

THE UNIVERSITY OF CHICAGO

ON MULTIPLE-PATHS SCHRAMM-LOEWNER EVOLUTION

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To my father

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## ABSTRACT

In this thesis, we study the properties of multiple-paths Schramm-Loewner Evolution ( $SLE_\kappa$ ). One of the main objectives is to study this process in multiply-connected domains, which requires discussing single path  $SLE_\kappa$  in such domains first.

While for some applications it is appropriate to consider  $SLE_\kappa$  as a probability measure, there are several cases where it is more natural to consider it as a non-probability measure. In this work, we take the second approach to study multiple-paths  $SLE_\kappa$ . The Brownian loop measure is one of the main ingredients of this work and we use it to give the Radon-Nikodym derivatives between several versions of  $SLE_\kappa$ .

First, we discuss multiple-paths  $SLE_\kappa$  in simply connected domains. In particular, we give a definition using the Brownian loop measure and show that the partition function is smooth. These results are based on a joint work with Greg Lawler.

Next, we recall  $SLE_\kappa$  in multiply-connected domains defined in a work of Lawler. As before, the Brownian loop measure is used to define  $SLE_\kappa$  by describing particular Radon-Nikodym derivatives. In addition, we give an argument comparing  $SLE_\kappa$  in annuli and radial  $SLE_\kappa$ .

Then, we define multiple-paths  $SLE_\kappa$  in multiply-connected domains and prove that its partition function is smooth using the Hörmander's theorem. While the definition is similar to multiple-paths  $SLE_\kappa$  in simply-connected domains, the proof of smoothness of the partition functions in simply-connected domains cannot be easily extended to multiply-connected domains. This is because in multiply-connected domains, the explicit form of the partition function for two  $SLE_\kappa$  paths is unknown.

Finally, we use two independent radial  $SLE_\kappa$  curves to give a construction of two-sided  $SLE_\kappa$  measure growing simultaneously from the marked points. We show that this measure is comparable to the distribution of two  $SLE_\kappa$  paths in an annulus as the inner circle shrinks.

# CHAPTER 1

## INTRODUCTION

The Schramm-Loewner evolution ( $SLE_\kappa$ ) is a one parameter family of probability measures on planar curves discovered by Oded Schramm [24]. It was proposed as a candidate for the scaling limits of loop-erased random walk and uniform spanning tree. In particular, Schramm proved that  $SLE_\kappa$  measures are the only measures on two-dimensional continuous curves that are conformally invariant and satisfy the domain Markov property.

In this dissertation, we discuss two main variants of  $SLE_\kappa$ , namely chordal  $SLE_\kappa$  and radial  $SLE_\kappa$ . Chordal  $SLE_\kappa$  is a family of measures supported on curves connecting two distinct boundary points of a domain, whereas radial  $SLE_\kappa$  measures are supported on curves connecting a boundary point to an interior point. Both chordal and radial  $SLE_\kappa$  are characterized by the Loewner equations with Brownian motion as the driving functions. We will mostly restrict our study to  $\kappa \leq 4$ , for which  $SLE_\kappa$  measures are supported on simple curves.

The Brownian loop measure was proved to be useful in understanding the properties of  $SLE_\kappa$  [13, 12, 14, 15]. An example is the *boundary perturbation property*, which indicates that if  $D' \subset D$  are simply connected domains that agree in neighborhoods of analytic boundary points  $z, w$ , then  $SLE_\kappa$  from  $z$  to  $w$  in  $D'$  is absolutely continuous with respect to  $SLE_\kappa$  from  $z$  to  $w$  in  $D$  and the Radon-Nikodym derivative can be expressed in terms of the Brownian loop measure of certain loops in  $D$ . One of the properties that make the Brownian loop measure essentially useful is the conformal invariance. Combined with the conformal invariance of  $SLE_\kappa$ , it is used in the proof of many results about  $SLE_\kappa$ .

Multiple-paths  $SLE_\kappa$  can be considered as the scaling limit of multiple-paths discrete models at criticality (as an example, see [20] for multiple-paths loop-erased random walk). It is a measure on multiple continuous curves connecting marked points in a domain. It has properties similar to those of regular  $SLE_\kappa$  such as the domain Markov property, conformal invariance, reversibility, boundary perturbation property, etc. Using the Brownian loop measure is the most natural way to define multiple-paths  $SLE_\kappa$ , as for loop-erased random walk the corresponding measure is defined

using the random walk loop measure. Using this approach, it is more convenient to consider  $SLE_\kappa$  as a non-probability measure and this is a reasonable choice because  $SLE_\kappa$  is known to be the scaling limit of various two-dimensional discrete models at criticality, which are considered as measures with partition functions. The total mass of chordal  $SLE_\kappa$  in simply connected domains is given by the Poisson kernel. For the measure on two curves, the total mass can be expressed using the hypergeometric function. For more than two curves, there is no explicit formula for the partition function. Instead, we show that it is at least  $C^2$  and describe it using a particular PDE.

$SLE_\kappa$  in multiply-connected domains can be defined in several ways. In contrast to simply connected domains, domain Markov property and conformal invariance do not uniquely determine the  $SLE_\kappa$  measure in multiply-connected domains. Therefore, additional assumptions need to be made. The approach of Dapeng Zhan in [28] was to require the measure to be reversible. He established a version of Loewner equation in multiply-connected domains and defined  $SLE_\kappa$  using a family of appropriate driving functions. On the other hand, in [17] Lawler gave a definition using the Brownian loop measure, requiring the measure to satisfy the boundary perturbation property. More precisely, for any simply-connected subdomain  $D$ , the Radon-Nikodym derivative between the restriction of the measure to  $D$  and the  $SLE_\kappa$  measure in  $D$  is given in terms of the Brownian loop measure of certain loops. This is similar to the boundary perturbation property of  $SLE_\kappa$  in simply-connected domains. It is the simplest to study chordal  $SLE_\kappa$  in annulus. Restricted to a fixed winding number,  $SLE_\kappa$  in an annulus can be compared to chordal  $SLE_\kappa$  in an appropriate horizontal strip in the upper-half plane. The relation can be described by the Radon-Nikodym derivative, which can be expressed in terms of the Brownian loop measure. As it turned out, the partition function of  $SLE_\kappa$  in multiply-connected domains is not given by a power of the Poisson kernel. But at least in annuli, one can show that the partition function is smooth and give a characterization using a partial differential equation. In addition, Lawler [17] found an asymptotic form for the partition function in annulus as the inner radius goes to 0. From this and properties of the Brownian loop measure, it follows that radial  $SLE_\kappa$  and annulus  $SLE_\kappa$  are absolutely continuous before reaching the boundary.

Similar to simply-connected domains, multiple-paths  $SLE_\kappa$  in multiply-connected domains is an interesting process to study. The definition is similar to the process in simply-connected domains. Characterization of the partition function is one of the interesting questions about this process. In this work, we show that the partition function is smooth and satisfies a specific partial differential equation. As mentioned before, annulus  $SLE_\kappa$  and radial  $SLE_\kappa$  are absolutely continuous before hitting the boundary. We will show that the distribution of two  $SLE_\kappa$  paths in annulus is comparable to the distribution of two-sided  $SLE_\kappa$  growing simultaneously from the endpoints. For a simply connected domain  $0 \in D$  with boundary points  $z, w$ , two-sided  $SLE_\kappa$  from  $z$  to  $w$  is usually considered as chordal  $SLE_\kappa$  from  $z$  to  $w$  conditioned to go through  $0$  (although this is an event with zero probability, there are ways to make this precise). This involves weighting the chordal  $SLE_\kappa$  by the Green's function until the curve hits the origin (the fact that two-sided  $SLE_\kappa$  is continuous at the origin is proved in [18]). The rest of the curve has the distribution of chordal  $SLE_\kappa$  from  $0$  to  $w$  in the slit domain. We give a construction that allows us to grow the curves from  $z$  and  $w$  at the same time.

## 1.1 Structure of the Dissertation

In chapter 2, we establish our notation and give a review of  $SLE_\kappa$  in simply connected domains. In addition, we discuss certain deterministic estimates using techniques from complex analysis. The main tools are Koebe-1/4 theorem and the distortion estimates. The Brownian loop measure is a major part of this study and we review its definition and the basics properties in this chapter.

In Chapter 3, we discuss multiple-paths  $SLE_\kappa$  in simply connected domains. In particular, we give a definition using the Brownian loop measure and show that the partition function is smooth. This chapter is based on a joint work with Lawler [9].

$SLE_\kappa$  in multiply-connected domains is discussed in Chapter 4. In this chapter, Brownian loop measure is used to define  $SLE_\kappa$  by describing particular Radon-Nikodym derivatives. In addition, we give an argument comparing  $SLE_\kappa$  in annuli and radial  $SLE_\kappa$ . Finally, we discuss Annulus Loewner equation, which provides a differential equation describing slit mapping in annuli. The

techniques used here are similar to the techniques used in the studies of the Loewner equations in simply-connected domains.

In Chapter 5, we define multiple-paths  $SLE_\kappa$  in annuli and prove that their partition functions are smooth. Then, we use this definition to show that two-sided  $SLE_\kappa$  measure can be estimated by the distribution of two  $SLE_\kappa$  paths in an annulus. Along the way, we prove the boundary perturbation property for radial  $SLE_\kappa$  in the unit disk.

## CHAPTER 2

### PRELIMINARIES

Properties of Brownian motion and estimates about one-to-one conformal transformations are crucial in the study of  $SLE_\kappa$ . After establishing our notation, we recall the Koebe-1/4 theorem and the distortion estimates in this chapter and prove a few deterministic results. These estimates will be used in the next chapters to prove our main theorems. In addition, we review the Brownian loop measure and the properties of  $SLE_\kappa$  in simply-connected domains.

#### 2.1 Notation

In this thesis, unless mentioned otherwise, we will use the following notation. We denote the unit disk by  $\mathbb{D}$  and the upper-half plane by  $\mathbb{H}$ . For  $r > 0$ , define

$$C_r = \{z \in \mathbb{C}; |z| = e^{-r}\}, \quad \mathbb{D}_r = e^{-r} \mathbb{D},$$

$$A_r = \{z \in \mathbb{C}; e^{-r} < |z| < 1\}, \quad S_r = \{z \in \mathbb{H}; \text{Im}(z) < r\}.$$

Let  $D' \subset D$  be domains in  $\mathbb{C}$ . Assume  $z \in D$  and  $w \in \partial D \cap \partial D'$  is an analytic boundary point of  $D, D'$  with  $\text{dist}(w, D \setminus D') > 0$ . That is, there exists an analytic function  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $f(0) = z$  and  $f(\mathbb{D}) \cap D = f(\mathbb{H} \cap \mathbb{D})$ . We denote by  $H_D(z, w)$  the Poisson kernel in  $D$  normalized so that  $H_{\mathbb{D}}(0, 1) = 1/2$ . Define

$$Q_D(z, w; D') = \frac{H_{D'}(z, w)}{H_D(z, w)}.$$

In other words,  $Q_D(z, w; D')$  is the probability that Brownian motion started from  $z$  and conditioned to exit  $D$  at  $w$  does not hit  $D \setminus D'$ . If  $w' \in \partial D \cap \partial D'$  is another analytic boundary point with  $\text{dist}(w', D \setminus D') > 0$ , then

$$H_D(w', w) := \partial_{\mathbf{n}_{w'}} H_D(w', w),$$

where  $\partial_{\mathbf{n}_{w'}}$  denotes the inward normal derivative at  $w'$ . Similarly, we will write

$$Q_D(w', w; D') = \frac{H_{D'}(w', w)}{H_D(w', w)},$$

which is the probability that a Brownian excursion from  $w'$  to  $w$  in  $D$  does not intersect  $D \setminus D'$ .

Suppose  $f : D \rightarrow f(D)$  is a conformal transformation such that  $f(w), f(w')$  are analytic boundary points of  $f(D)$ . Then the Poisson kernel satisfies the conformal covariance property

$$\begin{aligned} H_D(z, w) &= |f'(w)| H_{f(D)}(f(z), f(w)), \\ H_D(w', w) &= |f'(w')| |f'(w)| H_{f(D)}(f(w'), f(w)). \end{aligned}$$

Therefore,  $Q_D$  is conformally invariant and

$$\begin{aligned} Q_D(z, w; D') &= Q_{f(D)}(f(z), f(w); f(D')), \\ Q_D(w', w; D') &= Q_{f(D)}(f(w'), f(w); f(D')). \end{aligned}$$

Suppose  $V_1, V_2 \subset \partial D$  are segments of the boundary and assume  $V_1$  is smooth. For any  $z \in V_1$ , let  $\rho_D(z; V_2)$  denote the probability that a Brownian excursion starting from  $z$  exits  $D$  at  $V_2$ . In other words

$$\rho_D(z; V_2) = \partial_{\mathbf{n}_z} \mathbb{P}^z[B_{\tau_D} \in V_2],$$

where  $\tau_D$  is the first time the Brownian motion  $B_t$  exits  $D$  and  $\mathbf{n}_z$  denotes the inward normal derivative at  $z$ . Let  $f : D \rightarrow \tilde{D}$  be a conformal transformation such that  $f(z)$  is an analytic boundary point. Using conformal invariance of Brownian motion, we have

$$\rho_D(z; V_2) = |f'(z)| \rho_{\tilde{D}}(f(z); f(V_2)). \quad (2.1)$$

The *excursion measure* between  $V_1, V_2$  is defined by

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \rho_D(z; V_2) |dz|.$$

(Excursion measure is normally defined as a measure induced by Brownian motion on curves connecting different points on the boundary. What we call the excursion measure is the total mass of the aforementioned measure). If  $V_1, V_2$  are both smooth, then we can write

$$\mathcal{E}_D(V_1, V_2) = \frac{1}{\pi} \int_{V_1} \int_{V_2} H_D(w_1, w_2) |dw_2| |dw_1|.$$

Using conformal covariance of the Poisson kernel, we can see that if  $f : D \rightarrow \tilde{D}$  is a conformal transformation such that  $f(V_1), f(V_2)$  are smooth, then

$$\mathcal{E}_D(V_1, V_2) = \mathcal{E}_{f(D)}(f(V_1), f(V_2)).$$

This equality can be used to define  $\mathcal{E}_D(V_1, V_2)$  even if  $V_1, V_2$  are rough.

Suppose  $\gamma : (0, t_\gamma) \rightarrow A_r$  is a simple curve with  $\gamma(0+) = \bar{u} \in C_0$ ,  $\gamma(t_\gamma-) = \bar{w} \in C_r$ . We will write  $\gamma_t$  for  $\gamma((0, t])$ . For any  $t < t_\gamma$ , there is a unique  $0 < r(t) \leq r$  and a conformal transformation  $\tilde{h}_t$  such that

$$\tilde{h}_t : A_r \setminus \gamma_t \rightarrow A_{r(t)}, \quad \tilde{h}_t(C_r) = C_{r(t)}. \quad (2.2)$$

See [16] for a proof. Note that

$$\mathcal{E}_{A_r}(C_0, C_r) = \int_{C_0} \rho_{A_r}(z, C_r) |dz| = \int_{C_0} \frac{1}{r} |dz| = \frac{2\pi}{r}.$$

Therefore, by conformal invariance of the excursion measure,  $r(t) = 2\pi / \mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r)$ . It is easy to see that the function  $\tilde{h}_t$  is unique up to a rotation. We will uniquely specify our choice of rotation in section 4.2.

Define

$$\psi(z) = e^{iz}$$

and choose  $0 \leq u, w < 2\pi$  such that  $\psi(u) = \bar{u}$ ,  $\psi(w + ir) = \bar{w}$ . Let  $\eta_t \subset S_r$  be the continuous curve satisfying  $\gamma_t = \psi \circ \eta_t$ ,  $\eta(0) = u$  and let

$$\tilde{\eta}_t = \bigcup_{k \in \mathbb{Z}} [\eta_t + 2k\pi]. \quad (2.3)$$

Define

$$S_{r,t} = S_r \setminus \tilde{\eta}_t$$

and note that the transformation  $\tilde{h}_t$  can be raised to the covering space  $S_{r,t}$  to yield a conformal transformation

$$h_t : S_{r,t} \rightarrow S_{r(t)}, \quad \psi \circ h_t = \tilde{h}_t \circ \psi. \quad (2.4)$$

Similarly to  $\tilde{h}_t$ , the function  $h_t$  is unique up to a translation. However, regardless of the choice of translations,  $h_t(z) - z$  is  $2\pi$ -periodic.

Let  $\bar{g}_t : \mathbb{D} \setminus \gamma_t \rightarrow \mathbb{D}$  be the unique conformal transformation with  $\bar{g}_t(0) = 0$ ,  $\bar{g}'_t(0) > 0$  and let

$$\bar{\xi}_t = \bar{g}_t(\gamma(t)).$$

Let  $\bar{U}_t = \tilde{h}_t(\gamma(t))$  and define  $\bar{\phi}_t$  to be the unique conformal transformation satisfying

$$h_t = \bar{\phi}_t \circ \bar{g}_t, \quad \bar{\phi}_t(\bar{\xi}_t) = \bar{U}_t. \quad (2.5)$$

Similarly to  $\tilde{h}_t$ , we can raise  $\bar{g}_t$  to the covering space  $\mathbb{H} \setminus \tilde{\eta}_t$  to get a conformal transformation

$$\tilde{g}_t : \mathbb{H} \setminus \tilde{\eta}_t \rightarrow \mathbb{H},$$

such that  $\xi_t = \tilde{g}_t(\eta(t))$  is continuous and  $\psi(\xi_t) = \tilde{\xi}_t$ . Define the conformal transformation  $\phi_t$  with

$$h_t = \phi_t \circ \tilde{g}_t.$$

We let  $g_t : \mathbb{H} \setminus \eta_t \rightarrow \mathbb{H}$  be the unique conformal transformation with  $g_t(z) = z + o(1)$  as  $z \rightarrow \infty$ .

For a simply connected domain  $D$ , chordal  $SLE_\kappa$  from  $w$  to  $w'$  is considered as a measure with partition function  $\Psi_D(w, w') = \|\mu_D(w, w')\| = H_D(w', w)^b$  satisfying

$$\Psi_D(w, w') = |f'(w)|^b |f'(w')|^b \Psi_{f(D)}(f(w), f(w')). \quad (2.6)$$

Here,

$$a = \frac{2}{\kappa}, \quad b = \frac{3a - 1}{2} = \frac{6 - \kappa}{2\kappa}.$$

While (2.6) holds even if  $D$  is not simply connected, it is no longer true that  $\Psi_D(w, w') = H_D(w', w)^b$  for multiply connected domains.

For a function of the form  $f_t(z)$  we use the dot derivative  $\dot{f}_t(z)$  to denote the derivative with respect to  $t$ , while  $f'_t(z)$  denotes the derivative with respect to  $z$ .

## 2.2 Brownian Loop Measure

The Brownian loop measure was first defined and studied in [13]. We briefly review the results of [13] in this section.

For  $z \in \mathbb{C}$ , let  $\nu(z, \cdot, t)$  be the distribution of complex Brownian motion  $B_s$ ,  $0 \leq s \leq t$  with  $B_0 = z$ . Let  $\nu(z, w, t)$  be the measure on curves induced by a Brownian motion conditioned to be at  $w$  at time  $t$ . To be more precise,  $\nu(z, w, t)$  is the distribution of Brownian motion tilted by  $p_{t-s}(B_s, w)$  using the Girsanov's theorem. Let

$$\nu(z, w) = \int_0^\infty \nu(z, w, t) dt,$$

which is an infinite (but sigma-finite) measure. For a domain  $D$ , we define  $\nu_D(z, w)$  to be the restriction of  $\nu(z, w)$  to the paths that stay entirely in  $D$ . If  $D$  is a regular domain and  $z \neq w$ , then  $\nu_D(z, w)$  is a finite measure with total mass  $G_D(z, w)$ , where  $G_D$  denotes the Green's function in  $D$ .

If  $z \in D$  and  $w \in \partial D$  is an analytic boundary point, then define  $\nu_D(z, w)$  to be the distribution of Brownian motion tilted by  $H_D(B_t, w)$ . In other words,  $\nu_D(z, w)$  is the distribution of  $h$ -process in  $D$  conditioned to exit the domain at  $w$ . If  $z, w \in \partial$  are analytic boundary points, then

$$\nu_D(z, w) = \partial_{\mathbf{n}_z} \nu_D(z, w),$$

where  $\partial_{\mathbf{n}_z}$  denotes the inward normal derivative at  $z$ . The measure  $\nu_D(z, w)$  has total mass  $H_D(z, w)$ .

Let  $z, w \in D$  and assume  $u, v \in \partial D$  are analytic boundary points. Let  $f : D \rightarrow D'$  be a conformal transformation such that  $f(u), f(v)$  are analytic boundary points of  $D'$ . Using the conformal invariance of Brownian motion,

$$f \circ \nu_D(z, w) = \nu_{D'}(f(z), f(w)),$$

where  $f \circ \nu_D(z, w)$  denotes the pushforward measure. Using this, we get

$$f \circ \nu_D(z, u) = |f'(u)| \nu_{D'}(f(z), f(u)), \quad f \circ \nu_D(u, v) = |f'(u)| |f'(v)| \nu_{D'}(f(u), f(v)).$$

For  $z \in \partial D$ , the Brownian bubble measure is defined to be  $\pi \nu_D(z, z)$ . Note that this is an infinite measure. Suppose  $D \subset D'$  and  $z \in \partial D$  such that  $D, D'$  agree in a neighborhood of  $z$ . Then define

$$\Gamma_{D'}(z, D' \setminus D) := \pi[\nu_{D'}(z, z) - \nu_D(z, z)].$$

That is,  $\Gamma_{D'}(z, D' \setminus D)$  is the bubble measure of loops in  $D'$  that do not stay in  $D$ . Let  $f : D' \rightarrow U'$  be a conformal transformation and let  $U = f(D)$ . Using the conformal covariance properties of

$v_D$ , we have

$$\Gamma_{D'}(z, D' \setminus D) = |f'(z)|^2 \Gamma_{U'}(f(z), U' \setminus U).$$

Moreover, the strong Markov property implies that

$$\Gamma_{D'}(z, D' \setminus D) = \int_{D' \cap \partial D} H_{D'}(w, z) \mathcal{E}_D(z, dw).$$

Using this, it is not hard to see that if  $f(z) = -1/z$ ,  $D' = \mathbb{H}$  and  $D$  is a simply connected domain such that  $K := \mathbb{H} \setminus D$  is closed and  $\text{dist}(0, K) > 0$ , then

$$\Gamma_{\mathbb{H}}(0, K) = \text{hcap}[f(K)].$$

From this equality, it follows that if  $K = \mathbb{H} \setminus D$  and  $\Phi : D \rightarrow \mathbb{H}$  is a conformal transformation with  $\Phi(0) = 0$ , then

$$\Gamma_{\mathbb{H}}(0, K) = \frac{-1}{6} S\Phi(0), \tag{2.7}$$

where

$$S\Phi(z) = \frac{\Phi'''(z)}{\Phi'(z)} - \frac{3\Phi''(z)^2}{2\Phi'(z)^2}$$

is the Schwarzian derivative.

We say two curves are equivalent  $\gamma^1 \sim \gamma^2$  if  $t_{\gamma^1} = t_{\gamma^2}$  and there exists  $\theta > 0$  such that  $\gamma^1(t + \theta) = \gamma^2(t)$  for all  $t \in \mathbb{R}$  (while curves are considered as  $t_{\gamma^1}$ -periodic functions). The equivalence classes of loops under this equivalence relation are called unrooted loops. There are three equivalent ways of defining the Brownian loop measure on unrooted loops.

- A loop  $\gamma \subset \mathbb{C}$  can be described by a triple  $(z, t, \bar{\gamma})$ , where  $z$  is the root  $z = \gamma(0)$ ,  $t$  is the time duration of the loop and  $\bar{\gamma}(s) := (\gamma(ts) - z)/\sqrt{t}$  is a loop of time duration 1. The Brownian loop measure  $\mu_{\mathbb{C}}$  is the measure induced on unrooted loops by

$$(\text{Lebesgue measure}) \times \frac{1}{2\pi t^2} dt \times (\text{Brownian bridge distribution})$$

on triples  $(z, t, \bar{\gamma})$ . For a domain  $D$ ,  $\mu_D$  is restriction of  $\mu_{\mathbb{C}}$  to the loops in  $D$ .

- Consider the measure  $\nu_D = \int_D \nu_D(z, z) dA(z)$ , which is a sigma-finite (but infinite) measure.

Define the measure  $\bar{\mu}_D$  with

$$\frac{\bar{\mu}_D(\gamma)}{\nu_D} = \frac{1}{t_\gamma},$$

for any loop  $\gamma$  with time duration  $t_\gamma$ . Then the Brownian loop measure  $\mu_D$  is the measure induced on unrooted loops by  $\bar{\mu}_D$ .

- As a measure on unrooted loops, define

$$\mu_{\mathbb{C}} = \int_{\mathbb{C}} \nu_{\mathbb{H}+z}(z, z) dA(z).$$

The Brownian loop measure  $\mu_D$  is  $\mu_{\mathbb{C}}$  restricted to the loops that stay in  $D$ .

## 2.3 Conformal Transformations

We first state the Koebe-1/4 theorem and the distortion estimates (see [16] for proofs).

**Proposition 2.1.** *Suppose  $f : \mathbb{D} \rightarrow D$  is a conformal transformation. Then*

$$\text{dist}(f(0), \partial D) \leq |f'(0)| \leq 4 \text{dist}(f(0), \partial D).$$

**Proposition 2.2.** *Let  $f : \mathbb{D} \rightarrow D$  be a conformal transformation with  $f(0) = 0$ . Then*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq \frac{|f'(z)|}{|f'(0)|} \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

*In particular, as  $|z| \rightarrow 0$ ,*

$$|f'(z)| = |f'(0)| [1 + \mathcal{O}(|z|)], \quad |f(z)| = |z| |f'(0)| [1 + \mathcal{O}(|z|)].$$

Suppose  $K$  is a compact subset of  $\mathbb{H}$  such that  $\mathbb{H} \setminus K$  is simply connected. The half-plane

capacity of  $K$  is defined as

$$\text{hcap}(K) = \lim_{y \rightarrow \infty} y \mathbb{E}^{iy}[\text{Im}(B_\tau)],$$

where  $\tau$  is the first time a Brownian motion  $B_t$  exits  $\mathbb{H} \setminus K$ . For a positive constant  $c > 0$ ,  $\text{hcap}[cK] = c^2 \text{hcap}[K]$  and  $\text{hcap}[K + c] = \text{hcap}[K]$ .

**Proposition 2.3.** *Suppose  $\gamma \subset \mathbb{H}$  is a simple curve with  $\gamma(0+) = 0$ . Let  $V \subset \mathbb{H}$  be a neighborhood of 0 and assume  $\Phi$  is a conformal transformation defined on  $V$  with  $\Phi(V) \subset \mathbb{H}$  and  $\Phi(\mathbb{R} \cap \bar{V}) \subset \mathbb{R}$ . Then at  $t = 0$ ,*

$$\partial_t \text{hcap}[\Phi(\gamma)] = \Phi'(0)^2 \partial_t \text{hcap}[\gamma].$$

*Proof.* Without the loss of generality, assume  $\Phi(0) = 0$ ,  $\Phi'(0) = 1$ . We prove that  $\text{hcap}[\Phi(\gamma)] = \text{hcap}[\gamma][1 + o(1)]$ . By Schwarz reflection principle, we can extend  $\Phi$  to a disk centered at 0. Let  $r_t = \text{Rad}(\gamma)$  and choose  $t$  small enough so that  $\Phi$  is defined on the disk of radius  $r_t$  centered at the origin. Let  $\gamma_t^* = \Phi(\gamma)$ . Let  $f_t(z)$  be the unique bounded harmonic function defined on  $\mathbb{H} \setminus \gamma$  with boundary values  $\text{Im}(z)$ . Similarly, let  $f_t^*(z)$  be the unique bounded harmonic function defined on  $\mathbb{H} \setminus \gamma_t^*$  with boundary values  $\text{Im}(z)$ . Let  $D_{r_t} \subset \mathbb{H}$  be the half circle of radius  $2r_t$  centered at the origin. Using the exact form of the Poisson kernel in the half disk, we can see that

$$\text{hcap}[\gamma] = \frac{2}{\pi} \int_{D_{r_t}} h_t(r_t e^{i\theta}) \sin \theta d\theta,$$

$$\text{hcap}[\gamma_t^*] = \frac{2}{\pi} \int_{D_{r_t}} h_t^*(r_t e^{i\theta}) \sin \theta d\theta.$$

If  $|z| \leq r_t$ , then distortion estimates imply that  $\Phi(z) = z + O(|z|^2)$ . Moreover, since  $\Phi(\mathbb{R} \cap \bar{V}) \subset \mathbb{R}$ ,  $\text{Im}[\Phi(z)] = \text{Im}[z](1 + O(|z|))$ . Let  $B_t$  be a standard Brownian motion and define  $\tau, \tau_*$  to be the first time  $B_t$  leaves  $\mathbb{H} \setminus \gamma, \mathbb{H} \setminus \gamma_t^*$ . It follows from conformal invariance of Brownian motion that if

$|z| = 2r_t$ , then

$$\begin{aligned}
h_t(z) &= \mathbb{E}^z[\text{Im}(B_\tau)] \\
&= \mathbb{E}^{\Phi(z)}[\text{Im}[\Phi^{-1}(B_{\tau_*})]] \\
&= \mathbb{E}^z[\text{Im}[\Phi^{-1}(B_{\tau_*})]] [1 + O(r_t)] \\
&= \mathbb{E}^z[\text{Im}[B_{\tau_*}]] [1 + O(r_t)] = h_t^*(z)[1 + O(r_t)].
\end{aligned}$$

Therefore,

$$\text{hcap}[\gamma_t^*] = \text{hcap}[\gamma_t] [1 + O(r_t)]$$

and the proof is complete. □

**Lemma 2.1.** *Let  $z' \in C_0$ ,  $z \in C_r$ ,  $\bar{z} \in A_r$ . Then*

$$\begin{aligned}
H_{A_r}(\bar{z}, z') &= \frac{r + \log |\bar{z}|}{2r} + O(|\bar{z}|), \\
H_{A_r}(\bar{z}, z) &= \frac{-e^r \log |\bar{z}|}{2r} + O(|\bar{z}|^{-1}), \\
H_{A_r}(z', z) &= e^r \left[ \frac{1}{2r} + O(e^{-r}) \right].
\end{aligned}$$

*Proof.* By symmetry, it is enough to prove the lemma for  $z' = 1$ . Note that

$$H_{A_r}(1, z) = \frac{1}{\pi} \int_{C_{r-1}} H_{A_r \cap \mathbb{D}_{r-1}}(z, v) H_{A_r}(v, 1) |dv|. \quad (2.8)$$

If  $v \in C_{r-1}$ , we can see from the strong Markov property that

$$H_{A_r}(v, 1) = H_{\mathbb{D}}(v, 1) - \frac{1}{\pi} \int_{C_r} H_{A_r}(v, w) H_{\mathbb{D}}(w, 1) |dw|.$$

Using the exact form of the Poisson kernel in  $\mathbb{D}$ , we can see that for any  $w \in \mathbb{D}$

$$H_{\mathbb{D}}(w, 1) = \frac{1}{2}[1 + \mathcal{O}(|w|)].$$

Therefore,

$$\begin{aligned} H_{A_r}(v, 1) &= \frac{1}{2}[1 + \mathcal{O}(e^{-r})] - \frac{1}{2\pi}[1 + \mathcal{O}(e^{-r})] \int_{C_r} H_{A_r}(v, w) |dw| \\ &= \frac{1}{2}[1 + \mathcal{O}(e^{-r})] - \frac{r-1}{2r}[1 + \mathcal{O}(e^{-r})] \\ &= \frac{1}{2r} + \mathcal{O}(e^{-r}). \end{aligned}$$

Similarly,

$$H_{A_r}(\bar{z}, 1) = \frac{1}{2}[1 + \mathcal{O}(|\bar{z}|)] + \frac{\log |\bar{z}|}{2r}[1 + \mathcal{O}(|\bar{z}|)] = \frac{r + \log |\bar{z}|}{2r} + \mathcal{O}(|\bar{z}|),$$

which proves the first estimate. The second estimate follows from this and the transformation  $z \mapsto e^{-r}/z$ . Using (2.8),

$$\begin{aligned} H_{A_r}(1, z) &= \left[ \frac{1}{2r} + \mathcal{O}(e^{-r}) \right] \frac{1}{\pi} \int_{C_{r-1}} H_{A_r \cap \mathbb{D}_{r-1}}(z, v) |dv| \\ &= \left[ \frac{1}{2r} + \mathcal{O}(e^{-r}) \right] \mathcal{E}_{A_r \cap \mathbb{D}_{r-1}}(z, C_{r-1}) \\ &= e^r \left[ \frac{1}{2r} + \mathcal{O}(e^{-r}) \right], \end{aligned}$$

which proves the last estimate. □

**Lemma 2.2.** *Suppose  $\gamma: (0, t_\gamma) \in \mathbb{D} \setminus \{0\}$  is a simple curve with  $\gamma(0+) = 1$ . Let  $\Phi$  be a conformal transformation in a neighborhood of 1 that locally maps  $\mathbb{D}$  to  $\mathbb{H}$  and  $C_0$  to  $\mathbb{R}$ . Let  $B_t$  be a complex Brownian motion and define  $\tau_t$  to be the first time  $B_t$  exits  $\mathbb{D} \setminus \gamma$ . Then at  $t = 0$ ,*

$$\partial_t \mathbb{E}^0[\log |B_{\tau_t}|] = -\partial_t \log \bar{g}'_t(0) = \frac{-1}{2|\Phi'(1)|^2} \partial_t \text{hcap}[\Phi(\gamma)].$$

*Proof.* By Schwarz lemma,  $f_t(z) = \log(\bar{g}_t(z)/z)$  is a well-defined bounded analytic function on

$\mathbb{D} \setminus \gamma_t$  with  $f_t(0) = \log \bar{g}'_t(0) \in \mathbb{R}$ . Hence,  $\operatorname{Re}[f_t(z)]$  is a bounded harmonic function and

$$\operatorname{Re}[f_t(z)] = \mathbb{E}^z[\operatorname{Re}[f_t(B_{\tau_t})]] = -\mathbb{E}^z[\log |B_{\tau_t}|].$$

In particular,

$$\log \bar{g}'_t(0) = -\mathbb{E}^0[\log |B_{\tau_t}|].$$

Note that  $\phi(z) = i(1-z)/(1+z)$  is a conformal transformation from  $\mathbb{D}$  to  $\mathbb{H}$  with  $\phi(0) = i$ . Let  $\tilde{\gamma}_t = \phi(\gamma_t)$  and define  $g_t : \mathbb{H} \setminus \eta_t \rightarrow \mathbb{H}$  to be the unique conformal transformation with  $g_t(z) = z + o(1)$  as  $z \rightarrow \infty$ . Let

$$\phi_t(z) = \frac{z - \operatorname{Re}[g_t(i)]}{\operatorname{Im}[g_t(i)]}, \quad \phi_t : \mathbb{H} \rightarrow \mathbb{H}$$

be a conformal transformation with  $\phi_t \circ g_t(i) = i$ . Then

$$\bar{g}'_t(0) = |\partial_z \phi^{-1} \circ \phi_t \circ g_t \circ \phi(0)| = \frac{g'_t(i)}{\operatorname{Im}[g_t(i)]}. \quad (2.9)$$

Let  $r_t = \operatorname{diam}[\eta_t]$  and  $O_t = \mathbb{H} \setminus 2r_t\mathbb{D}$ . Then for  $|z| > 2r_t$ ,  $\theta \in (0, \pi)$ , we have

$$H_{O_t}(z, r_t e^{i\theta}) = 2\operatorname{Im}[-1/z] \sin(\theta) [1 + O(r_t/|z|)].$$

Therefore,

$$\operatorname{Im}[z] - \operatorname{Im}[g_t(z)] = \operatorname{Im}[-1/z] \operatorname{hcap}[\eta_t] [1 + O(r_t/|z|)]. \quad (2.10)$$

Let  $f_t(z) = g_t(z) - z - \operatorname{hcap}[\eta_t]/z$  and  $v_t(z) = \operatorname{Im}[f_t(z)]$ . Using (2.10), we can see that there exists a constant  $c$  such that for every  $|z| > 3/2r_t$ ,

$$|v_t(z)| \leq c r_t \operatorname{hcap}[\eta_t] |z|^{-2}.$$

By the mean value property of harmonic functions we can see that for  $|z| > 2r_t$ ,

$$|f'_t(z)| = |g'_t(z) - 1 + \text{hcap}[\eta_t]/z^2| \leq c r_t \text{hcap}[\eta_t] |z|^{-3}.$$

Using this and (2.10) gives

$$\text{Im}[g_t(i)] = 1 - \text{hcap}[\eta_t] + O(r_t \text{hcap}[\eta_t]),$$

$$|g'_t(i)| = 1 + \text{hcap}[\eta_t] + O(r_t \text{hcap}[\eta_t]).$$

Substituting these into (2.9) gives

$$\bar{g}'_t(0) = 1 + \text{hcap}[\eta_t] [2 + O(\text{hcap}[\eta_t])].$$

In addition, proposition 2.3 implies that at  $t = 0$ ,

$$\partial_t \text{hcap}[\Phi(\gamma_t)] = (\Phi \circ \phi^{-1})'(0)^2 \partial_t \text{hcap}[\eta_t],$$

from which the result follows. □

**Lemma 2.3.** *Suppose  $D \subset \mathbb{D}$  is a simply connected domain with  $\mathbb{D} \setminus D \subset A_r$ . Let  $K = \mathbb{D} \setminus D$  and  $\hat{r} = 2\pi/\mathcal{E}_{A_r \cap D}(C_r, \mathbb{D} \cap \partial D)$ . Define  $g : D \rightarrow \mathbb{D}$  to be the unique conformal transformation with  $g(0) = 0, g'(0) > 0$ . Then*

$$\hat{r} = r - \log g'(0) + O(e^{-r + \log g'(0)}), \tag{2.11}$$

and

$$m_{\mathbb{D}}(K, C_r) = \log(r/\hat{r}) + O(e^{-\hat{r}}).$$

*Proof.* Let  $s = \log(4g'(0))$  and note that by Koebe-1/4 theorem,  $C_s \subset D$ . Without the loss of generality, assume  $s < r$ . Distortion theorem for the transformation  $g$  restricted to  $\mathbb{D}_s$  implies that

there exists a constant  $c$  such that for any  $|w| \leq e^{-r}$ ,

$$|g'(w) - g'(0)| \leq c g'(0) e^{-r+s}.$$

By taking integral of this we get

$$|g(w) - w g'(0)| \leq c |w| g'(0) e^{-r+s},$$

from which (2.11) follows.

For  $x \in C_t$ , let  $\Gamma_{A_t}(x, K)$  be the bubble measure of loops rooted at  $x$  that intersect  $K$  (That is, the measure induced by Brownian excursions rooted at  $x$  in  $A_t$  restricted to the loops that intersect  $k$ . See [13] or Section 5.5 in [16] for more details). Then

$$m_{\mathbb{D}}(K, C_r) = \frac{1}{\pi} \int_r^\infty \int_0^{2\pi} e^{-2t} \Gamma_{A_t}(e^{-t+i\theta}, K) d\theta dt. \quad (2.12)$$

For  $z \in A_r$ , the probability that a Brownian motion starting at  $z$  exits  $A_r$  at  $C_0$  is  $1 + \log |z|/t$ . Using this and the strong Markov property for Brownian motion, we can see that

$$\frac{2\pi}{t} = \mathcal{E}_{A_t}(C_t, C_0) = \int_{\partial D} \left[ 1 + \frac{\log |z|}{t} \right] \mathcal{E}_{A_t \cap D}(C_t, dz).$$

Using equation (2.11), we can write

$$\mathcal{E}_{A_t \cap D}(C_t, \partial D) = \int_{\partial D} \mathcal{E}_{A_t \cap D}(C_t, dz) = \frac{2\pi}{t - \log g'(0)} \left[ 1 + O(e^{-t + \log g'(0)}) \right].$$

Therefore,

$$\int_{\mathbb{D} \cap \partial D} \frac{-\log |z|}{t} \mathcal{E}_{A_t \cap D}(C_t, dz) = \frac{2\pi}{t - \log g'(0)} - \frac{2\pi}{t} + O\left(\frac{e^{-t + \log g'(0)}}{t - \log g'(0)}\right).$$

Let  $V \subset \partial D$  and recall that for any  $z \in D$ ,  $\rho_D(z, V)$  denotes the probability that a Brownian motion

starting at  $z$  exits  $D$  at  $V$ . Since  $C_s \subset D$ , we have for any  $w \in C_t$

$$\begin{aligned} \mathcal{E}_{A_t \cap D}(C_t, V) &= \frac{1}{\pi} \int_{C_t} \int_{C_s} H_{A_t \cap \mathbb{D}_s}(v, u) \rho_D(u, V) |du| |dv| \\ &= \frac{[1 + O(e^{-t+s})]}{\pi} \int_{C_t} \int_{C_s} H_{A_t \cap \mathbb{D}_s}(w, u) \rho_D(u, V) |du| |dv| \\ &= e^{-t} \mathcal{E}_{A_t \cap D}(w, V) [1 + O(e^{-t+s})], \end{aligned}$$

where the second equality follows from lemma 2.1.

Using this, lemma 2.1 and the fact that  $\text{dist}(0, \partial D) > 4g'(0)$ , we can see that for  $w \in C_t$ ,

$$\begin{aligned} \Gamma_{A_t}(w, K) &= \int_{\mathbb{D} \cap \partial D} H_{A_t}(z, w) \mathcal{E}_{A_t \cap D}(w, dz) \\ &= \frac{e^{2t}}{2\pi} [1 + O(e^{-t+s})] \int_{\mathbb{D} \cap \partial D} \frac{-\log |z|}{2t} \mathcal{E}_{A_t \cap D}(C_t, dz) \\ &= \frac{e^{2t}}{2} [1 + O(e^{-t+s})] \left[ \frac{1}{t - \log g'(0)} - \frac{1}{t} + O\left(\frac{e^{-t + \log g'(0)}}{t - \log g'(0)}\right) \right]. \end{aligned}$$

Plugging this into (2.12), we get

$$m_{\mathbb{D}}(K, C_r) = \log\left(\frac{t}{t - \log g'(0)}\right) + O(e^{-r+s}).$$

The result follows from this and (2.11). □

**Lemma 2.4.** *Suppose  $D \subset \mathbb{D}$  is a simply connected domain such that  $\mathbb{D} \setminus D \subset A_r$  and let  $K = D \cap A_r$ . Define  $f : K \rightarrow A_{\hat{r}}$  to be a conformal transformation satisfying  $f(C_r) = C_{\hat{r}}$  and let  $g : D \rightarrow \mathbb{D}$  be the unique conformal transformation with  $g(0) = 0$ ,  $g'(0) > 0$ . Then for  $z \in C_r$ ,*

$$|f'(z)| = e^{r-\hat{r}} \left[1 + O\left(e^{-\hat{r}}\right)\right] = g'(0) \left[1 + O\left(e^{-\hat{r}}\right)\right],$$

where the error term is independent of  $z$ .

*Proof.* Let  $s = r + \log(4) - \log g'(0)$  and  $\mathbb{D}_r = e^{-r}\mathbb{D}$ . Koebe-1/4 theorem implies that  $C_s \subset g(\mathbb{D}_r)$ .

By conformal invariance of the excursion measure,

$$\frac{2\pi}{\hat{r}} = \mathcal{E}_{A_{\hat{r}}}(C_0, C_{\hat{r}}) = \mathcal{E}_{g(D)}(C_0, g(C_r)) = \mathcal{E}_D(\partial D, C_r). \quad (2.13)$$

Hence,  $s \leq \hat{r}$  and  $g'(0) \leq 4e^{r-\hat{r}}$ . Using Koebe-1/4 again, if  $\hat{s} = r - \hat{r} + 2\ln(4)$ , then  $C_{\hat{s}} \subset D$ .

For  $z \in C_r$ , recall that  $\rho_K(z; \partial D)$  is the probability that the Brownian excursion starting from  $z$  in  $K$  exits at  $\partial D$ . Using equation (2.13),

$$\frac{2\pi}{\hat{r}} = \int_{C_r} \rho_K(z; \partial D) |dz|.$$

Moreover, (2.1) implies

$$\rho_K(z; \partial D) = |f'(z)| \frac{e^{\hat{r}}}{\hat{r}}. \quad (2.14)$$

Since  $C_{\hat{s}} \subset D$ , we can also write

$$\rho_D(z; \partial D) = \int_{C_{\hat{s}}} H_{\mathbb{D}_{\hat{s}} \cap A_r}(z, w) \rho_K(w, \partial D) |dw|.$$

It follows from lemma 2.1 that for any  $z_1, z_2 \in C_r$ ,  $w \in C_{\hat{s}}$ ,

$$H_{\mathbb{D}_{\hat{s}} \cap A_{\hat{r}}}(z_2, w) = H_{\mathbb{D}_{\hat{s}} \cap A_{\hat{r}}}(z_1, w) \left[1 + O(e^{-\hat{r}})\right].$$

Therefore,

$$\rho_D(z_1; \partial D) = \rho_D(z_2; \partial D) \left[1 + O(e^{-\hat{r}})\right] = \frac{e^r}{\hat{r}} \left[1 + O(e^{-\hat{r}})\right]$$

and the first equality follows from (2.14). The second equality follows from lemma 2.3.  $\square$

**Corollary 2.1.** *Suppose  $D \subset \mathbb{D}$  is a simply connected domain containing the origin. Let  $K = \mathbb{D} \setminus \bar{D}$  and choose  $\hat{r}$  such that there exists a conformal transformation  $f : K \rightarrow A_{\hat{r}}$  satisfying  $f(C_0) = C_0$ . Then for  $z \in C_0$ ,*

$$|f'(z)| = \left[1 + O(e^{-\hat{r}})\right].$$

*Proof.* Since  $0 \in D$ , we can find  $0 < r < \infty$  such that  $C_r \subset D$ . Let

$$g_1(z) = \frac{e^{-r}}{z}, \quad g_2(z) = \frac{e^{-\hat{r}}}{z},$$

and define  $\hat{f} = g_2 \circ f \circ g_1$  to be a conformal transformation from  $g_1(D)$  onto  $A_{\hat{r}}$  with  $\hat{f}(C_r) = C_{\hat{r}}$ .

The result follows from chain rule and lemma 2.4.  $\square$

**Corollary 2.2.** *Recall the assumptions of lemma 2.4. Let  $\theta \in [0, 2\pi)$ ,  $z = e^{-r+i\theta}$  and define the function  $\Theta(\theta) = \arg f(z)$ . Then*

$$\Theta'(\theta) = 1 + O(e^{-\hat{r}}).$$

*Proof.* Define the function  $\varphi : C_0 \rightarrow C_0$  with  $\varphi(w) = e^{\hat{r}} f(e^{-r} w)$ . Using lemma 2.4,

$$|\varphi'(w)| = 1 + O(e^{-\hat{r}}).$$

If  $w = e^{i\theta}$ , then  $\Theta(\theta) = \arg \varphi(w)$  and  $|\varphi'(w)| = |\partial_w \arg \varphi(w)|$ . Therefore,

$$\Theta'(\theta) = \partial_\theta \arg \varphi(e^{i\theta}) = ie^{i\theta} \partial_w \arg \varphi(w)$$

and

$$|\Theta'(\theta)| = |\partial_w \arg \varphi(w)| = |\varphi'(w)| = 1 + O(e^{-\hat{r}}).$$

Since  $f(C_r) = C_{\hat{r}}$ , we have  $\Theta'(\theta) > 0$  and the result follows.  $\square$

**Lemma 2.5.** *Suppose  $\gamma_t \subset A_r$  is a simple curve with  $\gamma(0+) \in C_0$ . If  $\gamma_t$  has radial parametrization (i.e.  $\log \bar{g}_t'(0) = -at/2$ ), then*

$$\dot{r}(t) = -\frac{a\bar{\phi}_t'(\bar{\xi}_t)^2}{2}.$$

*Proof.* We first prove the lemma for the derivative at  $t = 0$ . Since  $\mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r) = 2\pi/r(t)$ , we have

$$\partial_t \mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r) = \dot{r}(t) \frac{-2\pi}{r^2}. \quad (2.15)$$

Since

$$\rho_{A_r \setminus \gamma_t}(z; C_r) = \rho_{A_{r(t)}}(\tilde{h}(z); C_{r(t)}),$$

we can write

$$\mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r) = \int_{C_r} \partial_{n_z} \left( 1 + \frac{\log |\tilde{h}_t(z)|}{r(t)} \right) |dz|,$$

where  $n_z$  denotes the inward normal derivative. Suppose  $\tau_t$  is the first time a Brownian motion  $B_t$  exits  $A_r \setminus \gamma_t$  and  $\sigma_t$  is the first time  $B_t$  exits  $\mathbb{D} \setminus \gamma_t$ . Since  $\log |\tilde{h}_t(z)| - \log |z|$  is a bounded harmonic function on  $A_r \setminus \gamma_t$ , we have

$$\begin{aligned} \log |\tilde{h}_t(z)| - \log |z| &= (r - r(t)) \rho_{A_r \setminus \gamma_t}(z; C_r) - \mathbb{E}^z[\log |B_{\tau_t}|; \tau_t = \sigma_t] \\ &= -(r - r(t)) \frac{\log |\tilde{h}_t(z)|}{r(t)} - \mathbb{E}^z[\log |B_{\tau_t}|; \tau_t = \sigma_t]. \end{aligned}$$

Hence,

$$r \mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r) = 2\pi - \int_{C_r} \partial_{n_z} \mathbb{E}^z[\log |B_{\tau_t}|; \tau_t = \sigma_t] |dz|. \quad (2.16)$$

Assume  $t$  is small enough so that  $\text{diam}[\gamma_t] < 1/10$  and therefore  $\gamma_t \subset A_1$ . By conditioning on the first time the Brownian motion hits  $C_1$  we get

$$\int_{C_r} \partial_{n_z} \mathbb{E}^z[\log |B_{\tau_t}|; \tau_t = \sigma_t] = \frac{e}{r-1} \int_{C_1} \mathbb{E}^z[\log |B_{\tau_t}|; \tau_t = \sigma_t] |dz|,$$

Here,  $\mathbb{E}$  is expectation with respect to a Brownian motion starting uniformly on  $C_1$ . Note that

$$\mathbb{E}[\log |B_{\tau_t}|; \sigma_t = \tau_t] = \mathbb{E}[\log |B_{\sigma_t}|] - \mathbb{E}[\log |B_{\sigma_t}|; \tau_t < \sigma_t].$$

Let  $d_t = 2 \text{diam}[\gamma_t]$  and note that for small enough  $t$ ,  $\gamma_t \cap \mathbb{D}_{d_t} = \emptyset$ . If  $T_s$  denotes the first time that a

Brownian motion started uniformly on  $C_1$  exits  $\mathbb{D}_s$ , then

$$\mathbb{E}[\log |B_{\sigma_t}|; \tau_t < T_{d_t}] \leq \mathbb{E}[\log |B_{\sigma_t}|; \tau_t < \sigma_t] \leq \mathbb{E}[\log |B_{\sigma_t}|; \tau_t < T_0].$$

Conditioned on  $\tau_t < T_{d_t}$  or  $\tau_t < T_0$ ,  $B_{\tau_t}$  is uniformly distributed on  $C_r$ . Moreover,

$$\mathbb{P}[\tau_t < T_{d_t}] = \frac{1}{r} [1 + \mathcal{O}(d_t)], \quad \mathbb{P}[\tau_t < T_0] = \frac{1}{r}.$$

Therefore,

$$\mathbb{E}[\log |B_{\sigma_t}|; \tau_t < \sigma_t] = \frac{1}{r} \mathbb{E}[\log |B_{\sigma_t}|] [1 + \mathcal{O}(d_t)].$$

and

$$\mathbb{E}[\log |B_{\tau_t}|; \sigma_t = \tau_t] = \mathbb{E}[\log |B_{\sigma_t}|] \left[ \frac{r-1}{r} + \mathcal{O}(d_t) \right].$$

It follows from (2.15) and (2.16) that at  $t = 0$ ,

$$\dot{r}(t) = \frac{-r^2}{2\pi} \partial_t \mathcal{E}_{A_r \setminus \gamma_t}(C_0 \cup \gamma_t, C_r) = \partial_t \mathbb{E}[\log |B_{\sigma_t}|] = \partial_t \mathbb{E}^0[\log |B_{\sigma_t}|].$$

The claim for  $t = 0$  follows from this and lemma 2.2. Using proposition 2.3, we can see that at  $s = 0$ ,

$$\partial_s \text{hcap}[h_t(\eta_{t+s})] = |\bar{\phi}_t(\bar{\xi}_t)|^2 \partial_s \text{hcap}[\tilde{g}_t(\eta_{t+s})],$$

from which we conclude the proof for general  $t$ . □

## 2.4 $SLE_\kappa$ in Simply Connected Domains

For a domain  $D$ , we use  $\mu_D^\#(z, w)$  to denote the  $SLE_\kappa$  probability measure on curves starting from  $z$  and ending at  $w$  in  $D$ . We use  $\mu_D(z, w)$  when we are considering  $SLE_\kappa$  as a nonprobability measure and use  $\Psi_D(z, w)$  to denote the partition function  $|\mu_D(z, w)|$ . These measures are usually considered on curves  $\gamma$  modulo the parametrization. But it is often convenient to parametrize the

curves when we are discussing the properties of the measures.

We recall that  $SLE_\kappa$  is a one parameter family of measures having the following properties.

- **Conformal Invariance** Suppose  $D, D'$  are simply connected domains and let  $f : D \rightarrow D'$  be a conformal transformation. If  $z \in \partial D, w \in \bar{D}$  are distinct points, then

$$\mu_D^\#(z, w) = \mu_{D'}^\#(f(z), f(w)).$$

Moreover, if  $f$  extends to an analytic function in the neighborhoods of  $z, w$ , then the non-probability measures have the conformal covariance property:

- If  $w \in \partial D$  (chordal  $SLE_\kappa$ ),

$$\mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{D'}(f(z), f(w)), \quad b = \frac{6 - \kappa}{2\kappa}.$$

- If  $w \in D$  (radial  $SLE_\kappa$ ),

$$\mu_D(z, w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \mu_{D'}(f(z), f(w)), \quad \tilde{b} = \frac{b(\kappa - 2)}{4}.$$

The constants  $b, \tilde{b}$  are called the *scaling exponents*. In the case of chordal  $SLE_\kappa$ ,  $\Psi_D(z, w) = H_D(z, w)^b$ , where  $H_D(z, w)$  denote the Poisson kernel in  $D$  normalized so that  $H_{\mathbb{H}}(0, \infty) = 1$ . However in the radial case, the partition function cannot be given as a power of the Poisson kernel. Instead,  $\Psi_{\mathbb{D}}(1, 0) = 1$  and the other cases are obtained using the conformal covariance relation given above.

- **Domain Markov Property** Given an initial segment of the curve  $\gamma$ , the conditional distribution of the rest of the curve is  $\mu_{D \setminus \gamma}^\#(\gamma(t), w)$ .

It is proved by Dapeng Zhan [27, 28] that  $SLE_\kappa$  is reversible when  $\kappa \leq 4$ . That is,  $\mu_D(w, z)$  can be obtained from  $\mu_D(z, w)$  by reversing the paths. In [3] Vincent Beffara proved that with

probability one the Hausdorff dimension of  $SLE_\kappa$  is  $\min\{2, 1 + \kappa/8\}$ . Moreover, Mohammad Rezaei [23] proved that for  $\kappa < 8$ ,  $d = 1 + \kappa/8$ , the Hausdorff  $d$ -measure of the path is zero.

### 2.4.1 Loewner equations

It is easiest to study chordal  $SLE_\kappa$  in the upper-half plane  $\mathbb{H}$ . Suppose  $K$  is a compact subset of  $\mathbb{H}$  such that  $\mathbb{H} \setminus K$  is simply connected. If  $g : \mathbb{H} \setminus K \rightarrow \mathbb{H}$  is the unique conformal transformation with  $g(z) = z + o(1)$ , then

$$g(z) = z + \frac{\text{hcap}[K]}{z} + O(|z|^{-2}).$$

Let  $a = 2/\kappa$  and assume  $\gamma$  is a simple continuous curve from 0 to  $\infty$  in  $\mathbb{H}$ . It can be shown (see for example [16]) that  $\text{hcap}[\gamma]$  is a continuous and strictly increasing function of  $t$ . Therefore if necessary, we can reparametrize the curve so that  $\text{hcap}[\gamma] = at$ . Let  $g_t : \mathbb{H} \setminus \gamma \rightarrow \mathbb{H}$  be the unique conformal transformation with  $g_t(z) = z + o(1)$  as  $z \rightarrow \infty$ . Since  $\gamma(t)$  has only one prime end and  $\mathbb{H}$  is locally connected,  $U_t := g_t(\gamma(t))$  is well defined and continuous in  $t$  (see [16] for more details). For every  $z \in \mathbb{H}$ , the transformation  $g_t(z)$  satisfies

$$\dot{g}_t(z) = \frac{\partial_t \text{hcap}[\gamma]}{g_t(z) - U_t}, \quad 0 \leq t < T_z, \quad g_0(z) = z, \quad (2.17)$$

where  $T_z = \inf\{s < t_\gamma : z \in \gamma_s\}$  and the dot derivative denotes the derivative with respect to  $t$ . Equation (2.17) is called the *Chordal Loewner Equation*. This deterministic result can be applied to realizations of  $SLE_\kappa$  curves. By conformal invariance, one can see that with probability one  $\text{hcap}[\gamma] \rightarrow \infty$  as  $t \rightarrow \infty$  and  $U_t$  is a standard Brownian motion if  $\gamma$  is parametrized with  $\text{hcap}[\gamma] = at$ . This parametrization is often called the half-plane capacity parametrization.

Since the radial Loewner equation has the simplest form in  $\mathbb{D}$ , radial  $SLE_\kappa$  is usually studied in  $\mathbb{D}$ . Suppose  $\gamma : (0, t_\gamma) \rightarrow \mathbb{D} \setminus \{0\}$  is a simple curve with  $\gamma(0+) = 1$ . For any  $t < t_\gamma$ , let  $D_t = \mathbb{D} \setminus \gamma$  and define  $g_t : D_t \rightarrow \mathbb{D}$  to be the unique conformal transformation with  $g_t(0) = 0$  and  $g'_t(0) > 0$ .

Let  $a(t) = \log g_t'(0)$  and assume  $a(t)$  is differentiable. Then for every  $z \in \mathbb{D}$  and  $0 \leq t < T_z$ ,

$$\dot{g}_t(z) = \dot{a}(t) g_t(z) \frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}, \quad g_0(z) = z, \quad (2.18)$$

where  $T_z = \inf\{s < t_\gamma : z \in \gamma_s\}$  and  $U_t$  is a continuous function with

$$e^{iU_t} = g_t(\gamma(t)).$$

For  $z \in \mathbb{H}$  with  $e^{iz} \in D_t$ , define  $h_t(z) = -i \log g_t(e^{iz})$ . We can uniquely specify  $h_t(z)$  by requiring  $h_0(z) = z$  and continuity of  $h_t(z)$  in  $t$ . From (2.18), we get

$$\dot{h}_t(z) = \dot{a}(t) \cot_2(h_t(z) - U_t),$$

where  $\cot_2(x) := \cot(x/2)$ . If  $\gamma_t$  has the distribution of radial  $SLE_\kappa$  and it is parametrized so that  $\log g_t'(0) = at/2$  (we call this the radial parametrization), then  $U_t$  is a standard Brownian motion.

#### 2.4.2 Boundary Perturbation Property

In this section, we discuss the boundary perturbation property (restriction property) of chordal  $SLE_\kappa$ . More details can be found in [12, 14]. Let  $D \subset \mathbb{H}$  be a simply connected domains such that  $K = \mathbb{H} \setminus D$  is bounded and  $\text{dist}(0, K) > 0$ . Let  $\kappa \leq 4$  and assume  $\gamma_t$  is a chordal  $SLE_\kappa$  from 0 to  $\infty$  with respect to a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . That is, if  $\gamma_t$  has half-plane parametrization and  $g_t : \mathbb{H} \setminus \gamma_t \rightarrow \mathbb{H}$  is the unique conformal transformation satisfying  $g_t(z) = z + o(1)$  as  $z \rightarrow \infty$ , then  $U_t := g_t(\gamma(t))$  is a standard Brownian motion with respect to  $\mathbb{P}$ . Let  $\tau = \inf\{t; \gamma_t \cap K \neq \emptyset\}$  and assume  $t < \tau$ .

Our goal is to find a martingale  $M_t$  such that if we weight  $\mathbb{P}$  by  $M_t$ , then under the new measure  $\gamma_t$  has the distribution of chordal  $SLE_\kappa$  from 0 to  $\infty$  in  $D$ . Let  $\Phi : D \rightarrow \mathbb{H}$  be the unique conformal transformation with  $\Phi(0) = 0$ ,  $\Phi'(\infty) = 1$ . Let  $\gamma_t^* = \Phi(\gamma_t)$  and define  $g_t^* : \mathbb{H} \setminus \gamma_t^* \rightarrow \mathbb{H}$  to be our usual conformal transformation. Finally, define the conformal transformation  $\Phi_t : g_t(D \setminus \gamma_t) \rightarrow \mathbb{H}$

with  $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$  and let

$$U_t^* := g_t^*(\gamma^*(t)) = \Phi_t(U_t).$$

By the chain rule,

$$\dot{\Phi}_t(z) = \dot{g}_t^* \circ \Phi \circ g_t^{-1}(z) + [g_t^{*'} \circ \Phi \circ g_t^{-1}(z)][\Phi' \circ g_t^{-1}(z)] \partial_t g_t^{-1}(z).$$

It follows from proposition 2.3 that  $\partial_t \text{hcap}[\gamma_t^*] = a \Phi_t'(U_t)^2$ . Moreover, the chain rule implies that

$$[g_t^{*'} \circ \Phi \circ g_t^{-1}(z)][\Phi' \circ g_t^{-1}(z)] = \Phi_t'(z) g_t'(g_t^{-1}(z))$$

and

$$\partial_t g_t^{-1}(z) = \frac{-\dot{g}_t(g_t^{-1}(z))}{g_t'(g_t^{-1}(z))}.$$

Therefore, using the Loewner equation

$$\dot{\Phi}_t(z) = a \left[ \frac{\Phi_t'(U_t)^2}{\Phi_t(z) - U_t^*} - \frac{\Phi_t'(z)}{z - U_t} \right].$$

We can also take derivatives of both sides to find an expression for  $\dot{\Phi}_t'(z)$ . Taking the limit as  $z \rightarrow U_t$  gives

$$\dot{\Phi}_t(U_t) = -3a\Phi_t''(U_t)/2, \quad \dot{\Phi}_t'(U_t) = a \left[ \frac{\Phi_t''(U_t)^2}{4\Phi_t'(U_t)} - \frac{4\Phi_t'''(U_t)}{3} \right]. \quad (2.19)$$

Now we can use the Itô's formula and write

$$dU_t^* = -b\Phi_t''(U_t) dt + \Phi_t'(U_t) dU_t.$$

A little bit of calculation using the Itô's formula and (2.19) implies that

$$M_t = \Phi_t'(U_t)^b \exp \left\{ -\frac{ac}{12} \int_0^t S\Phi_s(U_s) ds \right\}, \quad t < \tau,$$

is a local martingale satisfying

$$dM_t = b \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)} M_t dU_t.$$

Using (2.7), we can re-write the last expression as

$$M_t = \Phi_t'(U_t)^b \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma_t, K) \right\}, \quad t < \tau.$$

Let  $\mathbb{P}^*$  be the probability measure obtained from weighting  $\mathbb{P}$  by  $M_t$ . Then Girsanov's theorem implies that with respect to  $\mathbb{P}^*$ ,  $dU_t = b \Phi_t''(U_t) / \Phi_t'(U_t) dt + dW_t$ , where  $W_t$  is a Brownian motion. In particular,

$$dU_t^* = \Phi_t'(U_t) dW_t.$$

Therefore, with respect to the measure  $\mathbb{P}^*$ ,  $\gamma_t^*$  is a  $SLE_{\kappa}$  in  $\mathbb{H}$ . In particular,  $\gamma_t$  is a  $SLE_{\kappa}$  from 0 to  $\infty$  in  $D$ . Note that if  $\sigma(t)$  satisfies

$$\int_0^{\sigma(t)} \Phi_s'(U_s)^2 ds = t,$$

then  $\gamma_{\sigma(t)}^*$  has half-plane capacity parametrization.

Note that  $\Phi_t'(U_t)$  is the probability that Brownian excursion from  $U_t$  to  $\infty$  in  $\mathbb{H}$  does not intersect  $K$ . One can show that if  $\tau = \infty$ , then  $\gamma(t) \rightarrow \infty$  and  $\Phi_t'(U_t) \rightarrow 1$  [12]. Moreover, with respect to the measure  $\mathbb{P}^*$ ,  $\gamma_t$  is a  $SLE_{\kappa}$  in  $D$ . Therefore, with probability one  $\text{dist}(\gamma, K) > 0$  and  $m_{\mathbb{D}}(\gamma, K) < \infty$ . In particular,

$$\frac{\mu_D^{\#}(0, \infty)}{\mu_{\mathbb{H}}^{\#}(0, \infty)}(\gamma) = \frac{1}{\Phi'(0)^b} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, K) \right\} 1_{\{\gamma \subset D\}}.$$

Considering  $SLE_{\kappa}$  as measures with partition functions, we get

$$\frac{\mu_D(0, \infty)}{\mu_{\mathbb{H}}(0, \infty)}(\gamma) = \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, K) \right\} 1_{\{\gamma \subset D\}}.$$

More generally, if  $D' \subset D$  are domains and  $z, w \in \partial D$  are analytic boundary points such that  $D', D$

agree in a neighborhood of  $z, w$ , then

$$\frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = \exp\left\{\frac{\mathbf{c}}{2}m_D(\gamma, D \setminus D')\right\} 1_{\{\gamma \cap D \setminus D' = \emptyset\}}. \quad (2.20)$$

## CHAPTER 3

### MULTIPLE PATHS $SLE_\kappa$ IN SIMPLY CONNECTED DOMAINS

#### 3.1 Introduction

The definition of the measure on multiple  $SLE_\kappa$  paths immediately gives a partition function defined as the total mass of the measure. The measure on multiple  $SLE_\kappa$  paths

$$\gamma = (\gamma^1, \dots, \gamma^n)$$

has been constructed in [5, 11, 14]. Even though the definition in [11] is given for the so-called “rainbow” arrangement of the boundary points, it can be easily extended to the other arrangements [5, 15]. One can see that unlike  $SLE_\kappa$  measure on single curves, conformal invariance and domain Markov property do not uniquely specify the measure when  $n \geq 2$ . This definition makes it unique by requiring the measure to satisfy the restriction property, which is explained in Section 3.2.

Study of the multiple  $SLE_\kappa$  measure involves characterizing the partition function. For  $n = 2$ , the partition function is explicitly given in terms of the hypergeometric function. There is no explicit formula for the partition function when  $n \geq 3$ . Instead, the goal is to characterize it by a particular second-order PDE. That involves constructing a martingale  $M_t$  by conditioning on an adapted sigma-algebra generated by the curves up to time  $t$ . Itô’s formula can be used next to find the PDE that the partition function satisfies in a straightforward manner. See formula (1.2) in [10]. However, Itô’s formula can only be used if we know the functions are at least  $C^2$  and in our case, it does not directly follow from the definition that the partition function is  $C^2$ . There are two main approaches to address this problem. One approach is to show that the PDE system has a solution and use it to describe the partition function. In [5], it is shown that a family of integrals taken on a specific set of cycles satisfy the required PDE system. In [22], conformal field theory and partial differential equation techniques such as Hörmander’s theorem are used to show that the partition function satisfies the PDE system. The other approach, which is the one we take in

this work, is to directly prove that the partition function is  $C^2$ . Then Itô's formula can be used to show that the partition function satisfies the PDEs. Our approach has the advantage of only using standard techniques in probability. The basic idea of our proof is to interchange derivatives and expectations in expressions for the partition function. This interchange needs justification and we prove an estimate about  $SLE_\kappa$  to justify this. Another potential advantage of our approach is that it can be generalized to prove similar smoothness results for other  $SLE_\kappa$  measures or in multiply-connected domains.

In this chapter, we prove the following result.

**Theorem 3.1.** *If  $\kappa < 4$  and  $D$  is a simply connected domain, then the partition function of multiple-paths  $SLE_\kappa$  in  $D$  is a smooth function of the marked boundary points.*

We finish this introduction by reviewing examples of partition functions for  $SLE_\kappa$ . Definitions and properties of multiple  $SLE_\kappa$  and the outline of the proof are given in Section 3.2. Section 3.3 includes an estimate for  $SLE_\kappa$  using techniques similar to the ones in [19]. Proof of Lemma 3.2, which explains estimates for derivatives of the Poisson kernel is given in Section 3.4.

### 3.1.1 Examples

- **$SLE_\kappa$  in a subset of  $\mathbb{H}$ .** Let  $\kappa \leq 4$  and suppose  $D \subset \mathbb{H}$  is a simply connected domain such that  $K = \mathbb{H} \setminus D$  is bounded and  $\text{dist}(0, K) > 0$ . Also, assume that  $\gamma$  is parameterized with half-plane capacity. By the restriction property we have

$$\frac{d\mu_D(0, \infty)}{d\mu_{\mathbb{H}}(0, \infty)}(\gamma) = 1_{\{\gamma \cap K = \emptyset\}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma, K) \right\}, \quad (3.1)$$

where  $m_{\mathbb{H}}(\gamma, D)$  denotes the Brownian loop measure of the loops that intersect both  $\gamma$  and  $K$  and

$$\mathbf{c} = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}$$

is the *central charge*. Without mentioning Brownian loop measure and using Schwarzian derivative instead, a formula similar to equation (3.1) was proved for  $SLE_\kappa$  measures in [12]. In [14],  $SLE_\kappa$  was considered as a measure with partition function and formula (3.1) was proved. In Section 3.2, we will give a brief review of the definition of Brownian loop measure, which was first defined in [13].

We normalize the partition functions, so that  $\Psi_{\mathbb{H}}(0, \infty) = 1$ . For an initial segment of the curve  $\gamma$ , let  $g_t : \mathbb{H} \setminus \gamma \rightarrow \mathbb{H}$  be the unique conformal transformation with  $g_t(z) = z + o(1)$  as  $z \rightarrow \infty$ . Then

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t},$$

where  $a = 2/\kappa$  and  $U_t$  is a standard Brownian motion. Suppose  $\gamma \cap K = \emptyset$  and let  $D_t = g_t(D \setminus \gamma)$ . One can see that

$$m_{\mathbb{H}}(\gamma, K) = -\frac{a}{6} \int_0^t S\Phi_s(U_s) ds,$$

where  $S$  denotes the Schwarzian derivative and  $\Phi_s(U_s) = H_{D_s}(U_s, \infty)$ . See [13] for more details. It follows from conditioning on  $\gamma$  that

$$M_t = \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\gamma, K) \right\} \Psi_{D_t}(U_t, \infty)$$

is a martingale. We assume the function  $V(t, x) = \Psi_{D_t}(x, \infty)$  is  $C^2$  for a moment. Therefore, we can apply Itô's formula and we get

$$-\frac{a\mathbf{c}}{12} V(t, U_t) S\Phi_t(U_t) + \partial_t V(t, U_t) + \frac{1}{2} \partial_{xx} V(t, U_t) = 0.$$

Straightforward calculation shows that  $V(t, x) = H_{D_t}(x, \infty)^b$  is  $C^2$  and satisfies this PDE. Here,  $b$  is the *boundary scaling exponent*

$$b = \frac{6 - \kappa}{2\kappa}.$$

See [14] for more details.

- **Other examples.** Similar ideas were used in [11] to describe the partition function of two  $SLE_\kappa$  curves with a PDE. Differentiability of the partition function was justified using the explicit form of the solution in terms of the hypergeometric function. The PDE system in [17] characterizes the partition function of the annulus  $SLE_\kappa$ . That PDE is more complicated and one cannot find an explicit form for the solution. In fact, it is not easy to even show that the PDE has a solution. Instead, it was directly proved that the partition function is  $C^2$  and Itô's formula was used to derive the PDE.

### 3.2 Definitions and Preliminaries

We will consider the multiple  $SLE_\kappa$  measure only for  $\kappa \leq 4$  on simply connected domains  $D$  and distinct locally analytic boundary points  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n)$ . The measure is supported on  $n$ -tuples of curves

$$\gamma = (\gamma^1, \dots, \gamma^n),$$

where  $\gamma^j$  is a curve connecting  $x_j$  to  $y_j$  in  $D$ . If  $\mathbf{x}, \mathbf{y}$  are arranged in a way that there exist no non-intersecting curves  $\gamma^1, \dots, \gamma^n$ , then the multiple  $SLE_\kappa$  measure is a zero measure. If  $n = 1$ , then  $\mu_D(x_1, y_1)$  is  $SLE_\kappa$  from  $x_1$  to  $y_1$  in  $D$  with total mass  $H_D(x_1, y_1)^b$  whose corresponding probability measure  $\mu_D^\#(x_1, y_1) = \mu_D(x_1, y_1) / H_D(x_1, y_1)^b$  is (a time change of)  $SLE_\kappa$  from  $x_1$  to  $y_1$  as defined by Schramm.

**Definition 3.1.** *If  $\kappa \leq 4$  and  $n \geq 1$ , then  $\mu_D(\mathbf{x}, \mathbf{y})$  is the measure absolutely continuous with respect to  $\mu_D(x_1, y_1) \times \dots \times \mu_D(x_n, y_n)$  with Radon-Nikodym derivative*

$$Y(\gamma) := I(\gamma) \exp \left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^n m [K_j(\gamma)] \right\}.$$

Here  $\mathbf{c} = (6 - \kappa)(3\kappa - 8)/2\kappa$  is the central charge,  $I(\gamma)$  is the indicator function of the event

$$\{\gamma^j \cap \gamma^k = \emptyset, 1 \leq j < k \leq n\},$$

and  $m[K_j(\gamma)]$  denotes the Brownian loop measure of loops that intersect at least  $j$  of the paths  $\gamma^1, \dots, \gamma^n$ .

Similar definitions are also given in [11, 22]. Brownian loop measure was first defined in [13]. It is a measure on (continuous) curves  $\eta : [0, t_\eta] \rightarrow \mathbb{C}$  with  $\eta(0) = \eta(t_\eta)$ . Let  $\mathbf{v}^\#(0, 0; 1)$  be the law of the Brownian bridge starting from 0 and returning to 0 at time 1. Brownian loop measure can be considered as the image of the measure

$$m = \text{area} \times \left( \frac{1}{2\pi t^2} dt \right) \times \mathbf{v}^\#(0, 0; 1)$$

on the triples  $(z, t_\eta, \tilde{\eta})$  under the map  $(z, t_\eta, \tilde{\eta}) \mapsto \eta$  such that  $\eta$  is defined on  $[0, t_\eta]$ , and  $\eta(t) = z + t_\eta^{1/2} \tilde{\eta}(t/t_\eta)$  for  $t \in [0, t_\eta]$ . For a domain  $D \subset \mathbb{C}$ , we denote the restriction of  $m$  to the loops  $\eta \subset D$  by  $m_D$ . One important property of  $m_D$  is conformal invariance. More precisely, if  $f : D \rightarrow f(D)$  is a conformal transformation, then

$$f \circ m_D = m_{f(D)},$$

where  $f \circ m_D$  is the pushforward measure.

Note that if  $\sigma$  is a permutation of  $\{1, \dots, n\}$  and  $\gamma_\sigma = (\gamma^{\sigma(1)}, \dots, \gamma^{\sigma(n)})$ , then  $Y(\gamma) = Y(\gamma_\sigma)$ .

The partition function is the total mass of this measure

$$\Psi_D(\mathbf{x}, \mathbf{y}) = \|\mu_D(\mathbf{x}, \mathbf{y})\|.$$

We also write

$$\tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \frac{\Psi_D(\mathbf{x}, \mathbf{y})}{\prod_{j=1}^n H_D(x_j, y_j)^b},$$

which can also be written as

$$\tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \mathbb{E}[Y],$$

where the expectation is with respect to the probability measure  $\mu_D^\#(x_1, y_1) \times \cdots \times \mu_D^\#(x_n, y_n)$ . Note that  $\tilde{\Psi}_D(\mathbf{x}, \mathbf{y})$  is a conformal invariant,

$$f \circ \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \tilde{\Psi}_{f(D)}(f(\mathbf{x}), f(\mathbf{y})),$$

and hence is well defined even if the boundaries are rough. Since  $SLE_\kappa$  is reversible [27], interchanging  $x_j$  and  $y_j$  does not change the value.

To compute the partition function we use an alternative description of the measure  $\mu_D(\mathbf{x}, \mathbf{y})$ . We will give a recursive definition.

- For  $n = 1$ ,  $\mu_D(x_1, y_1)$  is the usual  $SLE_\kappa$  measure with total mass  $H_D(x_1, y_1)^b$ .
- Suppose the measure has been defined for all  $n$ -tuples of paths. Suppose  $\mathbf{x} = (x', x_{n+1})$ ,  $\mathbf{y} = (y', y_{n+1})$  are given and write an  $(n + 1)$ -tuple of paths as  $\gamma = (\gamma', \gamma^{(n+1)})$ .
  - The marginal measure on  $\gamma'$  induced by  $\mu_D(\mathbf{x}, \mathbf{y})$  is absolutely continuous with respect to  $\mu_D(x', y')$  with Radon-Nikodym derivative  $H_{\tilde{D}}(x_{n+1}, y_{n+1})^b$ . Here  $\tilde{D}$  is the component of  $D \setminus \gamma'$  containing  $x_{n+1}, y_{n+1}$  on its boundary. (If there is no such component, then we set  $H_{\tilde{D}}(x_{n+1}, y_{n+1}) = 0$  and  $\mu_D(\mathbf{x}, \mathbf{y})$  is the zero measure.)
  - Given  $\gamma'$ , the curve  $\gamma^{n+1}$  is chosen using the probability distribution  $\mu_{\tilde{D}}^\#(x_{n+1}, y_{n+1})$ .

One could try to use this description of the measure as the definition, but it is not obvious that it is consistent. However, one can see that the first definition satisfies this description using equation (3.1) and the following lemma.

**Lemma 3.1.** *Let  $\gamma$  denote a  $(n + 1)$ -tuple of paths which we write as  $\gamma = (\gamma', \gamma^{(n+1)})$ , and let  $\tilde{D}$  be*

the connected component of  $D \setminus \gamma'$  containing the end points of  $\gamma^{(n+1)}$  on its boundary. Then

$$\sum_{j=2}^{n+1} m [K_j(\gamma)] = \sum_{j=2}^n m [K_j(\gamma')] + m_D(\gamma^{(n+1)}, D \setminus \tilde{D}).$$

*Proof.* Let  $K_j^1(\gamma)$  denote the set of loops in  $K_j(\gamma)$  that intersect  $\gamma^{(n+1)}$  and let  $K_j^2(\gamma) := K_j(\gamma) \setminus K_j^1(\gamma)$  denote the set of loops that do not intersect  $\gamma^{(n+1)}$ . Then

$$m [K_2^1(\gamma)] = m_D(\gamma^{(n+1)}, D \setminus \tilde{D}). \quad (3.2)$$

Note that  $K_j^1(\gamma)$  is the set of loops in  $D$  that intersect  $\gamma^{(n+1)}$  and at least  $j-1$  paths of  $\gamma'$ . Moreover,  $K_j^2(\gamma)$  is the set of loops that intersect at least  $j$  paths of  $\gamma'$ , but do not intersect  $\gamma^{(n+1)}$ . Therefore,

$$K_j(\gamma') = K_{j+1}^1(\gamma) \cup K_j^2(\gamma).$$

The result follows from this, the fact that  $K_{n+1}^2(\gamma) = \emptyset$ , and (3.2). □

We can also take the marginals in a different order. For example, we could have defined the recursive step above as follows.

- The marginal measure on  $\gamma^{n+1}$  induced by  $\mu_D(\mathbf{x}, \mathbf{y})$  is absolutely continuous with respect to  $\mu_D(x_{n+1}, y_{n+1})$  with Radon-Nikodym derivative  $\Psi_{\tilde{D}}(\mathbf{x}', \mathbf{y}')$  where  $\tilde{D} = D \setminus \gamma$ . (It is possible that  $\tilde{D}$  has two separate components in which case we multiply the partition functions on the two components.)

We will consider boundary points on the real line. We write just  $H, \Psi, \tilde{\Psi}, \mu, \mu^\#$  for  $H_{\mathbb{H}}, \Psi_{\mathbb{H}}, \tilde{\Psi}_{\mathbb{H}}, \mu_{\mathbb{H}}, \mu_{\mathbb{H}}^\#$ ; and note that

$$\tilde{\Psi}(\mathbf{x}, \mathbf{y}) = \mathbb{E}[Y] = \Psi(\mathbf{x}, \mathbf{y}) \prod_{j=1}^n |y_j - x_j|^{2b},$$

where the expectation is with respect to the probability measure

$$\mu^\#(x_1, y_1) \times \cdots \times \mu^\#(x_n, y_n).$$

- If  $n = 1$ , then  $Y \equiv 1$  and  $\tilde{\Psi}(\mathbf{x}, \mathbf{y}) = 1$ .
- For  $n = 2$  and  $\gamma = (\gamma^1, \gamma^2)$ , then by equation (3.1),

$$\mathbb{E}[Y \mid \gamma^1] = \left[ \frac{H_{D \setminus \gamma^1}(x_2, y_2)}{H_D(x_2, y_2)} \right]^b.$$

The right-hand side is well defined even for non smooth boundaries provided that  $\gamma^1$  stays a positive distance from  $x_2, y_2$ . In particular,

$$\mathbb{E}[Y] = \mathbb{E} \left[ \mathbb{E}(Y \mid \gamma^1) \right] = \mathbb{E} \left[ \left( \frac{H_{D \setminus \gamma^1}(x_2, y_2)}{H_D(x_2, y_2)} \right)^b \right] \leq 1.$$

If  $8/3 < \kappa \leq 4$ , then  $\mathbf{c} > 0$  and  $Y > 1$  on the event  $I(\gamma)$  so the inequality  $\mathbb{E}[Y] \leq 1$  is not obvious.

- More generally, if  $\gamma = (\gamma', \gamma^{n+1})$ , then by equation (3.1) and Lemma 1,

$$\mathbb{E}[Y \mid \gamma'] = Y(\gamma') \left[ \frac{H_{D \setminus \gamma'}(x_{n+1}, y_{n+1})}{H_D(x_{n+1}, y_{n+1})} \right]^b \leq Y(\gamma').$$

Using this we see that  $\tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) \leq 1$ .

- For  $n = 2$ , if  $x_1 = 0, y_1 = \infty, y_2 = 1$  and  $x_2 = x$  with  $0 < x < 1$ , we have (see, for example, [11, (3.7)])

$$\tilde{\Psi}(\mathbf{x}, \mathbf{y}) = \phi(x) := \frac{\Gamma(2a)\Gamma(6a-1)}{\Gamma(4a)\Gamma(4a-1)} x^a F(2a, 1-2a, 4a; x), \quad (3.3)$$

where  $F = {}_2F_1$  denotes the hypergeometric function and  $a = 2/\kappa$ . This is computed by

finding

$$\mathbb{E} \left[ H_{\mathbb{H} \setminus \gamma^1}(x, 1)^b \right].$$

In fact, this calculation is valid for  $\kappa < 8$  if it is interpreted as

$$\mathbb{E} \left[ H_{\mathbb{H} \setminus \gamma^1}(x, 1)^b; H_{\mathbb{H} \setminus \gamma^1}(x, 1) > 0 \right].$$

It will be useful to write the conformal invariant (3.3) in a different way. If  $V_1, V_2$  are two arcs of a domain  $D$ , let

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \int_{V_2} H_D(z, w) |dz| |dw|.$$

This is  $\pi$  times the usual excursion measure between  $V_1$  and  $V_2$ ; the factor of  $\pi$  comes from our choice of Poisson kernel. Note that

$$\mathcal{E}_{\mathbb{H}}((-\infty, 0], [x, 1]) = \int_x^1 \int_{-\infty}^0 \frac{dr ds}{(s-r)^2} = \int_x^1 \frac{dr}{r} = \log(1/x),$$

Hence we can write (3.3) as  $\phi(\exp\{-\mathcal{E}_{\mathbb{H}}((-\infty, 0], [x, 1])\})$ . More generally, if  $x_1 < y_1 < x_2 < y_2$ ,

$$\tilde{\Psi}(\mathbf{x}, \mathbf{y}) = \phi(\exp\{-\mathcal{E}_{\mathbb{H}}([x_1, y_1], [x_2, y_2])\}) = \phi\left(\exp\left\{-\int_{x_1}^{y_1} \int_{x_2}^{y_2} \frac{dr ds}{(s-r)^2}\right\}\right),$$

and if  $D$  is a simply connected subdomain of  $\mathbb{H}$  containing  $x_1, y_2, x_2, y_1$  on its boundary, then

$$\tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi(\exp\{-\mathcal{E}_D([x_1, y_1], [x_2, y_2])\}) = \phi\left(\exp\left\{-\int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, s) dr ds\right\}\right). \quad (3.4)$$

This expression is a little bulky but it allows for easy differentiation with respect to  $x_1, x_2, y_1, y_2$ .

At this point we can state the main proposition.

**Proposition 3.1.** *If  $\kappa < 4$ ,  $\Psi$  and  $\tilde{\Psi}$  are  $C^2$  functions.*

It clearly suffices to prove this for  $\tilde{\Psi}$ . The idea is simple — we will write the partition function as an expectation and differentiate the expectation by interchanging the expectation and the

derivatives. This interchange requires justification and this is the main work of this chapter.

Before proving Proposition 3.1, we will state the following fact which is an analogue of derivative estimates for positive harmonic functions. The proof is straightforward but we delay it to Section 3.4.

**Lemma 3.2.** *For every  $x_1 < y_1 < x_2 < y_2$ , there exists a constant  $c < \infty$  such that the following hold.*

- *Suppose  $D \subset \mathbb{H}$  is a simply connected domain whose boundary contains an open real neighborhood of  $[x_1, y_1]$  and suppose that*

$$\delta := \min\{|x_1 - y_1|, \text{dist}[\{x_1, y_1\}, \partial D \cap \mathbb{H}]\} > 0.$$

*Then if  $z_1, z_2 \in \{x_1, y_1\}$ ,*

$$|\partial_{z_1} H_D(x_1, y_1)| \leq c \delta^{-1} H_D(x_1, y_1).$$

$$|\partial_{z_1 z_2} H_D(x_1, y_1)| \leq c \delta^{-2} H_D(x_1, y_1).$$

- *Suppose  $D \subset \mathbb{H}$  is a simply connected domain whose boundary contains open real neighborhoods of  $[x_1, y_1]$  and  $[x_2, y_2]$  and suppose that*

$$\delta := \min\{\{|w_1 - w_2|; w_1 \neq w_2 \text{ and } w_1, w_2 \in \{x_1, x_2, y_1, y_2\}\}, \text{dist}[\{x_1, y_1, x_2, y_2\}, \partial D \cap \mathbb{H}]\}.$$

*Then if  $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), z_1 \in \{x_1, y_1\}, z_2 \in \{x_2, y_2\}$ ,*

$$|\partial_{z_1 z_2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y})| \leq c \delta^{-2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}).$$

*Moreover, the constant can be chosen uniformly in neighborhoods of  $x_1, y_1, x_2, y_2$ .*

While we allow the constant  $c$  to depend on the boundary points  $x_1, x_2, y_1, y_2$ , the important

thing for us is that it does not depend on  $D$ . We will also need to show that expectations do not blow up when paths get close to starting points. We prove this lemma in Section 3.3. Let

$$\Delta_{j,k}(\gamma) = \text{dist} \left\{ \{x_k, y_k\}, \gamma^j \right\},$$

$$\Delta(\gamma) = \min_{j \neq k} \Delta_{j,k}(\gamma).$$

**Lemma 3.3.** *If  $\kappa < 4$ , then for every  $n$  and every  $(\mathbf{x}, \mathbf{y})$ , there exists  $c < \infty$  such that for all  $\varepsilon > 0$ ,*

$$\mathbb{E}[Y; \Delta \leq \varepsilon] \leq c \varepsilon^{\frac{12}{\kappa}-1}.$$

*In particular,*

$$\mathbb{E}[Y \Delta^{-2}] \leq \sum_{m=-\infty}^{\infty} 2^{-2m} \mathbb{E}[Y; 2^m \leq \Delta < 2^{m+1}] < \infty.$$

*Proof.* It suffices to show that for each  $j \neq k$ ,

$$\mathbb{E}[Y; \Delta_{j,k} \leq \varepsilon] \leq c \varepsilon^{\frac{12}{\kappa}-1},$$

and by symmetry we may assume  $j = 1, k = 2$ . If we write  $\gamma = (\gamma^1, \gamma^2, \gamma')$ , then the event  $\{\Delta_{1,2} \leq \varepsilon\}$  is measurable with respect to  $(\gamma^1, \gamma^2)$  and

$$\mathbb{E}[Y \mid \gamma^1, \gamma^2] \leq Y(\gamma^1, \gamma^2).$$

Hence it suffices to prove the result when  $n = 2$ . This will be done in Section 3.3; in that section we consider  $\kappa < 8$ . □

*Proof of Proposition 1.* For  $n = 1, 2$ , it is clear that  $\tilde{\Psi}$  is  $C^\infty$  from the exact expression, so we will assume that  $n \geq 3$ . By invariance under permutation of indices, it suffices to consider second order derivatives involving only  $x_1, x_2, y_1, y_2$ . We will only consider the configurations of  $x_1, x_2, y_1, y_2$ , for which  $\mu_D(\mathbf{x}, \mathbf{y})$  is not a zero measure (since the result is trivial for other configurations). We will

assume  $x_j < y_j$  for  $j = 1, 2$  and  $x_1 < x_2$  (otherwise we just relabel the vertices). The configuration  $x_1 < x_2 < y_1 < y_2$  is impossible for topological reasons. If  $x_1 < x_2 < y_2 < y_1$ , we can find a Möbius transformation taking a point  $y' \in (y_2, y_1)$  to  $\infty$  and then the images would satisfy  $y'_1 < x'_1 < x'_2 < y'_2$  and this reduces to above. So we may assume that

$$x_1 < y_1 < x_2 < y_2.$$

**Case I:** Derivatives involving only  $x_j, y_j$  for some  $j$ .

We assume  $j = 1$ . We will write  $\mathbf{x} = (x, \mathbf{x}'), y = (y, \mathbf{y}'), \gamma = (\gamma^1, \gamma')$ , and let  $D$  be the connected component of  $\mathbb{H} \setminus \gamma'$  containing  $x, y$  on the boundary. Then

$$\mathbb{E}[Y \mid \gamma'] = Y(\gamma') \left[ \frac{H_D(x, y)}{H(x, y)} \right]^b = Y(\gamma') Q_D(x, y)^b,$$

where  $Q_D(x, y)$  is the probability that a (Brownian) excursion in  $\mathbb{H}$  from  $x$  to  $y$  stays in  $D$ . Hence

$$\tilde{\Psi}(\mathbf{x}, \mathbf{y}) = \mathbb{E} \left[ Y(\gamma') Q_D(x, y)^b \right].$$

Let  $\delta = \delta(\gamma') = \text{dist}\{\{x, y\}, \gamma'\}$ . Using Lemma 3.2, we see that

$$\left| \partial_x [Q_D(x, y)^b] \right| \leq c \delta^{-1} Q_D(x, y)^b \tag{3.5}$$

$$\left| \partial_{xy} [Q_D(x, y)^b] \right| + \left| \partial_{xx} [Q_D(x, y)^b] \right| \leq c \delta^{-2} Q_D(x, y)^b.$$

(Recall that  $c$  may depend on  $x, y$  but not on  $D$ ). Hence

$$\mathbb{E} \left[ Y(\gamma') \left| \partial_x [Q_D(x, y)^b] \right| \right] \leq c \mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-1} Q_D(x, y)^b \right],$$

and if  $z = x$  or  $y$ ,

$$\mathbb{E} \left[ Y(\gamma') \left| \partial_{xz} [Q_D(x,y)^b] \right| \right] \leq c \mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-2} Q_D(x,y)^b \right].$$

Since by Lemma 3.3

$$\mathbb{E} \left[ Y(\gamma') \delta(\gamma')^{-2} Q_D(x,y)^b \right] = \mathbb{E} \left[ \mathbb{E} \left( Y \delta^{-2} \mid \gamma' \right) \right] = \mathbb{E}[Y \delta^{-2}] \leq \mathbb{E}[Y \Delta^{-2}] < \infty,$$

the interchange of expectation and derivative is valid,

$$\partial_x \tilde{\Psi}(\mathbf{x}, y) = \mathbb{E} \left[ Y(\gamma') \partial_x [Q_D(x,y)^b] \right], \quad \partial_{xz} \tilde{\Psi}(\mathbf{x}, y) = \mathbb{E} \left[ Y(\gamma') \partial_{xz} [Q_D(x,y)^b] \right].$$

**Case 2:** The partial  $\partial_{z_1 z_2}$  where  $z_1 \in \{x_j, y_j\}, z_2 \in \{x_k, y_k\}$  with  $j \neq k$ .

We assume  $j = 1, k = 2$ . We will write  $\mathbf{x} = (x_1, x_2, \mathbf{x}')$ ,  $y = (y_1, y_2, \mathbf{y}')$ ,  $\gamma = (\gamma^1, \gamma^2, \gamma')$ . We will write  $D' = D \setminus \gamma'$  and let  $D_1, D_2$  be the connected components of  $D'$  containing  $\{x_1, y_1\}$  and  $\{x_2, y_2\}$  on the boundary. It is possible that  $D_1 = D_2$  or  $D_1 \neq D_2$ .

- If  $D_1 \neq D_2$ , then

$$\mathbb{E}[Y \mid \gamma'] = Y(\gamma') Q_{D_1}(x_1, y_1)^b Q_{D_2}(x_2, y_2)^b.$$

- If  $D_1 = D_2 = D$ , then

$$\mathbb{E}[Y \mid \gamma'] = Y(\gamma') Q_D(x_1, y_1)^b Q_D(x_2, y_2)^b \tilde{\Psi}_D((x_1, x_2), (y_1, y_2)),$$

where  $\tilde{\Psi}_D$  is defined as in (3.4).

In either case we have written

$$\mathbb{E}[Y \mid \gamma'] = Y(\gamma') \Phi(\mathbf{z}; \gamma'),$$

where  $\mathbf{z} = (x_1, y_1, x_2, y_2)$  and we can use equation (3.5) and Lemma 3.2 to see that

$$|\partial_{z_1 z_2} \Phi(\mathbf{z}; \gamma')| \leq c \Delta(\gamma, \mathbf{z})^{-2} \Phi(\mathbf{z}, \gamma'), \quad \Delta(\gamma, \mathbf{z}) = \text{dist}\{\gamma, \{x_1, y_1, x_2, y_2\}\}.$$

As in the previous case, we can now interchange the derivatives and the expectation.  $\square$

### 3.3 Estimate

In this section we will derive an estimate for  $SLE_\kappa$ ,  $\kappa < 8$ . While the estimate is valid for all  $\kappa < 8$ , it is not strong enough to prove our main result for  $\kappa = 4$ . We follow the ideas in [19] where careful analysis was made of the boundary exponent for  $SLE$ . Let  $g_t$  denote the usual conformal transformation associated to the  $SLE_\kappa$  path  $\gamma$  from 0 to  $\infty$  parametrized so that

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \tag{3.6}$$

where  $a = 2/\kappa$  and  $U_t = -W_t$  is a standard Brownian motion. Throughout, we assume that  $\kappa < 8$ , so that  $D = D_\infty = \mathbb{H} \setminus \gamma$  is a nonempty set. If  $0 < x < y < \infty$  and  $\partial D$  contains an open real neighborhood of  $[x, y]$ , we let

$$\Phi = \Phi(x, y) = \frac{H_D(x, y)}{H_{\mathbb{H}}(x, y)},$$

where  $H$  denotes the boundary Poisson kernel. Otherwise (which can only happen for  $4 < \kappa < 8$ ), we define  $\Phi(x, y) = 0$ . As usual, we let

$$b = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}.$$

As a slight abuse of notation, we will write  $\Phi^b$  for  $\Phi^b 1\{\Phi > 0\}$  even if  $b \leq 0$ .

**Proposition 3.2.** *For every  $\kappa < 8$  and  $\delta > 0$ , there exists  $0 < c < \infty$  such that for all  $\delta \leq x < y \leq$*

$1/\delta$  and all  $0 < \varepsilon < (y-x)/10$ ,

$$\mathbb{E} \left[ \Phi^b; \text{dist}(\{x, y\}, \gamma) < \varepsilon \right] \leq c \varepsilon^{6a-1}.$$

It is already known that

$$\mathbb{P} \{ \text{dist}(\{x, y\}, \gamma) < \varepsilon \} \asymp \varepsilon^{4a-1},$$

and hence we can view this as the estimate

$$\mathbb{E} \left[ \Phi^b \mid \text{dist}(\{x, y\}, \gamma) < \varepsilon \right] \leq c \varepsilon^{2a}.$$

Using reversibility [21, 27] and scaling of  $SLE_\kappa$  we can see that to prove the proposition it suffices to show that for every  $\delta > 0$  there exists  $c = c_\delta$  such that if  $\delta \leq x < 1$ ,

$$\mathbb{E} \left[ \Phi^b; \text{dist}(1, \gamma) < \varepsilon \right] \leq c \varepsilon^{6a-1}.$$

This is the result we will prove.

**Proposition 3.3.** *If  $\kappa < 8$ , there exists  $c < \infty$  such that if  $\gamma$  is an  $SLE_\kappa$  curve from 0 to  $\infty$ ,  $0 < x < 1$ ,  $\Phi = \Phi(x, 1)$ ,  $0 < \varepsilon \leq 1/2$ ,*

$$\mathbb{E} \left[ \Phi^b; \text{dist}(\gamma, 1) < \varepsilon(1-x) \right] \leq c x^a (1-x)^{4a-1} \varepsilon^{6a-1}.$$

*Proof.* We will relate the distance to the curve to a conformal radius. In order to do this, we will need 1 to be an interior point of the domain. Let  $D_t^*$  be the unbounded component of

$$K_t = \mathbb{C} \setminus [(-\infty, x] \cup \gamma \cup \{\bar{z} : z \in \gamma\}],$$

and let  $T = T_1 = \inf\{t : 1 \notin D_t^*\}$ . Then for  $t < T$ , the distance from 1 to  $\partial D_t^*$  is the minimum of  $1-x$  and  $\text{dist}(1, \gamma)$ . In particular, if  $t < T$  and  $\varepsilon < 1-x$ , then  $\text{dist}(\gamma, 1) \leq \varepsilon$  if and only if

$\text{dist}(1, \partial D_t^*) < \varepsilon$ . We define  $\Upsilon_t$  to be  $[4(1-x)]^{-1}$  times the conformal radius of 1 with respect to  $D_t^*$  and  $\Upsilon = \Upsilon_\infty$ . Note that  $\Upsilon_0 = 1$ , and if  $\text{dist}(1, \partial D_t^*) \leq \varepsilon(1-x)$ , then  $\Upsilon \leq \varepsilon$ . It suffices for us to show that

$$\mathbb{E} \left[ \Phi^b; \Upsilon < \varepsilon \right] \leq c x^a (1-x)^{4a-1} \varepsilon^{6a-1}.$$

We set up some notation. We fix  $0 < x < 1$  and assume that  $g_t$  satisfies (3.6). Let

$$X_t = g_t(1) - U_t, \quad Z_t = g_t(x) - U_t, \quad Y_t = X_t - Z_t, \quad K_t = \frac{Z_t}{X_t},$$

and note that the scaling rule for conformal radius implies that

$$\Upsilon_t = \frac{Y_t}{(1-x)g_t'(1)}.$$

The Loewner equation implies that for some Brownian motion  $W_t$ ,

$$\begin{aligned} dX_t &= \frac{a}{X_t} dt + dW_t, & dZ_t &= \frac{a}{Z_t} dt + dW_t, \\ \partial_t g_t'(1) &= -\frac{a g_t'(1)}{X_t^2}, & \partial_t g_t'(x) &= -\frac{a g_t'(x)}{Z_t^2}, & \partial_t Y_t &= -\frac{a Y_t}{X_t Z_t}. \\ \partial_t \Upsilon_t &= \Upsilon_t \left[ \frac{a}{X_t^2} - \frac{a}{X_t Z_t} \right] = -a \Upsilon_t \frac{1}{X_t^2} \frac{1-K_t}{K_t}. \end{aligned}$$

Let  $D_t$  be the unbounded component of  $\mathbb{H} \setminus \gamma_t$  and let

$$\Phi_t = \frac{H_{D_t}(x, 1)}{H_{D_0}(x, 1)} = (1-x)^2 \frac{g_t'(x) g_t'(1)}{Y_t^2},$$

where we set  $\Phi_t = 0$  if  $x$  is not on the boundary of  $D_t$ , that is, if  $x$  has been swallowed by the path (this is relevant only for  $4 < \kappa < 8$ ). Note that  $\Phi = \Phi_\infty$  and

$$\partial_t \Phi_t^b = \Phi_t^b \left[ -\frac{ab}{X_t^2} - \frac{ab}{Z_t^2} + \frac{2ab}{X_t Z_t} \right] = -ab \frac{\Phi_t^b}{X_t^2} \left( \frac{1-K_t}{K_t} \right)^2,$$

$$\Phi_t^b = \exp \left\{ -ab \int_0^t \frac{1}{X_s^2} \left( \frac{1-K_s}{K_s} \right)^2 ds \right\}.$$

Itô's formula implies that

$$d\frac{1}{X_t} = -\frac{1}{X_t^2} dX_t + \frac{1}{X_t^3} d\langle X \rangle_t = \frac{1}{X_t} \left[ \frac{1-a}{X_t^2} dt - \frac{1}{X_t} dW_t \right],$$

and the product rule gives

$$d[1-K_t] = [1-K_t] \left[ \frac{1-a}{X_t^2} dt - \frac{a}{X_t Z_t} dt - \frac{1}{X_t} dW_t \right] = \frac{1-K_t}{X_t^2} \left[ (1-a) - \frac{a}{K_t} \right] dt - \frac{1-K_t}{X_t} dW_t.$$

which can be written as

$$dK_t = \frac{1-K_t}{X_t^2} \left[ \frac{a}{K_t} + a - 1 \right] dt + \frac{1-K_t}{X_t} dW_t.$$

As in [19], we consider the local martingale

$$M_t^* = (1-x)^{1-4a} X_t^{1-4a} g_t'(1)^{4a-1} = (1-x)^{1-4a} (1-K_t)^{4a-1} \Upsilon_t^{1-4a},$$

which satisfies

$$dM_t^* = \frac{1-4a}{X_t} M_t^* dW_t, \quad M_0^* = 1$$

If we use the Girsanov's theorem and tilt by the local martingale, we see that

$$dK_t = \frac{1-K_t}{X_t^2} \left[ \frac{a}{K_t} - 3a \right] dt + \frac{1-K_t}{X_t} dW_t^*.$$

where  $W_t^*$  is a standard Brownian motion in the new measure  $\mathbb{P}^*$ . We reparametrize so that  $\log \Upsilon_t$  decays linearly. More precisely, we let  $\sigma(t) = \inf\{t : \Upsilon_t = e^{-at}\}$  and define  $\hat{X}_t = X_{\sigma(t)}$ ,  $\hat{Y}_t = Y_{\sigma(t)}$ ,

etc. Since  $\hat{Y}_t := \Upsilon_{\sigma(t)} = e^{-at}$ , and

$$-a\hat{Y}_t = \partial_t \hat{Y}_t = -a\hat{Y}_t \frac{1}{\hat{X}_t^2} \frac{1-\hat{K}_t}{\hat{K}_t} \dot{\sigma}(t),$$

we see that

$$\dot{\sigma}(t) = \frac{\hat{X}_t^2 \hat{K}_t}{1-\hat{K}_t},$$

Therefore,

$$\hat{\Phi}_t^b := \Phi_{\sigma(t)}^b = \exp \left\{ -ab \int_0^t \frac{1-\hat{K}_s}{\hat{K}_s} ds \right\} = e^{abt} \exp \left\{ -ab \int_0^t \frac{1}{\hat{K}_s} ds \right\},$$

$$\begin{aligned} d\hat{K}_t &= [a - 3a\hat{K}_t] dt + \sqrt{\hat{K}_t(1-\hat{K}_t)} dB_t^* \\ &= \hat{K}_t \left[ \left( \frac{a}{\hat{K}_t} - 3a \right) dt + \sqrt{\frac{1-\hat{K}_t}{\hat{K}_t}} dB_t^* \right]. \end{aligned}$$

for a standard Brownian motion  $B_t^*$  (in the measure  $\mathbb{P}^*$ ).

Let  $\lambda = 2a^2$ , and

$$N_t = e^{\lambda t} \hat{\Phi}_t^b \hat{K}_t^a = \exp \left\{ \frac{a(7a-1)}{2} t \right\} \exp \left\{ -\frac{a(3a-1)}{2} \int_0^t \frac{1}{\hat{K}_s} ds \right\} \hat{K}_t^a.$$

Itô's formula shows that  $N_t$  is a local  $\mathbb{P}^*$ -martingale satisfying

$$dN_t = N_t a \sqrt{\frac{1-\hat{K}_t}{\hat{K}_t}} dB_t^*, \quad N_0 = x^a$$

One can show it is a martingale by using the Girsanov's theorem to see that

$$d\hat{K}_t = [2a - 4a\hat{K}_t] dt + \sqrt{\hat{K}_t(1-\hat{K}_t)} d\tilde{B}_t,$$

where  $\tilde{B}_t$  is a Brownian motion in the new measure  $\tilde{\mathbb{P}}$ . By comparison with a Bessel process, we

see that the solution exists for all time. Equivalently, we can say that

$$\hat{M}_t := \hat{M}_t^* N_t,$$

is a  $\mathbb{P}$ -martingale with  $\hat{M}_0 = x^a$ . (Although  $M_t^*$  is only a local martingale, the time-changed version  $\hat{M}_t^* := M_{\sigma(t)}^*$  is a martingale.)

Using (3.3) we see that  $\mathbb{E} \left[ \Phi_\infty^b \mid \gamma_{\sigma(t)} \right] \leq c \hat{K}_t^a \hat{\Phi}_t^b$ . If  $\varepsilon = e^{-at}$ , then

$$\begin{aligned} \mathbb{E} \left[ \Phi^b; \sigma(t) < \infty \right] &= c \mathbb{E} \left[ \mathbb{E}(\Phi^b \mathbf{1}\{\sigma(t) < \infty\} \mid \gamma_{\sigma(t)}) \right] \\ &\leq c \mathbb{E} \left[ \hat{K}_t^a \hat{\Phi}_t^b; \sigma(t) < \infty \right] \\ &= c e^{-\lambda t} e^{(1-4a)at} (1-x)^{4a-1} \hat{M}_0^{-1} \mathbb{E} \left[ \hat{M}_t (1 - \hat{K}_t)^{1-4a}; \sigma(t) < \infty \right] \\ &= c e^{a(1-6a)t} x^a (1-x)^{4a-1} \tilde{\mathbb{E}} \left[ (1 - \hat{K}_t)^{1-4a} \right] \\ &= c \varepsilon^{6a-1} x^a (1-x)^{4a-1} \tilde{\mathbb{E}} \left[ (1 - \hat{K}_t)^{1-4a} \right]. \end{aligned}$$

So the result follows once we show that

$$\tilde{\mathbb{E}} \left[ (1 - \hat{K}_t)^{1-4a} \right] < \infty$$

is uniformly bounded for  $t \geq t_0$ . The argument for this proceeds as in [19]. If we do the change of variables  $\hat{K}_t = [1 - \cos \Theta_t]/2$ , then Itô's formula shows that

$$d\Theta_t = \left( 4a - \frac{1}{2} \right) \cot \Theta_t dt + dB_t.$$

This is a radial Bessel process that never reaches the boundary. It is known that

$$\Phi(\theta) = c_{8a-1} \sin^{8a-1} \theta, \quad c_{8a-1} = \left[ \int_0^\pi \sin^{8a-1} \theta d\theta \right]^{-1}$$

is the invariant density of  $\Theta_t$ . If  $p_t(\theta_0, \theta)$  denotes the transition density of the process  $\Theta_t$ , then it is also known that for any  $t_0 > 0$  there exist constants  $0 < c_1(t_0) \leq c_2(t_0) < \infty$  such that for all  $0 < \theta_0, \theta < \pi$  and  $t_0 \leq t$ ,

$$c_1(t_0)\Phi(\theta) \leq p_t(\theta_0, \theta) \leq c_2(t_0)\Phi(\theta).$$

In particular, there exist  $0 < \beta(t_0), c(t_0) < \infty$  such that

$$\Phi(\theta)[1 - c(t_0)e^{-\beta(t_0)t}] \leq f_t(\theta_0, \theta) \leq \Phi(\theta)[1 + c(t_0)e^{-\beta(t_0)t}].$$

See section 4 of [19] for more details. Therefore,  $\phi(x) = c_{8a-1}2^{1-8a}x^{4a-1}(1-x)^{4a-1}$  is the invariant density of  $\hat{K}_t$  and if  $q_t(x_0, x)$  denotes the transition density of  $\hat{K}_t$ , then for any  $t > t_0$ ,

$$c_1(t_0)\phi(x) \leq q_t(x_0, x) \leq c_2(t_0)\phi(x).$$

In particular,  $(1 - \hat{K}_t)^{1-4a}$  is integrable. □

### 3.4 Proof of Lemma 3.2

We prove the first part of Lemma 3.2 for  $x_1 = 0, y_1 = 1$ . Other cases follow from this and a Möbius transformation sending  $x_1, y_1$  to  $0, 1$ .

**Lemma 3.4.** *There exists  $c < \infty$  such that if  $D$  is a simply connected subdomain of  $\mathbb{H}$  containing  $0, 1$  on its boundary, then*

$$|\partial_x H_D(0, 1)| + |\partial_y H_D(0, 1)| \leq c \delta^{-1} H_D(0, 1),$$

$$|\partial_{xx} H_D(0, 1)| + |\partial_{xy} H_D(0, 1)| + |\partial_{yy} H_D(0, 1)| \leq c \delta^{-2} H_D(0, 1),$$

where  $\delta = \min\{1, \text{dist}(\{0, 1\}, \partial D \cap \mathbb{H})\}$ .

*Proof.* Let  $g : D \rightarrow \mathbb{H}$  be a conformal transformation with  $g(0) = 0, g(1) = 1, g'(0) = 1$ . Then if  $|x| < \delta, |y - 1| < \delta$ ,

$$H_D(x, y) = \frac{g'(x)g'(y)}{[g(y) - g(x)]^2}. \quad (3.7)$$

In particular  $g'(0)g'(1) = H_D(0, 1) \leq H_{\mathbb{H}}(0, 1) = 1$  and hence  $g'(1) \leq 1$ . Using Schwartz reflection we can extend  $g$  to be a conformal transformations of disks of radius  $\delta$  about 0 and 1. By the distortion estimates (the fact that  $|a_2| \leq 2, |a_3| \leq 3$  for schlicht functions) we have

$$|g''(0)| \leq 4\delta^{-1}g'(0) \leq 4\delta^{-1}, \quad |g'''(0)| \leq 18\delta^{-2}g'(0) \leq 18\delta^{-2},$$

and similarly  $|g''(1)| \leq 4\delta^{-1}g'(1)$  and  $|g'''(1)| \leq 18\delta^{-2}g'(1)$ . By direct differentiation of the right-hand side of (3.7) we get the result.  $\square$

**Lemma 3.5.** *For any*

$$x_1 < y_1 \leq 0 < 1 \leq x_2 < y_2, \quad \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2),$$

*there exists  $c < \infty$  such that if  $\tilde{\Psi}_D(\mathbf{x}, \mathbf{y})$  is as in (3.4) and  $z_1 \in \{x_1, y_1\}, z_2 \in \{x_2, y_2\}$ , then*

$$|\partial_{z_1} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y})| + |\partial_{z_2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y})| \leq c\delta^{-1} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}),$$

$$|\partial_{z_1 z_2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y})| \leq c\delta^{-2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}),$$

*where*

$$\delta := \min \{ \{|w_1 - w_2|; w_1 \neq w_2 \text{ and } w_1, w_2 \in \{x_1, x_2, y_1, y_2\}\}, \text{dist}[\{x_1, y_1, x_2, y_2\}, \partial D \cap \mathbb{H}] \}.$$

*Proof.* According to equation (3.4),

$$\tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi(u_D(\mathbf{x}, \mathbf{y})),$$

where

$$u_D(\mathbf{x}, \mathbf{y}) = e^{-\mathcal{E}_D(\mathbf{x}, \mathbf{y})}, \quad \mathcal{E}_D(\mathbf{x}, \mathbf{y}) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, s) dr ds.$$

Using the Harnack inequality we can see that for  $j = 1, 2$ ,

$$H_D(x, s) \asymp H_D(x_j, s), \quad H_D(r, y) \asymp H_D(r, y_j)$$

if  $|x - x_j| \leq \delta/2, |y - y_j| \leq \delta/2$ . Therefore,

$$c\delta \int_{x_2}^{y_2} H_D(x_1, s) ds \leq \int_{x_2}^{y_2} \int_{x_1}^{x_1 + \delta/2} H_D(r, s) dr ds \leq \mathcal{E}_D(\mathbf{x}, \mathbf{y}),$$

$$c\delta H_D(x_1, x_2) \leq \int_{x_2}^{x_2 + \delta/2} H_D(x_1, s) ds \leq \int_{x_2}^{y_2} H_D(x_1, s) ds.$$

More generally, if we let  $z_1$  be  $x_1$  or  $y_1$  and let  $z_2$  be  $x_2$  or  $y_2$ , then we see that

$$\int_{x_2}^{y_2} H_D(z_1, s) ds + \int_{x_1}^{y_1} H_D(r, z_2) dr \leq c\delta^{-1} \mathcal{E}_D(\mathbf{x}, \mathbf{y}),$$

$$H_D(z_1, z_2) \leq c\delta^{-2} \mathcal{E}_D(\mathbf{x}, \mathbf{y}).$$

Hence,

$$\partial_{z_1} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi'(u_D(\mathbf{x}, \mathbf{y})) \partial_{z_1} u_D(\mathbf{x}, \mathbf{y}),$$

$$\partial_{z_1 z_2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi''(u_D(\mathbf{x}, \mathbf{y})) [\partial_{z_1} u_D(\mathbf{x}, \mathbf{y})] [\partial_{z_2} u_D(\mathbf{x}, \mathbf{y})] + \phi'(u_D(\mathbf{x}, \mathbf{y})) \partial_{z_1 z_2} u_D(\mathbf{x}, \mathbf{y}),$$

$$\partial_{z_1} u_D(\mathbf{x}, \mathbf{y}) = \left[ \pm \int_{x_2}^{y_2} H_D(z_1, s) ds \right] u_D(\mathbf{x}, \mathbf{y}),$$

$$\partial_{z_2} u_D(\mathbf{x}, \mathbf{y}) = \left[ \pm \int_{x_1}^{y_1} H_D(r, z_2) ds \right] u_D(\mathbf{x}, \mathbf{y}),$$

$$\partial_{z_2 z_1} u_D(\mathbf{x}, \mathbf{y}) = \left[ \pm \int_{x_1}^{y_1} \int_{x_2}^{y_2} H_D(r, z_2) dr H_D(z_1, s) ds \pm H_D(z_1, z_2) \right] u_D(\mathbf{x}, \mathbf{y}).$$

This gives

$$|\partial_{z_1} u_D(\mathbf{x}, \mathbf{y})| + |\partial_{z_2} u_D(\mathbf{x}, \mathbf{y})| \leq c\delta^{-1} \mathcal{E}_D(\mathbf{x}, \mathbf{y}) u_D(\mathbf{x}, \mathbf{y}),$$

$$|\partial_{z_2 z_1} u_D(\mathbf{x}, \mathbf{y})| \leq c \delta^{-2} [\mathcal{E}_D(\mathbf{x}, \mathbf{y}) + \mathcal{E}_D(\mathbf{x}, \mathbf{y})^2] u_D(\mathbf{x}, \mathbf{y}).$$

Note that

$$\partial_{z_i} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi'(u_D(\mathbf{x}, \mathbf{y})) \partial_{z_i} u_D(\mathbf{x}, \mathbf{y}),$$

$$\partial_{z_1, z_2} \tilde{\Psi}_D(\mathbf{x}, \mathbf{y}) = \phi'(u_D(\mathbf{x}, \mathbf{y})) \partial_{z_1 z_2} u_D(\mathbf{x}, \mathbf{y}) + \phi''(u_D(\mathbf{x}, \mathbf{y})) \partial_{z_1} u_D(\mathbf{x}, \mathbf{y}) \partial_{z_2} u_D(\mathbf{x}, \mathbf{y}).$$

The result will follow if we show that for all  $x_0 > 0$

$$\frac{x e^{-x} \phi'(e^{-x})}{\phi(e^{-x})}, \quad \frac{x^2 e^{-2x} \phi''(e^{-x})}{\phi(x)},$$

are uniformly bounded for  $0 < x \leq x_0$ . Recall that  $\phi(x) = cx^a F(x)$ , where  $F(x) = {}_2F_1(2a, 1 - 2a, 4a; x)$ . We also have

$$\frac{\phi'(x)}{\phi(x)} = \left[ \frac{a}{x} + \frac{F'(x)}{F(x)} \right],$$

$$\frac{\phi''(x)}{\phi(x)} = \left[ \left( \frac{a}{x} + \frac{F'(x)}{F(x)} \right)^2 - \frac{a}{x^2} + \frac{F''(x)}{F(x)} - \frac{F'(x)^2}{F(x)^2} \right].$$

Since  $F(x)$  is analytic in the unit disk, it suffices to show that

$$\frac{x F'(e^{-x})}{F(e^{-x})}, \quad \frac{x^2 F''(e^{-x})}{F(e^{-x})}$$

are uniformly bounded as  $x \downarrow 0$ . The hypergeometric function  $F$  has the power series expansion

$$F(x) = 1 + \sum_{n=1}^{\infty} b_n x^n, \quad b_n = C n^{-4a} [1 + O(n^{-1})],$$

in the unit disk. If  $\kappa < 8$ , then  $4a > 1$  and as  $x \downarrow 0$ ,

$$F(e^{-x}) = O(1), \quad x F'(e^{-x}) = o(1), \quad x^2 F''(e^{-x}) = o(1).$$

That is because for some constants  $c_1, c_2$ ,

$$\begin{aligned}x|F'(e^{-x})| &\leq c_1 x \sum_{n=1}^{\infty} n^{-4a+1} e^{-nx} \\ &\leq c_2 x \int_1^{\infty} y^{-4a+1} e^{-xy} dy \\ &= c_2 x^{4a-1} \int_x^{\infty} t^{-4a+1} e^{-t} dt = o(1)\end{aligned}$$

and

$$\begin{aligned}x^2|F''(e^{-x})| &\leq c_1 x^2 \sum_{n=1}^{\infty} n^{-4a+2} e^{-nx} \\ &\leq c_2 x^2 \int_1^{\infty} y^{-4a+2} e^{-xy} dy \\ &= c_2 x^{4a-1} \int_x^{\infty} t^{-4a+2} e^{-t} dt = o(1).\end{aligned}$$

□

## CHAPTER 4

### $SLE_\kappa$ IN MULTIPLY CONNECTED DOMAINS

In this chapter, we review  $SLE_\kappa$  in multiply connected domains for  $\kappa \leq 4$  [17, 15, 28, 25, 1]. Contents of this chapter are used in the next chapter to discuss multiple-path  $SLE_\kappa$  in multiply connected domains. We will closely follow [17, 15], whose approach is to use the Brownian loop measure to define  $SLE_\kappa$ . In particular, some of the results in this chapter are directly taken from [17].

#### 4.1 Definition

Suppose  $D$  is a domain and  $z, w \in \partial D$  are analytic boundary points. The  $d$ -dimensional Minkowski content of the curve is defined by

$$\text{Cont}_d(\gamma) = \lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} \text{Area} \{z; \text{dist}(z, \gamma) < \varepsilon\}.$$

We say  $\gamma_t$  has *natural  $d$ -parametrization* if  $\text{Cont}_d(\gamma_t) = t$  for all  $t \leq t_\gamma$ . Let  $d = 1 + \kappa/8$ . In this section, we construct  $SLE_\kappa$  measures  $\mu_D(z, w)$  as a positive measure on continuous curves  $\gamma(0, t_\gamma) \subset D$  with natural  $d$ -parametrization and  $\gamma(0+) = z$ ,  $\gamma(t_\gamma-) = w$ ,  $t_\gamma < \infty$ . As before, if  $\Psi_D(z, w) = \|\mu_D(z, w)\| < \infty$ , we use  $\mu_D^\#(z, w) = \mu_D(z, w)/\Psi_D(z, w)$  to denote the probability measure.

Suppose  $D' \subset D$  is a simply connected domain that agrees with  $D$  in neighborhoods of  $z, w$ . Define the measure  $\mu_D(z, w; D')$  with the Radon-Nikodym derivative

$$\frac{d\mu_D(z, w; D')}{d\mu_D(z, w)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D') \right\}. \quad (4.1)$$

**Definition 4.1.** Define  $\mu_D(z, w)$  to be the measure on continuous curves  $\gamma$  connecting  $z, w$  in  $D$  such that for every simply connected domain  $D' \subset D$ ,  $\mu_D(z, w)$  restricted to the curves  $\gamma \subset D'$  is  $\mu_D(z, w; D')$ .

Before showing that  $\mu_D(z, w)$  has the properties we expect from a  $SLE_\kappa$  measure, we should verify that this definition is consistent. That is, if  $D_1, D_2 \subset D$  are simply connected, then restriction of  $\mu_D(z, w; D_1)$  to the curves in  $D_1 \cap D_2$  is the same as restriction of  $\mu_D(z, w; D_2)$  to the curves in  $D_1 \cap D_2$ .

**Proposition 4.1.** *Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic boundary points. Let  $D_1, D_2$  be simply connected subdomains of  $D$  that agree with  $D$  in neighborhoods of  $z, w$ . For  $j = 1, 2$ , let  $\nu_j$  be  $\mu_D(z, w; D_j)$  restricted to curves  $\gamma \subset D_1 \cap D_2$ . Then  $\nu_1 = \nu_2$ .*

*Proof.* Suppose  $\gamma \subset D_1 \cap D_2$ . Then there exists simply connected  $\hat{D} \subset D_1 \cap D_2$  that agrees locally with  $D$  near  $z, w$  such that  $\gamma \subset \hat{D}$ . Hence it suffices to show that for every simply connected domain  $\hat{D}$ ,  $\nu_1$  and  $\nu_2$ , restricted to curves in  $\hat{D}$ , agree. Suppose  $\gamma \subset \hat{D}$ . Since  $D_j, \hat{D}$  are simply connected,

$$\frac{d\mu_{D_j}(z, w)}{d\mu_{\hat{D}}(z, w)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{D_j}(\gamma, D_j \setminus \hat{D}) \right\}.$$

Combining this with (4.1), we get

$$\frac{d\mu_D(z, w; D_j)}{d\mu_{\hat{D}}(z, w)}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_D(\gamma, D \setminus \hat{D}) \right\}.$$

Here we use the fact that the loops in  $D$  that intersect  $\gamma$  and  $D \setminus \hat{D}$  can be partitioned into two sets: those that intersect  $D \setminus D_1$  and those that are contained in  $D_1$ . □

Now we can verify that  $\mu_D(z, w)$  has the properties that we are expecting.

- **Conformal Invariance** Suppose  $D$  is a domain and  $z, w$  are distinct  $\partial D$ -analytic points and  $D_1 \subset D$  is a simply connected domain that agrees with  $D$  in neighborhoods of  $z, w$ . Suppose  $f : D \rightarrow f(D)$  is a conformal transformation. Then  $f : D_1 \rightarrow f(D_1)$  is also a conformal transformation, and hence

$$f \circ \mu_{D_1}(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D_1)}(f(z), f(w)).$$

Conformal invariance of the loop measure then implies that

$$f \circ \mu_D(z, w; D_1) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(z, w; f(D_1)).$$

Since this is true for every simply connected  $D_1$ , the family  $\{\mu_D(z, w)\}$  satisfies

$$\mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w))$$

Moreover, if  $\Psi_D(z, w) < \infty$ , then

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

- **Boundary Perturbation** Suppose  $D_1 \subset D_2$  are domains that agree in neighborhoods of analytic boundary points  $z, w$ . Let  $D \subset D_1$  be a simply connected domain. It follows from (4.1) that

$$\frac{d\mu_{D_1}(z, w; D)}{d\mu_{D_2}(z, w; D)}(\gamma) = \left\{ \frac{\mathbf{c}}{2} m_{D_2}(\gamma, D_2 \setminus D_1) \right\}.$$

Since this is true for all  $D$ , then

$$\frac{d\mu_{D_1}(z, w)}{d\mu_{D_2}(z, w)}(\gamma) = \left\{ \frac{\mathbf{c}}{2} m_{D_2}(\gamma, D_2 \setminus D_1) \right\}. \quad (4.2)$$

- **Finiteness of Partition Function** If  $z, w$  are on the same connected component of  $D$ , then we can find a simply connected  $D'$  such that  $D \subset D'$ ,  $z, w \in \partial D'$ . If  $\kappa \leq 8/3$ , then  $\mathbf{c} \leq 0$  and (4.2) gives

$$\Psi_D(z, w) \leq \Psi_{D'}(z, w) < \infty.$$

If  $z, w$  are on different connected components, then we can find a doubly connected domain  $D'$  with  $D \subset D'$ ,  $z, w \in \partial D'$ . We will later show that for doubly connected domains

$\Psi_{D'}(z, w) < \infty$ . Hence for  $\kappa \leq 8/3$ ,

$$\Psi_D(z, w) \leq \Psi_{D'}(z, w) < \infty.$$

It is still an open problem to show that  $\Psi_D(z, w) < \infty$  for  $8/3 < \kappa \leq 4$ . However, if  $D_1 \subset D$  is a simply connected domain, we can see from (4.1) that  $\|\mu_D(z, w; D_1)\| < \infty$  for  $8/3 < \kappa \leq 4$ .

- **Domain Markov Property** Let  $D_1 \subset D$  be a simply connect domain that agrees with  $D$  in neighborhoods of  $z, w$ . Recall that  $\Psi_D(z, w; D_1) = \|\mu_D(z, w; D_1)\| < \infty$ . Let  $t$  be a finite stopping time and denote by  $\mathcal{F}_t$  the sigma-algebra generated by  $\gamma_t$ . For  $\gamma \subset D_1$ , let

$$Y(\gamma) = \frac{\mu_D(z, w; D_1)}{\mu_{D_1}(z, w)}(\gamma) = \exp\left\{\frac{\mathbf{c}}{2} m_D(\gamma, D \setminus D_1)\right\}.$$

Let  $\mathbb{P}, \mathbb{E}$  denote probability and expectation with respect to the probability measure  $\mu_{D_1}^\#(z, w)$ .

Then,

$$\Psi_D(z, w; D_1) = \Psi_{D_1}(z, w) \mathbb{E}[Y].$$

Let  $\tau = \inf\{t; \gamma(t) = w\}$ . By the domain Markov property for  $SLE_\kappa$  in simply connected domains,

$$\mathbb{E}[Y | \mathcal{F}_t] = \exp\left\{\frac{\mathbf{c}}{2} m_D(\gamma_t, D \setminus D_1)\right\} \mathbb{E}_t^* \left[ m_{D \setminus \gamma_t}(\gamma(t, \tau), D \setminus \{D_1 \cup \gamma_t\}) \right],$$

where  $\mathbb{E}_t^*$  denotes expectation with respect to  $\mu_{D_1 \setminus \gamma_t}^\#(\gamma(t), w)$ . In other words, conditioned on  $\gamma_t$ , distribution of the remainder of the curve with respect to  $\mu_{D_1}^\#(z, w; D_1)$  is that of  $\mu_{D \setminus \gamma_t}^\#(\gamma(t), w; D_1 \setminus \gamma_1)$ .

Using this we can see that if  $\Psi_D(z, w) < \infty$ , then conditioned on  $\gamma_t$ , distribution of the remainder of the curve with respect to  $\mu_D^\#(z, w)$  is that of  $\mu_{D \setminus \gamma_t}^\#(\gamma(t), w)$ .

- **Reversibility** It follows from reversibility of chordal  $SLE_\kappa$  in simply connected domains (proved in [27]) and (4.1) that annulus  $SLE_\kappa$  is reversible.

When discussing  $SLE_\kappa$  in annuli, it is often convenient to consider annulus parametrization. That is, a time parametrization for which  $r(t) = r - t$  (recall the definition of  $r(t)$  in section 2.1).

## 4.2 Loewner equation

Let  $\gamma_t \subset A_r$ ,  $\gamma(0+) = \bar{u}$  be a simple curve with annulus parametrization. Let  $\bar{h}_t : A_r \setminus \gamma_t \rightarrow A_{r-t}$  and  $h_t : S_{r,t} \rightarrow S_{r-t}$  be as in section 2.1. Using conformal invariance, one can see that for  $z \in S_r$ ,

$$H_{S_r}(z, 0) = -\frac{\pi}{2r} \operatorname{Im} \coth \left( \frac{\pi z}{2r} \right).$$

For  $z_0 \in \mathbb{R}$  define

$$\bar{H}_{S_r}(z, 0) := H_{A_r}(e^{iz}, 1) = \sum_{k \in \mathbb{Z}} H_{S_r}(z, 2k\pi),$$

$$\mathcal{H}_{S_r}(z, 0) := -\frac{\pi}{2r} \coth \left( \frac{\pi z}{2r} \right),$$

$$\bar{\mathcal{H}}_{S_r}(z, 0) := \sum_{PP} \mathcal{H}_{S_r}(z, 2k\pi) = \mathcal{H}_{S_r}(z, 0) + \sum_{k=1}^{\infty} [\mathcal{H}_{S_r}(z, 2k\pi) + \mathcal{H}_{S_r}(z, -2k\pi)],$$

$$\mathcal{H}_{S_r}(z, z_0) := \mathcal{H}_{S_r}(z - z_0, 0), \quad \bar{\mathcal{H}}_{S_r}(z, z_0) := \bar{\mathcal{H}}_{S_r}(z - z_0, 0), \quad \bar{H}_{S_r}(z, z_0) := \bar{H}_{S_r}(z - z_0, 0).$$

We had to be a little careful with the definition of  $\bar{\mathcal{H}}_{S_r}(z, 0)$  because the real parts are not absolutely convergent. While  $\bar{H}_{S_r}(z, 0)$  is a  $2\pi$ -periodic function, it is not hard to see that

$$\bar{\mathcal{H}}_{S_r}(z + 2\pi, 0) = \bar{\mathcal{H}}_{S_r}(z, 0) + \frac{\pi}{r}.$$

**Lemma 4.1.** *Suppose  $D_t \subset S_r$  is a half disk of radius  $d_t$  centered at the origin. If  $x \in \bar{S}_r$ ,  $x \neq 0$ ,  $\theta \in (0, \pi)$ , then*

$$H_{S_r \setminus D_t}(x, d_t e^{i\theta}) = 2H_{S_r}(x, 0) \sin \theta [1 + O(d_t)],$$

where the error term is independent of  $\theta$ .

*Proof.* We prove the lemma for the case  $x \in \mathbb{R}$ . The case  $x \in S_r$  can be proved in a similar way.

Define  $f_t : \mathbb{H} \setminus D_t \rightarrow \mathbb{H}$  with  $f_t(z) = z + d_t^2/z$ . Then

$$\begin{aligned} H_{S_r \setminus D_t}(x, d_t e^{i\theta}) &= |f'(x)| |f'(d_t e^{i\theta})| H_{f_t(S_r \setminus D_t)}(f_t(x), 2d_t \cos \theta) \\ &= 2 \sin \theta H_{f_t(S_r \setminus D_t)}(f_t(x), 2d_t \cos \theta) [1 + O(d_t^2)]. \end{aligned}$$

Note that  $S_{r-d_t^2/r} \subset f_t(S_r \setminus D_t) \subset S_r$ . Therefore,

$$H_{f_t(S_r \setminus D_t)}(f_t(x), 2d_t \cos \theta) = H_{S_r}(f_t(x), 2d_t \cos \theta) [1 + O(d_t^2)] = H_{S_r}(x, 0) [1 + O(d_t)].$$

□

**Lemma 4.2.** *Let  $T$  be the first time a Brownian motion  $B$  exits  $S_{r,t}$ . Then for any  $z \in S_{r,t}$ ,*

$$\mathbb{E}^z[\text{Im}[B_T] 1\{B_T \in \tilde{\eta}_t\}] = \text{hcap}[\eta_t] \bar{H}_{S_r}(z, u) [1 + O(d_t)],$$

as  $t \rightarrow 0$ . Moreover, for any  $\varepsilon > 0$ , the error term is uniform on  $\{z; \forall k \in \mathbb{Z}, |z - 2k\pi| > \varepsilon\}$ .

*Proof.* Without the loss of generality we assume  $u = 0$ . Define  $d_t = 10 \text{diam}[\eta_t]$  and let  $C_t \subset S_r$  be a half circle of radius  $d_t$  centered at the origin. Let  $\tau$  be the first time the Brownian motion exits  $S_r \setminus \eta_t$ . Define the function  $f$  on  $C_t$  with

$$f(w) = \mathbb{E}^w[\text{Im}[B_\tau] 1\{B_\tau \in \eta_t\}].$$

Let  $D_t$  denote the unbounded connected component of  $S_r \setminus C_t$ . Then

$$\mathbb{E}^z[\text{Im}(B_\tau) 1\{B_\tau \in \eta_t\}] = \int_{C_t} H_{D_t}(z, w) f(w) |dw|. \quad (4.3)$$

Using lemma 4.1, for  $w \in C_t$  we have

$$H_{D_t}(z, w) = 2 \sin \theta_w H_{S_r}(z, 0) [1 + O(d_t)],$$

where  $\theta_w = \arg w$ . Let  $\sigma$  be the first time the Brownian motion exits  $\mathbb{H} \setminus \eta_t$  and define  $\tilde{f}(w) = \mathbb{E}^w[\text{Im}[B_\sigma]]$ . Note that

$$\tilde{f}(w) - f(w) = \mathbb{E}^w[\text{Im}[B_\sigma] \mathbf{1}\{\tau < \sigma\}].$$

Since  $w \in C_t$ ,

$$\mathbb{E}^w[\text{Im}[B_\sigma] \mathbf{1}\{\tau < \sigma\}] = \text{hcap}[\eta_t] \mathcal{O}(d_t).$$

Therefore, (4.3) implies that

$$\mathbb{E}^z[\text{Im}[B_\tau] \mathbf{1}\{B_\tau \in \eta_t\}] = H_{S_r}(z, 0) \text{hcap}[\eta_t] [1 + \mathcal{O}(d_t)].$$

Since this is true for any  $z$ , we have

$$\begin{aligned} \mathbb{E}^z[\text{Im}[B_T] \mathbf{1}\{B_\tau \in \tilde{\eta}_t\}] &= \text{hcap}[\eta_t] [1 + \mathcal{O}(d_t)] \sum_{k \in \mathbb{Z}} H_{S_r}(z, 2k\pi) \\ &\quad + \mathcal{O}(\text{hcap}[\eta_t]^2) \sum_{k \in \mathbb{Z}} \sum_{k' \neq k} H_{S_r}(z, 2k\pi) H_{S_r}(2k\pi, 2k'\pi). \end{aligned}$$

Since the double sum is finite for any  $z \in S_r$ , the result follows.  $\square$

**Lemma 4.3.** *Suppose  $\gamma_t$  has annulus parametrization. If  $\bar{U}_t = \bar{h}_t(\gamma(t))$ ,  $U_t = h_t(\eta(t))$ , then for any  $z \in S_{r,t}$*

$$\partial_t \text{Im}[h_t(z)] = -\frac{\text{Im}[h_t(z)]}{r-t} + 2\bar{H}_{S_{r-t}}(h_t(z), U_t).$$

*Moreover, choose  $w \in S_r$  such that  $w \notin \tilde{\eta}$ . If for all  $t < r$ ,  $\text{Re}[h_t(w)]$  is differentiable with respect to  $t$ , then*

$$\partial_t h_t(z) = -\frac{h_t(z)}{r-t} + 2\bar{\mathcal{H}}_{S_{r-t}}(h_t(z), U_t) + \beta_t,$$

*for some  $\beta_t$  independent of  $z$ .*

*Proof.* It suffices to prove this for  $t = 0$ . The function

$$I_t(z) = \text{Im}[z - h_t(z)]$$

is a bounded harmonic function on  $S_{r,t}$ . Considering the values of  $I_t(z)$  at the boundaries, we have

$$\begin{aligned} I_t(z) &= t \mathbb{P}^z[B_\tau \in \mathbb{R} + ir] + \mathbb{E}^z[\text{Im}[B_\tau] 1\{B_\tau \in \tilde{\eta}_t\}] \\ &= t \frac{\text{Im}[h_t(z)]}{r-t} + \mathbb{E}^z[\text{Im}[B_\tau] 1\{B_\tau \in \tilde{\eta}_t\}], \end{aligned} \quad (4.4)$$

where  $\tilde{\eta}_t$  is defined in (2.3) and  $\tau$  is the first time Brownian motion  $B$  exits  $S_{r,t}$ . It follows from lemma 2.5 that with annulus parametrization

$$\partial_t \text{hcap}[\eta_t]|_{t=0} = -2.$$

The first equality follows from this, lemma 4.2 and (4.4).

To see the second equality in the statement of the lemma, define

$$f_t(z) = \frac{r h_t(z)}{r-t} - z + \text{hcap}[\eta_t] \bar{\mathcal{H}}_{S_r}(z, u),$$

and let  $v_t(z) = \text{Im}[f_t(z)]$ . Then by using lemma 4.2 and (4.4) we can see that for any  $\varepsilon > 0$ , there exists a constant  $c_*$  such that for all  $\{z \in S_r; \forall k, |z - 2k\pi - u| > \varepsilon\}$ ,

$$|v_t(z)| < c_* d_t \text{hcap}[\eta_t] \bar{H}_{S_r}(z, u).$$

Since  $v_t(z)$  is harmonic, there exists a constant  $c$  such that

$$|v'_t(z)| < \frac{c}{\sqrt{2}} d_t \text{hcap}[\eta_t] \bar{H}_{S_r}(z, u).$$

Therefore,

$$|f'_t(z)| \leq \sqrt{2} |v'_t(z)| < c d_t \text{hcap}[\eta_t] \bar{H}_{S_r}(z, u).$$

Define  $a_k = w + 2k\pi$  and let  $k^* = \arg \min_k |z - a_k|$ . Since  $\bar{H}_{S_r}(z, u)$  is uniformly bounded on  $\{z \in S_r; \forall k, |z - 2k\pi - u| > \varepsilon\}$  and  $h_t(z), \bar{\mathcal{H}}_{S_r}(z, 0)$  are quasi-periodic functions, for some constant  $C$

$$|f_t(z) - f_t(w) - \frac{2k^* t \pi}{r-t} - \frac{k^* \pi}{r} \text{hcap}[\eta_t]| < C d_t \text{hcap}[\eta_t].$$

Since  $f_t(w)$  is differentiable with respect to  $t$ , we get

$$\partial_t f_t(z) = \lim_{t \downarrow 0} \frac{f_t(z) - f_t(w)}{t} = \partial_t f_t(w).$$

Therefore,  $h_t(z)$  is differentiable with respect to  $t$  and the second equality follows. □

**Lemma 4.4.** *For any  $\gamma_t$  with the annulus parametrization, there exists a collection of transformations  $h_t : S_{r,t} \rightarrow S_{r-t}$  such that*

$$\partial_t h_t(z) = -\frac{h_t(z) - U_t}{r-t} + 2 \bar{\mathcal{H}}_{S_{r-t}}(h_t(z), U_t).$$

*Proof.* Choose  $w \in S_r$  such that  $w \notin \tilde{\eta}$ . Let  $h_t^* : S_{r,t} \rightarrow S_{r-t}$  be a conformal transformation. We can assume  $h_t^*(w)$  is differentiable with respect to  $t$  (otherwise, we consider  $h_t^*(w) + c_t$  for an appropriate  $c_t \in \mathbb{R}$ ). Define

$$h_t(z) = h_t^*(z) - \text{Re}[h_t^*(w)] + \int_0^t \frac{h_s^*(\eta(s)) - \text{Re}[h_s^*(w)]}{r-s} + 2 \text{Re}[\bar{\mathcal{H}}_{S_{r-s}}(h_s^*(w), U_s)] ds.$$

Note that this is well defined for all  $t < r$ . Using lemma 4.3, we have

$$\partial_t h_t^*(z) = \partial_t h_t^*(w) + \frac{h_t^*(w) - h_t^*(z)}{r-t} + 2[\bar{\mathcal{H}}_{S_{r-t}}(h_t^*(z), U_t) - \bar{\mathcal{H}}_{S_{r-t}}(h_t^*(w), U_t)].$$

This and the first equality in lemma 4.3 give the result.  $\square$

In [10, 2],  $h_t(z)$  is specified by requiring  $\mathfrak{h}_t(e^{-r}) = e^{-(r-t)}$  and  $\beta_t$  is determined according to this condition. Instead, we uniquely specify  $h_t(z)$  by requiring  $\beta_t = U_t/(r-t)$  and  $h_0(z) = z$ . This is equivalent to requiring that  $\partial_t \operatorname{Re}[h_t(z)] = 0$  for  $\{z \in S_{r,t}; \exists k \in \mathbb{Z} \operatorname{Re}[h_t(z)] = U_t + 2k\pi\}$ . This is analogous to the usual conditions for chordal Loewner equation in  $\mathbb{H}$ . We summarize our discussion with the following proposition.

**Proposition 4.2.** *For  $z \in S_r$ ,  $\bar{z} = e^{iz}$ ,  $x \in \mathbb{R}$  define*

$$\mathbf{H}_r(z) = -\frac{z}{r} + 2\mathcal{H}_{S_r}^{\bar{z}}(z, 0), \quad (4.5)$$

$$\mathbf{H}_r^R(z) = \operatorname{Re}[\mathbf{H}_r(z + ir)] = -\frac{z}{r} + \frac{\pi}{r} \sum_k^{PP} \tanh\left(\frac{\pi(z + 2k\pi)}{2r}\right),$$

$$\bar{\mathbf{H}}_r(\bar{z}) = i\mathbf{H}_r(z).$$

Then

$$\partial_t h_t(z) = \mathbf{H}_{r-t}(h_t(z) - U_t), \quad h_0(z) = z,$$

$$\partial_t \operatorname{Re}[h_t(x + ir)] = \mathbf{H}_{r-t}^R(\operatorname{Re}[h_t(x + ir)] - U_t),$$

$$\partial_t \mathfrak{h}_t(z) = \mathfrak{h}_t(z) \bar{\mathbf{H}}_{r-t}(\mathfrak{h}_t(z)/\bar{U}_t), \quad \mathfrak{h}_0(z) = z. \quad (4.6)$$

*Proof.* This is an straightforward consequence of lemma 4.4.  $\square$

The function  $\mathbf{H}_r(z)$  has several interesting properties.

- $\mathbf{H}_r(z)$  is an odd elliptic function. In other words, it is a meromorphic doubly periodic function in  $\mathbb{C}$ , with periods  $2\pi, 2ir$ .
- Let  $\Gamma(r)$  be the measure of Brownian bubbles in  $\mathbb{D}$  that are rooted at 1 and intersect  $\mathbb{D} \setminus A_r$ . Since  $\mathbb{D}$  and  $A_r$  have smooth boundaries, we can write

$$\Gamma(r) = \frac{1}{\pi} \int_{C_r} H_{A_r}(1, w) H_{\mathbb{D}}(w, 1) |dw|.$$

(The constant  $1/\pi$  in the last equation is because of our choice of normalization for the Poisson kernel. It is normalized so that  $H_{\mathbb{D}}(0,1) = 1/2$ .) Starting from the definition (4.5), one can show that

$$\mathbf{H}_r(z) = \frac{2}{z} + z \left( 2\Gamma(r) - \frac{1}{r} - \frac{1}{6} \right) + O(|z|^3), \quad z \rightarrow 0. \quad (4.7)$$

See Lemma 3.16 in [17] for more details. It follows that  $\mathbf{H}_r(z)$  is analytic on  $\mathbb{C} \setminus \{2k\pi + i2mr; k, m \in \mathbb{Z}\}$  and has poles of degree 1 at points  $\{2k\pi + i2mr; k, m \in \mathbb{Z}\}$ .

- Let  $\wp$  be the Weierstrass elliptic function with periods  $2\pi, i2r$ . Then

$$\partial_z \mathbf{H}_r(z) = -2\wp(z) + \zeta_r,$$

where  $\zeta_r$  is a constant depending on  $r$  [25].

### 4.3 Properties

In this section, we review the key properties of crossing  $SLE_\kappa$  in annuli. More details can be found in [17].

#### 4.3.1 Shrinking Domains

Let  $\eta_t$  be a chordal  $SLE_\kappa$  from 0 to  $\infty$  in  $\mathbb{H}$  and assume  $w \in \mathbb{H}$  with  $\text{Im}[w] = r$ . Let  $\tau = \inf\{t; \eta_t \not\subset S_r\}$ . For  $t < \tau$ , define  $g_t : \mathbb{H} \setminus \eta_t \rightarrow \mathbb{H}$  to be the usual conformal transformation for the chordal Loewner equation and let  $\xi_t = g_t(\eta(t))$ ,  $w_t = g_t(w)$ ,  $H_t(z) = H_{g_t(S_r \setminus \eta_t)}(z, w_t)$ . It follows from Section 2.4.2 that for  $t < \tau$ ,

$$M_t^1 = H_t(\xi_t)^b |g_t'(w)|^b \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{H}}(\eta_t, \mathbb{H} \setminus S_r) \right\}$$

is a martingale satisfying

$$\frac{d\mu_{S_r}(0, w)}{d\mu_{\mathbb{H}}(0, \infty)}(\eta_t) = M_t^1, \quad dM_t^1 = \frac{bH_t'(\xi_t)}{H_t(\xi_t)} M_t^1 d\xi_t.$$

Now suppose  $\eta_t$  is a  $SLE_\kappa$  from 0 to  $w$  in  $S_r$ . Let  $\hat{\eta}_t = \cup_{k \in \mathbb{Z} \setminus \{0\}} \eta + 2k\pi$ ,  $\tilde{\eta}_t = \hat{\eta}_t \cup \eta_t$  be as in (2.3) and define  $\tau_0 = \inf\{t; \eta_t \cap \hat{\eta}_t \neq \emptyset\}$ . We want to describe the process that at each time  $t < \min\{\tau_0, \tau\}$ , evolves like  $SLE_\kappa$  from  $\eta(t)$  to  $w$  in  $S_{r,t} = S_r \setminus \tilde{\eta}_t$ . We will call this process *locally chordal  $SLE_\kappa$  in  $S_r$* . Let  $Q_t(z) = H_{g_t(S_{r,t})}(z, w_t) / H_{g_t(S_r \setminus \eta_t)}(z, w_t)$ , which is the probability that Brownian excursion from  $z$  to  $w_t$  in  $g_t(S_r \setminus \eta_t)$  does not hit  $g_t(\hat{\eta}_t)$ . Define

$$A(r, x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{H_{S_r}(0, 2k\pi) H_{S_r}(2k\pi, x + ir)}{H_{S_r}(0, x + ir)}.$$

Let  $h_t : S_{r,t} \rightarrow S_{r(t)}$  be as in (2.4) and define  $\phi_t$  to be the conformal transformation satisfying  $h_t = \phi_t \circ g_t$ . Let  $U_t = \phi_t(\xi_t)$  and define  $\bar{H}_t(z) = Q_t(z) H_t(z) = H_{g_t(S_{r,t})}(z, w_t)$ . Estimates using hitting probabilities for Brownian motion indicate that

$$\frac{\partial_t \bar{H}_t'(\xi_t)}{\bar{H}_t'(\xi_t)} = a \phi_t'(\xi_t)^2 A(r(t), \text{Re}[h_t(w)]) - U_t + a \left[ \frac{\bar{H}_t''(\xi_t)^2}{4\bar{H}_t'(\xi_t)^2} - \frac{2\bar{H}_t'''(\xi_t)}{3\bar{H}_t'(\xi_t)} \right].$$

This equality follows from arguments similar to those in section edit, so we omit the details here.

Using this, the Itô's formula, lemma 2.2 and lemma 2.5, we can see that for  $t < \tau_0$ ,

$$M_t^2 = Q_t'(\xi_t)^b \exp \left\{ \frac{\mathbf{c}}{2} m_{S_r}(\eta_t) + 2b \int_0^t A(r(s), \text{Re}[h_s(w)]) - U_s \dot{r}(s) ds \right\} 1_{\{\eta_t \cap \hat{\eta}_t = \emptyset\}} \quad (4.8)$$

is a local martingale. Here,  $m_{S_r}(\eta_t)$  denotes the Brownian loop measure of loops  $\ell$  in  $S_r$  that have the following properties.

- $\ell$  intersects both  $\eta_t$  and  $\hat{\eta}_t$ .
- If  $T$  is the first time  $\ell$  intersects  $\eta_t$ , then  $\ell \cap \hat{\eta}_T \neq \emptyset$ .

Let  $R_t = \text{Re}[h_t(w)] - U_t$  and define  $L(r, x) = (\pi/r) \tanh(\pi x/2r)$ . Assuming  $\eta_t$  has annulus parametriza-

tion, we get

$$dU_t = b\kappa L(r-t, R_t) dt - \sqrt{\kappa} dB_t,$$

where  $B_t$  is a Brownian motion. Using proposition 4.2, we can see that for  $z \in (0, 2\pi)$ ,  $Z_t = h_t(z) - U_t$ ,

$$dR_t = \left[ \mathbf{H}_{r-t}^R(R_t) - b\kappa L(r-t, R_t) \right] dt + \sqrt{\kappa} dB_t, \quad (4.9)$$

$$dZ_t = \left[ \mathbf{H}_{r-t}(Z_t) - b\kappa L(r-t, R_t) \right] dt + \sqrt{\kappa} dB_t.$$

To show that  $M_t^2$  is a martingale for all  $t < r$ , we need to show that  $0 < Z_t < 2\pi$  for all time  $t < r$ . In that case, with probability one with respect to the distribution of locally chordal  $SLE_\kappa$ ,  $\eta_\tau \cap \hat{\eta}_\tau = \emptyset$  and  $M_t^2, t < r$  is a martingale. To prove the claim, note that  $H_{r-t}(z) \sim 1/z$  as  $z \rightarrow 0$  and for any  $\varepsilon > 0$ ,  $L(r, x)$  is bounded for all  $r > \varepsilon$ . By comparing to a Bessel process,  $0 < Z_t$  for all  $t > \varepsilon$ . The result for all  $t > 0$  follows from the fact that  $R_t \rightarrow 0$  as  $t \uparrow r$ .

### 4.3.2 Partition Function

Let  $D' \subset A_r$  be a simply connected domain that agrees with  $A_r$  in neighborhoods of 1 and  $\bar{w} = e^{-r+iw}$ . If  $\gamma_t$  is a  $SLE_\kappa$  from 1 to  $\bar{w}$  in  $A_r$ , then

$$\frac{d\mu_{A_r}(1, \bar{w}; D')}{d\mu_{D'}(1, \bar{w})}(\gamma) = \exp \left\{ -\frac{\mathbf{c}}{2} m_{A_r}(\gamma, A_r \setminus D') \right\}.$$

We can write

$$m_{A_r}(\gamma, A_r \setminus D') = \hat{m}_{A_r}(\gamma, A_r \setminus D') + m^*(r), \quad (4.10)$$

where  $m^*(r)$  denotes the measure of the set of loops in  $A_r$  of nonzero winding number and  $\hat{m}_{A_r}(\gamma, A_r \setminus D')$  is the measure of the set of loops of zero winding number that intersect both  $\eta$  and  $A_r \setminus D'$ .

Let  $\eta_t$  be the continuous pre-image of  $\gamma_t$  under  $\psi$  (as discussed in section 2.1). For any loop  $\ell' \subset A_r$  that intersects  $\gamma$  and has zero winding number, there exists a unique unrooted loop  $\ell$  in  $S_r$

such that  $\ell \cap \eta \neq \emptyset$  and if  $T$  is the first time  $\eta_t$  hits  $\ell$ , then  $\ell \cap \hat{\eta}_T = \emptyset$ . We call these loops  $\eta$ -good. This gives a bijection between loops  $\ell'$  in  $A_r$  that intersect  $\gamma$  and  $\eta$ -good loops  $\ell$  in  $S_r$ . Let  $D \subset S_r$  be the unique simply connected domain with  $\eta \subset D$ ,  $\psi(D) = D'$ . Then the Brownian loop measure of  $\eta$ -good loops in  $S_r$  that intersect  $S_r \setminus D$  is equal to  $\hat{m}_{A_r}(\gamma, A_r \setminus D')$ .

We call a loop  $\ell$  in  $S_r$  a *bad* loop if it is not a  $\eta$ -good loop and it intersects both  $\eta$  and  $\hat{\eta}$ . If  $m_{S_r}(\eta_t)$  denotes the Brownian measure of bad loops that intersect  $\eta_t$ , then

$$m_{S_r}(\eta_t, S_r \setminus D) = \hat{m}_{A_r}(\gamma, A_r \setminus D') + m_{S_r}(\eta_t).$$

By conformal invariance,  $\psi \circ \mu_D^\#(0, w) = \mu_{D'}^\#(1, \bar{w})$ . Therefore, we are interested in studying the measure  $\nu_{S_r}(0, w + ir)$  satisfying

$$\frac{d\nu_{S_r}(0, w + ir; D)}{d\mu_D(0, w + ir)}(\eta) = \exp\left\{-\frac{\mathbf{c}}{2}\hat{m}_{A_r}(\gamma, A_r \setminus D')\right\}.$$

Here,  $\nu_{S_r}(0, w + ir; D)$  is  $\nu_{S_r}(0, w + ir)$  restricted to the paths staying in  $D$ . Recall that

$$\frac{d\mu_D(0, w + ir)}{d\mu_{S_r}(0, w + ir)}(\eta) = \exp\left\{\frac{\mathbf{c}}{2}m_{S_r}(\eta, S_r \setminus D)\right\} 1\{\eta \subset D\}.$$

Therefore, we can define  $\nu_{S_r}(0, w + ir)$  by

$$\frac{d\nu_{S_r}(0, w + ir)}{d\mu_{S_r}(0, w + ir)}(\eta) = \exp\left\{\frac{\mathbf{c}}{2}m_{S_r}(\eta)\right\} 1\{\eta \subset S_r \setminus \hat{\eta}\}.$$

We can relate  $\nu_{S_r}(0, w + ir)$  to  $SLE_\kappa$  in  $A_r$  by conformal covariance. We define  $\nu_{A_r}(1, w)$  by

$$\begin{aligned} \nu_{A_r}(1, w) &= |\psi'(0)|^{-b} |\psi'(w + ir)|^{-b} e^{-\mathbf{c}m^*(r)/2} \psi \circ \nu_{S_r}(0, w + ir) \\ &= e^{br} e^{-\mathbf{c}m^*(r)/2} \psi \circ \nu_{S_r}(0, w + ir), \end{aligned} \tag{4.11}$$

We think of this as annulus  $SLE_\kappa$  from 1 to  $\bar{w} = e^{-r+iw}$  restricted to curves of a particular winding

number. Annulus  $SLE_\kappa$  is obtained by summing over all winding numbers

$$\mu_{A_r}(1, \bar{w}) = \sum_{k \in \mathbb{Z}} \nu_{A_r}(1, w + 2\pi k). \quad (4.12)$$

Now assume with respect to a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\eta_t$  is a chordal  $SLE_\kappa$  from 0 to  $w + ir$  in  $S_r$ . Let  $V(r, w) = \|\nu_{S_r}(0, w + ir)\| / \Psi_{S_r}(0, w + ir)$  and note that  $V(0, w) = 1$ . Let  $\mathcal{F}_t$  be the sigma algebra generated by  $\eta_t$  and denote by  $\mathbb{E}$  the expectation with respect to the distribution of  $\eta_t$ . Then

$$M_t^3 = \mathbb{E} \left[ \exp \left\{ \frac{\mathbf{c}}{2} m_{S_r}(\eta) \right\} 1_{\{\eta \subset S_r \setminus \hat{\eta}\}} \middle| \mathcal{F}_t \right]$$

is a martingale. For each  $t < r$ , denote by  $m_t^1 = m_{S_r}(\eta_t)$  the Brownian measure of bad loops  $\ell$  such that  $\ell \cap \eta_t \neq \emptyset$ . Let  $m_t^2$  denote the measure of bad loops  $\ell$  such that  $\ell \subset S_r \setminus \eta_t$ ,  $\ell \cap \hat{\eta}_t \neq \emptyset$  and let  $m_t^3$  denote the measure of bad loops  $\ell$  with  $\ell \subset S_{r,t}$ . Note that

$$m_{S_r}(\eta) = m_t^1 + m_t^2 + m_t^3.$$

Weighting  $\eta(t, r)$  by

$$\exp \left\{ \frac{\mathbf{c}}{2} m_t^2 \right\} 1_{\{\eta(t, r) \subset S_{r,t}\}}$$

gives  $SLE_\kappa$  from  $\eta(t)$  to  $w + ir$  in  $S_{r,t}$ . Let  $\mathbb{E}^2$  denote expectation with respect to distribution of this process. Let  $R_t = \text{Re}[h_t(w + ir)] - U_t$ . Using conformal invariance,

$$\mathbb{E}^2[\exp\{\mathbf{c}m_t^3/2\} 1_{\{\eta(t, r) \cap \hat{\eta}(t, r) = \emptyset\}}] = V(r-t, R_t).$$

Therefore,

$$M_t^3 = Q_t'(\xi_t)^b \exp \left\{ \frac{\mathbf{c}}{2} m_{S_r}(\eta_t) \right\} 1_{\{\eta_t \cap \hat{\eta}_t = \emptyset\}} V(r-t, R_t)$$

is a martingale, where  $Q_t(\xi_t)$  is defined in subsection 4.3.1. Recall the martingale  $M_t^2$  defined in (4.8). Then

$$M_t^3 = M_t^2 V(r-t, R_t) \exp \left\{ -2b \int_0^t A(r-s, R_s) ds \right\}.$$

If  $\mathbb{P}^*$  denotes the probability measure obtained from weighting  $\mathbb{P}$  by  $M_t^2$ , then with respect to  $\mathbb{P}^*$ ,  $\eta_t$  has the distribution of locally chordal  $SLE_\kappa$  and using (4.9),

$$dR_t = \left[ \mathbf{H}_{r-t}^R(R_t) - b\kappa L(r-t, R_t) \right] dt + \sqrt{\kappa} dB_t,$$

where  $B_t$  is a standard Brownian motion. If

$$N_t = V(r-t, R_t) \exp \left\{ -2b \int_0^t A(r-s, R_s) ds \right\},$$

then  $N_t$  is a martingale under  $\mathbb{P}^*$ . Therefore,

$$\exp \left\{ -2b \int_0^r A(r-s, R_s) ds \right\} = \mathbb{E}^*[N_r] = \mathbb{E}^*[N_0] = V(r, w).$$

Moreover, if we knew that  $V(r, w)$  is at least  $C^1$  in  $r$  and  $C^2$  in  $w$ , then we could use the Itô's formula and the fact that  $N_t$  is a martingale to derive a PDE that  $V(r, w)$  satisfies. One way to prove the smoothness of  $V(r, w)$  is to interchange the derivative and expectation and show that the expectations of the derivatives are finite. In fact, this is the approach taken in [17, 28] to prove smoothness of  $V(r, w)$ .

### 4.3.3 Comparing Radial $SLE_\kappa$ to Annulus $SLE_\kappa$

Let  $\bar{h}_t, \bar{g}_t, \bar{\phi}_t$  be conformal transformations as in section 2.1. Suppose  $\gamma_t$  has radial parametrization.

Using the definition of  $SLE_\kappa$  in annuli, it is shown in [17] that for an annulus  $A_r$ ,

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma_t) = M_t := \frac{|\bar{h}'_t(\bar{w})|^b |\bar{\phi}'_t(\bar{\xi}_t)|^b}{\bar{g}'_t(0)^{\tilde{b}}} \exp \{m_{\mathbb{D}}(C_r, \gamma_t)\} \Psi_{A_{r(t)}}(\bar{\phi}_t(\bar{\xi}_t), \bar{h}_t(\bar{w})).$$

Moreover, it is shown that there exist positive constants  $c, q$  such that as  $r \rightarrow \infty$ ,

$$\Psi_{A_r}(1, \bar{w}) = cr^{c/2} e^{(b-\tilde{b})r} [1 + O(e^{-qr})].$$

Using this, we can prove the following theorem from [17].

**Theorem 4.1.** *Suppose  $\gamma_t$  has radial parametrization. Then there exist  $q, c_* > 0$  such that uniformly over  $t > 0, r \geq \frac{ta}{2} + 2$  and all initial segments  $\gamma_t$ ,*

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma_t) = c_* e^{r(b-\tilde{b})} r^{c/2} [1 + O(e^{-qu})],$$

where  $u = r - \frac{ta}{2}$ . In particular, there exists  $c > 0$  such that

$$\left| \frac{d\mu_{A_r}^\#(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}^\#(\bar{u}, 0)}(\gamma_t) - 1 \right| \leq ce^{-qu}.$$

*Proof. (sketch)* Using lemma 2.4 and lemma 2.3, we can see that

$$|\bar{\phi}'_t(\bar{\xi}_t)| = 1 + O(e^{-u}),$$

$$|\bar{h}'_t(\bar{w})| = \bar{g}'_t(0)[1 + O(e^{-u})],$$

$$r(t) = u + O(e^{-u}).$$

Moreover, deterministic estimates show that

$$\exp\{m_{\mathbb{D}}(C_r, \gamma_t)\} = (r/u)^{c/2} [1 + O(e^{-u})].$$

□

## CHAPTER 5

### MULTIPLE-PATH $SLE_\kappa$ IN ANNULI

#### 5.1 Introduction

The Schramm-Loewner evolution ( $SLE_\kappa$ ) is a one parameter family of measures on planar curves discovered by Oded Schramm [24]. He proved that in simply connected domains,  $SLE_\kappa$  describe the only measures satisfying the domain Markov property and conformal invariance. Schramm's construction benefits from the fact that a simply connected domain  $D$  with a simple curve starting from the boundary removed is conformally equivalent to  $D$ . Since this does not hold in multiply-connected domains, Schramm's construction does not readily extend to those cases. However, other authors have studied  $SLE_\kappa$  in multiply-connected domains using different methods. Bauer and Friedrich ([1, 2]) used a generalization of the Loewner equation for multiply-connected domains to describe the driving function of  $SLE_\kappa$ . Zhan's approach in [28] was similar, in that he used a generalization of Loewner equation to define annulus  $SLE_\kappa$ . However, in addition to conformal invariance and Markov property, he required  $SLE_\kappa$  to be reversible and used that to uniquely determine the driving function. These articles are based on the work of Komatu [10] in 1950, who studied an analogue of the Loewner equation in multiply-connected domains. In [17], Lawler used Brownian loop measure to define  $SLE_\kappa$  by giving its Radon-Nikodym derivative with respect to the measures in simply connected subdomains. It turned out that his definition agrees with the process defined by Zhan.

Interest in questions regarding multiple interfaces of various discrete models has led to the study of  $SLE_\kappa$  measure on multiple paths

$$\gamma = (\gamma^1, \dots, \gamma^n).$$

Unlike  $SLE_\kappa$  measure on single curves, conformal invariance and domain Markov property do not uniquely specify the measure when  $2 \leq n$ . In [6, 5], Dubédat characterized multiple  $SLE_\kappa$  paths

in simply connected domains using a commutation relation for some differential operators related to the driving functions. He also gave a discussion about multiply-connected domains, but did not give a complete classification. In [11], Kozdron and Lawler required the process to satisfy the restriction property in addition to the domain Markov property and conformal invariance and gave a global construction using the Brownian loop measure for  $\kappa \leq 4$ .

Since two-dimensional discrete models whose limit in continuum are known to be  $SLE_\kappa$  are considered as measures with partition functions, it is natural to define  $SLE_\kappa$  as a measure with partition function. For regular chordal  $SLE_\kappa$  in simply connected domains, the partition function is given by a power of the Poisson kernel. In simply-connected domains, the partition function is shown to be smooth and is described by a particular differential equation ([17, 28]). For multiple  $SLE_\kappa$  paths, Dubédat [5] proved that the partition function can be described as a family of Euler integrals taken on a specific set of cycles. In [22], Peltola and Wu used conformal field theory and partial differential equations techniques such as Hörmander’s theorem to show that the partition function satisfies a particular PDE when  $\kappa \leq 4$ . Only using techniques from probability, it was proved in [9] that the partition function satisfies the same PDE as in [5, 22] when  $\kappa < 4$ .

Our definition of multiple  $SLE_\kappa$  paths in multiply-connected domains is similar to the approach of [11, 9] in simply connected domains. That is, we define it to be the measure absolutely continuous with respect to the product of single  $SLE_\kappa$  measures with a particular Radon-Nikodym derivative involving the Brownian loop measure. To that end, we build on Lawler’s definition of annulus  $SLE_\kappa$  in [17]. We find these definitions the most natural ones because they provide a clear consistency with  $SLE_\kappa$  in simply connected domains.

The chapter is organized as follows. In section 5.2, we establish our notation, prove a few deterministic estimates, and state our main result about smoothness of the partition function. In section 4.2, we discuss an analog of the Loewner equation in annuli. In particular, we give an intuition about our choices in annuli by referring to the usual assumptions pertaining to the Loewner equations in the upper-half plane and the unit disk. We prove that the partition function is smooth in section 5.3. The main idea of the proof is to define appropriate martingales and use the Hörmander’s

theorem. Finally, section 5.4 consists of some results about two-sided  $SLE_\kappa$ . In particular, we show that two-sided  $SLE_\kappa$  can be constructed by weighting two independent radial  $SLE_\kappa$ s by an appropriate martingale. We use this to show that two-sided  $SLE_\kappa$  can be approximated by two  $SLE_\kappa$  paths in an annulus.

## 5.2 Preliminaries

### 5.2.1 Multiple-Paths $SLE_\kappa$

Although our definitions in this section can easily be extended to any finitely connected domains, we restrict to annuli for simplicity. Suppose  $\mathbf{z} = (z^1, z^2, \dots, z^n)$  and  $\mathbf{w} = (w^1, w^2, \dots, w^n)$  are distinct boundary points of  $A_r$ . For  $1 \leq j \leq n$ , let  $\gamma^j$  be a  $SLE_\kappa$  path from  $z^j$  to  $w^j$  in  $A_r$  with corresponding  $SLE_\kappa$  measure  $\mu_{A_r}(z^j, w^j)$ . Recall that

$$\Psi_{A_r}(z^j, w^j) = \|\mu_{A_r}(z^j, w^j)\|.$$

For a measure  $\mu$ , we will use  $\mu^\#$  to represent the probability measure  $\mu/\|\mu\|$ . Define the *central charge*

$$\mathbf{c} = \frac{(6-k)(3k-8)}{2k} = \frac{2b(3-4a)}{a}.$$

We define multiple-paths  $SLE_\kappa$  measure in  $A_r$  similar to [9, 11].

**Definition 5.1.** For  $\kappa \leq 4$ , we define the  $\mu_{A_r}(\mathbf{z}, \mathbf{w})$  to be the measure on  $n$ -tuple of paths  $\gamma = (\gamma^1, \dots, \gamma^n)$  that is absolutely continuous with respect to the product measure  $\mu_{prod}(\mathbf{z}, \mathbf{w}) := \mu_{A_r}(z^1, w^1) \times \dots \times \mu_{A_r}(z^n, w^n)$  with Radon-Nikodym derivative

$$Y(\gamma) = I(\gamma) \exp \left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^n m[K_j(\gamma)] \right\}.$$

Here  $I(\gamma)$  is the indicator function of the event that for all  $i \neq j$ ,  $\gamma^i \cap \gamma^j = \emptyset$  and  $m[K_j(\gamma)]$  is the Brownian loop measure of loops that intersect at least  $j$  of the paths. As before, the partition

function of the measure  $\mu_{A_r}(\mathbf{z}, \mathbf{w})$  is the total mass

$$\Psi_{A_r}(\mathbf{z}, \mathbf{w}) = \|\mu_{A_r}(\mathbf{z}, \mathbf{w})\|.$$

Note that  $\mu_{A_r}(\mathbf{z}, \mathbf{w}) = 0$  if there exists  $k$  such that  $z^k, w^k$  are not on the boundary of the same connected component of

$$A_r \setminus \bigcup_{j \neq k} \gamma^j.$$

Moreover, it is clear from the definition that if  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  is a permutation, then  $Y(\gamma) = Y(\gamma^\sigma)$  and  $\Psi_{A_r}(\mathbf{z}, \mathbf{w}) = \Psi_{A_r}(\mathbf{z}^\sigma, \mathbf{w}^\sigma)$ .

Let  $\gamma = (\gamma', \gamma^n)$ ,  $\mathbf{z}' = (z^1, z^2, \dots, z^{n-1})$ ,  $\mathbf{w}' = (w^1, w^2, \dots, w^{n-1})$ , and

$$D_k = A_r \setminus \bigcup_{j=1}^{k-1} \gamma^j.$$

**Lemma 5.1.** *We have*

$$\sum_{j=2}^n m[K_j(\gamma)] = \sum_{j=2}^n m_{A_r}(\gamma^j, \gamma^1 \cup \dots \cup \gamma^{j-1}). \quad (5.1)$$

*Proof.* We can see that

$$\sum_{j=2}^n m[K_j(\gamma)] = \sum_{j=2}^{n-1} m[K_j(\gamma')] + m_{A_r}(\gamma^n, \gamma^1 \cup \dots \cup \gamma^{n-1})$$

(This is straightforward. See lemma 1 in [9] for a proof). The proof follows from this and induction.  $\square$

Lawler used the RHS of (5.1) to define the multiple-paths  $SLE_\kappa$  measure in [17]. The last lemma shows that the two definitions are equivalent. The following important properties can be seen from Lemma 5.1. Kozdron and Lawler stated similar properties in [11] for the multiple-paths  $SLE_\kappa$  in simply connected domains.

- **Marginal Measure:** Let  $\mu'_{A_r}(\mathbf{z}', \mathbf{w}')$  be the marginal measure on  $\gamma'$  induced by  $\mu_{A_r}(\mathbf{z}, \mathbf{w})$ .

Then

$$\frac{d\mu'_{A_r}(\mathbf{z}', \mathbf{w}')}{d\mu_{A_r}(\mathbf{z}', \mathbf{w}')}(\gamma') = H_{D_n}(z^n, w^n)^b.$$

More generally, if  $k < n$  and  $\mu'_{A_r}((z^1, \dots, z^{k-1}), (w^1, \dots, w^{k-1}))$  is the marginal measure on  $(\gamma^1, \dots, \gamma^{k-1})$  induced by  $\mu_{A_r}(\mathbf{z}, \mathbf{w})$ , then

$$\frac{d\mu'_{A_r}((z^1, \dots, z^{k-1}), (w^1, \dots, w^{k-1}))}{d\mu_{A_r}((z^1, \dots, z^{k-1}), (w^1, \dots, w^{k-1}))}(\gamma^1, \dots, \gamma^{k-1}) = \Psi_{D_k}((z^k, \dots, z^n), (w^k, \dots, w^n)).$$

- **Conditional distribution:** Given  $\gamma'$ , the conditional distribution of  $\gamma^n$  is  $\mu_{D_n}^\#(z^n, w^n)$ . More generally, conditioned on  $(\gamma^1, \gamma^2, \dots, \gamma^{k-1})$ , the probability distribution of  $(\gamma^k, \dots, \gamma^n)$  is  $\mu_{D_k}^\#((z^k, \dots, z^n), (w^k, \dots, w^n))$ .

Let

$$\tilde{\Psi}_{A_r}(\mathbf{z}, \mathbf{w}) = \frac{\Psi_{A_r}(\mathbf{z}, \mathbf{w})}{\prod_{j=1}^n \Psi_{A_r}(z^j, w^j)}.$$

We will write  $\mathbb{E}_{\text{prod}}$  for expectation with respect to the product measure  $\mu_{\text{prod}}^\#$ . It is easy to see that

$$\mathbb{E}_{\text{prod}} [Y(\gamma) | \gamma'] = Y(\gamma') \frac{H_{D_n}(z^n, w^n)^b}{\Psi_{A_r}(z^n, w^n)}. \quad (5.2)$$

Here, if  $\gamma'$  is just a single curve, then  $Y(\gamma') = 1$ . It is proved in [17] that

$$\Psi_{A_r}(z^n, w^n) \leq e^{-\frac{\mathbf{c}}{2} m^*(r)} H_{A_r}(z^n, w^n)^b,$$

where  $0 < m^*(r) < r/6$  is the Brownian loop measure of the loops in  $A_r$  with nonzero winding number. Since  $D_n \subset A_r$  is simply connected,

$$\frac{d\mu_{D_n}(z^n, w^n)}{d\mu_{A_r}(z^n, w^n)}(\gamma^n) = 1 \{ \gamma^n \subset D_n \} \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r}(\gamma, A_r \setminus D_n) \right\}.$$

If  $\kappa \leq 8/3$ , then  $\mathbf{c} \leq 0$  and

$$e^{-\frac{\mathbf{c}}{2}m^*(r)}\Psi_{D_n}(z^n, w^n) = e^{-\frac{\mathbf{c}}{2}m^*(r)}H_{D_n}(z^n, w^n)^b \leq \Psi_{A_r}(z^n, w^n). \quad (5.3)$$

In this case, we can use (5.2) to see

$$\tilde{\Psi}_{A_r}(\mathbf{z}, \mathbf{w}) = \mathbb{E}_{\text{prod}} \left[ \mathbb{E}_{\text{prod}} [Y(\gamma) | \gamma'] \right] \leq e^{\frac{\mathbf{c}}{2}m^*(r)} \tilde{\Psi}_{A_r}(\mathbf{z}', \mathbf{w}') \leq e^{(n-1)\frac{\mathbf{c}}{2}m^*(r)}.$$

We still do not know if (5.3) holds for  $8/3 < \kappa \leq 4$ .

### 5.3 The case $n = 2$

We consider the measure on paths  $(\gamma, \gamma')$ , where  $\gamma, \gamma'$  are  $SLE_\kappa$  paths from  $|\bar{u}| = 1$  to  $|\bar{w}| = e^{-r}$  and from  $|\bar{u}'| = 1$  to  $|\bar{w}'| = e^{-r}$ , respectively. Let  $\bar{\mathbf{u}} = (\bar{u}, \bar{u}')$ ,  $\bar{\mathbf{w}} = (\bar{w}, \bar{w}')$ . By (5.2), the partition function can be written as

$$\Psi_{A_r}(\bar{\mathbf{u}}, \bar{\mathbf{w}}) = \Psi_{A_r}((u, u'), (w, w')) = \Psi_{A_r}(\bar{u}, \bar{w}) \mathbb{E} \left[ H_{A_r \setminus \gamma}(\bar{u}', \bar{w}')^b \right],$$

where  $\mathbb{E}$  denotes the expectation with respect to the distribution of  $\gamma$ . Let  $\psi(z) = e^{iz}$  and choose  $0 \leq u, u', w, w' < 2\pi$  such that

$$\bar{u} = \psi(u), \bar{u}' = \psi(u'), \bar{w} = \psi(w + ir), \bar{w}' = \psi(w' + ir).$$

Consider the function

$$V(r, \mathbf{u}, \mathbf{w}) = \frac{\Psi_{A_r}(\bar{\mathbf{u}}, \bar{\mathbf{w}})}{\Psi_{A_r}(\bar{u}, \bar{w})H_{A_r}(\bar{u}', \bar{w}')^b} = \mathbb{E} \left[ Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma)^b \right]. \quad (5.4)$$

We will show that  $V$  is a smooth function of  $r, u, u', w, w'$ . It is clear that  $V(r, \mathbf{u}, \mathbf{w})$  can be written as a function of  $r, (u' - u), (w - u), (w' - u)$ . However, it is easier to prove the smoothness if we

consider it as  $V(r, \mathbf{u}, \mathbf{w})$ .

Let  $\eta_t \subset S_r$  be the unique continuous curve satisfying  $\psi(\eta_t) = \gamma_t$ ,  $\eta(0+) = u$ . Define  $\hbar_t, \bar{g}_t, h_t, g_t$ , etc. for  $\gamma_t$  as in section 2.1. Define,

$$U_t = h_t(\eta(t)), U'_t = h_t(u'), W_t = h_t(w + ir), W'_t = h_t(w' + ir),$$

$$\bar{U}_t = \hbar_t(\gamma(t)), \bar{U}'_t = \hbar_t(\bar{u}'), \bar{W}_t = \hbar_t(\bar{w}), \bar{W}'_t = \hbar_t(\bar{w}').$$

Note that for  $t < \tau_r$ , we can write  $H_{A_r \setminus \gamma_t}(\bar{u}', \bar{w}')$  in terms of the Poisson kernels in the covering space  $S_{r,t}$

$$H_{A_r \setminus \gamma_t}(\bar{u}', \bar{w}') = e^r \sum_{k \in \mathbb{Z}} H_{S_{r,t}}(u', w' + 2k\pi).$$

For now, suppose  $\gamma_t$  has the radial parametrization.

**Lemma 5.2.** *Suppose  $t < \tau_r$  and let  $z \in \mathbb{R}, z' \in \mathbb{R} + ir$  and  $Z_t = h_t(z), Z'_t = h_t(z')$ . Then*

$$\partial_t \log Q_{S_r}(z, z'; S_{r,t}) = 2\dot{r}(t) \mathbf{F}(r(t), Z_t - U_t, Z'_t - U_t), \quad (5.5)$$

where  $U_t = h_t(\eta(t))$ ,  $r(t)$  are defined in (2.2) and

$$\mathbf{F}(r, z, z') := \sum_{k \in \mathbb{Z}} \frac{H_{S_r}(z, 2k\pi) H_{S_r}(2k\pi, z')}{H_{S_r}(z, z')}.$$

*Proof.* We only need to prove the claim for the right derivative with respect to  $t$  since the right-hand side of (5.5) is continuous in  $t$ . Moreover, we only need to prove the claim for the derivative at  $t = 0$  because

$$\begin{aligned} \partial_t \log Q_{S_r}(z, z'; S_{r,t}) &= \lim_{s \downarrow 0} \frac{1}{s} \log Q_{S_{r,t}}(z, z'; S_{r,t+s}) \\ &= \lim_{s \downarrow 0} \frac{1}{s} \log Q_{S_{r(t)}}(Z_t, Z'_t; S_{r(t)} \setminus h_t \circ \tilde{\eta}(t, t+s)). \end{aligned}$$

First, note that by using proposition 2.3 and lemma 2.5 we get

$$\lim_{s \downarrow 0} \frac{1}{s} \text{hcap}[h_t \circ \eta(t, t+s)] = a\phi_t'(\xi_t)^2 = -2\dot{r}(t).$$

At  $t = 0$  we have

$$\partial_t \log Q_{S_r}(z, z'; S_{r,t}) = \partial_t Q_{S_r}(z, z'; S_{r,t}) = -\partial_t [1 - Q_{S_r}(z, z'; S_{r,t})].$$

The term  $1 - Q_{S_r}(z, z'; S_{r,t})$  is the probability that a Brownian excursion from  $z$  to  $z'$  in  $S_r$  hits  $\tilde{\eta}_t$ .

We first calculate the probability that the Brownian excursion hits  $\eta_t$ . We can write

$$1 - Q_{S_r}(z, z'; S_r \setminus \eta_t) = \frac{E^z[H_{S_r}(B_\tau, z')1\{B_\tau \in \eta_t\}]}{H_{S_r}(z, z')},$$

where  $B_s$  is a Brownian excursion in  $\mathbb{H}$  starting from  $z$  and  $\tau$  is the first time  $B_s$  exits  $S_r \setminus \eta_t$ . Let  $D \subset S_r$  be a half disk centered at  $u$  such that  $z \notin \bar{D}$ . Assume  $t$  is small enough so that  $\eta_t \subset D$  and let  $d_t = 2\text{diam}[\eta_t]$ . Using the exact form of the Poisson kernel in  $D$ , we can see that if  $B_\tau \in \eta_t$ , then for any  $w \in \partial D \cap \mathbb{H}$

$$H_D(B_\tau, w) = \text{Im}(B_\tau)H_D(u, w)[1 + O(d_t)].$$

Using this, we get

$$H_{S_r}(B_\tau, z')1\{B_\tau \in \eta_t\} = \text{Im}(B_\tau)1\{B_\tau \in \eta_t\}H_{S_r}(u, z')[1 + O(d_t)]. \quad (5.6)$$

Let  $f(w)$  be the unique bounded harmonic function on  $S_r \setminus \eta_t$  with boundary condition  $\text{Im}(w)1\{w \in \eta_t\}$ . Similar to lemma 4.2, we can see that

$$E^z[f(B_\tau)] = \text{hcap}[\eta_t]H_{S_r}(z, u)[1 + O(d_t)]. \quad (5.7)$$

To be more precise, suppose  $D_t \subset S_r$  is a half disk of radius  $d_t$  centered at  $u$  and let  $w \in S_r \cap \partial D_t$ .

Lemma 4.1 implies that

$$H_{S_r \setminus \bar{D}_t}(z, w) = 2 \sin \theta_w H_{S_r}(z, u) [1 + O(d_t)],$$

where  $\theta_w = \arg w$ . Using this, we get

$$E^z[f(B_\tau)] = \frac{1}{\pi} H_{S_r}(z, u) [1 + O(d_t)] \int_{S_r \cap \partial D_t} 2 \sin \theta_w E^w[f(B_\tau)] |dw|. \quad (5.8)$$

Let  $\sigma$  be the first time  $B_s$  exits  $\mathbb{H} \setminus \eta_t$ . Note that  $\tau \leq \sigma$ ,

$$E^w[f(B_\tau)] = E^w[\text{Im}(B_\sigma)] - E^w[\text{Im}(B_\sigma) 1\{\tau < \sigma\}],$$

for  $w \in S_r \cap \partial D_t$ ,

$$\begin{aligned} E^w[\text{Im}(B_\sigma) 1\{\tau < \sigma\}] &= O(d_t) \text{hcap}[\eta_t], \\ \frac{1}{\pi} \int_{S_r \cap \partial D_t} 2 \sin \theta_w E^w[\text{Im}(B_\sigma)] |dw| &= \text{hcap}[\eta_t]. \end{aligned}$$

Plugging this into (5.8) proves (5.7). Therefore we can see that at  $t = 0$ , for any  $k \in \mathbb{Z}$

$$\partial_t [1 - Q_{S_r}(z, z'; S_r \setminus \eta_t + 2k\pi)] = a \frac{H_{S_r}(z, u + 2k\pi) H_{S_r}(u + 2k\pi, z')}{H_{S_r}(z, z')}. \quad (5.9)$$

One can use a similar argument to see that the probability of the Brownian excursion hitting at least two copies of  $\eta_t$  is of order  $O(\text{hcap}[\eta_t]^2)$ . Therefore,

$$1 - Q_{S_r}(z, z'; S_{r,t}) = \sum_{k=-\infty}^{\infty} [1 - Q_{S_r}(z, z'; S_r \setminus \eta_t + 2k\pi)] + O(\text{hcap}[\eta_t]^2),$$

and

$$\partial_t [1 - Q_{S_r}(z, z'; S_{r,t})] = \partial_t \sum_{k \in \mathbb{Z}} [1 - Q_{S_r}(z, z'; S_r \setminus \eta_t + 2k\pi)].$$

The result for general  $t$  follows from this and lemma 2.5. □

We can reparametrize  $\gamma_t$  so that  $r(t) = r - t$ . This is called the annulus parametrization. In this case,  $\tau_r = r$  and for any  $t < r$  we can write (5.5) as

$$\partial_t \log Q_{S_r}(z, z'; S_{r,t}) = -2\mathbf{F}(r-t, h_t(z) - U_t, h_t(z') - U_t). \quad (5.10)$$

**Proposition 5.1.** *For  $s > 0$ ,  $z, z' \in \mathbb{R}$ , define*

$$A(s, z, z') = \frac{H_{A_s}(\psi(z), 1)H_{A_s}(1, \psi(z' + is))}{H_{A_s}(\psi(z), \psi(z' + is))}.$$

*If  $\gamma_t$  has annulus parametrization, then for  $t < r$ ,*

$$\partial_t \log Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t) = -2\mathbf{A}(r-t, U_t' - U_t, W_t' - U_t). \quad (5.11)$$

*Proof.* We only need to prove the claim for the right derivative and  $t = 0$ , since the right-hand side of (5.11) is continuous in  $t$  and

$$\lim_{s \downarrow 0} \frac{1}{s} [\log Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_{t+s}) - \log Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t)] = \partial_s \log Q_{A_{r-t}}(\bar{U}_t', \bar{W}_t'; A_{r-t} \setminus \bar{\gamma}_s)|_{s=0},$$

where  $\bar{\gamma}_s = h_t \circ \gamma(t, t+s]$  is a  $SLE_\kappa$  curve in  $A_{r-t}$  starting from  $\bar{U}_t^1$ .

At  $t = 0$  we have

$$\begin{aligned} \partial_t \log Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t) &= \partial_t \log H_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t) \\ &= \frac{\partial_t H_{A_r \setminus \gamma_t}(\bar{u}', \bar{w}')}{H_{A_r}(\bar{u}', \bar{w}')} \end{aligned} \quad (5.12)$$

By using (5.10) we get

$$\begin{aligned}
\partial_t H_{A_r \setminus \gamma}(\bar{u}', \bar{w}') &= \partial_t e^r \sum_{k' \in \mathbb{Z}} H_{S_{r,t}}(u', w' + 2k'\pi) \\
&= -2e^r \sum_{k' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} H_{S_r}(u' - u, 2k\pi) H_{S_r}(2k\pi, w' + 2k'\pi - u) \\
&= -2 \sum_{k \in \mathbb{Z}} H_{S_r}(u' - u, 2k\pi) \sum_{k' \in \mathbb{Z}} e^r H_{S_r}(2k\pi, w' + 2k'\pi - u) \\
&= -2H_{A_r}(1, \bar{w}'/\bar{u}) \sum_{k \in \mathbb{Z}} H_{S_r}(u' - u, 2k\pi) \\
&= -2H_{A_r}(1, \bar{w}'/\bar{u}) H_{A_r}(\bar{u}'/\bar{u}, 1).
\end{aligned}$$

The result follows from this and (5.12). □

### 5.3.1 Radial and Annulus

Suppose with respect to a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\gamma_t$  is a radial  $SLE_\kappa$  curve from  $\bar{u}$  to 0 in  $\mathbb{D}$ . We assume  $\gamma_t$  has radial parametrization  $\log \bar{g}'_t(0) = at/2$ . With respect to  $\mathbb{P}$ ,  $\xi_t$  is a Brownian motion and  $\bar{\xi}_t = \bar{g}_t(\gamma(t)) = e^{i\xi_t}$ . Denote by  $\mu_{\mathbb{D}}(\bar{u}, 0)$  the distribution of  $\gamma_t$  and let  $\mu_{A_r}(\bar{u}, \bar{w})$  be the distribution of  $SLE_\kappa$  from  $\bar{u}$  to  $\bar{w}$  in  $A_r$ . The following result is stated in section 7.2 of [17]. Here, we give a proof with more details.

**Proposition 5.2.** *Let  $\tau_r = \inf\{t; \gamma_t \cap C_r \neq \emptyset\}$  and suppose  $t < \tau_r$ . Then*

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma) = \frac{|\bar{h}'_t(\bar{w})|^b |\bar{\phi}'_t(\bar{\xi}_t)|^b}{\bar{g}'_t(0)^b} \exp\{m_{\mathbb{D}}(C_r, \gamma_t)\} \Psi_{A_{r(t)}}(\bar{\phi}_t(\bar{\xi}_t), \bar{W}_t). \quad (5.13)$$

*Proof.* Let  $D \subset A_r$  be a simply connected domain that agrees with  $A_r$  in neighborhoods of  $\bar{u}, \bar{w}$ . It is easy to see that

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w}; D)}{d\mu_{A_r}(\bar{u}, \bar{w})}(\gamma) = 1\{\gamma \subset D\} \frac{\Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t; D_t)}{\Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)}.$$

Let  $|\bar{z}| = 1$  be another analytic boundary point of  $D$ . Then by definition of  $SLE_\kappa$  in  $A_r$ ,

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w}; D)}{d\mu_D(\bar{u}, \bar{w})}(\gamma_t) = \exp\left\{-\frac{\mathbf{c}}{2}m_{A_r}(\gamma_t, A_r \setminus D)\right\} 1\{\gamma_t \subset D\} \frac{\Psi_{A_r(t)}(\bar{U}_t, \bar{W}_t; D_t)}{\Psi_{D_t}(\bar{U}_t, \bar{W}_t)}.$$

Here,  $D_t = \mathfrak{h}_t(D \setminus \gamma_t)$  is a simply connected domain and  $\Psi_{A_r(t)}(\bar{U}_t, \bar{W}_t; D_t) = \|\mu_{A_r(t)}(\bar{U}_t, \bar{W}_t; D_t)\|$ .

Let  $\bar{Z}_t = \mathfrak{h}_t(\bar{z})$ . Since  $D$  is simply connected, we have

$$\frac{d\mu_D(\bar{u}, \bar{w})}{d\mu_D(\bar{u}, \bar{z})}(\gamma_t) = \frac{|\mathfrak{h}'_t(\bar{w})|^b \Psi_{D_t}(\bar{U}_t, \bar{W}_t)}{|\mathfrak{h}'_t(\bar{z})|^b \Psi_{D_t}(\bar{U}_t, \bar{Z}_t)}.$$

Moreover, by comparing chordal  $SLE_\kappa$  in  $D$  and  $\mathbb{D}$  we get

$$\frac{d\mu_D(\bar{u}, \bar{z})}{d\mu_{\mathbb{D}}(\bar{u}, \bar{z})}(\gamma_t) = \exp\left\{\frac{\mathbf{c}}{2}m_{\mathbb{D}}(\gamma_t, \mathbb{D} \setminus D)\right\} 1\{\gamma_t \subset D\} \frac{\Psi_{\bar{g}_t(D \setminus \gamma_t)}(\bar{\xi}_t, \bar{g}_t(\bar{w}))}{\Psi_{\mathbb{D}}(\bar{\xi}_t, \bar{g}_t(\bar{z}))}.$$

Finally, comparing chordal and radial  $SLE_\kappa$  in  $\mathbb{D}$  gives

$$\frac{d\mu_{\mathbb{D}}(\bar{u}, \bar{z})}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma_t) = \frac{|\bar{g}'_t(\bar{z})|^b \Psi_{\mathbb{D}}(\bar{\xi}_t, \bar{g}_t(\bar{z}))}{\bar{g}'_t(0)^{\bar{b}}}.$$

Note that

$$\Psi_{D_t}(\bar{U}_t, \bar{Z}_t) = |\bar{\phi}'_t(\bar{\xi}_t)|^{-b} |\bar{\phi}'_t(\bar{g}_t(\bar{z}))|^{-b} \Psi_{\bar{g}_t(D \setminus \gamma_t)}(\bar{\xi}_t, \bar{g}_t(\bar{z}))$$

and

$$m_{\mathbb{D}}(\gamma_t, \mathbb{D} \setminus D) = m_{A_r}(\gamma_t, A_r \setminus D) + m_{\mathbb{D}}(\gamma_t, C_r).$$

Therefore,

$$\frac{d\mu_{A_r}(\bar{u}, \bar{w}; D)}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma_t) = \frac{|\mathfrak{h}'_t(\bar{w})|^b |\bar{\phi}'_t(\bar{\xi}_t)|^b}{\bar{g}'_t(0)^{\bar{b}}} \exp\left\{\frac{\mathbf{c}}{2}m_{\mathbb{D}}(\gamma_t, C_r)\right\} 1\{\gamma_t \subset D\} \Psi_{A_r(t)}(\bar{U}_t, \bar{W}_t; D_t).$$

The result follows since this is true for any simply connected domain  $D$ . □

**Lemma 5.3.** For  $s > 0, x \in \mathbb{R}$ , define

$$L_s(x) = -\kappa \frac{\partial_x \Psi_{A_s}(\psi(x), e^{-s})}{\Psi_{A_s}(\psi(x), e^{-s})}.$$

If  $\gamma_t$  has the distribution of  $SLE_\kappa$  from  $\bar{u}$  to  $\bar{w}$  in  $A_r$ , then

$$dU_t = L_{r-t}(W_t - U_t)dt + \sqrt{\kappa}dB_t, \quad (5.14)$$

where  $B_t$  is a Brownian motion.

*Proof.* Let  $M_t$  denote the Radon-Nikodym derivative given in the statement of proposition 5.2 and note that  $|\bar{\phi}'(\bar{\xi}_t)| = \phi'(\xi_t)$ . With respect to  $\mathbb{P}$ ,  $M_t$  is a martingale and

$$dM_t = M_t \left[ b \frac{\phi''(\xi_t)}{\phi'(\xi_t)} + i\phi'(\xi_t)\bar{\phi}_t(\bar{\xi}_t) \frac{\partial_1 \Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)}{\Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)} \right] d\xi_t.$$

Here,  $\partial_1$  denotes the derivative with respect to the first argument. Using the Girsanov's theorem, there exists a probability measure  $\mathbb{P}'$  such that

$$d\xi_t = \left[ b \frac{\phi''(\xi_t)}{\phi'(\xi_t)} + i\phi'(\xi_t)\bar{\phi}_t(\bar{\xi}_t) \frac{\partial_1 \Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)}{\Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)} \right] dt + dB_t, \quad (5.15)$$

where  $B_t$  is a standard Brownian motion with respect to  $\mathbb{P}'$ .

For  $z \in S_{r,t}$ , the transformation  $\tilde{g}_t$  satisfies the radial Loewner equation

$$\partial_t \tilde{g}_t(z) = \frac{a}{2} \cot \left( \frac{\tilde{g}_t(z) - \xi_t}{2} \right).$$

By using the chain rule we get

$$\partial_t h_t(z) = \dot{\phi}_t(\tilde{g}_t(z)) + \phi'_t(\tilde{g}_t(z))\partial_t \tilde{g}_t(z).$$

Hence,

$$-\dot{r}(t)\mathbf{H}_{r(t)}(h_t(z) - U_t) - \frac{a}{2}\phi_t'(\tilde{g}_t(z)) \cot\left(\frac{\tilde{g}_t(z) - \xi_t}{2}\right) = \dot{\phi}_t(\tilde{g}_t(z)).$$

From lemma 2.5 we know  $\dot{r}(t) = -a\phi_t'(\xi_t)^2/2$ . Moreover,  $\cot(z) = 1/z + O(|z|)$  as  $z \rightarrow 0$ . By using equation (4.7) we can see that

$$\dot{\phi}_t(\xi_t) = -\frac{3a}{2}\phi_t''(\xi_t) = -\left(\frac{1}{2} + b\right)\phi_t''(\xi_t).$$

Therefore, (5.15) implies that

$$\begin{aligned} dU_t &= d\phi_t(\xi_t) = -b\phi_t''(\xi_t)dt + \phi_t'(\xi_t)d\xi_t \\ &= \left[ i\phi_t'(\xi_t)^2\bar{\phi}_t(\bar{\xi}_t) \frac{\partial_1 \Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)}{\Psi_{A_{r(t)}}(\bar{U}_t, \bar{W}_t)} \right] dt + \phi_t'(\xi_t)dB_t. \end{aligned}$$

We can use lemma 2.5 one more time to see that with annulus parametrization the last equation can be written as

$$dU_t = i\kappa\bar{U}_t \frac{\partial_1 \Psi_{A_{r-t}}(\bar{U}_t, \bar{W}_t)}{\Psi_{A_{r-t}}(\bar{U}_t, \bar{W}_t)} dt + \sqrt{\kappa}dB_t.$$

Using conformal covaraince,  $\Psi_{A_{r-t}}(\bar{U}_t, \bar{W}_t) = \Psi_{A_{r-t}}(\psi(W_t - U_t), e^{-r+t})$ . Moreover,

$$\partial_{(W_t - U_t)} \Psi_{A_{r-t}}(\psi(W_t - U_t), e^{-r+t}) = i\psi(W_t - U_t) \partial_1 \Psi_{A_{r-t}}(\psi(W_t - U_t), e^{-r+t})$$

and

$$\partial_1 \Psi_{A_{r-t}}(\bar{U}_t, \bar{W}_t) = -\psi(W_t - 2U_t) \partial_1 \Psi_{A_{r-t}}(\psi(W_t - U_t), e^{-r+t}),$$

which completes the proof. □

It is not hard to verify that for  $x \in (-\pi, \pi)$ ,  $\Psi_{A_r}(\psi(x), e^{-r})$  is an even function that is decreasing in  $|x|$ . Hence,  $\mathbf{L}_r(x)$  is an odd function with  $\mathbf{L}_r(x) \geq 0$  for  $x \in [0, \pi)$ . For more details see section 5 of [17].

### 5.3.2 Theorem

Before we prove our main result, we recall the Hörmander's theorem. Let  $\Omega \subset \mathbb{R}^n$  be an open set. A linear differential operator  $\mathcal{L}$  with  $C^\infty$  coefficients on  $\Omega$  is called *hypoelliptic* if for every distribution  $u$  on  $\Omega$ ,  $u$  is  $C^\infty$  when  $\mathcal{L}u$  is  $C^\infty$ . Assume  $X_0, X_1, \dots, X_k$  are first order homogeneous differential operators with  $C^\infty$  coefficients on  $\Omega$ . Let

$$\mathcal{L} = \sum_{j=1}^k X_j^2 + X_0 + c,$$

where  $c$  is a smooth function on  $\Omega$ . In [8], Hörmander established a characterization of hypoelliptic second order differential operators with  $C^\infty$  coefficients. In particular, he proved the following theorem.

**Theorem 5.1.** *If at all point in  $\Omega$  the rank of the Lie algebra generated by the vector fields  $X_0, X_1, \dots, X_k$  equals  $n$ , then  $\mathcal{L}$  is hypoelliptic.*

We say  $\mathcal{L}$  satisfies the *Hörmander's condition* if it satisfies the requirements of theorem 5.1. Having this result, we prove the smoothness of the partition function.

**Theorem 5.2.** *If  $\mathbf{u} = (u, u')$ ,  $\mathbf{w} = (w, w')$ , then  $V(r, \mathbf{u}, \mathbf{w})$  is a positive smooth function satisfying*

$$\begin{aligned} \partial_r V &= -2b\mathbf{A}V + \mathbf{L}_r(w - u)\partial_u V + \mathbf{H}_r(u' - u)\partial_{u'} V + \mathbf{H}_r(w' - u)\partial_{w'} V \\ &+ \mathbf{H}_r(w - u)\partial_w V + \frac{\kappa}{2}\partial_{uu} V = 0. \end{aligned} \quad (5.16)$$

*Proof.* For  $t < r$ , we have

$$Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t) = \exp \left\{ \int_0^t \partial_t \log Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_s) ds \right\}.$$

This is also true for  $t = r$  because  $Q_{A_r}(\bar{u}', \bar{w}'; A_r \setminus \gamma_t)$  is continuous at  $t = r$ . Proposition 5.1 gives us

$$V(r, \mathbf{u}, \mathbf{w}) = \mathbb{E} \left[ \exp \left\{ -2b \int_0^r \mathbf{A}(r-s, U'_s - U_s, W'_s - U_s) ds \right\} \right].$$

Denote by  $\mathcal{F}_t$  the  $\sigma$ -algebra generated by  $\gamma_t$ . Then

$$\begin{aligned} M_t &:= \mathbb{E} \left[ \exp \left\{ -2b \int_0^r \mathbf{A}(r-s, U'_s - U_s, W'_s - U_s) ds \right\} \middle| \mathcal{F}_t \right] \\ &= \exp \left\{ -2b \int_0^t \mathbf{A}(r-s, U'_s - U_s, W'_s - U_s) ds \right\} V(r-t, U_t, U'_t, W_t, W'_t) \end{aligned}$$

is a martingale. Recall that as in proposition 4.2,  $\mathbf{H}_t^R(z) = \text{Re}[\mathbf{H}_t(z+it)]$  for  $z \in \mathbb{R}$ . Using lemma 5.3 and proposition 4.2,

$$\begin{aligned} dU_t &= \mathbf{L}_{r-t}(W_t - U_t)dt + \sqrt{\kappa}dB_t \\ dU'_t &= \mathbf{H}_{r-t}(U'_t - U_t)dt \\ dW_t &= \mathbf{H}_{r-t}^R(W_t - U_t)dt \\ dW'_t &= \mathbf{H}_{r-t}^R(W'_t - U_t)dt. \end{aligned}$$

Consider the process

$$Z_t = (r-t, Q_{A_r}(u', w'; A_r \setminus \gamma_t), U_t, U'_t, W_t, W'_t).$$

This is a diffusion process with infinitesimal generator

$$\begin{aligned} A\phi(\mathbf{z}) &= -2b\mathbf{A}\partial_{z_2}\phi - \partial_{z_1}\phi + \mathbf{L}_{z_1}(z_5 - z_3)\partial_{z_3}\phi + \mathbf{H}_{z_1}(z_4 - z_3)\partial_{z_4}\phi \\ &\quad + \mathbf{H}_{z_1}^R(z_5 - z_3)\partial_{z_5}\phi + \mathbf{H}_{z_1}^R(z_6 - z_3)\partial_{z_6}\phi + \frac{\kappa}{2}\partial_{z_3z_3}, \end{aligned} \tag{5.17}$$

for any  $C^2$  function  $\phi \in \mathcal{D}_A$  and suitable  $\mathbf{z} = (z_1, \dots, z_6) \in \mathbb{R}^6$ . Here,  $\mathcal{D}_A$  is the domain of the generator  $A$ . Using the Fokker–Planck equation, we can see that (5.17) holds for any  $\phi \in \mathcal{D}_A$  as long as the derivatives are interpreted in the weak sense. Let

$$f(\mathbf{z}) = z_2V(z_1, z_3, z_4, z_5, z_6).$$

Since  $M_t = f(Z_t)$  is a martingale, we have  $\mathbb{E}(f(Z_t)) - f(Z_0) = 0$  for all  $Z_0$  and  $t \leq r$ . In particular,  $f \in \mathcal{D}_A$  and  $Af = 0$ . Therefore, at least in the weak sense  $\mathcal{L}V = 0$  for all  $r, u, u', w', w$ , where

$$\mathcal{L} = -2b\mathbf{A} - \partial_r + \mathbf{L}_r(w-u)\partial_u + \mathbf{H}_r(u'-u)\partial_{u'} + \mathbf{H}_r^R(w-u)\partial_w + \mathbf{H}_r^R(w'-u)\partial_{w'} + \frac{\kappa}{2}\partial_{uu}.$$

We now prove that  $\mathcal{L}$  satisfies the Hörmander's condition, from which we conclude  $\mathcal{L}$  is hypoelliptic and  $V$  is a smooth function using theorem 5.1. Note that

$$\mathcal{L} = \frac{1}{2}A_1^2 + A_0 + C,$$

where

$$\begin{aligned} A_1 &= \sqrt{\kappa}\partial_u \\ A_0 &= -\partial_r + \mathbf{L}_r(w-u)\partial_u + \mathbf{H}_r(u'-u)\partial_{u'} + \mathbf{H}_r^R(w-u)\partial_w + \mathbf{H}_r^R(w'-u)\partial_{w'} \\ C &= -2b\mathbf{A}. \end{aligned}$$

We will show that the Lie algebra generated by the vector fields

$$A_0, A_1, [A_1, A_0], [A_1, [A_1, A_0]], \dots \tag{5.18}$$

has rank 5 for every  $r, u, u', w, w'$ . Recall that for two vector fields  $X = \sum_{i=1}^n X_i \partial_{x_i}$ ,  $Y = \sum_{i=1}^n Y_i \partial_{x_i}$  on a smooth manifold with coordinate system  $x_1, \dots, x_n$ , we have

$$[X, Y] = \sum_{i=1}^n (X(Y_i) - Y(X_i)) \partial_{x_i}.$$

Using this, we can see that for every  $n \in \mathbb{N}$ , the  $(n+2)$ -th term in (5.18) can be written as

$$[\partial_{u^{(n)}} L_r(w-u)]\partial_u + [\partial_{u^{(n)}} \mathbf{H}_r(u'-u)]\partial_{u'} + [\partial_{u^{(n)}} \mathbf{H}_r^R(w-u)]\partial_w + [\partial_{u^{(n)}} \mathbf{H}_r^R(w'-u)]\partial_{w'},$$

where  $\partial_{u^{(n)}}$  denotes the  $n$ -th derivative with respect to  $u$ . It is not hard to see that among the vector fields given in (5.18),  $A_0$  is the only vector field with nonzero coefficient for  $\partial_r$ . Also for  $A_1$ , only  $\partial_u$  has nonzero coefficient. Hence it is enough to show that the span of the vector fields  $\partial_{u'}, \partial_w, \partial_{w'}$  is a subspace of the span of

$$[A_1, A_0], [A_1, [A_1, A_0]], [A_1, [A_1, [A_1, A_0]]], \dots$$

for every  $r, u, u', w, w'$ . For a fixed  $r > 0$ , define the functions  $f_0(z) = \mathbf{H}_r(z)$ ,  $f_1(z) = \mathbf{H}_r(z + ir)$ . Note that  $f_1'(z) = \partial_z \mathbf{H}_r^R(z)$  for  $z \in \mathbb{R}$ , since  $\text{Im}[\mathbf{H}_r(z + ir)] = 1$ . We want to show that for all  $0 < z_1 < 2\pi$  and  $0 \leq z_2 < z_3 < 2\pi$ , there exist three linearly independent vectors among

$$v_k := (f_0^{(k)}(z_1), f_1^{(k)}(z_2), f_1^{(k)}(z_3)), \quad k \geq 1.$$

Suppose the claim is not true and there exist constants  $a_j, j \in \{1, 2, 3\}$  such that they are not all equal to 0 and  $a_1 f_0^{(k)}(z_1) + a_2 f_1^{(k)}(z_2) + a_3 f_1^{(k)}(z_3) = 0$  for all  $k \geq 1$ . Consider the function

$$\tilde{f}(\varepsilon) = a_1 f_0(z_1 + \varepsilon) + a_2 f_1(z_2 + \varepsilon) + a_3 f_1(z_3 + \varepsilon).$$

Note that  $\tilde{f}^{(k)}(0) = 0$  for all  $k \geq 0$ . Let

$$\varepsilon_0 := \min\{z_1, (2\pi - z_1), |z_2 + ir|, |z_2 + ir - 2\pi|, |z_3 + ir|, |z_3 + ir - 2\pi|\}$$

and let  $B_{\varepsilon_0}(0)$  be the open ball of radius  $\varepsilon_0$  around the origin. Since  $\mathbf{H}_r(z)$  is an elliptic function with periods  $2\pi, 2ir$  and poles at  $2k\pi + i2mr$ , the function  $\tilde{f}(\varepsilon)$  is analytic on  $B_{\varepsilon_0}(0)$ . Therefore, for some constant  $c$ ,  $\tilde{f} \equiv c$  on  $B_{\varepsilon_0}(0)$ . But this is a contradiction because  $\tilde{f}$  is continuous and there exists  $\varepsilon \in \partial B_{\varepsilon_0}(0)$  such that  $\tilde{f}(\varepsilon) = \infty$ .  $\square$

### 5.3.3 The case $n > 2$

In this section we explain how the proof of theorem 5.2 can be extended to the case  $n > 2$ . Let  $\mathbf{z}, \mathbf{w}, \gamma$  be as in section 5.2.1 and define

$$V(r, \mathbf{z}, \mathbf{w}) = \frac{\Psi_{A_r}(\mathbf{z}, \mathbf{w})}{\Psi_{A_r}(z^1, w^1) \prod_{j=2}^n H_{A_r}(z^j, w^j)^b} = \frac{\prod_{j=2}^n \Psi_{A_r}(z^j, w^j)}{\prod_{j=2}^n H_{A_r}(z^j, w^j)^b} \mathbb{E}[Y(\gamma)],$$

where  $\mathbb{E}$  denotes expectation with respect to  $\mu_{A_r}^\#(z^1, w^1) \times \dots \times \mu_{A_r}^\#(z^n, w^n)$ .

**Lemma 5.4.** *Let  $m_t$  denote the Brownian loop measures of loops in  $A_r \setminus \gamma_t^1$  and let  $m = m_0$ . Let  $\mathbf{z}_t = (\gamma^1(t), z_2, \dots, z_n)$  and define  $\gamma_t$  to be  $n$ -tuples of curves connecting  $\mathbf{z}_t$  to  $\mathbf{w}$  in  $A_r \setminus \gamma_t$ . Then*

$$\sum_{j=2}^n m[K_j(\gamma)] = \sum_{j=2}^n m_t[K_j(\gamma_t)] + \sum_{j=2}^n m(\gamma_t^1, \gamma^j).$$

*Proof.* We prove this by induction. It is easy to see that claim is true for  $n = 2$ . Assuming the claim holds for  $n - 1$ , we prove it for  $n$ . Let  $\gamma = (\gamma', \gamma_n)$  and  $\gamma_t = (\gamma_t', \gamma_n)$ . Using lemma 5.1,

$$\sum_{j=2}^n m[K_j(\gamma)] = \sum_{j=2}^{n-1} m[K_j(\gamma')] + m(\gamma_t^1, \gamma^n) + m_t(\gamma_t', \gamma^n).$$

Hence, by the induction hypothesis for  $n - 1$ ,

$$\sum_{j=2}^n m[K_j(\gamma)] = \sum_{j=2}^{n-1} m_t[K_j(\gamma_t')] + \sum_{j=2}^n m(\gamma_t^1, \gamma^j) + m_t(\gamma_t', \gamma^n).$$

Using lemma 5.1 one more time completes the result. □

Let  $\tilde{h}_t : A_r \setminus \gamma_t^1 \rightarrow A_{r(t)}$  be our usual conformal transformation. Let  $Z_t^j = \tilde{h}_t(z^j)$ ,  $W_t^j = \tilde{h}_t(w^j)$ ,  $\mathbf{Z}_t = (Z_t^1, \dots, Z_t^n)$ ,  $\mathbf{W}_t = (W_t^1, \dots, W_t^n)$ . Using lemma 5.4, we can see that

$$M_t := \frac{\prod_{j=2}^n \Psi_{A_r}(z^j, w^j)}{\prod_{j=2}^n H_{A_r}(z^j, w^j)^b} \mathbb{E}[Y(\gamma) | \gamma_t^1] = \prod_{j=2}^n Q_{A_r}(z^j, w^j; A_r \setminus \gamma_t^1) V(r-t, \mathbf{Z}_t, \mathbf{W}_t)$$

is a martingale. At least in the weak sense,  $M_t$  satisfies a differential equation related to the infinitesimal generator of the process

$$(r-t, Q_{A_r}(z^2, w'; A_r \setminus \gamma_t^1), \dots, Q_{A_r}(z^n, w^n; A_r \setminus \gamma_t^1), \mathbf{Z}_t, \mathbf{W}_t).$$

Proposition 5.1 gives the time derivative of  $Q_{A_r}(z^j, w^j; A_r \setminus \gamma_t^1)$ . The derivatives of  $V(r-t, \mathbf{Z}_t, \mathbf{W}_t)$  can be described using proposition 4.2. Using these, we can write a differential equation describing  $V(r, \mathbf{z}, \mathbf{w})$ . Verifying the Hörmander conditions can be done similar to the proof of theorem 5.2. Therefore, we have justified the following theorem.

**Theorem 5.3.** *If  $\kappa \leq 4$ , then  $\Psi_{A_r}(z, w)$  is a smooth function of  $r, z^1, \dots, z^n, w^1, \dots, w^n$ .*

## 5.4 Two-sided

In this section, we start with reviewing the boundary perturbation property for radial  $SLE_\kappa$ . This is similar to the same property for chordal  $SLE_\kappa$ , which is described in (2.20). Similar to the chordal case, this allows to obtain radial  $SLE_\kappa$  in a smaller domain by a change of measure.

Next, we construct two-sided  $SLE_\kappa$  using two independent radial  $SLE_\kappa$ . For a simply connected domain  $0 \in D$  with boundary points  $z, w$ , two-sided  $SLE_\kappa$  from  $z$  to  $w$  is usually considered as chordal  $SLE_\kappa$  from  $z$  to  $w$  conditioned to go through  $0$  (although this is an event with zero probability, there are ways to make this precise). This involves weighting the chordal  $SLE_\kappa$  by the Green's function until the curve hits the origin (the fact that two-sided  $SLE_\kappa$  is continuous at the origin is proved in [18]). The rest of the curve has the distribution of chordal  $SLE_\kappa$  from  $0$  to  $w$  in the slit domain. Our construction allows us to grow the curves from  $z$  and  $w$  at the same time.

Finally, we make a connection between two-sided  $SLE_\kappa$  in  $\mathbb{D}$  and two chordal  $SLE_\kappa$  paths in annuli. In particular, we show that before reaching the boundary, the two measures are absolutely continuous and we describe the Radon-Nikodym derivative.

### 5.4.1 Boundary perturbation

Suppose  $D \subset \mathbb{D}$  is a simply connected domain such that  $0 \in D$  and  $D$  agrees with  $\mathbb{D}$  in a neighborhood of 1. Let  $K = \mathbb{D} \setminus D$ . Let  $\bar{G} : D \rightarrow \mathbb{D}$  be the unique conformal transformation with  $\bar{G}(0) = 0$  and  $\bar{G}'(0) > 0$ . Let  $\gamma_t$  be a radial  $SLE_\kappa$  from 1 to 0 in  $\mathbb{D}$  (continuity at 0 is shown in [18]) and let  $\bar{g}_t : \mathbb{D} \setminus \gamma_t \rightarrow \mathbb{D}$  be the unique conformal transformation satisfying  $\bar{g}_t(0) = 0$ ,  $\bar{g}_t'(0) > 0$ . Let  $\bar{\gamma}_t = \bar{g}_t(\gamma_t)$ , define  $\bar{\varphi}_t : \mathbb{D} \setminus \bar{\gamma}_t \rightarrow \mathbb{D}$  to be the conformal transformation satisfying  $\bar{\varphi}_t(0) = 0$ ,  $\bar{\varphi}_t'(0) > 0$  and let  $\bar{\Phi}_t := \bar{\varphi}_t \circ \bar{G} \circ \bar{g}_t^{-1}$ . Let  $g_t$  be the unique conformal transformation that is continuous in  $t$  and satisfies

$$\bar{g}_t(e^{iz}) = e^{ig_t(z)}, \quad g_0(z) = z.$$

Define  $G$ ,  $\varphi_t$ ,  $\Phi_t$  in a similar way (see figure 5.1). As before, consider the transformation  $\psi(z) = e^{iz}$ . Let  $U_t$  be the continuous process satisfying  $\psi(U_t) = \bar{g}_t(\gamma(t))$ ,  $U_0 = 0$ . We know that  $\bar{g}_t$  satisfies the radial Loewner equation

$$\partial_t \bar{g}_t(z) = \frac{a}{2} \bar{g}_t(z) \frac{e^{iU_t} + \bar{g}_t(z)}{e^{iU_t} - \bar{g}_t(z)}, \quad \bar{g}_0(z) = z,$$

for any  $z \in \mathbb{D} \setminus \gamma_t$ . Equivalently, if  $\cot_2(z) := \cot(z/2)$ , then

$$\partial_t g_t(z) = \frac{a}{2} \cot_2(g_t(z) - U_t), \quad g_0(z) = z.$$

If  $\gamma_t$  does not have radial parameterization, then the term  $a/2$  is substituted with  $\partial_t \log \bar{g}_t'(0)$ .

**Lemma 5.5.** *The Brownian loop measure of loops with nonzero winding number in  $\mathbb{D}$  that intersect  $\mathbb{D} \setminus D$  is*

$$\frac{\log \bar{G}'(0)}{6}.$$

*Proof.* See corollary 3.12 in [17]. □

**Lemma 5.6.** *Let  $\gamma_t \subset D$  be a deterministic curve such that  $\log \bar{g}_t'(0) = at/2$  and  $\gamma(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

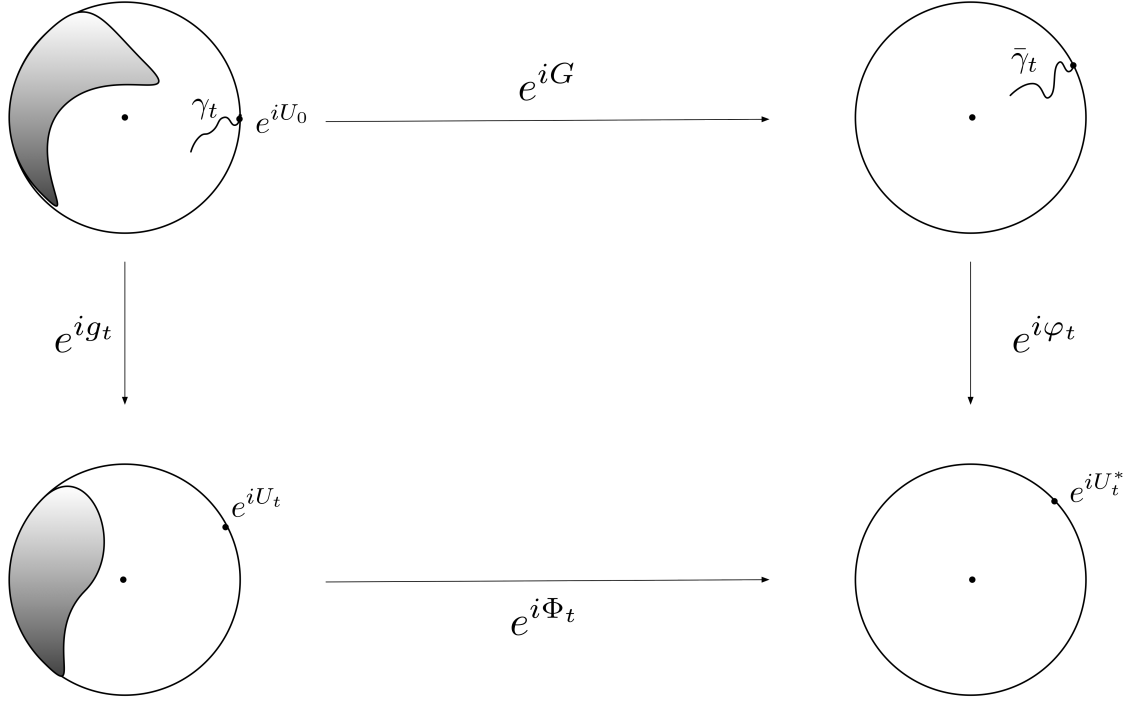


Figure 5.1: Shaded area in unit disk on the top left represents  $K = \mathbb{D} \setminus D$ .

Define  $\tau_r = \inf\{t; \gamma_t \cap C_r \neq \emptyset\}$ . Then

$$\lim_{t \rightarrow \infty} \bar{\Phi}'_t(0) = 1, \quad \lim_{r \rightarrow \infty} \Phi'_{\tau_r}(U_{\tau_r}) = 1.$$

*Proof.* Schwarz lemma and Koebe-1/4 theorem imply

$$1 \leq \bar{\Phi}'_t(0) \leq \frac{1}{\text{dist}(0, \bar{g}_t(K))}.$$

Hence, to prove the first equality it suffices to show that  $\text{dist}(0, \bar{g}_t(K)) \rightarrow 1$  as  $t \rightarrow \infty$ . Let  $T_t$  denote the first time a Brownian motion  $B_t$  starting at 0 exits  $D \setminus \gamma_t$ . It is not hard to see that since  $\lim_{t \rightarrow \infty} \gamma(t) = 0$ , then

$$\lim_{t \rightarrow \infty} \mathbb{P}[B_{T_t} \in \gamma_t] = 1.$$

Using the conformal invariance of Brownian motion, if  $\bar{T}_t$  is the first time a Brownian motion  $B_t, B_0 = 0$  exits  $\bar{g}_t(D \setminus \gamma_t)$ , then  $\mathbb{P}[B_{\bar{T}_t} \in C_0] \rightarrow 1$  as  $t \rightarrow \infty$ . From this, we get  $\text{dist}(0, \bar{g}_t(K)) \rightarrow 1$

as  $t \rightarrow \infty$ .

To prove the second equality, we need the following fact (see Lemma 2.6 in [18] for more details and a proof). Choose  $d$  such that  $e^{-d} \leq \text{dist}(0, K)$  and let  $r > d$ . There exists a unique open connected arc  $\eta(0, 1) \subset C_d$  such that  $\eta(0+), \eta(1-) \in \gamma_{\tau_r}$  and  $\eta(0, 1)$  disconnects  $K$  from 0 in  $\mathbb{D} \setminus \gamma_{\tau_r}$ . Consider the disjoint union  $C_0 = l_1 \cup l_2 \cup l_3 \cup \{\psi(U_{\tau_r})\}$ , where  $l_3$  is the unique closed connected arc with endpoints  $\bar{g}_{\tau_r}(\eta(0+)), \bar{g}_{\tau_r}(\eta(1-))$  and  $l_1, l_2$  are open connected arcs. Then

$$\text{diam}[\bar{g}_{\tau_r} \circ \eta(0, 1)] \leq c_0 e^{-(r-d)/2} \min\{|l_1|, |l_2|\},$$

where  $|\cdot|$  denotes length and  $c_0$  is an absolute constant. Let  $\bar{K}_r = \bar{g}_{\tau_r}(K)$ . Since  $\eta(0, 1)$  disconnects  $K$  from 0 in  $\mathbb{D} \setminus \gamma_{\tau_r}$ , the curve  $\bar{g}_{\tau_r} \circ \eta(0, 1)$  disconnects  $\bar{K}_r$  from 0 in  $\mathbb{D}$ . Therefore,

$$\text{diam}(\bar{K}_r) \leq c e^{-(r-d)/2} \min\{|l_1|, |l_2|\}, \quad \min\{|l_1|, |l_2|\} \leq \pi \text{dist}(\psi(U_{\tau_r}), \bar{K}_r), \quad (5.19)$$

where  $c$  is an absolute constant. Note that  $\Phi'_{\tau_r}(\psi(U_{\tau_r}))$  is the probability that a Brownian motion from 0 to  $\psi(U_{\tau_r})$  in  $\mathbb{D}$  does not hit  $\bar{K}_r$ . One can verify that for some constant  $c_*$ ,

$$1 - \Phi'_{\tau_r}(U_{\tau_r}) \leq c_* \text{diam}(\bar{K}_r)^2 \text{dist}(U_{\tau_r}, \bar{K}_r)^{-2}.$$

To see this let  $\sigma$  be the first time Brownian motion  $B$  starting from the origin exits  $\mathbb{D} \setminus \bar{K}_r$ . Then

$$1 - \Phi'_{\tau_r}(U_{\tau_r}) = \frac{\mathbb{E}[H_{\mathbb{D}}(B_{\sigma}, U_{\tau_r}) 1\{B_{\sigma} \in \mathbb{D}\}]}{H_{\mathbb{D}}(0, U_{\tau_r})}.$$

It is not hard to see that

$$\mathbb{P}[B_{\sigma} \in \mathbb{D}] = O(\text{diam}(\bar{K}_r)).$$

Using the exact form of Poisson kernel in  $\mathbb{D}$ , there exists a constant  $c$  such that for any  $w \in \bar{K}_r$ ,

$$H_{\mathbb{D}}(w, U_{\tau_r}) < c \frac{\text{diam}(\bar{K}_r)}{\text{dist}(\psi(U_{\tau_r}), \bar{K}_r)^2}.$$

The result follows from this and equation (5.19).  $\square$

**Proposition 5.3.** *Define  $\tau = \inf\{t; \gamma_t \not\subset D\}$  and let  $U_t$  be the continuous process satisfying  $e^{iU_t} = \bar{g}_t(\gamma(t))$ ,  $U_0 = 0$ . Then*

$$M_t = 1\{t < \tau\} \Phi_t'(U_t)^b \bar{\Phi}_t'(0)^{\tilde{b}} \exp\left\{\frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, K)\right\}, \quad \tilde{b} = \frac{b}{6} + \frac{\mathbf{c}}{12}, \quad (5.20)$$

is a uniformly integrable martingale and

$$\frac{d\mu_D(1,0)}{d\mu_{\mathbb{D}}(1,0)}(\gamma) = M_t.$$

Moreover,

$$M_\infty := \lim_{t \rightarrow \infty} M_t = 1\{\gamma \subset D\} \exp\left\{\frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma, \mathbb{D} \setminus D)\right\}.$$

*Proof.* Let  $\gamma_t$  be a radial  $SLE_K$  with respect to the probability space  $(\mathbb{P}, \Gamma, \mathcal{F})$ . Under this assumptions,  $U_t$  is a standard Brownian motion. Let  $\tau = \inf\{t; \gamma_t \cap K \neq \emptyset\}$  and assume  $t < \tau$ . We can see from lemma 2.5 that

$$\log \bar{\Phi}_t'(0) = \frac{a}{2} \int_0^t \Phi_s'(U_s)^2 ds. \quad (5.21)$$

Note that if  $h_t := g_t^{-1}$ , then  $\Phi_t = \varphi_t \circ G \circ h_t$ . Using this, the chain rule and radial Loewner equation, we can see that

$$\begin{aligned} \dot{\Phi}_t(z) &= \dot{\varphi}_t(G \circ h_t(z)) + \varphi_t'(G \circ h_t(z)) G'(h_t(z)) \dot{h}_t(z), \\ &= \frac{a\Phi_t'(U_t)^2}{2} \cot_2(\Phi_t(z) - U_t^*) + \Phi_t'(z) g_t'(h_t(z)) \dot{h}_t(z), \\ &= \frac{a\Phi_t'(U_t)^2}{2} \cot_2(\Phi_t(z) - U_t^*) - \frac{a\Phi_t'(z)}{2} \cot_2(z - U_t). \end{aligned}$$

Taking limit as  $z \rightarrow U_t$  gives

$$\dot{\Phi}_t(U_t) = -3a\Phi_t''(U_t)/2. \quad (5.22)$$

Moreover, we can take derivative (with respect to  $z$ ) of the right-hand-side of the equation above

and let  $z \rightarrow U_t$  to get

$$\dot{\Phi}'_t(U_t) = \frac{a}{2} \left[ \frac{\Phi''_t(U_t)^2}{2\Phi'_t(U_t)} - \frac{4\Phi'''_t(U_t)}{3} \right]. \quad (5.23)$$

Therefore, an application of the Itô's formula gives

$$dU_t^* = d\Phi_t(U_t) = -b\Phi''_t(U_t)dt + \Phi'_t(U_t)dU_t. \quad (5.24)$$

If  $\gamma_t$  is a  $SLE_\kappa$  in  $D$ , then  $\bar{\gamma}_t$  is a (time change of)  $SLE_\kappa$  in  $\mathbb{D}$  and  $U_t^*$  is a Brownian motion (with an appropriate time change). Let  $Z_t = \Phi'_t(U_t)^b$ . Using the Itô's formula and (5.23), we get

$$\frac{dZ_t}{Z_t} = \left[ \frac{a\mathbf{c}}{12}S\Phi_t(U_t) + \frac{ab}{12}(1 - \Phi'_t(U_t)^2) \right] dt + b\frac{\Phi''_t(U_t)}{\Phi'_t(U_t)}dU_t.$$

Here,  $S$  denotes the Schwarzian derivative

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3f''(z)^2}{2f'(z)^2}.$$

If  $K = \mathbb{D} \setminus D$ , then

$$\hat{m}_{\mathbb{D}}(\gamma_t, K) := \frac{-a}{6} \int_0^t S\Phi_s(U_s)ds$$

is the Brownian loop measure of loops in  $\mathbb{D}$  that have zero winding number and intersect both  $\gamma_t$  and  $K$  (see [13] for details). Using lemma 5.5 and an straightforward inclusion-exclusion argument, we can see that the Brownian loop measure of loops with nonzero winding number that intersect both  $\gamma_t$  and  $K$  is

$$\frac{1}{6} [\log \bar{G}'(0) - \log \bar{\Phi}'_t(0)] = \frac{a}{12} \int_0^t (1 - \Phi'_s(U_s)^2)ds.$$

Here for the second equality, we have used (5.21) and the fact  $\bar{\Phi}_t \circ \bar{g}_t = \bar{\varphi}_t \circ \bar{G}$ . Therefore, for  $t < \tau$ ,

$$\left[ \frac{a\mathbf{c}}{12}S\Phi_t(U_t) + \frac{ab}{12}(1 - \Phi'_t(U_t)^2) \right] = \frac{\mathbf{c}}{2}m_{\mathbb{D}}(\gamma_t, K) + \tilde{b} [\log \bar{\Phi}'_t(0) - \log \bar{G}'(0)]$$

and

$$M_t = \Phi'_t(U_t)^b \bar{\Phi}'_t(0)^{\bar{b}} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma_t, K) \right\}, \quad t < \tau,$$

is a local martingale satisfying

$$dM_t = b \frac{\Phi''_t(U_t)}{\Phi'_t(U_t)} M_t dU_t, \quad M_0 = G'(0)^b \bar{G}'(0)^{\bar{b}}.$$

Let  $\tau_n = \min(n, \inf\{t; \text{dist}(\gamma_t, K) < e^{-n}\})$ . Note that for  $t < \tau$ ,  $\Phi'_t(U_t)$  is the probability that Brownian motion starting at 0 conditioned to exit  $\mathbb{D}$  at  $\psi(U_t)$  does not hit  $\bar{g}_t(K)$ . Using Koebe-1/4 theorem, it is not hard to see that  $\bar{\Phi}'_t(0)$  is uniformly bounded for all  $t \leq \tau_n$ . Finally,  $m_{\mathbb{D}}(\gamma_t, K)$  is uniformly bounded for all  $t \leq \tau_n$  since  $\text{dist}(\gamma_t, K) \geq 2^{-n}$ . Therefore, for all  $n \in \mathbb{Z}^+$ , the local martingale  $M_{t \wedge \tau_n}$  is uniformly bounded and hence is a martingale.

Let  $\hat{\mathbb{P}}_n$  be the probability measure obtained from weighting  $\mathbb{P}$  by  $M_{\tau_n}/M_0$ . With respect to  $\hat{\mathbb{P}}_n$ , equation (5.24) becomes

$$dU_t^* = \Phi'_t(U_t) dW_t, \quad t < \tau_n,$$

where  $W_t$  is a standard Brownian motion. That is, with respect to  $\hat{\mathbb{P}}_n$  the curve  $\gamma_{t \leq \tau_n}$  has the distribution of radial  $SLE_{\kappa}$  from 1 to 0 in  $D$ . Let  $\hat{\mathbb{P}} = \hat{\mathbb{P}}_{\infty}$  denote the probability measure obtained from applying the Kolmogorov extension theorem to the consistent measures  $\hat{\mathbb{P}}_n$ . Since  $\tau_n \uparrow \tau$ , with respect to  $\hat{\mathbb{P}}$  and for any  $t < \tau$  the curve  $\gamma_t$  has the distribution of radial  $SLE_{\kappa}$  in  $D$ . Since radial  $SLE_{\kappa \leq 4}$  in  $D$  stays at a positive distance from  $K$  and goes to 0, we have  $m_{\mathbb{D}}(\gamma_{\infty}, K) < \infty$ . It follows from this and lemma 5.6 that with  $\hat{\mathbb{P}}$ -probability one  $M_{\tau} = M_{\infty} < \infty$ . From this, it is not hard to see that

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{M_{\tau}}{M_0}$$

(e.g. see theorem 5.3.3 in [7]). Since  $\kappa \leq 4$ , we have  $M_{\tau} = 0$  in case  $\tau < \infty$ . If  $\tau = \infty$ , then result follows from lemma 5.6. □

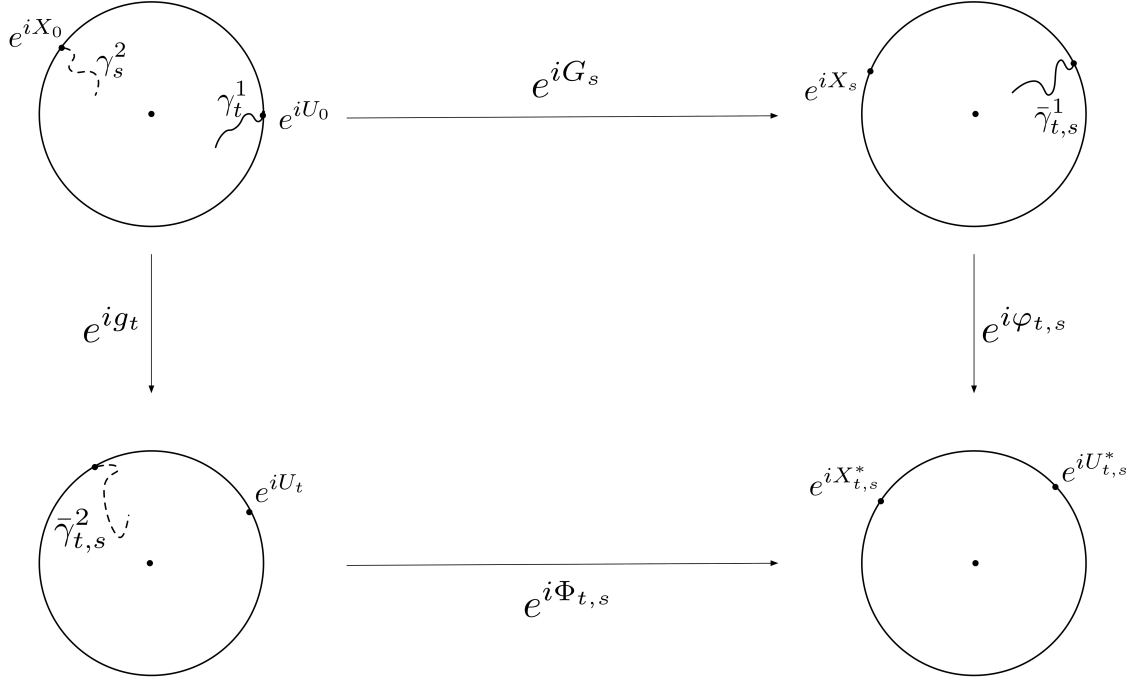


Figure 5.2: Two-sided  $SLE_\kappa$

#### 5.4.2 Two-sided SLE

Suppose  $\gamma_t^1, \gamma_s^2$  are two independent radial  $SLE_\kappa$  in  $\mathbb{D}$ , as shown in figure 5.2. Define  $\tau = \inf\{t; \gamma_t^1 \cap \gamma_t^2 \neq \emptyset\}$  to be the first time the curves intersect. Let  $t, s < \tau$  and define  $\bar{g}_t : \mathbb{D} \setminus \gamma_t^1 \rightarrow \mathbb{D}$  be the unique conformal transformation satisfying  $\bar{g}_t(0) = 0, \bar{g}_t'(0) > 0$ . Let  $g_t$  be the conformal transformation that is continuous in  $t$  and

$$\bar{g}_t(e^{iz}) = e^{ig_t(z)}, \quad g_0(z) = z.$$

Let  $\bar{\gamma}_{t,s}^2 = \bar{g}_t(\gamma_s^2)$ . Similar to  $\bar{g}_t, g_t$ , define the conformal transformations  $\bar{G}_s, G_s$  for  $\gamma_s^2$  and let  $\bar{\gamma}_{t,s}^1 = G_s(\gamma_t^1)$ . Finally, in a similar way to  $\bar{g}_t, g_t$ , define the transformations  $\bar{\varphi}_{t,s}, \varphi_{t,s}$  for  $\bar{\gamma}_{t,s}^1$  and define the transformations  $\bar{\Phi}_{t,s}, \Phi_{t,s}$  for  $\bar{\gamma}_{t,s}^2$  (see figure 5.2). For now, we assume  $\gamma_t^1, \gamma_t^2$  have radial parametrization and  $\log \bar{g}_t'(0) = at/2, \log \bar{G}_s'(0) = as/2$ . Under this parametrization,  $U_t, X_s$  are independent standard Brownian motions in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let

$$Z_{t,s} = \Phi_{t,s}'(U_t)^b \varphi_{t,s}'(X_s)^b. \quad (5.25)$$

We will write  $\Phi_t, \varphi_t, U_t^*, X_t^*, Z_t$ , etc. for  $\Phi_{t,t}, \varphi_{t,t}, U_{t,t}^*, X_{t,t}^*, Z_{t,t}$ , etc.. Let  $\Theta_t = X_t^* - U_t^*$  and  $K_t = \Phi_t'(U_t)^2 + \varphi_t'(X_t)^2$ .

Before taking our next steps, we briefly recall the definition of two-sided  $SLE_\kappa$ . As mentioned before, two-sided  $SLE_\kappa$  from  $\bar{u} = e^{iU_0}$  to  $\bar{x} = e^{iX_0}$  is chordal  $SLE_\kappa$  conditioned to go through the origin. In particular, it can be defined as chordal  $SLE_\kappa$  weighted by the Green's function, which is proportional to  $\sin_2(g_t(X_0) - U_t)^{4a-1}$ . Equivalently, it can be considered as radial  $SLE_\kappa$  from  $\bar{u}$  to 0 weighted by  $\sin_2(g_t(X_0) - U_t)^a$ . After reaching the origin (say at time  $T$ ), the rest of the process has the distribution of chordal  $SLE_\kappa$  from 0 to  $\bar{x}$  in  $\mathbb{D} \setminus \gamma_T$  (see [18] for more details). Straightforward calculation using the Itô's formula show that

$$\sin_2(g_t(X_0) - U_t)^a g_t'(X_0)^b e^{3a^2t/8} \quad (5.26)$$

is a martingale and weighting radial  $SLE_\kappa$  by this martingale gives two-sided  $SLE_\kappa$ .

**Lemma 5.7.** *The Brownian loop measure of loops in  $\mathbb{D}$  that intersect both  $\gamma_t^1$  and  $\gamma_s^2$  and have nonzero winding number is*

$$\frac{\log \bar{G}'_s(0) - \log \bar{\Phi}'_{t,s}(0)}{6} = \frac{\log \bar{g}'_t(0) - \log \bar{\varphi}'_{t,s}(0)}{6}.$$

*Proof.* This follows from lemma 5.5 and an easy inclusion-exclusion argument. □

**Lemma 5.8.** *Let  $\tilde{b} = b/6 + c/12$  and assume  $t < \tau$ . Then*

$$M_t = Z_t \bar{\Phi}'_t(0)^{\tilde{b}} e^{-\tilde{b}at/2} \exp \left\{ \frac{c}{2} m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) + \frac{ab}{4} \int_0^t \frac{K_r}{\sin_2(\Theta_r)^2} dr \right\},$$

*is a local martingale satisfying*

$$dM_t = bM_t \left[ \frac{\Phi_t''(U_t)}{\Phi_t'(U_t)} dU_t + \frac{\varphi_t''(X_t)}{\varphi_t'(X_t)} dX_t \right].$$

*Proof.* The Itô's formula and similar calculations as in the last section give

$$\frac{d\Phi'_{t,s}(U_t)^b}{\Phi'_{t,s}(U_t)^b} = \left[ \frac{ac}{12} S\Phi_{t,s}(U_t) + \frac{ab}{12} (1 - \Phi'_{t,s}(U_t)^2) \right] dt - \frac{ab \varphi'_{t,s}(X_s)^2}{4 \sin_2(U_{t,s}^* - X_{t,s}^*)^2} ds + b \frac{\Phi''_{t,s}(U_t)}{\Phi'_{t,s}(U_t)} dU_t.$$

Here,  $\sin_2(x) = \sin(x/2)$ . We can derive a similar formula holds for  $\varphi'_{t,s}(X_s)^b$ . Using the two formulas, we have

$$\begin{aligned} \frac{dZ_t}{Z_t} &= \left[ \frac{ac}{12} S\Phi_t(U_t) + \frac{ab}{12} (1 - \Phi'_t(U_t)^2) - \frac{ab \Phi'_t(U_t)^2}{4 \sin_2(\Theta_t)^2} \right] dt \\ &+ \left[ \frac{ac}{12} S\varphi_t(X_t) + \frac{ab}{12} (1 - \varphi'_t(X_t)^2) - \frac{ab \varphi'_t(X_t)^2}{4 \sin_2(\Theta_t)^2} \right] dt \\ &+ b \frac{\Phi''_t(U_t)}{\Phi'_t(U_t)} dU_t + b \frac{\varphi''_t(X_t)}{\varphi'_t(X_t)} dX_t. \end{aligned}$$

Here, we are also using the fact that  $\Phi'_{t,s}(U_t)$ ,  $\varphi'_{t,s}(X_s)$  are  $C^1$  in  $t, s$ . Note that

$$m_1(t) := \frac{-a}{6} \int_0^t S\Phi'_r(U_r) dr$$

is the Brownian loop measure of loops  $l$  in  $\mathbb{D}$  that have the following properties:

- Winding number of  $l$  is zero.
- $l$  intersects both  $\gamma_t^1$  and  $\gamma_t^2$ .
- If  $T \leq t$  is the first time  $l$  intersects  $\gamma_t^1$ , then  $l \cap \gamma_T^2 \neq \emptyset$ .

The term

$$m_2(t) := \frac{-a}{6} \int_0^t S\varphi'_r(X_r) dr$$

has a similar interpretation for  $\gamma_t^2$ . Hence,  $\hat{m}_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) := m_1(t) + m_2(t)$  is the Brownian loop measure of loops in  $\mathbb{D}$  that have zero winding number and intersect  $\gamma_t^1, \gamma_t^2$  (we are using the fact

that measure of the loops hitting  $\gamma_t^1, \gamma_t^2$  at the same time is zero). Moreover,

$$\frac{a}{12} \int_0^t (1 - \Phi'_r(U_r)^2) dr$$

is the Brownian loop measure of loops  $l$  in  $\mathbb{D}$  with the following properties:

- $l$  has nonzero winding number.
- $l$  intersects both  $\gamma_t^1$  and  $\gamma_t^2$ .
- If  $T \leq t$  is the first time  $l$  intersects  $\gamma_t^1$ , then  $l \cap \gamma_T^2 \neq \emptyset$ .

To see this, let  $\tilde{m}_1(t)$  be the measure of loops having the properties above. Using lemma 5.7 and equation (5.21), the Brownian loop measure of loops in  $\mathbb{D} \setminus \gamma_t^1$  that intersect both  $\gamma_t^2$  and  $\gamma^1(t, t + \varepsilon)$  is

$$\frac{a}{12} \left[ \varepsilon - \int_0^\varepsilon \Phi'_{t+r,t}(U_{t+r})^2 dr \right].$$

Moreover, the Brownian measure of loops in  $\mathbb{D} \setminus \{\gamma_t^1 \cup \gamma_t^2\}$  that intersect both  $\gamma^1(t, t + \varepsilon)$  and  $\gamma^2(t, t + \varepsilon)$  is  $O(\varepsilon^2)$ . Therefore,

$$\partial_t \tilde{m}_1(t) = \frac{a(1 - \Phi'_t(U_t)^2)}{12},$$

and by integrating the claim follows. Using a similar argument for  $\gamma_t^2$ , we can see that

$$\tilde{m}_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) = \frac{a}{12} \left[ \int_0^t (1 - \Phi'_r(U_r)^2) dr + \int_0^t (1 - \phi'_r(X_r)^2) dr \right] \quad (5.27)$$

is the Brownian loop measure of loops that intersect both  $\gamma_t^1$  and  $\gamma_t^2$  and have nonzero winding number. Note that  $ab/12 = a\tilde{b}/2 - ac/24$ . It follows from (5.27) and lemma 5.7 that

$$\frac{ab}{12} \left[ \int_0^t (2 - \Phi'_r(U_r)^2 - \phi'_r(X_r)^2) dr \right] = -\frac{c}{2} \tilde{m}_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) + \frac{\tilde{b}at}{2} - \tilde{b} \bar{\Phi}'_t(0).$$

Therefore,

$$M_t = Z_t \bar{\Phi}'_t(0)^{\bar{b}} e^{-\bar{b}at/2} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) + \frac{ab}{4} \int_0^t \frac{K_r}{\sin_2(\Theta_r)^2} dr \right\}$$

is a local martingale satisfying the claim.  $\square$

Let

$$\tau_n = \inf\{t; \text{dist}(\gamma_t^1, \gamma_t^2) \leq e^{-n}\}$$

and recall that  $\tau = \tau_\infty$ . Although  $M_{t \wedge \tau}$  is only a supermartingale (positive local martingale), one can see that  $M_{t \wedge \tau_n}$  is actually a martingale. To see this, note that  $Z_t \leq 1$ ,  $K_t \leq 2$ ,  $\bar{\Phi}'_t(0) \leq e^{at/2}$  and  $m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2)$  is uniformly bounded for all  $t \leq \tau_n$ . Let  $\mathbb{P}^*$  be the probability measure obtained from weighting  $\mathbb{P}$  by  $M_{t \wedge \tau_n}$ . Using the radial Loewner equation and equations (5.21), (5.22), we can see that

$$\begin{aligned} dU_{t,s}^* &= -b \Phi''_{t,s}(U_t) dt + \frac{a}{2} \phi'_{t,s}(X_s)^2 \cot_2(U_{t,s}^* - X_{t,s}^*) ds + \Phi'_{t,s}(U_t) dU_t, \\ dX_{t,s}^* &= -b \phi''_{t,s}(X_s) ds + \frac{a}{2} \Phi'_{t,s}(U_t)^2 \cot_2(X_{t,s}^* - U_{t,s}^*) ds + \phi'_{t,s}(X_s) dX_s. \end{aligned}$$

Using the Girsanov's theorem and the Itô's formula, we have

$$d\Theta_t = \frac{a}{2} K_t \cot_2(\Theta_t) dt + \sqrt{K_t} dW_t,$$

where  $W_t$  is a Brownian motion with respect to  $\mathbb{P}^*$ . By comparing to a radial Bessel process, we can see that with  $\mathbb{P}^*$ -probability one,  $\sin_2(\Theta_t) > 0$  for all times  $t < \infty$ . Therefore,  $M_{t \wedge \tau_n} < \infty$  with  $\mathbb{P}^*$ -probability one and the claim follows.

The curves  $\gamma_t^1, \gamma_t^2$  have interesting distributions under the measure  $\mathbb{P}^*$ . Fix  $t > 0$  and assume  $\varepsilon > 0$  is small. With respect to measure  $\mathbb{P}$ , the curve  $\gamma^1(t, t + \varepsilon)$  grows like radial  $SLE_\kappa$  from  $\gamma^1(t)$  to 0 in  $\mathbb{D} \setminus \gamma_t^1$ . Equivalently,  $\bar{g}_t(\gamma^1(t, t + \varepsilon))$  has the distribution of radial  $SLE_\kappa$  from  $U_t$  to 0 in  $\mathbb{D}$ . According to proposition 5.3, weighing this process by  $\Phi'_t(U_t)^b$  yields radial  $SLE_\kappa$  from  $\gamma^1(t)$  to 0 in  $\mathbb{D} \setminus \{\gamma_t^2 \cup \gamma_t^1\}$ . Similarly, weighing  $\gamma^2(t, t + \varepsilon)$  by  $\phi'_t(X_t)^2$  gives radial  $SLE_\kappa$  from  $\gamma^2(t)$  to 0 in

$\mathbb{D} \setminus \{\gamma_t^2 \cup \gamma_t^1\}$ . Therefore under the probability measure  $\mathbb{P}^*$ , at each time  $t$  the curves  $\gamma_t^1, \gamma_t^2$  grow like independent radial  $SLE_\kappa$  in  $\mathbb{D} \setminus \{\gamma_t^2 \cup \gamma_t^1\}$ .

We consider weighting the measure  $\mathbb{P}^*$  by  $\sin_2(\Theta_t)^a$ . Straightforward calculation using the Itô's formula shows that

$$N_t = \sin_2(\Theta_t)^a \exp \left\{ \frac{3a^2}{8} \int_0^t K_r dr - \frac{ab}{4} \int_0^t \frac{K_r}{\sin_2(\Theta_r)^2} dr \right\}$$

is a local martingale satisfying

$$dN_t = \frac{a}{2} N_t \cot_2(\Theta_t) \sqrt{K_t} dW_t.$$

Since  $K_r \leq 2$  for any  $r$ ,  $N_{t \wedge \tau}$  is actually a martingale. Let  $\hat{\mathbb{P}}$  be the probability measure obtained from weighting  $\mathbb{P}^*$  by  $N_t$ . Equivalently,  $\hat{\mathbb{P}}$  is the probability measure obtained from weighting  $\mathbb{P}$  by  $O_t := M_t N_t$ . Using lemma 5.7 and equation (5.27), we can see that

$$O_t = Z_t \sin_2(\Theta_t)^a \bar{\Phi}'_t(0)^{(\tilde{b}+3a/4)} \exp \left\{ \frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) + \frac{3a^2 t}{8} - \frac{a\tilde{b}t}{2} \right\}. \quad (5.28)$$

Using the Girsanov's theorem, we can see that there exists a Brownian motion  $B_t$  such that with respect to the measure  $\hat{\mathbb{P}}$ ,

$$d\Theta_t = aK_t \cot_2(\Theta_t) dt + \sqrt{K_t} dB_t.$$

**Proposition 5.4.** *Let  $\bar{u} = e^{iU_0}$ ,  $\bar{x} = e^{iX_0}$  and define the measure  $\nu_t(\bar{u}, \bar{x})$  with*

$$\frac{d\nu_t(\bar{u}, \bar{x})}{d\mu_{\mathbb{D}}(\bar{u}, 0) \times \mu_{\mathbb{D}}(\bar{x}, 0)}(\gamma_t^1, \gamma_t^2) = O_t 1\{t < \tau\},$$

where  $O_t$  is defined in (5.28). Then with respect to the measure  $\nu_t(\bar{u}, \bar{x})$ ,

- Marginal distribution of  $\gamma_t^1$  is two-sided  $SLE_\kappa$  from  $\bar{u}$  to  $\bar{x}$ .
- Marginal distribution of  $\gamma_t^2$  is two-sided  $SLE_\kappa$  from  $\bar{x}$  to  $\bar{u}$ .

- Conditional on  $\gamma_t^1$ , the process  $\gamma_t^2$  has the distribution of two-sided  $SLE_\kappa$  from  $\bar{x}$  to  $\gamma^1(t)$  in  $\mathbb{D} \setminus \gamma_t^1$ .
- Conditional on  $\gamma_t^2$ , the process  $\gamma_t^1$  has the distribution of two-sided  $SLE_\kappa$  from  $\bar{u}$  to  $\gamma^2(t)$  in  $\mathbb{D} \setminus \gamma_t^2$ .

*Proof.* It suffices to show that marginal measure induced on  $\gamma_t^2$  by  $v_t(\bar{u}, \bar{x})$  is the same as two-sided  $SLE_\kappa$  and conditioned on  $\gamma_t^2$ , the distribution of  $\gamma_t^1$  is the same as two-sided  $SLE_\kappa$  from  $\bar{x}$  to  $\gamma^2(t)$  in  $\mathbb{D} \setminus \gamma_t^2$ . This is because the local martingale given in (5.28) is symmetric with respect to  $\gamma_t^1, \gamma_t^2$ .

From proposition 5.3 we know that weighting  $\mu_{\mathbb{D}}(\bar{u}, 0)$  by

$$G'_t(U_0)^{-b} \bar{\Phi}'_t(0)^{\bar{b}} \Phi'_t(U_t)^b e^{-a\bar{b}t/2} \exp\left\{\frac{\mathbf{c}}{2} m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2)\right\} 1_{\{t < \tau\}}$$

yields  $\mu_{\mathbb{D} \setminus \gamma_t^2}(\bar{u}, 0)$  on curves up to time  $t$ . Moreover, we can see from an appropriate time change of (5.26) that weighting  $\mu_{\mathbb{D} \setminus \gamma_t^2}(\bar{u}, 0)$  by

$$\bar{\varphi}'_t(0)^{3a/4} \varphi'_t(X_t)^b \sin_2(\Theta_t)^a$$

gives two-sided  $SLE_\kappa$  from  $\bar{u}$  to  $\gamma^2(t)$  in  $\mathbb{D} \setminus \gamma_t^2$ . Let  $\mathbb{E}, \hat{\mathbb{E}}$  be expectations with respect to  $\mu_{\mathbb{D}}(\bar{u}, 0), \mu_{\mathbb{D} \setminus \gamma_t^2}(\bar{u}, 0)$ .

Then using the fact that  $\bar{\varphi}'_t(0) = \bar{\Phi}'_t(0)$  and that the process given in (5.26) is a martingale,

$$\begin{aligned} \mathbb{E}[O_t] &= \hat{\mathbb{E}} \left[ G'_t(U_t)^b \varphi'_t(X_t)^b \bar{\varphi}'_t(0)^{3a/4} \sin_2(\Theta_t)^a e^{3a^2t/8} \right] \\ &= \sin_2(X_t - G_t(U_0))^a G'_t(U_t)^b e^{3a^2t/8}. \end{aligned}$$

The proof follows from comparing this to (5.26). □

### 5.4.3 Two Annulus $SLE_\kappa$

Let  $\bar{u}, \bar{x} \in C_0$  be distinct boundary points of  $\mathbb{D}$  and let  $\gamma \subset \mathbb{D}$  be a simple curve with  $\gamma(0) = \bar{u}$ .

Define  $\bar{g}_t, g_t$  to be our usual transformations for  $\gamma$ . Define  $0 \leq u, x < 2\pi$  such that  $\bar{u} = \psi(u), \bar{x} =$

$\psi(x)$  and let  $U_t, X_t$  be the unique  $t$ -continuous processes satisfying  $U_0 = u, X_0 = x$  and  $\bar{g}_t(\gamma(t)) = \psi(U_t), \bar{g}_t(\bar{x}) = \psi(X_t)$ .

**Lemma 5.9.** *Suppose  $\kappa < 8$  and  $\gamma_t$  is a radial  $SLE_\kappa$  from  $\bar{u}$  to 0 in  $\mathbb{D}$  with radial parametrization. Then there exist constants  $C, \beta > 0$  such that for  $t > 1$ ,*

$$\mathbb{E} \left[ H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x})^b \right] = C \sin_2(x - u)^a e^{-3a^2 t/8} [1 + O(e^{-\beta t})],$$

where  $\sin_2(\theta) = \sin(\theta/2)$ .

*Proof.* Although we will only use it when  $\kappa \leq 4$ , we will prove this result for  $\kappa < 8$ . Let  $\Theta_t = X_t - U_t$  and define  $\tau = \inf\{t; \sin_2(\Theta_t) = 0\}$  be the first time that  $\Theta_t \in \{0, 2\pi\}$ . Since  $\gamma_t$  is a radial  $SLE_\kappa$ ,  $U_t = -B_t$  is a standard Brownian motion and

$$d\Theta_t = \frac{a}{2} \cot_2(\Theta_t) dt + dB_t, \quad t < \tau,$$

where as before  $\cot_2(\theta) = \cot(\theta/2)$ . Note that when  $\kappa \leq 4$ , with probability one  $\tau = \infty$  and the last equation is well-defined for all times  $t$ . As discussed in (5.26), let

$$M_t = \sin_2(\Theta_t)^a g'_t(x)^b e^{3a^2 t/8} \tag{5.29}$$

be a martingale satisfying

$$dM_t = \frac{a}{2} \cot_2(\Theta_t) M_t dB_t.$$

Let  $\hat{\mathbb{P}}$  be the probability measure obtained from using the Girsanov theorem with the martingale  $M_t$ . Under the probability measure  $\hat{\mathbb{P}}$ , there exists a Brownian motion  $W_t$  such that

$$d\Theta_t = a \cot_2(\Theta_t) dt + dW_t. \tag{5.30}$$

Since  $\kappa < 8$ , we have  $2a > 1/2$ . Comparing this to a radial Bessel process, we can see that with

$\hat{\mathbb{P}}$ -probability one  $\tau = \infty$ . Since  $H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x}) = g'_t(x)$  we can write

$$\begin{aligned} \mathbb{E} \left[ H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x})^b \right] &= \mathbb{E} \left[ g'_t(x)^b; t < \tau \right] \\ &= \mathbb{E} \left[ M_t \sin_2(\Theta_t)^{-a} e^{-3a^2 t/8}; t < \tau \right] \\ &= M_0 e^{-3a^2 t/8} \hat{\mathbb{E}} \left[ \sin_2(\Theta_t)^{-a}; t < \tau \right] \\ &= \sin_2(\Theta_0)^a e^{-3a^2 t/8} \hat{\mathbb{E}} \left[ \sin_2(\Theta_t)^{-a} \right]. \end{aligned}$$

The last equation holds because  $\hat{\mathbb{P}}\{\tau = \infty\} = 1$ . It only remains to compute  $\hat{\mathbb{E}} \left[ \sin_2(\Theta_t)^{-a} \right]$ . The function

$$f(x) = c_{4a} \sin_2(x)^{4a}, \quad c_{4a} = \left[ \int_0^{2\pi} \sin_2(x)^{4a} dx \right]^{-1}$$

satisfies the adjoint equation of (5.30) and therefore is the invariant density of  $\Theta_t$ . Let  $\tilde{f}(\theta, x)$  be the density of  $\Theta_t$  starting at  $\Theta_0 = \theta$ . It follows from properties of the radial Bessel equation (see section 4 of [19]) that there exists  $\beta > 0$  such that for all  $\theta, x$  and  $t > 1$

$$\tilde{f}(\theta, x) = f(x)[1 + O(e^{-\beta t})].$$

Hence,

$$\mathbb{E} \left[ H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x})^b \right] = C \sin_2(\Theta_0)^a e^{-3a^2 t/8} [1 + O(e^{-\beta t})],$$

where

$$C = \left[ \int_0^{2\pi} \sin_2(x)^{4a} dx \right]^{-1} \int_0^{2\pi} \sin_2(x)^{3a} dx.$$

□

**Lemma 5.10.** *For every  $\varepsilon_0 > 0$  and  $r_0 > \pi$ , there exists  $c_0 > 0$  such that the following holds.*

*Assume  $0 \leq u, x, w, y < 2\pi$  and  $\pi < r < r_0$ . Let*

$$\varepsilon = \min\{|u - x + 2k\pi|, |w - y + 2m\pi|; m, k \in \{-1, 0, 1\}\}.$$

Recall the partition function  $V$  defined in (5.4). If  $\varepsilon > \varepsilon_0$ , then  $V(r, u, x, w, y) > c_0$ .

*Proof.* Assume  $u = 0$  and let  $\gamma_t$  be a  $SLE_\kappa$  curve from 1 to  $\bar{w} = \psi(w + ir)$  in  $A_r$  with annulus parametrization. We can assume  $0 < x$ ,  $0 \leq w \leq \pi$  and  $w < y < 2\pi + w$ , since the other cases can be proved in a similar way. Let  $\bar{y} = \psi(y + ir)$ . From the definition,

$$V(r, u, x, w, y) = \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_r)^b \right],$$

where  $\mathbb{E}$  denotes the expectation with respect to the distribution of  $\gamma_t$ . The goal is to show that there exist  $p^* > 0$  and  $c^* > 0$  such that  $Q_{A_r}(\bar{x}, \bar{y}; \gamma_r) > c^*$  with probability at least  $p^*$ .

Let  $\eta_t \subset S_r$  be the unique continuous curve starting from 0, ending at  $w + 2k\pi + ir$  and satisfying  $\gamma_t = \psi(\eta_t)$  for  $0 \leq t \leq r$ . Here,  $k$  is uniquely determined by the winding number of  $\gamma$ . Let  $D_w$  denote the parallelogram created by the intersections of  $\mathbb{R}$ ,  $\mathbb{R} + ir$ , the line connecting  $\varepsilon_0/2$ ,  $w + \varepsilon_0/2 + ir$  and the line connecting  $-\varepsilon_0/2$ ,  $w - \varepsilon_0/2 + ir$ . First, there exists  $p_1 > 0$  such that for all  $0 \leq w \leq \pi$ , the probability that  $\gamma_r \subset D_w$  is at least  $p_1$ . This is because uniformly over  $0 \leq w \leq \pi$  and  $\pi \leq r \leq r_0$ , there is a positive probability that the winding number of  $SLE_\kappa$  from  $\bar{u}$  to  $\bar{w}$  in  $A_r$  is zero and therefore,  $\eta(r) = w + ir$ . Given this, the distribution of  $\eta_t$  is absolutely continuous with respect to the distribution of a chordal  $SLE_\kappa$  from 0 to  $w + ir$  in  $S_r$ . The Radon-Nikodym is bounded if  $\eta_r \subset D_w$ . Moreover, there exists  $p_0 > 0$  such that for all  $0 \leq w \leq \pi$  and  $\pi \leq r \leq r_0$ , the probability that chordal  $SLE_\kappa$  from 0 to  $w + ir$  in  $S_r$  does not exit  $D_w$  is at least  $p_0$ . To see this, let  $f_w : S_r \rightarrow \mathbb{H}$  be a conformal transformation with  $f_w(w + ir) = \infty$ ,  $f_w(0) = 0$ . The domain  $f_w(D_w)$  is a simply connect subdomain of  $\mathbb{H}$  and  $\mathbb{H} \setminus f_w(D_w)$  is bounded. Note that if  $\eta_t^*$  is a chordal  $SLE_\kappa$  from 0 to  $\infty$  in  $\mathbb{H}$ , then  $|\eta^*(t)| \rightarrow \infty$  as  $t \rightarrow \infty$ . Hence, for each  $w \in [0, \pi]$ , the  $SLE_\kappa$  curve  $\eta^*$  has a positive probability of staying in  $f_w(D_w)$  (see [16] for a proof). In addition, we can see that this probability is a continuous function of  $w$ . From this, the claim follows.

Let  $K_w$  denote the connected component of  $S_r \setminus \{D_w \cup D_w + 2\pi\}$  that has  $x, y + ir$  on its boundary. Then there exists a constant  $c_1 > 0$  such that for all  $\gamma_r \subset \psi(D_w)$ ,  $0 \leq w \leq \pi$ ,  $\varepsilon_0 \leq x \leq 2\pi - \varepsilon_0$

and  $w + \varepsilon_0 \leq y \leq w + \varepsilon_0 + 2\pi$ ,

$$Q_{A_r}(\bar{x}, \bar{y}; \gamma_r) \geq c_1 Q_{S_r}(x, y + ir; K_w).$$

Finally, there exists a constant  $c_2$  such that

$$H_{K_w}(x, y + ir) > c_2.$$

□

Recall that theorem 4.1 provides a comparison between radial  $SLE_\kappa$  and crossing  $SLE_\kappa$  in annuli. Our goal is to prove a similar result for two  $SLE_\kappa$  curves in annuli. First, we need an estimate for the partition function.

**Lemma 5.11.** *Let  $\gamma_t \subset A_r$  be a simple curve with  $\gamma(0+) = 1$ ,  $|\bar{g}'_t(0)| = e^{at/2}$ ,  $t \leq \frac{2(r-4)}{a}$  and let  $\bar{x} \in C_0, \bar{y} \in C_r$ . If  $r_t = at/2$ , then*

$$Q_{A_r}(\bar{y}, \bar{x}; \gamma_t) = \frac{r}{r-r_t} H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x}) \left[ 1 + O\left(\frac{1}{r-r_t}\right) \right]. \quad (5.31)$$

*Proof.* We can write

$$H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x}) = \frac{1}{\pi} \int_{C_r} G_{\mathbb{D} \setminus \gamma_t}(0, \bar{z}) H_{A_r \setminus \gamma_t}(\bar{z}, \bar{x}) |d\bar{z}|. \quad (5.32)$$

By Koebe-1/4 theorem,  $|r - at/2 + \log |\bar{g}_t(\bar{z})| \leq \log(4)$  for any  $\bar{z} \in C_r$ . In particular,  $C_{r_t+2} \subset \mathbb{D} \setminus \gamma_t$ .

Hence,

$$\begin{aligned} G_{\mathbb{D} \setminus \gamma_t}(0, \bar{z}) &= (r - r_t) \left[ 1 + O\left(\frac{1}{r - r_t}\right) \right], \\ H_{A_r \setminus \gamma_t}(\bar{z}, \bar{x}) &= H_{A_r \setminus \gamma_t}(\bar{y}, \bar{x}) \left[ 1 + O(e^{r_t - r}) \right], \end{aligned}$$

where the second equality follows from lemma 2.1. Using this and equation (5.32) we get

$$H_{\mathbb{D} \setminus \gamma_t}(0, \bar{x}) = H_{A_r \setminus \gamma_t}(\bar{x}, \bar{y})(r - r_t)e^{-r} \left[ 1 + \mathcal{O}\left(\frac{1}{r - r_t}\right) \right]$$

Using lemma 2.1 one more time, we get

$$H_{A_r}(\bar{x}, \bar{y}) = \frac{e^r}{r} [1 + \mathcal{O}(re^{-r})].$$

□

**Proposition 5.5.** *Let  $V(r, u, x, w, y)$  be as in (5.4). There exists a constant  $c$  such that for all  $0 \leq u, x, w, y < 2\pi$  and sufficiently large  $r$*

$$V(r, u, x, w, y) \leq cr^b e^{-3ar/4} \sin_2(x - u)^a.$$

Furthermore, for any  $\varepsilon > 0$ , there exists a constant  $c_\varepsilon > 0$  such that if  $\min\{|y - w + 2k\pi|; k \in \{-1, 0, 1\}\} > \varepsilon$ , then

$$c_\varepsilon r^b e^{-3ar/4} \sin_2(x - u)^a \leq V(r, u, x, w, y).$$

*Proof.* Suppose  $\gamma_t$  is a  $SLE_\kappa$  curve from  $\bar{u}$  to  $\bar{w}$  and let  $\mathcal{F}_t$  be the  $\sigma$ -algebra generated by  $\gamma_t$ . We assume  $\gamma_t$  has radial parametrization and let  $\tau$  be the hitting time of  $C_r$ . Let  $\bar{X}_t = \bar{h}_t(\bar{x})$ ,  $\bar{Y}_t = \bar{h}_t(\bar{y})$  and as before let  $X_t, Y_t$  be the unique continuous functions with  $\bar{X}_t = \psi(X_t)$ ,  $\bar{Y}_t = \psi(Y_t)$  and  $X_0 = x, Y_0 = y$ . Define

$$\varepsilon_t^x = \min\{|U_t - X_t + 2k\pi|; m, k \in \{-1, 0, 1\}\},$$

$$\varepsilon_t^y = \min\{|W_t - Y_t + 2m\pi|; m, k \in \{-1, 0, 1\}\},$$

$$\varepsilon_t = \min\{\varepsilon_t^x, \varepsilon_t^y\}.$$

For a fixed  $t < \tau$ , let  $\bar{\gamma} = \bar{h}_t(\gamma(t, \tau))$ . We can write

$$\begin{aligned} V(r, u, x, w, y) &= \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_\tau)^b \right] \\ &= \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_t)^b \mathbb{E} \left[ Q_{A_{r(t)}}(\bar{X}_t, \bar{Y}_t; \bar{\gamma})^b \middle| \mathcal{F}_t \right] \right]. \end{aligned}$$

Here,  $r(t)$  is as in (2.2) and conditioned on  $\mathcal{F}_t$ ,  $\bar{\gamma}$  is a  $SLE_\kappa$  from  $\bar{U}_t$  to  $\bar{W}_t$  in  $A_{r(t)}$ . Using corollary 2.2 and theorem 4.1, we can find a constant  $s_0 > 5$  such that if  $t = \frac{2(r-s_0)}{a}$ , then for all  $z \in C_r$ ,

$$|\partial_z \arg \bar{h}_t(z) - 1| < \varepsilon_0/2\pi \quad (5.33)$$

and

$$\left| \frac{d\mu_{A_r}^\#(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}^\#(\bar{u}, 0)}(\gamma_t) - 1 \right| \leq \frac{1}{2}. \quad (5.34)$$

Koebe-1/4 theorem implies that  $|s_0 - r(t)| \leq \ln(4)$  and  $C_{r-2} \subset \mathbb{D} \setminus \gamma$ . Using lemma 5.10, we can see that there exists  $c_0 = c_{\varepsilon_0/2}$  such that  $1 > c_0 > 0$  and

$$c_0 1 \left\{ \varepsilon_t > \frac{\varepsilon_0}{2} \right\} < \mathbb{E} \left[ Q_{A_{r(t)}}(\bar{U}'_t, \bar{W}'_t; \bar{\gamma})^b \middle| \mathcal{F}_t \right] < 1. \quad (5.35)$$

Let  $\bar{\mathbb{P}}$  be a probability measure under which  $\gamma$  is a radial  $SLE_\kappa$  and let  $\bar{\mathbb{E}}$  denote the expectation with respect to  $\bar{\mathbb{P}}$ . Using (5.31) and (5.34), there exists a constant  $c$  such that

$$\frac{1}{c} \leq \frac{V(r, u, x, w, y)}{r^b \bar{\mathbb{E}} \left[ H_{\mathbb{D} \setminus \gamma}(0, \bar{x})^b 1_{\{2\varepsilon_t > \varepsilon_0\}} \right]} \leq c.$$

Let  $g_s, \xi_s$  be as in section 2.1 and define  $\tilde{X}_s = g_s(x)$ . Equation (5.33) implies that  $\varepsilon_t^y > \varepsilon_0/2$  and

$$\left\{ \varepsilon_t > \frac{\varepsilon_0}{2} \right\} = \left\{ \varepsilon_t^x > \frac{\varepsilon_0}{2} \right\}.$$

Let

$$\tilde{\varepsilon}_t = \min\{|\tilde{X}_t - \xi_t + 2k\pi|; k \in \{-1, 0, 1\}\}.$$

Considering our choice of  $t$ , corollary 2.1 implies

$$\{\tilde{\varepsilon}_t > \varepsilon_0\} \subset \left\{ \varepsilon_t^x > \frac{\varepsilon_0}{2} \right\}.$$

Let  $\Theta_s = \tilde{X}_s - \xi_s$  and note that

$$d\Theta_s = \frac{a}{2} \cot_2(\Theta_s) ds + dB_s,$$

where  $B_s$  is a standard Brownian motion with respect to  $\bar{\mathbb{P}}$ . Consider the martingale

$$M_s = \sin_2(\Theta_s)^a H_{\mathbb{D} \setminus \gamma}(0, \bar{x})^b e^{3a^2 t/8}$$

defined in (5.29). Let  $\tilde{\mathbb{E}}$  be the expectation with respect to the probability measure obtained from weighing  $\bar{\mathbb{P}}$  by  $M_t$ . It follows from the Girsanov's theorem that with respect to the new measure, there exists a Brownian motion  $W_t$  such that

$$d\Theta_s = a \cot_2(\Theta_s) ds + dW_s.$$

Using properties of radial Bessel processes similar to the proof of lemma 5.9, we can see that

$$f(x) = c_{4a} \sin_2(x)^{4a}, \quad c_{4a} = \left[ \int_0^{2\pi} \sin_2(x)^{4a} dx \right]^{-1}$$

is the invariant density of  $\Theta_s$  and if  $\tilde{f}_s(\theta, x)$  is the density of  $\Theta_s$  starting at  $\Theta_0 = \theta$  then exists  $\beta > 0$  such that for all  $\theta, x$  and  $s > 1$

$$\tilde{f}_s(\theta, x) = f(x)[1 + O(e^{-\beta s})]. \tag{5.36}$$

Hence,

$$\begin{aligned}\tilde{\mathbb{E}} \left[ H_{\mathbb{D} \setminus \gamma}(0, \bar{x})^b 1\{\tilde{\varepsilon}_t > \varepsilon_0\} \right] &= \sin_2(x-u)^a e^{-3a^2t/8} \tilde{\mathbb{E}} \left[ \frac{M_t}{M_0} \sin_2(\Theta_t)^{-a} 1\{\tilde{\varepsilon}_t > \varepsilon_0\} \right] \\ &= \sin_2(x-u)^a e^{-3a^2t/8} \tilde{\mathbb{E}} \left[ \sin_2(\Theta_t)^{-a} 1\{\tilde{\varepsilon}_t > \varepsilon_0\} \right].\end{aligned}$$

Now we can use (5.36) to see that there exists a constant  $c_0 > 0$  such that for large enough  $t$

$$\frac{1}{c_0} < \tilde{\mathbb{E}} \left[ \sin_2(\Theta_t)^{-a} 1\{\tilde{\varepsilon}_t > \varepsilon_0\} \right] < c_0.$$

Hence, there exist constants  $c_1, c_2 > 0$  such that

$$c_1 \leq \frac{V(r, u, x, w, y)}{r^b e^{-3ar/4} \sin_2(x-u)^a} \leq c_2.$$

Finally, since only the lower bound in (5.35) depends on  $\varepsilon_0$ , we can see  $c_2$  in the last equation does not depend on  $\varepsilon_0$ . □

**Corollary 5.1.** *There exists a constant  $c$  such that for all  $0 \leq u, x, w, y < 2\pi$  and sufficiently large  $r$ ,*

$$V(r, u, x, w, y) \leq c r^b e^{-3ar/4} \sin_2(y-w)^a.$$

*Furthermore, for any  $\varepsilon > 0$ , there exists a constant  $c_\varepsilon > 0$  such that if  $\min\{|x-u+2k\pi|; k \in \{-1, 0, 1\}\} > \varepsilon_0$ , then*

$$c_\varepsilon r^b e^{-3ar/4} \sin_2(y-w)^a \leq V(r, u, x, w, y).$$

*Proof.* This follows from proposition 5.5, reversibility of  $SLE_\kappa$  and Brownian motion along with the fact that  $f(z) = -e^r/z$  is a conformal transformation mapping  $A_r$  to itself. □

**Proposition 5.6.** *Let  $V(r, u, x, w, y)$  be as in (5.4). There exists a constant  $c > 0$  such that for all*

$0 \leq u, x, w, y < 2\pi$  and sufficiently large  $r$

$$\frac{1}{c} \leq \frac{V(r, u, x, w, y)}{r^b e^{-3ar/4} \sin_2(x-u)^a \sin_2(w-y)^a} \leq c.$$

*Proof.* Fix  $1/2 > \varepsilon_0 > 1/3$  and let  $\gamma_t$  be a  $SLE_\kappa$  curve from  $\bar{u}$  to  $\bar{w}$  in  $A_r$ . As before, define

$$\varepsilon_t^y = \min\{|Y_t - W_t + 2k\pi|; k \in \{-1, 0, 1\}\}$$

$$\varepsilon_t^x = \min\{|X_t - U_t + 2k\pi|; k \in \{-1, 0, 1\}\}.$$

If  $\varepsilon_0^y > \varepsilon_0$ , then the result follows from proposition 5.5. So we assume  $\varepsilon_0^y \leq \varepsilon_0$ . Let  $\bar{Y}_t = \bar{h}_t(\bar{y})$ ,  $\bar{W}_t = \bar{h}(\bar{w}_t)$  and  $Y_t = h_t(y)$ ,  $W_t = h_t(w)$ . Using theorem 4.1 and corollary 2.2, for sufficiently large  $r$  and  $t = r/2$ ,

$$\left| \frac{d\mu_{A_r}^\#(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}^\#(\bar{u}, 0)}(\gamma_t) - 1 \right| \leq \frac{1}{2} \quad (5.37)$$

and

$$\frac{1}{2} < \frac{\varepsilon_t^y}{\varepsilon_0^y} < 2. \quad (5.38)$$

Define  $\tau$  to be the hitting time of  $C_r$  by  $\gamma_t$ . Let  $t = r/2$ . Recall that  $r(t)$  is the unique number satisfying  $\bar{h}_t(A_r \setminus \gamma_t) = A_{r(t)}$ . Let  $\tilde{\gamma}$  be a  $SLE_\kappa$  curve from  $\bar{U}_t$  to  $\bar{W}_t$  in  $A_{r(t)}$ . Then

$$\begin{aligned} V(r, u, x, w, y) &= \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_\tau)^b \right] \\ &= \mathbb{E} \left[ \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_\tau)^b \middle| \mathcal{F}_t \right] \right] \\ &= \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_t)^b \mathbb{E}_{\tilde{\gamma}} \left[ Q_{A_{r(t)}}(\bar{X}_t, \bar{Y}_t; \tilde{\gamma})^b \middle| \mathcal{F}_t \right] \right] \end{aligned}$$

Corollary 5.1 and equation (5.38) imply that there exists a constant  $c = c(\varepsilon_0)$  such that

$$\frac{1}{c} \mathbb{1}\{\varepsilon_t^x > \varepsilon_0\} \leq \frac{\mathbb{E}_{\tilde{\gamma}} \left[ Q_{A_{r(t)}}(\bar{X}_t, \bar{Y}_t; \tilde{\gamma})^b \middle| \mathcal{F}_t \right]}{r(t)^b e^{-3ar(t)/4} \sin_2(y-w)^a} \leq c.$$

Moreover, lemma 5.9 and equations (5.31), (5.37) gives

$$\mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_t)^b \right] \leq c \left( \frac{r}{r - at/2} \right)^b e^{-3a^2t/8} \sin_2(x - u)^a.$$

We can see from an argument similar to the proof of lemma 5.9 that

$$\frac{1}{c} \left( \frac{r}{r - at/2} \right)^b e^{-3a^2t/8} \sin_2(x - u)^a \leq \mathbb{E} \left[ Q_{A_r}(\bar{x}, \bar{y}; \gamma_t)^b \mathbf{1}_{\{\varepsilon_t^x > \varepsilon_0\}} \right].$$

Moreover, Koebe-1/4 theorem implies that

$$|r - at/2 - r(t)| \leq \log(4),$$

from which the result follows. □

**Theorem 5.4.** *Let  $\gamma_t^1, \gamma_t^2$  be SLE $_{\kappa}$  curves from  $\bar{u}$  to  $\bar{w}$  and from  $\bar{x}$  to  $\bar{y}$  in  $A_r$  with radial parametrization. Then there exists a constant  $c$  such that for all  $r > 11$  and  $\frac{2(r-11)}{a} > t$ , then*

$$\frac{1}{c} r^{c/2} e^{r(2b-\tilde{b}-3a/4)} \sin_2(y - w)^a < \frac{d\mu_{A_r}((\bar{u}, \bar{x}), (\bar{w}, \bar{y}))}{d\nu_t(\bar{u}, \bar{x})}(\gamma_t) < c r^{c/2} e^{r(2b-\tilde{b}-3a/4)} \sin_2(y - w)^a.$$

Here,  $\gamma_t = (\gamma_t^1, \gamma_t^2)$  and  $\nu_t(\bar{u}, \bar{x})$  is the measure defined in proposition 5.4.

*Proof.* Let  $\gamma = (\gamma^1, \gamma^2)$  and recall that

$$\frac{d\mu_{A_r}((\bar{u}, \bar{x}), (\bar{w}, \bar{y}))}{d\mu_{A_r}(\bar{u}, \bar{w}) \times \mu_{A_r}(\bar{x}, \bar{y})}(\gamma) = e^{\frac{c}{2}m_{A_r}(\gamma^1, \gamma^2)} \mathbf{1}_{\{\gamma^1 \cap \gamma^2 = \emptyset\}}.$$

For  $i \in \{1, 2\}$ , let  $\tau_i$  be the time  $\gamma^i$  hits  $C_r$ . For  $t < \min\{\tau_1, \tau_2\}$ , define the conformal transformations  $\tilde{h}_t^1 : A_r \setminus \gamma_t^1 \rightarrow A_{r_1(t)}$ ,  $\tilde{h}_t^2 : A_r \setminus \gamma_t^2 \rightarrow A_{r_2(t)}$  and let  $\tilde{\gamma}_t^2 = \tilde{h}_t^1(\gamma_t^2)$ ,  $\tilde{\gamma}_t^1 = \tilde{h}_t^2(\gamma_t^1)$ . In addition, define the conformal transformations  $\tilde{h}_t^1 : A_{r_2(t)} \setminus \tilde{\gamma}_t^1 \rightarrow A_{r(t)}$ ,  $\tilde{h}_t^2 : A_{r_1(t)} \setminus \tilde{\gamma}_t^2 \rightarrow A_{r(t)}$ . Let

$$\bar{U}_t = \tilde{h}_t^1(\gamma^1(t)), \quad \bar{x}_t = \tilde{h}_t^2(\gamma^2(t)), \quad \bar{W}_t = \tilde{h}_t^1(\bar{w}), \quad \bar{Y}_t = \tilde{h}_t^2(\bar{y}),$$

$$\tilde{U}_t = \tilde{h}_t^2(\tilde{U}_t), \quad \tilde{X}_t = \tilde{h}_t^1(\tilde{X}_t), \quad \tilde{W}_t = \tilde{h}_t^2(\tilde{W}_t), \quad \tilde{Y}_t = \tilde{h}_t^1(\tilde{Y}_t).$$

Note that

$$\begin{aligned} m_{A_r}(\gamma^1, \gamma^2) &= m_{A_r}(\gamma_t^1, \gamma_t^2) + m_{A_r \setminus \gamma_t^1}(\gamma^1(t, \tau_1), \gamma_t^2) + m_{A_r \setminus \gamma_t^2}(\gamma_t^1, \gamma^2(t, \tau_2)) \\ &\quad + m_{A_r \setminus \{\gamma_t^1 \cup \gamma_t^2\}}(\gamma^1(t, \tau_1), \gamma^2(t, \tau_2)) \end{aligned}$$

and

$$\begin{aligned} \{\gamma^1 \cap \gamma^2 = \emptyset\} &= \{\gamma_t^1 \cap \gamma_t^2 = \emptyset\} \cap \{\gamma_t^1 \cap \gamma^2(t, \tau_2) = \emptyset\} \cap \{\gamma^1(t, \tau_1) \cap \gamma_t^2 = \emptyset\} \\ &\quad \cap \{\gamma^1(t, \tau_1) \cap \gamma^2(t, \tau_2) = \emptyset\}. \end{aligned}$$

Let

$$\begin{aligned} Y_1 &= \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r \setminus \gamma_t^1}(\gamma^1(t, \tau_1), \gamma_t^2) \right\} 1_{\{\gamma^1(t, \tau_1) \cap \gamma_t^2 = \emptyset\}}, \\ Y_2 &= \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r \setminus \gamma_t^2}(\gamma_t^1, \gamma^2(t, \tau_2)) \right\} 1_{\{\gamma_t^1 \cap \gamma^2(t, \tau_2) = \emptyset\}}, \\ Y_{1,2} &= \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r \setminus \{\gamma_t^1 \cup \gamma_t^2\}}(\gamma^1(t, \tau_1), \gamma^2(t, \tau_2)) \right\} 1_{\{\gamma^1(t, \tau_1) \cap \gamma^2(t, \tau_2) = \emptyset\}}. \end{aligned}$$

Let  $\mathcal{F}_t = \sigma(\gamma_t^1, \gamma_t^2)$  be the  $\sigma$ -algebra generated by the curves up to time  $t$  and denote by  $\mathbb{E}$  the expectation with respect to the product measure  $\mu_{A_r}(\bar{u}, \bar{w}) \times \mu_{A_r}(\bar{x}, \bar{y})$ . Conditioning on  $F_t$ , equation (2.20) implies that  $\gamma^1(t, \tau_1)$  weighted by  $Y_1$  has the distribution of  $SLE_\kappa$  from  $\gamma^1(t)$  to  $\bar{w}$  in  $A_r \setminus \{\gamma_t^1 \cup \gamma_t^2\}$ . Similarly,  $\gamma^2(t, \tau_2)$  weighted by  $Y_2$  has the distribution of  $SLE_\kappa$  from  $\gamma^2(t)$  to  $\bar{y}$  in  $A_r \setminus \{\gamma_t^1 \cup \gamma_t^2\}$ . Let  $\mathbb{E}$  denote the expectation with respect to  $\mu_{A_r}^\#(\bar{u}, \bar{w}) \times \mu_{A_r}^\#(\bar{x}, \bar{y})$ . If  $\mathbb{E}^1, \mathbb{E}^2$  denote the expectations with respect to  $\mu_{A_r}^\#(\bar{u}, \bar{w}), \mu_{A_r}^\#(\bar{x}, \bar{y})$ , then equations (2.6), (2.20) give us

$$\begin{aligned} \mathbb{E}^1[Y_1 | F_t] &= \frac{|\tilde{h}_t^2'(\tilde{U}_t)|^b |\tilde{h}_t^2'(\tilde{W}_t)|^b \Psi_{A_{r(t)}}(\tilde{U}_t, \tilde{W}_t)}{\Psi_{A_{r_1(t)}}(\tilde{U}_t, \tilde{W}_t)}, \\ \mathbb{E}^2[Y_2 | F_t] &= \frac{|\tilde{h}_t^1'(\tilde{X}_t)|^b |\tilde{h}_t^1'(\tilde{Y}_t)|^b \Psi_{A_{r(t)}}(\tilde{X}_t, \tilde{Y}_t)}{\Psi_{A_{r_2(t)}}(\tilde{X}_t, \tilde{Y}_t)}. \end{aligned}$$

Using this and definition 5.1, if the martingale  $M_t$  is defined by

$$M_t = \mathbb{E} \left[ \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r}(\gamma^1, \gamma^2) \right\} 1_{\{\gamma^1 \cap \gamma^2 = \emptyset\}} \middle| \mathcal{F}_t \right],$$

then

$$\begin{aligned} M_t &= \Psi_{A_{r(t)}}((\tilde{U}_t, \tilde{X}_t), (\tilde{W}_t, \tilde{Y}_t)) \exp \left\{ \frac{\mathbf{c}}{2} m_{A_r}(\gamma_t^1, \gamma_t^2) \right\} 1_{\{\gamma_t^1 \cap \gamma_t^2 = \emptyset\}} \\ &\quad \times \frac{|\tilde{h}_t^{2'}(\tilde{U}_t)|^b |\tilde{h}_t^{2'}(\tilde{W}_t)|^b |\tilde{h}_t^{1'}(\tilde{X}_t)|^b |\tilde{h}_t^{1'}(\tilde{Y}_t)|^b \Psi_{A_r}(\bar{u}, \bar{w}) \Psi_{A_r}(\bar{x}, \bar{y})}{\Psi_{A_{r_1(t)}}(\tilde{U}_t, \tilde{W}_t) \Psi_{A_{r_2(t)}}(\tilde{X}_t, \tilde{Y}_t)}. \end{aligned} \quad (5.39)$$

Define  $\bar{g}_t^1 : \mathbb{D} \setminus \gamma_t^1 \rightarrow \mathbb{D}$  to be the unique conformal transformation with  $\bar{g}_t^1(0) = 0$ ,  $\bar{g}_t^{1'}(0) > 0$  and let  $\bar{\phi}_t^1 : \bar{g}_t^1(A_r \setminus \gamma_t^1) \rightarrow A_{r_1(t)}$  be the conformal transformation satisfying  $\bar{h}_t^1 = \bar{\phi}_t^1 \circ \bar{g}_t^1$ . Define  $\bar{g}_t^2, \bar{\phi}_t^2$  for  $\gamma_t^2$  in a similar manner. Let  $\xi_t^1 = \bar{g}_t^1(\gamma^1(t))$ ,  $\xi_t^2 = \bar{g}_t^2(\gamma^2(t))$ . Let

$$\begin{aligned} M_t^1 &:= \frac{d\mu_{A_r}(\bar{u}, \bar{w})}{d\mu_{\mathbb{D}}(\bar{u}, 0)}(\gamma_t^1), \\ M_t^2 &:= \frac{d\mu_{A_r}(\bar{u}', \bar{y})}{d\mu_{\mathbb{D}}(\bar{x}, 0)}(\gamma_t^2). \end{aligned} \quad (5.40)$$

Define  $\hat{M}_t$  to be the martingale satisfying

$$\frac{d\mu_{A_r}((\bar{u}, \bar{x}), (\bar{w}, \bar{y}))}{d\mu_{\mathbb{D}}(\bar{u}, 0) \times \mu_{\mathbb{D}}(\bar{x}, 0)}(\gamma_t) = \hat{M}_t.$$

That is,  $\hat{M}_t$  is the martingale obtained from multiplying the martingales given in (5.39) and (5.40)

$$\hat{M}_t = M_t^1 M_t^2 M_t.$$

From lemma 2.3 we have

$$r_1(t) = r - \frac{at}{2} + \mathcal{O}\left(e^{-r+at/2}\right), \quad r_2(t) = r - \frac{at}{2} + \mathcal{O}\left(e^{-r+at/2}\right).$$

It is known [17] that there exist absolute constants  $0 < c, c^*, q < \infty$  such that for all  $z_1 \in C_0, z_2 \in C_r$ ,

$$\Psi_{A_r}(z_1, z_2) = c r^{c/2} e^{(b-\tilde{b})r} [1 + \mathcal{O}(e^{-qr})]$$

and

$$M_t^1 = c^* r^{c/2} e^{(b-\tilde{b})r} [1 + \mathcal{O}(e^{-q(r-at/2)})], \quad M_t^2 = c^* r^{c/2} e^{(b-\tilde{b})r} [1 + \mathcal{O}(e^{-q(r-at/2)})].$$

Lemma 2.4 implies that

$$|\tilde{h}_t^{2'}(\tilde{W}_t)| = e^{r_1(t)-r(t)} [1 + \mathcal{O}(e^{-r(t)})], \quad |\tilde{h}_t^{1'}(\tilde{Y}_t)| = e^{r_2(t)-r(t)} [1 + \mathcal{O}(e^{-r(t)})].$$

Recall the process  $Z_t$  defined in (5.25). Using corollary 2.1, we can see that

$$|\tilde{h}_t^{2'}(\tilde{U}_t)|^b |\tilde{h}_t^{1'}(\tilde{X}_t)|^b = Z_t [1 + \mathcal{O}(e^{-br(t)})].$$

Proposition 5.6 implies that

$$\Psi_{A_{r(t)}}((\tilde{U}_t, \tilde{X}_t), (\tilde{W}_t, \tilde{Y}_t)) \asymp r^{c/2} e^{r(t)(-3a/4+2b-\tilde{b})} \sin_2(u-x)^a \sin_2(y-w)^a.$$

Using lemma 2.3,

$$m_{\mathbb{D}}(\gamma_t^1, \gamma_t^2) - m_{A_r}(\gamma_t^1, \gamma_t^2) = \log \frac{r}{r_1(t)} + \log \frac{r}{r_2(t)} - \log \frac{r}{r(t)} + \mathcal{O}(e^{-r(t)}).$$

Using these and comparing to the martingale  $O_t$  described in proposition 5.4 give the result.  $\square$

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