

THE UNIVERSITY OF CHICAGO

THE BERNSTEIN-SATO B-FUNCTION FOR THE COMPLEMENT OF THE OPEN  $SL_n$ -ORBIT ON  
A TRIPLE FLAG VARIETY

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## Abstract

We calculate Bernstein-Sato b-functions for  $SL_n$ -invariant sections of line bundles on  $SL_n/B \times SL_n/B \times \mathbb{P}^{n-1}$  whose zero-set is the complement of the open  $SL_n$ -diagonal orbit. The proof uses a similar calculation by Kashiwara of the b-function for  $B^-$ -semiinvariant sections of line bundles on  $SL_n/B$  whose zero-set is the complement of the big Bruhat cell.

## 2 Introduction

### 2.1 D-modules

Let  $X$  be a smooth variety over  $\mathbb{C}$ .

**Definition** Define the sheaf  $\mathcal{D}(X)$  of differential operators on  $X$  inductively as follows:

$$\mathcal{D}^0(X) = \mathcal{O}(X)$$

$$\mathcal{D}^i(X) = \{P \in \text{End}_{\mathbb{C}}(\mathcal{O}(X)) \mid \forall f \in \Gamma(X, \mathcal{O}(x)) [P, f] \in \mathcal{D}^{i-1}(X)\}$$

$$\text{Then } \mathcal{D}(X) = \cup_i \mathcal{D}^i(X).$$

**Example** Let  $X = \mathbb{A}^n$ . Then  $\mathcal{D}(X) = \mathbb{C}\langle x_1, x_2, \dots, x_n, \partial_1, \partial_2, \dots, \partial_n \rangle / \langle [x_i, x_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij} \rangle$

This is a sheaf of filtered algebras where multiplication comes from composition of operators, as if  $P \in \mathcal{D}^i(X), Q \in \mathcal{D}^j(X)$ , then  $PQ \in \mathcal{D}^{i+j}(X)$ . As with any algebra (or sheaf of algebras), one of the standard methods of study is through its modules, which are called D-modules.

**Example** The structure sheaf  $\mathcal{O}(X)$  is naturally a  $\mathcal{D}(X)$ -module, with  $P \cdot f = Pf$ .

An equivalent definition for D-modules is given by flat connections:

**Definition** A D-module is a sheaf of vector spaces  $E$  on  $X$  and a flat connection, i.e. a map  $\Delta : \Gamma(E) \rightarrow \Gamma(E \otimes T^*X)$  such that  $\Delta(f\sigma) = f\Delta(\sigma) + \sigma \otimes df$  for any function  $f$  and section  $\sigma$ .

A common class of D-modules is line bundles with flat connections, i.e. rank 1 sheaves of vector spaces with a flat connection. From now on, when we say line bundles, we are referring to line bundles with flat connections.

The filtration  $\mathcal{D}^i$  allows us to construct the associated graded algebra  $\text{gr}\mathcal{D}(X)$ . This graded algebra is naturally isomorphic (as a graded algebra) to  $\mathbb{C}[T^*X]$ , the algebra of functions on the cotangent bundle  $T^*X$ . This identification can be extended to coherent  $\mathcal{D}(X)$ -modules  $M$  by adding a filtration (though such a filtration is not unique); this results in a sheaf on  $T^*X$ . A filtration is called "good" if  $\text{gr}M$  is coherent as a module over  $\mathcal{O}(T^*X)$ .

If we started with a finitely generated  $\mathcal{D}(X)$ -module  $M$  and end up with a sheaf  $\text{gr}M$ , then as long as the filtration is "good", the support of  $\text{gr}M$  turns out not to depend on the filtration. The support can then be useful to learn about  $M$  itself. For example:

**Theorem 1.** *Let  $X = \mathbb{A}^n$ . Then define the Bernstein filtration on  $\mathcal{D}(X)$  based on the grading  $\deg(x_i) = \deg(\partial_i) = 1$  in  $\mathbb{C}\langle x_1, x_2, \dots, x_n, \partial_1, \partial_2, \dots, \partial_n \rangle$ . Let  $M$  be a  $\mathcal{D}(X)$  module with a compatible filtration, i.e. such that  $\mathcal{D}(X)^i M^j \subseteq M^{i+j}$ . Then there is some polynomial, called the Hilbert polynomial  $p \in \mathbb{Q}[t]$ , such that  $\dim M^i = p(i)$  for large  $i$ . Further,  $\deg(p) = \dim \text{supp } \text{gr}M$ . Finally,  $\deg(p) \geq n$ .*

So by learning about  $\text{gr}M$  geometrically, we can learn about the algebraic properties of  $M$  - e.g. how quickly a good filtration can grow in dimension.

If the inequality is exact, that is,  $\dim \text{supp } \text{gr}M = n$ , then  $M$  is called a holonomic D-module. These modules were studied by Kashiwara [10] and Bernstein [1], who proved several finiteness properties.

## 2.2 Quantum Hamiltonian reduction

Often, we will want to deal with differential operators on quotient varieties  $X/G$  for a connected Lie group  $G$ , especially when they are the quotient of an affine variety by the free action of a group. The standard method to do this is quantum Hamiltonian reduction. If  $P$  is a differential operator on  $X$  and  $Q$  is a differential operator on  $X/G$  (or corresponding affine subvarieties), then  $P$  is said to induce  $Q$  if given a function  $f$  on an affine subvariety of  $X/G$ ,  $P(\tilde{f}) = \widetilde{Q(f)}$ , where  $\tilde{f}$  denotes the composition of  $f$  with the map  $X \rightarrow X/G$  (or the appropriate restriction). It turns out that all differential operators on  $X/G$  are induced by operators on  $X$ .

First, the action of  $G$  induces a map  $\Phi : \mathfrak{g} \rightarrow \Gamma(X, TX)$ . If  $f$  is a function on  $X/G$  (or on any affine subvariety), then it can be pulled back to a  $G$ -invariant function  $\tilde{f}$  on  $X$  (or the appropriate subvariety). The fact that  $\tilde{f}$  is  $G$ -invariant implies that  $(\Phi(\xi))(\tilde{f}) = 0$  for any  $\xi \in \mathfrak{g}$ . As such, any differential operator which can be expressed in the form  $P\Phi(\xi)$  for some differential operator  $P$  reduces to the 0 differential operator on  $X/G$ . In other words, given the above statement that all differential operators on  $X/G$  are induced by operators on  $X$ , they must in fact be induced by elements of  $\mathcal{D}(X)/(\mathcal{D}(X)\mathfrak{g})$ , where we abuse notation to write  $\Phi(\mathfrak{g})$  as  $\mathfrak{g}$ .

Second, if  $P$  is a differential operator on  $X$  that induces a differential operator  $Q$  on  $X/G$ , then for any  $g \in G$ , the differential operator  $P^g$  also induces  $Q$ . This can be rewritten as the condition that the image of  $P$  in  $\mathcal{D}(X)/(\mathcal{D}(X)\mathfrak{g})$  must be  $G$ -invariant (although  $P$  itself does not need to be  $G$ -invariant). These two conditions turn out to be sufficient for a differential operator  $P \in \mathcal{D}(X)$  to induce a differential operator on  $X/G$ . In other words,  $\mathcal{D}(X/G) = (\mathcal{D}(X)/(\mathcal{D}(X)\mathfrak{g}))^{\mathfrak{g}}$ .

We often wish to replace functions with sections of line bundles on  $X/G$ . These sections can be induced by  $G$ -semiinvariant functions on  $X$ , rather than  $G$ -invariant functions; the line bundles  $\mathcal{L}_\lambda$  can be labeled with the character of the semiinvariant functions. Such line bundles have a flat connection induced by the connection on the algebra of functions on  $X$ .

The appropriate generalization deals with differential operators that operate between line bundles.

**Definition** Let  $\mathcal{L}, \mathcal{M}$  be two line bundles on any variety  $Y$ . Then define  $\mathcal{D}(\mathcal{L}, \mathcal{M})$  inductively:

$$\mathcal{D}^0(\mathcal{L}, \mathcal{M}) = \text{Hom}_{\mathcal{O}(Y)}(\mathcal{L}, \mathcal{M})$$

$$\mathcal{D}^i(\mathcal{L}, \mathcal{M}) = \{P \in \text{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{M}) \mid \forall f \in \mathcal{O}(Y)[P, f] \in \mathcal{D}^i(\mathcal{L}, \mathcal{M})\}$$

$$\mathcal{D}(\mathcal{L}, \mathcal{M}) = \cup_i \mathcal{D}^i(\mathcal{L}, \mathcal{M})$$

Let  $\lambda, \mu$  be characters of  $G$ . We then want to characterize the sheaf of differential operators  $\mathcal{D}(\mathcal{L}_\lambda, \mathcal{L}_\mu)$  in terms of the sheaf  $\mathcal{D}(X)$ , in a manner similar to the characterization of  $\mathcal{D}(X/G)$ . The functions  $\tilde{f}$  no longer have to be  $G$ -invariant, but rather semiinvariant with character  $\lambda$ ; as such, if  $\xi \in \mathfrak{g}$ , then  $\Phi(\xi)$  no longer takes  $\tilde{f}$  to 0. Instead,  $(\Phi(\xi))(f) = \lambda(\xi)f$ , and elements of  $\mathcal{D}(\mathcal{L}, \mathcal{M})$  are induced by elements of  $\mathcal{D}(X)/(\mathcal{D}(X)\text{span}(\Phi(\xi) - \lambda(\xi)))$ . The requirement that these elements be  $G$ -invariant is also changed; they must instead be  $G$ -semiinvariant with character  $\mu\lambda^{-1}$ . Putting these together:

$$\mathcal{D}(\mathcal{L}_\lambda, \mathcal{L}_\mu) = (\mathcal{D}(X)/(\mathcal{D}(X)\text{span}(\Phi(\xi) - \lambda(\xi))))^{G_{\mu\lambda^{-1}}} \quad (1)$$

where the superscript  $G_\lambda$  denotes the subset of elements that are  $G$ -semiinvariant with character  $\lambda$ .

We will generalize this definition to get a sheaf of differential operators which can be applied to any line bundle on  $X/G$ , and which “twist” by a line bundle corresponding to a character of  $G$ .

**Definition** Let  $\mu$  be a character of  $G$ . Define the sheaf of differential operators on  $X/G$  that twist by  $\mu$

as  $\mathcal{D}_\mu(X/G) := (\mathcal{D}(X)/(\mathcal{D}(X)[\mathfrak{g}, \mathfrak{g}]))^{G_{-\mu}}$  where  $\mathfrak{g}$  is the Lie algebra of  $G$ ,  $[\mathfrak{g}, \mathfrak{g}]$  denotes the commutator subalgebra of  $\mathfrak{g}$ , and the superscript  $G_{-\mu}$  denotes the subset of elements that are  $G$ -semiinvariant with character  $-\mu$ .

This construction mimics Kashiwara's use of the universal Verma module [9].

Note that for any  $\lambda$ , there is a natural map  $j_\lambda : \mathcal{D}_\mu(X/G) \rightarrow \mathcal{D}(\mathcal{L}_{\lambda+\mu}, \mathcal{L}_\lambda)$ , based on the fact that  $\lambda([\mathfrak{g}, \mathfrak{g}]) = 0$ . If  $P \in \Gamma(X/G, \mathcal{D}_\mu(x))$ ,  $f \in \Gamma(X/G, \mathcal{L}_\lambda)$  for some line bundle  $\mathcal{L}_\lambda$  on  $X/G$ , we will abuse notation to write  $Pf = (j_\lambda(P))(f)$ .

**Example** The differential operator  $\partial_x$  on  $\mathbb{A}^2 \setminus \{(0, 0)\}$  descends to a differential operator (which we also call  $\partial_x$  by abuse of notation) on  $\mathbb{P}^1$  that twists by  $\mathcal{O}(-1)$ .

### 2.3 Bernstein-Sato polynomials on affine varieties

Among their other uses, D-modules were found to be useful in studying singularity invariants of functions. This was accomplished by studying the Sato-Bernstein polynomials of those functions.

For the rest of this section, assume  $X$  is affine. Let  $f$  be a regular function on  $X$ . We can consider the space generated by differential operators starting with the formal term  $f^s$ :

**Definition** Define  $\mathcal{O}_X[s] = \mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{O}_X$ ,  $\mathcal{D}(X)[s] = \mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{D}(X)$ . Then define the sheaf of vector spaces  $U \rightarrow \mathcal{O}_U[f^{-1}][s]f^s = (\oplus_{k \in \mathbb{Z}} \mathcal{O}_U[s]f^{s-k}) / \{ff^{s-k} = f^{s-k+1}\}$ . This sheaf then has a flat connection induced by the equation  $\xi(gf^{s-k}) = \xi(g)f^{s-k} + (s-k)g\xi(f)f^{s-k-1}$ , and so  $\mathcal{O}_X[f^{-1}][s]f^s$  has a  $\mathcal{D}(X)[s]$ -module structure.

It turns out that  $\mathcal{O}_X[f^{-1}][s]f^s$  is a holonomic D-module. By using the rate of growth of the filtered vector spaces, we can find the following theorem, first proven by Bernstein [1] for functions on  $\mathbb{A}^n$  and later generalized by Kashiwara [7] to arbitrary varieties:

**Theorem 2.** *There is some nonzero  $b \in \mathbb{Q}[s]$  and some  $P \in \mathcal{D}(X)[s]$  such that  $Pf^{s+1} = b(s)f^s$ .*

**Example** If  $f = x$  on  $\mathbb{A}^1$ , then we can choose  $b(s) = s + 1$ . This is given by the differential operator  $\partial_x$ .

Note that the differential operator is allowed to depend on  $s$ , and sometimes must. For example:

**Example** For  $f = x(x + 1)$  on  $\mathbb{A}^1$ , we can choose  $b(s) = s + 1$ , based on the differential operator

$$P = (2s + 1)(2x + 1)\partial_x - 2x(x + 1)\partial_x^2.$$

The set of such  $b(s)$  forms an ideal in  $\mathbb{C}[s]$  (and the ideal is nontrivial by theorem 2), and therefore is generated by a unique nonzero monic polynomial. This is called the Bernstein-Sato polynomial, or b-function, associated with  $f$ . In both of the above examples, we have actually calculated the b-function. Note that in both these examples, the b-function is the same. This is because when the zero-set of  $f$  is smooth, the b-function of  $f$  is  $s + 1$  (and in fact, the converse is also true [2]). More generally,  $s + 1$  always divides  $b(s)$  as long as  $f$  is somewhere zero.

In fact, the b-function of  $f$  is equal to the least common multiple of "local" b-functions at points in the zero-set of  $f$  ([5]), which implies the case of smoothness. The b-function can then be seen as a singularity invariant of the zero-set of  $f$ .

Both the b-function and the differential operator are often more difficult to calculate than these examples, even in low-degree cases. For example [8]:

**Example** If  $f = x^2 + y^3$ , then  $b(s) = (s + 1)(s + \frac{5}{6})(s + \frac{7}{6})$ , based on the differential operator

$$P = \frac{1}{12}y\partial_x^2\partial_y + \frac{1}{27}\partial_y^3 + s\frac{1}{4}\partial_x + \frac{3}{8}\partial_x^2$$

One surprising result about b-functions, proven by Kashiwara, is that all of the roots of b-functions are rational and negative. [7]

There are some cases where the differential operator can be found relatively easily, by exploiting symmetries in the relevant space. Most prominently, b-functions were originally defined to study prehomogeneous vector spaces, by Sato and Shintani [13].

**Definition** A prehomogeneous vector space is a vector space  $V$  with a linear group action  $G \curvearrowright V$  such that the action has an open orbit.

Following Shintani [14], we make two further assumptions. First, we assume that  $G$  is a reductive algebraic group defined over  $\mathbb{R}$ , and that  $V$  has a real structure. Second, we assume that the complement of the open orbit is a hypersurface, so that it is the zero-set of an irreducible homogeneous polynomial

$f \in \mathbb{C}[V] = \text{Sym}V^*$  ([14] Prop 1.1). Because the zero-set is  $G$ -invariant,  $f$  would have to be  $G$ -semiinvariant for some character  $\chi$ . Sato and Shintani were interested in finding the b-function of  $f$  in such a case.

The extra structure afforded by the vector space and the group action gives a natural choice of differential operator for the b-function functional equation. Using the assumptions, the dual space  $V^*$  is also a prehomogeneous vector space ([14] lemma 1.1); the assumption about the complement of the open orbit also transfers. This gives a corresponding irreducible homogeneous polynomial  $f^* \in \mathbb{C}[V^*] = \text{Sym}V$ ;  $f^*$  is  $G$ -semiinvariant with character  $\chi^*$  ([14]). But as elements of  $V$  can be interpreted as translation-invariant vector fields on  $V^*$ , polynomials in those elements can be interpreted as translation-invariant differential operators, so  $f^*$  gives a  $G$ -semiinvariant differential operator  $P$ . It's not hard to see that  $Pf^{s+1} = b(s)f^s$  for some  $b \in \mathbb{C}[s]$ , and so  $P$  is the relevant differential operator that gives a b-function for  $f$ .

Parts of this method can be applied somewhat more generally, and will provide the general technique for the calculation of the relevant differential operator in multiple later chapters.

## 2.4 Extensions of b-functions

The methodology of b-functions can be extended in a few ways.

The first generalization uses multiple  $f$ . This allows the study of how multiple hypersurfaces interact. There were multiple generalizations of the functional equation; the one that is most helpful for this result is:

Let  $f_i$  be nonzero regular functions on  $X$ . For any  $\{m_1, m_2, \dots, m_n\} \in \mathbb{Z}^n$ , there is some  $P_{m_1, m_2, \dots, m_n} \in \mathcal{D}(X)[s_1, s_2, \dots, s_n]$  and nonzero  $b_{m_1, m_2, \dots, m_n} \in \mathbb{C}[s_1, s_2, \dots, s_n]$  such that

$$P_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \prod_i f_i^{s_i + m_i} = b_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \prod_i f_i^{s_i} \quad (2)$$

for all  $(s_1, s_2, \dots, s_n)$ .

While this still gives an ideal of such  $b_{m_1, m_2, \dots, m_n}$ , this ideal is no longer necessarily principal, so this functional equation doesn't lead to a unique b-function. This ideal was studied by Sabbah [11] and Gyoja [6]; one of the most striking results from Gyoja was a generalization of Kashiwara's result for b-functions for a single  $f$ :

**Theorem 3.** *For given  $m_1, m_2, \dots, m_n$  there is some  $b_{m_1, m_2, \dots, m_n}$  in the above ideal that can be expressed*

in the form  $\prod L_j$ , where each  $L_j$  is of the form  $c_j + \sum_i a_{ij}s_i$ , an affine function in terms of the  $s_i$ , with  $a_{ij} \in \mathbb{Z}, c_j \in \mathbb{Q}$ .

The second generalization is to allow  $X$  to not be affine. For the purposes of this thesis, we take  $X = Y/G$  to be the quotient of an affine variety by the free action of a connected Lie group.

We first replace our functions  $f_i$  with sections of line bundles on  $X$ . Second, we reconsider our differential operators. As each of the  $f_i$  “twists” by the line bundle it is a section of, our differential operators will also need to “twist”. The definition of the b-function generalizes immediately when we use  $\mathcal{D}_\mu(X)$ :

Let  $f_i \in \Gamma(X, \mathcal{L}_{\lambda_i})$  be sections of line bundles on  $X$ . For any  $\{m_1, m_2, \dots, m_n\} \in \mathbb{Z}^n$ , there is some  $P \in \Gamma(X, \mathcal{D}_{\sum_i m_i \lambda_i}(X)[s_1, s_2, \dots, s_n])$  and nonzero  $b_{m_1, m_2, \dots, m_n} \in \mathbb{C}[s_1, s_2, \dots, s_n]$  such that

$$P_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \prod_i f_i^{s_i + m_i} = b_{m_1, m_2, \dots, m_n}(s_1, s_2, \dots, s_n) \prod_i f_i^{s_i} \quad (3)$$

for all  $(s_1, s_2, \dots, s_n)$ .

Note that with this definition of b-functions, some of the conclusions about b-functions on affine varieties are no longer necessarily true. Even with a single section, global b-functions are no longer necessarily least common multiples of local b-functions:

**Example** The b-function of  $xy \in \Gamma(\mathbb{P}^1, \mathcal{O}(2))$  is  $(s+1)^2$ , given by the differential operator  $\partial_x \partial_y$ .

The fact that  $(s+1)$  must divide the b-function does remain true:

**Lemma 4.** *Assume that the zero-sets of the  $f_i$  are irreducible and distinct. Then for each  $i$ , for any  $b$  satisfying the functional equation (equations 2 or 3),  $s_i + 1$  is a factor of  $b_{(0,0,\dots,0,1,0,\dots,0)}$ , where the 1 is in the  $i$ th position.*

## 2.5 Mirabolic space

We now turn to specifics: the space, the line bundles, and the sections of those line bundles for which we wish to calculate the b-function.

For this paragraph, let  $G$  be a connected semisimple Lie group, with a Borel subgroup  $B$  and opposite Borel subgroup  $B^-$ . Then  $B^-$  acts on the flag variety  $\mathcal{F} = G/B$ . This action gives a decomposition,

called the Bruhat decomposition, of the flag variety  $G/B = \sqcup_{w \in W_G} B^- w B/B$  into orbits of  $B^-$ . In this decomposition, the largest cell is  $B^- B/B$ , and the complement of that orbit is a union of subvarieties of codimension 1. Each of these subvarieties is the zero-set of a  $B^-$ -semiinvariant section of a line bundle on  $\mathcal{F}$ . Kashiwara [9] studied the b-function given by these sections by using the universal Verma module.

Let  $G = SL_n$ , which naturally acts on a vector space  $V$  of dimension  $n$ ; let  $B$  be a Borel subgroup (such as the subgroup of upper triangular matrices),  $B^-$  an opposite Borel subgroup (such as the subgroup of lower triangular matrices). Together, these determine a basis of  $V$  (up to scaling). Let  $P$  be a parabolic subgroup of  $G$  that stabilizes a line in  $V$ ; this is called a mirabolic subgroup. We further assume that  $P$  is in general position with respect to the basis given by  $B$  and  $B^-$ . Note that this implies that  $B \cap B^- \cap P = \{cI | c^n = 1\}$ , a finite set.

Let  $X$  be the triple flag variety  $G/B \times G/B^- \times G/P = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$ . This is a projective variety of dimension  $\frac{n^2-n}{2} + \frac{n^2-n}{2} + (n-1) = n^2 - 1 = \dim G$ . We can consider the diagonal action of  $G$  on  $X$ . The point  $(eB, eB^-, eP)$  is only fixed by elements of  $B \cap B^- \cap P$  - which we already know to be finite. Therefore, the orbit of  $(eB, eB^-, eP)$  must have dimension  $\dim G = n^2 - 1$  - so it is the unique open orbit on  $X$ .

By Borel-Weil, a line bundle on  $X$  can be labeled with a triple  $(\lambda_1, \lambda_2, l)$  with  $\lambda_1, \lambda_2$  dominant weights and  $l$  a nonnegative integer. Further, the global sections for the line bundle  $L_{(\lambda_1, \lambda_2, l)}$  form a representation of  $G \times G \times G$  isomorphic to  $W_{\lambda_1} \otimes W_{\lambda_2} \otimes \text{Sym}^l \mathbb{C}^n$ . As shown by Ginzburg and Finkelberg [4], the line bundle then has a  $G$ -invariant global section if and only if it is in the subsemigroup generated by elements of the form  $\alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0), 1 \leq i \leq n-1$  and  $\beta_j = (\wedge^{j-1} \omega, \wedge^{n-j} \omega, 1), 1 \leq j \leq n$ , where  $\omega$  is the fundamental representation of  $G = SL_n$ . Therefore, each of these triples gives a  $G$ -invariant global section of a line bundle on  $X$ , and therefore a  $G$ -invariant hypersurface on  $X$ . The union of these subvarieties gives the complement of the open  $G$ -orbit on  $X$ . The purpose of this dissertation is to find a b-function for these global sections of line bundles on  $X$ .

This triple flag variety is closely related to the  $SL_n$ -variety  $\mathbb{M}^n \times \mathbb{P}^{n-1}$ , where  $\mathbb{M}^n$  denotes the space of  $n \times n$  matrices, and the diagonal  $SL_n$  action comes from the usual action of  $SL_n$  on  $\mathbb{P}^{n-1}$  and the conjugation action on  $\mathbb{M}^n$ . The set of cyclic pairs (i.e. pairs  $(A, \vec{v})$  such that  $\vec{v}, A\vec{v}, \dots, A^{n-1}\vec{v}$  form a basis) is a dense open set within the space; its complement is a subvariety of codimension 1, which consists of pairs with

nontrivial stabilizer. This space was studied by Robin Walters [15], who found the b-function corresponding to the section which has as zero-set the subvariety with nontrivial stabilizer. The relationship between these two spaces extends to a pair of adjoint functors that relate (a certain category of derived) D-modules on  $X$  with (a certain category of derived) D-modules on  $\mathbb{M}^n \times \mathbb{P}^{n-1}$  [4].

### 3 Notation

1.  $G = SL_n$
2.  $e \in G$  is the identity element.
3.  $B$  is a Borel subgroup of  $G$ .
4.  $B^-$  is a Borel subgroup of  $G$  opposite to  $B$ .
5.  $H = B \cap B^-$  is a maximal torus of  $G$ .
6.  $P$  is a mirabolic subgroup of  $G$ , that is, a subgroup of  $G$  that stabilizes a line in  $\mathbb{C}^n$ .
7.  $\mathfrak{g}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{h}, \mathfrak{p}$  are the corresponding Lie algebras.
8.  $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$  is the maximal nilpotent subalgebra of  $\mathfrak{b}$ .
9.  $U\mathfrak{g}$  denotes the universal enveloping algebra of  $\mathfrak{g}$ .
10.  $R_+$  denotes the set of positive roots of  $G$ .
11. If  $\alpha \in R_+$ , then  $h_\alpha$  denotes its corresponding coroot.
12.  $\mathcal{F} = G/B$  is the flag variety of  $G$ , thought of as the moduli space of Borel subgroups.
13.  $X = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$  is the moduli space of triples of two Borel subgroups and a mirabolic subgroup.
14.  $X' = \mathcal{F} \times \mathcal{F} \times (\mathbb{P}^{n-1})^\vee$  is the moduli space of triples of two Borel subgroups and a dual-mirabolic subgroup.
15.  $W_{\lambda_1}$  is the finite-dimensional irreducible representation of  $G$  with highest weight  $\lambda_1$ .

16. If  $\lambda = (\lambda_1, \lambda_2, l)$ , then  $V_\lambda := W_{\lambda_1} \otimes W_{\lambda_2} \otimes \text{Sym}^l \mathbb{C}^n$  is a finite-dimensional irreducible representation of  $G^3 = G \times G \times G$ .
17.  $B_1 \times B_2 \times P$  is a triple of subgroups of  $G$ , with  $B_1, B_2$  Borel and  $P$  mirabolic.
18.  $B'_1 \times B'_2 \times P'$  is a triple of subgroups of  $G$ , with  $B'_1, B'_2$  Borel and  $P'$  dual-mirabolic (i.e. stabilizing a line in  $\mathbb{C}^{n*}$  where the  $*$  denotes the dual action of  $G$ ).
19.  $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{p}, \mathfrak{b}'_1, \mathfrak{b}'_2, \mathfrak{p}'$  are the corresponding Lie algebras.
20.  $v_\lambda$  is a highest weight vector with respect to  $B_1 \times B_2 \times P$  in  $V_\lambda$ .
21.  $v_{-\lambda}$  is a highest weight vector in  $V_\lambda^*$  with respect to  $B'_1 \times B'_2 \times P'$  such that  $\langle v_{-\lambda}, v_\lambda \rangle = 1$ .
22.  $\Omega$  is the set of triples  $\lambda$  such that there is a diagonally  $G$ -invariant vector  $u_{-\lambda} \in V_\lambda^*$  with  $\langle u_{-\lambda}, v_\lambda \rangle = 1$ .
23.  $u_\lambda \in V_\lambda$  is the diagonally  $G$ -invariant vector with  $\langle v_{-\lambda}, u_\lambda \rangle = 1$ .
24.  $M_{\lambda_1}$  is the line bundle on  $G$  whose global sections are induced by  $B$ -semiinvariant functions on  $G$  with character  $\lambda_1$ .
25.  $f_G^{\lambda_1} \in \Gamma(\mathcal{F}, M_{\lambda_1})$  is the unique global section of  $M_{\lambda_1}$  that is  $B^-$ -semiinvariant.
26.  $\text{Mer}(\mathbb{C}\Omega, \mathbb{C})^\times$  is the multiplicative group of the field of meromorphic functions on the vector space  $\mathbb{C}\Omega$ .
27.  $L_\lambda$  is the line bundle on  $X$  whose global sections are induced by  $B_1 \times B_2 \times P$ -semiinvariant functions on  $G^3$  with character  $\lambda$ .
28.  $f_K^\lambda \in \Gamma(X, L_\lambda)$  is the unique global section of  $L_\lambda$  that is  $B'_1 \times B'_2 \times P'$ -semiinvariant.
29.  $f_{G^3}^\lambda \in \Gamma(X, L_\lambda)$  is the unique global section of  $L_\lambda$  that is  $G$ -invariant under the diagonal action of  $G$ .
30.  $H(\lambda) = \frac{\langle v_{-\lambda}, v_\lambda \rangle \langle u_{-\lambda}, u_\lambda \rangle}{\langle v_{-\lambda}, u_\lambda \rangle \langle u_{-\lambda}, v_\lambda \rangle}$
31.  $\omega$  is the standard representation  $\mathbb{C}^n$  of  $G$ .
32.  $\wedge^i \omega$  is the  $i$ th fundamental representation of  $SL_n$ .
33.  $\omega_j$  is the standard representation of  $SL_j \subset G$ .

$$34. \alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0), 1 \leq i \leq n-1 \in \Omega.$$

$$35. \beta_j = (\wedge^{j-1} \omega, \wedge^{n-j} \omega, 1), 1 \leq j \leq n \in \Omega.$$

## 4 Result

The main result of the thesis reads as follows: Let  $\Gamma$  be the gamma function.

**Theorem 5.** *Let  $\lambda = (\sum_i a_i \alpha_i) + (\sum_j b_j \beta_j)$ ,  $\mu$  be two triples of weights that both lie in  $\Omega$ . Let  $R_+$  denote the set of positive roots of  $G$  and let*

$$A(\lambda) = \prod_j \Gamma(b_j + 1) \cdot \prod_{\gamma \in R_+} \Gamma \left( h_\gamma(\rho) + \sum_{\omega^i < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{and } \omega^j < \gamma}} b_j \right) \cdot \Gamma \left( h_\gamma(\rho) + \sum_{\omega^j < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{or } \omega^j < \gamma}} b_j + 1 \right)$$

*Then there exists a differential operator  $P_{\mu, G^3} \in \mathcal{D}_\mu(X)$  such that  $P_{\mu, G^3} f_{G^3}^{\lambda+\mu} = b_{\mu, G^3}(\lambda) f_{G^3}^{\lambda+\mu}$ , where*

$$b_{\mu, G^3}(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)}.$$

## 5 Summary

We start in section 6 by defining the  $G$ -invariant global sections  $f_{G^3}^\lambda$  of line bundles on  $X$ , and the relevant functional equation for the type of b-functions we wish to use. In section 7, we explain some of Kashiwara's techniques.

**Theorem 6.** [9] *For any dominant integral weight  $\lambda_1$  of  $G$ , recall that the line bundle  $M_{\lambda_1}$  on  $\mathcal{F}$  has global sections induced from  $B$ -semiinvariant functions of character  $\lambda$  on  $G$ . Let  $f_G^\lambda \in \Gamma(\mathcal{F}, M_\lambda)$  denote the unique (up to scaling) global section that is  $B^-$ -semiinvariant. Then there exists a differential operator  $P_{\mu, G} \in \mathcal{D}_\mu(\mathcal{F})$  that satisfies the b-function equation for the  $f_G^\lambda$ :*

$$P_{\mu, G} f_G^{\lambda+\mu} = b_{\mu, G}(\lambda) f_G^\lambda \tag{4}$$

where  $b_{\mu, G}(\lambda) = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha(\mu)} (h_\alpha(\lambda) + h_\alpha(\rho) + i) \in \mathbb{C}[\mathfrak{h}]$ .

In doing so, we prove the following lemma relating the differential operators:

**Lemma 7.** *For all  $\mu, \nu$  integral dominant weights of  $G$ , the following equation holds:*

$$P_{\mu, G} P_{\nu, G} = P_{\mu + \nu, G} \tag{5}$$

In section 8, we use a generalization of equation 5 to determine that the b-functions satisfy a cocycle equation

$$b_{\mu}(\lambda) b_{\nu}(\lambda + \mu) = b_{\mu + \nu}(\lambda) \tag{6}$$

for  $\mu, \nu$  elements of a lattice  $\Lambda$  (in the case of equation 5, the weight lattice) and  $\lambda \in \mathbb{C}\Lambda$ .

We use equation 6 to prove that the b-functions can be expressed as products of affine factors on  $\mathbb{C}\Lambda$ , roughly following [12].

**Lemma 8.** *Let  $\{b_{\mu}\}$  satisfy equation 6. Then  $b_{\mu} = \prod_i (\alpha_i(\lambda) + k_i)$  for some  $\alpha_i \in \mathfrak{h}, k_i \in \mathbb{C}$ .*

We also prove some relationships between the b-functions. In section 9, we use Kashiwara's differential operator to find the differential operator  $P_{\mu, G^3}$  on  $X$  that satisfies the functional equation, and a relationship between the b-function Kashiwara found for a different global section and the b-function for the global section examined in this dissertation, through the function  $H(\lambda) : \Omega \rightarrow \mathbb{C}$ . We also show that the differential operators satisfy a similar equation to equation 5:

**Lemma 9.** *For our differential operators  $P_{\mu, G^3}$ , the following equation holds:*

$$P_{\mu, G^3} P_{\nu, G^3} = P_{\mu + \nu, G^3} \tag{7}$$

We can then combine the previous two lemmas, where the relevant lattice is  $\Omega$ . In section 10, we examine  $H(\lambda)$  more closely and find its values for some  $\lambda \in \Omega$ . In section 11, we use the values given in section 10 with the relationships in section 8 to determine the affine factors, and therefore find the b-functions and prove the above theorem.

## 6 The relevant line bundle and section

Choose two Borel subgroups  $B_1, B_2$  and a mirabolic subgroup  $P$  of  $G = SL_n$ . The corresponding subgroup  $B_1 \times B_2 \times P \subset G \times G \times G$  is the stabilizer of a unique point  $x \in X$ , so  $X$  can be seen as the moduli space of such triples. We further require that  $x$  in general position; that is, that  $x$  lie in the open  $G$ -diagonal orbit on  $X$ . Let  $\Gamma$  be the lattice of triples  $(\lambda_1, \lambda_2, l)$  of pairs of integral weights and an integer; let  $\Gamma_{\geq 0}$  be the subcone where the weights are dominant and the integer is nonnegative.

For  $\lambda = (\lambda_1, \lambda_2, l) \in \Gamma_{\geq 0}$ , recall  $V_\lambda = W_{\lambda_1} \otimes W_{\lambda_2} \otimes \text{Sym}^l \mathbb{C}^n$ ; also recall  $L_\lambda$  as the line bundle with global sections induced by functions  $f : G \times G \times G \rightarrow \mathbb{C}$  such that for any  $(g_1, g_2, g_3) \in G^3$ ,  $(b_1, b_2, p) \in B_1 \times B_2 \times P$ ,  $f(g_1 b_1, g_2 b_2, g_3 p) = e^{2\pi i \lambda(b_1, b_2, p)} f(g_1, g_2, g_3)$ . Recall that  $\Omega$  is the set of  $\lambda \in \Gamma_{\geq 0}$  such that  $V_\lambda^*$  has a nontrivial  $G$ -invariant element.

**Lemma 10.**  *$\Omega$  is closed under addition.*

*Proof.* By the Borel-Weil theorem, the global sections of the line bundle  $L_\lambda$  on  $X = \mathcal{F} \times \mathcal{F} \times \mathbb{P}^{n-1}$  form a representation of  $G^3$  isomorphic to  $V_\lambda^*$ . As such,  $V_\lambda^*$  contains a nontrivial  $G$ -invariant element under the diagonal action of  $G$  if and only if there is a nontrivial  $G$ -invariant global section of  $L_\lambda$ . Then because  $X$  is an irreducible variety (and as such has a multi-homogeneous structure sheaf without zero divisors), if there is a nontrivial  $G$ -invariant element of  $V_\lambda^*$  and another of  $V_{\lambda'}^*$ , then there necessarily is a nontrivial  $G$ -invariant element of  $V_{\lambda+\lambda'}^*$  corresponding to the product of the two invariant sections in  $L_{\lambda+\lambda'}$ .  $\square$

For  $\lambda \in \Gamma_{\geq 0}$ , recall that  $v_\lambda \in V_\lambda$  is a nonzero highest-weight element with respect to  $B_1 \times B_2 \times P$ . Because we have chosen our subgroups in general position, the  $G$ -invariant projection  $v_\gamma^G \neq 0$  for all  $\gamma \in \Gamma_{\geq 0}$ . If we choose  $w_{-\lambda} \in V_\lambda^*$  such that  $w_{-\lambda}$  is  $G$ -invariant and nonzero, then  $\langle w_{-\lambda}, v_\lambda \rangle = 0$  is the defining equation of a closed union of components of the complement of the open orbit. In fact, any defining equation of such a union comes from some such  $\lambda$ . Therefore  $x$  being in the open orbit is equivalent to the existence for any  $\lambda \in \Omega$  of a  $w_{-\lambda} \in V_\lambda^*$  that is  $G$ -invariant such that  $\langle w_{-\lambda}, v_\lambda \rangle = 1$  (by scaling  $w_{-\lambda}$  appropriately).

Recall the variety  $X' = \mathcal{F} \times \mathcal{F} \times (\mathbb{P}^{n-1})^\vee$ , which has a  $G^3$  action from the  $G$ -action on each factor. Then  $X'$  corresponds to the moduli space of triples of two Borel subgroups  $B'_1, B'_2$  and a dual-mirabolic subgroup  $P'$  (i.e. a mirabolic subgroup stabilizing a line in the dual of the standard representation). We then say that

$x' \in X'$  is in general position if it is in the open  $G$ -orbit on  $X'$ .

Finally, a point  $(x, x') \in X \times X'$  is in general position if each subgroup and its primed counterpart are opposite, and both  $x$  and  $x'$  are in general position. Equivalently (for similar reasoning to the above), for any  $\lambda \in \Gamma_{\geq 0}$ ,  $(x, x')$  is in general position if we can choose a highest weight vector  $v_{-\lambda} \in V_{\lambda}^*$  with respect to  $B_1' \times B_2' \times P'$  such that  $\langle v_{-\lambda}, v_{\lambda} \rangle = 1$ , and a  $G$ -invariant element  $u_{\lambda} \in V_{\lambda}$  such that  $\langle v_{-\lambda}, u_{\lambda} \rangle = 1$ .

Recall that  $L_{\Lambda}$  is the line bundle whose global sections correspond to right  $B_1 \times B_2 \times P$ -semiinvariant functions on  $G^3$ . Borel-Weil implies that the correspondence between  $V_{\lambda}^*$  and  $\Gamma(X, L_{\lambda})$  can be given by  $w \rightarrow [g \rightarrow \langle w, gv_{\lambda} \rangle]$  for  $g \in G^3$ . We therefore have two sections of  $L_{\lambda}$ ,  $f_{G^3}^{\lambda}$  and  $f_K^{\lambda}$ , such that  $f_{G^3}^{\lambda}$  is diagonally  $G$ -invariant, while  $f_K^{\lambda}$  is left  $B_1' \times B_2' \times P'$  semiinvariant. The former subscript denotes that the section is  $G$ -invariant; the latter denotes that the section is closely related to the functions studied by Kashiwara [9].

Because there is both a unique open  $G$ -orbit and a unique open  $B_1' \times B_2' \times P'$ -orbit on  $X$ , these sections are the unique ones with this property up to scaling. Then as products of semiinvariants are semiinvariant, and by looking at the image of  $(e, e, e)$  in  $X$ , we can see that  $f_K^{\lambda} f_K^{\mu} = f_K^{\lambda+\mu}$  and  $f_{G^3}^{\lambda} f_{G^3}^{\mu} = f_{G^3}^{\lambda+\mu}$ .

The b-function we want to find is that of  $f_{G^3}^{\lambda}$ ; to be more precise, we want to find  $P_{\mu, G}, b_{\mu, G}$  satisfying the following functional equation:

$$P_{\mu, G} f_{G^3}^{\lambda+\mu} = b_{\mu, G}(\lambda) f_{G^3}^{\lambda} \quad (8)$$

where  $P_{\mu, G}$  is a differential operator on  $\mathcal{F}$  (equivalently, a differential operator on  $G$  such that for any right  $B$ -semiinvariant  $f$ ,  $P_{\mu} f$  is also right  $B$ -semiinvariant).

This corresponds to a b-function using the definition given by Gyoja [6], where the  $f_i$  are  $f_{\lambda}$  for  $\lambda$  a generator of  $\Omega$ . We will describe those generators in section 10.

## 7 Kashiwara's argument and the differential operator

Let  $G$  be any semisimple complex simply connected Lie group with Lie algebra  $\mathfrak{g}$ ; let  $B$  be a Borel subgroup and  $B^-$  be an opposite Borel subgroup with common torus  $H = B \cap B^-$ . Let  $\mathfrak{h}$  be the Lie algebra of  $H$ . Then Kashiwara found [9] the b-function of the  $B^-$ -semiinvariant section  $f_G^{\lambda}$  of the line bundle  $M_{\lambda}$  over

$\mathcal{F} = G/B$ , using its preimage  $f_G^\lambda : G \rightarrow \mathbb{C}, g \rightarrow \langle v_{-\lambda}, gv_\lambda \rangle$  (which we will also call  $f_G^\lambda$  through abuse of notation) for any dominant integral weight  $\lambda$ .

Recall that  $M_\lambda$  denotes the line bundle on  $\mathcal{F}$  which has global sections induced by semiinvariant functions on  $G$  with character  $\lambda$ .

Recall that  $R_+$  is the set of positive roots of  $G$  with respect to  $B$ ,  $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ .

**Definition** For  $\mu \in \mathfrak{h}^*$ , recall the sheaf  $\mathcal{D}_\mu(\mathcal{F})$  of differential operators on  $\mathcal{F}$  that twist by  $-\mu$ . In other words, if  $P \in \Gamma(\mathcal{F}, \mathcal{D}_\mu(\mathcal{F}))$ , then  $P$  induces a linear map  $P : \Gamma(\mathcal{F}, M_\lambda) \rightarrow \Gamma(\mathcal{F}, M_{\lambda-\mu})$  for every  $\lambda$ .

**Theorem 11.** (Kashiwara) [9] For any integral dominant  $\mu \in \mathfrak{h}^*$ , there is a differential operator  $P_{\mu,G} \in \mathcal{D}_\mu(\mathcal{F})$  on  $\mathcal{F}$  such that for any nonnegative integral  $\lambda$ :

$$P_{\mu,G} f_G^{\lambda+\mu} = b_{\mu,G}(\lambda) f_G^\lambda \quad (9)$$

$$\text{where } b_{\mu,G}(\lambda) = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha(\mu)} (h_\alpha(\lambda) + h_\alpha(\rho) + i) \in \mathbb{C}[\mathfrak{h}]$$

We will find a differential operator satisfying the functional equation, as the method will be useful later; however, we will not reproduce the proof of the value of  $b_{\mu,G}$ .

*Proof.* Following Kashiwara, we can trivialize the sheaf of differential operators on  $G$  using right translation:  $\mathcal{D}(G) = \mathbb{C}[G] \otimes R(U\mathfrak{g})$ , where  $R : U\mathfrak{g} \rightarrow \mathcal{D}(G)$  is the extension to  $U\mathfrak{g}$  of the map  $R : \mathfrak{g} \rightarrow TG$  given by right translation. Then by quantum Hamiltonian reduction [3],  $\mathcal{D}_\mu(\mathcal{F}) = \left( \frac{\mathcal{D}(G)}{\mathcal{D}(G)\mathfrak{n}} \right)^{\mathfrak{b}-\mu}$ , where the subscript  $-\mu$  on  $\mathfrak{b}$  denotes the subset of elements of weight  $-\mu$ . The quotient  $\frac{\mathcal{D}(G)}{\mathcal{D}(G)\mathfrak{n}} \simeq \mathbb{C}[G] \otimes (U\mathfrak{g}/((U\mathfrak{g})\mathfrak{n}))$ . Using the well-known decomposition  $\mathbb{C}[G] = \oplus (V_\nu \otimes V_\nu^*)$  as  $G$ -bimodules and rearranging tensor products, we get that  $\mathcal{D}_\mu(\mathcal{F}) = \oplus (V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}))^{\mathfrak{b}-\mu}$ . Because we trivialized by right translation, and because the decomposition separates the actions of left and right translation,  $\mathcal{D}_\mu(\mathcal{F}) = \oplus V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ , where the action of  $G$  on  $\mathcal{F}$  induces the usual action on  $V_\nu$ .

We will also use two results on the  $U\mathfrak{h}$ -module structure of  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  for a  $\mathfrak{b}$ -representation  $V$ ; the proofs are given in the appendix. Note that the universal enveloping algebra  $U\mathfrak{h}$  is also the symmetric algebra of  $\mathfrak{h}$ .

**Theorem 12.** *Let  $V$  be a finite dimensional  $\mathfrak{b}$ -module. Then  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is a  $S\mathfrak{h}$  module of rank equal to the dimension of the  $\mu$  weight subspace of  $V$ .*

**Theorem 13.** *Assume that the dimension of  $V^\mu$  is 1. Then  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is free as a right  $S\mathfrak{h}$ -module.*

Theorem 12 implies that if  $\nu \not\geq \mu$  (where  $\geq$  denotes the usual partial inequality on weights), then  $(V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is trivial. The second then shows that in the minimal case of  $\nu = \mu$ ,  $(V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is a free rank 1 right  $S\mathfrak{h}$ -module. Call the generating element  $P \in (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ . Then by attaching a highest weight vector  $v_\mu \in V_\mu$ , we get a differential operator  $P_\mu = v_\mu \otimes P \in \mathcal{D}_\mu(\mathcal{F})$ . Note that  $P_\mu$  varies under left translation as  $v_\mu$  does, that is,  $gP_\mu = (gv_\mu) \otimes P$  where  $g$  acts on differential operators by left translation, for  $g \in G$ .

By the definition of  $\mathcal{D}_\mu(\mathcal{F})$ ,  $P_{\mu,G}f_G^{\lambda+\mu} \in \Gamma(\mathcal{F}, M_\lambda)$ . Further, as the map of application of differential operators  $\mathcal{D}_\mu(\mathcal{F}) \otimes \Gamma(\mathcal{F}, M_{\lambda+\mu}) \rightarrow \Gamma(\mathcal{F}, M_\lambda)$  is  $G$ -equivariant,  $P_{\mu,G}f_G^{\lambda+\mu}$  will be of weight  $-\lambda$ . By Borel-Weil,  $\Gamma(\mathcal{F}, M_\lambda) \simeq V_\lambda^*$ , so there is only one element of weight  $-\lambda$  (up to scaling). But  $f_G^\lambda$  is already of weight  $-\lambda$  - so we get that  $P_{\mu,G}f_G^{\lambda+\mu} = b_{\mu,G}(\lambda)f_G^\lambda$ .  $\square$

Kashiwara further found an explicit formula:

$$b_{\mu,G}(\lambda) = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha(\mu)} (h_\alpha(\lambda) + h_\alpha(\rho) + i) \quad (10)$$

[9] (main theorem). and that this is the generator of the ideal of all solutions to the functional equation.

One result was not stated explicitly by Kashiwara but which will be useful for us is the following:

**Lemma 14.** *For any  $\mu, \nu \in \mathfrak{h}^*$  integral dominant weights, we have  $P_{\mu,G}P_{\nu,G} = P_{\mu+\nu,G}$ .*

*Proof.* Consider the decomposition of  $\mathbb{C}[G]$  into  $\oplus V_\mu \otimes V_\mu^*$  as an increasing algebra filtration,  $\mathbb{C}[G]_{\leq \mu} = \oplus_{\nu \leq \mu} V_\nu \otimes V_\nu^*$  with  $\mathbb{C}[G]_{\leq \mu} \mathbb{C}[G]_{\leq \nu} \subseteq \mathbb{C}[G]_{\leq \mu+\nu}$ . Then by using the trivial filtration on  $U\mathfrak{g}$ , we obtain an increasing filtration on  $\mathcal{D}(G) \simeq \mathbb{C}[G] \otimes U\mathfrak{g}$ ; as the action of  $\mathfrak{g}$  respects the filtration on  $\mathbb{C}[G]$ , this filtration is an increasing algebra filtration. This filtration then descends to  $\mathcal{D}_\mu(\mathcal{F})$  for any character  $\mu$  of  $B$ , so we obtain an increasing filtration  $\mathcal{D}_\mu^{\leq \eta}(\mathcal{F}) = \oplus_{\nu \leq \eta} V_\nu \otimes (V_\nu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ .

The highest weight in  $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$  is 0, so if  $\nu < \mu$ , then  $\mathcal{D}_\mu^{\leq \nu} = \{0\}$ , and  $\mathcal{D}_\mu^{\leq \mu}(\mathcal{F}) = V_\mu \otimes (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$ . By definition,  $P_\mu \in \mathcal{D}_\mu^{\leq \mu}(\mathcal{F})$ . Therefore,  $P_{\mu,G}P_{\nu,G} \in \mathcal{D}_{\mu+\nu}^{\leq (\mu+\nu)}(\mathcal{F})$ .

The left translation action of  $G$  acts on the first factor  $V_\mu$ , so  $P_{\mu,G} = v_\mu \otimes P$  has weight  $\mu$  under left translation. Therefore,  $P_{\mu,G}P_{\nu,G}$  has weight  $\mu + \nu$ . But  $P_{\mu,G}P_{\nu,G} \in \mathcal{D}_{\mu+\nu}^{\leq \mu+\nu}(\mathcal{F}) = V_{\mu+\nu} \otimes (V_{\mu+\nu}^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b_{-\mu-\nu}}$ , where left translation acts on the  $V_{\mu+\nu}$ . As  $v_{\mu+\nu} \in V_{\mu+\nu}$  is the unique element of weight  $\mu + \nu$  up to scaling,  $P_{\mu,G}P_{\nu,G}$  can be expressed as  $v_{\mu+\nu} \otimes P'$  for some  $P' \in (V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b_{-\mu}}$ ; similarly,  $P_{\mu+\nu,G} = v_{\mu+\nu} \otimes P$ . Then as  $P$  generates  $(V_\mu^* \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b_{-\mu}}$  over  $U\mathfrak{h}$ ,  $P' = Pa$  for some  $a \in U\mathfrak{h}$ . Using Kashiwara's calculation, we can check that  $P_{\mu+\nu,G}f_G^{\lambda+\mu+\nu} = P_{\mu,G}P_{\nu,G}f_G^{\lambda+\mu+\nu}$  for any  $\lambda$ , which implies that  $a = 1$  and  $P' = P$ , so  $P_{\mu,G}P_{\nu,G} = P_{\mu+\nu,G}$ .  $\square$

## 8 Polynomial Cocycles

We have proven that for the operators  $P_\mu$  that give the b-function on  $G$ ,  $P_{\mu,G}P_{\nu,G} = P_{\mu+\nu,G}$ . This situation is common enough that it is useful to study it on its own; it will turn out to be true for the two relevant families of differential operators studied in this dissertation. This equation on differential operators implies the following equation on b-functions (with subscripts omitted because it works for all relevant subscripts):

$$b_\mu(\lambda)b_\nu(\lambda + \mu) = b_{\mu+\nu}(\lambda) \tag{11}$$

This section generally follows the appendix in [12].

*Remark 1.* As the b-functions are nonzero polynomials on  $\mathbb{C}\Omega$ , we can consider  $b$  to be a function with values in the multiplicative group of meromorphic functions over  $\mathbb{C}\Omega$ ,  $b : \Omega \rightarrow \text{Mer}(\mathbb{C}\Omega, \mathbb{C})^\times$ . In that context, the above equation can be rephrased as the statement that  $b$  is a twisted homomorphism, under the action of translation  $(T_\mu(f))(\lambda) = f(\lambda + \mu)$  on  $\mathbb{C}(\mathbb{C}\Omega)^\times$ . Twisted homomorphisms appear in the study of group cohomology as 1-cocycles in the Hochschild complex, so we can view these b-functions as cocycles in the same complex. This complex is acyclic, so any such  $b$  is a coboundary; remark 2 will expand on a convenient

$A \in \text{Mer}(\mathbb{C}\Omega, \mathbb{C})^\times$  such that  $b = \delta A$ , that is,  $b_\mu(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)}$

**Lemma 15.** *Let  $\mu \rightarrow b_\mu$  be a cocycle (i.e. satisfy equation (11)) such that for  $\mu \in \Omega$ ,  $b_\mu$  is a polynomial (not just a rational function). Then for any  $\mu \in \Omega$ ,  $b_\mu = \prod_i (\alpha_i(\lambda) + k_i)$  for some set of  $\alpha_i \in \text{Hom}(\Lambda, \mathbb{Z}), k_i \in \mathbb{C}$ .*

Similar results have been found for b-functions of multiple functions; see Sabbah, theorem 4.2.1 [11] and Gyoja [6].

*Proof.* From the cocycle equation, we can switch  $\mu, \nu$  to get that

$$b_\mu T_\mu(b_\nu) = b_\nu T_\nu(b_\mu) \quad (12)$$

Assume  $p|b_\mu$  for some irreducible  $p$ ; we need to prove that  $p$  is of the form  $\alpha(\lambda) + k$  for some  $\alpha \in \text{Hom}(\Lambda, \mathbb{Z}), k \in \mathbb{C}$ . We first check that for any  $\nu \in \Omega$ , either there is a nonnegative integer  $a$  such that  $p|T_{\nu-a\mu}b_\mu$  or  $p$  is invariant under translation by  $\mu$ .

By equation (12), we know that  $p|b_\nu T_\nu b_\mu$ . As polynomials over  $\mathbb{C}\Lambda$  form a UFD and as we have assumed that  $p$  is irreducible, we know that either  $p|b_\nu$  or  $p|T_\nu b_\mu$ . In the latter case, we are done, so assume we are in the former case. Then by translation, we also have that  $T_\mu p|T_\mu b_\nu|b_\mu T_\mu b_\nu = b_\nu T_\nu b_\mu$ . By repeating this, we either have infinitely many  $a$  such that  $p(\lambda + a\mu)|b_\nu(\lambda)$ , or for some  $a$  that  $T_{a\mu}p|T_\nu b_\mu$ . As before, in the latter case, we are done, so assume we are in the former case. Then as  $b_\nu(\lambda)$  has finitely many factors, there can only be finitely many distinct  $T_{a\mu}p$ ; therefore they must be equal for some  $a_0, a_1$ , and therefore,  $T_{(a_1-a_0)\mu}p = p$ . This means that for any  $\lambda$ , the function  $t \rightarrow p(\lambda + t\mu)$  is periodic. But  $p$  is a polynomial, so the function is a polynomial, which can only be periodic if it is constant, so  $p(\lambda + t\mu) = p(\lambda)$  for any  $t$ , and  $p$  is invariant under translation by any multiple of  $\mu$ .

Assume we are in the first case, that is, that  $T_{a\mu}p|T_\nu b_\mu$ . Then we know that  $T_{a\mu-\nu}p|b_\mu$ . As the former is a translation of  $p$ , it is also irreducible - so we can repeat the above process. As  $b_\mu$  has finitely many factors, there can only be finitely many distinct such translations; therefore, for some  $m_0 < m_1, n_0 < n_1$  we get that  $T_{m_1\mu-n_1\nu}p = T_{m_0\mu-n_0\nu}p$ . Therefore there are nonnegative integers  $m, n$  with  $n > 0$  such that  $T_{m\mu-n\nu}p = p$ . Dropping the assumption that  $T_{a\mu}p|T_\nu b_\mu$  and going back to the general case (i.e. allowing  $T_{a\mu}p = p$  as in the above paragraph), we get that there are integers  $m, n$  nonnegative and not both 0 such that  $T_{m\mu-n\nu}p = p$ .

Choose generators  $\beta_i$  of  $\Lambda$ ; then as  $p$  is a nonconstant polynomial, for some  $i_0$  we get that  $T_{\beta_{i_0}}p \neq p$ . Then for all  $i \neq i_0$ , we know that there are  $m_i, n_i$  such that  $p(\lambda + m_i\beta_{i_0} - n_i\beta_i) = p(\lambda)$ . Hence  $n_i \neq 0$ . For any  $\mu \in \text{span}(\{m_i\beta_{i_0} - n_i\beta_i\})$ , we have that  $p(\lambda + \mu) = p(\lambda)$ . This is a subspace of codimension 1 in  $\mathbb{C}\Lambda$ , so there is some  $\alpha \in \text{Hom}(\Lambda, \mathbb{C})$  such that  $\alpha(\mu) = 0$  if and only if  $p(\lambda + \mu) = p(\lambda)$  for all  $\lambda$ ; further, it's clear that

$\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$ . We then have that for any  $\mu$  such that  $\langle \alpha, \mu \rangle = 0$ ,  $p(\lambda + \mu) = p(\lambda)$ . Choose a point  $\lambda_0$  such that  $p(\lambda_0) = 0$ . Then for any point where  $\alpha(\lambda) - \alpha(\lambda_0) = 0$ , we have that  $p(\lambda) = p(\lambda + (\lambda_0 - \lambda)) = 0$ . But  $\alpha(\lambda) - \alpha(\lambda_0)$  is an irreducible polynomial, so  $\alpha(\lambda) - \alpha(\lambda_0)$  divides  $p$ . Since  $p$  is irreducible,  $p = \alpha(\lambda) - \alpha(\lambda_0)$ . Setting  $k = -\alpha(\lambda_0)$ , we have proven the lemma. □

Consider  $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$  positive and indivisible (that is,  $\alpha \neq 0$  and for any  $\mu \geq 0$ ,  $\langle \alpha, \mu \rangle \geq 0$ , and there is no nontrivial  $a \in \mathbb{Z}$  with  $\frac{\alpha}{a} \in \text{Hom}(\Lambda, \mathbb{Z})$ ). For each  $\mu \in \Lambda$ , let  $K_{\alpha, \mu} = \{k \in \mathbb{C} \text{ s.t. } (\alpha(\mu) + k) | b_\mu\}$  with multiplicity (including negative multiplicity for factors in the denominator). Then for any  $\mu, \nu \in \Lambda$

$$K_{\alpha, \mu} \cup (K_{\alpha, \nu} + \langle \alpha, \mu \rangle) = K_{\alpha, \nu} \cup (K_{\alpha, \mu} + \langle \alpha, \nu \rangle) \quad (13)$$

where  $A + k = \{a + k | a \in A\}$ ; this follows directly from unique factorization and the cocycle equation.

**Corollary 16.** *Let  $\alpha \neq 0 \in \text{Hom}(\Lambda, \mathbb{Z})$ . Then there is some  $c \in \mathbb{Z}_{\geq 0}$  such that  $|K_{\alpha, \mu}| = c \langle \alpha, \mu \rangle$ .*

*Proof.* Let  $\nu \in \Lambda$ ,  $\langle \alpha, \nu \rangle \neq 0$ , and let  $c = \frac{|K_{\alpha, \nu}|}{\langle \alpha, \nu \rangle}$ . Then for any  $\mu \in \Lambda$ , by summing equation (13) over the sets with multiplicity, we get that:

$$\sum K_{\alpha, \mu} + \sum K_{\alpha, \nu} + |K_{\alpha, \nu}| \langle \alpha, \mu \rangle = \sum K_{\alpha, \nu} + \sum K_{\alpha, \mu} + |K_{\alpha, \mu}| \langle \alpha, \nu \rangle. \quad (14)$$

We can subtract and divide to get  $c = \frac{|K_{\alpha, \nu}|}{\langle \alpha, \nu \rangle}$ ; then as  $\alpha$  is indivisible and positive,  $c$  must be a nonnegative integer. □

**Corollary 17.** *For any  $\alpha \neq 0 \in \text{Hom}(\Lambda, \mathbb{Z})$ ,  $\mu \in \Lambda$  with  $\langle \alpha, \mu \rangle = 0$ ,  $K_{\alpha, \mu} = \emptyset$ .*

**Corollary 18.** *Let  $\mu, \nu \in \Lambda$ , and let  $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$  such that  $\alpha(\mu) = \alpha(\nu) = a \neq 0$ . Then  $K_{\alpha, \mu} = K_{\alpha, \nu}$ .*

*Proof.* Assume otherwise. Then equation (13) implies that

$$(K_{\alpha, \mu} + a) - K_{\alpha, \mu} = (K_{\alpha, \nu} + a) - K_{\alpha, \nu}$$

But the map  $S \rightarrow (S + a) - S$  is a  $\mathbb{Z}$ -linear map where every nontrivial element of the kernel is a multiset containing infinitely many elements, and both  $K_{\alpha, \mu}$  and  $K_{\alpha, \nu}$  are finite multisets - so they must be the same. □

As we chose  $\alpha$  primitive, every integer can be expressed as  $\langle \alpha, \mu \rangle$  for some  $\mu \in \Lambda$ . We can therefore (by abuse of notation) relabel our  $K$  with subscripts  $K_{\alpha,a}$  for  $a \in \mathbb{Z}$ , such that  $K_{\alpha,\mu} = K_{\alpha,\langle \alpha, \mu \rangle}$ .

*Remark 2.* Let

$$A(\lambda) = \prod_{\substack{\alpha \in \text{Hom}(\Lambda, \mathbb{Z}) \\ \alpha \text{ positive, indivisible}}} \prod_{k \in K_{\alpha,1}} \Gamma(\langle \alpha, \lambda \rangle). \quad (15)$$

Then  $A(\lambda)$  will have  $b$  as a coboundary, that is,  $b_\mu(\lambda) = \frac{A(\mu + \lambda)}{A(\lambda)}$  for every  $\mu, \lambda$ .

Note that it is possible for each  $b_\mu$  to be a polynomial for  $\mu \in \Lambda^+$  while still having negative multiplicities for  $K_{\alpha,1}$ .

**Example** Let  $\Lambda = \mathbb{Z}^2$  with generators  $e_1, e_2$ . Let  $\alpha \in \text{Hom}(\Lambda, \mathbb{Z})$  such that  $\alpha(e_1) = 2, \alpha(e_2) = 3$ . Let  $A(\lambda) = \frac{\Gamma(\langle \alpha, \lambda \rangle + 2)\Gamma(\langle \alpha, \lambda \rangle)}{\Gamma(\langle \alpha, \lambda \rangle + 1)}$ . The coboundary  $b_\mu(\lambda) = \frac{A(\mu + \lambda)}{A(\lambda)}$  is clearly a cocycle, and for  $\mu \in \Lambda^+$ ,  $b_\mu$  is a polynomial. This corresponds to the fact that the multiset  $K = \{0, 2\} - \{1\}$  has an element with negative multiplicity, but that for every  $n$  that can be expressed as  $\langle \alpha, \mu \rangle$  (i.e. for all  $n \geq 2$ ),  $\sum_{i=0}^{n-1} K + i$  has no elements with negative multiplicity.

## 9 The differential operator for $f_{G^3}$

Let  $\mu = (\mu_1, \mu_2, m) \in \Omega$ . Remember that  $f_{G^3}^\lambda \in \Gamma(X, L_\lambda)$  is the unique diagonally  $G$ -invariant section.

We now want to find the differential operators  $P_{\mu, G^3}$  that give us the functional equation

$$P_{\mu, G^3} f_{G^3}^{\lambda + \mu} = b_{\mu, G^3}(\lambda) f_{G^3}^\lambda \quad (16)$$

for all  $\lambda \in \Omega$ .

**Definition** Define  $H(\mu) = \langle u_{-\mu}, u_\mu \rangle$ .

**Proposition 19.** *There is a  $P_{\mu, G^3} \in \mathcal{D}_\mu(X)$  that satisfies the functional equation*

$$P_{\mu, G^3} f_{G^3}^{\lambda + \mu} = b_{\mu, G^3}(\lambda) f_{G^3}^\lambda \quad (17)$$

where  $b_{\mu, G^3}(\lambda) = \frac{H(\lambda + \mu)}{H(\lambda)} b_{\mu, K}(\lambda)$ .

To prove this, we first need a lemma about  $H(\mu)$ .

**Lemma 20.**  $u_\mu = H(\mu)v_\mu^G, u_{-\mu} = H(\mu)v_{-\mu}^G$ .

*Proof.* Because  $u_\mu$  is the unique  $G$ -invariant up to scaling,  $u_\mu = cv_\mu^G$  for some  $c$ . Then  $H(\mu) = \langle u_{-\mu}, u_\mu \rangle = \langle u_{-\mu}, cv_\mu^G \rangle = c\langle u_{-\mu}, v_\mu \rangle = c$ , so  $u_\mu = H(\mu)v_\mu^G$ . Similarly,  $u_{-\mu} = H(\mu)v_{-\mu}^G$ .  $\square$

*Proof of Proposition 19.* Consider differential operators of the form  $v \otimes P$  for  $v \in V_\mu$ , and the restriction of the map of application of differential operators  $\mathcal{D}_\mu(X) \otimes \Gamma(X, L_{\lambda+\mu}) \rightarrow \Gamma(X, L_\lambda)$  to only include such differential operators. Then this is equivalent to a  $G \times G \times G$ -equivariant map  $V_\mu \otimes V_{\lambda+\mu}^* \rightarrow V_\lambda^*$ , where the equivalence takes  $P_{\mu,K} \rightarrow v_\mu, P_{\mu,G} \rightarrow u_\mu, f_K^\lambda \rightarrow v_{-\lambda}, f_G^\lambda \rightarrow u_{-\lambda}$ . But up to scaling, there is only one such map. Under this interpretation, we have  $v_\mu \otimes v_{-\lambda-\mu} \rightarrow b_{K,\mu}(\lambda)v_{-\lambda}$  and  $u_\mu \otimes u_{-\lambda-\mu} \rightarrow b_{G^3,\mu}(\lambda)u_{-\lambda}$ . By pairing with the respective vectors and considering the fact that this map is  $G \times G \times G$ -invariant, we can think of this as a map  $V_\mu \otimes V_\lambda \rightarrow V_{\mu+\lambda}$  that takes  $v_\mu \otimes v_\lambda \rightarrow v_{\mu+\lambda}, H(\lambda+\mu)u_\mu \otimes u_\lambda \rightarrow H(\lambda)b_{G^3,\mu}(\lambda)u_{\lambda+\mu}$ .

But by Borel-Weil, we also have an isomorphism between  $\Gamma(X', L_{-\lambda})$  and  $V_\lambda$ ; as such, we can consider the multiplication map on sections as a map  $V_\mu \otimes V_\lambda \rightarrow V_{\mu+\lambda}$ . The multiplication map takes  $v_\mu \otimes v_\lambda$  to  $v_{\mu+\lambda}$  so the map we want is the multiplication map scaled by  $b_{K,\mu}$ . The multiplication map also takes  $u_\mu \otimes u_\lambda$  to  $u_{\mu+\lambda}$ . Therefore, the map we want takes  $H(\lambda+\mu)u_\mu \otimes u_\lambda$  to  $H(\lambda+\mu)u_{\lambda+\mu}$ . So  $b_{\mu,G^3}(\lambda) = \frac{H(\lambda+\mu)}{H(\lambda)}b_{\mu,K}$ .  $\square$

*Remark 3.* Proposition 19 is easier to understand in terms of the lift of the cocycle, as explained in remark 2. If  $A_K(\lambda)$  is the lift of  $b_{\mu,K}$  and  $A_{G^3}(\lambda)$  is the lift of  $b_{\mu,G^3}$ , then for  $\lambda \in \Omega$ ,  $A_{G^3}(\lambda) = H(\lambda)A_K(\lambda)$ .

Similar reasoning applied to the map  $V_\mu \rightarrow \mathcal{D}_\mu(X), v \rightarrow v \otimes P$  and the map of composition of differential operators  $\mathcal{D}_\mu(X) \otimes \mathcal{D}_\nu(X) \rightarrow \mathcal{D}_{\mu+\nu}(X)$  also implies that  $P_{\mu,G^3}P_{\nu,G^3} = P_{\mu+\nu,G^3}$ . As such, all of the conclusions from section 8 apply to the b-function we are trying to find.

We now only need to know  $H(\mu)$  and the b-function for  $f_K$ , the  $B_1 \times B_2 \times P$ -semiinvariant section, using Kashiwara's calculation, and then find the relationship between  $b_{\mu,G^3}$  and  $b_{\mu,K}$ . As  $f_K = f_G \otimes f_G \otimes f_{\mathbb{P}}$  is a product of three functions on the three factors of  $X$ , we can set  $P_{\mu,K} = P_{\mu_1,G} \otimes P_{\mu_2,G} \otimes P_{\mathbb{P},m}$  and get the functional equation

$$P_{\mu,K}f_K^{\lambda+\mu} = (P_{\mu_1,G}f_G^{\lambda_1+\mu_1}) \otimes (P_{\mu_2,G}f_G^{\lambda_2+\mu_2}) \otimes (P_{\mathbb{P},m}f_{\mathbb{P}}^{l+m}) = b_{\mu_1,G}(\lambda_1)b_{\mu_2,G}(\lambda_2)b_{m,\mathbb{P}}(l)f_K^\lambda \quad (18)$$

where  $b_{m,\mathbb{P}}$  is the b-function for a hyperplane in  $\mathbb{P}^{n-1}$ . It is easy to see that  $b_{m,\mathbb{P}}(l) = \prod_{i=1}^m (l+i)$ ; combining that with the calculation in section 7, we get that:

$$P_{\mu,K} f_K^{\lambda+\mu} = b_{\mu,K}(\lambda) f_K^\lambda \quad (19)$$

where

$$b_{\mu,K} = \prod_{\alpha \in R_+} \prod_{i=1}^{h_\alpha \mu_1} (h_\alpha(\lambda_1) + h_\alpha(\rho) + i) \prod_{j=1}^{h_\alpha \mu_2} (h_\alpha(\lambda_2) + h_\alpha(\rho) + j) \prod_{k=1}^m (l + i)$$

Let  $P_{\mu,G^3} = u_\mu \otimes P \in \mathcal{D}_\mu(X)$ . Then as  $P_{\mu,K} = v_\mu \otimes P$ , the last lemma shows that  $P_{\mu,G^3} = H(\mu) P_{\mu,K}^G$ .

## 10 The functions $H(\lambda)$

Ginzburg and Finkelberg found the generators of  $\Omega$  in the proof of [4] lemma 5.5.1; there are two families of generators. We recall them here:

$$\alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0), 1 \leq i \leq n-1$$

$$\beta_j = (\wedge^{j-1} \omega, \wedge^{n-j} \omega, 1), 1 \leq j \leq n$$

Define the following subsemigroups of  $\Omega$ :

$$\Delta = \text{span}(\{\alpha_i\}_{i=1}^n)$$

$$\Delta_{<j} = \text{span}(\{\beta_j\} \cup \{\alpha_i\}_{i < j})$$

$$\Delta_{\geq j} = \text{span}(\{\beta_j\} \cup \{\alpha_i\}_{i \geq j})$$

Let  $R_+$  be the set of positive roots of  $G = SL_n$ . For each  $j$ , define the following function on positive roots:

For  $\gamma \in R_+$ , define

$$\chi_j(\gamma) = \begin{cases} 1 & \text{if } \gamma > \wedge^j \omega \\ 1 & \text{if } \gamma > \wedge^{j-1} \omega \\ 0 & \text{else} \end{cases} \quad (20)$$

Define a function  $\hat{H}(\lambda)$  on part of  $\Omega$  as follows:

$$\hat{H}(\lambda) = \begin{cases} \prod_{\gamma \in R_+} \frac{h_\gamma(\lambda_1 + \rho)}{h_\gamma(\rho)} & \text{if } \lambda \in \Delta \\ \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + \sum r_i \wedge^i \omega) + s_j \chi_j(\gamma))}{\prod_{\gamma \in R_+} h_\gamma(\rho)} & \text{if } \lambda \in \Delta_{\geq j} \text{ or } \lambda \in \Delta_{< j} \end{cases} \quad (21)$$

Recall that we defined  $H(\lambda) = \langle u_{-\lambda}, u_\lambda \rangle$ .

**Theorem 21.** *If  $\mu$  and  $\lambda$  are in the same cone ( $\Delta, \Delta_{\geq j}$  for some  $j$ , or  $\Delta_{< j}$  for some  $j$ ), then*

$$\frac{H(\lambda + \mu)}{H(\lambda)H(\mu)} = \frac{\hat{H}(\lambda + \mu)}{\hat{H}(\lambda)\hat{H}(\mu)}.$$

To prove the theorem, we first need to show how the choice of the subgroups  $B_1, B_2, P, B'_1, B'_2, P'$  changes  $H(\lambda)$ . In the last section,  $u_\lambda$  and  $u_{-\lambda}$  were normalized with  $\langle u_{-\lambda}, v_\lambda \rangle = \langle v_{-\lambda}, u_\lambda \rangle = \langle v_{-\lambda}, v_\lambda \rangle = 1$ ; however, it is sometimes easier to work with an "un-normalized" version. Define  $\tilde{H}(\lambda) = \frac{\langle u_{-\lambda}, u_\lambda \rangle \langle v_{-\lambda}, v_\lambda \rangle}{\langle u_{-\lambda}, v_\lambda \rangle \langle v_{-\lambda}, u_\lambda \rangle}$ . In this form, we can instead choose arbitrary nonzero  $G$ -invariant  $u_\lambda \in V_\lambda, u_{-\lambda} \in V_\lambda^*, B_1 \times B_2 \times P$ -semiinvariant  $v_\lambda \in V_\lambda$ , and  $B'_1 \times B'_2 \times P'$ -semiinvariant  $v_{-\lambda} \in V_\lambda^*$ ; as these are unique up to scalar, and that scalar cancels in the "un-normalized form",  $\tilde{H}(\lambda)$  is independent of the choices. We can then regard  $\tilde{H}(\lambda)$  as a meromorphic function  $\tilde{H}_\lambda(x \times x')$  on  $X \times X'$ , by considering  $X \times X'$  as a moduli space of 6-tuples of subgroups.

The action of  $G^3$  on  $X$  is transitive, as is the action of  $G^3$  on  $X'$ , so for any point  $x_0 \times x'_0$  there is some  $g = (g_1, g_2, g_3, g_4, g_5, g_6)$  such that  $x_0 \times x'_0 = (g_1, g_2, g_3, g_4, g_5, g_6)(x \times x')$ .

**Lemma 22.** *The function  $\frac{\tilde{H}_{\lambda+\mu}}{\tilde{H}_\lambda \tilde{H}_\mu}$  is constant on  $X \times X'$ .*

*Proof.* We have

$$\tilde{H}_\lambda(x_0 \times x'_0) = \frac{\langle u_{-\lambda}, u_\lambda \rangle \langle (g_4, g_5, g_6)v_{-\lambda}, (g_1, g_2, g_3)v_\lambda \rangle}{\langle u_{-\lambda}, (g_1, g_2, g_3)v_\lambda \rangle \langle (g_4, g_5, g_6)v_{-\lambda}, u_\lambda \rangle} \quad (22)$$

But using the correspondence  $v_\lambda \rightarrow f_K^\lambda, u_\lambda \rightarrow f_{G^3}^\lambda$ , it's clear that each of the three products other than

$\langle u_\lambda, u_{-\lambda} \rangle$  cancels in the expression  $\frac{\tilde{H}_{\lambda+\mu}}{\tilde{H}_\lambda \tilde{H}_\mu}$ . We therefore get that for any  $x_0 \times x'_0$ , we have

$$\frac{\tilde{H}_{\lambda+\mu}}{\tilde{H}_\lambda \tilde{H}_\mu} = \frac{\langle u_{-\mu-\lambda}, u_{\mu+\lambda} \rangle}{\langle u_{-\mu}, u_\mu \rangle \langle u_{-\lambda}, u_\lambda \rangle}$$

Therefore, the function  $\frac{\tilde{H}_{\lambda+\mu}}{\tilde{H}_\lambda \tilde{H}_\mu}$  is constant.  $\square$

*Proof of Theorem 21.* Lemma 22 shows that for each cone  $\Delta, \Delta_{<j}$  for  $1 \leq j \leq n$ , and  $\Delta_{\geq j}$  for  $1 \leq j \leq n$ , it is sufficient to choose some  $x_0 \times x'_0$  such that  $H_\lambda(x_0 \times x'_0) = \hat{H}(\lambda)$  for any  $\lambda$  within that cone.

If  $\lambda \in \Delta$ , then  $\lambda = (\lambda_1, \lambda_1^*, 0)$ , where  $W_{\lambda_1^*} = W_{\lambda_1}^*$ . Choose  $B_1, B'_2$  to be the subgroup of upper triangular matrices,  $B'_1, B_2$  to be the subgroup of lower triangular matrices. Then by choosing  $u_\lambda \in V_\lambda$  corresponding to the identity map on  $W_{\lambda_1}$ ,  $u_{-\lambda} \in V_\lambda^*$  corresponding to the trace map,  $v_\lambda = v_{\lambda_1} \otimes v_{-\lambda_1}$  and  $v_{-\lambda} = v_{-\lambda_1} \otimes v_{\lambda_1}$ , we can see that:

$$H(\lambda) = \text{tr}(Id_{W_{\lambda_1}}) = \dim(W_{\lambda_1}) = \prod_{\gamma \in R_+} \frac{h_\gamma(\lambda_1 + \rho)}{h_\gamma(\rho)} \quad (23)$$

by the Weyl dimension formula.

If  $\lambda \in \Delta_{\geq j}$ , then let

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, l) = s_j \beta_j + \sum_{i \geq j} r_i \alpha_i \\ \lambda' &= (\lambda'_1, \lambda'_2, l') = s_j \alpha_j + \sum_{i \geq j} r_i \alpha_i \\ \lambda^0 &= (\lambda_1^0, \lambda_2^0, l^0) = \sum_{i \geq j} r_i \alpha_i \end{aligned}$$

Choose  $B_1, B'_2$  to be the subgroup of upper triangular matrices,  $B'_1, B_2$  to be the subgroup of lower triangular matrices,  $P$  to be the subgroup of matrices that stabilize the  $j$ th basis element  $e_j$  up to scaling, and  $P'$  to be the subgroup of matrices that stabilize  $j$ th dual basis element  $e'_j$  up to scaling.

We now define several maps to be used later. Note that  $W_\omega = \mathbb{C}^n$ . Let us denote the wedge map  $\wedge_j : W_{\wedge^{j-1}\omega} \otimes W_\omega \rightarrow W_{\wedge^j\omega}$ . Note that the wedge map is  $G$ -equivariant. By copying this map  $s_j$  times, we get a map  $\wedge_j^{s_j} : W_{s_j \wedge^{j-1}\omega} \otimes W_{s_j \omega} \rightarrow W_{s_j \wedge^j \omega}$  taking  $v_{s_j \wedge^{j-1}\omega} \otimes v_{s_j \omega} \rightarrow v_{s_j \wedge^j \omega}$ . Then by tensoring with  $Id_{W_{\lambda_0}}$ , we have a map  $f : (W_{s_j \wedge^{j-1}\omega} \otimes W_{\lambda_1^0}) \otimes W_{\lambda_2^0} \otimes W_{s_j \omega} \rightarrow (W_{s_j \wedge^j \omega} \otimes W_{\lambda_1^0}) \otimes W_{\lambda_2^0}$ . Then let  $g : V_\lambda \rightarrow V_{\lambda'}$  be defined by the composition:

$$V_\lambda \xrightarrow{i_{s_j \wedge^{j-1} \omega, \lambda_1^0}} (W_{s_j \wedge^{j-1} \omega} \otimes W_{\lambda_1^0}) \otimes W_{\lambda_2^0} \otimes W_{s_j \omega} \xrightarrow{f} (W_{s_j \wedge^j \omega} \otimes W_{\lambda_1^0}) \otimes W_{\lambda_2^0} \xrightarrow{p_{s_j \wedge^j \omega, \lambda_1^0}} V'_\lambda \quad (24)$$

where  $i$  is the natural inclusion map that takes  $v_\lambda$  to  $v_{s_j \wedge^{j-1} \omega} \otimes v_{\lambda_1^0}$  and  $p$  is the natural quotient map that takes  $v_{s_j \wedge^j \omega} \otimes v_{\lambda_1^0}$  to  $v'_\lambda$ . Then  $g$  is  $G$ -equivariant and  $g(v_\lambda) = v'_\lambda$ .

Because  $g$  is  $G$ -equivariant, we have that  $g(u_\lambda) = g(H(\lambda)v_\lambda^G) = H(\lambda)g(v_\lambda)^G = H(\lambda)v_{\lambda'}^G = \frac{H(\lambda)}{H(\lambda')}u_{\lambda'}$ . Therefore, if we define  $c$  by  $g(u_\lambda) = cu_{\lambda'}$ , then  $H(\lambda) = cH(\lambda')$ ; as  $\lambda' \in \Delta$ , we know  $H(\lambda')$ , so we only need to find  $c$ .

Let  $g^* : V_{\lambda'}^* \rightarrow V_\lambda^*$  be the transpose of  $g$ . By the definition of  $g$ , we have

$$g^*(v_{-\lambda'}) = g^*(v_{-\lambda'_1} \otimes v_{-\lambda_2}) = (p \circ f)^*((v_{-\lambda_1^0} \otimes v_{-s_j \wedge^j \omega}) \otimes v_{-\lambda_2})$$

Using the definition of  $f$ , we want to know  $(\wedge_j^{s_j})^* v_{-s_j \wedge^j \omega}$ , where  $(\wedge_j^{s_j})^*$  is the transpose of  $\wedge_j^{s_j}$ . Let  $SL_j \subset G$  act on the first  $j$  coordinates in both  $\mathbb{C}^n$  and its dual. By restricting our focus to the subspaces where  $SL_j$  acts nontrivially (and doing the same for the dual spaces), we obtain a  $SL_j$ -equivariant map  $\wedge_j^{s_j} : W_{s_j \wedge^{j-1} \omega_j} \otimes W_{s_j \omega_j} \rightarrow W_{s_j \wedge^j \omega_j} = \mathbb{C}$  taking  $v_{s_j \wedge^{j-1} \omega_j} \otimes v_{s_j \omega_j} \rightarrow 1$ ; in other words, a  $SL_j$ -invariant element  $w \in W_{s_j \wedge^{j-1} \omega_j}^* \otimes W_{s_j \omega_j}^*$  such that  $\langle w, v_{s_j \wedge^{j-1} \omega_j} \otimes v_{s_j \omega_j} \rangle = 1$ .

But this corresponds to the conditions we've already studied if we take  $G' = SL_j$ ,

$\delta = (s_j \wedge^{j-1} \omega_j, s_j \omega_j, 0) \in \Delta_{G'}$ ; by definition,  $w = u_{-\delta}$ . Then by the uniqueness up to scalar of the  $G'$ -invariant, we have  $w = c_1 v_{-\delta}^{G'}$  for some  $c_1$ . By pairing with  $v_\delta$ , we get that  $1 = \langle w, v_\delta \rangle = \langle c_1 v_{-\delta}^{G'}, v_\delta \rangle = c_1 \langle v_{-\delta}, v_\delta^{G'} \rangle$ . Then as  $u_\delta = H_{SL_j}(\lambda)v_\delta^{G'}$ , we get that  $H_{SL_j}(\delta) = c_1 \langle v_{-\delta}, u_\delta \rangle = c_1$ .

We therefore have that  $w = H_{SL_j}(\delta)v_{-\delta}^{G'}$ . By expanding our focus again to all  $n$  coordinates, we have that  $\wedge_j^{s_j*} v_{-s_j \wedge^j \omega} = H_{SL_j}(\delta)(v_{-s_j \wedge^{j-1} \omega} \otimes v_{-s_j \omega})^{SL_j}$ . Therefore, as  $v_{-\lambda_1^0}$  and  $v_{-\lambda_2}$  are  $SL_j$ -invariant,  $g^*(v_{-\lambda'}) = H_{SL_j}(\delta)v_{-\lambda}^{SL_j}$ . Then by pairing with  $u_\lambda$ , we get that:

$$\begin{aligned} c &= \langle v_{-\lambda'}, cu_{\lambda'} \rangle = \langle v_{-\lambda'}, g(u_\lambda) \rangle = \langle g^*(v_{-\lambda'}), u_\lambda \rangle \\ &= \langle H_{SL_j}(\delta)v_{-\lambda}^{SL_j}, u_\lambda \rangle = H_{SL_j}(\delta) \langle v_{-\lambda}, u_\lambda \rangle = H_{SL_j}(\delta) \end{aligned}$$

where we use the  $G$ -invariance of  $u$  to say that  $\langle v_{-\lambda}^{SL_j}, u_\lambda \rangle = \langle v_{-\lambda}, u_\lambda \rangle$ .

We therefore have that  $H(\lambda) = H_{SL_j}(\delta)H(\lambda')$ . By the calculation of  $H(\lambda)$  on  $\Delta$  (equation (23)), we get that:

$$H(\lambda') = \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + (\sum_i r_i \wedge^i \omega) + s_j \wedge^j \omega))}{\prod_{\gamma \in R_+} (h_\gamma(\rho))}$$

$$H_{SL_j}(\delta) = \frac{\prod_{\gamma \in R_{SL_j+}} (h_\gamma(\rho_{SL_j} + s_j \wedge^{j-1} \omega_j))}{\prod_{\gamma \in R_{SL_j+}} h_\gamma(\rho_{SL_j})}$$

By considering  $R_{SL_j+}$  as a subset of  $R_+$ , we get the second half of the theorem.  $\square$

For  $\gamma \in R_+$ , define

$$\chi'_j(\gamma) = \begin{cases} 1 & \text{if } \gamma > \wedge^j \omega + \wedge^{j-1} \omega \\ 0 & \text{else} \end{cases} \quad (25)$$

Combining this calculation of  $H(\lambda)$  and  $b_{K,\mu}(\lambda)$  (equation (18)), we get:

**Corollary 23.** *If  $\mu = \alpha_j, \lambda \in \Delta$  (so  $\lambda = (\lambda_1, \lambda_1^*, 0)$ ), then*

$$b_\mu(\lambda) = \frac{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho + \lambda_1) + 1)(h_\gamma(\rho + \lambda))}{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho) + 1)(h_\gamma(\rho))}$$

up to constant depending only on  $\mu$  (not on  $\lambda$ ).

If  $\mu = \beta_j, \lambda \in \Delta_{<j}$  or  $\lambda \in \Delta_{\geq j}$  with  $\lambda = (\sum_i r_i \alpha_i) + s_j \beta_j$ , then

$$b_\mu(\lambda) = \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} (h_\gamma(\rho + \sum r_i \alpha_i) + s_j) \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} (h_\gamma(\rho + \sum r_i \alpha_i) + s_j + 1)$$

up to constant depending only on  $\mu$  (not on  $\lambda$ ).

If  $\mu = \alpha_k, k < j, \lambda \in \Delta_{<j}$  or  $k \geq j, \lambda \in \Delta_{\geq j}$ , then

$$b_\mu(\lambda) = \prod_{\gamma > \wedge^k \omega} (h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi'_j(\gamma)) (h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi_j(\gamma) + 1)$$

up to constant depending only on  $\mu$  (not on  $\lambda$ ).

## 11 The b-function

We have now assembled enough results to fully determine  $b_{G^3, \mu}(\lambda)$  for any  $\mu \in \Omega, \lambda \in \mathbb{C}\Omega$ . By equation (11), it is enough to find  $b_{G^3, \mu}$  for  $\mu$  generators of  $\Omega$ . Write  $\lambda = \sum a_i \alpha_i + \sum b_j \beta_j$ . We therefore want to find a set of polynomials  $\{b_{\alpha_i}\}_{1 \leq i \leq n-1}, \{b_{\beta_j}\}_{0 \leq j \leq n-1}$  satisfying the following properties:

1. Each of the polynomials factors as a product of affine factors with integer coefficients, as in lemma 15.
2. For each  $i$ ,  $(a_i + 1)|b_{\alpha_i}$ ; this is analogous to the fact that  $s + 1|b(s)$ . Similarly, for each  $j$ ,  $(b_j + 1)|b_{\beta_j}$ . (lemma 4)
3. If  $(\delta + k)|b_{\alpha_i}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$ , then  $\langle \delta, \alpha_i \rangle \neq 0$ . Similarly, if  $\delta + k|b_{\beta_j}$ , then  $\langle \delta, \beta_j \rangle \neq 0$  (corollary 17).
4. If  $(\delta + k)|b_{\alpha_i}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$ , and  $\langle \delta, \alpha_{i'} \rangle \neq 0$ , then for some  $k'$ ,  $(\delta + k')|b_{\alpha_{i'}}$ . Similarly, if  $(\delta + k)|b_{\alpha_i}$  and  $\langle \delta, \beta_j \rangle \neq 0$ , then for some  $k'$ ,  $(\delta + k')|b_{\beta_j}$ . Similarly, if  $(\delta + k)|b_{\beta_j}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$ , and  $\langle \delta, \alpha_i \rangle \neq 0$ , then for some  $k'$ ,  $(\delta + k')|b_{\alpha_i}$ . Similarly, if  $(\delta + k)|b_{\beta_j}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$ , and  $\langle \delta, \beta_{j'} \rangle \neq 0$ , then for some  $k'$ ,  $(\delta + k')|b_{\beta_{j'}}$ . All of these result from corollary 16.
5. If  $(\delta + k)|b_{\alpha_i}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$  and  $\langle \delta, \alpha_i \rangle = \langle \delta, \alpha_{i'} \rangle$ , then  $(\delta + k)|b_{\alpha_{i'}}$ . Similarly, if  $(\delta + k)|b_{\alpha_i}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$  and  $\langle \delta, \alpha_i \rangle = \langle \delta, \beta_j \rangle$ , then  $(\delta + k)|b_{\beta_j}$ . Similarly, if  $(\delta + k)|b_{\beta_j}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$  and  $\langle \delta, \beta_j \rangle = \langle \delta, \alpha_i \rangle$ , then  $(\delta + k)|b_{\alpha_i}$ . Similarly, if  $(\delta + k)|b_{\beta_j}$  for  $\delta \in Hom(\Lambda, \mathbb{Z})$  and  $\langle \delta, \beta_j \rangle = \langle \delta, \beta_{j'} \rangle$ , then  $(\delta + k)|b_{\beta_{j'}}$ . All of these result from corollary 18.
6. If  $b_j = 0$  for all  $j$ , we have

$$b_{\alpha_i} = \frac{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho + \lambda_1) + 1)(h_\gamma(\rho + \lambda_1))}{\prod_{\gamma > \wedge^j \omega} (h_\gamma(\rho) + 1)(h_\gamma(\rho))}.$$

7. If  $s_k = 0$  for all  $j \neq k$  and either  $a_i = 0$  for either all  $i < j$  or all  $i \geq j$ , we have

$$b_{\alpha_i} = \prod_{\gamma > \wedge^k \omega} \left( h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi'_j(\gamma) \right) \left( h_\gamma(\rho + \sum r_i \alpha_i) + s_j \chi_j(\gamma) + 1 \right)$$

$$b_{\beta_k} = \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} \left( h_\gamma(\rho + \sum r_i \alpha_i) + s_j \right) \prod_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} \left( h_\gamma(\rho + \sum r_i \alpha_i) + s_j + 1 \right).$$

These will determine the b-functions entirely. Let  $\Gamma$  be the gamma function.

**Theorem 24.** Write  $\lambda = (\sum_i a_i \alpha_i) + (\sum_j b_j \beta_j)$  and let

$$A(\lambda) = \prod_j \Gamma(b_j + 1) \cdot \prod_{\gamma \in R_+} \Gamma \left( h_\gamma(\rho) + \sum_{\omega^i < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{and } \omega^j < \gamma}} b_j \right) \cdot \Gamma \left( h_\gamma(\rho) + \sum_{\omega^j < \gamma} a_i + \sum_{\substack{\omega^{j-1} < \gamma \\ \text{or } \omega^j < \gamma}} b_j + 1 \right)$$

$$\text{Then we have } b_{\mu, G^3}(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)}.$$

From here on, we drop the subscript  $G^3$ .

*Proof.* Let  $\alpha_i = (\wedge^i \omega, \wedge^{n-i} \omega, 0) \in \Delta$ . By property 1,  $b_{\alpha_i}(\lambda) = \prod_k \langle \eta_k, \lambda \rangle + m_k$  for some  $\eta_k, m_k$  such that  $\langle \eta_k, \alpha_i \rangle \neq 0$ . By property 6, we have that  $b_{\alpha_i}(\lambda) = \frac{\prod_{\gamma \in R_+} (h_\gamma(\rho + \lambda_1) + 1) h_\gamma(\rho + \lambda)}{\prod_{\gamma \in R_+} (h_\gamma(\rho) + 1) h_\gamma(\rho)}$  if  $\lambda \in \mathbb{C}\Delta$ . By combining these, we get that we can partition the factors of  $b_{\alpha_i}$  into two families of affine factors. The families are both indexed by the roots  $\gamma \in R_+$  such that  $\gamma > \wedge^i \omega$ ; define  $L_{\gamma, \alpha_i}$  as the affine factor that, on  $\Delta$ , is equal to  $h_\gamma(\lambda_1) + h_\gamma(\rho) + 1$ , and  $L'_{\gamma, \alpha_i}$  as the affine factor that, on  $\Delta$ , is equal to  $h_\gamma(\lambda_1) + h_\gamma(\rho)$ .

By property 5, if  $\gamma > \wedge^i \omega, \wedge^{i'} \omega$ , then  $L_{\gamma, \alpha_i} | b_{\alpha_{i'}}$ ; the only affine factor it could match is  $L_{\gamma, \alpha_{i'}}$ , so  $L_{\gamma, \alpha_i} = L_{\gamma, \alpha_{i'}}$ . Similarly,  $L'_{\gamma, \alpha_i} = L'_{\gamma, \alpha_{i'}}$ . Thus, we can ignore the subscripts of  $\alpha$  and just denote the affine factors as  $L_\gamma, L'_\gamma$ .

Choose some  $j$ . Then for each  $\gamma$ , there is some  $i$  such that  $\gamma > \wedge^i \omega$ . If  $i < j$ , then  $\alpha_i$  and  $\beta_j$  are both in  $\Delta_{< j}$ , while if  $i \geq j$ , then they are both in  $\Delta_{\geq j}$ . In either case, by property 7, the coefficient of  $b_j$  in  $L_\gamma$  and  $L'_\gamma$  must either be 0 or equal to the coefficient of  $a_i$ . As the coefficients of all of the  $a_i$  are either 0 or 1, this implies that the coefficients of all of the  $b_j$  are either 0 or 1. By properties 3, 4, and 5, this implies that any affine factor of  $b_{\beta_j}$  is either  $L_\gamma$  or  $L'_\gamma$  for some  $\gamma$ , or that its coefficients of  $a_i$  are all 0.

Let  $S_j$ , resp.  $S'_j$  be the set of  $\gamma$  such that the coefficient of  $b_j$  in  $L_\gamma$ , resp.  $L'_\gamma$  is 1. Let  $S_{i,j}$ , resp.  $S'_{i,j}$  be the subset of  $S_j$ , resp.  $S'_j$  such that  $\gamma > \wedge^i \omega$ . As for each  $\gamma$  there is some  $i$  such that  $\gamma > \wedge^i \omega$ ,  $S_j = \cup S_{i,j}, S'_j = \cup S'_{i,j}$ .

Assume without loss of generality that  $i < j$ , and therefore that  $\alpha_i, \beta_j$  are in  $\Delta_{< j}$ . Then by property 7,  $|S_{i,j}| + |S'_{i,j}| = i(n-j) + i(n-j-1)$ . Also by property 6, if  $\gamma \in S_{i,j}$ , then  $h_\gamma(\rho) + 1 \geq i-j+1$ , while if  $\gamma \in S'_{i,j}$ , then  $h_\gamma(\rho) \geq i-j+1$ . Finally, also by property 7, if  $\gamma \in S_{i,j}$  or  $S'_{i,j}$ , then  $\gamma > \wedge^{j-1} \omega$ . Then  $\gamma \in S_{i,j}$  or  $S'_{i,j}$  implies that  $\gamma > \sum_{k=i}^{j-1} \wedge^k \omega$ .

Assume  $\sum_{k=i}^{j-1} \wedge^k \omega \in S'_j$  for some  $i$ . Then it is also in  $S'_{i,j}$ ; therefore,  $\gamma(\rho) \geq i - j + 1$ . But  $\gamma(\rho) = i - j$ , so

this is impossible - so no root of the form  $\sum_{k=i}^{j-1} \wedge^k \omega \in S'_j$ . Therefore, if  $\gamma \in S'_{i,j}$ , then  $\gamma \geq \wedge^j \omega$ .

We therefore have that if  $\gamma \in S_{i,j}$ , then  $\gamma > \wedge^{j-1} \omega$ , while if  $\gamma \in S'_{i,j}$ , then  $\gamma > \wedge^j \omega$ . But then  $|S_{i,j}| \leq i(n-j)$ , while  $|S'_{i,j}| \leq i(n-j-1)$ . We know that  $|S_{i,j}| + |S'_{i,j}| = i(n-j) + i(n-j-1)$ ; therefore, we must have that all of the elements that could be in either must be, and therefore that  $S_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^{j-1} \omega\}$ , while  $S'_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^j \omega\}$ .

Correspondingly, if  $i \geq j$  instead, then  $S_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^j \omega\}$  while  $S'_{i,j} = \{\gamma | \gamma > \wedge^i \omega, \wedge^{j-1} \omega\}$ . Then as  $S_j = \cup S_{i,j}, S'_j = \cup S'_{i,j}$ , we get that

$$S_j = \{\gamma | \gamma > \wedge^{j-1} \omega \text{ or } \gamma > \wedge^j \omega\}$$

and

$$S'_j = \{\gamma | \gamma > \wedge^{j-1} \omega \text{ and } \gamma > \wedge^j \omega\}$$

Therefore

$$L_\gamma = \sum_{\gamma > \wedge^i \omega} a_i + \sum_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{or } \gamma > \wedge^j \omega}} b_j + h_\gamma(\rho) + 1$$

$$L'_\gamma = \sum_{\gamma > \wedge^i \omega} a_i + \sum_{\substack{\gamma > \wedge^{j-1} \omega \\ \text{and } \gamma > \wedge^j \omega}} b_j + h_\gamma(\rho)$$

This gives  $b_{\alpha_i}$  as the product of the  $L_\gamma, L'_\gamma$ . We now only need to find  $b_{\beta_j}$ .

By property 6, the degree of  $b_{\beta_j}$  is

$$(j-1)(n-j) + j(n-j-1) + 1 = (j-1)(n-j-1) + (j(n-j)-1) + 1$$

. But  $|S'_j| = (j-1)(n-j-1)$ , and  $|S_j| = j(n-j)-1$ , so only one factor is unaccounted for - and that factor, by property 2, is  $b_j + 1$ . By inspection, then, we get the formula in the theorem.  $\square$

## 12 Appendix 1: Structure of $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$

**Theorem 25.** *Let  $V$  be a finite dimensional  $\mathfrak{b}$ -module. Then  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is a  $U\mathfrak{h} = S\mathfrak{h}$  module of rank equal to the dimension of the  $\mu$  weight subspace of  $V$ .*

*Proof.* This proof follows [9] (though we discard the assumption that  $V$  be generated by a weight vector).

As  $V$  is finite-dimensional, we can define an increasing filtration by  $\mathfrak{b}$ -submodules  $F_i$  on  $V^*$  such that  $F_{-1} = \{0\}$ ,  $F_{i+1}/F_i$  has a unique weight  $\lambda_i$  (note that  $F_{i+1}/F_i$  is not necessarily 1-dimensional), and the  $\lambda_i$  are distinct.

We have an isomorphism  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu} \simeq \text{Hom}_{\mathfrak{b},-\mu}(V^*, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  where the subscript  $-\mu$  denotes that the homomorphism twists by  $-\mu$ ; in other words, if  $w \in W$  of weight  $\lambda$  and  $f \in \text{Hom}_{\mathfrak{b},-\mu}(W, U)$ , then  $f(w)$  has weight  $\lambda - \mu$ . This homomorphism set has a residual right  $U\mathfrak{h}$  action from its action on  $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ . We proceed by induction on the filtration.

We can start with the base case of  $F_{-1}$ , the trivial  $\mathfrak{b}$ -module; by definition,  $\text{Hom}_{\mathfrak{b},-\mu}(F_{-1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) = \{0\}$ .

Assume  $\lambda_{i+1} \neq \mu$ . Because  $F_i$  is a submodule of  $F_{i+1}$ , we get a map  $\iota_i : \text{Hom}_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \rightarrow \text{Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ . Kashiwara [9] proved (equation (1.4)) that  $\exists g_i \neq 0 \in S\mathfrak{h}$  such that  $g_i \text{Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \in \text{Im}(\iota_i)$ . Therefore  $\text{rk Hom}_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) \geq \text{rk Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  as  $U\mathfrak{h}$ -modules.

Let  $f \in \text{Hom}_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  with  $\iota_i(f) = 0$ . Then  $f(F_i) = 0$ , so  $f$  descends to a map  $\bar{f} : F_{i+1}/F_i \rightarrow \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$  which twists by  $-\mu$ . For any  $v \in F_{i+1}/F_i$ ,  $\mathfrak{n}v = 0$ , so  $\mathfrak{n}\bar{f}(v) = 0$ . But the only elements of  $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$  with this property are elements of  $U\mathfrak{h}$ , that is, elements of weight 0. As  $\bar{f}$  twists by  $-\mu$ , and  $\lambda_{i+1} \neq \mu$ , this is impossible unless  $\bar{f} = 0$ , and therefore  $f = 0$  - so  $\iota$  is injective. Therefore, we get that the ranks of  $\text{Hom}_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  and  $\text{Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  must in fact be equal.

Assume  $\lambda_{i+1} = \mu$ . As the  $\lambda_i$  are distinct, by the above,  $\text{Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$  is trivial. The map  $\iota_i$  is not injective in this case; we need to find its kernel. Choose a basis  $\{v_j\} \in F_{i+1}/F_i$ ; then for each  $j$ , we can define  $\bar{f}_j : F_{i+1}/F_i \rightarrow \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$  with  $\bar{f}_j(v_k) = 1$  if  $j = k$ , 0 else. By inspection, these twist by  $-\mu$ . We can therefore extend these to  $f_j \in \text{Hom}_{\mathfrak{b},-\mu}(F_{i+1}, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})$ , so  $\text{rk Hom}_{\mathfrak{b},-\mu}(F_i, \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}) = \dim F_{i+1}/F_i = \dim V^\mu$ .  $\square$

**Corollary 26.** *Assume  $V = V_\nu^*$  is the dual of a finite-dimensional irreducible representation of  $U\mathfrak{g}$  with highest weight  $\nu$ . If  $\nu < \mu$  or  $\nu - \mu$  is not integral, then  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is trivial.*

**Theorem 27.** *Assume that the dimension of  $V^\mu$  is 1. Then  $(V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{\mathfrak{b}-\mu}$  is free as a right  $U\mathfrak{h}$ -module.*

In order to prove this, we will need a lemma about when a submodule of a free submodule is free. Let  $A$

be a UFD, and let  $N$  be an  $A$ -module. Define  $A^{-1}N = \text{Frac}(A) \otimes_A N$ .

**Lemma 28.** *Let  $A$  be a UFD,  $M$  a free  $A$ -module, and  $N$  a finitely generated submodule of  $M$  of rank 1. If  $A^{-1}N \cap M = N$ , then  $N$  is free.*

*Proof.* Note that  $A^{-1}N \cap M = \{m \in M \mid \exists a \text{ s.t. } am \in N\}$ . Let  $\{n_i\}$  be a minimal generating set of  $N$ . Assume it contains at least two elements  $n, n'$ . If we can find some  $n'' \in n$  that generates both  $n, n'$ , then by induction, we can show that  $N$  is free.

Because  $N$  is rank 1, there must be some  $b, b'$  relatively prime with  $bn = b'n'$ . Choose a basis  $\{v_j\} \in M$ ; then  $n = \sum a_j v_j, n' = \sum a'_j v_j$ . As they are a basis, we get that  $ba_j = b'a'_j$ . Then as  $A$  is a UFD,  $b' \mid a_j$ . Let  $a''_j = \frac{a_j}{b'}$ , and let  $n'' = \sum a''_j v_j = \frac{n}{b'} = \frac{n'}{b}$ . As  $a''_j \in A$  for each  $j$ ,  $n'' \in M$ , so  $n'' \in A^{-1}N \cap M = N$ . Therefore, the induction step is done, and  $N$  is free.  $\square$

*Proof of Theorem 27.* Let  $N = (V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}})^{b-\mu}$ ; by the first theorem,  $N$  is a rank 1  $U\mathfrak{h}$ -module. As  $\frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$  is a free right  $U\mathfrak{h}$ -module, so is  $M = V \otimes \frac{U\mathfrak{g}}{(U\mathfrak{g})\mathfrak{n}}$ . We then only need to prove that  $N(U\mathfrak{h})^{-1} \cap M = N$ .

By definition,  $N(U\mathfrak{h})^{-1} \cap M = \{m \in M \mid \exists h \in U\mathfrak{h} \text{ } mh \in N\}$ . Let  $m \in M, h \in U\mathfrak{h}$  such that  $mh \in N$ . We need to prove two things in order to prove that  $m \in N$ : that for any  $n \in \mathfrak{n}, nm = 0$ , and that for any  $h' \in \mathfrak{h}, h'm = m(h' - \mu)$ . If  $mh \in N$ , then  $nmh = 0$ . But  $M$  is free as a right  $U\mathfrak{h}$ -module, so  $nm = 0$ . If  $mh \in N$ , then  $h'mh = mh(h' - \mu) = m(h' - \mu)h$ . Again as  $M$  is free,  $h'm = m(h' - \mu)$ . Therefore, by the lemma,  $N$  is a free rank 1  $U\mathfrak{h}$ -module.  $\square$

## 13 Appendix 2: Techniques for calculating polynomial cocycles

Much of the relevant reasoning for the differential operator equation  $P_\mu P_\nu = P_\nu P_\mu$  can be applied more generally, especially when trying to find the b-function of zero-sets of semiinvariant sections of line bundles under actions of a group with dense orbit. As such, all of the corrolaries from section 8 apply. These corrolaries lend themselves to certain techniques that facilitate the calculation of b-functions. Here, we provide as an example the calculation for the b-function of the section studied in this thesis for  $G = SL_3$  on the variety  $\mathcal{F}_{SL_3} \times \mathcal{F}_{SL_3} \times \mathbb{P}^2$ .

In this setting, we have a lattice  $\mathbb{Z}^n$  generated by the standard elements  $\epsilon_i$ . We then have an unknown

cocycle  $b \in H^1(\mathbb{Z}^n, \mathbb{C}(\mathbb{C}^n)^\times)$  in the Hochschild complex for the group cohomology of  $\mathbb{Z}^n$ , where  $\mathbb{Z}^n$  acts on  $\mathbb{C}(\mathbb{C}^n)^\times$  by translation, such that for  $\mu$  in the "positive" subsemigroup  $\mathbb{Z}_+^n$  generated by some  $\epsilon_i$ , we have that the b-functions  $b_\mu \in \mathbb{C}[\mathbb{C}^n]$  are polynomials. For the example case, as shown by Ginzburg and Finkelberg [4], these generators are  $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ . Our goal is then to find a way to calculate these b-functions on some set of limited information.

For the rest of this section, we write  $\lambda \in \Omega$  in the form  $\lambda = \sum_{i=1}^2 a_i \alpha_i + \sum_{j=1}^3 b_j \beta_j$

The final result from this calculation is the following:

$$b_{\alpha_1}(\lambda) = (a_1 + 1)(b_1 + a_1 + b_2 + 2)(a_1 + b_2 + a_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3)$$

$$b_{\alpha_2}(\lambda) = (a_2 + 1)(b_2 + a_2 + b_3 + 2)(a_1 + b_2 + a_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3)$$

$$b_{\beta_1}(\lambda) = (b_1 + 1)(b_1 + a_1 + b_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3)$$

$$b_{\beta_2}(\lambda) = (b_2 + 1)(b_1 + a_1 + b_2 + 2)(a_1 + b_2 + a_2 + 2)(b_2 + a_2 + b_3 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3)$$

$$b_{\beta_3}(\lambda) = (b_3 + 1)(b_2 + a_2 + b_3 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \quad (26)$$

### 13.1 Step 1: Table Creation

The b-functions for an arbitrary  $\mu$  can be determined by the b-functions  $b_{\epsilon_i}$  for the generators. Further, by lemma 15, these generating b-functions can be decomposed as a product of affine factors.

For each generator, we make a table. Each column in the table for  $\epsilon_i$  corresponds to an affine factor of  $b_{\epsilon_i}$ ; each entry in the table corresponds to a coefficient. As such, there should be one row in each table for each generator, plus one row for constants. The coefficient of  $\epsilon_i$  in a factor of  $b_{\epsilon_i}$  cannot be 0 (lemma 17), so the number of columns will be the degree of the coefficient of  $\epsilon_i$  in  $b_{\epsilon_i}$ . For our example case, from the calculations in section 10, we know that the degrees of the generators are as follows:  $\alpha_1$  has degree 4,  $\alpha_2$  has degree 4,  $\beta_1$  has degree 3,  $\beta_2$  has degree 5, and  $\beta_3$  has degree 3.

$\alpha_1$
Const ? ? ? ?
$\alpha_1$ ? ? ? ?
$\alpha_2$ ? ? ? ?
$\beta_1$ ? ? ? ?
$\beta_2$ ? ? ? ?
$\beta_3$ ? ? ? ?

$\alpha_2$
Const ? ? ? ?
$\alpha_1$ ? ? ? ?
$\alpha_2$ ? ? ? ?
$\beta_1$ ? ? ? ?
$\beta_2$ ? ? ? ?
$\beta_3$ ? ? ? ?

$\beta_1$
Const ? ? ?
$\alpha_1$ ? ? ?
$\alpha_2$ ? ? ?
$\beta_1$ ? ? ?
$\beta_2$ ? ? ?
$\beta_3$ ? ? ?

$\beta_2$
Const ? ? ? ? ?
$\alpha_1$ ? ? ? ? ?
$\alpha_2$ ? ? ? ? ?
$\beta_1$ ? ? ? ? ?
$\beta_2$ ? ? ? ? ?
$\beta_3$ ? ? ? ? ?

$\beta_3$
Const ? ? ?
$\alpha_1$ ? ? ?
$\alpha_2$ ? ? ?
$\beta_1$ ? ? ?
$\beta_2$ ? ? ?
$\beta_3$ ? ? ?

Table 1: A starting blank table for each generator for example  $SL_3$ , with as many columns as  $deg(b_i)$

### 13.2 Step 2: Filling in $s + 1$

The fact that for a single-function b-function,  $b(s)$  is divisible by  $s + 1$  can be generalized to the multiple-function b-function, as long as the generating sections are irreducible. For the generator  $\epsilon_i$ , the first column in the table can be filled with 0s for  $\epsilon_j, j \neq i$ , and 1 for the constant and for  $\epsilon_i$ . For our example, we can choose  $\epsilon_i = \beta_2$ . Then if we write  $\mu = (\sum_{i=1}^2 a_i \alpha_i) + (\sum_{j=1}^3 b_j \beta_j)$ , then  $b_2 + 1 | b_{\beta_2}$ . That gives us the following for our example case:

$\alpha_1$				
Const	1	?	?	?
$\alpha_1$	1	?	?	?
$\alpha_2$	0	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\alpha_2$				
Const	1	?	?	?
$\alpha_1$	0	?	?	?
$\alpha_2$	1	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\beta_1$			
Const	1	?	?
$\alpha_1$	0	?	?
$\alpha_2$	0	?	?
$\beta_1$	1	?	?
$\beta_2$	0	?	?
$\beta_3$	0	?	?

$\beta_2$				
Const	1	?	?	?
$\alpha_1$	0	?	?	?
$\alpha_2$	0	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	1	?	?	?
$\beta_3$	0	?	?	?

$\beta_3$			
Const	1	?	?
$\alpha_1$	0	?	?
$\alpha_2$	0	?	?
$\beta_1$	0	?	?
$\beta_2$	0	?	?
$\beta_3$	1	?	?

Table 2: Filling in the factors  $\epsilon_i + 1$

### 13.3 Step 3: Using $\Delta$

In order to fill out the table further, more information specific to the relevant sections of line bundles is necessary. In our example, we use more calculations from section 10. The calculation on  $\Delta$ , which is generated by  $\alpha_1, \alpha_2$  allows us to fill out a large amount of those two tables:

$\alpha_1$				
Const	1	2	2	3
$\alpha_1$	1	1	1	1
$\alpha_2$	0	0	1	1
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\alpha_2$				
Const	1	2	2	3
$\alpha_1$	0	0	1	1
$\alpha_2$	1	1	1	1
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\beta_1$			
Const	1	?	?
$\alpha_1$	0	?	?
$\alpha_2$	0	?	?
$\beta_1$	1	?	?
$\beta_2$	0	?	?
$\beta_3$	0	?	?

$\beta_2$				
Const	1	?	?	?
$\alpha_1$	0	?	?	?
$\alpha_2$	0	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	1	?	?	?
$\beta_3$	0	?	?	?

$\beta_3$			
Const	1	?	?
$\alpha_1$	0	?	?
$\alpha_2$	0	?	?
$\beta_1$	0	?	?
$\beta_2$	0	?	?
$\beta_3$	1	?	?

Table 3: Conclusions from  $\Delta$

### 13.4 Step 4: Using $\Delta_{\geq j}$ and $\Delta_{< j}$ for $b_{\beta_1}$ and $b_{\beta_3}$

The calculations on  $\Delta_{\geq j}$  and  $\Delta_{< j}$  are similar for  $j = 1, 3$ ; however, for  $j = 2$ , the match between the two subsemigroups is ambiguous. As such, we first "fill out" the tables for  $\beta_1, \beta_3$  using  $\Delta_{\geq 1}$  and  $\Delta_{< 3}$ , and save the  $j = 2$  cases for later.

$\alpha_1$				
Const	1	2	2	3
$\alpha_1$	1	1	1	1
$\alpha_2$	0	0	1	1
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\alpha_2$				
Const	1	2	2	3
$\alpha_1$	0	0	1	1
$\alpha_2$	1	1	1	1
$\beta_1$	0	?	?	?
$\beta_2$	0	?	?	?
$\beta_3$	0	?	?	?

$\beta_1$			
Const	1	2	3
$\alpha_1$	0	1	1
$\alpha_2$	0	0	1
$\beta_1$	1	1	1
$\beta_2$	0	?	?
$\beta_3$	0	?	?

$\beta_2$				
Const	1	?	?	?
$\alpha_1$	0	?	?	?
$\alpha_2$	0	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	1	?	?	?
$\beta_3$	0	?	?	?

$\beta_3$			
Const	1	2	3
$\alpha_1$	0	0	1
$\alpha_2$	0	1	1
$\beta_1$	0	?	?
$\beta_2$	0	?	?
$\beta_3$	1	1	1

Table 4: Preliminary conclusions from  $\Delta_{\geq j}$  and  $\Delta_{< j}$

### 13.5 Step 5: Matching one factor

Using lemma 18, we can now start to "match" the entries between the tables. In our example, it's clear that all of the columns with a constant of 3 must be the same. That allows us to fill out much of the tables:

$\alpha_1$				
Const	1	2	2	3
$\alpha_1$	1	1	1	1
$\alpha_2$	0	0	1	1
$\beta_1$	0	1	0	1
$\beta_2$	0	?	?	?
$\beta_3$	0	0	0	1

$\alpha_2$				
Const	1	2	2	3
$\alpha_1$	0	0	1	1
$\alpha_2$	1	1	1	1
$\beta_1$	0	0	0	1
$\beta_2$	0	?	?	?
$\beta_3$	0	1	0	1

$\beta_1$			
Const	1	2	3
$\alpha_1$	0	1	1
$\alpha_2$	0	0	1
$\beta_1$	1	1	1
$\beta_2$	0	?	?
$\beta_3$	0	0	1

$\beta_2$				
Const	1	?	?	?
$\alpha_1$	0	?	?	?
$\alpha_2$	0	?	?	?
$\beta_1$	0	?	?	?
$\beta_2$	1	?	?	?
$\beta_3$	0	?	?	?

$\beta_3$			
Const	1	2	3
$\alpha_1$	0	0	1
$\alpha_2$	0	1	1
$\beta_1$	0	0	1
$\beta_2$	0	?	?
$\beta_3$	1	1	1

Table 5: Matching Factors for  $\beta_1, \beta_3$

### 13.6 Step 6: Filling factors for $\beta_2$

In the case of  $\beta_2$ , we can start to sort the affine factors by the constant; using both  $\Delta_{\geq 2}$  and  $\Delta_{< 2}$ , we get the constants 1, 2, 2, 2, 3. Because there is only one factor with constant 3, and by using the same "matching" as above, we know that all of the b-functions have the same last column full of 1s. Finally, some information from  $\Delta_{\geq 2}$  and  $\Delta_{< 2}$  implies that we can fill out the  $\beta_2$  row in the  $\beta_2$  table with 1s.

$\alpha_1$
Const 1 2 2 3
$\alpha_1$ 1 1 1 1
$\alpha_2$ 0 0 1 1
$\beta_1$ 0 1 0 1
$\beta_2$ 0 ? ? 1
$\beta_3$ 0 0 0 1

$\alpha_2$
Const 1 2 2 3
$\alpha_1$ 0 0 1 1
$\alpha_2$ 1 1 1 1
$\beta_1$ 0 0 0 1
$\beta_2$ 0 ? ? 1
$\beta_3$ 0 1 0 1

$\beta_1$
Const 1 2 3
$\alpha_1$ 0 1 1
$\alpha_2$ 0 0 1
$\beta_1$ 1 1 1
$\beta_2$ 0 ? 1
$\beta_3$ 0 0 1

$\beta_2$
Const 1 2 2 2 3
$\alpha_1$ 0 ? ? ? 1
$\alpha_2$ 0 ? ? ? 1
$\beta_1$ 0 ? ? ? 1
$\beta_2$ 1 1 1 1 1
$\beta_3$ 0 ? ? ? 1

$\beta_3$
Const 1 2 3
$\alpha_1$ 0 0 1
$\alpha_2$ 0 1 1
$\beta_1$ 0 0 1
$\beta_2$ 0 ? 1
$\beta_3$ 1 1 1

Table 6: Matching factors for  $\beta_2$

### 13.7 Step 7: Final matching

Some more information from  $\Delta_{\geq 2}$  and  $\Delta_{< 2}$  tells us that two of the factors with constant 2 in the  $\beta_2$  table have a 1 for  $\alpha_1$ , and that two of the factors with constant 2 have a 1 for  $\alpha_2$ . But by once again matching the factors for  $\alpha_1$ , we can see that at most one factor with constant 2 can have 1 as a coefficient for both  $\alpha_1, \alpha_2$ . Therefore, we can fill out the tables by elimination:

$\alpha_1$
Const 1 2 2 3
$\alpha_1$ 1 1 1 1
$\alpha_2$ 0 0 1 1
$\beta_1$ 0 1 0 1
$\beta_2$ 0 1 1 1
$\beta_3$ 0 0 0 1

$\alpha_2$
Const 1 2 2 3
$\alpha_1$ 0 0 1 1
$\alpha_2$ 1 1 1 1
$\beta_1$ 0 0 0 1
$\beta_2$ 0 1 1 1
$\beta_3$ 0 1 0 1

$\beta_1$
Const 1 2 3
$\alpha_1$ 0 1 1
$\alpha_2$ 0 0 1
$\beta_1$ 1 1 1
$\beta_2$ 0 1 1
$\beta_3$ 0 0 1

$\beta_2$
Const 1 2 2 2 3
$\alpha_1$ 0 1 1 0 1
$\alpha_2$ 0 0 1 1 1
$\beta_1$ 0 1 0 0 1
$\beta_2$ 1 1 1 1 1
$\beta_3$ 0 0 0 1 1

$\beta_3$
Const 1 2 3
$\alpha_1$ 0 0 1
$\alpha_2$ 0 1 1
$\beta_1$ 0 0 1
$\beta_2$ 0 1 1
$\beta_3$ 1 1 1

Table 7: Finished table

From this table, we can conclude that the b-functions are:

$$\begin{aligned}
b_{\alpha_1}(\lambda) &= (a_1 + 1)(b_1 + a_1 + b_2 + 2)(a_1 + b_2 + a_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \\
b_{\alpha_2}(\lambda) &= (a_2 + 1)(b_2 + a_2 + b_3 + 2)(a_1 + b_2 + a_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \\
b_{\beta_1}(\lambda) &= (b_1 + 1)(b_1 + a_1 + b_2 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \\
b_{\beta_2}(\lambda) &= (b_2 + 1)(b_1 + a_1 + b_2 + 2)(a_1 + b_2 + a_2 + 2)(b_2 + a_2 + b_3 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \\
b_{\beta_3}(\lambda) &= (b_3 + 1)(b_2 + a_2 + b_3 + 2)(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \quad (27)
\end{aligned}$$

Note that this exact "matching", including by constants, only worked because all nonzero coefficients were 1. In a more general setting, where some of the coefficients can be larger, the constants no longer necessarily have to match. The technique can still be used by matching the coefficients, keeping in mind lemma 16.

An alternative technique that could be used depends on the "lift" referred to in section 8. The "lift" takes the form of a product of gamma functions of affine factors. As such, it allows the analysis to use only one table; however, the number of columns is not easily determined. The advantage of this method is that the "matching" is done automatically. The main disadvantage is that determining which column new information belongs to can be difficult. Further, determining constants when coefficients are at least 2 can be difficult. We illustrate using the same example.

The final result from this calculation is that if we let:

$$\begin{aligned}
A(\lambda) &= \Gamma(b_1 + 1)\Gamma(a_1 + 1)\Gamma(b_2 + 1)\Gamma(a_2 + 1)\Gamma(b_3 + 1) \\
&\Gamma(b_1 + a_1 + b_2 + 2)\Gamma(a_1 + b_2 + a_2 + 2)\Gamma(b_2 + a_2 + b_3 + 2)\Gamma(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \quad (28)
\end{aligned}$$

then the b-functions generated by equation 27 can be written as  $b_\mu(\lambda) = \frac{A(\mu + \lambda)}{A(\mu)}$  for any  $\mu \in \Omega$ .

### 13.8 Step 1: Empty "Lift" Table

We start by making a single table with a row for each generator and a constant row. The number of columns is difficult to determine beforehand, so we may need to use a large number of columns and reduce as necessary.

Const	?	?	?	?	?	?	?	?	?	?
$\alpha_1$	?	?	?	?	?	?	?	?	?	?
$\alpha_2$	?	?	?	?	?	?	?	?	?	?
$\beta_1$	?	?	?	?	?	?	?	?	?	?
$\beta_2$	?	?	?	?	?	?	?	?	?	?
$\beta_3$	?	?	?	?	?	?	?	?	?	?

Table 8: Empty "lift" table. The necessary number of columns is unknown, so we will correct the number of columns in a later table.

### 13.9 Step 2: Filling $s + 1$ factors

The first few columns can be filled by the generalization of the fact that  $s + 1$  is a factor of  $b(s)$ .

Const	1	1	1	1	1	?	?	?	?	?
$\alpha_1$	1	0	0	0	0	?	?	?	?	?
$\alpha_2$	0	1	0	0	0	?	?	?	?	?
$\beta_1$	0	0	1	0	0	?	?	?	?	?
$\beta_2$	0	0	0	1	0	?	?	?	?	?
$\beta_3$	0	0	0	0	1	?	?	?	?	?

Table 9:  $s + 1$  factors filled

### 13.10 Step 3: Filling the final column

Some information from the  $\Delta, \Delta_{<j}, \Delta_{\geq j}$  implies that all coefficients are either 0 or 1. Further information implies that the constants in the currently unfilled columns must be 2 or 3. From  $\Delta$ , there is exactly one column with a constant of 3 and a coefficient of 1 for  $\alpha_1$  and  $\alpha_2$ ; all others must have coefficients of 0 for  $\alpha_1, \alpha_2$ . From the  $\Delta_{<j}, \Delta_{\geq j}$ , all columns with a constant of 3 must have a coefficient of 1 for at least one of  $\alpha_1, \alpha_2$ , so there is only one column with a constant of 3, and it has all coefficients 1.

Const	1	1	1	1	1	?	?	?	?	3
$\alpha_1$	1	0	0	0	0	?	?	?	?	1
$\alpha_2$	0	1	0	0	0	?	?	?	?	1
$\beta_1$	0	0	1	0	0	?	?	?	?	1
$\beta_2$	0	0	0	1	0	?	?	?	?	1
$\beta_3$	0	0	0	0	1	?	?	?	?	1

Table 10: Final column filled

### 13.11 Step 4: Filling rows for $\alpha_1, \alpha_2$

From  $\Delta$ , we can fill out the coefficients of  $\alpha_1, \alpha_2$  in three columns with constant 2.

Const	1	1	1	1	1	2	2	2	?	3
$\alpha_1$	1	0	0	0	0	1	1	0	?	1
$\alpha_2$	0	1	0	0	0	0	1	1	?	1
$\beta_1$	0	0	1	0	0	?	?	?	?	1
$\beta_2$	0	0	0	1	0	?	?	?	?	1
$\beta_3$	0	0	0	0	1	?	?	?	?	1

Table 11: Conclusions from  $\Delta$

### 13.12 Step 5: Filling rows for $\beta_1, \beta_3$

From  $\Delta_{<3}$  and  $\Delta_{\geq 1}$ , we can fill out the coefficients for  $\beta_1, \beta_3$ .

Const	1	1	1	1	1	2	2	2	?	3
$\alpha_1$	1	0	0	0	0	1	1	0	?	1
$\alpha_2$	0	1	0	0	0	0	1	1	?	1
$\beta_1$	0	0	1	0	0	1	0	0	?	1
$\beta_2$	0	0	0	1	0	?	?	?	?	1
$\beta_3$	0	0	0	0	1	0	0	1	?	1

Table 12: Conclusions from  $\Delta_{<3}, \Delta_{\geq 1}$

### 13.13 Step 6: Filling rows for $\beta_2$

From  $\Delta_{<2}$ , two of the factors which have coefficient 1 for  $\beta_2$  must have coefficient 1 for  $\alpha_1$ . Similarly, two of the factors which have coefficient 1 for  $\beta_2$  must have coefficient 1 for  $\alpha_2$ . That fills out the two columns:

Const	1	1	1	1	1	2	2	2	?	3
$\alpha_1$	1	0	0	0	0	1	1	0	?	1
$\alpha_2$	0	1	0	0	0	0	1	1	?	1
$\beta_1$	0	0	1	0	0	1	0	0	?	1
$\beta_2$	0	0	0	1	0	1	1	1	?	1
$\beta_3$	0	0	0	0	1	0	0	1	?	1

Table 13: Finished matching

### 13.14 Step 7: Final Table

Finally, by checking the degree of the coefficient of  $\beta_i$  in  $b_{\beta_i}$  and similar considerations for  $\alpha_i$ , we can see that there are no more factors. As such, our final table is

Const	1	1	1	1	1	2	2	2	3
$\alpha_1$	1	0	0	0	0	1	1	0	1
$\alpha_2$	0	1	0	0	0	0	1	1	1
$\beta_1$	0	0	1	0	0	1	0	0	1
$\beta_2$	0	0	0	1	0	1	1	1	1
$\beta_3$	0	0	0	0	1	0	0	1	1

Table 14: Finished "lift" table

From this, we conclude that if

$$A(\lambda) = \Gamma(b_1 + 1)\Gamma(a_1 + 1)\Gamma(b_2 + 1)\Gamma(a_2 + 1)\Gamma(b_3 + 1)$$

$$\Gamma(b_1 + a_1 + b_2 + 2)\Gamma(a_1 + b_2 + a_2 + 2)\Gamma(b_2 + a_2 + b_3 + 2)\Gamma(b_1 + a_1 + b_2 + a_2 + b_3 + 3) \quad (29)$$

then  $b_\mu(\lambda) = \frac{A(\lambda + \mu)}{A(\lambda)}$  are the b-functions for the  $G$ -invariant global sections of line bundles on  $X$ .

This method has the additional disadvantage that although the cocycles must be products of affine factors, with no denominator, the "lift" cannot always be expressed as a product of gamma functions of affine factors with no denominator, as explained in remark 8. This can still be expressed in a table-like form by including "denominator" columns.

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