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HYDRODYNAMICS OF WEYL SEMIMETALS

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"What then is truth? A movable host of metaphors, metonymies, and anthropomorphisms: in short, a sum of human relations which have been poetically and rhetorically intensified, transferred, and embellished, and which, after long usage, seem to a people to be fixed, canonical, and binding. Truths are illusions which we have forgotten are illusions- they are metaphors that have become worn out and have been drained of sensuous force, coins which have lost their embossing and are now considered as metal and no longer as coins."

— Friedrich Nietzsche

"Once you free your mind about a concept of harmony, and of music being correct, you can do whatever you want."

— Giorgio Moroder

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

Twelve years is a long time to spend in graduate school. However, the memory of my first morning in Chicago, when I stepped out of O'Hare international airport into a bleak and bitterly cold December morning, and the initial pangs of regret accompanying my first steps in this alien world, are still fresh in my mind. The initial thoughts that popped into my head (as I stopped feeling any sensation in my frozen hands) were along the lines of "need to finish PhD as quickly as I can; won't survive long in this cold". Needless to say, I was wrong on both counts; I did get used to the cold pretty quickly, so obviously I also ended up doing the opposite of finishing quickly. It has been a long time since that morning, and during this long adventure, I have lived through a few more mornings when I have been similarly blue. However, this morning, I feel glad that I landed in this beautiful city, which just happens to be a little chilly, and at the University of Chicago. I can confidently say now, with the benefit of hindsight, that I landed up in exactly the right place for me; for that, I am very grateful.

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Above all, I am grateful to Mrinalini for the fact that she has been the chief source of color in my life in this last decade. She has shown me things, taken me to places, and given me experiences that would be beyond my (fairly unimaginative) brain to conjure up, which have invariably left my life immesurably enriched; This she has done untiringly, even though I have (regrettably) been often a less than ideal partner to her. For this I am truly grateful. Finally, I would like to thank Mrinalini for persuading me to postpone plans to open a tea shop in the Himalayas, an idea I had become enamored with on a trip to McLeod Ganj shortly after we got married. Her point was that the tea shop can wait, but if I did not get back to Physics at that moment, I would regret it for the rest of my life. She was right about that; and quite literally, this would not have been possible without her.

ABSTRACT

Weyl semimetals are three dimensional topological states of matter whose band structures are characterized by the presence of points of degeneracy between bands near the Fermi energy. These points can appear in three-dimensional materials that break at least one of inversion or time reversal symmetry. Their presence in a band structure, and the consequent emergence of chiral species of particles that are mirror images of one another, leads to macroscopic behavior that is qualitatively different from normal metals, including Chiral Anomaly and the related phenomenon of Negative Magnetoresistance. In this thesis, we develop a hydrodynamic description for Weyl semimetals, suitable for modelling slow non-equilibrium situations such as are commonly encountered in typical transport experiments. Our analysis is based on macroscopic reasoning such as symmetry considerations and the 2^{nd} law of thermodynamics. Using this phenomenological model, we explain magnetoresistance, and the related effect of magnetic enhancement to the thermoelectric conductivity. We also touch upon some general aspects of formulation of hydrodynamic theories for solid state electronic systems, and motivate their construction in clean electronic systems.

CHAPTER 1

INTRODUCTION

1.1 Hydrodynamics

Hydrodynamics[1] is a theory that describes the dynamics of a physical system at long length and time scales. The physical system is assumed to be composed of interacting microscopic constituents, with microscopic processes occuring on some typical timescale τ_{micro} and length l_{micro}. For example, if we envision our physical system to be composed of microscopic particles colliding with each other, $\tau_{micro} \sim \tau_{mft}$, the mean free time that elapses between collisions. Processes that evolve over times $\tau >> \tau_{micro}$ and over lengths $l >> l_{micro}$ may be described by a physical theory that is *universal*, or independent of the details of the microscopic constituents and their interactions. Such a theory is called a theory of Hydrodynamics.

It is assumed that if such a system is left isolated from external influences for a very long time, the system approaches a state of global thermal equilibrium. Under such circumstances, the system is described by thermodynamics. In thermodynamics, a macroscopic system is described by a few conserved charges, such as total energy of the system E or total number of particles N (along with the total volume V), and their thermodynamically conjugate variables, such as temperature T, chemical potential μ and pressure p.

The 'conserved' charges are so called because they represent extensive macroscopic variables that remain constant in an isolated system. The thermodynamic variables are constrained by an equation of state whose form is dependent upon the details of the microscopic theory. This may be written in the form $S = S(E, N, V...)$, where S is the total entropy of the system. Alternately, we may represent the equation of state in terms of pressure

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[1] This section is based on lecture notes from PHYS 429: *Advanced Hydrodynamics* taught during Winter 2021 at the University of Chicago by D. T. Son

as $p = p(T, \mu, ...)$, or in terms of densities: $S/V \equiv s = s(\varepsilon, n, ..)$, where $\varepsilon = E/V$ is the energy density, n is the number density etc. Other than this relation, thermodynamics is not dependent on the details of how the microscopic constituents behave; the relations of thermodynamics are completely general and universal.

Thermodynamics represents a universal theory to describe states of global equilibrium; hydrodynamics can be seen as an extension of thermodynamics to describe situations of *local* equilibrium. Let us imagine a situation in which the macroscopic medium above has been perturbed off equilibrium by a sudden change. We assume that within a few τ_{micro}, the system is locally thermalized. By this, we mean that we may assign a nearly constant T, μ over macroscopically large regions of the medium, which may nevertheless be small relative to the dimensions of the overall system. Now, the state of the overall system may be represented by smoothly varying functions of space and time: $T(\mathbf{x}, t)$, $\mu(\mathbf{x}, t)$, which we call hydrodynamic variables, and conserved densities, $\varepsilon(\mathbf{x}, t)$ or $n(\mathbf{x}, t)$. The resolution of $\mathbf{x}$ and t required to describe the state of the system are much larger than l_{micro} and τ_{micro}. This separation of scales between microscopic processes and macroscopic phenomena are an essential assumption in any hydrodynamic theory.

The equations of hydrodynamics consist of differential equations that take the form of local conservation laws for the conserved charges. For example, in an electrically charged medium, the electric charge satisfies the continuity equation $\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$, where $\mathbf{j}$ is the associated electric current. This must be supplemented by the equation of state $n = n(T, \mu, ..)$, as well as the 'constitutive relations', which describe the dependence of the currents on the hydrodynamic variables, $\mathbf{j} = \mathbf{j}[T, \mu ..]$.

The constitutive relations are functionals of the hydrodynamic variables: the currents depend not just on the local values, but also on the derivatives of the hydrodynamic variables. Roughly speaking, $O(\partial^n)$ contribution to the current $\mathbf{j}$ is of the order of $(l_{micro}/\ L)^n$ in importance, where L is the typical lengthscale of external perturbations driving the system off equilibrium. It follows that when the non-equilibrium processes being described occur on much larger time and length scales relative to the microscopic scales, we may retain only the lowest derivative orders in the hydrodynamic theory. This is the situation where the hydrodynamic theory is useful in describing the physics of the system.

In this situation, one may write the currents or 'fluxes' as $\mathbf{j} = \sum_i \alpha_i \mathbf{X}_i$, where $\mathbf{X}_i$ are $O(\partial^1)$ 'forces' that are driving the currents, such as gradients of hydrodynamic variables, $\boldsymbol{\nabla} T$, $\boldsymbol{\nabla} \mu$, or external electric field $\mathbf{E}$ etc. The coefficients α are called 'Kinetic coefficients'. Like the densities, the kinetic coefficients are also functions of the hydrodynamic variables.

The most familiar hydrodynamic theory is what is traditionally called 'fluid dynamics' [35], describing the dynamics of everyday fluids (such as water). In this hydrodynamic theory, there are three conserved charges: ε, mass density ρ and momentum density $\boldsymbol{\pi}$. Correspondingly, there are three hydrodynamic variables: T, μ and flow velocity, $\mathbf{v}$, or equivalently, T, $\mathbf{v}$ and p. The presence of $\mathbf{v}$, a vector, in the set of hydrodynamics variable, means that currents now have the form $\mathbf{j} = n\mathbf{v} + \sum_i \alpha_i \mathbf{X}_i$, where n is a thermodynamic function. The $O(\partial^0)$ contribution to the current corresponds to 'convection', whereby the flow of the fluid advects a conserved charge even in the absence of any external forces driving the currents.

From thermodynamics, we know that any change in the state of a system is reversible if the process is adiabatic, i.e. where the total entropy of the universe is unchanged during the process, whereas, 'dissipative' processes, where entropy production takes place, are irreversible. Since hydrodynamics is the description of how non-equilibrium processes that bring about change in the state of the system unfold, it should not be surprising that this link between dissipation and reversibility can be explicitly seen in hydrodynamics.

In conventional fluid dynamics, the $O(\partial^1)$ currents correspond to irreversible flows, whereas the convective currents are reversible. Let us imagine that at some time $t = t_0$, the state of the fluid is described by $T(\mathbf{x}, t_0) = T_0(\mathbf{x})$, $p(\mathbf{x}, t_0) = p_0(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}, t_0) = \mathbf{v}_0(\mathbf{x})$. This evolves to $T_1(\mathbf{x})$, $p_1(\mathbf{x})$ and $\mathbf{v}_1(\mathbf{x})$ at some time $t_1 > t_0$. The flow is said to be 'reversible' if the reverse flow is also possible. This means if we describe the initial configuration at time $-t_1$ as the time reversed version of the original final state, i.e. $T_1^{\mathcal{T}}(\mathbf{x}) = T_1(\mathbf{x})$, etc. but $\mathbf{v}_1^{\mathcal{T}}(\mathbf{x}) = -\mathbf{v}_1(\mathbf{x})$, then this evolves to the time reversed version of the original initial state, $T_0(\mathbf{x})$, $p_0(\mathbf{x})$, $-\mathbf{v}_0(\mathbf{x})$ at time $-t_0 > -t_1$.

This is only possible if the currents are also reversed in going from the final state at t_1 to the time reversed initial state prepared at time $-t_1$, $\mathbf{j}^{\mathcal{T}}(\mathbf{x}, -t_1) = -\mathbf{j}(\mathbf{x}, t_1)$. Clearly, this is true when $\mathbf{j} \sim \mathbf{v}$, as in the time reversed state, $\mathbf{v}^{\mathcal{T}}(\mathbf{x}) = -\mathbf{v}(\mathbf{x})$. However, if we have a current that depends on a gradient of some thermodynamic variable, e.g. $\mathbf{j} \sim \boldsymbol{\nabla} T$, then in the time reversed state, this does not reverse, as $T^{\mathcal{T}}(\mathbf{x}, -t) = T(\mathbf{x}, t)$, so $\boldsymbol{\nabla} T^{\mathcal{T}} = \boldsymbol{\nabla} T$. So, advective currents are reversible, whereas the $O(\partial^1)$ currents are irreversible.

It can be explicitly seen that only the irreversble flows contribute to local production of entropy. For this, we need to write down the continuity equation for local entropy density s. The 2^{nd} law of thermodynamics states that entropy of the universe can only increase. In the context of hydrodynamics, this means that

$$\partial_t s + \boldsymbol{\nabla} \cdot \mathbf{j}^s \equiv \mathcal{R} \geq 0 \tag{1.1}$$

where $\mathbf{j}^s$ is the entropy current; i.e. entropy can only be locally produced at the rate $\mathcal{R} \geq 0$. In order to express $\mathcal{R}$ in terms of the conserved charge currents, we may use the 1^{st} law of thermodynamics, which, in the case of conventional fluid dynamics, where energy density ε, number (or 'mass') density n and momentum density $\boldsymbol{\pi}$, takes the form:

$$Tds = d\varepsilon - \mu dn - \mathbf{v}.d\boldsymbol{\pi} \tag{1.2}$$

This may be used to relate $\partial_t s$ to $\partial_t \varepsilon$, $\partial_t n$ and $\partial_t \boldsymbol{\pi}$, and hence, to the conserved currents. Doing this exercise in detail shows us that the contribution to the convective parts of the currents to $\partial_t s$ is precisely $-\boldsymbol{\nabla} \cdot (s\mathbf{v})$, i.e. $-$ the convective part of the entropy current $\mathbf{j}^s$. Hence, these do not contribute to $\mathcal{R}$. On the other hand, the irreversible flows remain in the expression for $\mathcal{R}$, which takes a form $\mathcal{R} = \sum_i \mathbf{j}_i^{(\mathcal{R})} \cdot \mathbf{X}_i$, i.e. inner products of dissipative fluxes and thermodynamically conjugate forces.

The currents that appear in the expression for $\mathcal{R}$ are themselves linear combinations of the forces, $\mathbf{j}_i = \alpha_{ij}\mathbf{X}_\mathbf{j}$, eventually $\mathcal{R}$ may be expressed in a quadratic form involving only forces: $\mathcal{R} = \alpha_{ij}\mathbf{X}_i \cdot \mathbf{X}_j$. The fact that $\mathcal{R} \geq 0$ for arbitrary (externally induced) forces imposes constraints on the allowed structure of $\mathbf{j}$. For the kinetic coefficients that appear in the above expression for $\mathcal{R}$, the constraints take the form of inequalities. These are the dissipative kinetic coefficients. These are the only form of kinetic coefficients we meet in text-book fluid dynamics.

It is not always necessary that all currents controlled by external forces lead to dissipation. For example, in the presence of a force that behaves like $\mathbf{v}$ under time reversal, then, the corresponding kinetic coefficients would be 'non-dissipative', i.e. instead of appearing in the final expression for $\mathcal{R}$, they would instead cancel out by some special arrangement, much like the convective parts of the currents entering the expression for $\partial_t s$ must exactly cancel out the convective part of $\boldsymbol{\nabla} \cdot \mathbf{j}^s$. We shall explore in this thesis electronic systems, where magnetic field $\mathbf{B}$ is a relevant forcing, which also flips under time reversal like $\mathbf{v}$. The associated kinetic coefficients will indeed be seen to have the structure we are describing here.

Conventional fluid dynamics is a hydrodynamic theory describing slow deformations off equilibrium for a material medium. It is intuitively clear why a certain quantity of water possesses mass or energy; We can see and feel it. However, hydrodynamics has much wider applicability. For example, we may apply hydrodynamics to situations where these conserved charges are emergent quantities, and may not be as easy to intuitively motivate as in the case of a material medium such as water. Consider an electronic system, such as a metal or a semiconductor, which is our system of interest in this thesis. Here too, one may see that there are certain conserved charges (electric charge density n, energy density ε, and in the case of very clean electronic conductors, also momentum density $\boldsymbol{\pi}$) that are transported by the underlying electrons.

In the existing literature on electronic systems, the word 'hydrodynamics' is usually used in the context of physical systems which have 'flow' $\mathbf{v}$, that is, in systems where momentum is conserved. This is the case in exceptionally clean metals at low temperatures, where the dominant mode of relaxation is momentum conserving collisions between electrons, rather than momentum non-conserving collisions between electrons and impurities, or electrons and phonons [21], [16]. Recent examples in the literature of momentum conserving hydrodynamics include the observation of Dirac fluid in clean graphene [15], [41], and evidence of viscous dynamics in wires made of $\mathrm{PdCoO_2}$ [42]. However, the machinery of hydrodynamics is equally well applicable to systems without momentum conservation [34].

Figure 1.1: Crowds of people often show hydrodynamic behavior. Coherent and cooperative behaviour at the macroscopic level emerges from the actions of individual constituents interacting with their neighbors at the microscopic level. (Photograph courtesy: Mrinalini Pandey [48])

A hydrodynamic theory must be *universal*, i.e. independent of the details of the underlying microscopic theory. As such, the construction of hydrodynamic theory for a given physical system makes use of macroscopic organizing principles to constrain the structure of the constitutive relations, with minimal input from the underlying microscopics. The positivity of entropy production discussed earlier is one such ingredient that goes into the construction of the hydrodynamic theory.

Another such ingredient are the symmetries obeyed by the system. The presence of rotational invariance, or the presence of boost invariance, either Lorentz or Galilean, simplifies the structure of hydrodynamics. For example, consider the flow of heat in response to a temperature gradient in a fluid such as water. The heat current $\mathbf{Q}$ and the gradient of temperature $\boldsymbol{\nabla} T$ are both vectors, and in general, the kinetic coefficient is a rank-2 object: $\mathbf{Q}_i = -\kappa_{ij} \boldsymbol{\nabla}_j T$, where i, j are now spatial indices. κ_{ij} is the thermal conductivity tensor.

Now, because from any point in the fluid, all directions look the same (local isotropy), κ_{ij} can only be constructed from rotationally invariant tensors and any thermodynamic degrees of freedom that are vectors. In 3 dimensions, there are 2 rotationally invariant tensors: Kronecker Delta $\boldsymbol{\delta}_{ij}$, and totally antisymmetric Levi-Civita symbol $\boldsymbol{\epsilon}_{ijk}$. In the hydrodynamic theory of fluids, one also has $\mathbf{v}$, which is an $O(\partial^0)$ vector, that can be used to construct higher rank objects. Using these, one may come up with 3 distinct tensor structures, which therefore, correspond to 3 kinetic coefficients: $\kappa_{ij} = \kappa_1\boldsymbol{\delta}_{ij} + \kappa_2\mathbf{v}_i\mathbf{v}_j + \widetilde{\kappa}\boldsymbol{\epsilon}_{ijk}\mathbf{v}_k$. where the scalar coefficients can be functions of the thermodynamic variables, i.e. $\kappa_1 = \kappa_1(T, p, |\mathbf{v}|)$ etc.

However, the hydrodynamic theory describing fluids such as water must also obey invariance under Galilean boosts, i.e. the equations should look the same in two frames of reference related by a Galilean boost. Clearly, in such a fluid, one can only have one coefficient of thermal conductivity κ_1, and that too has to be independent of $|\mathbf{v}|$. Another physical way to motivate the same result is by using the fact that pure convection, that is, a fluid moving with a constant velocity at all points, is dissipationless. Hence, the rate of entropy production in a non-equilibrium process, $\mathcal{R}$ must be the same irrespective of the frame it is measured from. From our previous discussion, we know that we may write $\mathcal{R} = \alpha_{ij}\mathbf{X}_i\mathbf{X}_j$, so, one contribution to $\mathcal{R}$ would be of the form $\kappa_{ij}\boldsymbol{\nabla}_iT\boldsymbol{\nabla}_jT$. The rate $\mathcal{R}$ can only be independent of the frame in which it was measured if it is independent from $\mathbf{v}$. This leads to the same conclusion.

Time reversal invariance of the underlying microscopic system is a further principle that can be used to limit the constitutive relations. The macroscopic manifestation of reversibility of motion at the microscopic level are the Onsager reciprocal relations [40]. These are relations between dissipative kinetic coefficients that reduce the number of independent kinetic coefficients required to characterize the system. The simplest example is that of a rank-2 kinetic coefficient, such as the thermal conductivity, in an anisotropic medium. In a general system without isotropy, one cannot reduce the κ_{ij} into a single coefficient. However, due to Onsager reciprocity, one may still relate the off diagonal elements, i.e. $\kappa_{ij} = \kappa_{ji}$, where i, j are spatial degrees of freedom. So, in a 3 dimensional anisotropic material that can conduct heat, one may only have 6 independent coefficients of thermal conductivity, instead of 9. Generally, the Onsager reciprocal relations take the form $\alpha_{ij} = \alpha_{ji}$, where i, j can be spatial or thermodynamic degrees of freedom.

1.2 Topological phases of matter

One learns in elementary school that matter exists in different forms, or 'phases'. The first classification we are exposed to is that of solids, liquids, and gases. Solids are rigid, liquids and gases can flow. Liquid water, when in a jar, will sit at the bottom, but in vapor form, it will fill the jar, and even escape if allowed. It makes sense to classify things as solid and liquids because *all* liquids can flow, and *all* solids are rigid. Ice may be the same substance as water, but as far as its response to an external force is concerned, it would have more in common with a log of wood than its liquid form. We are also made to understand that the transition between phases, such as melting, is a sharp process, accompanied by qualitative change in physical properties.

Some time later, we are taught that all matter is ultimately composed of atoms and molecules, which are nothing but electrons revolving around nuclei made of protons and neutrons. One therefore begins to understand that even though the basic constituents of all matter is the same, they can organize in ways that can lead to very different observable behavior on our (macroscopic) scale of observation. At around the same time, the suspicion also starts taking root that the story of phases as taught in elementary school must be incomplete. What is a magnet, for example? Surely, it is a solid, but there is something different about a magnet from ordinary solids. Not all solids can attract a bar of iron for example.

If one's curiosity towards physics holds a few more years, one finally begins to understand that the story as taught in elementary school is indeed incomplete. Matter can exist in three forms specifically in reference to its mechanical properties, and there too, the story is quite murky. There can be amorphous substances (like glass) that can look quite solid on our (human) scale of observation, but really, are not 'rigid' in the same way as crystalline solids; something that eludes the simple categorization of solid vs liquid. Or, the fact that beyond a certain critical temperature and pressure, the phenomenon of boiling, accompanied by a discontinuous jump in density between the liquid and vapor phase, disappers entirely, so perhaps, they *aren't* distinct phases of matter after all in any well defined manner. One also learns that it is indeed acceptable to refer to magnets as a distinct phase of matter, and indeed, it is one of many distinct forms of macroscopic behavior that matter can display with respect to its electromagnetic properties. Materials that can conduct electricity (conductors), compared to materials that cannot (insulators). Materials that can conduct electricity without resistance (superconductors) compared to materials that can conduct electricity, but not dissipationlessly (ordinary conductors) etc.

A substance that can exist in more than one phase will undergo phase transitions. Many phase transitions are accompanied by a change in the symmetries obeyed by the thermodynamically favorable configuration on either side of the phase transition. For example, a liquid enjoys the full symmetry of the dynamics of its microscopic constituents, and thus, obeys continuous translation symmetry. However, at low temperatures, the liquid solidifies into a state of lower symmetry (crystal), that only obeys discrete translation symmetry. Similarly, below the Curie temperature, the thermodynamically stable configuration for a ferromagnet to exist in is one with a non-zero magnetic moment, spontaneously chosen, that breaks rotational symmetry. This is the general pattern of spontaneous symmetry breaking, a phenomenon whereby at high temperatures a substance exists in a symmetric phase, enjoying the full set of microscopic symmetries, whereas at lower temperatures, it exists in a symmetry breaking phase, characterized by an order parameter (like magnetization) that breaks the symmetry explicitly. These ideas were first developed by Landau and then later expanded upon by development of Renormalization Group analysis of critical phenomena [33] [68].

For a long time, it was believed that symmetry was the only organizing principle by which different types of order in which matter may exist may be classified. However, the discovery of the Quantum Hall Effect led to the realization that symmetry and symmetry breaking alone do not encompass all the distinct phases of matter than exist. In QHE, a 2D electronic system subject to a strong perpendicular magnetic field shows quantization of Hall conductance σ_{xy} (defined as the ratio of current I_x flowing in the x direction and the Hall voltage V_y that develops perpendicular to the current; $\mathbf{B} = B\hat{z}$). As one sweeps through a range of values for the magnetic field, the Hall conductance shows a staircase like variation; Sudden changes followed by plateaux where σ_{xy} is constant. Moreover, the longitudinal conductance σ_{xx} is zero on each plateau, and only shows a spike in the narrow ranges of magnetic field where the Hall conductance jumps from one step to the next.

At any of these plateaux, the value of the Hall conductance is precisely quantized to be $\sigma_{xy} = \nu e^2/h$, where ν can be an integer [66], or a rational fraction [62]. Moreover, this phenomenon is insenitive to details such as the amount of impurities present, or the geometric shape of the experimental sample. Each plateau characterized by a distinct ν is a distinct state of matter, that exists for some range of magnetic fields. However, all these phases enjoy the same symmetries, so symmetry cannot be the principle that classifies them. Rather, the emergence of integers and rational fractions suggests that the basis of classification in this case is topological.

There is an intuitive connection between integers and topology. Indeed, the popular perception of the word 'topology' has to do with how a coffee cup is equivalent to a donut, and how these are distinct from a ball. A coffee cup can be deformed into a donut, but not a ball, because of differing number of holes. If we now imagine a generalization of this idea, the number of holes, or the 'genus' of a closed surface is topological information, as opposed to its precise shape, which is a matter of geometry. Obviously, any closed surface can only have an integer number of holes.

Formally, the emergence of topological information from geometry in this context is encoded in the famous Gauss-Bonnet theorem, which relates the surface integral of the local curvature over a closed surface to the genus of the surface. This is instructive, because it shows us that topological information about a geometric space emerges when we consider its global properties. The local curvature is a locally defined quantity, sensitive to geometric details. However, its surface integral is a global property of the object, and hence, should be insensitive to smooth changes in geometric details. This is why the surface integral carries topological information about the object.

The topology referred to in the case of topological phases is that of the Hilbert space of states of the underlying quantum mechanical system. In this sense, electronic states of matter referred to as 'topological' are macroscopically quantum mechanical, and the underlying quantum mechanical nature is reflected in macroscopic properties of the system. This does not make these states of matter special though. Even the phenomenologically simplest electronic state of matter, ordinary insulator, requires quantum mechanics to explain its insulating behavior. Electrons in crystalline substances reside in energy bands. Due to the quantum mechanical phenomenon of level repulsion, these bands usually are separated in energy, much like the energy levels in an atom. In a spatially finite sample, each band contains a finite number of states. Pauli exclusion principle implies that at most, only one electron can occupy each energy state in the band. So, it is possible for band to be completely full. Insulators are precisely substances where the last occupied band (and the ones below in energy) are completely full, and the next unoccupied band has a large energy gap. In essence, insulators cannot conduct electricity because there is nowhere for the electrons to go. In contrast, conductors are substances that have a partially filled band.

What instead makes topological phases special is that they arise when the underlying Hilbert space of states has a topologically non-trivial structure. Since the discovery of the QHE, it has been realized that topological effects are more widely spread than in the 2D electron systems subjected to strong perpendicular magnetic fields. Some highlights have included the work of Haldane [23], who presented a model of realizing QHE with a magnetic field that is zero on average (the so called Quantum Anomalous Hall Effect), and the recent prediction and discovery of Topological insulators, time reversal invariant states of matter that can exist in 2 or 3 dimensions. Like ordinary insulators, topological insulators are insulating in the bulk, but have metallic surfaces [24] [51].

In each case, the topological phases are characterized by the presence of ‘topological invariants’, that are dependent on the global properties of the state space. In IQHE, the topological invariant is called the ‘Chern number’, which can take up values belonging to the set of integers, $\mathbb{Z}$. In the TI state, the integer (or set of integers, dependent upon the dimensionality of the system) can only take up two values, even or odd, in trivial and topological insulators respectively; accordingly, the topological invariant is called the $\mathbb{Z}_2$ invariant.

The topological invariants characterizing these states of matter are physical observables. The physical consequences of the Chern number is the quantization of σ_{xy} into integer multiples of e^2/h [61]. The $\mathbb{Z}_2$ invariant, on the other hand, counts the number of propagating boundary modes localized at the edge of an insulator. Only an odd number of modes ensures that the surface conducts. Hence, the relevant physical observable in this case is the metallicity of the surface. In each case, the topological invariant is insensitive to smooth changes of the system. In order to change the topological invarant characterizing a certain topological state, one would have to change the topology of the underlying space of states, leading the system through a quantum phase transition.

Quantum Hall Effect, Topological insulators, are all insulating states of matter, i.e. their band structure is gapped, and the lack of conduction is a result of the finite energy gap separating the conduction and the valence bands. However, topological effects are not restricted to gapped systems alone, and may also be observed in conductors. Weyl semimetals, the subject of this thesis, is such a gapless phase of matter that are nevertheless topologically distinct in their macroscopic properties than ordinary conductors.

1.3 Topological band theory

1.3.1 Berry phase

The basic concept that is useful in describing the topological character of the state space of a quantum mechanical system is Berry phase, which we will review briefly here. A comprehensive review to the subject may be found in [70], [64].

Berry phase is the name given to the geometric phase accumulated for a quantum mechanical state evolving adiabatically through some parameter space. To make things concrete, we consider a system with a Hamiltonian that depends on a set of parameters $\mathbf{R} = (R_1, R_2, ...)$; $H = H(\mathbf{R})$. For any given $\mathbf{R}$, we assume that the Schroedinger equation has a discrete spectrum:

$$H(\mathbf{R})|n(\mathbf{R})\rangle = \epsilon_n(\mathbf{R})|n(\mathbf{R})\rangle \tag{1.3}$$

The label n label the energy bands of the system. The spectrum is assumed to be non-degenerate, and gapped at a generic value of $\mathbf{R}$, although, there may be isolated band touchings. We imagine slowly evolving the system through some curve C in parameter space; $\mathbf{R} = \mathbf{R}(t)$. By 'slow', we mean such that the typical timescale of variation $1/\omega >> \hbar/\Delta\epsilon$, where $\Delta\epsilon$ is the typical size of the energy gap between successive bands. Then, the quantum adiabatic theorem guarantees that if a system is prepared at time $t = 0$ in the state $|n(\mathbf{R}(0)\rangle$, then at any successive time $t > 0$, the system remains in eigenstate $|n(\mathbf{R}(t)\rangle$; i.e. no interband transitions happen when the evolution is slow enough.

This, however, is still not enough information to determine the state at time t; we are yet to address the issue of its phase. In general, the quantum mechanical state vector $|\psi\rangle$ signifies the same physical state as $e^{i\phi}|\psi\rangle$, for an arbitrary real phase ϕ. This applies to the eigenstates $|n(\mathbf{R})\rangle$ as well; solving the eigenvalue equation does not fix their phase. Since we are considering a variation through parameter space, we need to specify some convention for fixing this phase, such that its variation is smooth and single valued along the trajectory the system traverses. This is called gauge choice. Ideally, we would like to make a gauge choice that works over the entire parameter space of $\mathbf{R}$. In topologically non-trivial band structures however, this may not be possible. Even in such cases though, it can be assumed that one can make a consistent gauge choice in a finite neighborhood of any given $\mathbf{R}$; the whole parameter space may be covered by overlapping patches where consistent gauge choice may be made.

Assuming, therefore, that $|n(t)\rangle = |n(\mathbf{R}(t))\rangle$ has a well defined phase relation with $|n(0)\rangle$, we may then write the state at time t as:

$$|\psi(t)\rangle = e^{i\gamma_n(t)} e^{-\frac{i}{\hbar}\int_0^t dt' \epsilon_n(\mathbf{R}(t'))} |n(t)\rangle \tag{1.4}$$

The second factor is the dynamical phase that is associated with Hamiltonian evolution. The first phase factor is the so called Berry phase. Plugging this into the Schroedinger equation $i\hbar\partial_t|\psi(t)\rangle = \epsilon_n(\mathbf{R}(t))|\psi(t)\rangle$, we find that the Berry phase $\gamma_n(t)$ satisfies:

$$\gamma_n(t) = i\int_0^t dt' \langle n, \mathbf{R}(t')|\partial_{t'}|n, \mathbf{R}(t')\rangle \equiv \int_C d\mathbf{R}\cdot\mathcal{A}_n(\mathbf{R}) \tag{1.5}$$

where $\mathcal{A}_n(\mathbf{R})$ is called the 'Berry connection', or the Berry vector potential, and is given by:

$$\mathcal{A}_n(\mathbf{R}) = i\langle n(\mathbf{R})|\boldsymbol{\nabla}_{\mathbf{R}}|n(\mathbf{R})\rangle \tag{1.6}$$

The Berry connection is not gauge invariant: making the gauge transformation $|n(\mathbf{R})\rangle \rightarrow e^{i\phi(\mathbf{R})}|n(\mathbf{R})\rangle$ results in $\mathcal{A}_n(\mathbf{R}) \rightarrow \mathcal{A}_n(\mathbf{R}) - \boldsymbol{\nabla}_{\mathbf{R}}\phi(\mathbf{R})$. This led to the belief in the early years of the development of quantum mechanics that the Berry phase is unimportant, as one can always make a suitable gauge choice so that $\gamma_n(t) \rightarrow \gamma_n(t) + \phi(\mathbf{R}(0)) - \phi(\mathbf{R}(t))$ can be be cancelled out over the path C.

This belief persisted till the work of Berry [9], who considered the case of a closed contour $\mathbf{R}(t) = \mathbf{R}(0)$. Once a gauge is chosen, a unique phase factor $e^{i\phi(\mathbf{R})}$ is associated with each $\mathbf{R}$. For closed contour C, $\mathbf{R}(t) = \mathbf{R}(0)$ is the same point in parameter space. It follows that $\phi(0) - \phi(t)$ is restricted to be $2\pi m$, where $m \in \mathbb{Z}$. So, when the contour C is closed, the accumulated Berry phase γ_n can only change by an integer multiple of 2π under a gauge transform. Since we are only interested in $e^{i\gamma_n}$, this makes γ_n accumulated along a closed contour C a gauge invariant (and hence, physically relevant) quantity.

The Berry phase is a physical quantity, which encodes information about topology of the parameter space, but needs a contour C to be defined. This is inconvenient; it would be convenient to have a local (in parameter space) observable that encodes topological information. This can be found by using Stoke's theorem to convert the contour integral over C into an area integral over a 2-dimensional surface S bounded by C. Then, we may write the Berry phase as:

$$\gamma_n = \oint_C d\mathbf{R} \cdot \mathcal{A}_n(\mathbf{R}) \equiv \frac{1}{2} \int_S d\mathbf{R}_i \wedge d\mathbf{R}_j \boldsymbol{\Omega}^n_{ij} \tag{1.7}$$

where, the 'Berry curvature' $\boldsymbol{\Omega}_n$ is given as:

$$\boldsymbol{\Omega}^n_{ij} = \frac{\partial \mathcal{A}^n_j}{\partial R_i} - (i \leftrightarrow j) = i\Big(\langle\, \partial_{R_i} n(\mathbf{R}) \,|\, \partial_{R_j} n(\mathbf{R}) \,\rangle - (i \leftrightarrow j)\Big) \tag{1.8}$$

If the parameter space is 3-dimensional, then we may convert the Berry curvature tensor to its dual vector, $\boldsymbol{\Omega}^i_n$, which is related to the antisymmetric tensor by the relation $\boldsymbol{\Omega}^n_{ij} = \boldsymbol{\epsilon}_{ijk} \boldsymbol{\Omega}^i_n$. The vector Berry curvature, like Magnetic field, can be written in cross product form in terms of the vector potential:

$$\begin{aligned} \boldsymbol{\Omega}_n(\mathbf{R}) &= \boldsymbol{\nabla}_{\mathbf{R}} \times \mathcal{A}_n \\ \gamma_n &= \int_S d\mathbf{S} \cdot \boldsymbol{\Omega}_n(\mathbf{R}) \end{aligned} \tag{1.9}$$

The Berry curvature is a gauge invariant, locally defined observable quantity that encodes information about the topology of the energy bands in parameter space. As such, the berry curvature $\boldsymbol{\Omega}_n$ has observable consequences on the microscopic dynamics of the system under consideration.

We started this construction with the assumption of adiabatic evolution. This is an approximation restricting dynamics of the system to be contained in a single band n. The Berry curvature encodes information about the residual effects of the other bands $n' \neq n$, which can still affect the dynamics within the n-th band through virtual transitions. Indeed, it is possible to write the expression for Berry Curvature tensor in an alternate form that suggests this interpretation [70]. As a result, if we study the full problem (without the approximation of projecting to a single band n), we would find that the Berry Curvatures of all bands at any given $\mathbf{R}$ sum to 0:

$$\sum_n \boldsymbol{\Omega}_n(\mathbf{R}) = 0 \tag{1.10}$$

1.3.2 Band theory of electronic solids

In this thesis, we are concerned with the properties of electronic crystalline matter. The basic properties of electronic crystals are explained by band theory of solids. The cornerstone of band theory is the Bloch theorem, that tells the structure of energy eigenstates available to an electron residing in a crystalline solid. Solving the single particle Schroedinger equation with Hamiltonian $H = \mathbf{p}^2/2m + V_L(\mathbf{x})$, where V_L is the lattice potential, the eigenfunctions organize into discrete bands n, and within each band, may be labelled by wavevector $\mathbf{k}$. For a given n, $\mathbf{k}$, the eigenfunctions take the form:

$$\psi_{n,\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} u_{n,\mathbf{k}}(\mathbf{x}) \tag{1.11}$$

where $u_{n,\mathbf{k}}$ is periodic with the periodicity of the lattice. The $e^{i\mathbf{k}\cdot\mathbf{x}}$ term is the term that leads to variation across primitive cells of the lattice, whereas u looks the same in every unit cell.

The extent of each band is finite in $\mathbf{k}$-space, and is called the 'Brillouin Zone'. Let

$\{\mathbf{a}_i\}$ be the set of primitive vectors for the lattice. The set of vectors $\{\mathbf{b}_j\}$ that satisfies $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\boldsymbol{\delta}_{ij}$ are the primitive vectors of the 'Reciprocal Lattice'; Fourier Transform of a cell-periodic function in real space $\mathbf{x}$ can only have non-zero Fourier components for $\mathbf{k}$ that lie on the Reciprocal lattice. For any lattice translation $\mathbf{R} = \sum_i n_i \mathbf{a}_i$ and reciprocal lattice translation $\mathbf{G} = \sum_j m_j \mathbf{b}_j$ for some $\{n_i\}$, $\{m_j\} \in \mathbb{Z}$, $\mathbf{R} \cdot \mathbf{G} = 2\pi n$, for some $n \in \mathbb{Z}$.

Bloch functions may only be uniquely defined for $\mathbf{k}$ modulo arbitrary reciprocal lattice translation $\mathbf{G}$. To see this, one needs to solve the Schroedinger equation in Fourier space [31]. Since the potential is cell-periodic, its Fourier transform will include contributions at all $\mathbf{G}$; $V_L(\mathbf{x}) = \sum_{\mathbf{G}} V_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{x}}$. Expanding $\psi(\mathbf{x}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}$, the Schroedinger equation takes the form $(\hbar^2\mathbf{k}^2/2m - \epsilon)\psi_{\mathbf{k}} + \sum_{\mathbf{G}} V_{\mathbf{G}} \psi_{\mathbf{k}-\mathbf{G}} = 0$, which generically will have a discrete spectrum of eigensolutions ϵ_n for a given $\mathbf{k}$. The lattice potential mixes the mode $\psi_{\mathbf{k}}$ with all $\psi_{\mathbf{k}+\mathbf{G}}$. As a result of this, an energy eigenstate at $\mathbf{k}$ naturally contains contributions from Fourier modes at all $\mathbf{k}+\mathbf{G}$. Energy eigenstates are defined uniquely for each $\mathbf{k}$ only if we restrict ourselves to $\mathbf{k}$ belonging to a single primitive cell in the Reciprocal space. Conventionally, this is chosen to be the set of points in $\mathbf{k}$-space that are closer to the point $\mathbf{k} = 0$ than any $\mathbf{G}$, in which case, it is called the 'First Brillouin Zone', or simply 'Brillouin Zone'.

Because the wavevector $\mathbf{k}$ is physically equivalent to the wavevector for $\mathbf{k}+\mathbf{G}$, it follows that the edges of the Brillouin Zone physically represent the same physical state. Hence, the Brillouin Zone of a d-dimensional crystal is topologically equivalent to a d-torus.

Wavevector $\mathbf{k}$ takes the role of the parameter $\mathbf{R}$ from the previous section. In order to adapt the machinery of Berry phase to this context, we define the 'Bloch Hamiltonian' $H(\mathbf{k})$:

$$\hat{H}(\mathbf{k}) = e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} \hat{H} e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} = \frac{(\hat{\mathbf{p}} + \hbar\mathbf{k})^2}{2m} + V_L(\hat{\mathbf{x}}) \tag{1.12}$$

where ˆ notation has been introduced to demarcate the operators from the parameter $\mathbf{k}$. The cell periodic part of the Bloch function, $|u_{n\mathbf{k}}\rangle$ is the energy eigenstate corresponding to the Bloch Hamiltonian: $H(\mathbf{k})|u_{n\mathbf{k}}\rangle = \epsilon_{n,\mathbf{k}}|u_{n\mathbf{k}}\rangle$. Using the Bloch states, we may define the Berry curvature vector $\boldsymbol{\Omega}_n$ as:

$$\boldsymbol{\Omega}_n = i\ \boldsymbol{\nabla}_{\mathbf{k}} \times \langle u_{n\mathbf{k}}|\boldsymbol{\nabla}_{\mathbf{k}}|u_{n\mathbf{k}}\rangle \tag{1.13}$$

which works in 3-dimensions, as well as 2 (where Berry curvature can have only a single component, $\perp$ to the plane).

1.3.3 Spin and symmetry

So far, we haven't said anything about the spin of electrons. Electrons possess a quantum mechanical degree of freedom, which may be interpreted as an intrinsic form of angular momentum. Electrons are spin-$\frac{1}{2}$ particles. Measurement of spin angular momentum $\mathbf{s}$ along an arbitrary direction $\hat{\mathbf{n}}$, $\mathbf{s} \cdot \hat{\mathbf{n}}$, yields two possible results of measurements: $\pm\frac{\hbar}{2}$. Charged matter with angular momentum generally possesses magnetic moment, and can couple to and be manipulated by a magnetic field. Spinful charged particles like the electron, therefore, possess magnetic moment just as a result of their intrinsic spin degree of freedom.

Under the assumption that there are no magnetic fields, and none of the orbital degrees of freedom interact with the spin degree of freedom (basically, nothing interacts with spin), $[H, \mathbf{s}] = 0$; i.e. electronic energy levels are 2-fold degenerate. The 2-fold degeneracy is accounted for by an extra quantum number s_z, the projection of $\mathbf{s}$ along an arbitrarily chosen z axis. Accordingly, the Bloch functions should now be written as $|u^{\uparrow}_{n,\mathbf{k}}\rangle$ and $|u^{\downarrow}_{n,\mathbf{k}}\rangle$. In general, electrons will be found in states that are linear superpositions of both ↑ and ↓ states. Hence, states vectors representing the physical state of a spin-$\frac{1}{2}$ should be 2-dimensional ‘spinors’, instead of the 1-dimensional objects we have been dealing with so far. In this spin space, the spin angular momentum acts as $\frac{\hbar}{2}\boldsymbol{\sigma} = \mathbf{s}$ is the spin angular momentum operator, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

In the presence of a non-zero magnetic field along the z direction $\mathbf{B} = B\hat{z}$, the $\uparrow$ and $\downarrow$ states are still the energy eigenstates, however, the degeneracy is lifted by the coupling of spin magnetic moment with the magnetic field, which takes the form $H_{Zeeman} \sim Bs_z$. However, now, the choice of z direction is not arbitrary; it is decided by the $\mathbf{B}$ field. If instead, the spin degree of freedom enters the Hamiltonian in the form $\mathbf{h}(\mathbf{k}) \cdot \mathbf{s}$, not only will the degeneracy be lifted, but also, s_z along any fixed direction z becomes a bad quantum number to label energy eigenstates by. We may call these states $|u^+_{n,\mathbf{k}}\rangle$ and $|u^-_{n,\mathbf{k}}\rangle$. It may be seen that the energy eigenstates are still ‘spin polarized’, i.e., in eigenstates of spin along some particular direction. However, this direction is locally defined in momentum space as the direction of $\mathbf{h}$.

Generically, we should expect the spin degrees of freedom to be coupled to the orbital degrees of freedom. A rough picture of how this coupling comes about is to realize that the orbital motion of the electron gives rise to an effective magnetic field, which then couples with the spin of the electron. The general form taken by this ‘Spin-Orbit’ interaction is $H_{so} \sim (\mathbf{s} \times \mathbf{p}) \cdot \boldsymbol{\nabla} V_L$ [53]. Once again, this makes $[H, \mathbf{s}] \neq 0$, as a result of which s_z, is not a good quantum number.

It is immediately clear how the Zeeman interaction leads to the breaking of 2-fold spin degeneracy. However, it is not immediately clear if (or when) the spin-orbit interaction lifts this degeneracy. Note that $\mathbf{p}$ and V_L are operators, unlike B and $\mathbf{k}$, which are parameters. More broadly, we would like to know when the Bloch bands are two fold degenerate, and when this degeneracy is lifted, and we get two spin-polarized energy bands with energies

$\epsilon^+_{n,\mathbf{k}} \neq \epsilon^-_{n,\mathbf{k}}$. In order to answer this question, we need to consider the behavior of our system under two discrete symmetry operations, time reversal $\mathcal{T}$ and spatial inversion $\mathcal{I}$

The action of $\mathcal{I}$ on observable quantities is: $\mathcal{I}\mathbf{x}\mathcal{I}^{-1} = -\mathbf{x}$, $\mathcal{I}\mathbf{p}\mathcal{I}^{-1} = -\mathbf{p}$, and $\mathcal{I}\mathbf{s}\mathcal{I}^{-1} = \mathbf{s}$. Spatial inversion flips $\mathbf{x}$ and $\mathbf{p}$, but leaves angular momentum $\mathbf{s}$ invariant. The system is said to possess inversion symmetry if : $\mathcal{I}H\mathcal{I}^{-1} = H$. For the single particle hamiltonian considered in presence of lattice potential, this implies $V_L(\mathbf{x}) = V_L(-\mathbf{x})$. For an inversion symmetric hamiltonian, the Bloch hamiltonian $H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}}He^{i\mathbf{k}\cdot\mathbf{x}}$ satisfies: $\mathcal{I}H(\mathbf{k})\mathcal{I}^{-1} = H(-\mathbf{k})$. Since $H(\mathbf{k})|u^{\pm}_{n,\mathbf{k}}\rangle = \epsilon^{\pm}_{n,\mathbf{k}}|u^{\pm}_{n,\mathbf{k}}\rangle$, it follows that $\mathcal{I}|u^{\pm}_{n,\mathbf{k}}\rangle = |u^{\pm}_{n,-\mathbf{k}}\rangle$, and $\epsilon^{\pm}_{n,\mathbf{k}} = \epsilon^{\pm}_{n,-\mathbf{k}}$.

The action of $\mathcal{T}$ is more subtle. The basic action of $\mathcal{T}$ is to reverse the direction of unitary evolution in time. This implies that for a time-reversal symmetric system, $\mathcal{T}U(t)|\psi\rangle = U(-t)\mathcal{T}|\psi\rangle$, for an arbitrary state ψ, where $U(t) = e^{-\frac{i}{\hbar}\int_0^t dt' H(t')}$ is the unitary time evolution operator. That is, the time reversed version of a state $|\psi\rangle$ evolved to time t is the same as the time reversed version of the initial state, $\mathcal{T}|\psi\rangle$ evolved to time $-t$. For infinitesimal time δt, $U(\delta t) \simeq 1 - \frac{i}{\hbar}H\delta t$, so in order to reverse the direction of time evolution, $\mathcal{T}(iH) = -iH\mathcal{T}$. Naively, we may expect that this imples that $\mathcal{T}H = -H\mathcal{T}$. However, this is an unphysical imposition, as this would imply that if we have an energy eigenstate of energy ϵ, then, its time reversed state would also be an energy eigenstate, with energy $-\epsilon$. This would signify that the system has no ground state, and hence, is unstable.

The solution is that for a time-reversal symmetric system, $\mathcal{T}H = H\mathcal{T}$, but $\mathcal{T}i = -i$. To see how this is possible, note that $\mathcal{T}$ and $\mathcal{I}$ are both ‘symmetry operations’. This means that under these transforms, probabilities are preserved. Probabilities in quantum mechanics take the form $|\langle\alpha|\beta\rangle|^2$. An old theorem by Wigner says that any symmetry operation must be either unitary, leaving the amplitude $\langle\alpha|\beta\rangle$ invariant, or anti-unitary, transforming the amplitide $\langle\alpha|\beta\rangle$ to $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$. Time reversal $\mathcal{T}$ is of the second variety. This leads to the required behavior. In general, we may write $\mathcal{T}$ as $U_{\mathcal{T}}\mathcal{K}$, where $U_{\mathcal{T}}$ is a unitary matrix, and $\mathcal{K}$ is complex conjugation operator.

$\mathcal{T}\mathbf{x}\mathcal{T}^{-1} = \mathbf{x}$. However, $\mathcal{T}\mathbf{p}\mathcal{T}^{-1} = -\mathbf{p}$. Time reversal should reverse the direction of motion, and hence, the momentum. Alternately, writing $\mathbf{p} = -i\hbar\boldsymbol{\nabla}_{\mathbf{x}}$ leads to the same conclusion. Finally, $\mathcal{T}$ should flip spin, $\mathcal{T}\mathbf{s}\mathcal{T}^{-1} = -\mathbf{s}$. Within the constraint of the anti-unitary structure that $\mathcal{T}$ must possess, this can be accomplished by choosing $U_{\mathcal{T}} = e^{i\delta}\sigma_y$, where σ_y is the only Pauli matrix with imaginary elements, and δ is an arbitrary phase. Commonly, this phase factor is chosen to be $-i$.

From this it follows that for a time-reversal invariant system, $\mathcal{T}|u^{\pm}_{n,\mathbf{k}}\rangle = \mp|u^{\mp}_{n,-\mathbf{k}}\rangle$, and $\epsilon^{\pm}_{n,\mathbf{k}} = \epsilon^{\mp}_{n,-\mathbf{k}}$. This is the so called 'Kramer degeneracy'. As an aside, let us note that for spin-$\frac{1}{2}$ particles, $\mathcal{T}^2 = -1$.

If a system has both $\mathcal{T}$ and $\mathcal{I}$ symmetries, $\epsilon^{+}_{n,\mathbf{k}} = \epsilon^{-}_{n,-\mathbf{k}} = \epsilon^{-}_{n,\mathbf{k}}$, i.e. the energy bands are everywhere doubly degenerate. If one of these two symmetries are broken, then generically, we should expect these bands to be split in energy.

How does one break $\mathcal{T}$ or $\mathcal{I}$ symmetry? We have already seen that $\mathcal{I}$ is broken when V_L is not inversion symmetric. If we can assume that $\mathcal{T}$ symmetry holds, we can still say that $\epsilon^{+}_{n,\mathbf{k}} = \epsilon^{-}_{n,-\mathbf{k}}$. However, if $H_{so} = 0$, the energy bands will still look doubly degenerate; nothing interacts with spin, so automatically $\epsilon^{+}_{n,\mathbf{k}} = \epsilon^{-}_{n,\mathbf{k}}$. If $H_{so} \neq 0$, it will break $\mathcal{I}$ symmetry (because of its dependence on V_L) and lift the ± degeneracy, because of its coupling to $\mathbf{s}$. The simplest example of a situation where $\mathcal{T}$ symmetry is broken is in the presence of an external magnetic field $\mathbf{B}$. H_{Zeeman} explicitly breaks $\mathcal{T}$ symmetry (provided we define the $\mathcal{T}$ operation to be limited to the system alone, and not external sources giving rise to the $\mathbf{B}$ field). In ferromagnetic materials, below the Curie temperature, a non-zero spontaneous magnetization $\mathbf{m}$ can play the same role as an external $\mathbf{B}$ field.

1.3.4 Accidental degeneracies

So far, we have motivated that it is reasonable to expect the splitting of bands n into 2 spin-polarized sub-bands $n, +$ and $n, -$ in non-centrosymmetric crystals and ferromagnetic materials. These bands should be close energetically, compared to $\Delta\epsilon_n$, the typical bandgap between successive n, provided that the influence of the degeneracy lifting perturbation is small enough relative to V_L. It is The most general hamiltonian that can be written down for this 2-level system is:

$$H(\mathbf{k}) = h_0(\mathbf{k})\mathbb{1} + \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma} \tag{1.14}$$

where $\mathbb{1}$ is the identity matrix, and h_0 and the components of $\mathbf{h}$ are real.

The question now arises when these bands can touch, i.e. when can we expect to encounter points $\mathbf{k}_0$ in $\mathbf{k}$-space where $\mathbf{h}(\mathbf{k}_0) = 0$? Since there are 3 Pauli Matrices, one in general needs to tune three parameters to tune all 3 components of $\mathbf{h}$ to 0. Thus, in a 3-dimensional crystal, we can expect to generically encounter bands touching at isolated points. These band touchings are *accidental*. In graphene, which is a 2 dimensional material, bands (that arise from sublattice degree of freedom, instead of spin) touch at K and K' points at the corners of the hexagonal Brillouin zone. However, these band touchings cannot be accidental by the argument above; instead, they only arise under the assumption of $\mathcal{T}$ and $\mathcal{I}$ symmetries, and are lifted by perturbations that break either symmetry [8].

In contrast, accidental degeneracies are stable [26]. One can imagine a smooth deformation to the system that takes $\mathbf{h}(\mathbf{k}) \to \mathbf{h} + \boldsymbol{\delta}\mathbf{h}(\mathbf{k})$, will only shift the point of degeneracy from

$\mathbf{k}_0$ to some $\mathbf{k}_0 + \boldsymbol{\delta}\mathbf{k}_0$, but will not lift the degeneracy. Once created, accidental degeneracies are stable; their existence is insensitive to smooth deformations. This suggests that they are actually topological objects.

To investigate, consider the direction specified by $\mathbf{h}$, given as $\hat{\mathbf{h}}$. Clearly, $\hat{\mathbf{h}}$ is not defined at $\mathbf{k}_0$. Let us now imagine surrounding $\mathbf{k}_0$ by a sphere in momentum space. The unit vector $\hat{\mathbf{h}}$ is then a map from the 2-sphere to itself: $\hat{\mathbf{h}} : S^2 \to S^2$. Such a map has a 'winding number' $\mathtt{w}$ which counts the number of times the the map wraps the first sphere around the second. This has to an integer. Note that the trivial map $\mathbf{h} = \mathbf{k}$ has a winding number $\mathtt{w} = 1$. The fact that each point of degeneracy is associated with an integer is a further sign of the topological origins of its stability. Essentially, creation of a $\mathbf{k}_0$ 'wraps' the eigenstates on an S^2 around S^2; To lift the degeneracy, we would have to 'unwrap' this wrapping.

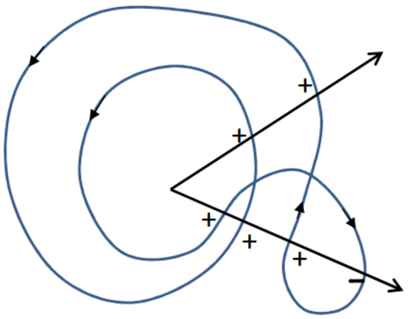


Figure 1.2: Winding number of map $S^1 \to S^1$. Note that the choice of segment does not matter in evaluating the winding number. Image from [12]

For a general $\mathbf{h}$, $\mathtt{w}$ about a general point $\mathbf{k}$ (not necessarily a point of degeneracy) is specified by the following surface integral defined on a S^2 containing $\mathbf{k}$:

$$\mathtt{w}_{\mathbf{k}} = \frac{1}{4\pi} \oint_{S^2} d^2\mathbf{a}_i \boldsymbol{\epsilon}_{abc} \boldsymbol{\epsilon}_{ijk} \hat{\mathbf{h}}_a \frac{\partial \hat{\mathbf{h}}_b}{\partial \mathbf{k}_j} \frac{\partial \hat{\mathbf{h}}_c}{\partial \mathbf{k}_k} \tag{1.15}$$

where $\mathbf{a}$ is the area element on S^2. Now, it is possible to explicitly compute the Berry curvature for the two band hamiltonian specified by $\mathbf{h}$ explicitly, by diagonalizing the Hamiltonian and then finding the eigenstates $|u^{\pm}_{n,\mathbf{k}}\rangle$ [8]. The Berry curvature is given as:

$$\boldsymbol{\Omega}^{\pm}_i = \mp \frac{1}{2} \boldsymbol{\epsilon}_{abc} \boldsymbol{\epsilon}_{ijk} \hat{\mathbf{h}}_a \frac{\partial \hat{\mathbf{h}}_b}{\partial \mathbf{k}_j} \frac{\partial \hat{\mathbf{h}}_c}{\partial \mathbf{k}_k} \tag{1.16}$$

for the upper and the lower bands. We see that the winding number of $\mathbf{h}$ around $\mathbf{k}_0$ is nothing but $\mathtt{w} = \frac{1}{2\pi} \oint_{S^2} d\mathbf{a} \cdot \boldsymbol{\Omega}$ for some S^2 containing $\mathbf{k}_0$, that is points of degenracy are monopoles of Berry curvature! This is an explicit demostration of the fact that the Berry curvature encodes the topology of the Hilbert space of states. For linear band touchings $\mathbf{h}(\mathbf{k}) = \pm \mathsf{v}_{\mathsf{F}}(\mathbf{k} - \mathbf{k}_0)$, the winding number is $\mathtt{w}_{\mathbf{k}_0} = \pm 1$. There is no topological reason why points of degeneracy with $|\mathtt{w}| > 1$ should not split up into several points of $|\mathtt{w}| = 1$. So generic perturbations can split up monopoles with higher charge w. Hence, generally, one encounters monopoles with charge $\mathtt{w} = \pm 1$.

So, does this all mean that once a $\mathbf{k}_0$ is created in the Brillouin Zone of a crystalline solid, nothing can ever destroy it, just move it around in the BZ? Not quite. The reason is the Nielsen Ninomiya theorem [44], which states that the sum of winding numbers of all $\mathbf{k}_0$ in the Brillouin Zone is zero:

$$\sum_{\mathbf{k}_0 \in BZ} \mathtt{w}_{\mathbf{k}_0} = 0 \tag{1.17}$$

There must be an equal number of Berry monopoles and anti-monopoles in the Brillouin Zone. It is possible to imagine that a strong enough perturbation to the system can move around a monopole $\mathbf{k}_0$ with $\mathtt{w} = 1$ across the Brillouin Zone till it meets with a monopole of charge $\mathtt{w} = -1$. This would then accomplish the unwrapping that we mentioned above. The basic reason for the Nielsen Ninomiya theorem is that the Brillouin Zone is a torus, $S^1 \times S^1 \times S^1$ in 3 dimensions. An elegant proof due to Witten [69] goes as follows.

Consider $\boldsymbol{\nabla}_i(\boldsymbol{\epsilon}_{ijk}\hat{\mathbf{h}}\cdot(\boldsymbol{\nabla}_j\hat{\mathbf{h}}\times\boldsymbol{\nabla}_k\hat{\mathbf{h}})$, where the vector operations $\cdot$ and $\times$ are acting on the $\hat{\mathbf{h}}$'s and $\boldsymbol{\nabla} = \partial/\partial\mathbf{k}$. This reduces to $\boldsymbol{\epsilon}_{ijk}\boldsymbol{\nabla}_i\hat{\mathbf{h}}\cdot(\boldsymbol{\nabla}_j\hat{\mathbf{h}}\times\boldsymbol{\nabla}_k\hat{\mathbf{h}}) = 0$ identically, because $\boldsymbol{\nabla}_{1,2,3}\hat{\mathbf{h}}$ are 3 vectors that lie tangent to the 2 dimensional sphere S^2 on which $\hat{\mathbf{h}}$ belongs. Now, consider BZ', which we get by removing small balls $S_{\mathbf{k}_0}$ around every monopole in the BZ. $\hat{\mathbf{h}}$ is well defined everywhere on BZ'. So, we may write $0 = \int_{BZ'}\boldsymbol{\nabla}_i(\boldsymbol{\epsilon}_{ijk}\hat{\mathbf{h}}\cdot(\boldsymbol{\nabla}_j\hat{\mathbf{h}}\times\boldsymbol{\nabla}_k\hat{\mathbf{h}})$. By Stoke's theorem, this reduces to $\oint_{\partial BZ'} d\mathbf{a}_i\boldsymbol{\epsilon}_{ijk}\hat{\mathbf{h}}\cdot(\boldsymbol{\nabla}_j\hat{\mathbf{h}}\times\boldsymbol{\nabla}_k\hat{\mathbf{h}})$, where $\partial BZ'$ is the 2-d boundary of BZ'. But the fact that BZ is a torus means that the BZ has no boundary, and consequently, the boudary of BZ' is just $\cup_{\mathbf{k}_0}S_{\mathbf{k}_0}$. Hence, it follows that the sum of winding numbers for all the $\mathbf{k}_0$'s is zero.

So, the minimum number of monopoles that need to exist in the BZ is 2, of opposite charge. In a $\mathcal{I}$-symmetric ferromagnet, this conclusion still holds, because under $\mathcal{I}$, $\sigma \to \sigma$, and $\mathbf{k} \to -\mathbf{k}$, so $\mathtt{w} \to -\mathtt{w}$; $\mathcal{I}$ transforms a monopole of charge $\mathtt{w}$ at $\mathbf{k}_0$ to a monopole of charge $-\mathtt{w}$ at $-\mathbf{k}_0$. In other words, $\mathtt{w}$ and $-\mathtt{w}$ are 'chiral' partners, two inequivalent objects (such as hands) that transform into one another under reflection. Hence, $\mathcal{I}$-symmetry guarantees that a monopole will be accompanied by an anti-monopole in the BZ. However, in a $\mathcal{T}$-symmetric anisotropic crystal, the minimum number of monopoles required is 4, as $\mathcal{T}$ does not flip $\mathtt{w}$. Note that it follows that a material with both $\mathcal{T}$ and $\mathcal{I}$ cannot have a monopole of Berry flux. This ties in neatly with our previous understanding; essentially, when $\mathcal{T}$ and $\mathcal{I}$ are both symmetries, both bands are glued together everywhere, so there is no question of a point degeneracy.

1.3.5 Semiclassical approximation

The discussion so far has been within the frramework of single-particle quantum mechanics. The energy bands that we have found are solutions to the problem of a single particle moving in a periodic lattice. In reality though, electrons do interact on account of their mutual Coulomb repulsion. Does this imply that everything we have discussed so far needs to be discarded? Surprisingly, the answer is no. It turns out that the approximation of almost free electrons works to explain the properties of most (but not all) electronic materials. This may roughly be understood in the following manner. Let us say that interaction related processes are occuring at some characteristic timescale τ_{micro}. The effect on the single particle states that we have been envisioning so far is that now, we can no longer pretend that these are exact stationary energy eigenstates. Instead, their energy acquires a width, $\Delta\epsilon \sim 1/\tau_{micro}$, which follows from the uncertainty principle for energy. In other words, instead of corresponding to plane waves, $e^{-i(\epsilon_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}$ that propagate forever, it is more appropriate to imagine the single particle states as being described by a wavefunction $\psi_{\mathbf{p}}(\mathbf{x},t) \sim e^{-t/\tau_{micro}}e^{-i(\epsilon_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}$; objects that decay with timescale τ_{micro}.

Now, let us consider a ‘Fermi gas’, a collection of fermions assumed to be so weakly interacting that it can be modelled as a gas of independent particles that suffer occasional collisions. By virtue of obeying Fermi-Dirac statistics, fermions must obey the Pauli exclusion principle. Accordingly, in equilibrium, the occupancy of the single particle energy levels $\epsilon(\mathbf{p})$ is given by the Fermi distribution

$$f(\mathbf{p}) = \frac{1}{e^{(\epsilon(\mathbf{p})-\mu)/k_B T}+1} \tag{1.18}$$

Macroscopic densities are given by integrals of f over $\mathbf{p}$- space:

$$n = \int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p}) \tag{1.19a}$$

$$\varepsilon = \int \frac{d^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p})\epsilon(\mathbf{p}) \tag{1.19b}$$

For $\epsilon - \mu >> T$, then $f \to 0$. When $\mu - \epsilon >> T$, $f \to 1$. We can therefore see that only in a band of energy T around μ host particle states that are capable of participating in collisions. Much above μ, there are no electrons, and much below, there are no nearby vacant states for an electron to move to as a result of the collision. This also means that the typical collision between two electrons will change their energy by $\epsilon_f - \epsilon_i \sim T$.

When can an interacting collection of fermions be treated like a gas? When the collisional width $\Delta\epsilon << T$. In this situation, the 'Fermi liquid' can be said to possess 'Quasiparticles', elementary excitations that can be labelled by the same charge, spin, momentum as the original electron, and which have a *sharply defined* energy relative to T. Note that although the quasiparticles look like electrons, they are emergent objects, which emerge as an effective single particle description from an underlying theory of correlated electrons. This effective description is called 'Fermi Liquid theory' and was first developed by Landau to explain the properties of helium-3 in its normal state (above the T for superfluid transition) [39] [49].

The quasiparticle energy dispersion is not the same as that of the underlying 'bare' electrons, but are instead 'renormalized' to take into account interactions. In general, determination of the energy dispersion of quasiparticles is a problem that requires the full machinery of many-body quantum theory [1] [54] [50]. However, the point is that we can treat this problem of quantum field theory (which is the quantum theory of many-body systems) with an effectively classical picture of particles with a well defined $\mathbf{p}$ and ϵ. This is the 'Semiclassical' description. In 3 dimensions, a simple dimensional argument suggests that $\tau_{micro} \sim T^{-2}$. This explains why the Fermi liquid theory works at low temperatures, $T \to 0$, which is where helium-3 exists in its liquid phase (boiling point of He-3 $\simeq 3$ K). Note also that Helium atoms are neutral, and only interact via short-range forces.

In contrast, electrons interact via long range Coulomb interaction. However, that is at the level of the bare ingredients that go into the field theoretic formulation. The effective quasiparticles that emerge only interact via short range interactions; i.e. the influence of far-off charges is effectively 'screened' out. Also, in typical crystalline conductors, room temperature is actually low when compared to the $\Delta\epsilon_{band}$, the typical energetic spread of bands. So, typically, these are also well described by semiclassical theory (see [52] for exceptions).

When we have a semiclassical description at our disposal, we can utilize it to describe non-equilibrium processes in this gas of quasiparticles. These quasiparticles reside in an effectively classical $\mathbf{x} - \mathbf{p}$ phase space. In this space, there are two main processes that displace quasiparticles. One is the Hamiltonian evolution $\dot{\mathbf{x}}(t)$ and $\dot{\mathbf{p}}(t)$ of a particle tracing trajectory $(\mathbf{x}(t), \mathbf{p}(t))$ through phase space. The second is the effect of collisions. The result is the *Boltzmann equation* [40], or 'kinetic theory', after the Kinetic Theory proposed by Boltzmann as the description of the *classical* many-body state. This equation takes the form of a conservation law for distribution $f(\mathbf{x}, \mathbf{p}, t)$:

$$\partial_t f + \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f = \mathcal{W}[f] \tag{1.20}$$

where the collision term $\mathcal{W}$ tracks the rate at which collisions deflect particles into a phase space element $d\mathbf{x}d\mathbf{p}$ around $(\mathbf{x}, \mathbf{p})$. Most generally, it has to track how many particles are being collided into the element $d\mathbf{p}$, and how many are being collided out of it. $\mathcal{W}$ is generally a complex functional of f even at the classical level. Under the assumption that collisions are only pairwise, and that a state of 'molecular chaos' exists, that is, 2 (and higher)-particle probability distributions reduce to products of 1-particle distributions f, $\mathcal{W}$ may be written as:

$$\mathcal{W}(\mathbf{x},\mathbf{p}) = \int \frac{d^d\mathbf{p}'}{(2\pi)^d}\frac{d^d\mathbf{p}''}{(2\pi)^d}\frac{d^d\mathbf{p}'''}{(2\pi)^d}(2\pi)^{d+1}\delta_\epsilon\delta_{\mathbf{p}}\mathsf{w}\{f_{\mathbf{p}''}f_{\mathbf{p}'''}(1-f_{\mathbf{p}})(1-f_{\mathbf{p}'}) - (\mathbf{p}\mathbf{p}' \leftrightarrow \mathbf{p}''\mathbf{p}''')\} \tag{1.21}$$

where w is the transition probability due to collisions between 2-particle states at $(\mathbf{p},\mathbf{p}')$ and $(\mathbf{p}'',\mathbf{p}''')$, δ_ϵ is short hand for $\delta(\epsilon+\epsilon'-\epsilon''-\epsilon''')$, and $\delta_{\mathbf{p}}$ is similarly defined; i.e. total energy and momentum are conserved by collisions. Pauli exclusion principle is reflected by the factors of $(1-f)$, which would be absent in a truly 'classical' gas. The molecular chaos hypothesis is an assumption that 'collisions' are essentially instantaneous, and local in $\mathbf{x}$ 'events' suffered by otherwise 'free' particles whose motion is uncorrelated.

The transition probability w is the same for both $\mathbf{p},\mathbf{p}' \to \mathbf{p}'',\mathbf{p}'''$ and the reverse process. This principle of 'detailed balance' is just an expression of $\mathcal{T}$ invariance at the microscopic level. The probability of a microscopic reaction to occur is the same as the reverse reaction. That does not, however guarantee that there will be no net flow from $(\mathbf{p},\mathbf{p}')$ to $(\mathbf{p}'',\mathbf{p}''')$. This can still occur if the $\{\#_{in} - \#_{out}\}$ in the expression above does not cancel out. This is precisely the case in generic non-equilibrium situations. Essentially, the fact that in non-equilibrium, $\nabla_{\mathbf{x}}f$, or $\partial_t f$, or $\dot{\mathbf{p}} \neq 0$ forces $\mathcal{W} \neq 0$. Thus, f is deformed into $f^{eq} + \delta f$, where f^{eq} is the local equilibrium distrbution function Eq.(1.18), that may be derived from the condition $\mathcal{W}[f] = 0$, and δf is the non-equilibrium deformation of the distribution function. Of course, now, $\epsilon = \epsilon[n,\varepsilon..]$, so strictly speaking now Eq.(1.18) represents a complicated implicit functional relationship.

The limit where hydrodynamics is useful (when the gradient approximation works) is when t, the non-equilibrium scale, is much longer than τ_{micro}; Since δf is the response of the system to non-equilibrium perturbation to the system, it follows that $\delta f << f^{eq}$ as well. In particular, we may solve for δf for a given non-equilibrium situation (specified by $T(\mathbf{x}, t)$, $\mu(\mathbf{x}, t)$ etc.) using:

$$\partial_t f^{eq} + \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f^{eq} + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_{\mathbf{p}} f^{eq} = \mathcal{W}[\delta f] \tag{1.22}$$

where $f^{eq} = f^{eq}(\mathbf{x}, t)$ is the local equilibrium distribution, and $\mathcal{W}$ is now *linearized* to leading order in δf. This is more tractable than Eq.(1.20), but still hard to solve. The approximation that is simplest is the so called 'Relaxation Time Approximation' which collects all the non-equilibrium processes contained in $\mathcal{W}$ into a charactersitic timescale τ_{micro}. Also, broadly, the effect of of $\mathcal{W}$ is to establish local equilibrium, i.e. to decay any $\delta f \neq 0$ to 0. Putting these two together, we obtain $\mathcal{W} \sim -\delta f / \tau_{micro}$.

What about the rates of evolution of the quasiparticle in phase space, $\dot{\mathbf{x}}$ and $\dot{\mathbf{p}}$? If these are truely classical particles, then, we should have [4] ($e = c = \hbar = 1$):

$$\dot{\mathbf{x}} = \boldsymbol{\nabla}_{\mathbf{p}} \epsilon_{\mathbf{p}} \tag{1.23a}$$

$$\dot{\mathbf{p}} = \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B} \tag{1.23b}$$

However, a more careful analysis involving wavepackets of Bloch states [13] [60] [70] reveals that this classical intuition misses the effect of the Berry curvature $\boldsymbol{\Omega}_{\mathbf{p}}$ experienced by the quasiparticles due to the bandstructure. This is the 'Anomalous velocity', a term that has been known for a long time [30], but whose geometric significance was not appreciated till Berry's work. Including this, we obtain:

$$\dot{\mathbf{x}} = \boldsymbol{\nabla}_{\mathbf{p}} \epsilon_{\mathbf{p}} + \dot{\mathbf{p}} \times \boldsymbol{\Omega}_{\mathbf{p}} \tag{1.24a}$$

$$\dot{\mathbf{p}} = \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B} \tag{1.24b}$$

We can see from the form of the above equation that Berry curvature can be interpreted as a magnetic field in momentum space. A direct consequence is the fact the phase space element occupied by a wavepacket, $d\mathbf{x}d\mathbf{p}(t)$, is no longer conserved as it evolves in time, but instead acquires a factor of $(1+\mathbf{B}\cdot\mathbf{\Omega})^{-1}$, which generally depends on $\mathbf{x}$ and $\mathbf{p}$. This is problematic though, because, under the assumption that the element $d\mathbf{x}d\mathbf{p}$ contains the same number of states irrespective of its location in phase space (a reasonable assumption to make in a gas of classical particles), this implies that the number of states covered by the w-p is not a constant as it evolves; in other words, there is no defininite particle number associated with a quasiparticle. So, the assumption of uniformity of density of states does not hold in our quantum gas of quasiparticles; instead, it acquires a factor:

$$\rho(\mathbf{x},\mathbf{p}) = (1+\mathbf{B}\cdot\mathbf{\Omega}) \tag{1.25}$$

This allows us to associate quasiparticles with a definite number, which is a conserved quantity (as a result of the $U(1)$ gauge symmetry at the field theoretic level). Now, computation of macroscopic densities must that ρ into account as well, e.g. $n(\mathbf{x}) = \int_{\mathbf{p}} \rho(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p})$.

Note that strictly speaking, this treatment is only valid under 'weak' $\mathbf{B}$ fields; The solution to the quantum mechanical problem of a particle in a magnetic field is the emergence of 'Landau Levels' as single particle eigenstates. $\mathbf{B}$ enters the hamiltonian H through the 'minimal substitution' of replacing $\mathbf{p}$ by $\mathbf{p}-e\mathbf{A}$, where $\mathbf{A}$ is the vector potential associated with $\mathbf{B}$. So, $H_{\mathbf{B}} = \mathtt{w}\,\mathsf{v_F}\boldsymbol{\sigma}\cdot(\mathbf{p}-e\mathbf{A})$ for a single Berry monopole of charge $\mathtt{w}$. The implication is that now $[\mathbf{p}, H_{\mathbf{B}}] \neq 0$; momentum is no longer a good quantum number to label energy eigenstates by. However, since $\mathbf{B} = \mathbf{\nabla}\times\mathbf{A}$, $\mathbf{A}$ can only be non-zero in the plane $\perp$ to $\mathbf{B}$; so $[\mathbf{p}\cdot\hat{\mathbf{B}}, H_{\mathbf{B}}] = 0$. That is, only momentum along the direction of $\mathbf{B}$, $\mathbf{p}\cdot\hat{\mathbf{B}}$, is still a simulataneous observable with $H_{\mathbf{B}}$, and accordingly, there is only one component of momentum, and correspondingly, one momentum quantum number given by wave-number k.

The 2-dimensional momentum degree of freedom $\perp$ to the $\mathbf{B}$ field collapses into states labelled by a new quantum number $\mathtt{n}$, which takes values $\in \mathbb{Z}$. Accordingly, each Landau Level $\mathtt{n}$ has enormous degeneracy. In essence, there is a hidden quantum number that does not appear in the dispersion relation, which labels the degenerate states within each Landau Level. The dispersion relation for $H_{\mathbf{B}}$ may be explicitly computed. For $\mathtt{n} \neq 0$, $\epsilon^{\pm}_{\mathtt{n},k} = \pm\sqrt{\hbar^2\omega_c^2\mathtt{n} + \mathsf{v_F}^2\hbar^2k^2}$, where ω_c is the 'cyclotron frequency' given as $\omega_c = \mathsf{v_F}\sqrt{2eB/\hbar c}$. That is, $\epsilon^{\pm}_{k_\perp,k} = \pm\sqrt{\mathsf{v_F}^2\hbar^2k_\perp^2 + \mathsf{v_F}^2\hbar^2k^2}$, a continuous spectrum in $k_\perp$, breaks up into a discrete spectrum labelled by $\mathtt{n}$ as we turn on $\mathbf{B}$. The lowest Landau level is special; the $\mathtt{n} = 0$ eigenstates are chiral, with dispersion $\epsilon_{\mathtt{n}=0,k} = \mathtt{w}\,\mathsf{v_F}\hbar k$. There is only one, and it propagates either parallel or anti-parallel to $\mathbf{B}$ depending upon the sign of $\mathtt{w}$.

The Landau bands have the following feature: for fixed k, the band gaps get smaller and smaller as we go to higher and higher $\mathtt{n}$. For high $\mathtt{n}$, therefore, the energy levels approximate a continuous spectrum, i.e. the energy levels start resembling the spectrum at $\mathbf{B} = 0$. This in particular means that if we only have to deal with high-$\mathtt{n}$ levels, then, it may be a good approximation to replace $\epsilon_{\mathtt{n}}$ by $\epsilon_{k_\perp}$ in our semiclassical analysis (and capture the effect of the $\mathbf{B}$ field through the coupling of $\mathbf{\Omega}$ and $\mathbf{B}$ in the modified $\dot{\mathbf{x}}$ above. This is precisely the case when $\epsilon >> \sqrt{B}$, where ϵ is the typical energy of a particle. The typical energy ϵ (of particles free to participate in collisions) lies in the range $\mu \pm T$. Hence, we get the condition(s) that for kinetic theory to be valid one needs to have $\mu >> \sqrt{B}$, and $\mu >> T$. Note that the second condition is a separate constraint from the first, but also ensuring that states at $\epsilon \sim B$ cannot participate in collisions, and hence, there is no need of describing the discrete character of these low lying states correctly. In the context of hydrodynamics, the first condition directiy translates to $\mathbf{B} \sim O(\partial)$; kinetic theory can only describe electrons under the influence of weak $\mathbf{B}$ fields.

The second condition is a general condition for the kinetic approach to work, not just in the presence of a $\mathbf{B}$ field. Quasiparticles participating in collisional processes should come for a thin shell of width T around the 'Fermi Surface', defined by $\epsilon_{\mathbf{p}} = \mu$. This is the 'low' temperature mentioned earlier required for a picture of interacting quasiparticles to be valid.

1.4 Weyl semimetals

So far, we have seen that points of accidental degeneracy $\mathbf{k}_0$'s can appear in bandstructures of 3-dimensional materials, provided at least one of $\mathcal{T}$ or $\mathcal{I}$ symmetries is broken. Once formed, these points are stable. This, as we saw, makes sense because they are topological objects, monopoles of Berry curvature with integer topological charge $\mathtt{w}$. The question now is, when do these Berry monopoles have observable effects? The answer is: when they lie 'near' the Fermi level.

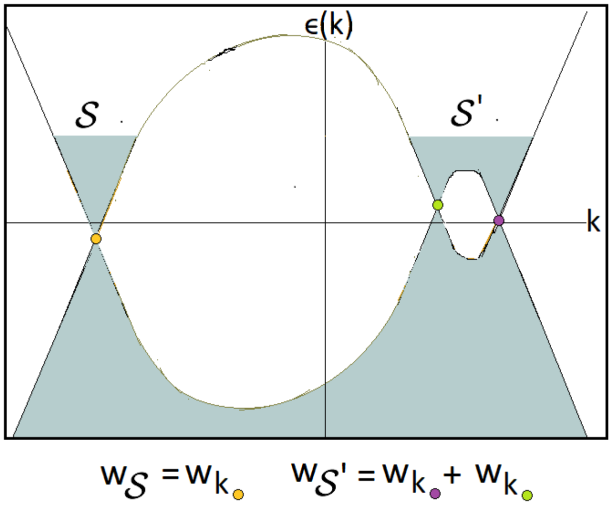


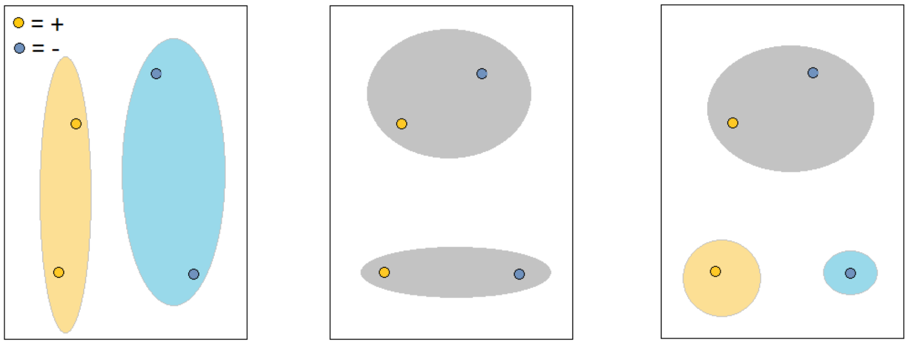
Figure 1.3: Winding number $\mathtt{w}_{\mathcal{S}}$ associated with a Fermi surface $\mathcal{S}$ is the sum of winding numbers $\mathtt{w}_{\mathbf{k}_0}$ for all Berry monopoles $\mathbf{k}_0 \in \mathcal{S}$

When $\mu \pm T \sim \epsilon_{\mathbf{k}_0}$, the energy of the degeneracy points, then the Fermi surface is no longer connected. Instead, it splits up into a collection of balls, each enclosing one (or more) monopole, reducing to points when $\mu = \epsilon_{\mathbf{k}_0}$. These disconnected pieces all enclose topologically charged objects with charge $\mathtt{w}$; moreover, this charge is associated with the F.S. itself, through Gauss' law for Berry curvature. This a situation which is *topologically distinct* (from the point of view of F.S. topology) from a 'normal' metal, in which either, there are no charges $\mathtt{w}$, or, each disconnected piece $\mathcal{S}$ of the Fermi surface is such that $\mathtt{w}_{\mathcal{S}} \equiv \sum_{\mathtt{w} \in \mathcal{S}} \mathtt{w} = 0$. This topologically distinct phase is called the 'Weyl semimetal' (WSM) [3] [63] [27] [11].

1.4.1 Chiral anomaly

What makes WSM a distinct phase, therefore, is the presence of closed *Fermi surfaces* $\mathcal{S}$ with non-zero topological charge $\mathtt{w}_{\mathcal{S}}$. The Nielsen Ninomiya theorem now reads $\sum_{\mathcal{S} \subset BZ} \mathtt{w}_{\mathcal{S}} = 0$. One suspects that the F.S. charges $\mathtt{w}_{\mathcal{S}}$ should somehow be able to classify WSMs in much the same way as Hall conductance classifies the different QHE states, i.e. in the form of a physical response. As it turns out, this is precisely the case. The relevant number here is $\frac{1}{2} \sum_{\mathcal{S} \subset BZ} |\mathtt{w}_{\mathcal{S}}|$; i.e. a count of the total number of Berry monopoles of of given sign of $\mathtt{w}$ contained in the BZ. The physical response that is characterized by this number is called the 'Chiral anomaly'.

In a WSM, electrons are trapped in 'valleys' defined by the disconnected Fermi surfaces $\mathcal{S}$. Each valley is defined by the topological charge $\mathtt{w}_{\mathcal{S}}$. The magnitude, $|\mathtt{w}_{\mathcal{S}}|$, defines the degree of mapping $\hat{\mathbf{h}}$ from $\mathcal{S}$ to the S^2 on which the map lies. Generically, when $|\mathtt{w}_{\mathcal{S}}| > 1$, we expect to find several monopoles of charge 1 contained within $\mathcal{S}$. The sign, $\frac{\mathtt{w}_{\mathcal{S}}}{|\mathtt{w}_{\mathcal{S}}|}$, which can take up values $\pm$, is a quantity that inverts under the action of spatial inversion $\mathcal{I}$ on the band structure. Physically, this means that there are effectively two chiral species of particles, L and R, that are mirror images of one another. The presence of L/R-handed species is the defining characteristic of a WSM.



a) WSM b) Normal metal c) WSM

Figure 1.4: The defining characteristic of WSMs is the presence of Fermi surfaces $\mathcal{S}$ with $\mathtt{w}_{\mathcal{S}} \neq 0$, leading to emergence of chiral valleys.

Experimental signatures of the WSM phase, therefore, would be observed by introducing a perturbation that couples differently to the L/R species. Such a perturbation indeed exists: $\mathbf{E} \cdot \mathbf{B}$. Without thinking in detail, the key fact that motivates us to suspect that this can somehow be used to differentiate between L/R species is that $\mathbf{E} \cdot \mathbf{B}$ flips sign under $\mathcal{I}$, and so does $n^5 \equiv n^L - n^R$, where n^L and n^R are the number densities associated with the L/R species; both are pseudoscalars under $\mathcal{I}$.

To see how, let us imagine the system at $T \sim 0$, and $\mu >> \sqrt{B}$, so that the semiclassical picture is valid. Now, consider the typical microscopic collision, between either two electrons, or, elastic (energy conserving, but momentum nonconserving) collisions between electrons and impurities, whose collision rate takes the form $\mathcal{W}(\mathbf{p}) = \int_{\mathbf{p}'} \rho(\mathbf{p}')\delta(\epsilon - \epsilon')\mathrm{w}_{\mathbf{p}\to\mathbf{p}'}(f_{\mathbf{p}'} - f_{\mathbf{p}})$. In both types of collisions suffered by electrons, the typical change of momentum of an e^- will be 'small' relative to the size of the overall Brillouin Zone. So, rarely will an electron suffer a collision so violent that it ends up from one valley to another. Some may end up random walking across the empty states available across the BZ (in an energetically unfavorable landscape; the empty states are all higher in energy), but again, this will take a long time. This is why, the electrons in each valley are effectively ‘trapped’.

This means that if we could create a ‘chiral imbalance’, i.e. $n^L - n^R \neq 0$, it would persist for a much longer time τ than the timescale on which all other (intra-valley) processes occur, τ_{micro}. Recall that τ_{micro} is the timescale on which local equilibriation occurs within a valley; if we have a nonequilibrium deformation δf to the equlibrium distribution f^{eq}, then $\delta f \to 0$ in a few τ_{micro}’s. However, if δf instead signifies an imbalance between different valleys, then the *inter-valley* equilibriation will occur at rate τ^{-1}, which is significantly slower.

But how to create the chiral imbalance in the first place? This is where the $\mathbf{E} \cdot \mathbf{B}$ comes in. One can show that in the presence of $\mathbf{E} \cdot \mathbf{B} \neq 0$, chiral imbalance is produced at a constant rate proportional to $\mathbf{E} \cdot \mathbf{B}$. This can be shown directly from the kinetic approach; we will reproduce here the calculation of Son and Spivak [57]:

In order to get conserved densities, one has to integrate out the $\mathbf{p}$ dependence of the quasiparticle distribution $f(\mathbf{p}, \mathbf{x}, t)$ to produce conserved densities such as $n(\mathbf{x}, t)$ etc. It makes sense to separate f out into the distribution functions for the individual valleys $\mathcal{S}$, $f^{\mathcal{S}}$. We will introduce a new variable, ‘valley momentum’ $\mathbf{q}$, defined relative to an origin that lies within $\mathcal{S}$. For example, if there is only one monopole in $\mathcal{S}$, $\mathbf{q}$ can represent the distance of a point in the valley from the monopole. So a $\mathbf{p}$ in the full BZ is represented by $(\mathbf{q}, \mathcal{S})$. Accordingly, $f(\mathbf{p}) = f^{\mathcal{S}}(\mathbf{q})$. We can write down a Boltzmann equation for the $f^{\mathcal{S}}$’s separately. This takes the form:

$$\partial_t f^{\mathcal{S}} + \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f^{\mathcal{S}} + \dot{\mathbf{q}} \cdot \boldsymbol{\nabla}_{\mathbf{q}} f^{\mathcal{S}} = \mathcal{W}_{\mathcal{S} \to \mathcal{S}} + \sum_{\mathcal{S} \neq \mathcal{S}'} \mathcal{W}_{\mathcal{S}' \to \mathcal{S}} \tag{1.26}$$

where we have separated out the intra and intervalley collision terms in the first and second terms.

Valley density $n^{\mathcal{S}}$ is given as $\int_{\mathbf{q}} \rho_{\mathbf{q}} f^{\mathcal{S}}_{\mathbf{q}}$, so to get the hydrodynamic equation for $n^{\mathcal{S}}$, we need to integrate the above equation with respect to $\int_{\mathbf{q}} \rho_{\mathbf{q}}$. First, let us focus on $\int_{\mathbf{q}} \rho_{\mathbf{q}} \mathcal{W}_{\mathbf{q}}$. The first term, $\mathcal{W}_{\mathcal{S}\to\mathcal{S}}$, gives 0. The intra-valley collision rate counts the rate at which particles from other parts of the valley are being deflected into the momentum space element at $\mathbf{q}$. Since 'number' is a conserved quantity in each collision, any particle deflecting into a given element must have been collided out of another one. It follows that if we integrate over $\mathbf{q}$, we should get 0. It can in general be said that $\mathcal{W}$ that takes particles from a space $\mathcal{S}$ to itself never (directly) contributes to the hydrodynamic equation for any conserved quantity that is defined as an integral over $\mathcal{S}$. For example, in the case of energy density $\varepsilon = \int_{\mathbf{p}} \rho_{\mathbf{p}} \epsilon_{\mathbf{p}} f_{\mathbf{p}}$, where $\mathbf{p} \in BZ$, the $e-i$ collision term contributes $\int_{\mathbf{p}} \int_{\mathbf{p}'} \rho_{\mathbf{p}} \rho_{\mathbf{p}'} \epsilon_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'})(f_{\mathbf{p}'} - f_{\mathbf{p}}) = 0$ due to the δ_ϵ.

On the other hand, inter-valley collisions do not average to 0 automatically. Assuming a model of elastic collisions, we may write $\int_{\mathcal{S}} d^3\mathbf{q} \mathcal{W}_{\mathcal{S}'\to\mathcal{S}} = \int_{\mathcal{S}} \int_{\mathcal{S}'} d^3\mathbf{q} d^3\mathbf{q}' \rho_{\mathcal{S}}(\mathbf{q}) \rho_{\mathcal{S}'}(\mathbf{q}')$ $\Big[\mathsf{w}_{\mathcal{S}\to\mathcal{S}'} \delta(\epsilon_{\mathcal{S},\mathbf{q}} - \epsilon_{\mathcal{S}',\mathbf{q}'}) \Big(f^{\mathcal{S}'}(\mathbf{q}') - f^{\mathcal{S}}(\mathbf{q}) \Big) \Big]$, which is not automatically guaranteed to be equal to 0. For this term to contribute exactly 0, one would have to ensure that there is no chiral imbalance, i.e. $f^{\mathcal{S}}(\epsilon) = f^{\mathcal{S}'}(\epsilon)$. However, if we imagine each valley is associated with a different chemical potential $\mu^{\mathcal{S}}$, this condition will not be met. In this case, $\int_{\mathbf{q}} \sum_{\mathcal{S}' \neq \mathcal{S}} \mathcal{W}_{\mathcal{S}'\to\mathcal{S}} \sim (n_0^{\mathcal{S}} - n^{\mathcal{S}})/\tau$, where $n_0^{\mathcal{S}}$ is the density that would be associated with valley $\mathcal{S}$ if $\mu^{\mathcal{S}} = \mu^{\mathcal{S}'} \, \forall \, \mathcal{S} \neq \mathcal{S}'$, i.e. when the Fermi surface is at the same energy over the entire BZ.

To simplify our life, we will assume that the system is in a spatially homogenous state, i.e. $\boldsymbol{\nabla}_{\mathbf{x}} f = 0$. This means that hydrodynamic variables such as T are constant in $\mathbf{x}$. Since the situation we are trying to describe is the production of chiral imbalance in the presence of $\mathbf{E}$ parallel to $\mathbf{B}$, this is enough to isolate the relevant response. This means that $\int_{\mathbf{q}} \rho_{\mathbf{q}} \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f^{\mathcal{S}} = 0$ as well. Formally, it may be seen that this term integrates to $\boldsymbol{\nabla}_{\mathbf{x}} \cdot \mathbf{j}^{\mathcal{S}}$, where $\mathbf{j}^{\mathcal{S}}$ is the current density associated with $n^{\mathcal{S}}$, as it is the only source of $\boldsymbol{\nabla}_x$ in the equation.

This leaves us with the $\dot{\mathbf{q}}$ term. Before proceeding further, it is helpful to decouple Eq.(1.24) into uncoupled equations for $\dot{\mathbf{x}}$ and $\dot{\mathbf{q}}$ ($e = c = \hbar = 1$):

$$\dot{\mathbf{x}} = \rho_{\mathbf{q}}^{-1} \Big(\mathbf{v} + \mathbf{E} \cdot \boldsymbol{\Omega}_{\mathbf{q}} + (\boldsymbol{\Omega}_{\mathbf{q}} \cdot \mathbf{v}) \mathbf{B} \Big) \tag{1.27a}$$

$$\dot{\mathbf{q}} = \rho_{\mathbf{q}}^{-1} \Big((\mathbf{E} + \mathbf{v} \times \mathbf{B}) + (\mathbf{E} \cdot \mathbf{B}) \Omega_{\mathbf{q}} \Big) \tag{1.27b}$$

where $\mathbf{v}(\mathbf{q}) \equiv \boldsymbol{\nabla}_{\mathbf{q}} \epsilon_{\mathbf{q}}$. We need to find the integral $\int_{\mathbf{q}} \rho_{\mathbf{q}} \dot{\mathbf{q}} \cdot \boldsymbol{\nabla}_{\mathbf{q}} f$. We can immediately see that $\rho_{\mathbf{q}}$ cancels the $\rho_{\mathbf{q}}^{-1}$. Also, let us recall that we are in the linearized limit, and f on the LHS is to be interpreted as the local equilibrium function f^{eq}.

Now, $f^{eq}(\mathbf{q}) = f^{eq}(\epsilon_{\mathbf{q}})$; the local equilibrium function only depends on $\mathbf{q}$ through the dependence of the q-p energy $\epsilon_{\mathbf{q}}$ on it. This means that $\boldsymbol{\nabla}_{\mathbf{q}} f = (\partial f / \partial \epsilon) \mathbf{v}(\mathbf{q})$. The first term in $\dot{\mathbf{q}}$ gives: $\int_{\mathbf{q}} \mathbf{E} \cdot \mathbf{v}(\mathbf{q}) (\partial f / \partial \epsilon)$. We will break up the volume integral into surface integrals over surfaces of constant energy ϵ. Then we may write $d^3\mathbf{q}$ as $d^2\mathbf{a} dq_{\perp}$, where $d^2\mathbf{a}$ is the area element of some constant energy surface $\mathcal{S}_\epsilon$, and $dq_{\perp}$ is the infinitesimal element in the direction $\perp$ to the surface $\mathcal{S}_\epsilon$ at $\mathbf{q}$. This may alternately be written as $dq_{\perp} = d\epsilon / |\mathbf{v}|$. So, writing, $\mathbf{v} = |\mathbf{v}| \hat{\mathbf{v}}$, the integral may be represented as $\int d\epsilon (\partial f / \partial \epsilon) \oint_{\mathcal{S}_\epsilon} |d\mathbf{a}| \hat{\mathbf{v}} \cdot \mathbf{E} = \int d\epsilon (\partial f / \partial \epsilon) \oint_{\mathcal{S}_\epsilon} d\mathbf{a} \cdot \mathbf{E} = 0$, where the last equality follows from the fact that integral of a constant vector $\mathbf{E}$ over a closed surface $\mathcal{S}_\epsilon$ is 0 by the divergence theorem. Similarly, the second term also contributes 0 because $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$.

So, we are only left with the final term, which contributes : $\int \frac{d^3\mathbf{q}}{(2\pi)^3}(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_{\mathbf{q}}\cdot\mathsf{v}(\mathbf{q})\frac{\partial f}{\partial \epsilon}$. Once again, the $\mathbf{E}\cdot\mathbf{B}$ is a constant in $\mathbf{q}$ and may be pulled out of the integral. Splitting up the integral into constant energy surfaces $\mathcal{S}_\epsilon$, we get: $\frac{1}{(2\pi)^3}(\mathbf{E}\cdot\mathbf{B})\int d\epsilon\frac{\partial f}{\partial \epsilon}\oint_{\mathcal{S}_\epsilon} d\mathbf{a}\cdot\boldsymbol{\Omega}$. Now, the surface integral just yields $(2\pi)\mathtt{w}_{\mathcal{S}}$ (when ϵ lies in the upper band, or $\mu > 0$). The energy integral on the other hand, yield -1; f varies from 1 to 0 as we cross μ. So, putting everything together, we get:

$$\partial_t n^{\mathcal{S}} + \boldsymbol{\nabla}\cdot\mathbf{j}^{\mathcal{S}} = \mathtt{w}_{\mathcal{S}}\frac{1}{(2\pi)^2}(\mathbf{E}\cdot\mathbf{B}) + \frac{n_0^{\mathcal{S}} - n^{\mathcal{S}}}{\tau} \tag{1.28}$$

The total charge density $n = \sum_{\mathcal{S}} n^{\mathcal{S}}$ obeys:

$$\partial_t n + \boldsymbol{\nabla}\cdot\mathbf{j} = \sum_{\mathcal{S}}\mathtt{w}_{\mathcal{S}}\frac{1}{(2\pi)^2}(\mathbf{E}\cdot\mathbf{B}) + \frac{1}{\tau}\sum_{\mathcal{S}}(n_0^{\mathcal{S}} - n^{\mathcal{S}}) = 0 \tag{1.29}$$

owing to the Nielsen Ninomiya theorem, and the fact that $\sum_{\mathcal{S}} n_0^{\mathcal{S}} = \sum_{\mathcal{S}} n^{\mathcal{S}}$; particles can change the valley they belong to, but their total number should be conserved. The chiral charge density, $n^5 = \sum_{\mathcal{S}} \frac{\mathtt{w}_{\mathcal{S}}}{|\mathtt{w}_{\mathcal{S}}|} n^{\mathcal{S}}$, on the other hand, is not conserved;

$$\begin{aligned}\partial_t n^5 + \boldsymbol{\nabla}\cdot\mathbf{j}^5 &= \Big(\sum_{\mathcal{S}}|\mathtt{w}_{\mathcal{S}}|\Big)\frac{1}{(2\pi)^2}(\mathbf{E}\cdot\mathbf{B}) \;+\; \frac{1}{\tau}\sum_{\mathcal{S}}\frac{\mathtt{w}_{\mathcal{S}}}{|\mathtt{w}_{\mathcal{S}}|}(n_0^{\mathcal{S}} - n^{\mathcal{S}}) \\ &= \mathsf{C}\mathbf{E}\cdot\mathbf{B} - \frac{n^5}{\tau}\end{aligned} \tag{1.30}$$

Where we have defined the ‘Anomaly coefficient’:

$$\mathsf{C} \equiv \frac{1}{(2\pi)^2}\sum_{\mathcal{S}}|\mathtt{w}_{\mathcal{S}}| \tag{1.31}$$

The second equality follows from the fact that when all the μ’s are the same, $\sum_{\mathcal{S}^+} n_0^{\mathcal{S}} = \sum_{\mathcal{S}^-} n_0^{\mathcal{S}}$, where $\mathcal{S}^+$ and $\mathcal{S}^-$ are the sets of valleys of L or R type; densities of L and R species should be equal in chiral equilibrium.

Note that a mechanism for the Chiral anomaly can be motivated in the strong-$\mathbf{B}$ regime as well, using Landau Levels. This was indeed the original setting in which it was shown possible to realize the chiral anomaly in a 3-dimensional crystal [45]. The basic picture, as we saw, is that in the presence of $\mathbf{B}$, the $\mathtt{n} = 0$ level becomes chiral, only carrying particles that move parallel or anti-parallel to $\mathbf{B}$ depending upon the sign of $\mathtt{w}$ of the nearby monopole. Now, an $\mathbf{E}$ can only displace these particles if it is along $\mathbf{B}$, since the Landau levels only propagate (possess momentum) along the direction of $\mathbf{B}$. In the presence of an $\mathbf{E}$, therefore 'charge pumping' occurs between valleys of ± chirality at a rate $\sim \mathbf{E}\cdot\mathbf{B}$.

In the context of field theory (where the concept originated), an *Anomaly* is a symmetry of a classical field theory that does not survive quantization. Since symmetries lead to conservation laws, an anomaly will be associated with 'anomalous non-conservation' of some classically conserved density. An example is relativistic Dirac fermions in the massless limit, also known as Weyl fermions, which have an identical dispersion as that we have encountered (two bands touching at a point, with linear dispersion, with velocity c, the speed of light, instead of $\mathsf{v_F}$, which is a material dependent parameter for a given WSM). Weyl fermions are locked into eigenstates of 'helicity', the projection of spin along angular momentum. Accordingly, there are L/R species of Weyl fermions.

At the classical level, both the total charge current j_α and the chiral charge density j_μ^5 are exactly conserved quantities. However, when the theory is quantized, it turns out that after including quantum corrections, j_5^μ is no longer conserved when the fermion is coupled to an electromagnetic field. Instead, $\partial_\mu j_5^\mu = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. The right hand side reduces to $\mathbf{E}\cdot\mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in some reference frame. Therefore, in the presence of parallel electric and magnetic fields, the chiral charge is no longer conserved. This was first discovered by Adler, Bell and Jackiw, and proposed to decay the fact that the neutral pi meson, π^0, decays predominantly by decaying into two photons, a process that should be forbidden classically [2] [7]. Note however, that the analogy to our case is only approximate; even in the absence of $\mathbf{E} \cdot \mathbf{B}$, the chiral density in a WSM is only approximately conserved, owing to intervalley relaxation process happenning at rate τ^{-1}.

When dealing with hot ultrarelativistic charged matter, a situation may be encountered when the masses of the interacting fermions may be negligible, in which case, one may observe macroscopic effects of the anomaly. Experimentally, this is realized in the 'Quark Gluon Plasma', a hot, dense, deconfined form of nuclear matter thought to be realized for extremely brief periods of time during high energy nuclear collision experiments performed in particle accelerators. In such a situation, one observes macroscopic manifestations of the chiral anomaly. These effects are *universal*, in the sense that they are fixed to a definite functional dependence just as a result of positivity of entropy production, and hence, independent of the precise nature of the microscopics. This was shown by Son and Surowka [59] in the context of relativistic hydrodynamics; these effects take the form of non-dissipative currents along magnetic field and fluid vorticity, the 'Chiral Magnetic Effect' (CME) and the 'Chiral vortical effect', in relativistic hydrodynamics. The main work in this thesis is adapting the algorithm presented in [59] to the condensed matter context of WSMs.

The CME, and a related effect, the large positive enhancement to conductance in the presence of a $\mathbf{B}$ along the direction of current flow (known as "Large negative magnetoresistance' in the literature), may be worked out from the kinetic theory [57]. However, as we will show, this is a direct consequence of the fact that the semiclassical machinery *automatically* guarantees $\mathcal{R} \geq 0$. In fact, the first rigorous expression of this principle was given by Boltzmann himself, the famous 'H-theorem' of Boltzmann(e.g. see [40]). The crux of the matter is that *any* microscopic description compatible with positivity of entropy production will always produce these auxilliary effects whenever there is a chiral anomaly. This suggests that we may isolate these dependences without any further recourse to the microscopic theory, based on purely macroscopic principles. This is the hydrodynamic approach we shall work through in detail in the following chapters.

'Large negative magnetoresistance' (NMR) encompasses two related physical effects. First is the positive enhancement of conductivity in the presence of a $\mathbf{B}$ field; and second is the fact that this enhancement falls off roughly as $cos^2\phi$, where ϕ is the angle between the current carrying direction and $\mathbf{B}$. This is known as 'current locking': $\sigma \to \sigma + \sigma_{\mathbf{B}}$, where $\sigma_{\mathbf{B}} \sim \# B^2 cos^2\phi$, where $\# > 0$. These effects have been claimed to have been observed [28] [72] [46]. Note, however, that magnetoresistance effects seem to be extremely sensitive to the experimental geometry, and shows non-trivial effects related to contact geometry, which go by the name of 'current jetting artifacts', which makes it difficult to confirm Weyl semi-metallic behavior using NMR experiments [17] [38] [47].

A related effect is Magneto-thermoelectric conductivity. In this, the measured response is electric current $\mathbf{j}$ in response to a temperature gradient $\boldsymbol{\nabla} T$, which also shows a B^2 dependence. This has been explained as an effect related to 'Mixed Axial-gravitational anomaly, which is an anomolous non-conservation of the axial current j_5 in a relativistic fluid on curved spacetime [18]. It turns out that in this situation, remnant effects of the gravitational anomaly can be observed even in flat space-time. We will present phenomenological explanations for both effects based on the hydrodynamic theory we will formulate in the next chapter.

1.4.2 Anomalous Hall Effect & other effects

As we have mentioned earlier, one of the classes of materials in which WSM behavior is possible is $\mathcal{T}$-breaking ferromagnets. There is a host of magnetization effects that are observed in magnets, chief among which is the so called 'Anomalous Hall Effect' [43], which is an observation of a Hall effect in ferromagnetic materials that goes as $\mathbf{j} \sim \mathbf{m} \times \mathbf{E}$, where $\mathbf{m}$ is the magnetic polarization density. These magnetization related effects are sister effects that are present in $\mathcal{T}$ breaking WSMs. However, as our hydrodynamic analysis will reveal, these effects do not interact with the anomaly related effects, and reside in a separate sector of the hydrodynamic theory we will develop.

Finally, there is another effect observed in WSMs that is considered a good experimental diagnostic for Weyl semi-metallic behavior. This is the presence of special surface states called 'Fermi Arcs', which are surface localized states that lie along 1-dimensional arcs connecting the projections of degeneracy points $\mathbf{k}_0$'s on the surface Brillouin Zone [67] [71]. These may be observed using surface spectroscopy (ARPES) techniques, which map out the occupied surface states. Fermi arcs lie at the Fermi-level. When the Fermi-level is above $\epsilon_{\mathbf{k}_0}$, the energy at which the degeneracy exists, then these take the form of arcs connecting the projections of the bulk Fermi-surfaces on the surface Brillouin Zone [22]. Their existence may be postulated simply from the condition of conservation of spln angular momentum [69]. A particle reflecting off a boundary has its momentum $\mathbf{p}$ reversed; however, its spin $\boldsymbol{\sigma}$ does not change. It follows that the reflection off the boundary flips the chirality of particles $L \leftrightarrow R$, from which it may be worked out that the surface must provide a conduit between valleys of different chirality. The main effect of Fermi Arcs in the context of non-equilibrium phenomenon seems to be that $n^5 = 0$ at the surfaces of a WSM. Any chiral imbalance produced in the bulk is neutralized when it its the boundary.

CHAPTER 2

CONSTRUCTION OF THE HYDRODYNAMIC THEORY FOR DIRTY WEYL SEMIMETAL

In this chapter, we will construct a hydrodynamic theory for a system where there are three conserved charges (in the absence of external electromagnetic fields). Energy density ε, total charge density n and chiral charge density n^5. The conservation of n^5 is assumed to be exact in the absence of external electromagnetic fields, which is an idealization in a real Weyl semimetal, where there are always intervalley scattering processes that relax a chiral imbalance. This idealization is applicable when $\tau_{micro} \ll t \ll \tau$, where τ_{micro} represents the typical timescale over which intra-valley processes occur, τ is the typical timescale of inter-valley scattering processes, and t is the timescale over which the macroscopic phenomena are evolving. That is, non-equilibrium situations that are 'fast' relative to intervalley relaxation, (but slow relative to *other* timescales emerging from the microscopic theory). In this case, we may write down hydrodynamic equations where the density at each valley is separately conserved. The conservation of n^5 is anomalous, breaking in the presence of parallel $\mathbf{E}$ and $\mathbf{B}$ fields.

This hydrodynamic theory is expected to be applicable when one has a Weyl semimetal where the primary mode of relaxation is elastic collisions of electrons with random impurities, in other words, a 'dirty' Weyl semimetal. These collisions do not conserve the momentum of the electron system. Hence, momentum is not one of the conserved quantities in this system.

With the development of fabrication technologies, it has become possible to grow materials from single crystals, resulting in what are exceptionally 'clean' materials, which are almost entirely free of external dopants or impurities. A hydrodynamic theory for a clean system, while being an ideal limit, may indeed be a good starting point in such a case.

However, even in the case that the material is built by crystal growth, a situation may be imagined where the dirty hydrodynamic theory may be applicable. Namely, if the crystal growth process is sufficiently fast, and therefore uncontrolled, we may end up with a macroscopic material that has a grained structure, with domain boundaries separating small volumes which have a consistent crystal structure. The overall material would then be a patchwork with no consistent crystal structure. The interaction of charge carriers with these domain walls, which are oriented randomly, can plausibly be modelled in the same way as the presence of random impurities: by momentum non-conserving collisions. Therefore, even in this case, the dirty hydrodynamic theory developed here might find application. Note also that generally, 'clean' experimental samples are usually what would be called *mesoscopic*, i.e. not truly macroscopic (on our human scale). Typically, this is of the order of μm's (for example, see [18]). It may be reasonable to imagine that in the fabrication of larger samples, uncontrolled growth may be unavoidable.

By corollary then, we will also assume that the system is rotationally invariant; a dirty system is isotropic, as no information about special crystalline directions survives at the macroscopic level. We also assume microscopic $\mathcal{T}$ symmetry (broken by the presence of $\mathbf{B}$ and/or $\mathbf{m}$), which will allow us to simplify the dissipative structure of the constitutive relations via Onsager relations. As we discussed earlier, in order to realize the WSM phase, at least one of spatial inversion symmetry or Time-reversal symmetry must be broken. Our hydrodynamic theory encompasses both cases. Note that in both cases, the separate conservation of valley species, and the fact that the sign of the Berry index of a valley flips under $\mathcal{I}$, means that we will have the emergence of a density (n^5) that is a pseudo-scalar under $\mathcal{I}$, and a current ($\mathbf{j}^5$) that is a pseudo-vector, in our hydrodynamic theory. However, it is important to note that these signals will be observed even in WSMs with $\mathcal{I}$ symmetry (i.e. ferromagnetic WSMs). Their presence does not imply that the system necessarily breaks $\mathcal{I}$

symmetry at the microscopic level. It only implies that there are 2 mirror-species that are separately conserved, that interchange their identities under $\mathcal{I}$.

We shall start off by assuming that the background magnetic field $\mathbf{B}$ is $O(\partial^0)$ in order of derivatives, i.e. a 'strong' magnetic field. As such, $\mathbf{B}$ would be at par with temperature T, chemical potential μ and chiral chemical potential μ^5, in the sense that $\mathbf{B}$ now needs to be considered as an intensive thermodynamic variable; thermodynamics depends on $\mathbf{B}$. A natural consequence is that we need to worry about magnetization effects (even in the non-ferromagnetic case), because nearly all materials exhibit magnetic response at strong enough magnetic fields. We will see that this allows us to glean useful information about the structure of the hydrodynamic theory in the ferromagnetic case.

When $\mathbf{B} \sim O(\partial^0)$, the kinetic coefficients will also in general depend on the $\mathbf{B}$ field. However, $\mathbf{E} \sim O(\partial^1)$, the reason for which will become apparent when we write down the expression for entropy production rate $\mathcal{R}$. This is the regime that is usually referred to as 'Magnetohydrodynamics' [25]. We will find that reconciling the anomaly with the 2^{nd} law produces macroscopic effects similar to the ones found in [59] (where $\mathbf{B} \sim O(\partial^1)$). Namely, there is are currents along $\mathbf{B}$ field; the Chiral Magnetic Effect. It is also seen that the effect of the anomaly on the constitutive relation is entirely captured by the weak-$\mathbf{B}$ part of the constitutive relations. There are no additional currents related to the anomaly when the $\mathbf{B}$ field is made strong. Instead, we see that in strong $\mathbf{B}$ field, the currents do acquire additional terms, but they are all related to the magnetization effects. There are no hydrodynamic responses that are dependent on both the anomaly and the magnetization.

2.1 Basic ingredients

2.1.1 Thermodynamics

We first discuss the thermodynamics of the system. For generality we assume that the magnetic field $\mathbf{B}$ can be both time- and space-dependent. The grand potential $\Omega = -pV$, where V is the volume and the pressure p is a function of temperature, chemical potentials, and the magnetic field, specified by the equation of state:

$$p = p(T, \mu^L, \mu^R, \mathbf{B}). \tag{2.1}$$

Differentiating p we obtain thermodynamically conjugate variables,

$$dp = sdT + n^L d\mu^L + n^R d\mu^R + \mathbf{m} \cdot d\mathbf{B} \tag{2.2}$$

where s is the volume density of entropy, $n^{L,R}$ are the densities of left- and right-handed particles, and $\mathbf{m}$ is the magnetic polarization density.

Instead of dealing with separate densities for the separately conserved L/R species, it is convenient to work with total and chiral charge densities, which are defined as:

$$n = n^L + n^R, \qquad n^5 = n^L - n^R, \tag{2.3}$$

The corresponding total and chiral chemical potentials are given as:

$$\mu = \frac{1}{2}\left(\mu^L + \mu^R\right), \qquad \mu^5 = \frac{1}{2}\left(\mu^L - \mu^R\right), \tag{2.4}$$

Formulating equation (2.2) in this new basis gives us:

$$dp = sdT + nd\mu + n^5 d\mu^5 + \mathbf{m} \cdot d\mathbf{B}. \tag{2.5}$$

The energy density ε is related to p by a Legendre transform:

$$\varepsilon = Ts + \mu n + \mu^5 n^5 - p \tag{2.6}$$

Its differential gives us the 1^{st} law of thermodynamics for this system:

$$d\varepsilon = Tds + \mu dn + \mu^5 dn^5 - \mathbf{m} \cdot d\mathbf{B} \tag{2.7}$$

2.1.2 Conservation laws

To write down the hydrodynamic equations, we start from the conservation laws. They have the following form:

$$\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{2.8a}$$

$$\partial_t n^5 + \boldsymbol{\nabla} \cdot \mathbf{j}^5 = \mathsf{C}\mathbf{E} \cdot \mathbf{B} \tag{2.8b}$$

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \mathbf{j}^\varepsilon = \mathbf{j} \cdot \mathbf{E} \tag{2.8c}$$

where $\mathbf{j}$, $\mathbf{j}^5$, and $\mathbf{j}^\varepsilon$ are the currents of the total particle number, chiral particle number, and energy. C is the anomaly coefficient. We add to this system of equations one of Maxwell's equations:

$$\partial_t \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{E} = 0 \tag{2.9}$$

which has the form of a conservation law and for convenience will be treated on equal footing as the other three equations (2.8).

Using the 1^{st} law of thermodynamics (2.7), we get:

$$\partial_t s = \frac{1}{T}\partial_t \varepsilon - \frac{\mu}{T}\partial_t n - \frac{\mu^5}{T}\partial_t n^5 + \frac{\mathbf{m}}{T}\partial_t \mathbf{B} \tag{2.10}$$

which, by using the conservation laws, can be rewritten as:

$$\partial_t s = \frac{1}{T}(-\boldsymbol{\nabla}\cdot\mathbf{j}^\varepsilon + \mathbf{j}\cdot\mathbf{E}) + \frac{\mu}{T}\boldsymbol{\nabla}\cdot\mathbf{j} + \frac{\mu^5}{T}\Big(\boldsymbol{\nabla}\cdot\mathbf{j}^5 - \mathsf{C}\mathbf{E}\cdot\mathbf{B}\Big) - \frac{1}{T}\mathbf{m}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) \tag{2.11}$$

Instead of the energy current, it is convenient to use the 'heat current', defined as:

$$\mathbf{Q} = \mathbf{j}^\varepsilon - \mu\mathbf{j} - \mu^5\mathbf{j}^5 \tag{2.12}$$

in terms of which the equation for time evolution of the entropy has the form:

$$\partial_t s = -\frac{1}{T}\Big(\boldsymbol{\nabla}\cdot\mathbf{Q} + \mathbf{j}\cdot(\boldsymbol{\nabla}\mu - \mathbf{E}) + \mathbf{j}^5\cdot\boldsymbol{\nabla}\mu^5 + \mathsf{C}\mu^5\mathbf{E}\cdot\mathbf{B} + \mathbf{m}\cdot(\boldsymbol{\nabla}\times\mathbf{E})\Big) \tag{2.13}$$

2.2 Constitutive relations

2.2.1 Chiral magnetic effect and Magnetization current

Our goal is to write down the most general constitutive relations that express the currents $\mathbf{Q}$, $\mathbf{j}$, $\mathbf{j}^5$ in terms of the thermodynamic variables T, μ, μ^5, and $\mathbf{B}$ and their derivatives. The constraint that the structure of the currents is subject to is positivity of entropy production.

The general structure of the constitutive relations are as follows: Each of the currents has a dissipative contribution, that participates in local entropy production. The dissipative part of the currents contain kinetic coefficients that are subject to Onsager reciprocity and inequalities that ensure the positivity of entropy production.

In addition, there are non-dissipative currents that do not participate in entropy production. These come in two varieties: The first is currents that *must* be present to ensure the compatibility of the anomaly and magnetization terms with positivity of entropy production. These terms can be constrained, and may be evaluated as exact functions of the hydrodynamic variables.

The second variety is non-dissipative currents that *may* be present, but are not to required to in order to cancel the anomaly and magnetization dependent contributions. These unconstrained non-dissipative currents enter the constitutive relations in special arrangements to ensure compatibility with 2^{nd} law. We will leave these aside in this section.

Isolating the fixed non-dissipative terms, terms, we may write the constitutive relations as:

$$\mathbf{j} = \mathsf{C}\mu^5\mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m} + \boldsymbol{\nu}, \tag{2.14a}$$

$$\mathbf{j}^5 = \mathsf{C}\mu\mathbf{B} + \boldsymbol{\nu}^5, \tag{2.14b}$$

$$\mathbf{Q} = -\mathsf{C}\mu\mu^5\mathbf{B} + \mathbf{m} \times (\mathbf{E} - \boldsymbol{\nabla}\mu) + \boldsymbol{\nu}^Q \tag{2.14c}$$

We have isolated in the above constitutive relations the Chiral magnetic effect(s), which take the form of non-dissipative currents along the **B** field. In addition, we have also isolated the magnetization current, the contribution $\boldsymbol{\nabla} \times \mathbf{m}$ in the electric current **j**, and the related contribution to the heat current **Q**. The dissipative and the unconstrained non-dissipative contributions to the currents are contained within the $\boldsymbol{\nu}$'s. Inserting these into Eq. (2.13), we find after many cancellations,

$$\frac{\partial s}{\partial t} = -\frac{1}{T}\boldsymbol{\nabla} \cdot \boldsymbol{\nu}^Q + \frac{1}{T}\boldsymbol{\nu} \cdot (\mathbf{E} - \boldsymbol{\nabla}\mu) - \frac{1}{T}\boldsymbol{\nu}^5 \cdot \boldsymbol{\nabla}\mu^5 \tag{2.15}$$

As promised, the anomaly and magnetization dependent contributions have cancelled out.

Finally, identifying the relation between entropy and heat current $\mathbf{j}_s$ as

$$\mathbf{j}^s = \frac{\boldsymbol{\nu}^Q}{T} \tag{2.16}$$

allows us to finally write the following expression for the rate of entropy production:

$$\mathcal{R} = \frac{\partial s}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}^s = -\Big(\frac{\boldsymbol{\nu}^Q}{T^2} \cdot \boldsymbol{\nabla} T + \frac{\boldsymbol{\nu}}{T} \cdot \boldsymbol{\nabla} \bar{\mu} + \frac{\boldsymbol{\nu}^5}{T} \cdot \boldsymbol{\nabla} \mu^5\Big) \tag{2.17}$$

We have defined a shorthand $\boldsymbol{\nabla}\bar{\mu} \equiv \boldsymbol{\nabla}\mu - \mathbf{E}$. Note that $\mathbf{E}$ and $\boldsymbol{\nabla}\mu$ appear together in the expression, which makes sense as that first signifies the 'external' electric field, whereas the latter signifies the effective electric field induced by a gradient in chemical potential. This is where their influence on the system enters together into the constitutive relations. This is the reason why for reasons of consistency, $\mathbf{E} \sim O(\partial)$, as alluded to earlier [25]. This is generally always going to be the case in the hydrodynamic theory for a system with itinerant charges.

2.2.2 Dissipative structure

We are now in a position to analyize the structure of the dissipative currents. In general, we expect that in the 'linear response' or first order gradient approximation regime, there should be currents that are linear in gradients of hydrodynamic variables, or electric field $\mathbf{E}$.

We will expand $\boldsymbol{\nu}$'s as linear combinations of gradients as:

$$\boldsymbol{\nu}_i = \lambda^T_{ia}\boldsymbol{\nabla}_a T + \lambda^\mu_{ia}\boldsymbol{\nabla}_a\bar{\mu} + \lambda^c_{ia}\boldsymbol{\nabla}_a\mu^5 \tag{2.18a}$$

$$\boldsymbol{\nu}^5_i = \xi^T_{ia}\boldsymbol{\nabla}_a T + \xi^\mu_{ia}\boldsymbol{\nabla}_a\bar{\mu} + \xi^c_{ia}\boldsymbol{\nabla}_a\mu^5 \tag{2.18b}$$

$$\boldsymbol{\nu}^Q_i = \kappa^T_{ia}\boldsymbol{\nabla}_a T + \kappa^\mu_{ia}\boldsymbol{\nabla}_a\bar{\mu} + \kappa^c_{ia}\boldsymbol{\nabla}_a\mu^5 \tag{2.18c}$$

where the indices i, a are spatial indices. In general, kinetic coefficients are rank-2 objects. Therefore, in a 3-dimensional anisotropic medium, each kinetic coefficient is composed of 6 scalar coefficients (accounting for Onsager reciprocity) that can each be functions of the hydrodynamic variables: T, μ, μ^5 and $B \equiv |\mathbf{B}|$.

However, when we further impose the requirement of isotropy, we can see that we are only allowed to build up higher rank objects using the isotropic tensors, Kronecker delta, $\boldsymbol{\delta}_{ia}$, the Levi-Civita symbol $\boldsymbol{\epsilon}_{iab}$ and any available hydrodynamic variables which possesses indices. In this case, that is magnetic field $\mathbf{B}_a$. Out of these, there are 3 combinations that may be built, which are:

$$\lambda^\mu_{ia} = \lambda^\mu_1\boldsymbol{\delta}_{ia} + \lambda^\mu_2\mathbf{B}_i\mathbf{B}_a + \widetilde{\lambda^\mu}\boldsymbol{\epsilon}_{iab}\mathbf{B}_b \tag{2.19}$$

where, the scalar coefficients (such as λ^μ_1) are functions of the hydrodynamic variables; $\lambda^\mu_1 = \lambda^\mu_1(T, \mu, \mu^5, B)$. All the kinetic coefficients in Eq.(2.18) can be decomposed in an identical manner.

We will now invoke Onsager reciprocity to limit the number of independent dissipative coefficients. In order to do so, it is helpful to first reduce the expression for $\mathcal{R}$ into a quadratic form by plugging Eq.(2.18) into Eq.(2.17):

$$\begin{aligned} -\mathcal{R} = &\boldsymbol{\nabla}_i T \boldsymbol{\nabla}_a T \left(\frac{\kappa^T_{ia}}{T^2}\right) + \boldsymbol{\nabla}_i \bar{\mu} \boldsymbol{\nabla}_a \bar{\mu} \left(\frac{\lambda^\mu_{ia}}{T}\right) + \boldsymbol{\nabla}_i \mu^5 \boldsymbol{\nabla}_a \mu^5 \left(\frac{\xi^c_{ia}}{T}\right) \\ &+ \boldsymbol{\nabla}_i T \boldsymbol{\nabla}_a \bar{\mu} \left(\frac{\kappa^\mu_{ia}}{T^2} + \frac{\lambda^T_{ai}}{T}\right) + \boldsymbol{\nabla}_i T \boldsymbol{\nabla}_a \mu^5 \left(\frac{\kappa^c_{ia}}{T^2} + \frac{\xi^T_{ai}}{T}\right) \\ &+ \boldsymbol{\nabla}_i \bar{\mu} \boldsymbol{\nabla}_a \mu^5 \left(\frac{\lambda^c_{ia}}{T} + \frac{\xi^\mu_{ai}}{T}\right) \end{aligned} \tag{2.20}$$

Onsager's original reciprocity relations assume that there are no time reversal symmetry breaking effects at the microscopic level [40]. This of course precludes the presence of a **B** field. However, we may still write Onsager relations in the presence of **B** fields, by extending the action of $\mathcal{T}$ operation to not just the electronic system at hand, but also the external sources that give rise to **B** in the first place. Magnetic fields are produced by currents; in this extended definition of $\mathcal{T}$, we need to flip these currents, thus flipping the direction of **B** as well. The reciprocal relations for the diagonal terms are:

$$\begin{aligned} \kappa^T_{ia}(\mathbf{B}) &= \kappa^T_{ai}(-\mathbf{B}) \\ \lambda^\mu_{ia}(\mathbf{B}) &= \lambda^\mu_{ai}(-\mathbf{B}) \\ \xi^c_{ia}(\mathbf{B}) &= \xi^c_{ai}(-\mathbf{B}) \end{aligned} \tag{2.21}$$

which are identically true for kinetic coefficients of the form Eq.(2.19), and hence, are trivial. On the other hand, the Onsager relations between the off-diagonal terms takes the form:

$$\begin{aligned} \kappa^\mu_{ia}(\mathbf{B}) &= T\lambda^T_{ai}(-\mathbf{B}) \\ \kappa^c_{ia}(\mathbf{B}) &= T\xi^T_{ai}(-\mathbf{B}) \\ \lambda^c_{ia}(\mathbf{B}) &= \xi^\mu_{ai}(-\mathbf{B}) \end{aligned} \tag{2.22}$$

which is a non-trivial constraint that reduces the number of independent coefficients. For example, the first equation above implies that $\kappa^{\mu}_{1,2} = T\lambda^{T}_{1,2}$ and $\widetilde{\kappa^{\mu}} = T\widetilde{\lambda^{T}}$.

Finally, we are able to write $\mathcal{R}$ in a form that involves just the dissipative coefficients. Using the decomposition discussed earlier, and the Onsager relations, we get:

$$\begin{aligned}
-\mathcal{R} = {}& \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T \left(\frac{\kappa^{T}_{1}}{T^2}\right) + \boldsymbol{\nabla}\bar{\mu} \cdot \boldsymbol{\nabla}\bar{\mu} \left(\frac{\lambda^{\mu}_{1}}{T}\right) + \boldsymbol{\nabla}\mu^5 \cdot \boldsymbol{\nabla}\mu^5 \left(\frac{\xi^{c}_{1}}{T}\right) \\
& + \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla}\bar{\mu} \left(\frac{2\lambda^{T}_{1}}{T}\right) + \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla}\mu^5 \left(\frac{2\xi^{T}_{1}}{T}\right) + \boldsymbol{\nabla}\bar{\mu} \cdot \boldsymbol{\nabla}\mu^5 \left(\frac{2\lambda^{c}_{1}}{T}\right) \\
& + (\boldsymbol{\nabla} T \cdot \mathbf{B})^2 \left(\frac{\kappa^{T}_{2}}{T^2}\right) + (\boldsymbol{\nabla}\bar{\mu} \cdot \mathbf{B})^2 \left(\frac{\lambda^{\mu}_{2}}{T}\right) + (\boldsymbol{\nabla}\mu^5 \cdot \mathbf{B})^2 \left(\frac{\xi^{c}_{2}}{T}\right) \\
& + (\boldsymbol{\nabla} T \cdot \mathbf{B})(\boldsymbol{\nabla}\bar{\mu} \cdot \mathbf{B}) \left(\frac{2\lambda^{T}_{2}}{T}\right) + (\boldsymbol{\nabla} T \cdot \mathbf{B})(\boldsymbol{\nabla}\mu^5 \cdot \mathbf{B}) \left(\frac{2\xi^{T}_{2}}{T}\right) \\
& + (\boldsymbol{\nabla}\bar{\mu} \cdot \mathbf{B})(\boldsymbol{\nabla}\mu^5 \cdot \mathbf{B}) \left(\frac{2\lambda^{c}_{2}}{T}\right)
\end{aligned} \tag{2.23}$$

There are 12 independent scalar dissipative coefficients. The positivity of entropy production $\mathcal{R} \geq 0$ imposes inequalities that must be satisfied by these dissipative coefficients. On the other hand, we can see that the coefficients that involve $\boldsymbol{\epsilon}_{ijk}$, e.g. $\widetilde{\lambda^{T}}$, identically cancel out from the expression, as a result of the off-diagonal Onsager relations Eq.(2.22). This is just as well, as these terms give contributions of the type $\boldsymbol{\nabla}\bar{\mu} \cdot (\boldsymbol{\nabla} T \times \mathbf{B})$ to $\mathcal{R}$, that cannot be constrained to be positive definite by imposing inequalities.

This is the first example of how unconstrained non-dissipative currents come about in the constitutive relations. They require some special arrangement that ensures cancellation of offending terms. Here, this is arranged for by Onsager reciprocity. These non-dissipative currents are subject to no further constraints. So, the $\boldsymbol{\nu}$'s may be separated into dissipative and non-dissipative parts,

$$
\begin{aligned}
\boldsymbol{\nu} &= \boldsymbol{\nu}^{(\mathcal{R})} + \widetilde{\boldsymbol{\nu}} \\
\boldsymbol{\nu}^{Q} &= \boldsymbol{\nu}^{Q(\mathcal{R})} + \widetilde{\boldsymbol{\nu}^{Q}} \\
\boldsymbol{\nu}^{5} &= \boldsymbol{\nu}^{5(\mathcal{R})} + \widetilde{\boldsymbol{\nu}^{5}}
\end{aligned}
\tag{2.24}
$$

The non-dissipative part has 6 independent coefficients, and is given by:

$$
\begin{aligned}
\widetilde{\boldsymbol{\nu}} &= \widetilde{\lambda^{\mu}}(\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \widetilde{\lambda^{T}}(\boldsymbol{\nabla}T \times \mathbf{B}) + \widetilde{\lambda^{c}}(\boldsymbol{\nabla}\mu^{5} \times \mathbf{B}) \\
\widetilde{\boldsymbol{\nu}^{Q}} &= T\widetilde{\lambda^{T}}(\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \widetilde{\kappa^{T}}(\boldsymbol{\nabla}T \times \mathbf{B}) + \widetilde{\kappa^{c}}(\boldsymbol{\nabla}\mu^{5} \times \mathbf{B}) \\
\widetilde{\boldsymbol{\nu}^{5}} &= \widetilde{\lambda^{c}}(\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \frac{\widetilde{\kappa^{c}}}{T}(\boldsymbol{\nabla}T \times \mathbf{B}) + \widetilde{\xi^{c}}(\boldsymbol{\nabla}\mu^{5} \times \mathbf{B})
\end{aligned}
\tag{2.25}
$$

Note, that $\widetilde{\lambda^{\mu}}$ in particular, and the $\widetilde{\boldsymbol{\nu}}$'s in general, correspond to what is commonly known as the 'Hall Effect'. This belongs in the category of non-dissipative currents that cannot be fixed by positivity of $\mathcal{R}$.

2.3 General non-dissipsative structure

In this section, we will systematically explore non-dissipative currents that cannot be fixed by anomaly or magnetization. So far, the constitutive relations read:

$$
\mathbf{j} = \mathsf{C}\mu^{5}\mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m} + \boldsymbol{\nu}^{(\mathcal{R})} + \widetilde{\boldsymbol{\nu}} \tag{2.26a}
$$

$$
\mathbf{j}^{5} = \mathsf{C}\mu\mathbf{B} + \boldsymbol{\nu}^{5(\mathcal{R})} + \widetilde{\boldsymbol{\nu}^{5}} \tag{2.26b}
$$

$$
\mathbf{Q} = -\mathsf{C}\mu\mu^{5}\mathbf{B} + \mathbf{m} \times (\mathbf{E} - \boldsymbol{\nabla}\mu) + \boldsymbol{\nu}^{Q(\mathcal{R})} + \widetilde{\boldsymbol{\nu}^{Q}} \tag{2.26c}
$$

$$
\mathbf{j}^{s} = \frac{\boldsymbol{\nu}^{Q(\mathcal{R})}}{T} + \frac{\widetilde{\boldsymbol{\nu}^{Q}}}{T} \tag{2.26d}
$$

These are not the most general constitutive relations that are compatible with positivity of entropy production. There are yet more unconstrained non-dissipative currents aside from that $\widetilde{\boldsymbol{\nu}}$'s that may be presesnt in a general situation. Some of these may be discovered easily. For example, from Eq.(2.13), we have:

$$\mathcal{R} = \boldsymbol{\nabla}\cdot\mathbf{j}^s - \frac{1}{T}\Big(\boldsymbol{\nabla}\cdot\mathbf{Q} + \mathbf{j}\cdot(\boldsymbol{\nabla}\mu - \mathbf{E}) + \mathbf{j}^5\cdot\boldsymbol{\nabla}\mu^5 + \mathsf{C}\mu^5\mathbf{E}\cdot\mathbf{B} + \mathbf{m}\cdot(\boldsymbol{\nabla}\times\mathbf{E})\Big) \tag{2.27}$$

By inspection it may be seen that because of the appearance of $\boldsymbol{\nabla}\cdot\mathbf{Q}$ and $\boldsymbol{\nabla}\cdot\mathbf{j}^s$ in the expression, we may add magnetization type currents, $\boldsymbol{\nabla}\times\widetilde{\mathbf{m}^Q}$ and $\boldsymbol{\nabla}\times\widetilde{\mathbf{m}^s}$ to $\mathbf{Q}$ and $\mathbf{j}^s$ respectively. The 'Entropy Magnetization' and 'Heat Magnetization', $\widetilde{\mathbf{m}^s}$ and $\widetilde{\mathbf{m}^Q}$, are constrained by rotational symmetry to be parallel to $\mathbf{B}$ (just like magnetization $\mathbf{m}$ is). However, the magnitudes, $|\widetilde{\mathbf{m}^s}|$ and $|\widetilde{\mathbf{m}^Q}|$ are unconstrained.

There are two more sets of unconstrained non-dissipative currents that may be present in a general case, but which are harder to realize by mere inspection. To uncover these, we follow an alternate algorithmic approach to derive the non-dissipative contributions to the currents.

Let us focus first on the case when $\mathbf{B}\sim O(\partial)$ first. In the weak-$\mathbf{B}$ field case, the analysis is much simpler; thermodynamics does not depend upon $\mathbf{B}$. So we may ignore magnetization related effects. The kinetic coefficients do not depend upon $\mathbf{B}$ as well, and hence, are just scalar functions of T, μ and μ^5. Eq.(2.13) now reduces to

$$\mathcal{R} = \boldsymbol{\nabla}\cdot\mathbf{j}^s - \frac{1}{T}\Big(\boldsymbol{\nabla}\cdot\mathbf{Q} + \mathbf{j}\cdot(\boldsymbol{\nabla}\mu - \mathbf{E}) + \mathbf{j}^5\cdot\boldsymbol{\nabla}\mu^5 + \mathsf{C}\mu^5\mathbf{E}\cdot\mathbf{B}\Big) \tag{2.28}$$

To ensure positivity of $\mathcal{R}$, we must ensure that all terms of the form $\#.\mathbf{B}$ should cancel out. In order to build a minimal set of constitutive relations compatible with $\mathcal{R}\geq 0$, we may identify the non-dissipative currents required through inspection. We can see that the

$\mathsf{C}\mu^5\mathbf{E}.\mathbf{B}$ term necessitates a non-dissipative contribution to $\mathbf{j}$ equal to $\mathsf{C}\mu^5\mathbf{B}$. Now, adding $\mathsf{C}\mu\mathbf{B}$ to the chiral current $\mathbf{j}^5$, we may combine the left over terms dependent on $\mathbf{B}$ from $\mathbf{j}$ and $\mathbf{j}^5$ into $\boldsymbol{\nabla}\cdot(\mathsf{C}\mu\mu^5\mathbf{B})$. In order to cancel this, we need a non-dissipative current in $\mathbf{Q}$ that goes as $-\mathsf{C}\mu\mu^5\mathbf{B}$. This is what we have so far from our discussion earlier.

The algorithmic method would proceed by realizing that in order to accomodate the $\mathbf{E}.\mathbf{B}$ term, we should have non-dissipative currents along the $\mathbf{B}$ field. That is the only piece of information we are allowed to use. Then, we would then write the constitutive relations as:

$$\mathbf{j} = \boldsymbol{\nu} + \lambda^B\mathbf{B} \tag{2.29a}$$

$$\mathbf{j}^5 = \boldsymbol{\nu}^5 + \xi^B\mathbf{B} \tag{2.29b}$$

$$\mathbf{Q} = \boldsymbol{\nu}^Q + \kappa^B\mathbf{B} \tag{2.29c}$$

$$\mathbf{j}^s = \frac{\mathbf{Q}}{T} + D^B\mathbf{B} \tag{2.29d}$$

Since we have $\mathbf{B}\sim O(\partial)$, the $\boldsymbol{\nu}$'s are entirely dissipative. For example, $\boldsymbol{\nu} = \lambda^\mu\boldsymbol{\nabla}\bar{\mu} + \lambda^T\boldsymbol{\nabla}T + \lambda^c\boldsymbol{\nabla}\mu^5$. The dissipative and the non-dissipative coefficients are only allowed to be functions of T, μ and μ^5. Plugging into Eq.(2.28) the condition that terms involving $\mathbf{B}$ should cancel out translates to:

$$\mathbf{B}.\Big(\frac{\kappa^B}{T^2}\boldsymbol{\nabla}T + \frac{\lambda^B}{T}\boldsymbol{\nabla}\mu + \frac{\xi^B}{T}\boldsymbol{\nabla}\mu^5 - \boldsymbol{\nabla}D^B\Big) + \mathbf{B}.\mathbf{E}\Big(\frac{\mathsf{C}\mu^5 - \lambda^B}{T}\Big) = 0 \tag{2.30}$$

The second term fixes $\lambda^B = \mathsf{C}\mu^5$. In the first term, we can expand $\boldsymbol{\nabla}D^B = D^B_T\boldsymbol{\nabla}T + D^B_\mu\boldsymbol{\nabla}\mu + D^B_{\mu^5}\boldsymbol{\nabla}\mu^5$, where $D^B_T = \partial_T D^B|_{\mu,\mu^5}$ etc. Solving $TD^B_\mu = \lambda^B = \mathsf{C}\mu^5$ gives us $D^B = \frac{\mathsf{C}\mu\mu^5}{T} + \widetilde{\mathsf{F}}(T,\mu^5)$. The only constraint on $\widetilde{\mathsf{F}}$ is that it cannot be a function of μ. Apart from this requirement, $\widetilde{\mathsf{F}}$ is completely unconstrained. Using this to solve for ξ^B and κ^B, it follows that:

$$\mathbf{j} = \boldsymbol{\nu} + \mathsf{C}\mu^5\mathbf{B} \tag{2.31a}$$

$$\mathbf{j}^5 = \boldsymbol{\nu}^5 + (\mathsf{C}\mu + T\widetilde{\mathsf{F}}_{\mu^5})\mathbf{B} \tag{2.31b}$$

$$\mathbf{Q} = \boldsymbol{\nu}^Q + (-\mathsf{C}\mu\mu^5 + T^2\widetilde{\mathsf{F}}_T)\mathbf{B} \tag{2.31c}$$

$$\mathbf{j}^s = \frac{\boldsymbol{\nu}^Q}{T} + (\widetilde{\mathsf{F}} + T\widetilde{\mathsf{F}}_T)\mathbf{B} \tag{2.31d}$$

where, once again, $\widetilde{\mathsf{F}}_T = \partial_T\widetilde{\mathsf{F}}|_{\mu^5}$ etc. This represents the most general constitutive relations allowed when $\mathbf{B} \sim O(\partial)$.

Note that $\widetilde{\mathsf{F}}$ is not totally unconstrained. Some information about it may be gleaned by considering its behavior under the action of $\mathcal{I}$. Under spatial inversion, all the terms present in any of the currents must behave the same way; either they all flip, or they all do not. In order for this to be so, it may be seen that $\widetilde{\mathsf{F}}$ can only be an *odd function* of μ^5, which is a pseudoscalar under the action of $\mathcal{I}$. We should have made similar remarks in our analysis of dissipative structures as well; ξ^T and ξ^μ are odd functions of μ^5, and ξ^c is an even function.

There is another set of unconstrained non-dissipative currents that is present at strong-$\mathbf{B}$ fields. These may be discovered in a similar algorithmic manner. When $\mathbf{B} \sim O(1)$, retaining terms in the constitutive relations up to $O(\partial^1)$, we should include spatial derivatives of the $\mathbf{B}$ field as well. In addition, expressing $\mathbf{m} = \chi\mathbf{B}$, where $\chi = \chi(T, \mu, \mu^5, B)$ is the magnetic susceptibility, it can be seen that we must have terms proportional to the external electric field $\mathbf{E}$ as well in the non-disspative parts of the currents. In order to see this, it is sufficient to expand the magnetization dependent term in Eq.(2.27). We have $-\frac{\mathbf{m}\cdot(\boldsymbol{\nabla}\times\mathbf{E})}{T}$, which can be written as $-\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{m}}{T}\times\mathbf{E}\right)+\mathbf{E}\cdot\left(\boldsymbol{\nabla}\times\frac{\mathbf{m}}{T}\right)$, i.e. a total derivative part (that can be absorbed into the entropy current $\mathbf{j}^s$) and a leftover piece of the form $\#\cdot\mathbf{E}$, which will have to cancel with other terms entering Eq.(2.27)

We can express the second term explicitly in terms of $\mathbf{E}$, $\mathbf{B}$, and the gradients of the hydrodynamic variables by separating out the susceptibility. Doing this gives us:

$$\begin{aligned}\mathbf{E}\cdot\left(\boldsymbol{\nabla}\times\frac{\mathbf{m}}{T}\right) = \left(\frac{\chi}{T}\right)_T\mathbf{E}\cdot(\boldsymbol{\nabla}T\times\mathbf{B}) + \frac{\chi_\mu}{T}\mathbf{E}\cdot(\boldsymbol{\nabla}\mu\times\mathbf{B}) \\ + \frac{\chi_{\mu^5}}{T}\mathbf{E}\cdot(\boldsymbol{\nabla}\mu^5\times\mathbf{B}) + \frac{\chi_B}{BT}\boldsymbol{\epsilon}_{ijk}\mathbf{E}_i\mathbf{B}_a\mathbf{B}_k(\boldsymbol{\nabla}_j\mathbf{B}_a)\end{aligned} \tag{2.32}$$

To cancel the above, one clearly needs non-dissipative currents along $\mathbf{E}$.

We will initially assume that $\chi_B = 0$, i.e. magnetization $\mathbf{m}$ is linear in $\mathbf{B}$, to simplify the algebra. In analogy with what we did in the low-$\mathbf{B}$ case, we may write the constitutive relations as:

$$\mathbf{j}_i = \boldsymbol{\nu}_i + \lambda^B\mathbf{B}_i + \lambda^E_{ia}\mathbf{E}_a + \lambda^{\partial B}_{iab}\boldsymbol{\nabla}_a\mathbf{B}_b \tag{2.33a}$$

$$\mathbf{j}^5_i = \boldsymbol{\nu}^5_i + \xi^B\mathbf{B}_i + \xi^E_{ia}\mathbf{E}_a + \xi^{\partial B}_{iab}\boldsymbol{\nabla}_a\mathbf{B}_b \tag{2.33b}$$

$$\mathbf{Q}_i = \boldsymbol{\nu}^Q_i + \kappa^B\mathbf{B}_i + \kappa^E_{ia}\mathbf{E}_a + \kappa^{\partial B}_{iab}\boldsymbol{\nabla}_a\mathbf{B}_b \tag{2.33c}$$

$$\mathbf{j}^s = \frac{\mathbf{Q}}{T} + D^B\mathbf{B} + D^E(\mathbf{E}\times\mathbf{B}) + D^{\partial B}(\boldsymbol{\nabla}\times\mathbf{B}) \tag{2.33d}$$

where we have pre-empted something about the form of λ^E, $\lambda^{\partial B}$ etc. in the constitutive relation for $\mathbf{j}^s$. Plugging these constitutive relations into Eq.(2.27), and demanding that only the dissipative part (involving the $\boldsymbol{\nu}$'s) survives, we arrive at the following expressions for the currents:

$$\mathbf{j} = \boldsymbol{\nu} + \lambda^B \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m} - \Big(\chi_\mu (\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \chi_T (\boldsymbol{\nabla} T \times \mathbf{B}) + \chi_{\mu^5} (\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \Big) \tag{2.34a}$$

$$\mathbf{j}^5 = \boldsymbol{\nu}^5 + \xi^B \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m}^5 - \Big(\chi^5_\mu (\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \chi^5_T (\boldsymbol{\nabla} T \times \mathbf{B}) + \chi^5_{\mu^5} (\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \Big) \tag{2.34b}$$

$$\mathbf{Q} = \boldsymbol{\nu}^Q + \kappa^B \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m}^Q - \Big(\chi^Q_\mu (\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \chi^Q_T (\boldsymbol{\nabla} T \times \mathbf{B}) + \chi^Q_{\mu^5} (\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \Big) \tag{2.34c}$$

$$\mathbf{j}^s = \frac{\mathbf{Q}}{T} + D^B \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m}^s - \Big(\chi^s_\mu (\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \chi^s_T (\boldsymbol{\nabla} T \times \mathbf{B}) + \chi^s_{\mu^5} (\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \Big) \tag{2.34d}$$

where the anomaly dependent coefficients λ^B etc. are identical to the ones we found in the low-$\mathbf{B}$ case in Eq.(2.31). The 'magnetizations' $\mathbf{m}^5$, $\mathbf{m}^Q$ and $\mathbf{m}^s$ are given as $\mathbf{m}^s = \chi^s \mathbf{B}$ etc. where the 'susceptibilities' can be solved in terms of the magnetic susceptibility χ. We obtain: $\chi^s = \int_\mu \Big(\frac{\chi}{T}\Big) + f(T, \mu^5)$, where f is an undetermined integration constant. For the rest of the susceptibilities, we have $\chi^Q = T^2 \chi^s_T$ and $\chi^5 = T \chi^s_{\mu^5}$.

Of course, in general, $\chi_B \neq 0$. However, (seemingly) quite miraculously, even though these expressions are derived for the case when magnetization χ is not a function of B, they still work even in the case when $\chi_B \neq 0$.

However, from the form of Eq.(2.34), it is also clear that there are redundancies, and many of the terms may be absorbed into the unconstrained, non-dissipative parts of the $\boldsymbol{\nu}$'s from Eq.(2.25), and $\widetilde{\mathbf{m}^Q}$ and $\widetilde{\mathbf{m}^s}$ that we discussed earlier. Starting with $\mathbf{j}$, we can see that all the extra terms of the form $\boldsymbol{\nabla}(\#) \times \mathbf{B}$ can be absorbed into $\widetilde{\boldsymbol{\nu}}$. Combining the two, we see that we may replace $\widetilde{\boldsymbol{\nu}}$ by $\widetilde{\boldsymbol{\nu}}'$, which may be expanded as

$$\widetilde{\boldsymbol{\nu}}' = \widetilde{\lambda^\mu}' (\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + \widetilde{\lambda^T}' (\boldsymbol{\nabla} T \times \mathbf{B}) + \widetilde{\lambda^c}' (\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \tag{2.35}$$

where the new coefficients are given as $\widetilde{\lambda^\mu}' = \widetilde{\lambda^\mu} - \chi_\mu$, $\widetilde{\lambda^T}' = \widetilde{\lambda^T} - \chi_T$ and $\widetilde{\lambda^c}' = \widetilde{\lambda^c} - \chi_{\mu^5}$.

So, we see that all the extra terms in $\mathbf{j}$ have been absorbed into a redefinition of the unconstrained $\widetilde{\boldsymbol{\nu}}$

Turning our attention next to the heat current $\mathbf{Q}$, we see that the term $\mathbf{m}^Q$ can be absorbed into the unconstrained $\widetilde{\mathbf{m}^Q}$. Collecting the terms of the form $\boldsymbol{\nabla}(\#) \times \mathbf{B}$ together with the terms from $\widetilde{\boldsymbol{\nu}^Q}$, we have:

$$(\widetilde{\kappa^\mu} - \chi^Q_\mu)(\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + (\widetilde{\kappa^T} - \chi^Q_T)(\boldsymbol{\nabla}T \times \mathbf{B}) + (\widetilde{\kappa^c} - \chi^Q_{\mu^5})(\boldsymbol{\nabla}\mu^5 \times \mathbf{B}) \tag{2.36}$$

we are free to absorb χ^Q_T and $\chi^Q_{\mu^5}$ into redifinitions of $\widetilde{\kappa^T}$ and $\widetilde{\kappa^c}$, by defining $\widetilde{\kappa^T}' = \widetilde{\kappa^T} - \chi^Q_T$ and $\widetilde{\kappa^c}' = \widetilde{\kappa^c} - \chi^Q_{\mu^5}$. However, by Onsager reciprocity, $\widetilde{\kappa^\mu} = T\widetilde{\lambda^T}$, and we have already redefined λ^T to $\widetilde{\lambda^T}' = \widetilde{\lambda^T} - \chi_T$. The new primed coefficients must also obey Onsager reciprocity, so we are no longer free to redefine $\widetilde{\kappa^\mu}$.

From $\chi^Q = T^2\chi^s_T$, and $\chi^s = \int_\mu \left(\frac{\chi}{T}\right) + f(T, \mu^5)$, we get $\chi^Q_\mu = T\chi_T - \chi$. So, it follows that $(\widetilde{\kappa^\mu} - \chi^Q_\mu) = T(\widetilde{\lambda^T} - \chi_T) + \chi = T\widetilde{\lambda^T}' + \chi$, where $\widetilde{\lambda^T}'$ is as previously redefined. So, after redefinition of $\widetilde{\boldsymbol{\nu}^Q}$, we still have a left over piece, $\chi(\boldsymbol{\nabla}\mu - \mathbf{E}) \times \mathbf{B}$. Combining χ and $\mathbf{B}$ into $\mathbf{m}$, we get:

$$\mathbf{Q} = \boldsymbol{\nu}^{Q(\mathcal{R})} + \kappa^B\mathbf{B} + \widetilde{\boldsymbol{\nu}^Q}' + \mathbf{m} \times (\mathbf{E} - \boldsymbol{\nabla}\mu) \tag{2.37}$$

Once again, all the extra terms found in $\mathbf{Q}$ in Eq.(2.34) have been absorbed into consistent redefinitions of the unconstrained terms present in the currents.

Finally, we focus on $\mathbf{j}^5$. Collecting the extra terms together with $\widetilde{\boldsymbol{\nu}^5}$, we have:

$$\boldsymbol{\nabla} \times \mathbf{m}^5 + \Big((\widetilde{\xi^\mu} - \chi^5_\mu)(\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}) + (\widetilde{\xi^T} - \chi^5_T)(\boldsymbol{\nabla}T \times \mathbf{B}) + (\widetilde{\xi^c} - \chi^5_{\mu^5})(\boldsymbol{\nabla}\mu^5 \times \mathbf{B})\Big) \tag{2.38}$$

We are free to define $\widetilde{\xi^c}' = \widetilde{\xi^c} - \chi^5_{\mu^5}$. However, by Onsager reciprocity, $\widetilde{\xi^\mu} = \widetilde{\lambda^c}$ and $\widetilde{\xi^T} = \frac{\widetilde{\kappa^c}}{T}$, and we already have defined $\widetilde{\lambda^c}' = \widetilde{\lambda^c} - \chi_{\mu^5}$ and $\widetilde{\kappa^c}' = \widetilde{\kappa^c} - \chi^Q_{\mu^5}$. So, in effect, we have already defined $\widetilde{\xi^\mu}'$ and $\widetilde{\xi^T}'$.

Focussing first on $\widetilde{\xi^\mu}'$, we can write $\widetilde{\xi^\mu}' = \widetilde{\lambda^c}' = \widetilde{\lambda^c} - \chi_{\mu^5}$. Furthermore, we have $\chi^5 = T\chi^s_{\mu^5}$, and plugging in χ^s, we find that $\chi^5_\mu = \chi_{\mu^5}$. So, it follows that $\widetilde{\xi^\mu}' = \widetilde{\xi^\mu} - \chi^5_\mu$, which is what we have prefixing $\boldsymbol{\nabla}\bar{\mu} \times \mathbf{B}$ in the expression above.

The final term we consider does lead to something non-trivial; $\boldsymbol{\nabla}T \times \mathbf{B}$ term is prefixed by $\widetilde{\xi^T} - \chi^5_T$. We can write $\widetilde{\xi^T}' = \widetilde{\kappa^c}'/T = (\widetilde{\kappa^c} - \chi^Q_{\mu^5})/T$. Expressing χ^Q and χ^5 in terms of χ^s, it can be seen that $\chi^5_T = \chi^Q_{\mu^5}/T + \chi^5/T$. So, it follows that $\widetilde{\xi^T} - \chi^5_T = (\widetilde{\xi^T} - \chi^Q_{\mu^5}/T) - \chi^5/T = \widetilde{\xi^T}' - \chi^5/T$.

To recap; our algorithmic method led to Eq.(2.34). After absorbing all trivial contributions into the currents already present, we still have two pieces left over in $\mathbf{j}^5$: These are coupled: $\boldsymbol{\nabla} \times \mu^5$, a 'Chiral Magnetization' current, as well as a 'Chiral Thermal Hall' effect which is controlled by the same chiral magnetization μ^5, as $\chi^5\mathbf{B} = \mu^5$. So, after all possible redefinitions, we obtain:

$$\mathbf{j}^5 = \boldsymbol{\nu}^{5(\mathcal{R})} + \xi^B\mathbf{B} + \widetilde{\boldsymbol{\nu}^5}' + \boldsymbol{\nabla} \times \mathbf{m}^5 - \frac{\boldsymbol{\nabla}T \times \mathbf{m}^5}{T} \tag{2.39}$$

We do not need to do this analysis with $\mathbf{j}^s$ in Eq.(2.34) to find which terms survive after redefinition of the existing unconstrained contributions. For that, we examine the contribution of the two terms containing $\mathbf{m}^5$ in $\mathbf{j}^5$ to the rate of entropy production $\mathcal{R}$. From Eq.(2.27), the contribution of $\mathbf{j}^5$ to $\mathcal{R}$ is $-\mathbf{j}^5 \cdot \boldsymbol{\nabla}\mu^5/T$. Plugging in $\mathbf{j}^5 = \boldsymbol{\nabla} \times \mathbf{m}^5 - (\boldsymbol{\nabla}T \times \mathbf{m}^5)/T$, it can be seen that the contribution to $\mathcal{R}$ reduces to $-\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{m}^5 \times \boldsymbol{\nabla}\mu^5}{T}\right)$. Clearly, to cancel this, we must have a compensating piece in the entropy current $\mathbf{j}^s$ that is $(\mathbf{m}^5 \times \boldsymbol{\nabla}\mu^5)/T$. Moreover, we can see that this arrangement works out for arbitrary $\mathbf{m}^5$, and hence, is also an unconstrained term allowed in the constitutive relations. The only restriction on this term is that it must be parallel to $\mathbf{B}$.

In fact, this illustrates why this arrangement works. All the unconstrained terms we have discovered so far essentially cancel out through general vector identities; hence, they encompass a certain amount of generality. The $\widetilde{\mathbf{m}^Q}$ and $\widetilde{\mathbf{m}^s}$ too *need not* be along $\mathbf{B}$ to not contribute to $\mathcal{R}$. Anything will work, because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \#) = 0$. It is the assumption of isotropy that forces them to. In a non-cubic crystal, we should expect these to also be controlled by the crystal directions, for example.

This implies that if we now consider an (locally) isotropic Ferromagnet with spontaneous magnetization $\mathbf{m}_0$, subject to *weak* $\mathbf{B}$ fields, then too, the constitutive relations we have uncovered should accurately describe the possible currents. The entire dependence of the dissipative currents and the magnetization currents on $\mathbf{B}$ is transferred onto $\mathbf{m}_0$. The only exception are the anomally related terms, and $\widetilde{\mathsf{F}}$, which belong in the weak $\mathbf{B}$ sector of the hydrodynamic theory we have developed.

Therefore, the most general constitutive relations, encompassing all possible currents are

$$\mathbf{j} = \boldsymbol{\nu}^{(\mathcal{R})} + \mathsf{C}\mu^5 \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{m} + \widetilde{\boldsymbol{\nu}} \tag{2.40a}$$

$$\mathbf{j}^5 = \boldsymbol{\nu}^{5(\mathcal{R})} + (\mathsf{C}\mu + T\widetilde{\mathsf{F}}_{\mu^5})\mathbf{B} + \widetilde{\boldsymbol{\nu}^5} + \boldsymbol{\nabla} \times \widetilde{\mathbf{m}^5} - \frac{\boldsymbol{\nabla} T \times \widetilde{\mathbf{m}^5}}{T} \tag{2.40b}$$

$$\mathbf{Q} = \boldsymbol{\nu}^{Q(\mathcal{R})} + (-\mathsf{C}\mu\mu^5 + T^2\widetilde{\mathsf{F}}_T)\mathbf{B} + \mathbf{m} \times (\mathbf{E} - \boldsymbol{\nabla}\mu) + \widetilde{\boldsymbol{\nu}^Q} + \boldsymbol{\nabla} \times \widetilde{\mathbf{m}^Q} \tag{2.40c}$$

$$\mathbf{j}^s = \frac{\boldsymbol{\nu}^{Q(\mathcal{R})}}{T} + (\widetilde{\mathsf{F}} + T\widetilde{\mathsf{F}}_T)\mathbf{B} + \frac{\widetilde{\boldsymbol{\nu}^Q}}{T} + \boldsymbol{\nabla} \times \widetilde{\mathbf{m}^s} + \frac{\widetilde{\mathbf{m}^5} \times \boldsymbol{\nabla}\mu^5}{T} \tag{2.40d}$$

where, all terms with the $\sim$ on top are unconstrained terms. The only condition on the Hall response $\widetilde{\boldsymbol{\nu}}$'s are the Onsager reciprocal relations.The only conditions on $\widetilde{\mathsf{F}}$ are that it cannot be a function of μ, and it can only be an odd function of μ^5. Finally, the sole constraint on $\widetilde{\mathbf{m}^s}$, $\widetilde{\mathbf{m}^Q}$ and $\widetilde{\mathbf{m}^5}$ is that these terms must be parallel to the only $O(\partial^0)$ vector available to us, which is $\mathbf{B}$, or $\mathbf{m}$, in Ferromagnet. Other than these conditions, these terms are totally unconstrained. In the ferromagnetic case, λ^μ is now to be interpreted as the Anomalous Hall effect.

For the rest of this thesis, we will restrict ourselves to exploring the consequences of the low-$\mathbf{B}$ theory, when the currents are given by Eq.(2.31).

CHAPTER 3

STEADY STATE TRANSPORT OF WEYL SEMIMETALS

In this chapter, we will explore the consequences of the low-**B** part of the hydrodynamic theory of the previous section in the context of 'slow' transport. By slow, we mean non-equilibrium processes that are slow enough that the chiral charge density n^5 can no longer be though of being exactly conserved; i.e., when $t \sim \tau$, where t is the timescale over which the non-equilibrium phenomena are evolving, and τ is the timescale of chiral relaxation. This encompasses the situation of DC transport, which is the main subject of interest in this chapter.

First, we will have to examine if our hydrodynamic theory works in the slow case. As we have seen earlier, the appropriate hydrodynamic equations are in this limit are:

$$\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{3.1a}$$

$$\partial_t n^5 + \boldsymbol{\nabla} \cdot \mathbf{j}^5 = \mathsf{C}\mathbf{E} \cdot \mathbf{B} - \frac{n^5}{\tau} \tag{3.1b}$$

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \mathbf{j}^\varepsilon = \mathbf{j} \cdot \mathbf{E} \tag{3.1c}$$

Assuming that $\mathbf{B} \sim O(\partial^1)$, we can neglect magnetization effects. In this case, we may write:

$$\begin{aligned}\partial_t s + \boldsymbol{\nabla} \cdot \mathbf{j}^s &= \boldsymbol{\nabla} \cdot \mathbf{j}^s + \frac{1}{T}\partial_t \varepsilon - \frac{\mu}{T}\partial_t n - \frac{\mu^5}{T}\partial_t n^5 \\ &= \Big[\boldsymbol{\nabla} \cdot \mathbf{j}^s + \frac{1}{T}(\boldsymbol{\nabla} \cdot \mathbf{j}^\varepsilon + \mathbf{j} \cdot \mathbf{E}) + \frac{\mu}{T}\boldsymbol{\nabla} \cdot \mathbf{j} + \frac{\mu^5}{T}(\boldsymbol{\nabla} \cdot \mathbf{j}^5 - \mathsf{C}\mathbf{E} \cdot \mathbf{B})\Big] + \frac{\mu^5 n^5}{T\tau}\end{aligned} \tag{3.2}$$

where, we have separated out the extra term that is present in $\partial_t s$ when we include chiral relaxation.

We have already seen in the previous chapter that the constitutive relations for the currents Eq.(2.31), along with inequalities constraining the dissipative parts of the currents, guarantee that the contribution of the terms within the $\Big[\quad \Big]$ to $\mathcal{R}$ is ≥ 0. The contribution of the relaxation term $\mu^5 n^5/T\tau$ to $\mathcal{R}$ is guaranteed to be ≥ 0 provided that

$$\chi^5 \equiv n^5/\mu^5 \geq 0 \tag{3.3}$$

that is, n^5 is an odd function of μ^5. No additional constraints on the constitutive relations are required to ensure $\mathcal{R} \geq 0$. We shall later see that the fact that $\chi^5 \geq 0$ plays a crucial role in explaining the positivity of magnetoconductance in Weyl semimetals using our theoretical framework.

3.1 Definitions

There are three standard steady-state transport experiments that are usually performed on materials that conduct heat and electricity, to measure 3 kinetic coefficients required to characterize the transport properties of the material: (*i*) Electrical conductivity σ, where $\mathbf{j} = \sigma\mathbf{E}$ is the electric current in response to an external electric field $\mathbf{E}$, (*ii*) Thermoelectric conductivity α, where $\mathbf{j} = \alpha\boldsymbol{\nabla}T$ is the electric current induced by a temperature gradient $\boldsymbol{\nabla}T$, and (*iii*) Thermal conductivity κ, where $\mathbf{j}^{\varepsilon} = -\kappa\boldsymbol{\nabla}T$, when $\mathbf{j} = 0$, i.e. measurement of the energy current in response to a gradient of temperature in the situation when electric current $\mathbf{j} = 0$. When a material supports a thermoelectric current, this is only possible if there is an electric field $\mathbf{E} \neq 0$ as well present, so that the electric currents driven by $\mathbf{E}$ and $\boldsymbol{\nabla}T$ cancel out. Note that owing to Onsager reciprocity, the fourth transport experiment that can be imagined (flow of energy in response to an $\mathbf{E}$ field) is not independent.

We will first formulate these standard steady-state transport experiments in a normal metal [34]. In a normal metal, when there are only two conserved charges n and ε, the equations of motion are given by:

$$\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{3.4a}$$

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \mathbf{j}^{\varepsilon} = \mathbf{j} \cdot \mathbf{E} \tag{3.4b}$$

Adopting our naming scheme from the previous chapter, the constitutive relations would read:

$$\mathbf{j} = \lambda^{\mu} \boldsymbol{\nabla} \bar{\mu} + \lambda^{T} \boldsymbol{\nabla} T \tag{3.5a}$$

$$\mathbf{Q} = \mathbf{j}^{\varepsilon} - \mu \mathbf{j} = \kappa^{\mu} \boldsymbol{\nabla} \bar{\mu} + \kappa^{T} \boldsymbol{\nabla} T \tag{3.5b}$$

By Onsager reciprocity $\kappa^{\mu} = T\lambda^{T}$; there are only three independent dissipative coefficients. Using $d\varepsilon = Tds + \mu dn$, we may evaluate entropy production rate to be:

$$\mathcal{R} = -\Big(\frac{\kappa^T}{T^2} \boldsymbol{\nabla} T \cdot \boldsymbol{\nabla} T + \frac{\lambda^{\mu}}{T} \boldsymbol{\nabla} \bar{\mu} \cdot \boldsymbol{\nabla} \bar{\mu} + 2\frac{\lambda^T}{T} \boldsymbol{\nabla} \bar{\mu} \cdot \boldsymbol{\nabla} T\Big) \tag{3.6}$$

The condition $\mathcal{R} \geq 0$ imposes the following conditions on the kinetic coefficients:

$$\begin{gathered} \kappa^T \leq 0 \\ \lambda^{\mu} \leq 0 \\ T(\lambda^T)^2 \leq \kappa^T \lambda^{\mu} \end{gathered} \tag{3.7}$$

From the definition of the standard kinetic coefficients, in the presence of $\mathbf{E}$ and $\boldsymbol{\nabla} T$, the electric current is given as $\mathbf{j} = \sigma \mathbf{E} + \alpha \boldsymbol{\nabla} T$. Comparing with the constitutive relations above, we see:

$$\begin{aligned} \sigma &= -\,\lambda^{\mu} \\ \alpha &= +\,\lambda^{T} \end{aligned} \tag{3.8}$$

Moreover, from the fact that $\lambda^\mu \leq 0$, it follows that:

$$\sigma \geq 0 \tag{3.9}$$

Electrical conductivity must be positive to ensure the positivity of entropy production.

To work out the constraint imposed on thermal conductivity κ, we note that in order for $\mathbf{j} = 0$, we need to apply an electric field $\mathbf{E} = -\alpha \boldsymbol{\nabla} T / \sigma$ in addition to the temperature gradient. In this case, energy current and heat current are equal; $\mathbf{j}^\varepsilon = \mathbf{Q} = \kappa^T \boldsymbol{\nabla} T - \kappa^\mu \mathbf{E} = \kappa^T \boldsymbol{\nabla} T - T\alpha \mathbf{E}$. Plugging in $\mathbf{E}$ in terms of $\boldsymbol{\nabla} T$ into $\mathbf{j}^\varepsilon$, we get $\left(\kappa^T + T\alpha^2/\sigma\right)\boldsymbol{\nabla} T$. From this, we get:

$$\kappa = -\left(\kappa^T + \frac{T\alpha^2}{\sigma}\right) \tag{3.10}$$

The last inequality imposed by positivity of entropy production can be written as $-\kappa^T \geq T\alpha^2/\sigma$. From this, we have:

$$\kappa \geq 0 \tag{3.11}$$

Thermal conductivity must also be positive to ensure $\mathcal{R} \geq 0$.

Now, we will try to develop some intution about how steady state experiments work. To measure σ, the system is subjected to a constant $\mathbf{E}$ field. We assume that the $\mathbf{E}$ field can be treated as an 'external' field, and satisfies $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. A long time after the $\mathbf{E}$ field is switched on, the state of the system is described by a steady state solution to Eq.(3.4). In this steady state, charge should not accumulate. This implies $\partial_t n = 0$. Moreover, the external $\mathbf{E}$ field is spatially constant, which should lead us to expect that the resulting state is spatially homogenous, i.e. $\boldsymbol{\nabla} T = \boldsymbol{\nabla} \mu = 0$. We can check that the charge conservation equation is satisfied, as $\boldsymbol{\nabla} \cdot \mathbf{j} = \boldsymbol{\nabla} \cdot (\sigma \mathbf{E}) = 0$, as both $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, and $\boldsymbol{\nabla} \sigma = 0$. The latter condition follows from the fact that $\sigma = \sigma(T, \mu)$, and the spatial homogeneity of the state we are describing.

Condunting materials get hot due to passage of electric current under an $\mathbf{E}$ field (Joule heating). The energy current is given as $\mathbf{j}^{\varepsilon} = \mathbf{Q}+\mu\mathbf{j} = (\mu\sigma - T\alpha)\mathbf{E}$. So, $\boldsymbol{\nabla}\cdot\mathbf{j}^{\varepsilon} = 0$ for the same reason as $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$. The energy conservation equation reads: $\partial_t\varepsilon = \mathbf{j}\cdot\mathbf{E} = \sigma\mathbf{E}\cdot\mathbf{E} \neq 0$. That is, even though we are in a 'steady state', the internal energy (and hence, the temperature etc.) are rising as a result of the heat dissipated due to conduction of electric current through the metal. In real experiments though, the material is finite, and in contact with the external environment. If we wait long enough after turning on $\mathbf{E}$, the sample reaches an equilibrium with its surroundings such that its temperature etc. are constant, but is radiating heat into its surroundings a constant rate:

$$P_{Joule} = V\left(\sigma\mathbf{E}\cdot\mathbf{E}\right) \tag{3.12}$$

Note that joule heating is only possible in the presence of an $\mathbf{E}$ field.

The analysis of transport under $\boldsymbol{\nabla}T$ is more subtle, as the assumption of spatial homogeneity no longer holds. T is obviously not constant in the material; $\boldsymbol{\nabla}T \neq 0$. The naive expectation though, would be that μ may be held constant independetly in the sample. However, this is not true in the typical experiment for thermoelectric transport. Recall that a in standard thermoelectric setup, the system is only sibjected to $\boldsymbol{\nabla}T$, i.e. there is no $\mathbf{E}$ field. Assuming that the steady state is homogenous with respect to μ, we may write $\mathbf{j} = \alpha\boldsymbol{\nabla}T$. This must satisfy $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$, which gives us $\alpha\boldsymbol{\nabla}^2T + \alpha_T|\boldsymbol{\nabla}T|^2 = 0$, where $\alpha_T \equiv \partial_T\alpha|_{\mu}$. This reduces to a linear profile in T, satisfying $\boldsymbol{\nabla}^2T = 0$, when $\alpha_T = 0$. Now, since there is no $\mathbf{E}$ field, $\partial_t\varepsilon = 0$ as well in the steady state. The energy current $\mathbf{j}^{\varepsilon} = \mathbf{Q} + \mu\mathbf{j} = (\kappa^T + \mu\lambda^T)\boldsymbol{\nabla}T$. Plugging in the standard coefficients, we get: $\mathbf{j}^{\varepsilon} = \Big(\mu\alpha - (\kappa + T\alpha^2/\sigma)\Big)\boldsymbol{\nabla}T \equiv \alpha'\boldsymbol{\nabla}T$. So, $\boldsymbol{\nabla}\cdot(\alpha' T) = 0$ *and* $\boldsymbol{\nabla}\cdot(\alpha T) = 0$ need to be simultaneously satisfied, which is only the case when $\alpha\alpha'_T - \alpha'\alpha_T = 0$. However, this represents a very strong constraint on the kinetic coefficients, and in general, will not be true. So it looks like we have an inconsistency.

The solution to this conundrum is that the statement $\boldsymbol{\nabla}\mu = 0$ is only true to leading order in ∂. For the purposes of describing the leading order behavior of the currents, it is sufficient to write $\mathbf{j} = \alpha\boldsymbol{\nabla}T$. However, if we wish to evaluate the profile of the temperature within the sample, $T(\mathbf{x})$, we would have to retain the $\boldsymbol{\nabla}\mu$ dependence as well; just the leading order approximation is insufficient.

3.2 DC transport with B-field in WSMs

Now, we turn our attention to WSMs. We want to model steady state experiments involving the flow of a steady current in the presence of an $\mathbf{E}$ field or a temperature gradient $\boldsymbol{\nabla}T$. From Eq.(2.31), in the presence of a $\mathbf{B}$ field, the currents are given as:

$$\mathbf{j} = \sigma\mathbf{E} + \alpha\boldsymbol{\nabla}T + \mathsf{C}\mu^5\mathbf{B} \tag{3.13a}$$

$$\mathbf{j}^5 = \sigma^5\mathbf{E} + \alpha^5\boldsymbol{\nabla}T + (\mathsf{C}\mu + T\widetilde{\mathsf{F}}_{\mu^5})\mathbf{B} \tag{3.13b}$$

$$\mathbf{Q} = \kappa^T\boldsymbol{\nabla}T - T\alpha\mathbf{E} + (-\mathsf{C}\mu\mu^5 + T^2\widetilde{\mathsf{F}}_T)\mathbf{B} \tag{3.13c}$$

3.2.1 *Positive magneto-electric conductivity*

First, we model the experiment to measure Electrical conductivity in Weyl semimetal. Again, in this case it is reasonable to assume that the steady state is spatially homogenous. The electric current is given by:

$$\mathbf{j} = \sigma\mathbf{E} + \mathsf{C}\mu^5\mathbf{B} \tag{3.14}$$

Once again, we can check that this current satisfies the continuity equation, as $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$, as a result of the spatial homogeneity of the state, and the fact that $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

The latter is one of the homogenous Maxwell's equations, and is an exact relation.

In the equation of motion for chiral charge n^5, there are two competing effects. The anomaly $\mathsf{C}\mathbf{E}\cdot\mathbf{B}$ has the effect of 'pumping' chiral imbalance, whereas the relaxation term $-n^5/\tau$ relaxes it. In the steady state, $\partial_t n^5 = 0$, so, we have $\boldsymbol{\nabla}\cdot\mathbf{j}^5 = \mathsf{C}\mathbf{E}\cdot\mathbf{B} - n^5/\tau$. The chiral current in a spatially homogenous state in the presence of external $\mathbf{E}$ and $\mathbf{B}$ fields is $\mathbf{j}^5 = \sigma^5\mathbf{E} + (\mathsf{C}\mu + T\widetilde{\mathsf{F}}^5_\mu)\mathbf{B}$. Once again, we can see that $\boldsymbol{\nabla}\cdot\mathbf{j}^5 = 0$. So, the equation of motion for n^5 reduces to:

$$\mu^5 = \frac{\tau\mathsf{C}\mathbf{E}\cdot\mathbf{B}}{\chi^5} \tag{3.15}$$

where, we have used $n^5 = \chi^5\mu^5$. Plugging this back into the expression for $\mathbf{j}$, we obtain:

$$\mathbf{j}_i = \Big(\sigma\boldsymbol{\delta}_{ij} + \frac{\tau\mathsf{C}^2}{\chi^5}\mathbf{B}_i\mathbf{B}_j\Big)\mathbf{E}_j \tag{3.16}$$

When the $\mathbf{B}$ field is parallel to the $\mathbf{E}$ field, the electric current is $\mathbf{j} = \sigma_{\parallel}\mathbf{E}$, where:

$$\sigma_{\parallel} = \sigma + \frac{\tau\mathsf{C}^2B^2}{\chi^5} \tag{3.17}$$

The extra term is quadratic in B. Also, the magnetoconductivity effect is positive. As we showed earlier, $\chi^5 \geq 0$ in order to ensure that the hydrodynamic equations including the relaxation term are compatible with positivity of entropy production. It follows the enhancement over the Drude conductivity is $\sigma_{\parallel} - \sigma \geq 0$

When $\mathbf{B}$ is not $\parallel$ to $\mathbf{E}$, the additional piece in $\mathbf{j}$ is $\Big(\tau\mathsf{C}^2\mathbf{E}\cdot\mathbf{B}/\chi^5\Big)\mathbf{B}$, i.e. the additional piece in the current density is along $\mathbf{B}$, not $\mathbf{E}$. However, in a typical experiment, the direction of $\mathbf{E}$ defines the conducting direction of the sample. So, the effective enhancement to the conductivity, $\sigma_{\mathbf{B}}$, is given as:

$$\sigma_{\mathbf{B}} = \frac{\tau \mathsf{C}^2 B^2 cos^2\phi}{\chi^5} \tag{3.18}$$

where ϕ is the angle between $\mathbf{E}$ and $\mathbf{B}$. This is the 'locking' phenomenon alluded to earlier.

In order to compute the energy dissipated into the environment, we look at the equation of motion for ε: $\partial_t \varepsilon = \mathbf{j} \cdot \mathbf{E} = \sigma \mathbf{E} \cdot \mathbf{E} + \tau \mathsf{C}^2 (\mathbf{E} \cdot \mathbf{B})^2 / \chi^5$. For a sample of Weyl semimetal of volume V supporting a steady electric current under an $\mathbf{E}$ and a $\mathbf{B}$, the rate of heat dissipated into the environment is:

$$P = V\Big(\sigma \mathbf{E} \cdot \mathbf{E} + \frac{\tau \mathsf{C}^2 (\mathbf{E} \cdot \mathbf{B})^2}{\chi^5}\Big) \tag{3.19}$$

The additional term in P is precisely $TV\mathcal{R}_5$, where $\mathcal{R}_5$ is the contribution to the local entropy production rate $\mathcal{R}$ by the relaxation term $\mu^5 n^5 / T\tau$.

3.2.2 Magneto-thermoelectric conductivity

Next, we turn our attention to the experiment for thermo-electric conductivity. In this case, we can write:

$$\mathbf{j} = \alpha \boldsymbol{\nabla} T + \mathsf{C}\mu^5 \mathbf{B} \tag{3.20}$$

We will show that similar enhancement to the longitudinal conductivity occurs in the thermoelectric case as well. The mechanism is once again a saturation of μ^5 to a non-zero value in the steady state; however, now, this is driven by $\boldsymbol{\nabla} T \cdot \mathbf{B}$ instead of $\mathbf{E} \cdot \mathbf{B}$.

Let us focus on the chiral current $\mathbf{j}^5$. This is given by: $\mathbf{j}^5 = \alpha^5 \boldsymbol{\nabla} T + (\mathsf{C}\mu + T\widetilde{\mathsf{F}}_{\mu^5})\mathbf{B}$. In the steady state, this must satisfy:

$$\boldsymbol{\nabla} \cdot \mathbf{j}^5 = -\frac{n^5}{\tau} \tag{3.21}$$

Note that there is no anomaly term on the RHS, as $\mathbf{E} = 0$. The LHS reduces to $\boldsymbol{\nabla}\cdot(\alpha^5\boldsymbol{\nabla}T)+\mathsf{C}\mathbf{B}\cdot\boldsymbol{\nabla}\mu + (\mathbf{B}\cdot\boldsymbol{\nabla}T)(T\widetilde{\mathsf{F}}_{\mu^5})_T$.

Now, let us recall some facts we have encountered. First, as we mentioned after Eq.(2.31), the dissipative coefficient $\xi^T = \alpha^5$ is an odd function of μ^5. Based on what we have experienced so far, the leading order contribution to μ^5 is $O(\partial^2)$. So, the first term $\boldsymbol{\nabla}\cdot(\alpha^5\boldsymbol{\nabla}T)$ is $O(\partial^4)$ to leading order. The leading order contribution to $\boldsymbol{\nabla}\mu$ is $O(\partial^2)$. Therefore, the second term is $O(\partial^3)$. For the final term, let us recall that $\widetilde{\mathsf{F}}$ is an odd function of μ^5 as well, and may in general be written as:

$$\widetilde{\mathsf{F}} = \widetilde{\mathsf{F}}^{(1)}(T)\mu^5 + \widetilde{\mathsf{F}}^{(3)}(T)(\mu^5)^3 + \ldots \tag{3.22}$$

Retaining the leading order term, we see that the final term has a leading order contribution, $(\mathbf{B}\cdot\boldsymbol{\nabla}T)(T\widetilde{\mathsf{F}}^{(1)}(T))_T$, that is $O(\partial^2)$. So, to leading order, the equation for $\mathbf{j}^5$ reduces to:

$$\mu^5 = -\frac{\tau\widetilde{\mathsf{C}}}{\chi^5}(\mathbf{B}\cdot\boldsymbol{\nabla}T) \tag{3.23}$$

where $\widetilde{\mathsf{C}}(T) \equiv (T\widetilde{\mathsf{F}}^{(1)}_T + \widetilde{\mathsf{F}}^{(1)})$ is a pure function of T alone. Plugging this back into the expression for electric current $\mathbf{j}$, we obtain:

$$\mathbf{j}_i = \Big(\alpha\boldsymbol{\delta}_{ij} - \frac{\tau\mathsf{C}\widetilde{\mathsf{C}}}{\chi^5}\mathbf{B}_i\mathbf{B}_j\Big)\boldsymbol{\nabla}_jT \tag{3.24}$$

Note that the presence of both C and $\widetilde{\mathsf{C}}$ means that we cannot say anything definite about the sign of the magnetic enhancement to α.

3.3 Alternate considerations

3.3.1 Kinetic theory

It would be satisfying at this stage to identify a microscopic mechanism for the term $\widetilde{\mathsf{F}}$ and the associated $\widetilde{\mathsf{C}}$. To investigate, let us turn now to kinetic theory, from which currents can be directly computed. We are interested in the coefficients of $\mathbf{B}$ in the currents, κ^B, λ^B and ξ^B. So it will be sufficient to assume that $\mathbf{E} = 0$ and the steady state is spatially homogenous. Recall from our earlier discussion in (1.4.1) that kinetic theory is valid in the limit $\mu >> T$. To leading order, therefore, the currents will not involve T, which then precludes the possibility of finding $\widetilde{\mathsf{F}}$. Our strategy therefore will be to isolate next to leading order corrections. Also, we will imagine that it is possible to create left and right-handed magnetic fields, $\mathbf{B}^L$ and $\mathbf{B}^R$ to couple separately to particles belonging to valleys with $\hat{\mathtt{w}}_{\mathcal{S}} \equiv \frac{\mathtt{w}_{\mathcal{S}}}{|\mathtt{w}_{\mathcal{S}}|} = \pm$. We will discuss the implications of this assumption in the next section.

Recall that we mentioned in (1.4.1) that macroscopic currents are obtained by integrating the $\dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f^{\mathcal{S}}$ term in Eq.(1.26) with respect to valley momentum $\mathbf{q}$. Then, by shifting the $\boldsymbol{\nabla}_{\mathbf{x}}$ outside, the current for valley $\mathcal{S}$, $\mathbf{j}_{\mathcal{S}}$ may be computed as ($e = \hbar = c = 1$):

$$\mathbf{j}_{\mathcal{S}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3} (\mathsf{\mathbf{v}} + (\boldsymbol{\Omega}_{\mathbf{q}} \cdot \mathsf{\mathbf{v}})\mathbf{B}^{\mathcal{S}}) f^{\mathcal{S}}_{\mathbf{q}} \tag{3.25}$$

where $f^{\mathcal{S}}_{\mathbf{q}} = (e^{(\epsilon_{\mathbf{q}} - \mu^{\mathcal{S}})/T} + 1)^{-1}$ is the local equilibrium function, and $\mu^{\mathcal{S}} = \mu^{L/R}$ when $\hat{\mathtt{w}}_{\mathcal{S}} = +/-$. Following what we did in (1.4.1), we break up the volume integral into surface integrals over surfaces $\mathcal{S}_\epsilon$ of constant energy ϵ. So, we may write the volume integral $d^3\mathbf{q} = |d\mathbf{a}|dq_\perp$, where $d\mathbf{a}$ is the area element on $\mathcal{S}_\epsilon$ and $dq_\perp$ is an infinitesimal perpendicular to it. This may be transformed into an integral over energy; $dq_\perp|\mathsf{\mathbf{v}}| = d\epsilon$, where $\mathsf{\mathbf{v}}_{\mathbf{q}} = \boldsymbol{\nabla}_{\mathbf{q}}\epsilon_{\mathbf{q}}$. Also, since $\mathsf{\mathbf{v}}$ is $\perp$ to $\mathcal{S}_\epsilon$, $d\mathbf{a} = |d\mathbf{a}|\hat{\mathsf{\mathbf{v}}}$.

We can immediately see that the first term contributes 0. The integral $\int_{\mathbf{q}} \mathsf{\mathbf{v}} f_{\mathbf{q}} = \int d\epsilon\, f(\epsilon) \oint_{\mathcal{S}_\epsilon} d\mathbf{a} = 0$, because $\oint_{\mathcal{S}} d\mathbf{a} = 0$ for any closed surface $\mathcal{S}$ by the divergence theorem. The second term gives a non-zero contribution. In this case, the surface integral is $\oint_{\mathcal{S}_\epsilon} d\mathbf{a} \cdot \mathbf{\Omega} = 2\pi \mathtt{w}_{\mathcal{S}}$ when the Fermi-surface $\mathcal{S}$, defined by μ, lies in the upper band. The energy integral is $\int d\epsilon f(\epsilon)$. We need to think carefully about the limits of this integral, as the bands have a finite spread in energy. The upper and lower limits should be the highest and lowest energy states in the 2-band structure lie at energy ϵ_{max} and $\epsilon_{min} = 0$. The integral can be explicitly computed by representing $f = (e^{(\epsilon-\mu)/T} + 1)^{-1}$ as $-T \frac{\partial}{\partial \epsilon} ln\Big(1 + e^{-(\epsilon-\mu)/T}\Big)$ [5]. It can be seen that if $(\epsilon_{max} - \mu) >> T$ and $(\mu - \epsilon_{min}) >> T$, then, the integral gives $\mu - \epsilon_{min} + O(e^{-(\epsilon_{max}-\epsilon_{min})/T})$. The exponentially suppressed corrections may be neglected if the Fermi level lies far from the extremeties of the band. So, putting everything together, we obtain:

$$\mathbf{j}_{\mathcal{S}} = \frac{1}{(2\pi)^2} \mathtt{w}_{\mathcal{S}} \mu^{\mathcal{S}} \mathbf{B}^{\mathcal{S}} \tag{3.26}$$

where $\mu^{\mathcal{S}}\ \forall\ \mathcal{S}$ are measured from the lowest energy in the bandstructure, ϵ_{min}. One may wonder if it matters if the Fermi surface $\mathcal{S}$ lies in the upper or the lower band. The answer is it does not matter, as the integrand remains the same; $\mathbf{\Omega} \to -\mathbf{\Omega}$ and $\mathsf{\mathbf{v}} \to -\mathsf{\mathbf{v}}$ in the lower band. We want the total currents $\mathbf{j} = \sum_{\mathcal{S}} \mathbf{j}^{\mathcal{S}}$ and $\mathbf{j}^5 = \sum_{\mathcal{S}} \hat{\mathtt{w}}_{\mathcal{S}} \mathbf{j}^{\mathcal{S}}$. Using $\sum_{\mathcal{S}} \frac{1}{(2\pi)^2} |\mathtt{w}_{\mathcal{S}}| = \mathsf{C}$ for the anomaly coefficient, and defining the 'magnetic field' $\mathbf{B} \equiv \frac{\mathbf{B}^L + \mathbf{B}^R}{2}$ and 'Pseudo-magnetic field' $\mathbf{B}^5 \equiv \frac{\mathbf{B}^L - \mathbf{B}^R}{2}$ we get:

$$\mathbf{j} = \mathsf{C}\mu^5 \mathbf{B} + \mathsf{C}\mu \mathbf{B}^5 \tag{3.27a}$$

$$\mathbf{j}^5 = \mathsf{C}\mu \mathbf{B} + \mathsf{C}\mu^5 \mathbf{B}^5 \tag{3.27b}$$

Focusing on just the $\mathbf{B}$ dependent part, we can see that there is no evidence of a T dependent correction. Recall that the form that we are trying to uncover is $\mathbf{j}^5 = (\mathsf{C} + T\widetilde{\mathsf{F}}^{(1)}(T))\mathbf{B}$.

For completeness, we should also compute the energy current $\mathbf{j}^\varepsilon$ and heat current $\mathbf{Q}$. For $\mathbf{j}^\varepsilon$, the contribution of each valley $\mathcal{S}$ to the total energy current is given as:

$$\mathbf{j}^\varepsilon_{\mathcal{S}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\epsilon_{\mathbf{q}}(\mathsf{v} + (\mathbf{\Omega}_{\mathbf{q}}\cdot\mathsf{v})\mathbf{B}^{\mathcal{S}})f^{\mathcal{S}}_{\mathbf{q}} \tag{3.28}$$

Once again, only the second term contributes. As above, the surface integral gives $2\pi\mathtt{w}_{\mathcal{S}}$. The energy integral is now $\int d\epsilon f(\epsilon)\epsilon$, which may be broken up as $f^{\mathcal{S}}(\epsilon) = \Theta(\mu^{\mathcal{S}}-\epsilon)+\Big(f(\epsilon)-\Theta(\mu^{\mathcal{S}}-\epsilon)\Big)$, i.e. a $T=0$ part that is a step function and a $T\neq 0$ part. The $T=0$ part gives $\frac{(\mu^{\mathcal{S}})^2}{2}$, whereas the $T\neq 0$ part gives $\frac{\pi^2T^2}{6}$ upto exponentially suppressed corrections. So, we may write:

$$\mathbf{j}^\varepsilon_{\mathcal{S}} = \frac{\mathtt{w}_{\mathcal{S}}}{(2\pi)^2}\Big(\frac{(\mu^{\mathcal{S}})^2}{2}+\frac{\pi^2T^2}{6}\Big)\mathbf{B}^{\mathcal{S}} \tag{3.29}$$

The total energy current is given as $\mathbf{j}^\varepsilon = \sum_{\mathcal{S}}\mathbf{j}^\varepsilon_{\mathcal{S}}$. This reduces to:

$$\mathbf{j}^\varepsilon = \mathsf{C}\mu\mu^5\mathbf{B} + \mathsf{C}\Big(\frac{(\mu^2+(\mu^5)^2)}{2}+\frac{\pi^2T^2}{3}\Big)\mathbf{B}^5 \tag{3.30}$$

The heat current $\mathbf{Q} = \mathbf{j}^\varepsilon - \mu\mathbf{j} - \mu^5\mathbf{j}^5$ then reduces to:

$$\mathbf{Q} = -\mathsf{C}\mu\mu^5\mathbf{B} + \mathsf{C}\Big(-\frac{(\mu^2+(\mu^5)^2)}{2}+\frac{\pi^2T^2}{3}\Big)\mathbf{B}^5 \tag{3.31}$$

Once again, we see that there seems to be no freedom in κ^B. Recall that the form we are trying to find in this case should look like $\mathbf{Q} = \big(-\mathsf{C}\mu\mu^5 + T^2\widetilde{\mathsf{F}}^{(1)}(T)\mu^5\big)\mathbf{B}$ upto leading order in μ^5.

3.3.2 Pseudo-Electromagnetic fields

Note that we may derive the basic form of the currents derived using kinetic theory using hydrodynamic analysis. In particular, using the algorithmic method of (2.3), we may derive the basic form of the allowed corrections to the $T=0$ expressions for the currents. But first, let us assume that it is possible for L/R species of particles to experience different electric and magnetic fields $\mathbf{E}^{L/R}$ and $\mathbf{B}^{L/R}$. Accordingly, in additional to the usual $\mathbf{E} = \frac{\mathbf{E}^L + \mathbf{E}^R}{2}$ and $\mathbf{B}$, one also has a 'Pseudo-electric field' $\mathbf{E}^5 = \frac{\mathbf{E}^L - \mathbf{E}^R}{2}$ and $\mathbf{B}^5$. These pseudo-electromagnetic fields now appear in the equations of motion:

$$\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = \mathsf{C}(\mathbf{E}^5 \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{B}^5) \tag{3.32a}$$

$$\partial_t n^5 + \boldsymbol{\nabla} \cdot \mathbf{j}^5 = \mathsf{C}(\mathbf{E} \cdot \mathbf{B} + \mathbf{E}^5 \cdot \mathbf{B}^5) \tag{3.32b}$$

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \mathbf{j}^\varepsilon = \mathbf{j} \cdot \mathbf{E} + \mathbf{j}^5 \cdot \mathbf{E}^5 \tag{3.32c}$$

Accordingly, we may write our trial constitutive relations as

$$\mathbf{j} = \boldsymbol{\nu} + \lambda^B \mathbf{B} + \lambda^{Bc} \mathbf{B}^5 \tag{3.33a}$$

$$\mathbf{j}^5 = \boldsymbol{\nu}^5 + \xi^B \mathbf{B} + \xi^{Bc} \mathbf{B}^5 \tag{3.33b}$$

$$\mathbf{Q} = \boldsymbol{\nu}^Q + \kappa^B \mathbf{B} + \kappa^{Bc} \mathbf{B}^5 \tag{3.33c}$$

$$\mathbf{j}^s = \frac{\mathbf{Q}}{T} + D^B \mathbf{B} + D^{Bc} \mathbf{B}^5 \tag{3.33d}$$

Plugging this into the expression for $\partial_t s + \boldsymbol{\nabla} \cdot \mathbf{j}^s$, and ensuring all terms dependent on $\mathbf{B}$ and $\mathbf{B}^5$ cancel out, we get: $\lambda^B = \xi^{Bc} = \mathsf{C}\mu^5$, $\lambda^{Bc} = \xi^B = \mathsf{C}\mu$, $\kappa^B = -\mathsf{C}\mu\mu^5 + \widetilde{\mathsf{G}}^{(1)}(T)$ and $\kappa^{Bc} = -\mathsf{C}\Big(\frac{\mu^2 + (\mu^5)^2)}{2}\Big) + \widetilde{\mathsf{G}}^{(2)}(T)$, where $\widetilde{\mathsf{G}}^{(1,2)}$ are arbitrary functions of T. Note however that all parts of the currents must transform the same under spatial inversion $\mathcal{I}$. Since μ^5 flips under $\mathcal{I}$ and $\mathbf{B}$ does not, $\widetilde{\mathsf{G}}^{(1)}(T)$ has the wrong parity, and hence is forbidden. So, we are left with just $\kappa^B = -\mathsf{C}\mu\mu^5$ and $\kappa^{Bc} = -\mathsf{C}\Big(\frac{\mu^2 + (\mu^5)^2)}{2}\Big) + \widetilde{\mathsf{G}}(T)$, which captures the basic form derived using kinetic theory. This is not surprising, that a microscopic picture that allows for $\mathbf{E}^5$ and $\mathbf{B}^5$ is compatible with a hydrodynamic formulation allowing the same freedom. However, the hydrodynamic description does show beyond doubt that if it were possible to turn on $\mathbf{E}^5$ and $\mathbf{B}^5$, this would preclude the existence of the $\widetilde{\mathsf{F}}$ term (and hence, the phenomenological explanation we have offered for magneto-theormoelectric conductivity involving $\widetilde{\mathsf{C}}$) based on the genral requirement of positivity of $\mathcal{R}$. So the question is, can these fields exist?

The most basic objection to the presence of $\mathbf{E}^5$ and $\mathbf{B}^5$ is that total electric charge n is no longer a conserved quantity. However, there is a recent proposal in the literature about how this problem may be circumvented. This involves the use of so called 'Bardeen counterterms', which have their origin in the field theoretic treatment of anomalies. The basic construction in the case of WSMs may be found in [29] [6], which we outline here. We begin by considering the Hamiltonian of a WSM with just two points of degenracy, located in momentum space at $\pm\mathbf{b}$ and separated in energy by $2b_0$. The hamiltonian would be represented as:

$$H(\mathbf{p}) = \mathtt{w}\,\mathsf{v}_{\mathsf{F}}\boldsymbol{\sigma} \cdot (\mathbf{p} - \mathtt{w}\mathbf{b}) + \mathtt{w}b_0 \tag{3.34}$$

One can see that the $\mathbf{b}$ and b_0 appear in H in a similar manner to how the scalar and vector potential $(A_0, \mathbf{A})$ associated with electromagnetic fields would appear, except that there is now a $\mathtt{w}$ in front. So, it's as if this correspomds to a 'pseudo' set of electromagnetic fields that couples differently to the L/R species. This motivates us to define pseudo-electromagnetic fields given as:

$$\mathbf{E}^5 = \partial_t\mathbf{b} - \boldsymbol{\nabla} b_0 \tag{3.35a}$$

$$\mathbf{B}^5 = -\boldsymbol{\nabla} \times \mathbf{b} \tag{3.35b}$$

Real electric fields $\mathbf{E}$ flip under the action of spatial inversion $\mathcal{I}$ and are unaffected by $\mathcal{T}$, whereas $\mathbf{B}$ flips under $\mathcal{T}$ alone. In contrast, $\mathbf{E}^5$ remains unchanged under both $\mathcal{I}$ and $\mathcal{T}$, whereas $\mathbf{B}^5$ flips under both; The action of $\mathcal{I}$ is: $\boldsymbol{\nabla} \to -\boldsymbol{\nabla}$, $\mathbf{b} \to \mathbf{b}$ and $b_0 \to -b_0$, whereas under $\mathcal{T}$: $\partial_t \to -\partial_t$, $\mathbf{b} \to -\mathbf{b}$ and $b_0 \to b_0$. It follows that : $\mathcal{I} : \mathbf{E}^5 \to \mathbf{E}^5$, $\mathcal{T} : \mathbf{E}^5 \to \mathbf{E}^5$,

$\mathcal{I} : \mathbf{B}^5 \to -\mathbf{B}^5$, and $\mathcal{T} : \mathbf{B}^5 \to -\mathbf{B}^5$.

But when can thes $\mathbf{E}^5$ and $\mathbf{B}^5$ fields be observed? Obviously only when $\mathbf{b} \to \mathbf{b}(t, \mathbf{x})$ etc. That is, b_0 and $\mathbf{b}$ are functions of space and time. This is argued to happen if the material is subjected to elastic deformations. Note however that this $(\mathbf{x}, t)$ dependence should not be interpreted as signalling a 'non-equilibrium' situation in the hydrodynamic sense, as this inhomogeneity affects the system at the level of the structure of the Bloch space. Recall that in our treatment, the structure of the energy bands in $\mathbf{p}$ space is a constant in space and time; it is the *occupancy* of these bands that changes as a function of $(\mathbf{x}, t)$.

To get around the non-conservation of n in the presence of $\mathbf{E}^5$ and $\mathbf{B}^5$, the use of the Bardeen counterterm construction is made [19] [20]. It is argued that the charge density and the current n and $\mathbf{j}$ actually are composed of an observable part and an unobservable part. The observable, or 'transport' [14] currents and charges are actually measurable in an experimental setup, whereas the extra Bardeen counterterms are unobservable. In this context, the full currents are called the 'Covariant' currents, whereas the observable part is called the 'Consistent' form of the currents [36]. The relation of the full currents with the observable parts may be written as:

$$n = n_{trans} + \mathsf{C}\mathbf{b} \cdot \mathbf{B} \tag{3.36a}$$

$$\mathbf{j} = \mathbf{j}_{trans} - \mathsf{C}b_0\mathbf{B} - \mathsf{C}\mathbf{b} \times \mathbf{E} \tag{3.36b}$$

The full n and $\mathbf{j}$ satisfy an equation that dislays non-conservation of charge. However, using the definitions of $\mathbf{E}^5$ and $\mathbf{B}^5$ introduced above, it may be seen that the transport parts *do not* violate charge conservation;

$$\partial_t n_{trans} + \boldsymbol{\nabla} \cdot \mathbf{j}_{trans} = 0 \tag{3.37}$$

So it seems that pseudo-electromagnetic fields can exist without catastrophic consequences. However, we would like to point out a few concerns we have regarding the validity of this construction below:

- *Frame choice in non-relativistic systems*: In the construction above, the n differs from its $\mathbf{B} = 0$ value by an $O(\partial)$ quantity $(\mathbf{b} \cdot \mathbf{B})$. This seems to be a matter of choice of a 'hydrodynamic frame', which generically exists in the formulation of any hydrodynamic theory (see [32] [10] for a discussion of these ideas within the context of relativistic fluid dynamics). Let us consider a system with just two conserved charges, n and ε. Complementarily, we have two intensive variables T and μ. T and μ are well defined in equilibrium, however, out of equilibrium, these are no longer well defined. This is in the sense that whereas ε and n correspond to phsyically measurable quantities, T and μ are merely parameters. Now, given a parametrization T, μ, we are not guaranteed that ε and n will be represented by the same expressions ε^{eq} and n^{eq} as they were in equilibrium. In general,

$$\begin{aligned} \varepsilon(T,\mu) &= \varepsilon^{eq}(T,\mu) + \Delta_\varepsilon(\partial T, \partial\mu) \\ n(T,\mu) &= n^{eq}(T,\mu) + \Delta_n(\partial T, \partial\mu) \end{aligned} \tag{3.38}$$

where Δ_ε etc. are $O(\partial)$ corrections (including both ∂_t and $\boldsymbol{\nabla}$ in general) that must be included out of equilibrium. This means that we are free to redefine $T \to T' = T + \delta T(\partial T, \partial\mu)$, and $\mu \to \mu' = \mu + \delta\mu$ with $O(\partial)$ deformations δT and $\delta\mu$ as long as the measured densities are the same in both parametrizations; $n(T,\mu) = n'(T',\mu')$ etc. So, the first order corrections in the primed and unprimed parametrizations are related:

$$\begin{aligned} \Delta_\varepsilon &= \Delta'_\varepsilon + \frac{\partial\varepsilon^{eq}}{dT}\delta T + \frac{\partial\varepsilon^{eq}}{\partial\mu}\delta\mu \\ \Delta_n &= \Delta'_n + \frac{\partial n^{eq}}{dT}\delta T + \frac{\partial n^{eq}}{\partial\mu}\delta\mu \end{aligned} \tag{3.39}$$

Since there are 2 conserved densities and 2 degrees of freedom available via frame redefinition, we can always choose a parametrization in which $\Delta_\varepsilon = \Delta_n = 0$, or, n and ε can be represented by the same functions of hydrodynamic variables out of equilibrium as they were in equilibrium. Note that the form of the constitutive relations we have been using (only using $\boldsymbol{\nabla}$, and not ∂_t) follows as a corollary of this choice. This is the choice we have been making so far, so it seems that in our framework, there is no room for the $\mathbf{b} \cdot \mathbf{B}$ type term. In other words, we can always choose a frame in which this is not present.

- *Net flux of currents*: In general, a current may be called 'unobservable' if it has zero net flux across any surface [14]. This is clearly the case in a magnetization current, where $\mathbf{j} \sim \boldsymbol{\nabla} \times \mathbf{m}$, so tha $\int_{\mathtt{A}} d\mathtt{A} \cdot \mathbf{j} = 0$ for any spatial 2-surface $\mathtt{A}$. Here however, this cannot be the case, as the current is $\mathsf{C}b_0\mathbf{B}$. While $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, but $b_0 = b_0(\mathbf{x}, t)$ is deemed to be possible, in which case the current $\mathbf{j} \neq \boldsymbol{\nabla} \times \#$. So this is not a current that will have a zero net flux across any surface, and hence, is an observable current.

- *Equilibrium CME*: This problem is no longer there if b_0 is a constant in space and time. In this case, we may write $\mathbf{j} = \mathsf{C}b_0\mathbf{B} \sim \boldsymbol{\nabla} \times \mathbf{A}$. This would 'unobservable' in the sense defined above, but suffers from several conceptual issues. First, such a current would imply a 'magnetization' (presumably a physically well defined quantity) $\sim$ the magnetic vector potential $\mathbf{A}$, a quantity that is not gauge invariant [37]. It has also been pointed out that a band whose Fermi surface is an equal energy surface cannot lead to a a non-zero current. Non-zero currents are produced by *deformations* of the Fermi-surface from its equilibrium configuration [65]. Indeed, this term is an artifact of where we measure $\mu^{L/R}$ from. Since all valleys are part of the same bandstructure, it is appropriate that they should all be measured from the lowest energy state in the band ϵ_{min}. Lastly, if an equilibrium CME were allowed, it would be possible to extract energy from the system at $T = 0$, which is impossible physically,

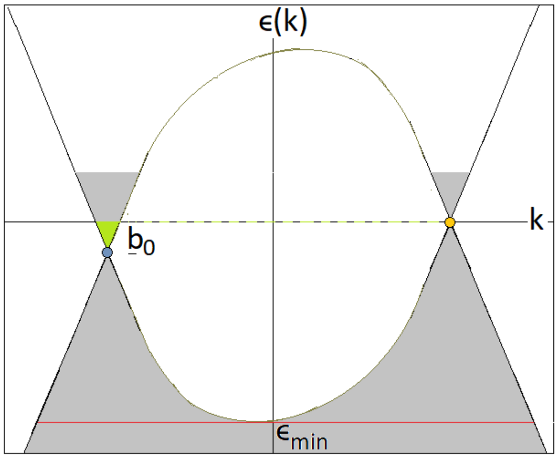


Figure 3.1: Proposed mechanism for b_0. Note that the valleys are joined below, and hence, μ^L and μ^R should be measured from ϵ_{min}, the energetic bottom of the band.

since at $T = 0$, the system is already in its ground state [5], although, this last critique may not apply if we imagine only the transport part of the current to participate in Joule heating.

- *Anomalous Hall effect*: $\mathbf{b}$ has the same properties as $\mathbf{m}$ under the action of $\mathcal{I}$ and $\mathcal{T}$. As such, $\mathbf{b}$ may be interpred as a magnetization. It follows that the term $\mathbf{b} \times \mathbf{E}$ is an anomalous hall effect, which also appears in our hydrodynamic theory, but in a sector independent of the anomaly. The main concern here is that $\mathbf{b}$ is *crystal information*; it defines the momentum displacement of the Weyl nodes in Brillouin zone. As such, information about $\mathbf{b}$ should not survive in a hydrodynamic theory formulated for a dirty WSM.

- *Gravitational anomaly and 2-fluid model*: If it is indeed possible to physically realize $\mathbf{E}^5$, $\mathbf{B}^5$, then, our phenomenological model for magneto-thermoelectric conductivity cannot work. There is another proposal recently of how this phenomenon arises in a WSM [18]. In this proposal, the thermoelectric conductivity is related to the so called 'Mixed Axial Gravitational anomaly', an anomalous nonconservation of chiral charge in a relativistic fluid on a curved spacetime. At the level of transport phenomena, the model seems to suggest that in Weyl semimetals, it is not just n^L and n^R that are separately conserved, but also ε^L and ε^R. The explanation of the magnetoconductivity experiments then relies on the saturation of both n^5 and ε^5 in the steady state. The implication is that the two species L and R are allowed to be at different temperatures T^L and T^R). Note that even at the field theoretic level, the action has two separate $U(1)$ symmetries that lead to the separate conservation of j_μ and j^5_μ, but there seems to be no basis on which to motivate that the stress energy tensor $T^{\mu\nu}$ may be similarly broken up into separately conserved L and R parts.

CHAPTER 4

HYDRODYNAMICS OF CLEAN WEYL SEMIMETALS

So far, we have been discussing the case of a dirty Weyl semimetal. In this final section, we will finally start imagining what happens in the limit that the Weyl semimetal sample can be approximated as being 'clean'. This will mean that the momentum non-conserving $e - i$ collisions may be neglected, and consequently, momentum density $\boldsymbol{\pi}$ is a conserved quantity. So, we have to include the flow velocity $\mathbf{v}$ among the hydrodynamic variables. However, the fluid that emerges is qualitatively different from conventional fluids. Conventional fluids have no special frame of reference. Hence, transforming from one inertial frame to another, the equations of hydrodynamics remain invariant. In other words, hydrodynamics possesses *boost invariance*. In contrast, the electronic fluid does possess a special frame, that is, the static frame of the background lattice of ions. There is no need for the equations of motion for an electron fluid to appear the same across an equivalence class of frames related by boost transform.

The general philosophy of hydrodynamics we have seen so far is, that whatever forms are not forbidden by considerations of symmetry or the 2^{nd} law of thermodynamics are in general allowed, and may be present in a particular system displaying the given symmetries. Accordingly, boost invariance is constraint, just like isotropy or the Onsager relations on the structure of hydrodynamics. Removing this constraint leads to a larger set of possible structures. This is what we explore first, in the context of a normal metal.

4.1 Fluids without boost invariance

In a normal metal in its clean limit, the three conserved quantities are ε, n and momentum density $\boldsymbol{\pi}$. We will assume that the system is isotropic. This immediately means that $\boldsymbol{\pi} = \rho\mathbf{v}$, where $\rho = \rho(T, \mu, |\mathbf{v}|)$ is the 'mass' density (but does not possess the usual connotations of the term in the absence of boost invariance). Note that in the clean limit, the crystal is not grained; instead, it should have a consistent crystal structure on the macroscopic scale. As such, the assumption of isotropy is only valid in cubic crystals. The three conserved quantities are related via the thermodynamic relation:

$$dp = Tds + nd\mu + \boldsymbol{\pi} \cdot d\mathbf{v} \tag{4.1}$$

The energy density ε is now $\varepsilon = sT + n\mu + \boldsymbol{\pi} \cdot \mathbf{v} - p$, so, the infinitesimal form of the first law is given as:

$$ds = \frac{d\varepsilon}{T} - \frac{\mu dn}{T} - \frac{\mathbf{v} \cdot d\boldsymbol{\pi}}{T} \tag{4.2}$$

$\partial_t s$ may be computed by using the equations of motion, which now are:

$$\partial_t \varepsilon + \boldsymbol{\nabla} \cdot \mathbf{j}^{\varepsilon} = \mathbf{j} \cdot \mathbf{E} \tag{4.3a}$$

$$\partial_t n + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{4.3b}$$

$$\partial_t \boldsymbol{\pi}_j + \boldsymbol{\nabla}_i \mathbf{j}^{\boldsymbol{\pi}}_{ij} = n\mathbf{E}_j + (\mathbf{j} \times \mathbf{B})_j \tag{4.3c}$$

In hydrodynamics with $\mathbf{v}$, the possibility exists of convection, i.e. non-dissipative $O(\partial^0)$ currents. The ideal parts of the currents are completely fixed by thermodynamics. Isolating these, one may write:

$$\mathbf{j} = \boldsymbol{\nu} + n\mathbf{v} \tag{4.4a}$$

$$\mathbf{j}^{\varepsilon} = \boldsymbol{\nu}^{\varepsilon} + (\varepsilon + p)\mathbf{v} \tag{4.4b}$$

$$\mathbf{j}^{\boldsymbol{\pi}}_{ij} = \boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij} + (\rho\mathbf{v}_i\mathbf{v}_j + p\boldsymbol{\delta}_{ij}) \tag{4.4c}$$

When collected in the expression for $\partial_t s$, the ideal parts contribute exactly $-\boldsymbol{\nabla}\cdot(s\mathbf{v})$, i.e. exactly cancel with the ideal part of the entropy current. This is just evidence of the fact that the form of the 1^{st} law in a momentum-conserving hydrodynamic system completely fixes hydrodynamics at the ideal order. This connection can be put on a rigorous footing with the use of the Poisson Bracket Formalism (e.g. see [55] [58]), that treats the conserved densities as classical limits of quantum operators, and makes use of this correspondence to establish Poisson bracket relations between the densities. Using this approach, the ideal hydrodynamics of any momentum conserving system can be systematically worked out, given a description of how its thermodynamic degrees of freedom interact via the 1^{st} law.

The $\boldsymbol{\nu}$'s contain the $O(\partial^1)$ currents. Defining heat current as $\mathbf{Q}_i = \boldsymbol{\nu}^{\varepsilon}_i - \mu\boldsymbol{\nu}_i - \mathbf{v}_j\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij}$, the usual manipulations give:

$$\partial_t s + \boldsymbol{\nabla}\cdot(s\mathbf{v}) = -\frac{1}{T}\Big(\boldsymbol{\nabla}\cdot\mathbf{Q} + \mathbf{j}\cdot\boldsymbol{\nabla}\bar{\mu} + \mathbf{j}^{\boldsymbol{\pi}}_{ij}\boldsymbol{\nabla}_i\mathbf{v}_j\Big) \tag{4.5}$$

where we have defined $\boldsymbol{\nabla}\bar{\mu} \equiv \boldsymbol{\nabla}\mu - \mathbf{E} - \mathbf{v}\times\mathbf{B}$. We can see that choosing the dissipative part of the entropy current, $\boldsymbol{\nu}^s$ to be $\frac{\mathbf{Q}}{T}$ reduces $\mathcal{R}$ to the manifestly positive definite form:

$$\mathcal{R} = -\left[\frac{\mathbf{Q}\cdot\boldsymbol{\nabla}T}{T^2} + \frac{\boldsymbol{\nu}\cdot\boldsymbol{\nabla}\bar{\mu}}{T} + \frac{(\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij})(\boldsymbol{\nabla}_i\mathbf{v}_j)}{T}\right] \tag{4.6}$$

We may now examine the dissipative structure of the currents. The most general constitutive relations one can write down to $O(\partial)$ is:

$$\mathbf{Q}_i = \kappa^T_{ia} \boldsymbol{\nabla}_a T + \kappa^{\mu}_{ia} \boldsymbol{\nabla}_a \bar{\mu} + \kappa^{\mathbf{v}}_{iab} \boldsymbol{\nabla}_a \mathbf{v}_b \tag{4.7a}$$

$$\boldsymbol{\nu}_i = \lambda^T_{ia} \boldsymbol{\nabla}_a T + \lambda^{\mu}_{ia} \boldsymbol{\nabla}_a \bar{\mu} + \lambda^{\mathbf{v}}_{iab} \boldsymbol{\nabla}_a \mathbf{v}_b \tag{4.7b}$$

$$\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij} = \gamma^T_{ija} \boldsymbol{\nabla}_a T + \gamma^{\mu}_{ija} \boldsymbol{\nabla}_a \bar{\mu} + \gamma^{\mathbf{v}}_{ijab} \boldsymbol{\nabla}_a \mathbf{v}_b \tag{4.7c}$$

The constraints on the kinetic coefficients are: *(i)* Rotational symmetry, which means that they can only be composed of $\mathbf{v}_i$, $\boldsymbol{\delta}_{ij}$, and $\boldsymbol{\epsilon}_{ijk}$; *(ii)* Onsager reciprocity, of which, the non-trivial ones are: $\kappa^{\mu}_{ij}(\mathbf{v}) = T\lambda^T_{ji}(-\mathbf{v})$, $\kappa^{\mathbf{v}}_{iab}(\mathbf{v}) = -T\gamma^T_{abi}(-\mathbf{v})$, $\lambda^{\mathbf{v}}_{iab}(\mathbf{v}) = -\gamma^{\mu}_{abi}(-\mathbf{v})$, and $\gamma^{\mathbf{v}}_{ijab}(\mathbf{v}) = \gamma^{\mathbf{v}}_{abij}(-\mathbf{v})$; *(iii)* the requirement of conservation of total angular momentum of the system, which is a consequence of the assumption of isotropy. This condition translates to the requirement that $\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij} = \boldsymbol{\nu}^{\boldsymbol{\pi}}_{ji}$, i.e. the stress tensor is symmetric. This follows from representing the total (orbital) angular momentum of the system as $\mathbf{L}_{ij}(t) = \int d\mathbf{x} \left(\mathbf{x}_i \boldsymbol{\pi}_j(\mathbf{x},t) - \mathbf{x}_j \boldsymbol{\pi}_i(\mathbf{x},t) \right)$ and imposing that $\frac{d\mathbf{L}}{dt} = 0$.

It may be seen that 17 scalar dissipative coefficients are compatible with the constraints listed above. There are: *(i)*3 independent rank-2 kinetic coefficients (κ^T, λ^{μ} and λ^T) that correspond to 2 scalar coefficients each:

$$\lambda^{\mu}_{ia} = \lambda^{\mu}_1 \boldsymbol{\delta}_{ia} + \lambda^{\mu}_2 \mathbf{v}_i \mathbf{v}_j \tag{4.8}$$

(ii) 2 independent rank-3 kinetic coefficients (γ^T and γ^{μ}) that correspond to 3 scalar coefficients each:

$$\gamma^T_{iab} = \gamma^T_1 \mathbf{v}_i \boldsymbol{\delta}_{ab} + \gamma^T_2 \mathbf{v}_i \mathbf{v}_a \mathbf{v}_b + \gamma^T_3 (\boldsymbol{\delta}_{ia} \mathbf{v}_b \boldsymbol{\delta}_{ib} \mathbf{v}_a) \tag{4.9}$$

(iii) 1 rank-4 object ($\gamma^{\mathbf{v}}$) that corresponds to 5 scalar coefficients:

$$\begin{aligned}\gamma^{\mathbf{v}}_{ijab} =& \gamma^{\mathbf{v}}_1 \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ab} + \gamma^{\mathbf{v}}_2(\boldsymbol{\delta}_{ia}\boldsymbol{\delta}jb + \boldsymbol{\delta}_{ib}\boldsymbol{\delta}_{ja}) + \gamma^{\mathbf{v}}_3 \mathbf{v}_i\mathbf{v}_j\mathbf{v}_a\mathbf{v}_b + \gamma^{\mathbf{v}}_4(\mathbf{v}_i\mathbf{v}_j\boldsymbol{\delta}_{ab} + \mathbf{v}_a\mathbf{v}_b\boldsymbol{\delta}_{ij}) \\ &+ \gamma^{\mathbf{v}}_5(\mathbf{v}_i\mathbf{v}_a\boldsymbol{\delta}_{jb} + \mathbf{v}_j\mathbf{v}_b\boldsymbol{\delta}_{ia} + \mathbf{v}_i\mathbf{v}_b\boldsymbol{\delta}_{ja} + \mathbf{v}_j\mathbf{v}_a\boldsymbol{\delta}_{ib})\end{aligned} \tag{4.10}$$

Plugging all of this in, after some algebra, the rate of entropy production $\mathcal{R}$ may be reduced to:

$$\begin{aligned}\mathcal{R} =& -\Big[\frac{\kappa^T_1}{T^2}(\boldsymbol{\nabla}T)^2 + \frac{\kappa^T_2}{T^2}(\mathbf{v}\cdot\boldsymbol{\nabla}T)^2 + \frac{\lambda^\mu_1}{T}(\boldsymbol{\nabla}\bar{\mu})^2 + \frac{\lambda^\mu_2}{T}(\mathbf{v}\cdot\boldsymbol{\nabla}\bar{\mu})^2 \\ &+ \frac{\gamma^{\mathbf{v}}_1}{T}(\boldsymbol{\nabla}\cdot\mathbf{v})^2 + \frac{\gamma^{\mathbf{v}}_2}{2T}(\boldsymbol{\nabla}_i\mathbf{v}_j + \boldsymbol{\nabla}_j\mathbf{v}_i)^2 + \frac{\gamma^{\mathbf{v}}_3}{T}\left(\mathbf{v}_i(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}_i\right)^2 \\ &+ \frac{2\gamma^{\mathbf{v}}_4}{T}(\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{v}_i(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}_i) + \frac{\gamma^{\mathbf{v}}_5}{T}\big(\mathbf{v}_j(\boldsymbol{\nabla}_i\mathbf{v}_j + \boldsymbol{\nabla}_j\mathbf{v}_i)\big)^2 \\ &+ \frac{2\lambda^T_1}{T}(\boldsymbol{\nabla}\bar{\mu}\cdot\boldsymbol{\nabla}T) + \frac{2\lambda^T_2}{T}(\mathbf{v}\cdot\boldsymbol{\nabla}\bar{\mu})(\mathbf{v}\cdot\boldsymbol{\nabla}T) \\ &+ \frac{2\gamma^\mu_1}{T}(\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{v}\cdot\boldsymbol{\nabla}\bar{\mu}) + \frac{2\gamma^\mu_2}{T}(\mathbf{v}_i(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}_i)(\mathbf{v}\cdot\boldsymbol{\nabla}\bar{\mu}) + \frac{2\gamma^\mu_3}{T}(\boldsymbol{\nabla}_i\bar{\mu})(\mathbf{v}_j(\boldsymbol{\nabla}_i\mathbf{v}_j + \boldsymbol{\nabla}_j\mathbf{v}_i)) \\ &+ \frac{2\gamma^T_1}{T}(\boldsymbol{\nabla}\cdot\mathbf{v})(\mathbf{v}\cdot\boldsymbol{\nabla}T) + \frac{2\gamma^T_2}{T}(\mathbf{v}_i(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}_i)(\mathbf{v}\cdot\boldsymbol{\nabla}T) + \frac{2\gamma^T_3}{T}(\boldsymbol{\nabla}_iT)(\mathbf{v}_j(\boldsymbol{\nabla}_i\mathbf{v}_j + \boldsymbol{\nabla}_j\mathbf{v}_i))\Big]\end{aligned} \tag{4.11}$$

where the scalar kinetic coefficients are functions of μ, T, and $|\mathbf{v}|$ subject to inequalities arising from $\mathcal{R} \geq 0$.

The set of coefficients is enlarged from the usual Galilean hydrodynamics. When, the fluid enjoys Galilean boost symmetry, $\mathcal{R}$ should be independent of the frame of reference the measurement takes place in. So, all the terms with free $\mathbf{v}$'s drop out. Only 5 survive; κ^T_1, λ^μ_1, λ^T_1, and 2 coefficients of viscosity, $\gamma^{\mathbf{v}}_1$ and $\gamma^{\mathbf{v}}_2$. At this stage, this represents a Galilean hydrodynamics for a 2-species fluid [35].

To get conventional fluid dynamics for a one species fluid, we need to take into account effect of Galilean invariance, that is, that the number current is fixed to be proportional to the momentum density. The obvious effect is that λ^T and λ^μ drop out of the set of allowed coefficients, and hydrodynamics is characterized by just 3 kinetic coefficients: thermal conductivity and 2 coefficients of viscosity. The second effect is more subtle [56]. Writing the thermodynamic relations for $d\varepsilon$ and dp as:

$$d\left(\varepsilon - \frac{\rho\mathbf{v}^2}{2}\right) = Tds + \mu dn + \frac{\mathbf{v}^2}{2}d\rho \tag{4.12a}$$

$$dp = sdT + nd\mu + \rho d\frac{\mathbf{v}^2}{2} \tag{4.12b}$$

The condition for Galilean invariance implies that $\rho = mn$, where m is the mass of a single microscopic particle. This is mass density in the usual sense. We can see that in this case, $p = p(T, \mu + \frac{m\mathbf{v}^2}{2})$, where $\mu + \frac{m\mathbf{v}^2}{2} \equiv \mu_0$ can be interpreted as chemical potential in the rest frame of the fluid element. Complementarily, $\varepsilon_0 \equiv \varepsilon - \frac{\rho\mathbf{v}^2}{2}$ may be interpreted as the energy density in the rest frame. In other words, this gives us a definite law of transformation law for thermodynamic quantities under Galilean boost transform.

4.2 WSM with momentum conservation

In a Weyl semimetal, in addition to the three conserved quantities considered so far, we have the chiral charge n^5 as well that is conserved:

$$\partial_t n^5 + \boldsymbol{\nabla}\cdot\mathbf{j}^5 = \mathsf{C}\mathbf{E}\cdot\mathbf{B} \tag{4.13}$$

The currents will now have a non-dissipative response that is forced by the anomaly. To uncover this response, we will use the algorithmic method of (2.3). Similar to Eq.(4.14), we will assume the most general form that these can take. Once again, the anomaly nessecitates a response controlled by the $\mathbf{B}$ field. Accordingly, the coefficients κ^B, λ^B and ξ^B are now rank-2 objects (due to the presence of $\mathbf{v}$, an $O(1)$ vector, among the list of hydrodynamic variables). Similarly, γ^B is a rank-3 object. The most general constitutive relations one may write are:

$$\boldsymbol{\nu}_i = \boldsymbol{\nu}_i^{(\mathcal{R})} + \lambda^B_{ia}\mathbf{B}_a \tag{4.14a}$$

$$\boldsymbol{\nu}^5_i = \boldsymbol{\nu}^{5(\mathcal{R})}_i + \xi^B_{ia}\mathbf{B}_a \tag{4.14b}$$

$$\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij} = \boldsymbol{\nu}^{\boldsymbol{\pi}(\mathcal{R})}_{ij} + \gamma^B_{ija}\mathbf{B}_a \tag{4.14c}$$

$$\mathbf{Q} = \boldsymbol{\nu}^{Q(\mathcal{R})}_i + \kappa^B_{ia}\mathbf{B}_a \tag{4.14d}$$

$$\boldsymbol{\nu}^s = \frac{\mathbf{Q}}{T} + D^B\mathbf{B} \tag{4.14e}$$

where the $(\mathcal{R})$ superscript emphasizes that we have separated out the dissipative parts. All the coefficients introduced are now functions of T, μ, μ^5 and $|\mathbf{v}|$. Note that we haven't elevated D^B to a rank-2 object as well, unlike the other coefficients. This is because if we did, then $\boldsymbol{\nabla}\cdot\mathbf{j}^s$ would give us $\boldsymbol{\nabla}_i(D^B_{ia}\mathbf{B}_a) = \mathbf{B}_a\boldsymbol{\nabla}_i(D^B_{ia}) + D^B_{ia}(\boldsymbol{\nabla}_i\mathbf{B}_a)$; i.e. because of the presence of off-diagonal components of D^B, we can no longer use $\boldsymbol{\nabla}\cdot\mathbf{B}$ to exclude terms that involve $(\boldsymbol{\nabla}_i\mathbf{B}_a)$, which are $O(\partial^2)$ when $\mathbf{B}\sim O(\partial)$. Plugging everything into the expression for $\mathcal{R}$ and demanding that all terms dependent on $\mathbf{B}$ cancel, we obtain:

$$\mathbf{B}_a\left[\frac{\kappa^B_{ia}}{T^2}(\boldsymbol{\nabla}_iT) + \frac{\lambda^B_{ia}}{T}(\boldsymbol{\nabla}_i\mu) + \frac{\xi^B_{ia}}{T}(\boldsymbol{\nabla}_i\mu^5) + \frac{\gamma^B_{ija}}{T}(\boldsymbol{\nabla}_i\mathbf{v}_j)\right] + (\mathsf{C}\mu^5\boldsymbol{\delta}_{ab} - \lambda^B_{ab})\frac{\mathbf{E}_a\mathbf{B}_b}{T} - \mathbf{B}\cdot\boldsymbol{\nabla}D^B = 0 \tag{4.15}$$

we can immediately see that the $\mathbf{E}_a\mathbf{B}_b$ term forces $\lambda^B_{ab} = \mathsf{C}\mu^5\boldsymbol{\delta}_{ab}$, just like in the dirty case. Expanding $\boldsymbol{\nabla}_iD^B = D^B_T\boldsymbol{\nabla}_iT + D^B_\mu\boldsymbol{\nabla}_i\mu + D^B_{\mu^5}\boldsymbol{\nabla}_i\mu^5 + D^B_{|\mathbf{v}|}\frac{\mathbf{v}_a}{|\mathbf{v}|}\boldsymbol{\nabla}_i\mathbf{v}_a$, we can see that this implies $D^B = \frac{\mathsf{C}\mu\mu^5}{T} + \widetilde{\mathsf{F}}(T,\mu^5,|\mathbf{v}|)$. From the $\mathbf{B}_a\boldsymbol{\nabla}_iT$ and $\mathbf{B}_a\boldsymbol{\nabla}_i\mu^5$ terms, we get $\kappa^B_{ia} = (-\mathsf{C}\mu\mu^5 + T^2\widetilde{\mathsf{F}}_T)\boldsymbol{\delta}_{ia}$ and $\xi^B_{ia} = (\mathsf{C}\mu + T\widetilde{\mathsf{F}}_{\mu^5})\boldsymbol{\delta}_{ia}$. This is exactly what we had in the dirty case.

The new coefficient is γ^B. This, as we can see, only recieves contributions from the unconstrained $\widetilde{\mathsf{F}}$ part, as the constrained part of D^B does not depend upon $|\mathbf{v}|$. $\gamma^B_{ija} = T\boldsymbol{\delta}_{ai}\frac{\mathbf{v}_j}{|\mathbf{v}|}\widetilde{\mathsf{F}}_{|\mathbf{v}|}$. However, this gives a contribution to the stress tensor $\mathbf{j}^{\boldsymbol{\pi}}_{ij} \sim T\frac{\mathbf{B}_i\mathbf{v}_j}{|\mathbf{v}|}\widetilde{\mathsf{F}}_{|\mathbf{v}|}$. As we have mentioned earlier, the stress tensor must be symmetric so that angular momentum is conserved. So, this term is forbidden. We seem to have isolated all the anomaly related responses within the dirty sector alone.

This shows us that the boost free hydrodynamic setup we are considering is qualitatively different from conventional fluids, and is pretty much the dirty hydrodynamics with an extra degree of freedom that doesn't interact with the anomaly at all. Another example that shows this fact is the absence of anomaly enforced Chiral Vortical Effect in boost free hydrodynamics. It was shown in the case of a Relativistic fluid by Son and Surowka [59], that the presence of an anomaly forces not just a chiral magnetic effect, but also a chiral *vortical* effect; a non-dissipative current along the flow vorticity. In our context that would translate to $\boldsymbol{\nu} = \boldsymbol{\nu}^{(\mathcal{R})} + \lambda^B\mu^5\mathbf{B} + \lambda^v\boldsymbol{\nabla}\times\mathbf{v}$, where like λ^B, λ^v is also fixed to be some definite function of the hydrodynamic variables. However, we can see that this is forbidden in our case, as Onsager reciprocity would then imply a contribution to the stress tensor that goes as $\boldsymbol{\nu}^{\boldsymbol{\pi}}_{ij} \sim \boldsymbol{\epsilon}_{ijk}(\boldsymbol{\nabla}_k\bar{\mu})$ which is again forbidden by the constraint that the stress tensor must be symmetric.

CHAPTER 5

CONCLUSION

In this thesis, we have formulated a hydrodynamic theory for Weyl semimetals. Weyl semimetals are topological conductors that possess chiral species of charge L/R that interchange under the action of spatial inversion $\mathcal{I}$. Hydrodynamical theories are unversal theories that describe macroscopic non-equilibrium phenomena. They are *universal* in the sense that their construction does not rely on the precise details of the microscopic constituents of the system and their dynamics, but rather, on macroscopic principles such as symmetries the system obeys, and the 2^{nd} law of thermodynamics. As such, identifying all the degrees of freedom allowed in the hydrodynamic constitutive relations is equivalent to identifying all possible transport phenomena in a given system. We have performed this exercise in detail for a *dirty* WSM, where only n, n^5 and ε are conserved (approximately, in the case of chiral charge density n^5), but momentum density $\boldsymbol{\pi}$ is not, owing to the presence of electron-impurity collisions.

We have also proposed a phenomenological model, based on the hydrodynamic theory we have formulated for a dirty WSM, for the phenomena of Positive magnetoconductivity and magneto-thermoelectric conductivity. Our explanation of the thermoelectric effect in WSMs is reliant on an allowed (but unconstrained) contribution to the chiral current density $\mathbf{j}^5$ that emerges from our hydrodynamic theory. We discuss in detail the conditions of validity of this contribution.

Finally, we have constructed of a hydrodynamic theory for *clean* metals, and motivated how the 'fluid' that emerges in this case is qualitatively different from conventional fluids that enjoy symmetry with respect to Boost transformation. Using this construction, we have shown that in a clean WSM, all interesting (anomaly related) responses are contained within the dirty sector of the theory itself; addition of the extra $\boldsymbol{\pi}$ degree of freedom to the list of conserved quantities produces no extra anomaly related transport effects.

One interesting future investigation would be to carefully analyze the boundaries of the system, given that all experimental samples are finite (and hence, posess boundaries). While some things are understood about the surfaces of Weyl semimetals (the presence of Fermi Arc surface states), a careful analysis of the surface at the level of boundary conditions imposed on solutions of the hydrodynamic equations is required to be able to model the experimental behavior of realistic samples of Weyl semimetals.

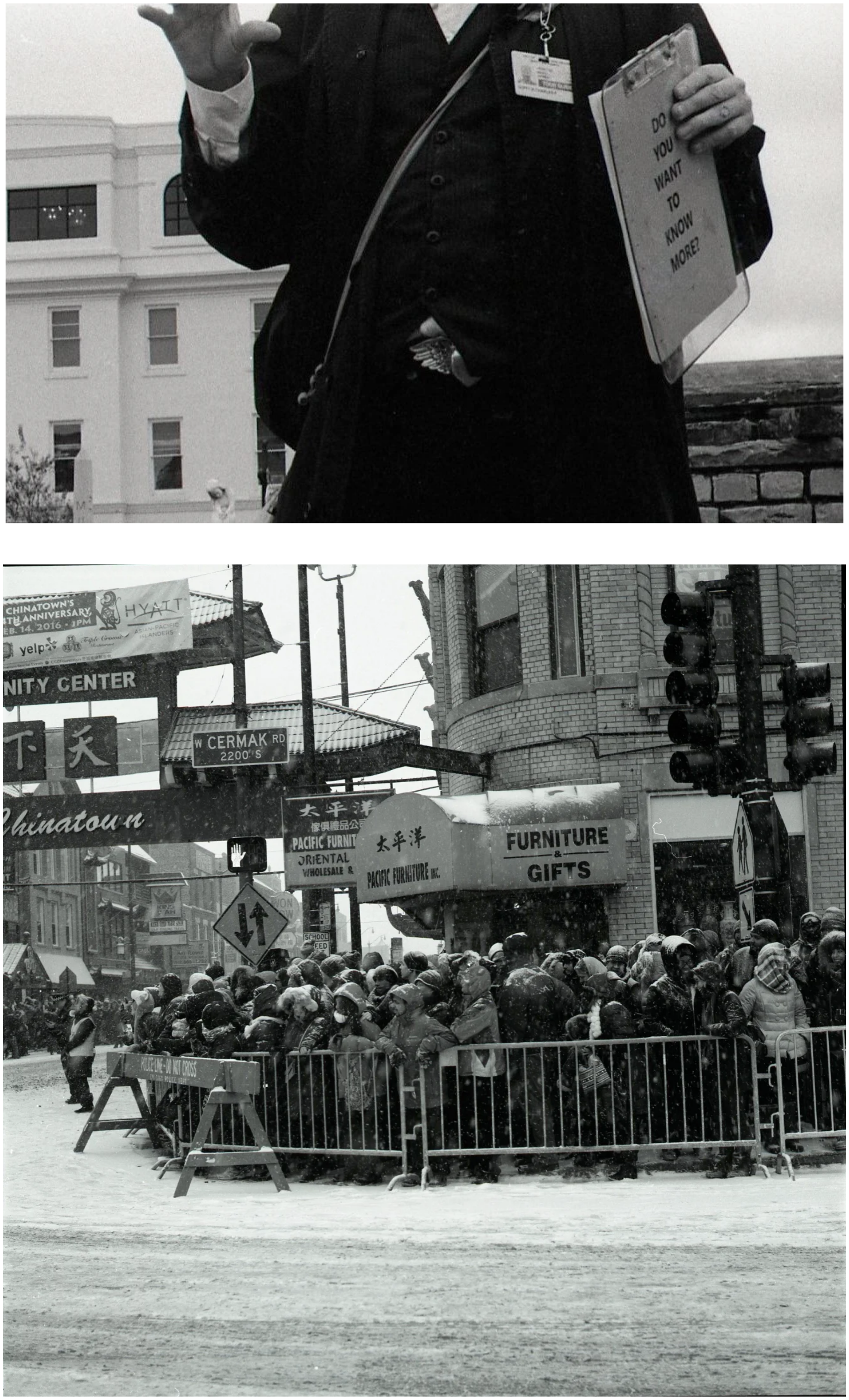


Figure 5.1: The full picture (Images courtesy : Mrinalini Pandey [48])

Figure 5.2: City by night

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