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REPRESENTATION THEORY AND ARITHMETIC STATISTICS OF GENERALIZED CONFIGURATION SPACES

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ABSTRACT

In this thesis, we extend the work of Church-Ellenberg-Farb [CEF15] [CEF14] on FI-modules, representation stability of configuration spaces, and arithmetic statistics. We study two generalizations of the category FI: namely FI_G for G a group, first studied by Sam-Snowden [SSb], and FI^m, first studied by Gadish [Gada]. We use these to study two types of generalized configuration spaces: the orbit configuration spaces $\mathrm{Conf}_n^G(M)$ associated to a G-cover M, and the space of ordered 0-cycles $\widetilde{\mathcal{Z}}_n^{(d_1,\ldots d_m)}(X)$ introduced by Farb-Wolfson-Wood [FWW]. After establishing basic properties of FI_G- and FI^m-modules, we obtain representation stability results for the cohomology of these generalized configuration spaces. We establish subexponential bounds on the growth of unstable cohomology, and the Grothendieck-Lefschetz trace formula then allows us to translate these topological stability phenomena to stabilization of arithmetic statistics for generalized configuration spaces over finite fields.

CHAPTER 1

INTRODUCTION

In [CEF15], Church-Ellenberg-Farb developed the theory of FI-*modules*. Recall that FI is the category whose objects are finite sets and whose morphisms are injections, and that an FI-module is a functor from FI to the category of k-modules (where k is usually a field). One of their prominent examples of an FI-module is the cohomology of configuration spaces. Recall that for a space X, the ordered configuration space is

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j\}$$

$\mathrm{Conf}_n(X)$ carries an action of S_n, making the rational cohomology $H^i(\mathrm{Conf}_n(X); \mathbb{Q})$ into an S_n-representation for each i. For each inclusion $[m] \hookrightarrow [n]$, there is a "forgetting points" map $\mathrm{Conf}_n(M) \to \mathrm{Conf}_m(M)$,

Church-Ellenberg-Farb [CEF15] developed a theory of FI-modules and used this theory to obtain strong stability results about the cohomology of configuration spaces of manifolds. In [CEF14] and [FW], the Grothendieck-Lefschetz trace formula was applied to transform these topological stability properties of configuration spaces into stability results for arithmetic statistics of configuration spaces over finite fields.

In this thesis we extend this work, by studying two generalizations of the category FI and two types of generalized configuration spaces. The first is the category FI_G introduced by Sam-Snowden [SSa]. Just as FI is a way of bundling together the symmetric groups S_n for all n, the category FI_G is a way of bundling together the groups $G^n \rtimes S_n$ for all n.

Suppose M is a manifold with free and properly discontinuous action of a group G. Define the *orbit configuration space* $\mathrm{Conf}_n^G(M)$ as

$$\mathrm{Conf}_n^G(M) := \{(m_1, \ldots, m_n) \in M^n \mid m_i \neq gm_j \ \forall g \in G\}$$

The cohomology $H^i(\mathrm{Conf}_n^G(M);\mathbb{Q})$ is a representation of $G^n \rtimes S_n$. When we consider all n at once, then $H^i(\mathrm{Conf}^G(M);\mathbb{Q})$ forms an FI_G-module.

The second is the product category $\mathrm{FI}^m = \mathrm{FI} \times \cdots \times \mathrm{FI}$, which was studied by Gadish [Gada]. In [FWW], Farb-Wolfson-Wood studied a general class of spaces of 0-cycles on a manifold X that they call $\mathcal{Z}_n^{(d_1,\ldots d_m)}(X)$, which is the subspace of $\prod_i \mathrm{Sym}^{d_i}(X)$ where no point of X appears n (or more) times in every $\mathrm{Sym}^{d_i}(X)$. (Here $\mathrm{Sym}^d(X) := X^d/S_d$). They analyze these spaces in detail, and among other results, prove that they satisfy homological stability.

In order to study the spaces $\mathcal{Z}_n^{(d_1,\ldots d_m)}(X)$, Farb-Wolfson-Wood pass to an ordered version that they denote $\widetilde{\mathcal{Z}}_n^{(d_1,\ldots d_m)}(X)$, which is the subspace of $\prod_i X^{d_i}$ where no point of X appears n or more times in every X^{d_i}. The space $\widetilde{\mathcal{Z}}_n^{(d_1,\ldots d_m)}(X)$ has an action of

$$S_{\mathbf{d}} := S_{d_1} \times \cdots \times S_{d_m}$$

with quotient $\mathcal{Z}_n^{(d_1,\ldots d_m)}(X)$. When we consider all such $\mathbf{d}$ simultaneously, then $H^i(\mathcal{Z}_n^{\mathbf{d}}(X);\mathbb{Q})$ forms an FI^m*-module.*

1.1 Representation stability

By analyzing the Leray spectral sequences associated to the inclusions $\mathrm{Conf}_n^G(M) \hookrightarrow M^n$ and $\mathcal{Z}_n^{\mathbf{d}}(X) \hookrightarrow X^{\mathbf{d}}$, we prove that $H^i(\mathrm{Conf}^G(M);\mathbb{Q})$ is a *finitely generated* FI_G-modules, and that $H^*(\mathcal{Z}_n^{\mathbf{d}}(X);\mathbb{Q})$ is a finitely generated FI^m-module. Our work implies the following about these two spaces.

First, recall that the irreducible representations of $G^n \rtimes S_n$ are given by partition-valued functions $\underline{\lambda}$ on the irreducible representations of G with $\|\underline{\lambda}\| = n$. Let $c(G)$ be the set of conjugacy classes of G. Define an FI_G *character polynomial* to be a polynomial in $c(G)$-labeled cycle-counting functions. This is a class function on W_n.

Next, recall that the irreducible representations of $S_{\mathbf{d}}$ are parameterized by lists of partitions $\lambda = (\lambda^1, \ldots, \lambda^m)$, where λ^i is a partition of d_i. For each i, j, let X^i_j be the class function on $\bigcup_{\mathbf{d}} S_{\mathbf{d}}$ that counts the number of j-cycles on S_{d_i}. Define a FI^m *character polynomial* to be an element of $\mathbb{Q}[\{X^i_j\}]$. (We will define this and other terminology more precisely in §2).

Theorem 1.1.1 (**Polynomiality of characters and representation stability for** $\mathrm{Conf}^G(M)$**).** *Let M be a connected manifold with a free action of a finite group G. Assume that* $\dim M \geq 2$ *and that each* $\dim H^i(M; \mathbb{Q}) < \infty$*. Then:*

1. *The characters of $H^i(\mathrm{Conf}^G_n(M); \mathbb{C})$ are given by a single FI_G character polynomial for all $n \gg 0$.*
2. *The multiplicity of each irreducible W_n-representation in $H^i(\mathrm{Conf}^G_n(M); \mathbb{C})$ is eventually independent of n, and $\dim H^i(\mathrm{Conf}^G_n(M); \mathbb{C})$ is given by a single polynomial for all $n \gg 0$.*

Theorem 1.1.2 (**Polynomiality of characters and representation stability for** $\mathcal{Z}^{\mathbf{d}}_n(X)$**).** *Fix n and i, let X be a connected manifold of dimension at least 2, and let $V_{\mathbf{d}} = H^i(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n; \mathbb{Q})$. Then:*

1. *There is a single FI^m character polynomial $P \in \mathbb{Q}[\{X^i_j\}]$ such that the character $\chi_{V_{\mathbf{d}}} = P$ for all $d_i \gg 0$. In particular, there is a polynomial $Q \in \mathbb{Q}[y_1, \ldots, y_m]$ such that* $\dim V_{\mathbf{d}} = Q(d_1, \ldots, d_m)$ *for all $d_i \gg 0$.*
2. *The multiplicity of each irreducible $S_{\mathbf{d}}$-representation in $V_{\mathbf{d}}$ is independent of* $\mathbf{d}$ *when all $d_i \gg 0$.*

Plugging in $\mathbf{d} = (d)$ and $n = 2$ into Theorem 1.1.2 recovers representation stability for $\mathrm{Conf}_d(X)$, as proven by Church [Chu12, Thm 1] and Church-Ellenberg-Farb [CEF15, Thm 1.8]. Furthermore, just looking at the multiplicity of the trivial representation in Theorem 1.1.2.2 recovers [FWW, Thm 1.6].

1.2 Stability of arithmetic statistics

We consider the case where we replace the manifold M with a scheme X over $\mathbb{Z}[1/N]$. Thus we can consider the complex points $X(\mathbb{C})$, but also the finite field points $X(\mathbb{F}_q)$. We generalize the results of Farb-Wolfson [FW] on $\mathrm{Conf}_n(X)$ regarding *étale representation stability*.

Theorem 1.2.1 (Étale representation stability). *Let X be a smooth scheme over $\mathbb{Z}[1/N]$ with geometrically connected fibers that is smoothly compactifiable. Let K be either a number field or a finite field over $\mathbb{Z}[1/N]$.*

1. *For each n and i, the $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}^m$-module $H^i_{\acute{e}t}(\widetilde{\mathcal{Z}}^\bullet_n(X); \mathbb{Q}_l)$ is finitely generated.*
2. *Suppose a finite group G acts freely on X, such that X is smoothly compactifiable as a G-scheme. For each $i \geq 0$, the $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}_G$-module $H^i_{\acute{e}t}(\mathrm{Conf}^G(X)_{/\overline{K}}; \mathbb{Q}_l)$ is finitely generated.*

The next result concerns bounds on $H^i(\mathrm{Conf}^G_n(X))$ and $H^i(\widetilde{\mathcal{Z}}^\bullet_n(X))$ as i varies, which are necessary to ensure convergence of the point-counts we are interested in.

Theorem 1.2.2 (Convergent cohomology). *Let X be a connected manifold of dimension at least 2 with $\dim H^*(X) < \infty$.*

1. *For each FI^m character polynomial P, the inner product $|\langle P, H^i(\mathcal{Z}^{\mathbf{d}}_n(X))\rangle|$ is bounded subexponentially in i and uniformly in n.*
2. *Suppose a finite group G acts freely on X. For each FI_G character polynomial P, the inner product $|\langle P, H^i(\mathrm{Conf}^G_n(X))\rangle|$ is bounded subexponentially in i and uniformly in n.*

Finally, we use the Grothendieck-Lefschetz trace formula, along with Theorem 1.2.2 and Theorem 1.2.3, to obtain the following results on arithmetic statistics.

Theorem 1.2.3 (Stability of arithmetic statistics). *Let X be a smooth quasiprojective scheme over $\mathbb{Z}[1/N]$ with geometrically connected fibers.*

1. *For any n and any FI^m character polynomial P,*

$$\lim_{\mathbf{d}\to\infty} q^{-|\mathbf{d}|\dim X} \sum_{y\in\mathcal{Z}^{\mathbf{d}}_n(X)(\mathbb{F}_q)} P(y) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)), P\rangle\right)$$

2. *Suppose a finite group G acts freely on X, such that X is a smoothly compactifiable G-scheme. Then for any FI_G character polynomial P,*

$$\lim_{n\to\infty} q^{-n\dim X} \sum_{y\in\mathrm{UConf}_n(X/G)(\mathbb{F}_q)} P(\sigma_y) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i(\mathrm{Conf}^G(X(\mathbb{C})); L), P\rangle\right) \tag{1.1}$$

Namely, both the limit on the left and the series on the right converge, and they converge to the same limit.

In particular, Theorems Theorem 1.2.2 and Theorem 1.2.3 recover the results for $\mathrm{Conf}_d(X)$ proven by Farb-Wolfson [FW, Thm C], either by taking G trivial for the FI_G case, or by taking $\mathbf{d} = (d)$ and $n = 2$ for the FI^m.

1.2.1 Gauss Sums

For the specific case $G = \mathbb{Z}/d\mathbb{Z}$, the resulting automorphism groups $W_n = (\mathbb{Z}/d\mathbb{Z})^n \wr S_n$ are the so-called *main series of complex reflection groups*. These directly generalize the Weyl groups of type BC_n of Wilson's paper, for which $d = 2$. In particular, we can apply thref-arith-stats to the action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{C}^*$ by rotation. In this case, the sum on the left-hand side of (ref-groth) is over $\mathrm{UConf}_n(\mathbb{G}_m)(\mathbb{F}_q) = \mathrm{Poly}_n(\mathbb{F}_q^*)$, the space of square-free polynomials over $\mathbb{F}_q$ that do not have 0 as a root. We thus obtain the following result, generalizing the work of Church-Ellenberg-Farb [CEF14] (see §4.5 for definitions).

Theorem 1.2.4 (**Gauss sums for** $\mathrm{Conf}^G(\mathbb{C}^*)$ **stabilize).** *For any prime power q, any $d \mid q-1$, and any character polynomial P for $\mathrm{FI}_{\mathbb{Z}/d\mathbb{Z}}$,*

$$\lim_{n\to\infty} q^{-n} \sum_{f\in \mathrm{Poly}_n(\mathbb{F}_q^*)} P(f) = \sum_{i=0}^{\infty}(-1)^i \frac{\langle P^*, H^i(\mathrm{Conf}^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{C})\rangle}{q^i} \tag{1.2}$$

In particular both the limit on the left and the series on the right converge, and they converge to the same limit.

Theorem 1.2.4 essentially says that the *average value* of certain Gauss sums across all polynomials in $\mathrm{Poly}_n(\mathbb{F}_q^*)$ always converges to the series in q^{-1} on the right. For example, let χ be a character of $\mathbb{Z}/(q-1)\mathbb{Z}$. Define the character polynomial $X_i^\chi := \sum_{g\in G}\chi(g)X_i^g$. Then

$$\begin{aligned}\lim_{n\to\infty} q^{-n} &\sum_{f\in \mathrm{Poly}_n(\mathbb{F}_q^*)} \sum_{\substack{\alpha\neq\beta\in\mathbb{F}_q\\ f(\alpha)=f(\beta)=0}} \chi(\alpha)\chi(\beta)^{-1}\\ &= \sum_i (-1)^i \frac{\langle X_1^{\overline{\chi}}X_1^{\chi}, H^i(\mathrm{Conf}^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q}(\zeta_{q-1}))\rangle}{q^i} = -\frac{1}{q}+\frac{5}{q^2}+\cdots\end{aligned} \tag{1.3}$$

That is, the average value of the Gauss sum obtained by applying χ to each quotient of pairs of linear factors of f, across all $f\in\mathrm{Poly}_n(\mathbb{F}_q^*)$, is equal to the series on the right-hand side of (1.3) obtained by looking at the inner product of the character polynomial $X_1^{\overline{\chi}}X_1^{\chi}$ with $H^i(\mathrm{Conf}^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q}(\zeta_{q-1}))$. As another example, suppose q is odd and let $\psi = \left(\frac{-}{q^2}\right)$ be the Legendre symbol in $\mathbb{F}_q^2$, which is 1 or -1 according to whether its argument is a square or nonsquare in $\mathbb{F}_{q^2}$. Then

$$\begin{aligned}\lim_{n\to\infty} q^{-n} &\sum_{f\in \mathrm{Poly}_n(\mathbb{F}_q^*)} \sum_{\substack{p\mid f\\ \deg(p)=2}} \psi(\mathrm{root}(p))\\ &= \sum_i (-1)^i \frac{\langle X_2^{\psi}, H^i(\mathrm{Conf}^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q}(\zeta_{q-1}))\rangle}{q^i} = -\frac{1}{q}+\frac{3}{q^2}+\cdots\end{aligned} \tag{1.4}$$

where root(p) denotes a root of p, an irreducible degree 2 factor of f, lying in $\mathbb{F}_{q^2}$; the value $\psi(\text{root}(p))$ turns out not to depend on the choice of root. Thus, (1.4) says that the average value of the Gauss sum obtained by applying ψ to the *quadratic* factors of f, across all $f \in \text{Poly}_n(\mathbb{F}_q^*)$, is equal to the series on the right obtained by looking at the inner product of the character polynomial X_2^ψ with $H^i(\text{Conf}^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*); \mathbb{Q}(\zeta_{q-1}))$.

Remark 1.2.5. Just as in [CEF14, §4.3], it is quite likely that one can compute the left-hand side of (1.2) using twisted L-functions of the form

$$L(P, s) = \sum_n \sum_{f \in \text{Poly}_n(\mathbb{F}_q^*)} P(f) q^{-ns}$$

or by other analytic methods. Zeev Rudnick sketched such an argument for the case of (1.3) in a private communication. However, Theorem 1.2.4 gives a topological interpretation to the left-hand side of (1.2). More to the point, the fact that representation stability holds for orbit configuration spaces is what suggests Theorem 1.2.4 in the first place. In fact, the bridge to topology provided by the Grothendieck-Lefschetz trace formula gives further motivation to study such L-functions. One can often go in the other direction, and prove representation stability by means of counting points over finite fields. Chen [Che] has done this for the usual configuration spaces, and it would be interesting to investigate this for orbit configuration spaces as well.

1.2.2 Point-counts for rational maps

In the specific case where $\mathbf{d} = (\overbrace{d, \ldots, d}^{m})$, $n = 1$, and $X = \mathbb{A}^1$, we have $\mathcal{Z}_1^{(d,\ldots,d)}(\mathbb{A}^1) = \text{Rat}_d^*(\mathbb{P}^{m-1}) = \text{Rat}_d^*(\mathbb{CP}^{m-1})$, the space of degree d, based rational maps $\mathbb{CP}^1 \to \mathbb{CP}^{m-1}$ with $f(\infty) = [1 : \cdots : 1]$. We therefore obtain the following result about rational maps over finite fields.

Theorem 1.2.6. *For any prime power q and any character polynomial P,*

$$\lim_{d\to\infty} q^{-md} \sum_{f\in \mathrm{Rat}_d^*(\mathbb{P}^{m-1})(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i(\widetilde{\mathrm{Rat}}_d^*(\mathbb{CP}^{m-1})), P\rangle\right)$$

Namely, both the limit on the left and the series on the right converge, and they converge to the same limit.

In general, the space $\widetilde{\mathrm{Rat}}_d^*(\mathbb{CP}^{m-1})$ is the complement of a linear subspace arrangement, one that is not well-behaved enough for us to say what the eigenvalues of Frobenius acting on its étale cohomology are. However, in the case $m = 2$ the space $\widetilde{\mathrm{Rat}}_d^*(\mathbb{CP}^1)$ is the complement of a hyperplane arrangement, and therefore by [BE97] the action of Frob_q on $H^i(\widetilde{\mathrm{Rat}}_d^*(\mathbb{CP}^1))$ is multiplication by q^{-i}. So we obtain

$$\lim_{d\to\infty} q^{-2d} \sum_{f\in \mathrm{Rat}_d^*(\mathbb{P}^1)(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty} (-1)^i \langle H^i(\widetilde{\mathrm{Rat}}_d^*(\mathbb{CP}^1)), P\rangle\, q^{-i}$$

As an example of Theorem 1.2.6, in the case $P = 1$, we obtain

$$\lim_{d\to\infty} q^{-md} \left|\mathrm{Rat}_d^*(\mathbb{P}^m)(\mathbb{F}_q)\right| = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : H^i(\mathrm{Rat}_d^*(\mathbb{P}^{m-1}))\right)$$

so that the number of such rational maps, as the degree goes to infinity, stabilizes to the series on the right. As another example, if $P = X_1^i$, then $X_1^i(f)$ counts the number (with multiplicity) of $\mathbb{F}_q$-rational intersection points of the image of f in $\mathbb{P}^m$ with the hyperplane $\{x_i = 0\}$. Thus,

$$\lim_{d\to\infty} q^{-md} \sum_{f\in \mathrm{Rat}_d^*(\mathbb{P}^m)(\mathbb{F}_q)} \#\{f^{-1}\{x_i = 0\}\} = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle X_1^i, H^i(\widetilde{\mathrm{Rat}}_d^*(\mathbb{P}^{m-1}))\rangle\right)$$

so that the *average* number of intersection points, across all such rational maps, stabilizes as the degree goes to infinity to the series on the right.

Remark 1.2.7 (Related work). After distributing the first version of this thesis, we learned of new work of Rolland-Wilson [RW] investigating the specific case of the type B/C hyperplane arrangement. This corresponds to the case $d = 2$ in Theorem 1.2.4. Their proof of the necessary convergence results are by completely different methods. Our proof of the general result Theorem 1.2.2.1 is by the same argument as Farb-Wolfson [FW]. Rolland-Wilson's result, [RW, Thm 3.8], is proven using a novel graph-theoretic argument. It would be very interesting to see if their method could be strengthened to prove the stronger polynomial bounds of our Theorem 1.2.2.2.

Much of what we prove in Chapter 3 was independently proven by Li-Yu [LY]. In particular, they obtain Theorem 2.2.5 and Theorem 2.2.6. Their results work in the greater generality of FI^m-modules over arbitrary Noetherian rings. Li-Yu also probe deeply into the homological algebra of FI^m-modules, and in particular study the "FI^m-homology" of FI^m-modules, following Church-Ellenberg's [CE] theory of FI-homology, which we do not explore at all in this thesis.

CHAPTER 2

FI_G-MODULES AND FI^M-MODULES

In this chapter, we define the categories FI_G and FI^m and study their modules in detail.

2.1 FI_G-modules and their properties

If G is a group, and R and S sets, define a *G-map* $(a, (g_i)) : R \to S$ to be a pair $a : R \to S$ and $(g_i) \in G^R$. If $(b, (h_j)) : S \to T$ is another G-map, their composition is $(b \circ a, (g_i \cdot h_{a(i)}))$. Let FI_G be the category with objects finite sets and morphisms G-maps with the function a injective. This is clearly equivalent to the full subcategory with objects the sets $[n] = \{1, \ldots, n\}$. Note that the automorphism group of $[n]$ is

$$W_n := G \wr S_n = G^n \rtimes S_n$$

An *FI_G-module over k* is just a functor $V : \mathrm{FI}_G \to k$-Mod; when k is clear, we simply write FI_G-modules. These form a category, called FI_G-Mod. Thus an FI_G-module V is a sequence of W_n-representations V_n, with maps $V_n \to V_{n+1}$ satisfying certain coherency conditions. Say that V is *finitely generated* if there is a finite set of elements $v_1, \ldots, v_n \in V$ such that the smallest FI_G-submodule containing the v_i is all of V.

For $m \geq 0$, define the "free" FI_G-module $M(m)$ by setting

$$M(m)_n = \begin{cases} 0 & n < m \\ k[\mathrm{Hom}_{\mathrm{FI}_G}([m], [n])] & n \geq m \end{cases}$$

Recall that for an FI_G-module V and an element $v \in V_m$, we can also characterize the submodule generated by v as the image of the map

$$M(m) \to V, \quad f \in \mathrm{Hom}_{\mathrm{FI}_G}([m], [n]) \mapsto f_* v$$

We can thus characterize finitely generated FI_G-modules as those V that admit a surjection $\bigoplus_{i=1}^N M(m_i) \twoheadrightarrow V$.

2.1.1 Noetherianity and representation stability

Recall that a group is *polycyclic* if it has a composition series with cyclic factors, and is *virtually polycyclic* if it has a polycyclic subgroup of finite index. Virtually polycyclic groups are of interest, among other things, because they are the only known groups to have Noetherian group rings, and are conjectured to be the only such groups. In [SSb, Cor 1.2.2], Sam-Snowden proved that if G is a virtually polycyclic group, then FI_G-Mod is Noetherian over any Noetherian ring k. That is, they proved that any finitely generated FI_G-module has all its submodules finitely generated. The crucial property used was that the group ring $k[G]$ is Noetherian.

For FI-modules, the most important consequence of finite generation is representation stability, and for G finite, Gan-Li proved in [GL15, Thm 1.10] that this holds for FI_G as well. Technically there are three parts to representation stability according to the definition given in [CF13], but the first two parts ("surjectivity" and "injectivity") follow straightforwardly from the definition of being finitely generated. It is the third part, "multiplicity stability", that is really the most interesting, and which we will now describe.

Let us briefly review the representation theory of a finite wreath product (e.g, [Mac95, Ch. I, Appendix A]). Take G finite and k a *splitting field* of characteristic 0 for G, that is, a field over which all its irreducible representations over $\mathbb{C}$ are defined. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ denote the set of isomorphism classes of irreducible representations of G. Let $\underline{\lambda}$ be a partition-valued function on $\mathrm{Irr}(G)$. Put $|\underline{\lambda}| = (|\underline{\lambda}(\chi_1)|, \ldots, |\underline{\lambda}(\chi_r)|)$, and $\|\underline{\lambda}\| = |\underline{\lambda}(\chi_1)| + \cdots + |\underline{\lambda}(\chi_r)|$. Also, let $E(\mu)$ denote the irreducible representation of $S_{|\mu|}$ corresponding to a partition μ. Then if $\|\underline{\lambda}\| = n$, there is an associated irreducible representation of W_n:

$$L(\underline{\lambda}) = \mathrm{Ind}_{W_{|\underline{\lambda}|}}^{W_n} \left(\chi_1^{\otimes \underline{\lambda}(\chi_1)} \otimes E(\underline{\lambda}(\chi_1))\right) \otimes \cdots \otimes \left(\chi_r^{\otimes \underline{\lambda}(\chi_r)} \otimes E(\underline{\lambda}(\chi_1))\right)$$

where $W_\mu = W_{\mu(1)} \times \cdots W_{\mu(l)}$, and these comprise all the irreducible representations of W_n up to isomorphism. Extend $\underline{\lambda}$ to $n \geq \|\lambda\| + \underline{\lambda}(\chi_0)_1$ as follows:

$$\underline{\lambda}[n](\chi) = \begin{cases} (n - \|\underline{\lambda}\|, \underline{\lambda}(\chi_0)) & \text{if } \chi = \chi_0 \\ \underline{\lambda}(\chi) & \text{otherwise} \end{cases}$$

Writing $L(\underline{\lambda})_n$ for the irreducible representation corresponding to $\underline{\lambda}[n]$, multiplicity stability for an FI_G-module V says that the decomposition into irreducibles has multiplicities independent of n for large n:

$$V_n = \bigoplus_{\underline{\lambda}} L(\underline{\lambda})_n^{\oplus c(\underline{\lambda})} \text{ for all } n \geq N$$

where we call N the *stability degree* of V. In particular, when G is trivial, we recover (uniform) multiplicity stability for FI-modules in the sense of [CF13, Defn 2.6], and when $G = \mathbb{Z}/2\mathbb{Z}$, we recover (uniform) multiplicity stability for $\mathrm{FI}_{\mathrm{BC}}$-modules in the sense of [Wil13, Defn 2.6].

2.1.2 Projective resolutions and character polynomials

For G finite, [SSb] and [GL15] actually obtain a deeper structural result that implies representation stability, and this result has other important consequences for us. To state it, define a *torsion* FI_G-module to be one with $V_n \neq 0$ for only finitely many n, and say that an FI_G-module is projective if it is a projective object in the category FI_G-Mod. Gan-Li's result can then be stated as follows.

Proposition 2.1.1 ([GL15, Thm 1.6]). *For any finitely generated* FI_G*-module* V*, with* G *finite, there is a finite resolution of* FI_G *modules*

$$0 \to V \to T^1 \oplus P^1 \to T^2 \oplus P^2 \to \cdots \to T^n \oplus P^k \to 0$$

with each T^i torsion and each P^i projective.

In particular, this resolution's existence means that for $n \gg 0$ there is a resolution of W_n-representations

$$0 \to V_n \to P_n^1 \to \cdots \to P_n^k \to 0 \tag{2.1}$$

This is powerful because, as in [CEF15], we have strong control over the structure of projective FI_G-modules. Namely, let $\mathrm{Res} : \mathrm{FI}_G\text{-Mod} \to W_i\text{-Mod}$ be the restriction to a single group. This functor has a left adjoint $\mathrm{Ind}^{\mathrm{FI}_G} : W_i\text{-Mod} \to \mathrm{FI}_G\text{-Mod}$ given by

$$\mathrm{Ind}^{\mathrm{FI}_G}(V)_{i+j} = \mathrm{Ind}_{W_i \times W_j}^{W_{i+j}} V \boxtimes k$$

Then $\mathrm{Ind}^{\mathrm{FI}_G}(V)$ is a projective FI_G-module whenever V is a projective W_i-module, and tom Dieck [tD87, Prop 11.18] proved that any projective FI_G-module is of this form (in fact, for a large class of category representations). Following Ramos [Rama], define a *relatively projective* module to be a direct sum of any such induced modules, even if the V's are not projective W_n-representations. Finitely-generated (relatively) projective FI_G-modules thus have a compact description, as direct sums of induced modules from a finite list of W_n-representations, even for infinite G.

In fact, Ramos [Rama] has found an effective bound for when n occurs in (2.1). To describe it, we need to refine our notion of finite generation. As we said, finitely generated modules are those that admit a surjection $\bigoplus_{i=1}^N M(d_i) \twoheadrightarrow V$. Say that such a V is *generated in degree* $\leq d = \max_i\{d_i\}$. Next, suppose there is an exact sequence

$$0 \to K \to M \to V \to 0$$

with M relatively projective and K generated in degree $\leq r$. Then say that V is *related in degree* $\leq r$.

Ramos [Rama, Thm C] then says that if V is a finitely-generated FI_G-module generated in degree d and related in degree r, then the above resolution (2.1) holds whenever $n \geq r + \min\{r, d\}$. Notice that if V is already relatively projective, then $r = 0$ and this is sharp.

In particular, when G is finite, the resolution (2.1) implies representation stability: as Gan-Li verify in [GL15, Thm 1.10], the individual projective modules $\mathrm{Ind}^{\mathrm{FI}_G}(V_m)$ satisfy representation stability, with stability degree $\leq 2m$. By semisimplicity of each $k[G_n]$, V therefore satisfies representation stability as well. Furthermore, this resolution provides a quick proof of the stabilization of character polynomials for FI_G-modules, as follows.

Recall that a *character polynomial* for S_n is an element of $\mathbb{Q}[X_1, X_2, \ldots]$, which we think of as a class function on S_n, where X_i counts the number of i-cycles of a permutation. Church-Ellenberg-Farb [CEF15, Thm 3.3.4] prove that any finitely generated FI-module is eventually given by a single character polynomial. We generalize this and Wilson's [Wil13, Thm 5.15] result for $G = \mathbb{Z}/2$ as follows.

Theorem 2.1.2 (Character polynomials for FI_G). *Let G be a finite group and k a splitting field for G of characteristic 0. If V is a finitely generated FI_G-module over k, generated in degree m and related in degree r, then there is a character polynomial*

$$P_V \in k\left[\{X_i^C \mid i \geq 1, C \textit{ is a conjugacy class of } G\}\right]$$

of degree $\leq m$, so that for all $n \geq r + \min(m, r)$

$$\chi_{V_n}(g) = P_V(g).$$

Proof. If V is a representation of W_m, we can explicitly compute the character of the projective FI_G-module $\mathrm{Ind}^{\mathrm{FI}_G}(V)$. This calculation is done in [Wil13, Lem 5.14] for $G = \mathbb{Z}/2$, and her proof applies essentially verbatim, but enough small details are different that it is easier to just give the adapted proof than to describe the necessary changes.

The character of the induced representation $\mathrm{Ind}^{\mathrm{FI}_G}(V)_n = \mathrm{Ind}_{W_m \times W_{n-m}}^{W_n} V \boxtimes k$ is

$$\begin{aligned}
\chi_{\mathrm{Ind}^{\mathrm{FI}_G}(V)_n}(w) &= \sum_{\substack{\{\text{cosets } C \,|\, w \cdot C = C\} \\ \text{any } s \in C}} \chi_{V \boxtimes k}(s^{-1} w s) \\
&= \sum_{\substack{\{\text{cosets } C \,|\, w \cdot C = C\} \\ \text{any } s \in C}} \chi_V(p_m(s^{-1} w s))
\end{aligned}$$

where $p_m : W_m \times W_{n-m} \to W_m$ is the projection, and where the sum is over all cosets in $W_n/(W_m \times W_{n-m})$ that are stabilized by w, equivalently, those cosets C such that $s^{-1}ws \in W_m \times W_{n-m}$ for any $s \in C$.

An element $w \in W_n$ can be conjugated in $W_m \times W_{n-m}$ precisely when its $c(G)$-labeled cycles can be split into a set of cycles of total length m, and a set of cycles of total length $(n-m)$. If we fix a labeled partition $\underline{\lambda}$ of m, then the cycles of w can be factored into an element w_m of labeled cycle type $\underline{\lambda}$ and its complement w_{n-m} in the following number of ways (possibly 0):

$$\prod_{C \in c(G)} \binom{X_1^C}{n_1(\underline{\lambda}(C))} \binom{X_2^C}{n_2(\underline{\lambda}(C))} \cdots \binom{X_m^C}{n_m(\underline{\lambda}(C))}$$

where $n_r(\mu)$ is the number of r's in μ. Each such factorization of w corresponds to a coset $C \in W_n/(W_m \times W_{n-m})$ that is stabilized by w. For any representative $s \in C$, $p_m(s^{-1}ws)$ has labeled cycle type $\underline{\lambda}$. So we conclude that

$$\chi_{\mathrm{Ind}^{\mathrm{FI}_G}(V)_n} = \sum_{\|\underline{\lambda}\| = m} \chi_V(\underline{\lambda}) \prod_C \binom{X_1^C}{n_1(\underline{\lambda}(C))} \binom{X_2^C}{n_2(\underline{\lambda}(C))} \cdots \binom{X_m^C}{n_m(\underline{\lambda}(C))}$$

where the left-hand-side is manifestly a fixed character polynomial independent of n. Notice that this character polynomial has degree m. The general result therefore again follows from (2.1) and Ramos [Rama, Thm C] by semisimplicity. $\square$

In particular, by taking $g = e$ in Theorem 2.1.2, we see that $\dim V_n$ is given by a single polynomial for $n \geq r + \min(m, r)$, which is [Rama, Thm D].

2.1.3 Arbitrary G and finite presentation degree

Until recently, the Noetherian property was seen as the lynchpin of the theory of FI-modules and related categories. However, coming out of the work of Church-Ellenberg [CE] on homological properties of FI-modules, a new perspective has emerged that shifts the emphasis from finite generation of modules to finite *presentation degree* of modules, e.g. in the work of Ramos [Ramc] and Li [Li]. In one sense, this perspective is more "constructive", because possessing the knowledge of both the degree of generation *and* the degree of relation of a module gives us quantitative control over various stability properties of the module, as we have seen. The Noetherian property then tells us that *any* finitely generated module is necessarily finitely presented, which is an important fact but no longer at the absolute center of the theory. Appealing to Noetherianity is also necessarily ineffective, since we are no longer able to say what the relation degree is, and thus lose effective bounds on stability.

At the same time, this shift in perspective allows us to expand our scope to situations where there is no hope of finite generation. For example, we will see later on examples of FI_G-modules V that are not finitely generated simply because the individual pieces V_n are not finitely-generated W_n-representations. Nevertheless, we will be able to prove that V is still generated in finite degree, in the sense that all the generators of V (an infinite number) live only in $V_1, \ldots V_m$ for some m, *and* that V is related in finite degree. Of course, in a sense such V are much *less* constructive then anything in the finitely-generated world.

This shift also allows us to leave behind the requirement that G be virtually polycyclic. Indeed, as we have seen, this requirement is based in the fact that, for any kind of Noetherianity to get off the ground, we certainly need the ring $k[G]$ to be Noetherian, which as far as we know is only true when G is virtually polycyclic. However, once we have accepted that modules can be infinitely generated, and only care about the *degree* that that they are generated (and related) in, we can leave this need behind.

The central result that informs this perspective is the following, proved simultaneously by Ramos and Li:

Theorem 2.1.3 ([Ramc, Thm B], [Li, Prop 3.4]). *For any group G, the category of* FI_G-*modules presented in finite degree is abelian.*

This is the analogue of Noetherianity for finite presentation degree, since it allows us to argue that kernels of maps between FI_G-modules presented in finite degree are still presented in finite degree, and therefore for example to chase being presented in finite degree through a spectral sequence. Indeed, from a certain point of view Theorem 2.1.3 is the fundamental fact, and Noetherianity as we have seen it so far is just a consequence of Theorem 2.1.3 and the fact that the individual group rings $k[G]$ are Noetherian.

What we lose at this level of generality is any ability to refer back to stability results in terms of things like "representation stability" or "stability of character polynomials" that are not couched explicitly in terms of FI_G-modules. All we can say is that the FI_G-modules in question are presented in finite degree, which perhaps is less interesting to someone who only cares about the individual W_n-representations V_n. At the same time, since these are infinitely-generated representations of infinite groups, it is hard to say much about the individual representations.

2.1.4 $\mathrm{FI}_G\,\sharp$-modules

The classification of projective modules provided above means that even when G is infinite, if an FI_G module V is projective, then there is still a compact description of the representation theory of V. We would therefore like to be able to determine when an FI_G-module is projective, so that it has such a description. [CEF15] provide just such a method: they define a category $\mathrm{FI}\,\sharp$ with an embedding $\mathrm{FI} \hookrightarrow \mathrm{FI}\,\sharp$ so that $\mathrm{FI}\,\sharp$-Mod is precisely the category of projective FI-modules, so a module is projective just when it extends to an $\mathrm{FI}\,\sharp$-module. Their construction and proof of equivalence carry over to the setting of FI_G, as Wilson [Wil13] proved for the case $G = \mathbb{Z}/2$.

So define $\mathrm{FI}_G\sharp$ to be the category of *partial morphisms* of FI_G: the objects are still finite sets, but a map $X \to Y$ is given by a pair (Z, f), where $Z \subset X$ and $f : Z \to Y$ is a G-map. Composition of morphisms is defined by pullback, i.e. with the domain the largest set on which the composition is defined. Then there is a natural structure of $\mathrm{Ind}^{\mathrm{FI}_G}(V)$ as an $\mathrm{FI}_G\sharp$-module, as follows.

First we define this structure for $M(m) = k[\mathrm{Hom}_{\mathrm{FI}_G}(\mathbf{m}, \bullet)]$. Let $(Z, f) : X \to Y$ be an $\mathrm{FI}_G\sharp$ morphism, where $Z \subset X$ and $f : Z \to Y$ is a G-map. Then for $g \in \mathrm{Hom}_{\mathrm{FI}_G}([m], X)$ we put

$$(Z, f) \cdot g = \begin{cases} f \circ g & \text{if } \operatorname{im} g \subset Z \\ 0 & \text{otherwise} \end{cases} \in k[\mathrm{Hom}_{\mathrm{FI}_G}([m], Y)]$$

and extend by linearity to all of $M(m)$. Next, we note that if V is a representation of W_m, then $\mathrm{Ind}^{\mathrm{FI}_G}(V) = M(m) \otimes_{W_m} V$. Since we have just defined an action of $\mathrm{FI}_G\sharp$ on $M(m)$, this gives an action of $\mathrm{FI}_G\sharp$ on $\mathrm{Ind}^{\mathrm{FI}_G}(V)$.

Proposition 2.1.4. *Any* $\mathrm{FI}_G\sharp$ *module is isomorphic to* $\bigoplus_i \mathrm{Ind}^{\mathrm{FI}_G}(W_i)$ *for some representations* W_i *of* G_i*.*

Proof. This was proved for G trivial in [CEF15, Thm 4.1.5], and for $G = \mathbb{Z}/2$ in [Wil13, Thm 4.42]. As Wilson explains, the proof in [CEF15] applies almost verbatim: the only change that needs to be made is to the definition of the endomorphism $E : V \to V$, which should be defined as follows, for $m \geq n$,

$$E_m : V_m \to V_m$$

$$E_m = \sum_{\substack{S \subset [m] \\ |S| = n}} I_S, \quad \text{where } I_S = (S, \iota) \in \mathrm{Hom}_{\mathrm{FI}_G\sharp}([m], [m]) \text{ with } \iota : S \hookrightarrow [m] \text{ the inclusion.}$$

□

Corollary 2.1.5. *If* V *is an* $\mathrm{FI}_G\sharp$*-module generated in degree* m*, then* χ_V *is given by a single character polynomial of degree* $\leq m$*, and satisfies representation stability with stability* *degree* $\leq 2m$.

Proof. This follows from [GL15, Thm 1.10] and Theorem 2.1.2. $\square$

2.1.5 Tensor products and FI_G-algebras

Here we proceed to generalize the notions introduced in [CEF15, §4.2] from FI to FI_G.

Given FI_G-modules V and V', their tensor product $V \otimes V'$ is the FI_G-module with $(V \otimes V')_n = V_n \otimes V'_n$, where FI_G acts diagonally.

A *graded* FI_G*-module* is a functor from FI_G to graded modules, so that each piece is graded, and the induced maps respect the grading. If V is graded, each graded piece V^i is thus an FI_G-module. If V and W are graded, the tensor product $V \otimes W$ is graded in the usual way. Say that V is of *finitely-generated type* if each V^i is finitely generated. Say that V is of *finite type* if it is of finitely-generated type and furthermore each V^i_n is finite-dimensional. Notice if G is a finite group, these two notions coincide.

Similarly, an FI_G-algebra is a functor from FI_G to k-algebras, which can also be graded. We can also define graded co-FI_G-modules and algebras, as functors from $\mathrm{FI}_G^{\mathrm{op}}$. A (co-)$\mathrm{FI}_G$-algebra A is *generated (as an* FI_G*-algebra)* by a submodule V when each A_n is generated as an algebra by V_n.

Finally, there is another type of tensor product that we will need. Suppose V is graded G-module with $V^0 = k$. Then the space $V^{\otimes \bullet}$ defined by $(V^{\otimes \bullet})_n = V^{\otimes n}$ has the structure of an $\mathrm{FI}_G\,\sharp$-module, as in [CEF15, Defn 4.2.5], with the morphisms permuting and acting on the tensor factors.

The following theorem characterizes the above constructions.

Theorem 2.1.6. *Let G be any group.*

1. *Let V and V' be* FI_G*-modules generated in degree m and m'. Then $V \otimes V'$ is generated in degree $m + m'$. If V is finitely generated and V' is finite type, then $V \otimes V'$ is finitely generated.*

2. *Let A be a graded (co-)FI_G-algebra generated by a graded submodule V, where $V^0 = 0$ and V is generated as an FI_G-module in degree m. Then the i-th grades piece A^i is generated in degree $m \cdot i$. If V is of finite type, then A is of finite type.*

3. *If V is a graded G-module with $V^0 = k$, then $V^{\otimes \bullet}$ is an $\mathrm{FI}_G\sharp$-module whose i-th graded piece is generated in degree i. If V is of finite type, then $V^{\otimes \bullet}$ is of finite type.*

4. *If X is a connected G-space, then $H^*(X^\bullet; k)$ is an $\mathrm{FI}_G\sharp$-algebra whose i-th graded piece is generated in degree i. If $H^*(X; k)$ is of finite type, then $H^*(X^\bullet; k)$ is of finite type.*

Proof.

1. This was proved by Sam-Snowden [SSb, Prop 3.1.6] for G finite, and their proof applies verbatim even when G is infinite to show that $V \otimes V'$ is always generated in degree $m + m'$, though not always finitely.

   For the second part, for each $k \leq m + m'$, we know $(V \otimes V')_k = V_k \otimes V'_k$ is a finitely-generated W_k-module, since the tensor product of a finitely-generated W_k-module and a finite-dimensional module is finitely generated. The result follows.

2. If A is generated as an algebra by V, then A is a quotient of the free algebra $k\langle V \rangle$, so A^i is a quotient of $k\langle V \rangle^i = V^{\otimes i}$. By (1), $V^{\otimes i}$ is generated in degree $m \cdot i$, and therefore A^i is generated in degree $m \cdot i$. The second part follows from the second part of (1).

3. This was proved for G trivial in [CEF15, Prop 4.2.7], but here we have simplified the assumptions, by having V just be a single G-module rather than a whole FI_G-module, and so the proof is simpler. To wit, the i-th graded piece is

$$(V^{\otimes n})^i = \bigoplus_{k_1 + \cdots + k_n = i} V^{k_1} \otimes \cdots \otimes V^{k_n}$$

At most i of the nonnegative integers $k_1, \ldots, k_n$ can be nonzero. Let $k_{j_1}, \ldots, k_{j_l}$ be the subsequence of nonzero integers, so $l \leq i$. Then for any pure tensor

$$v = v_1 \otimes \cdots \otimes v_n \in V^{k_1} \otimes \cdots \otimes V^{k_n},$$

since $V^{k_m} = V^0 = k$ for $m \notin \{j_1, \ldots, j_l\}$, we can take

$$w = \left(\prod_{m \notin \{j_1, \ldots\}} v_{k_m} \right) \cdot v_{k_{j_1}} \otimes \cdots v_{k_{j_l}}$$

and then the inclusion $j : [l] \hookrightarrow [n]$ clearly induces $j_* w = v$. So by linearity, $(V^{\otimes n})^i$ is generated in degree i.

If each V^i is finite-dimensional, then each $V^{k_1} \otimes \cdots \otimes V^{k_l}$ is finite-dimensional for $l = 1, \ldots, i$. Therefore $(V^{\otimes n})^i$ is an FI_G-module of finite type.

4. This was proved for G trivial in [CEF15, Prop 6.1.2], and their proof applies verbatim. It essentially follows from (3) and the Künneth formula. As [CEF15] explain, technically sometimes a sign is introduced in permuting the order of tensor factors, but this does not change the proof of (3). The assumption that $V^0 = k$ holds by connectivity of X.

$\square$

2.2 FI^m-modules and their properties

FI is the category introduced by Church-Ellenberg-Farb [CEF15] whose objects are finite sets and whose morphisms are injections. Here, we consider the m-fold product category FI^m for some fixed m. Thus, FI^m has as objects m-tuples of finite sets $(S_1, \ldots, S_m)$, and morphisms $f : S \to T$ given by tuples $(f_1, \ldots, f_m)$ of injections $f_i : S_i \hookrightarrow T_i$. This clearly has a skeleton with objects indexed by tuples $(c_1, \ldots, c_m)$ of natural numbers, and morphisms $\mathbf{c} \to \mathbf{d}$ given by tuples $(f_1, \ldots, f_m)$ of injections $f_i : [c_i] \hookrightarrow [d_i]$, where $[n] = \{1, \ldots, n\}$. We can define a partial order on such $\mathbf{d}$ by saying that $\mathbf{c} \leq \mathbf{d}$ if each $c_i \leq d_i$, and then $\mathrm{Hom}(\mathbf{c}, \mathbf{d}) \neq \emptyset$ just when $\mathbf{c} \leq \mathbf{d}$. We see that $\mathrm{End}(\mathbf{d}) = \mathrm{Aut}(\mathbf{d}) = S_{\mathbf{d}} = S_{d_1} \times \cdots \times S_{d_m}$.

An FI^m-module is just a functor $\mathrm{FI}^m \to k$-Mod, where k is some ring, which we will always take here to be a field of characteristic 0. If V is an FI^m-module, then each $V_{\mathbf{d}}$ is an $S_{\mathbf{d}}$-representation. FI^m-modules form an abelian category, with maps given by natural transformations of functors. If V is an FI^m-module and $v_1, \ldots v_m \in V$, the submodule generated by the v_i's is the smallest submodule that contains them. V is finitely generated if it is generated by a finite subset. For any $\mathbf{c}$, we define the "free" module $M(\mathbf{c}) = k[\mathrm{Hom}_{\mathrm{FI}^m}(\mathbf{c}, -)]$. Thus for any $\mathbf{d} \geq \mathbf{c}$, we have $M(\mathbf{c})_{\mathbf{d}} = k[\mathrm{Hom}_{\mathrm{FI}^m}(\mathbf{c}, \mathbf{d})]$. Furthermore, if V is an FI^m-module and $v \in V_{\mathbf{d}}$, there is a natural map $M(\mathbf{d}) \to V$ taking f to $f_* v$, whose image is the submodule of V generated by v. Thus, V is finitely generated just when there is a surjection $\bigoplus_i M(\mathbf{d}_i) \twoheadrightarrow V$.

Gadish [Gada] proved the Noetherian property for FI^m-modules:

Theorem 2.2.1 ([Gada, Prop 6.3])**.** *If V is a finitely generated FI^m-module, then any submodule is finitely generated.*

2.2.1 Projective FI^m-modules and $\mathrm{FI}\sharp^m$

Let $\mathrm{Res} : \mathrm{FI}^n \to S_{\mathbf{d}}$ be the restriction to a single group. This functor has a left adjoint $\mathrm{Ind}^{\mathrm{FI}^n} : S_{\mathbf{d}} \to \mathrm{FI}^n$ given by

$$\mathrm{Ind}^{\mathrm{FI}^m}(V)_{\mathbf{d}+\mathbf{c}} = \mathrm{Ind}_{S_{\mathbf{d}} \times S_{\mathbf{c}}}^{S_{\mathbf{d}+\mathbf{c}}} V \boxtimes k$$

Then $\mathrm{Ind}^{\mathrm{FI}^n}(V)$ is a projective FI^n-module, and tom Dieck [tD87, Prop 11.18] proved that any projective FI^n-module is of this form (in fact, for a large class of category representations). Furthermore, any $S_{\mathbf{d}}$-representation V is a direct sum of external tensor products of

S_{d_i}-representations, and we have

$$\begin{aligned}
\operatorname{Ind}^{\mathrm{FI}^m}(V_1 \boxtimes \cdots \boxtimes V_m)_{\mathbf{d}+\mathbf{c}} &= \operatorname{Ind}_{S_{d_1}\times\cdots\times S_{d_m}\times S_{c_1}\times\cdots\times S_{c_m}}^{S_{d_1+c_1}\times\cdots\times S_{d_m+c_m}} (V_1 \boxtimes \cdots \boxtimes V_m) \boxtimes (k \boxtimes \cdots \boxtimes k) \\
&= (\operatorname{Ind}_{S_{d_1}\times S_{c_1}}^{S_{d_1+c_1}} V_1 \boxtimes k) \boxtimes (\operatorname{Ind}_{S_{d_m}\times S_{c_m}}^{S_{d_m+c_m}} V_m \boxtimes k) \\
&= \Big(\operatorname{Ind}^{\mathrm{FI}}(V_1) \boxtimes \cdots \boxtimes \operatorname{Ind}^{\mathrm{FI}}(V_m)\Big)_{\mathbf{d}+\mathbf{c}}
\end{aligned} \tag{2.2}$$

so that any projective FI^m-module is a direct sum of tensor product of projective FI^m-modules.

[CEF15] give a convenient way of determining when an FI-module is projective: they defined a category $\mathrm{FI}\,\sharp$ in which FI embeds, such that $\mathrm{FI}\,\sharp$-modules are exactly the projective FI-modules. Thus a given FI-module is a direct sum of $\operatorname{Ind}^{\mathrm{FI}}(W)$'s just when it extends to an $\mathrm{FI}\,\sharp$-module.

The same construction works for FI^m: we just take $\mathrm{FI}\,\sharp^m$. One description of $\mathrm{FI}\,\sharp^m$ is as the category whose objects are those of FI^m, and whose morphisms $\mathbf{x} \to \mathbf{y}$ are given by pairs $(\mathbf{z}, f)$, where $\mathbf{z} \subset \mathbf{x}$ and $f : \mathbf{z} \hookrightarrow \mathbf{y}$. Thus, it is the category of "partial morphisms" of FI^m. We then have the following.

Theorem 2.2.2. *Every* $\mathrm{FI}\,\sharp^m$*-module is isomorphic to* $\bigoplus_{\mathbf{d}} \operatorname{Ind}^{\mathrm{FI}^m}(W_{\mathbf{d}})$ *for some representations* $W_{\mathbf{d}}$ *of* $S_{\mathbf{d}}$*.*

Proof. [CEF15] prove that $\mathrm{FI}\,\sharp$-modules are precisely the direct sums of $\operatorname{Ind}^{\mathrm{FI}}(W)$'s, and thus that $\mathrm{FI}\,\sharp$-Mod is semisimple. This implies that $(\mathrm{FI}\,\sharp \times \mathrm{FI}\,\sharp)$-Mod is semisimple, and by induction that $\mathrm{FI}\,\sharp^m$-Mod is semisimple, and that its simples are just external tensor products of the simples of $\mathrm{FI}\,\sharp$-Mod, which as we said are just the $\operatorname{Ind}^{\mathrm{FI}}(W)$. The claim follows by (1).

□

So $\mathrm{FI}\,\sharp^m$-modules are always sums of tensor products of $\mathrm{FI}\,\sharp$-modules. Note, however, that this is not true for general FI^m-modules. Indeed, take the FI^2-module M with

$$M_{a,b} = \begin{cases} 0 & \text{if } (a,b) \in \{(0,0),(1,0),(0,1)\} \\ k & \text{otherwise} \end{cases}$$

and where all morphisms of FI^2 induce the identity $k \to k$ or the unique map $0 \to k$. Then M is not the direct sum of external tensor products of FI-modules.

First, we know M cannot be decomposed as the direct sum of two nonzero FI^2-modules. Indeed, if we could write $M = V \oplus V'$, then without loss of generality there would be nonzero elements $v \in V_{\mathbf{c}}$, $v' \in V'_{\mathbf{d}}$ with $\mathbf{c} \leq \mathbf{d}$, and thus no morphism of FI^2 could have f^*v be a nonzero multiple of v'. But for any nonzero $v \in M_{\mathbf{c}}, v' \in M_{\mathbf{d}}$ with $\mathbf{c} \leq \mathbf{d}$, we have that f^*v is always a nonzero multiple of v' for any $f \in \mathrm{Hom}_{\mathrm{FI}^2}(\mathbf{c}, \mathbf{d})$.

But then if we had $M = V \boxtimes W$, we would have $V_1 \boxtimes W_0 = 0, V_0 \boxtimes W_1 = 0, V_1 \boxtimes W_1 = k$, which is impossible.

Finally, (1) lets us compute the character of an $\mathrm{FI}\sharp^m$-module. First, define a *character polynomial for* FI^m to be a polynomial in $k[X_1^{(1)}, \dots X_1^{(m)}, X_2^{(1)}, \dots]$, where $X_i^{(k)}$ is the class function on $S_{\mathbf{d}}$ that counts the number of i-cycles in S_{d_k}. We then have the following.

Proposition 2.2.3. *If V is a finitely generated $\mathrm{FI}\sharp^m$-module, then χ_{V_n} is given by a single character polynomial P for all n.*

Proof. If $V = \mathrm{Ind}^{\mathrm{FI}^m}(V_1 \boxtimes \cdots \boxtimes V_m)$, then by (1), $V = \mathrm{Ind}^{\mathrm{FI}}(V_1) \boxtimes \cdots \boxtimes \mathrm{Ind}^{\mathrm{FI}}(V_m)$. By [CEF15, Thm 4.1.7], the character of $\mathrm{Ind}^{\mathrm{FI}}(V_i)_n$ is given by a single character polynomial $P_i \in k[X_1^{(i)}, X_2^{(i)}, \dots]$ for all m. Then $\chi_V = P_1 \cdots P_m$. By Theorem 2.2.2, a general $\mathrm{FI}\sharp^m$-module is a direct sum of such V's, so the claim follows. $\square$

2.2.2 Shift functors and representation stability

Another basic operation on FI^m-modules are the shift functors. For $\mathbf{a} \in \mathrm{FI}^m$, let

$$S_{+\mathbf{a}} : \mathrm{FI}^m\text{-Mod} \to \mathrm{FI}^m\text{-Mod}$$

be the functor defined by $S_{+\mathbf{a}}(V)_{\mathbf{d}} = V_{\mathbf{d}+\mathbf{a}}$. Following [CEFN14] and [Nag15], we will use this functor to establish representation stability for FI^m.

Notice that we have $S_{+\mathbf{a}} \circ S_{+\mathbf{b}} = S_{+\mathbf{b}} \circ S_{+\mathbf{a}} = S_{+(\mathbf{a}+\mathbf{b})}$. In particular, if we decompose $\mathbf{a}$ into "unit vectors" as $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$, where $\mathbf{e}_i = (0, \dots, 1_i, \dots, 0)$, then $S_{\mathbf{a}} = (S_{+\mathbf{e}_n})^{a_n} \circ \cdots \circ (S_{+\mathbf{e}_1})^{a_1}$. The following fundamental proposition describes the effect of shift functors on the "free" modules $M(\mathbf{d})$, generalizing [CEFN14, Prop 2.12]:

Proposition 2.2.4. *For any $\mathbf{a}, \mathbf{d} \in \mathrm{FI}^m$, there is a natural decomposition*

$$S_{+\mathbf{a}}M(\mathbf{d}) = M(\mathbf{d}) \oplus Q_a$$

where Q_a is a free FI^m-module generated in degree $\leq d - 1$.

Proof. It is enough to prove this for the case $\mathbf{a} = \mathbf{e}_i$, since for a general $\mathbf{a}$, we know that $S_{+\mathbf{a}}$ is a composition of the $S_{+\mathbf{e}_i}$'s. A basis for $S_{+\mathbf{e}_i}M(\mathbf{d})_{\mathbf{c}}$ is the set of tuples of injections $\mathbf{f}$, where

$$\begin{gathered} f_1 : [d_1] \hookrightarrow [c_1] \\ \vdots \\ f_i : [d_i] \hookrightarrow [c_1] \sqcup \{\star\} \\ \vdots \\ f_m : [d_m] \hookrightarrow [c_m] \end{gathered}$$

This set can be partitioned into d_i+1 subsets, according to $f_i^{-1}(\star)$—that is, by which element of $[d_i]$ (or possibly none) gets mapped by f_i to $\star$. Notice that $f_i^{-1}(\star)$ is not affected by post-composing with an FI-morphism. Thus this partition actually defines a decomposition of $S_{+\mathbf{e}_i}M(\mathbf{d})$ as a direct sum of FI^m-modules.

For $T \subset [d_i]$ of size at most 1, let M^T be the submodule of $S_{+\mathbf{e}_i}M(\mathbf{d})$ spanned by those $\mathbf{f}$ with $f_i^{-1}(\star) = T$. These $\mathbf{f}$ are distinguished by the restrictions $f|_{\mathbf{d}-T}$, and we have $(g_* f)|_{\mathbf{d}-T} = g \circ f|_{\mathbf{d}-T}$. We therefore have $M^{\emptyset} \cong M(\mathbf{d})$, and

$$M^{\{t\}} \cong M(\mathbf{d} - \mathbf{e}_i) = M(d_1, \ldots, d_i - 1, \ldots, d_m).$$

So we have a decomposition

$$S_{+\mathbf{e}_i}M(\mathbf{d}) = M^{\emptyset} \oplus \bigoplus_{t \in [d_i]} M^{\{t\}} = M(\mathbf{d}) \oplus \bigoplus_{t \in [d_i]} M(\mathbf{d} - \mathbf{e}_i).$$

$\square$

Following [Nag15], say that a finitely generated FI^m-module V is *filtered* if it admits a surjection

$$\Pi : \bigoplus_{i=1}^{g} M(\mathbf{d}_i) \twoheadrightarrow V$$

such that the filtration $0 = V^0 \subset V^1 \subset \cdots \subset V^g = V$ given by

$$V^r := \Pi\left(\bigoplus_{i=1}^{r} M(\mathbf{d}_i)\right),\ 0 \le r \le d$$

has successive quotients V^r/V^{r-1} which are projective FI^m-modules.

Theorem 2.2.5. *For any finitely generated* FI^m*-module* V*, there is some* $\mathbf{a} \in FI^m$ *such that* $S_{+\mathbf{a}}V$ *is filtered.*

Furthermore, there are filtered FI^m*-modules* $J^0, \ldots, J^N$ *and a sequence*

$$0 \to V \to J^0 \to \cdots \to J^N \to 0$$

which is exact in high enough degree. That is, the sequence

$$0 \to V_{\mathbf{d}} \to J^0_{\mathbf{d}} \to \cdots \to J^N_{\mathbf{d}} \to 0$$

is exact for sufficiently large $\mathbf{d}$*.*

Our proof follows the one given by Nagpal [Nag15, Thm A] and Ramos [Ramb, Thm 3.1] for the case $m = 1$. As we mentioned in the introduction, Theorem 2.2.5 was independently proven by Li-Yu [LY, Thm 1.5, Thm 4.10].

Proof. Let V be an FI^m-module generated in degree $\mathbf{D}$ and related in degree $\mathbf{r}$. This means there is an exact sequence

$$0 \to K \to \bigoplus_{i=1}^{g} M(\mathbf{d}_i) \to V \to 0$$

where each $\mathbf{d}_i \leq \mathbf{D}$. We put $\widetilde{V} := \bigoplus_{i=1}^{g} M(\mathbf{d}_i)$. Notice that by Proposition 2.2.3, we may write

$$S_{+\mathbf{a}}\widetilde{V} = \bigoplus_{i=1}^{g} M(\mathbf{d}_i) \oplus Q_i^{\mathbf{a}}$$

where each $Q_i^{\mathbf{a}}$ is free and generated in degree $< \mathbf{d}_i$. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{U}^{\mathbf{a}} \arrow[r] \arrow[d, "\Pi^{\mathbf{a}}"'] & S_{+\mathbf{a}}\widetilde{V} \arrow[r] \arrow[d, "S_{+\mathbf{a}}\Pi"'] & M(\mathbf{d}_i) \arrow[r] \arrow[d, "\phi^{\mathbf{a}}"'] & 0 \\
0 \arrow[r] & U^{\mathbf{a}} \arrow[r] \arrow[d] & S_{+\mathbf{a}}V \arrow[r] \arrow[d] & A^{\mathbf{a}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Looking at the FI^m-module $K^{\mathbf{a}} = \ker(\phi^{\mathbf{a}})$, we observe that the $S_{\mathbf{d}_i}$-modules $K^{\mathbf{a}}_{\mathbf{d}_i}$ are increasing in $\mathbf{a}$, and therefore must stabilize for large $\mathbf{a}$. In fact, we can take $\mathbf{a} = \mathbf{r}$.

Fixing such an $\mathbf{a}$, it must be the case that $K^{\mathbf{a}}$ is generated in degree $\leq \mathbf{d}_i$. By exactness of $\mathrm{Ind}^{\mathrm{FI}^m}$, it follows that $A^{\mathbf{a}} = \mathrm{Ind}^{\mathrm{FI}^m}(W)$ for some $S_{\mathbf{d}_i}$-module W. So we are left with the exact sequence

$$0 \to U^{\mathbf{a}} \to S_{+\mathbf{a}}V \to \mathrm{Ind}^{\mathrm{FI}^m}(W) \to 0$$

By induction on degree, $S_{+\mathbf{b}}U^{\mathbf{a}}$ is filtered for sufficiently large $\mathbf{b}$. Since shifting is exact, we obtain

$$0 \to S_{+\mathbf{b}}U^{\mathbf{a}} \to S_{+(\mathbf{a}+\mathbf{b})}V \to S_{+\mathbf{b}}\,\mathrm{Ind}^{\mathrm{FI}^m}(W) \to 0$$

We conclude that $S_{+(\mathbf{a}+\mathbf{b})}V$ must be filtered. This completes the first part of the theorem.

For the second part, let $\mathbf{a}$ be large enough so that $S_{+\mathbf{a}}V$ is filtered. Continuing the notation of the first part, we have

$$S_{+\mathbf{a}}\widetilde{V} = \bigoplus_i M(\mathbf{d}_i) \oplus \widetilde{Q}$$

where $\widetilde{Q}$ is generated in degree $< D$. We thus have an exact sequence

$$0 \to V \to S_{+\mathbf{a}}V \to Q \to 0$$

where Q is generated in degree $< D$. By induction, the claim is true for Q, say with filtered modules $K^0, \ldots, K^M$. If we form the sequence

$$0 \to V \to S_{+\mathbf{a}}V \to K^0 \to \cdots K^M \to 0$$

the claim then follows. $\square$

Recall that the irreducible representations of $S_{\mathbf{d}}$ are just given by tensor products of irreducible representations of each S_{d_i}, which are indexed by partitions of d_i. If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is a list of partitions of $\mathbf{d} = (d_1, \ldots, d_m)$, then write $\mathrm{Irr}(\boldsymbol{\lambda}) = \mathrm{Irr}(\lambda_1) \boxtimes \cdots \boxtimes \mathrm{Irr}(\lambda_m)$ for the irreducible representation indexed by $\boldsymbol{\lambda}$. Extend $\boldsymbol{\lambda}$ to $\mathbf{c} \geq \mathbf{d} + (\lambda_1^{(1)}, \ldots, \lambda_m^{(1)})$ as follows:

$$\boldsymbol{\lambda}[\mathbf{c}] = ((c_1 - |\lambda_1|, \lambda_1), \ldots, (c_m - |\lambda_m|, \lambda_m))$$

Then we obtain the following.

Theorem 2.2.6 (**Representation stability for** FI^m**).** *Let V be a finitely-generated FI^m-module. Then there is a character polynomial P such that for all $\mathbf{d} \gg 0$, the character $\chi_{V_\mathbf{d}} = P_\mathbf{d}$. In particular, the dimension $\dim V_\mathbf{d}$ is eventually given by a polynomial in the d_i's. Furthermore, the decomposition into irreducibles has multiplicities independent of $\mathbf{d}$ for $\mathbf{d}$ large:*

$$V_\mathbf{d} = \bigoplus_{\boldsymbol{\lambda}} \mathrm{Irr}(\boldsymbol{\lambda}[\mathbf{d}])^{c_{\boldsymbol{\lambda}}} \textit{ for all } \mathbf{d} \gg 0$$

As we mentioned in the introduction, the claim about multiplicity stability was independently proven by Li-Yu [LY, Thm 1.8].

Proof. Theorem 2.2.5 gives us filtered FI^m-modules $J^0, \ldots, J^N$ and a sequence

$$0 \to V \to J^0 \to \cdots \to J^N \to 0$$

which is exact in high enough degree. By semisimplicity of $\mathbb{Q}[S_\mathbf{d}]$, it is therefore enough to prove the claim for the J^i. But since each J^i is filtered, meaning it has a filtration whose graded pieces are projective, we can reduce to the case of $\mathrm{FI}\sharp^m$-modules, again by semisimplicity. By Proposition 2.2.3, the character of $\mathrm{FI}\sharp^m$-modules is given by a single character polynomial. Finally, by [CEF15, Prop 3.26] we know that each $\mathrm{Ind}^{\mathrm{FI}}(V_i)$ satisfies representation stability, so since an $\mathrm{FI}\sharp^m$-module is just a direct sum of tensor products of these, it therefore satisfies representation stability. □

2.2.3 Tensor products and FI^m-algebras

Here we proceed to generalize the notions introduced in [CEF15, §4.2] from FI to FI^m. Given FI^m-modules V and V', their tensor product $V \otimes V'$ is the FI^m-module with $(V \otimes V')_{\mathbf{d}} = V_{\mathbf{d}} \otimes V'_{\mathbf{d}}$, where FI^m acts diagonally.

A *graded* FI^m*-module* is a functor from FI^m to graded modules, so that each piece is graded, and the induced maps respect the grading. If V is graded, each graded piece V^i is thus an FI^m-module. If V and W are graded, their tensor product $V \otimes W$ is graded in the usual way. Say that V is *finite type* if each V_n^i is finitely generated.

Similarly, an FI^m-algebra is a functor from FI^m to k-algebras, which can also be graded. Here our algebras will always be graded-commutative. We can also define graded co-FI^m-modules and algebras, as functors from $(\mathrm{FI}^m)^{\mathrm{op}}$. A (co-)$\mathrm{FI}^m$ algebras is *generated (as an* FI^m*-algebra)* by a submodule V when each $A_{\mathbf{d}}$ is generated as an algebra by $V_{\mathbf{d}}$.

Finally, there is another type of tensor product that we will need. Suppose V is a graded vector space with $V^0 = k$. Then the space $V^{\otimes \bullet}$ defined by $(V^{\otimes \bullet})_{\mathbf{d}} = \boxtimes_{i=1}^m V^{\otimes d_i}$ has the structure of an $\mathrm{FI}\sharp^m$-module, as in [CEF15, Defn 4.2.5], with the morphisms permuting and acting on the tensor factors.

The following theorem characterizes the above constructions.

Theorem 2.2.7.

1. *If V and V' are finitely generated* FI^m*-modules, then $V \otimes V'$ is finitely generated.*

2. *Let A be a graded* FI^m*-algebra generated by a graded submodule V, where $V^0 = 0$. If V is finite type, then A is finite type.*

3. *Let V be a graded vector space with $V^0 = k$. If V is finite type as a graded vector space, then $V^{\otimes \bullet}$ is finite type as a graded* FI^m*-module.*

4. *Let X be a connected space such that $H^*(X; k)$ is finite type. Then $H^*(X^\bullet; k)$ is an* $\mathrm{FI}\sharp^m$*-algebra of finite type.*

Proof.

1. It is enough to prove the theorem in the case where V and W are projective. But this is [Gada, Thm B(1)].

2. Let $T(V)$ be the tensor algebra on V. It is of finite-type, since

$$(T(V))^k = \bigoplus_{i_1+\cdots+i_m=k} V^{i_1} \otimes \cdots \otimes V^{i_m}$$

where each summand on the right is finitely generated by (1). But since A is an FI^m-algebra generated by V, there is an FI^m-algebra surjection $T(V) \twoheadrightarrow A$. Therefore A is of finite type.

3. We have $(V^{\otimes \bullet})_{\mathbf{d}} = (V^{\otimes \bullet})_{d_1} \boxtimes \cdots \boxtimes (V^{\otimes \bullet})_{d_m}$. Since the $\mathrm{FI}\,\sharp$-algebra $V^{\otimes \bullet}$ is finitely generated [CEF15, Prop 4.2.7], we conclude that the $\mathrm{FI}\,\sharp^m$-algebra $V^{\otimes \bullet}$ is finitely generated.

4. As [CEF15, Prop 6.1.2] explain for the case $m = 1$, this essentially follows from (3) and the Künneth formula; technically sometimes a sign is introduced when permuting the order of tensor factors, but this does not change the proof of (3). The degree 0 part is k by connectivity.

□

CHAPTER 3

GENERALIZED CONFIGURATION SPACES

In this chapter we define the orbit configuration space $\mathrm{Conf}_n^G(X)$ and space of ordered 0-cycles $\widetilde{\mathcal{Z}}_n^{\mathbf{d}}(X)$ in detail, analyze their cohomology as an FI_G-module, resp. FI^m-module, and prove that these are finitely generated.

3.1 Orbit configuration spaces

Let M be a manifold with a free and properly discontinuous action of a group G, so that $M \to M/G$ is a cover. Define the *orbit configuration space* by:

$$\mathrm{Conf}_n^G(M) = \{(m_i) \in M^n \mid Gm_i \cap Gm_j = \emptyset \text{ for } i \neq j\}$$

This was first considered by Xicoténcatl in [Xic97] and later investigated in e.g. [Coh01], [FZ02], [CX02], [CCX03]. There is a covering map $\mathrm{Conf}_n^G(M) \to \mathrm{Conf}_n(M/G)$ with deck group G^n, given by $(m_i) \mapsto (Gm_i)$. Thus another way to think of $\mathrm{Conf}_n^G(M)$ is as the space of configurations in M that do not degenerate upon projection to M/G, or as configurations in M/G which also keep track of a lift in M of each point in the configuration.

If N is a normal subgroup of G, there is an intermediate cover

$$\mathrm{Conf}_n^G(M) \to \mathrm{Conf}_n^{G/N}(M/N) \to \mathrm{Conf}_n(M/G)$$

where the first map has deck group N^n, and the second has deck group $(G/N)^n$. Also, notice that if G is finite, there is an embedding

$$\mathrm{Conf}_n^G(M) \hookrightarrow \mathrm{Conf}_{|G|n}(M), \quad (m_1, \ldots m_n) \mapsto (g_1m_1, g_2m_1, \ldots, g_{|G|}m_n)$$

3.1.1 FI_G-structure and finite generation for finite groups

For any G acting discretely and properly discontinuously on M, if we write $\left(\mathrm{Conf}^G(M)\right)_n = (\mathrm{Conf}_n^G(M))$, then $\mathrm{Conf}^G(M)$ is a co-FI_G-space: given $a : [m] \hookrightarrow [n]$ and $(g_i) \in G^m$, there is a map

$$\begin{aligned}(a,g)^* : \mathrm{Conf}_n^G(M) &\to \mathrm{Conf}_m^G(M)\\ (m_i) &\mapsto \left(g_i m_{a(i)}\right)\end{aligned}$$

In particular, if G is trivial we recover the usual ordered configuration space, and if $M = \mathbb{C}^*$ and $G = \mathbb{Z}/2\mathbb{Z}$ acting as multiplication by -1, we obtain the type BC hyperplane complement from [Wil13]. Composing with the contravariant cohomology functor, we see that $H^*(\mathrm{Conf}^G(M), k)$ has the structure of an FI_G-module over the field k.

We are therefore interested in orbit configuration spaces for which G is virtually polycyclic, so that we can apply the results on FI_G-modules from §2.1. Interesting examples include:

- $G = \mathbb{Z}/2$ acting antipodally on $M = S^m$, so that $M/G = \mathbb{RP}^m$. This was analyzed by Feichtner and Ziegler in [FZ02]. Their computation of the cohomology in Thm 17 shows that $\dim H^i(\mathrm{Conf}_n^G(S^m); \mathbb{Q})$ is bounded by a polynomial in n. We strengthen this by proving that it is in fact equal to a polynomial for $n \gg 0$. Furthermore, the G^n action on $H^i(\mathrm{Conf}_n^G(S^m))$ was analyzed in [GGSX15]: in particular, their Prop 6.6 is a sort of weak form of representation stability.

- $G = \mathbb{Z}/2$ acting by a hyperelliptic involution on a $2g+2$-punctured Σ_g, with quotient a $2g+2$-punctured sphere

- Any finite cover $\Sigma_h \to \Sigma_g$ (so that $h = |G|(g-1)+1$), with G the deck group of the covering.

- $M = \mathbb{R}^2$, with G a lattice isomorphic to $\mathbb{Z}^2$, and M/G a torus

- $M = \mathrm{Conf}_d(\mathbb{C})$ for some fixed d, with $G = S_d$, so that we are looking at the iterated configuration space $\mathrm{Conf}_n^{S_d}(\mathrm{Conf}_d(\mathbb{C}))$ and its quotient $\mathrm{UConf}_n(\mathrm{UConf}_d(\mathbb{C}))$.

- $M = S^3$, with G any finite subgroup of $\mathrm{SO}(4)$, and M/G a spherical 3-manifold

- Baues [Bau04] proved that every torsion-free virtually polycyclic group G acts discretely, properly discontinuously, and cocompactly on $\mathbb{R}^d$ for some d, and furthermore, the quotient spaces $\mathbb{R}^d/G$ precisely comprise the *infra-solvmanifolds*. Thus for any such G, consider $\mathrm{Conf}_n^G(\mathbb{R}^d)$.

A straightforward transversality argument (e.g., [Bir69, Thm 1]) shows that if $\dim M \geq 3$, the map $\mathrm{Conf}_n(M) \hookrightarrow M^n$ induces an isomorphism on π_1. Therefore $\mathrm{Conf}_n^{\pi_1(M)}(\widetilde{M})$ is in fact the universal cover of $\mathrm{Conf}_n(M)$ in dimension ≥ 3, which provides further motivation as to why orbit configuration spaces are natural to study. Note that this does not make the last example trivial (i.e., contractible), since if $\dim M > 2$, $\mathrm{Conf}_n(M)$ need not be aspherical: its homotopy groups only agree with M^n up to $\dim(M) - 2$.

We first consider the case where G is finite.

Theorem 3.1.1 (Cohomology of orbit configuration spaces). *Let k be a field, let M be a connected, orientable manifold of dimension at least 2 with $\dim H^*(M;k) < \infty$, and let G be a finite group acting freely on M. Then the FI_G-algebra $H^*(\mathrm{Conf}^G(M);k)$ is of finite type.*

Following Church-Ellenberg-Farb-Nagpal [CEFN14], we could adapt Theorem 3.1.1 to handle $\mathbb{Z}$ coefficients. As [CEFN14] mention, the proof with $\mathbb{Z}$ coefficients is essentially identical to the one with field coefficients: the difference is that one of the inputs, the analogue of our Theorem 2.1.6.4, becomes harder to prove. However, their proof of this analogue, [CEFN14, Lemm. 4.1], is readily adaptable to our context. We do not make use of this except in §3.1.3.

Proof. The proof is based on modifying the argument of Totaro in [Tot96, Thm 1]. Following Totaro, consider the Leray spectral sequence associated to the inclusion $\iota : \mathrm{Conf}_n^G(M) \hookrightarrow$

M^n. This spectral sequence has the form

$$H^i(M^n; R^j\iota_* k) \implies H^{i+j}(\mathrm{Conf}_n^G(M); k) \tag{3.1}$$

where $R^j\iota_* k$ is the sheaf on M^n associated to the presheaf

$$U \mapsto H^j(U \cap \mathrm{Conf}_n^G(M); k)$$

As in Totaro's proof, the sheaf $R^j\iota_* k$ vanishes outside the appropriate "fat diagonal", which in this case is the union of the subspaces $\Delta_{a,g,b} = \{(m_i) \in M^n \mid m_a = g \cdot m_b\}$, for $1 \leq a < b \leq n$ and $g \in G$. Consider a point in the fat diagonal,

$$x = (x_1, g_{11}x_1, \ldots g_{1i_1}x_1, \ldots x_s, g_{s1}x_s, \ldots g_{si_s}x_s).$$

Since G acts properly discontinuously, take a neighborhood of each x_j small enough to be disjoint from all translates of all the other neighborhoods. Then use a Riemannian metric to identify each of these with the tangent space $T_{x_j}X$, $T_{g_{j1}x_j}X$, etc. $dg_{jk}(x_j)$ is then an isomorphism from $T_{x_j}X$ to $T_{g_{jk}x_j}X$, and the condition that a point m_1 near x and m_2 near gx satisfy $m_2 = gm_1$ becomes, upon passing to the tangent space and under the isomorphism dg, simply the condition that $(v, w) \in (T_xX)^2$ satisfy $v = w$. Thus, for a neighborhood U of x small enough so that the inverse exponential map is a diffeomorphism,

$$(R^j\iota_* k)_x = H^j(U \cap \mathrm{Conf}_n^G(M); k) = H^j(\mathrm{Conf}_{i_1}(T_{x_1}X) \times \cdots \times \mathrm{Conf}_{i_s}(T_{x_s}X))$$

Thus, the local picture looks exactly the same as in Conf_n, which it should since there is a covering map $\mathrm{Conf}_n^G(M) \to \mathrm{Conf}_n(M/G)$. So as in Totaro, we get generators of $\mathrm{Conf}_n(\mathbb{R}^d)$, where $\dim M = d$. However, for $\mathrm{Conf}_n(M)$, we got just one copy of each generator e_{ab}, coming from the diagonal $\{m_a = m_b\}$. Here, however, we get a generator $e_{a,g,b}$ for each

$g \in G$, coming from each $\Delta_{a,g,b}$. The permutation action of W_n on the $\{\Delta_{a,g,b}\}$ induces an action on the $\{e_{a,g,b}\}$, which is given by

$$(\sigma, \vec{h}) \cdot e_{a,g,b} = e_{\sigma(a), h_a g h_b^{-1}, \sigma(b)} \tag{3.2}$$

As in Totaro, we can explicitly write down the relations that these $e_{a,g,b}$ satisfy:

$$\begin{aligned} e_{a,g,b} &= (-1)^d e_{b,g^{-1},a} \\ e_{a,g,b}^2 &= 0 \\ e_{a,g,b} \wedge e_{b,h,c} &= (e_{a,g,b} - e_{b,h,c}) \wedge e_{a,gh,c} \end{aligned} \tag{3.3}$$

To conclude, we use the argument from [CEF15, Thm 6.2.1]. To wit, because the Leray spectral sequence is functorial, all of the spectral sequences of $\mathrm{Conf}_n^G(M) \hookrightarrow M^n$, for each n, collected together form a spectral sequence of FI_G-modules. As we just described, the E_2 page is generated by $H^*(M^n; k)$ and the FI_G-module spanned by the $e_{a,g,b}$. This latter is evidently finitely-generated, since it is just generated in degree 2 by $e_{1,e,2}$, and the former is of finite type by Theorem 2.1.6.4, so therefore the E_2 page as a whole is of finite type. The E_∞ page is a subquotient of E_2, so by Noetherianity it is of finite type, and therefore $H^*(\mathrm{Conf}^G(M); k)$ is of finite type. $\square$

We pause briefly to dwell on the permutation action (3.2) of W_n on the module V_n spanned by $\{e_{a,g,b}\}$, since it will come up repeatedly. Let $k[G]$ be the representation of $G \times G$ where the first factor of G acts by multiplication on the left, and the second by multiplication on the right by the inverse (two commuting left actions). Therefore

$$V_n = \mathrm{Ind}_{W_2 \times W_{n-2}}^{W_n} k[G] \otimes k = \mathrm{Ind}^{\mathrm{FI}_G}(k[G])_n$$

Recall that, as a $(k[G], k[G]^{\mathrm{op}})$-bimodule, the regular representation has the following decomposition into irreducibles:

$$k[G] = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi \boxtimes (V_\chi)^*$$

Thus if we turn this into a $k[G] \otimes k[G]$-module by having the right factor act by g^{-1}, this becomes

$$k[G] = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi \boxtimes V_\chi = \bigoplus_{\chi \in \mathrm{Irr}(G)} L((2)_\chi)$$

as $k[G] \otimes k[G] = k[G \times G]$-modules. Therefore

$$V = \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{Ind}^{\mathrm{FI}_G}((2)_\chi). \tag{3.4}$$

as an FI_G-module, so it is manifestly an $\mathrm{FI}_G\sharp$-module. In particular, if G is trivial, we obtain $\mathrm{Ind}^{\mathrm{FI}}((2)) = \mathrm{Sym}^2 k^n/k^n$, consistent with the computation done in [CEF15].

Corollary 3.1.2. *Let M be a connected, orientable manifold of dimension at least 2 with each $H^i(M;\mathbb{Q})$ finite-dimensional, let G be a finite group acting freely on M, and let k be a splitting field for G of characteristic 0. Then for each i, the characters of the W_n-representations $H^i(\mathrm{Conf}_n^G(M);k)$ are given by a single character polynomial for all $n \gg 0$.*

Theorem 3.1.1 has another consequence, which as far as we can tell is a new result (recall that [CEF15, Thm 6.2.1] only applied to orientable manifolds).

Corollary 3.1.3. *Let M be a connected, non-orientable manifold of dimension at least* 2 *with $H^*(M;\mathbb{Q})$ of finite type. Then the* FI-*algebra $H^*(\mathrm{Conf}(M);\mathbb{Q})$ is of finite type.*

Proof. Consider the orientation cover $\widetilde{M} \to M$, which has deck group $G = \mathbb{Z}/2$. Since there is a covering map $\mathrm{Conf}_n^G(\widetilde{M}) \to \mathrm{Conf}_n(M)$, with deck group G^n, then by transfer there is an isomorphism

$$H^*(\mathrm{Conf}_n(M);\mathbb{Q}) \cong \left(H^*(\mathrm{Conf}_n^G(\widetilde{M});\mathbb{Q})\right)^{G^n}$$

By Theorem 3.1.1, $H^*(\mathrm{Conf}_n^G(\widetilde{M});\mathbb{Q})$ is a finite type FI_G-algebra, and therefore $H^*(\mathrm{Conf}_n(M);\mathbb{Q})$ is a finite type FI-algebra. □

Homological stability for unordered configuration spaces of non-orientable manifolds, which is a consequence of this corollary, was proven by Randal-Williams in [RW11].

3.1.2 Dealing with infinite groups

We pause to explain the complications that arise when G is infinite, before proceeding to at least partially resolve them. As a toy example, forgetting for a moment the setting of FI_G and sequences of spaces, consider the space $X = S^1 \vee S^1 = K(F_2, 1)$, with one loop called a and the other b. Let Y be the $G := \mathbb{Z}$ cover associated to the kernel of the map $F_2 \to \mathbb{Z}$, $a \mapsto 0, b \mapsto 1$. Hence Y is an infinite sequence of line segments labeled b joining loops labeled a. So Y is homotopy equivalent to a wedge of infinitely many circles, and thus $H_1(Y;k)$ has infinite rank. However, notice that the covering group G acts on Y, and that $H_1(Y;k) \cong k[G] = k[b^{\pm 1}]$ as G-modules.

In particular, $H_1(Y;k)$ is finitely-generated as a G-module. However, looking at cohomology, $H^1(Y;k) = \mathrm{Hom}_k(H_1(Y),k) = \mathrm{Hom}_k(k[G],k) = k^G$. Thus, $H_1(Y;k)$ consists of finite linear combinations of elements of G, while $H^1(Y;k)$ consists of infinite linear combinations of elements of G. In particular, $H^1(Y;k)$ is no longer finitely generated as a G-module. However, notice that it contains a dense submodule isomorphic to $H_1(Y;k)$.

Now we see why the proof of Theorem 3.1.1 does not suffice when G is infinite: in general, the E_2 page is not just be generated by the $e_{a,g,b}$, which is to say, by finite linear combinations of them, but instead it is generated by all infinite linear combinations of them. Therefore the E_2 page is in general not a finitely-generated FI_G-module.

If we are willing to settle for "presented in finite degree"—for example, if G is not virtually polycyclic—then this is good enough:

Theorem 3.1.4. *Let M be a connected, orientable manifold of dimension at least 2. Then the* FI_G*-algebra* $H^*(\mathrm{Conf}_n^G(M); k)$ *is presented in finite degree.*

Proof. The argument from Theorem 3.1.1 carries over essentially directly. The E_2 page of the spectral sequence (3.1) is generated as a k-algebra by $H^*(M^n; k)$ and the dual space to $\langle e_{a,g,b} \rangle$, that is, the space of infinite linear combinations

$$\sum_{a,g,b} v_{a,g,b} e_{a,g,b} = \sum_{1 \le a < b \le n} \left(\sum_{g \in G} v_{a,g,b} e_{a,g,b} \right)$$

This space is evidently generated as an FI-module by those sums of the form $\sum_{g \in G} v_g e_{1,g,2}$, since we can get all other a, b by appropriate permutations. These generators evidently live in degree 2, and the relations (3.3) all live in degree 3, so that the E_2 page is presented in finite degree. By Theorem 2.1.3, taking successive pages in the spectral sequences preserves being presented in finite degree, as does passing from the E_∞ page to the final cohomology. So we conclude that $H^*(\mathrm{Conf}_n^G(M); k)$ is presented in finite degree. □

However, if we want to preserve finite generation, the analysis at the beginning of this subsection suggests that the correct thing to look at is actually $H_\bullet(\mathrm{Conf}^G(M); k)$ instead. Unfortunately, on the face of it, $H_\bullet(\mathrm{Conf}^G(M); k)$ is a co-FI_G-module, so since the maps go "in the wrong way", it is never finitely generated.

However, when the quotient M/G is an *open* manifold, we obtain the following generalization of [CEF15, Prop 6.4.2].

Theorem 3.1.5 (Orbit configuration spaces of open manifolds). *Let N be the interior of a connected, compact manifold $\overline{N}$ of dimension ≥ 2 with nonempty boundary $\partial\overline{N}$, and let $\pi : M \to N$ be a G-cover, so that G acts freely and properly discontinuously on M. Then* $\mathrm{Conf}^G(M)$ *has the structure of a homotopy* $\mathrm{FI}_G\sharp$*-space, that is, a functor from* $\mathrm{FI}_G\sharp$ *to hTop, the category of spaces and homotopy classes of maps.*

Proof. We follow the argument in [CEF15]. Fix a collar neighborhood S of one component of

$\partial\overline{N}$, let $R = \pi^{-1}(S)$ and let R_0 be a connected component of R, and fix a homeomorphism $\Phi : M \cong M \setminus \overline{R}$ isotopic to the identity (Φ and the isotopy both exist by lifting). For any inclusion of finite sets $X \subset Y$, define a map

$$\Psi_X^Y : \mathrm{Conf}_X^G(M) \to \mathrm{Conf}_Y^G(M)$$

up to homotopy, as follows. First, if $Y = X$, set $\Psi_X^Y = \mathrm{id}$. Next, note that a configuration in $\mathrm{Conf}_X^G(M)$ is just an embedding $X \hookrightarrow M$ (that stays injective upon composition with π). So fix an element $q_X^Y : (Y - X) \hookrightarrow R_0$ of $\mathrm{Conf}_{Y-X}^G(R_0)$. Then any embedding $f : X \hookrightarrow M$ in $\mathrm{Conf}_X^G(M)$ extends to an embedding $\Psi_X^Y(f) : Y \hookrightarrow M$ by

$$\Psi_X^Y(f)(t) = \begin{cases} \Phi(f(t)) & t \in X \\ q_X^Y(t) & t \notin X \end{cases}$$

The image of $\pi \circ \Phi$ is disjoint from S, while the image of $\pi \circ q_X^Y$ is contained in S, so the above map never send points in X into the same G-orbit that it sends points outside of X into, and therefore this does give an element of $\mathrm{Conf}_Y^G(M)$. Furthermore, since $\mathrm{Conf}_{Y-X}^G(R_0)$ is connected (since R_0 is, and $\dim R_0 \geq 2$), different choices of q_X^Y give homotopic maps, so Φ_X^Y is well-defined up to homotopy.

Now, an $\mathrm{FI}_G\sharp$ morphism $Z \to Y$ consists of an injection $X \hookrightarrow Z$, an injection $X \hookrightarrow Y$, and a tuple $g : X \to G$. Normally if we were extending from an FI_G structure, we would think of X as being a subset of Z, but since we are extending from a *co*-FI_G structure, it is more natural to think of X as a subset of Y, with an explicit map $a : X \to Z$. The induced map is then given by

$$\begin{aligned} \mathrm{Conf}_Z^G(M) \to \mathrm{Conf}_X^G(M) &\xrightarrow{\Psi_X^Y} \mathrm{Conf}_Y^G(M) \\ (m_i) \mapsto (g_i m_{a(i)}) & \end{aligned}$$

It is straightforward to verify that this is functorial up to homotopy, as [CEF15, Prop 6.4.2] do for trivial G. $\square$

In particular, when the conditions of Theorem 3.1.5 hold, then $H_*(\mathrm{Conf}^G(M);k)$ is an $\mathrm{FI}_G\sharp$-module. We want to argue that, when G is virtually polycyclic, $H_*(\mathrm{Conf}^G(M);k)$ is of finitely-generated type. To do this, we need the following.

Proposition 3.1.6. *Let $A(G,d)$ be the $\mathrm{FI}_G\sharp$-algebra where $A(G)_n$ has generators $\{e_{a,g,b} \mid 1 \leq a \neq b \leq n, g \in G\}$ of degree $d-1$ and action given by (3.2), modulo the following relations:*

$$\begin{aligned} e_{a,g,b} &= (-1)^d e_{b,g^{-1},a} \\ e_{a,g,b}^2 &= 0 \\ e_{a,g,b} \wedge e_{b,h,c} &= (e_{a,g,b} - e_{b,h,c}) \wedge e_{a,gh,c} \end{aligned}$$

Then $A(G,d)$ is an $\mathrm{FI}_G\sharp$-module of finitely-generated type.

Proof. First, by construction $A(G,d)$ is presented in finite degree, so it remains to show that each $A(G,d)_n$ is of finitely-generated type. For convenience, put $D = d-1$, so that $A(G,d)_n$ is only nonzero in degree divisible by D. It is straightforward to verify (e.g., see Arnold [Arn69]) that $A(G,d)_n^{iD}$ is spanned as a vector space by all products of the form

$$v = e_{a_1,g_1,b_1} \wedge e_{a_2,g_2,b_2} \wedge \cdots \wedge e_{a_i,g_i,b_i} \quad \text{where } a_s < b_s, \quad b_1 < b_2 < \cdots < b_i, \quad g_s \in G \tag{3.5}$$

We will describe an explicit procedure which, given such a v, finds an element $\vec{h} \in G^n$ so that

$$v' := \vec{h} \cdot v = e_{a_1,e,b_1} \wedge e_{a_2,e,b_2} \wedge \cdots \wedge e_{a_i,e,b_i}$$

that is, so that each $g_i' = e$ in (3.5). We construct $\vec{h}$ inductively. To begin, we put a copy of g_1 in the b_1-th coordinate of $\vec{h}$, which cancels out the g_1 in (3.5). We multiply v by this partial $\vec{h}$, which may have the effect of modifying later g_i's. We then proceed and put a copy of (the new, modified) g_2 in the b_2-th coordinate of $\vec{h}$, which cancels the g_2 in (3.5). We can do this with no trouble because we know $b_1 < b_2$. Again, we use this to modify v and proceed to b_3, etc. Eventually we have constructed our $\vec{h}$ and modified v so that each $g_s = e$.

We therefore conclude that $A(G,d)_n^{iD}$ is finitely generated as a G^n-module, so *a fortiori* as a W_n-module. Therefore $A(G,d)$ is of finitely-generated type. □

We therefore obtain the following.

Theorem 3.1.7 (Homology of orbit configuration spaces). *Let N be the interior of a connected, compact manifold of dimension ≥ 2 with nonempty boundary. Let $M \to N$ be a G-cover, with G virtually polycyclic, such that $H_*(M)$ is of finite type. Then $H_*(\mathrm{Conf}^G(M); k)$ is a finitely-generated type $\mathrm{FI}_G\,\sharp$-module.*

Proof. We still need to appeal to cohomology, in order to make use of the cup product structure. So the proof follows that of Theorem 3.1.1, but only considering the sub-FI_G-module of the E_2 page that actually is generated by $H^*(M^n)$ and the $e_{a,g,b}$, and not the infinite linear combinations of them. Notice that this is isomorphic to the E^2 page associated to the appropriate spectral sequence computing the homology of $\mathrm{Conf}^G(M)$ (to be technical, this comes from *cosheaf* homology).

This submodule of the E_2 page is precisely the algebra described in Proposition 3.1.6. Thus it is of finitely-generated type, and therefore the E^2 page for homology is an FI_G-module of finitely-generated type. The same final argument from Theorem 3.1.1 (which uses Noetheriantiy, since G is virtually polycyclic) thus shows that $H_*(\mathrm{Conf}^G(M); k)$ is of finitely-generated type. □

3.1.3 Homotopy groups of configuration spaces

The main application of the theory of FI_G-modules until this point has been in Kupers-Miller's [KM15] work on the homotopy groups of configuration spaces. They prove that, for M a simply-connected manifold of dimension at least 3, the dual homotopy groups $\mathrm{Hom}(\pi_i(\mathrm{Conf}_n(M)), \mathbb{Z})$ form a finitely-generated FI-module. In [KM15, §5.2], they sketch an extension of this result to the non-simply connected case. Kupers-Miller are naturally led to consider $\mathrm{Conf}_n^{\pi_1(M)}(\widetilde{M})$, since as we have said it is the universal cover of $\mathrm{Conf}_n(M)$ once $\dim M \geq 3$, and so has the same higher homotopy groups as $\mathrm{Conf}_n(M)$.

Our results on orbit configuration spaces are able to confirm most of Kupers-Miller's sketch, while also clarifying some oversights. As stated, their Prop 5.8 is not correct, since as we have seen, if G is infinite, in general we cannot conclude that *co*homology is finitely generated. Instead, the best we can do is our Theorem 3.1.1 and Theorem 3.1.7, where we either assume that G is finite or that M is an open manifold. We therefore obtain the following. Note that, following Kupers-Miller, we work here with $\mathbb{Z}$ coefficients, since as we mentioned, we could rework Theorem 3.1.1 to use $\mathbb{Z}$ coefficients.

Theorem 3.1.8 (Homotopy groups of configuration spaces, take 1). *Let M be a connected manifold of dimension ≥ 3 such that $G := \pi_1(M)$ is finite and such that $H^*(M)$ is finite-dimensional. For $i \geq 2$, the dual homotopy groups $\mathrm{Hom}(\pi_i(\mathrm{Conf}_n(M)), \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\pi_i(\mathrm{Conf}_n(M)), \mathbb{Z})$ are finitely generated FI_G-modules.*

In particular, if k is a splitting field for G of characteristic 0, then for each i, the characters of the W_n-representations $\mathrm{Hom}(\pi_1(\mathrm{Conf}_n(M)), k)$ are given by a single character polynomial for $n \gg 0$, and $\{\mathrm{Hom}(\pi_1(\mathrm{Conf}_n(M)), k)\}$ satisfies representation stability.

Applying Theorem 3.1.4, we obtain the following.

Theorem 3.1.9 (Homotopy groups of configuration spaces, take 2). *Let M be a connected, orientable manifold of dimension at least 2. Then the dual homotopy groups*

$\mathrm{Hom}(\pi_i(\mathrm{Conf}_n(M)),\mathbb{Z})$ *and* $\mathrm{Ext}^1_{\mathbb{Z}}(\pi_i(\mathrm{Conf}_n(M)),\mathbb{Z})$ *are* FI_G*-modules presented in finite degree.*

3.1.4 Complex reflection groups

One classical example of a finite wreath product is the *complex reflection group* $G(d,1,n)$, where $G=\mathbb{Z}/d\mathbb{Z}$ and so $W_n=\mathbb{Z}/d\mathbb{Z}\wr S_n$, also sometimes referred to as the *full monomial group*. Because G acts on $\mathbb{C}$ as multiplication by d-th roots of unity, W_n acts on $\mathbb{C}^n$ by generalized permutation matrices whose entries are d-th roots of unity. Note that since G is abelian, a splitting field for G is just the same as a field containing all the character values of G. So in this case there is a minimal splitting field of characteristic zero, namely the field generated by the character values, which is $\mathbb{Q}(\zeta_d)$.

We can then consider the orbit configuration space where $M=\mathbb{C}^*$ and $G=\mathbb{Z}/d\mathbb{Z}$ acting, as just described, as multiplication by d-th roots of unity. Thus

$$\mathrm{Conf}_n^G(\mathbb{C}^*)=\left\{(v_i)\in\mathbb{C}^n \mid v_i\neq\zeta_d^k v_j \text{ for } i\neq j, v_i\neq 0 \text{ for all } i\right\}$$

$\mathrm{Conf}_n^G(\mathbb{C}^*)$ is thus the complement of the hyperplanes fixed by the standard complex-reflection generators of W_n. This arrangement, called the *complex reflection arrangement*, is much studied. For instance, Bannai [Ban76] proved that the complement is aspherical; its fundamental group is referred to as the *pure monomial braid group*, and sometimes denoted $P(d,n)$. Thus the cohomology of $\mathrm{Conf}_n^G(\mathbb{C}^*)$ is isomorphic to the group cohomology of $P(d,n)$.

Orlik-Solomon [OS80] calculated the cohomology of the complement of any hyperplane arrangement, as follows. Let $\mathcal{A}$ be a collection of linear hyperplanes in $\mathbb{C}^n$, and put $M(\mathcal{A})=\mathbb{C}^n\setminus\bigcup\mathcal{A}$. Say that a subset $\{H_1,\ldots,H_p\}\subset\mathcal{A}$ is *dependent* if $H_1\cap\cdots\cap H_p=H_1\cap\cdots\widehat{H_i}\cdots\cap H_p$; alternately, if the linear forms defining the H_i are linearly dependent. Now let

$$E(\mathcal{A}) = \bigwedge \langle e_H \mid H \in \mathcal{A} \rangle = \bigwedge H^1(M(\mathcal{A}); \mathbb{Q})$$

and let $I(\mathcal{A}) \subset E(\mathcal{A})$ be the ideal

$$I(\mathcal{A}) = \left(\sum_{i=1}^{p} (-1)^i e_{H_1} \wedge \cdots \wedge \widehat{e_{H_i}} \wedge \cdots \wedge e_{H_p} \,\middle|\, H_1, \ldots, H_p \text{ are dependent} \right).$$

Then $H^*(M(\mathcal{A}); \mathbb{Q}) = E(\mathcal{A})/I(\mathcal{A})$. We then obtain the following.

Theorem 3.1.10 (Cohomology of complex reflection arrangements). $H^*(\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*); \mathbb{Q}) = H^*(P(d,1,n); \mathbb{Q})$ *is a finite type* $\mathrm{FI}_G\sharp$*-algebra. For each* i*, the characters of the* W_n*-representations* $H^i(P(d,1,n); \mathbb{Q}(\zeta_d))$ *are given by a single character polynomial of degree* $2i$ *for all* n*, and therefore* $H^i(P(d,1,n; \mathbb{Q}(\zeta_d))$ *satisfies representation stability with stability degree* $\leq 4i$*.*

Proof. Since $\mathbb{C}^*$ is the interior of a compact orientable 2-manifold with boundary, Theorem 3.1.7 says that $H^*(\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*); \mathbb{Q})$ is a finitely-generated $\mathrm{FI}_G\sharp$-module. To determine the degree this FI_G-module is generated in, we use the Orlik-Solmon presentation. The hyperplane arrangement is:

$$\mathcal{A} = \{z_i \mid 1 \leq i \leq n\} \cup \{e_{i,a,j} \mid 1 \leq i \neq j \leq n; a \in G\}$$

where z_i is the hyperplane $\{v_i = 0\}$ and $e_{i,a,j}$ is the hyperplane $\{v_i = \zeta^a v_j\}$, so that $e_{i,a,j} = e_{j,a^{-1},i}$. To understand $I(\mathcal{A})$, it is helpful to consider the embedding mentioned in §3.1.1,

$$\mathrm{Conf}_n^G(\mathbb{C}^*) \hookrightarrow \mathrm{Conf}_{|G|n}(\mathbb{C}), \;\; (v_1, \ldots, v_n) \mapsto (v_1, \zeta v_1, \zeta^2 v_1, \ldots, \zeta^{d-1} v_n)$$

This induces a map $H^*(\mathrm{Conf}_{|G|n}(\mathbb{C}); \mathbb{Q}) \to H^*(\mathrm{Conf}_n^G(\mathbb{C}^*); \mathbb{Q})$. Orlik-Solomon likewise determines the cohomology of $\mathrm{Conf}_{|G|n}(\mathbb{Q})$. The hyperplane arrangement for $\mathrm{Conf}_{|G|n}(\mathbb{C})$

is

$$\mathcal{B} = \{f_{i,g;j,h} \mid 1 \le i,j \le n;\ g,h \in G;\ (i,g) \neq (j,h)\}$$

The induced map $H^1(\mathrm{Conf}_{|G|n}(\mathbb{C});\mathbb{Q}) \to H^1(\mathrm{Conf}_n^G(\mathbb{C}^*);\mathbb{Q})$ is just given by

$$f_{i,g;j,h} \mapsto \begin{cases} e_{i,h/g,j} & \text{if } i \neq j \\ z_i & \text{if } i = j \end{cases}$$

and is thus evidently surjective. Since $H^*(\mathrm{Conf}_n^G(\mathbb{C}^*);\mathbb{Q})$ is generated by H^1, this means that the total induced map $H^*(\mathrm{Conf}_{|G|n}(\mathbb{C});\mathbb{Q}) \to H^*(\mathrm{Conf}_n^G(\mathbb{C}^*);\mathbb{Q})$ is surjective. We can therefore describe the ideal $I(\mathcal{A})$ in terms of the simple relations generating the ideal $I(\mathcal{B})$ of the braid arrangement. We obtain the following presentation:

$$H^*(M(\mathcal{A});\mathbb{Q}) = \bigwedge \langle e_{i,a,j}, z_k \rangle \Big/ \left\langle \begin{array}{c} e_{i,a,j} \wedge e_{j,b,k} = (e_{i,a,j} - e_{j,b,k}) \wedge e_{i,ab,k} \\ z_i \wedge z_j = (z_i - z_j) \wedge e_{i,a,j} \\ e_{i,a,j} \wedge e_{i,b,j} = (e_{i,a,j} - e_{i,b,j}) \wedge z_i \end{array} \right\rangle$$

Another way to view these is by writing $e_{i,a,i} = z_i$ for any $a \neq 1 \in G$, as suggested by the induced map on H^1 above: then the first relation, if i, j, k are allowed to equal one another, implies the others. As described in §3.1.1, $G^n \rtimes S_n$ acts on H^1 as follows: on the $\{z_i\}$, G^n acts trivially and S_n acts in the standard way, and on the $\{e_{i,a,j}\}$ the action is:

$$\left(\sigma, \zeta^{\vec{b}}\right) \cdot e_{i,a,j} = e_{\sigma(i),\, a-b_i+b_j,\, \sigma(j)}$$

So we see that $H^*(\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q})$ is generated as an algebra by $H^1(\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q})$, and that H^1 is generated as an FI_G-module by $\{e_{1,a,2}\}$ and z_1, and is therefore generated in degree 2. By Theorem 2.1.6, H^i is generated in degree $2i$, and so the stable range follows from Corollary 2.1.5. $\square$

Wilson [Wil13, Thm 7.14] proved the case $d = 2$, for which the arrangement is the type

BC braid arrangement. The decomposition for H^1, following (3.4), is

$$H^1(P(d,1,n);\mathbb{Q}) = \mathrm{Ind}^{\mathrm{FI}_G}((1)_{\chi_0}) \oplus \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{Ind}^{\mathrm{FI}_G}((2)_\chi)$$

where χ_0 is the trivial character of G. That is, $\mathrm{Ind}^{\mathrm{FI}_G}((1)_{\chi_0})$ picks out the submodule spanned by the $\{z_i\}$, and $\bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{Ind}^{\mathrm{FI}_G}((2)_\chi)$ the submodule spanned by the $\{e_{i,a,j}\}$. We can also compute the decomposition for H^2:

$$H^2(P(d,1,n);\mathbb{Q}) = \mathrm{Ind}^{\mathrm{FI}_G}((2,1)_{\chi_0})^2 \oplus \mathrm{Ind}^{\mathrm{FI}_G}((3)_{\chi_0}) \oplus \bigoplus_{\chi \in \mathrm{Irr}(G)} \mathrm{Ind}^{\mathrm{FI}_G}((3,1)_\chi) \oplus \mathrm{Ind}^{\mathrm{FI}_G}((2)_\chi)$$
$$\oplus \bigoplus_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \chi_0}} \mathrm{Ind}^{\mathrm{FI}_G}((2)_\chi) \oplus \mathrm{Ind}^{\mathrm{FI}_G}((1)_{\chi_0},(2)_\chi)^2 \oplus \mathrm{Ind}^{\mathrm{FI}_G}((1)_{\chi_0},(1,1)_\chi) \oplus \mathrm{Ind}^{\mathrm{FI}_G}((2)_{\chi_0},(2)_\chi)$$

These calculations agree with Wilson's [Wil13, p. 123] for the case $d = 2$.

3.2 Spaces of 0-cycles

Let M be a manifold, n an integer, and $\mathbf{d} = (d_1, \ldots, d_m)$. Define $M^{\mathbf{d}} := \prod_i M^{d_i}$. Then $M^\bullet$ forms a co-FI^m-space, where an FI $\sharp^m$ morphism $\mathbf{f} : \mathbf{c} \hookrightarrow \mathbf{d}$ acts on $(v_1, \ldots, v_m) \in M^{\mathbf{d}}$ by

$$f^*(v_1, \ldots, v_m) = (f_1^*(v_1), \ldots, f_m^*(v_m))$$

where the action of $f : [c] \hookrightarrow [d]$ on $v = (m_1, \ldots, m_d) \in M^d$ is the usual co-FI action:

$$f^*(m_1, \ldots m_d) = (m_{f(1)}, \ldots, m_{f(c)})$$

Therefore by applying the contravariant functor of taking cohomology, we obtain an FI^m-algebra $H^*(M^\bullet)$. We proved in Theorem 2.2.7.4 that $H^*(M^\bullet)$ is of finite type. Next, define

$$\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(M) = \left\{ (v_1, \dots, v_m) \in \prod_i M^{d_i} \;\middle|\; \text{no } m \in M \text{ appears } n \text{ or more times in each } v_i \right\}$$

Then $\widetilde{\mathcal{Z}}^\bullet_n(M)$ is a co-FI^m-subspace of $M^\bullet$. Indeed if $\mathbf{f} : \mathbf{c} \hookrightarrow \mathbf{d}$ is an FI^m-morphism and $(v_1, \dots, v_m) \in \widetilde{\mathcal{Z}}^{\mathbf{d}}_n(M)$, then $f^*(v_1, \dots, v_m) = (f_1^*(v_1), \dots, f_m^*(v_m)) \in \widetilde{\mathcal{Z}}^{\mathbf{c}}_n(M)$. This holds because the coordinates of $f_i^*(v_i)$ are just drawn from the coordinates of v_i, so the number of times any $m \in M$ appears in $f_i^*(v_i)$ is bounded by the number of times m appears in v_i.

Thus we obtain an FI^m-algebra $H^*(\widetilde{\mathcal{Z}}^\bullet_n(M))$. Our first main theorem is the following.

Theorem 3.2.1. *Let k be a field and let M a connected, oriented manifold of dimension at least 2 with $\dim H^*(M; k) < \infty$. Then the FI^m-algebra $H^*(\widetilde{\mathcal{Z}}^\bullet_n(M); k)$ is of finite type.*

In the case of usual configuration space, where $m = 1$ and $n = 2$, Theorem 3.2.1 was proven by Church-Ellenberg-Farb [CEF15, Thm 6.2.1], based on an analysis of the Leray spectral sequence studied by Totaro [Tot96].

Proof. Following [FWW, §5], the basic strategy for understanding $H^*(\widetilde{\mathcal{Z}}^\bullet_n(M); k)$ is to analyze the Leray spectral sequence associated to the inclusion $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(M) \hookrightarrow M^{\mathbf{d}}$. This is a spectral sequence with E_2 page

$$E^{p,q}_{2,\mathbf{d}} = H^p(M^{\mathbf{d}}; U \mapsto H^q(U \cap \widetilde{\mathcal{Z}}^{\mathbf{d}}_n(M)))$$

and converging to $H^*(\widetilde{\mathcal{Z}}^\bullet_n(M))$. Since, as we saw, the inclusion $\widetilde{\mathcal{Z}}^\bullet_n(M) \hookrightarrow M^\bullet$ is a map of co-FI^m-spaces, and the Leray spectral sequence is functorial, we actually obtain a spectral sequence $E_2^{p,q}$ of FI^m-modules.

Now, the bottom row $E_2^{*,0}$ is just isomorphic to $H^*(M^\bullet)$ as an FI^m-module. Furthermore, as Farb-Wolfson-Wood prove [FWW, Thm 5.6], the leftmost column $E_2^{0,*}$ is isomorphic to $H^*(\widetilde{\mathcal{Z}}_n(\mathbb{R}^N))$, where $N = \dim M$. Furthermore, they show that the E_2 page is generated as an FI^m-algebra by $E_2^{*,0}$ and $E_2^{0,*}$. As we showed in Theorem 3.2.1, $E_2^{*,0}$ is finite-type. And $\widetilde{\mathcal{Z}}_n(\mathbb{R}^N)$ is a subspace arrangement of the type studied by Gadish [Gadb]. He proves [Gadb, Thm B] that $H^*(\widetilde{\mathcal{Z}}_n(\mathbb{R}^N))$ is a finite-type $\mathrm{FI}\sharp^m$-module. Since it is generated as an algebra by a finite-type FI^m-module, the E_2 page as a whole is a finite-type FI^m-module. The E_∞ page is a subquotient of the E_2 page, so by Noetherianity it is finite type, and therefore $H^*(\widetilde{\mathcal{Z}}_n^{\bullet}(M))$ is finite type. □

Proof of Theorem 1.1. This is a direct corollary of Proposition 3.2.2 after applying Theorem 2.2.6. □

Next, as in [CEF15], if M is the interior of a compact manifold with boundary, we obtain the following generalization of [CEF15, Prop 6.1.2].

Proposition 3.2.2. *Let M be the interior of a connected compact manifold $\overline{M}$ with nonempty boundary $\partial\overline{M}$. Then $\widetilde{\mathcal{Z}}_n(M)$ has the structure of a homotopy $\mathrm{FI}\sharp^m$-space, that is, a functor $\mathrm{FI}\sharp^m \to \mathrm{hTop}$, the category of spaces and homotopy classes of maps.*

Proof. We follow the argument in [CEF15]. Fix a collar neighborhood R of one component $\partial\overline{M}$, and fix a homeomorphism $\Phi : M \cong M - \overline{R}$ isotopic to the identity. For any m-tuple of inclusions of m-tuples of finite sets $\mathbf{X} \subset \mathbf{Y}$, define a map

$$\Psi_X^Y : \widetilde{\mathcal{Z}}_n^{\mathbf{X}}(M) \to \widetilde{\mathcal{Z}}_n^{\mathbf{Y}}(M)$$

up to homotopy, as follows. First, if $\mathbf{Y} = \mathbf{X}$, set $\Psi_{\mathbf{X}}^{\mathbf{Y}} = \mathrm{id}$. Next, note that $\mathrm{Conf}_{\sqcup X_i}(M) \hookrightarrow \widetilde{\mathcal{Z}}_n^{\mathbf{X}}(M)$. So fix an embedding $q_{\mathbf{X}}^{\mathbf{Y}} : \sqcup_i(Y_i - X_i) \hookrightarrow R$ of $\mathrm{Conf}_{\sqcup Y_i - X_i}(M)$. Then any element $f : \sqcup X_i \to M$ in $\widetilde{\mathcal{Z}}_n^{\mathbf{X}}(M)$ extends to a map $\Psi_{\mathbf{X}}^{\mathbf{Y}}(f) : \sqcup Y_i \to M$ by

$$\Psi_{\mathbf{X}}^{\mathbf{Y}}(f)(t) = \begin{cases} \Psi(f(t)) & t \in \sqcup_i X_i \\ q_{\mathbf{X}}^{\mathbf{Y}}(t) & t \notin \sqcup_i X_i \end{cases}$$

The image of Φ is disjoint from R, while the image of $q_{\mathbf{X}}^{\mathbf{Y}}$ is contained in R, so the above map does not have any more coincidences of points than f itself did, and therefore $\Psi_{\mathbf{X}}^{\mathbf{Y}}(f)$ does give an element of $\widetilde{\mathcal{Z}}_n^{\mathbf{Y}}(M)$. Furthermore, since $\mathrm{Conf}_{\sqcup Y_i - X_i}(M)$ is connected (since R is, and $\dim R \geq 2$), different choices of $q_{\mathbf{X}}^{\mathbf{Y}}$ give homotopic maps, so $\Phi_{\mathbf{X}}^{\mathbf{Y}}$ is well-defined up to homotopy.

Now, an $\mathrm{FI}\sharp^m$ morphism $\mathbf{Z} \to \mathbf{Y}$ consists of a map $\mathbf{X} \hookrightarrow \mathbf{Z}$ and a map $X \hookrightarrow Y$. Normally if we were extending from an FI^m-structure, we would think of $\mathbf{X}$ as being a subset of $\mathbf{Z}$, but since we are extending from a *co*-FI^m-structure, it is more natural to think of $\mathbf{X}$ as a subset of $\mathbf{Y}$, with an explicit map $\mathbf{a} : \mathbf{X} \to \mathbf{Z}$. The induced map is then given by

$$\begin{aligned} \widetilde{\mathcal{Z}}_n^{\mathbf{Z}}(M) \to \widetilde{\mathcal{Z}}_n^{\mathbf{X}}(M) &\xrightarrow{\Psi_{\mathbf{X}}^{\mathbf{Y}}} \widetilde{\mathcal{Z}}_n^{\mathbf{Y}}(M) \\ (v_1, \dots, v_m) \mapsto (a_1^*(v_1), \dots, a_m^*(v_m)) & \end{aligned}$$

It is straightforward to verify that this is functorial up to homotopy, as [CEF15, Prop 6.4.2] do for $m = 1$. $\square$

In particular, when the conditions of Proposition 3.2.2 hold, then $H^*(\widetilde{\mathcal{Z}}_n(M))$ is an $\mathrm{FI}\sharp^m$-module. We therefore obtain the following.

Corollary 3.2.3. *Let M be a connected orientable manifold of dimension at least 2 which is the interior of a compact manifold with nonempty boundary. Then for each i, the characters of the $S_{\mathbf{d}}$-representations $H^i(\widetilde{\mathcal{Z}}_n^{\mathbf{d}}(M;\mathbb{Q})$ are given by a single character polynomial for* all $\mathbf{d}$.

CHAPTER 4

ARITHMETIC STATISTICS

In this chapter, we study generalized configuration spaces as schemes, prove stability and convergence properties about them, and use the Grothendieck-Lefschetz formula to conclude results about the stability of arithmetic statistics.

4.1 Étale homological stability

To begin, we extend the theory of étale representation stability, developed by Farb-Wolfon [FW] for the category FI, to the categories FI_G and FI^m, and apply them to our generalized configuration spaces.

4.1.1 Smoothly compactifiable I-schemes

Farb-Wolfson define a notion of *étale represention stability* for a co-FI scheme Z over $\mathbb{Z}[1/N]$. They show that if there is a uniform way of normally compactifying the Z_n, then the base-change maps commute with the induced FI maps. This allows them to pass from knowing that $H^i(Z(\mathbb{C}))$ is a finitely-generated FI-module to knowing that $H^i_{\acute{e}t}(Z_{/\overline{\mathbb{F}}_p};\mathbb{Q}_l)$ is a finitely-generated $\mathrm{Gal}(\overline{K}/K)$-*FI-module*. Since Fulton-Macpherson constructed a normal compactification of $\mathrm{Conf}(X)$, this allows them to conclude that $\mathrm{Conf}(X)$ satisfies étale representation stability.

We want to generalize this to the orbit configuration space $\mathrm{Conf}^G(X)$. However, to do so, we need to allow for the possibility that Z is only defined over some finite Galois extension of $\mathbb{Q}$. For example, the orbit configuration space

$$\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{G}_m) = \{(x_i) \in \mathbb{G}_m^n \mid x_i \neq \zeta^m x_j\}$$

where ζ is a primitive d-th root of unity and the group $G = \mathbb{Z}/d\mathbb{Z}$ acts on $\mathbb{G}_m$ by multiplication by ζ, is naturally defined over $\mathbb{Z}[\zeta]$. Thus, rather than reducing modulo primes p, we now have to reduce modulo prime ideals $\mathfrak{p}$ of $\mathcal{O}_k[1/N]$. We then have $\mathcal{O}_k/\mathfrak{p} = \mathbb{F}_q$, where $q = p^f$ with f the inertia degree of $\mathfrak{p}$ over p.

We therefore make the following definition, following [FW].

Definition 4.1.1 (**Smoothly compactifiable I-scheme over k**). Let I be a category and k a Galois number field. A *smooth I-scheme over k* is a functor $Z : I \to$ **Schemes** consistings of smooth schemes over $\mathcal{O}_k[1/N]$ for some N independent of i. A smooth I-scheme is *smoothly compactifiable at* $\mathfrak{p} \nmid N$ if there is a smooth projective I-scheme $\overline{Z}$ and a natural transformation $Z \to \overline{Z}$ so that for all $i \in I$, $Z_i \to \overline{Z}_i$ is an open embedding and $\overline{Z}_i - Z_i$ is a normal crossings divisor with good reduction at $\mathfrak{p}$.

We then have the following, generalizing [FW, Thm 2.6]:

Theorem 4.1.2 (**Base change for I-schemes over k**). *Let l be a prime, and $Z : I \to$ **Schemes** be a smooth I-scheme over k which is smoothly compactifiable at $\mathfrak{p} \nmid N \cdot l$. Then for all morphisms $i \to j$ in I, the following diagram of ring homomorphisms commutes:*

$$\begin{tikzcd}
H^*_{\acute{e}t}(Z_{j/\overline{\mathbb{F}}_q}; \mathbb{Z}_l) \arrow[r, "\sim"] \arrow[d] & H^*_{sing}(Z_j(\mathbb{C}); \mathbb{Z}_l) \arrow[d] \\
H^*_{\acute{e}t}(Z_{i/\overline{\mathbb{F}}_q}; \mathbb{Z}_l) \arrow[r, "\sim"] & H^*_{sing}(Z_i(\mathbb{C}); \mathbb{Z}_l)
\end{tikzcd}$$

Proof. The proof of [FW, Thm 2.6] applies verbatim, since as they say, theirs is essentially an I-scheme version of [EVW16, Thm 7.7], which is more than general enough to include our extension to number fields. Just replace p with $\mathfrak{p}$ and $\mathbb{F}_p$ with $\mathbb{F}_q$ everywhere in the proof of [FW, Thm 2.6]. $\square$

Now, let Z be a co-FI$_G$ scheme over k and l a prime. For each $i \geq 0$ the étale cohomology $H^i_{\acute{e}t}(Z_{/\overline{k}}, \mathbb{Q}_l)$ is an FI$_G$-module equipped with an action of $\mathrm{Gal}(\overline{k}/k)$ commuting with the FI$_G$

action. Following [FW], we call such an object a $\mathrm{Gal}(\overline{k}/k)$-*$FI_G$-module*. Likewise, if $\mathfrak{p} \nmid N \cdot l$ is a prime ideal of k, and we put $\mathbb{F}_q = \mathcal{O}_k/\mathfrak{p}$, then $H^i_{\acute{e}t}(Z_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-FI-module.

For K a number field or finite field of characteristic p, a $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}_G$-module is finitely generated if it has a finite set of element not contained in any proper $\mathrm{Gal}(\overline{K}/K)$-FI-module. We can therefore use Theorem 4.1.2 to pass from knowing finite generation of $H^i(Z(\mathbb{C}))$ as an FI_G-module to finite generation of $H^i(Z_{/K}; \mathbb{Q}_l)$ as a $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}_G$-module.

4.1.2 Étale representation stability of $\mathrm{Conf}^G(X)$

Let X be a scheme over $\mathcal{O}_k[1/N]$ with a free algebraic action of a finite group G. The *orbit configuration space* $\mathrm{Conf}_n^G(X)$ is defined as the functor of points:

$$\mathrm{Conf}_n^G(X)(R) = \{(x_i) \in X(R)^n \mid x_i \neq gx_j \ \forall g \in G\}$$

for any $\mathcal{O}_k[1/N]$-algebra R. Thus $\mathrm{Conf}_n^G(X)$ has the structure of a scheme over $\mathbb{Z}[1/N]$. If X is smooth, then $\mathrm{Conf}_n^G(X)$ is smooth.

The group $W_n := G^n \rtimes S_n$ acts freely on $\mathrm{Conf}_n^G(X)$, where the copies of G act on each coordinate and S_n permutes the coordinates. In fact, $\mathrm{Conf}^G(X)$ forms a co-FI_G-scheme: for an injection $a : [m] \hookrightarrow [n]$ and $\vec{g} \in G^m$, the action on $(x_i) \in \mathrm{Conf}_n^G(X)$ is given by

$$(\vec{g}, a)(x_i) = (g_i \cdot x_{a(i)})$$

It thus follows that $H^i(\mathrm{Conf}^G(X))$ is an FI_G-module. In [Casb, Thm 3.1], we proved that $H^i(\mathrm{Conf}^G(X)$ is finitely generated. We would like to use Theorem 4.1.2 to deduce étale representation stability for $\mathrm{Conf}^G(X)$. To do so, we need to know that $\mathrm{Conf}^G(X)$ is smoothly compactifiable at all but finitely many primes. Luckily, there are general constructions of smooth compactifications for the complement of an arrangement of subvarieties. The basic construction is due to Macpherson-Procesi [MP98] in the complex-analytic setting, generalizing Fulton-Macpherson's [FM94] compactification of $\mathrm{Conf}_n(X)$. This work was extended to the scheme-theoretic setting by Hu [Hu03] and Li [Li09]. Following Li, define the *wonderful compactification* $\overline{\mathrm{Conf}_n^G}(X)$ to be the closure of the image of the embedding

$$\mathrm{Conf}_n^G(X) \hookrightarrow \prod_{\substack{i \neq j \\ g \in G}} \mathrm{Bl}_{\Delta_{i,g,j}}(X^n) \tag{4.1}$$

of the product of the blowups of X^n along the diagonals $\Delta_{i,g,j} := \{(x_i) \mid x_i = gx_j\}$.

To carry this out, we first need to know that X itself can be smoothly compactified in a manner consistent with the action of G. For a smooth scheme X equipped with an action of a finite group G, say that X is a *smoothly compactifiable G-scheme* if there is a scheme $\overline{X}$ with an action of G and a G-equivariant embedding $X \hookrightarrow \overline{X}$ such that $\overline{X} - X$ is a normal crossings divisor. Note that this is always possible in characteristic 0 by resolution of singularities, so this assumption is only needed to ensure such a compactification exists in finite characteristic.

We then have the following.

Proposition 4.1.3 ($\overline{\mathrm{Conf}^G}(X)$ **is smoothly compactifiable).** *Let X be a smooth scheme over $\mathcal{O}_k[1/N]$ with a free action of a finite group G so that X is a smoothly compactifiable G-scheme. Then $\overline{\mathrm{Conf}_n^G}(X)$ is a smoothly compactifiable co-*FI_G*-scheme at any prime ideal $\mathfrak{p} \nmid N$ of $\mathcal{O}_k$.*

Proof. First, note that X^n is a smoothly compactifiable G^n-scheme, since X is smoothly compactifiable G-scheme. We want to use Li's [Li09] work on wonderful compactifications, but to do so we need to check that our arrangement satisfies his hypotheses, that the arrangement is a *building set* in the sense of [Li09, §2]. This follows from the fact that G acts freely on X, so that diagonals $\Delta_{i,g,j}$ and $\Delta_{i,h,j}$ are disjoint for $g \neq h$.

By definition (3) of $\overline{\mathrm{Conf}_n^G}(X)$ as the closure in the blowup, we see that W_n acts on $\overline{\mathrm{Conf}_n^G}(X)$, and that $\overline{\mathrm{Conf}^G}(X)$ forms a co-FI_G-space. Furthermore, since X is a scheme over $R := \mathcal{O}_k[1/N]$, then (3) defines $\overline{\mathrm{Conf}^G}(X)$ as a scheme over R. There is a natural open embedding $\mathrm{Conf}^G(X) \hookrightarrow \overline{\mathrm{Conf}^G}(X)$, again just given by (3). To see that the complement $\overline{\mathrm{Conf}^G}(X) - \mathrm{Conf}^G(X)$ is a normal crossings divisor, it is enough to check this at the geometric fibers over the points of R. This holds by [Li09, Thm 1.2]. Likewise, it is enough to check smoothness of $\overline{\mathrm{Conf}^G}(X)$ at the geometric fibers over the points of R. This holds by [Li09, Thm 1.2]. Since $\overline{\mathrm{Conf}^G}(X)$ is smooth and $\overline{\mathrm{Conf}^G}(X) - \mathrm{Conf}^G(X)$ is a normal crossings divisor relative to $\operatorname{Spec} R$, in the sense of [EVW16, Prop 7.7], we conclude that $\overline{\mathrm{Conf}^G}(X) - \mathrm{Conf}^G(X)$ has good reduction at each prime $\mathfrak{p}$ of R. $\square$

Theorem 4.1.4 (Étale representation stability for orbit configuration spaces). *Let X be a smooth scheme over $\mathcal{O}_k[1/N]$ with geometrically connected fibers. Let G be a finite group acting freely on X, such that X is smoothly compactifiable as a G-scheme. Let K be either a number field or an unramified finite field over $\mathcal{O}_k[1/N]$. For each $i \geq 0$, the* $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}_G$*-module* $H^i_{\acute{e}t}(\mathrm{Conf}^G(X)_{/\overline{K}}; \mathbb{Q}_l)$ *is finitely generated.*

Proof. Again, we follow [FW, Thm 2.8]. Since X has geometrically connected fibers, $X(\mathbb{C})$ is connected. Since X is smoothly compactifiable, $H^*(X; \mathbb{Q})$ is finitely generated. Since $X(\mathbb{C})$ is a complex manifold, it is orientable and has real dimension at least 2. Thus X satisfies the hypotheses of [Casb, Thm 3.1], and so $H^i(X(\mathbb{C}); \mathbb{Q})$ is a finitely generated FI_G-module.

If K is a number field, we conclude immediately from Artin's comparison theorem that $H^i_{\acute{e}t}(\mathrm{Conf}^G(X)_{/K}; \mathbb{Q}_l)$ is a finitely generated $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}_G$-module. If K is a finite field, by Proposition 4.1.3 $\mathrm{Conf}^G(X)$ is a smoothly compactifiable co-FI_G-scheme, and so by Theorem 4.1.2, the conclusion likewise follows. $\square$

4.1.3 Étale representation stability of $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$

In this section we consider the case where we replace the manifold M with a scheme X over $\mathbb{Z}[1/N]$, as in Farb-Wolfson [FW]. Here the $\mathbb{C}$-points $X(\mathbb{C})$ take the place of M. However, now we can also consider the points $X(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$.

So, if X is a scheme over $\mathbb{Z}[1/N]$, define $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$ as the functor of points

$$\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)(R) = \left\{ (v_1, \ldots, v_m) \in \prod_i X(R)^{d_i} \;\middle|\; \text{no } x \in X(R) \text{ appears } n \text{ or more times in each } v_i \right\}$$

for any $\mathbb{Z}[1/N]$-algebra R. Thus $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$ has the structure of a scheme over $\mathbb{Z}[1/N]$. If X is smooth, then $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$ is smooth. Treated as a single object, $\widetilde{\mathcal{Z}}^{\bullet}_n(X)$ forms a co-FI^m-scheme.

Let K be a number field or an finite field over $\mathbb{Z}[1/N]$, and let l be a prime number invertible in K. Then we can base-change to $\overline{K}$ and consider the etale cohomology $H^i_{ét}(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)_{/\overline{K}}, \mathbb{Q}_l)$. Thus $H^i_{ét}(\widetilde{\mathcal{Z}}^{\bullet}_n(X)_{/\overline{K}}, \mathbb{Q}_l)$ is an FI^m-module equipped with an action of $\mathrm{Gal}(\overline{K}/K)$ commuting with the FI^m action. Following [FW], we call such an object a $\mathrm{Gal}(\overline{K}/K)$-FI^m*-module.*

We would like to prove Theorem 1.2, that $H^i_{ét}(\widetilde{\mathcal{Z}}^{\bullet}_n(X)_{/\overline{K}}, \mathbb{Q}_l)$ is finitely generated as an $\mathrm{Gal}(\overline{K}/K)$-FI^m-module. One method of doing this, the one used in Farb-Wolfson [FW] and Casto [Casa], would be to find an appropriate compactification of $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$, use this to argue that the étale cohomology $H^i_{ét}(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)_{/\overline{K}}, \mathbb{Q}_l)$ is isomorphic to the singular cohomology $H^i(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)(\mathbb{C}), \mathbb{Q}_l)$ of the complex points, and then conclude by Proposition 3.2.2. We are confident that this approach could be made to work. However, constructing and proving the requisite properties about the desired compactification would involve a fair amount of technical work that we want to avoid. (In the special case of configuration spaces, this technical work was done by Fulton-Macpherson [FM94].) Instead, we will reprove Proposition 3.2.2 for étale cohomology, by directly computing with the same Leray spectral sequence for the inclusion $\widetilde{\mathcal{Z}}^{\bullet}_n(X) \hookrightarrow X^{\bullet}$.

Proof of Theorem 1.2. This argument is essentially the one given in the proof of the second part of [FWW, Thm 5.6], on page 28. However, Farb-Wolfson-Wood do not quite state the conclusion in terms of étale cohomology, so we will redo it. Given a variety X, define the "big diagonal"

$$\Delta_n^{\mathbf{d}}(X) := \left\{ (v_1, \ldots, v_m) \in \prod_i X^{d_i} \;\middle|\; \text{some } x \in X \text{ appears at least } n \text{ times in each } v_i \right\}$$

so that $\widetilde{\mathcal{Z}}_n^{\mathbf{d}}(X) = X^{\mathbf{d}} - \Delta_n^{\mathbf{d}}(X)$. Let $m := \dim X$. By [Mil13, Lem 16.8], the pair $(\Delta_n^{\mathbf{d}}(X), X^{\mathbf{d}})$ is locally isomorphic (for the étale topology) to the pair $(\Delta_n^{\mathbf{d}}(\mathbb{A}^m), (\mathbb{A}^m)^{\mathbf{d}})$. Indeed, we can describe this isomorphism explicitly. Choose regular functions $f_1, \ldots, f_m$ defined on an open neighborhood V in X, giving an étale map $F : V \to \mathbb{A}^m$. This gives us an étale map $F^{\mathbf{d}} : V^{\mathbf{d}} \to (\mathbb{A}^m)^{\mathbf{d}}$, and under this map, the image of $\Delta_n^{\mathbf{d}}(X)$ is $\Delta_n^{\mathbf{d}}(\mathbf{A}^m)$.

The point is the following: in the classical topology (as in [Tot96]) we were able to argue about the sheaf $U \mapsto H^q(U \cap \widetilde{\mathcal{Z}}_n^{\mathbf{d}}(X))$ at a point x in the big diagonal by picking an open set U small enough that it only intersects the irreducible component of $\Delta_n^{\mathbf{d}}(X)$ containing x. In the étale topology, we do not have enough fine-grained control over neighborhoods to find one that only intersects one component. However, the argument still goes through, because the point is that, as we have just seen, we can find an étale neighborhood V of x that étale-locally looks like $\widetilde{\mathcal{Z}}_n^{\mathbf{d}}(\mathbb{A}^m)$, and this is enough.

Indeed, from here we can basically follow the proof of the first part of [FWW, Thm 5.6]. As Farb-Wolfson-Wood say (and using their notation), it is enough to give an $S_{\mathbf{d}}$-equivariant isomorphism of sheaves

$$R^q j_{X*}\mathbb{Z} \cong \bigoplus_{I \in \Pi_n^{\mathbf{d}}} \epsilon_I(q) \tag{4.2}$$

where $j_X : \widetilde{\mathcal{Z}}_n^{\mathbf{d}}(X) \hookrightarrow X^{\mathbf{d}}$ and

$$\epsilon_I(q) := \tilde{H}_{\mathrm{cd}(I,X)-q-2}(\Delta(\overline{\Pi_n^{\mathbf{d}}(\leq I)}); \mathbb{Z}) \otimes \mathrm{coor}(X_I)$$

But by restricting to the étale neighborhoods V mentioned above, we obtain for each $x \in$

$\Delta_n^{\mathbf{d}}(X)$, an isomorphism of stalks

$$(R^q j_{X*}\mathbb{Z})_x \cong (R^q j_{\mathbb{A}^m *}\mathbb{Z})_y$$

where the right hand side denotes the stalk at a generic y in the component of $\Delta_n^{\mathbf{d}}(\mathbb{A}^m)$ containing $F(x)$. This isomorphism of stalks is explicitly mentioned on [FWW, p. 28]. But now, since [FWW] already verified (4.2) for $\mathbb{A}^m$, we conclude that it holds for X.

Our argument from the proof of Theorem 3.2.1 therefore applies directly, since we have reproven the necessary tools of [FWW, Thm 5.6]. To wit, we have just shown that the E_2 page of the spectral sequence is generated as a $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}^m$-algebra by $H^*_{\acute{e}t}(X^\bullet)$ and $H^*_{\acute{e}t}(\widetilde{\mathcal{Z}}_n(\mathbb{A}^d))$. Again, the first is finite-type by Theorem 3.2.1, and second is finite-type by [Gada, Thm B]—note that Gadish specifically addresses the étale cohomology and $\mathrm{Gal}(\overline{K}/K)$-action of this subspace arrangement. Thus the E_2 page as a whole is a finite-type $\mathrm{Gal}(\overline{K}/K)$-$\mathrm{FI}^m$-module. So again by Noetherianity, we conclude that $H^*_{\acute{e}t}(\widetilde{\mathcal{Z}}^\bullet_n(X))$ is finite type. □

4.2 Convergence

Recall from Theorem 2.2.6 that if V is a finitely-generated FI_G-module, then the characters χ_{V_n} are eventually given by a single character polynomial for all large n. Furthermore, for another FI_G character polynomial P, we know that the inner product $\langle P_n, V_n\rangle_{W_n}$ is eventually independent of n. We put

$$\langle P, V\rangle = \lim_{n\to\infty} \langle P, V_n\rangle_{W_n}$$

for the limiting multiplicity.

Thus if Z is a co-FI_G-scheme that satisfies étale representation stability, there is a stable inner product $\langle P, H^i_{\acute{e}t}(Z;\mathbb{Q}_l)\rangle$, eventually independent of n. Likewise, if Z is a co-FI^m-scheme that satisfies étale representation stability, and Q is an FI^m character polynomial there is a stable inner product $\langle Q, H^i_{\acute{e}t}(Z;\mathbb{Q}_l)\rangle$, eventually independent of $\mathbf{d}$.

However, for our applications we need to bound how these inner products grow in i. The following proposition is helpful in doing so:

Proposition 4.2.1. *For any graded FI_G-module V^*, the following are equivalent:*

1. *For each character polynomial P, $|\langle P_n, V^i_n\rangle|$ is bounded subexponentially in i and uniformly bounded in n.*
2. *For every a, the dimension $\dim(V^i_n)^{W_{n-a}}$ is bounded subexponentially in i and uniformly bounded in n.*

Proposition 4.2.2. *For any graded FI^m-module V^*, the following are equivalent:*

1. *For each character polynomial P, $|\langle P_{\mathbf{d}}, V^i_{\mathbf{d}}\rangle|$ is bounded subexponentially in i and uniformly in $\mathbf{d}$.*
2. *For every $\mathbf{a}$, the dimension $\dim\left((V^i_{\mathbf{d}})^{S_{\mathbf{d}-\mathbf{a}}}\right)$ is bounded subexponentially in i and uniformly in $\mathbf{d}$.*

Proof. First, note that $\dim\left((V^i_{\mathbf{d}})^{S_{\mathbf{d}-\mathbf{a}}}\right) = \langle M(\mathbf{a})_{\mathbf{d}}, V_{\mathbf{d}}\rangle_{S_{\mathbf{d}}}$. For any irreducible $S_{\mathbf{a}}$-representation W, we have $\mathrm{Ind}^{\mathrm{FI}^m}(W) \subset \mathrm{Ind}^{\mathrm{FI}^m}(k[S_{\mathbf{a}}]) = M(\mathbf{a})$, and therefore

$$\langle \mathrm{Ind}^{\mathrm{FI}^m}(W)_{\mathbf{d}}, V_{\mathbf{d}}\rangle_{S_{\mathbf{d}}} < \dim\left((V^i_{\mathbf{d}})^{S_{\mathbf{d}-\mathbf{a}}}\right)$$

The second condition for arbitrary P follows, since any P is a finite linear combination of the $\chi_{\mathrm{Ind}^{\mathrm{FI}^m}(W)}$ for some irreducible representations W.

The exact same proof works for FI_G, replacing $S_{\mathbf{d}}$ with W_n. □

If a graded FI_G or FI^m algebra V^* satisfies these two equivalent conditions, we say it is *convergent*. Theorem 1.2.2 thus states that, under appropriate conditions, the FI_G algebra $H^*(\mathrm{Conf}^G(X);\mathbb{Q})$ is convergent, and the FI^m-algebra $H^*(\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X);\mathbb{Q})$ is convergent.

4.2.1 Convergence for $\mathrm{Conf}^G(M)$

Proof of Theorem 1.2.2.1. Farb-Wolfson [FW, Thm 3.4] proved this for $G = 1$ by using the Leray spectral sequence to reduce to convergence of $H^*(X^n) \otimes H^*(\mathrm{Conf}(\mathbb{R}^d))$. We will use the same technique to reduce to their result.

Let $A(G, d)_n$ be the graded commutative $\mathbb{Q}$-algebra generated by $\{e_{a,g,b} \mid 1 \leq a \neq b \leq n, g \in G\}$, each of degree $d - 1$, modulo the following relations:

$$
\begin{aligned}
e_{a,g,b} &= (-1)^d e_{b,g^{-1},a} \\
e_{a,g,b}^2 &= 0 \\
e_{a,g,b} \wedge e_{b,h,c} &= (e_{a,g,b} - e_{b,h,c}) \wedge e_{a,gh,c}
\end{aligned}
$$

where the action of $(\vec{h}, \sigma) \in W_n$ is given by

$$(\vec{h}, \sigma) \cdot e_{a,g,b} = e_{\sigma(a), h_a g h_b^{-1}, \sigma(b)}$$

If we let $A(d)_n = A(1, d)_n$, then there is an embedding $A(d)_n \hookrightarrow A(G, d)_n$ given by $e_{a,b} \mapsto e_{a,e,b}$. In [Casb, Thm 3.6], we proved that $A(G, d)_n$ is spanned by the G^n-translates of $A(d)_n$.

In [Casb, Thm 3.1], we proved that $H^*(\mathrm{Conf}_n^G(X); \mathbb{Q})$ is isomorphic as a graded W_n-representation, to a subquotient of

$$H^*(X^n; \mathbb{Q}) \otimes A(G, d)_n.$$

Thus, it suffices to bound

$$(H^*(X^n; \mathbb{Q}) \otimes A(G, d)_n)^{W_{n-a}} = \left((H^*(X^n; \mathbb{Q}) \otimes A(G, d)_n)^{G^{n-a}} \right)^{S_{n-a}}$$

We know that $(H^*(X^n; \mathbb{Q}) \otimes A(G, d)_n)^{G^{n-a}}$ is a subquotient of $H^*(X^n; \mathbb{Q}) \otimes (A(G, d)_n)^{G^{n-a}}$

as S_n-representations. Next, since $A(G,d)_n$ is spanned by the G^n translates of $A(d)_n$, we know that $(A(G,d)_n)^{G^{n-a}}$ will be spanned by G^a translates of $A(d)_n$. In conclusion,

$$\dim\left(H^*(X^n;\mathbb{Q})\otimes A(G,d)_n\right)^{W_{n-a}} \le |G|^a \dim\left(H^*(X^n;\mathbb{Q})\otimes A(d)_n\right)^{S_{n-a}} \tag{4.3}$$

But the right-hand side of (4.3) is just (a constant times) exactly the term that Farb-Wolfson considered in [FW, Thm 3.4] and proved was convergent. Therefore $H^*(\mathrm{Conf}^G(X);\mathbb{Q})$ is convergent. $\square$

4.2.2 Convergence for $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$

Proof of Theorem 1.2.2.2. Our proof proceeds along the lines of the proofs of [FWW, Thm 3.1 and Lem 7.1]. In order to follow their proofs, we will need to recall some of the definitions they use. Recall [FWW, Defn 4.1] that the *n-equals partition lattice* $\Pi^{\mathbf{d}}_n$, for $\mathbf{d}\in \mathrm{FI}^m$, is the poset of partitions of $\mathbf{d}$ such that each block of the partition either has size 1, or contains at least n elements from each of the m columns. These are ordered by (reverse) refinement: $I \le J$ if and only if I refines J. We refer to blocks of size 1 as "singleton blocks", and the others (of size at least $m\times n$) as "non-singleton blocks". Recall that an *edge* in a poset P is a pair $a,b\in P$ with $a<b$ and no elements between them, and a *chain* of length r in P is a string $a_0<\cdots<a_r$ with $a_i\in P$. Finally, recall that in [FWW, Thm 4.9], Farb-Wolfson-Wood determine the three types of edges in $\Pi^{\mathbf{d}}_n$, which we will make reference to:

Block creation: A new non-singleton block with n elements each from the m columns is created from singletons.

Singleton adding A singleton block is merged with a non-singleton block.

Block merging Two non-singleton blocks are merged.

Now, recall that our goal is to show that

$$\dim H^*_{\acute{e}t}\left(\widetilde{Z}^{\mathbf{d}}_n(X)_{/\overline{K}};\mathbb{Q}_l\right)^{S_{\mathbf{d}-\mathbf{a}}} \leq F_{\mathbf{a}}(i)$$

where $F_{\mathbf{a}}(i)$ is a polynomial in i (and independent of $\mathbf{d}$). Recall from the proof of ?THM? **??** that there is a spectral sequence $E^{p,q} \implies H^{p+q}_{\acute{e}t}(\widetilde{Z}^{\mathbf{d}}_n(X)_{/\overline{K}};\mathbb{Q}_l)$ with

$$E_2^{p,q} \cong \bigoplus_{I\in\Pi^{\mathbf{d}}_n} H^p(X_I;\epsilon_I(q))$$

We know that $H^i_{\acute{e}t}(\widetilde{Z}^{\mathbf{d}}_n(X)_{/\overline{K}};\mathbb{Q}_l)$ is a subquotient of $\bigoplus_{p+q=i} E_2^{p,q}$, so it is enough to prove that $\dim(E_2^{p,q})^{S_{\mathbf{d}-\mathbf{a}}}$ is bounded by a polynomial in p and q (uniformly in $\mathbf{d}$).

Given $I \in \Pi^{\mathbf{d}}_n$, let $S_I := S_{I_1}\times S_{I_2}\times\cdots$. Now, we have

$$\begin{aligned}(E_2^{p,q})^{S_{\mathbf{d}-\mathbf{a}}} &= \left(\bigoplus_{I\in\Pi^{\mathbf{d}}_n} H^p(X_I;\epsilon_I(q))\right)^{S_{\mathbf{d}-\mathbf{a}}} = \left(\bigoplus_{I\in\Pi^{\mathbf{d}}_n} H^p(X_I;\epsilon_I(q))^{S_I\cap S_{\mathbf{d}-\mathbf{a}}}\right)^{S_{\mathbf{d}-\mathbf{a}}} \\ &= \left(\bigoplus_{I\in\Pi^{\mathbf{d}}_n} H^p\left(X_I;\epsilon_I(q)^{S_I\cap S_{\mathbf{d}-\mathbf{a}}}\right)\right)^{S_{\mathbf{d}-\mathbf{a}}}\end{aligned} \tag{4.4}$$

Now, following the argument of [FWW, Lem 7.1], we have that $\epsilon_I(q)^{S_I\cap S_{\mathbf{d}-\mathbf{a}}} = 0$ unless I consists of exactly $k := q/(d(mn-1)-1)$ non-singleton blocks, such that $\#\left(I|_{\mathbf{d}_i-\mathbf{a}_i}\right) \leq n$. That is, we know each of the (non-singleton) blocks I_j has at least n elements in each column, but for these invariants to be nonzero, any extras need to be in the $\mathbf{a}$ coordinates. . Denote by $\Pi' \subset \Pi^{\mathbf{d}}_n$ the subset of partitions satisfying this condition. It is therefore enough to take the sum in (3) over Π'.

For a fixed $\mathbf{a}$, it is clear that there are only a bounded number of ways to distribute the $|\mathbf{a}|$ "extra" coordinates to the k blocks. This shows that $P := \Pi'/S_{\mathbf{d}-\mathbf{a}}$, the set of all "shapes" of such partitions, has bounded size. We thus have

$$\begin{aligned}
(E_2^{p,q})^{S_{\mathbf{d}-\mathbf{a}}} &= \left(\bigoplus_{I\in\Pi'} H^p(X_I;\epsilon_I(q))\right)^{S_{\mathbf{d}-\mathbf{a}}} = \bigoplus_{\rho\in P}\left(\bigoplus_{I\in\rho} H^p(X_I;\epsilon_I(q))\right)^{S_{\mathbf{d}-\mathbf{a}}} \\
&= \bigoplus_{\rho\in P}\left(\bigoplus_{I\in\rho} H^p(X_I;\epsilon_I(q))\right)^{S_{\mathbf{d}-\mathbf{a}}} = \bigoplus_{\rho\in P}\left(\mathrm{Ind}_{\mathrm{stab}\, I_\rho}^{S_{\mathbf{d}-\mathbf{a}}} H^p(X_{I_\rho};\epsilon_{I_\rho}(q))\right)^{S_{\mathbf{d}-\mathbf{a}}} \quad (4.5)\\
&= \bigoplus_{\rho\in P} H^p(X_{I_\rho};\epsilon_{I_\rho}(q))^{\mathrm{stab}\, I_\rho}
\end{aligned}$$

again as in [FWW, Thm 3.1, p. 38-39], where the last equality is by Frobenius reciprocity. Here I_ρ is some partition chosen from the class ρ, and the stabilizer is taken inside the group $S_{\mathbf{d}-\mathbf{a}}$. We have

$$\begin{aligned}
H^p(X_I;\epsilon_I(q))^{\mathrm{stab}\, I} &= \left(H^p(X_I;\epsilon_I(q))^{S_I\cap S_{\mathbf{d}-\mathbf{a}}}\right)^{\mathrm{stab}\, I/(S_I\cap S_{\mathbf{d}-\mathbf{a}})} \\
&= H^p(X_I;\epsilon_I(q)^{S_I\cap S_{\mathbf{d}-\mathbf{a}}})^{\mathrm{stab}\, I/(S_I\cap S_{\mathbf{d}-\mathbf{a}})}
\end{aligned}$$

since S_I acts trivially on X_I. Next, we claim that $\dim \epsilon_I(q)^{S_I\cap S_{\mathbf{d}-\mathbf{a}}}$ is bounded by a polynomial q and uniformly bounded in $\mathbf{d}$. Indeed, since $\mathbf{a}$ is bounded, we know that only a finite number of the non-singleton I_j's can be larger than an $m\times n$ block. All the rest are $m\times n$ blocks and singletons. Thus, among all the other I_j's of size $m\times n$, there are only two partitions refined by I_j: the complete partitions $\hat{0}_{I_j}$, and I_j itself.

Now, the elements of $\epsilon_I(q)$ are chains of $\Pi_n^{\mathbf{d}}(\leq I)$ of length $2r(|\mathbf{d}|-1)-q$. So the size of the invariants $\epsilon_I(q)^{S_I\cap S_{\mathbf{d}-\mathbf{a}}}$ is bounded by the the number of orbits of these chains under the action of $S_I\cap S_{\mathbf{d}-\mathbf{a}}$. But up to this group action, such chains just look like a sequence of block formations, interspersed with a *bounded* number of singleton-mergers and block-mergers in the blocks larger than $m\times n$. Notice, first of all, that this number only depends on the number of non-singleton blocks $k = q/(d(mn-1)-1)$ and is independent of $\mathbf{d}$, since all the extra singletons don't refine any nontrivial partitions. Furthermore, the number of ways to intersperse the extra moves is bounded by $\binom{k}{a}a!$, which is a polynomial in q. So our claim is proven.

Now, we have

$$\dim H^p(X_I; \epsilon_I(q)^{S_I \cap S_{\mathbf{d}-\mathbf{a}}})^{\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}})} \leq \dim H^p(X_I; \mathbb{Q})^{\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}})} \cdot \dim \epsilon_I(q)^{S_I \cap S_{\mathbf{d}-\mathbf{a}}}$$

Among the non-singleton blocks of any $I \in \Pi'$, only a bounded number will have fall within the $\mathbf{a}$ coordinates; denote this number by b. As discussed earlier, all of the other non-singleton blocks must have size exactly $m \times n$. Likewise there are only a bounded number of singletons in I that fall within the $\mathbf{a}$ coordinates. Denote the number of singletons in the i-th column of I by l_i, and the number of these in the $\mathbf{a}$ coordinates by c_i. Thus

$$X_I = X^b \times X^{k-b} \times \prod_i X^{l_i - c_i} \times X^{c_i}$$

Now, $\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}})$ consists of those permutations that blockwise permute the $k - b$ blocks without any $\mathbf{a}$ coordinates, as well as permutations of the non-$\mathbf{a}$ singletons. Thus

$$\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}}) \cong S_{k-b} \times S_{l_1 - c_1} \times S_{l_2 - c_2} \times \dots$$

So we have

$$\begin{aligned} \dim H^p(X_I; \mathbb{Q})^{\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}}} &= \dim H^p(X_I/(\operatorname{stab} I/(S_I \cap S_{\mathbf{d}-\mathbf{a}})); \mathbb{Q}) \\ &= \dim H^p\left(X^b \times \operatorname{Sym}^{k-b} X \times \prod_i X^{c_i} \times \operatorname{Sym}^{l_i - c_i} X; \mathbb{Q}\right) \end{aligned}$$

Since b and c_i are bounded, the important terms are the $\operatorname{Sym}^{k-b} X$ and $\operatorname{Sym}^{l_i - c_i}$. Note that, *a priori*, the second seems concerning, since it depends on $\mathbf{d}$, which we need our bound to be independent of. However, Macdonald [Mac62] proved that $\dim H^i(\operatorname{Sym}^n X; \mathbb{Q})$ is eventually independent of n, and in fact has Poincaré polynomial given by a rational function in i, with poles at roots of unity. Such a rational function is known to be bounded by a polynomial. In particular, we conclude that $\dim H^p(X_I;\mathbb{Q})^{\mathrm{stab}\, I/(S_I \cap S_{\mathbf{d}-\mathbf{a}}}$ is bounded by a polynomial in p, uniformly in $\mathbf{d}$. Since we already knew this was likewise true of $\dim \epsilon_I(q)^{S_I \cap S_{\mathbf{d}-\mathbf{a}}}$, we conclude the theorem.

□

4.3 Arithmetic statistics for FI^m

Let Z be a smooth quasiprojective scheme over $\mathbb{Z}[1/N]$. Suppose that $S_{\mathbf{d}}$ acts generically freely on Z by automorphisms, and let $Y = Z/S_{\mathbf{d}}$ be the quotient, which is known to be a scheme.

For any prime power $q \nmid N$, we can base-change Y to $\overline{\mathbb{F}}_q$. The geometric Frobenius Frob_q then acts on $Y_{/\overline{\mathbb{F}}_q}$. The fixed-point set of Frob_q is exactly $Y(\mathbb{F}_q)$.

Fix a prime $l \nmid q$. Since all the irreducible representations of $S_{\mathbf{d}}$ are defined over $\mathbb{Q}$, *a fortiori* over $\mathbb{Q}_l$, there is a natural correspondence between finite-dimensional representations of $S_{\mathbf{d}}$ over $\mathbb{Q}_l$ and finite-dimensional constructible l-adic sheaves on Y that become trivial when pulled back to Z.

Given a representation V of $S_{\mathbf{d}}$, let χ_V be its character and let $\mathcal{V}$ the associated sheaf on Y. For any point $y \in Y(\mathbb{F}_q)$, since Frob_q fixes y, then Frob_q acts on the fiber $p^{-1}(y)$. Now $S_{\mathbf{d}}$ acts transitively on $p^{-1})(y)$ with some stabilizer H, and so we can identify $p^{-1}(y)$ with $S_{\mathbf{d}}/H$. The Frob_q action on $p^{-1}(y)$ commutes with this $S_{\mathbf{d}}$ action, and so it is determined by its action on a single basepoint, which we choose once and for all to be H. Now $\mathrm{Frob}_q(H) = \sigma_y H$ for some $\sigma_y \in S_{\mathbf{d}}$. Following Gadish [Gadc], for any $S_{\mathbf{d}}$-representation V and any coset σ_H of $S_{\mathbf{d}}$, we set

$$\chi_V(\sigma H) = \frac{1}{|H|}\sum_{h\in H} \chi_v(\sigma h)$$

More generally, for any class function P and $y \in Y(\mathbb{F}_q)$, we define

$$P(y) := \frac{1}{|H|} \sum_{h \in H} P(\sigma_y h)$$

It is straightforward to show that this is independent of the choice of coset H, since the action of $S_{\mathbf{d}}$ is transitive on fibers. With this notation we have

$$\operatorname{tr}(\operatorname{Frob}_q : \mathcal{V}_y) = \chi_V(\sigma_y H)$$

The Grothendieck-Lefschetz trace formula says that

$$\sum_{y \in Y(\mathbb{F}_q)} \operatorname{tr}(\operatorname{Frob}_q : \mathcal{V}_y) = \sum_i (-1)^i \operatorname{tr}\left(\operatorname{Frob}_q : H^i_{\acute{e}t,c}(Y_{/\overline{\mathbb{F}}_q}; \mathcal{V})\right)$$

We then have the following chain of equalities, as in [CEF14], [FW], and [Casa]:

$$\begin{aligned}
H^i_{\acute{e}t,c}(Y_{/\overline{\mathbb{F}}_q}; \mathcal{V}) &\cong (H^i_{\acute{e}t,c}(Z; \pi^*\mathcal{V}))^{S_{\mathbf{d}}} && \text{by transfer} \\
&\cong \left(H^i_{\acute{e}t,c}(Z; L) \otimes V\right)^{S_{\mathbf{d}}} && \text{by triviality of pullback} \\
&\cong \left(H^{2\dim Z - i}_{\acute{e}t}(Z; L(-\dim Z))^* \otimes V\right)^{S_{\mathbf{d}}} && \text{by Poincaré duality} \\
&\cong \langle H^{2\dim Z - i}_{\acute{e}t}(Z; L(-\dim Z)), V\rangle_{S_{\mathbf{d}}}
\end{aligned}$$

and so we obtain

$$\sum_{y \in Y(\mathbb{F}_q)} \chi_V(\sigma_y H) = q^{\dim Z} \sum_i (-1)^i \operatorname{tr}\left(\operatorname{Frob}_q : \langle H^i_{\acute{e}t}(Z; L), V\rangle_{S_{\mathbf{d}}}\right) \tag{4.6}$$

We would like to apply (4.8) to a collection of schemes $Z_{\mathbf{d}}$ that form a co-FI^m-scheme, and then let $\mathbf{d} \to \infty$. To make this work, we need to know that Z satisfies étale representation stability, and that $H^*(Z)$ is convergent in the sense of §3.2. Following [FW] and [Casa], we have the following.

Theorem 4.3.1. *Suppose that Z is a smooth quasiprojective co-*FI^m*-scheme over* $\mathbb{Z}[1/N]$ *such that $H^i(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_l)$ is a finitely-generated* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-$\mathrm{FI}^m$*-module, and that $H^*(Z;\mathbb{Q}_l)$ is convergent. Then for any* FI^m *character polynomial P,*

$$\lim_{\mathbf{d}\to\infty} q^{-\dim Z_{\mathbf{d}}} \sum_{y\in Y_{\mathbf{d}}(\mathbb{F}_q)} P(y) = \sum_{i=0}^{\infty}(-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(Z;L), P\rangle\right) \tag{4.7}$$

Proof. Since each $Z_{\mathbf{d}}$ is smooth quasiprojective, we can apply (4.8) to it. By linearity, we can replace a representation V of $S_{\mathbf{d}}$ with a virtual representation given by a character polynomial P, so we obtain

$$q^{-\dim Z_{\mathbf{d}}} \sum_{y\in Y_{\mathbf{d}}(\mathbb{F}_q)} P(y) = \sum_{i=0}^{2\dim Z_{\mathbf{d}}} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(Z_{\mathbf{d}};L), P\rangle_{S_{\mathbf{d}}}\right)$$

Call this sum $A_{\mathbf{d}}$. Furthermore, let

$$B_{\mathbf{d}} = \sum_{i=0}^{2\dim Z_{\mathbf{d}}} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(Z;L), P\rangle\right)$$

Our goal is thus to show that

$$\lim_{\mathbf{d}\to\infty} A_{\mathbf{d}} = \lim_{\mathbf{d}\to\infty} B_{\mathbf{d}},$$

that is, first of all to show that both sides converge, and that their limits are equal.

By the assumption that $H^*(Z)$ is convergent, we know that there is a function $F_P(i)$ which is subexponential in i, such that for all $\mathbf{d}$,

$$|\langle H^i_{\acute{e}t}(Z_{\mathbf{d}};L), P\rangle_{S_{\mathbf{d}}}| \leq F_P(i)$$

and so by taking $\mathbf{d}$ large enough,

$$|\langle H^i_{\acute{e}t}(Z;L), P\rangle| \leq F_P(i)$$

Furthermore, by [Del80, Thm 1.6], we know that

$$\left|\mathrm{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(Z_{\mathbf{d}}; L), P\rangle_{S_{\mathbf{d}}}\right)\right| \leq q^{-i/2}\left|\langle H^i_{\acute{e}t}(Z_{\mathbf{d}}; L), P\rangle_{S_{\mathbf{d}}}\right|$$

We therefore have

$$\begin{aligned}
|A_{\mathbf{d}}| &\leq \sum_{i=0}^{2\dim Z_{\mathbf{d}}} \left|\mathrm{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(Z_{\mathbf{d}}; L), P\rangle_{S_{\mathbf{d}}}\right)\right| \\
&\leq \sum_{i=0}^{2\dim Z_{\mathbf{d}}} \left|\langle H^i_{\acute{e}t}(Z_{\mathbf{d}}; L), P\rangle_{S_{\mathbf{d}}}\right| \\
&\leq \sum_{i=0}^{2\dim Z_{\mathbf{d}}} q^{-i/2} F_P(i).
\end{aligned}$$

By exactly the same argument, $|B_{\mathbf{d}}| \leq \sum_{i=0}^{2\dim Z_{\mathbf{d}}} q^{-i/2} F_P(i)$. Since $F_P(i)$ is subexponential in i, this means that both $A_{\mathbf{d}}$ and $B_{\mathbf{d}}$ converge.

It remains to show that $\lim_{\mathbf{d}\to\infty} A_{\mathbf{d}} - B_{\mathbf{d}} = 0$. Let $N(\mathbf{d}, P)$ be the number such that

$$\langle H^i_{\acute{e}t}(Z_{\mathbf{d}}), P\rangle_{S_{\mathbf{d}}} = \langle H^i_{\acute{e}t}(Z), P\rangle \quad \text{for all } i \leq N(\mathbf{d}, P).$$

We thus have

$$\begin{aligned}
|B_{\mathbf{d}} - A_{\mathbf{d}}| &\leq \sum_{i=0}^{2\dim Z_{\mathbf{d}}} q^{-i/2}\left|\langle H^i_{\acute{e}t}(Z), P\rangle - \langle H^i_{\acute{e}t}(Z_{\mathbf{d}}), P\rangle_{S_{\mathbf{d}}}\right| \\
&= \sum_{i=N(\mathbf{d},P)+1}^{2\dim Z_{\mathbf{d}}} q^{-i/2}\left|\langle H^i_{\acute{e}t}(Z), P\rangle - \langle H^i_{\acute{e}t}(Z_{\mathbf{d}}), P\rangle_{S_{\mathbf{d}}}\right| \\
&\leq \sum_{i=N(\mathbf{d},P)+1}^{2\dim Z_{\mathbf{d}}} 2q^{-i/2} F_P(i)
\end{aligned}$$

Since $N(\mathbf{d}, P) \to \infty$ as $\mathbf{d} \to \infty$, and $F_p(i)$ is subexponential in i, we conclude that $|B_{\mathbf{d}} - A_{\mathbf{d}}|$ becomes arbitrarily small as $\mathbf{d} \to \infty$. $\square$

We can now apply this to $\widetilde{\mathcal{Z}}^{\mathbf{d}}_n(X)$ to obtain Theorem 1.2.3.2.

Proof of Theorem 1.2.3.2. By Theorem 1.2, $H^i_{\acute{e}t}(\widetilde{\mathcal{Z}}^{\bullet}_n(X)_{/\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is a finitely generated $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-$\mathrm{FI}^m$-module. By Theorem 1.2.2, $H^*_{\acute{e}t}(\widetilde{\mathcal{Z}}^{\bullet}_n(X)_{/\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is convergent. We thus conclude by Proposition 4.2.1.2 $\square$

4.4 Arithmetic statistics for FI_G

Let Z be a smooth quasiprojective scheme over $\mathcal{O}_k[1/N]$. Suppose that $W_n = G^n \rtimes S_n$ acts freely on Z by automorphisms, and let $Y = Z/W_n$ be the quotient, which is known to be a scheme.

For any prime $\mathfrak{p}$ of $\mathcal{O}_k[1/N]$, we have $\mathcal{O}_k/\mathfrak{p} \cong \mathbb{F}_q$, where q is a power of the rational prime p under $\mathfrak{p}$. We can then base-change Y to $\overline{\mathbb{F}}_q$. The geometric Frobenius Frob_q then acts on $Y_{/\overline{\mathbb{F}}_q}$. The fixed-point set of Frob_q is exactly $Y(\mathbb{F}_q)$.

Fix a prime $l \neq p$, and let L be a splitting field for G over $\mathbb{Q}_l$. Since all the irreducible representations of G are defined over L, there is a natural correspondence between finite-dimensional representations of W_n over L and finite-dimensional constructible L-sheaves on Y that become trivial when pulled back to Z.

Given a representation V of W_n, let χ_V be its character and $\mathcal{V}$ the associated sheaf on Y. For any point $y \in Y(\mathbb{F}_q)$, since $Frob_q$ fixes y, then Frob_q acts on the stalk $\mathcal{V}_y$, which is isomorphic to V. This action determines an element $\sigma_y \in W_n$ up to conjugacy, so that $\mathrm{tr}(\mathrm{Frob}_q : \mathcal{V}_y) = \chi_V(\sigma_y)$. The Grothendieck-Lefschetz trace formula says that

$$\sum_{y \in Y(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q : \mathcal{V}_y) = \sum_i (-1)^i \,\mathrm{tr}\left(\mathrm{Frob}_q : H^i_{\acute{e}t,c}(Y_{/\overline{\mathbb{F}}_q}; \mathcal{V})\right)$$

We then have the following chain of equalities, as in [CEF14] and [FW]:

$$\begin{aligned}
H^i_{\acute{e}t,c}(Y_{/\overline{\mathbb{F}}_q};\mathcal{V}) &\cong (H^i_{\acute{e}t,c}(Z;\pi^*\mathcal{V}))^{W_n} && \text{by transfer}\\
&\cong \left(H^i_{\acute{e}t,c}(Z;L)\otimes V\right)^{W_n} && \text{by triviality of pullback}\\
&\cong \left(H^{2\dim Z-i}_{\acute{e}t}(Z;L(-\dim Z))^*\otimes V\right)^{W_n} && \text{by Poincaré duality}\\
&\cong \langle H^{2\dim Z-i}_{\acute{e}t}(Z;L(-\dim Z)),V\rangle_{W_n}
\end{aligned}$$

and so we obtain

$$\sum_{y\in Y(\mathbb{F}_q)}\chi_V(\sigma_y)=q^{\dim Z}\sum_i(-1)^i\operatorname{tr}\left(\mathrm{Frob}_q:\langle H^i_{\acute{e}t}(Z;L),V\rangle_{W_n}\right) \tag{4.8}$$

We would like to apply (4.8) to a sequence of schemes Z_n that form a co-FI_G-scheme, and then let $n\to\infty$. To make this work, we need to know that Z satisfies étale representation stability, and that $H^*(Z)$ is convergent in the sense of §4.2. Following [FW], we have the following.

Theorem 4.4.1. *Suppose that Z is a smooth quasiprojective co-FI_G-scheme over $\mathcal{O}_k[1/N]$ such that $H^i(Z_{/\overline{\mathbb{F}}_q};\mathbb{Q}_l)$ is a finitely-generated $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-$\mathrm{FI}_G$-module, and that $H^*(Z;\mathbb{Q}_l)$ is convergent. Then for any FI_G character polynomial P,*

$$\lim_{n\to\infty}q^{-\dim Z_n}\sum_{y\in Y_n(\mathbb{F}_q)}P(\sigma_y)=\sum_{i=0}^{\infty}(-1)^i\operatorname{tr}\left(\mathrm{Frob}_q:\langle H^i_{\acute{e}t}(Z;L),P\rangle\right) \tag{4.9}$$

Proof. Since each Z_n is smooth quasiprojective, we can apply (4.8) to it. By linearity, we can replace a representation V of W_n with a virtual representation given by a character polynomial P, so we obtain

$$q^{-\dim Z_n}\sum_{y\in Y_n(\mathbb{F}_q)}P(\sigma_y)=\sum_{i=0}^{2\dim Z_n}(-1)^i\operatorname{tr}\left(\mathrm{Frob}_q:\langle H^i_{\acute{e}t}(Z_n;L),P\rangle_{W_n}\right) \tag{4.10}$$

Call this sum A_n. Furthermore, let

$$B_n = \sum_{i=0}^{2\dim Z_n} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{ét}(Z; L), P\rangle\right)$$

Our goal is thus to show that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$: that is, first of all that both sequences converge, and that their limits are equal.

By our assumption that $H^*(Z)$ is convergent, we know that there is a function $F_P(i)$ which is subexponential in i, such that for all n,

$$|\langle H^i_{ét}(Z_n; L), P\rangle_{W_n}| \leq F_P(i)$$

and thus in particular (taking n large enough)

$$|\langle H^i_{ét}(Z; L), P\rangle| \leq F_P(i)$$

Furthermore, by Deligne [Del80, Thm 1.6], we know that

$$\left|\operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{ét}(Z_n; L), P\rangle\right)\right| \leq q^{-i/2}\left|\langle H^i_{ét}(Z_n; L), P\rangle\right|$$

We thus have

$$\begin{aligned}
|A_n| &\leq \sum_{i=0}^{2\dim Z_n} \left|\operatorname{tr}\left(\mathrm{Frob}_q : \langle H^i_{ét}(Z_n; L), P\rangle_{W_n}\right)\right| \\
&\leq \sum_{i=0}^{2\dim Z_n} q^{-i/2}\left|\langle H^i_{ét}(Z_n; L), P\rangle\right| \\
&\leq \sum_{i=0}^{2\dim Z_n} q^{-i/2} F_P(i)
\end{aligned}$$

For exactly the same reason, $|B_n| \leq \sum_{i=0}^{2\dim Z_n} q^{-i/2} F_P(i)$. Since $F_P(i)$ is subexponential in i, this means that both A_n and B_n converge.

It remains to show that $\lim_{n\to\infty} A_n - B_n = 0$. Let $N(n,P)$ be the number such that

$$\langle H^i(Z_n), P\rangle_{W_n} = \langle H^i(Z), P\rangle \quad \text{for all } i \leq N(n,P).$$

We thus have

$$\begin{aligned} |B_n - A_n| &\leq \sum_{i=0}^{2\dim Z_n} q^{-i/2} \left|\langle H^i(Z), P\rangle - \langle H^i(Z_n), P\rangle_{W_n}\right| \\ &= \sum_{i=N(n,P)+1}^{2\dim Z_n} q^{-i/2} \left|\langle H^i(Z), P\rangle - \langle H^i(Z_n), P\rangle_{W_n}\right| \\ &= \sum_{i=N(n,P)+1}^{2\dim Z_n} 2q^{-i/2} F_P(i) \end{aligned}$$

Since $N(n,P) \to \infty$ as $n \to \infty$, and $F_P(i)$ is subexponential in i, we conclude that $|B_n - A_n|$ becomes arbitrarily small as $n \to \infty$. $\square$

We can now apply this to $\mathrm{Conf}_n^G(X)$ to obtain Theorem 1.2.3.1.

Proof of Theorem 1.2.3.1. By Theorem 4.1.4, $H^i(\mathrm{Conf}^G(X)_{/\overline{\mathbb{F}}_q}$ is a finitely-generated $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-$\mathrm{FI}_G$-module. By Theorem 1.2.2 $H^*(\mathrm{Conf}^G(X))$ is convergent. We thus conclude by Theorem 4.4.1. $\square$

4.5 Point-counts for polynomials and Gauss sums

For special choices of X in Theorem 1.2.3.1, we can give an interpretation to the left-hand side of (1.1) in terms of point-counts of polynomials over $\mathbb{F}_q$. The example we consider here is where $X = \mathbb{G}_m$ and $G = \mathbb{Z}/d\mathbb{Z}$ acting by multiplication by a d-th root of unity. In order for this action to be well-defined, we need to consider X as a scheme over $\mathbb{Z}[\zeta_d]$. However, notice that here the action is not free: if we look at the fiber of X over a prime dividing d, then $\mathbb{Z}/d\mathbb{Z}$ will act trivially, because these are the primes that ramify in $\mathbb{Z}[\zeta_d]$. Thus, to satisfy our hypotheses that we have a free action of G, we consider X as a scheme over

$\mathcal{O}_k[1/d]$, where $k = \mathbb{Q}(\zeta)$ is the cyclotomic field. Thus the finite fields $\mathbb{F}_q$ we consider will satisfy $q \equiv 1 \mod d$, since $\mathbb{F}_q$ is a residue field of $\mathbb{Z}[\zeta]$.

In this case, we have

$$\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{G}_m)(R) = \{(x_i) \in R^n \mid x_i \neq 0, x_i \neq \zeta^k x_j\}$$

so that $\mathrm{Conf}_n^G(X)$ is the complement of a hyperplane arrangement. The arithmetic and étale cohomology of this arrangement was studied by Kisin-Lehrer in [**?**], where they obtained formulas for the equivariant Poincaré polynomial of $\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{G}_m)$.

By Björner-Ekedahl [BE97], the action of Frob_q on $H^i_{\acute{e}t,c}(Z;\mathbb{Q}_l)$ is given by multiplication by q^i, and thus (again, by Poincaré duality)

$$\mathrm{tr}\left(\mathrm{Frob}_q : \langle H^i_{\acute{e}t}(\mathrm{Conf}_n^G(X)), P\rangle\right) = q^{-i}\langle H^i_{\acute{e}t}(\mathrm{Conf}_n^G(X)), P\rangle$$

This lets us compute the right-hand side of (1.1) explicitly in this case:

$$\lim_{n\to\infty} q^{-n} \sum_{f\in \mathrm{Poly}_n(\mathbb{F}_q^*)} P(y) = \sum_{i=0}^{\infty} (-1)^i q^{-i} \langle H^i(\mathrm{Conf}^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{C}), P\rangle \tag{4.11}$$

since we know by Theorem 4.1.4 that $H^i_{\acute{e}t}(\mathrm{Conf}^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{G}_m)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_l) \cong H^i(\mathrm{Conf}^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q}_l)$.

Notice that $d \mid q-1$, so $\mathbb{Z}/d\mathbb{Z}$ is a quotient of $\mathbb{Z}/(q-1)\mathbb{Z}$. Thus any representation of $\mathbb{Z}/d\mathbb{Z} \wr S_n$ lifts to a representation of $\mathbb{Z}/(q-1)\mathbb{Z} \wr S_n$, so we can always interpret the left-hand side of (4.11) as a statement about representations of the single group $\mathbb{Z}/(q-1)\mathbb{Z} \wr S_n$. On the other hand, as remarked in [Casb, §3], there is a Galois cover

$$\mathrm{Conf}_n^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*) \to \mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*)$$

with deck group $(\mathbb{Z}/\frac{q-1}{r}\mathbb{Z})^n$ and so by transfer,

$$H^i(\mathrm{Conf}_n^{\mathbb{Z}/d\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q}) = \left(H^i(\mathrm{Conf}_n^{\mathbb{Z}/(q-1)\mathbb{Z}}(\mathbb{C}^*);\mathbb{Q})\right)^{(\mathbb{Z}/\frac{q-1}{r}\mathbb{Z})^n}$$

and thus the right-hand side of (4.11) is the same whether we consider V as a representation of $\mathbb{Z}/d\mathbb{Z} \wr S_n$ or $\mathbb{Z}/(q-1)\mathbb{Z} \wr S_n$. Therefore we lose nothing if we simply assume that in fact $d = q - 1$.

Now we give some number-theoretic meaning to the left-hand side of (4.11). An element $f \in \mathrm{Poly}_n(\mathbb{F}_q^*)$ is a polynomial in $\mathbb{F}_q[T]$ that does not have 0 as a root. The roots of $f(T)$ are sitting in some extension field of $\mathbb{F}_q$, and the d-th roots of those roots possibly in some even higher extension field. The permutation σ_f is the action of Frob_q on all these d-th roots, which permutes the actual roots (think of these as the columns each containing a set of d-th roots), and then further permutes the d-th roots cyclically. This precisely gives an element of $G^n \rtimes S_n$ (up to conjugacy).

Recall that a *Gauss sum* is a certain sum of roots of unity obtained by summing values of a character of the unit group of a finite ring. Now, suppose that χ is an irreducible character of G. Write

$$X_i^\chi = \sum_{g\in G} \chi(g) X_i^g.$$

For each i, the $\{X_i^\chi\}_{\chi\in\widehat{G}}$ have the same span as $\{X_i^g\}_{g\in G}$, and the $\{X_i^\chi\}$ are more natural to use here.

Since $d = q-1$, there is an isomorphism $\mathbb{F}_q^* \cong G$, and in fact such a surjection $\mathbb{F}_{q^k}^* \to G$ for any k. It therefore makes sense to talk about applying χ to elements of $\overline{\mathbb{F}}_q^*$. For a given $f \in \mathrm{Poly}_n(\mathbb{F}_q^*)$, consider first decomposing f into irreducibles factors over $\mathbb{F}_q$, and then each of these into linear factors over $\overline{\mathbb{F}}_q$; since none are zero, all the roots actually lie in $\overline{\mathbb{F}}_q^*$. For a given irreducible degree-i factor p of f, since Frob_q acts transitively on the roots of p over $\overline{\mathbb{F}}_q$, and since $q \equiv 1 \mod d$, then χ takes the same value on each of the roots. It is then straightforward to calculate that

$$X_i^\chi(\sigma_f) = \sum_{\deg(p)=i} \chi(\mathrm{root}(p))$$

where the sum is taken over all irreducible factors p of f of degree i, and $\mathrm{root}(p)$ denotes any of the roots of p in $\overline{\mathbb{F}}_q$. Thus, X_i^χ is a Gauss sum of χ applied to the roots of degree-i irreducible factors of f. A general character polynomial is generated as a ring by the X_i^χ's, so this says how to interpret the left-hand-side. Thus, (4.11) says that the *average value* of any such Gauss sum across all polynomials in $\mathrm{Poly}_n(\mathbb{F}_q^*)$ always converges to the series in q^{-1} on the right-hand side of (4.11). In particular, the decomposition of H^1 and H^2 determined in [Casb, §4.1] allows us to compute the examples (1.3) and (1.4) from the introduction.

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