THE UNIVERSITY OF CHICAGO

TRACE METHODS AND SINGULAR SUPPORT

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To my family and friends

“Estar contigo o no estar contigo es la medida de mi tiempo.” – Jorge Luis Borges.

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ABSTRACT

In this thesis we explore the use of categorical methods in Algebraic Geometry. The notion of dualizable objects and traces in symmetric monoidal categories provide a good framework to study trace formulas for different sheaf theories. These techniques allow us to give a novel proof of the Lefschetz trace formula in Stable Motivic Homotopy theory. Moreover, following ideas of Beilinson and Lu, Zheng, we provide a general definition of Singular Support for any sheaf theory satisfying a six-functor formalism.

CHAPTER 1

INTRODUCTION

The process of categorification involves, broadly speaking, working at different levels of abstraction: the zeroth level is numerical in nature, the first level is given by objects (which live inside of a 1-category), the second level is given by categories (which live in the world of 2-categories), and so on. For example, we can recover a number from a vector space by looking at its dimension. In general, the theory of traces in symmetric monoidal categories allows us to move between these categorical levels. For a symmetric monoidal $(\infty, 1)$-category $(\mathcal{C}, \otimes, \mathbf{1})$, the set of endomorphisms $\Omega\mathcal{C}$ of the unit object $\mathbf{1} \in \mathcal{C}$ will allow us to go down one categorical level. Indeed, we obtain a functorial construction taking values in $\Omega\mathcal{C}$,

$$(X, f : X \to X) \mapsto \mathrm{ev} \circ (f \times \mathrm{id}) \circ \mathrm{coev} =: \mathrm{tr}(f)$$

where $X \in \mathcal{C}$ is a dualizable object with dual $X^\vee$ and

$$\mathrm{ev} : X^\vee \otimes X \to \mathbf{1}, \;\; \mathrm{coev} : \mathbf{1} \to X \otimes X^\vee$$

are the evaluation and coevaluation maps respectively. This construction can be extended to any (∞, n)-category, where traces will now take values in an $(\infty, n-1)$-category, allowing us effectively to move from the n-th categorical level to the $(n-1)$-th one (see [HSS17]). When applied to the world of 2-categories, we recover Hochschild homology as the trace of the identity endofunctor (i.e., as the Euler characteristic of the category). A fundamental property of traces is that they are localizing invariants.

Characteristic classes are prone to categorification. Categories of sheaves on a smooth and proper variety X over k contain a lot of cohomological information about X. The Hochschild-Kostant-Rosenberg theorem (assuming that k is a field of characteristic 0) provides us with this link. Indeed, the Euler characteristic of the category IndCoh(X) is

$$\chi(\mathrm{IndCoh}(X)) = R\Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}) \simeq \bigoplus_{r} \bigoplus_{q-p=r} H^p(X, \Omega_X^q)$$

where $\mathcal{L}X$ is the derived loop space of X. Moreover, we can think of a compact object $\mathcal{F} \in \mathrm{IndCoh}(X)$ (i.e., a perfect complex) as a continuous functor $\mathrm{Vect}_k \to \mathrm{IndCoh}(X)$ mapping k to $\mathcal{F}$. The functoriality of traces yields a map

$$k \to \bigoplus_{r} \bigoplus_{q-p=r} H^p(X, \Omega_X^q), \quad 1 \mapsto \mathrm{ch}(\mathcal{F})$$

sending 1 to the Chern character of $\mathcal{F}$. Notice that derived algebraic geometry comes very naturally into the scene, since the calculation of traces leaves the world of classical schemes, involving the derived loop space.

The goal of this work is to understand these categorical principles in the context of algebraic geometry, and apply them to different sheaf theories. In Chapter 2, we recall some necessary definitions on ∞-categories, symmetric monoidal categories and dualizable objects. In Chapter 3, we give a concrete definition of what we mean by a sheaf theory with a six-functor formalism in terms of the category of correspondences, and we explain how to calculate traces for such sheaf theories using the ideas from [BZN19]. We give a description of residues in local cohomology via a local version of the Hochschild-Kostant-Rosenberg theorem. We also explain how to prove Lefschetz theorems for sheaf theories satisfying six-functor formalism, and in particular we give an alternate proof of a quadratic refinement of the Grothendieck-Lefschetz formula in stable motivic homotopy theory (see [Hoy14]). Finally, in Chapter 4 we explain how to define a notion of singular support for any sheaf theory satisfying a six-functor formalism. Following ideas of Beilinson [Bei16], the singular support controls the directions in which a sheaf is not locally constant. To define this in a more general context, we need a notion of universal local acyclicity, which is obtained via duality in the category of cohomological correspondences following Lu and Zheng [LZ20]. We compare our notion of singular support with previously known notions for different sheaf theories, such as étale sheaves (see [Bei16]), Ind-coherent sheaves (see [AG15]) and sheaves of categories (see [Fio19]).

Our main contribution is the use of categorical methods for three different purposes. First, to identify the residue map via local Hochschild-Kostant-Rosenberg theorem (3.4.2). Secondly, to give a new proof of the Lefschetz Theorem in Stable Motivic Homotopy Theory (3.6.3). Finally, to prove the existence theorem (4.6.3) for the general definition of Singular Support for any sheaf theory with a six functor formalism, together with the comparison between our notion and previously defined notions for some sheaf theories.

CHAPTER 2

CATEGORICAL PRELIMINARIES

2.1 Reminder on ∞-categories

In this thesis, every time we mention ∞-categories, we mean $(\infty, 1)$-categories, and we will not make a particular choice of model. We will denote by $\mathcal{S}$ the ∞-category of ∞-groupoids (or *spaces*) and we will take for granted that the category of ∞-categories is enriched in $\mathcal{S}$.

Definition 2.1.1. We say that an ∞-category is *stable* if it has finite limits and colimits, and pushout and pullback squares coincide. We denote by $\mathrm{Cat}^{\mathrm{ex}}$ the (pointed) ∞-category of small stable ∞-categories and exact functors (that is, functors that preserve finite limits and colimits).

Although the categories we will be dealing with will be large, they will be determined by some small categories. Roughly speaking, *presentable ∞-categories* are large ∞-categories that are generated under sufficiently large filtered colimits by some small ∞-category. To make this discussion precise, we need to introduce the Ind-category.

Given any small ∞-category $\mathcal{C}$, we can form the the ∞-category $\mathrm{Pre}(\mathcal{C})$ of presheaves of simplicial sets on $\mathcal{C}$. We can think of the presheaf category as the formal closure of $\mathcal{C}$ under colimits via the Yoneda embedding $\mathcal{C} \to \mathrm{Pre}(\mathcal{C})$. For any regular cardinal κ, we can define the Ind-category $\mathrm{Ind}_\kappa(\mathcal{C})$ as the formal closure under κ-filtered colimits of $\mathcal{C}$, and as such it will be a subcategory of $\mathrm{Pre}(\mathcal{C})$. This category is characterized by the property that it has κ-small filtered colimits, admits a functor $\mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$, and this functor induces an equivalence

$$\mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for any $\mathcal{D}$ which admits κ-filtered colimits and $\mathrm{Fun}_\kappa(\cdot, \cdot)$ denotes the ∞-category of functors that preserve κ-small filtered colimits.

Definition 2.1.2. We say that an ∞-category $\mathcal{C}$ is *presentable* if it arises as $\mathrm{Ind}_\kappa(\mathcal{D})$ where $\mathcal{D}$ is a small ∞-category that admits κ-small colimits. We denote the ∞-category of presentable ∞-categories and colimit-preserving functors by $\mathcal{P}r^{\mathrm{L}}$.

Note that the adjoint functor theorem implies that a functor between presentable ∞-categories is a left adjoint if and only if it preserves colimits.

We will restrict ourselves to the case κ is the first infinite ordinal ω. Recall that an object c of an ∞-category $\mathcal{C}$ is *compact* if the functor

$$\mathcal{C}^{\mathrm{op}} \to \mathcal{S}, \quad x \mapsto \mathrm{Hom}_{\mathcal{C}}(c, x)$$

preserves filtered colimits. We will denote by $\mathcal{C}^\omega$ the full subcategory of $\mathcal{C}$ consisting of compact objects. We say that a presentable ∞-category $\mathcal{C}$ is *compactly generated* if the natural functor

$$\mathrm{Ind}(\mathcal{C}^\omega) \to \mathcal{C}$$

sending a filtered diagram in $\mathcal{C}^\omega$ to its colimit in $\mathcal{C}$, is an equivalence.

Notice that many examples of a geometric nature are compactly generated, such as the category of quasi-coherent sheaves or D-modules on sufficiently nice stacks. However many other important objects, also of geometric nature, are not compactly generated but merely dualizable. We will discuss this notion at length in the following section.

We say that a category $\mathcal{C}$ is *idempotent complete* if the image of $\mathcal{C}$ under the Yoneda embedding is closed under retracts. Let us denote $\mathrm{Cat}^{\mathrm{perf}}$ the ∞-category of small, stable, and idempotent complete ∞-categories and exact functors, and by $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ the ∞-category of stable, presentable ∞-categories and left adjoint functors.

The ind-completion functor $\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ induces an equivalence between $\mathrm{Cat}^{\mathrm{perf}}$ and the subcategory of $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ whose objects are the compactly generated stable ∞-categories and whose morphisms are the left adjoint functors preserving compact objects. The idempotent completion functor Idem : $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Cat}^{\mathrm{perf}}$ arises as the left adjoint to the inclusion of categories. One can also prove that $\mathrm{Idem}(\mathcal{C}) \simeq \mathrm{Ind}(\mathcal{C})^{\omega}$, since compact objects of the ind-completion are given by retracts of objects in $\mathcal{C}$.

For any ∞-category $\mathcal{C}$ with finite limits, we can form the stabilization $\mathrm{Stab}(\mathcal{C})$, together with a pair of adjoint functors called *looping* and *suspension* functors

$$\Omega^{\infty} : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}, \;\; \Sigma^{\infty}_{+} : \mathcal{C} \to \mathrm{Stab}(\mathcal{C}).$$

We can describe the stabilization explicitly via spectra. A *spectrum object* of a pointed ∞-category $\mathcal{C}$ consists of a functor $N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$, or in other words, a family of objects $A_{i,j}$ together with maps $A_{i,j} \to A_{i+1,j}$ and $A_{i,j} \to A_{i,j+1}$ such that $A_{i,j}$ is the zero object whenever $i \neq j$ and the square

$$\begin{array}{ccc} A_{i,i} & \longrightarrow & A_{i,i+1} \\ \downarrow & & \downarrow \\ A_{i+1,i} & \longrightarrow & A_{i+1,i+1} \end{array}$$

is Cartesian for all i.

2.2 Traces in Symmetric Monoidal Categories

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. We say that an object $X \in \mathcal{C}$ is *dualizable* if there is an object $X^{\vee} \in \mathcal{C}$ together with maps

$$\mathrm{coev} : \mathbf{1} \to X \otimes X^{\vee}, \;\; \mathrm{ev} : X^{\vee} \otimes X \to \mathbf{1}$$

such that the compositions

$$X \xrightarrow{\sim} \mathbf{1} \otimes X \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} X \otimes X^{\vee} \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} X \otimes \mathbf{1} \xrightarrow{\sim} X$$

$$X^{\vee} \xrightarrow{\sim} X^{\vee} \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} X^{\vee} \otimes X \otimes X^{\vee} \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \mathbf{1} \otimes X^{\vee} \xrightarrow{\sim} X^{\vee}$$

are both identity morphisms.

Recall that an *internal hom* in $\mathcal{C}$ is a functor

$$\underline{\mathrm{Hom}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that for every object $X \in \mathcal{C}$ we have a pair of adjoint functors

$$(\cdot) \otimes X : \mathcal{C} \leftrightarrows \mathcal{C} : \underline{\mathrm{Hom}}(X, \cdot).$$

Whenever the internal hom exists, we can describe the evaluation map

$$\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$$

as the map that corresponds to id $\in \mathrm{Hom}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$ under the adjunction. Moreover, we can give the following well-known description of dualizable objects.

Lemma 2.2.1. *An object X is dualizable if and only if the internal hom objects $\underline{\mathrm{Hom}}(X, \mathbf{1})$ and $\underline{\mathrm{Hom}}(X, X)$ exist and the morphism $X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \to \underline{\mathrm{Hom}}(X, X)$ adjoint to*

$$X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes X \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} X$$

is a split epimorphism.

Proof. Suppose that X is dualizable and $X^\vee$ is its dual. The coevaluation and evaluation maps give an adjunction between the functors $(\cdot) \otimes X^\vee$ and $(\cdot) \otimes X$. This means that, for any object $Y \in \mathcal{C}$, the internal hom $\underline{\mathrm{Hom}}(X, Y)$ exists and is equal to $Y \otimes X^\vee$. In particular, when $Y = \mathbf{1}$ is the unit object we obtain the desired conclusion.

For the reverse implication, we can exhibit $\underline{\mathrm{Hom}}(X, \mathbf{1})$ as the dual of X by defining the coevaluation map $\mathrm{coev}_X : \mathbf{1} \to X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1})$ as the composition of a section of

$X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \to \underline{\mathrm{Hom}}(X, X)$ and the map $\mathbf{1} \to \underline{\mathrm{Hom}}(X, X)$ corresponding to the identity morphism. □

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $X \in \mathcal{C}$ a dualizable object. The *trace* $\mathrm{tr}_{\mathcal{C}}(f)$ of a morphism $f : X \to X$ is defined as the composition

$$\mathbf{1} \longrightarrow X \otimes X^{\vee} \xrightarrow{f \otimes \mathrm{id}_{X^{\vee}}} X \otimes X^{\vee} \xrightarrow{\sim} X^{\vee} \otimes X \longrightarrow \mathbf{1}.$$

One can verify that whenever we have morphisms $f : X \to Y$ and $g : Y \to X$, the trace satisfies the *cyclic property*:

$$\mathrm{tr}_{\mathcal{C}}(fg) = \mathrm{tr}_{\mathcal{C}}(gf).$$

Whenever we are dealing with a symmetric monoidal $(\infty, 2)$-category $\mathcal{C}$, we have very nice functorial properties. A morphism between endomorphisms $f : X \to X$, $g : Y \to Y$ consists of a morphism $\varphi : X \to Y$ together with natural transformation $\alpha : f \to g$

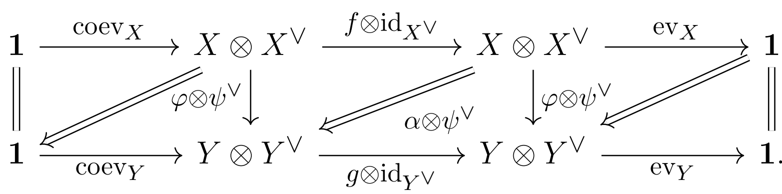
$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[g]{} & Y. \end{array}$$

We can consider an $(\infty, 1)$-category of arrows $\mathrm{Arr}(\mathcal{C})$ in which objects are arrows $f : X \to Y$ and morphisms between arrows are natural transformations. It is straightforward to see that an object in the arrow category $\varphi : X \to Y \in \mathrm{Arr}(\mathcal{C})$ is *dualizable* if and only if X and Y are dualizable and φ admits a right adjoint ψ. More explicitly, the dual is given by $\psi^{\vee} : X^{\vee} \to Y^{\vee}$ the dual of the right adjoint of φ.

Let $\varphi : X \to Y$ be dualizable. Then, a triple (f, g, α) as before gives us an endomorphism of φ in the arrow category $\mathrm{Arr}(\mathcal{C})$. More importantly, the trace in the arrow category $\mathrm{tr}_{\mathrm{Arr}(\mathcal{C})}(F, G, \alpha)$ gives us a map

$$\mathrm{tr}(\varphi, \alpha) : \mathrm{tr}_{\mathcal{C}}(F) \to \mathrm{tr}_{\mathcal{C}}(G)$$

which is functorial, and can be described by the following diagram



Example 2.2.2. When $\mathcal{C} = \mathrm{Vect}_k$ the category of vector spaces over k, V is dualizable if and only if it has finite-dimensional cohomology spaces nonzero in finitely many degrees. For $f : V \to V$, we see that

$$\mathrm{tr}_{\mathrm{Vect_k}}(f) = \sum_i (-1)^i \mathrm{tr}(f_* : H^i(V) \to H^i(V)).$$

Indeed, it suffices to verify this for complexes concentrated in degree 0. Pick a basis $v_1, \ldots, v_n$ of V and $v_1^*, \ldots, v_n^*$ dual basis of V^*. The coevaluation map is then given by

$$\mathrm{coev} : k \to V \otimes_k V^*, \;\; 1 \mapsto \sum_i v_i \otimes v_i^*,$$

and so the composition

$$k \xrightarrow{\qquad} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}} V \otimes V^* \xrightarrow{\sim} V^* \otimes V \xrightarrow{\qquad} k$$

is given by

$$1 \mapsto \sum_i v_i \otimes v_i^* \to \sum_i f(v_i) \otimes v_i^* \mapsto \sum_i v_i^*(f(v_i)) = \mathrm{tr}(f).$$

Example 2.2.3. When $\mathcal{C} = \mathrm{Sp}$ is the category of *spectra*, the sphere spectrum $\mathbb{S}$ is the unit object with respect to smash product. Given X compact CW complex, the suspension $\Sigma^\infty_+ X$ is a dualizable object whose dual is the Spanier-Whitehead dual DX.

$$\mathbb{S} \xrightarrow{\qquad} \Sigma^\infty_+ X \wedge DX \xrightarrow{f \wedge \mathrm{id}} \Sigma^\infty_+ X \wedge DX \xrightarrow{\sim} DX \wedge \Sigma^\infty_+ X \xrightarrow{\qquad} \mathbb{S}$$

gives the Lefschetz number of f in $\mathrm{End}_{\mathrm{Sp}}(\mathbb{S}) = \pi_0 \mathbb{S} = \mathbb{Z}$.

Even considering the identity map of an object yields interesting information. In the setting of vector spaces we obtain the dimension (or more generally, the Euler characteristic for a complex) and in the case of spectra, the identity on $\Sigma^\infty_+ X$ gives us the Euler characteristic of X. In the following section we will see that (topological) Hochschild homology also arises in this way.

2.3 The Lurie tensor product

Recall that $\mathcal{P}r^{\mathrm{L}}$ admits a symmetric monoidal structure, which is characterized by the following *universal property.*

Proposition 2.3.1 ([Lur17, Section 4.8]). *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be presentable ∞-categories, then any functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ preserving small colimits separately in each variable factors uniquely through $\mathcal{C} \otimes \mathcal{D}$. Moreover, we have a canonical equivalence*

$$\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

where $\mathrm{RFun}(\cdot, \cdot)$ *are functors that admit a left adjoint. The unit object of $\mathcal{P}r^{\mathrm{L}}$ is given by the category of spaces $\mathcal{S}$.*

We will say that a stable presentable ∞-category is *dualizable* if it is a dualizable with respect to such monoidal structure on $\mathcal{P}r^{\mathrm{L}}$. One of the advantages of dealing with compactly generated categories is that they are dualizable for the symmetric monoidal structure of $\mathcal{P}r^{\mathrm{L}}$, and the dual can be found explicitly.

Proposition 2.3.2. *If $\mathcal{C}$ is compactly generated, then it is dualizable as an object of $\mathcal{P}r^{\mathrm{L}}$. In particular, if $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^{\omega})$, then $\mathcal{C}^{\vee} = \mathrm{Ind}(\mathcal{C}^{\omega,\mathrm{op}})$, and the evaluation map is given by ind-completion via universal properties of the Yoneda pairing $\mathrm{Map}_{\mathcal{C}}(\cdot,\cdot) : \mathcal{C}^{\omega,\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. Furthermore, we have isomorphisms*

$$\mathrm{LFun}(\mathcal{C}, \mathcal{C}) \simeq \mathrm{LFun}(\mathcal{C}, \mathcal{S}) \otimes \mathcal{C} \simeq \mathcal{C}^{\vee} \otimes \mathcal{C}.$$

Many times, it will be convenient to study objects which are linear over an E_∞-ring R. An *R-linear ∞-category* is a presentable ∞-category $\mathcal{C}$ which is tensored over the monoidal ∞-category LMod_k of left k-vector spaces, such that $\otimes : \mathrm{LMod}_k \otimes \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. The category LMod_R can be identified with an associative algebra object of $\mathcal{P}r^{\mathrm{L}}$, and we can think of $\mathrm{Cat}_R = \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}r^{\mathrm{L}})$ as the ∞-category of R-linear ∞-categories. As in Proposition 2.3.2, we can see that compactly generated R-linear ∞-categories are dualizable as objects of Cat_R.

When k is a field, we can describe a down-to-earth way to compute of traces in Cat_k. We notice that for $A, B \in \mathrm{CAlg}_k$ we have

$$A\text{-mod} \otimes B\text{-mod} = (A \otimes B)\text{-mod}.$$

The unit object of this category is Vect_k. A continuous functor $F : \mathrm{Vect}_k \to \mathrm{Vect}_k$ is determined by the image $F(k)$. From this, it is clear that we have an identification

$$\mathrm{End}_{\mathrm{Cat_k}}(\mathrm{Vect}_k) \simeq \mathrm{Vect}_k, \quad F \mapsto F(k).$$

In a similar fashion, a functor $F : A\text{-mod} \to B\text{-mod}$ is determined by $F(A)$. Indeed, this follows since A-mod is the free cocompletion of $\mathrm{B}A^{\mathrm{op}}$, which tells us that $F(A)$ is an (A, B)-bimodule. Therefore, any such functor F looks like

$$F(X) = M \otimes_A X, \quad \text{for } M \in (A, B)\text{-bimod}.$$

For such objects $A\text{-mod} \in \mathrm{Cat}_k$ we can verify that the dual is given by A^{op}-mod, where the evaluation and coevaluation maps are given as follows:

- The coevaluation map is defined as

$$\mathrm{Vect}_k \to A\text{-mod} \otimes A^{\mathrm{op}}\text{-mod} = (A \otimes A^{\mathrm{op}})\text{-mod}$$

  given by $k \mapsto A$, that is, $V \mapsto V \otimes_k A$.

- The evaluation map is defined as

$$(A \otimes A^{\mathrm{op}})\text{-mod} \to \mathrm{Vect}_k$$

  given by the bimodule A, that is,

$$M \mapsto M \otimes_{A \otimes A^{\mathrm{op}}} A$$

Putting these things together shows that the trace of the endofunctor F given by the (A, B)-bimodule M on A-mod in Cat_k is the composition

$$k \mapsto M \mapsto M \otimes_{A \otimes A^{\mathrm{op}}} A$$

which is the usual Hochschild homology $\mathrm{HH}_\bullet(A, M)$ of A with coefficients in M. This motivates the following definition.

Definition 2.3.3. Given a k-linear presentable ∞-category $\mathcal{C}$ we define its *Hochschild homology* as

$$\mathrm{HH}(\mathcal{C}) = \mathrm{tr}_{\mathrm{Cat_k}}(\mathrm{id} : \mathcal{C} \to \mathcal{C}).$$

If we work with an arbitrary presentable ∞-category $\mathcal{C}$, its *Topological Hochschild homology* is

$$\mathrm{THH}(\mathcal{C}) = \mathrm{tr}_{\mathcal{P}r^{\mathrm{L}}}(\mathrm{id} : \mathcal{C} \to \mathcal{C}).$$

When $\mathcal{C}$ is a small dg-category, we have the *cyclic bar complex*, that is, the simplicial set $C^\bullet(\mathcal{C})$ whose n-simplices are given by

$$C^n(\mathcal{C}) = \bigsqcup_{X_0,\ldots,X_n \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X_0, X_n) \otimes \mathrm{Hom}_{\mathcal{C}}(X_n, X_{n-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(X_1, X_0)$$

and the face maps are given by composition and the degeneracy maps by the identity homomorphism. Notice that our definition as a categorical trace recovers the cyclic bar complex.

Moreover if $\mathcal{C}$ is a k-linear presentable monoidal ∞-category, the Hochschild homology acquires extra structure. Indeed, the tensor product functor

$$\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$$

is continuous in each factor and preserves compact objects. This yields a lax monoidal structure

$$\mathrm{HH}(\mathcal{C}) \otimes \mathrm{HH}(\mathcal{C}) \to \mathrm{HH}(\mathcal{C})$$

which gives an algebra structure on $\mathrm{HH}(\mathcal{C})$.

2.4 Localizing invariants and K-theory

Suppose that $\mathcal{B}$ is a presentable stable ∞-category, and $\mathcal{C} \subset \mathcal{B}$ is a full presentable subcategory, closed under direct sums. If the inclusion functor admits a left adjoint $\ell : \mathcal{B} \to \mathcal{C}$, called a *localization functor*, we can consider the full subcategory $\mathcal{A}$ of $\mathcal{C}$ consisting of objects $X \in \mathcal{C}$ for which $\ell(X) \simeq 0$. Because of the adjunction, we can think of $\mathcal{A}$ as the *left orthogonal* of $\mathcal{C}$, that is, the full subcategory consisting of objects $X \in \mathcal{C}$ such that if $\mathrm{Hom}_{\mathcal{B}}(X, C) = 0$ for any $C \in \mathcal{C}$ implies that $X \simeq 0$. Notice that the inclusion $\mathcal{A} \subset \mathcal{B}$ is continuous, and hence admits a right adjoint functor $c : \mathcal{B} \to \mathcal{A}$ which is moreover continuous. In such a situation, we say that

$$\mathcal{A} \hookrightarrow \mathcal{B} \xrightarrow{\ell} \mathcal{C}$$

is a *short exact sequence* of categories, or a *localization sequence*. In the case when we are dealing with small, stable, idempotent complete ∞-categories, we say that a sequence is exact if the ind-completion is exact.

Moreover, we say that a short exact sequence is *split* if we have right adjoint splittings. In other words, if we can realize $\mathcal{A}$ as a subcategory of objects $X \in \mathcal{B}$ for which $r(X) \simeq 0$ where $r : \mathcal{C} \to \mathcal{B}$ is the right adjoint of the localization functor $\ell : \mathcal{B} \to \mathcal{C}$. Similarly, whenever we deal with small, stable, idempotent complete ∞-categories, the notion of a split exact sequence is obtained by taking the ind-completion.

Definition 2.4.1. We say that a functor $E : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ or $E : \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}} \to \mathrm{Sp}$ is an *additive invariant* if it sends split exact sequences to cofiber sequences. We say that E is a *localizing invariant* if it sends exact sequences to cofiber sequences.

Remark 2.4.2. In the previous definition, we can add R-linear structure to the categories under consideration, and changing the target of E to be $\mathrm{Mod}(R)$, to obtain similar notions.

The main theorem of [BGT13] tells us that we can think of K-theory as a functor

$$K : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$$

which is the *universal additive invariant*.

Consider the symmetric monoidal $(\infty, 2)$-category $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$. Consider $X, Y \in \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ together with endofunctors $f : X \to X$, $g : Y \to Y$. We say that $(\varphi, \alpha) : (X, f) \to (Y, g)$ is *right adjointable* if φ has a right adjoint ψ and the associated push-pull transformation $\alpha^\flat : f \circ \psi \to \psi \circ g$ is an equivalence. We care about such maps since they induce morphisms on traces, as seen in Section 2.2.

For each dualizable object $X \in \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ we can look at the restricted trace functor

$$\mathrm{tr} : \mathrm{End}_{\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}}(X) \to \mathrm{End}_{\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}}(\mathrm{Sp}) = \mathrm{Sp}$$

preserves cofiber sequences. Indeed, we can write it as the composite

$$\mathrm{End}_{\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}}(X) \xrightarrow{(\cdot)\otimes \mathrm{id}_{X^\vee}} \mathrm{End}_{\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}}(X \otimes X^\vee) \xrightarrow{(\cdot)\circ \mathrm{coev}} \mathrm{LFun}(\mathrm{Sp}, X \otimes X^\vee) \xrightarrow{\mathrm{ev}\circ(\cdot)} \mathrm{Sp}$$

where each functor in that composition is exact. In particular, this means that the restricted trace preserves cofiber sequences. Using this fact, and the functoriality of traces we can prove the following.

Theorem 2.4.3 ([HSS17, Theorem 3.4])**.** *Consider a localization sequence*

$$(X, f) \overset{(\iota,\alpha)}{\longrightarrow} (Y, g) \overset{(\pi,\beta)}{\longrightarrow} (Z, h).$$

Then

$$\mathrm{tr}(f) \to \mathrm{tr}(g) \to \mathrm{tr}(h)$$

is a cofiber sequence in Sp.

Proof. The idea is to construct a cofiber sequence

$$\mathrm{tr}(f') \to \mathrm{tr}(g) \to \mathrm{tr}(h')$$

that is homotopy equivalent to the sequence

$$\mathrm{tr}(f) \to \mathrm{tr}(g) \to \mathrm{tr}(h)$$

and the functors f', h' are endofunctors of Y. To do this, consider $f' = \iota\iota^r g$ and $h' = \pi^r \pi g$

and consider the natural transformations

$$\alpha' : f' = \iota\iota^r g \xrightarrow{\epsilon} g, \quad \beta' : g \xrightarrow{\eta} \pi^r \pi g = h',$$

$$\overline{\alpha} : \iota f \xrightarrow{\eta} \iota f \iota^r \iota \xrightarrow{\alpha^\flat} \iota\iota^r g\iota = f'\iota, \quad \overline{\beta} : \pi h' = \pi\pi^r \pi g \xrightarrow{\epsilon} \pi g \xrightarrow{\beta} h\pi.$$

These maps fit together into the following diagram

$$\begin{array}{ccccc} (X, f) & \xrightarrow{(\iota,\alpha)} & (Y, g) & \xrightarrow{(\pi,\beta)} & (Z, h) \\ {\scriptstyle(\iota,\overline{\alpha})}\downarrow & & \| & & \uparrow{\scriptstyle(\pi,\overline{\beta})} \\ (Y, f') & \xrightarrow[(\mathrm{id},\alpha')]{} & (Y, g) & \xrightarrow[(\mathrm{id},\beta')]{} & (Y, h') \end{array}$$

Moreover, since $X \xrightarrow{\iota} Y \xrightarrow{\pi} Z$ is a localization sequence, we know that

$$\iota\iota^r \to \mathrm{id}_X \to \pi^r \pi$$

is a cofiber sequence, from which

$$f' \to g \to h'$$

is itself a cofiber sequence. Therefore, since the relative trace preserves cofiber sequences we see that

$$\mathrm{tr}(f') \to \mathrm{tr}(g) \to \mathrm{tr}(h')$$

is a cofiber sequence. Finally, we only need to verify that the maps induced on traces by $(\iota, \overline{\alpha})$ and $(\pi, \overline{\beta})$ are equivalences. For the first one, since (ι, α) is right adjointable, $\overline{\alpha}$ is an equivalence and $\eta : \mathrm{id}_X \to \iota^r \iota$ is also an equivalence because it is part of a localizing sequence. For the second one, $\epsilon : \pi\pi^r \to \mathrm{id}_Z$ is an equivalence because it is part of a localizing sequence, and $\overline{\beta}^\flat$ is an equivalence because (π, β) is right-adjointable, so it boils down to proving that the 2-morphism

$$\mathrm{tr}(h') = \mathrm{ev}_Y(h' \otimes \mathrm{id})\,\mathrm{coev}_Y \xrightarrow{\eta} \mathrm{ev}_Y(h' \otimes \mathrm{id})(\pi^r \pi \otimes \mathrm{id})\mathrm{coev}_Y = \mathrm{tr}(h'\pi^r \pi)$$

is an equivalence, which is true because $\eta h'$ is an equivalence and the cyclic property of traces. □

A very important corollary, which follows by applying the previous theorem in the case of the identity endofunctors, is the following.

Corollary 2.4.4. *Given a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ of stable categories with Verdier quotient $\mathcal{B}/\mathcal{A}$, there is a cofiber sequence of spectra*

$$\mathrm{THH}(\mathcal{A}) \to \mathrm{THH}(\mathcal{B}) \to \mathrm{THH}(\mathcal{B}/\mathcal{A}).$$

Moreover, the same results hold true, with the same proofs, if instead of working over $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ we work over Cat_k for some field k, replacing THH by HH in the last corollary.

2.5 Dualizable categories

We saw that compactly generated categories are dualizable for the symmetric monoidal structure on $\mathcal{P}r^{\mathrm{L}}$. However, we can give a complete characterization of dualizable categories, following Lurie [Lur18, D.7.0.7].

Proposition 2.5.1. *Let R be an E_∞-ring and $\mathcal{C}$ a stable R-linear ∞-category. Then $\mathcal{C}$ is dualizable if and only if it is a retract of a compactly generated category.*

Another equivalent condition for $\mathcal{C}$ to be dualizable is that the colimit functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{y} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Since the colimit functor is left adjoint to the Yoneda embedding $y : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$, this exhibits $\mathcal{C}$ as a retract of a compactly generated category. In the case that $\mathcal{C}$ is compactly generated, $\hat{y}$ becomes the Ind-extension of the inclusion $\mathcal{C}^\omega \subset \mathcal{C}$.

We could also introduce the notion of a small, stable, idempotent complete ∞-category being dualizable, since $\mathrm{Cat}^{\mathrm{perf}}$ is a symmetric monoidal category as well. Notice that for *any* $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, the ind-completion is a compactly generated presentable category and so

$\mathrm{Ind}(\mathcal{C})$ is dualizable in the above sense. Dualizability in $\mathrm{Cat}^{\mathrm{perf}}$ is related to two *geometric* notions. We say that $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ is *proper* if the evaluation functor

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \mathrm{Ind}(\mathcal{C}) \to \mathrm{Sp}$$

preserves compact objects. On the other hand, we say that $\mathcal{C}$ is *smooth* if the coevaluation functor

$$\mathrm{Sp} \to \mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$$

preserves compact objects.

Proposition 2.5.2 ([Toe09])**.** *A small, stable, idempotent complete ∞-category $\mathcal{C}$ is dualizable if and only if it is smooth and proper.*

Suppose that we write $\mathcal{C}$ as a retract of a compactly generated category $\mathcal{A}$, and consider the quotient $\mathcal{B} = \mathcal{A}/\mathcal{C}$. Then $\mathcal{B}$ is also compactly generated and we have a localization sequence

$$\mathcal{C} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}.$$

Following Efimov, we define the continuous K-theory of $\mathcal{C}$ as

$$K^{\mathrm{cont}}(\mathcal{C}) := \mathrm{hofib}(K(\mathcal{A}^{\omega}) \to K(\mathcal{B}^{\omega})).$$

This is independent of the category $\mathcal{A}$ we choose to write $\mathcal{C}$ as a retract of, as a consequence of the excision theorem in [Tam18, Theorem 18]. Notice that when $\mathcal{C}$ is compactly generated itself, we have that $K^{\mathrm{cont}}(\mathcal{C}) \simeq K(\mathcal{C}^{\omega})$. However, it is not evident how to construct the class of a non-compact object $X \in \mathcal{C}$ in continuous K-theory, and interesting examples have no compact objects. We do not know if there is a class (resembling Wall's finiteness obstruction) in K-theory whose vanishing determines if a dualizable category is compactly generated.

2.6 Determinant functors and the trace formalism

A *Picard groupoid* is a symmetric monoidal category in which every object is invertible (together with certain commutativity and associativity constraints).

There is an equivalence of categories between the category of Picard groupoids and the category of $[0,1]$-connected spectra. Let us briefly recall how the equivalence is constructed. If $\mathrm{Mon}_{\mathbb{E}_\infty}$ is the category of $\mathbb{E}_\infty$-monoids and $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}$ is the full subcategory of group-like $\mathbb{E}_\infty$-monoids (that is, $X \in \mathrm{Mon}_{\mathbb{E}_\infty}$ such that $\pi_0(X)$ is a group). The category of group-like $\mathbb{E}_\infty$-monoids is equivalent to the category of connective spectra. The equivalence is given by the *infinite loop-space machine*, that is, the functor $\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty} \to \mathrm{Sp}^{\geq 0}$ defined by taking X to the connective spectrum $X, BX, B^2X, \ldots$. One can see that a symmetric monoidal groupoid $\mathcal{C}$ is a Picard groupoid if and only if the $\mathbb{E}_\infty$-monoid given by the nerve $N(\mathcal{C})$ is group-like. Notice that in such situation, $N(\mathcal{C})$ is 1-truncated, and therefore this establishes the equivalence between Picard groupoids and 1-truncated connective spectra.

In particular, if $\mathcal{C}$ is a small, stable, idempotent complete ∞-category, we can associate to it a Picard groupoid $\mathrm{Pic}(\mathcal{C})$ by taking $\tau_{\leq 1}K(\mathcal{C})$, or if $\mathcal{C}$ is a dualizable category, we can instead take $\mathrm{Pic}(\mathcal{C}) = \tau_{\leq 1}K^{\mathrm{cont}}(\mathcal{C})$. We would be interested in having a notion of a *universal determinant functor* $\det : \mathcal{C} \to \mathrm{Pic}(\mathcal{C})$ as in [Del87].

Example 2.6.1. Consider the Picard groupoid $\mathrm{Pic}^{\mathbb{Z}}(X)$ whose objects are graded lines, that is, $(\mathcal{L}, \alpha)$ for $\mathcal{L}$ a line bundle on X and $\alpha : X \to \mathbb{Z}$ a continuous function. The set of morphisms from $(\mathcal{L}, \alpha)$ to $(\mathcal{L}', \alpha')$ is defined to be the set of isomorphisms $\mathcal{L} \to \mathcal{L}'$ if $\alpha = \alpha'$ and the empty set otherwise. Given any vector bundle V on X, we can define an object $\det(V) \in \mathrm{Pic}^{\mathbb{Z}}(X)$.

When $X = \mathrm{Spec}(k)$ and V is a finite dimensional vector space over k, an automorphism $f : V \to V$ yields a map $\det(f) : (\det(V), \dim(V)) \to (\det(V), \dim(V))$ in $\mathrm{Pic}^{\mathbb{Z}}(\mathrm{Spec}(k))$, whose categorical trace is given by the usual determinant of f.

One could possibly play the same game as in the previous example, provided we have a determinant functor $\mathcal{C} \to \mathrm{Pic}(\mathcal{C})$. Indeed, to get a notion of the determinant of an automorphism $f : V \to V$ in $\mathcal{C}$ we can apply the universal determinant $\det(f) : \det(V) \to \det(V)$ to obtain an endomorphism of $\det(V) \in \mathrm{Pic}(\mathcal{C})$, and we can look at $\mathrm{tr}_{\mathrm{Pic}(\mathcal{C})}(\det(V))$ which lives in $K_1(\mathcal{C})$.

CHAPTER 3

TRACE METHODS IN ALGEBRAIC GEOMETRY

3.1 Categories of correspondences

In this section we will deal with the $(\infty, 2)$-category of correspondences, which will allow us to encode the six functor formalism in various theories in a conceptual manner. This idea, due to Lurie, is explained by Gaitsgory and Rozenblyum in [GR17], and we follow their exposition closely.

Let us first give a brief summary of what the six functor formalism usually entails. Suppose that $\mathcal{C}$ is a category of geometric objects, and for each geometric object X we can associate a stable, presentable $(\infty, 1)$-category of sheaves

$$X \mapsto \mathrm{Sh}(X) \in \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}.$$

For instance, when $\mathcal{C} = \mathrm{Sch}_{\mathrm{aft}}$ is the $(\infty, 1)$-category of schemes of almost finite type, we can think of ind-coherent sheaves $\mathrm{IndCoh}(X)$ or $\mathcal{D}$-modules $\mathcal{D}\text{-mod}(X)$. This assignment comes with the following additional data:

- **Functoriality** For every map $f : X \to Y \in \mathcal{C}$, there are functors

$$f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X), \quad f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y).$$

- **Proper adjunction** Given $f : X \to Y \in \mathcal{C}$, there is a natural transformation

$$\mathrm{id} \to f^! \circ f_*$$

which is the unit of an adjunction when f is proper.

- **Open adjunction** If $f : X \to Y \in \mathcal{C}$ is an open immersion, there is a natural isomorphism

$$f^! \circ f_* \simeq \mathrm{id},$$

which is the counit of an adjunction.

- **Proper base change** For a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

in $\mathcal{C}$, there is a natural base change isomorphism

$$f'_* \circ g'^! \simeq g^! \circ f_*.$$

In the case that f is proper (resp. open), this isomorphism is given by the natural transformation arising from the proper (resp. open) adjunction above.

- **Tensor structure** For each $X, Y \in \mathcal{C}$ we have a functor

$$\boxtimes : \mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y),$$

natural in X and Y. Moreover, we have a tensor structure on $\mathrm{Sh}(X)$ which is given by $\mathcal{F} \overset{!}{\otimes} \mathcal{G} = \Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ where Δ is the diagonal embedding and requiring that $f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is symmetric monoidal with respect to this tensor structure.

- **Projection formula** Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Since $f^!$ is a tensor functor with respect to the $\overset{!}{\otimes}$-product, we have that $\mathrm{Sh}(X)$ is a module category over the tensor category $\mathrm{Sh}(Y)$. We require that the functor f_* is a functor of module categories over $\mathrm{Sh}(Y)$ with respect to the tensor structure $\overset{!}{\otimes}$. In particular, we have the projection formula

$$f_*(\mathcal{F} \overset{!}{\otimes} f^!\mathcal{G}) \simeq f_*\mathcal{F} \overset{!}{\otimes} \mathcal{G}.$$

When f_* and $f^!$ have left adjoints (denoted f^* and $f_!$ respectively) we say that we have the full six functor formalism. The six functors that the name refers to are $f_*, f^*, f_!, f^!, \otimes, \underline{\mathrm{Hom}}$. Not every sheaf theory as described before has left adjoints $f_!$ and f^*, for example, in the cases of Ind-coherent sheaves or $\mathcal{D}$-modules we only have functors f_* and $f^!$. In the case of constructible ℓ-adic sheaves or stable motivic homotopy theory we have the full six functors.

If the left adjoints exist, we can give slightly more familiar expressions for the previous properties. The proper adjunction can be understood as a natural transformation $f_! \to f_*$ which is natural in f and is an isomorphism when f is proper. The open adjunction can be understood as an isomorphism $f^! \simeq f^*$ when f is an open immersion. We can define a dual tensor structure by $*$-pullback as $\mathcal{F} \otimes \mathcal{G} \simeq \Delta^*(\mathcal{F} \boxtimes \mathcal{G})$ along the diagonal $\Delta : X \to X \times X$ which gives a closed symmetric monoidal structure on $\mathrm{Sh}(X)$ and f^* is a symmetric monoidal functor for that tensor structure. This means that $\mathrm{Sh}(X)$ comes equipped with an internal hom functor $\underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}$. Finally, we have a projection formula for this tensor structure defined by $*$-pullback. This amounts to $f_!$ being a functor of module categories, which gives the projection formula

$$f_!(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_!\mathcal{F} \otimes \mathcal{G}.$$

We also have an interesting interplay with the self-duality of the category of sheaves $\mathrm{Sh}(X)$: the functors $f^!$ and $f_!$ are dual to f_* and f^* respectively.

The $(\infty, 2)$-category of correspondences allows us to provide a very clean set-up for the six functor formalism. Indeed, all such properties of a sheaf theory Sh will be encoded in the form of a symmetric monoidal functor of $(\infty, 2)$-categories

$$\mathrm{Sh} : \mathrm{Corr} \to 2\text{-Cat}.$$

We will explain how the categories of correspondences look like, leaving the details of the $(\infty, 2)$-categorical constructions via Segal spaces to [GR17, Chapter 7].

Suppose that $\mathcal{C}$ is an $(\infty, 1)$-category together with three distinguished classes of horizontal, vertical and admissible morphisms. We shall denote by $\mathcal{C}_{\text{ver}}, \mathcal{C}_{\text{hor}}$ and $\mathcal{C}_{\text{adm}}$ the corresponding full subcategories of $\mathcal{C}$. These classes need to be closed under composition and verify the following properties:

1. The identity maps of objects of $\mathcal{C}$ belong to all three classes.

2. If a morphism belongs to a given class, then so do all isomorphic morphisms.

3. Suppose that $\alpha_1 : c_{1,1} \to c_{1,0}$ is a horizontal morphism and $\beta_0 : c_{0,0} \to c_{1,0}$ is a vertical morphism, then the Cartesian square

$$\begin{array}{ccc} c_{0,1} & \xrightarrow{\alpha_0} & c_{0,0} \\ \beta_1 \downarrow & & \downarrow \beta_0 \\ c_{1,1} & \xrightarrow[\alpha_1]{} & c_{1,0} \end{array}$$

   exists, α_0 is horizontal and β_1 is vertical. Moreover, if α_1 (resp. β_0) is admissible, then so is α_0 (resp. β_1).

4. The class of admissible morphisms satisfies the '2 out of 3' property: if

$$c_1 \xrightarrow{\alpha} c_2 \xrightarrow{\beta} c_3$$

   are maps such that β and $\beta \circ \alpha$ are admissible, then α is also admissible.

In the case that $\mathcal{C}$ has fiber products for all morphisms, we can take the classes of horizontal, vertical and admissible morphisms to simply be all morphisms. Another important example comes when $\mathcal{C}$ is the $(\infty, 1)$-category of schemes of almost finite type, the admissible morphisms are proper maps, and the horizontal and vertical morphisms are any maps.

The category of correspondences $\mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}}$ can be described as follows. Its objects are the objects of $\mathcal{C}$, the 1-morphisms are diagrams

$$\begin{array}{ccc} c_{0,1} & \longrightarrow & c_0 \\ \big\downarrow & & \\ c_1 & & \end{array}$$

where the vertical arrow is in $\mathcal{C}_{\mathrm{ver}}$ and the horizontal arrow is in $\mathcal{C}_{\mathrm{hor}}$. The composition of such diagrams is given by taking the fibered product

$$\begin{array}{ccccc} c_{1,2} \times_{c_1} c_{0,1} & \dashrightarrow & c_{0,1} & \longrightarrow & c_0 \\ \big\downarrow & & \big\downarrow & & \\ c_{1,2} & \longrightarrow & c_1 & & \\ \big\downarrow & & & & \\ c_2. & & & & \end{array}$$

The 2-morphisms between a pair of correspondences $(c_{0,1}, \alpha, \beta)$ and $(c'_{0,1}, \alpha', \beta')$ are given by diagrams of the form

$$\begin{array}{ccccc} c_{0,1} & & & & \\ & \overset{\gamma}{\searrow} & & \overset{\alpha'}{\searrow} & \\ & \underset{\beta'}{\searrow} & c'_{0,1} & \xrightarrow{\ \alpha\ } & c_0 \\ & & \big\downarrow{\scriptstyle\beta} & & \\ & & c_1 & & \end{array}$$

where γ is an admissible map.

In the case where $\mathcal{C}$ is a category of geometric objects like $\mathcal{C} = \mathrm{Sch}_{\mathrm{aft}}$, we can see that functoriality and proper base change in a sheaf theory is equivalent to the data of a functor of $(\infty, 1)$-categories

$$\mathrm{Sh} : \mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}.$$

In other words, an object $X \in \mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$ maps to $\mathrm{Sh}(X)$ and a morphism

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle g}\big\downarrow & & \\ Y & & \end{array}$$

gets mapped to $g_* \circ f^! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Let us see how to recover the proper and open adjunctions from the $(\infty, 2)$-categorical point of view. Suppose that $f : X \to Y \in \mathcal{C}$ is proper. The functor $f_* \circ f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(Y)$ is the image under Sh of the morphism

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \\ Y & & \end{array}$$

in $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$. Similarly, $f^! \circ f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$ is given by the image of the composite

$$\begin{array}{ccccc} X \times_Y X & \longrightarrow & X & = & X \\ \big\downarrow & & \big\downarrow & & \\ X & \longrightarrow & Y & & \\ \| & & & & \\ X & & & & \end{array}$$

If the diagonal morphism $X \to X \times_Y X$ is proper, we obtain the desired natural transformation

$$\mathrm{id} \to f^! \circ f_*.$$

Furthermore, if the map $f : X \to Y$ is proper, we also obtain a natural transformation $f_* \circ f^! \to \mathrm{id}$, and it's easy to see that the two natural transformations give the unit and counit of an adjunction.

For the case where $f : X \to Y$ is an open embedding, we have that

$$X \times_Y X \simeq X$$

and therefore we obtain the desired isomorphism

$$f^! \circ f_* \simeq \mathrm{id}\,.$$

The assertion that this isomorphism gives a counit of an adjunction is an additional *condition*.

As for duality, a key feature of the symmetric monoidal category $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$ is that every object X is self-dual. In particular, it is easy to see that the morphisms

$$\begin{array}{ccc} X & \longrightarrow & * \\ {\scriptstyle \Delta}\downarrow & & \\ X \times X & & \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \downarrow & & \\ * & & \end{array}$$

give the unit and counit maps, respectively. Applying the symmetric monoidal functor Sh, we obtain that the spectral category $\mathrm{Sh}(X)$ is self-dual. Even more so, we can check that for $f : X \to Y \in \mathcal{C}$, the morphisms

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \\ X & & \end{array} \qquad\qquad \begin{array}{ccc} X & = & X \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

are dual to each other in $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$. This implies that the functors $f^!$ and f_* are dual to each other.

The symmetric monoidal structure on the functor Sh gives a natural isomorphism

$$\boxtimes : \mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \xrightarrow{\sim} \mathrm{Sh}(X \times Y).$$

When Sh is only right-lax symmetric monoidal, we would only have a functor in that direction.

Moreover, the construction of the symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$ gives us a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$$

by horizontal morphisms, which is symmetric monoidal (and the symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$ is given by the coproduct). In particular, for every object $X \in \mathcal{C}^{\mathrm{op}}$, the diagonal map provides us with a commutative algebra structure and so $\mathrm{Sh}(X)$ carries naturally a symmetric monoidal structure $\overset{!}{\otimes}$ by !-restriction along the diagonal.

As for the projection formula, suppose that we have an object $Y \in \mathcal{C}$. Then, as we have seen, Y has a commutative algebra object structure in $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$. Furthermore, if $f : X \to Y$ is a morphism in Y, we have that X has a canonical structure of a module over Y. It is straightforward to see that in this case, the morphism

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

has the structure of a morphism of Y-modules in $\mathrm{Corr}(\mathcal{C})^{\mathrm{proper}}_{\mathrm{all,all}}$. In particular, applying the symmetric monoidal functor Sh shows that the functor

$$f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$

is a morphism of $\mathrm{Sh}(Y)$-modules. This concludes the proof of the projection formula.

3.2 Calculations of traces via correspondences

In this section we are going to use the category of correspondences to calculate certain traces of endofunctors in geometric categories. In order to do so, we first need to discuss calculations of traces in the categories of correspondences.

If we have classes of horizontal, vertical and admissible morphisms as in the previous section, we can notice that all the objects $c \in \mathcal{C}$ for which $c \to *$ is both horizontal and vertical are self-dual for the symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}}$. Indeed, it is easy to see that if $c \to *$ is both horizontal and vertical, then the diagonal map $\Delta : c \to c \times c$ is also both horizontal and vertical. Then, it is easy to verify that the evaluation and coevaluation maps given by

$$\begin{array}{ccc} c & \xrightarrow{\Delta} & c \times c \\ \downarrow & & \\ * & & \end{array} \qquad\qquad \begin{array}{ccc} c & \longrightarrow & * \\ {\scriptstyle \Delta}\downarrow & & \\ c \times c & & \end{array}$$

exhibit c as a self-dual object of $\mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}}$. Moreover, we can see that in that case the trace of the identity morphism id_c is given by the free loop space of c,

$$\mathcal{L}c := c \times_{c\times c} c.$$

More generally, for any 1-endomorphism of c in correspondences

$$\begin{array}{ccc} d & \xrightarrow{f} & c \\ {\scriptstyle g}\downarrow & & \\ c & & \end{array}$$

we can see that the trace is given by the object $c^{f=g}$ that is defined by the pullback diagram

$$\begin{array}{ccc} c^{f=g} & \longrightarrow & d \\ \downarrow & & \downarrow{\scriptstyle (f,g)} \\ c & \xrightarrow[\Delta]{} & c\times c. \end{array}$$

Indeed, this follows by looking at the diagram associated to the composition

$$\begin{array}{ccccccc} c^{f=g} & \dashrightarrow & d & \overset{f}{\dashrightarrow} & c & \longrightarrow & * \\ \vdots & (3) & \vdots{\scriptstyle (\mathrm{id},f)} & (1) & \downarrow{\scriptstyle \Delta} & & \\ d & \underset{(\mathrm{id},g)}{\dashrightarrow} & d\times c & \xrightarrow[f\times\mathrm{id}]{} & c\times c & & \\ {\scriptstyle g}\vdots & (2) & \downarrow{\scriptstyle g\times\mathrm{id}} & & & & \\ c & \xrightarrow[\Delta]{} & c\times c & & & & \\ \downarrow & & & & & & \\ * & & & & & & \end{array}$$

where the squares (1) and (2) are readily seen to be pullback squares and (3) is a pullback square by the definition of $c^{f=g}$ and simply noticing that the composition of the vertical arrows in the second column is $(f,g) : d \to c\times c$.

The category of correspondences allows us to encode *bivariant functors* $\Phi : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is some fixed $(\infty,1)$-category. This means that for each $c \in \mathcal{C}$ we have an object $\Phi(c)$, and for each 1-morphism $\gamma : c_1 \to c_2$ in $\mathcal{C}$ we have a 1-morphism

$$\Phi(\gamma) : \Phi(c_1) \to \Phi(c_2)$$

if γ is a vertical morphism, and a 1-morphism

$$\Phi^!(\gamma) : \Phi(c_2) \to \Phi(c_1)$$

if γ is a horizontal morphism, together with compatibility of $\Phi(\cdot)$ and $\Phi^!(\cdot)$ with compositions of 1-morphisms in $\mathcal{C}$, and for any Cartesian square

$$\begin{array}{ccc} c_{0,1} & \xrightarrow{\alpha_0} & c_{0,0} \\ {\scriptstyle\beta_1}\big\downarrow & & \big\downarrow{\scriptstyle\beta_0} \\ c_{1,1} & \xrightarrow[\alpha_1]{} & c_{1,0} \end{array}$$

where α_0, α_1 are in the class of horizontal arrows and β_0, β_1 are in the class of vertical arrows, we have a *base-change morphism*

$$\Phi(\beta_1) \circ \Phi^!(\alpha_0) \simeq \Phi^!(\alpha_0) \circ \Phi(\beta_0).$$

Moreover, this data must satisfy a homotopy-coherent system of compatibilities. All of this information is conveniently encoded by the category $\mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}}$ in the form of a functor

$$\Phi : \mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}} \to \mathcal{D}.$$

In particular, if $\mathcal{D}$ has a symmetric monoidal structure and Φ is a monoidal functor, we can compute traces in Φ by first computing the trace in the category of correspondences and then applying Φ. That is, if we have a correspondence $(c \xleftarrow{g} d \xrightarrow{f} c)$ and c is a dualizable object as in the previous discussion, the trace of

$$\Phi_*(g)\Phi^!(f) : \Phi(c) \to \Phi(c)$$

is given by the composition

$$\Phi_*(p)\Phi^!(p) : \Phi(*) \longrightarrow \Phi(c^{f=g}) \longrightarrow \Phi(*).$$

Example 3.2.1. These techniques allow us to calculate the Euler characteristic of $\mathrm{IndCoh}(X)$ for some smooth projective variety over a field of characteristic 0. Indeed, applying the previous discussion to the ind-coherent sheaves as a functor from correspondences, we have that

$$\mathrm{tr}_{\mathrm{Cat}_k}(\mathrm{id} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)) = R\Gamma(\mathcal{L}X, \omega_{\mathcal{L}X}).$$

Since we can identify $\mathcal{O}_{\mathcal{L}X} \simeq \omega_{\mathcal{L}X}$, the Hochschild-Kostant-Rosenberg theorem tells us that

$$\mathrm{HH}_r(\mathrm{IndCoh}(X)) \simeq \bigoplus_{q-p=r} H^q(X, \Omega^p_X).$$

Moreover, the functoriality of this construction shows that $R\Gamma : \mathrm{IndCoh}(X) \to \mathrm{Vect}$ induces the usual *integration* map in Hochschild homology

$$\int : \mathrm{HH}(\mathrm{IndCoh})(X) \simeq \bigoplus_r \bigoplus_{q-p=r} H^q(X, \Omega^p_X) \to k.$$

Remark 3.2.2. Even though the more familiar category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ is isomorphic to the category of ind-coherent sheaves $\mathrm{IndCoh}(X)$ when X is a smooth variety, we prefer the use of ind-coherent sheaves. There is a certain trade-off between using $\mathrm{QCoh}(X)$ or $\mathrm{IndCoh}(X)$. On the side of $\mathrm{QCoh}(X)$, the identification of the induced morphism on Hochschild homology by pullback $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ with the classical pullback of global sections is simple whereas the induced map by the pushforward is not easy to understand. On the side of $\mathrm{IndCoh}(X)$ the identification on Hochschild homology of the pushforward $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ with the classical pushforward is simple whereas the induced map by the pullback is not easy to understand. It is important to notice that the identification $\mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ does not respect the monoidal structure. Therefore, even though we can identify both $\mathrm{HH}(\mathrm{QCoh}(X))$ and $\mathrm{HH}(\mathrm{IndCoh}(X))$ with Hodge cohomology as described in the previous example, the isomorphism between $\mathrm{HH}(\mathrm{QCoh}(X))$ and $\mathrm{HH}(\mathrm{IndCoh}(X))$ is not trivial: it is given by multiplication by the Todd class td_X (see [KP19, Corollary 4.4.4.])

3.3 Chern character via categorical traces

Let $\mathcal{C}$ be a k-linear $(\infty,1)$-category. We know that functors $\varphi : \mathrm{Vect} \to \mathcal{C}$ in Cat_k are classified by the image of k. Moreover, one can verify that such a morphism is dualizable if and only if the image of k is a compact object of $\mathcal{C}$. Suppose that we have an endofunctor $F : \mathcal{C} \to \mathcal{C}$, a functor $\varphi : \mathrm{Vect} \to \mathcal{C}$ classified by a compact object $\varphi(k) = E$ and a 2-morphism

$$\begin{array}{ccc} \mathrm{Vect} & = & \mathrm{Vect} \\ {\scriptstyle\varphi}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle\varphi} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}, \end{array}$$

which simply boils down to some morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(E, F(E))$. In this setting, we can apply the functoriality of traces to define a morphism

$$\mathrm{tr}(\varphi, \alpha) : k = \mathrm{tr}_{\mathrm{Cat}_k}(\mathrm{Vect}) \longrightarrow \mathrm{tr}_{\mathrm{Cat}_k}(F : \mathcal{C} \to \mathcal{C}).$$

We call the image of $1 \in k$ under this map the *Chern character* $\mathrm{ch}(E, \alpha)$. When $\alpha = \mathrm{id}$ and $F = \mathrm{id}$ we simply write the corresponding Chern character as $\mathrm{ch}(E) \in \mathrm{HH}(\mathcal{C})$ and lives in the Hochschild homology of $\mathcal{C}$. Furthermore, a proper functor $F : \mathcal{C} \to \mathcal{D}$ (that is, it maps compact objects to compact objects) induces a morphism $F_* := \mathrm{tr}(F, \mathrm{id}) : \mathrm{HH}(\mathcal{C}) \to \mathrm{HH}(\mathcal{D})$ and by the functoriality of traces, we can see that

$$F_*(\mathrm{ch}(E)) = \mathrm{ch}(F(E)).$$

Example 3.3.1. When $\mathcal{C} = \text{Vect}$ and F is the identity endofunctor, φ corresponds to a finite dimensional vector space V and α corresponds to a map $\alpha : V \to V$. In this case,

$$\mathrm{ch}(V, \alpha) = \mathrm{tr}_k(\alpha : V \to V).$$

Example 3.3.2. Let X be a smooth, proper variety over an algebraically closed field k of characteristic 0. For a dualizable $\mathcal{E} \in \mathrm{QCoh}(X)$ (i.e., a perfect complex of sheaves on X), the Chern character we just described $\mathrm{ch}(\mathcal{E}, \mathrm{id}_{\mathcal{E}})$ agrees with the classical Chern character in Hodge cohomology. Under the identification between $\mathrm{QCoh}(X)$ and $\mathrm{IndCoh}(X)$ explained in Remark 3.2.2, we see that the categorical Chern character in $\mathrm{IndCoh}(X)$ is $\mathrm{ch}(\mathcal{E})\mathrm{td}_X$.

Example 3.3.3 ([KP19, Proposition 6.1.6.])**.** Consider X as in the previous example, and suppose that $\mathcal{E}$ is a perfect complex of sheaves on X. Then, by the functoriality of traces, the description of the Chern character and the identification of Hochschild-Kostant-Rosenberg together with the diagram

$$\begin{array}{ccccc} \text{Vect} & \longrightarrow & \mathrm{IndCoh}(X) & \longrightarrow & \text{Vect} \\ k & \longmapsto & \mathcal{E} & \longmapsto & R\Gamma(X, \mathcal{E}), \end{array}$$

yield the Hirzebruch-Riemann-Roch formula

$$\int_X \mathrm{ch}(\mathcal{E})\mathrm{td}_X = \chi(R\Gamma(X, \mathcal{E})).$$

3.4 Residues via Local Cohomology

Suppose that X is a smooth variety and let $Z \hookrightarrow X$ be a closed subscheme which is proper as a k-scheme. The functor Γ_Z of sections with support in Z is left exact. The derived functors are called the *local cohomology* groups. In the affine case, that is when $X = \mathrm{Spec}(A)$ and

$Z = V(I)$, we can calculate

$$R^p\Gamma_Z(M) = \operatorname*{colim}_r \operatorname{Ext}^p_A(A/I^r, M).$$

There is an exact triangle

$$0 \longrightarrow R\Gamma_Z(\cdot) \longrightarrow R\Gamma(\cdot) \longrightarrow R\Gamma(\cdot|_U) \longrightarrow 0.$$

The notion of residue goes back to [Har66]. Suppose that Z consists of isolated (maybe non-reduced) points. Take a coordinate patch $x_1, \ldots, x_n$ centered at one of those points and suppose that Z is given by equations $a_1, \ldots, a_n$ forming a regular sequence. Elements in the top local cohomology groups $R^n\Gamma_Z(\Omega^n_X)$ can be written as *generalized fractions*

$$\begin{pmatrix} f\, \mathrm{d}x_1 \wedge \cdots \wedge x_n \\ x_1^{e_1}, \ldots, x_n^{e_n} \end{pmatrix}.$$

For a generalized fraction we define the *residue symbol* as

$$\operatorname{Res}\begin{pmatrix} f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ x_1^{e_1}, \ldots, x_n^{e_n} \end{pmatrix} = b_{e_1-1,\ldots,e_n-1}$$

where $f = \sum b_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$. Because of the Nullstellensatz, we can write

$$x_i^{m_i} = \sum_j c_{ij} a_j.$$

Therefore,

$$\operatorname{Res}\begin{pmatrix} f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ a_1, \ldots, a_n \end{pmatrix} = \operatorname{Res}\begin{pmatrix} \det(c_{ij}) f\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ x_1^{m_1}, \ldots, x_n^{m_n} \end{pmatrix}.$$

Using this formula, we see that

$$\operatorname{Res}\begin{pmatrix} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ x_1 - f_1, \cdots, x_n - f_n \end{pmatrix} = \frac{1}{\det(\mathrm{id} - \mathrm{d}_x f)}$$

in the case that the fixed point is the transversal intersection of the diagonal and the graph of f. The previous discussion yields a map $\operatorname{Res} : R^n\Gamma_Z(\Omega^n_X) \to \mathbb{C}$ called the *residue map*.

Theorem 3.4.1 ([Lip84, Theorem 10.2])**.** *Let X be a smooth projective complex variety and $Z \hookrightarrow X$ a closed immersion. The following diagram commutes*

$$\begin{array}{ccc} R^n\Gamma_Z(\Omega^n_X) & \xrightarrow{\qquad\qquad} & H^n(X, \Omega^n_X) \\ & \searrow^{\operatorname{Res}} \quad \swarrow_{\int_X} & \\ & \mathbb{C} & \end{array}$$

In our setting, if $U = X \smallsetminus Z$ is the open complement of the closed subscheme Z, we have a localizing sequence given by

$$\operatorname{IndCoh}(X)_Z \longrightarrow \operatorname{IndCoh}(X) \longrightarrow \operatorname{IndCoh}(U).$$

We can understand $\operatorname{IndCoh}(X)_Z$ as the ind-completion of the category of coherent sheaves on X with support on Z, or equivalently, as the category of ind-coherent sheaves on the formal completion $\mathfrak{X}^\wedge_Z$ of X along Z.

Theorem 3.4.2. *We have an identification*

$$\mathrm{HH}(\operatorname{IndCoh}(\mathfrak{X}^\wedge_Z)) \simeq \bigoplus_r \bigoplus_{q-p=r} R^q\Gamma_Z(\Omega^p_X).$$

Moreover, the induced map on Hochschild homology by the global sections functor $R\Gamma$ identifies with the residue map in local cohomology.

Proof. Since HH is a localizing invariant (see Theorem 2.4.3), we get the desired identification thanks to the exact triangle that arises in local cohomology. Moreover, the theorem by

Lipman and our calculation of the induced map on traces by $R\Gamma : \mathrm{IndCoh}(X) \to \mathrm{Vect}$, together with the functoriality of traces, yield the identification with the residue map. □

3.5 Kernel construction

In many interesting examples, our sheaf theory viewed as a functor

$$\mathrm{Sh} : \mathrm{Corr}(\mathcal{C}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$$

is only lax symmetric monoidal. This is the case of the category of ind-constructible ℓ-adic sheaves and the stable motivic homotopy category. Even in those cases, we can calculate traces in the same way as before by using the following construction (communicated to us by N. Rozenblyum).

Construction 3.5.1. Let $\Phi : \mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}} \to \mathcal{D}$ be a lax symmetric monoidal functor. Consider the category ker_Φ whose objects are the same as those for $\mathcal{C}$ and the morphisms between two objects X and Y are given by $\Phi(X \times Y)$. This category carries a natural symmetric monoidal structure and its formal closure under colimits $\mathrm{Ker}_\Phi = \mathrm{Pre}(\mathrm{ker}_\Phi)$ also carries a natural symmetric monoidal structure via the Day convolution. The functor Φ factors through ker_Φ since given a correspondence $(f, g) : C \to X \times Y$ we get a map $\Phi(C) \to \Phi(X \times Y)$ and the image of the unit of $\Phi(C)$ is our desired kernel.

The upshot of such a factorization of Φ, is that $\Phi : \mathrm{Corr}(\mathcal{C})^{\mathrm{adm}}_{\mathrm{vert,hor}} \to \mathrm{Ker}_\Phi$ is now a symmetric monoidal functor. Hence, we can calculate traces in the exact same way as we did before. Moreover, the trace we calculate will live in the endomorphisms of the unit of Ker_Φ, which is given by $\Phi(\mathrm{Spec}\ k)$. In other words, the trace lives in the place we expect it to, and even when Φ is just lax symmetric monoidal we can carry out the same procedure as in Section 3.2.

3.6 Archetypical Lefschetz theorems

Using the formalism of categorical traces, we can obtain various *Lefschetz theorems* in different situations. Suppose that we have a commutative diagram of the form

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{C}, \end{array}$$

and assume that f and g both admit continuous right adjoints. Then we have induced maps on traces $g_* : \mathrm{tr}(f) \to \mathrm{tr}(f)$ and $f_* : \mathrm{tr}(g) \to \mathrm{tr}(g)$. The main theorem in [CP19] tells us that $\mathrm{tr}(f_*) = \mathrm{tr}(g_*)$.

Example 3.6.1. When $\mathcal{C} = \mathrm{IndCoh}(X)$ and $f : X \to X$, we can consider the commutative square

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{f_*} & \mathrm{IndCoh}(X) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{IndCoh}(X) & \xrightarrow{f_*} & \mathrm{IndCoh}(X). \end{array}$$

The previous discussion, together with the ideas from Section 3.2, tells us that

$$\mathrm{tr}(f_* : \mathrm{HH}(X) \to \mathrm{HH}(X)) = \chi(R\Gamma(X^f, \mathcal{O}_{X^f})).$$

Example 3.6.2. When X is a compact topological space and $f : X \to X$ a continuous map, we can apply this to the category of parametrized spectra $\mathcal{C} = \mathrm{Sp}^X = \mathrm{Fun}(C_{\mathrm{Sing}}(X), \mathrm{Sp})$ to obtain the usual Lefschetz formula.

The kernel construction (Section 3.5) provides a general way of (de)categorifying in any geometric context. In particular, we can apply this philosophy to the context of stable motivic homotopy theory. Indeed, the Euler characteristic of the category $\mathrm{SH}(X)$ is an object of $\mathrm{SH}(k)$. Moreover, a theorem of Morel shows that $\mathrm{End}_{\mathrm{SH}(k)}(\mathbf{1}) \simeq \mathrm{GW}(k)$, which means that taking traces again will yield an element in $\mathrm{GW}(k)$. We can use these ideas to give an alternative proof of a refinement of the Grothendieck-Lefschetz-Verdier trace formula in stable motivic homotopy theory.

Before doing so, let us recall some basic constructions from stable motivic homotopy theory. If ω is an endomorphism of $\mathbf{1}_X$ in $\mathrm{SH}(X)$ the integral is defined as the endomorphism $\int_X \omega \,\mathrm{d}\chi = \mathrm{tr}(p_{\#}\omega) \in \mathrm{End}_{\mathrm{SH}(S)}(\mathbf{1}_S)$ where $p_{\#}$ is the left adjoint to p^*. For a vector bundle V over X with projection $\pi : V \to X$ and zero section $s : X \to V$, we have the Thom endomorphisms $\Sigma^V = s^*\pi^!$ and $\Sigma^{-V} = s^!p^*$ of $\mathrm{SH}(X)$. We can associate to each automorphism $\phi : V \to V$ an automorphism $\langle\phi\rangle$ of the identity functor on $\mathrm{SH}(X)$. If $\mathcal{N}_i$ is the conormal bundle of a closed immersion $i : Z \hookrightarrow X$ and $p : X \to S$, $q : Z \to S$ are smooth, the *purity isomorphism* gives us an equivalence between $p_{\#}s_* \simeq q_{\#}\Sigma^{\mathcal{N}_i}$. The purity isomorphism is natural.

Theorem 3.6.3 ([Hoy14, Theorem 1.3])**.** *Let X be a smooth and proper S-scheme and $f : X \to X$ an S-morphism with regular fixed points. Then*

$$\mathrm{tr}(\Sigma^\infty_+ f) = \int_{X^f} \langle\phi\rangle \,\mathrm{d}\chi,$$

where ϕ is the automorphism of the conormal sheaf of the immersion $X^f \hookrightarrow X$ induced by $\mathrm{id} - i^*(\mathrm{d}f)$.

Proof. Just like in the previous examples, we obtain an identification

$$\mathrm{tr}\left(f_* : \chi(\mathrm{SH}(X)) \to \chi(\mathrm{SH}(X))\right) = \chi\left(\mathrm{tr}(f_* : \mathrm{SH}(X) \to \mathrm{SH}(X))\right).$$

The calculations of traces via correspondences, now taking place in the kernel construction, show that $\chi(\mathrm{SH}(X)) = \Sigma^\infty_+ X \in \mathrm{SH}(S)$ and $\mathrm{tr}\left(f_* : \mathrm{SH}(X) \to \mathrm{SH}(X)\right) = \Sigma^\infty_+ X^f \in \mathrm{SH}(S)$. Taking traces again we obtain in $\mathrm{End}_{\mathrm{SH}(S)}(\mathbf{1}_S)$, an equality

$$\mathrm{tr}\left(\Sigma^\infty_+ f : \Sigma^\infty_+ X \to \Sigma^\infty_+ X\right) = \chi\left(\Sigma^\infty_+ X^f\right).$$

The only remaining point is to identify $\chi\left(\Sigma^\infty_+ X^f\right)$ with $\int_{X^f} \langle\phi\rangle \, \mathrm{d}\chi$. The automorphism $\langle\phi\rangle$ on $q_\# \Sigma^{\mathcal{N}_i}$ becomes the identity on $p_\# i_* \mathbf{1}_{X^f}$ under the naturality of the purity isomorphism, concluding the proof. $\square$

CHAPTER 4

SINGULAR SUPPORT

4.1 Grothendieck's Construction

In this section we imitate the construction of cohomological correspondences in [LZ20] in the general setting of a lax symmetric monoidal sheaf theory

$$\mathrm{Sh} : \mathrm{Corr}(\mathrm{Sch}_S)^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$$

where Sch_S is the category of schemes over S. The Grothendieck construction provides us with a category $\mathcal{C}_{\mathrm{Sh}}$ whose objects are pairs $(X, \mathcal{F})$ with X a scheme over S and $\mathcal{F} \in \mathrm{Sh}(X)$, and a morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ in the category $\mathcal{C}_{\mathrm{Sh}}$ is a pair $((f, g), u)$ consisting of a correspondence $(f, g) : C \to X \times_S Y$ over S and a morphism $u : \mathcal{F} \to \mathrm{Sh}((f, g))(\mathcal{G}) = f_* g^! \mathcal{G}$ in $\mathrm{Sh}(X)$. When we have the full six functors, we favor the more symmetrical point of view of associating the map $u : f^*\mathcal{F} \to g^!\mathcal{G}$ in $\mathrm{Sh}(C)$ via the adjunction (f^*, f_*). The composition of morphisms

$$(X, \mathcal{F}) \xrightarrow{((f,g),u)} (Y, \mathcal{G}) \xrightarrow{((f',g'),u')} (Z, \mathcal{H})$$

is obtained by the correspondence composition $C \times_Y C' \to X \times_S Z$ and the morphism is obtained by pre and post-composition with u,u' and the base-change morphism on the diagram

$$\begin{array}{ccc} C \times_Y C' & \longrightarrow & C' \\ \big\downarrow & & \big\downarrow{\scriptstyle f'} \\ C & \xrightarrow[g]{} & Y. \end{array}$$

Moreover, a 2-morphism $((f, g), u) \to ((f', g'), u')$ in $\mathcal{C}_{\mathrm{Sh}}$ can be described using the 2-structure of Sh. When we have the full six functors, we can describe it more explicitly as a proper map $p : C \to C'$ satisfying $f' \circ p = f$ and $g' \circ p = g$ and such that u' is equal to the composition

$$(f')^*\mathcal{F} \xrightarrow{\text{adj}} p_*p^*(f')^*\mathcal{F} \simeq p_!f^*\mathcal{F} \xrightarrow{u} p_!g^!\mathcal{G} \simeq p_!p^!(g')^!\mathcal{G} \xrightarrow{\text{adj}} g^!\mathcal{G},$$

where we used the fact that $p_! \simeq p_*$ since p is a proper map.

The 2-category $\mathcal{C}_{\mathrm{Sh}}$ just described admits a symmetric monoidal structure since Sh is a lax symmetric monoidal functor. When we have the full six functors, it is described explicitly by setting the tensor product as $(X,\mathcal{F})\otimes(X',\mathcal{F}') = (X\times_S X', \mathcal{F}\boxtimes_S \mathcal{F}')$, and on morphisms $((f,g),u)\otimes((f',g'),u')$ is defined as $((f'',g''),u'')$ where $f'' = f\times_S f'$, $g'' = g\times_S g'$ and u'' is given as the composition

$$(f'')^*(\mathcal{F}\boxtimes_S \mathcal{F}') \simeq f^*\mathcal{F}\boxtimes_S (f')^*\mathcal{F}' \xrightarrow{u\boxtimes_S u'} g^!\mathcal{G}\boxtimes_S (g')^!\mathcal{G}' \longrightarrow (g'')^!(\mathcal{G}\boxtimes_S \mathcal{G}'),$$

where the last map is given by the adjoint to the Kunneth map.

The monoidal unit of $\mathcal{C}_{\mathrm{Sh}}$ is given by the pair $(S,\mathbf{1})$ where $\mathbf{1}\in \mathrm{Sh}(S)$ is the monoidal unit. We can describe the endomorphisms $\mathrm{End}_{\mathcal{C}_{\mathrm{Sh}}}(S,\mathbf{1}_S)$, where $\mathbf{1}_S\in\mathrm{Sh}(S)$ is the unit object. Indeed, it consists of pairs (X,α) where X is an S-scheme and α is an element of the set $p_*p^!(\mathbf{1}_S)$ where $p : X\to S$ is the structure map. If we denote the *dualizing sheaf* by $K_X = p^!\mathbf{1}_S$, we see that α is an element of p_*K_X.

Suppose that $(X,\mathcal{F})$ is a dualizable object. We define the *characteristic class* $\mathrm{cc}_{X/S}(\mathcal{F})$ to be the element in p_*K_X corresponding to the Euler characteristic of $(X,\mathcal{F})$ in $\mathcal{C}_{\mathrm{Sh}}$. If $f : X\to Y$ is a proper morphism, we can show that $f_*\mathrm{cc}_{X/S}(\mathcal{F}) = \mathrm{cc}_{Y/S}(f_*\mathcal{F})$.

We introduced two different ways of talking about traces for sheaf theories equipped with a six-functor formalism, and as such, understanding the relation between the notion of dualizable objects in those categories is fundamental. The following theorem connects both points of view.

Theorem 4.1.1. *An object $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}}$ is dualizable if and only if $\mathcal{F} \in \mathrm{Hom}_{\mathrm{Ker}_{\mathrm{Sh}}}(S, X)$ is a right adjoint.*

Proof. Consider $\mathcal{F} \in \mathrm{Sh}(X)$ as an object of $\mathrm{Hom}_{\mathrm{ker}_{\mathrm{Sh}/S}}(S, X)$. If $(X, \mathcal{F})$ is dualizable in $\mathcal{C}_{\mathrm{Sh}}$, then the dual is an object $(X, \mathcal{F}^\vee)$ and we can identify $\mathcal{F}^\vee$ with a functor $\mathrm{Hom}_{\mathrm{ker}_{\mathrm{Sh}}}(X, S)$. The dualizability of $(X, \mathcal{F})$ provides us with evaluation and coevaluation maps that can be used to define unit and counit maps that exhibit $\mathcal{F}$ as the right adjoint of $\mathcal{F}^\vee$. Conversely, if $\mathcal{G} \in \mathrm{Hom}_{\mathrm{ker}_{\mathrm{Sh}}}(X, S)$ is the left adjoint of $\mathcal{F}$, the unit and counit maps of the adjunction provide us with evaluation and coevaluation maps that exhibit $(X, \mathcal{G})$ as the dual of $(X, \mathcal{F})$. □

Notice that for both descriptions we only need the two functors f_* and $f^!$ and not the full six functors. This is useful when we deal with sheaf theories like IndCoh and $\mathcal{D}$-mod where $f_!$ and f^* are not always defined.

Remark 4.1.2. When $\mathrm{Sh} : \mathrm{Corr}(\mathrm{Sch}_S)^{\mathrm{proper}}_{\mathrm{all,all}} \to \mathcal{P}^{\mathrm{L}}_{\mathrm{St}}$ is symmetric monoidal, we can further characterize dualizable objects $(X, \mathcal{F})$ by looking at the corresponding $\mathrm{Sh}(S)$-linear functor $\mathrm{Sh}(S) \to \mathrm{Sh}(X)$ given by the integral transform. By the equivalence with the kernel construction, we see that this functor is a right adjoint if and only if $(X, \mathcal{F})$ is dualizable. In general it is not easy to describe when such functors are right adjoints. However, in the case where $p : X \to S$ is proper and Sh is QCoh, such kernels must be $\mathcal{F} \in \mathrm{Perf}(X)$ by [BZNP17, Theorem 1.2.4.].

However, to give an explicit description of the internal hom in $\mathcal{C}_{\mathrm{Sh}}$ we need to assume that we have the full six functors. This will be the case when Sh is the category of ℓ-adic constructible sheaves or the motivic stable homotopy category $\mathrm{SH}(X)$.

Lemma 4.1.3. *The symmetric monoidal structure $\otimes$ on $\mathcal{C}_{\mathrm{Sh}}$ is closed, with internal mapping object*

$$\underline{\mathrm{Hom}}((X, \mathcal{F}), (Y, \mathcal{G})) = \left(X \times_S Y, \underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S Y)}(p_X^* \mathcal{F}, p_Y^! \mathcal{G})\right).$$

Proof. Let $c : C \to X \times_S Y \times_S Z$ and denote the composition of c with the corresponding projection $p_?$ by $c_?$. Notice that the internal hom adjunction for $\mathrm{Sh}(C)$ provides us with an isomorphism

$$\underline{\mathrm{Hom}}_{\mathrm{Sh}(C)}(c_X^*\mathcal{F} \otimes c_Y^*\mathcal{G}, c_Z^!\mathcal{H}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(c_X^*\mathcal{F}, \underline{\mathrm{Hom}}_{\mathrm{Sh}(C)}(c_Y^*\mathcal{G}, c_Z^!\mathcal{H})).$$

We need to construct an isomorphism of categories between

$$\mathrm{Hom}((X,\mathcal{F}) \otimes (Y,\mathcal{G}), (Z,\mathcal{H})) \ \text{ and } \ \mathrm{Hom}\left((X,\mathcal{F}), (Y \times_S Z, \underline{\mathrm{Hom}}_{\mathrm{Sh}(Y\times_S Z)}(p_Y^*\mathcal{G}, p_Z^!\mathcal{H}))\right).$$

An object of $\mathrm{Hom}((X,\mathcal{F}) \otimes (Y,\mathcal{G}), (Z,\mathcal{H}))$ is a pair $(c : C \to X \times_S Y \times_S Z, u)$ where c is a correspondence and u is a morphism $u : c_X^*\mathcal{F} \otimes c_Y^*\mathcal{G} \to c_Z^!\mathcal{H}$.

Similarly, an object of $\mathrm{Hom}((X,\mathcal{F}), (Y \times_S Z, \underline{\mathrm{Hom}}_{\mathrm{Sh}(Y\times_S Z)}(p_Y^*\mathcal{G}, p_Z^!\mathcal{H})))$ amounts to a correspondence $c : C \to X \times_S Y \times_S Z$ and a morphism $u' : c_X^*\mathcal{F} \to c_{Y\times_S Z}^!\underline{\mathrm{Hom}}_{\mathrm{Sh}(Y\times_S Z)}(p_Y^*\mathcal{G}, p_Z^!\mathcal{H})$.

Composing u with the adjunction map yields a morphism $c_X^*\mathcal{F} \to \underline{\mathrm{Hom}}_{\mathrm{Sh}(C)}(c_Y^*\mathcal{G}, c_Z^!\mathcal{H})$. Notice that compatibility of the internal hom with shriek pullback gives us

$$c_{Y\times_S Z}^!\underline{\mathrm{Hom}}_{\mathrm{Sh}(Y\times_S Z)}(p_Y^*\mathcal{G}, p_Z^!\mathcal{H}) = \underline{\mathrm{Hom}}_{\mathrm{Sh}(C)}(c_Y^*\mathcal{G}, c_Z^!\mathcal{H}).$$

These identifications yield the desired isomorphism. □

4.2 Universal local acyclicity

In [LZ20, Theorem 2.16] it is shown that when $\mathrm{Sh}(X)$ is the category of ℓ-adic sheaves over X, an object $(X, \mathcal{F})$ is dualizable in $\mathcal{C}_{\mathrm{Sh}}$ precisely when $\mathcal{F}$ is universally locally acyclic over S. It is therefore reasonable to consider the following.

Definition 4.2.1. Suppose that $\mathrm{Sh} : \mathrm{Corr}(\mathrm{Sch}/S) \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ is a sheaf theory admitting six-functor formalism. A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is *universally locally acyclic* over S if $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}}$

is a dualizable object.

Remark 4.2.2. Notice that in the case where $X = S$, a sheaf $\mathcal{F} \in \mathrm{Sh}(S)$ is universally locally acyclic over S if and only if $\mathcal{F} \in \mathrm{Sh}(S)$ is a dualizable object.

Remark 4.2.3. Remark 4.1.2 can be rephrased as saying that when Sh is symmetric monoidal, $\mathcal{F}$ is universally locally acyclic with respect to $f : X \to S$ if and only if the associated $\mathrm{Sh}(S)$-linear functor $\mathrm{Sh}(S) \to \mathrm{Sh}(X)$ is a right adjoint.

From now on, unless explicitly mentioned, we will assume that we have the full six functors. This means that we have a characterization of the internal hom in the category of cohomological correspondences, which allows us to see that if an object $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}}$ is dualizable, then its dual must be

$$\begin{aligned}(X, \mathcal{F})^{\vee} &= \underline{\mathrm{Hom}}((X, \mathcal{F}), (S, \mathbf{1}_S)) \\ &= \Big(X \times_S S, \underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S S)}(p_X^* \mathcal{F}, p_S^! \mathbf{1}_S)\Big) \\ &\simeq (X, \underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(\mathcal{F}, K_X)).\end{aligned}$$

For this reason, we denote $\underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(\mathcal{F}, K_X)$ as $D_{X/S}(\mathcal{F})$.

Dualizable objects in the category of cohomological correspondences are well-behaved with respect to the operations of the six-functor formalism. For an object $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}}$ and a morphism $f : X \to Y$ of separated schemes of finite type over S, we consider the map $(X, \mathcal{F}) \to (Y, f_! \mathcal{F})$ where the correspondence is given by (id, f) and the morphism $\mathcal{F} \to f^! f_! \mathcal{F}$ is the adjuntion map. Notice that this morphism admits a right adjoint $(Y, f_* \mathcal{F}) \to (X, \mathcal{F})$ where the correspondence is given by (f, id) and the morphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is the adjunction map.

This allows us to prove that the notion of universal local acyclicity is well-behaved with respect to functorial properties.

Proposition 4.2.4. *Suppose that $f : X \to Y$ is a morphism of schemes separated of finite type over S. Then,*

1. *If $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over S, then $f_*\mathcal{F}$ is also universally locally acyclic over S if f is proper.*

2. *If $\mathcal{G} \in \mathrm{Sh}(Y)$ is universally locally acyclic over S, then $f^*\mathcal{G}$ is also universally locally acyclic over S if f is smooth.*

Proof. For part (1), suppose that $(X, \mathcal{F})$ is a dualizable object of $\mathcal{C}_{\mathrm{Sh}}$. This implies that for any object $(Z, \mathcal{H})$ we have an isomorphism

$$D_{X/S}\mathcal{F} \boxtimes_S \mathcal{H} \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S Z)}(p_X^* \mathcal{F}, p_Z^! \mathcal{H}),$$

where p_X, p_Z are the projection maps. We want to show that we have an isomorphism

$$D_{Y/S}(f_! \mathcal{F}) \boxtimes_S \mathcal{H} \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh}(Y \times_S Z)}(q_Y^* \mathcal{F}, q_Z^! \mathcal{H}),$$

where q_Y, q_Z are the projection maps. Applying the functor $(f \times_S \mathrm{id})_*$ we obtain an isomorphism

$$(f \times_S \mathrm{id})_* \left(D_{X/S}\mathcal{F} \boxtimes_S \mathcal{H} \right) \simeq (f \times_S \mathrm{id})_* \left(\underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S Z)}(p_X^* \mathcal{F}, p_Z^! \mathcal{H}) \right).$$

By the six-functor formalism, we can identify

$$D_{Y/S}(f_! \mathcal{F}) \boxtimes_S \mathcal{H} \simeq (f \times_S \mathrm{id})_*(D_{X/S}\mathcal{F} \boxtimes \mathcal{H}).$$

Similarly, the six-functor formalism implies that

$$(f \times_S \mathrm{id})_* \underline{\mathrm{Hom}}(p_X^* \mathcal{F}, p_Z^! \mathcal{H}) \simeq \underline{\mathrm{Hom}}((f \times_S \mathrm{id})_! p_X^* \mathcal{F}, q_Z^! \mathcal{H}),$$

and by proper base change we have that $q_Y^* f_! \mathcal{F} \simeq (f \times_S \mathrm{id})_! p_X^* \mathcal{F}$. This yields the desired isomorphism. The proof of part (2) is analogous. □

Remark 4.2.5. In the case where f is a closed immersion, f_* is a conservative functor. Inspecting the proof of the previous proposition, we can see that this implies that if $f_*\mathcal{F}$ is universally locally acyclic over S, then $\mathcal{F}$ must be universally locally acyclic over S.

The notion of universally locally acyclic sheaves is also well-behaved with respect to functorial properties on the base. More precisely, for a morphism $g : S \to T$ we can construct a symmetric monoidal functor $g^* : \mathcal{C}_{\mathrm{Sh}/T} \to \mathcal{C}_{\mathrm{Sh}/S}$ taking an object $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}/T}$ to $(X \times_T S, g_X^* \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}/S}$ where $g_X : X \times_T S \to X$ is the projection map. Since symmetric monoidal functors preserve dualizable objects, this proves the following.

Proposition 4.2.6. *Suppose that X is a scheme of finite type over T and $g : S \to T$ is a morphism of schemes. If $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over T, then the pullback $g_X^* \mathcal{F} \in \mathrm{Sh}(X \times_S T)$ is universally locally acyclic over S, where $g_X : X \times_S T \to X$ is the projection map.*

The following properties will be important for us in the upcoming sections.

Proposition 4.2.7. *Let X be a scheme separated of finite type over S. If $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over S, then $\mathcal{F}$ is a compact object of $\mathrm{Sh}(X)$ provided that the unit object $\mathbf{1}_X \in \mathrm{Sh}(X)$ is compact.*

Proof. If $(X, \mathcal{F}) \in \mathcal{C}_{\mathrm{Sh}}$ is dualizable, then Lemma 2.2.1 implies that

$$(X, \mathcal{F})^\vee \otimes (Y, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}_{\mathrm{Sh}}}((X, \mathcal{F}), (Y, \mathcal{G}))$$

for any $(Y, \mathcal{G}) \in \mathcal{C}_{\mathrm{Sh}}$. Unraveling the definitions, this simply means that

$$\Big(X \times_S Y, \pi_X^* D_{X/S} \mathcal{F} \otimes \pi_Y^* \mathcal{G}\Big) \simeq \Big(X \times_S Y, \underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S Y)}(\pi_X^* \mathcal{F}, \pi_Y^! \mathcal{G})\Big).$$

In particular, setting $Y = X$, we obtain

$$\pi_X^* D_{X/S}\mathcal{F} \otimes \pi_X^* \mathcal{G} \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh}(X\times_S X)}(\pi_X^*\mathcal{F}, \pi_X^! \mathcal{G}).$$

Therefore, we see that if $\Delta : X \to X \times_S X$ is the diagonal embedding, then

$$\Delta^! \left(\pi_X^* D_{X/S}(\mathcal{F}) \otimes \pi_X^* \mathcal{G}\right) \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{G}).$$

Notice that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{1}_X, \underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{G}))\\ &= \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbf{1}_X, \Delta^!(\pi_X^* D_{X/S}\mathcal{F} \otimes \pi_X^* \mathcal{G}))\\ &= \mathrm{Hom}_{\mathrm{Sh}(X)}(\Delta_*\mathbf{1}_X, \pi_X^* D_{X/S}\mathcal{F} \otimes \pi_X^* \mathcal{G}).\end{aligned}$$

Finally, since $\Delta_* = \Delta_!$ for X is separated over S, we see that $\Delta_*\mathbf{1}_X$ is compact and colimits in $\mathcal{G}$ are preserved. This shows that $\mathcal{F}$ is compact as desired. □

Proposition 4.2.8. *Let X be a scheme separated of finite type over S. The property of being universally locally acyclic over S is local in the Zariski topology on X.*

Proof. Suppose that $j : U \hookrightarrow X$ is an open embedding and $i : Z = X \smallsetminus U \hookrightarrow X$ is the complement. Notice that if two out of the three objects in a cofiber sequence are dualizable, then the third is dualizable as well. The localization sequence

$$i_* i^! \longrightarrow \mathrm{id} \longrightarrow j_* j^!$$

and Proposition 4.2.4 yield the desired result. □

4.3 Generic Local Acyclicity

In this section we prove the following theorem, imitating the proof by Deligne on [Del77, Th. finitude 2.11]. We are going to assume that any sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is dualizable when restricted to some open subset $U \subseteq X$.

Theorem 4.3.1. *Let X be a scheme separated of finite type over S. Then, for any sheaf $\mathcal{F} \in \mathrm{Sh}(X)$, there exists some open subset $U \subseteq S$ so that $\mathcal{F}|_{X_U}$ is universally locally acyclic over U, where $X_U = X \times_S U$.*

The proof starts by reducing to S being integral with generic point η, and proceeds by induction on $\dim X_\eta$. We can deduce in this way that, up to shrinking S, there exists a closed embedding $Z \subseteq X$ finite over S such that $\mathcal{F}|_{X \smallsetminus Z}$ is universally locally acyclic over S. This follows from the local nature of universal local acyclicity (see 4.2.8) and factoring $X \to S$ through $\mathbb{A}^1_S$. The inductive hypothesis applies for the map $X \to \mathbb{A}^1_S$ and we can use the following lemma.

Lemma 4.3.2. *Let $f : X \to S$ be separated of finite type. If $g : S \to T$ is smooth and $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over S, then it is also universally locally acyclic over T.*

Proof. We want to prove that $(X, \mathcal{F})$ is a dualizable object of $\mathcal{C}_{\mathrm{Sh}/T}$. This is equivalent to showing that

$$\pi_X^* D_{X/T}(\mathcal{F}) \otimes \pi_Y^* \mathcal{G} \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh(X \times_T Y)}}(\pi_X^* \mathcal{F}, \pi_Y^! \mathcal{G})$$

for any $(Y, \mathcal{G}) \in \mathcal{C}_{\mathrm{Sh}/T}$. Notice that $D_{X/T}(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(\mathcal{F}, f^! g^! \mathbf{1}_T)$, and the smoothness of g provides us with an isomorphism $g^! \mathbf{1}_T \simeq \mathbf{1}_S$, which in turn gives an isomorphism $D_{X/S}(\mathcal{F}) \simeq D_{X/T}(\mathcal{F})$. We know that $(X, \mathcal{F})$ is dualizable as an object of $\mathcal{C}_{\mathrm{Sh}/S}$, and in particular, if $p : Y_S = Y \times_T S \to Y$ is the canonical projection, we have an isomorphism

$$\pi_X^* D_{X/S}(\mathcal{F}) \otimes \pi_{Y_T}^* (p^* \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathrm{Sh(X \times_S Y_T)}}(\pi_X^* \mathcal{F}, \pi_{Y_T}^! (p^* \mathcal{G})).$$

These are sheaves on $X \times_S Y_T = X \times_T Y$. The smoothness of p yields an isomorphism $p^*\mathcal{G} \simeq p^!(\mathcal{G})$ which gives exactly what we want. $\square$

Shrinking S so that $f_*\mathcal{F}$ becomes dualizable, combining the previous lemma with the following lemma concludes the proof of the main theorem in this section.

Lemma 4.3.3. *Let $f : X \to S$ be a proper map. If $\mathcal{F} \in \mathrm{Sh}(X)$ is such that $\mathcal{F}|_{X \smallsetminus Z}$ is universally locally acyclic over S for some closed subscheme $Z \subseteq X$ that is finite over S, and $f_*\mathcal{F}$ is dualizable in $\mathrm{Sh}(S)$, then $\mathcal{F}$ is universally locally acyclic over S.*

Proof. It suffices to see that the map

$$\pi_X^* D_{X/S}(\mathcal{F}) \otimes \pi_X^* \mathcal{F} \to \underline{\mathrm{Hom}}_{\mathrm{Sh}(X \times_S X)}(\pi_X^* \mathcal{F}, \pi_X^! \mathcal{F})$$

is an isomorphism. Restricting to $(X \smallsetminus Z) \times_S X$ and $X \times_S (X \smallsetminus Z)$, the dualizability of $(X \smallsetminus Z, \mathcal{F}|_{X \smallsetminus Z})$ implies that the map we want is an isomorphism outside of $Z \times_S Z$. Therefore it suffices to show that restricted to $Z \times_S Z$ the map is an isomorphism, which follows from the dualizability of $f_*\mathcal{F}$ and the fact that pushforward via $Z \times_S Z \to S$ is a conservative functor. $\square$

4.4 Geometric properties of conical subsets

Suppose that $f : X \to S$ is a smooth morphism. In this section we will study some geometric properties of subsets of the relative cotangent bundle $\mathbf{T}^*(X/S)$.

We say that a constructible subset $C \subseteq \mathbf{T}^*(X/S)$ is *conical* if it is invariant under the canonical $\mathbb{G}_m$-action on the relative cotangent bundle. For a closed conical subset, its projectivization $\mathbf{P}(C)$ is a closed subset of the projectivized relative cotangent bundle $\mathbf{P}(\mathbf{T}^*(X/S))$. We say that a closed conical subset C is *strict* if none of its irreducible components lies in the zero section $\mathbf{T}^*_X(X/S)$. The map $C \mapsto \mathbf{P}(C)$ is a bijective correspondence between strict closed conical subsets of $\mathbf{T}^*(X/S)$ and closed subsets of $\mathbf{P}(\mathbf{T}^*(X/S))$. For a closed conical subset $C \subseteq \mathbf{T}^*(X/S)$, we denote by C^+ the union of C and the zero section of the cotangent bundle.

Definition 4.4.1. Suppose that X is smooth over S and C is a closed conical subset in $\mathbf{T}^*(X/S)$.

1. If U is smooth over S, we say that an S-morphism $h : U \to X$ is C-transversal at a geometric point $u \in U$ if $\ker(\mathrm{d}_u h) \cap C_{h(u)} \setminus \{0\}$ is empty.
2. If Y is smooth over S, we say that an S-morphism $f : X \to Y$ is C-transversal at a geometric point $x \in X$ if $(\mathrm{d}_x f)^{-1}(C_x) \setminus \{0\}$ is empty.

We say that f or h are C-transversal if they are C-transversal at every geometric point.

Let X be smooth over S, and C be a closed conical subset of $\mathbf{T}^*(X/S)$. Given an S-morphism $h : U \to X$ or $f : X \to Y$, the set of points $x \in X$ such that f is C-transversal and the set of points $u \in U$ such that h is C-transversal are open subsets. In other words, C-transversality is an open property.

If $h : U \to X$ is C-transversal and $C_U = C \times_X U$, the image $\mathrm{im}\left(\mathrm{d}h|_{C_U}\right)$ is denoted by $h^\circ C$. This is a closed conical subset of $\mathbf{T}^*(U/S)$.

The idea behind the proof of the following lemma is going to be helpful to understand the relationship between smoothness and C-transversality.

Lemma 4.4.2. *Let U be smooth S-schemes of finite type. If $h : U \to X$ is a smooth morphism, then h is C-transversal for any closed conical subset C of $\mathbf{T}^*(X/S)$. Moreover, in this case we have $h^\circ C = C \times_X U$.*

Proof. The smoothness of h is equivalent to the injectivity of $h^*\Omega^1_{X/S} \to \Omega^1_{U/S}$, because it is in turn equivalent to the vanishing of the cotangent bundle. The injectivity of $\mathrm{d}h$ implies that $\ker(\mathrm{d}_u h) = \{0\}$ for any closed point u, and so it implies that h is C-transversal for any conical subset C. □

Definition 4.4.3. The *base* of a conical subset $C \subseteq \mathbf{T}^*(X/S)$ is the scheme-theoretic image of C under the projection map $\mathbf{T}^*(X/S) \to X$. We will denote it as $B(C)$.

Suppose that C is a conical subset of $\mathbf{T}^*(X/S)$ and $f : X \to Y$ is a C-transversal morphism. We can see that the induced map $\mathrm{d}f : \mathbf{T}^*_{f(x)}(Y/S) \to \mathbf{T}^*_x(X/S)$ is injective for any closed point $x \in B(C)$. Since smoothness is a local property, this implies that f must be smooth on a Zariski neighborhood of x. Hence, any C-transversal morphism $f : X \to Y$ must be smooth on a Zariski neighborhood of $B(C)$.

Definition 4.4.4. A *test pair* (h, f) is a pair of S-morphisms $h : U \to X$ and $f : U \to Y$ where U, Y are smooth over S. We say that a test pair (h, f) is C-transversal at $u \in U$ if h is C-transversal at u and f is $h^\circ C$-transversal at u.

Lemma 4.4.5. *Let X be smooth over S.*

1. *A test pair (h, f) is $\mathbf{T}^*_X(X/S)$-transversal if and only if f is smooth.*

2. *A test pair (h, f) is $\mathbf{T}^*(X/S)$-transversal if and only if $h \times f : U \to X \times_S Y$ is smooth.*

Proof. For the first part, notice that if $h : U \to X$ is $\mathbf{T}^*_X(X/S)$-transversal then $h^\circ \mathbf{T}^*_X(X/S)$ is the zero-section $\mathbf{T}^*_U(U/S)$. Now, $f : U \to Y$ is $\mathbf{T}^*_U(U/S)$-transversal precisely when the induced map $\mathrm{d}f$ on cotangent bundles is injective, which amounts to the injectivity of the morphism $f^*\Omega^1_{Y/S} \to \Omega^1_{U/S}$. This is equivalent to the vanishing of the cotangent complex and holds precisely when $f : U \to Y$ is smooth (see the proof of Lemma 4.4.2).

For the second part, if $h \times f$ is smooth, then both f and h are smooth and Lemma 4.4.2 tells us that $h^\circ \mathbf{T}^*(X/S) = \mathbf{T}^*(X/S) \times_X U$. As in the previous part, the smoothness of $h \times f$ amounts to the injectivity of the induced morphism $\mathrm{d}(h \times f)$ on cotangent bundles. Composing with the induced maps on cotangent bundles by the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$, we see that the intersection of the images of $\mathrm{d}h$ and $\mathrm{d}f$ is contained in the zero section, which is precisely the $h^\circ \mathbf{T}^*(X/S)$-transversality of f. The converse is similar. □

If $r : X \to Z$ is an S-morphism between smooth schemes and C is a conical subset of $\mathbf{T}^*(X/S)$. The map r induces a morphism $\mathrm{d}r : \mathbf{T}^*(Z/S) \times_Z X \to \mathbf{T}^*(X/S)$ and we define $r_\circ C$ as the image through the projection $\mathbf{T}^*(Z/S) \times_Z X \to \mathbf{T}^*(Z/S)$ of the conical subset given by $(\mathrm{d}r)^{-1}(C)$.

For projective spaces, we can give a useful characterization of transversality of test pairs. Recall that the universal hyperplane $Q \subseteq \mathbb{P}^n_S \times \mathbb{P}^{n\vee}_S$ is the hypersurface defined by the incidence correspondence with projections $p : Q \to \mathbb{P}^n_S$ and $p^\vee : Q \to \mathbb{P}^{n\vee}_S$. The projectivization of the contangent bundle $\mathbf{P}(\mathbf{T}^*(\mathbb{P}^n_S/S))$ can be identified with the universal hyperplane Q. Indeed, a point $(x, x^\vee) \in Q$ can be identified with the conormal line to the hyperplane $Q_{x^\vee}$ at x. Analogously, we can also identify Q with $\mathbf{P}(\mathbf{T}^*(\mathbb{P}^{n\vee}_S/S))$. Such identifications are known as the *Legendre transforms*.

Suppose that $C \subseteq \mathbf{T}^*(\mathbb{P}^n_S/S)$ is a strict conical subset. A test pair (h, f) fits in the following commutative diagram

$$\begin{array}{ccccc}
Q_U = U \times_{\mathbb{P}^n_S} Q & \xrightarrow{h_U} & Q & \xrightarrow{p^\vee} & \mathbb{P}^{n\vee}_S \\
\downarrow{\scriptstyle p_U} & & \downarrow{\scriptstyle p} & & \\
U & \xrightarrow{\quad h \quad} & \mathbb{P}^n_S & & \\
\downarrow{\scriptstyle f} & & & & \\
Y. & & & &
\end{array}$$

In this way, we obtain a test pair $(p^\vee h_U, f p_U)$ on $\mathbb{P}^{n\vee}_S$. Let E be the image of $\mathbf{P}(C)$ in Q by the Legendre transform. We can give a criterion to decide if the given test pair (h, f) is C-transversal in terms of the test pair for the dual projective space.

Lemma 4.4.6. *Let (h, f) be a test pair on $\mathbb{P}^n_S$. Then, the test pair (h, f) is C-transversal if and only if $(p^\vee h_U, f p_U) : Q_U \to \mathbb{P}^{n\vee}_S \times Y$ is smooth at $E_U = E \times_{\mathbb{P}^n_S} U$.*

Proof. It is enough to prove the claim when $S = \mathrm{Spec}(k)$ by looking at the fibers above the points of S. The Legendre transform identifies the projectivization of the cotangent spaces of $\mathbb{P}^n$ and $\mathbb{P}^{n\vee}$ with Q. For a point $(x, x^\vee) \in Q$, the point $(\lambda_{x^\vee x}) \in \mathbf{P}(\mathbf{T}^*\mathbb{P}^n)_x$ that corresponds to the conormal line $\lambda_{x^\vee x} \subset \mathbf{T}^*_x\mathbb{P}^n$ gets mapped to the hyperplane $Q_{x^\vee}$ at x.

Suppose that $u \in U$ is a geometric point and $x = h(u)$, $y = f(u)$. The C-transversality of (h, f) implies that for every $(x, x^\vee) \in E_x$ the differential map $\mathrm{d}_u h + \mathrm{d}_u f : \lambda_{x^\vee x} \oplus \mathbf{T}^*_y Y \to \mathbf{T}^*_u U$ is injective, or equivalently, the map $\mathrm{d}_{(u,e)}(p^\vee h_U) + \mathrm{d}_{(u,e)}(f p_U) : \mathbf{T}^*_{x^\vee} \oplus \mathbf{T}^*_y Y \to \mathbf{T}^*_{(u,e)} Q_U$ is injective. Indeed, the cotangent space $\mathbf{T}^*_{(u,e)} Q_U$ can be described as the pushout

$$\begin{array}{ccc} \mathbf{T}^*_x\mathbb{P}^n & \xrightarrow{\mathrm{d}p} & \mathbf{T}^*_e Q \\ \downarrow{\scriptstyle \mathrm{d}h} & & \downarrow \\ \mathbf{T}^*_u U & \longrightarrow & \mathbf{T}^*_{(u,e)} Q_U, \end{array}$$

and the Legendre transform allows us to identify $\mathbf{T}^*_e Q$ as the direct sum $\mathbf{T}^*_x\mathbb{P}^n$ and $\mathbf{T}^*_{x^\vee}\mathbb{P}^{n\vee}$ with the lines $\lambda_{x^\vee x}$ and $\lambda_{x x^\vee}$ identified. This shows that the map $(p^\vee h_U, f p_U)$ must be smooth at $(u, e) \in E_U$.

□

4.5 Singular Support

Using ideas of Beilinson [Bei16] and [HY18] in the relative setting, we can give a definition of singular support for any sheaf theory having a six-functor formalism.

The main idea is to consider, at each closed point $x \in X$, the directions $\xi \in \mathbf{T}^*_x(X/S)$ in which $\mathcal{F}$ is not universally locally acyclic. More precisely, we can consider test pairs (h, f) where $h : U \to X$ is an open immersion around x and $f : U \to \mathbb{A}^1_S$ is such that $\mathrm{d}_x f = \xi$. Such a test pair is called a *weak test pair*.

Definition 4.5.1. Let X be smooth over S and C a closed conical subset of $\mathbf{T}^*(X/S)$. We say that $\mathcal{F}$ is *weakly micro-supported* on a conical subset C if $\mathcal{F}|_U$ is universally locally acyclic with respect to f for any C-transversal weak test pair (h, f). We denote by

$$\mathrm{MS}^w_S(\mathcal{F}) = \{C \subseteq \mathbf{T}^*(X/S) : C \text{ is closed conical and } \mathcal{F} \text{ is weakly micro-supported on } C\}.$$

This notion is well-defined for any sheaf theory coming from the point of view of the category of correspondences since we only need a notion of universal local acyclicity and the restriction to an open subset (i.e., upper shriek pullback via the open inclusion). However, if our sheaf theory has the full six functors, we can give a stronger notion.

Definition 4.5.2. Let X be smooth over S and C a closed conical subset of $\mathbf{T}^*(X/S)$. We say that a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is *micro-supported* on C if, for every C-transversal test pair (h, f), the sheaf $h^*\mathcal{F}$ is universally locally acyclic with respect to f. We denote by

$$\mathrm{MS}_S(\mathcal{F}) = \{C \subseteq \mathbf{T}^*(X/S) : C \text{ is closed conical and } \mathcal{F} \text{ is micro-supported on } C\}.$$

Remark 4.5.3. Suppose that X is smooth over S and $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over S. Then $\mathrm{MS}_S(\mathcal{F})$ is not empty. More precisely, $\mathcal{F}$ is micro-supported on $\mathbf{T}^*(X/S)$. To prove this, notice that by Lemma 4.4.5, any test pair (h, f) which is $\mathbf{T}^*(X/S)$-transversal we must have that $h \times f$ is smooth. Composing with $(h \times f)^*$, it suffices to show that $p_X^*(\mathcal{F}) \in \mathrm{Sh}(X \times_S Y)$ is universally locally acyclic with respect to $p_Y : X \times_S Y \to Y$ due to Proposition 4.2.4. This is precisely the content of Proposition 4.2.6.

Remark 4.5.4. By Lemma 4.4.5 and Remark 4.2.2, we see that $\mathcal{F} \in \mathrm{Sh}(X)$ is micro-supported on the zero section $\mathbf{T}^*_X(X/S)$ if and only if $\mathcal{F}$ is a dualizable object of $\mathrm{Sh}(X)$.

Definition 4.5.5. The *singular support* of $\mathcal{F} \in \mathrm{Sh}(X)$ is the unique smallest element of $\mathrm{MS}_S(\mathcal{F})$. We denote it by $\mathrm{SS}(\mathcal{F})$.

The remainder of this section will be devoted to the proof that the notion of singular support is actually well-defined.

Remark 4.5.6. An advantage of dealing with the notion of weak micro-support is that if we have conical subsets $C_1, C_2 \in \mathrm{MS}^w_S(\mathcal{F})$ then $C_1 \cap C_2 \in \mathrm{MS}^w_S(\mathcal{F})$. Indeed, if a weak test pair (h, f) is $C_1 \cap C_2$ is transversal, locally on U it must be either C_1-transversal or C_2-transversal (this need not be true if the codomain of f has dimension higher than 1). This implies that, as long as $\mathrm{MS}_S^w(\mathcal{F})$ is not empty, it must have a minimal element that we denote $\mathrm{SS}^w(\mathcal{F})$. In the case that we have the full six functors, we know that $\mathrm{MS}_S(\mathcal{F}) \subseteq \mathrm{MS}_S^w(\mathcal{F})$, and so, if the singular support of $\mathcal{F}$ exists, then $\mathrm{SS}^w(\mathcal{F}) \subseteq \mathrm{SS}(\mathcal{F})$.

Lemma 4.5.7. *If $\mathcal{F} \in \mathrm{Sh}(X)$ is universally locally acyclic over S, then the base of $\mathrm{SS}^w(\mathcal{F})$ is equal to the support of $\mathcal{F}$.*

Proof. It's straightforward to see that the support of $\mathcal{F}$ contains the base of $\mathrm{SS}^w(\mathcal{F})$. For the converse, since the base $B = B(\mathrm{SS}^w(\mathcal{F}))$ is a closed subset, it is enough to show that for $\mathcal{F} \neq 0$ we have that $\mathrm{SS}^w(\mathcal{F})$ is not empty, up to replacing X by $X \smallsetminus B$. Proceeding similarly as in Remark 4.5.3, the conclusion follows. □

The conical sets on which a sheaf is micro-supported enjoy good functorial properties. In particular, we show in the following lemmas, that they behave well with respect to push-forward, pullbacks, and possess additivity properties.

Lemma 4.5.8. *Suppose that U is a smooth S-scheme and $h : U \to X$ an S-morphism. If h is C-transversal and $\mathcal{F}$ is micro-supported on C, then $h^*\mathcal{F}$ is micro-supported on $h^\circ C$.*

Proof. If (g, f) is a $h^\circ C$-transversal test pair, then $(g, h \circ f)$ is a C-transversal test pair. Since $\mathcal{F}$ is micro-supported on C, it follows that $(h \circ f)^*\mathcal{F}$ is universally locally acyclic with respect to g. This is equivalent to $f^*(h^*\mathcal{F})$ is universally locally acyclic with respect to g. This means that $h^*\mathcal{F}$ is micro-supported on $h^\circ C$ as desired. □

Lemma 4.5.9. *Suppose that Z is a smooth S-scheme and $r : X \to Z$ an S-morphism. If $\mathcal{F} \in \mathrm{Sh}(X)$ is micro-supported on C and $B(C)$ is proper over Z then $r_*\mathcal{F}$ is micro-supported on $r_\circ C$.*

Proof. Let $h' : V \to Z$, $f' : V \to Y$ be a $r_\circ C$-transversal test pair. We want to see that $h'^* r_*\mathcal{F}$ is universally locally acyclic with respect to f'. Consider $r' : X_V := X \times_Z V \to V$ and $h : X_V \to X$ the projections. Since $\mathrm{SS}^w(\mathcal{F}) \subseteq C$, the image of the support of $\mathcal{F}$ by the zero section is contained in C by Lemma 4.5.7. Hence, by Lemma 4.4.2, X_V is smooth on some neighborhood U of the support of $h^*\mathcal{F}$. We know that $h^*\mathcal{F}$ is universally locally acyclic with respect to $f'r'$ because the test pair $(h|_U, f'r'|_U)$ is C-transversal. Because of Proposition 4.2.6 we obtain that $r'_*h^*\mathcal{F}$ is universally locally acyclic with respect to f'. Applying proper base change, we see that $h'^*r_*\mathcal{F} = r'_*h^*\mathcal{F}$ since r' is proper on the support of $h^*\mathcal{F}$. This concludes the proof. □

Lemma 4.5.10. *Suppose that $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is a distinguished triangle in* $\mathrm{Sh}(X)$*. If C' and C'' are conical subsets in $\mathbf{T}^*(X/S)$ such that $\mathcal{F}'$ is micro-supported on C' and $\mathcal{F}''$ is micro-supported on C'', then $\mathcal{F}$ is micro-supported on $C' \cup C''$.*

Proof. Consider a $(C' \cup C'')$-transversal test pair (h, f). In particular, it must be both C' and C''-transversal. Therefore, $h^*\mathcal{F}'$ and $h^*\mathcal{F}''$ are universally locally acyclic with respect to f. Since dualizable objects satisfy a two-out-of-three property on distinguished triangles, we see that $h^*\mathcal{F}$ must be universally locally acyclic with respect to g, as desired. □

Consider the set $\mathrm{MS}_S^{\min}(\mathcal{F})$ of minimal elements of $\mathrm{MS}_S(\mathcal{F})$. Assuming our schemes are Noetherian, we see that any $C \in \mathrm{MS}_S(\mathcal{F})$ must contain some minimal element $C' \subseteq C$, $C' \in \mathrm{MS}_S^{\min}(\mathcal{F})$.

Lemma 4.5.11. *Suppose that $h : U \to X$ is a smooth and surjective morphism of S-schemes. If C is a closed conical subset of $\mathbf{T}^*(X/S)$ and $h^\circ C \in \mathrm{MS}_S^{\min}(h^*\mathcal{F})$, then $C \in \mathrm{MS}_S^{\min}(\mathcal{F})$.*

Proof. Let C' be a minimal element of $\mathrm{MS}_S(\mathcal{F})$ that is contained in C. Then, by Lemma 4.5.8, we see that $h^*\mathcal{F}$ is micro-supported on $h^\circ C'$. Since $h^\circ C$ is a minimal element, the natural inclusion $h^\circ C' \subseteq h^\circ C$ must be an isomorphism. Moreover, given that h is smooth, Lemma 4.4.2 implies that we have an isomorphism $C' \times_X U \simeq C \times_X U$. Finally, the surjectivity of h implies that $C = C'$. □

The following result will be key in showing the existence of singular support.

Lemma 4.5.12. *Suppose that $i : X \hookrightarrow P$ is a closed embedding of smooth varieties over S. If $i_*\mathcal{F}$ has singular support over P, then $\mathcal{F}$ must have singular support over X.*

Proof. First, let us explore the situation for weak singular support. By Remark 4.2.5 we know that $i_*\mathcal{F}$ is universally locally acyclic with respect to g for some $g : P \to Y$ if and only if $\mathcal{F}$ is universally locally acyclic with respect to $g|_X$. Now, if we have a weak test pair on X, we can always extend it to a weak test pair on P. In other words, for any $f : X \to \mathbb{A}^1_S$, any closed point $x \in X$ and any covector $\xi \in \mathbf{T}^*_x(P/S)$ such that $\mathrm{d}_x f = \xi$, we can find Zariski locally a function $g : P \to \mathbb{A}^1_S$ such that $g|_X = f$ and $\mathrm{d}_x g = \xi$. Combining these two pieces of information it follows that $\mathrm{SS}^w(i_*\mathcal{F}) = i_\circ \mathrm{SS}^w(\mathcal{F})$.

The claim is Zariski local. We can assume that P is affine and we have an étale map where we can view $X \hookrightarrow P$ as the inclusion $\mathbb{A}^n_S \hookrightarrow \mathbb{A}^m_S$. This yields $s : \mathbf{T}^*(X/S) \to \mathbf{T}^*(P/S) \times_P X$ a splitting of $\mathrm{d}i : \mathbf{T}^*(P/S) \times_P X \to \mathbf{T}^*(X/S)$. If $\phi : \widetilde{P} \to P$ is such an étale map and $\rho : \widetilde{P} \to X$ is a retraction for which $\mathrm{d}\rho|_X = s$. We know that the base of $\mathrm{SS}(i_*\mathcal{F})$ lies in X, from which we have a closed conical subset $C = \rho_\circ \phi^\circ \mathrm{SS}(i_*\mathcal{F}) = s^{-1}\mathrm{SS}(i_*\mathcal{F})$ inside of $\mathbf{T}^*(X/S)$. This C just constructed is precisely $\mathrm{SS}(\mathcal{F})$. Indeed, we know that $\mathcal{F}$ is micro-supported on C since $\rho_*\phi^*(i_*\mathcal{F}) = \mathcal{F}$ and it must be the smallest element of $\mathrm{MS}_S(\mathcal{F})$ since for any $C' \in \mathrm{MS}_S(\mathcal{F})$ we have that $i_*\mathcal{F}$ is micro-supported on $i_\circ C'$ and Lemma 4.5.11 implies that $i_\circ C'$ contains $\mathrm{SS}(i_*\mathcal{F})$ and so $C' = \rho_\circ \phi^\circ(i_\circ C')$ contains C. □

Remark 4.5.13. In some special circumstances we can show that $\mathrm{SS}(i_*\mathcal{F}) = i_\circ \mathrm{SS}(\mathcal{F})$, but we only care about the existence of singular support in this section.

4.6 Radon Transform

The results from the previous section show that it is enough to prove that singular support exists and is unique in the case of $X = \mathbb{P}^n_S$. In this section we apply the Radon transform to do so.

The *Radon transform* functor is defined as

$$R : \mathrm{Sh}(\mathbb{P}^n_S) \to \mathrm{Sh}(\mathbb{P}^{n\vee}_S), \quad R(\mathcal{F}) = p^\vee_* p^* \mathcal{F}.$$

Similarly, the dual Radon transform is defined as $R^\vee = p_* p^{\vee *}$. It is easy to see that R and $R^\vee$ are adjoint to each other, both left and right.

Lemma 4.6.1. *Let $\mathcal{F} \in \mathrm{Sh}(\mathbb{P}^n_S)$ and C be a closed conical subset of $\mathbf{T}^*(\mathbb{P}^n_S/S)$. If $\mathcal{F}$ is micro-supported on C^+ then the Radon transform $R(\mathcal{F})$ is micro-supported on $C^{\vee +}$. Conversely, if $R(\mathcal{F})$ is micro-supported on $C^{\vee +}$ then $\mathcal{F}$ is micro-supported on C, possibly after replacing S by a Zariski open subset.*

Proof. By the functoriality properties of Lemma 4.5.8 and Lemma 4.5.9, we see that $R(\mathcal{F})$ is micro-supported on $p^\vee_\circ p^\circ(C^+) = C^{\vee +}$. Conversely, if $R(\mathcal{F})$ is micro-supported on $C^{\vee +}$, then $R^\vee(R(\mathcal{F}))$ is micro-supported on $p_\circ p^{\vee \circ}(C^{\vee +}) = C^+$. Shrinking S we may assume that the cone of the adjunction $\mathcal{F} \to R^\vee(R(\mathcal{F}))$ is dualizable. By Lemma 4.5.10, $\mathcal{F}$ must also be micro-supported on C^+. □

We are now in conditions of showing the main theorem.

Theorem 4.6.2. *Suppose that $\mathcal{F} \in \mathrm{Sh}(\mathbb{P}^n_S)$. The singular support $\mathrm{SS}(\mathcal{F})$ exists and is unique up to possibly replacing S by a Zariski open subset.*

Proof. Replacing S by a Zariski open subset, we may assume that $\mathcal{F} = R^\vee(\mathcal{G})$ for some $\mathcal{G} \in \mathrm{Sh}(\mathbb{P}^{n\vee}_S)$. Let E be the smallest closed subset in $\mathbb{P}^n_S$ such that $p^{\vee *}\mathcal{G}$ is universally locally acyclic with respect to p over $\mathbb{P}^n_S \smallsetminus E$. It's enough to show that $C^+ = C'^+$ for any minimal element $C' \in \mathrm{MS}_S(\mathcal{F})$.

Let $E' \subseteq Q$ be the Legendre transform of $\mathbf{P}(C')$. We have a $C'^{\vee +}$-transversal test pair on $\mathbb{P}^\vee$ defined by the projections $(p|_{Q \smallsetminus E'}, p^\vee|_{Q \smallsetminus E'})$ due to Lemma 4.4.6. Since $\mathcal{G}$ is micro-supported on $C'^{\vee +}$ by Lemma 4.6.1, it follows that $p^{\vee *}\mathcal{G}$ is universally locally acyclic with respect to $p|_{Q \setminus E'}$. This implies that $E \subseteq E'$, or equivalently, $C^+ \subseteq C'^+$. Therefore, we only need to show that $\mathcal{F}$ is micro-supported on C^+.

In order to do so, let (h, f) be a C^+-transversal test pair on $\mathbb{P}^n_S$. We need to check that $h^*\mathcal{F}$ is universally locally acyclic with respect to f. By Lemma 4.4.6, we know that $(p^\vee h_U)^*\mathcal{G}|_{E_U}$ is universally locally acyclic with respect to fp_U. Therefore, it must also be true for a Zariski neighborhood of E_U. By definition, $p^{\vee *}\mathcal{G}|_{Q \setminus E}$ is universally locally acyclic with respect to p, and so $(p^\vee h_U)^*\mathcal{G}|_{Q_U \setminus E_U}$ is universally locally acyclic with respect to p_U. Since C^+ contains the zero section, f must be smooth and so by Lemma 4.3.2 we see that $(p^\vee h_U)^*\mathcal{G}|_{Q_U \setminus E_U}$ is universally locally acyclic with respect to fp_U. This implies that $(p^\vee h_U)^*\mathcal{G}$ is universally locally acyclic with respect to fp_U. By Proposition 4.2.4, it follows that $(p^\vee h_U)_*(p^\vee h_U)^*\mathcal{G} = h^*\mathcal{F}$ is universally locally acyclic with respect to f, which is what we wanted. □

Corollary 4.6.3. *For any smooth variety X over S and $\mathcal{F} \in \mathrm{Sh}(X)$, up to possibly replacing S by a Zariski open subset, the singular support $\mathrm{SS}(\mathcal{F})$ exists and is well-defined.*

4.7 Singular Support of Sheaves of Categories

In this section, we show that our notion of singular support for sheaves of categories agree with the one defined by di Fiore and Stefanich. Let us recall the definition of sheaves of categories.

Definition 4.7.1. Given a scheme X, a quasi-coherent sheaf of categories is a small idempotent complete stable category with a monoidal action of $\mathrm{Perf}(X)$. Moreover, a quasi-coherent sheaf of categories is *perfect* if it is proper and smooth over $\mathrm{Perf}(X)$ and *coherent* if it is proper over $\mathrm{Perf}(X)$ and smooth over $\mathrm{Perf}(k)$.

The theory of quasi-coherent sheaves of categories has enough functoriality so that we can define singular support using our general methods. The only caveat is that we do not know whether a sheaf of categories can be restricted to an open subset where it is dualizable, although we suspect it is true under reasonable hypothesis. Notice that by Proposition 2.5.2, a sheaf of categories is dualizable if and only if it is smooth and proper. In particular, for a proper and smooth quasi-coherent sheaf of categories, we can see that its singular support must be contained in the zero section.

As in the previous section, the definition of singular support for sheaves of categories is cohomological in nature. Fortunately, we also have a description of microlocal nature. For a proper map $p : Y \to X$ with finite tor-dimension and a quasi-coherent sheaf of categories $\mathcal{F}$ on X, the category $\mathrm{MF}_p(\mathcal{F})$ of matrix factorizations of $\mathcal{F}$ with respect to p is defined as the cofiber of $p^*\mathcal{F} \to p^!\mathcal{F}$. When X is smooth and $\mathcal{F}$ is $\mathrm{Perf}(X) = \mathrm{Coh}(X)$, the category of matrix factorizations $\mathrm{MF}_p(\mathcal{F})$ is equal to the singularity category $\mathrm{Sing}(Y)$. If we have $f : X \to \mathbb{A}^1$ and $i_f : X_0 \to X$ is the inclusion of the closed fiber, we interpret $\mathrm{MF}_{i_f}(\mathcal{F})$ as a categorical version of vanishing cycles.

Theorem 4.7.2 ([Fio19, Proposition 10.7])**.** *Let X be a smooth variety, and (x, ξ) a covector. Suppose that we have a coherent sheaf of categories $\mathcal{F}$ and a function $f : X \to \mathbb{A}^1$ with $f(x) = 0$ and $\mathrm{d}_x f = \xi$. If the fiber $i^*_{x,X_0}(\mathrm{MF}_{i_f}(\mathcal{F}))$ is non-trivial, then (x, ξ) is in $\mathrm{SS}(\mathcal{F})$. If $i^!_{x,X_0}(\mathcal{F})$ has a compact generator, the converse holds.*

Our description of weak singular support tells us that a point (x, ξ) is in $\mathrm{SS}^w(\mathcal{F})$ if for any function $f : X \to \mathbb{A}^1$ with $f(x) = 0$ and $\mathrm{d}_x f = \xi$, the sheaf $\mathcal{F}|_U$ is not universally locally acyclic with respect to f on a Zariski neighborhood of x. To show that our notion agrees, we would like to see that if $i^*_{x,X_0}(\mathrm{MF}_{i_f}(\mathcal{F}))$ is non-trivial, then there is an open set U containing x for which $\mathcal{F}|_U$ is not universally locally acyclic with respect to f. At the present moment we do not know how to prove this.

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