

for p, q in $U'(r) = U(r;1)$ and $d(p,q) < 2r$

In fact, from [18], if $B(p;r), B(q;r)$ are two balls with radius r on a d-dimensional compact manifold with no boundary and $d(p,q) \leq 2r$. Then, the volume

$$\mathrm{vol}(B(q;r) \setminus B(p;r)) \geq Cr^{d-1} d(p,q). \tag{2.293}$$

In our case, for any two points p, q in $U'(r)$ and $d(p,q) < 2r$, both balls $B(p;r)$ and

$B(q;r)$ are in the interior. Therefore, the result from [18] applies and (2.293) above holds. On the other hand, from the definition of $U'(r)$, (2.1) and (2.7), the probability density is bounded below by $m/2$ for sufficiently small r, so

$$\nu(B(q;r) \setminus B(p;r)) \geq \frac{m}{2}\mathrm{vol}(B(q;r) \setminus B(p;r)) \geq Cr^{d-1}d(p,q). \tag{2.294}$$

It concludes that

$$\nu(B(p;r) \cup B(q;r)) = \nu(B(p;r)) + \nu(B(p;r) \cup B(q;r)) \tag{2.295}$$

$$\geq m\omega_d r^d(1 + Cr^2) + Cr^{d-1}d(p,q) \tag{2.296}$$

$$\geq m(1 + C_1 r^{-1}d(p,q) - C_2\epsilon). \tag{2.297}$$

Since our model satisfies all the required conditions, by the main theorem (Theorem 1),

Corollary 2. *Suppose M is a d-dimensional Riemannian manifold with no boundary with total volume 1. Suppose ν is a probability measure on M with Radon–Nikodym derivative with respect to the Riemannian volume f. Suppose p_m on M is the unique minimum point of f and $f(p_m) = \min_M f = m > 0$. Furthermore, we assume that there exists some constant ξ such that*

$$f(p) \leq m(1 + Cd(p_m,p)^{\xi}). \tag{2.298}$$

Let N_r be the number of balls with radii r necessary to cover M under the probability measure ν. Then

$$\mathbb{E}\left[N_r\right] = \frac{1}{m\alpha}\left[\log\frac{1}{\alpha} + d\left(1 - \frac{1}{\xi}\right)\log\log\frac{1}{\alpha} + O(1)\right], \quad \text{as } r \to 0. \tag{2.299}$$

In particular, if the density function f is a Morse function with positive lower bound, i.e.

the Hessian matrix of f at the minimum point p_m is non-singular, then we have that

$$f(p) \leq m(1 + Cd(p_m, p)^2). \tag{2.300}$$

According to the corollary above, we have that

$$\mathbb{E}[N_r] = \frac{1}{m\alpha}\left[\log\frac{1}{\alpha} + \frac{d}{2}\log\log\frac{1}{\alpha} + O(1)\right], \quad \text{as } r \to 0. \tag{2.301}$$

2.3.4 n-Dimensional Cubes

Consider the unit cube $\mathcal{C}^d = [0,1]^d$ in the Euclidean space $\mathbb{R}^d$. Suppose $d \geq 2$. We use the Euclidean measure restricted on $\mathcal{C}^d$ as the probability measure.

The boundary of the cube is not a smooth sub-manifold but rather a union of singularities with various dimensions. We will use $\mathcal{C}^k$ to denote those singularities on $\partial\mathcal{C}^d$ that is isometric to $\mathcal{C}^k = [0,1]^k \times \{0\}^{d-k}$. For each k, the number of isometric copies of $\mathcal{C}^k$ in total is $2^{d-k}\binom{d}{k}$ in the form of Cartesian products of k intervals $[0,1]$ and $d-k$ sets of single points: $\{0\}$ or $\{1\}$. Let S_k be the union of all such isometric copies of $\mathcal{C}^k$. Now, we recursively define $U_k(r)$ for $k = 0, \ldots d$ as

$$U_0(r) = T(S_0; (d-k+1)r), \tag{2.302}$$

$$U_k(r) = T(S_k \setminus T(S_0; (d-k)r); (d-k)r) \tag{2.303}$$

$$\text{for } k = 1, \ldots, d,$$

where $T(S;r)$ is the tubular neighborhood of S with radius r as per Section 2.3.1. We know from the section that the volumes satisfy

$$C_1 r^{d-k} \leq \mathrm{vol}(U_k(r)) \leq C_2 r^{d-k}. \tag{2.304}$$

For $p \in U_k(r)$,

$$c(p;r) = \frac{\mathrm{vol}(B(p;r))}{\omega_d r^d} \geq 2^{-d+k}. \tag{2.305}$$

As a summery, the parameters in those regions $U_k(r)$ are

$$m_k = 2^{-d+k}, \tag{2.306}$$

$$a_k = d - k, \text{ and} \tag{2.307}$$

$$\frac{d - a_k}{m_k} = 2^{d-k} \cdot k. \tag{2.308}$$

The maximum of $\frac{d-a_k}{m_k}$ is achieved when $k = 1$ and $k = 2$ with value 2^{d-1}. Both $k = 1$ and $k = 2$ lead to the same result so we will stick to $k = 1$. In this case

$$m = m_1 = 2^{-d+1}, \; a = a_1 = d - 1. \tag{2.309}$$

Furthermore, in $U_1(r)$, define $U(r;l) = U_k(lr\epsilon) = U_1\left(\frac{lr}{|\log r|}\right)$.

Then,

$$C_1 r^{d-1}(l\epsilon)^{d-1} = C_1(lr\epsilon)^{d-1} \tag{2.310}$$

$$\leq \mathrm{vol}(U(r;l)) \leq \tag{2.311}$$

$$C_2(lr\epsilon)^{d-1} = C_2 r^{d-1}(l\epsilon)^{d-1}. \tag{2.312}$$

Here we have $b = d - 1$.

For $p \in U(r;l)$, $d(p, S_1) \le lr\epsilon = \frac{lr}{|\log r|}$

$$\mathrm{vol}(B(p;r)) - 2^{d-1}\omega_d r^d \le C_2 r^{d-1} d(p, S_1) = C_2 r^d l\epsilon. \tag{2.313}$$

That is,

$$c(p;r) \le 2^{d-1}(1 + Cl\epsilon). \tag{2.314}$$

For $p \in U_1(r) \setminus U(r;l)$, $d(p, S_1) \ge lr\epsilon = \frac{lr}{|\log r|}$

$$\mathrm{vol}(B(p;r)) - 2^{d-1}\omega_d r^d \ge C_2 r^{d-1} d(p, S_1) = C_2 r^d l\epsilon \ge C_2' r^d (l+1)\epsilon. \tag{2.315}$$

That is,

$$c(p;r) \ge 2^{d-1}(1 + C(l+1)\epsilon). \tag{2.316}$$

The volume of union of two balls, (2.10), can be verified similarly to Section 2.3.2.

It concludes that the number of balls with radii r necessary to cover the d-dimensional cube $\mathcal{C}^d$ under the Euclidean volume, N_r, satisfies

$$\mathbb{E}\left[N_r\right] = \frac{2^{d-1}}{\alpha}\left[\frac{1}{d}\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + O(1)\right], \quad \text{as } r \to 0. \tag{2.317}$$

It turns out that for $d = 2$, when $\mathcal{C}^d$ is a square on the plane,

$$\mathbb{E}\left[N_r\right] = \frac{2}{\alpha}\left[\frac{1}{2}\log\frac{1}{\alpha} + \log\log\frac{1}{\alpha} + O(1)\right] \tag{2.318}$$

$$= \frac{1}{\alpha}\left[\log\frac{1}{\alpha} + 2\log\log\frac{1}{\alpha} + O(1)\right], \quad \text{as } r \to 0. \tag{2.319}$$

It has the same form as the formula of a 2-dimensional manifold without boundaries from

[18].

CHAPTER 3

HOMOLOGY OF ČECH COMPLEXES ON A FLAT TORUS $\mathbb{T}^D$

3.1 Main Results

Suppose P_n is a set of n points sampled on a flat torus $\mathbb{T}^d$ with volume 1 generated by a Poisson process. Set $\Lambda = n\omega_d r^d$ (ω_d is the volume of a d-dimensional ball). Let $C(n, r)$ be the Čech complex at radius r.

Denote the k-th Betti number of $\mathbb{T}^d$ by $\beta_k = \beta_k(\mathbb{T}^d)$. Denote the k-th homology group of $C(n, r)$ by $H_k(r)$. Let $\beta_k(r)$ be the k-th Betti number of $C(n, r)$.

For $\mathbf{p} = (p_0, \ldots, p_k) \in (\mathbb{T}^d)^{k+1}$, define the following objects:

1. $\Delta(\mathbf{p})$ is the k-simplex with $\mathbf{p}$ as its vertexes.
2. $B(\mathbf{p})$ is the smallest closed ball that passes the $k+1$ points $\mathbf{p}$.
3. $C(\mathbf{p})$ is the center of $B(\mathbf{p})$.
4. $R(\mathbf{p})$ is the radius of $B(\mathbf{p})$.
5. $A_\varphi(\mathbf{p})$ is the annulus $B(C(\mathbf{p}); R(\mathbf{p})) \setminus B(C(\mathbf{p}); \varphi R(\mathbf{p}))$, for $0 < \varphi < 1$.
6. $\varphi(\mathbf{p}) := \max \left\{\varphi : \partial\Delta(\mathbf{p}) \subset A_\varphi(\mathbf{p})\right\}$.
7. Denote $A(\mathbf{p}) := A_{\varphi(\mathbf{p})}(\mathbf{p})$.

Let $U(P; r) := \bigcup_{p \in P} B(p; r)$ be the union of all the balls centered at points from P with radii r.

3.1.1 $H_0(\mathbb{T}^d)$ and $H_d(\mathbb{T}^d)$

$H_0(\mathbb{T}^d)$ is equivalent to the connectivity of $U(P; r)$. In [30], Penrose uses the Erdös-Rényi random graph model to get that the threshold for connectivity is $\Lambda = 2^{-d} \log n$.

Since $\mathbb{T}^d$ is a connected closed manifold, the top homology $H_d(\mathbb{T}^d)$ is correct if and only if the whole torus is covered. The threshold from the coverage problem from [18] and the previous part is $\Lambda = \log n + (d-1)\log\log n$.

3.1.2 Expectation of $H_k(\mathbb{T}^d)$

Theorem 2. *Suppose that $w(n) - \log\log\log n \to \infty$ as $n \to \infty$. Then, for $1 \le k \le d-2$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\beta_k(r)\right] = \begin{cases} \infty & \Lambda = \log n + (k-2)\log\log n - w(n) \\ \beta_k & \Lambda = \log n + (k-1)\log\log n + w(n) \end{cases} \tag{3.1}$$

3.1.3 Upper Bound of Probability that $H_k(\mathbb{T}^d)$ is Correct

Theorem 3. *Suppose that $w(n) - \log\log\log n \to \infty$ as $n \to \infty$. Then, for $1 \le k \le d-2$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{\beta_k(r) = \beta_k\right\} = \begin{cases} 0 & \Lambda = \log n + (k-2)\log\log n - w(n) \\ 1 & \Lambda = \log n + (k-1)\log\log n + w(n) \end{cases} \tag{3.2}$$

3.2 Critical Points

3.2.1 Critical Points of the Distance Function

The distance function is an example of Minimum-type functions which are studied in [20] in general. The critical points in [20] is consistent with those defined in Section 3.2. A critical point of index k, $c \in \mathbb{T}^d$, is the center of the smallest sphere that passes $k+1$ points $\mathbf{p}$ from P_n where the center is in the simplex $\Delta(\mathbf{p})$ and $B(\mathbf{p})$ contains only points in $\mathbf{p}$ and no other points from the process P_n. That is, for some $\mathbf{p} = (p_0, \ldots, p_k) \in (P_n)^{k+1}$, $c = C(\mathbf{p}) \in \Delta(\mathbf{p})$ and $B(\mathbf{p}) \cap P_n = \{p_0, \ldots, p_k\}$. We say that the critical point c is *formed* at radius $r = R(\mathbf{p})$.

3.2.2 Morse Theory

By Morse theory, the homology group $H_k(r)$ of the Čech complex at radius r changes only when a critical point of index k or $k+1$ is formed at r. Furthermore, a critical point of index k may create a new k-th homology class increasing the Betti number $\beta_k(r)$ by 1 or, alternatively, a critical point of index k may kill an existing $(k-1)$-th homology class as the Betti number $\beta_{k-1}(r)$ is decreased by 1.

Define $\hat{K}_k(r)$ to be the number of critical points of index $k+1$ that are formed for some radius $\rho > r$ which kill some k-th homology classes. Then,

$$\beta_k(r) - \beta_k(\mathbb{T}^d) \leq \hat{K}_k(r). \tag{3.3}$$

3.2.3 Θ-Cycles

A k-th Θ-cycle is a critical point generated by $\mathbf{p} \in (\mathbb{T}^d)^{k+1}$ where the annulus that contains the boundary of the simplex $\Delta(\mathbf{p})$ is covered by balls centered at points perhaps other than $\mathbf{p}$. We have that $A(\mathbf{p}) \subset U(P_n; R(\mathbf{p}))$.

It follows from the proofs in [7] that if $0 \leq k \leq d-1$, a Θ-cycle is guaranteed to create a new homology class when it is formed. Therefore, none of the killers is a Θ-cycle.

3.3 Re-parametrization of Configurations p on Sphere by $\varphi(\mathbf{p})$

To give an estimate of killer critical points, we need to re-parametrize configurations $\mathbf{p} = (p_0, \ldots, p_{k+1}) \in (S^k)^{k+2}$ by $\varphi(\mathbf{p})$. We will construct a coordinate transformation and calculate its Jacobian in terms of φ. In this section for simplicity, we embed the sphere S^k into $\mathbb{R}^{k+1}$ as the unit sphere and still denote the unit sphere by S^k. We regard S^{k-1} as $\mathbb{R}^k \times \{0\} \cap S^k$.

Let $e_1, \dots, e_{k+1}$ be the standard orthonormal basis of $\mathbb{R}^{k+1}$. Without loss of generality, we assume that $\varphi = \varphi(\mathbf{p}) = \varphi((p_0, \dots, p_{k+1}))$ equals the distance from the origin to the k-face $\Delta(p_1, p_2, \dots, p_{k+1})$. Let $n \in S^k$ be the unit normal vector orthogonal to the k-dimensional hyperplane spanned by $p_1, \dots, p_{k+1}$. For $n \in S^k \setminus \{-e_{k+1}\}$, let $\sigma_n \in SO(k+1)$ be the rotation though the unique geodesic from the north pole e_{k+1} to n. We have that σ_n is smooth in n. Set $\tau(\varphi, \cdot) : S^{k-1} \to S^k$ to be

$$\tau(\varphi, p') = \varphi \cdot e_{k+1} + (1 - \varphi^2)^{1/2} \cdot p', \text{ for } p' \in S^{k-1}. \tag{3.4}$$

That is, $\tau(\varphi, \cdot)$ is a map that "lift" the equator S^{k-1} by height φ. Let $\psi(\varphi, n, p') := \sigma_n(\tau(\varphi, p'))$. Then $\psi(\varphi, n, \cdot) : S^{k-1} \to S^k$ maps S^{k-1} smoothly to the intersection of S^k and the plane orthogonal to n with distance φ to the origin.

Then, we can define the parametrization map Ψ on some subregion of $\mathbb{R} \times (S^k \setminus \{-e_{k+1}\}) \times S^k \times (S^{k-1})^{k+1}$ to $(S^k)^{k+2}$ by

$$\begin{aligned}\Psi(\varphi, n, p_0, \mathbf{p}'_0) &= \Psi(\varphi, n, p_0, (p'_1, p'_2, \dots, p'_{k+1})) \\ &= (p_0, \psi(\varphi, n, p'_1), \dots, \psi(\varphi, n, p'_{k+1})) = \mathbf{p}.\end{aligned} \tag{3.5}$$

We first calculate the tangent map of τ. Given $p \in S^{k-1}$ and $\varphi \in \mathbb{R}$, let $v'_1, \dots, v'_{k-1}$ to be an orthonormal frame of $T_p S^{k-1}$. Let $v_1, \dots, v_{k-1}, v_k$ to be an orthonormal frame of $T_{\tau(\varphi, p')} S^k$ satisfying that v_i points the same direction as $d\tau|_{\varphi, p'}(v'_i)$ for $1 \le i \le k-1$ and as the projection of e_{k+1} for $i = k+1$. Denote the unit vector in $T_\varphi \mathbb{R}$ as $\partial\varphi$. Then we have

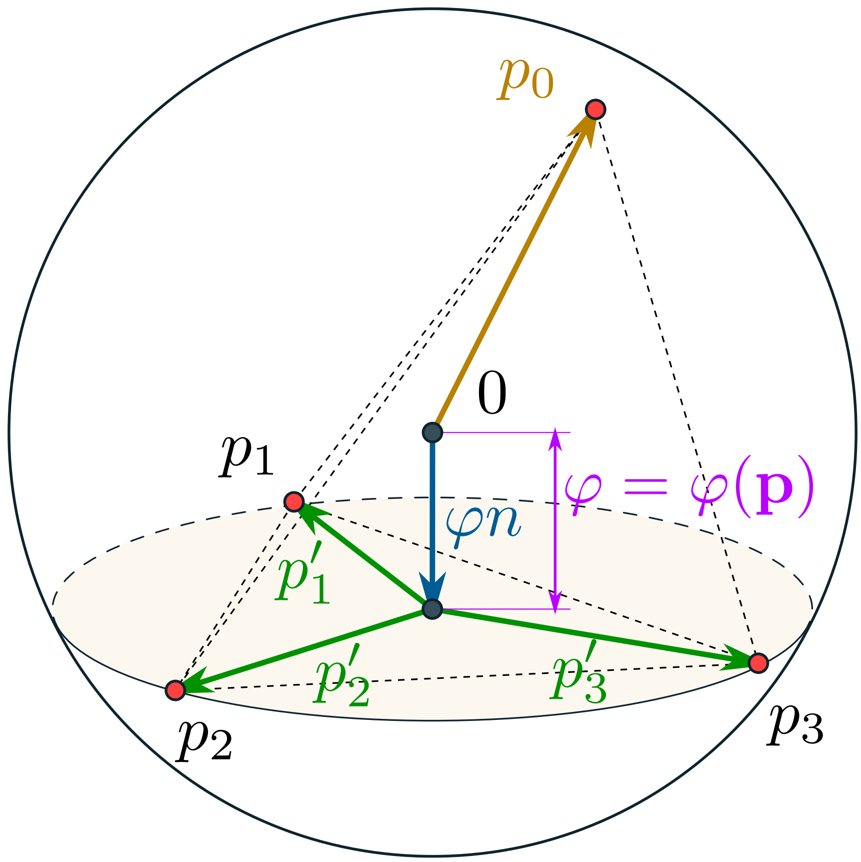


Figure 3.1
Re-parametrization
$\Psi(\varphi, n, p_0, \mathbf{p}'_0) = \Psi(\varphi, n, p_0, (p'_1, p'_2, \ldots, p'_{k+1})) = (p_0, p_1, \ldots, p_{k+1})$

the tangent map:

$$d\tau|_{\varphi,p'}(v'_i) = (1-\varphi^2)^{1/2} v_i, \text{ for } 1 \le i \le k-1 \tag{3.6}$$

$$d\tau|_{\varphi,p'}(\partial\varphi) = e_{k+1} - \varphi(1-\varphi^2)^{-1/2} p'. \tag{3.7}$$

Note that the orthogonal projection of p' onto $T_{\tau(\varphi,p')}S^k$ is $-\varphi v_k$ and the orthogonal projection of e_{k+1} onto $T_{\tau(\varphi,p')}S^k$ is $(1-\varphi^2)^{1/2}v_k$. Therefore, (3.7) becomes

$$d\tau|_{\varphi,p'}(\partial\varphi) = (1-\varphi^2)^{1/2} v_k - \varphi(1-\varphi^2)^{-1/2}(-\varphi v_k) \tag{3.8}$$

$$= (1-\varphi^2)^{-1/2} v_k. \tag{3.9}$$

For $\psi(\varphi, n, p') = \sigma_n(\tau(\varphi, p'))$, since $\sigma_\cdot : S^k \setminus \{-e_{k+1}\} \to SO(k+1)$, its derivative with respect to n is in $so(k+1)$, the Lie algebra corresponding to $SO(k+1)$. Write this derivative at n of any tangent vector $\partial n \in T_n(S^k \setminus \{-e_{k+1}\})$ as $\Sigma_n(\partial n) \in so(k+1)$. Then,

$$d\psi|_{\varphi,n,p'}(\partial n) = \Sigma_n(\partial n) \cdot \tau(\varphi, p') \tag{3.10}$$

$$= \varphi \cdot \Sigma_n(\partial n) e_{k+1} + (1-\varphi^2)^{1/2} \cdot \Sigma_n(\partial n) p', \tag{3.11}$$

for any $\partial n \in T_n(S^k \setminus \{-e_{k+1}\})$ where $\dim T_n(S^k \setminus \{-e_{k+1}\}) = k$.

And from what we already have in (3.9) and (3.6):

$$d\psi|_{\varphi,n,p'}(\partial \varphi) = (1-\varphi^2)^{-1/2} \cdot \sigma_n v_k, \tag{3.12}$$

$$d\psi|_{\varphi,n,p'}(v_i') = (1-\varphi^2)^{1/2} \cdot \sigma_n v_i, \text{ for } 1 \le i \le k-1 \tag{3.13}$$

Now, to calculate the Jacobian of Ψ for fixed variables $\varphi, n, p_0, \mathbf{p}_0'$ and $\mathbf{p} = \Psi(\varphi, n, p_0, \mathbf{p}_0')$ with respect to φ, we have that the Jacobian is

$$J_\Psi(\varphi, n, p_0, \mathbf{p}_0') = \left((1-\varphi^2)^{1/2}\right)^{(k-1)(k+1)} (1-\varphi^2)^{-1/2} \cdot f(\varphi, (1-\varphi^2)^{1/2}) \tag{3.14}$$

$$= (1-\varphi^2)^{(k^2-2)/2} \cdot f(\varphi, (1-\varphi^2)^{1/2}), \tag{3.15}$$

where f is some polynomial with degree k and coefficients only depending on $n, p_0, \mathbf{p}_0'$ which derives from (3.11).

Since $0 < \varphi < 1$, it concludes that for some constant $\widetilde{J_\Psi}(n, p_0, \mathbf{p}_0')$ that does not depend on φ,

$$\left|J_\Psi(\varphi, n, p_0, \mathbf{p}_0')\right| \le \widetilde{J_\Psi}(n, p_0, \mathbf{p}_0'). \tag{3.16}$$

3.4 Proofs: Upper Bound of Expectation of $H_k(\mathbb{T}^d)$ via Estimation of Killers

In this section, we are going prove the following lemma:

Lemma 11. *Suppose that $w(n) - \log\log\log n \to \infty$ as $n \to \infty$. Then, for $1 \le k \le d-2$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\hat{K}_k(r)\right] = 0, \tag{3.17}$$

as $n \to \infty$ if $\Lambda = \log n + (k-1)\log\log n + w(n)$.

Define the follow characteristic functions for $\mathbf{p} = (p_0, \ldots, p_{k+1}) \in (\mathbb{T}^d)^{k+2}$ and $P_n \subset \mathbb{T}^d$:

1. $\chi_r(\mathbf{p}) = \mathbb{1}\{C(\mathbf{p}) \in \Delta(\mathbf{p})\}\, \mathbb{1}\{R(\mathbf{p}) \ge r\}$.

2. $\chi'_r(\mathbf{p}; P_n) = \chi_r(\mathbf{p})\mathbb{1}\{B(\mathbf{p}) \cap P_n = \{p_0, \ldots, p_{k+1}\}\}$
   $\cdot \mathbb{1}\{A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p}))\}$.

It follows that if $\mathbf{p}$ is a $(k+1)$-critical point which kills a k-th homology class, then $\chi'_r(\mathbf{p}; P_n) = 1$. Therefore,

$$\hat{K}_k(r) \le \sum_{\mathbf{p}\in(P_n)^{k+2}} \chi'_r(\mathbf{p}; P_n). \tag{3.18}$$

In order to estimate the expectation, we may apply Palm theorem [30].

Lemma 12 (Palm theorem)**.** *Suppose P_n is a set of n points generated by a Poisson process on a manifold M. Let integer $k \ge 0$. Suppose $h(\mathbf{p}; P)$ is a bounded measurable function defined for all $\mathbf{p} \in P^k$ where P is any finite subset of M. Then*

$$\mathbb{E}\left[\sum_{\mathbf{p}\in P_n^k} h(\mathbf{p}; P_n)\right] = \frac{n^k}{k!}\mathbb{E}\left[h(\mathbf{p}'; \mathbf{p}' \cup P_n)\right], \tag{3.19}$$

where on the right-hand side, $bp' \in M^k$ is given by k points i.i.d and independent of the Poisson process P_n.

Therefore, we have

$$\mathbb{E}\left[\hat{K}_k(r)\right] \leq \mathbb{E}\left[\sum_{\mathbf{p}\in(P_n)^{k+2}} \chi_r'(\mathbf{p};P_n)\right] \tag{3.20}$$

$$= \frac{n^{k+2}}{(k+2)!}\mathbb{E}\left[\chi_r'(\mathbf{p};\mathbf{p}\cup P_n)\right], \tag{3.21}$$

where $\mathbf{p} \in (\mathbb{T}^d)^{k+2}$ is a set of $(k+2)$ i.i.d. points independent of the Poisson process P_n. Moreover, we have

$$\mathbb{1}\left\{B(\mathbf{p}) \cap (\mathbf{p}\cup P_n) = \{p_0,\ldots,p_{k+1}\}\right\} = \mathbb{1}\left\{B(\mathbf{p}) \cap P_n = \emptyset\right\} \tag{3.22}$$

almost surely. Then, with probability 1,

$$\chi_r'(\mathbf{p};\mathbf{p}\cup P_n)$$

$$= \chi_r(\mathbf{p})\mathbb{1}\left\{B(\mathbf{p})\cap P_n = \emptyset\right\}\mathbb{1}\left\{A(\mathbf{p}) \not\subset U(\mathbf{p}\cup P_n; R(\mathbf{p}))\right\} \tag{3.23}$$

$$\leq \chi_r(\mathbf{p})\mathbb{1}\left\{B(\mathbf{p})\cap P_n = \emptyset\right\}\mathbb{1}\left\{A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p}))\right\} \tag{3.24}$$

For the first part, from the Poisson process,

$$\mathbb{P}\left\{B(\mathbf{p})\cap P_n = \emptyset\right\} = e^{-n\omega_d R(\mathbf{p})^d}. \tag{3.25}$$

For $\mathbb{1}\left\{A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p}))\right\}$, set

$$p_{A(\mathbf{p})} = \mathbb{P}\left\{A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p})) | B(\mathbf{p})\cap P_n = \emptyset\right\}. \tag{3.26}$$

It implies that

$$\mathbb{E}\left[\hat{K}_k(r)\right] \leq \frac{n^{k+2}}{(k+2)!}\mathbb{E}\left[\chi'_r(\mathbf{p};\mathbf{p}\cup P_n)\right] \tag{3.27}$$

$$= \frac{n^{k+2}}{(k+2)!}\int_{(\mathbb{T}^d)^{k+2}} \chi_r(\mathbf{p})\mathbb{P}\left\{B(\mathbf{p})\cap P_n = \emptyset\right\}$$
$$\cdot\mathbb{P}\left\{A(\mathbf{p}) \not\subset U(P_n;R(\mathbf{p}))|B(\mathbf{p})\cap P_n=\emptyset\right\}\, d\mathbf{p} \tag{3.28}$$

$$\leq Cn^{k+2}\int_{(\mathbb{T}^d)^{k+2}} \chi_r(\mathbf{p})e^{-n\omega_d R(\mathbf{p})^d}\cdot p_{A(\mathbf{p})}\, d\mathbf{p} \tag{3.29}$$

When n is sufficiently large, $R(\mathbf{p})$ is less than the injectivity radius of $\mathbb{T}^d$. Then, we can define a change of variables as follows: $\mathbf{p} = \Phi(c,\rho,V,\mathbf{p}_0)$ where $c = C(\mathbf{p}) \in \mathbb{T}^d, \rho = R(\mathbf{p}) \in \mathbb{R}$, V in the Grassmannian $G(k+1;d)$ is the $(k+1)$ dimensional real vector space in $\mathbb{R}^d$ in which the points in $\mathbf{p}$ is, and $\mathbf{p}_0 \in (S^k)^{k+2}$ which is a tuple of $(k+2)$ points on the unit sphere. The Jacobian of Φ satisfies the following scaling property:

$$J_\Phi(c,\rho,V,\mathbf{p}_0) = C_\Phi\rho^{d(k+1)-1}, \tag{3.30}$$

where C_Φ is the Jacobian of Φ at a fixed $c, V, \mathbf{p}_0$ and radius 1.

Note that

$$\chi_r(\mathbf{p}) = \mathbb{1}\left\{C(\mathbf{p})\in\Delta(\mathbf{p})\right\}\mathbb{1}\left\{R(\mathbf{p})\geq r\right\} \tag{3.31}$$

$$= \mathbb{1}\left\{\mathbf{0}\in\Delta(\mathbf{p}_0)\right\}\mathbb{1}\left\{\rho\geq r\right\} = \chi_{\mathbf{p}_0}\cdot\mathbb{1}\left\{\rho\geq r\right\}, \tag{3.32}$$

where $\chi_{\mathbf{p}_0}$ only depends on $\mathbf{p}_0$.

We have

$$\mathbb{E}\left[\hat{K}_k(r)\right] \le Cn^{k+2} \int_{(\mathbb{T}^d)^{k+2}} \chi_r(\mathbf{p}) e^{-n\omega_d R(\mathbf{p})^d} \cdot p_{A(\mathbf{p})}\, d\mathbf{p} \tag{3.33}$$

$$= Cn^{k+2} \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} \int_{c,V,\mathbf{p}_0} \chi_{\mathbf{p}_0} \cdot p_{A(\mathbf{p})}\, dc dV d\mathbf{p}_0 d\rho. \tag{3.34}$$

Now apply the change of variables from Section 3.3. We re-parametrize the points on the unit sphere $\mathbf{p}_0$ by $\mathbf{p}_0 = \Psi(\varphi, n, p_0, \mathbf{p}_0')$ where $\varphi = \varphi(\mathbf{p}_0) = \varphi(\mathbf{p}) \in \mathbb{R}$, $n \in S^k$ is the unit normal vector of the linear subspace of the largest k-face of $\Delta(\mathbf{p}_0)$, $p_0 \in S^k$ is the point in $\mathbf{p}_0$ that is not on the largest k-face, and $\mathbf{p}_0 \in (S^{k-1})^{k+1}$ gives the positions of the points in the largest face. According to the estimate in Section 3.3 (3.16), the Jacobian of Ψ satisfies:

$$\left|J_\Psi(\varphi, n, p_0, \mathbf{p}_0')\right| \le \widetilde{J_\Psi}(n, p_0, \mathbf{p}_0'), \tag{3.35}$$

where $\widetilde{J_\Psi}$ does not depend on φ.

Finally,

$$\mathbb{E}\left[\hat{K}_k(r)\right]$$

$$\le Cn^{k+2} \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} \int_{c,V,\mathbf{p}_0} \chi_{\mathbf{p}_0} \cdot p_{A(\mathbf{p})}\, dc dV d\mathbf{p}_0 d\rho \tag{3.36}$$

$$\begin{aligned} &= Cn^{k+2} \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} \\ &\cdot \left(\int_0^{\varphi_{\max}} \int_{c,V,n,p_0,\mathbf{p}_0'} \chi_{\mathbf{p}_0} p_{A(\mathbf{p})} \right. \\ &\left. \cdot \left|J_\Psi(\varphi, n, p_0, \mathbf{p}_0')\right|\, dc dV dn dp_0 d\mathbf{p}_0' d\varphi \right) d\rho \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\le Cn^{k+2} \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} \\ &\cdot \left(\int_0^{\varphi_{\max}} \int_{c,V,n,p_0,\mathbf{p}_0'} \chi_{\mathbf{p}_0} p_{A(\mathbf{p})} \widetilde{J_\Psi}\, dc dV dn dp_0 d\mathbf{p}_0' d\varphi \right) d\rho. \end{aligned} \tag{3.38}$$

Set $\varphi_0 = \frac{C \log\log n}{\Lambda}$ such that for $\rho \geq r$ and $\varphi \geq \varphi_0$

$$\varphi^{-d} e^{C_0 n \omega_d \rho^d \varphi} \leq \frac{\Lambda^d}{(C \log\log n)^d} \cdot e^{-C_0 \cdot C \log\log n} \tag{3.39}$$

$$= \frac{\Lambda^d}{(C \log\log n)^d} \cdot (\log n)^{-C_0 \cdot C} < \Lambda^{-1}. \tag{3.40}$$

For $\varphi(\mathbf{p}) = \varphi \geq \varphi_0$, take an ε-net S on the annulus $A(\mathbf{p})$ with $\varepsilon = \frac{1}{2}\varphi(\mathbf{p})R(\mathbf{p})$ and $|S| \leq C\varphi(\mathbf{p})^{-d}$ for some constant C, that is, for any point $p \in A(\mathbf{p})$, there exist a point $p_0 \in S$, such that

$$\mathrm{dist}(p_0, p) \leq \varepsilon = \frac{1}{2}\varphi(\mathbf{p})R(\mathbf{p}). \tag{3.41}$$

Therefore, $A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p}))$ implies that $S \not\subset U(P_n; R(\mathbf{p}) - \varepsilon)$. Now we have,

$$p_{A(\mathbf{p})} = \mathbb{P}\left\{A(\mathbf{p}) \not\subset U(P_n; R(\mathbf{p})) | B(\mathbf{p}) \cap P_n = \emptyset\right\} \tag{3.42}$$

$$\leq \mathbb{P}\left\{S \not\subset U(P_n; (1-\varepsilon)R(\mathbf{p})) | B(\mathbf{p}) \cap P_n = \emptyset\right\} \tag{3.43}$$

$$\leq \sum_{p \in S} \mathbb{P}\left\{p \notin U(P_n; R(\mathbf{p}) - \varepsilon) | B(\mathbf{p}) \cap P_n = \emptyset\right\} \tag{3.44}$$

$$= \sum_{p \in S} \mathbb{P}\left\{B(p; R(\mathbf{p}) - \varepsilon) \cap P_n = \emptyset | B(\mathbf{p}) \cap P_n = \emptyset\right\} \tag{3.45}$$

$$\leq |S| \, e^{-n \mathrm{vol}(B(p_0; R(\mathbf{p})(1-\varphi(\mathbf{p}))) \setminus B(\mathbf{p}))} \tag{3.46}$$

$$\leq C\varphi(\mathbf{p})^{-d} e^{C_0 n \omega_d R(\mathbf{p})^d \varphi(\mathbf{p})}, \tag{3.47}$$

where p_0 is a point on the sphere $\partial B(C(\mathbf{p}); \varphi(\mathbf{p})R(\mathbf{p}))$. Therefore,

$$\int_{\varphi_0}^{\varphi_{\max}} \int_{c,V,n,p_0,\mathbf{p}_0'} \chi_{\mathbf{p}_0} p_{A(\mathbf{p})} \widetilde{J_\Psi} \, dc dV dn dp_0 d\mathbf{p}_0' d\varphi \tag{3.48}$$

$$\leq \int_{\varphi_0}^{\varphi_{\max}} \int_{c,V,n,p_0,\mathbf{p}_0'} C\varphi^{-d} e^{C_0 n \omega_d \rho^d \varphi} \widetilde{J_\Psi} \, dc dV dn dp_0 d\mathbf{p}_0' d\varphi \tag{3.49}$$

$$\leq C \int_{\varphi_0}^{\varphi_{\max}} \varphi^{-d} e^{C_0 n \omega_d \rho^d \varphi} \, d\varphi. \tag{3.50}$$

Moreover, for $\varphi \leq \varphi_0$, we have

$$\int_0^{\varphi_0} \int_{c,V,n,p_0,\mathbf{p}_0'} \chi_{\mathbf{p}_0} p_{A(\mathbf{p})} \widetilde{J_\Psi}\, dcdV dndp_0 d\mathbf{p}_0' d\varphi \leq \int_0^{\varphi_0} 1\, d\varphi = \varphi_0 \tag{3.51}$$

Then, from (3.50) and (3.51),

$$\mathbb{E}\left[\hat{K}_k(r)\right] \tag{3.52}$$

$$\leq Cn^{k+2} \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} \left(\varphi_0 + C \int_{\varphi_0}^{\varphi_{\max}} \varphi^{-d} e^{C_0 n\omega_d \rho^d \varphi}\, d\varphi\right) d\rho \tag{3.53}$$

$$\leq Cn^{k+2} \Lambda^{-1} \log\log n \int_r^\infty e^{-n\omega_d \rho^d} \rho^{d(k+1)-1} d\rho \tag{3.54}$$

$$\leq Cn\Lambda^{k-1} e^{-\Lambda} \log\log n. \tag{3.55}$$

When $\Lambda = \log n + (k-1)\log\log n + w(n)$ where $w(n) - \log\log\log n \to \infty$, then $\mathbb{E}\left[\hat{K}_k(r)\right] \to 0$ as $n \to \infty$.

3.5 Proofs: Upper Bound of Probability

3.5.1 Estimation on the Hole Size

In this section, we will prove that the complement of the union of random balls centered at points from P_n only has components of small diameters with high probability.

Proposition 1. *Suppose the radius $r < r_{max}/3$ where r_{max} is the injectivity radius of the torus. Then, there exists some constant $1/2 < \kappa < 1$ and for $\Lambda = n\omega_d r^d \geq \kappa \log n$, the probability of each component of the complement being contained in a ball with radius $2r$ tends to 1 as $n \to \infty$.*

Note that if for any point p in the complement, the sphere $\partial B(p; 2r)$ is covered by the union of balls, then all the components of the complement are contained in balls with radii

$2r$. We need to show that

$$\mathbb{P}\left\{\begin{array}{c}\exists p \in \mathbb{T}^d \setminus U(P_n; r) \\ \text{such that } \partial B(p; 2r) \not\subset U(P_n; r)\end{array}\right\} \to 0, \text{as } n \to \infty. \tag{3.56}$$

Fix the point p in the complement. Let $\mathcal{N}$ be a 1/3-net on the unit sphere $\partial B(0; 1)$ in the Euclidean space $\mathbb{R}^d$. Let $\mathcal{N}_r$ be the rescaled $\mathcal{N}$ on $\partial B(p; 2r)$. We have that for any point s on the sphere $\partial B(p; 2r)$, there exists $s_0 \in \mathcal{N}_r$ satisfying $\text{dist}(s, s_0) \leq 2r/3$. It follows that if $s \notin U(P_n; r)$, then $s_0 \notin U(P_n; r/3)$. We may obtain that,

$$\begin{aligned}
&\mathbb{P}\left\{\partial B(p; 2r) \not\subset U(P_n; r) | p \in \mathbb{T}^d \setminus U(P_n; r)\right\} \\
&= \mathbb{P}\left\{\exists s \in \partial B(p; 2r), s \notin U(P_n; r) | p \in \mathbb{T}^d \setminus U(P_n; r)\right\} \\
&\leq \mathbb{P}\left\{\exists s_0 \in \mathcal{N}_r, s_0 \notin U(P_n; 2r/3) | p \in \mathbb{T}^d \setminus U(P_n; r)\right\} \\
&\leq \mathbb{P}\left\{\exists s_0 \in \mathcal{N}_r, P_n \cap B(s_0; r/3) = \emptyset | p \in \mathbb{T}^d \setminus U(P_n; r)\right\} \\
&\leq |\mathcal{N}_r| \cdot e^{-n\omega_d r^d / 3^d} = |\mathcal{N}| \cdot e^{-n\omega_d r^d / 3^d}
\end{aligned} \tag{3.57}$$

To extend the result to the entire torus, we need an (ϵr)-net on the torus $\mathbb{T}^d$, $\mathcal{N}'$, such that there exists a point p_0 for any point $p \in \mathbb{T}^d$ such that $\text{dist}(p, p_0) \leq \epsilon r$, where ϵ is a constant such that $1-(1+1/3^d)^{-d} < \epsilon < 1$. We may choose the (ϵr)-net such that the number of points $|\mathcal{N}'| \leq C/(\epsilon r)^d$ for some constant C. Set $r' = (1-\epsilon)r > 0$. Similar to the arguments above, we have that for each $p \in \mathbb{T}^d$, there exists $p_0 \in \mathcal{N}'$ satisfying $\text{dist}(p, p_0) \leq \epsilon r$ and furthermore, if $p \notin U(P_n; r)$ and $\partial B(p; 2r) \not\subset U(P_n; r)$, then $p_0 \notin U(P_n; r')$ and $\partial B(p_0; 2r') \not\subset U(P_n; r')$.

Therefore,

$$
\begin{aligned}
&\mathbb{P}\left\{\exists p \in \mathbb{T}^d \setminus U(P_n;r) \text{ such that } \partial B(p;2r) \not\subset U(P_n;r)\right\} \\
&= \mathbb{P}\left\{\exists p \in \mathbb{T}^d, p \notin U(P_n;r) \text{ and } \partial B(p;2r) \not\subset U(P_n;r)\right\} \\
&\leq \mathbb{P}\left\{\exists p_0 \in \mathcal{N}', p_0 \notin U(P_n;r') \text{ and } \partial B(p_0;2r') \not\subset U(P_n;r')\right\} \\
&\leq \left|\mathcal{N}'\right| \cdot \mathbb{P}\left\{p_0 \notin U(P_n;r') \text{ and } \partial B(p;2r) \not\subset U(P_n;r')\right\} \\
&\leq \left|\mathcal{N}'\right| \mathbb{P}\left\{p_0 \notin U(P_n;r')\right\} \mathbb{P}\left\{\partial B(p;2r) \not\subset U(P_n;r') | p_0 \notin U(P_n;r')\right\} \\
&\leq \left|\mathcal{N}'\right| e^{-n\omega_d r'^d} \cdot \left|\mathcal{N}\right| e^{-n\omega_d r^d/3^d} \leq C_1 r^{-d} e^{-n\omega_d r^d (1-\epsilon)^d (1+1/3^d)} \\
&= C_1 n \Lambda^{-1} e^{-\Lambda(1-\epsilon)^d (1+1/3^d)},
\end{aligned}
\tag{3.58}
$$

for some positive constants C_1.

Note that $\epsilon > 1 - (1+1/3^d)^{-d}$ and so $1 < (1-\epsilon)^d(1+1/3^d) < 2$. Pick κ such that $\kappa(1-\epsilon)^d(1+1/3^d) > 1$ for some $c > 0$. Then, for $\Lambda \geq \kappa \log n$, $\Lambda(1-\epsilon)^d(1+1/3^d) > \log n$. It follows that the probability

$$
\begin{aligned}
C_1 n \Lambda^{-1} e^{-\Lambda(1-\epsilon)^d(1+1/3^d)} &\leq C_1 \kappa^{-1} n (\log n)^{-1} e^{-\log n} \\
&= C_1 \kappa^{-1} (\log n)^{-1} \to 0, \text{ as } n \to \infty.
\end{aligned}
\tag{3.59}
$$

3.5.2 Extra Homology Classes

From the previous section, the for large n, the components of the complement are contained in small balls when the radius r is small. It leads to the results that the homology classes of $\mathbb{T}^d$ are all represented in the union of balls $U(P_n;r)$. That is, we have

Theorem 4. *For a fixed and sufficiently small r, the map*
$H_k(U(P_n;r)) \to H_k(\mathbb{T}^d)$ when $k < d$ is surjective with high probability as $n \to \infty$.

In this section, we denote $U = U(P_n; r)$ for short. Let $\mathbb{T}^d \setminus U = \sqcup_i X_i$ where each X_i is a component of the complement. When the radius r is not a critical value of the distance function to P_n, we have the following long exact sequence by Mayer-Vietoris

$$
\begin{aligned}
&\cdots \to H_{k+1}(\mathbb{T}^d) \to \oplus_i H_k(\partial X_i) \\
&\to H_k(U) \oplus (\oplus_i H_k(X_i)) \to H_k(\mathbb{T}^d) \to \cdots
\end{aligned} \tag{3.60}
$$

For sufficiently small radius r, using the results from the previous section, the component X_i has a diameter less than $2r$ and thus the map $H_k(X_i) \to H_k(\mathbb{T}^d)$ is 0. It remains to show that $H_{k+1}(\mathbb{T}^d) \to \oplus_i H_k(\partial X_i)$ is also 0 which holds if each $H_{k+1}(\mathbb{T}^d) \to H_k(\partial X_i)$ is 0. Let D be the d-dimensional disk with radius $2r$ in $\mathbb{T}^d$ that contains X_i. Then, both from Mayer-Vietoris sequence we have

$$
H_{k+1}(D) \to H_k(\partial X_i) \to H_k(D \setminus X_i) \oplus H_k(X_i) \to H_k(D) \tag{3.61}
$$

and

$$
\begin{aligned}
&H_k(S^d) \to H_k(D \setminus X_i) \oplus H_k(\mathbb{T}^d \setminus D) \\
&\to H_k(\mathbb{T}^d \setminus X_i) \to H_{k-1}(S^d)
\end{aligned} \tag{3.62}
$$

are exact. It follows that

$$
H_k(\partial X_i) \xrightarrow{\cong} H_k(D \setminus X_i) \oplus H_k(X_i) \tag{3.63}
$$

and that

$$
H_k(D \setminus X_i) \to H_k(\mathbb{T}^d \setminus X_i) \tag{3.64}
$$

is injective if $k < d$.

Therefore, the composition $H_k(\partial X_i) \to H_k(\mathbb{T}^d \setminus X_i) \oplus H_k(X_i)$

$$\begin{tikzcd} H_k(\partial X_i) \arrow[rr] \arrow[d, "\cong"] & & H_k(\mathbb{T}^d \setminus X_i) \oplus H_k(X_i) \\ H_k(D \setminus X_i) \oplus H_k(X_i) \arrow[urr, hook] & & \end{tikzcd} \tag{3.65}$$

is injective.

Apply Mayer-Vietoris sequence to X_i and $\mathbb{T}^d \setminus X_i$ and we will get

$$\begin{aligned} \cdots \to H_{k+1}(\mathbb{T}^d) \to H_k(\partial X_i) \\ \to H_k(\mathbb{T}^d \setminus X_i) \oplus H_k(X_i) \to H_k(\mathbb{T}^d) \to \cdots \end{aligned} \tag{3.66}$$

It follows that $H_{k+1}(\mathbb{T}^d) \to H_k(\partial X_i)$ is 0.

3.5.3 Probability of Correct Homology

Theorem 4 implies that with high probability, all the k-th homology classes of $\mathbb{T}^d$ can be found in the homology of the union of balls $U(P_n; r)$. It follows that the Betti numbers $\beta_k(U(P_n; r)) \geq \beta_k(\mathbb{T}^d)$ when $k < d$ with high probability as $n \to \infty$. Furthermore, we have

$$0 \leq \beta_k(U(P_n; r)) - \beta_k(\mathbb{T}^d) \leq \hat{K}_k(r), \tag{3.67}$$

where $\hat{K}_k(r)$ is the number of critical points of index $k+1$ which kill some k-th homology classes defined in Section 3.2. Therefore,

$$\mathbb{E}\left[\hat{K}_k(r)\right] \geq \mathbb{E}\left[\beta_k(U(P_n; r)) - \beta_k(\mathbb{T}^d)\right] \geq \mathbb{P}\left\{\beta_k(U(P_n; r)) \neq \beta_k(\mathbb{T}^d)\right\}. \tag{3.68}$$

From (3.55), if $\Lambda = \log n + (k-1)\log\log n + w(n)$ where $w(n) - \log\log\log n \to \infty$, then

$$\mathbb{P}\left\{\beta_k(U(P_n;r)) \neq \beta_k(\mathbb{T}^d)\right\} \leq \mathbb{E}\left[\hat{K}_k(r)\right] \tag{3.69}$$

$$\leq Cn\Lambda^{k-1}e^{-\Lambda}\log\log n = Ce^{-w(n)+\log\log\log n} \to 0. \tag{3.70}$$

It concludes that the Betti number is correct with high probability for $\Lambda = \log n + (k-1)\log\log n + w(n)$.

3.6 Simulation

3.6.1 Delaunay Triangulation and Čech Persistent Homology

In our settings, when the homology groups are getting correct, $\Lambda = n\omega_d r^d$ is around $\log n$, which means that for each point on the torus $\mathbb{T}^d$, there are $\log n$ points in expectation within the range of r. To construct the Čech complex from the points, we have to examine the intersections among about $\log n$ points for each of the n points in P_n. The computation of checking intersections of several balls is also expensive. The result spaces of complexes may have even higher dimension. Therefore, directly computing Čech complex to get the persistent homology groups is computationally difficult. Discussion about computability of Čech complexes can be found in [29].

Let $P_n \subset \mathbb{T}^d$ be a set of n points. For any point p in P_n, define function $f_p(x) = d(x,p)$. We have that f_p is differentiable at any points on $\mathbb{T}^d$ except p. The distance function to the whole set P_n is $f : \mathbb{T}^d \to \mathbb{R}$ defined as, for any $x \in \mathbb{T}^d$,

$$f(x) = \min\left\{d_p(x) : p \in P_n\right\}. \tag{3.71}$$

For each fixed point x on $\mathbb{T}^d$, in general, not all the n functions f_i are necessary to get

$f(x)$ in a neighborhood of x. For a subset P of P_n, define the set

$$G(P) = \left\{ x \in \mathbb{T}^d : f(x) = f_p(x) \text{ for all } p \in P. \right\}. \tag{3.72}$$

One may prove a simple property of the sets G:

$$G(P \cup Q) = G(P) \cap G(Q). \tag{3.73}$$

Suppose that the points in P_n are in general positions. That is, for any subset $P \subset P_n$ of $(k+1)$ points with $k \leq d$, there does not exist any point x on $\mathbb{T}^d$ such that the gradients ∇f_p for all p in P do not lie in a $(k-1)$-dimensional affine space. Then, it can be prove that if $G(P)$ is non-empty, it is a compact submanifold of dimension $d - k = d - |P| + 1$.

Furthermore, for every P with exactly $d+1$ points and $G(P)$ non-empty, since the dimension of $G(P)$ is $d - (d+1) + 1 = 0$, $G(P)$ is a single point. The ball centered at the point in $G(P)$ with radius given by $f|_{G(P)}$ is the smallest one that contains the simplex $\Delta(P)$, $B(P)$. It does not contain any points other than P. $\Delta(P)$ is called a Delaunay simplex. If we collect all such simplexes $\Delta(P)$ where $G(P) \neq \emptyset$, we will get a *Delaunay triangulation* or Delaunay complex of the torus $\mathbb{T}^d$ with vertexes given by points in P_n. A *Delaunay triangulation* is the unique triangulation of a set of vertexes where the circumsphere of each top simplexes has no points in its interior. One of the advantages of Delaunay triangulation is that there are relatively efficient algorithms to construct. We may use Delaunay complex to compute Čech homology groups. Discussions of Delaunay triangulation can be found in [19], [33].

To define a persistent homology on Delaunay complex, each simplex or sub-simplex $\Delta(P)$ for some set of vertexes P is assigned to the value

$$F(\Delta(P)) = \min f|_{G(P)}. \tag{3.74}$$

Then the persistent homology of the filtration given by the Delaunay complex and the function F is isomorphic to the Čech persistent homology with respect to the changing radius, since $\bigcap_{p \in P} B(p;r) \neq \emptyset$ if and only if $r \geq F(\Delta(P))$ which is also equivalent to that the simplex $\Delta(P)$ is covered by $B(p;r)$ for $p \in P$.

We want to find a way to evaluate F effectively. In fact, the restriction $f|_{G(P)}$ is a smooth Morse function on $G(P)$. In our case, the set $G(P)$ is an intersection of several convex $G(p)$ and therefore, and thus $G(P)$ is convex. This implies that the distance function $f|_{G(P)}$ always has a unique local minimum, either in the interior or on the boundary. If $f|_{G(P)}$ has no interior minimum point, the minimum point is on the boundary of $G(P)$. In this case, we have that the minimum point will also be a minimum point of $G(P')$ for some set $P' \supset P$; that is,

$$F(\Delta(P)) = \min_{P' \supset P} F(\Delta(P')). \tag{3.75}$$

Therefore, the value of the simplex $\Delta(P)$ when $G(P)$ has no interior minimum point is equal to the minimum of the values of parent simplexes that contain $\Delta(P)$.

Meanwhile, we know that the center $C(P)$ of the minimal circumsphere satisfying

$$d(C(P), p) = R(P), \quad \text{for any } p \in P. \tag{3.76}$$

. If the center $C(P)$ is in $G(P)$, it is the minimum point of $f|_{G(P)}$. In this case, we have that $F(\Delta(P)) = \min f|_{G(P)} = R(P)$ is the radius of the circumsphere. On the other hand, if the center $C(P)$ is not in $G(P)$, the minimum point of $\min f|_{G(P)}$ is on the boundary.

As a result, we may evaluate F recursively starting from the top simplexes $\Delta(P)$ of dimension d. For each of the (sub-)simplex $\Delta(P)$, we may first find the center of the circumsphere of P, $C(P)$, then examine if $C(P)$ is in $G(P)$ by checking if there are other points in the circumsphere $B(C(P), R(P))$ (which is always true for top simplexes). We have that

$$F(\Delta(P)) = \begin{cases} R(P), & \text{if } C(P) \in G(P); \\ \\ \min_{P' \supset P} F(\Delta(P')), & \text{otherwise.} \end{cases} \tag{3.77}$$

3.6.2 Relation to Critical Points

As mentioned in Section 3.2, we used critical points of the distance function to the point sets P_n to find the homology groups by Morse theory.

We know that the critical points are defined to be the centers $C(P)$ that lies inside of $\Delta(P)$. They are interior minimum points of $f|_{G(P)}$. According to [20], the critical point of the distance function f is a critical point of $f|_{G(P)}$ and the index is $(|P|-1)+\mathrm{Ind}(f|_{G(P)})$. The critical point of $f|_{G(P)}$ can only be the unique interior minimum point. The minimum point has index 0 and so the critical point has index $|P|-1$.

The critical points of index d are corresponding to the sets P with $d+1$ points, the convex hulls of which are just the top simplexes in the Delaunay complex. Furthermore, for any P such that $G(P)$ is non-empty, the maximum point of $f|_{G(P)}$ must be in $G(P')$ for some $P' \supset P$ and $\left|P'\right| = d+1$. That means $\Delta(P)$ is a sub-simplex of $\Delta(P')$ and thus every critical point is corresponding to a simplex in the Delaunay complex.

However, not all the Delaunay simplexes or even those with an interior minimum are critical points according to [20] and Section 3.2. Those interior minimal points of $f|_{G(P)}$ that fall outside of the convex hull can be problematic since they do not create or kill any new cycles. This is the reason why we exclude them in the previous proofs. However, this will not be a problem in Delaunay persistent homology defined before.

For the critical points that fall outside of the simplex $\Delta(P)$, there must exist a boundary sub-simplex $\Delta(P')$ where $P' \subset P$ and $\left|P'\right| = |P|-1$ such that the minimum of $f|_G(P')$

is not in the interior of $G(P')$ but rather on $G(P) \subset \partial G(P')$. We have that $F(\Delta(P')) = \min f|_{G(P')} = \min f|_{G(P)} = F(\Delta(P))$; that is, $\Delta(P)$ has the same value as one of its boundary simplex. It follows that such critical point will not create or kill a homology class in the persistent settings.

3.6.3 Construction of Delaunay Triangulation

Algorithms of construction of Delaunay triangulation are discussed in [19], [33]. Incremental algorithm is one of the basic ones. It assumes the points are all contained in a large simplex. Then, it adds points one by one. In each step, an existing simplex is broken into smaller simplexes by the newly added point. Then, delete and update the sub-simplexes whose circumsphere encompasses the new point. However, this algorithm does not apply to our case since on the torus, there is no such maximal simplex that contains all the points and circumsphere of a large scale simplex is not easy to spot because of the global manifold structure. There are indeed some extensions of this method to allow in torus case as known as periodic Delaunay triangulation. Those methods essentially construct triangulation on the universal covering of $\mathbb{R}^d$. [12] gives an optimization of the 3-dimensional case.

Divide-and-conquer algorithm is another one to construct Delaunay triangulation. The original algorithm can be found in [14]. The algorithm uses hyperplanes to split the point sets into two. It first stitches the two halves by constructing triangulation along the hyperplane using incremental algorithm. Then, it applies the same divide-and-conquer algorithm to the two subsets but starting with $(d-1)$-faces from the crossing simplexes in the previous step. This algorithm, according to [14], has a $O(n^{\lceil d/2 \rceil + 1})$ worst case time complexity. The limitation is that the original algorithm only considers $\mathbb{R}^d$. However, due to the nature of this algorithm, we can adapt it to the problems of any Riemannian manifold since we can partition the manifold into pieces diffeomorphic to Euclidean spaces.

In our settings, the flat torus is $\mathbb{R}^d$ modulo the integer lattice $\mathbb{Z}^d$. The first step of the divide-and-conquer algorithm is to cut it along the boundary of the unit cube $\mathcal{C} = [0,1]^d$. We set F_{interior} to be the set of the other $(d-1)$-simplexes that are in $\mathcal{C}$ and F_{active} to be the set of the $(d-1)$-simplexes across the cut $\partial\mathcal{C}$ which remain to find the next d-simplex adjacent to it.

To simplify some operations in the algorithm, we say switching an element e in a set S if we add e to S if $e \notin S$ and remove e from S if $e \in S$.

Algorithm 3.1: Divide-and-conquer algorithm to construct the Delaunay triangulation on $\mathbb{T}^d$

Input: Points P_n of $\mathbb{T}^d$ as coordinates in $\mathcal{C} = [0,1]^d$.

Output: Delaunay triangulation T as a set of d-dimension of simplexes.

Data: Sets F_{active}, F_{interior}.

1 Find any pair (p,q) from P_n such that one is the nearest point to the other and the shortest geodesic between them (as a line segment in $\mathbb{R}^d$) crosses the cut $\partial\mathcal{C}$. Let $P = \{p,q\}$.

2 Repeat the following steps to build the first Delaunay simplex $\Delta(P)$ across the cut $\partial\mathcal{C}$:

   1. Find the next point p' from P_n such that the radius of the circumsphere $R(P \cup \{p'\})$ is minimal.

   2. Add p' to set P.

3 Add the simplex $\Delta(P)$ to the Delaunay triangulation T.

4 For each $(d-1)$-dimensional sub-simplex f of $\Delta(P)$:

   1. If f lies entirely in $\mathcal{C}$, add f to F_{interior}.

   2. Otherwise, if f crosses the cut $\partial\mathcal{C}$, switch f in F_{active}.

Algorithm 3.1 (cont.): Divide-and-conquer algorithm to construct the Delaunay triangulation on $\mathbb{T}^d$

5 Enumerate all the $(d-1)$-dimensional sub-simplex f of F_{active} until the set becomes empty,

1. Find the next d-dimensional simplex $\Delta(P)$ such that
   f is a face of $\Delta(P)$ and $\Delta(P)$ is a Delaunay simplex.
2. Add the simplex $\Delta(P)$ to the Delaunay triangulation T.
3. Do the same as Line 4 with $\Delta(P)$ to update F_{interior} and F_{active}.

6 Now that we have the triangulation T along the cut $\partial\mathcal{C}$ and a set of interior faces F_{interior}. We may just apply the original divide-and-conquer algorithm from [14] to the cube $\mathcal{C} = [0,1]^d$. Notice that we no more need initializing steps because we already have F_{interior}. Add the simplexes from the original algorithm to T.

[14] uses a uniform grid to index the point for fast query of the points. However, this method is not efficient when the dimension is large as the grid size grows exponentially in dimension. Instead, we utilize a binary space partitioning tree called k-d tree to organizing the points in querying nearest points and the minimal circumsphere. In general the nodes in the k-d tree represents a partition of points usually divided by a hyperplane. Its technical details can be found in [1]. In fact, the k-d tree cooperate with our divide-and-conquer algorithm so well since we may reuse the partition of the tree in the construction of the Delaunay triangulation.

3.6.4 Numerical Results and Graphs

		$(\Lambda - \log n)/\log\log n =$				
	k	$(k-3)$	$(k-2)$	$(k-1)$	k	$(k+1)$
Remaining critical point numbers	0	0.0	0.0	0.0	0.0	0.0
	1	724.1	74.0	**7.6**	1.1	0.1
	2	1384.6	176.9	**23.0**	3.6	1.0
	3	833.8	129.9	**22.2**	4.8	0.8
	4	173.7	31.7	**5.9**	1.0	0.3
Betti numbers	0	0.0	0.0	0.0	0.0	0.0
	1	35.0	3.0	0.2	0.0	0.0
	2	78.6	6.4	0.6	0.3	0.1
	3	278.0	65.3	12.3	1.3	0.0

Table 3.1
Averages of critical points and Betti numbers in $\mathbb{T}^4$ with number of points $n = 15000$ in 10 experiments.
The columns are the numbers when $\Lambda = n\omega_d r^d = \log n + l \log\log n$ for respective coefficients l. The bold numbers are corresponding to $l = k-1$ which are the theoretical vanishing thresholds in [7].

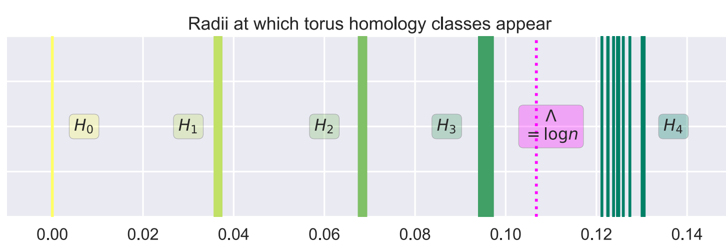
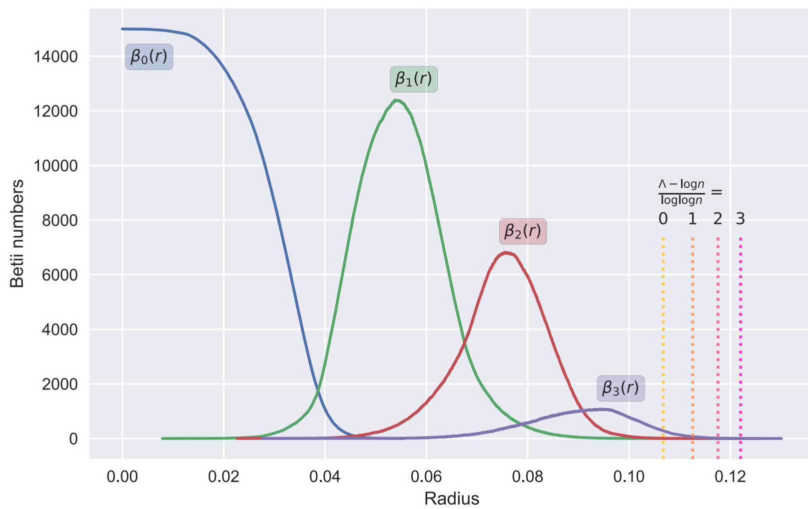
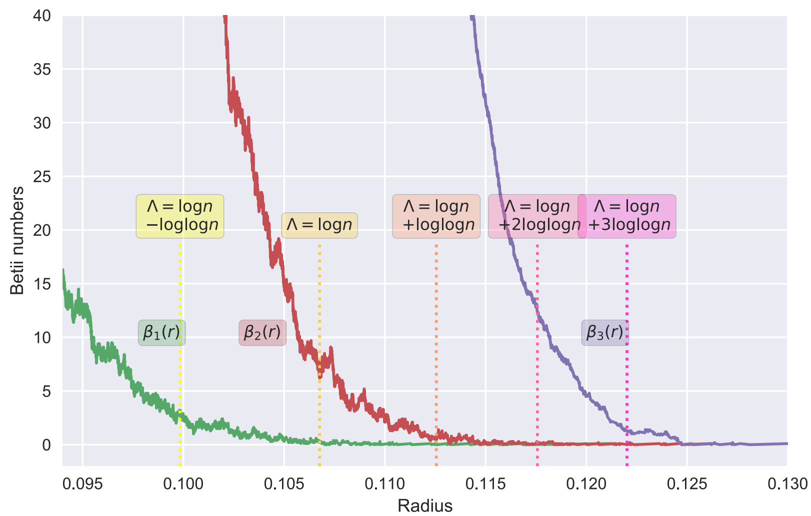


Figure 3.2
Radii of global torus homology classes of $\mathbb{T}^4$ in 10 experiments. Number of points $n = 15000$. The radius when $\Lambda = n\omega_d r^d = \log n$ is also marked. The torus homology classes are all before this point which is guaranteed by Theorem 4 for sufficiently large n.

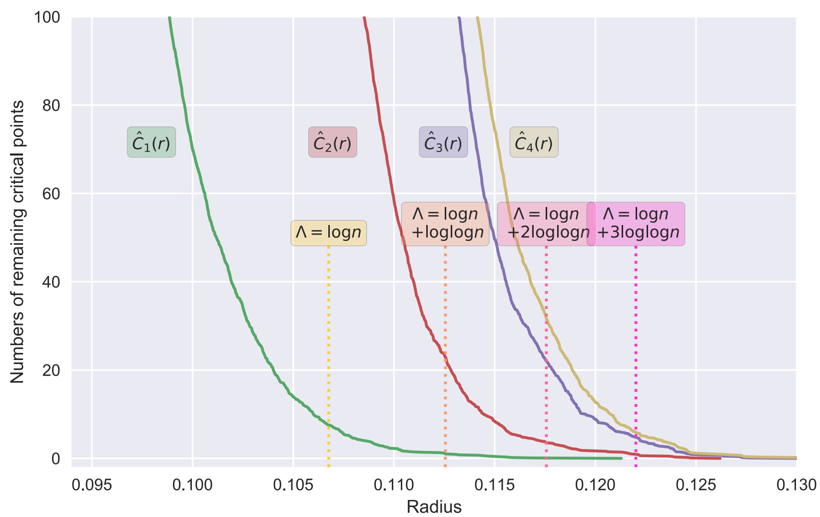


(a) Overview of Betti numbers in $\mathbb{T}^4$ with number of points $n = 15000$.



(b) Betti numbers near the thresholds in $\mathbb{T}^4$ with number of points $n = 15000$.

Figure 3.3
Betti numbers and critical points of $\mathbb{T}^4$ with number of points $n = 15000$ in 10 experiments.



(c) Numbers of remaining critical points in $\mathbb{T}^4$ with number of points $n = 15000$. The vertical lines indicated by values of Λ are theoretical vanishing thresholds from [7].

Figure 3.3 (cont.)
Betti numbers and critical points of $\mathbb{T}^4$ with number of points $n = 15000$ in 10 experiments.

From the simulation results in Figure 3.3 (cont.), we can see that the Betti numbers $\beta_k(r)$ for $1 \leq k \leq d-2$ are very small at $\Lambda = \log n + (k-1)\log\log n$, while $\beta_{d-1}(r)$ remains relatively large at $\Lambda = \log n + (d-2)\log\log n$ and only vanishes around $\Lambda = \log n + (d-1)\log\log n$ which is the theoretical threshold for the coverage. We may conjecture that

Conjecture 1. *Suppose that $w(n) \to \infty$ as $n \to \infty$. Then, for $1 \leq k \leq d-2$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{\beta_k(r) = \beta_k(\mathbb{T}^d)\right\} = \begin{cases} 0 & \Lambda = \log n + (k-1)\log\log n - w(n) \\ 1 & \Lambda = \log n + (k-1)\log\log n + w(n). \end{cases} \tag{3.78}$$

And for $\beta_{d-1}(r)$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\beta_{d-1}(r) = \beta_{d-1}(\mathbb{T}^d)\right\} = \begin{cases} 0 & \Lambda = \log n + (d-1)\log\log n - w(n) \\ 1 & \Lambda = \log n + (d-1)\log\log n + w(n). \end{cases} \tag{3.79}$$

CHAPTER 4

TOPOLOGICAL PERIODICITY

4.1 Introduction

Unlike the ordinary periodicity which requires that function values by a fixed offset are identical, topological periodicity allows distortion of the function's domain. We say that a function f is topologically periodic, if there exists some homeomorphism ϕ, as a re-parametrization of the domain, such that the composition $f \circ \phi$ is periodic in the common sense.

A traditional approach to the task is using Fourier transform. See [17]. However, Fourier transform does not work well when the domain is re-parametrized.

Here, we focus on H_0 of the persistent homology of f. Then, the topological periodicity should be preserved regardless of re-parametrization. In [16], the bottleneck distance between persistence diagrams that encodes persistent homology is defined. They show that

Theorem (Edelsbrunner & Harer)**.** *Let $\mathbb{X}$ be a topological space with two functions $f, g : \mathbb{X} \to \mathbb{R}$ satisfying some conditions. Then, for each dimension p, the bottleneck distance between the persistence homology is*

$$d_B(\mathrm{Dgm}_p(f), \mathrm{Dgm}_p(g)) \leq \|f - g\|_\infty = \sup_{x \in \mathbb{X}} |f(x) - g(x)| \,. \tag{4.1}$$

In this part, we are going to introduce a data structure called persistent dendrogram to represent one-dimensional persistent homology of a function on $\mathbb{R}$. The tree is ordered. The vertexes in such trees are corresponding to critical points of the function. We will also introduce a distance function defined between two trees. Finally, we can derive that the distance between persistent dendrograms is connected to the distance between functions considering a reparametrization of the domain, which resembles the result in [16].

4.2 Persistent Dendrogram for Functions and Distance of Trees

4.2.1 Ordered Trees with Edge Lengths

Ordered Trees

We use a pair of sets (V, E) to indicate a tree T, where V is the set of vertexes and E is the set of edges. The degree of a vertex v in the tree T is the number of other vertexes connected to v, denoted by $\deg_T(v)$.

Ordered trees are rooted trees where an order is assigned for the children of each connector (the vertex with degree more than 1). By iteration, an order can be assigned for every leaves (the vertex with degree 1) in an ordered tree such that the vertexes in the branch of a smaller child are always smaller than the vertexes in the larger child branch. Since the trees we talk about have finitely many vertexes, we can give integer labels to all the vertexes of the tree in accordance with the order.

A map $f : V_1 \to V_2$ is called a *homomorphism between ordered trees* $T_i = (V_i, E_i)$ for $i = 1, 2$ if

1. f maps the root of T_1 to the root of T_2;
2. for any vertex $u \in V_1$, if v is a child of u, then $f(u) = f(v)$ or $f(v)$is a child of $f(u)$.
3. f preserves the order of the subtrees and leaves.

In this case we write $f : T_1 \to T_2$ or simply $T_1 \to T_2$.

Ordered Trees with Edge Lengths

The *lengths of edges* in a tree are given by a function $l : E \to \mathbb{R}$ and $l \geq 0$. The *length of a path* is the sum of the lengths of all the edges on the path. Since the path joining any two vertexes in the tree is unique, the assignment of edge lengths gives a distance between two vertexes u, v by the length of the path, denoted by $d_l(u, v)$. The *depth* of a vertex v is defined to be the distance from the vertex to the root, denoted by $\text{depth}_l(v)$.

If a tree with edge lengths has neither edges with 0 length nor vertexes with degree 2 except the root, then the tree with edge lengths is called *reduced*.

Suppose we have two trees with edge lengths $(T_1; l_1), (T_2; l_2)$ and a surjective homomorphism $f : T_1 \to T_2$. We call that f is a *shrink* from $(T_1; l_1)$ to $(T_2; l_2)$ if $\text{depth}_{l_2}(f(v)) = \text{depth}_{l_1}(v)$ for any v in T_1. It is equivalent to say that $d_{l_2}(f(u), f(v)) = d_{l_1}(u, v)$ for any u, v in T_1 for a shrink f. On the other hand, if for any u and it parent v such that $f(u) = f(v)$, we have $\deg_{T_1}(u) = 2$ and the depth is preserved for vertexes other than such u, then f is called a *merge*.

We can always use a shrink to map a tree with edge lengths to a tree with no 0-length edges, and then use a merge to map a tree to one that has no vertexes with degree 2. By doing so, we will get a reduced tree as a result.

If a tree can be reduced to another, then given any depth $t \geq 0$ the subgraphs that consist of vertexes with the depth greater than t and the edges connecting them have the equal number of components.

If two trees with edge lengths $(T_1; l_1), (T_2; l_2)$ can be reduced to the same tree with the same length assignment, they are considered as equivalent. Write $(T_1; l_1) \sim (T_2; l_2)$.

Now given two trees T_1 and T_2 and endow T_1 with edge lengths given by l_1. Suppose there is an embedding, an injective homomorphism $f : T_1 \to T_2$. We can always define an edge lengths function l_2 from the embedding. We assign the edge joining $f(p)$ and $f(q)$ to the length of edge joining p, q and all the other edges to 0. The tree $(T_2; l_2)$ can be shrunk to $(T_1; l_1)$. $(T_1; l_1)$ and $(T_2; l_2)$ are thus equivalent.

4.2.2 Distance between trees

Distance between assignments of edge lengths

Fix a combinatorial tree structure $T = (V, E)$. Let l_1 and l_2 be two edge length functions. The *distance* between the two assignments is denoted by $\delta_T(l_1, l_2)$. It is defined by

$$\delta_T(l_1, l_2) := \left\| \mathrm{depth}_{l_1} - \mathrm{depth}_{l_2} \right\|_\infty \tag{4.2}$$

$$= \max_{e \in E} \left| \mathrm{depth}_{l_1}(e) - \mathrm{depth}_{l_2}(e) \right|. \tag{4.3}$$

Distance between Two Trees with Edge Lengths

For two different combinatorial trees T_1 and T_2, we can always find a third tree T which embeds T_1 and T_2. It follows from the arguments in the previous section that there exists $\tilde{l_1}$ and $\tilde{l_2}$ as two edge length functions on T such that $(T; \tilde{l_1}) \sim (T_1; l_1)$ and $(T; \tilde{l_2}) \sim (T_2; l_2)$.

The *distance* between $(T_1; l_1)$ and $(T_2; l_2)$ is defined as

$$d((T_1; l_1), (T_2; l_2)) := \inf_{\substack{(T;\tilde{l_1}) \sim (T_1; l_1) \\ (T;\tilde{l_2}) \sim (T_2; l_2)}} \delta_T(\tilde{l_1}, \tilde{l_2}).$$

4.2.3 Trees and Functions

Let f be a continuous real-valued function defined on a compact interval $[a, b]$ with finitely many local extreme points. Let $M = \max f$. Then there exists an ordered tree with edge lengths $(T; l)$, such that the following two objects have the same number of components:

1. the sublevel set $f^{-1}((-\infty, M-t]))$;

2. the subgraph of T which consists of vertexes with the depth greater than t and edges connecting them.

A reduced tree with edge lengths $(T; l)$ satisfying the above condition is called the persistent dendrogram corresponding to the function f.

In particular, the minimum points of f are leaves of the persistent dendrogram and the maximum points are the connectors that joining several branches with smaller depths.

If two continuous functions with finitely many local extreme points f_1, f_2 both correspond to the same tree and $\max f_1 = \max f_2$, then because of the arguments above, f_1 and f_2 have the same sequence of extreme points with the equal extreme values. It follows that we can define a homeomorphism ϕ of the domains such that $f_1 \circ \phi = f_2$.

For a function f with isolated extreme points, we can construct the persistent dendrogram $(T; l)$ by the following procedure:

1. Find all the local minimal and the interior local maximal points and values $(x, f(x))$. Since the function is continuous, the minimal points and maximal points appear alternatively.

   Those local minimum points are leaves of T. The local maximum points are candidates for connectors.

2. For each minimum, compare the nearby maximum values:

   (a) if they are not equal, connect the minimum to the less one.

   (b) if they are the same, merge them into a single connector. Connect the minimum to the connector.

   Assign the length of the edge to the difference of the minimum value and the connected maximum value.

3. For each maximum, from the least to the greatest, connect it to the nearby maximum that with less value. Assign the length of the edge to the difference of the connected maxima.

On the other hand, for any tree with edge lengths there is a continuous function corresponding to the tree. The following procedure gives a piecewise linear function corresponding to a tree with length $(T; l)$. It generates the set S of the points that are joined by the piecewise linear function.

1. Mark each leaf with a pair of numbers (i, d_i), where i is an integer according to the order of the tree and d_i is the depth of the leaf. Add points (i, d_i) to the set S.

2. For each connector with depth d, suppose it has $(n+1)$ children and thus it has $(n+1)$ subtrees. Let (i_k, d_{i_k}) be the leftmost leaf and (j_k, d_{j_k}) be the rightmost leaf in its kth subtree, for $0 \leq k \leq n$. Add $(\frac{j_{k-1}+i_k}{2}, d)$ to the set S for $1 \leq k \leq n$. (That is, insert a "maximum" between two minimum points.)

If the tree is reduced, the piecewise linear function generated by the procedure above has only finitely many extreme points at integer points.

4.2.4 Distance of Functions

Suppose that two continuous functions f, g are defined on a compact interval $[a, b]$ with finitely many local extreme values. Define the distance between them up to a homeomorphism of $[a, b]$ and a constant as

$$d(f,g) := \inf_{\substack{\phi \text{ is homeomorphism,} \\ C \in \mathbb{R}}} \|f - g \circ \phi + C\|_{\infty}.$$

For functions corresponding to the same combinatorial ordered tree, we can always use a homeomorphism to align the extreme points of both functions and the differences between the two values of the functions are just the differences of the depths. Therefore, we have that

Lemma 13. *Suppose T is a tree. If two continuous functions with finitely many local extreme points f_1, f_2 have persistent dendrograms $(T; l_1)$ and $(T; l_2)$, then*

$$d(f_1, f_2) \leq \delta_T(l_1, l_2). \tag{4.4}$$

Then we have that

Proposition 2. *If two continuous functions are defined on a compact interval has finitely many local extreme points, the distance between two functions is less than the distance of their corresponding persistent dendrograms.*

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