THE UNIVERSITY OF CHICAGO

FINDING LARGE MINIMAL HYPERSURFACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
JAMES TAYLOR STEVENS

CHICAGO, ILLINOIS
JUNE 2022

Dedicated to my mom and grandparents.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I am extremely grateful to my advisor André Neves for his invaluable advice, guidance, and patience over the past five years. Also, I would like to thank Ao Sun for his help and encouragement during this project. I am also grateful to Matthew Macauley for his early support during my undergraduate studies. Finally, I want to thank my friends and family for all their love and for being so supportive and understanding since the beginning.

ABSTRACT

We show that for any closed Riemannian manifold with dimension between 3 and 7, either there are minimal hypersurfaces with arbitrarily large area, or the space of certain pathological-looking minimal hypersurfaces has a Cantor set structure. In particular, among the applications, we prove that there exist minimal hypersurfaces with arbitrarily large area in analytic manifolds. The proof uses the Almgren-Pitts min-max theory proposed by Marques-Neves, the ideas developed by Song in his proof of Yau's conjecture, and the resolution of the generic multiplicity-one conjecture by Zhou. Many of the results in this thesis are from joint work with Ao Sun [StSu].

CHAPTER 1

INTRODUCTION

Background and results

Minimal surfaces are critical points of the area functional. They are fundamental geometric models in the calculus of variations that play important roles in the study of geometry, topology, and general relativity. For instance, stable minimal surfaces found from area-minimization have been used by Schoen-Yau to study manifolds with nonnegative scalar curvature [SY2] and to settle the positive mass theorem [SY3]. More recently, the study of unstable minimal surfaces by min-max theory has been used by Colding-Minicozzi [CoM1] to give a proof finite extinction time of the Ricci flow on homotopy 3-spheres, and used by Marques-Neves [MN2] in the resolution of the Willmore conjecture.

In 1982, Yau [Ya] conjectured that there exist infinitely-many minimal hypersurfaces in a closed Riemannian manifold. This was inspired by the then-recent work of Almgren-Pitts [Pi] whom developed a min-max theory which at least guaranteed the existence of a single minimal hypersurface. After further developing this min-max theory in order to solve the Willmore conjecture, Marques-Neves moved on to studying Yau's problem. When the ambient manifold has dimension between 3 and 7 (due to the regularity theory of Schoen-Simon [ScSi]), this conjecture was recently solved by Song [So1], building on the work of Marques-Neves in [MN1].

The purpose of this thesis is to provide further delicate information on the space of minimal hypersurfaces. Throughout, let (M^{n+1}, g) be a closed Riemannian manifold with $3 \leq n+1 \leq 7$, and all minimal hypersurfaces we consider will be smooth, closed, and embedded. Our main result Theorem 1.2 shows that a manifold either admits minimal hypersurfaces with arbitrarily large area, or it carries a very pathological metric described below. In particular, analytic metrics will not occur in this pathological case.

Theorem 1.1. *Let (M^{n+1}, g) be a closed analytic Riemannian manifold with $3 \leq n+1 \leq 7$. Then there exists connected minimal hypersurfaces with arbitrarily large area.*

Although we now know there exist infinitely-many minimal hypersurfaces, it is not immediately clear whether they must have arbitrarily large area. For example, this is it not true for embedded geodesics on a surface ($n+1=2$). Prior to this work, Chodosh-Mantoulidis [ChM] showed there exist arbitrarily large minimal hypersurfaces in *bumpy metrics* by using min-max theory along with the multiplicity-one conjecture proven by Zhou [Zh]. Bumpy metrics are the class of metrics which make the area functional Morse in the sense that every critical point of the area functional (minimal hypersurface) is nondegenerate (in terms of the Jacobi operator).

White [Wh1] showed that bumpy metrics are generic (in the Baire sense) just like analytic metrics. However, we do not actually know of any explicit examples of bumpy metrics because it is such a difficult condition to verify. In fact, many metrics we care about will have symmetry which often creates degeneracy of minimal hypersurfaces. On the other hand, many metrics we care about are analytic, in which case, our result holds.

Now, we will state the main theorem for general metrics. It says that when our method of finding large minimal hypersurfaces fails, then not only does there exist pathological-looking minimal hypersurfaces as pictured in Figure 1 and defined in Section 5.3, the space of all these non-monotonic minimal hypersurfaces is pathological as well. In Appendix B, we give an example of a smooth metric where this pathological case happens.

Theorem 1.2. *Let (M^{n+1}, g) be a closed Riemannian manifold with $3 \leq n+1 \leq 7$. Either*

1. *there exists connected minimal hypersurfaces with arbitrarily large area, or*
2. *the space of non-monotonic minimal hypersurfaces is homeomorphic to a Cantor set.*

We will find our minimal hypersurfaces by studying the non-linear volume spectrum considered by Gromov [Gr1, Gr2] and later developed by Guth [Gu] and Marques-Neves

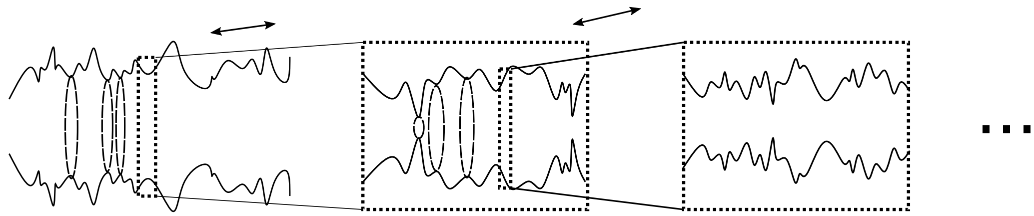

Figure 1.1: This diagram gives an idea of the fractal-like behavior demonstrated by these pathological non-monotonic minimal hypersurfaces.

[MN1] using min-max theory. This volume spectrum is a nondecreasing sequence of values ω_p tending to infinity which are realized as the area of some minimal hypersurface (possibly has multiple connected components each with a multiplicity) coming from a min-max process over certain homotopy classes of maps into the topologically-rich space of hypercycles.

Although these widths ω_p tend to infinity, the main issue is that the minimal hypersurfaces we find from min-max theory could possibly have many components and/or high multiplicities. We resolve the problem of many connected components by defining certain generalizations of the Frankel property from [MN1] and cores from [So1]. On the other hand, by the multiplicity-one theorem resolved by Zhou [Zh], higher multiplicities only occur for degenerate stable minimal hypersurfaces. So we closely study the space of stable minimal cycles in Section 3.2, and show that if min-max theory produces stable minimal hypersurfaces, then the widths will grow in a way which violates the Weyl law [LMN].

Organization

In Section 2, we give preliminary tools and definitions needed along with an overview of the proof. In Section 3, we find large minimal hypersurfaces in weakly Frankel manifolds by using Lusternik–Schnirelmann arguments on the space of stable minimal cycles. In Section 4, find large minimal hypersurfaces in weak cores by using the constrained min-max theory of Song. In Section 5, we discuss coreless manifolds, and prove the main Theorem 1.2. In Section 6, we sketch the standard technical modifications needed to handle one-sided minimal hypersurfaces. In Section 7, we discuss some applications to both analytic manifolds and positive scalar curvature 3-manifolds. Finally, in the appendix, we prove some lemmas about tubular neighborhoods of minimal hypersurfaces, and we also give an example of the pathological case from Theorem 1.2. Much of these results are joint work with Ao Sun from our paper [StSu].

CHAPTER 2

PRELIMINARIES

In Sections 2 through 5, we will assume that (M, g) contains no one-sided hypersurfaces to make the exposition clearer. But in Section 6, we describe the standard technical modifications necessary to handle the case when there are one-sided hypersurfaces.

2.1 Minimal hypersurfaces

Properties of minimal hypersurfaces

We briefly review some standard definitions and results on minimal hypersurfaces. More details on most of this material can be found in [CoM2]. Let Γ be a smooth, two-sided, closed, embedded hypersurface in M. We say Γ is a *minimal hypersurface* of (M^{n+1}, g) if it is a critical point for the n-dimensional area functional. This means that for every smooth vector field X on M,

$$\delta\Gamma(X) := \frac{d}{dt}\bigg|_{t=0} \mathrm{area}_g(\varphi_t(\Gamma)) = 0. \tag{2.1}$$

where $\{\varphi_t\}_{t\in\mathbb{R}}$ is a one-parameter family of diffeomorphisms on M generated by X. Moreover, there is a first variation formula which says that

$$\delta\Gamma(X) = -\int_\Gamma \langle X, \mathbf{H}\rangle$$

where $\mathbf{H}$ is the mean curvature vector of Γ. So the mean curvature vector $\mathbf{H}$ can be interpreted as the (negative) gradient of the area functional. In particular, Γ is a minimal hypersurface for (M, g) if and only if its mean curvature vector $\mathbf{H}$ vanishes identically.

We say that a minimal hypersurface Γ is *stable* if for all vector fields X,

$$\delta^2\Gamma(X, X) := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{area}_g(\varphi_t(\Gamma)) \geq 0.$$

Since we are assuming Γ is two-sided, we can pick a choice of unit normal ν on Γ. Consider *normal* variations which are generated by (extensions of) vector fields of the form $X = f\nu$ for $f \in C^\infty(\Gamma)$. The second variation formula says that for such normal variations,

$$\delta^2\Gamma(X,X) = \int_\Gamma |\nabla f|^2 - |A|^2 f^2 - \mathrm{Ric}_M(\nu,\nu) f^2 \tag{2.2}$$

where ∇ is the gradient for Γ, A is second fundamental form on Γ, and Ric_M denotes the Ricci curvature tensor on M. After integrating (2.2) by parts, we obtain

$$\delta^2\Gamma(X,X) = -\int_\Gamma f\Delta f + |A|^2 f^2 + \mathrm{Ric}_M(\nu,\nu) f^2 = \int_\Gamma f L_\Gamma(f)$$

where Δ is the Laplacian on Γ and $L_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is known as the *Jacobi operator*:

$$L_\Gamma(f) = -\Delta f - (|A|^2 + \mathrm{Ric}_M(\nu,\nu))f.$$

The Jacobi operator is a self-adjoint, second-order, linear, elliptic operator. Since L_Γ equals $-\Delta$ minus the zeroth-order term $|A|^2 + \mathrm{Ric}_M(\nu,\nu)$, then similar to the spectrum of the (negative) Laplacian, the spectrum of L_Γ is discrete, bounded from below, and each eigenspace is finite-dimensional and orthogonal to each other. In fact, the smooth eigenfunctions form an orthonormal basis for $L^2(\Gamma)$. For details, see [CoM2, Section 1.8.3].

We let $\mathrm{index}_g(\Gamma)$ denote the *Morse index* of a minimal hypersurface Γ which can be defined as the number of negative eigenvalues of L_Γ. So a stable minimal hypersurface has $\mathrm{index}_g(\Gamma) = 0$, while we call a minimal hypersurface with $\mathrm{index}_g(\Gamma) \geq 1$ *unstable*. A minimal hypersurface Γ is *degenerate* if the Jacobi operator L_Γ has a non-trivial kernel. The functions in the kernel of L_Γ are called *Jacobi fields* of Γ.

We say that a metric $(M, \tilde{g})$ is *bumpy* if no immersed minimal hypersurfaces are degenerate. So bumpy metrics are the metrics which make the $\mathrm{area}_{\tilde{g}}$ functional Morse in a sense.

Just like how a generic smooth function is Morse, White [Wh1] was able to show the space of bumpy metrics is Baire-generic (hence is dense) in the space of smooth metrics. Despite the abundance of such metrics, we do not know of any explicit examples of bumpy metrics because the defining condition is too difficult to check.

By standard elliptic theory [GT], we can say more about this spectrum. For instance, the least eigenvalue λ of the Jacobi operator has the following variational characterization:

$$\lambda = \inf_{W^{1,2}(\Gamma)\setminus\{0\}} \frac{\int_\Gamma |\nabla f|^2 - |A|^2 f^2 - \mathrm{Ric}_M(\nu,\nu) f^2}{\int_\Gamma f^2}. \tag{2.3}$$

Let $R_\Gamma(f)$ denote the expression inside the infimum above. One can show if $R_\Gamma(f) = \lambda$ so that f gives a weak solution to $L_\Gamma(f) = \lambda f$, then f must be smooth by a standard bootstrapping argument [CoM2, Lemma 1.34].

Lemma 2.1.1. *The least eigenvalue λ of L_Γ has multiplicity one. Moreover, if f is the corresponding eigenfunction so that $L_\Gamma(f) = \lambda f$, then f does not change sign.*

This follows because if $R_\Gamma(f) = \lambda$ for $f \in W^{1,2}(M)$, then have $R_\Gamma(|f|) = \lambda$ as well where also $|f| \in W^{1,2}(M)$. Thus, by the variational characterization (2.3), $|f|$ is smooth and satisfies $L_\Gamma(|f|) = \lambda|f|$. Since $|f| \geq 0$ is smooth, the Harnack inequality [CoM2, Lemma 1.46]) implies that $|f| > 0$. Thus, no eigenfunction of the least eigenvalue λ changes sign.

Finally, note the eigenspace must be one-dimensional because if not, we could take two L^2-orthogonal eigenfunctions f_1, f_2 for the least eigenvalue λ so that

$$\int_\Gamma f_1 f_2 = \langle f_1, f_2\rangle_{L^2(\Gamma)} = 0$$

which cannot happen if both f_1 and f_2 do not change sign.

Corollary 2.1.2. *If Γ has a Jacobi field which changes sign, then Γ is unstable.*

Note Lemma 2.1.1 depended on the celebrated Harnack inequality from elliptic theory.

The strong maximum principle [GT, Section 3.2] is another fundamental elliptic result which used to prove the following.

Proposition 2.1.3 (Maximum principle)**.** *Let* Γ_0 *and* Γ_1 *be connected minimal hypersurfaces in* (M, g)*. If* Γ_0 *lies on one side of* Γ_1 *locally at some point of intersection, then* $\Gamma_0 = \Gamma_1$*.*

The idea of this result is that we can locally express Γ_0 and Γ_1 as functions u_0 and u_1 such that $u_1 - u_0 \geq 0$ which satisfies a second-order linear elliptic PDE stemming the linearization of the non-linear elliptic minimal hypersurface equation (see [CoM2, Section 7.1]). Then the classical strong maximum principle implies that $u_1 - u_0 = 0$, and hence, $\Gamma_0 = \Gamma_1$. Note that there are more general maximum principles [Wh2] as well.

A consequence of the curvature estimates proven in [ScSi] is that the space of stable minimal hypersurfaces with bounded area is compact in the smooth topology. Motivated by this, Sharp [Sh] showed that replacing stable assumption with bounded index then gives a similar compactness result. Note stable minimal hypersurfaces are the case where $\text{index}_g(\Gamma) = 0$.

Proposition 2.1.4 ([Sh, Theroem 2.3])**.** *Let* $\Gamma_i \subset (M^{n+1}, g)$ *be a sequence of connected, closed, embedded, minimal hypersurfaces where* $3 \leq n+1 \leq 7$ *and such that for all* $i \in \mathbb{N}$*,*

$$\text{area}_g(\Gamma_i) \leq A \quad \textit{and} \quad \text{index}_g(\Gamma_i) \leq I.$$

Then after taking a subsequence, there exists a connected, closed, embedded, minimal hypersurface $\Gamma \subset (M, g)$ *along with an integer multiplicity* $m \in \mathbb{N}$ *such that*

$$m\,\text{area}_g(\Gamma) \leq A \quad \textit{and} \quad \text{index}_g(\Gamma) \leq I$$

and where $\Gamma_i \to m\Gamma$ *in the varifold sense. In fact, this convergence is smooth and graphical away from a finite set of size at most* I*. Moreover, if* Γ *is two-sided, then:*

- *if the multiplicity is* $m = 1$*, then the convergence is smooth everywhere, and:*

- *if Γ_i and Γ are disjoint eventually, then Γ must be degenerate stable*

- *if Γ_i and Γ are intersect eventually, then Γ must be unstable*

- *if the multiplicity is $m \geq 2$, then Γ must be degenerate stable.*

The last part follows from the smooth and graphical convergence which allows us to explicitly produce Jacobi fields on Γ by a suitable renormalization of the signed distance between the sheets in the graphs as done in [Si2]. Sharp [Sh, Theroem A.6] also showed a similar result for varying metrics, that is, it applies for sequences $\Gamma_i \subset M$ such that Γ_i is minimal in (M, g_i) and where the metrics $g_i \to g$ smoothly.

Proposition 2.1.5 (Monotonicity formula)**.** *There exists $c, r_0 > 0$ such that for any minimal hypersurface $\Gamma \subset (M, g)$ and point $p \in \Gamma$, the quantity given by*

$$\exp(cr^2)\frac{\text{area}_g(\Gamma \cap B_r(p))}{\text{vol}(B_r(0) \subset \mathbb{R}^n)}$$

is nondecreasing for all $r \leq r_0$.

As a consequence of the monotonicity formula, observe that

$$\exp(cr_0^2)\frac{\text{area}_g(\Gamma \cap B_{r_0}(p))}{\text{vol}(B_{r_0}(0) \subset \mathbb{R}^n)} \geq \lim_{r \to 0^+} \exp(cr^2)\frac{\text{area}_g(\Gamma \cap B_r(p))}{\text{vol}(B_r(0) \subset \mathbb{R}^n)} = 1$$

which then implies that for all minimal hypersurfaces Γ,

$$\text{area}_g(\Gamma) \geq \text{area}_g(\Gamma \cap B_{r_0}(p)) \geq \frac{\text{vol}(B_{r_0}(0) \subset \mathbb{R}^n)}{\exp(cr_0^2)}.$$

Thus, the area of closed minimal hypersurfaces $\Gamma \subset (M, g)$ is uniformly bounded from below.

Remark 2.1.6. More generally, the monotonicity formula holds for so called *stationary varifolds* (see [All, Si3]) which we will discuss in Section 2.2.

Tubular neighborhoods of minimal hypersurfaces

We will now be interested in certain nice foliations of a tubular neighborhood of a minimal hypersurface.

Definition 2.1.7. Let Γ be a minimal hypersurface in (M, g), and suppose that $\varphi : \Gamma \times [-\delta, \delta] \to M$ is a diffeomorphism onto its image such that $\varphi(x, 0) = x$ for all $x \in \Gamma$. Consider the foliation $\{\Gamma_t\}_{t\in[0,\delta]}$ of the closed halved tubular neighborhood $\varphi(\Gamma \times [0, \delta])$ given by $\Gamma_t := \varphi(\Gamma \times \{t\})$. We say the neighborhood $\varphi(\Gamma \times [0, \delta])$ is

- *contracting*: if $\mathbf{H}(\Gamma_t)$ is strictly non-vanishing and points towards Γ for all $t \in (0, \delta]$;
- *expanding*: if $\mathbf{H}(\Gamma_t)$ is strictly non-vanishing and points away from Γ for all $t \in (0, \delta]$;
- *foliated by minimal hypersurfaces*: if Γ_t is a minimal hypersurface all $t \in (0, \delta]$; and
- *accumulating*: if for each $t \in (0, \delta]$, either Γ_t is minimal or $\mathbf{H}(\Gamma_t)$ is strictly non-vanishing; and moreover, exist arbitrarily small $r, s \in (0, \delta]$ such that Γ_r is minimal and $\mathbf{H}(\Gamma_s)$ is strictly non-vanishing.

Similarly, we can define each of the above for the other closed halved tubular neighborhood $\varphi(\Gamma \times [-\delta, 0])$ by replacing each $(0, \delta]$ above with $[-\delta, 0)$.

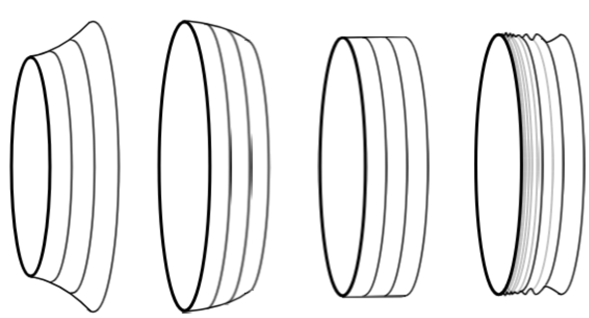

Figure 2.1: From left to right: model of a contracting neighborhood, an expanding neighborhood, a neighborhood foliated by minimal hypersurfaces, and an accumulating neighborhood.

It was noted in [So1, Lemma 10][1] that each side of a minimal hypersurface must be given by one the four neighborhoods above and is essentially generated by flowing Γ along the first eigenfunction of L_Γ (recall this eigenfunction is positive). For the reader's convenience, we give a proof of this below fact in Appendix A.

Lemma 2.1.8. *Let $\Gamma \subset (M, g)$ be a minimal hypersurface. There exists a diffeomorphism $\varphi : \Gamma \times [-\delta, \delta] \to M$ onto a tubular neighborhood of Γ such that $\varphi(x, 0) = x$ for all $x \in \Gamma$ and each closed halved tubular neighborhood is one of the types from Definition 2.1.7.*

As expected, unstable minimal hypersurfaces will always be expanding on both sides, while nondegenerate stable minimal hypersurfaces will always be contracting on both sides. However, in the case of degenerate stable minimal hypersurfaces, any of the four above neighborhood types in Definition 2.1.7 can happen independently on either side.

We will see that considering contracting minimal hypersurfaces are vital to our approach. The following lemma gives us a way to find these contracting minimal hypersurfaces.

Lemma 2.1.9. *Let (N^{n+1}, g) be a compact manifold with minimal boundary ∂N and $3 \leq n + 1 \leq 7$. If $\Sigma \subset \partial N$ is a non-contracting boundary component such that*

$$\text{area}_g(\Sigma) \leq \text{area}_g(\partial N \setminus \Sigma),$$

then exactly one of the following must be true: either

1. *there exists a minimal $\Gamma \subset \text{int}(N)$ which is contracting on a side; or*
2. *the manifold (N, g) is foliated by minimal hypersurfaces of the form $\Sigma_t = \varphi(\Sigma \times \{t\})$ for some diffeomorphism $\varphi : \Sigma \times [0, 1] \to N$ such that $\Sigma_0 = \Sigma$.*

1. Song only needed to define and consider expanding and contracting neighborhoods because he assumed his manifold only had finitely many stable minimal hypersurfaces.

Proof. Let $\Sigma \subset \partial N$ be a non-contracting minimal hypersurface with $\text{area}_g(\Sigma) \leq \text{area}_g(\partial N \setminus \Sigma)$. Note that if Σ is accumulating, then there must be a contracting minimal hypersurface nearby so that we are in the first case.

Thus, we can assume Σ is either expanding or locally foliated by minimal hypersurfaces. In either case, by compactness [ScSi] and Lemma 2.1.8, there exists some $\varepsilon \geq 0$ along with a diffeomorphism $\varphi : \Sigma \times [0, \varepsilon] \to N$ onto its image such that $\varphi(\Sigma \times \{0\}) = \Sigma$, the hypersurfaces $\varphi(\Sigma \times \{t\})$ are minimal for all $t \in [0, \varepsilon]$, and which cannot be extended as a minimal foliation any further. By the maximum principle, either $\Sigma_\varepsilon = \varphi(\Sigma \times \{\varepsilon\})$ is contained in $\text{int}(N)$ or is contained in ∂N. If $\Sigma_\varepsilon \subset \partial N$, then either $\varepsilon = 0$ or we must be in the second case where N is foliated by minimal hypersurfaces.

Now assume that either $\Sigma_\varepsilon \subset \text{int}(N)$ or $\Sigma_\varepsilon = \Sigma$. By construction, Σ_ε is not foliated by minimal hypersurfaces outside of $\varphi(\Sigma \times [0, \varepsilon])$. Moreover, if Σ_ε is contracting or accumulating on this side then we are in the first case. So finally assume that Σ_ε is expanding on this side. Thus, if we minimize area in the homology class $[\Sigma] = [\Sigma_\varepsilon] \in H_n(N; \mathbb{Z})$, we find a stable minimal hypersurface Γ (with possibly multiple components) in N such that

$$\text{area}_g(\Gamma) < \text{area}_g(\Sigma_\varepsilon) = \text{area}_g(\Sigma)$$

since Σ_ε is expanding on a side. By the maximum principle, each component of Γ is either contained in $\text{int}(N)$ or $\partial N \setminus \Sigma$. There must be at least one connected component Γ' of Γ which is contained in $\text{int}(N)$. Otherwise if $\Gamma \subseteq \partial N \setminus \Sigma$, then $\Gamma = \partial N \setminus \Sigma$ since $[\Gamma] = [\Sigma] = [\partial N \setminus \Sigma] \in H_n(N; \mathbb{Z})$ which gives a contradiction to

$$\text{area}_g(\Gamma) < \text{area}_g(\Sigma) \leq \text{area}_g(\partial N \setminus \Sigma).$$

Thus, let Γ' be a connected component of Γ contained in $\text{int}(N)$. Like above, there exists some $\delta \geq 0$ along with a diffeomorphism $\varphi' : \Gamma' \times [0, \delta] \to N$ onto its image such that

$\varphi'(\Gamma' \times \{0\}) = \Gamma'$, the hypersurfaces $\varphi'(\Gamma' \times \{t\})$ are minimal for all $t \in [0, \delta]$, and which cannot be extended as a minimal foliation any further. By construction, Γ'_δ must be locally area minimizing outside of $\varphi'(\Gamma' \times [0, \delta])$. Thus, Γ'_δ is either contracting or accumulating on this side putting us in the second case. □

2.2 Geometric measure theory

In this section, we cover the basics needed from geometric measure theory. For more details, see [Si3, MN2].

Hypercycles and the flat norm

Let $\mathcal{P}_k(\mathbb{R}^L)$ denote the set of *polyhedral k-chains mod 2*, that is, formal sums of compact, convex, k-dimensional polyhedra $\sigma \subset \mathbb{R}^L$ under the identification that $\sigma + \sigma = 0$ and that $\sigma = \sigma_1 + \sigma_2$ whenever σ is formed by gluing nonoverlapping σ_1 and σ_2. Let $\mathrm{spt}(P)$ denote the support of $P \in \mathcal{P}_k(\mathbb{R}^L)$. Given a compact set $K \subset \mathbb{R}^L$, define

$$\mathcal{P}_k(K) = \{P \in \mathcal{P}_k(\mathbb{R}^L) : \mathrm{spt}(P) \subseteq K\}.$$

We can define the *mass norm* $\mathbf{M}$ on $\mathcal{P}_k(K)$ to be the k-dimensional volume $\mathbf{M}(P) = \mathrm{vol}_k(\mathrm{spt}(P))$ of the chain $P \in \mathcal{P}_k(K)$. As natural as the mass norm seems, it is usually too strong of a topology to work with, as the following example illustrates.

Example 2.2.1. The boundary map $\partial : \mathcal{P}_k(\mathbb{R}^L) \to \mathcal{P}_{k-1}(\mathbb{R}^L)$ not $\mathbf{M}$-continuous. For $\varepsilon \geq 0$, consider the chain $P_\varepsilon \in \mathcal{P}_2(\mathbb{R}^2)$ given by $P_\varepsilon = [0,1] \times [0,\varepsilon]$. Observe that $P_0 = 0$ in $\mathcal{P}_2(\mathbb{R}^2)$ and so $\partial P_0 = 0$ in $\mathcal{P}_1(\mathbb{R}^2)$. However, $\mathbf{M}(P_\varepsilon) = \varepsilon$ while $\mathbf{M}(\partial P_\varepsilon) = 2 + 2\varepsilon$ which implies that

$$\mathbf{M}(P_\varepsilon - P_0) \to 0, \text{ but } \mathbf{M}(\partial P_\varepsilon - \partial P_0) \to 2.$$

So although P_ε converges to P_0, the boundaries ∂P_ε do not converge to ∂P_0 in the $\mathbf{M}$-norm.

A much better norm for geometric purposes is the *flat norm* defined as

$$\mathcal{F}(P) = \inf\left\{\mathbf{M}(R) + \mathbf{M}(P - \partial R) : R \in \mathcal{P}_{k+1}(\mathbb{R}^L)\right\}.$$

One can show that for any $P \in \mathcal{P}_k(\mathbb{R}^L)$, we have the inequalities

$$\mathcal{F}(\partial P) \leq \mathcal{F}(P) \leq \mathbf{M}(P).$$

The first inequality implies the boundary map $\partial : \mathcal{P}_k(\mathbb{R}^L) \to \mathcal{P}_{k-1}(\mathbb{R}^L)$ is $\mathcal{F}$-continuous, while the second inequality confirms that the flat topology is weaker than the mass topology.

We define the space of *flat k-chains mod 2* to be

$$\mathcal{F}_k(\mathbb{R}^L) = \bigcup_{\substack{K \subset \mathbb{R}^L \\ \text{compact}}} \overline{\mathcal{P}_k(K)}$$

where the closures are in the flat topology. By construction, flat chains have compact support and the notions of mass, boundary, and support of a flat chain can be defined by taking approximations by polyhedra chains. Now, define *rectifiable flat k-chains mod 2* as

$$\mathcal{I}_k(\mathbb{R}^L) = \{T \in \mathcal{F}_k(\mathbb{R}^L) : \mathbf{M}(T), \mathbf{M}(\partial T) < \infty\},$$

and similar to before, given a compact set $K \subset \mathbb{R}^L$, we denote

$$\mathcal{I}_k(K) = \{T \in \mathcal{I}_k(\mathbb{R}^L) : \operatorname{spt}(T) \subseteq K\}.$$

Proposition 2.2.2 ([FF]). *Let $C > 0$ and $K \subset \mathbb{R}^L$ compact. The set*

$$\{T \in \mathcal{I}_k(K) : \mathbf{M}(T), \mathbf{M}(\partial T) \leq C\}$$

is compact in the flat topology.

We define the *rectifiable flat k-cycles mod 2* to be the space $\tilde{\mathcal{Z}}_k(K) \subset \mathcal{I}_k(K)$ consisting of $T \in \mathcal{I}_k(K)$ such that $\partial T = 0$, that is, the kernel of $\partial : \mathcal{I}_k(K) \to \mathcal{I}_{k-1}(K)$. And we define the *rectifiable flat k-boundaries mod 2* to be the space $\mathcal{Z}_k(K) \subset \mathcal{I}_k(K)$ consisting of $T \in \mathcal{I}_k(K)$ such that $T = \partial R$ for some $R \in \mathcal{I}_{k+1}(K)$, that is, the image of $\partial : \mathcal{I}_{k+1}(K) \to \mathcal{I}_k(K)$. Since $\partial\partial T = 0$ for any chain T, we see that $\mathcal{Z}_k(K) \subseteq \tilde{\mathcal{Z}}_k(K)$. In fact, $\mathcal{Z}_k(K)$ is the connected component of $\tilde{\mathcal{Z}}_k(K)$ which contains the 0 cycle.

Now given some closed Riemannian manifold (M^{n+1}, g), we can isometrically embed $M \subset \mathbb{R}^L$ for some L by Nash's theorem. We will be interested in two particular spaces: $\mathcal{I}_{n+1}(M)$ and $\mathcal{Z}_n(M)$. The set of top-dimensional chains $\mathcal{I}_{n+1}(M)$ are equivalent to Caccioppoli sets (regions with finite perimeter), while we will think of the codimension-one unoriented boundaries $\mathcal{Z}_n(M)$ as being the space of hypersurfaces on M. We will refer to $\mathcal{Z}_n(M)$ as the space of *hypercycles*[2] and on M.

Lemma 2.2.3. *The space of top-dimensional rectifiable chains $\mathcal{I}_{n+1}(M)$ is contractible.*

Proof. We can define the flat continuous homotopy $H : \mathcal{I}_{n+1}(M) \times [0,1] \to \mathcal{I}_{n+1}(M)$ by

$$H(\Omega, t) = \Omega \llcorner \{f \leq t\}$$

where $f : M \to [0,1]$ is any choice of Morse function. □

2. We call this space of rectifiable boundaries the space of hypercycles and use the notation $\mathcal{Z}_k(K)$ (instead of $\mathcal{B}_k(K)$) is because this is standard in the literature.

Proposition 2.2.4 (Constancy theorem [Si3, Theorem 26.27])**.** *Let* $U \in \mathcal{I}_{n+1}(M)$*. Then*

$$\partial U = 0 \quad \textit{if and only if} \quad U = 0 \textit{ or } U = M.$$

The constancy theorem implies that the flat-continuous surjection $\partial : \mathcal{I}_{n+1}(M) \to \mathcal{Z}_n(M)$ is 2–to–1. In fact, we can show is a covering map using the following isoperimetric inequality.

Proposition 2.2.5 (Isoperimetric inequality [Si3, Theorem 30.1])**.** *There exists* $\delta, c > 0$ *where*

$$\mathbf{M}(U) \leq c\,\mathbf{M}(\partial U)^{\frac{n+1}{n}}$$

for every $U \in \mathcal{I}_{n+1}(M)$ *with* $\mathbf{M}(\partial U) \leq \delta$ *and* $\mathbf{M}(U) \leq \mathrm{vol}_g(M)/2$*.*

The next lemma is indispensable in understanding the topology of $\mathcal{Z}_n(M)$.

Lemma 2.2.6 ([MN3, Section 5])**.** *The boundary* $\partial : \mathcal{I}_{n+1}(M) \to \mathcal{Z}_n(M)$ *is a* 2*-cover.*

Proof. Consider a cycle $T \in \mathcal{Z}_n(M)$. By the Constancy Theorem, $\partial^{-1}(\{T\}) = \{U_1, U_2\}$ for some $U_1 \in \mathcal{I}_{n+1}(M)$ and where $U_2 = M - U_1$. We first will show that

$$\partial^{-1}(B_\varepsilon(T)) = B_\varepsilon(U_1) \cup B_\varepsilon(U_2) \tag{2.4}$$

for small enough $\varepsilon > 0$ where B_ε represent ε-balls in the flat metric. First, take $\varepsilon < \mathrm{vol}(M)/2$ so that $B_\varepsilon(U_1)$ and $B_\varepsilon(U_2)$ are forced to be disjoint since $\mathcal{F}(U_1 - U_2) = \mathcal{F}(M) = \mathrm{vol}_g(M)$.

Suppose $S \in B_\varepsilon(T)$ so that $\mathcal{F}(S - T) < \varepsilon$. Then there exists $W \in \mathcal{I}_{n+1}(M)$ such that $\mathbf{M}(W) + \mathbf{M}(S - T - \partial W) < \varepsilon$. Since $S - T - \partial W \in \mathcal{Z}_n(M)$, there is a $W' \in \mathcal{I}_{n+1}(M)$ with

$$\partial W' = S - T - \partial W \quad \text{where} \quad \mathbf{M}(W') \leq \mathrm{vol}_g(M)/2.$$

If we also take $\varepsilon \leq \min(\delta, c^{-n})$ where these are from the isoperimetric inequality,

$$\mathbf{M}(W') \leq c\,\mathbf{M}(S - T - \partial W)^{\frac{n+1}{n}} \leq \mathbf{M}(S - T - \partial W).$$

Take $W_i = W' + W + U_i$ so that $\partial W_i = \partial(W' + W + U_i) = S$ where

$$\mathcal{F}(W_i - U_i) \leq \mathbf{M}(W') + \mathbf{M}(W) \leq \mathbf{M}(S - T - \partial W) + \mathbf{M}(W) < \varepsilon,$$

that is, $W_i \in B_\varepsilon(U_i)$ for $i = 1, 2$. Conversely, if we take $W_i \in B_\varepsilon(U_i)$, then we have

$$\mathcal{F}(\partial W_i - T) = \mathcal{F}(\partial(W_i - U_i)) \leq \mathcal{F}(W_i - U_i) < \varepsilon$$

which shows the other inclusion holds. Thus, equation (2.4) holds. Finally, note we can show this flat-continuous bijection $\partial : B_\varepsilon(T) \to B_\varepsilon(U_1) \cup B_\varepsilon(U_2)$ is a homeomorphism using compactness Proposition 2.2.2. Therefore, $\partial : \mathcal{I}_{n+1}(M) \to \mathcal{Z}_n(M)$ is a covering map. $\square$

Corollary 2.2.7. *The space of hypercycles $\mathcal{Z}_n(M)$ is a $K(\mathbb{Z}_2, 1)$ Eilenberg–Maclane space.*

In particular, $\mathcal{Z}_n(M)$ is weakly homotopy equivalent to $\mathbb{RP}^\infty$. In fact, for any Morse function $f : M \to [0, 1]$, the map $\Lambda_f : \mathbb{RP}^\infty \to \mathcal{Z}_n(M)$ given by

$$\Lambda_f([a_0, a_1, \ldots, a_k, 0, 0, \ldots]) = \partial\{x \in M : a_0 + a_1 f(x) + \cdots + a_k f(x)^k \leq 0\}$$

gives an explicit weak homotopy equivalence.

Remark 2.2.8. More generally, Almgren proved in his PhD thesis [Alm1] that

$$\pi_\ell(\mathcal{Z}_k(M; G), 0) \simeq H_{k+\ell}(M; G)$$

where G is a normed abelian group. Indeed, in our case where $G = \mathbb{Z}_2$ and $k = n$,

$$\pi_\ell(\mathcal{Z}_n(M), 0) \simeq H_{n+\ell}(M^{n+1}; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \ell = 1, \\ 0 & \ell \geq 2, \end{cases}$$

which again confirms that $\mathcal{Z}_n(M)$ is a $K(\mathbb{Z}_2, 1)$ space.

In particular, this implies the cohomology ring $H^*(\mathcal{Z}_n(M); \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\bar{\lambda}]$ generated by what we call the *fundamental class* $\bar{\lambda} \in H^1(\mathcal{Z}_n(M); \mathbb{Z}_2)$.

Proposition 2.2.9. *Maps $\Phi_0, \Phi_1 : X \to \mathcal{Z}_n(M)$ are homotopic in the flat topology if and only if the pullbacks $\Phi_0^*(\bar{\lambda}) = \Phi_1^*(\bar{\lambda})$ in $H^1(X; \mathbb{Z}_2)$.*

Let $[X, \mathcal{Z}_n(M)]$ denote the (free) homotopy classes of maps $X \to \mathcal{Z}_n(M)$ in the flat topology. Thus, Proposition 2.2.9 says that the following map is a natural bijection:

$$[X, \mathcal{Z}_n(M)] \to H^1(X; \mathbb{Z}_2) \quad \text{given by} \quad [\Phi] \mapsto \Phi^*(\bar{\lambda})$$

Varifolds and the **F**-*norm*

We will need to more structure than just the flat norm. Although the mass norm is not flat-continuous, it is lower semicontinuous in the flat topology, that is,

$$\mathcal{F}(T_i - T) \to 0 \quad \text{implies} \quad \mathbf{M}(T) \leq \liminf_{i\to\infty} \mathbf{M}(T_i). \tag{2.5}$$

This possibility of losing mass in a flat-limit is not an issue if we are minimizing the mass of chains (e.g., solving the Plateau problem or minimizing area in a homology class). However, the hypersurfaces we want to find in minmax theory are necessarily non-minimizing. We expand on some of these issues with the flat topology in Remark 2.3.3.

The extra variational structure needed is that of *varifolds* (see [Si3, Chapter 4] for details) first introduced by Almgren [Alm2]. A n-varifold on M is a Radon measure on the

Grassmannian bundle $\mathbf{G}_n(M^{n+1})$ of hyperplanes tangent to M. The space of varifolds come equipped with the topology induced by *weak convergence*: we say $V_i \to V$ as varifolds if

$$\lim_{i\to\infty} \int_{\mathbf{G}_n(M)} \varphi(x,\pi)\ dV_i(x,\pi) = \int_{\mathbf{G}_n(M)} \varphi(x,\pi)\ dV(x,\pi)$$

for all compactly supported functions $\varphi \in C_c(\mathbf{G}_n(M))$. We let $\|V\|$ denote the induced measure on M, and call $\|V\|(M)$ the *mass* of V. Given a diffeomorphism $\psi : M \to M$ and a varifold V on M, we can define the pushforward varifold $\psi_\sharp V$ on M which satisfies

$$\int_{\mathbf{G}_n(M)} \varphi(y,\sigma)\ d(\psi_\sharp V)(y,\sigma) = \int_{\mathbf{G}_n(M)} J\psi(x,\pi)\varphi(\psi(x), d\psi_x(\pi))\ dV(x,\pi)$$

where $J\psi(x,\pi)$ denotes the Jacobian of ψ when restricted to the hyperplane $d\psi_x(\pi)$.

As for hypersurfaces (2.1), we can define the *first variation* for a vector field X on M as

$$\delta V(X) := \left.\frac{d}{dt}\right|_{t=0} \|(\psi_t)_\sharp V\|(M)$$

where ψ_t is the isotopy generated by X. We say that a varifold V is *stationary* if $\delta V(X) = 0$ for all vector fields X on M. This is a weak notion of a minimal hypersurface. As mentioned in Remark 2.1.6, stationary varifolds also have a monotonicity formula as in Proposition 2.1.5.

In particular, we are interested in the space of *integer rectifiable n-varifolds*, which we will denote as $\mathcal{V}_n(M)$, which are the set of n-varifolds of the following form: given a n-rectifiable set $R \subset M$ and measurable function $m : R \to \mathbb{N}$, we can define $V = V(R,m)$ so that

$$\int_{\mathbf{G}_n(M)} \varphi(x,\pi)\ dV(x,\pi) = \int_R m(x)\varphi(x, T_xR)\ d\mathcal{H}^n(x)$$

where T_xR is the tangent space at $x \in R$ which is defined almost everywhere for the Hausdorff measure $\mathcal{H}^n$. Integral stationary varifolds have the following compactness property due to

Allard.

Proposition 2.2.10 ([All]). *Given $C > 0$. The space of stationary integer rectifiable varifolds with mass uniformly bounded mass $\{V \in \mathcal{V}_n(M) : \|V\|(M) \leq C\}$ is compact.*

Finally, we note that the weak topology on varifolds is induced by the norm where

$$\mathbf{F}(V) := \sup \left\{ \int_{\mathbf{G}_n(M)} \varphi \, dV : \varphi \in C_c(\mathbf{G}_n(M)) \text{ with } |\varphi| \leq 1,\ \mathrm{Lip}(\varphi) \leq 1 \right\}$$

Recall we require more structure on the space of hypercycles $\mathcal{Z}_n(M)$. Given $T \in \mathcal{Z}_n(M)$, we define the integer rectifiable varifold $|T| \in \mathcal{V}_n(M)$ by viewing the rectifiable set $\mathrm{spt}(T)$ as a varifold with multiplicity-one. We define the $\mathbf{F}$-norm on hypercycles $\mathcal{Z}_n(M)$ by

$$\mathbf{F}(T) = \mathcal{F}(T) + \mathbf{F}(|T|). \tag{2.6}$$

We will let $\mathcal{Z}_n^{\mathbf{F}}(M)$ denote the space $\mathcal{Z}_n(M)$ with this stronger $\mathbf{F}$-topology.

Lemma 2.2.11 ([Pi, Section 2.1.18]). *Hypercycles T_i converges to T in the $\mathbf{F}$-topology if and only if the cycles T_i converge to T in the flat-topology and preserves mass in the sense*

$$\lim_{i \to \infty} \mathbf{M}(T_i) = \mathbf{M}(T). \tag{2.7}$$

Recall in general, mass is only lower semicontinuous (2.5) in the flat topology. In particular, a continuous map $\Phi : X \to \mathcal{Z}_n^{\mathbf{F}}(M)$ has *no concentration of mass* as defined in [MN1]. Again, we later explain in Remark 2.3.3 why this is a desired property.

2.3 Minmax theory and the volume spectrum

Continuous minmax theory

Originally, Pitts' min-max theory [Pi] was in terms of certain discrete maps for technical reasons. Thanks to the interpolation results developed by Marques-Neves in [MN1, MN2], we can state much of the theory in the continuous setting.

Definition 2.3.1. The *homotopy class* $\Pi = [\Phi]$ of a map $\Phi : X \to \mathcal{Z}_n^{\mathbf{F}}(M)$ to be

$$\Pi := \{\Psi : X \to \mathcal{Z}_n^{\mathbf{F}}(M) : \Psi \text{ is } \mathbf{F}\text{-continuous and is flat-homotopic to } \Phi\}.$$

Then we define the *width* of the homotopy class Π to be the minmax value

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \mathbf{M}(\Phi(x)).$$

Lemma 2.3.2. *A homotopy class* Π *is non-trivial if and only if* $\mathbf{L}(\Pi) > 0$*.*

Proof. Suppose that Π is non-trivial (the other direction is trivial). Thus, the induced map $\Phi_* : \pi_1(X) \to \pi_1(\mathcal{Z}_n(M))$ is nontrivial for any $\Phi \in \Pi$. In particular, there exists a loop $\gamma : [0,1] \to X$ so that $\gamma(0) = \gamma(1)$ where the corresponding loop $\Phi \circ \gamma$ is nontrivial in $\mathcal{Z}_n(M)$. Therefore, there exists a lift $\tilde{\gamma} : [0,1] \to \mathcal{I}_{n+1}(M)$ with

$$\partial \circ \tilde{\gamma} = \Phi \circ \gamma \quad \text{such that} \quad \tilde{\gamma}(0) = 0 \text{ and } \tilde{\gamma}(1) = M. \tag{2.8}$$

By the isoperimetric inequality, if $\Phi(\gamma(t))$ has sufficiently small mass (independent of Φ),

$$\mathbf{M}(\tilde{\gamma}(t)) \le c\, \mathbf{M}(\Phi(\gamma(t)))^{\frac{n+1}{n}}$$

where c is also independent of Φ. In particular, it cannot be true that $\mathbf{L}(\Pi) = 0$ because then we could find $\Phi \in \Pi$ to make $\mathbf{M}(\Phi(\gamma(t)))$ arbitrarily small uniformly in t, and hence $\mathbf{M}(\tilde{\gamma}(t))$ as well as long as $\mathbf{M}(\tilde{\gamma}(t)) \leq \mathrm{vol}_g(M)/2$. However, this cannot happen because always exists some $t_0 \in [0,1]$ such that $\mathbf{M}(\tilde{\gamma}(t_0)) = \mathrm{vol}_g(M)/2$ because $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = M$. □

Remark 2.3.3. We explain the need for varifolds as mentioned in Section 2.2. Our goal is to use a homotopy class to construct smooth minimal hypersurfaces Σ such that $\mathrm{vol}_g(\Sigma) = \mathbf{L}(\Pi)$ (each component counted with possible integer multiplicities). A naive approach would be to use compactness to find a convergent sequence $\Phi_{i_k}(x_k) \to T \in \mathcal{Z}_n(M)$ with

$$\lim_{i\to\infty} \mathbf{M}(\Phi_{i_k}(x_k)) = \mathbf{L}(\Pi).$$

However, there are some issues with this. In particular:

- Not necessarily true $\mathbf{M}(\Phi_{i_k}(x_k)) \to \mathbf{M}(T)$. In general, mass is lower-semicontinuous in the flat topology due to phenomena such as mass concentration [MN1].
- More importantly, there is no reason to expect T to be smooth or even stationary.

Both of these issues, can be remedied by using varifolds as well.

Given a sequence $\{\Phi_i\} \subset \Pi$, we similarly define its *width* as

$$\mathbf{L}(\{\Phi_i\}) = \liminf_{i\to\infty} \sup_{x\in X} \mathbf{M}(\Phi_i(x)).$$

We say the sequence $\{\Phi_i\}$ is *minimizing* if $\mathbf{L}(\{\Phi_i\}) = \mathbf{L}(\Pi)$. Given a minimizing sequence, we can always find a subsequence i_k and points $x_k \in X$ such that

$$\lim_{k\to\infty} \mathbf{M}(\Phi_{i_k}(x_k)) = \mathbf{L}(\Pi)$$

which we call *minmax sequence*. We define the *critical set* of $\{\Phi_i\}$ to be

$$\mathbf{C}(\{\Phi_i\}) := \{\, V \in \mathcal{V}_n(M) : \|V\|(M) = \mathbf{L}(\Pi) \text{ where } V = \lim_{i\to\infty} |\Phi_{i_k}(x_k)|$$

$$\text{for some sequence } x_k \in X \text{ and subsequence } i_k \in \mathbb{N}\,\}.$$

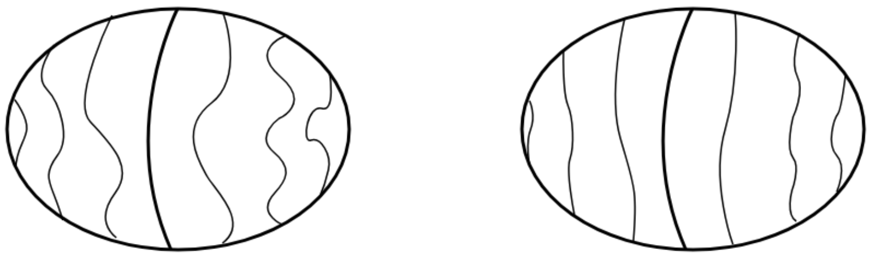

Figure 2.2: Here the central minimal hypersurface realizes a min-max width. But in the first sweepout, there are cycles with mass near that of the width which could unwantedly contribute to the critical set. To avoid this the pull-tight on the right lowers the area of these undesirable cycles.

Proposition 2.3.4 ([MN4]). *Given a minimizing sequence $\{\Phi_i\} \subset \Pi$, there exists $\{\Phi_i^*\} \subset \Pi$ where $\mathbf{C}(\{\Phi_i^*\}) \subset \mathbf{C}(\{\Phi_i\})$ is nonempty and every $V \in \mathbf{C}(\{\Phi_i^*\})$ is stationary.*

This follows by essentially constructing a gradient-like flow for the **M**-functional where

$$H : \{T \in \mathcal{Z}_n^{\mathbf{F}}(M) : \mathbf{M}(T) \leq C\} \times [0,1] \to \{T \in \mathcal{Z}_n^{\mathbf{F}}(M) : \mathbf{M}(T) \leq C\}$$

is such that $\mathbf{M}(H(T,t)) \leq \mathbf{M}(T)$ with equality if and only if $t = 0$ or $|T|$ is a stationary varifold. Furthermore, for any $\varepsilon > 0$, we can assure $\mathbf{F}(H(T,t),T) \leq \varepsilon$ holds. Although the **M**-functional is not smooth, we can create this map by hand by patching together deformations which decrease the mass of near-by non-stationary varifolds (see [Pi, Theorem 4.3]). Finally, the *pull-tight* sequence is defined as $\Phi_i^*(x) := H(\Phi_i(x), 1)$.

Therefore, for each non-trivial homotopy class Π, we can find a stationary varifold $V \in$

$\mathcal{V}_n(M)$ such that $\|V\|(M) = \mathbf{L}(M)$. However, not every stationary varifold is smooth. We need these varifolds to hold further properties to guarantee the regularity.

Definition 2.3.5. Given an open set $U \subseteq M$ and $\varepsilon, \delta > 0$. We say that a cycle $T_0 \in \mathcal{Z}_n(M)$ is (ε, δ)-*almost-minimizing* in U if for any sequence $T_1, \ldots, T_m \in \mathcal{Z}_n(M)$ such that

$$\mathrm{spt}(T_i - T_0) \subset U, \quad \mathcal{F}(T_i, T_{i-1}) \leq \delta, \quad \text{and} \quad \mathbf{M}(T_i) \leq \mathbf{M}(T_0) + \delta$$

for all $i = 1, \ldots, m$, then we must also have $\mathbf{M}(T_m) \geq \mathbf{M}(T) - \varepsilon$.

We say that a varifold $V \in \mathcal{V}_n(M)$ is *almost-minimizing* in U if for every $\varepsilon > 0$, there exists a $\delta > 0$ along with a (ε, δ)-almost-minimizing $T \in \mathcal{Z}_n(M)$ such that $\mathbf{F}(|T|, V) \leq \varepsilon$.

Moreover, a varifold $V \in \mathcal{V}_n(M)$ is *almost-minimizing in annuli* if for every $p \in \mathrm{spt}(\|V\|)$, there exists an $r > 0$ such that V is almost-minimizing in $B(p, r) \setminus B(p, s)$ for all $0 < s < r$.

Essentially V is almost minimizing in U if any deformation supported in U which decreases the mass of V must intermediately increase the mass by a definite amount at some point. The following regularity theorem was first proved by Pitts [Pi] for $n + 1 \leq 6$, and later for $n + 1 = 7$ by Schoen-Simon [ScSi].

Proposition 2.3.6 ([ScSi])**.** *Let (M^{n+1}, g) have $3 \leq n+1 \leq 7$. If $V \in \mathcal{V}_n(M)$ is stationary and almost-minimizing in annuli, then* $\mathrm{spt}(\|V\|)$ *is a smooth embedded minimal hypersurface.*

Remark 2.3.7. For $n + 1 > 8$, then [ScSi] actually shows $\mathrm{spt}(\|V\|)$ is a smooth embedded minimal hypersurface away from a singular set of Hausdorff codimension 7.

Pitt's was able to prove that there must exist some $V \in \mathbf{C}(\{\Phi_i^*(x)\})$ which is almost-minimizing in annuli with a technical combinatorial argument [Pi, Section 4.10] which uses the discrete min-max theory. Using the interpolation results of Marques-Neves [MN3], Pitt's result can be summarized as follows.

Proposition 2.3.8 ([Pi])**.** *If a pull-tight sequence $\{\Phi_i\} \subset \Pi$ has no $V \in \mathbf{C}(\{\Phi_i\})$ which is almost-minimizing in annuli, there exists sequence $\{\Phi_i^*\} \subset \Pi$ such that $\mathbf{L}(\{\Phi_i^*\}) < \mathbf{L}(\{\Phi_i\})$.*

The last three stated propositions above imply the following.

Corollary 2.3.9. *Given a minimizing sequence* $\{\Phi_i\} \subset \Pi$. *There exists a* $V \in \mathbf{C}(\{\Phi_i\})$ *whose support is a smooth embedded minimal hypersurface.*

But due to recent resolution by Zhou [Zh] of the *multiplicity-one conjecture*, we can say more. In a bumpy metric, the minimal hypersurfaces are always two-sided and have multiplicity-one. Then following min-max theorem is the culmination of the work of Almgren-Pitts [Pi], the work of Marques-Neves [MN3], the multiplicity-one conjecture proven by Zhou [Zh], the denseness of bumpy metrics [Wh1], along with the compactness theory of Sharp [Sh], and other important work.

Theorem 2.3.10 ([Zh] Theorem C)**.** *Let* (M^{n+1}, g) *with* $3 \leq n+1 \leq 7$ *and* $\{\Phi_i\} \subset \Pi$ *where*

$$\mathbf{L}(\{\Phi_i\}) = \mathbf{L}(\Pi) > 0.$$

Then there exists a disjoint collection of closed embedded minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_N$ *along with positive integer multiplicities* $m_1, \ldots, m_N$ *such that*

$$\mathbf{L}(\Pi) = \sum_{i=1}^{N} m_i \operatorname{area}_g(\Sigma_i)$$

where $m_i > 1$ *only if* Σ_i *is degenerate stable, and where as varifolds*

$$m_1\Sigma_1 + \cdots + m_N\Sigma_N \in \mathbf{C}(\{\Phi_i\}).$$

The volume spectrum

The volume spectrum was first introduced by Gromov [Gr1, Gr2] as a nonlinear analogue of the spectrum of the Laplacian. Guth [Gu] later proved some important estimates for their growth. This spectrum played a fundamental role in showing the existence of infinitely-many minimal hypersurfaces in [MN1, So1].

Definition 2.3.11. Let X be a k-dimensional cubical subcomplex of $I^m = [0,1]^m$. For $p \in \mathbb{N}$, we say that a flat-continuous map $\Phi : X \to \mathcal{Z}_n(M)$ is a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2)$$

where $\bar{\lambda}^p$ is the p-th cup product of the fundamental class $\bar{\lambda}$.

Note that if Φ is a p-sweepout and Ψ is flat-homotopic to Φ, then Ψ is also a p-sweepout. Given $p \in \mathbb{N}$, we use $\mathbb{S}_p := \mathbb{S}_p(M, g)$ to denote all p-sweepouts which are continuous in the $\mathbf{F}$-topology (so has no mass concentration). Finally, define the p-width of (M, g) as

$$\omega_p(M, g) := \inf_{\Phi \in \mathbb{S}_p} \sup_{x \in X_\Phi} \mathbf{M}(\Phi(x))$$

where X_Φ denotes the domain of $\Phi : X_\Phi \to \mathcal{Z}_n(M)$ (note this will vary in $\mathbb{S}_p$) and where $\mathbf{M}$ denotes the mass norm given the metric g.

Lemma 2.3.12 (Vanishing Lemma)**.** *Let $\Phi : X \to \mathcal{Z}_n(M)$ be a $(p+q)$-sweepout and $X = Y \cup Z$. If $\Phi|_Y$ is not an p-sweepout, then $\Phi|_X$ must be a q-sweepout.*

Proof. Let $\bar{\lambda} \in H^1(\mathcal{Z}_n(M); \mathbb{Z}_2)$ be the fundamental class, meaning $\lambda := \Phi^*(\bar{\lambda})$ satisfies $\lambda^{p+q} \neq 0$ in $H^{p+q}(X; \mathbb{Z}_2)$. Consider the exact sequence coming from the long exact sequence

$$H^p(X, Y; \mathbb{Z}_2) \xrightarrow{j_Y^*} H^p(X; \mathbb{Z}_2) \xrightarrow{i_Y^*} H^p(Y; \mathbb{Z}_2).$$

where $i_Y : Y \hookrightarrow X$ and $j_Y : (X, \emptyset) \hookrightarrow (X, Y)$ are the inclusion maps. Since $\Phi|_Y$ is not an p-sweepout, we have $\lambda_Y := (\Phi|_Y)^*(\bar{\lambda})$ satisfies $i_Y^*(\lambda^p) = \lambda_Y^p = 0$ in $H^p(Y; \mathbb{Z}_2)$. Thus, by exactness, there exists $\sigma_Y \in H^p(X, Y; \mathbb{Z}_2)$ such that $j_Y^*(\sigma_Y) = \lambda^p$.

Now, if $\Phi|_Z$ were not a q-sweepout, then we could similarly find $\sigma_Z \in H^q(X, Z; \mathbb{Z}_2)$ such that $j_Z^*(\sigma_Z) = \lambda^q$. Consider the relative cup product defined so that

$$\begin{array}{ccc} H^p(X,Y;\mathbb{Z}_2) \otimes H^q(X,Z;\mathbb{Z}_2) & \xrightarrow{\smile} & H^{p+q}(X, Y\cup Z;\mathbb{Z}_2) \\ \downarrow {\scriptstyle j_Y^* \otimes j_Z^*} & & \downarrow {\scriptstyle j_{Y\cup Z}^*} \\ H^p(X;\mathbb{Z}_2) \otimes H^q(X;\mathbb{Z}_2) & \xrightarrow{\smile} & H^{p+q}(X;\mathbb{Z}_2) \end{array}$$

commutes. Since $Y \cup Z = X$, we have $H^{p+q}(X, Y \cup Z; \mathbb{Z}_2) = 0$. However, this implies

$$\lambda^{p+q} = j_Y^*(\sigma_Y) \smile j_Z^*(\sigma_Z) = j_{Y\cup Z}^*(\sigma_Y \smile \sigma_Z) = 0$$

which contradicts that Φ is a $(p+q)$-sweepout. Therefore, $\Phi|_Z$ must be a q-sweepout. $\square$

Note that $\omega_p(M,g) \leq \omega_{p+1}(M,g)$ for all p because any $(p+1)$-sweepout must be a p-sweepout. These widths go to infinity as $p \to \infty$. In fact, we can quantify this growth by the Weyl law of volume spectrum, which plays an important role in the study of the min-max theory of minimal hypersurfaces.

Theorem 2.3.13 (Liokumovich-Marques-Neves [LMN])**.** *There exist a constant $a_n > 0$ such that for every closed $(n+1)$-dimensional Riemannian manifold (M,g),*

$$\lim_{p\to\infty} \omega_p(M,g)p^{-\frac{1}{n+1}} = a_n \operatorname{vol}_g(M)^{\frac{n}{n+1}}.$$

Theorem 2.3.14 ([Zh] Theorem C)**.** *Let (M^{n+1}, g) with $3 \leq n+1 \leq 7$ and $p \in \mathbb{N}$. Then there exists a disjoint collection of closed embedded minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_N$ along with positive integer multiplicities $m_1, \ldots, m_N$ such that*

$$\omega_p(M,g) = \sum_{i=1}^N m_i \operatorname{area}_g(\Sigma_i) \quad \textit{and} \quad \sum_{i=1}^N m_i \operatorname{index}_g(\Sigma_i) \leq p$$

where $m_i > 1$ only if Σ_i is degenerate stable.

2.4 Overview of proof

The main idea is trying to cut the manifold into simpler pieces as done by Song in [So1] along contracting minimal hypersurfaces. But in our case, the cutting process can get more complicated since there may be infinitely many minimal hypersurfaces to cut along. There are several cases.

Weakly Frankel case

The first possibility is that there are no minimal hypersurfaces which are contracting on a side. In this case, we say (M, g) is *weakly Frankel* generalizing the Frankel property from [MN1]. Recall any two minimal hypersurfaces in a Frankel manifold intersect with each other. But two minimal hypersurfaces can be disjoint in a weakly Frankel manifold, but these hypersurfaces turn out to be connected by a minimal foliation.

By standard Jacobi field arguments, in weakly Frankel manifolds, we show that

$$\{\operatorname{area}_g(\Gamma) : \Gamma \subset (M, g) \text{ connected stable minimal hypersurface with } \operatorname{area}_g(\Gamma) \leq \omega\}$$

is finite for any choice of $\omega > 0$. Thus, either we can find stable minimal hypersurfaces with arbitrarily large area, or there are only finitely-many areas of stable minimal hypersurfaces. In the latter case, we need to find unstable minimal hypersurfaces with arbitrarily large area.

Although the minimal hypersurface which realizes the p-width could have multiple components, but if so, these components are connected by a minimal foliation which implies they each are stable and have the same area. Therefore, we have that

$$\omega_p = m_p \operatorname{area}_g(\Gamma_p)$$

for some connected minimal hypersurface Γ_p and integer multiplicity m_p. Moreover, Zhou observed that the unstable minimal hypersurfaces must have multiplicity-one. Thus, we only need to show that the p-width can not be all realized by multiples of stable minimal hypersurfaces for large enough p.

To do this, we utilize Lusternik-Schnirelmann arguments developed in [Ai]. Here Aiex showed that the space of min-max minimal hypersurfaces in a manifold with positive Ricci curvature is noncompact. His idea is to study a generalization of the Lusternik-Schnirelmann category, and he can prove that under certain conditions, there is a strict jump of the width from $\omega_p < \omega_{p+N}$ for p large. Then if there are eventually only stable minimal hypersurfaces appearing in the volume spectrum, we can show the growth violates the Weyl law. In this paper, we generalize Aiex ideas to weakly Frankel manifolds. One important new case which arises is when the manifold is foliated by minimal hypersurfaces. These can complicate the space of stable minimal hypersurfaces, and so we need a more precise study of its topology.

Weak core case

The second possibility is that after cutting along finitely many contracting minimal hypersurfaces, we get a component that has minimal contracting boundary, and there are no contracting minimal hypersurfaces in the interior of this component. Motivated by Song's work, we call such a compact manifold a *weak core*. If we can find a weak core, then we can still use Song's results with slight modifications in this more general setting to show that there exist embedded minimal hypersurfaces with arbitrarily large area.

Coreless case

The last possibility is that we can not get a weak core after finitely many cutting steps. We call such manifolds *coreless*. Even for coreless manifolds, there are some relatively mild cases that we are able to handle. For example, there is a kind of minimal hypersurface which we will call a *monotonic saddle*, where the minimal hypersurface in the middle is non-isolated, and hypersurfaces have larger area the closer they are to the middle one (see Figure 5.2). Although we may not be able to cut the manifold to a weak core in finitely many steps, we can use the approximation argument to show that there exist minimal hypersurfaces with arbitrarily large area.

Finally, there is an extremely pathological case, that we can not get a weak core after finitely many cuttings, and we also can not find a monotonic saddle part. In this case, we can show that the space of certain pathological stable minimal hypersurfaces (see Figure 1) is homeomorphic to a Cantor set.

CHAPTER 3

WEAKLY FRANKEL MANIFOLDS

In this section, we will study manifolds with the following property.

Definition 3.0.1. A closed Riemannian manifold (M, g) is *weakly Frankel* there exist no minimal hypersurfaces which are contracting on a side.

The goal of this section is to find minimal hypersurfaces with arbitrarily large area in a weakly Frankel manifold, see Theorem 3.3.3. Weakly Frankel is a generalization of the Frankel property. Recall that a manifold is Frankel if any two minimal hypersurfaces intersect with each other. In [MN1] Marques-Neves proved that in a manifold with Frankel property, there exists infinitely many closed embedded minimal hypersurfaces.

The idea is that the Frankel property forces the min-max widths ω_p to be realized as $m_p \Sigma_p$ for some integer multiple m_p of some *connected* minimal hypersurface Σ_p. Then by using the growth of the widths along with some Lusternik-Schnirelmann arguments, they showed that min-max theory must give infinitely many distinct minimal hypersurfaces. We will use similar reasoning.

3.1 Manifolds without contracting minimal hypersurfaces

First, more generally, consider compact manifolds (N^{n+1}, g) with $3 \leq n+1 \leq 7$ such that:

- the (possibly empty) boundary ∂N is minimal and contracting, and
- $\text{int}(N)$ contains no minimal hypersurfaces which are contracting on a side.

Remark 3.1.1. The second condition also implies that N contains no minimal hypersurfaces which are accumulating on a side because such a hypersurface must be a limit of contracting minimal hypersurfaces.

In particular, these following lemmas apply for a weakly Frankel manifold N in the case when N has no boundary. The case where N has nonempty contracting minimal boundary ∂N will be relevant in the later sections.

Definition 3.1.2. We say connected closed hypersurfaces $\Sigma_0, \Sigma_1 \subset N$ are *connected by a minimal foliation* if there exists a map $\varphi : \Sigma \times [0, \delta] \to N$ such that

- $\varphi(\Sigma \times \{0\}) = \Sigma_0$ and $\varphi(\Sigma \times \{\delta\}) = \Sigma_1$,
- φ is a diffeomorphism onto its image, and
- $\varphi(\Sigma \times \{t\})$ is a minimal hypersurface for all $t \in [0, \delta]$.

Unlike Frankel manifolds, weakly Frankel manifolds can have disjoint minimal hypersurfaces, but these hypersurfaces must connected by a minimal foliation.

Lemma 3.1.3. *Assume* (N, g) *as above. If* $\Sigma_0, \Sigma_1 \subset \operatorname{int}(N)$ *are disjoint connected minimal hypersurfaces, then* Σ_0, Σ_1 *are connected by a minimal foliation.*

Proof. Assume that Σ_0, Σ_1 are disjoint minimal hypersurfaces. Consider the metric completion of $N \setminus (\Sigma_0 \cup \Sigma_1)$ and pick a connected component W which contains two non-contracting minimal boundary components Γ_0, Γ_1 (along with possibly other necessarily contracting minimal boundary components coming from N) which are isometric to Σ_0, Σ_1 respectively. Without loss of generality, we may assume $\operatorname{area}_g(\Gamma_0) \leq \operatorname{area}_g(\Gamma_1)$ so that

$$\operatorname{area}_g(\Gamma_0) \leq \operatorname{area}_g(\partial W \setminus \Gamma_0)$$

because $\Gamma_1 \subseteq \partial W \setminus \Gamma_0$. Finally, note since we are assuming that $\operatorname{int}(N)$ contains no contracting minimal hypersurfaces, then so does $\operatorname{int}(W)$. And therefore, W must be foliated by minimal hypersurfaces by Lemma 2.1.9. This gives the desired minimal foliation which connects Σ_0, Σ_1. $\square$

Remark 3.1.4. In particular, disjoint minimal hypersurfaces Σ_0, Σ_1 must be both degenerate stable, homologous to each other, and have $\text{area}_g(\Sigma_0) = \text{area}_g(\Sigma_1)$.

Consider the set $\mathcal{M}^S$ of all connected stable minimal hypersurfaces in $(\text{int}(N), g)$. For $\omega > 0$, it will be useful for us to define an equivalence relation on

$$\mathcal{M}^S_\omega = \{\Sigma \in \mathcal{M}^S : \text{area}_g(\Sigma) \leq \omega\}$$

by $\Sigma_0 \sim \Sigma_1$ if and only if Σ_0, Σ_1 are connected by a minimal foliation. In particular, this means that if $\Sigma_0 \not\sim \Sigma_1$, then Σ_0 and Σ_1 must intersect by Lemma 3.1.3.

Lemma 3.1.5. *Assume (N, g) as above. For each $\omega > 0$, the set $\mathcal{M}^S_\omega/\sim$ is finite.*

Proof. Suppose that $\mathcal{M}^S_\omega/\sim$ were infinite. Then we could find a sequence $\Sigma_k \in \mathcal{M}^S_\omega$ of stable minimal hypersurfaces each representing a different equivalence class. The areas being bounded implies that after relabeling, we can find a subsequence Σ_k which converges smoothly to some stable $\Sigma \in \mathcal{M}^S_\omega$ by [ScSi]. Moreover, since Σ is not connected by a minimal foliation to all but possibly one Σ_k, we can assume each Σ_k intersects Σ by Lemma 3.1.3. But by [Sh], this means we can construct a Jacobi field for Σ which changes sign which would contradict that Σ is stable. □

Corollary 3.1.6. *Assume (N, g) as above. For each $\omega > 0$, the set of stable areas*

$$\text{area}_g(\mathcal{M}^S_\omega) = \{\text{area}_g(\Sigma) : \Sigma \in \mathcal{M}^S_\omega\}$$

is finite. In particular, if the area of stable minimal hypersurfaces in (N, g) is uniformly bounded, then the set of all stable areas $\text{area}_g(\mathcal{M}^S)$ is finite.

Proof. This follows Lemma 3.1.5 and Remark 3.1.4 which tells us that any two stable minimal hypersurfaces in the same equivalence class must have the same area. □

Let $[\Sigma] \in \mathcal{M}^S/\sim$ denote the equivalence class of stable minimal hypersurfaces which are connected to Σ by a minimal foliation. There are three types of classes:

1. We say Σ is *isolated* if $[\Sigma] = \{\Sigma\}$.

2. We say Σ generates a *partial minimal foliation* if there exists a diffeomorphism $\varphi : \Sigma \times I \to M$ onto its image with $\{\Sigma_t\}_{t\in I} = [\Sigma]$ for $\Sigma_t = \varphi(\Sigma \times \{t\})$.

3. We say Σ generates a *(full) minimal foliation* if there exists a fiber bundle $\pi : M \to S^1$ such that $\{\Sigma_\theta\}_{\theta \in S^1} = [\Sigma]$ where $\Sigma_\theta = \pi^{-1}(\{\theta\})$.

3.2 Space of stable minimal hypersurfaces

Consider the subspace $\mathcal{S}_\omega \subseteq \mathcal{Z}_n(M)$ of all $T \in \mathcal{Z}_n(M)$ such that $T = 0$ or $\operatorname{spt}(T)$ is an embedded stable minimal hypersurface in (M, g) with $\mathbf{M}(T) \leq \omega$. Note T does *not* need to be connected.

Given $\Sigma \in \mathcal{M}^S_\omega$ which generates either a full or partial minimal foliation of (M, g), we define the space $\mathcal{F}_\omega \subseteq \mathcal{S}_\omega$ *associated to the foliation* to be the set of all $T \in \mathcal{S}_\omega$ such that each connected component of T equals some leaf in this foliation.

Lemma 3.2.1. *If Σ generates a (full) minimal foliation of (M, g), then the subspace $\mathcal{F}_\omega \subseteq \mathcal{S}_\omega$ associated to the foliation is homeomorphic to $\mathbb{RP}^m$ where m is the largest even number such that $m \leq \omega/\operatorname{area}_g(\Sigma)$.*

Proof. Suppose Σ generates a minimal foliation of (M, g) so that there exist a fiber bundle $\pi : M \to S^1$ where the fibers $\Sigma_\theta := \pi^{-1}(\{\theta\})$ parameterize the foliation. Note that each Σ_θ is non-separating and homologous to Σ, and so each $T \in \mathcal{F}_\omega$ must have an even number of components. Let m be the largest even number with $m \operatorname{area}_g(\Sigma) \leq \omega$. Then for each $T \in \mathcal{F}_\omega$, there exists $\theta_1, \ldots, \theta_m \in S^1$ such that

$$T = \Sigma_{\theta_1} + \Sigma_{\theta_2} + \cdots + \Sigma_{\theta_m}$$

where we are considering Σ_{θ_i} as cycles in $\mathcal{Z}_n(M)$. This means that the flat continuous map

$$q : (S^1)^m \to \mathcal{F}_\omega \quad \text{given by} \quad q(\theta_1, \ldots, \theta_m) = \Sigma_{\theta_1} + \cdots + \Sigma_{\theta_m}$$

is surjective. Note $\operatorname{spt}(T)$ will have fewer than m components when $\theta_i = \theta_j$ for some $i \neq j$, and $T = 0$ whenever $\theta_1 = \theta_i$ for all i. By identifying the fibers of this map, we get a continuous bijection (and hence a homeomorphism by compactness)

$$f : TP^m(S^1) \to \mathcal{F}_\omega \qquad \text{by} \qquad f[(\theta_1, \ldots, \theta_m)] = \Sigma_{\theta_1} + \cdots + \Sigma_{\theta_m}$$

where $TP^m(S^1) = (S^1)^m / \sim$ is given the quotient topology by the relation

$$(\theta_1, \ldots, \theta_m) \sim (\theta_1', \ldots, \theta_m') \quad \text{iff} \quad \Sigma_{\theta_1} + \cdots + \Sigma_{\theta_m} = \Sigma_{\theta_1'} + \cdots + \Sigma_{\theta_m'}.$$

The space $TP^m(S^1)$ is known as the m-th truncated symmetric product of S^1 and is homeomorphic to $\mathbb{RP}^m$ by [Mo, Theroem 2]. Therefore, $\mathcal{F}_\omega \cong \mathbb{RP}^m$. □

Lemma 3.2.2. *Suppose Σ generates a partial minimal foliation of (M, g), then the subspace $\mathcal{K}_\omega \subseteq \mathcal{S}_\omega$ associated to the foliation has two exactly components $\mathcal{K}_\omega^0, \mathcal{K}_\omega^1$ where $\mathcal{K}_\omega^0$ strongly deformation retracts to $0 \in \mathcal{K}_\omega^0$ and where $\mathcal{K}_\omega^1$ is contractible.*

Proof. Suppose Σ generates a partial minimal foliation of (M, g) so that there exists a diffeomorphism $\varphi : \Sigma \times [0, \delta] \to M$ onto its image where the slices $\Sigma_t := \varphi(\Sigma \times \{t\})$ parameterize the partial foliation.

First, note that Σ here must be separating. Otherwise, we can consider the metric completion N of $M \setminus \Sigma$ which must be connected and have exactly two boundary components both isometric to Σ. Moreover, if N is foliated by minimal hypersurfaces, then Σ must generate a (full) foliation of M by Lemma 2.1.9, which contradicts that this foliation is only partial.

Thus, each $T \in \mathcal{K}_\omega$ can have either an even or odd number of components. And, in fact, $\mathcal{K}_\omega$ has exactly two connected components $\mathcal{K}_\omega^0$, $\mathcal{K}_\omega^1$ consisting of all $T \in \mathcal{K}_\omega$ with an even number and odd number, respectively, of components. Consider the flat continuous homotopy $H : \mathcal{K}_\omega \times [0,1] \to \mathcal{K}_\omega$ given by

$$H(T,s) = \Sigma_{(1-s)t_1} + \cdots + \Sigma_{(1-s)t_m}$$

where $T = \Sigma_{t_1} + \cdots + \Sigma_{t_m}$. This map gives a strong deformation retract of $\mathcal{K}_\omega^0$ to the point $0 \in \mathcal{K}_\omega^0$ and a strong deformation retract of $\mathcal{K}_\omega^1$ to the point $\Sigma_0 \in \mathcal{K}_\omega^1$. □

Proposition 3.2.3. *There exist $C' > 0$ such that for all $\omega > 0$, we have*

$$H^m(\mathcal{S}_\omega, \mathbb{Z}_2) = 0 \quad \textit{for all} \quad m \geq C'\omega.$$

Proof. Let $\Sigma_F^{(1)}, \ldots, \Sigma_F^{(n_F)}$ generate all distinct full minimal foliations, $\Sigma_P^{(1)}, \ldots, \Sigma_P^{(n_P)}$ generate all distinct partial minimal foliations, and let $\Sigma_I^{(1)}, \ldots, \Sigma_I^{(n_T)}$ be all the isolated stable minimal hypersurfaces of (M, g) with area less than or equal to ω. Note there are indeed only finitely many such hypersurfaces by Lemma 3.1.5.

Given $\Sigma_F^{(i)}$ which generates a full minimal foliation, let $\mathcal{F}_\omega^{(i)}$ denote the associated space of cycles. Similarly, given $\Sigma_P^{(j)}$ which generates a partial minimal foliation, let $\mathcal{K}_\omega^{(j)}$ denote the associated space of cycles. Then

$$\mathcal{S}_\omega = \left(\bigvee_{i=1}^{n_F} \mathcal{F}_\omega^{(i)} \vee \bigvee_{j=1}^{n_P} \mathcal{K}_\omega^{(j),0} \right) \sqcup \bigsqcup_{j=1}^{n_P} \mathcal{K}_\omega^{(j),1} \sqcup \bigsqcup_{k=1}^{n_I} \{\Sigma_I^{(k)}\}$$

where the wedge sums are all taken at $0 \in \mathcal{Z}_n(M)$. Since each $\mathcal{K}_\omega^{(j),0}$ strongly deformation retracts to 0 and each $\mathcal{K}_\omega^{(j),1}$ is contractible by Lemma 3.2.2, we can construct a deformation retraction of $\mathcal{S}_\omega$ onto the subspace

$$\bigvee_{i=1}^{n_F} \mathcal{F}_\omega^{(i)} \sqcup \bigsqcup_{j=1}^{n_P} \{\Sigma_P^{(j)}\} \sqcup \bigsqcup_{k=1}^{n_I} \{\Sigma_I^{(k)}\} \subseteq \mathcal{S}_\omega\,.$$

Now recall that each $\mathcal{F}_\omega^{(i)}$ is homeomorphic to $\mathbb{R}\mathrm{P}^{m_i}$ where m_i is the largest even number less than or equal to $\omega/\operatorname{area}_g(\Sigma_F^{(i)})$ by Lemma 3.2.1. Thus, for all $m \geq 1$,

$$H^m(\mathcal{S}_\omega;\mathbb{Z}_2) = H^m(\bigvee_{i=1}^{n_F} \mathcal{F}_\omega^{(i)};\mathbb{Z}_2) = \prod_{i=1}^{n_F} H^m(\mathcal{F}_\omega^{(i)};\mathbb{Z}_2) = \prod_{i=1}^{n_F} H^m(\mathbb{R}\mathrm{P}^{m_i};\mathbb{Z}_2).$$

Finally, by the monotonicity formula, there exists $C' > 0$ such that $1/C' < \operatorname{area}_g(\Sigma)$ for all minimal hypersurfaces in (M, g). Note C' is defined independent of ω and that $m_i < C'\omega$ for all $i = 1, \ldots, n_F$. Therefore, $H^m(\mathcal{S}_\omega;\mathbb{Z}_2) = 0$ for all $m \geq C'\omega$ because then $H^m(\mathbb{R}\mathrm{P}^{m_i};\mathbb{Z}_2) = 0$ for each i. $\square$

3.3 Lusternik–Schnirelmann theory

Let Λ_ω^S denote the subspace of varifolds $V \in \mathcal{V}_n(M)$ with $\|V\|(M) \leq \omega$ and where $\operatorname{spt}(V)$ is a stable minimal hypersurface. Again, $V \in \Lambda_\omega^S$ may have multiple connected components where each component comes with some positive integer multiplicity.

The following lemma follows from the results proven by Aiex in [Ai] which extend the arguments used in [MN1]. It says that a map which stays near stable minimal hypersurfaces cannot be a m-sweepout for m large enough because we have control on the topology of the space of stable minimal hypersurfaces.

Lemma 3.3.1. *For every $\omega > 0$, there exists $\varepsilon > 0$ with the following property:*

If Y is a cubical subcomplex and $\Psi : Y \to \mathcal{Z}_n^{\mathbf{F}}(M)$ is a map such that

$$\mathbf{F}(|\Psi(y)|, \Lambda_\omega^S) < \varepsilon \quad \textit{for all} \quad y \in Y,$$

then Ψ is not an m-sweepout for $m \geq C'\omega$ where C' given by Proposition 3.2.3.

Proof. By inspection, note that we can still apply Lemma 2 and Proposition 2 from [Ai] to the subspace $\mathcal{S}_\omega \subset \mathcal{Z}_n(M)$ so that

$$\mathcal{N}_1\text{-cat}_{\mathcal{Z}_n}(\mathcal{S}_\omega) \leq \text{cat}(\mathcal{S}_\omega) \leq \dim(\mathcal{S}_\omega) \leq m$$

for $m \geq C'\omega$ by Lemma 3.2.3 where $\mathcal{N}_1\text{-cat}_{\mathcal{Z}_n}$ is a form of a relative Lusternik-Schnirelmann Category as defined in Section 3 of [Ai], $\text{cat}(\mathcal{S}_\omega)$ denotes the ordinary Lusternik-Schnirelmann Category, and $\dim(\mathcal{S}_\omega)$ denotes the cohomological dimension. Finally, since $\mathcal{S}_\omega \subset \mathcal{Z}_n(M)$ is closed with $\mathcal{N}_1\text{-cat}_{\mathcal{Z}_n}(\mathcal{S}_\omega) \leq m$, the desired result follows from Lemmas 3 and 4 in [Ai]. □

Proposition 3.3.2. *There exists $C > 0$ such that for each $p \in \mathbb{N}$, either*

1. *there exists a connected unstable minimal hypersurface $\Sigma \subset (M, g)$ with*

$$\text{area}_g(\Sigma) \geq \omega_p(M, g),$$

2. *or for all $m \geq Cp^{\frac{1}{n+1}}$, we have that*

$$\omega_{p-m}(M, g) < \omega_p(M, g).$$

Proof. Let C' be given in Proposition 3.2.3 and Lemma 3.3.1, and pick $C > 0$ such that $C'\omega_p(M, g) \leq Cp^{\frac{1}{n+1}}$. Now assume that the second case above does not happen for this choice of C, that is, there exists an $m \geq Cp^{\frac{1}{n+1}}$ such that

$$\omega_{p-m}(M, g) = \omega_p(M, g).$$

We will consider two separate cases.

Case 1: Suppose there exists a homotopy class Π of p-sweepouts with

$$\mathbf{L}(\Pi) = \omega_p(M, g).$$

Thus, there exist a cubical subcomplex X along with a min-max sequence of flat homotopic p-sweepouts $\Phi_i : X \to \mathcal{Z}_n^{\mathbf{F}}(M)$ such that

$$\mathbf{L}(\{\Phi_i\}) = \mathbf{L}(\Pi) = \omega_p(M, g).$$

Moreover, can assume that the sequence is *pull-tight* by [MN3, Section 2.8], that is, so that every $V \in \mathbf{C}(\{\Phi_i\})$ is stationary. Choose $\varepsilon > 0$ as in Lemma 3.3.1 for $\omega = \omega_p(M, g)$. Let Y_i be the cubical subcomplex consisting of all cells α in X with

$$\mathbf{F}(|\Phi_i(x)|, \Lambda_\omega^S) < \varepsilon \quad \text{for all } x \in \alpha,$$

and consider the cubical subcomplex $Z_i := \overline{X \setminus Y_i}$. Note that $\mathbf{F}(|\Phi_i(x)|, \Lambda_\omega^S) \geq \varepsilon/2$ for all $x \in Z_i$. By the choice of ε and C, the maps $\Phi_i|_{Y_i}$ are not m-sweepouts for all $m \geq Cp^{\frac{1}{n+1}}$ by Lemma 3.3.1. Therefore, by the Vanishing Lemma, the maps $\Phi_i|_{Z_i}$ must be $(p-m)$-sweepouts. In particular, $\mathbf{L}(\{\Phi_i|_{Z_i}\}) \geq \omega_{p-m}(M, g)$. Moreover, we must have equality $\mathbf{L}(\{\Phi_i|_{Z_i}\}) = \omega_{p-m}(M, g)$ because

$$\mathbf{L}(\{\Phi_i|_{Z_i}\}) \leq \mathbf{L}(\{\Phi_i\}) = \omega_p(M, g) = \omega_{p-m}(M, g).$$

Since the sequence Φ_i is pull-tight, every $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\})$ is stationary as well. Furthermore, note if there were no $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\})$ such that $\mathrm{spt}(V)$ is a smooth embedded minimal hypersurface, then by the regularity theory of Pitts [Pi], no $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\})$ is almost minimizing in annuli. However, Pitts' combinatorial argument[1] (see [Ai, Theroem

1. It is important that all Z_i here are cubical subcomplexes of $X \subset I^m$ where m is fixed.

5]) would allow us to find a sequence of $(p-m)$-sweepouts $\Psi_i : Z_i' \to \mathcal{Z}_n^{\mathbf{F}}(M)$ such that

$$\mathbf{L}(\{\Psi_i\}) < \mathbf{L}(\{\Phi_i|_{Z_i}\}) = \omega_{p-m}(M,g)$$

which contradicts the definition of the $(p-m)$-width.

Therefore, there must exist a $V \in \mathbf{C}(\{\Phi_i|_{Z_i}\})$ such that

$$\|V\|(M) = \mathbf{L}(\{\Phi_i|_{Z_i}\}) = \omega_p(M,g)$$

and where $\operatorname{spt}(V)$ is a smooth embedded minimal hypersurface. However, by construction, $V \notin \Lambda_\omega^S$. Thus, $\Gamma := \operatorname{spt}(V)$ must be unstable, and hence, Γ is connected by Lemma 3.1.3 and has $\operatorname{area}_g(\Gamma) = \omega_p(M,g)$.

Case 2: Now, suppose for every homotopy class Π of p-sweepouts, we have

$$\omega_p(M,g) < \mathbf{L}(\Pi).$$

In particular, we can find homotopy classes Π_i of p-sweepouts such that

$$\omega_p(M,g) < \cdots < \mathbf{L}(\Pi_i) < \cdots < \mathbf{L}(\Pi_2) < \mathbf{L}(\Pi_1) \leq \omega_p(M,g) + 1.$$

Note each $\mathbf{L}(\Pi_i) = m_i \operatorname{area}_g(\Gamma_i)$ for some minimal hypersurfaces Γ_i where $m_i = 1$ if Γ_i is unstable. Note that some Γ_i must be unstable because there would be infinitely many distinct areas of stable minimal hypersurfaces of bounded area which violates Lemma 3.1.6. So take $\Gamma = \Gamma_i$ unstable so that $m_1 = 1$. Therefore, we have $\operatorname{area}_g(\Gamma) = \mathbf{L}(\Pi_i) > \omega_p(M,g)$. □

Theorem 3.3.3. *If (M,g) is weakly Frankel, then there exists connected minimal hypersurfaces of arbitrarily large area.*

Proof. Assume the area of unstable minimal hypersurfaces in (M,g) is bounded. Take $C > 0$

from Proposition 3.3.2 and pick $p_0 \in \mathbb{N}$ such that $\omega_{p_0}(M, g)$ is strictly larger than the area of any unstable minimal hypersurface so that (1) does not happen in Proposition 3.3.2 for p large. Then for all $p \geq p_0$, we have

$$\omega_{p-m_0}(M, g) < \omega_p(M, g)$$

where $m_0 = \lceil C p^{\frac{1}{n+1}} \rceil$. Note that this implies that

$$\omega_{p-\ell m_0}(M, g) < \cdots < \omega_{p-2m_0}(M, g) < \omega_{p-m_0}(M, g) < \omega_p(M, g)$$

where $\ell = \lfloor (p - p_0)/m_0 \rfloor$. Moreover, by the choice of m_0, there exists $q_0 \geq p_0$ and $0 < C' < C$ such that $\ell \geq C' p^{\frac{n}{n+1}}$ for all p. Therefore, for all $p \geq q_0$,

$$\#\{\omega_k(M, g) : p_0 \leq k \leq p\} \geq \ell \geq C' p^{\frac{n}{n+1}}.$$

Now, we will show that this forces the area of stable minimal hypersurfaces to be unbounded. Otherwise, then by Lemma 3.1.6, there are only finitely many possible stable areas $\{\alpha_1, \ldots, \alpha_N\}$. By the choice of p_0, we know that each width with $p \geq p_0$ is of the form $\omega_p(M, g) = m \, \mathrm{area}_g(\Sigma)$ for some $m \in \mathbb{N}$ and some stable minimal hypersurface Σ by Lemma 3.1.3. However, this implies

$$\#\{\omega_k(M, g) : p_0 \leq k \leq p\} \leq \#\{m\alpha_i : m \in \mathbb{N}, 1 \leq i \leq N, m\alpha_i \leq \omega_p(M, g)\}.$$

Let $\alpha = \min_i \alpha_i$ so that if $m\alpha_i \leq \omega_p(M, g)$, then $m \leq \frac{\omega_p(M,g)}{\alpha} \leq \frac{C}{\alpha} p^{\frac{1}{n+1}}$. Thus,

$$\#\{m\alpha_i : m \in \mathbb{N}, 1 \leq i \leq N, m\alpha_i \leq \omega_p(M, g)\} \leq C'' p^{\frac{1}{n+1}}$$

where $C'' = \frac{CN}{\alpha}$. But this gives a contradiction. $\square$

CHAPTER 4

WEAK CORE MANIFOLDS

Song resolved Yau's conjecture by proving in [So1] that for any (M^{n+1}, g) with $3 \le n+1 \le 7$, there exists infinitely many distinct minimal hypersurfaces. Song did so by building upon the work of Marques-Neves [MN1] in the Frankel case.

His idea is that if (M, g) not Frankel and only has finitely many stable minimal hypersurfaces, then we can cut M along some contracting minimal hypersurfaces to construct a compact manifold U with contracting boundary which satisfies the Frankel property for minimal hypersurfaces in $\mathrm{int}(U)$. He then introduced a novel min-max theory which generates minimal hypersurfaces in $\mathrm{int}(U)$ by essentially doing min-max on a non-compact manifold formed by gluing cylindrical ends to U.

Although the widths $\widetilde{\omega}_p$ in Song's min-max theory grow linearly (instead of sublinearly), by the nature of his construction, Song was able to say more delicate information about the growth of these widths which allows him to find infinitely many minimal hypersurfaces.

In this section we adapt Song's ideas, but using our more general notion of weak cores. Many Song's results carry over to this case without much more work.

4.1 Song's constrained minmax theory

First, we will briefly overview Song's min-max theory introduced in [So1]. Let (U, g) be compact with nonempty minimal contracting boundary. Consider the complete non-compact cylindrical extension

$$\mathcal{C}(U) = U \cup (\partial U \times [0, \infty))$$

where we identify the boundaries $\partial U \subset U$ and $\partial U \times \{0\} \subset \partial U \times [0, \infty)$ to each other in the obvious way. Give $(\mathcal{C}(U), h)$ the metric g on U and the product metric on $\partial U \times [0, \infty)$. Note that $\mathcal{C}(U)$ may not be smooth where we glued at.

Let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of $\mathcal{C}(U)$ by compact regions with smooth boundary. We can consider the corresponding ordinary min-max widths[1] to define

$$\widetilde{\omega}_p(U,g) := \omega_p(\mathcal{C}(U),h) := \lim_{i\to\infty} \omega_p(K_i,h)$$

which Song shows is independent of the choice of exhaustion. Although we are considering sweepouts of the complete non-compact space $\mathcal{C}(U)$, Song shows that we realize the widths by minimal hypersurfaces with support in $\operatorname{int}(U) \subset \mathcal{C}(U)$. Since the time Song gave his proof, the multiplicity-one conjecture has been settled in [Zh] and [SWZ] which gives the following result.

Proposition 4.1.1 ([So1] Theorem 9, [SWZ] Theorem 1.1)**.** *Let (U,g) be compact with nonempty minimal contracting boundary, and let $p \in \mathbb{N}$ be fixed. Then there exist a collection of disjoint connected minimal hypersurfaces $\Gamma_1,\ldots,\Gamma_N \subset \operatorname{int}(U)$ along with positive integer multiplicities $m_1,\ldots,m_N$ such that*

$$\widetilde{\omega}_p(U,g) = \sum_{i=1}^{N} m_i \operatorname{area}_g(\Gamma_i) \quad \textit{and} \quad \sum_{i=1}^{N} \operatorname{index}_g(\Gamma_i) \le p$$

where $m_i = 1$ whenever the component Γ_i is unstable.

Proof. The proof is the same as [So1, Theorem 9], except at the time of Song's proof, the index bounds and multiplicity-one result in [SWZ, Theorem 1.1] were not available. □

4.2 Volume spectrum of a weak core

We need to generalize the notion of Song's cores.

1. Although K_i has boundary, we can still define the widths $\omega_p(K_i,h)$ and by the min-max theory of Li-Zhou [LZ] (see also [SWZ]), we can find minimal hypersurfaces realizing the widths. However, these may have boundary which touches the boundary of K_i.

Definition 4.2.1. We say a compact manifold (U, g) with nonempty minimal contracting boundary is a *weak core* if there exist no minimal hypersurfaces contained in $\text{int}(U)$ which are contracting on a side.

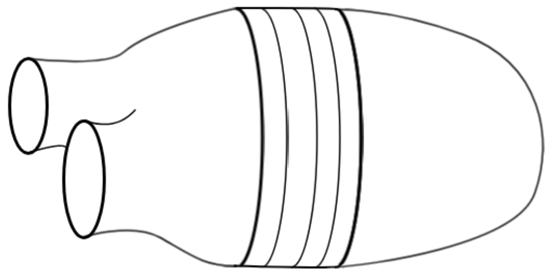

Figure 4.1: Unlike Song's notion of a core, these weak core can have disjoint minimal hypersurfaces, but they must be connected by a minimal foliation.

Corollary 4.2.2. *Let (U, g) be a weak core. For $p \in \mathbb{N}$, there exists a connected minimal hypersurface Γ_p in the interior of U and an positive integer m_p such that*

$$\widetilde{\omega}_p(U, g) = m_p \, \text{area}_g(\Gamma_p) \quad \textit{and} \quad \text{index}_g(\Gamma_p) \leq p$$

where $m_p = 1$ whenever Γ_p is unstable.

Proof. Fix $p \in \mathbb{N}$. By Proposition 4.1.1, there exist disjoint connected minimal hypersurfaces $\Gamma_1, \ldots, \Gamma_N \subset \text{int}(U)$ with positive integers $m_1, \ldots, m_N$ such that

$$\widetilde{\omega}_p(U, g) = \sum_{i=1}^{N} m_i \, \text{area}_g(\Gamma_i).$$

Since (U, g) is a weak core and $\Gamma_1, \ldots, \Gamma_N$ are disjoint, we must have that

$$\text{area}_g(\Gamma_1) = \text{area}_g(\Gamma_2) = \cdots = \text{area}_g(\Gamma_N).$$

by Remark 3.1.4. Now, let $\Gamma = \Gamma_1$ and $m = m_1 + \cdots + m_N$ so that

$$\widetilde{\omega}_p(U, g) = \sum_{i=1}^{N} m_i \operatorname{area}_g(\Gamma_i) = m \operatorname{area}_g(\Gamma).$$

Finally, note if $m > 1$, then either $N > 1$ or $m_1 > 1$. In the first case, Γ is stable by Remark 3.1.4, while $\Gamma = \Gamma_1$ must be stable by Proposition 4.1.1 in the second case. □

The largest area boundary component of U plays an important role in Song's proof. This lemma is proved exactly like [So1, Lemma 12], but using Lemma 2.1.9.

Lemma 4.2.3 ([So1])**.** *Let (U, g) be a weak core, and let A be the area of the largest connected component of ∂U. For any minimal hypersurface $\Gamma \subset \operatorname{int}(U)$,*

$$\operatorname{area}_g(\Gamma) > A.$$

Theorem 4.2.4. *If (U, g) is a weak core, then there exist connected minimal hypersurfaces in U of arbitrarily large area.*

Proof. Assume that the area of stable minimal hypersurfaces in (U, g) is uniformly bounded. Let A denote the area of the largest component of ∂U. Then by [So1, Thereom 8], there exists some constant B (which is independent of p) such that

$$Ap \leq \widetilde{\omega}_p(U, g) \leq Ap + Bp^{\frac{1}{n+1}} \qquad \text{and} \qquad \widetilde{\omega}_{p+1}(U, g) \geq \widetilde{\omega}_p(U, g) + A$$

for all p. By Proposition 4.2.2, we can find a connected minimal hypersurface Γ_p contained the interior of U along with a positive integer m_p such that

$$\widetilde{\omega}_p(U, g) = m_p \operatorname{area}_g(\Gamma_p)$$

where $m_p = 1$ whenever Γ_p is unstable. Moreover, by Lemma 4.2.3, we have that $\operatorname{area}_g(\Gamma_p) >$

A for each p. Consider the set of areas (ignoring multiplicities) which appear in this sequence, that is, the set

$$\{\text{area}_g(\Gamma_p) : p \in \mathbb{N}\}.$$

Note this set must be infinite because if not, then by [So1, Lemma 13], there would exist an $\varepsilon > 0$ such that for all p large enough,

$$\widetilde{\omega}_p(U, g) > (A + \varepsilon)p$$

which contradicts the growth of the widths. Recall there are only finitely many distinct areas of stable minimal hypersurfaces in (U, g) by Lemma 3.1.6 and our assumed stable area bound. Therefore, we can find an increasing sequence p_k such that each Γ_{p_k} is unstable so that $m_{p_k} = 1$, and thus,

$$Ap_k \leq \widetilde{\omega}_{p_k}(U, g) = \text{area}_g(\Gamma_{p_k})$$

for all k. And so, $\text{area}_g(\Gamma_{p_k}) \to \infty$ as $k \to \infty$. □

CHAPTER 5

CORELESS MANIFOLDS

In this section, we will assume that (M, g) is not weakly Frankel, does not contain a weak core, and has a uniform area bound for connected stable minimal hypersurfaces. We say that such a manifold is *coreless*. Note if (M, g) were not coreless, then we can find minimal hypersurfaces of arbitrarily large area by Sections 3 and 4.

The following lemma is proven just like [So1, Lemma 11], but using Lemma 2.1.9.

Lemma 5.0.1 ([So1, Lemma 11])**.** *Let (N, g) be compact manifold with (possibly empty) contracting minimal boundary, and let $\Gamma \subset \operatorname{int}(N)$ be a minimal hypersurface which is contracting on a side. We can cut N along Γ (and possibly finitely many other minimal hypersurfaces) to obtain a compact manifold (N', g) with contracting minimal boundary which contains a component $\Sigma \subseteq \partial N'$ which is isometric to Γ.*

Definition 5.0.2. We say that U is a *Song region* of (M, g) if (U, g) is a compact manifold with nonempty contracting boundary formed by *cutting M* along some collection of disjoint (stable) minimal hypersurfaces $\Sigma_1, \ldots, \Sigma_N$, that is, we consider some connected component of the metric completion of $M \setminus (\Sigma_1 \cup \cdots \cup \Sigma_N)$.

5.1 Accumulating minimal hypersurfaces

Like in Section 3.1, we define

$$\mathcal{M}^S(U) = \{\Sigma \subset \operatorname{int}(U) : \Sigma \text{ connected stable minimal hypersurface in } (U, g)\}$$

for Song regions U of (M, g). Since we are assuming a uniform area bound for stable minimal hypersurfaces, this space is strongly compact in the smooth topology [ScSi] (see also [Sh]). We will be interested in various subspaces coming from different Song regions.

Observe our assumption that no Song region U of (M, g) is a weak core implies

$$\mathcal{M}^C(U) = \{\Sigma \in \mathcal{M}^S(U) : \Sigma \text{ is contracting on a side}\}.$$

is always nonempty. Note that we can consider $\mathcal{M}^C(U)$ as a subspace of the compact space $\mathcal{M}^S(M)$ of all connected stable minimal hypersurfaces in (M, g). However, note that this subspace is not closed. In fact, if we denote

$$\mathcal{M}^A(U) = \{\Sigma \in \mathcal{M}^S(U) : \Sigma \text{ is accumulating on a side}\},$$

then we have that $\mathcal{M}^A(U)$ is precisely the limit points of $\mathcal{M}^C(U)$ so that the closure

$$\overline{\mathcal{M}}^C(U) = \mathcal{M}^C(U) \cup \mathcal{M}^A(U).$$

By iteratively applying Lemma 5.0.1, we can see that $\mathcal{M}^A(U)$ must always be nonempty as well. In fact, we will look at the largest area minimal hypersurface in $\mathcal{M}^A(U)$, and either, it will be very pathological, or it will have a nice enough local structure will allow us to find large minimal hypersurfaces.

Lemma 5.1.1. *Let U be a Song region of a coreless (M, g). There exists a minimal hypersurface $\Gamma \subset \operatorname{int}(U)$ that is accumulating on a side and not contracting on the other side.*

Proof. Since M has no weak core, the set $\mathcal{M}^C(U)$ is nonempty. By compactness of stable minimal hypersurfaces, we can find Γ in the closure of $\mathcal{M}^C(U)$ such that

$$\text{area}_g(\Gamma) = \sup\{\text{area}_g(\Sigma) : \Sigma \in \mathcal{M}^C(U)\}.$$

Note since Γ is in the closure of $\mathcal{M}^C(U)$, it suffices to show that Γ is not contracting on a side because then Γ must be a limit point of $\mathcal{M}^C(U)$ and hence also accumulating on a side.

So suppose otherwise that Γ is contracting on a side. Let W_0 be the connected component of the metric completion of $U \setminus \Gamma$ which has a contracting boundary component Σ isometric to Γ given by Lemma 2.1.9. We will inductively cut W_0 along finitely many minimal hypersurfaces to find weak core $W = W_N$ which would give a contradiction.

Given W_i containing this same boundary component Σ. By compactness, there is a stable minimal hypersurface $\Sigma_i \in \overline{\mathcal{M}}^C(W_i)$ such that

$$d_H(\Sigma, \Sigma_i) = \inf\{d_H(\Sigma, \Sigma') : \Sigma' \in \mathcal{M}^C(W_i)\}$$

where d_H denotes Hausdorff distance in W_i. Note that this minimizer Σ_i has a positive distance from Σ by the maximum principle applied to the contracting neighborhood of Σ. Now, we form W_{i+1} by cutting W_i along Σ_i and picking the connected component containing Σ as a boundary component.

We will now see that the above process must eventually terminate. Otherwise, we obtain a sequence of disjoint stable $\Sigma_i \in \overline{\mathcal{M}}^C(W_i) \subseteq \overline{\mathcal{M}}^C(W_0)$. By compactness, there is a smoothly and graphically convergent subsequence. Therefore, we can find $N_0 < N_1 < N_2$ and closed cylindrical regions R_0 and R_1 such that

$$\partial R_0 = \Sigma_{N_0} \cup \Sigma_{N_1}, \qquad \partial R_1 = \Sigma_{N_1} \cup \Sigma_{N_2}, \qquad R_0 \cap R_1 = \Sigma_{N_1}.$$

Observe that W_{N_1} contains a boundary component isometric to Σ_{N_0} which has the cylindrical neighborhood $R_0 \cup R_1$ contained in W_{N_1}. Since the R_0 part touches a boundary component not equal to Σ, then Σ_{N_2} must strictly closer than Σ_{N_1} to Σ inside W_{N_1}. However, this would contradict our choice of Σ_{N_1}.

Therefore, this inductive procedure must give a compact region $W = W_N$ such that $\mathcal{M}^C(W)$ is empty. Finally, to show that W is weak core—in order to reach a contradiction to our assumption that Γ is contracting on a side—we just need to show that the other components of ∂W are also contracting.

So consider any other boundary component $\Sigma' \subseteq \partial W \setminus \Sigma$. Note that

$$\text{area}_g(\Sigma') \leq \text{area}_g(\Gamma) = \text{area}_g(\Sigma) \leq \text{area}_g(W \setminus \Sigma')$$

by the choice of Γ. By the construction of W, there are no contracting minimal hypersurfaces in $\text{int}(W)$ and W is not foliated by minimal hypersurface (because $\Sigma \subseteq \partial W$ is contracting). Therefore, we reach a contradiction by Lemma 2.1.9. $\square$

5.2 Monotonic saddle minimal hypersurfaces

As mentioned, minimal hypersurfaces found by Lemma 5.1.1 may have a nice enough structure.

Definition 5.2.1. We say a minimal hypersurface $\Gamma \subset (M, g)$ is a *monotonic saddle* if there exists a foliation $\Gamma_t := \varphi(\Gamma \times \{t\})$ given by $\varphi : \Gamma \times [-\delta, \delta] \to M$ such that

- φ is a diffeomorphism onto its image with $\Gamma_0 = \Gamma$,
- Γ_t either is minimal or has entirely non-vanishing mean curvature for all t,
- $\Gamma_{-\delta}$ and Γ_δ have entirely non-vanishing mean curvature, and
- $\text{area}_g(\Gamma_t)$ is nondecreasing for $t \in [-\delta, 0]$ and is nonincreasing for $t \in [0, \delta]$.

We will show that if (M, g) has a monotonic saddle which is accumulating on a side, then we can find a Song region which looks like a weak core after small conformal perturbations.

Proposition 5.2.2. *If (M, g) has a monotonic saddle minimal hypersurface Γ which is accumulating on a side, then M has minimal hypersurfaces of arbitrarily large area.*

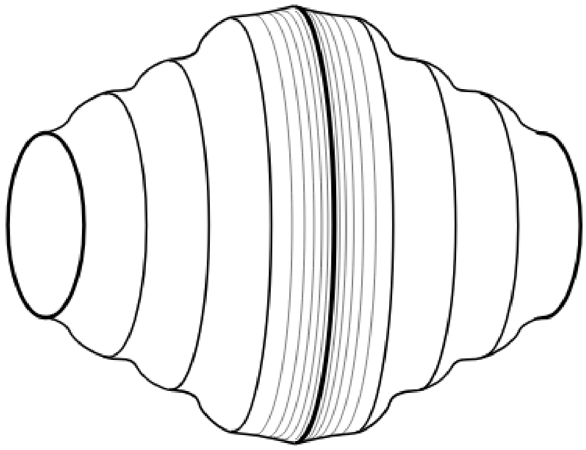

Figure 5.1: This is a model of a monotonic saddle minimal hypersurface. There are infinitely many minimal hypersurfaces approaching the center one. However, after a small conformal perturbation, this metric becomes a weak core.

Proof. If Γ is a monotonic saddle, then there exists a foliation Γ_t as described in Definition 5.2.1. Moreover, if Γ is accumulating on a side, we can also assume that there exist arbitrarily small $s \in (-\delta, 0)$ such that Γ_s is minimal and contracting inside $\phi(\Gamma \times [s, 0])$.

Following [So2, Section 3.2][1], we can run level set flow on $\phi(\Gamma \times [-\delta, \delta])$ to find a compact region W containing $\phi(\Gamma \times [-\delta, \delta])$ where ∂W is (possibly empty) minimal and contracting. Note, by the maximum principle, the only minimal hypersurfaces disjoint from Γ inside $\mathrm{int}(W)$ are of the form Γ_t for some $t \in (-\delta, \delta)$.

First, we will show that there exists a $t_0 \in [-\delta, 0)$ such that any stable minimal hypersurface Σ which intersects Γ must then also intersect Γ_{t_0}. Otherwise, we can find a sequence of stable minimal hypersurfaces Σ_k intersecting Γ but disjoint from Γ_{t_k} for some negative sequence $t_k \to 0$. This implies that Σ_k converges to Γ by the maximum principle. However, since Σ_k intersects Γ for all k, we can construct a Jacobi field for Γ that changes sign which contradicts the fact that Γ is stable.

So let U be the connected component containing Γ of the metric completion of $W \setminus \Gamma_{s_0}$

1. In [So2], Song works with manifolds which are *thick-at-infinity*, but such manifolds include the class of compact manifolds with minimal boundary.

where $s_0 \in (t_0, 0)$ such that Γ_{s_0} is minimal and contracting inside $\phi(\Gamma \times [s_0, 0])$. Note U is a compact region with nonempty minimal contracting boundary. Furthermore, by construction, U contains no stable minimal hypersurfaces which intersect Γ. This implies—by a standard area minimization argument—that any two minimal hypersurfaces in U which intersect Γ must also intersect each other.

Now although this (U, g) is not a weak core, we will see that from the point of view of min-max, can think of it as being so. By [So2, Lemma 23], we can find a sequence of metrics $h^{(i)}$ converging smoothly to g such that with respect to $h^{(i)}$,

- $g \equiv h^{(i)}$ on $U \setminus \phi(\Gamma \times [s_0, \delta])$,
- ∂U is still minimal and contracting,
- $\Gamma = \Gamma_0$ is an unstable minimal hypersurface, and
- Γ_t has nonzero mean curvature pointing away from Γ for $t \in (s_0, 0) \cup (0, \delta)$.

The maximum principle implies that for all i, any minimal hypersurface in $(\mathrm{int}(U), h^{(i)})$ must intersect Γ. So by Proposition 4.1.1, for each p and i, we can find a $h^{(i)}$-stationary integral varifold $V_p^{(i)}$ contained in $\mathrm{int}(U)$ with

$$\widetilde{\omega}_p(U, h^{(i)}) = \|V_p^{(i)}\|(U)$$

and whose support is a smooth minimal hypersurface with index at most p such that if the multiplicity of any component is more than 1, then that component is stable. For fixed p, by [Sh, Theorem A.6], after picking a subsequence, $V_p^{(i)}$ converges to some g-stationary integral varifold V_p with smooth minimal support and

$$\|V_p\|(U) = \lim_{i \to \infty} \|V_p^{(i)}\|(U) = \lim_{i \to \infty} \widetilde{\omega}_p(U, h^{(i)}) = \widetilde{\omega}_p(U, g)$$

because the widths vary continuously in the metric. Since each component of $V_p^{(i)}$ intersects Γ, then so does each component of V_p which implies that the support of V_p is connected. Moreover, V_p must either be an integer multiple m_p of Γ or is unstable and hence has multiplicity $m_p = 1$ by [Sh, Theorem A.6]. Therefore, there must be minimal hypersurfaces of arbitrarily large area by the same argument in Theorem 4.2.4. □

5.3 Pathological case

As mentioned, the minimal hypersurfaces found by Lemma 5.1.1 may be very pathological in the following sense.

Definition 5.3.1. We say a minimal hypersurface $\Gamma \subset (M, g)$ is *non-monotonic* if there *does not* exist foliation $\Gamma_t := \varphi(\Gamma \times \{t\})$ given by a diffeomorphism $\varphi : \Gamma \times [-\delta, \delta]$ onto its image with $\Gamma_0 = \Gamma$ such that $\text{area}_g(\Gamma_t)$ is weakly monotonic on each $[-\delta, 0]$ and $[0, \delta]$.

Note that non-monotonic minimal hypersurfaces are necessarily accumulating on a side. Note such minimal hypersurfaces can possibly exist. See Appendix B for an example. So given a Song region U, consider

$$\mathcal{M}^N(U) = \{\Sigma \in \mathcal{M}^A(U) : \Sigma \text{ is non-monotonic}\},$$

and let $\mathcal{M}^N$ be the set of all non-monotonic minimal hypersurfaces in (M, g).

Proposition 5.3.2. *Let (M, g) be coreless with no monotonic saddle minimal hypersurfaces. The space $\mathcal{M}^N$ of non-monotonic minimal hypersurfaces is homeomorphic to a Cantor set.*

Proof. First, we will show that $\mathcal{M}^N$ is nonempty. In fact, we will show that $\mathcal{M}^N(U)$ is non-empty for all Song regions U. Consider a minimal hypersurface Γ given by Lemma 5.1.1 so that Γ is accumulating on a side but not contracting on the other. If Γ is not non-monotonic, then must have a minimal foliation on a side. So if we consider the longest minimal foliation Γ_t for $t \in [0, \varepsilon]$ generated by Γ, then Γ_ε must be non-monotonic, or else Γ would be a monotone saddle minimal hypersurface. Thus, $\mathcal{M}^N(U)$ is nonempty.

We will now show that $\mathcal{M}^N$ is a perfect set. Let $\Gamma \in \mathcal{M}^N$. Then there exists $\phi : \Gamma \times [0, \delta] \to M$ onto a closed halved tubular neighborhood $\varphi(\Gamma \times [0, \delta])$ where $\Gamma_0 = \Gamma$ and there exists sequences $0 < s_k < t_k$ with $t_k \to 0$ such that $U_k = \varphi(\Gamma \times [s_k, t_k])$ is a Song region. By the above, there exists $\Gamma_k \in \mathcal{M}^N(U) \subseteq \mathcal{M}^N$. Since $s_k, t_k \to 0$, we must have that Γ_k converges to Γ by the maximum principle.

Finally, note that $\mathcal{M}^N$ is a metric space (from the flat metric) and is totally disconnected. Therefore, by Brouwer's characterization [Br], $\mathcal{M}^N$ is homeomorphic to a Cantor set. □

Proof of Theorem 1.2. Let (M^{n+1}, g) be an arbitrary closed Riemannian manifold with $3 \leq n + 1 \leq 7$. In the case, where (M, g) is weakly Frankel, we can find connected minimal hypersurfaces with arbitrarily large area by Theorem 3.3.3.

If (M, g) is not weakly Frankel, the by Lemma 5.0.1, we can find Song regions U of (M, g). If one of these Song regions is a weak core, then we can find connected minimal hypersurfaces with arbitrarily large area by Theorem 4.2.4.

Finally, assume that (M, g) is not weakly Frankel, does not contain a weak core, and that the area of stable minimal hypersurfaces is uniformly bounded. If (M, g) contains a monotonic saddle minimal hypersurface, then we can find connected minimal hypersurfaces with arbitrarily large area by Proposition 5.2.2. Otherwise, by Proposition 5.3.2, we are in the second case of Theorem 1.2. □

CHAPTER 6

MODIFICATIONS FOR ONE-SIDED HYPERSURFACES

In the previous sections, we assumed that (M, g) contains no one-sided minimal hypersurfaces for simplicity. However, our results still hold without this assumption. The purpose of this section is to explain some of the technical modifications needed to handle when (M, g) possibly contains one-sided minimal hypersurfaces.

So suppose $\Gamma \subset (M, g)$ is a one-sided minimal hypersurface. We say that Γ is *contracting* (note there is only one side here) if the two-sided double cover is contracting. Equivalently, this says that if N is the metric completion of $M \setminus \Gamma$, then N has a boundary component Σ which is contracting and is isometric to the double cover. Also, note if this double cover of Γ is merely stable, our Jacobi field arguments used throughout still apply by lifting things to this double cover.

Finally, note that Zhou [Zh] showed min-max theory does not produce one-sided minimal hypersurfaces for bumpy metrics. In particular, this implies for a general metric, if a one-sided component does appear from min-max, then it will have even multiplicity. Moreover, when the multiplicity is strictly larger than 2, then the double cover must be degenerate stable.

6.1 Weakly Frankel case

Now we discuss the specific modifications needed in Section 3. The definition for weakly Frankel remains the same by using the notion of contracting above. Then all the results follow by considering the double cover whenever working with a one-sided minimal hypersurface. In particular, we can still define when disjoint minimal hypersurfaces Γ_0, Γ_1 (each possibly one-sided) are connected by a minimal foliation by cutting along them and using Lemma 2.1.9. Again, there are three different types of one-sided minimal hypersurfaces $\Gamma \subset (M, g)$:

1. We say Γ is *isolated* it has no local minimal foliation.

2. We say Γ generates a *partial minimal foliation* if it is connected to other hypersurfaces by minimal foliations where we can parameterize them all as $\Sigma_t = \varphi(\Sigma \times \{t\})$ for some $\varphi : \Sigma \times [0,1] \to M$ where $\varphi|_{\Sigma\times(0,1]}$ is a diffeomorphism onto its image where $\varphi|_{\Sigma\times\{0\}}$ is a double covering onto Γ.

3. We say Σ generates a *(full) minimal foliation* if it is connected to other hypersurfaces by minimal foliations which union to M and where we can parameterize them all as $\Sigma_t = \varphi(\Sigma \times \{t\})$ for some $\varphi : \Sigma \times [0,1] \to M$ where $\varphi|_{\Sigma\times(0,1)}$ is a diffeomorphism onto its image and where both $\varphi|_{\Sigma\times\{0\}}$ and $\varphi|_{\Sigma\times\{1\}}$ are double covering maps onto one-sided minimal hypersurfaces.

We can still consider the cycles defined in Section 3.2 associated to the above:

1. We get the zero cycle in $\mathcal{Z}_n(M)$ since Γ occurs with even multiplicity.

2. Here the space of cycles $\mathcal{K}_\omega$ is similar as in Lemma 3.2.2, but it will be connected in this case and strongly deformation retracts to the zero cycle (which represents the double cover Σ_0 of Γ).

3. Here the space of cycles $\mathcal{F}_\omega$ is similar as in Lemma 3.2.1. From the above, we get map $S^1 \to \mathcal{Z}_n(M)$ parametrizing the foliation by considering Σ_t as cycles and identifying the double covers Σ_0, Σ_1 with the zero cycle. Again, by taking products of this map and quotienting, we obtain a homeomorphism $TP^m(S^1) \to \mathcal{F}_\omega$ where now m is the largest integer with $2m \leq \omega/\operatorname{area}_g(\Gamma)$. Thus, $\mathcal{F}_\omega \cong \mathbb{RP}^m$.

In particular, we still can describe the topology of the space of all stable cycles $\mathcal{S}_\omega \subset \mathcal{Z}_n(M)$ to show that $H^m(\mathcal{S}_\omega, \mathbb{Z}_2) = 0$ for $m \geq C'\omega$. Then the rest of the arguments follow directly.

6.2 Weak core case

As before, the one-sided components which appear from Song's min-max occur with even multiplicity, and all our Jacobi field arguments used still apply by lifting things to double covers if necessary. Although, all the results we use from [So1] are stated for two-sided hypersurfaces, in [So1], Song handles the one-sided cases (with appropriate modifications). So everything still follows in this case.

6.3 Coreless case

Again, we can still define when a one-sided minimal hypersurface $\Gamma \subset (M, g)$ is accumulating or non-monotonic, by considering the two-sided double cover (or equivalently, by cutting M along Γ and considering the metric completion). Also, there is no issue in defining one-sided monotonic saddles.

CHAPTER 7

APPLICATIONS

7.1 Analytic manifolds

Łojasiewicz-Simon inequality is a powerful tool to study the local behavior of a critical point of an analytic elliptic integrand functional, see [Si1]. In the special case that the analytic functional is chosen to be the area functional in an analytic manifold, the result implies that:

Theorem 7.1.1 (Łojasiewicz-Simon inequality [Si1])**.** *Suppose (M, g) is an analytic manifold and Σ is a minimal hypersurface. If a sequence of minimal hypersurfaces $\{\Sigma_i\}$ converges to Σ smoothly, then when i is sufficiently large,* $\text{area}(\Sigma_i) = \text{area}(\Sigma)$*.*

However, we only need to know this for stable minimal hypersurfaces, so we give an explicit proof this fact in Corollary A.2 using the implicit function theorem. In particular, we show if Σ is a non-isolated stable minimal hypersurface, then Σ must locally be a minimal foliation on both sides[1], As a consequence, we have that:

Corollary 7.1.2. *In an analytic metric, there exist no minimal hypersurfaces which are accumulating on a side.*

Note the same holds for bumpy metrics simply because accumulating minimal hypersurfaces are degenerate. However, unlike bumpy metrics, analytic metrics can (and often will) have minimal foliations. Such foliations can complicate the space of minimal hypersurfaces, but as shown in Section 3, we can control such things.

Proof of Theorem 1.1. Let (M^{n+1}, g) with $3 \leq n+1 \leq 7$ have an analytic metric. Note that in Section 5, we showed if a manifold is not weakly Frankel, has no weak core, and has a uniform area bound for stable minimal hypersurfaces, then there must exist some

1. In fact, compactness and the maximum principle tell us that this local foliation extends to a (full) minimal foliation of (M, g).

accumulating minimal hypersurface (for example, see Lemma 5.1.1). But by Corollary A.2, there exist no accumulating minimal hypersurfaces in analytic metrics.

Therefore, either (M, g) must either have stable minimal hypersurfaces of arbitrarily large area, be weakly Frankel, or has a weak core. In the later cases, we can find arbitrarily large minimal hypersurfaces by Theorem 3.3.3 and Theorem 4.2.4. □

7.2 Positive scalar curvature 3-manifolds

In a 3-manifold with positive scalar curvature, the area of a minimal surface is controlled by the index. This is a consequence of the following theorem proved by Chodosh-Ketover-Maximo:

Theorem 7.2.1 ([CKM] Theorem 1.3)**.** *Suppose (M^3, g) is a closed 3-manifold with positive scalar curvature. For any $I \in \mathbb{N}$, there exist $A_0 = A_0(M, g, I) > 0$ such that if Σ is a closed embedded minimal surface in (M, g), then*

$$\text{area}(\Sigma) \leq A_0 \quad \textit{whenever} \quad \text{index}_g(\Sigma) \leq I.$$

From this, we obtain the following result.

Proposition 7.2.2. *If (M^3, g) has positive scalar curvature and is analytic, there exist connected minimal surfaces of arbitrarily large area and index.*

Proof. Suppose (M^3, g) is analytic with positive scalar curvature. Since the metric is analytic, we can find minimal surfaces of arbitrarily large area by Theorem 1.1. Therefore, the result follows from [CKM, Theorem 1.3]. □

Moreover, from our methods along with some facts about minimal foliations proved in the next Section 7.3 give the following results.

Proposition 7.2.3. *If (M^3, g) has positive scalar curvature and is foliated by minimal surfaces, there exist connected minimal surfaces of arbitrarily large area and index.*

Proof. Suppose (M^3, g) has positive scalar curvature and admits a foliation by closed embedded minimal surfaces. We will show that (M, g) must be weakly Frankel. Note that Lemma 7.3.1 shows that any minimal surface is either a leaf in some minimal foliation, or intersects each leaf of every minimal foliation. Because the manifold (M, g) has positive scalar curvature, a classical result of Schoen-Yau [SY1] shows that any stable minimal surfaces must be topologically either a sphere or real protective plane (in particular, has finite fundamental group). Thus, the second part of Lemma 7.3.1 shows that any stable minimal surface in (M, g) must be a leaf of a foliation. Therefore, (M^3, g) must be weakly Frankel, and hence has minimal hypersurfaces of arbitrarily large area by Theorem 3.3.3. □

Corollary 7.2.4. *If $S^2 \times S^1$ has the product metric where (S^2, g) has positive curvature, then there exist connected minimal surfaces of arbitrarily large area and genus.*

7.3 Foliations by minimal hypersurfaces

A special case of a weakly Frankel manifold when the whole manifold is foliated by closed embedded minimal hypersurfaces. In general, even without the weakly Frankel property, the minimal hypersurfaces satisfy some nice properties.

Note here the foliation is given in the sense of Lemma 2.1.9: if we cut a manifold (M, g) along a minimal hypersurface Σ to get a manifold with boundary (N, g), then there is a diffeomorphism $\varphi : \Sigma \times [0, 1] \to N$ such that $\varphi(\Sigma \times \{t\})$ is a minimal hypersurface for all $t \in [0, 1]$. In particular, this allows one to construct a fiber bundle $\pi : M \to S^1$ where the fibers $\pi^{-1}(\{\theta\}) = \Sigma_\theta$ parameterize this foliation.

Lemma 7.3.1. *Suppose (M, g) is foliated by closed embedded minimal hypersurfaces. Then any minimal hypersurface Γ must either be a leaf of some foliation or intersects every leaf* *of any minimal foliation. Moreover, when Γ is not a leaf of some foliation, the fundamental group of Γ is infinite.*

Proof. Fix a foliation of (M, g) by viewing M as a fiber bundle $\pi : M \to S^1$ where the fibers $\pi^{-1}(\{\theta\}) = \Sigma_\theta$ are connected minimal hypersurfaces parameterizing the foliation. By unraveling the base circle, there exists a Riemannian cover $p : \Sigma \times \mathbb{R} \to M$ where each slice $\Sigma \times \{t\}$ is closed minimal hypersurface which projects isometrically to some Σ_θ in our foliation.

Suppose Γ is a minimal hypersurface in (M, g) which does not intersect every leaf Σ_θ in the foliation. Then any loop contained in Γ can be homotoped inside M to a loop contained in some fixed slice Σ_{θ_0}. By the lifting criterion, we can lift to obtain a minimal hypersurface $\hat{\Gamma} \subset \Sigma \times \mathbb{R}$ which projects isometrically to Γ. Since $\hat{\Gamma}$ is compact, it must touch some minimal leaf $\Sigma \times \{t\} \subset \Sigma \times \mathbb{R}$ from one side. Therefore, by the maximum principle, $\hat{\Gamma}$ must equal some slice $\Sigma \times \{t\}$, and hence, Γ must equal some leaf Σ_θ.

Finally, suppose Γ is a minimal hypersurface which is not equal to some leaf of the foliation. We can lift the universal cover $\tilde{\Gamma}$ of Γ to get a minimal immersion $\tilde{\Gamma} \to \Sigma \times \mathbb{R}$. But if $\pi_1(\Gamma)$ were finite, then $\tilde{\Gamma}$ is also compact which would again give contradiction by the maximum principle. □

Remark 7.3.2. The second part of the lemma can be made stronger. For instance, if Γ is a minimal hypersurface which is not equal to some leaf of the foliation, one can show $\pi_* i_*(\pi_1(\Gamma)) \subseteq \pi_1(S^1)$ must be infinite where $i : \Gamma \to M$ is the inclusion.

APPENDIX A

NICE NEIGHBORHOOD LEMMA

Here we prove two lemmas used in Lemma 2.1.8 and Corollary 7.1.2.

Lemma A.1. *Let Γ be a closed two-sided minimal hypersurface in (M, g). There exists a foliation $\{\Gamma_t\}_{t\in[-\delta,\delta]}$ of some tubular neighborhood N such that $\Gamma_0 = \Gamma$ and where for each fixed $t \in [-\delta, \delta]$, the mean curvature of Γ_t is either identically zero, positive everywhere, or negative everywhere. Moreover, the foliation is parameterized by a diffeomorphism*

$$\Gamma_t = \varphi(\Gamma \times \{t\}) \quad \textit{where} \quad \varphi : \Gamma \times [-\delta, \delta] \to N$$

such that $\varphi(x, 0) = x$ for all $x \in \Gamma$.

Proof. For $\phi \in C^\infty(\Gamma)$, consider the smooth hypersurfaces given by

$$\Gamma_\phi = \{\exp_x(\phi(x)\nu(x)) : x \in \Gamma\}.$$

Pick a neighborhood U around $0 \in C^\infty(\Gamma)$ such that Γ_ϕ is embedded for every $\phi \in U$. Consider the smooth map $H : U \to C^\infty(\Gamma)$ where $H(\phi)$ is the mean curvature of Γ_ϕ (pulled back to be a function on Γ). The differential at $0 \in C^\infty(\Gamma)$

$$DH_0 : C^\infty(\Gamma) \to C^\infty(\Gamma)$$

is given by the Jacobi operator of Γ, that is, $DH_0 = L_\Gamma$ where

$$L_\Gamma = -\Delta - |A|^2 - \mathrm{Ric}_M(\nu, \nu).$$

Let λ be the least eigenvalue for L_Γ. Recall the corresponding eigenspace for λ is spanned by a single eigenfunction ϕ_0 which we can assume to be positive (see Lemma 2.1.1). Note

$$H(t\phi_0) = H(0) + DH_0(t\phi_0) + O(t^2) = tL_\Gamma(\phi_0) + O(t^2) = t\lambda\phi_0 + O(t^2)$$

by Taylor expansion. Therefore, if $\lambda \neq 0$, then for small enough $\delta > 0$,

$$\varphi : \Gamma \times [-\delta, \delta] \to M \quad \text{given by} \quad \varphi(x,t) = \exp_x(t\phi_0(x)\nu(x))$$

is the desired local foliation. Moreover, we indeed see that when Γ is unstable (equivalent to $\lambda < 0$), this foliation is expanding on both sides. Likewise, when Γ is strictly stable (equivalent to $\lambda > 0$), this foliation is contracting on both sides.

So now, assume that Γ is degenerate stable, that is, $\lambda = 0$. Note that we have

$$K := \ker(L_\Gamma) = \operatorname{span}(\phi_0).$$

Let $\pi : C^\infty(\Gamma) \to K^\perp$ be the projection onto the L^2 orthogonal complement $K^\perp \subset C^\infty(\Gamma)$ of the kernel, and then consider the map $H^\perp : C^\infty(\Gamma) \to K^\perp$ given by $H^\perp = \pi \circ H$. Decompose the domain as $C^\infty(\Gamma) = K \oplus K^\perp$, and note for $\phi \in K^\perp$,

$$DH^\perp_{(0,0)}(0,\phi) = D\pi_{(0,0)}(DH_{(0,0)}(0,\phi)) = \pi(L_\Gamma(\phi)) = L_\Gamma(\phi)$$

is invertible as map $K^\perp \to K^\perp$ by the Fredholm alternative, and the inverse is bounded (because the spectrum of L_Γ is discrete). Therefore, by the implicit function theorem, there exists an $\varepsilon > 0$ and a neighborhood $W \subset U$ around 0 along with a map $\Phi : (-\varepsilon, \varepsilon) \to K^\perp$ with $\Phi'(0) = 0$ such that for all $\phi \in W$,

$$H^\perp(\phi) = 0 \quad \text{if and only if} \quad \phi = t\phi_0 + \Phi(t)$$

for some $t \in (-\varepsilon, \varepsilon)$. In particular, there exists $c : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$H(t\phi_0 + \Phi(t)) = c(t)\phi_0.$$

Therefore, for $\delta > 0$ sufficiently small, the map

$$\varphi : \Gamma \times [-\delta, \delta] \to M \quad \text{given by} \quad \varphi(x,t) = \exp_x((t\phi_0(x) + \Phi(t)(x))\nu(x))$$

gives the desired local foliation. □

Corollary A.2. *If Γ is a two-sided non-isolated stable minimal hypersurface in an analytic metric (M, g), then the local foliation given by Lemma A.1*

$$\Gamma_t = \varphi(\Gamma \times \{t\}) \quad \textit{where} \quad \varphi : \Gamma \times [-\delta, \delta] \to N$$

is a minimal foliation, that is, Γ_t is minimal for all $t \in [-\delta, \delta]$.

Proof. Since Γ is assumed to be non-isolated, Γ is degenerate stable, that is, we are in the case where $\lambda = 0$ from the previous proof. In that notation, for all $\phi \in W$,

$$H(\phi) = 0 \quad \text{if and only if} \quad \phi = t\phi_0 + \Phi(t) \quad \text{and} \quad c(t) = 0$$

for some $t \in (-\delta, \delta)$. Since the metric is analytic, the map $H : W \to C^\infty(\Gamma)$ is analytic, and thus, the map $c : (-\delta, \delta) \to \mathbb{R}$ is analytic as well. Since Γ is non-isolated, the function $c(t)$ has an accumulation of zeros. Therefore, $c = 0$ identically by analyticity, and so, Γ_t is minimal for all $t \in (-\delta, \delta)$. □

APPENDIX B

EXAMPLE OF A PATHOLOGICAL MANIFOLD

We give examples of smooth manifolds where the second case happens from Theorem 1.2.

Proposition B.1. *For $n \geq 2$, let (Σ^n, g_0) be any closed Riemannian manifold. There exists a smooth metric $(\Sigma \times S^1, g)$ such that the space $\mathcal{M}^N$ of non-monotonic minimal hypersurfaces is homeomorphic to the Cantor set C.*

We will construct a warped product metric using a pathological function.

Lemma B.2. *There exists a smooth positive periodic function $f : \mathbb{R} \to \mathbb{R}$ where:*

1. *each critical point is either non-isolated or a strict local minimum;*
2. *for each non-isolated critical point $p \in \mathbb{R}$, we have that $f(x)$ is not weakly monotone on at least one side $[p - \varepsilon, p]$ or $[p, p + \varepsilon]$ for every $\varepsilon > 0$.*

Proof. Recall the standard Cantor set construction where we start with $C_0 = [0, 1]$ and where given C_{n-1} which consists of 2^{n-1} disjoint closed intervals centered at the points $m_{n,1}, m_{n,2} \ldots, m_{n,2^{n-1}}$, then we form C_n by removing open intervals of length $1/3^n$ centered about those midpoints. Then the Cantor set C is defined to be the intersection of all C_n.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be the smooth bump function given by $\exp(1/(x^2 - 1))$ on $(-1, 1)$ and zero elsewhere, and let $\Psi_{n,k} : \mathbb{R} \to \mathbb{R}$ be this map translated and rescaled as

$$\Psi_{n,k}(x) = \Psi(2 \cdot 3^n (x - m_{n,k}))$$

so that $\Psi_{n,k}$ is non-zero exactly on the open middle third centered at $m_{n,k}$. Define

$$h(x) = \sum_{m \in \mathbb{Z}} \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} c_n \Psi_{n,k}(x - m)$$

and pick the coefficients c_n to decay quick enough to make the function smooth and satisfy $h(x) < 1$. We define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = 1 - h(x)$, and we claim that such an f satisfies item (2) and (3). Since the function has period 1, it suffices to look just at f on $[0, 1]$.

To prove item (2), notice that the critical points $p \in [0, 1]$ are precisely either in C or is some middle third midpoint $m_{n,k}$. Note that $p \in C$ must be non-isolated because C is a perfect set and every point in C is a critical point. And, if p is some midpoint $m_{n,k}$, then p is a strict local minimum because $m_{n,k}$ is the strict maximum of $\Psi_{n,k}$.

Next, we prove item (3). From the above, the set C is all of the non-isolated critical points in $[0, 1]$. By construction, for all $\varepsilon > 0$, we can find some midpoint such that $0 < |p - m_{n,k}| < \varepsilon$. Note that f is not weakly monotone at $m_{n,k}$ Since $m_{n,k}$ is the strict maximum of $\Psi_{n,k}$. □

Proof of Proposition B.1. We consider the smooth warped product metric

$$g = f(t)^2 g_0 + dt^2 \quad \text{on} \quad \Sigma \times S^1$$

where we are identifying S^1 here as $\mathbb{R} / \mathbb{Z}$. Consider the slices $\Sigma_\theta = \Sigma \times \{\theta\}$. Note Σ_θ is a minimal hypersurface of $(\Sigma \times S^1, g)$ if and only if $\theta \in S^1$ is a critical point of f. Moreover, the minimal slices Σ_θ are non-monotonic if and only if $\theta \in S^1$ is a non-isolated critical point of f which is given by the Cantor set by Lemma B.2. □

Remark B.3. It is not known that whether or not such a metric admits minimal hypersurfaces with arbitrarily large area.

REFERENCES

[Ai] Aiex, Nicolau S. *Non-compactness of the space of minimal hypersurfaces*. Math. Ann. 370 (2018), no. 1-2, 191–208.

[All] Allard, William K. “On the First Variation of a Varifold.” Annals of Mathematics 95, no. 3 (1972): 417–91.

[Alm1] Almgren, Frederick Justin, Jr. *The homotopy groups of the integral cycle groups*. Topology 1 (1962), 257–299.

[Alm2] Almgren, Frederick Justin, Jr. *The Theory of Varifolds: A Variational Calculus in the Large for the k-Dimensional Area Integrand*. Princeton; Institute for Advanced Study, 1965.

[Br] Brouwer, L. E. J. *On the structure of perfect sets of points*. Proc. Koninklijke Akademie van Wetenschappen, 12:785–794, 1910.

[ChM] Chodosh, Otis; Mantoulidis, Christos *Minimal hypersurfaces with arbitrarily large area*. Int. Math. Res. Not. IMRN 2021, no. 14, 10841–10847.

[CKM] Chodosh, Otis; Ketover, Daniel; Maximo, Davi *Minimal hypersurfaces with bounded index*. Invent. Math. 209 (2017), no. 3, 617–664.

[CoM1] Colding, Tobias H.; Minicozzi, William P., II *Width and finite extinction time of Ricci flow*. Geometry & Topology 12 (2008), 2537–2586.

[CoM2] Colding, Tobias H.; Minicozzi, William P., II *A course in minimal surfaces*. Graduate Studies in Mathematics, 121, AMS, (2011).

[FF] Federer, Herbert, and Wendell H. Fleming. *Normal and Integral Currents*. Annals of Mathematics 72, no. 3 (1960): 458–520.

[GT] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[Gr1] Gromov, M. *Isoperimetry of waists and concentration of maps*. Geom. Funct. Anal. 13 (2003), no. 1, 178–215.

[Gr2] Gromov, M. *Dimension, nonlinear spectra and width*, Geometric aspects of functional analysis, 132–184, Lecture Notes in Math., 1317, Springer, Berlin, 1988.

[Gu] Guth, Larry *Minimax problems related to cup powers and Steenrod squares*. Geom. Funct. Anal. 18 (2009), no. 6, 1917–1987.

[LZ] Li, Martin Man-Chun; Zhou, Xin *Min-max theory for free boundary minimal hypersurfaces I—Regularity theory*. J. Differential Geom. 118 (2021), no. 3, 487–553.

[LMN] Liokumovich, Yevgeny; Marques, Fernando C.; Neves, André *Weyl law for the volume spectrum*. Ann. of Math. (2) 187 (2018), no. 3, 933–961.

[MN1] Marques, Fernando C.; Neves, André. *Existence of infinitely many minimal hypersurfaces in positive Ricci curvature*. Invent. Math. 209 (2017), no. 2, 577–616.

[MN2] Marques, Fernando C.; Neves, André. *Min-Max theory and the Willmore conjecture*. Ann. of Math. 179 (2014), no. 2, 683-782

[MN3] Marques, Fernando C.; Neves, André *Morse index of multiplicity one min-max minimal hypersurfaces*. Adv. Math. 378 (2021), Paper No. 107527, 58

[MN4] Marques, Fernando C.; Neves, André *Morse index and multiplicity of min-max minimal hypersurfaces*. Camb. J. Math. 4 (2016), no. 4, 463–511.

[Mo] Mostovoy, Jacob *Geometry of truncated symmetric products and real roots of real polynomials*. Bull. London Math. Soc. 30 (1998), no. 2, 159–165.

[Pi] Pitts, Jon T. *Existence and regularity of minimal surfaces on Riemannian manifolds*. Mathematical Notes, 27. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981. iv+330 pp. ISBN: 0-691-08290-1

[ScSi] Schoen, Richard; Simon, Leon *Regularity of stable minimal hypersurfaces*. Comm. Pure Appl. Math. 34 (1981), no. 6, 741–797.

[SY1] Schoen, R.; Yau, S. T. *On the structure of manifolds with positive scalar curvature*. Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[SY2] Schoen, R.; Yau, S. T. *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*. Ann. of Math. 110(1) (1979), 127.

[SY3] Schoen, Richard; Yau, S. T. *On the proof of the positive mass conjecture in general relativity*. Comm. Math. Phys. 65 (1979), no. 1, 45–76.

[Sh] Sharp, Ben *Compactness of minimal hypersurfaces with bounded index*. J. Differential Geom. 106 (2017), no. 2, 317–339.

[Si1] Simon, Leon. *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*. Ann. of Math. (2) 118 (1983), no. 3, 525–571.

[Si2] Simon, Leon. *A strict maximum principle for area minimizing hypersurfaces*. J. Differential Geom. 26 (2) 327 - 335, 1987.

[Si3] Simon, Leon. *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, (1983). vii+272 pp.

[SWZ] Sun, Ao, Zhichao Wang, and Xin Zhou. *Multiplicity one for min-max theory in compact manifolds with boundary and its applications*. arXiv preprint arXiv:2011.04136 (2020).

[So1] Song, Antoine. *Existence of infinitely many minimal hypersurfaces in closed manifolds*. arXiv preprint arXiv:1806.08816 (2018).

[So2] Song, Antoine. *A dichotomy for minimal hypersurfaces in manifolds thick at infinity.* arXiv preprint arXiv:1902.06767 (2019).

[StSu] Stevens, James, and Ao Sun. *Existence of minimal hypersurfaces with arbitrarily large area in full generality*. arXiv preprint arXiv:2112.02389 (2021).

[Wh1] White, B. *On the bumpy metrics theorem for minimal submanifolds*. Amer. Journal of Math, 139(4):1149-1155. 2017.

[Wh2] White, B. *The maximum principle for minimal varieties of arbitrary codimension.* Comm. Anal. Geom. 18 (2010), no. 3, 421–432. MR 2747434

[Ya] Yau, S.-T., *Problem section. Seminar on Differential Geometry*, pp. 669706, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

[Zh] Zhou, Xin *On the multiplicity one conjecture in min-max theory*. Ann. of Math. (2) 192 (2020), no. 3, 767–820.