THE UNIVERSITY OF CHICAGO

A UNIQUENESS THEOREM FOR A DISCRETE CALDERÓN PROBLEM

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Abstract

The work in this thesis provides a proof of uniqueness of the potentials for which a discrete Calderón problem can be formulated. The Calderón problem will be defined with an associated Schrödinger equation and studied in a discrete setting. Specifically, the derivatives used to define the Schrödinger equation and the Dirichlet-to-Neumann map are determined using finite difference approximations, and the functions are discrete functions defined on a uniform grid in dimensions $n \geq 3$. In the continuous setting, the uniqueness proof of the Calderón problem utilizes a Carleman estimate and a particular form of solutions known as Complex Geometrical Optics solutions, or CGO solutions (as presented in the well-known paper by J. Sylvester and G. Uhlmann). S. Ervedoza and F. de Gournay presented a discrete version of this Carleman estimate and a construction of discrete CGO solutions to the Schrödinger equation. This paper expands on the constructions in this previous work to define a specific set of CGO solutions. These particular constructions will then be used to complete the uniqueness theorem.

1 Introduction

Inverse problems are mathematical problems concerned with making conclusions about the form of a differential equation using knowledge of the differential equation's solutions on the boundary of a certain domain. The study of these problems has several potential applications, one of the most notable being their application to medical technology. As an example of its medical uses, Electrical Impedence Tomography (EIT) may be able to noninvasively identify different types of human tissues (potentially tumors) due to their different conductivities (see also [14]).

Mathematically, the problem can be stated as follows. Let $\Omega \subseteq \mathbb{R}^n$ be a domain with smooth boundary, where $n \geq 2$. For a fixed conductivity $\gamma \in L^\infty(\overline{\Omega})$ that is bounded from below by a positive constant, define the Dirichlet-to-Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_\gamma(f) = \gamma \frac{\partial}{\partial \eta} u,$$

where u solves the conductivity equation

$$\nabla \cdot (\gamma \nabla u) = 0 \tag{1}$$

$$u\restriction_{\partial\Omega} = f. \tag{2}$$

The goal is to identify the conductivity γ of the domain's interior using measurements taken from the boundary - namely, evaluations of the function Λ_γ. In order to determine whether or not this problem is solveable, it is reasonable to first ask: does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply that $\gamma_1 = \gamma_2$? This question will be from here on referred to as the Calderón problem. It was introduced by Calderón in [2], where he was able to solve an associated linearized problem.

When Ω has a C^∞ boundary and $n \geq 2$, Kohn and Vogelius were able to show in [5] the following result. If γ_1 and γ_2 were two strictly positive $C^\infty(\overline{\Omega})$ functions such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $D^{\mathbf{k}}\gamma_1\restriction_{\partial\Omega} = D^{\mathbf{k}}\gamma_2\restriction_{\partial\Omega}$ for all $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n)$ ($\mathbf{k}_j \in \mathbb{N}, \mathbf{k}_j \geq 0$ for every $j = 1, 2, ..., n$).

They extended this result to include piecewise real-analytic conductivities γ_1 and γ_2 in [6].

The full proof the Calderón problem where $\gamma \in C^2(\overline{\Omega})$ and $n \geq 3$ was given in [11], which utilized solutions to (1) with specific properties. These solutions are called Complex Geometrical Optics (CGO) solutions, and take on the form

$$u(x, \boldsymbol{\eta}) = \gamma^{-1/2} e^{x \cdot \boldsymbol{\eta}} \left(1 + r(x)\right), \tag{3}$$

where $\boldsymbol{\eta} \in \mathbb{C}^n$ satisfies $\boldsymbol{\eta} \cdot \boldsymbol{\eta} = 0$ and $r \to 0$ as $|\boldsymbol{\eta}| \to \infty$.

This proof makes use of a special relationship between the conductivity equation $\nabla \cdot (\gamma \nabla u) = 0$ and the Schrödinger equation $(\Delta + q)u = 0$, where q is the potential function satisfying $q = \frac{\Delta\left(\gamma^{1/2}\right)}{\gamma^{1/2}}$. If q is a potential for which the Dirichlet problem for $\Delta + q$ is uniquely solvable for any boundary condition $f \in H^{1/2}(\partial\Omega)$, the corresponding Dirichlet-to-Neumann map is defined by $\Lambda_q(f) = \frac{\partial u}{\partial \eta}$, where u solves the Schrödinger equation

$$(\Delta + q)\, u = 0 \tag{4}$$

$$u\restriction_{\partial\Omega} = f. \tag{5}$$

Just as before, one can ask if $\Lambda_{q_1} = \Lambda_{q_2}$ implies that $q_1 = q_2$. In fact, an affirmative result for the Calderón problem in the case of the conductivity equation for a given conductivity γ is implied by proving the same in the case of the Schrödinger equation with potential $q = \frac{\Delta\left(\gamma^{1/2}\right)}{\gamma^{1/2}}$. See M. Salo's work [13] for another proof of the Calderón problem for the conductivity equation using this relationship. See also [14] for a selection of important results made in the study of the Calderón problem.

This paper will consider the Calderón problem for the Schrödinger equation in a discrete setting. Significant progress on this inverse problem defined on grids in dimensions $n \geq 3$ was made in [3]. Given two potentials q_1 and q_2 defined on a uniform mesh such that the associated

Schrödinger equations are uniquely solveable, one of the results presented in [3] showed that the discrete integrals of the potentials along lines parallel to the coordinate axes were equal if the two potentials had the same Dirichlet-to-Neumann maps.

The domains used in this paper's results will be limited to uniform discrete grids. However, these results provide a full uniqueness theorem of the Calderón problem for the Schrödinger equation for potential functions for which the Calderón problem can be formulated.

There are two difficulties that make the arguments presented in [11] seem incompatible with the discrete setting. One is the fundamentally different ways that the discrete differential operators behave compared to the continuous operators. As they are defined in this paper, the discrete derivatives do not satisfy an exact product rule. In fact, the range (respectively, the domain) of a discrete function is a different space than the range (respectively, the domain) of any one of the discrete function's discrete derivatives. Furthermore, the discrete derivative lacks any known version of the chain rule, which is crucial to the study of the continuous CGO solutions in [11] and [13]. For these reasons, the requirement that $\boldsymbol{\eta} \cdot \boldsymbol{\eta} = 0$ in the construction of the CGO solutions given in (3) does not appear to be sufficient to construct usable discrete CGO solutions.

The remainder of Section 1 covers the definitions necessary for studying the discrete Calderón problem for the Schrödinger equation as it is defined in this paper. These definitions include defining the discrete domain used for the problem's setup, discrete functions and the spaces in which they are defined, and certain discrete operators. This section will also state and prove important identities that will be used throughout the paper, such as a discrete version of Green's Theorem. Many of these definitions and results follow those given in [3].

Section 2 will cover several other preliminary results that are used to construct the CGO solutions used in the main uniqueness proof. Included among these results is a discrete Carleman estimate.

Theorem 10 then puts specific requirements on the vector $\boldsymbol{\eta}$ in order to ensure that the CGO solutions satisfy the needed decay properties. Section 2.3 proves this Theorem and completes the proof of the unqueness result.

1.1 Definitions

1.1.1 The domain

This paper's theorems, lemmas, and propositions will all use functions defined on subsets of what will be referred to as the uniform discrete grid of step size h, for a fixed step size $h > 0$. This section's purpose is to define the objects that will be used to describe the problem, and present some basic properties. Many of the definitions and results here are taken (with a few minor adjustments, as needed) from [3].

The precise definition of this grid follows.

Definition 1. *The uniform discrete grid of step size h for $h > 0$ is the set of vectors*

$$\{\mathbf{n} \in \mathbb{R}^n : \mathbf{n}_k = M_k h \textit{ for some } M_k \in \mathbb{Z} \textit{ for } k = 1, 2, ..., n\} = h\mathbb{Z}^n.$$

The positive integer n in this definition is the dimension of the grid. Throughout this paper, it will be assumed that $n \geq 3$.

An alternative definition uses a set of basis vectors, which will be referred to in this paper as $\mathbf{e}^k$ for $k = 1, 2, ..., n$.

Definition 2. *The n basis vectors $\mathbf{e}^1$, $\mathbf{e}^2$,...,$\mathbf{e}^n$ for the n-dimensional grid are defined to be the vectors with coordinates*

$$\mathbf{e}^k_j = \delta_{kj},$$

where δ_{kj} is the Kroneker delta function. An alternative definition for the uniform discrete grid of *step size $h > 0$ is the set of vectors*

$$h\mathbb{Z}^n = \left\{ \mathbf{n} = \sum_{k=1}^{n} hM_k \mathbf{e}^k : M_k \in \mathbb{Z} \right\}.$$

Let $\mathcal{B} \subseteq h\mathbb{Z}^n$ be a bounded subset of the uniform discrete grid with step size $h > 0$. The goal of this paper is to present a uniqueness result for a discrete Calderón problem on this domain $\mathcal{B}$. To begin to define the discrete operators used in these inverse problems, we will need definitions for the edges of the domain $\mathcal{B}$ (denoted $\mathcal{B}^E$), as well as the edges of $\mathcal{B}$ in the k direction (denoted $\mathcal{B}^k$) for each $k = 1, 2, ..., n$.

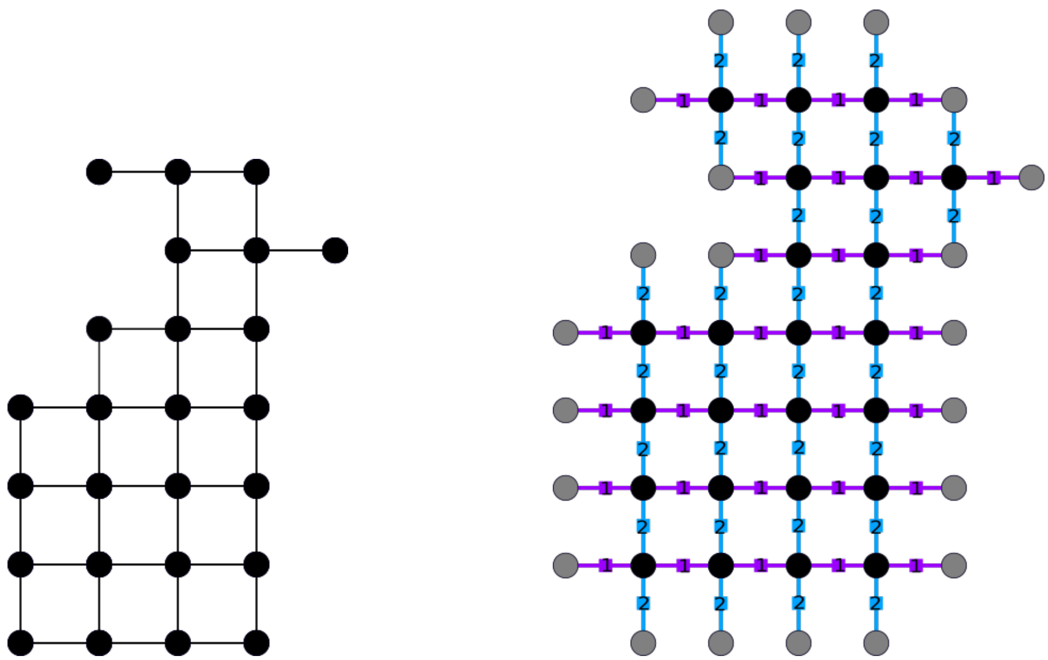

Figure 1: *Left*: A sample 2-dimensional domain $\mathcal{B}$.
Right: The grid $\mathcal{B}$ in black, the set $\mathcal{B}^1$ in purple rectangles that are labeled with the number 1 (where 1 designates the horizontal direction), and the set $\mathcal{B}^2$ in blue rectangles that are labeled with the number 2 (where 2 designates the vertical direction). The gray dots represent the boundary points $\partial\mathcal{B}$ of $\mathcal{B}$, which will be defined later.

Definition 3. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$.*

1. *For each index $k \in \{1, 2, ..., n\}$, the set of edges of $\mathcal{B}$ in the k-direction is defined to be the*

set

$$\mathcal{B}^k = \left\{x + \frac{h}{2}\mathbf{e}^k : x \in \mathcal{B}\right\} \bigcup \left\{x - \frac{h}{2}\mathbf{e}^k : x \in \mathcal{B}\right\}.$$

Sometimes it will be useful to distinguish between the positive and negative k-directions. For this reason, we define the two subsets $\mathcal{B}^k_\pm$ of $\mathcal{B}^k$,

$$\mathcal{B}^k_+ = \left\{x + \frac{h}{2}\mathbf{e}^k : x \in \mathcal{B}\right\}$$
$$\mathcal{B}^k_- = \left\{x - \frac{h}{2}\mathbf{e}^k : x \in \mathcal{B}\right\}.$$

2. *The set of edges of $\mathcal{B}$ is the set $\mathcal{B}^E$, defined to be*

$$\mathcal{B}^E = \bigcup_{k=1}^{n} \mathcal{B}^k.$$

(See figure 1)

3. *Fix a $k \in \{1, 2, ..., n\}$. The set of edges of $\mathcal{B}^k$ in the k-direction is defined to be*

$$\mathcal{B}^{kk} = \{x + h\mathbf{e}^k : x \in \mathcal{B}\} \bigcup \{x - h\mathbf{e}^k : x \in \mathcal{B}\} \bigcup \mathcal{B}.$$

Informally, $\mathcal{B}^{kk}$ can also be thought of as $\left(\mathcal{B}^k\right)^k$.

The set of k-neighbours of the set $\mathcal{B}$ in the positive and negative directions are subsets of the set $\mathcal{B}^{kk}$, and denoted $\mathcal{B}^{kk}_+$ and $\mathcal{B}^{kk}_-$, respectively. They are defined to be

$$\mathcal{B}^{kk}_+ = \{x + h\mathbf{e}^k : x \in \mathcal{B}\}$$
$$\mathcal{B}^{kk}_- = \{x - h\mathbf{e}^k : x \in \mathcal{B}\}.$$

Informally, the sets $\mathcal{B}^{kk}_+$ and $\mathcal{B}^{kk}_-$ can respectively be thought of as $\left(\mathcal{B}^k_+\right)^k_+$ and $\left(\mathcal{B}^k_-\right)^k_-$.

(See Figure 2).

In this paper, points in a domain $\mathcal{B}$ and its edge set $\mathcal{B}^E$ may also be referred to as vertices.

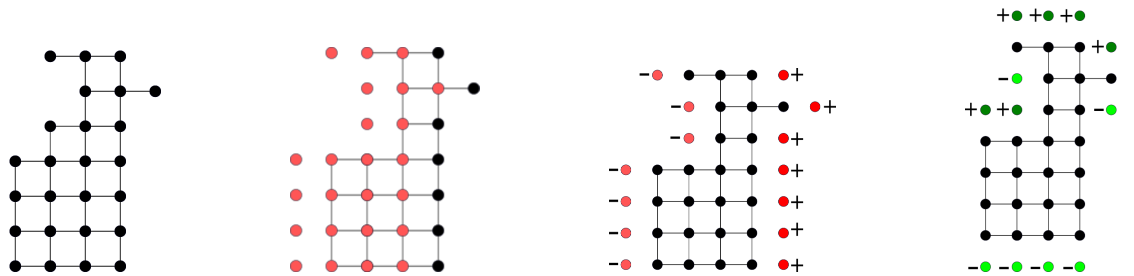

Figure 2: *Left*: A sample 2-dimensional domain $\mathcal{B}$.
Center-Left: The set $\mathcal{B}_-^{11}$ in pink, where the horizontal direction is designated as direction 1.
Center-Right: Points contained in $\mathcal{B}_-^{11}\backslash\mathcal{B}$ are in pink and marked with negative signs on their left sides. Points contained in $\mathcal{B}_+^{11}\backslash\mathcal{B}$ are in red and marked with positive signs on their right sides.
Right : The domain $\mathcal{B}$ in black, $\mathcal{B}_-^{22}\backslash\mathcal{B}$ in light green with negative signs on their left sides, and $\mathcal{B}_+^{22}\backslash\mathcal{B}$ in darker green with positive signs on their left sides.

Definition 4. *Given $\mathcal{B} \subset h\mathbb{Z}^n$ a bounded subset of the uniform discrete grid with step size $h > 0$, the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is defined to be*

$$\partial\mathcal{B} := \left(\bigcup_{k=1}^{n} \mathcal{B}^{kk}\right) \backslash\mathcal{B}.$$

The closure of $\mathcal{B}$ will then be defined as

$$\overline{\mathcal{B}} := \mathcal{B} \cup \partial\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}^{kk}.$$

Note that the dimension n of the surrounding uniform discrete grid $h\mathbb{Z}^n$ plays a part in determining which points are included within the set $\overline{\mathcal{B}}$. Fix two dimensions n_1 and n_2 with $1 \leq n_1 < n_2$. Then a domain $\mathcal{B} \subseteq h\mathbb{Z}^{n_1}$ may also be considered a domain within the larger uniform discrete grid $h\mathbb{Z}^{n_2}$ under the inclusion map $\iota : h\mathbb{Z}^{n_1} \to h\mathbb{Z}^{n_2}$ given by

$$\iota : x = (x_1, x_2, ..., x_{n_1}) \mapsto (x_1, ..., x_{n_1}, 0, ..., 0).$$

However, when $n_2 > n_1$ the closure of $\mathcal{B}$ within the domain $h\mathbb{Z}^{n_2}$ is strictly larger than the closure of $\mathcal{B}$ within the domain $h\mathbb{Z}^{n_1}$ in the sense that

$$\iota\left(\overline{\mathcal{B}}\right) \subsetneq \overline{\iota\left(\mathcal{B}\right)}. \tag{6}$$

Later, the dimension n of the surrounding uniform discrete grid $h\mathbb{Z}^n$ will use the fact that $n \geq 3$ in order for the constructions in this paper to be valid. Equation (6) seems to prevent the proof of the uniqueness problem in dimensions $n \geq 3$ from immediately applying to the same uniqueness problem in dimensions $n < 3$.

The bounded domains $\mathcal{B}$ that are used in this paper all will have an associated strictly larger set $\mathcal{K}$ that is described in the following Proposition.

Proposition 1. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$. Then there exists a bounded subset $\mathcal{K}$ of the uniform discrete grid of step size h such that $\overline{\mathcal{B}} \subseteq \mathcal{K}$. This larger set $\mathcal{K}$ will be called a covering set of $\mathcal{B}$ throughout this paper.*

Proof. Because $\mathcal{B}$ is bounded, there is a real integer $M \in \mathbb{N}^+$ such that

$$\mathcal{B} \subset \{x \in \mathbb{R}^n : |x_j| < hM \; \forall j = 1, 2, ..., n\}.$$

Let $M' = M + 2$, and define

$$\mathcal{K} := \{\mathbf{n} \in h\mathbb{Z}^n : |\mathbf{n}_j| \leq M'h \; \forall j = 1, 2, ..., n\}.$$

Now if $x \in \overline{\mathcal{B}}$, then by Definition 4, there exists an $x_0 \in \mathcal{B}$ and $k \in \{1, 2, ..., n\}$ such that either $x = x_0 + h\mathbf{e}^k$ or $x = x_0 - h\mathbf{e}^k$. Because $x_0 \in \mathcal{B}$, we get that $|(x_0)_j| < hM$ for every $j \in \{1, 2, ..., n\}$. Therefore, for every $j \in \{1, 2, ..., n\} \backslash k$, we have that $|(x)_j| = |(x_0)_j| < hM < h(M + 1)$. Furthermore, $|x_k| \leq |(x_0)_k| + h < hM + h = h(M + 1)$.

In particular, $\overline{\mathcal{B}} \subseteq \{x \in \mathbb{R}^n : |x_j| < (M+1)h \; \forall j = 1, 2, ..., n\}$. Since $M + 1 < M'$, it follows that $\overline{\mathcal{B}} \subset \mathcal{K}$. $\square$

Note that Proposition 1 implies that if $\mathcal{B}$ is a bounded subset of the uniform discrete grid of step size $h > 0$, then so is $\overline{\mathcal{B}}$ and $\mathcal{B}^{jj}$ for each $j \in \{1, 2, ..., n\}$.

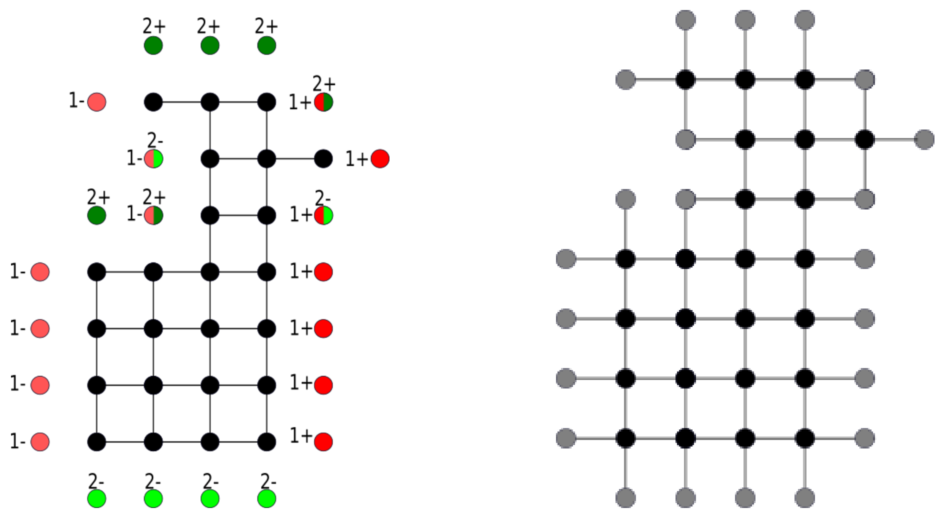

Figure 3: A sample domain $\mathcal{B}$ is in black. On the left, the boundary points are designated with colors and symbols. Points in the set $\mathcal{B}_-^{11} \backslash \mathcal{B}$ are denoted with the color pink, with the symbol $1-$ directly on their left sides. Points in the set $\mathcal{B}_+^{11} \backslash \mathcal{B}$ are denoted with the color red, with the symbol $1+$ directly on their left sides. Points in the set $\mathcal{B}_-^{22} \backslash \mathcal{B}$ are denoted with light green, and sit directly below the symbol $2-$. Finally, points in the set $\mathcal{B}_+^{22} \backslash \mathcal{B}$ are denoted with dark green, and sit directly below the symbol $2+$.
On the right, $\mathcal{B}$ is represented by black points and its boundary $\partial\mathcal{B}$ by gray points. The edge set $\mathcal{B}^E$ is represented by lines connecting the points.

1.1.2 Discrete Functions and Derivatives

With the discrete domain of interest established, this subsection will elaborate on the function spaces and operators used throughout this paper.

Definition 5. *Let $\mathcal{W} \subseteq h\mathbb{Z}^n$ be a subset of the uniform discrete grid of step size $h > 0$.*

1. *The set of $\mathbb{F}$-valued functions defined (and finite) on a domain $\mathcal{W}$ will be denoted by $C(\mathcal{W}, \mathbb{F})$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. In this paper, the set $\mathcal{W}$ will generally be either the set $\mathcal{B}$, $\overline{\mathcal{B}}$, $\partial\mathcal{B}$, or $\mathcal{B}^E$*

for a given bounded subset $\mathcal{B}$ of the uniform discrete grid.

2. *The subset $C_c(\mathcal{W}, \mathbb{F}) \subset C(\overline{\mathcal{W}}, \mathbb{F})$ will be defined to be the set of functions u in $C(\overline{\mathcal{W}}, \mathbb{F})$ such that $u\restriction_{\partial \mathcal{W}} = 0$.*

Throughout this paper, when the output space $\mathbb{F}$ is omitted, it will be assumed that $\mathbb{F} = \mathbb{C}$. For example, $C(\mathcal{B}) = C(\mathcal{B}, \mathbb{C})$ and $C_c(\mathcal{B}) = C_c(\mathcal{B}, \mathbb{C})$.

At times, it will be stated that discrete functions defined on the grid are contained in certain L^p spaces to emphasize the use of certain norms. This paper will next define the integral of a discrete function over a bounded subset $\mathcal{B}$ of the uniform discrete grid. Then, this definition will be used to define some important discrete L^p norms.

Definition 6. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$.*

1. *Let $u \in C(\mathcal{B})$ and $v \in C(\mathcal{B}^E)$. Then*

$$\int_{\mathcal{B}} u(x) := h^n \sum_{x \in \mathcal{B}} u(x)$$
$$\int_{\mathcal{B}^E} v(x) := h^n \sum_{x \in \mathcal{B}^E} v(x).$$

Similarly, for every $k \in \{1, 2, ..., n\}$ and $v \in C(\mathcal{B}^k)$:

$$\int_{\mathcal{B}^k} v(x) := h^n \sum_{x \in \mathcal{B}^k} v(x).$$

Finally, if $u \in C(\partial \mathcal{B})$, then

$$\int_{\partial \mathcal{B}} u(x) := h^{n-1} \sum_{x \in \partial \mathcal{B}} u(x).$$

2. *If* $1 \leq p < \infty$ *and* $u \in C(\mathcal{B})$*, the norm* $||u||_{L^p(\mathcal{B})}$ *is given by*

$$||u||^p_{L^p(\mathcal{B})} := \int_{\mathcal{B}} |u(x)|^p$$

and if $p = \infty$*, then the norm* $||u||_{L^p(\mathcal{B})}$ *is given by*

$$||u||_{L^\infty(\mathcal{B})} := \sup_{x \in \mathcal{B}} |u(x)|.$$

3. *The spaces* $L^2(\mathcal{B})$ *and* $L^2(\overline{\mathcal{B}})$ *respectively have inner products* $(u, v)_{\mathcal{B}}$ *for and* $(u, v)_{\overline{\mathcal{B}}}$*, where*

$$(u, v)_{\mathcal{B}} = \int_{\mathcal{B}} u(x)\overline{v(x)}.$$
$$(u, v)_{\overline{\mathcal{B}}} = \int_{\overline{\mathcal{B}}} u(x)\overline{v(x)}.$$

Furthermore, the Cauchy-Schwarz inequality implies that for either $\mathcal{W} = \mathcal{B}$ *or* $\mathcal{W} = \overline{\mathcal{B}}$*,*

$$|(u, v)_{\mathcal{W}}| \leq ||u||_{L^2(\mathcal{W})}||v||_{L^2(\mathcal{W})} \tag{7}$$

Proof of equation (7). : Showing this inequality using the definitions given in Definition 6 is straightforward: let $\mathcal{W} = \mathcal{B}$ or $\overline{\mathcal{B}}$. Then

$$\begin{aligned} |(u, v)_{\mathcal{W}}| &= \frac{1}{h^n}\left|\sum_{x \in \mathcal{W}} u(x)v(x)\right| \leq \frac{1}{h^n}\sum_{x \in \mathcal{W}} |u(x)v(x)| \\ &\leq \frac{1}{h^n}\left(\sum_{x \in \mathcal{W}} |u(x)|^2\right)^{1/2}\left(\sum_{x \in \mathcal{W}} |v(x)|^2\right)^{1/2} \\ &= \left(\frac{1}{h^n}\sum_{x \in \mathcal{W}} |u(x)|^2\right)^{1/2}\left(\frac{1}{h^n}\sum_{x \in \mathcal{W}} |v(x)|^2\right)^{1/2} \\ &= ||u||_{L^2(\mathcal{W})}||v||_{L^2(\mathcal{W})}. \end{aligned}$$

□

With the function sets defined, this paper will now introduce some important operators that are necessary for defining the discrete differential equations.

Definition 7. *Let $\mathcal{B}$ be a bounded subset of the uniform grid with step size $h > 0$, and let $k \in \{1, 2, ..., n\}$.*

1. *Define the average and difference operators a_k, $\partial_k : C(\overline{\mathcal{B}}) \to C(\mathcal{B}^k)$ of step size h to be the linear operators given as follows: for $x \in \mathcal{B}^k$ and $u \in C(\overline{\mathcal{B}})$,*

$$a_k u(x) = \frac{u\left(x + \frac{h}{2}\mathbf{e}^k\right) + u\left(x - \frac{h}{2}\mathbf{e}^k\right)}{2}$$
$$\partial_k u(x) = \frac{u\left(x + \frac{h}{2}\mathbf{e}^k\right) - u\left(x - \frac{h}{2}\mathbf{e}^k\right)}{h}.$$

   Note that this definition makes sense because if $x \in \mathcal{B}^k$, then either $x = y + \frac{h}{2}\mathbf{e}^k$ or $x = y - \frac{h}{2}\mathbf{e}^k$ for some $y \in \mathcal{B}$. It would therefore follow respectively that either $\left\{x + \frac{h}{2}\mathbf{e}^k, x - \frac{h}{2}\mathbf{e}^k\right\} = \left\{y + h\mathbf{e}^k, y\right\}$, or that $\left\{x + \frac{h}{2}\mathbf{e}^k, x - \frac{h}{2}\mathbf{e}^k\right\} = \left\{y, y - h\mathbf{e}^k\right\}$. In either case, $\left\{x + \frac{h}{2}\mathbf{e}^k, x - \frac{h}{2}\mathbf{e}^k\right\} \subseteq \mathcal{B}^{kk} \subseteq \overline{\mathcal{B}}$.

2. *For a fixed $k \in \{1, 2, ..., n\}$, one can also define average and difference operators of step size h in the k direction on the set of functions $C(\mathcal{B}^k)$. Specifically, the operators $\tilde{a}_k$, $\tilde{\partial}_k$: $C\left(\mathcal{B}^k\right) \to C\left(\mathcal{B}\right)$ are defined by*

$$\tilde{a}_k v(x) = \frac{v\left(x + \frac{h}{2}\mathbf{e}^k\right) + v\left(x - \frac{h}{2}\mathbf{e}^k\right)}{2}$$
$$\tilde{\partial}_k v(x) = \frac{v\left(x + \frac{h}{2}\mathbf{e}^k\right) - v\left(x - \frac{h}{2}\mathbf{e}^k\right)}{h}$$

   for any $x \in \mathcal{B}$ and $v \in C(\mathcal{B}^k)$.

   This definition makes sense because (by (1) in Definition 3) $x + \frac{h}{2}\mathbf{e}^k \in \mathcal{B}^k$ and $x - \frac{h}{2}\mathbf{e}^k \in \mathcal{B}^k$ when $x \in \mathcal{B}$.

3. *It is therefore possible to define* a_k^2*,* $\partial_k^2 : C(\overline{\mathcal{B}}) \to C(\mathcal{B})$*. To be precise, the second-order average and difference operators* a_k^2 *and* $\partial_k^2 : C(\overline{\mathcal{B}}) \to C(\mathcal{B})$ *are defined as follows:*

$$a_k^2 u(x) := \tilde{a}_k \circ a_k u(x) = \frac{u\left(x + h\mathbf{e}^k\right) + 2u(x) + u\left(x - h\mathbf{e}^k\right)}{4}$$
$$\partial_k^2 u(x) := \tilde{\partial}_k \circ \partial_k u(x) = \frac{u\left(x + h\mathbf{e}^k\right) - 2u(x) + u\left(x - h\mathbf{e}^k\right)}{h^2}.$$

for any $x \in \mathcal{B}$*.*

The difference operator ∂_k is an approximation of the continuous derivative in the k direction for any $k = 1, 2, ..., n$. As the discrete partial derivatives are only an approximation to the continuous versions, not all properties of the continuous derivative are necessarily transferred to the discrete derivative. One such difference between discrete and continuous differentiation is their respective product rules.

Proposition 2. ***(The Discrete Product Rule)*** *Fix* $k \in \{1, 2, ..., n\}$*. Let* $\mathcal{B}$ *be a subset of the uniform discrete grid with step size* $h > 0$*, and let* u*,* $v \in C(\overline{\mathcal{B}})$*. Then for all* $x \in \mathcal{B}^k$*,*

$$\partial_k(uv)(x) = \partial_k u(x) a_k v(x) + a_k u(x) \partial_k v(x). \tag{8}$$

Proof. The proof of this is straightforward by applying Definition 7 above. For brevity, let $u^{\pm}(x) =$

$u\left(x \pm \frac{h}{2}\mathbf{e}^k\right)$ for any $u \in C(\overline{\mathcal{B}})$. Then if $x \in \mathcal{B}^k$,

$$\begin{aligned}
\partial_k u(x) a_k v(x) + a_k u(x) \partial_k v(x) &= \frac{1}{2h}\left(u^+(x) - u^-(x)\right)\left(v^+(x) + v^-(x)\right) \\
&\quad + \frac{1}{2h}\left(v^+(x) - v^-(x)\right)\left(u^+(x) + u^-(x)\right) \\
&= \frac{1}{2h}\big(u^+(x)v^+(x) + u^+(x)v^-(x) \\
&\quad - u^-(x)v^+(x) - u^-(x)v^-(x)\big) \\
&\quad + \frac{1}{2h}\big(v^+(x)u^+(x) + v^+(x)u^-(x) \\
&\quad - v^-(x)u^+(x) - v^-(x)u^-(x)\big) \\
&= \frac{1}{2h}\left(2v^+(x)u^+(x) - 2v^-(x)u^-(x)\right) \\
&= \frac{1}{h}\left(v^+(x)u^+(x) - v^-(x)u^-(x)\right) \\
&= \frac{v\left(x + \frac{h}{2}\mathbf{e}^k\right) u\left(x + \frac{h}{2}\mathbf{e}^k\right) - v\left(x - \frac{h}{2}\mathbf{e}^k\right) u\left(x - \frac{h}{2}\mathbf{e}^k\right)}{h} \\
&= \partial_k(vu)(x).
\end{aligned}$$

□

The discrete average and difference operators behave nicely with one another in the sense that they commute.

Proposition 3. ***(The Discrete Commutativity Property)*** *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$, fix $k \in \{1, 2, ..., n\}$ and let $u \in C(\overline{\mathcal{B}})$. Then for any $x \in \mathcal{B}$,*

$$\tilde{a}_k \partial_k u(x) = \tilde{\partial}_k a_k u(x).$$

Proof. For $x \in \mathcal{B}$,

$$\begin{aligned}
\tilde{\partial}_k a_k u(x) &= \frac{1}{h}\left(a_k u\left(x + \frac{h}{2}\mathbf{e}^k\right) - a_k u\left(x - \frac{h}{2}\mathbf{e}^k\right)\right) \\
&= \frac{1}{h}\left(\frac{u\left(x + h\mathbf{e}^k\right) + u\left(x\right)}{2} - \frac{u\left(x\right) + u\left(x - h\mathbf{e}^k\right)}{2}\right) \\
&= \frac{1}{2h}\left(\left[u(x + h\mathbf{e}^k) - u(x)\right] + \left[u(x) - u(x - h\mathbf{e}^k)\right]\right) \\
&= \frac{1}{2}\left(\frac{u(x + h\mathbf{e}^k) - u(x)}{h} + \frac{u(x) - u(x - h\mathbf{e}^k)}{h}\right) \\
&= \frac{1}{2}\left(\partial_k u\left(x + \frac{h}{2}\mathbf{e}^k\right) + \partial_k u\left(x - \frac{h}{2}\mathbf{e}^k\right)\right) \\
&= \tilde{a}_k \partial_k u(x).
\end{aligned}$$

□

These discrete difference and average operators are key to defining the discrete Laplacian and discrete normal derivative. In what follows are the definitions for the discrete Laplacian and the conjugated Laplacian operator, which will be used to define and analyze the inverse problem covered in this paper.

Definition 8. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$. Then for $u \in C(\overline{\mathcal{B}})$ and $x \in \mathcal{B}$, the linear operater $\Delta : C(\overline{\mathcal{B}}) \to C(\mathcal{B})$ is defined by*

$$\Delta u(x) = \sum_{k=1}^{n} \partial_k^2 u(x) = \frac{1}{h^2}\sum_{k=1}^{n}\left\{u\left(x + h\mathbf{e}^k\right) - 2u(x) + u\left(x - h\mathbf{e}^k\right)\right\}.$$

Definition 9. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$. Then for $u \in C(\overline{\mathcal{B}})$ and $\mathbf{s} \in \mathbb{R}^n$, the conjugate operator $\Delta_{\mathbf{s}} : C(\overline{\mathcal{B}}) \to C(\mathcal{B})$ is the linear operator defined* *by*

$$\begin{aligned}\Delta_{\mathbf{s}} u(x) &:= e^{-\mathbf{s}\cdot x}\Delta\left(e^{\mathbf{s}\cdot x}u\right) = e^{-\mathbf{s}\cdot x}\sum_{k=1}^{n}\partial_k^2\left(e^{\mathbf{s}\cdot x}u(x)\right)\\ &= e^{-\mathbf{s}\cdot x}\cdot\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}\cdot(x+h\mathbf{e}^k)}u\left(x+h\mathbf{e}^k\right) - 2e^{\mathbf{s}\cdot x}u(x) + e^{\mathbf{s}\cdot(x-h\mathbf{e}^k)}u\left(x-h\mathbf{e}^k\right)\right\}\\ &= \frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{h\mathbf{s}_k}u\left(x+h\mathbf{e}^k\right) - 2u(x) + e^{-h\mathbf{s}_k}u\left(x-h\mathbf{e}^k\right)\right\}\end{aligned}$$

for all $x \in \mathcal{B}$*.*

This paper will use two different representations for functions in $C(\mathcal{B})$. The first is fairly straightforward: a function $u \in C(\mathcal{B})$ that takes value $u_0 \in \mathbb{C}$ at $x \in \mathcal{B}$ will satisfy

$$u(x) = u_0.$$

An alternative but important representation of functions in $C(\mathcal{B})$ is a vector representation, which is described in the following definition.

Definition 10. *Let* N *be the number of vertices in the set* $\mathcal{W}$ *for a bounded subset* $\mathcal{W}$ *of the uniform discrete grid with step size* $h > 0$*. Then assign to each vertex* x *in* $\mathcal{W}$ *a unique index* $n(x) \in \{1, 2, ..., N\}$*, where* $n : \mathcal{W} \to \{1, 2, ..., N\}$ *is a bijective discrete function. Any function* $u \in C(\mathcal{W})$ *can be represented by a vector* $\mathbf{u}$ *of length* N *with indices*

$$\mathbf{u}_{n(x)} = u(x).$$

This definition will hold for $\mathcal{W} = \mathcal{B}$ *or* $\mathcal{W} = \overline{\mathcal{B}}$*, where* $\mathcal{B}$ *is a bounded subset of the uniform discrete grid of step size* h*.*

If $\mathcal{B}$ *is a bounded subset of the uniform discrete grid of step size* h *and has* $N \in \mathbb{N}$ *vertices,* *then under this representation we have that*

$$C(\mathcal{B}) = \mathbb{C}^N \qquad \textit{and} \qquad C(\mathcal{B}, \mathbb{R}) = \mathbb{R}^N.$$

Similarly, if N' is the number of vertices in $\overline{\mathcal{B}}$, then

$$C(\overline{\mathcal{B}}) = \mathbb{C}^{N'} \qquad \textit{and} \qquad C(\overline{\mathcal{B}}, \mathbb{R}) = \mathbb{R}^{N'}.$$

In this paper, any function $u \in C(\mathcal{B})$ may be equivalently referred to as the function u defined on vertices $x \in \mathcal{B}$, or as the vector $\mathbf{u}$ according to the identification just described.

Later, Definition 10 will be useful for studying the operators Δ and $\Delta + q$ for potential functions $q \in C(\mathcal{B})$, which under this representation can be characterized by matrices.

1.1.3 Normal Derivative

This subsection provides the definition of the discrete normal derivative, which will be used to formulate the discrete Green's theorem.

Definition 11. *Let $\mathcal{B} \subseteq h\mathbb{Z}^n$ be a bounded subset of the uniform discrete grid of step size $h > 0$.*

1. *Define the characteristic function $\chi_{\mathcal{B}} : h\mathbb{Z}^n \to \{0, 1\}$ of $\mathcal{B}$ by*

$$\chi_{\mathcal{B}}(x) = \begin{cases} 1 & \textit{if } x \in \mathcal{B} \\ 0 & \textit{otherwise} \end{cases}$$

   for $x \in h\mathbb{Z}^n$.

2. *Recall from Proposition 1 that given a bounded subset $\mathcal{B}$ of the uniform discrete grid of step size h, there can be found another bounded subset $\mathcal{K}$ of the uniform discrete grid of step size h such that $\overline{\mathcal{B}} \subset \mathcal{K}$. Given a covering set $\mathcal{K}$, define the extension operators $I : C(\mathcal{B}) \to C(\overline{\mathcal{K}})$,*

$\tilde{I} : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{K}})$, and $I_j : C(\mathcal{B}^j) \to C(\mathcal{K}^j)$ for $j \in \{1, 2, ..., n\}$ by the extension by zero operators.

Specifically, for any $u \in C(\overline{\mathcal{B}})$, $v \in C(\mathcal{B})$, and $x \in \overline{\mathcal{K}}$, define

$$I(v)(x) = \begin{cases} v(x) & \textit{when } x \in \mathcal{B} \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\tilde{I}(u)(x) = \begin{cases} u(x) & \textit{when } x \in \overline{\mathcal{B}} \\ 0 & \textit{otherwise .} \end{cases}$$

Similarly, for $v \in C(\mathcal{B}^j)$ and $x \in \mathcal{K}^j$, define for every $j \in \{1, 2, ..., n\}$

$$I_j(v)(x) = \begin{cases} v(x) & \textit{when } x \in \mathcal{B}^j \\ 0 & \textit{otherwise.} \end{cases}$$

The characteristic and extension operators defined above satisfy the following identity that will become useful shortly.

Proposition 4. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$, and $\mathcal{K}$ be a covering set whose existence is guaranteed by Proposition 1. Fix an index $j \in \{1, 2, ..., n\}$. For any $u \in C(\overline{\mathcal{B}})$ and $x \in \mathcal{K}^j$, the following equalities hold:*

$$I_j\left(\partial_j u\right)(x) = \chi_{\mathcal{B}^j}(x)\, \partial_j \left(\tilde{I}(u)\right)(x) \tag{9}$$

$$I_j\left(a_j u\right)(x) = \chi_{\mathcal{B}^j}(x)\, a_j \left(\tilde{I}(u)\right)(x) \tag{10}$$

Proof of Proposition 4. The proof will be shown for (9) only; the equality (10) follows by very similar arguments.

Let $x \in \mathcal{K}^j$.

If $x \notin \mathcal{B}^j$, then

$$I_j\left(\partial_j u\right)(x) = 0 = \chi_{\mathcal{B}^j}(x),$$

and so (9) follows easily.

Now assume that $x \in \mathcal{B}^j$. It follows that $x = y + \frac{h}{2}\mathbf{e}^j$ or $x = y - \frac{h}{2}\mathbf{e}^j$ for some $y \in \mathcal{B}$.

In the case of the former, $x - \frac{h}{2}\mathbf{e}^j = y \in \mathcal{B}$, and $x + \frac{h}{2}\mathbf{e}^j = y + h\mathbf{e}^j \in \overline{\mathcal{B}}$ for $y \in \mathcal{B}$; consequently, $\tilde{I}(u)\left(x \pm \frac{h}{2}\mathbf{e}^j\right) = u\left(x \pm \frac{h}{2}\mathbf{e}^j\right)$ for both choices of sign.

In the case of the latter, $x + \frac{h}{2}\mathbf{e}^j = y \in \mathcal{B}$, and $x - \frac{h}{2}\mathbf{e}^j = y - h\mathbf{e}^j \in \overline{\mathcal{B}}$ for $y \in \mathcal{B}$. Again, $\tilde{I}(u)\left(x \pm \frac{h}{2}\mathbf{e}^j\right) = u\left(x \pm \frac{h}{2}\mathbf{e}^j\right)$ for both choices of sign.

In either case:

$$\begin{aligned} I_j\left(\partial_j u\right)(x) &= \partial_j u(x) \\ &= \frac{u\left(x + \frac{h}{2}\mathbf{e}^j\right) - u\left(x - \frac{h}{2}\mathbf{e}^j\right)}{h} = \frac{\tilde{I}(u)\left(x + \frac{h}{2}\mathbf{e}^j\right) - \tilde{I}(u)\left(x - \frac{h}{2}\mathbf{e}^j\right)}{h} \\ &= \chi_{\mathcal{B}^j}(x) \cdot \partial_j\left(\tilde{I}(u)\right)(x), \end{aligned}$$

as needed.

□

It is now possible to state the definition of the normal derivative.

Definition 12. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$, and fix a covering set $\mathcal{K}$ of $\mathcal{B}$ that satisfies the description in Proposition 1. Let also $u \in C(\overline{\mathcal{B}})$. Then for all $x \in \partial\mathcal{B}$, the normal derivative of u at x is given by*

$$\partial_\eta u(x) = -h \sum_{j=1}^{n} \tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)$$

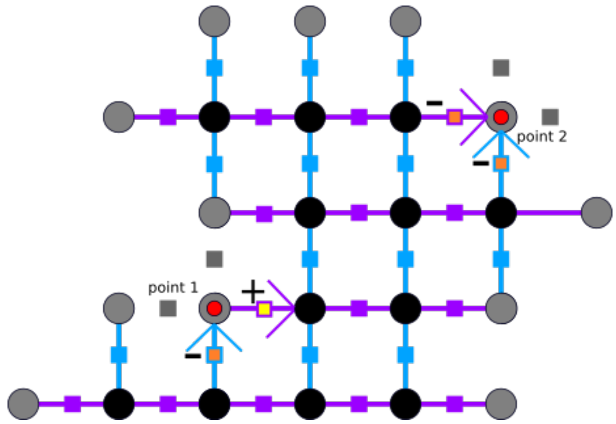

Figure 4: A visualization aid for Definition 12.

See figure 4 for an illustration in two dimensions of the definition of $\partial_\eta u$. If x is Point 1 marked in red in the Figure, then $\partial_\eta u(x)$ is given by

$$\begin{aligned}
\partial_\eta u(x) &= -h \sum_{j=1}^{n} \tilde{\partial}_j \left(I_j \left(\partial_j u \right) \right)(x) \\
&= -\left[I_1 \partial_1 u \left(x + \frac{h}{2} \mathbf{e}^1 \right) - I_1 \partial_1 u \left(x - \frac{h}{2} \mathbf{e}^1 \right) \right] \\
&\qquad\qquad - \left[I_2 \partial_2 u \left(x + \frac{h}{2} \mathbf{e}^2 \right) - I_2 \partial_2 u \left(x - \frac{h}{2} \mathbf{e}^2 \right) \right] \\
&= -\left[\partial_1 u \left(x + \frac{h}{2} \mathbf{e}^1 \right) - 0 \right] - \left[0 - \partial_2 u \left(x + \frac{h}{2} \mathbf{e}^2 \right) \right] \\
&= -\partial_1 u \left(x + \frac{h}{2} \mathbf{e}^1 \right) + \partial_2 u \left(x + \frac{h}{2} \mathbf{e}^2 \right).
\end{aligned} \tag{11}$$

The squares marked in gray that do not lie on any edge lines represent the points in the set $\{x \pm \frac{h}{2}\mathbf{e}^j\}$ where $I_j \partial_j u$ is 0. Orange squares that lie beside negative signs represent points x' where $\partial_j I_j \partial_j u$ contributes a factor of $-I_j \partial_j u(x')$ to the sum that defines $\partial_\eta u$. Yellow squares that lie beside positive signs, meanwhile, represent points x' where $\partial_j I_j \partial_j u$ contributes $I_j \partial_j u(x')$. See the contributions of each square in (11).

Similarly, when x is Point 2 marked in red in the Figure, then $\partial_\eta u(x)$ is given by

$$\partial_\eta u(x) = \partial_1 u\left(x - \frac{h}{2}\mathbf{e}^1\right) + \partial_2 u\left(x - \frac{h}{2}\mathbf{e}^2\right).$$

Note that the definition of ∂_η is independent of the choice of covering set $\mathcal{K}$. To see this, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be any two covering sets of $\mathcal{B}$ that satisfy the description given in Proposition 1. Also let $I_m^j : C(\mathcal{B}^m) \to C(\overline{\mathcal{K}_j^m})$ be the extension operator to $(\mathcal{K}_j)^m$ for $j = 1, 2$ and $m = 1, 2, ..., n$.

Now because $\partial\mathcal{B} \subset \mathcal{K}_j$ for $j = 1, 2$, it follows that if $x \in \partial\mathcal{B}$, then $x + \frac{h}{2}\mathbf{e}^m \in \mathcal{K}_j^m$ and $x - \frac{h}{2}\mathbf{e}^m \in \mathcal{K}_j^m$ for both $j = 1$ and $j = 2$ and for each $m = 1, 2, ..., n$. Therefore, for any fixed index $m \in \{1, 2, ..., n\}$, any function $v \in C(\mathcal{B}^m)$, and any $x \in \partial\mathcal{B}$:

$$\begin{aligned}\tilde{\partial}_m \circ I_m^1(v)(x) &= \frac{1}{h}\left(I_m^1(v)\left(x + \frac{h}{2}\mathbf{e}^m\right) - I_m^1(v)\left(x - \frac{h}{2}\mathbf{e}^m\right)\right) \\ &= \frac{1}{h}\left(I_m^2(v)\left(x + \frac{h}{2}\mathbf{e}^m\right) - I_m^2(v)\left(x - \frac{h}{2}\mathbf{e}^m\right)\right) = \tilde{\partial}_m \circ I_m^2(v)(x).\end{aligned}$$

By setting $v = \partial_m u$, it follows that the definitions provided in Definition 12 are independent of the choice of covering set $\mathcal{K}$.

1.2 Green's Formula

The discrete Green's Theorem is essential to analyzing the Dirichlet-to-Neumann maps that will be introduced later.

Lemma 1. *(Integration by Parts) Let $\mathcal{B} \subset h\mathbb{Z}^n$ be a uniform discrete grid of step size $h > 0$, and fix an index $k \in \{1, 2, ..., n\}$. Then for any $v \in C(\mathcal{B}^k)$ and $u \in C_c(\mathcal{B})$, the discrete integration by parts formula is:*

$$\int_{\mathcal{B}^k} \partial_k(u)v = -\int_{\mathcal{B}} u\tilde{\partial}_k(v). \tag{12}$$

There is another integration by parts formula that involves the average operator a_k. It is given by:

$$\int_{\mathcal{B}^k} a_k(u)v = \int_{\mathcal{B}} u\tilde{a}_k(v). \tag{13}$$

The integration by parts formula involving the average operator (given in equation (13)) is not used to prove the discrete Green's formula; however, it will be a useful identity later in this paper.

The proof of Lemma 1 will require a change of variables. To simplify the domains of integration after the change of variables, this paper will use a set identity provided in the proposition below.

Proposition 5. *Let $(\mathcal{B}^k)^k_\pm = \left\{x \pm \frac{h}{2}\mathbf{e}^k : x \in \mathcal{B}^k\right\}$ for $k \in \{1, 2, ..., n\}$, where $\mathcal{B}$ is a bounded subset of the uniform discrete grid. Then $(\mathcal{B}^k)^k_\pm \backslash \partial\mathcal{B} = \mathcal{B}$ for either choice of sign.*

Proof of proposition 5. For this proof, given any subset $A \subset \overline{\mathcal{B}}$, the set A^c will be used to refer to the set $\overline{\mathcal{B}} \backslash A$ (that is, A^c the complement of A within the domain $\overline{\mathcal{B}}$). For a source of set operations used in this proof, see for instance [9] (section 2.2).

It will be shown only that $(\mathcal{B}^k)^k_+ \backslash \partial\mathcal{B} = \mathcal{B}$; the equality $(\mathcal{B}^k)^k_- \backslash \partial\mathcal{B} = \mathcal{B}$ can be proved by a very similar argument.

Let $x \in \mathcal{B}$ and $y := x - \frac{h}{2}\mathbf{e}^k$. Then $y \in \mathcal{B}^k$, and so $x = y + \frac{h}{2}\mathbf{e}^k \in (\mathcal{B}^k)^k_+$ by definition. Furthermore, since $\partial\mathcal{B} \cap \mathcal{B} = \emptyset$, it follows that $x \in (\mathcal{B}^k)^k_+ \backslash \partial\mathcal{B}$. Therefore, $\mathcal{B} \subset (\mathcal{B}^k)^k_+ \backslash \partial\mathcal{B}$.

Now let $x \in (\mathcal{B}^k)^k_+ \backslash \partial\mathcal{B}$. Since $(\mathcal{B}^k)^k_+ \subset \mathcal{B}^{kk}$, it follows that $x \in \mathcal{B}^{kk} \backslash \partial\mathcal{B}$. Then

$$\begin{aligned} x \in \mathcal{B}^{kk} \backslash \partial\mathcal{B} &= \mathcal{B}^{kk} \backslash \left(\cup_j \mathcal{B}^{jj} \backslash \mathcal{B}\right) = \mathcal{B}^{kk} \cap \left(\cup_j \mathcal{B}^{jj} \cap \mathcal{B}^c\right)^c \\ &= \mathcal{B}^{kk} \cap \left(\left(\cup_j \mathcal{B}^{jj}\right)^c \cup \mathcal{B}\right) = \left(\mathcal{B}^{kk} \cap \left(\cup_j \mathcal{B}^{jj}\right)^c\right) \cup \left(\mathcal{B}^{kk} \cap \mathcal{B}\right) \\ &= \emptyset \cup \left(\mathcal{B}^{kk} \cap \mathcal{B}\right) \subseteq \mathcal{B}. \end{aligned}$$

Therefore $x \in \mathcal{B}$, completing the proof that $(\mathcal{B}^k)^k_+ = \mathcal{B}$. □

As previously mentioned, Proposition 5 will be used in the proof of Lemma 1:

Proof of Lemma 1. First this paper will prove equation (12).

$$
\begin{aligned}
\int_{\mathcal{B}^k} \partial_k u(x)v(x) &= \frac{1}{h}\int_{\mathcal{B}^k}\left(u\left(x+\frac{h}{2}\mathbf{e}^k\right)-u\left(x-\frac{h}{2}\mathbf{e}^k\right)\right)v(x)\\
&= \frac{1}{h}\int_{\mathcal{B}^k} u\left(x+\frac{h}{2}\mathbf{e}^k\right)v(x)-\frac{1}{h}\int_{\mathcal{B}^k} u\left(x-\frac{h}{2}\mathbf{e}^k\right)v(x)\\
&= \frac{1}{h}\int_{(\mathcal{B}^k)^k_+} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)-\frac{1}{h}\int_{(\mathcal{B}^k)^k_-} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right)\\
&= \frac{1}{h}\int_{(\mathcal{B}^k)^k_+\setminus\partial\mathcal{B}} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)-\frac{1}{h}\int_{(\mathcal{B}^k)^k_-\setminus\partial\mathcal{B}} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right) \qquad (14)\\
&= \frac{1}{h}\int_{\mathcal{B}} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)-\frac{1}{h}\int_{\mathcal{B}} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right) \qquad (15)\\
&= -\frac{1}{h}\int_{\mathcal{B}} u(y)\left(v\left(y+\frac{h}{2}\mathbf{e}^k\right)-v\left(y-\frac{h}{2}\mathbf{e}^k\right)\right)\\
&= -\int_{\mathcal{B}} u(y)\tilde{\partial}_k v(y),
\end{aligned}
$$

where (14) comes from the fact that $u = 0$ on $\partial\mathcal{B}$ and (15) follows from Proposition 5.

The proof of equation (13) is very similar:

$$
\begin{aligned}
\int_{\mathcal{B}^k} a_k(u)v &= \frac{1}{2}\int_{\mathcal{B}^k}\left(u\left(x+\frac{h}{2}\mathbf{e}^k\right)+u\left(x-\frac{h}{2}\mathbf{e}^k\right)\right)v(x)\\
&= \frac{1}{2}\int_{\mathcal{B}^k} u\left(x+\frac{h}{2}\mathbf{e}^k\right)v(x)+\frac{1}{2}\int_{\mathcal{B}^k} u\left(x-\frac{h}{2}\mathbf{e}^k\right)v(x)\\
&= \frac{1}{2}\int_{(\mathcal{B}^k)^k_+} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)+\frac{1}{2}\int_{(\mathcal{B}^k)^k_-} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right)\\
&= \frac{1}{2}\int_{(\mathcal{B}^k)^k_+\setminus\partial\mathcal{B}} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)+\frac{1}{2}\int_{(\mathcal{B}^k)^k_-\setminus\partial\mathcal{B}} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right)\\
&= \frac{1}{2}\int_{\mathcal{B}} u(y)v\left(y-\frac{h}{2}\mathbf{e}^k\right)+\frac{1}{2}\int_{\mathcal{B}} u(y)v\left(y+\frac{h}{2}\mathbf{e}^k\right)\\
&= \int_{\mathcal{B}} u(y)\tilde{a}_k v(y).
\end{aligned}
$$

□

Now for the discrete Green's theorem.

Theorem 2. *Let $\mathcal{B} \subset h\mathbb{Z}^n$ be a uniform discrete grid of step size $h > 0$. Then for u, $v \in C(\overline{\mathcal{B}})$,*

$$\sum_{k=1}^{n} \int_{\mathcal{B}^k} \partial_k u \partial_k v = -\int_{\mathcal{B}} \Delta(u) v + \int_{\partial \mathcal{B}} v \partial_\eta u. \tag{16}$$

Recall the definitions of Δ and ∂_η from Definition 8 and Definition 12, respectively.

Proof of Theorem 2. Let $\mathcal{K}$ be any covering set of $\mathcal{B}$. Let u, $v \in C(\overline{\mathcal{B}})$, and let the extension functions I_j and $\tilde{I}$ for the covering set $\mathcal{K}$ be as given in Definition 11.

A few notes to begin with:

1. If $x \in \mathcal{B}^j$ for some index j, then $x \pm \frac{h}{2}\mathbf{e}^j \in \overline{\mathcal{B}}$ for both choices of sign. Therefore,

$$\begin{aligned}(\partial_j \circ \tilde{I})v(x) &= \frac{1}{h}\left[\tilde{I}v\left(x + \frac{h}{2}\mathbf{e}^j\right) - \tilde{I}v\left(x - \frac{h}{2}\mathbf{e}^j\right)\right] \\ &= \frac{1}{h}\left[v\left(x + \frac{h}{2}\mathbf{e}^j\right) - v\left(x - \frac{h}{2}\mathbf{e}^j\right)\right] \\ &= \partial_j v(x)\end{aligned}$$

   whenever $v \in C(\overline{\mathcal{B}})$ and $x \in \mathcal{B}^j$ for any $j \in \{1, 2, ..., n\}$.

2. If $x \in \mathcal{B}^j$ for some index j, then $\partial_j u(x) = I_j\left(\partial_j u\right)(x)$; this follows easily from the definition of I_j.

3. For any $x \in \mathcal{K} \backslash \overline{\mathcal{B}}$, we get that $\tilde{I}(v) = 0$.

4. If $x \in \mathcal{B}$, then $x \pm \frac{h}{2}\mathbf{e}^j \in \mathcal{B}^j$ for both choices of sign. Therefore,

$$\begin{aligned}
\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)(x) &= \frac{1}{h}\left[I_j\left(\partial_j u\right)\left(x+\frac{h}{2}\mathbf{e}^j\right) - I_j\left(\partial_j u\right)\left(x-\frac{h}{2}\mathbf{e}^j\right)\right] \\
&= \frac{1}{h}\left[\left(\partial_j u\right)\left(x+\frac{h}{2}\mathbf{e}^j\right) - \left(\partial_j u\right)\left(x-\frac{h}{2}\mathbf{e}^j\right)\right] \\
&= \tilde{\partial}_j\left(\partial_j u\right)(x).
\end{aligned}$$

Using these observations, we get the following:

$$\begin{aligned}
\sum_{j=1}^{n}\int_{\mathcal{B}^j}\partial_j u \partial_j v &= \sum_{j=1}^{n}\int_{\mathcal{B}^j} I_j\left(\partial_j u\right)\partial_j\left(\tilde{I}v\right) = \sum_{j=1}^{n}\int_{\mathcal{K}^j} I_j\left(\partial_j u\right)\partial_j\left(\tilde{I}v\right) \\
&= -\sum_{j=1}^{n}\int_{\mathcal{K}}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)\tilde{I}v \quad \text{by Lemma 1} \\
&= -\sum_{j=1}^{n}\int_{\overline{\mathcal{B}}}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)v \\
&= -\sum_{j=1}^{n}\int_{\mathcal{B}}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)v - \sum_{j=1}^{n}h\int_{\partial\mathcal{B}}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)v \\
&= -\sum_{j=1}^{n}\int_{\mathcal{B}}\tilde{\partial}_j\left(\partial_j u\right)v - \sum_{j=1}^{n}h\int_{\partial\mathcal{B}}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)v \\
&= -\int_{\mathcal{B}}\Delta(u)v + \int_{\partial\mathcal{B}}\left(-h\sum_{j=1}^{n}\tilde{\partial}_j\left(I_j\left(\partial_j u\right)\right)\right)v \\
&= -\int_{\mathcal{B}}\Delta(u)v + \int_{\partial\mathcal{B}}v\partial_\eta u.
\end{aligned}$$

This completes the proof. □

1.3 The Discrete Time Fourier Transform

The discrete time Fourier transform will play an important role in this paper. The goal of this subsection is to prove a property of this transformation that will be particularly useful: the fact that the discrete time Fourier transform is injective. For an alternative source of this fact and for more information about the discrete time Fourier transform, see for example [8].

Definition 13. *Fix a step size $h > 0$. Denote by $\mathcal{F}$ the discrete time Fourier transform as follows. For any $u \in L^2(h\mathbb{Z}^n)$ and $\boldsymbol{\beta} \in \frac{1}{h}\mathbb{T}^n := [-\pi/h, \pi/h)^n$,*

$$\mathcal{F}(u)(\boldsymbol{\beta}) = \sum_{\mathbf{k}\in h\mathbb{Z}^n} u(\mathbf{k})e^{-i\boldsymbol{\beta}\cdot\mathbf{k}}.$$

Here, $\mathcal{F}(u)$ is a periodic function on $\frac{1}{h}\mathbb{T}^n$.

Let $\iota : C(\mathcal{B}) \to C(h\mathbb{Z}^n)$ and $\tilde{\iota} : C(\overline{\mathcal{B}}) \to C(h\mathbb{Z}^n)$ be extension by zero operators.

Given a bounded subset $\mathcal{B}$ of the uniform discrete grid of step size h, the Fourier transform of a function $u \in C(\overline{\mathcal{B}})$ is defined to be $\mathcal{F}(\tilde{\iota}(u))(\boldsymbol{\beta})$ for all $\boldsymbol{\beta} \in \frac{1}{h}\mathbb{T}^n$. Similarly, the Fourier transform of a function $u \in C(\mathcal{B})$ is defined to be $\mathcal{F}(\iota(u))(\boldsymbol{\beta})$ for all $\boldsymbol{\beta} \in \frac{1}{h}\mathbb{T}^n$.

This discrete time Fourier transform has several properties shared by the continuous Fourier transform. Two that will be used later in this paper are stated and proved below.

Theorem 3. *(Plancherel's theorem) Let $\mathcal{B}$ be bounded subset of the uniform discrete grid of step size $h > 0$. For any $u \in L^2(\mathcal{B})$,*

$$||\mathcal{F}u||_{L^2(\frac{1}{h}\mathbb{T}^n)} = C_{n,h}||u||_{L^2(\mathcal{B})}. \tag{17}$$

Proof. Fix $u \in L^2(\mathcal{B})$.

Since $\mathcal{B}$ is bounded, the series given by $\sum_{\mathbf{k}\in h\mathbb{Z}^n} \iota(u)(\mathbf{k})e^{-\boldsymbol{\beta}\cdot\mathbf{k}}$ is a finite sum.

Then

$$
\begin{aligned}
||\mathcal{F}u||^2_{L^2(\frac{1}{h}\mathbb{T}^n)} &= \int_{\frac{1}{h}\mathbb{T}^n} |\mathcal{F}u(\boldsymbol{\beta})|^2 d\boldsymbol{\beta} = \int_{\frac{1}{h}\mathbb{T}^n} \left| \sum_{\mathbf{k}\in h\mathbb{Z}^n} \iota(u)(\mathbf{k}) e^{-i\boldsymbol{\beta}\cdot\mathbf{k}} \right|^2 \, d\boldsymbol{\beta} \\
&= \int_{\frac{1}{h}\mathbb{T}^n} \sum_{\mathbf{k},\mathbf{j}} \iota(u)(\mathbf{k})\overline{\iota(u)(\mathbf{j})} e^{-i\boldsymbol{\beta}\cdot(\mathbf{k}-\mathbf{j})} \, d\boldsymbol{\beta} \\
&= \sum_{\mathbf{k},\mathbf{j}} \iota(u)(\mathbf{k})\overline{\iota(u)(\mathbf{j})} \int_{\frac{1}{h}\mathbb{T}^n} e^{-i\boldsymbol{\beta}\cdot(\mathbf{k}-\mathbf{j})} \, d\boldsymbol{\beta} \\
&= \sum_{\mathbf{k},\mathbf{j}} \iota(u)(\mathbf{k})\overline{\iota(u)(\mathbf{j})} \prod_{m=1}^{n} \int_{-\pi/h}^{\pi/h} e^{-i\boldsymbol{\beta}_m(\mathbf{k}_m-\mathbf{j}_m)} \, d\boldsymbol{\beta}_m. \qquad (18)
\end{aligned}
$$

Consider the product of integrals in (18) above. If $\mathbf{k}_m - \mathbf{j}_m = Nh \neq 0$ for some index $m \in \{1, 2, ..., n\}$ and some $N \in \mathbb{Z}$, then the corresponding integral is equal to

$$
\begin{aligned}
\int_{-\pi/h}^{\pi/h} e^{-i\boldsymbol{\beta}_m(\mathbf{k}_m-\mathbf{j}_m)} d\boldsymbol{\beta}_m &= \frac{1}{-i(\mathbf{k}_m - \mathbf{j}_m)} \left(e^{-i\frac{\pi}{h}(\mathbf{k}_m-\mathbf{j}_m)} - e^{i\frac{\pi}{h}(\mathbf{k}_m-\mathbf{j}_m)}\right) \\
&= \frac{1}{-i(\mathbf{k}_m - \mathbf{j}_m)} \left(e^{-i\frac{\pi}{h}\cdot Nh} - e^{i\frac{\pi}{h}\cdot Nh}\right) \\
&= \frac{1}{-i(\mathbf{k}_m - \mathbf{j}_m)} \left(e^{-i\pi\cdot N} - e^{i\pi\cdot N}\right) \\
&= \frac{1}{-i(\mathbf{k}_m - \mathbf{j}_m)} \left(\cos(-N\pi) - \cos(N\pi)\right) \\
&= 0.
\end{aligned}
$$

On the other hand, if $\mathbf{k}_m = \mathbf{j}_m$ for all $m \in \{1, 2, ..., n\}$, then

$$
\prod_{m=1}^{n} \int_{-\pi/h}^{\pi/h} e^{-i\boldsymbol{\beta}_m(\mathbf{k}_m-\mathbf{j}_m)} d\boldsymbol{\beta}_m \, d\boldsymbol{\beta}_m = \prod_{m=1}^{n} \int_{-\pi/h}^{\pi/h} 1 \, d\boldsymbol{\beta}_m = \prod_{m=1}^{n} \frac{2\pi}{h} = \left(\frac{2\pi}{h}\right)^n.
$$

Therefore

$$
\begin{aligned}
||\mathcal{F}u||^2_{L^2(\mathbb{T}^n)} &= \sum_{\mathbf{k}} \sum_{\mathbf{j}=\mathbf{k}} \iota(u)(\mathbf{k})\overline{\iota(u)(\mathbf{j})} \prod_{m=1}^{n} \left(\frac{2\pi}{h}\right) + \sum_{\mathbf{k}} \sum_{\mathbf{j}\neq\mathbf{k}} u(\mathbf{k})\overline{u(\mathbf{j})} \cdot 0 \\
&= \sum_{\mathbf{k}} \iota(u)(\mathbf{k})\overline{\iota(u)(\mathbf{k})} \prod_{m=1}^{n} \left(\frac{2\pi}{h}\right) \\
&= \left(\frac{2\pi}{h}\right)^n ||u||^2_{L^2(\mathcal{B})}.
\end{aligned}
$$

□

Theorem 4. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$. The Fourier transform $\mathcal{F}$ defined on $C(\mathcal{B})$ is injective.*

Proof. Let $u \in L^2(\mathcal{B})$ for a bounded subset $\mathcal{B}$ of the uniform discrete grid with step size h such that $\mathcal{F}(u) = 0$; that is, $\mathcal{F}(u)(\boldsymbol{\beta}) = 0$ for almost all $\boldsymbol{\beta} \in [-\pi/h, \pi/h)^n$. The proof that $u = 0$ is an application of Plancherel's theorem:

$$
0 = \int_{[-\pi/h,\pi/h)^n} |\mathcal{F}(u)(\boldsymbol{\beta})|^2 \ d\boldsymbol{\beta} = \left(\frac{2\pi}{h}\right)^n \sum_{\mathbf{k}\in h\mathcal{Z}^n} |u(\mathbf{k})|^2.
$$

Therefore u is necessarily 0 everywhere. □

2 Schrödinger Equation

2.1 Statement of the Problem

Fix a bounded subset $\mathcal{B}$ of the uniform discrete grid of step size $h > 0$. Given a potential $q \in C(\mathcal{B})$, let $\mathcal{C}(q) \subseteq C(\partial\mathcal{B})$ be the set of functions f such that there exists a unique $u \in C(\overline{\mathcal{B}})$ that solves $(\Delta + q)u = 0$ on $\mathcal{B}$ and $u\restriction_{\partial\mathcal{B}} = f$. Then define the Dirichlet-to-Neumann map Λ_q by

$$\begin{aligned}\Lambda_q :&\mathcal{C}(q) \to C(\partial\mathcal{B})\\ \Lambda_q :&f \mapsto \partial_\eta u,\end{aligned}$$

where u is the function solving

$$\begin{aligned}(\Delta + q)\, u &= 0\\ u\restriction_{\partial\mathcal{B}} &= f.\end{aligned}$$

The goal of this section is to prove the following theorem.

Theorem 5. *Let $\mathcal{B}$ be a bounded uniform grid of step size $h > 0$, and let q_1, $q_2 \in C(\mathcal{B}, \mathbb{R})$ be potentials such that $\mathcal{C}(q_1) = \mathcal{C}(q_2) = C(\partial\mathcal{B})$. If*

$$\Lambda_{q_1} = \Lambda_{q_2},$$

then

$$q_1 = q_2.$$

Before getting into the proof of this Theorem, it will first be shown that the Dirichlet-to-Neumann maps Λ_q are well-defined for almost every q.

Recall the relationship between a function $q \in C(\mathcal{B})$ and its vector representation $\mathbf{q}$ from

Definition 10.

Lemma 6. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$. Let also N be the number of vertices in $\mathcal{B}$. Then there is a set $X \subseteq \mathbb{R}^N$ of measure 0 such that for every $\mathbf{q} \in \mathbb{R}^N \backslash X$, we have $\mathcal{C}(q) = C(\partial\mathcal{B})$. In other words, for every $q \in C(\mathcal{B})\backslash X$, we have that for any $f \in C(\partial\mathcal{B})$ there exists a unique $u \in C(\overline{\mathcal{B}})$ such that*

$$(\Delta + q)\, u = 0 \text{ on } \mathcal{B}$$

$$u\restriction_{\partial\mathcal{B}} = f$$

Furthermore, there exists a neighbourhood of the constant function 1 contained entirely within $\mathbb{R}^N \backslash X$.

In the continuous case, a maximum principle can be used to prove uniqueness of solutions when the potential q is identically 0 (see for example [4], Theorem 4, p.27). The proof of Lemma 6 will also utilize a discrete maximum principle in a similar manner. This principle is detailed in the following proposition, which is proved in [1] (Theorem 2.2, p.10):

Proposition 6. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid with step size $h > 0$. If $u \in C(\overline{\mathcal{B}})$ and*

$$\Delta u \geq 0$$

on $\mathcal{B}$, then

$$\max_{\mathcal{B}} u \leq \max_{\partial\mathcal{B}} u.$$

Similarly, if

$$\Delta u \leq 0$$

on $\mathcal{B}$, then

$$\min_{\mathcal{B}} u \geq \min_{\partial\mathcal{B}} u.$$

Proof of Proposition 6. Let $\Delta u \geq 0$ on $\mathcal{B}$. Assume for contradiction that $\max_{\mathcal{B}} u > \max_{\partial\mathcal{B}} u$, and fix an $x \in \mathcal{B}$ such that $u(x) = \max_{y\in\mathcal{B}} u(y)$. Then

$$\begin{aligned}
2nu(x) &= \sum_{k=1}^{n} u\left(x + h\mathbf{e}^k\right) + \sum_{k=1}^{n} u\left(x - h\mathbf{e}^k\right) \\
&\qquad - \sum_{k=1}^{n} \left(u\left(x + h\mathbf{e}^k\right) - 2u(x) + u\left(x - h\mathbf{e}^k\right)\right) \\
&= \sum_{k=1}^{n} u\left(x + h\mathbf{e}^k\right) + \sum_{k=1}^{n} u\left(x - h\mathbf{e}^k\right) - h^2\Delta u(x) \\
&\leq \sum_{k=1}^{n} u\left(x + h\mathbf{e}^k\right) + \sum_{k=1}^{n} u\left(x - h\mathbf{e}^k\right) \qquad \text{(because } \Delta u \geq 0\text{)} \qquad (19) \\
&\leq \sum_{k=1}^{n} u\left(x\right) + \sum_{k=1}^{n} u\left(x\right) = 2nu(x).
\end{aligned}$$

By the assumption that $u(x) = \max_{\mathcal{B}} u > \max_{\partial\mathcal{B}} u$, we have that $u(x + \mathbf{e}^k) \leq u(x)$ and $u(x - h\mathbf{e}^k) \leq u(x)$ for each $k \in \{1, 2, ..., n\}$. It follows necessarily from (19) that $u(x + h\mathbf{e}^k) = u(x) = u(x - h\mathbf{e}^k)$ for all k.

Fix any $k \in \{1, 2, ..., n\}$. Because $\mathcal{B}$ is bounded,

$$N := \min\{m \in \mathbb{N} : x + mh\mathbf{e}^k \notin \mathcal{B}\} < \infty.$$

Furthermore, because $x \in \mathcal{B}$, we have that $N \geq 1$. Finally, by definition of N it follows that $x + Nh\mathbf{e}^k \in \partial\mathcal{B}$. Repeating the argument above for the point $x + mh\mathbf{e}^k$ for every $m \in \{0, 1, ..., N-1\}$ then implies that $x + Nh\mathbf{e}^k$ satisfies

$$u\left(x + Nh\mathbf{e}^k\right) = u\left(x + (N-1)h\mathbf{e}^k\right) = ... = u\left(x + h\mathbf{e}^k\right) = u(x).$$

Therefore

$$\max_{\partial\mathcal{B}} u < \max_{\mathcal{B}} u = u(x) = u\left(x + Nh\mathbf{e}^k\right) \leq \max_{\partial\mathcal{B}} u.$$

This contradicts the assumption that $\max_{\mathcal{B}} u > \max_{\partial\mathcal{B}} u$, and completes the proof.

The proof that u achieves its minimum value on $\partial\mathcal{B}$ when $\Delta u \le 0$ follows by a similar series of inequalities, with the inequalities reversed. □

The proof of Lemma 6 itself will use the vector representation of functions in $C(\overline{\mathcal{B}})$ (see Definition 10).

Proof of Lemma 6. See also [1] (Theorem 2.2, p.10).

Let N' be the number of vertices in the set $\overline{\mathcal{B}}$, and N the number of vertices in $\mathcal{B}$. Fix a bijective index function $n : \overline{\mathcal{B}} \to \{1, 2, ..., N'\}$. Define the mapping $T : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$ by

$$T\left(\mathbf{u}\right)_{n(x)} = \begin{cases} \mathbf{u}_{n(x)} & \text{if } x \in \partial\mathcal{B} \\ \Delta u(x) & \text{if } x \in \mathcal{B} \end{cases}.$$

This mapping is linear, and therefore is represented by a matix.
Now if $\mathbf{u}$ is a vector that satisfies $T(\mathbf{u}) = \mathbf{0}$, then $\mathbf{u}$ represents a discrete function u on $\overline{\mathcal{B}}$ that satisfies $\Delta u = 0$ and $u\restriction_{\partial\mathcal{B}} = 0$. The maximum/minimum principle (Proposition 6) then implies necessarily that

$$0 = \min_{\partial\mathcal{B}} \le \min_{\mathcal{B}} u \le \max_{\mathcal{B}} u \le \max_{\partial\mathcal{B}} u = 0.$$

Thterefore $u \equiv 0$, or (equivalently) $\mathbf{u} = \mathbf{0}$. This implies that the mapping T is injective.

Since T is an injective and linear mapping from an N'-dimensional space to itself, it is invertible, and $\det(T) \neq 0$. In particular, for any boundary value $f \in C(\partial\mathcal{B})$, there is a unique solution $\mathbf{u} = T^{-1}\left(\mathbf{f}, \mathbf{0}\right)$ that solves $u\restriction_{\partial\mathcal{B}} = f$, $\Delta u = 0$.

Now let $\mathcal{T} : \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$\mathcal{T}(\mathbf{q}) = \det(T'(q)),$$

where the matrix $T'(q)$ is the matrix associated to the linear transformation $T''(q) : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$

given by

$$[T''(q)\mathbf{u}]_{n(x)} = \begin{cases} \mathbf{u}_{n(x)} & \text{if } x \in \partial\mathcal{B} \\ (\Delta + q)u(x) & \text{if } x \in \mathcal{B} \end{cases}$$

for $x \in \overline{\mathcal{B}}$.

Note that because the determinant of a matrix is a polynomial with respect to its entries, $\mathcal{T}$ is continuous on $\mathbb{R}^N$. Because it was just shown that $\mathcal{T}(\mathbf{0}) \neq 0$, this means that if q is such that $||q||_{L^\infty(\mathcal{B})} < c$ for $c > 0$ small enough, then $\mathcal{T}(\mathbf{q}) \neq 0$.

Note also that because polynomials are real analytic, we have that $\mathcal{T}$ is an analytic function on $\mathbb{R}^N$. Therefore, because $\mathcal{T}(\mathbf{0}) \neq 0$, we get also that $\mathcal{T}(\mathbf{q}) \neq 0$ for almost every $\mathbf{q} \in \mathbb{R}^N$ (see [7]).

In particular, for almost every potential $\mathbf{q} \in \mathbb{R}^N$, and for every q that satisfies $||q||_{L^\infty(\mathcal{B})} < c$ for $c > 0$ small enough, we have that $\mathcal{T}(\mathbf{q}) \neq 0$. Equivalently, for these q we have that for any $f \in C(\partial\mathcal{B})$ there is a unique $\mathbf{u} = (T'(q))^{-1}(\mathbf{f}, \mathbf{0})$ such that $(\Delta + q)\, u = 0$ on $\mathcal{B}$ and $u|_{\partial\mathcal{B}} = f$. This completes the proof of Lemma 6. □

2.2 Background

The proof of Theorem 5 requires a few preliminary results. First, an equation will be derived from the assumption in Theorem 5 that $\Lambda_{q_1} = \Lambda_{q_2}$. After integrating by parts, this assumption results in an equality that involves the two potential functions q_1 and q_2 as well as solutions to the Schrödinger equations corresponding to these two potentials.

The next subsection derives a Carleman estimate, which is used to produce certain CGO solutions to the Schrödinger equation.

The third and final subsection will provide some precursory results that will be used to construct these CGO solutions.

Section 2.3 will then complete this construction and the proof of Theorem 5.

2.2.1 Dirichlet-to-Neumann map Identity

As mentioned previously, the goal of this subsection is to translate the assumption that $\Lambda_{q_1} = \Lambda_{q_2}$ into an expression involving the potential functions q_1 and q_2 themselves. This will be done using the discrete Green's theorem (16). The results presented in this section follow those found in [3] (Proposition 2.8, p.13).

Lemma 7. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$, and let $q_1 \in C(\mathcal{B}, \mathbb{R})$ and $q_2 \in C(\mathcal{B}, \mathbb{R})$ be two potentials that satisfy $\mathcal{C}(q_j) = C(\partial\mathcal{B})$ for $j \in \{1, 2\}$.*

If $\Lambda_{q_1} = \Lambda_{q_2}$, then for all $u_1, u_2 \in C(\overline{\mathcal{B}})$ such that $(\Delta + q_j)\, u_j = 0$,

$$0 = \int_{\mathcal{B}} (q_1 - q_2) u_1 u_2. \tag{20}$$

As in [3], the proof of this lemma will use the fact that the Dirichlet-to-Neumann maps are self-adjoint.

Proposition 7. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$ and let $q \in C(\mathcal{B})$ be such that $\mathcal{C}(q) = C(\partial\mathcal{B})$.*

Then for any $f, g \in C(\partial\mathcal{B})$,

$$\int_{\partial\mathcal{B}} g\Lambda_q f = \int_{\partial\mathcal{B}} f\Lambda_q g.$$

Proof. Let u solve $(\Delta + q)u = 0$ on $\mathcal{B}$ and $u\restriction_{\partial\mathcal{B}} = f$, and let v be the solution to $(\Delta + q)v = 0$ on

$\mathcal{B}$ and $v\restriction_{\partial\mathcal{B}} = g$. Then using Theorem 2, we get that

$$
\begin{aligned}
\int_{\partial\mathcal{B}} g\Lambda_q f - f\Lambda_q g &= \int_{\partial\mathcal{B}} v\partial_\eta u - u\partial_\eta v \\
&= \sum_k \int_{\mathcal{B}^k} \partial_k u \partial_k v + \int_{\mathcal{B}} \Delta(u)v - \sum_k \int_{\mathcal{B}^k} \partial_k u \partial_k v - \int_{\mathcal{B}} \Delta(v)u \\
&= \int_{\mathcal{B}} \Delta(u)v - \int_{\mathcal{B}} \Delta(v)u \\
&= \int_{\mathcal{B}} (-q)uv - \int_{\mathcal{B}} (-q)vu \\
&= 0.
\end{aligned}
$$

□

With that, it is now possible to prove (20)

Proof of Lemma 7. Let u_1 and u_2 be any two solutions to $(\Delta + q_j)u_j = 0$, and denote $f := u_1\restriction_{\partial\mathcal{B}}$ and $g := u_2\restriction_{\partial\mathcal{B}}$. From the assumption that $\Lambda_{q_1} = \Lambda_{q_2}$, it follows that

$$
\begin{aligned}
0 &= \int_{\partial\mathcal{B}} g\left(\Lambda_{q_1} - \Lambda_{q_2}\right) f = \int_{\partial\mathcal{B}} g\Lambda_{q_1} f - f\Lambda_{q_2} g \quad \text{(by Proposition 7)} \\
&= \int_{\partial\mathcal{B}} g\partial_\eta u_1 - f\partial_\eta u_2 = \int_{\partial\mathcal{B}} u_2\partial_\eta u_1 - u_1\partial_\eta u_2 \\
&= \sum_k \int_{\mathcal{B}^k} \partial_k u_2 \partial_k u_1 + \int_{\mathcal{B}} u_2\Delta(u_1) - \sum_k \int_{\mathcal{B}^k} \partial_k u_1 \partial_k u_2 - \int_{\mathcal{B}} \Delta(u_2)u_1 \\
&= \int_{\mathcal{B}} u_2\Delta(u_1) - \int_{\mathcal{B}} \Delta(u_2)u_1 \\
&= \int_{\mathcal{B}} (-q_1)u_1u_2 - \int_{\mathcal{B}} (-q_2)u_2u_1 \\
&= \int_{\mathcal{B}} (q_2 - q_1)u_1u_2.
\end{aligned}
$$

□

2.2.2 Carleman Estimate

As mentioned in the last subsection, the goal is to show that $\Lambda_{q_1} = \Lambda_{q_2}$ implies that the discrete time Fourier transform of $q_1 - q_2$ is 0. The proof will use solutions to $(\Delta + q_j)u_j = 0$ of a very particular form called CGO solutions. Products of these solutions will result in a function that approximates $x \mapsto e^{2i\boldsymbol{\beta}\cdot x}$ for almost every $\boldsymbol{\beta} \in [-\pi/2h, \pi/2h]$. Inserting products of CGO solutions into (20) will therefore imply that the discrete time Fourier transform of $q_1 - q_2$ is 0.

Essential to showing that these CGO solutions exist and satisfy certain decay properties is the following Carleman estimate. Its statement and proof closely follow the work in [3] (Theorem 3.1 on p.15 and Theorem 6.1 on p.31).

Theorem 8. *Fix a bounded subset $\mathcal{B}$ of the uniform discrete grid of step size $h > 0$, and let $\mathcal{K}$ be a covering set. Given any vector $\mathbf{s} \in \mathbb{R}^n$, let $\mathbf{b} = \mathbf{b}(\mathbf{s}) \in \mathbb{R}^n$ be the vector given by $\mathbf{b}_k = \frac{2}{h}\sinh(h\mathbf{s}_k)$.*
Then there exists a constant C depending only on $\mathcal{B}$ so that for any $u \in C_c(\mathcal{B})$,

$$|\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})} \leq C \left|\left|\Delta_{\mathbf{s}}\left(\tilde{I}(u)\right)\right|\right|_{L^2(\overline{\mathcal{B}})}. \tag{21}$$

Recall the definition of $\Delta_{\mathbf{s}}$ from Definition 9.

Note also that the choice of $\mathcal{K}$ does not change the value of $\Delta_{\mathbf{s}}\left(\tilde{I}(u)\right)(x)$ for any $x \in \overline{\mathcal{B}}$, and therefore it does not change the value of $||\Delta_{\mathbf{s}}\left(\tilde{I}(u)\right)||_{L^2(\overline{\mathcal{B}})}$. Consequently, (21) holds independently of the choice of covering set $\mathcal{K}$. This follows by reasoning very similar to that used in the note immediately after Definition 12.

To see that (21) holds independently of the choice of covering set, fix any two covering sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{B}$. If $x \in \overline{\mathcal{B}}$, it follows that $x \pm h\mathbf{e}^k \in \overline{\mathcal{K}_j}$ for all $k \in \{1, 2, ..., n\}$, for $j \in \{1, 2\}$, and for both choices of sign. This is due to the fact that $\overline{\mathcal{B}} \subseteq \mathcal{K}_j$ for $j \in \{1, 2\}$. Therefore if

$\tilde{I}_j : C(\overline{\mathcal{B}}) \to \overline{\mathcal{K}_j}$ are the two extension by zero operators:

$$\begin{aligned}
\Delta_{\mathbf{s}}\left(\tilde{I}_1(u)\right)(x) &= \frac{1}{h^2}\sum_{j=1}^{n}\left[e^{\mathbf{s}_j h}\tilde{I}_1 u\left(x+h\mathbf{e}^j\right) - 2\tilde{I}_1 u(x) + e^{-\mathbf{s}_j h}\tilde{I}_1 u\left(x-h\mathbf{e}^j\right)\right] \\
&= \frac{1}{h^2}\sum_{j=1}^{n}\left[e^{\mathbf{s}_j h}\tilde{I}_2 u\left(x+h\mathbf{e}^j\right) - 2\tilde{I}_2 u(x) + e^{-\mathbf{s}_j h}\tilde{I}_2 u\left(x-h\mathbf{e}^j\right)\right] \\
&= \Delta_{\mathbf{s}}\left(\tilde{I}_2(u)\right)(x)
\end{aligned}$$

for any $x \in \overline{\mathcal{B}}$.

The proof of the Carleman estimate given in Theorem 8 involves first splitting $\Delta_{\mathbf{s}}$ up into its symmetric and asymmetric parts, S and A respectively. Bounding $||\Delta_{\mathbf{s}}(\tilde{I}u)||_{L^2(\overline{\mathcal{B}})}$ is then reduced to studying $||S(u)||_{L^2(\overline{\mathcal{B}})}$ and $||A(u)||_{L^2(\overline{\mathcal{B}})}$.

Proposition 8. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$, and let $\mathcal{K}$ be a covering set. Let $\Delta_{\mathbf{s}}$ be the conjugated operator defined in Definition 9. Then $\Delta_{\mathbf{s}}\tilde{I} : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$ can be written as $\Delta_{\mathbf{s}}\tilde{I} = S + A$, where:*

1. *$S = \sum_{j=1}^{n} S_j$ is a symmetric operator, and $A = \sum_{j=1}^{n} A_j$ is skew-symmetric on the space $C_c(\mathcal{B})$ with inner product*
$$(u, v)_{\overline{\mathcal{B}}} := \int_{\overline{\mathcal{B}}} u\overline{v}.$$

2. *For any j, $m = 1, 2, ..., n$ and $u \in C_c(\mathcal{B})$, the operators S_j and A_m satisfy $S_j A_m u(x) = A_m S_j u(x)$ for all $x \in \mathcal{B}$.*

3. *For all j, $m = 1, 2, ..., n$ and any $u \in C_c(\mathcal{B}, \mathbb{R})$, the operators S_j and A_m satisfy*
$$(S_j u, A_m u)_{\overline{\mathcal{B}}} = 0 = (A_m u, S_j u)_{\overline{\mathcal{B}}}.$$

4. For any $\mathbf{s} \in \mathbb{R}^n$ *and* $u \in C_c(\mathcal{B})$,

$$||\Delta_{\mathbf{s}}\tilde{I}(u)||^2_{L^2(\overline{\mathcal{B}})} = ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||^2_{L^2(\overline{\mathcal{B}})}. \tag{22}$$

Proof of Proposition 8. See also [3]. Let $v \in C(\overline{\mathcal{B}})$ and $x \in \overline{\mathcal{B}}$. For notational simplicity, define for now $u = \tilde{I}(v)$. Then

$$\begin{aligned}
\Delta_{\mathbf{s}}u(x) &= e^{-\mathbf{s}\cdot x}\sum_{k=1}^{n}\partial_k^2\left(e^{\mathbf{s}\cdot x}u\right) = e^{-\mathbf{s}\cdot x}\sum_{k=1}^{n}\tilde{\partial}_k\left(\partial_k\left(e^{\mathbf{s}\cdot x}\right)a_k u + a_k\left(e^{\mathbf{s}\cdot x}\right)\partial_k u\right)\\
&= e^{-\mathbf{s}\cdot x}\sum_{k=1}^{n}\left[\partial_k^2\left(e^{\mathbf{s}\cdot x}\right)a_k^2 u + 2\tilde{\partial}_k a_k\left(e^{\mathbf{s}\cdot x}\right)\tilde{\partial}_k a_k u + a_k^2\left(e^{\mathbf{s}\cdot x}\right)\partial_k^2 u\right]\\
&= e^{-\mathbf{s}\cdot x}\sum_{k=1}^{n}\left\{\frac{1}{h^2}\left[e^{\mathbf{s}\cdot x}e^{\mathbf{s}_k h} - 2e^{\mathbf{s}\cdot x} + e^{\mathbf{s}\cdot x}e^{-\mathbf{s}_k h}\right]a_k^2 u]\right.\\
&\qquad + \frac{1}{h}\left[e^{\mathbf{s}\cdot x}e^{\mathbf{s}_k h} - e^{\mathbf{s}\cdot x}e^{-\mathbf{s}_k h}\right]\tilde{\partial}_k a_k u\\
&\qquad \left.+\frac{1}{4}\left[e^{\mathbf{s}\cdot x}e^{\mathbf{s}_k h} + 2e^{\mathbf{s}\cdot x} + e^{\mathbf{s}\cdot x}e^{-\mathbf{s}_k h}\right]\partial_k^2 u\right\}\\
&= \sum_{k=1}^{n}\left\{\frac{1}{h^2}\left[e^{\mathbf{s}_k h} - 2 + e^{-\mathbf{s}_k h}\right]a_k^2 u + \frac{1}{h}\left[e^{\mathbf{s}_k h} - e^{-\mathbf{s}_k h}\right]\tilde{\partial}_k a_k u\right.\\
&\qquad \left.+\frac{1}{4}\left[e^{\mathbf{s}_k h} + 2 + e^{-\mathbf{s}_k h}\right]\partial_k^2 u\right\}\\
&= \sum_{k=1}^{n}\left\{\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)a_k^2 u + \frac{2}{h}\sinh(\mathbf{s}_k h)\tilde{\partial}_k a_k u + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\partial_k^2 u\right\}.\\
&= \sum_{k=1}^{n}\left\{\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)a_k^2 + \frac{2}{h}\sinh(\mathbf{s}_k h)\tilde{\partial}_k a_k + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\partial_k^2\right\}\left(\tilde{I}v\right)
\end{aligned}$$

Then define the operators $S_k : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$ and $A_k : C(\overline{\mathcal{B}}) \to C(\overline{\mathcal{B}})$ by

$$\begin{aligned}
S_k v(x) &= \frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)a_k^2\left(\tilde{I}v\right)(x) + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\partial_k^2\left(\tilde{I}v\right)(x)\\
A_k v(x) &= \frac{2}{h}\sinh(\mathbf{s}_k h)\tilde{\partial}_k a_k\left(\tilde{I}v\right)(x)
\end{aligned}$$

for all $x \in \overline{\mathcal{B}}$ so that $\Delta_{\mathbf{s}} \tilde{I} = S + A = \sum_{k=1}^{n} S_k + \sum_{k=1}^{n} A_j$ according to the above calculations. All that is left to show are the three properties listed in Proposition 8.

Property 1

Let u, $v \in C_c(\mathcal{B})$. Then using the integration by parts formulas (12) and (13) from Lemma 1 as well as the fact that $\overline{v} = 0 = u$ on $\partial\mathcal{B}$,

$$
\begin{aligned}
(Su, v)_{\overline{\mathcal{B}}} &= \sum_{k=1}^{n} (S_k u, v)_{\overline{\mathcal{B}}} = \sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} \overline{v} S_k u = \sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} \tilde{I}(\overline{v}) S_k u \\
&= \sum_{k=1}^{n} \int_{\mathcal{B}} \tilde{I}(\overline{v}) \left(\frac{4}{h^2} \sinh^2\left(\frac{\mathbf{s}_k h}{2}\right) a_k^2 \tilde{I}(u) + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right) \partial_k^2 \tilde{I}(u) \right) \\
&= \sum_{k=1}^{n} \int_{\mathcal{B}^k} a_k \left(\overline{\frac{4}{h^2} \sinh^2\left(\frac{\mathbf{s}_k h}{2}\right) \tilde{I}(v)} \right) a_k \tilde{I}(u) - \sum_{k=1}^{n} \int_{\mathcal{B}^k} \partial_k \left(\overline{\cosh^2\left(\frac{\mathbf{s}_k h}{2}\right) \tilde{I}(v)} \right) \partial_k \tilde{I}(u) \\
&= \sum_{k=1}^{n} \int_{\mathcal{B}} a_k^2 \left(\overline{\frac{4}{h^2} \sinh^2\left(\frac{\mathbf{s}_k h}{2}\right) \tilde{I}(v)} \right) \tilde{I}(u) + \sum_{k=1}^{n} \int_{\mathcal{B}} \partial_k^2 \left(\overline{\cosh^2\left(\frac{\mathbf{s}_k h}{2}\right) \tilde{I}(v)} \right) \tilde{I}(u) \\
&= \sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} \left(\overline{\frac{4}{h^2} \sinh^2\left(\frac{\mathbf{s}_k h}{2}\right) a_k^2 \tilde{I}(v) + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right) \partial_k^2 \tilde{I}(v)} \right) u \\
&= \sum_{k=1}^{n} (u, S_k v)_{\overline{\mathcal{B}}} = (u, Sv)_{\overline{\mathcal{B}}} \,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(Au, v)_{\overline{\mathcal{B}}} &= \sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} A_k u \overline{v} = \sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} \left(\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \tilde{I}(u) \right) \overline{v} \\
&= \sum_{k=1}^{n} \int_{\mathcal{B}} \left(\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \tilde{I}(u) \right) \tilde{I}(\overline{v}) \\
&= -\sum_{k=1}^{n} \int_{\mathcal{B}^k} \left(a_k \tilde{I}(u) \right) \partial_k \left(\overline{\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{I}(v)} \right) \\
&= -\sum_{k=1}^{n} \int_{\mathcal{B}} \tilde{I}(u) \left(\tilde{a}_k \partial_k \left(\overline{\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{I}(v)} \right) \right) \\
&= -\sum_{k=1}^{n} \int_{\overline{\mathcal{B}}} u \left(\overline{\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \tilde{I}(v)} \right) \\
&= -\sum_{k=1}^{n} (u, A_k v)_{\overline{\mathcal{B}}} = -\, (u, Av)_{\overline{\mathcal{B}}} \, .
\end{aligned}
$$

Therefore S and A are symmetric and skew-symmetric, respectively.

Property 2

Fix k, $j \in \{1, 2, ..., n\}$. For $u \in C_c(\mathcal{B})$ and $x \in \mathcal{B}$:

$$
\begin{aligned}
S_k A_j u(x) &= \frac{4}{h^2} \sinh^2 \left(\frac{\mathbf{s}_k h}{2} \right) a_k^2 \tilde{I} \left(\frac{2}{h} \sinh(\mathbf{s}_j h) \tilde{\partial}_j a_j \tilde{I}(u) \right)(x) \\
&\quad + \cosh^2 \left(\frac{\mathbf{s}_k h}{2} \right) \partial_k^2 \tilde{I} \left(\frac{2}{h} \sinh(\mathbf{s}_j h) \tilde{\partial}_j a_j \tilde{I}(u) \right)(x) \\
&= \frac{2}{h} \sinh(\mathbf{s}_j h) \frac{4}{h^2} \sinh^2 \left(\frac{\mathbf{s}_k h}{2} \right) \left(a_k^2 \tilde{I} \left(\tilde{\partial}_j a_j \tilde{I}(u) \right) \right)(x) \\
&\quad + \frac{2}{h} \sinh(\mathbf{s}_j h) \cosh^2 \left(\frac{\mathbf{s}_k h}{2} \right) \left(\partial_k^2 \tilde{I} \left(\tilde{\partial}_j a_j \tilde{I}(u) \right) \right)(x).
\end{aligned}
\tag{23}
$$

It will be shown that for $u \in C_c(\mathcal{B})$ and $x \in \mathcal{B}$,

$$
a_k^2 \tilde{I} \left(\tilde{\partial}_j a_j \tilde{I}(u) \right)(x) = \tilde{\partial}_j a_j \tilde{I} \left(a_k^2 \tilde{I}(u) \right)(x) \tag{24}
$$

and

$$\partial_k^2 \tilde{I}\left(\tilde{\partial}_j a_j \tilde{I}(u)\right)(x) = \tilde{\partial}_j a_j \tilde{I}\left(\partial_k^2 \tilde{I}(u)\right)(x) \tag{25}$$

Then combining (23), (24), and (25) gives the commutativity result - namely, that

$$\begin{aligned}
S_k A_j u(x) &= \frac{2}{h}\sinh(\mathbf{s}_j h)\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)\left(a_k^2 \tilde{I}\left(\tilde{\partial}_j a_j \tilde{I}(u)\right)\right)(x) \\
&\qquad + \frac{2}{h}\sinh(\mathbf{s}_j h)\cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\left(\partial_k^2 \tilde{I}\left(\tilde{\partial}_j a_j \tilde{I}(u)\right)\right)(x) \\
&= \frac{2}{h}\sinh(\mathbf{s}_j h)\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)\left(\tilde{\partial}_j a_j \tilde{I}\left(a_k^2 \tilde{I}(u)\right)\right)(x) \\
&\qquad + \frac{2}{h}\sinh(\mathbf{s}_j h)\cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\left(\tilde{\partial}_j a_j \tilde{I}\left(\partial_k^2 \tilde{I}(u)\right)\right)(x) \\
&= \frac{2}{h}\sinh(\mathbf{s}_j h)\left(\tilde{\partial}_j a_j \tilde{I}\left(\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)a_k^2 \tilde{I}(u)\right)\right)(x) \\
&\qquad + \frac{2}{h}\sinh(\mathbf{s}_j h)\left(\tilde{\partial}_j a_j \tilde{I}\left(\cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\partial_k^2 \tilde{I}(u)\right)\right)(x) \\
&= \frac{2}{h}\sinh(\mathbf{s}_j h)\tilde{\partial}_j a_j \tilde{I}\left(\frac{4}{h^2}\sinh^2\left(\frac{\mathbf{s}_k h}{2}\right)a_k^2 \tilde{I}(u) + \cosh^2\left(\frac{\mathbf{s}_k h}{2}\right)\partial_k^2 \tilde{I}(u)\right)(x) \\
&= A_j S_k u(x)
\end{aligned}$$

for $u \in C_c(\mathcal{B})$ and $x \in \mathcal{B}$.

Now to prove (24). Let $v(x) := \tilde{\partial}_j a_j \tilde{I}(u)(x) = \frac{1}{2h}\left(\tilde{I}u\left(x + h\mathbf{e}^j\right) - \tilde{I}u\left(x - h\mathbf{e}^j\right)\right)$ for $x \in \mathcal{B}$.

Then because $x + h\mathbf{e}^j$, $x - h\mathbf{e}^j \in \overline{\mathcal{B}}$ for all $x \in \mathcal{B}$:

$$\begin{aligned}
a_k^2 \tilde{I}(v)(x) &= \frac{1}{4}\left[\tilde{I}v\left(x + h\mathbf{e}^k\right) + 2\tilde{I}v(x) + \tilde{I}v\left(x - h\mathbf{e}^k\right)\right] \\
&= \frac{1}{4}\left[v\left(x + h\mathbf{e}^k\right) + 2v(x) + v\left(x - h\mathbf{e}^k\right)\right] \\
&= \frac{1}{4}\cdot\frac{1}{2h}\left[\left(\tilde{I}u\left(x + h\mathbf{e}^k + h\mathbf{e}^j\right) - \tilde{I}u\left(x + h\mathbf{e}^k - h\mathbf{e}^j\right)\right)\right. \\
&\qquad + 2\left(\tilde{I}u(x + h\mathbf{e}^j) - \tilde{I}u(x - h\mathbf{e}^j)\right) \\
&\qquad \left. + \left(\tilde{I}u(x - h\mathbf{e}^k + h\mathbf{e}^j) - \tilde{I}u(x - h\mathbf{e}^k - h\mathbf{e}^j)\right)\right] \\
&= \frac{1}{2h}\cdot\frac{1}{4}\left[\left(\tilde{I}u(x + h\mathbf{e}^j + h\mathbf{e}^k) + 2\tilde{I}u(x + h\mathbf{e}^j) + \tilde{I}u(x + h\mathbf{e}^j - h\mathbf{e}^k)\right)\right. \\
&\qquad \left. - \left(\tilde{I}u(x - h\mathbf{e}^j + h\mathbf{e}^k) + 2\tilde{I}u(x - h\mathbf{e}^j) + \tilde{I}u(x - h\mathbf{e}^j - h\mathbf{e}^k)\right)\right] \\
&= \frac{1}{2h}\left[a_k^2\tilde{I}u(x + h\mathbf{e}^j) - a_k^2\tilde{I}u(x - h\mathbf{e}^j)\right].
\end{aligned}$$

Using again that $x + h\mathbf{e}^j$, $x - h\mathbf{e}^j \in \overline{\mathcal{B}}$ for any $x \in \mathcal{B}$:

$$\begin{aligned}
a_k^2 \tilde{I}(v)(x) &= \frac{1}{2h}\left[a_k^2\tilde{I}u(x + h\mathbf{e}^j) - a_k^2\tilde{I}u(x - h\mathbf{e}^j)\right] \\
&= \frac{1}{2h}\left[\tilde{I}a_k^2\tilde{I}u(x + h\mathbf{e}^j) - \tilde{I}a_k^2\tilde{I}u(x - h\mathbf{e}^j)\right] \\
&= \tilde{\partial}_j a_j \tilde{I}\left(a_k^2\tilde{I}u\right)(x).
\end{aligned}$$

The proof of (25) is very similar. Let $v(x) = \tilde{\partial}_j a_j \tilde{I}(u)(x) = \frac{1}{2h}\left(\tilde{I}u\left(x + h\mathbf{e}^j\right) - \tilde{I}u\left(x - h\mathbf{e}^j\right)\right)$

for $x \in \mathcal{B}$, as before. Then $x \pm h\mathbf{e}^k \in \overline{\mathcal{B}}$ and $x \pm h\mathbf{e}^j \in \overline{\mathcal{B}}$, and so

$$\begin{aligned}
\partial_k^2 \tilde{I}(v)(x) &= \frac{1}{h^2}\left[\tilde{I}v\left(x+h\mathbf{e}^k\right) - 2\tilde{I}v(x) + \tilde{I}v\left(x-h\mathbf{e}^k\right)\right] \\
&= \frac{1}{h^2}\left[v\left(x+h\mathbf{e}^k\right) - 2v(x) + v\left(x-h\mathbf{e}^k\right)\right] \\
&= \frac{1}{(2h)\cdot(h^2)}\Big[\tilde{I}u\left(x+h\mathbf{e}^k+h\mathbf{e}^j\right) - \tilde{I}u\left(x+h\mathbf{e}^k-h\mathbf{e}^j\right) \\
&\qquad\qquad - 2\left(\tilde{I}u\left(x+h\mathbf{e}^j\right) - \tilde{I}u\left(x-h\mathbf{e}^j\right)\right) \\
&\qquad\qquad + \tilde{I}u\left(x-h\mathbf{e}^k+h\mathbf{e}^j\right) - \tilde{I}u\left(x-h\mathbf{e}^k-h\mathbf{e}^j\right)\Big] \\
&= \frac{1}{(2h)\cdot(h^2)}\Big\{\left[\tilde{I}u\left(x+h\mathbf{e}^k+h\mathbf{e}^j\right) - 2\tilde{I}u\left(x+h\mathbf{e}^j\right) + \tilde{I}u\left(x-h\mathbf{e}^k+h\mathbf{e}^j\right)\right] \\
&\qquad\qquad - \left[\tilde{I}u\left(x+h\mathbf{e}^k-h\mathbf{e}^j\right) - 2\tilde{I}u\left(x-h\mathbf{e}^j\right) + \tilde{I}u\left(x-h\mathbf{e}^k-h\mathbf{e}^j\right)\right]\Big\} \\
&= \frac{1}{2h}\left\{\partial_k^2\left(\tilde{I}u\right)\left(x+h\mathbf{e}^j\right) - \partial_k^2\tilde{I}u\left(x-h\mathbf{e}^j\right)\right\} \\
&= \frac{1}{2h}\left\{\tilde{I}\partial_k^2\left(\tilde{I}u\right)\left(x+h\mathbf{e}^j\right) - \tilde{I}\partial_k^2\tilde{I}u\left(x-h\mathbf{e}^j\right)\right\} \\
&= \tilde{\partial}_j a_j \tilde{I}\left(\partial_k^2\tilde{I}u\right)(x).
\end{aligned}$$

Therefore $\partial_k^2\tilde{I}\left(\tilde{\partial}_j a_j \tilde{I}(h)\right)(x) = \tilde{\partial}_j a_j \tilde{I}\left(\partial_k^2 \tilde{I}u\right)(x)$ for any k, $j \in \{1, 2, ..., n\}$, $u \in C_c(\mathcal{B})$, and $x \in \mathcal{B}$.

This completes the proof of Property 2.

Property 3

Let $u \in C_c(\mathcal{B}, \mathbb{R})$, and let k, $j \in \{1, 2, ..., n\}$. Using both property 1 and property 2 above as well as the fact that $u = 0$ on $\partial\mathcal{B}$:

$$\begin{aligned}
(S_k u, A_j u)_{\overline{\mathcal{B}}} &= -\left(A_j S_k u, u\right)_{\overline{\mathcal{B}}} \\
&= -\left(A_j S_k u, u\right)_{\mathcal{B}} = -\left(S_k A_j u, u\right)_{\mathcal{B}} \\
&= -\left(S_k A_j u, u\right)_{\overline{\mathcal{B}}} = -\left(A_j u, S_k u\right)_{\overline{\mathcal{B}}}.
\end{aligned}$$

Because $(S_k u, A_j u)_{\overline{\mathcal{B}}} = \overline{(S_k u, A_j u)_{\overline{\mathcal{B}}}}$ when u is real-valued, it follows $(S_k u, A_j u)_{\overline{\mathcal{B}}} = -(S_k u, A_j u)_{\overline{\mathcal{B}}} = 0$.

The above equalities then also imply that $(A_j u, S_k u)_{\overline{\mathcal{B}}} = -(S_k u, A_j u)_{\overline{\mathcal{B}}} = 0$.

Property 4

Let $u \in C_c(\mathcal{B})$; write it as $u = u_1 + iu_2$, where $u_1, u_2 \in C_c(\mathcal{B}, \mathbb{R})$. Then because A is linear and its coefficients are real,

$$\begin{aligned}
||Au||^2_{L^2(\overline{\mathcal{B}})} &= \int_{\overline{\mathcal{B}}} |Au|^2 \\
&= \int_{\overline{\mathcal{B}}} \left| \sum_{k=1}^{n} \left[\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \left(\tilde{I} \left[u_1 + iu_2 \right] \right) \right] \right|^2 \\
&= \int_{\overline{\mathcal{B}}} \left| \sum_{k=1}^{n} \left[\frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \left(\tilde{I} u_1 \right) + i \sum_{j=1}^{n} \frac{2}{h} \sinh(\mathbf{s}_j h) \tilde{\partial}_j a_j \left(\tilde{I} u_2 \right) \right] \right|^2 \\
&= \int_{\overline{\mathcal{B}}} \left\{ \left| \sum_{k=1}^{n} \frac{2}{h} \sinh(\mathbf{s}_k h) \tilde{\partial}_k a_k \left(\tilde{I} u_1 \right) \right|^2 + \left| \sum_{j=1}^{n} \frac{2}{h} \sinh(\mathbf{s}_j h) \tilde{\partial}_j a_j \left(\tilde{I} u_2 \right) \right|^2 \right\} \\
&= \int_{\overline{\mathcal{B}}} \left\{ |Au_1|^2 + |Au_2|^2 \right\} \\
&= ||Au_1||^2_{L^2(\overline{\mathcal{B}})} + ||Au_2||^2_{L^2(\overline{\mathcal{B}})} .
\end{aligned}$$

The operators S and $\Delta_{\mathbf{s}} \tilde{I}$ are both also linear and (in the case of S) are defined using discrete average and difference operators multiplied by real coefficients. Therefore, using analysis very similar to the analysis used above for the operator A, we get that the following equalities hold:

$$\begin{aligned}
||Su||^2_{L^2(\overline{\mathcal{B}})} &= ||Su_1||^2_{L^2(\overline{\mathcal{B}})} + ||Su_2||^2_{L^2(\overline{\mathcal{B}})} \\
||Au||^2_{L^2(\overline{\mathcal{B}})} &= ||Au_1||^2_{L^2(\overline{\mathcal{B}})} + ||Au_2||^2_{L^2(\overline{\mathcal{B}})} \\
||\Delta_{\mathbf{s}} \tilde{I} u||^2_{L^2(\overline{\mathcal{B}})} &= ||\Delta_{\mathbf{s}} \tilde{I} u_1||^2_{L^2(\overline{\mathcal{B}})} + ||\Delta_{\mathbf{s}} \tilde{I} u_2||^2_{L^2(\overline{\mathcal{B}})}.
\end{aligned}$$

Therefore if it is shown that

$$||\Delta_{\mathbf{s}}\tilde{I}(u_j)||^2_{L^2(\overline{\mathcal{B}})} = ||Su_j||^2_{L^2(\overline{\mathcal{B}})} + ||Au_j||^2_{L^2(\overline{\mathcal{B}})} \tag{26}$$

for $j = 1, 2$, then it would follow that

$$\begin{aligned}
||\Delta_{\mathbf{s}}\tilde{I}(u)||^2_{L^2(\overline{\mathcal{B}})} &= ||\Delta_{\mathbf{s}}\tilde{I}(u_1)||^2_{L^2(\overline{\mathcal{B}})} + ||\Delta_{\mathbf{s}}\tilde{I}(u_2)||^2_{L^2(\overline{\mathcal{B}})} \\
&= \left(||Su_1||^2_{L^2(\overline{\mathcal{B}})} + ||Au_1||^2_{L^2(\overline{\mathcal{B}})}\right) + \left(||Su_2||^2_{L^2(\overline{\mathcal{B}})} + ||Au_2||^2_{L^2(\overline{\mathcal{B}})}\right) \\
&= ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||^2_{L^2(\overline{\mathcal{B}})},
\end{aligned}$$

which is the identity guaranteed by Property 4.

It therefore suffices to prove

$$||\Delta_{\mathbf{s}}\tilde{I}(u)||^2_{L^2(\overline{\mathcal{B}})} = ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||^2_{L^2(\overline{\mathcal{B}})}$$

for any $u \in C_c(\mathcal{B}, \mathbb{R})$. This would imply that (26) holds for $j = 1, 2$.

This identity is a consequence of Property 3 above.

$$\begin{aligned}
||\Delta_{\mathbf{s}}\tilde{I}(u)||^2_{L^2(\overline{\mathcal{B}})} &= ||(S+A)u||^2_{L^2(\overline{\mathcal{B}})} = ((S+A)u,(S+A)u)_{\overline{\mathcal{B}}} \\
&= (Su,Su)_{\overline{\mathcal{B}}} + (Au,Au)_{\overline{\mathcal{B}}} + (Su,Au)_{\overline{\mathcal{B}}} + (Au,Su)_{\overline{\mathcal{B}}} \\
&= (Su,Su)_{\overline{\mathcal{B}}} + (Au,Au)_{\overline{\mathcal{B}}} + \left(\sum_{k=1}^{n} S_k u, \sum_{j=1}^{n} A_j u\right)_{\overline{\mathcal{B}}} \\
&\quad + \left(\sum_{j=1}^{n} A_j u, \sum_{k=1}^{n} S_k u\right)_{\overline{\mathcal{B}}} \\
&= ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||_{L^2(\overline{\mathcal{B}})} + \sum_{k,j=1}^{n} (S_k u, A_j u)_{\overline{\mathcal{B}}} \\
&\quad + \sum_{k,j=1}^{n} (A_j u, S_k u)_{\overline{\mathcal{B}}} \\
&= ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||^2_{L^2(\overline{\mathcal{B}})}
\end{aligned}$$

for $u \in C_c(\mathcal{B},\mathbb{R})$. This completes the proof of Proposition 8. □

Property (4) of Proposition 8 means that to show equation (21), it suffices to show that

$$||Au||^2_{L^2(\overline{\mathcal{B}})} \geq |\mathbf{b}|^2 ||u||^2_{L^2(\overline{\mathcal{B}})}.$$

This is a key fact used in the following proof of Theorem 8.

Proof of Theorem 8. Given $\mathbf{b} \in \mathbb{R}^n$, let $A_{\mathbf{b}}$ denote the operator $A_{\mathbf{b}}u = \sum_{j=1}^{n} \mathbf{b}_j \tilde{a}_j \partial_j \tilde{I}(u) \in C(\overline{\mathcal{B}})$ for any $u \in C_c(\mathcal{B})$. Fix a $\mathbf{b} \neq 0$. The proof will first show that $u \mapsto ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})}$ is a norm on $C_c(\mathcal{B})$.

Because $||\cdot||_{L^2(\overline{\mathcal{B}})}$ is a norm, we get immediately that $||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} \geq 0$ for all $u \in C(\overline{\mathcal{B}})$. Furthermore, by linearity of the operator $A_{\mathbf{b}}$:

$$||A_{\mathbf{b}}(\alpha u)||_{L^2(\overline{\mathcal{B}})} = ||\alpha A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} = |\alpha| ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})}$$

for any $\alpha \in \mathbb{R}$, and

$$||A_{\mathbf{b}}(u+v)||_{L^2(\overline{\mathcal{B}})} = ||A_{\mathbf{b}}(u) + A_{\mathbf{b}}(v)||_{L^2(\overline{\mathcal{B}})} \leq ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} + ||A_{\mathbf{b}}v||_{L^2(\overline{\mathcal{B}})}$$

for any $u, v \in C_c(\mathcal{B})$.

The final property left to check is that $||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} = 0$ only when $u = 0$. To see this, let $u \in C_c(\mathcal{B})$ be a function that is not identically zero; that is, $\text{supp}(u) \neq \emptyset$. Make the following definitions:

1. Let $j_0 \in \{1, 2, ..., n\}$ be an index such that $\mathbf{b}_{j_0} \neq 0$.

2. Let $M := \min_{x \in \text{supp}(u)} x_{j_0}$ be the smallest value of the j_0-th index achieved over the support of u, denoted $\text{supp}(u)$. Because $\mathcal{B}$ is bounded, M is finite.

3. Define the set $\mathcal{S}_0$ to be the set of all x in the support of u whose j_0-th index value is this minimum value M. That is,

$$\mathcal{S}_0 = \{x \in \text{supp}(u) : x_{j_0} = M\} \subset \text{supp}(u) \subset \mathcal{B}.$$

Fix an $x \in \mathcal{S}_0$, and define $x_0 := x - h\mathbf{e}^{j_0}$. Since $(x_0)_{j_0} = M - h < M = x_{j_0}$, it follows by definition of M that $x_0 \notin \text{supp}(u)$; however, since $x \in \text{supp}(u)$, we do get that $x_0 \in \overline{\text{supp}(u)} \subset \overline{\mathcal{B}}$.

To prove that $||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} \neq 0$ when $u \neq 0$, it will be shown that $|A_{\mathbf{b}}u(x_0)| \neq 0$, where $x_0 \in \overline{\mathcal{B}}$ is the point described above.

First, note that $\mathbf{b}_{j_0}\tilde{a}_{j_0}\partial_{j_0}\tilde{I}u(x_0) = \frac{\mathbf{b}_{j_0}}{2h}\left[\tilde{I}u\left(x_0 + h\mathbf{e}^{j_0}\right) - \tilde{I}u\left(x_0 - h\mathbf{e}^{j_0}\right)\right]$. However, $\left(x_0 - h\mathbf{e}^{j_0}\right)_{j_0} = M - 2h < M$. Therefore by the definition of M, $x_0 - h\mathbf{e}^{j_0} \notin \text{supp}(u)$, and so $\tilde{I}u\left(x_0 - h\mathbf{e}^{j_0}\right) = 0$. However, $\mathbf{b}_{j_0} \neq 0$ and $\tilde{I}u\left(x_0 + h\mathbf{e}^{j_0}\right) = \tilde{I}u(x) \neq 0$, so

$$\left|\frac{\mathbf{b}_{j_0}}{2h}u\left(x_0 + h\mathbf{e}^{j_0}\right)\right| = |\mathbf{b}_{j_0}\tilde{a}_{j_0}\partial_{j_0}u\left(x_0\right)| > 0.$$

Now for any other index $j \in \{1, 2, ..., n\}$ such that $j \neq j_0$, we have that $u(x_0 + h\mathbf{e}^j) = 0$. For if $u(x_0 + h\mathbf{e}^j) \neq 0$, then $y := x_0 + h\mathbf{e}^j \in \text{supp}(u)$, but $y_{j_0} = (x_0)_{j_0} = M - h$, a contradiction to the definition of the constant M. Similarly, we have that $u(x_0 - h\mathbf{e}^j) = 0$ for any other $j \in \{1, 2, ..., n\}$ such that $j \neq j_0$. Thus

$$|A_{\mathbf{b}}u(x_0)| = \left| \sum_{j=1}^{n} \frac{\mathbf{b}_j}{2h} \left[\tilde{I}u\left(x_0 + h\mathbf{e}^j\right) - \tilde{I}u\left(x_0 - h\mathbf{e}^j\right) \right] \right| = \left| \frac{\mathbf{b}_{j_0}}{2h} \tilde{I}(u)(x_0 + h\mathbf{e}^{j_0}) \right| > 0,$$

and so $||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} > 0$.

Since $u \mapsto ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})}$ takes the value 0 only when $u = 0$, this map is a norm on $C_c(\mathcal{B})$. Norms on finite-dimensional spaces are equivalent ([10] Corollary 2.17, p.43), and so there exists a constant $C = C(\mathbf{b}, \overline{\mathcal{B}}) > 0$ such that

$$||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} \geq C(\mathbf{b}, \overline{\mathcal{B}})||u||_{L^2(\overline{\mathcal{B}})} \tag{27}$$

for every $u \in C_c(\mathcal{B})$.

If p is the number of points in $\overline{\mathcal{B}}$, define the map $F : \mathbb{R}^n \times \mathbb{C}^p \to \mathbb{R}$ by

$$F(\mathbf{b}, \mathbf{u}) = ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})},$$

where $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{u}$ is the vector form for a function $u \in C(\overline{\mathcal{B}})$. Because F is defined using absolute values, products, and sums of components of the vectors $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{u} \in C(\overline{\mathcal{B}})$, the map F is continuous on $\mathbb{R}^n \times \mathbb{C}^p$.

Let $K \subseteq \mathbb{R}^n \times \mathbb{C}^p$ be the compact set given by

$$K := \left\{(\mathbf{b}, \mathbf{u}) : |\mathbf{b}| = 1 \text{ and } ||u||_{L^2(\overline{\mathcal{B}})} = 1\right\}.$$

By the work done before, it is already known that for any $\mathbf{b} \in \mathbb{R}^n$ and $u \in C(\overline{\mathcal{B}})$,

$$F(\mathbf{b}, \mathbf{u}) \geq C(\mathbf{b}, \overline{\mathcal{B}})||u||_{L^2(\overline{\mathcal{B}})} = C(\mathbf{b}, \overline{\mathcal{B}}) > 0.$$

Since F is continuous on K and K is compact, it follows that there is a constant $c = c(\overline{\mathcal{B}}) > 0$ such that

$$F(\mathbf{b}, \overline{\mathcal{B}}) \geq c(\overline{\mathcal{B}}) > 0$$

on K.

Therefore for any $\mathbf{b} \in \mathbb{R}^n$ and $u \in C(\overline{\mathcal{B}})$ with $\mathbf{b} \neq \mathbf{0}$ and $u \not\equiv 0$,

$$\begin{aligned} ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} &= |\mathbf{b}|||A_{\mathbf{b}/|\mathbf{b}|}u||_{L^2(\overline{\mathcal{B}})} \\ &= |\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})} \left|\left| A_{\mathbf{b}/|\mathbf{b}|} \frac{u}{||u||_{L^2(\overline{\mathcal{B}})}} \right|\right|_{L^2(\overline{\mathcal{B}})} \\ &\geq |\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})} \cdot F\left(\frac{\mathbf{b}}{|\mathbf{b}|}, \frac{\mathbf{u}}{||u||_{L^2(\overline{\mathcal{B}})}} \right) \\ &\geq c(\overline{\mathcal{B}})|\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})}. \end{aligned}$$

In particular,

$$||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} \geq c(\overline{\mathcal{B}})|\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})} \tag{28}$$

whenever $\mathbf{b} \in \mathbb{R}^n$ and $u \in C(\overline{\mathcal{B}})$ satisfy $\mathbf{b} \neq \mathbf{0}$ and $u \not\equiv 0$. Furthermore, if either $\mathbf{b} = 0$ or $u = 0$, then

$$0 = ||A_{\mathbf{b}}u||_{L^2(\overline{\mathcal{B}})} \geq c(\overline{\mathcal{B}})|\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})} = 0.$$

Therefore (28) is satisfied for all $\mathbf{b} \in \mathbb{R}^n$ and $u \in C(\overline{\mathcal{B}})$.

Noting that $A = A_{\mathbf{b}}$ where $\mathbf{b}_j = \frac{2}{h}\sinh(\mathbf{s}_j h)$, we get that by property (4) of Theorem 8, it follows that for all $u \in C_c(\mathcal{B})$,

$$||\Delta_{\mathbf{s}}\tilde{I}(u)||^2_{L^2(\overline{\mathcal{B}})} = ||Su||^2_{L^2(\overline{\mathcal{B}})} + ||Au||^2_{L^2(\overline{\mathcal{B}})} \geq ||Au||^2_{L^2(\overline{\mathcal{B}})}$$
$$\geq c^2(\overline{\mathcal{B}})|\mathbf{b}|^2||u||^2_{L^2(\overline{\mathcal{B}})},$$

where $\mathbf{b}_j = \frac{2}{h}\sinh(\mathbf{s}_j h)$. □

2.2.3 Construction of CGO Solutions

The proof that $q_1 = q_2$ if $\Lambda_1 = \Lambda_2$ relies on the construction of discrete CGO solutions that are defined using an exponential term $e^{i\boldsymbol{\beta}\cdot x}$ and a remainder term r. The following Theorem provides the tools needed to define and analyze these remainder terms.

Theorem 9. *Let $q \in L^\infty(\mathcal{B}, \mathbb{R})$ be a potential function defined on a bounded and connected subset $\mathcal{B}$ of the uniform discrete grid. Then there exists an $s_0 = s_0(||q||_{L^\infty(\mathcal{B})}, \mathcal{B}) \geq 1$ sufficiently large such that the following holds. For any $f \in C(\mathcal{B})$ and any $\mathbf{s} \in \mathbb{R}^n$ such that $|\mathbf{s}| \geq s_0$, there is an $r \in C(\overline{\mathcal{B}})$ that solves*

$$(\Delta_{\mathbf{s}} + q)r = f \qquad (29)$$

on $\mathcal{B}$. Furthermore, r satisfies $|\mathbf{b}|\,||r||_{L^2(\overline{\mathcal{B}})} \leq c(\mathcal{B}, h)||f||_{L^2(\mathcal{B})}$, where $\mathbf{b}_j = \frac{2}{h}\sinh(h\mathbf{s}_j)$.

Proof of Theorem 9. This proof follows that in [3] (Lemma 4.2, p.24).

STEP 1: Solution Existence

Fix $q \in L^\infty(\mathcal{B}, \mathbb{R})$ and $f \in C(\mathcal{B}, \mathbb{R})$; the proof will later be extended for any $f \in C(\mathcal{B})$.

Then define the functional $J : C_c(\mathcal{B}, \mathbb{R}) \to \mathbb{R}$ by

$$J(v) = \int_{\overline{\mathcal{B}}} |P^*(\tilde{I}v)|^2 - \int_{\mathcal{B}} 2fv,$$

where the operator P^* is defined to be $P^*(v) = (\Delta_{-\mathbf{s}} + I(q))(v)$ for $v \in C_c(\mathcal{K}, \mathbb{R})$. In particular, for any $v, w \in C_c(\mathcal{K}, \mathbb{R})$:

$$\begin{aligned}
\int_{\mathcal{K}} vP^*w &= \int_{\mathcal{K}} v\left(\Delta_{-\mathbf{s}} + I(q)\right)w \\
&= \int_{\mathcal{K}} ve^{\mathbf{s}\cdot x}\Delta\left(e^{-\mathbf{s}\cdot x}w\right) + \int_{\mathcal{K}} I(q)vw \\
&= -\sum_{j=1}^{n} \int_{\mathcal{K}^j} \partial_j\left(ve^{\mathbf{s}\cdot x}\right)\partial_j\left(e^{-\mathbf{s}\cdot x}w\right) + \int_{\partial\mathcal{K}} \left(ve^{\mathbf{s}\cdot x}\right)\partial_\eta\left(e^{-\mathbf{s}\cdot x}w\right) + \int_{\mathcal{K}} I(q)vw \\
&= -\sum_{j=1}^{n} \int_{\mathcal{K}^j} \partial_j\left(ve^{\mathbf{s}\cdot x}\right)\partial_j\left(e^{-\mathbf{s}\cdot x}w\right) + \int_{\mathcal{K}} I(q)vw \\
&= \int_{\mathcal{K}} e^{-\mathbf{s}\cdot x}\Delta\left(ve^{\mathbf{s}\cdot x}\right)w - \int_{\partial\mathcal{K}} e^{-\mathbf{s}\cdot x}w\partial_\eta\left(ve^{\mathbf{s}\cdot x}\right) + \int_{\mathcal{K}} I(q)vw \\
&= \int_{\mathcal{K}} e^{-\mathbf{s}\cdot x}\Delta\left(ve^{\mathbf{s}\cdot x}\right)w + \int_{\mathcal{K}} I(q)vw \\
&= \int_{\mathcal{K}} w\left(\Delta_{\mathbf{s}} + I(q)\right)v =: \int_{\mathcal{K}} P(v)w.
\end{aligned}$$

The computations follow from two applications of the discrete Green's Theorem (Theorem 2) as well as the fact that $w \equiv 0 \equiv v$ on $\partial\mathcal{K}$. It follows that P^* is the transpose of the operator P given by $P(v) = (\Delta_{\mathbf{s}} + I(q))\tilde{I}(v)$ for $v \in C_c(\mathcal{K}, \mathbb{R})$.

Suppose that $v \in C_c(\mathcal{B}, \mathbb{R})$ minimizes the functional J. Then for any $w \in C_c(\mathcal{B}, \mathbb{R})$,

$$\begin{aligned}
0 = \left.\frac{d}{dt}\right|_{t=0} J(v + tw) &= \left.\frac{d}{dt}\right|_{t=0} \left[\int_{\overline{\mathcal{B}}} \left(P^*\left(\tilde{I}v\right) + tP^*\left(\tilde{I}w\right)\right)^2 - \int_{\mathcal{B}} 2f(v + tw)\right] \\
&= \int_{\overline{\mathcal{B}}} 2\left(P^*\left(\tilde{I}v\right) + tP^*\left(\tilde{I}w\right)\right)P^*(\tilde{I}w) - \left.\int_{\mathcal{B}} 2fw\right|_{t=0} \\
&= \int_{\overline{\mathcal{B}}} 2P^*(\tilde{I}w)P^*(\tilde{I}v) - \int_{\mathcal{B}} 2fw. \qquad (30)
\end{aligned}$$

Suppose for now that a minimizer $v \in C_c(\mathcal{B}, \mathbb{R})$ of J exists, and let $r := P^*(\tilde{I}v)$ on $\overline{\mathcal{B}}$. Using the fact that P^* is the transpose of P on $C_c(\mathcal{K}, \mathbb{R})$, it follows that (30) is equivalent to

$$\begin{aligned} 0 &= \int_{\overline{\mathcal{B}}} P^*(\tilde{I}w)r - \int_{\mathcal{B}} fw \\ &= \int_{\mathcal{K}} P^*\left(\tilde{I}w\right)\tilde{I}(r) - \int_{\mathcal{B}} fw \\ &= \int_{\mathcal{K}} \tilde{I}(w)P\left(\tilde{I}r\right) - \int_{\mathcal{B}} wf \\ &= \int_{\mathcal{B}} w\left(P(\tilde{I}r) - f\right) \end{aligned}$$

for all $w \in C_c(\mathcal{B}, \mathbb{R})$. Therefore if a minimizer $v \in C_c(\mathcal{B}, \mathbb{R})$ exists for the functional J, then it would follow that $r = P^*(\tilde{I}v)$ is a solution to $P(\tilde{I}r) = f$ on $\mathcal{B}$.

In fact, for any $x \in \mathcal{B}$, $x \pm h\mathbf{e}^k \in \overline{\mathcal{B}}$ for both choices of sign; therefore

$$\begin{aligned} f(x) = P(\tilde{I}r)(x) &= \frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}\tilde{I}r(x+h\mathbf{e}^k) - 2\tilde{I}r(x) + e^{-\mathbf{s}_k h}\tilde{I}r(x-h\mathbf{e}^k)\right\} + Iq(x)\tilde{I}r(x) \\ &= \frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}r(x+h\mathbf{e}^k) - 2r(x) + e^{-\mathbf{s}_k h}r(x-h\mathbf{e}^k)\right\} + q(x)r(x) \\ &= (\Delta_{\mathbf{s}} + q(x))r(x) \end{aligned}$$

Therefore this r satisfies (29).

Now to show the minimizer v exists on $C_c(\mathcal{B}, \mathbb{R})$. The Carleman estimate in (21) provides that $||\Delta_s(\tilde{I}(u))||_{L^2(\overline{\mathcal{B}})} \geq C|\mathbf{b}|||u||_{L^2(\overline{\mathcal{B}})}$ if $u \in C_c(\mathcal{B}, \mathbb{R})$. Therefore if s_0 is large enough compared to q so that $C\frac{|\mathbf{b}|}{2} \geq ||q||_{L^\infty(\mathcal{B})}$ whenever $|\mathbf{s}| > s_0$, then for any $\mathbf{s} \in \mathbb{R}^n$ such that $|\mathbf{s}| > s_0$:

$$\begin{aligned} ||P^*(\tilde{I}v)||_{L^2(\overline{\mathcal{B}})} &\geq ||\Delta_{-\mathbf{s}}(\tilde{I}(v))||_{L^2(\overline{\mathcal{B}})} - ||I(q)||_{L^\infty(\overline{\mathcal{B}})}||v||_{L^2(\overline{\mathcal{B}})} \\ &\geq C|-\mathbf{b}|||v||_{L^2(\overline{\mathcal{B}})} - ||q||_{L^\infty(\mathcal{B})}||v||_{L^2(\overline{\mathcal{B}})} \\ &\geq C|\mathbf{b}|||v||_{L^2(\overline{\mathcal{B}})} - \left(C\frac{|\mathbf{b}|}{2}\right)||v||_{L^2(\overline{\mathcal{B}})} \\ &\geq C'|\mathbf{b}|||v||_{L^2(\overline{\mathcal{B}})}, \end{aligned} \tag{31}$$

for all $v \in C_c(\mathcal{B}, \mathbb{R})$, where $C = C(\mathcal{B})$ is the constant given in Theorem 8. It follows that

$$J(v) = ||P^*(\tilde{I}v)||^2_{L^2(\overline{\mathcal{B}})} - \int_{\mathcal{B}} 2fv \geq C'|\mathbf{b}|||v||^2_{L^2(\overline{\mathcal{B}})} - 2||f||_{L^2(\mathcal{B})}||v||_{L^2(\overline{\mathcal{B}})} \to +\infty$$

as $||v||_{L^2(\overline{\mathcal{B}})} \to \infty$, indicating that J must have a minimum on $C_c(\mathcal{B}, \mathbb{R})$ (see Proposition 9 in Section 3 for more details).

STEP 2: Estimate

If $v \in C_c(\mathcal{B}, \mathbb{R})$ is the minimizer of J and $r = P^*\left(\tilde{I}v\right)$ as described in Step 1, then

$$\begin{aligned} ||r||^2_{L^2(\overline{\mathcal{B}})} = ||P^*(\tilde{I}v)||^2_{L^2(\overline{\mathcal{B}})} &= \int_{\overline{\mathcal{B}}} |P^*(\tilde{I}v)|^2 \\ &= \int_{\mathcal{B}} fv \leq ||f||_{L^2(\mathcal{B})}||v||_{L^2(\mathcal{B})} \\ &\leq ||f||_{L^2(\mathcal{B})} \cdot \frac{1}{C'|\mathbf{b}|}||P^*(\tilde{I}v)||_{L^2(\overline{\mathcal{B}})}. \end{aligned}$$

To get from the first to the second line, use (30) with $w = v$. To get from the second to the last line, use the fact that $v \in C_c(B, \mathbb{R})$ and (31). If $r \not\equiv 0$, then dividing both sides by $||P^*(\tilde{I}v)||_{L^2(\overline{\mathcal{B}})} = ||r||_{L^2(\overline{\mathcal{B}})}$ gives the desired inequality. If $r \equiv 0$, then the desired inequality follows trivially.

STEP 3: Complex-Valued f

Now the theorem will be proved for $f \in C(\mathcal{B}, \mathbb{C})$. Let $f_1 = \text{Re}(f)$ and $f_2 = \text{Im}(f)$. Let s_0 be large enough so that solutions r_1 and $r_2 \in C(\overline{\mathcal{B}}, \mathbb{R})$ as described by steps 1 and 2 above exist to both of the equations

$$(\Delta_{\mathbf{s}} + q)r_1 = f_1$$

$$(\Delta_{\mathbf{s}} + q)r_2 = f_2.$$

Then $r = r_1 + ir_2$ solves

$$(\Delta_{\mathbf{s}} + q)r = (\Delta_{\mathbf{s}} + q)(r_1 + ir_2) = (\Delta_{\mathbf{s}} + q)r_1 + i(\Delta_{\mathbf{s}} + q)r_2 = f_1 + if_2 = f,$$

and

$$\begin{aligned} |\mathbf{b}|^2 ||r||^2_{L^2(\overline{\mathcal{B}})} &= |\mathbf{b}|^2 \left(||r_1||^2_{L^2(\overline{\mathcal{B}})} + ||r_2||^2_{L^2(\overline{\mathcal{B}})} \right) \\ &\leq c^2 \left(||f_1||^2_{L^2(\mathcal{B})} + ||f_2||^2_{L^2(\mathcal{B})} \right) = c^2 ||f||_{L^2(\mathcal{B})}, \end{aligned}$$

where $c = c(\mathcal{B}, h)$. □

2.3 Proof

Theorem 10. *Let $\mathcal{B}$ be a bounded subset of the uniform discrete grid of step size $h > 0$ in dimension $n \geq 3$. Fix two potentials $q_j \in L^\infty(\mathcal{B}, \mathbb{R})$ such that $\mathcal{C}(q_j) = C(\partial\mathcal{B})$ for $j = 1, 2$ and a vector $\boldsymbol{\beta} \in \left(-\frac{\pi}{2h}, \frac{\pi}{2h}\right)^n$ such that $\boldsymbol{\beta}_j \neq 0$ for every $j \in \{1, 2, ..., n\}$. Then there exists a constant $s_0(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, \mathcal{B}, h) \geq 1$ such that for any $S > s_0$, there are vectors $\mathbf{s}$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$ and a pair of solutions u_+ and u_- to the discrete PDEs*

$$(\Delta + q_1)u_+ = 0$$

$$(\Delta + q_2)u_- = 0$$

on $\mathcal{B}$ that satisfy the following.

1. *The solutions u_+ and u_- are of the form*

$$u_\pm(x) = e^{i\boldsymbol{\beta}\cdot\boldsymbol{x}} e^{\pm\boldsymbol{\eta}\cdot\boldsymbol{x}}(1 + r_\pm(x)),$$

where $\boldsymbol{\eta} = \mathrm{s} + i\boldsymbol{\alpha}$.

2. *The real part* s *of* $\boldsymbol{\eta}$ *satisfies* $|\mathrm{s}| \geq S$ *and the imaginary part* $\boldsymbol{\alpha}$ *satisfies* $\boldsymbol{\alpha}_k \in \{0, -\pi/h, \pi/h\}$ *for all* $k \in \{1, 2, ..., n\}$. *The choices of* s *and* $\boldsymbol{\alpha}$ *depend only on* h, $\boldsymbol{\beta}$, *and* S.

3. *The remainder terms* $r_\pm$ *satisfy*

$$||r_\pm||_{L^2(\overline{\mathcal{B}})} \leq \frac{C}{\sum_k |\frac{2}{h}\sinh(\mathbf{s}_k h)|},$$

where $C = C(h, \mathcal{B}, q_1, q_2)$.

Proof of Theorem 10.

2.3.0.1 Solution Existence To simplify future expressions, let $\mathcal{S}$ be a function defined on the possible choices of $\boldsymbol{\alpha}_k$ such that $\mathcal{S}(0) = 1$ and $\mathcal{S}(\pm\pi/h) = -1$. This function is the sign change in front of the $\sin$ and $\cos$ terms after an angle shift of $\boldsymbol{\alpha}_k h$. Specifically, $\mathcal{S}$ is defined so that

$$\cos((\boldsymbol{\beta}_j \pm \boldsymbol{\alpha}_j)h) = \mathcal{S}(\boldsymbol{\alpha}_j)\cos(\boldsymbol{\beta}_j h)$$

and

$$\sin((\boldsymbol{\beta}_j \pm \boldsymbol{\alpha}_j)h) = \mathcal{S}(\boldsymbol{\alpha}_j)\sin(\boldsymbol{\beta}_j h)$$

for any $j \in \{1, 2, ..., n\}$ whenever $\boldsymbol{\alpha}_j \in \{0, \pm\pi/h\}$.

If a function u_+ of the form $u_+(x) = e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}(1 + r_+(x))$ solved $(\Delta + q_1)u_+ = 0$, then the following is equivalent:

$$\begin{aligned}0 = e^{-\mathbf{s}\cdot x}(\Delta + q_1)\, u_+ &= e^{-\mathbf{s}\cdot x}(\Delta + q_1)\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x} + e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}r_+\right)\\ &= e^{-\mathbf{s}\cdot x}(\Delta + q_1)\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) + e^{-\mathbf{s}\cdot x}(\Delta + q_1)\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}r_+\right).\end{aligned}$$

Define $r_+ = e^{-i(\boldsymbol{\beta}+\boldsymbol{\alpha})\cdot x}\tilde{r}_+$. Then the above becomes

$$\begin{aligned}0 &= e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) + e^{-\mathbf{s}\cdot x}q_1\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) + e^{-\mathbf{s}\cdot x}(\Delta + q_1)\left(e^{\mathbf{s}\cdot x}\tilde{r}_+\right)\\ &= e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) + q_1 e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x} + \left(\Delta_{\mathbf{s}} + q_1\right)\tilde{r}_+.\end{aligned}$$

By Theorem 9, given a s $\in \mathbb{R}^n$, a solution $\tilde{r}_+$ to

$$\left(\Delta_{\mathbf{s}} + q_1\right)\left(\tilde{r}_+\right) = -e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) - e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}q_1$$

exists when $|\mathbf{s}| \geq s_0' = s_0'(||q_1||_{L^\infty(\mathcal{B})}, \mathcal{B}, h)$. Equivalently, when $|\mathbf{s}| \geq s_0'$, there exists an $r_+ = e^{-i(\boldsymbol{\beta}+\boldsymbol{\alpha})\cdot x}\tilde{r}_+$ such that $u_+(x) = e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}(1 + r_+(x))$ solves $(\Delta + q_1)u_+ = 0$.

Furthermore, the remainder term r_+ satisfies

$$\begin{aligned}||r_+||_{L^2(\overline{\mathcal{B}})} = ||\tilde{r}_+||_{L^2(\overline{\mathcal{B}})} &\leq \frac{c(\mathcal{B}, h)}{|\mathbf{b}|}\left|\left|-e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) - e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}q_1\right|\right|_{L^2(\mathcal{B})}\\ &\leq \frac{c(\mathcal{B}, h)}{|\mathbf{b}|}\left(||e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} + c'(\mathcal{B})||q_1||_{L^\infty(\mathcal{B})}\right),\end{aligned}$$

where the vector $\mathbf{b}$ has entries $\mathbf{b}_k = \frac{2}{h}\sinh(\mathbf{s}_k h)$.

Similarly, a solution u_- to the equation $(\Delta+q_2)u_- = 0$ of the form $u_-(x) = e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\left(1 + r_-(x)\right)$ exists exactly when

$$0 = e^{\mathbf{s}\cdot x}\left(\Delta + q_2\right)u_- = e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right) + q_2 e^{i\boldsymbol{\beta}\cdot x}e^{-i\boldsymbol{\alpha}\cdot x} + \left(\Delta_{-\mathbf{s}} + q_2\right)\tilde{r}_-,$$

where $\tilde{r}_-(x) := e^{i(\boldsymbol{\beta}-\boldsymbol{\alpha})\cdot x}r_-(x)$. This is equivalent to saying that there exists an $\tilde{r}_-$ that solves

$$\left(\Delta_{-\mathbf{s}} + q_2\right)\tilde{r}_- = -e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right) - q_2 e^{i\boldsymbol{\beta}\cdot x}e^{-i\boldsymbol{\alpha}\cdot x}.$$

Therefore by Theorem 9, there exists an $s_0'' = s_0''(||q_2||_{L^\infty(\mathcal{B})}, \mathcal{B}, h)$ such that for any $S > s_0''$, if

$|-\mathbf{s}| = |\mathbf{s}| \geq S$ then a solution to $(\Delta + q_2)u_- = 0$ of the form $u_-(x) = e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}(1 + r_-(x))$ exists, and

$$||r_-||_{L^2(\overline{\mathcal{B}})} = ||\tilde{r}_-||_{L^2(\overline{\mathcal{B}})} \leq \frac{c(\mathcal{B},h)}{|-\mathbf{b}|}\left(||e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} + c'(\mathcal{B})||q_2||_{L^\infty(\mathcal{B})}\right)$$
$$= \frac{c(\mathcal{B},h)}{|\mathbf{b}|}\left(||e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} + c'(\mathcal{B})||q_2||_{L^\infty(\mathcal{B})}\right).$$

Therefore the proof is complete if there can be found a constant C such that for any

$$S > s_0'''(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \mathcal{B}, h) := \max\{s_0', s_0''\} \tag{32}$$

arbitrarily large, there is a vector $\boldsymbol{\eta} = \mathbf{s} + i\boldsymbol{\alpha}$ such that $|\mathbf{s}| \geq S$ and

$$||e^{\mp\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\pm\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} \leq C,$$

or equivalently (since $\mathcal{B}$ has a finite number of points):

$$||e^{\mp\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\pm\boldsymbol{\eta}\cdot x}\right)||_{L^\infty(\mathcal{B})} \leq C. \tag{33}$$

Note that (33) will need to be satisfied for both choices of sign in order to prove the Theorem.

Constructing the vectors $\mathbf{s}$ and $\boldsymbol{\alpha}$ that satisfy $|\mathbf{s}| \geq S$ and $||e^{\mp\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\pm\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} \leq C$ (for C independent of S) is therefore the goal of the remainder of the proof.

2.3.0.2 Estimates The domain $\mathcal{B}$, potentials q_1 and q_2, and vector $\boldsymbol{\beta}$ as given in the statement of Theorem 10 are all fixed.

Let $s_0 \geq 1$ be large enough so that $s_0 \geq s_0'''$ (where s_0''' is given in Part 2.3.0.1, equation (32)), and so that

$$\frac{|\sin(\boldsymbol{\beta}_1 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{|\sin(\boldsymbol{\beta}_3 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)} e^{s_0 h} \geq 1. \tag{34}$$

Note that this is possible because $\boldsymbol{\beta} \in (-\pi/2h, \pi/2h)^n$ satisfies $\boldsymbol{\beta}_j \notin \{0, \pm\frac{\pi}{2}\}$ for every $j \in \{1, 2, ..., n\}$ and therefore the left-hand side of (34) is finite and strictly positive.

Then given any $S > s_0$ fixed, define the vectors s, $\boldsymbol{\alpha} \in \mathbb{R}^n$ as follows.

1. Set $\mathrm{s}_k = 0$ and $\boldsymbol{\alpha}_k = 0$ for $k > 3$.

2. Let $\boldsymbol{\alpha}_1 = 0$, $\boldsymbol{\alpha}_3 = \pi/h$, and $|\mathrm{s}_1| = S > s_0$. The condition on s_1 implies then that $|\mathrm{s}| \geq S$.

3. Choose sign(s_j) for $j = 1, 2, 3$ so that

$$\text{sign}\,(\mathrm{s}_1) = \text{sign}\,(\sin(\boldsymbol{\beta}_1 h)) \tag{35}$$

$$\text{sign}\,(\mathrm{s}_2) = -\text{sign}\,(\sin(\boldsymbol{\beta}_2 h)) \tag{36}$$

$$\text{sign}\,(\mathrm{s}_3) = \text{sign}\,(\sin(\boldsymbol{\beta}_3 h))\,. \tag{37}$$

   These choices depend only on $\boldsymbol{\beta}$.

4. Because $s_0(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, h)$ is chosen to be large enough so that (34) is satisfied, the right-hand side of (38) below is also greater than or equal to 1. Since $|\mathrm{s}_1| = S > s_0$, we can therefore let $|\mathrm{s}_3|$ satisfy

$$e^{|\mathrm{s}_3|h} = \frac{|\sin(\boldsymbol{\beta}_1 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{|\sin(\boldsymbol{\beta}_3 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)} e^{|\mathrm{s}_1|h}. \tag{38}$$

   This choice depends only on $\boldsymbol{\beta}$ and S.

5. Let $|\mathrm{s}_2|$ satisfy

$$e^{|\mathbf{s}_2|h} = \left| \frac{\cos(\boldsymbol{\beta}_1 h) e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h) e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)} \right| + 1, \tag{39}$$

and $\boldsymbol{\alpha}_2$ satisfy

$$\mathcal{S}(\boldsymbol{\alpha}_2) = -\mathrm{sign}\left(\frac{\cos(\boldsymbol{\beta}_1 h) e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h) e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)} \right). \tag{40}$$

If the right-hand side of (40) is 0, set $\boldsymbol{\alpha}_2 = 0$. These choices depend only on $\boldsymbol{\beta}$ and previously-determined variables (s_1, s_3) that themselves depend only on S and $\boldsymbol{\beta}$.

By Part 2.3.0.1, the solutions u_+ and u_- given in Theorem 10 exist if (33) is shown for C independent of S. Proving this estimate is the goal of this part of the proof.

It will first be shown that (33) is satisfied with the $(+)\boldsymbol{\eta}$ sign choice; specifically, it will be shown that

$$||e^{-\mathbf{s}\cdot x} \Delta \left(e^{i\boldsymbol{\beta}\cdot x} e^{\boldsymbol{\eta}\cdot x} \right) ||_{L^\infty(\mathcal{B})} \leq C$$

for a C independent of S.

The quantity $e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right)$ can be equivalently written as

$$\begin{aligned}
e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right) &= e^{-\mathbf{s}\cdot x}\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{(i\boldsymbol{\beta}+\boldsymbol{\eta})\cdot(x+he_k)} - 2e^{i\boldsymbol{\beta}\cdot x+\boldsymbol{\eta}\cdot x} + e^{(i\boldsymbol{\beta}+\boldsymbol{\eta})\cdot(x-he_k)}\right\}\\
&= e^{-\mathbf{s}\cdot x}\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}e^{i\boldsymbol{\beta}_k h}e^{\boldsymbol{\eta}_k h} - 2e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x} + e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}e^{-i\boldsymbol{\beta}_k h}e^{-\boldsymbol{\eta}_k h}\right\}\\
&= e^{-\mathbf{s}\cdot x}\frac{1}{h^2}\cdot e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\sum_{k=1}^{n}\left\{e^{i\boldsymbol{\beta}_k h}e^{\boldsymbol{\eta}_k h} - 2 + e^{-i\boldsymbol{\beta}_k h}e^{-\boldsymbol{\eta}_k h}\right\}\\
&= \frac{1}{h^2}\cdot e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}e^{i(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k)h} - 2 + e^{-\mathbf{s}_k h}e^{-i(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k)h}\right\}\\
&= \frac{e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}}{h^2}\sum_{k=1}^{n}\Big\{e^{\mathbf{s}_k h}\left[\cos\left(\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right) + i\sin\left(\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right]\\
&\qquad -2 + e^{-\mathbf{s}_k h}\left[\cos\left(-\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right) + i\sin\left(-\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)\right)\right]\Big\}\\
&= \frac{e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}\cos\left(\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right) - 2 + e^{-\mathbf{s}_k h}\cos\left(-\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}\\
&\qquad + i\frac{e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}\sin\left(\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right) + e^{-\mathbf{s}_k h}\sin\left(-\left(\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}
\end{aligned} \tag{41}$$

$$\begin{aligned}
&= \frac{e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right) - 2 + e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right)\right\}\\
&\qquad + i\frac{e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right) - e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right)\right\}.
\end{aligned}$$

In particular, because $e^{i\boldsymbol{\beta}\cdot x}e^{i\boldsymbol{\alpha}\cdot x}$ is unitary, the goal is therefore to bound

$$\begin{aligned}
\left|e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right)\right| &= \left|e^{-i\boldsymbol{\beta}\cdot x}e^{-i\boldsymbol{\alpha}\cdot x}e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right)\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right)-2+e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right)\right\}\right. \\
&\qquad \left.+i\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right)-e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right)\right\}\right| \qquad (42) \\
&:= \left|\mathcal{A}+i\mathcal{B}\right| \qquad (43)
\end{aligned}$$

uniformally in $x \in \mathcal{B}$. To do this, the goal of the proof is as follows. There will be found constants C_1 and C_2 such that for any $S > s_0(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, h)$ arbitrarily large, there exist vectors s and $\boldsymbol{\alpha}$ such that

$$|\mathcal{A}| = \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right)-2+e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\cos\left(\boldsymbol{\beta}_k h\right)\right\}\right| \leq C_1 \qquad (44)$$

$$|\mathcal{B}| = \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right)-e^{-\mathbf{s}_k h}S\left(\boldsymbol{\alpha}_k\right)\sin\left(\boldsymbol{\beta}_k h\right)\right\}\right| \leq C_2, \qquad (45)$$

and $|\mathbf{s}| \geq S$. The constants C_1 and C_2 will not depend on S.

First set $\mathrm{s}_k = 0 = \boldsymbol{\alpha}_k$ for all k that satisfy $3 < k \leq n$ (recall that $n \geq 3$), set $\boldsymbol{\alpha}_3 = \pi/h$, and set $\boldsymbol{\alpha}_1 = 0$. Then the left-hand side of (44) becomes

$$\begin{aligned}
|\mathcal{A}| &= \frac{1}{h^2}\Big|\cos(\boldsymbol{\beta}_1 h)\left(e^{\mathbf{s}_1 h}+e^{-\mathbf{s}_1 h}\right)+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\left(e^{\mathbf{s}_2 h}+e^{-\mathbf{s}_2 h}\right) \\
&\qquad -\cos(\boldsymbol{\beta}_3 h)\left(e^{\mathbf{s}_3 h}+e^{-\mathbf{s}_3 h}\right)+2\sum_{j=4}^{n}\mathcal{S}(\boldsymbol{\alpha}_j)\cos(\boldsymbol{\beta}_j h)-2n\Big| \\
&\leq \frac{1}{h^2}\Big|\cos(\boldsymbol{\beta}_1 h)\left(e^{\mathbf{s}_1 h}+e^{-\mathbf{s}_1 h}\right)+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\left(e^{\mathbf{s}_2 h}+e^{-\mathbf{s}_2 h}\right) \\
&\qquad -\cos(\boldsymbol{\beta}_3 h)\left(e^{\mathbf{s}_3 h}+e^{-\mathbf{s}_3 h}\right)\Big|+\frac{2(n-3)}{h^2}+\frac{2n}{h^2}
\end{aligned}$$

Define $\boldsymbol{\alpha}_2$ and s_2 as in (40) and (39), respectively. These definitions give that

$$
\begin{aligned}
|\mathcal{A}| &\leq \frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)\left(e^{\mathbf{s}_1 h}+e^{-\mathbf{s}_1 h}\right)+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\left(e^{\mathbf{s}_2 h}+e^{-\mathbf{s}_2 h}\right)\right.\\
&\qquad \left.-\cos(\boldsymbol{\beta}_3 h)\left(e^{\mathbf{s}_3 h}+e^{-\mathbf{s}_3 h}\right)\right|+\frac{1}{h^2}\left(2(n-3)+2n\right)\\
&\leq \frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)e^{|\mathbf{s}_2|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}\right|\\
&\qquad +\frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)e^{-|\mathbf{s}_1|h}+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)e^{-|\mathbf{s}_2|h}-\cos(\boldsymbol{\beta}_3 h)e^{-|\mathbf{s}_3|h}\right|\\
&\qquad +\frac{1}{h^2}\left(2(n-3)+2n\right)\\
&\leq \frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)e^{|\mathbf{s}_2|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}\right|\\
&\qquad \frac{1}{h^2}\cdot 3+\frac{1}{h^2}\left(2(n-3)+2n\right)\\
&= \frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}\right.\\
&\qquad \left.+\mathcal{S}(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\left(\left|\frac{\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)}\right|+1\right)\right|\\
&\qquad +\frac{1}{h^2}\left(4n-3\right)\\
&= \frac{1}{h^2}\left|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}\right.\\
&\qquad \left.+\cos(\boldsymbol{\beta}_2 h)\mathcal{S}(\boldsymbol{\alpha}_2)\left|\frac{\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h}-\cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)}\right|+S(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\right|\\
&\qquad +\frac{1}{h^2}\left(4n-3\right).
\end{aligned}
$$

Using the fact that $\boldsymbol{\alpha}_2$ is defined to satisfy (40), the above computations become

$$
\begin{aligned}
|\mathcal{A}| \leq & \frac{1}{h^2} \Big|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h} \\
& - \cos(\boldsymbol{\beta}_2 h)\left(\frac{\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)}\right) + S(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\Big| \\
& + \frac{1}{h^2}(4n-3). \\
\leq & \frac{1}{h^2}\Big|\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h} \\
& - \left(\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}\right) + S(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)\Big| \\
& + \frac{1}{h^2}(4n-3). \\
= & \frac{1}{h^2}|S(\boldsymbol{\alpha}_2)\cos(\boldsymbol{\beta}_2 h)| + \frac{1}{h^2}(4n-3) \\
\leq & \frac{1}{h^2} + \frac{1}{h^2}(4n-3) = \frac{1}{h^2}(4n-2).
\end{aligned}
$$

Using again the fact that $\boldsymbol{\alpha}_k = \mathbf{s}_k = 0$ for all $3 < k \leq n$, the imaginary part of equation (42) is given by

$$
\begin{aligned}
|\mathcal{B}| = & \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathbf{s}_k h}S(\boldsymbol{\alpha}_k)\sin(\boldsymbol{\beta}_k h) - e^{-\mathbf{s}_k h}S(\boldsymbol{\alpha}_k)\sin(\boldsymbol{\beta}_k h)\right\}\right| \\
= & \frac{1}{h^2}\Big|e^{\mathbf{s}_1 h}\sin(\boldsymbol{\beta}_1 h) - e^{-\mathbf{s}_1 h}\sin(\boldsymbol{\beta}_1 h) \\
& + e^{\mathbf{s}_2 h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) - e^{-\mathbf{s}_2 h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) \\
& - \left(e^{\mathbf{s}_3 h}\sin(\boldsymbol{\beta}_3 h) - e^{-\mathbf{s}_3 h}\sin(\boldsymbol{\beta}_3 h)\right)\Big| \\
\leq & \frac{1}{h^2}\Big|\text{sign}(\mathbf{s}_1)e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) + \text{sign}(\mathbf{s}_2)e^{|\mathbf{s}_2|h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) - \text{sign}(\mathbf{s}_3)e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\Big| \\
& + \frac{1}{h^2}\Big|-\text{sign}(\mathbf{s}_1)e^{-|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) - \text{sign}(\mathbf{s}_2)e^{-|\mathbf{s}_2|h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) \qquad (46) \\
& \quad + \text{sign}(\mathbf{s}_3)e^{-|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\Big| \\
\leq & \frac{1}{h^2}\Big|\text{sign}(\mathbf{s}_1)e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) + \text{sign}(\mathbf{s}_2)e^{|\mathbf{s}_2|h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) \\
& \quad - \text{sign}(\mathbf{s}_3)e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\Big| + \frac{3}{h^2}.
\end{aligned}
$$

Now applying equation (39):

$$
\begin{aligned}
|\mathcal{B}| \leq{} & \frac{1}{h^2}\left|\text{sign}\,(\mathbf{s}_1)\, e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) + \text{sign}\,(\mathbf{s}_2)\, e^{|\mathbf{s}_2|h}\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h)\right. \\
& \qquad\qquad \left. -\text{sign}\,(\mathbf{s}_3)\, e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\right| + \frac{3}{h^2}. \\
={} & \frac{1}{h^2}\left|\text{sign}(\mathbf{s}_1)e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) - \text{sign}(\mathbf{s}_3)e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\right. \\
& \quad + \text{sign}(\mathbf{s}_2)\left|\frac{\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)}\right|\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h) \\
& \quad \left. +\text{sign}(\mathbf{s}_2)\mathcal{S}(\boldsymbol{\alpha}_2)\sin(\boldsymbol{\beta}_2 h)\right| + \frac{3}{h^2} \\
\leq{} & \frac{1}{h^2}\left|\text{sign}(\mathbf{s}_1)e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) - \text{sign}(\mathbf{s}_3)e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\right. \\
& \quad \left. -\text{sign}(\mathbf{s}_2)\left(\frac{\cos(\boldsymbol{\beta}_1 h)e^{|\mathbf{s}_1|h} - \cos(\boldsymbol{\beta}_3 h)e^{|\mathbf{s}_3|h}}{\cos(\boldsymbol{\beta}_2 h)}\right)\sin(\boldsymbol{\beta}_2 h)\right| + \frac{4}{h^2} \\
={} & \frac{1}{h^2}\left|\text{sign}(\mathbf{s}_1)e^{|\mathbf{s}_1|h}\sin(\boldsymbol{\beta}_1 h) - \text{sign}(\mathbf{s}_3)e^{|\mathbf{s}_3|h}\sin(\boldsymbol{\beta}_3 h)\right. \\
& \quad \left. -\text{sign}(\mathbf{s}_2)\frac{\cos(\boldsymbol{\beta}_1 h)\sin(\boldsymbol{\beta}_2 h)}{\cos(\boldsymbol{\beta}_2 h)}e^{|\mathbf{s}_1|h} + \text{sign}(\mathbf{s}_2)\frac{\cos(\boldsymbol{\beta}_3 h)\sin(\boldsymbol{\beta}_2 h)}{\cos(\boldsymbol{\beta}_2 h)}e^{|\mathbf{s}_3|h}\right| + \frac{4}{h^2} \\
\leq{} & \frac{4}{h^2} + \frac{1}{h^2}\left|e^{|\mathbf{s}_1|h}\left[\text{sign}(\mathbf{s}_1)\sin(\boldsymbol{\beta}_1 h) - \frac{\text{sign}(\mathbf{s}_2)\sin(\boldsymbol{\beta}_2 h)\cos(\boldsymbol{\beta}_1 h)}{\cos(\boldsymbol{\beta}_2 h)}\right]\right. \\
& \quad \left. -e^{|\mathbf{s}_3|h}\left[\text{sign}(\mathbf{s}_3)\sin(\boldsymbol{\beta}_3 h) - \frac{\text{sign}(\mathbf{s}_2)\sin(\boldsymbol{\beta}_2 h)\cos(\boldsymbol{\beta}_3 h)}{\cos(\boldsymbol{\beta}_2 h)}\right]\right|.
\end{aligned}
$$

Now letting

$$
\begin{aligned}
\rho_1 &= \text{sign}(\mathbf{s}_1)\sin(\boldsymbol{\beta}_1 h) - \text{sign}(\mathbf{s}_2)\frac{\sin(\boldsymbol{\beta}_2 h)\cos(\boldsymbol{\beta}_1 h)}{\cos(\boldsymbol{\beta}_2 h)} \\
\rho_2 &= -\left(\text{sign}(\mathbf{s}_3)\sin(\boldsymbol{\beta}_3 h) - \text{sign}(\mathbf{s}_2)\frac{\sin(\boldsymbol{\beta}_2 h)\cos(\boldsymbol{\beta}_3 h)}{\cos(\boldsymbol{\beta}_2 h)}\right),
\end{aligned}
$$

the above inequality becomes

$$
|\mathcal{B}| \leq \frac{1}{h^2}\left|e^{|\mathbf{s}_1|h}\rho_1 + e^{|\mathbf{s}_3|h}\rho_2\right| + \frac{4}{h^2}. \tag{47}
$$

Setting $\text{sign}(\mathbf{s}_2) = -\text{sign}(\sin(\boldsymbol{\beta}_2 h))$ as given by (36), these variables become

$$\rho_1 = \left(\text{sign}(\mathbf{s}_1)\sin(\boldsymbol{\beta}_1 h) + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{\cos(\boldsymbol{\beta}_2 h)}\right) \tag{48}$$

$$\rho_2 = -\left(\text{sign}(\mathbf{s}_3)\sin(\boldsymbol{\beta}_3 h) + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)}{\cos(\boldsymbol{\beta}_2 h)}\right). \tag{49}$$

Then using the assignments (35) and (37):

$$\rho_1 = |\sin(\boldsymbol{\beta}_1 h)| + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{\cos(\boldsymbol{\beta}_2 h)} \tag{50}$$

$$\rho_2 = -\left(|\sin(\boldsymbol{\beta}_3 h)| + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)}{\cos(\boldsymbol{\beta}_2 h)}\right) \tag{51}$$

Now by equations (50) and (51), the equality (38) is equivalent to

$$\begin{aligned} e^{|\mathbf{s}_3|h} &= \frac{|\sin(\boldsymbol{\beta}_1 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{|\sin(\boldsymbol{\beta}_3 h)|\cos(\boldsymbol{\beta}_2 h) + |\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)} e^{|\mathbf{s}_1|h} \\ &= \frac{|\sin(\boldsymbol{\beta}_1 h)| + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_1 h)}{\cos(\boldsymbol{\beta}_2 h)}}{|\sin(\boldsymbol{\beta}_3 h)| + \frac{|\sin(\boldsymbol{\beta}_2 h)|\cos(\boldsymbol{\beta}_3 h)}{\cos(\boldsymbol{\beta}_2 h)}} e^{|\mathbf{s}_1|h} \\ &= \frac{\rho_1}{-\rho_2} e^{|\mathbf{s}_1|h} \\ &= -\frac{\rho_1}{\rho_2} e^{|\mathbf{s}_1|h}. \end{aligned}$$

Then $\mathbf{s}_1$ and $\mathbf{s}_3$ solve $\rho_2 e^{|\mathbf{s}_3|h} = -\rho_1 e^{|\mathbf{s}_1|h}$, and so the bounds on equation (47) become:

$$\begin{aligned} |\mathcal{B}| &\leq \frac{4}{h^2} + \frac{1}{h^2}\left|e^{|\mathbf{s}_1|h}\rho_1 + e^{|\mathbf{s}_3|h}\rho_2\right| \\ &= \frac{4}{h^2}. \end{aligned}$$

This bounds the imaginary part of (42) for some $|\mathbf{s}| \geq S$ by a constant independent of S. Therefore there is a $C = C(\boldsymbol{\beta}, h)$ such that for any $S > s_0(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, \mathcal{B}, h)$, there are vectors $\mathbf{s} \in \mathbb{R}^n$ and $\boldsymbol{\alpha} \in \mathbb{R}^n$ so that $|\mathbf{s}| \geq S$ and $||e^{-\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{\boldsymbol{\eta}\cdot x}\right)||_{L^2} \leq C(\boldsymbol{\beta}, h)$. This was done by finding choices of s and $\boldsymbol{\alpha}$ so that (44) and (45) are satisfied uniformly in $|\mathrm{s}|$. Now it must be shown that (33) holds for the same vector $\boldsymbol{\eta}$ and the choice of a negative sign: namely, that

$$||e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right)||_{L^2(\mathcal{B})} \leq C$$

for C a constant independent of $S > s_0(||q_1||_{L^\infty(\mathcal{B})}, ||q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, \mathcal{B}, h)$.

The same computations used to produce equation (41) lead to the equality

$$\begin{aligned}
|e^{\mathbf{s}\cdot x}\ \Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right)| &= \left|e^{\mathbf{s}\cdot x}\Delta\left(e^{i\boldsymbol{\beta}\cdot x}e^{-\boldsymbol{\eta}\cdot x}\right)\right| \\
&= \left|\frac{1}{h^2}e^{\mathbf{s}\cdot x}\cdot e^{-\mathbf{s}\cdot x}e^{i(\boldsymbol{\beta}-\boldsymbol{\alpha})\cdot x}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}e^{i(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k)h} - 2 + e^{\mathbf{s}_k h}e^{-i(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k)h}\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}\cos\left(\left(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k\right)h\right) - 2 + e^{\mathbf{s}_k h}\cos\left(\left(-\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}\right. \\
&\qquad \left. + i\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}\sin\left(\left(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k\right)h\right) + e^{\mathbf{s}_k h}\sin\left(\left(-\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}\right| \\
&=: |\mathcal{A}' + i\mathcal{B}'|.
\end{aligned}$$

Using the same vectors s and $\boldsymbol{\alpha}$ as constructed for $+\boldsymbol{\eta}$, we have that

$$\begin{aligned}
|\mathcal{A}'| &= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}\cos\left(\left(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k\right)h\right) - 2 + e^{\mathbf{s}_k h}\cos\left(\left(-\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\cos\left(\boldsymbol{\beta}_k h\right) - 2 + e^{\mathbf{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\cos\left(-\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathbf{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\cos\left(\boldsymbol{\beta}_k h\right) - 2 + e^{\mathbf{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\cos\left(\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= |\mathcal{A}| \leq \frac{2}{h^2} + \frac{4n-2}{h^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\mathcal{B}'| &= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathrm{s}_k h}\sin\left(\left(\boldsymbol{\beta}_k-\boldsymbol{\alpha}_k\right)h\right)+e^{\mathrm{s}_k h}\sin\left(\left(-\boldsymbol{\beta}_k+\boldsymbol{\alpha}_k\right)h\right)\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)+e^{\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(-\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{-\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)-e^{\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= \left|-\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)-e^{-\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= \left|\frac{1}{h^2}\sum_{k=1}^{n}\left\{e^{\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)-e^{-\mathrm{s}_k h}\mathcal{S}(\boldsymbol{\alpha}_k)\sin\left(\boldsymbol{\beta}_k h\right)\right\}\right| \\
&= |\mathcal{B}| \leq \frac{4}{h^2}
\end{aligned}$$

Therefore u_- as defined in the statement of the theorem exists and its remainder term r_- satisfies the needed decay properties. $\square$

It is now possible to prove Theorem 5.

Proof of Theorem 5. Fix $\boldsymbol{\beta} \in \left(-\frac{\pi}{2h}, \frac{\pi}{2h}\right)^n$ so that $\boldsymbol{\beta}_j \notin \{0, -\pi/2h, \pi/2h\}$ for every $j \in \{1, 2, ..., n\}$. Let $s_0 = s_0(||q_1||_{L^\infty(\mathcal{B})}, ||, q_2||_{L^\infty(\mathcal{B})}, \boldsymbol{\beta}, \mathcal{B}, h)$ be the constant whose existence is guaranteed by Theorem 10. Then for any real number $S > s_0$, define $u_\pm(x) = e^{i\boldsymbol{\beta}\cdot x}e^{\pm i\boldsymbol{\eta}\cdot x}\left(1 + r_\pm(x)\right)$ to be the solutions to

$$(\Delta + q_1)u_+ = 0$$

$$(\Delta + q_2)u_- = 0$$

on $\mathcal{B}$ as constructed in Theorem 10. As before, $\boldsymbol{\eta} = \mathrm{s} + i\boldsymbol{\alpha}$, and $|\mathrm{s}| \geq S$. By the assumption that

$\Lambda_1 = \Lambda_2$, Lemma 7 implies that

$$0 = \int_{\partial \mathcal{B}} u_- \left(\Lambda_1 - \Lambda_2\right) u_+ = \int_{\mathcal{B}} u_+ u_- (q_1 - q_2).$$

Then

$$\begin{aligned} 0 &= \int_{\partial \mathcal{B}} u_- (\Lambda_1 - \Lambda_2) u_+ = \int_{\mathcal{B}} (q_1 - q_2) u_+ u_- \\ &= \int_{\mathcal{B}} (q_1 - q_2) e^{2i\boldsymbol{\beta} \cdot x} (1 + r_+(x))(1 + r_-(x)). \end{aligned}$$

Taking $S \to \infty$ turns the above equation into

$$0 = \int_{\mathcal{B}} (q_1 - q_2) e^{2i\boldsymbol{\beta} \cdot x}. \tag{52}$$

By continuity of the discrete time Fourier transform in $\boldsymbol{\beta}$, this equation holds for all $\boldsymbol{\beta} \in [-\pi/2h, \pi/2h)^n$. The injectivity of the discrete-time Fourier transform (Theorem 4) then implies that $q_1 = q_2$. □

3 Appendix

This subsection is to add more detail to the assertion in Theorem 9 that the functional J has a minimizer.

Proposition 9. *If $J : \mathbb{R}^N \to \mathbb{R}$ is a functional satisfying*

$$J(v) \geq b||v||^2 - p||v||$$

for some $b > 0$, $p \geq 0$, then there exists a finite $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ (not necessarily unique) such that

$$J(\mathbf{v}) = \min_{\mathbf{w} \in \mathbb{R}^n} J(\mathbf{w}) = c.$$

Proof. For a fixed $b > 0$ and $p \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be the quadratic equation $f(x) = bx^2 - px$. The function f has an extremum at the point $x = p/2b$, and because $b > 0$, this extremum is a global minimum.

Then fix any $\mathbf{v}_0 \in \mathbb{R}^n$. Let $C > 0$ be large enough so that for any x that satisfies $|x| > C$, we have that $f(x) > 2|J(\mathbf{v}_0)|$. If N is the neighbourhood of $\mathbf{0} \in \mathbb{R}^n$ defined by

$$N := \{\mathbf{v} \in \mathbb{R}^n : ||\mathbf{v}|| \leq C\},$$

then $J(\mathbf{v}) = f(||\mathbf{v}||) > 2J(v_0)$ whenever $\mathbf{v} \notin N$.

Now f is continuous on the compact interval $[0, C]$. Therefore there exists an $m \in [0, C]$ such that

$$f(m) = \min_{x \in [0,C]} f(x).$$

Let $\mathbf{v}$ be any vector in N such that $||\mathbf{v}|| = m$. Then for any other $\mathbf{w} \in N$, we get that

$$J(\mathbf{w}) = f\left(||\mathbf{w}||\right) \geq \min_{x \in [0,C]} f(x) = f(||\mathbf{v}||) = J(\mathbf{v}).$$

Therefore

$$J(\mathbf{v}) = \min_{\mathbf{w} \in N} J(\mathbf{w}).$$

Furthermore, since $\mathbf{v}_0$ must satisfy $||\mathbf{v}_0|| \leq C$ (because $J(\mathbf{v}_0) \leq 2|J(\mathbf{v}_0)|$), we have that for any $\mathbf{w} \in N^c$:

$$J(\mathbf{v}) = \min_{x \in [0,C]} f(x) \leq J(\mathbf{v}_0) \leq 2|J(\mathbf{v}_0)| \leq J(\mathbf{w}).$$

Therefore $\mathbf{v}$ is a minimizer of J. $\square$

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