THE UNIVERSITY OF CHICAGO

SUPERSYMMETRY AND IRRELEVANT DEFORMATIONS

A DISSERTATION SUBMITTED TO

THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

BY

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CHICAGO, ILLINOIS

AUGUST 2021

To my parents

Get confused. Solve problems. Repeat. The universe is waiting for you.

— Mark Eichenlaub

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ACKNOWLEDGMENTS

First, I would like to thank my advisor, Sav Sethi. Our discussions throughout my graduate school career have shaped my intellectual development and taught me about the process of doing research. I have learned that physics is an intrinsically social enterprise, driven as much by conversations as calculations, and as such I am very grateful to Sav for generously giving his time to talk physics with me over the past few years.

Similarly, I want to thank my other colleagues who collaborated with me on the papers described in this manuscript and on unrelated projects. I have greatly enjoyed discussing physics with Chih-Kai, Gabriele, Alessandro, Hongliang, Daniel, Emil, Gautam, Stephen, Hao-Yu, Zhengdi, and Sam. The opportunity to collaborate with them has enriched my thinking and exposed me to many new perspectives on physics.

I would also like to thank my parents, for supporting my interest in science from my childhood until the present day. I've dedicated this thesis to them because they have always encouraged me to challenge myself and pursue my goals. Their love and unshakeable confidence in my abilities has been a stabilizing influence throughout my academic journey.

I am indebted to the broader theory group at the University of Chicago for providing stimulating conversations over lunch and whiskey, and for making this such an exciting place to do physics. In particular, thanks to the efforts of Clay Córdova and others, we were able to safely continue these community-building events in the latter part of the coronavirus pandemic, which was extremely helpful for re-establishing a sense of normalcy.

Outside of research, I am grateful to the students and friends who have spoken to me about science and, through our discussions, helped me to understand many subjects more deeply. I'd like to thank Colin, Ben, Bruno and the other then-undergraduate students that I've met while teaching and organizing reading seminars at the University of Chicago. Likewise, I thank those that I've worked with as a teaching assistant, especially Mark, for helping to develop my pedagogy. I thank the students in the MS-PSD cohorts which I supported as co-director for the opportunity to watch their trajectories through the master's program, and hopefully to help in some small way. I thank Reynard for enduring many of my lectures and for co-teaching a very interesting class on representation theory with me. I thank all of the students and colleagues that I've interacted with through outreach activities like jtTalks, Splash, and others. And I thank Gregor, Stavros, Kevin, and the Gregorious Maths Discord community, for many stimulating discussions about mathematics and physics.

Finally, I would like to thank Michael for many enjoyable conversations and for companionship and support in the last years of my PhD.

ABSTRACT

The $T\overline{T}$ operator provides a universal irrelevant deformation of two-dimensional quantum field theories with remarkable properties, including connections to both string theory and holography beyond AdS spacetimes. In particular, it appears that a $T\overline{T}$-deformed theory is a kind of new structure, which is neither a local quantum field theory nor a full-fledged string theory, but which is nonetheless under some analytic control. On the other hand, supersymmetry is a beautiful extension of Poincaré symmetry which relates bosonic and fermionic degrees of freedom. The extra computational power provided by supersymmetry renders many calculations more tractable. It is natural to ask what one can learn about irrelevant deformations in supersymmetric quantum field theories.

In this work, we describe a presentation of the $T\overline{T}$ deformation in manifestly supersymmetric settings. We define a "supercurrent-squared" operator, which is closely related to $T\overline{T}$, in any two-dimensional theory with $(0,1)$, $(1,1)$, or $(2,2)$ supersymmetry. This deformation generates a flow equation for the superspace Lagrangian of the theory, which therefore makes the supersymmetry manifest. In certain examples, the deformed theories produced by supercurrent-squared are related to superstring and brane actions, and some of these theories possess extra non-linearly realized supersymmetries. Finally, we show that $T\overline{T}$ defines a new theory of both abelian and non-abelian gauge fields coupled to charged matter, which includes models compatible with maximal supersymmetry. In analogy with the Dirac-Born-Infeld (DBI) theory, which defines a non-linear extension of Maxwell electrodynamics, these models possess a critical value for the electric field.

CHAPTER 1
INTRODUCTION

A central goal of modern theoretical physics, broadly speaking, is to better understand the space of all quantum field theories and string theories. One familiar way to explore this space is to begin with a well-understood theory and deform it, for instance by adding an integrated local operator to the Lagrangian. It is convenient to adopt a Wilsonian perspective and organize such deformations into classes according to the dimension of the perturbing operator, as one does in effective field theory. If the scaling dimension of the deforming operator is Δ and the spacetime dimension is d, then there are three cases.

1. If $\Delta < d$, then we say that the operator is *relevant*. This class of deformations triggers a conventional renormalization group flow, which modifies the behavior of the theory in the infrared. A classic example is adding a mass term $m^2\phi^2$ to the theory of a free scalar field, which is relevant in any dimension. The term "relevant" refers to the fact that such operators become more important at low energies (and conversely, become negligible at high energies).

2. A (classically) *marginal* operator does not become more nor less important as one flows to low energies. In a general QFT, quantum corrections can introduce a weak scale dependence to such an operator which makes it either *marginally relevant* or *marginally irrelevant*. If the seed theory is conformal, however, there may exist certain exactly marginal operators whose scaling dimension $\Delta = d$ matches their mass dimension. In this case, addition of a marginal operator generates motion on the conformal manifold, and the coupling constants of these marginal operators parameterize the moduli space of the theory. For example, the kinetic term $\partial^\mu\phi\partial_\mu\phi$ is marginal in any dimension; in the $2D$ CFT of a single compact boson, the addition of this operator is interpreted as a change in the radius of the boson's target-space circle.

3. An *irrelevant deformation* is one for which $\Delta > d$, like the square of the kinetic term $(\partial^\mu\phi\partial_\mu\phi)^2$. Such an operator grows more important at high energies and less important at low energies, which means that this class of deformations modifies the ultraviolet behavior of the theory. Because the process of flowing from high to low energies is lossy (there can be many UV theories in the same universality class), it is not possible in general to uniquely reverse a renormalization group trajectory by adding an irrelevant operator. Mathematically, adding such an operator will generically turn on infinitely many other operators which leads to a loss of predictive power. For this reason, deformation by irrelevant operators is more difficult to understand.

In view of the technical and conceptual challenges associated with addition of an irrelevant operator, it is especially surprising that there exists a well-defined irrelevant deformation of any two-dimensional field theory with the property that the deformed theory remains under analytic control, in a sense which we will describe shortly. This observation was first made by Zamolodchikov in [1], where he studied the deformation of two-dimensional Euclidean quantum field theories by the combination

$$\det\left(T_{\mu\nu}\right) = T_{00}T_{11} - T_{01}^2 \, , \tag{1.0.1}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the theory defined in Euclidean signature with a flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. The determinant can be written in a manifestly Lorentz-invariant way using the property of 2×2 matrices that

$$\det\left(T_{\mu\nu}\right) = \frac{1}{2}\left(\left(T^{\mu}{}_{\mu}\right)^2 - T^{\mu\nu}T_{\mu\nu} \right) \, . \tag{1.0.2}$$

The expression (1.0.2) involves products of stress tensor operators, and products of local operators are generally divergent in quantum field theory. However, [1] showed that this particular combination gives a finite, well-defined local operator as the insertion points of the stress tensors are taken coincident (up to total derivative terms, which are not relevant inside of one-point functions or integrals over spacetimes without boundary). It is therefore possible to deform a quantum field theory by the local operator defined by $\det\left(T_{\mu\nu}\right)$ in this coincident point limit; this operator is commonly referred to as $T\overline{T}$ due to its expression in complex coordinates for a conformal field theory.

Because $T_{\mu\nu}$ has mass dimension d in d dimensions, a bilinear in the stress tensor such as (1.0.2) – and, therefore, the local $T\overline{T}$ operator defined by their coincident point limit – has dimension $2d$, and is irrelevant in any number of spacetime dimensions. As we saw above, deformation of quantum field theories by irrelevant operators is typically difficult to understand. However, for the very special case of deformations by the $T\overline{T}$ operator in two-dimensional quantum field theories, a flow equation of the form

$$\frac{\partial \mathcal{L}^{(\lambda)}}{\partial \lambda} = \det\left(T_{\mu\nu}^{(\lambda)} \right) \tag{1.0.3}$$

leads to a controlled one-parameter family of theories in which certain quantities can still be computed. One can think of the equation (1.0.3) as defining a curve in the space of two-dimensional quantum field theories. At each point along the curve, the "tangent vector" (or the local operator by which we deform to move along the curve) is the determinant of the stress-energy tensor $T_{\mu\nu}^{(\lambda)}$ computed from the Lagrangian $\mathcal{L}^{(\lambda)}$ of the theory at that point (not the stress-energy tensor of the seed theory at $\lambda = 0$).

The proof of well-definedness for the $T\overline{T}$ operator, as well as the derivations of the flow equation for the cylinder energy levels and the deformed Lagrangian for a free scalar, will be reviewed in more detail in Chapter 2. For now, we will simply quote some results by way of motivation. Consider a family of $T\overline{T}$-deformed quantum field theories defined on a cylinder of radius R, whose energy levels are denoted by $E_n(R, \lambda)$. These energies satisfy an equation of inviscid Burgers' type, namely

$$\frac{\partial E_n}{\partial \lambda} = E_n \frac{\partial E_n}{\partial R} + \frac{1}{R} P_n^2 \,, \tag{1.0.4}$$

where $P_n = P_n(R)$ is the momentum (which does not flow). If the seed theory is a CFT, then on dimensional grounds the undeformed energies $E_n^{(0)} \equiv E_n(R, 0)$ must scale as $E_n^{(0)} \sim \frac{1}{R}$. In this case, one can solve the differential equation (1.0.4) in closed-form to obtain

$$E_n(R, \lambda) = \frac{R}{2\lambda} \left(\sqrt{1 + \frac{4\lambda E_n^{(0)}}{R} + \frac{4\lambda^2 P_n^2}{R^2}} - 1 \right) . \tag{1.0.5}$$

This is an example of what we mean by saying that the deformed theory "remains under analytic control." More concretely, one can make precise statements about quantities in the deformed theory in terms of the corresponding quantities in the undeformed theory; here, the deformed energies $E_n(R, \lambda)$ are functions of the undeformed energies $E_n^{(0)}$. There are analogous statements about other properties of the deformed theory such as its torus partition function, S-matrix, and – at least perturbatively in λ – its correlation functions.

It turns out that the square-root structure of the energy levels (1.0.5) implies a surprising result about the theory's asymptotic density of states. Recall we have assumed that the UV behavior of the undeformed theory is described by a CFT. This implies that the high energy density of states has Cardy behavior

$$\rho(E_n^{(0)}) \sim \exp\left(\sqrt{\frac{c}{3} E_n^{(0)}} \right) . \tag{1.0.6}$$

On the other hand the high energy behavior ($E_n \gg \frac{R}{\lambda}$) of the deformed energy scales as

$$E_n(R, \lambda) \sim \sqrt{\frac{R E_n^{(0)}}{\lambda}} \qquad \Longrightarrow \qquad E_n^{(0)} \sim \frac{\lambda E_n^2}{R} . \tag{1.0.7}$$

It can be shown – by first demonstrating that a $T\overline{T}$-deformed CFT enjoys modular invariance, and then by using the modular S transformation to relate the high-temperature and low-temperature limits of the partition function – that the high energy density of states in the deformed theory grows as

$$\rho(E_n) \sim \exp\left(\sqrt{\frac{c\lambda}{3R}}E_n\right) . \tag{1.0.8}$$

A theory with the asymptotic behavior (1.0.8) cannot be a local quantum field theory, which would necessarily exhibit the Cardy growth (1.0.6). Rather, this Hagedorn growth is more characteristic of a string theory.

This result is the first hint that a $T\overline{T}$-deformed theory is some new and mysterious intermediate structure, neither a local quantum field theory nor a full-fledged string theory which includes dynamical gravity. This Hagedorn behavior was interpreted in [2] – at least for the so-called "single trace" version of $T\overline{T}$ – by exploring its connection to little string theory and asymptotically linear dilaton spacetimes. In particular, it seems that adding the single-trace $T\overline{T}$ to a boundary conformal field theory dual to a bulk AdS spacetime corresponds to a deformation of the bulk which changes the asymptotics to linear dilaton. Because asymptotically AdS spacetimes are qualitatively different from asymptotically linear dilaton spacetimes, which behave more like flat space, this result suggests that $T\overline{T}$ may be a promising avenue for understanding holography beyond AdS.

A more direct way to see that $T\overline{T}$-deformed theories share some properties of string theories is to directly deform the seed theory of a single free boson. Beginning from the undeformed Lagrangian

$$\mathcal{L}_0 = \partial^\mu \phi \partial_\mu \phi \, , \tag{1.0.9}$$

it was shown in [3] that applying $T\overline{T}$ to this theory leads to a deformed Lagrangian corresponding to a Nambu-Goto string in static gauge,

$$\mathcal{L}_\lambda = \frac{1}{2\lambda}\left(\sqrt{1 + 2\lambda \partial_\mu \phi \partial^\mu \phi + 1} - 1\right) . \tag{1.0.10}$$

This gives another piece of evidence, independent of the Hagedorn density of states, to suggest that $T\overline{T}$ is related to string theory.

On the other hand, it has been known since the 1970s that the introduction of supersymmetry resolves several unsatisfying properties of bosonic string theory, such as the presence of a tachyon and the absence of fermions in the spectrum. Given the tantalizing connections between $T\overline{T}$-deformed field theories and (bosonic) string theories, it is natural to ask whether some supersymmetric presentation of $T\overline{T}$ might similarly be related to the superstring.

In one sense, the interplay between $T\overline{T}$ and supersymmetry is trivial: the finite-volume spectrum of a $T\overline{T}$-deformed theory obeys a differential equation whereby each deformed energy level is determined in terms of the corresponding undetermined energy level. This means that any degeneracies of the undeformed energies – such as a pairing between the energy levels of bosonic and fermionic states, characteristic of supersymmetric theories – will also persist in the deformed theory. From this perspective, it is obvious that applying the $T\overline{T}$ deformation to a supersymmetric seed theory produces a deformed theory which is also supersymmetric.

However, the analytic control offered by supersymmetry is most powerful when the symmetry is made manifest, for instance by a superspace construction which geometrizes the supersymmetry transformations. Although the ordinary $T\overline{T}$ preserves the supersymmetry of the deformed theory, the supersymmetry transformations will generically flow, so that the action of the supersymmetry generators on the fields of the theory must be corrected order-by-order in the $T\overline{T}$ parameter λ. It is desirable to present a formulation of $T\overline{T}$ where the supersymmetry of the deformed theory acts in a simple way, for instance by formulating a flow equation for a superspace Lagrangian. This is one motivation for the present work, and we will propose such superspace flow equations for theories with various amounts of supersymmetry in Chapters 3 and 4.

Another motivation is the possibility of finding $T\overline{T}$-like deformations for higher-dimensional theories. We will briefly explain why the point-splitting definition of the local $T\overline{T}$ operator fails in $D > 2$ spacetime dimensions. In any D, there is some operator product expansion

$$T^{\mu\nu}(x)T_{\mu\nu}(y) - \frac{1}{D-1}T^{\mu}{}_{\mu}(x)T^{\nu}{}_{\nu}(y) = \sum_{\alpha} A_{\alpha}\left(|x-y|^2\right)\mathcal{O}_{\alpha}(y), \tag{1.0.11}$$

The argument of [1], to be reviewed in Chapter 2, showed that that in $D = 2$ the conservation equation $\partial^{\mu}T_{\mu\nu} = 0$ implies A_{α} is independent of coordinate unless the corresponding $\mathcal{O}_{\alpha}$ is a total derivative. This is sufficient to define a local operator by taking $x \to y$, modulo total derivative terms. When $D > 2$, however, the conservation equation imposes fewer constraints on this operator product expansion, and there can be additional non-derivative divergences on the right side of (1.0.11) which obstruct a universal definition of $T\overline{T}$.

However, in theories with supersymmetry, the stress tensor sits in a multiplet that contains additional fields which are related to $T_{\mu\nu}$ by supersymmetry. This multiplet of operators satisfies a larger collection of constraints than conservation, which correspondingly imposes more structure on the operator product expansions of fields in the multiplet. One might hope that, with a sufficiently large amount of supersymmetry (likely maximal), a particular combination of supercurrent bilinears might satisfy a property similar to that of $T\overline{T}$ in two dimensions, and therefore provide a universal irrelevant deformation of any sufficiently supersymmetric theory. Although there is no proof of this statement, we will discuss some classical properties of supersymmetric $T\overline{T}$-like deformations for four-dimensional theories in Chapter 5 and speculate about the possibility of defining a quantum operator in Section 5.5.

We close this introductory section by reiterating that irrelevant current-type deformations lie at an exciting intersection of many interesting directions in theoretical physics. Here we have focused on the relationship of $T\overline{T}$ to string theory and on tantalizing hints of holography beyond AdS. However, $T\overline{T}$ and related deformations also have connections to integrability, random geometry, topological gravity, the uniform light-cone gauge, and many other areas of interest. For an informative overview of the subject, see [4].

The layout of the remainder of this dissertation is as follows.

- In Chapter 2, we will review some facts about the ordinary $T\overline{T}$ deformation, such as the point-splitting definition of the operator and the flow equation for the energies on a cylinder. This chapter is a review of previous results, primarily from [1] and [3].

- Next, Chapter 3 presents a manifestly supersymmetric version of the $T\overline{T}$ operator for theories with $(0,1)$ or $(1,1)$ supersymmetry. This chapter is based on [5].

- In Chapter 4, we extend this definition to theories with $(2,2)$ supersymmetry and study some deformed models in detail. Here we follow the treatment in [6].

- Chapter 5 then identifies certain non-linearly realized symmetries of the $(2,2)$ deformed models that we first introduced in Chapter 4; we also discuss how these theories are related to certain models with $\mathcal{N}=1$ supersymmetry in four dimensions. This chapter is based on the paper [7].

- Finally, Chapter 6 uses the ordinary $T\overline{T}$ deformation to define a new theory of a non-abelian gauge field coupled to scalars in two dimensions; this deformed theory is compatible with maximal supersymmetry. The discussion of this chapter follows [8].

The details of various calculations which we use in the main body of the thesis are presented in Appendix A.

CHAPTER 2
THE $T\overline{T}$ DEFORMATION

In this chapter, we will review some aspects of the ordinary $T\overline{T}$ deformation with no supersymmetry required. We also set our conventions for the various superspaces that will be used in later chapters.

Nothing in Section 2.1 is new; our goal is to present results from earlier works on $T\overline{T}$ for context and motivation, focusing on [1] and [3]. Another useful review of the content in this chapter is [4].

2.1 Review of $T\overline{T}$

In this section, we will provide more detailed derivations of a few results concerning the ordinary $T\overline{T}$ deformation which were quoted in the introductory chapter.

2.1.1 Definition by point-splitting

As we mentioned in the introduction, the first remarkable property of $T\overline{T}$ is that it can be unambiguously defined by point-splitting (up to total derivatives), despite involving products of local operators. Here we will review the proof of this fact from the original paper of Zamolodchikov [1]. There he considered a Euclidean QFT with the following properties:

1. Local translation and rotation symmetry. This property implies the existence of a stress energy tensor $T_{\mu\nu}$ which is conserved,

$$\partial_\mu T^{\mu\nu} = 0\,, \tag{2.1.1}$$

   and symmetric, $T_{\mu\nu} = T_{\nu\mu}$. In the $2D$ parametrization $T_{zz} = T,\ \bar{T} = T_{\bar{z}\bar{z}},\ \Theta = T_{z\bar{z}}$, this can be written

$$\partial_{\bar{z}} T(z) = \partial_z \Theta(z)\ , \qquad \partial_z \bar{T}(z) = \partial_{\bar{z}} \Theta(z)\ . \tag{2.1.2}$$

2. Global translation symmetry. This property implies that any 1-point function is independent of position

$$\langle \mathcal{O}_i(z) \rangle = \langle \mathcal{O}_i(0) \rangle\ , \tag{2.1.3}$$

   and that any 2-point function

$$\langle \mathcal{O}_i(z) \mathcal{O}_j(z') \rangle = G_{ij}(z - z')\ , \tag{2.1.4}$$

depends only on the distance $|z - z'|$ between the insertion points.

3. Clustering. That there exists some direction of infinite length such that

$$\lim_{x\to\infty} \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \langle \mathcal{O}_i\rangle\langle \mathcal{O}_j\rangle \ . \tag{2.1.5}$$

4. UV CFT. The seed QFT which we begin with is assumed to be described by a CFT at short distances.

These conditions then require that we consider a theory on the flat plane or cylinder. Using these assumptions one can show that the coincident point limit

$$T\overline{T} := \lim_{z'\to z} \left(T(z')\bar{T}(z) - \Theta(z')\Theta(z) \right) \ , \tag{2.1.6}$$

defines a local operator.

First, note that the conservation of the stress tensor implies

$$\begin{aligned} \partial_{\bar{z}}(T(z)\bar{T}(z') - \Theta(z)\Theta(z')) &= (\partial_z + \partial_{z'})\Theta(z)\bar{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\Theta(z)\Theta(z') \ , \\ \partial_z(T(z)\bar{T}(z') - \Theta(z)\Theta(z')) &= (\partial_z + \partial_{z'})T(z)\overline{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})T(z)\Theta(z') \ . \end{aligned} \tag{2.1.7}$$

Now using the operator product expansions

$$\begin{aligned} T(z)\Theta(z') &= \sum_i A_i(z-z')\mathcal{O}_i(z') \ , \qquad & \Theta(z)\Theta(z') &= \sum_i C_i(z-z')\mathcal{O}_i(z') \ , \\ \Theta(z)\overline{T}(z') &= \sum_i B_i(z-z')\mathcal{O}_i(z') \ , \qquad & T(z)\overline{T}(z') &= \sum_i D_i(z-z')\mathcal{O}_i(z') \ , \end{aligned} \tag{2.1.8}$$

the equations (2.1.7) imply

$$\begin{aligned} \sum_i \partial_{\bar{z}} F_i(z-z')\mathcal{O}_i(z') &= \sum_i \Big(B_i(z-z')\partial_{z'}\mathcal{O}_i - C_i(z-z')\partial_{\bar{z}'}\mathcal{O}_i \Big) \ , \\ \sum_i \partial_z F_i(z-z')\mathcal{O}_i(z') &= \sum_i \Big(D_i(z-z')\partial_{z'}\mathcal{O}_i(z') - A_i(z-z')\partial_{\bar{z}'}\mathcal{O}_i(z') \Big) \ , \end{aligned} \tag{2.1.9}$$

where

$$F_i(z-z') = D_i(z-z') - C_i(z-z') \ . \tag{2.1.10}$$

This implies that any operator arising in the OPE

$$T(z)\overline{T}(z') - \Theta(z)\Theta(z') = \sum_i F_i(z-z')\mathcal{O}_i(z') \ , \tag{2.1.11}$$

must either have a coordinate independent coefficient function $F_i(z-z')$ or is itself the derivative of another local operator:

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \mathcal{O}_{T\overline{T}}(z') + \text{derivative terms} \ . \tag{2.1.12}$$

This allows us to define the composite operator

$$T\overline{T}(z) := \mathcal{O}_{T\overline{T}}(z) \ . \tag{2.1.13}$$

Note that we have only defined $T\overline{T}$ up to derivative terms, but by the assumption of global translation symmetry above, any one-point function of a total derivative vanishes.

Although it will not be used in this thesis, we note that assumption (2) of global translation symmetry can be weakened to the existence of a transitive global isometry. This can be used to define the $T\overline{T}$ deformation in AdS_2, as is done in [9]. In that case, one can show that the $T\overline{T}$ operator obeys a factorization property in the AdS_2-invariant ground state.

2.1.2 Deformed Lagrangian for free scalar

In this subsection, we will review the solution of the $T\overline{T}$ flow equation for the deformed Lagrangian $\mathcal{L}(\lambda)$ of a free scalar field ϕ. We stress that this is a purely classical result, unrelated to the preceding argument that the $T\overline{T}$ operator is well-defined by point-splitting. Indeed, the explicit solution for the deformed Lagrangian of scalars coupled to an arbitrary background metric was already written down in [10], which follows from the analysis in [3].

Consider a general λ-dependent Lagrangian for a real scalar ϕ coupled to a background metric $g_{\mu\nu}$. For simplicity, we assume that the Lagrangian reduces to the usual free kinetic Lagrangian for ϕ at $\lambda = 0$:

$$\mathcal{L}(\lambda = 0) = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{2.1.14}$$

The $T\overline{T}$ flow will not introduce dependence on the undifferentiated field ϕ, as one would have in a potential energy term, so the finite-λ Lagrangian can only depend on the scalar quantity $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and on the parameter λ. To ease notation, we define $\xi = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ so that

$$\mathcal{L}(\lambda) = f(\lambda, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) \equiv f(\lambda, \xi). \tag{2.1.15}$$

We now compute the components of the stress tensor,

$$T_{\mu\nu}^{(\lambda)} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(\lambda)}}{\delta g^{\mu\nu}}, \tag{2.1.16}$$

where $S^{(\lambda)}$ is the effective action of the deformed theory

$$S^{(\lambda)} = \int d^2x \sqrt{-g}\, \mathcal{L}(\lambda). \tag{2.1.17}$$

Taking the variation, one finds

$$T_{\mu\nu}^{(\lambda)} = g_{\mu\nu} f - 2\frac{\partial f}{\partial \xi} \partial_\mu \phi \partial_\nu \phi \tag{2.1.18}$$

Written in a manifestly diffeomorphism invariant form appropriate for a general background metric, the flow equation for the Lagrangian is

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{1}{2} \left(\left(g^{\mu\nu} T_{\mu\nu} \right)^2 - g^{\mu\rho} g^{\nu\sigma} T_{\mu\nu} T_{\rho\sigma} \right). \tag{2.1.19}$$

Note that, when $g_{\mu\nu} = \delta_{\mu\nu}$, the right side of (2.1.19) reduces to the ordinary determinant of a 2×2 matrix.

Evaluating (2.1.19) with the components (2.1.18), one finds

$$\frac{1}{2} \left(\left(g^{\mu\nu} T_{\mu\nu} \right)^2 - g^{\mu\rho} g^{\nu\sigma} T_{\mu\nu} T_{\rho\sigma} \right) = f^2 - 2f\xi \frac{\partial f}{\partial \xi}. \tag{2.1.20}$$

Thus the differential equation for the deformed Lagrangian becomes

$$\frac{df}{d\lambda} = f^2 - 2f\xi \frac{\partial f}{\partial \xi}. \tag{2.1.21}$$

To solve this differential equation, we begin by noting that the solution can depend only on the dimensional parameter λ and the dimensionless combination $\lambda\xi$. Since $\frac{1}{\lambda}$ has the same mass dimension as the Lagrangian, it must be consistent to make an ansatz of the form

$$\begin{aligned} f(\lambda, \xi) &= \frac{1}{\lambda} \cdot \tilde{f}(\lambda\xi) \\ &= \frac{1}{\lambda} \cdot \tilde{f}(x)\,, \end{aligned} \tag{2.1.22}$$

where we have defined $x = \lambda\xi$. With this ansatz, we have

$$\frac{df}{d\lambda} = \frac{1}{\lambda^2}\left(x\tilde{f}'(x) - \tilde{f}(x)\right) ,$$
$$\frac{\partial f}{\partial \xi} = \tilde{f}'(x) , \tag{2.1.23}$$

so the differential equation becomes

$$\tilde{f}'(x) = \frac{\tilde{f}(x) + \tilde{f}(x)^2}{x(1 + 2\tilde{f}(x))} . \tag{2.1.24}$$

The result is now an ordinary differential equation which can be solved by separation of variables. After replacing $x = \lambda\xi = \lambda g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and imposing the initial condition (2.1.14)), the result is

$$\mathcal{L}(\lambda) = \frac{1}{2\lambda}\left(\sqrt{1 + 2\lambda g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} - 1\right) . \tag{2.1.25}$$

Expanding about $\lambda = 0$, one finds

$$\mathcal{L}(\lambda) = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^2 + \mathcal{O}(\lambda^2) . \tag{2.1.26}$$

We emphasize that this is a purely classical result that is true for any conformally flat metric $g_{\mu\nu}$. The additional power which comes from working in (Lorentzian or Euclidean) flat space, with metric $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$ is that one can unambiguously define the local $T\overline{T}$ operator up to total derivatives, as described in Section 2.1.1. With a flat metric, one can also obtain a flow equation for the finite-volume spectrum, which we turn to next.[1]

2.1.3 Factorization and inviscid Burgers' equation

We conclude the review of $T\overline{T}$ by deriving the flow equation (1.0.4) for the cylinder energy levels of a $T\overline{T}$-deformed quantum field theory, which has square-root type solutions of the form (1.0.5) when the seed theory is conformal.

The first step is to show that the expectation value of the point-split $T\overline{T}$ operator is actually independent of the distance between the insertion points. Define the function

$$C(z, w) = \langle T(z)\overline{T}(w)\rangle - \langle \Theta(z)\Theta(w)\rangle , \tag{2.1.27}$$

1. As we mentioned earlier, one can also obtain a flow equation for the AdS-invariant ground state for $T\overline{T}$-deformed theories in AdS_2, as shown in [9].

where as in Section 2.1.1 we use the notation $T = T_{zz}$, $\overline{T} = T_{\bar{z}\bar{z}}$, $\Theta = T_{z\bar{z}}$. Following the steps we used in (2.1.7) above, we take a z derivative to write

$$\partial_{\bar{z}} C(z, w) = \langle \partial_{\bar{z}} T(z) \overline{T}(w) \rangle - \partial_{\bar{z}} \langle \Theta(z) \Theta(w) \rangle \tag{2.1.28}$$

and use the conservation equation $\partial_{\bar{z}} T = \partial_z \Theta$ to write this as

$$\partial_{\bar{z}} C(z, w) = \partial_z \langle \Theta(z) \overline{T}(w) \rangle - \partial_{\bar{z}} \langle \Theta(z) \Theta(w) \rangle \,. \tag{2.1.29}$$

On the other hand, by the assumption of global translation invariance, the two-point function $\langle \Theta(z) \Theta(w) \rangle$ can depend only on the separation $|z - w|^2$, which means that

$$\partial_{\bar{z}} \langle \Theta(z) \Theta(w) \rangle = -\partial_{\overline{w}} \langle \Theta(z) \Theta(w) \rangle \,. \tag{2.1.30}$$

Similarly in the first term,

$$\partial_z \langle \Theta(z) \overline{T}(w) \rangle = -\partial_w \langle \Theta(z) \overline{T}(w) \rangle \,. \tag{2.1.31}$$

Therefore we conclude that

$$\partial_{\bar{z}} C(z, w) = -\langle \Theta(z) \partial_w \overline{T}(w) \rangle + \langle \Theta(z) \partial_{\overline{w}} \Theta(w) \rangle = 0 \,, \tag{2.1.32}$$

where we have again used the conservation equation $\partial_w \overline{T} = \partial_{\overline{w}} \Theta$.

Since $C(z, w)$ can depend only on $|z - w|^2$, and since $\partial_{\bar{z}} C(z, w) = 0$, it follows that $C(z, w) = C$ is a constant. Thus we may evaluate it for any choice of z and w. On the one hand, when z and w are taken coincident, we have

$$\lim_{z \to w} C(z, w) = \langle T \overline{T} \rangle \,, \tag{2.1.33}$$

since we have established in Section 2.1.1 that this limit defines a local operator up to total derivative terms (which vanish inside of one-point functions). But on the other hand, for a theory defined on the cylinder, we may take the points z, w to be infinitely separated alone the non-compact cylinder direction:

$$C = \lim_{|z - w| \to \infty} C(z, w) = \langle T \rangle \langle \overline{T} \rangle - \langle \Theta \rangle^2 \,, \tag{2.1.34}$$

where in the last step we have used the fact that vacuum two-point functions cluster decompose into products of one-point functions at infinite separation. We therefore conclude that

$$\langle T\overline{T}\rangle = \langle T\rangle\langle\overline{T}\rangle - \langle\Theta\rangle^2 \,, \tag{2.1.35}$$

which is referred to as the factorization property of $T\overline{T}$.

The above derivation establishes factorization of the $T\overline{T}$ operator in the vacuum state, where cluster decomposition holds. To show that it also factorizes in any energy eigenstate, one can insert a complete set of states to write

$$\begin{aligned}\langle n \mid T(z)\overline{T}(w) \mid n\rangle = \sum_m \langle n \mid T(z) \mid m\rangle \langle m \mid \overline{T}(w) \mid n\rangle \\ \cdot \exp\Big((E_n - E_m)|\operatorname{Im} z - \operatorname{Im} w| + i(P_n - P_m)|\operatorname{Re} z - \operatorname{Re} w|\Big) \,,\end{aligned} \tag{2.1.36}$$

and likewise for $\langle n \mid \Theta(z)\Theta(w) \mid n\rangle$. Because the exponential factors contain explicit dependence on the coordinates z, w, but the function $C(z, w)$ is a constant when the correlation functions are taken in *any* energy eigenstate, all of the terms in the sum (2.1.36) must vanish except when $m = n$. In the coincident point limit, this implies that

$$\langle n \mid T\overline{T} \mid n\rangle = \langle n \mid T \mid n\rangle\langle n \mid \overline{T} \mid n\rangle - \langle n \mid \Theta \mid n\rangle^2 \,, \tag{2.1.37}$$

which is the statement that $T\overline{T}$ factorizes in all energy eigenstates, not just the vacuum state.

Once the factorization property of $T\overline{T}$ is established, the flow equation for the cylinder spectrum follows almost immediately. The interpretation of the components of the stress tensor for a theory on a cylinder of radius R is

$$\begin{aligned}\langle n \mid T_{yy} \mid n\rangle &= -\frac{1}{R}E_n(R, \lambda) \,, \\ \langle n \mid T_{xx} \mid n\rangle &= -\partial_R E_n(\lambda, R) \,, \\ \langle n \mid T_{xy} \mid n\rangle &= iP_n(R) \,.\end{aligned} \tag{2.1.38}$$

Here $x \sim x + R$ is the circular direction of the cylinder and y is the non-compact direction. Because the Euclidean Lagrangian density is the Hamiltonian density, and we are deforming by adding the $T\overline{T}$ operator to the Lagrangian, the flow equation for the energy levels is

$$\begin{aligned}\partial_\lambda E_n(\lambda, R) &= -R\langle n \mid \det\big(T_{\mu\nu}\big) \mid n\rangle \\ &= -R\Big(\langle n \mid T_{xx} \mid n\rangle\langle n \mid T_{yy} \mid n\rangle - \langle n \mid T_{xy} \mid n\rangle^2\Big) \ , \end{aligned} \tag{2.1.39}$$

where in the second step we have used factorization. Expressing the stress tensor components in terms of energies and momenta according to (2.1.38), we find

$$\partial_\lambda E_n = E_n \partial_R E_n + \frac{1}{R} P_n^2 \,. \tag{2.1.40}$$

This is the inviscid Burgers' equation (1.0.4) which we referred to in the introduction. Given any seed theory with energies E_n, this flow equation determines the energy levels of the $T\overline{T}$ deformed theory at finite λ, although for a general starting theory we cannot solve the differential equation in closed form. Matters simplify if the undeformed theory is a CFT, since in this case the energies and momenta satisfy

$$\begin{aligned} E_n &= \frac{1}{R}\left(n + \bar{n} - \frac{c}{12}\right) \ , \\ P_n &= \frac{1}{R}\left(n - \bar{n}\right) \ , \end{aligned} \tag{2.1.41}$$

where c is the central charge. Because of the especially simple R-dependence of the undeformed energies, in this case we can solve the flow equation to write

$$E_n(R,\lambda) = \frac{R}{2\lambda}\left(\sqrt{1 + \frac{4\lambda E_n^{(0)}}{R} + \frac{4\lambda^2 P_n^2}{R^2}} - 1\right) \ , \tag{2.1.42}$$

which is equation (1.0.5) that was quoted in the introduction.

One generic feature of $T\overline{T}$-deformed CFTs is apparently from the square-root structure of the energy levels. Let's restrict to the ground state, $n = \bar{n} = 0$, for which the flow equation becomes

$$E_0(R,\lambda) = \frac{R}{2\lambda}\left(\sqrt{1 - \frac{4\lambda c}{R^2}} - 1\right) \tag{2.1.43}$$

When λ exceeds $\frac{R^2}{4c}$, the argument of the square root becomes negative and the energies are complex. This signals that there is a maximum allowed value of the flow parameter λ, after which the theory appears to suffer some pathology. A complete understanding of this complex-energy behavior is still lacking, but it might be related to the non-locality of $T\overline{T}$ deformed theories; a possible interpretation is that this maximum value of λ represents the point at which the non-locality scale becomes comparable to the radius of the cylinder, and that this large non-locality causes the theory to become ill-defined [11].

2.2 Conventions

We consider two-dimensional field theories in Lorentzian signature with coordinates (x^0, x^1). It will be convenient to change coordinates to light-cone variables, defining

$$x^{\pm\pm} = \frac{1}{\sqrt{2}} \left(x^0 \pm x^1 \right) . \tag{2.2.1}$$

Here we have used the bi-spinor conventions, where coordinates and vector quantities are written with a pair of $\pm$ indices ($x^{\pm\pm}$) rather than a single index ($x^{\pm}$).

The derivatives corresponding to the coordinates (2.2.1) are written

$$\partial_{\pm\pm} = \frac{1}{\sqrt{2}} (\partial_0 \pm \partial_1) \tag{2.2.2}$$

In these conventions, we have $\partial_{\pm\pm} x^{\pm\pm} = 1$ and $\partial_{\pm\pm} x^{\mp\mp} = 0$.

Spinors in two dimensions carry a single index which is raised or lowered as follows:

$$\psi^+ = -\psi_-, \qquad \psi^- = \psi_+. \tag{2.2.3}$$

The advantage of writing all vector indices as pairs of spinor indices is that it allows us to more easily compare terms in equations which involve a combination of spinor, vector, spinor-vector, and tensor quantities. For instance, in this notation the derivatives with respect to light-cone coordinates carry two indices $\partial_{\pm\pm}$, whereas spinor quantities ψ_+ carry only a single index, which allows us to distinguish between spin-$\frac{1}{2}$ and spin-1 objects. Similarly, the supercurrent has components $S_{+++}, S_{---}, S_{+--}$, and S_{-++} – which we can immediately identify as a spinor-vector because it has three indices – and the stress-energy tensor carries two vector indices so its components will be written as $T_{++++}, T_{----}, T_{++--} = T_{--++}$.

(1, 1) and (0, 1) Superspace

When we consider $(1, 1)$ supersymmetric theories, we will introduce anticommuting coordinates $\theta^{\pm}$. The corresponding supercovariant derivatives are defined in our conventions as

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\pm \partial_{\pm\pm}, \tag{2.2.4}$$

which satisfy $D_\pm D_\pm = \partial_{\pm\pm}$ and $\{D_+, D_-\} = 0$. There are also two supercharges $Q_\pm$ given by

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} - \theta^\pm \partial_{\pm\pm}, \tag{2.2.5}$$

which satisfy $Q_\pm Q_\pm = -\partial_{\pm\pm}$. The Lagrangian for a field theory is written as an integral over this $(1,1)$ superspace,

$$\mathcal{L} = \int d^2\theta \, \mathcal{A}(\Phi) \, , \tag{2.2.6}$$

where $\mathcal{A}$ is the superspace Lagrangian, Φ represents some collection of $(1,1)$ superfields, and $d^2\theta = d\theta^- \, d\theta^+$.

We will briefly describe theories in $(0,1)$ superspace by truncating the conventions for $(1,1)$ superspace described above. Such a superspace has a single anticommuting coordinate θ^+ along with the associated supercovariant derivative D_+ and supercharge Q_+. The Lagrangian for a $(0,1)$ theory can be written as an integral

$$\mathcal{L} = \int d\theta^+ \, \mathcal{A}_+(\Phi) \, , \tag{2.2.7}$$

where now the superspace Lagrangian $\mathcal{A}_+$ must carry spin to ensure that the bosonic Lagrangian density $\mathcal{L}$ is a Lorentz scalar.

(2, 2) Superspace

When we study two-dimensional theories with $4 = 2 + 2$ supercharges, the four anticommuting coordinates will be written $\theta^\pm$ and $\overline{\theta}^\pm$. It will sometimes be convenient to collectively denote the superspace coordinates by $\zeta^M = (x^\mu, \, \theta^\pm, \, \bar{\theta}^\pm)$. In Chapter 4, we will use the supercovariant derivatives, collectively denoted by $D_A = (\partial_a, \, D_\pm, \overline{D}_\pm)$, given by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \frac{i}{2}\overline{\theta}^\pm \partial_{\pm\pm} \; , \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \frac{i}{2}\theta^\pm \partial_{\pm\pm} \; , \tag{2.2.8}$$

and which satisfy

$$\{D_\pm, \overline{D}_\pm\} = i\partial_{\pm\pm}, \tag{2.2.9}$$

with all other (anti-)commutators vanishing.

The supersymmetry transformations for an $\mathcal{N} = (2,2)$ superfield $\mathcal{F}(\zeta) = \mathcal{F}(x^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$ are given by

$$\delta_Q \mathcal{F} := i\epsilon^+ \mathcal{Q}_+ \mathcal{F} + i\epsilon^- \mathcal{Q}_- \mathcal{F} - i\bar{\epsilon}^+ \overline{\mathcal{Q}}_+ \mathcal{F} - i\bar{\epsilon}^- \overline{\mathcal{Q}}_- \mathcal{F} \ , \tag{2.2.10}$$

where on superfields the supercharges are represented by the following differential operators

$$\mathcal{Q}_\pm = \frac{\partial}{\partial \theta^\pm} + \frac{i}{2}\overline{\theta}^\pm \partial_{\pm\pm} \ , \qquad \overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial \overline{\theta}^\pm} - \frac{i}{2}\theta^\pm \partial_{\pm\pm} \ , \tag{2.2.11}$$

satisfying

$$\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} = -i\partial_{\pm\pm} \ , \tag{2.2.12}$$

and commuting with the covariant derivatives D_A.

In Chapter 5, to more easily facilitate comparison between $\mathcal{N} = (2,2)$ in $2D$ and $\mathcal{N} = 1$ theories in $4d$, we will use a slightly different convention for the supercovariant derivative which differs only by relative constant factors. There we will use supercovariant derivatives

$$D'_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm \partial_{\pm\pm} \, , \qquad \overline{D}'_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm \partial_{\pm\pm} \, , \tag{2.2.13}$$

which satisfy the rescaled algebra $\{D'_\pm, \overline{D}'_\pm\} = -2i\partial_{\pm\pm}$.

When we write integrals over $(2,2)$ superspace, we use the notation $d^2\theta = d\theta^- d\theta^+$, $d^2\bar{\theta} = d\bar{\theta}^+ d\bar{\theta}^-$ and $d^4\theta = d^2\theta d^2\bar{\theta}$.

Four Dimensional Theories

When we study theories in four spacetime dimensions, we will mostly follow the conventions of Wess and Bagger [12]. The $D = 4$, $\mathcal{N} = 2$ superspace is parametrised by bosonic coordinates x^μ and anticommuting coordinates $(\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, $(\tilde{\theta}^\alpha, \bar{\tilde{\theta}}^{\dot{\alpha}})$.

To discuss chirality constraints in these theories, it is convenient to introduce the coordinates

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} + i\tilde{\theta}\sigma^{\mu}\bar{\tilde{\theta}} \tag{2.2.14}$$

In terms of the y^{μ}, the supercovariant derivatives are given by

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^{\mu}}, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \ , \tag{2.2.15}$$

and similarly for $\tilde{D}_{\alpha}, \bar{\tilde{D}}_{\dot{\alpha}}$.

Our conventions differ from those of [12] in the conversion between vector and bi-spinor indices. We will use the normalization

$$v_{\alpha\dot{\alpha}} = -2\sigma^{\mu}_{\alpha\dot{\alpha}}v_{\mu}, \ v_{\mu} = \frac{1}{4}\bar{\sigma}^{\alpha\dot{\alpha}}v_{\alpha\dot{\alpha}} \, . \tag{2.2.16}$$

In these conventions, for instance, one has

$$\mathcal{J}_{\alpha\dot{\alpha}} = -2\sigma^{\mu}_{\alpha\dot{\alpha}}\mathcal{J}_{\mu}, \qquad \mathcal{J}^{\mu} = \frac{1}{4}\mathcal{J}_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}, \qquad \mathcal{J}^2 \equiv \eta^{\mu\nu}J_{\mu}J_{\nu} = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\mathcal{J}_{\alpha\dot{\alpha}}J_{\beta\dot{\beta}} \ . \tag{2.2.17}$$

CHAPTER 3
$(1,1)$ AND $(0,1)$ SUPERCURRENT-SQUARED DEFORMATIONS

In this chapter, we propose a solvable, irrelevant deformation that is built from bilinears in currents and which manifestly preserves supersymmetry. Just as the remarkable properties of the $T\overline{T}$ deformation follows from continuity equations, in the supersymmetric case, we will describe analogous relations based on the conservation laws in superspace. This chapter is based on the paper "Supersymmetry and $T\overline{T}$ Deformations" [5].

3.1 $T\overline{T}$ and Supersymmetry

3.1.1 Supercurrent-squared

Because the usual $T\overline{T}$ deformation discussed in section (2.1) is built from the Noether current for spatial translations, we will generalize this construction by writing a manifestly supersymmetric Noether current associated with translations in superspace. For concreteness, we work in the $(1,1)$ theory, but a similar calculation in $(0,1)$ will be described in section 3.3.

Consider a supersymmetric Lagrangian which is written as an integral over $(1,1)$ superspace as $\mathcal{L} = \int d^2\theta\, \mathcal{A}$. We allow $\mathcal{A}$ to depend on a superfield Φ and a particular set of Φ derivatives listed below:

$$\mathcal{A} = \mathcal{A}\left(\Phi, D_+\Phi, D_-\Phi, \partial_{++}\Phi, \partial_{--}\Phi, D_+D_-\Phi\right). \tag{3.1.1}$$

The supercovariant derivatives $D_\pm$ are defined in (2.2.4). The superspace equation of motion associated with this Lagrangian is

$$\begin{aligned}\frac{\delta\mathcal{A}}{\delta\Phi} = D_+\left(\frac{\delta\mathcal{A}}{\delta D_+\Phi}\right) + D_-\left(\frac{\delta\mathcal{A}}{\delta D_-\Phi}\right) + \partial_{++}\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right)\\ + \partial_{--}\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - D_+D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right).\end{aligned} \tag{3.1.2}$$

As in the derivation of the usual stress tensor T, we now consider a spatial translation of the form $\delta x^{\pm\pm} = a^{\pm\pm}$ for some constant $a^{\pm\pm}$. The variation $\delta\mathcal{A}$ of the superspace Lagrangian is given by

$$\begin{aligned}\delta\mathcal{A} = {} & D_+\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi}\right) + D_-\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi}\right) + \partial_{++}\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right)\\ & + \partial_{--}\left(\delta\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\left(D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\delta\Phi\right) + D_-\left(\delta\Phi D_+\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\right)\\ & - \frac{1}{2}\left(D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\delta\Phi\right) + D_+\left(\delta\Phi D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\right)\\ & - \delta\Phi\left(-\frac{\delta\mathcal{A}}{\delta\Phi} + D_+\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_-\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{++}\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi} + \partial_{--}\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right.\\ & \left. - D_+D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right).\end{aligned} \tag{3.1.3}$$

Here we have chosen to symmetrize the term involving $D_+D_-\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}$ using $\{D_+, D_-\} = 0$.

The last two lines of (3.1.3) are the superspace equation of motion; we now specialize to the case of on-shell variations, for which this last term vanishes. Further, the left side of (3.1.3) is $\delta\mathcal{A} = a^{++}\partial_{++}\mathcal{A} + a^{--}\partial_{--}\mathcal{A}$, which is a total derivative. We use $\partial_{\pm\pm} = D_\pm D_\pm$ to express (3.1.3) in the form

$$\begin{aligned}0 = {} & a^{++}D_+\Big[\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{++}\Phi\right)\\ & \qquad - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A}\Big]\\ & + a^{++}D_-\Big[\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + D_-\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{++}\Phi\right)\\ & \qquad + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\Big]\\ & + a^{--}D_+\Big[\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{--}\Phi\right)\\ & \qquad - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right)\Big]\\ & + a^{--}D_-\Big[\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + D_-\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{--}\Phi\right)\\ & \qquad + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}\Big].\end{aligned} \tag{3.1.4}$$

This equation gives a conservation law for a superfield $\mathcal{T}$ which we define by

$$
\begin{aligned}
\mathcal{T}_{++-} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{++}\Phi\right) \\
&\quad - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A}, \\
\mathcal{T}_{+++} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + D_-\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{++}\Phi\right) \\
&\quad + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{---} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_-\left(\partial_{--}\Phi\right) \\
&\quad - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right), \\
\mathcal{T}_{--+} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + D_-\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) - \frac{1}{2}\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_+\left(\partial_{--}\Phi\right) \\
&\quad + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}.
\end{aligned}
\tag{3.1.5}
$$

In terms of $\mathcal{T}$, then, equation (3.1.4) implies the superspace conservation laws:

$$
D_+\mathcal{T}_{++-} + D_-\mathcal{T}_{+++} = 0, \qquad\qquad D_+\mathcal{T}_{---} + D_-\mathcal{T}_{--+} = 0. \tag{3.1.6}
$$

We are now in a position to propose the supercurrent-squared deformation. Consider a one-parameter family of superspace Lagrangians labeled by t, which satisfy the ordinary differential equation

$$
\frac{\partial}{\partial t}\mathcal{A}^{(t)} = \mathcal{T}^{(t)}_{+++}\mathcal{T}^{(t)}_{---} - \mathcal{T}^{(t)}_{--+}\mathcal{T}^{(t)}_{++-}, \tag{3.1.7}
$$

where $\mathcal{T}^{(t)}$ is the supercurrent superfield (3.1.5) computed from the superspace Lagrangian $\mathcal{A}^{(t)}$. This uniquely defines the supercurrent-squared deformation of an initial Lagrangian $\mathcal{A}^{(0)}$ at finite deformation parameter t.

3.1.2 Reduction to components for a free theory

To illutrate the relationship between the flow equation (3.1.7) and the usual $T\overline{T}$ operator, let us explicitly compute the components of the supercurrent-squared deformation for a free

(1, 1) superspace Lagrangian

$$\mathcal{A} = D_+\Phi D_-\Phi, \tag{3.1.8}$$

where Φ is a superfield with component expansion

$$\Phi = \phi + i\theta^+\psi_+ + i\theta^-\psi_- + \theta^+\theta^- f. \tag{3.1.9}$$

The entries of $\mathcal{T}$, defined by (3.1.5), for the free theory are

$$\begin{aligned}
\mathcal{T}_{++-} &= \partial_{++}\Phi D_-\Phi - D_+\left(D_+\Phi D_-\Phi\right), \\
\mathcal{T}_{+++} &= -\partial_{++}\Phi D_+\Phi, \\
\mathcal{T}_{---} &= \partial_{--}\Phi D_-\Phi, \\
\mathcal{T}_{--+} &= -\partial_{--}\Phi D_+\Phi - D_-\left(D_+\Phi D_-\Phi\right).
\end{aligned} \tag{3.1.10}$$

In components, (3.1.10) is

$$\begin{aligned}
\mathcal{T}_{++-} =& -i\psi_+ f + \theta^+\left(-f\partial_{++}\phi + \psi_+\partial_{++}\psi_-\right) + \theta^-\left(-f^2 - \psi_+\partial_{--}\psi_+\right) \\
&+ i\theta^+\theta^-\left(-\partial_{++}\phi\partial_{--}\psi_+ - \partial_{++}\psi_+\partial_{--}\phi - f\partial_{++}\psi_- + \psi_-\partial_{++}f + \partial_{++}\left(\psi_+\partial_{--}\phi - \psi_- f\right)\right), \\
\mathcal{T}_{+++} =& -i\psi_+\partial_{++}\phi - \theta^+\left(\psi_+\partial_{++}\psi_+ + \left(\partial_{++}\phi\right)^2\right) - \theta^-\left(f\partial_{++}\phi + \psi_+\partial_{++}\psi_-\right) \\
&- i\theta^+\theta^-\left(2\partial_{++}\phi\partial_{++}\psi_- + \psi_+\partial_{++}f - f\partial_{++}\psi_+\right), \\
\mathcal{T}_{---} =& i\psi_-\partial_{--}\phi + \theta^+\left(\psi_-\partial_{--}\psi_+ - f\partial_{--}\phi\right) + \theta^-\left(\psi_-\partial_{--}\psi_- + \left(\partial_{--}\phi\right)^2\right) \\
&+ i\theta^+\theta^-\left(\psi_-\partial_{--}f - f\partial_{--}\psi_- - 2\partial_{--}\phi\partial_{--}\psi_+\right), \\
\mathcal{T}_{--+} =& -i\psi_- f + \theta^+\left(f^2 + \psi_-\partial_{++}\psi_-\right) + \theta^-\left(-f\partial_{--}\phi - \psi_-\partial_{--}\psi_+\right) \\
&+ i\theta^+\theta^-\left(-\partial_{--}\phi\partial_{++}\psi_- + f\partial_{--}\psi_+ - \partial_{--}\psi_-\partial_{++}\phi - \psi_+\partial_{--}f + \partial_{--}\left(\psi_+ f + \psi_-\partial_{++}\phi\right)\right).
\end{aligned} \tag{3.1.11}$$

To compare with the bosonic $T\overline{T}$ deformation, we identify the components of the usual stress tensor T for the theory of a free boson ϕ and fermions $\psi_\pm$ which one obtains by performing the integrals over $\theta^\pm$. In our conventions, these take the form:

$$\begin{aligned}
T_{++++} &= \left(\partial_{++}\phi\right)^2 + \psi_+\partial_{++}\psi_+, \\
T_{----} &= \left(\partial_{--}\phi\right)^2 + \psi_-\partial_{--}\psi_-.
\end{aligned} \tag{3.1.12}$$

We will also drop terms involving the auxiliary field f, since in the bosonic part of the supercurrent-squared deformation, these terms vanish after integrating out f using its equation of motion. Then the bilinears appearing in our flow equation (3.1.7) are

$$
\begin{aligned}
\mathcal{T}_{+++}\mathcal{T}_{---} &= \psi_+\psi_-\partial_{++}\phi\partial_{--}\phi + i\theta^+ \left(\psi_+\psi_-\partial_{++}\phi\partial_{--}\psi_+ - T_{++++}\psi_-\partial_{--}\phi\right) \\
&\quad + i\theta^- \left(\psi_+\partial_{++}\phi T_{----} + \psi_+\psi_-\partial_{++}\psi_-\partial_{--}\phi\right) - \theta^+\theta^- \big(T_{++++}T_{----} \\
&\quad + 2\partial_{++}\phi\partial_{--}\phi \left(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\right) - \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+\big), \\
\mathcal{T}_{++-}\mathcal{T}_{--+} &= -2\theta^+\theta^- \left(\psi_+\partial_{--}\psi_+\psi_-\partial_{++}\psi_-\right).
\end{aligned}
\tag{3.1.13}
$$

The superspace integral of the deformation $\mathcal{T}_{+++}\mathcal{T}_{---} + \mathcal{T}_{++-}\mathcal{T}_{--+}$ picks out the top component, which is

$$
\begin{aligned}
&\int d^2\theta \left(\mathcal{T}_{+++}\mathcal{T}_{---} + \mathcal{T}_{++-}\mathcal{T}_{--+}\right) = \\
&\quad - T_{++++}T_{----} - 2\partial_{++}\phi\partial_{--}\phi \left(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\right) - \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+.
\end{aligned}
\tag{3.1.14}
$$

We see that (3.1.14) contains the usual $T\overline{T}$ deformation, given in our bi-spinor notation by $-T_{++++}T_{----}$, along with extra terms which are all proportional to the fermion equations of motion, $\partial_{\pm\pm}\psi_{\mp} = 0$. These added terms vanish on-shell, which means that the supercurrent-squared deformation is on-shell equivalent to the usual bosonic $T\overline{T}$ deformation. In particular, this means that these additional terms do not affect the energy levels of the theory deformed by supercurrent-squared, which means that the same inviscid Burgers' relation between the deformed and undeformed energy levels holds for our supersymmetric deformation as for the ordinary $T\overline{T}$ flow. An explicit check that the energy levels are unaffected can be found in [5].

3.1.3 Relationship with the $\mathcal{S}$-multiplet

The $(1,1)$ superfield $\mathcal{T}$ contains the conserved stress-energy tensor $T_{\mu\nu}$ and the supercurrent $S_{\mu\alpha}$. Such current multiplets have received much attention in the literature; the first construction for four-dimensional theories was the FZ multiplet [13], which was later shown to be a special case of the more general $\mathcal{S}$-multiplet [14].

For the two-dimensional theories we consider here, it is known that the $\mathcal{S}$-multiplet is the most general multiplet containing the stress tensor and supercurrent, subject to assumptions that the multiplet be indecomposable and contain no other operators with spin greater than one. Since our supercurrent superfield $\mathcal{T}$ satisfies these properties, it must be equivalent to the $\mathcal{S}$-multiplet. As we will show, the four superfields contained in $\mathcal{T}$ are identical to the four superfields of the $\mathcal{S}$-multiplet, up to terms which vanish on-shell and therefore do not affect the conservation equations for the currents.

The $\mathcal{S}$-multiplet is a reducible but indecomposable set of two superfields $\mathcal{S}$ and χ satisfying the constraints

$$\begin{aligned} D_{\mp}\mathcal{S}_{\pm\pm\pm} &= D_{\pm}\chi_{\pm}, \\ D_{-}\chi_{+} &= D_{+}\chi_{-}. \end{aligned} \tag{3.1.15}$$

In components, the $\mathcal{S}$-multiplet for $(1,1)$ theories contains the usual stress tensor $T_{\mu\nu}$, the supercurrent $S_{\mu\alpha}$, and a vector Z_{μ} which is associated with a scalar central charge:

$$\begin{aligned} \mathcal{S}_{+++} &= S_{+++} + \theta^{+}T_{++++} + \theta^{-}Z_{++} - \theta^{+}\theta^{-}\partial_{++}S_{-++}, \\ \mathcal{S}_{---} &= S_{---} + \theta^{+}Z_{--} + \theta^{-}T_{----} + \theta^{+}\theta^{-}\partial_{--}S_{+--}, \\ \chi_{+} &= S_{-++} + \theta^{+}Z_{++} + \theta^{-}T_{++--} - \theta^{+}\theta^{-}\partial_{++}S_{+--}, \\ \chi_{-} &= S_{+--} + \theta^{+}T_{++--} + \theta^{-}Z_{--} + \theta^{+}\theta^{-}\partial_{--}S_{-++}. \end{aligned} \tag{3.1.16}$$

In terms of these component fields, the constraints (3.1.15) give conservation equations for the currents:

$$\begin{aligned} \partial_{++}T_{----} + \partial_{--}T_{++--} &= 0 = \partial_{++}T_{--++} + \partial_{--}T_{++++}, \\ \partial_{++}S_{+--} + \partial_{--}S_{+++} &= 0 = \partial_{++}S_{---} + \partial_{--}S_{-++}, \\ \partial_{++}Z_{--} + \partial_{--}Z_{++} &= 0. \end{aligned} \tag{3.1.17}$$

We claim that the components (3.1.11) of our superspace supercurrent are the same as those in the two superfields $\mathcal{S}$ and χ appearing in the $(1,1)$ $\mathcal{S}$-multiplet (3.1.16), up to signs and terms which vanish on-shell. In particular, after discarding terms which are proportional to the equations of motion, we find the identifications:

$$\mathcal{S}_{\pm\pm\pm} = \mp\mathcal{T}_{\pm\pm\pm}, \qquad \chi_{+} = \mathcal{T}_{++-}, \qquad \chi_{-} = \mathcal{T}_{--+}. \tag{3.1.18}$$

We will check this explicitly for the free theory, $\mathcal{A} = D_{+}\Phi D_{-}\Phi$, for which we computed the components of $\mathcal{T}$ in section (3.1.2). Writing only those terms that survive when the component equations of motion $f = 0$, $\partial_{++}\psi_{-} = 0 = \partial_{--}\psi_{+}$, and $\partial_{++}\partial_{--}\phi = 0$ are all satisfied, (3.1.11) becomes

$$\begin{aligned}
\mathcal{T}_{++-} &\overset{\text{on-shell}}{=} 0, \\
\mathcal{T}_{+++} &\overset{\text{on-shell}}{=} -i\psi_+\partial_{++}\phi - \theta^+ \left(\psi_+\partial_{++}\psi_+ + (\partial_{++}\phi)^2\right), \\
\mathcal{T}_{---} &\overset{\text{on-shell}}{=} i\psi_-\partial_{--}\phi + \theta^- \left(\psi_-\partial_{--}\psi_- + (\partial_{--}\phi)^2\right), \\
\mathcal{T}_{--+} &\overset{\text{on-shell}}{=} 0.
\end{aligned} \tag{3.1.19}$$

For the free $(1,1)$ superfield considered here, the supercurrent is given in our conventions by

$$\begin{aligned}
S_{+++} &= \psi_+\partial_{++}\phi, \\
S_{---} &= \psi_-\partial_{--}\phi, \\
S_{+--} &= 0 = S_{-++},
\end{aligned} \tag{3.1.20}$$

while the stress tensor components are as in (3.3.12). To find expressions for the scalar central charge current $Z_{\pm\pm}$, we use the supersymmetry algebra implied by the $\mathcal{S}$-multiplet constraints, which gives

$$\begin{aligned}
\{Q_\pm, S_{\pm\pm\pm}\} &= T_{\pm\pm\pm\pm}, \\
\{Q_\pm, S_{\pm\mp\mp}\} &= T_{\pm\pm\mp\mp}, \\
\{Q_\pm, S_{\mp\pm\pm}\} &= Z_{\pm\pm}, \\
\{Q_\pm, S_{\mp\mp\mp}\} &= Z_{\mp\mp}.
\end{aligned} \tag{3.1.21}$$

Note that the $\mathcal{S}$-multiplet constraints only hold when the conservation equations for the currents hold, so the relations (3.1.21) should be viewed as an on-shell algebra. Acting with the supercharges $Q_\pm$ on the stress tensor and supercurrent components, one finds that $Z_{--} \sim \psi_-\partial_{--}\psi_+$ and $Z_{++} \sim \psi_+\partial_{++}\psi_-$, both of which vanish when the fermion equations of motion are satisfied.

Thus, after imposing the equations of motion, we can write our supercurrent superfield components as

$$\begin{aligned}
&\mathcal{T}_{++-} = \chi_+ = 0, && \mathcal{T}_{--+} = 0 = \chi_-, \\
&\mathcal{T}_{+++} = -S_{+++} - \theta^+ T_{++++} = -\mathcal{S}_{+++}, && \mathcal{T}_{---} = S_{---} + \theta^- T_{----} = \mathcal{S}_{---}.
\end{aligned} \tag{3.1.22}$$

Since terms which vanish on-shell do not affect conservation equations, one can view $\mathcal{T}$ as an improvement transformation of the $\mathcal{S}$-multiplet. The constraint equation $D_\mp\mathcal{S}_{\pm\pm\pm} -$

$D_\pm\chi_\pm = 0$ is expressed by our conservation equations $D_+\mathcal{T}_{++-} + D_-\mathcal{T}_{+++} = 0$ and $D_+\mathcal{T}_{---} + D_-\mathcal{T}_{+--} = 0$.

3.2 Theories with (1, 1) Supersymmetry

In this section, we consider the supercurrent-squared deformation of a theory involving a single $(1,1)$ superfield Φ, both in the free case and with a superpotential.

3.2.1 Free $(1,1)$ *superfield*

First consider an undeformed superspace Lagrangian $\mathcal{A}^{(0)} = D_+\Phi D_-\Phi$. We make the following ansatz for the deformed Lagrangian at finite t:

$$\mathcal{A}^{(t)} = F\left(t\partial_{++}\Phi\partial_{--}\Phi,\, t\,(D_+D_-\Phi)^2\right) D_+\Phi D_-\Phi. \tag{3.2.1}$$

Here F may only depend on the two dimensionless combinations which we define by

$$x = t\,\partial_{++}\Phi\partial_{--}\Phi, \qquad y = t\,(D_+D_-\Phi)^2\,. \tag{3.2.2}$$

In order to reduce to the free theory as $t \to 0$, we must also impose the boundary condition $F(0,0) = 1$.

After computing the components of the supercurrent-squared deformation and simplifying, the flow equation (3.1.7) yields

$$\begin{aligned}\frac{\partial}{\partial t}F &= \left((D_+D_-\Phi)^2 - \partial_{++}\Phi\partial_{--}\Phi\right)F^2 \\ &\quad - 2F\,(\partial_{++}\Phi\partial_{--}\Phi)\left(\partial_{++}\Phi\partial_{--}\Phi + (D_+D_-\Phi)^2\right)\frac{\partial F}{\partial x}.\end{aligned} \tag{3.2.3}$$

In terms of the dimensionless variables x and y, equation (3.2.3) becomes

$$\frac{\partial F}{\partial x}x + \frac{\partial F}{\partial y}y = (y-x)F^2 - 2F\frac{\partial F}{\partial x}x(x+y). \tag{3.2.4}$$

Supplemented with the boundary condition $F(0,0) = 1$, the partial differential equation (3.2.3) uniquely determines the deformed Lagrangian at finite t.

As a check, we would like to verify that the bosonic structure of the solution to (3.2.3) reduces to the known results for the $T\overline{T}$-deformed theory of a free boson. We will argue that, in fact, it suffices to set $y = 0$ in (3.2.3) and note that the result agrees with the flow equation obtained in the purely bosonic case [15].

Indeed, let us write the components of the superfield Φ as $\Phi = \phi + i\theta^+\psi_+ + i\theta^-\psi_- + \theta^+\theta^- f$. To probe the bosonic structure, it suffices to set $\psi_\pm = 0$, perform the superspace integration, and then integrate out the auxiliary field f using its equation of motion. Thus consider an arbitrary superspace integral of the form

$$\mathcal{L}^{(t)} = \int d^2\theta F^{(t)}(x,y) D_+\Phi D_-\Phi. \tag{3.2.5}$$

The lowest component of the superfield $y = tD_+\Phi D_-\Phi$ is $-f$, and the higher components will not contribute to the bosonic part because they come multiplying $D_+\Phi D_-\Phi$, which is already proportional to $\theta^+\theta^-$ after setting the fermions to zero.

Thus the purely bosonic piece of the physical Lagrangian associated with a superspace Lagrangian $\mathcal{A}^{(t)} = F^{(t)}(x,y)D_+\Phi D_-\Phi$ is

$$\mathcal{L}^{(t)} = F^{(t)}\left(t\partial_{++}\phi\partial_{--}\phi, tf^2\right)\left(f^2 + 4\partial_{++}\phi\partial_{--}\phi\right). \tag{3.2.6}$$

The equation of motion for the auxiliary field f is

$$2tf\frac{\partial F}{\partial y}\left(f^2 + 4\partial_{++}\phi\partial_{--}\phi\right) + 2fF = 0, \tag{3.2.7}$$

which admits the solution $f = 0$. The Lagrangian for the bosonic field ϕ is then

$$\mathcal{L}^{(t)} = 4F^{(t)}\left(t\partial_{++}\phi\partial_{--}\phi, 0\right)\partial_{++}\phi\partial_{--}\phi. \tag{3.2.8}$$

Therefore, to determine the terms in the Lagrangian which involve only ϕ, we may solve the simpler partial differential equation

$$\frac{\partial F}{\partial x}x = -xF^2 - 2Fx^2\frac{\partial F}{\partial x}, \tag{3.2.9}$$

which holds upon setting $y = 0$ in (3.2.4). But this is precisely the equation discussed in section 2.1, whose solution is equation (2.1.25):

$$\mathcal{L}^{(t)} = \frac{\sqrt{1 + 4t\partial_{++}\phi\partial_{--}\phi} - 1}{2t}. \tag{3.2.10}$$

We see that the supercurrent-squared deformation of the free superfield is indeed a generalization of the $T\overline{T}$ deformation of a free boson, in the sense that it yields the same modification to the purely bosonic terms in the action but also includes additional terms which affect only the fermions.

3.2.2 Interacting $(1,1)$ *superfield*

Next, we consider the case with a superpotential: that is, we begin from the undeformed superspace Lagrangian

$$\mathcal{A}^{(0)} = D_+\Phi D_-\Phi + h(\Phi), \tag{3.2.11}$$

where $h(\Phi)$ is an arbitrary function (it need not give rise to a theory with infinitely many integrals of motion). After performing the superspace integral, the physical Lagrangian is

$$\mathcal{L}^{(0)} = \int d^2\theta\, \mathcal{A}^{(0)} = \partial_{++}\phi\partial_{--}\phi + \psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_- + f^2 + h'(\phi)f. \tag{3.2.12}$$

Integrating out the auxiliary field using its equation of motion $f = -\frac{1}{2}h'(\phi)$, we see that the physical potential V is given by $V = -\frac{1}{4}h'(\phi)^2$.

We might expect that both the kinetic and potential terms are modified by a finite supercurrent-squared deformation, which would lead us to make the ansatz

$$\mathcal{A}^{(t)} = F(x,y)D_+\Phi D_-\Phi + G(t,\Phi), \tag{3.2.13}$$

where G is a new function to be determined, and $x = t\partial_{++}\Phi\partial_{--}\Phi$, $y = t\left(D_+D_-\Phi\right)^2$ as above. However, the deformation does not induce any change in the potential h, so in fact we may put $G = h$ for all t. To see this, we can write down the supercurrent-squared deformation associated with the ansatz (3.2.13), which gives

$$\begin{aligned} &\frac{\partial}{\partial t}F(x,y)D_+\Phi D_-\Phi + \frac{\partial}{\partial t}G(t,\Phi) = \\ &\frac{1}{t}\left((y-x)\,F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(G'\right)^2 + 2G'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right) - 2\sqrt{y}xG'\frac{\partial F}{\partial y}\right)D_+\Phi D_-\Phi. \end{aligned} \tag{3.2.14}$$

The details of the calculation leading to (3.2.14) are discussed in Appendix A.1. We see that deformation is proportional to $D_+\Phi D_-\Phi$, so it does not source any change in the potential $h(\Phi)$; thus we may take $G(h,\Phi) = h(\Phi)$ in our ansatz. This leaves us with a single partial differential equation for F, namely

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = (y-x)\,F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(h'\right)^2 + 2h'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right) - 2\sqrt{y}xh'\frac{\partial F}{\partial y}. \tag{3.2.15}$$

In the second line, we have used the constraint that F can depend only on the dimensionless combinations $x = t\partial_{++}\Phi\partial_{--}\Phi$ and $y = t\,(D_+D_-\Phi)^2$.

As in the free case, we would like to study the purely bosonic terms in the physical Lagrangian resulting from (3.2.15) and compare them to known results. Here the auxiliary will play a more important role since $f = 0$ is no longer a solution.

We can expand both the Lagrangian $\mathcal{L} = \int d^2\theta\,(F(x,y)D_+\Phi D_-\Phi + h(\Phi))$ and the auxiliary field f as power series in t:

$$\mathcal{L} = \sum_{j=0}^{\infty} t^j \mathcal{L}^{(j)}, \qquad f = \sum_{j=0}^{\infty} t^j f^{(j)}, \tag{3.2.16}$$

and then integrate out the auxiliary order-by-order in t. Doing so to order t^3, we arrive at

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}h'(\phi)^2 + \frac{x}{t} + t\left(\frac{1}{16}h'(\phi)^4 - \left(\frac{x}{t}\right)^2\right) + t^2\left(-\frac{1}{4}\left(\frac{x}{t}\right)^2 h'(\phi)^2 - \frac{1}{64}h'(\phi)^6 + 2\left(\frac{x}{t}\right)^3\right)\\ &+ t^3\left(\left(\frac{x}{t}\right)^3 h'(\phi)^2 + \frac{1}{256}h'(\phi)^8 - 5\left(\frac{x}{t}\right)^4\right) + \mathcal{O}(t^4),\end{aligned} \tag{3.2.17}$$

after setting the fermions to zero. Up to conventions, this matches the Taylor expansion of the known result [15, 16] for the $T\overline{T}$ deformation of a boson with a generic potential V, which is given in our conventions as

$$\mathcal{L}^{(t)} = -\frac{1}{2t}\frac{1-2tV}{1-tV} + \frac{1}{2t}\sqrt{\frac{t\,(4V + \partial_{++}\phi\partial_{--}\phi)}{1-tV} + \frac{(1-2tV)^2}{(1-tV)^2}}. \tag{3.2.18}$$

Again the physical potential V is related to h via $V = -\frac{1}{4}h'(\phi)^2$. We have checked explicitly that the bosonic part of the series solution to the PDE (3.2.15) matches the Taylor expansion of (3.2.18) up to $\mathcal{O}(t^7)$.

3.3 Theories with $(0, 1)$ Supersymmetry

In this section we study the deformation of a theory with chiral $(0, 1)$ supersymmetry; a $(0, 1)$ scalar superfield Φ consists of a scalar and a real fermion, $\Phi = \phi + i\theta^+\psi_+$. The Lagrangian in superspace is a function of $D_+\Phi$, $\partial_{++}\Phi$, $\partial_{--}\Phi$, as well as Φ itself.

3.3.1 Free $(0,1)$ *superfield*

The free theory is defined by the Lagrangian,

$$\begin{aligned} \mathcal{L} &= \int d\theta^+ \; D_+\Phi\partial_{--}\Phi, \\ &= \partial_{++}\phi\partial_{--}\phi + \psi_+\partial_{--}\psi_+. \end{aligned} \tag{3.3.1}$$

Following the approach of section 3.1.1, we first look for conservation laws for a given superspace Lagrangian $\mathcal{A}$. They take the form,

$$\begin{aligned} \partial_{--}\mathcal{S}_{+++} + D_+\mathcal{T}_{++--} &= 0, \\ \partial_{--}\mathcal{S}_{--+} + D_+\mathcal{T}_{----} &= 0, \end{aligned} \tag{3.3.2}$$

where $\mathcal{S}_{\pm\pm+}$ and $\mathcal{T}_{\pm\pm--}$ are superfields given by:

$$\begin{aligned} \mathcal{S}_{+++} &= \frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\partial_{++}\Phi, \\ \mathcal{S}_{--+} &= \frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\partial_{--}\Phi - \mathcal{A}, \\ \mathcal{T}_{++--} &= \frac{\delta\mathcal{A}}{\delta D_+\Phi}\partial_{++}\Phi + D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\partial_{++}\Phi\right) - D_+\mathcal{A}, \\ \mathcal{T}_{----} &= \frac{\delta\mathcal{A}}{\delta D_+\Phi}\partial_{--}\Phi + D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\partial_{--}\Phi\right). \end{aligned} \tag{3.3.3}$$

We define the supercurrent-squared deformation as follows:

$$\frac{\partial}{\partial t}\mathcal{A}^{(t)} = \mathcal{S}_{+++}\mathcal{T}_{----} - \mathcal{S}_{--+}\mathcal{T}_{++--}. \tag{3.3.4}$$

To understand what the deformation (3.3.4) does to a $(0,1)$ theory, consider an undeformed Lagrangian in superspace

$$\mathcal{A}^{(0)} = g(\Phi)D_+\Phi\partial_{--}\Phi, \tag{3.3.5}$$

where $g(\Phi)$ is an arbitrary differentiable function of the superfield. A free theory corresponds to a constant $g(\Phi)$. To find the deformed theory $\mathcal{A}^{(t)}$, we first make a general ansatz for the deformed Lagrangian

$$\mathcal{A}^{(t)} = f(t\partial_{++}\Phi\partial_{--}\Phi)D_+\Phi\partial_{--}\Phi, \tag{3.3.6}$$

where $f(x)$ is some differentiable function. Using the expression for the supercurrents given in (3.3.3) and imposing the initial condition $f(x \to 0) = g(\Phi)$, we find the function $f(x)$ satisfies the same differential equation found in (3.2.9). Its solution is given by

$$f(x) = \frac{\sqrt{1 + 4xg(\Phi)} - 1}{2x}. \tag{3.3.7}$$

3.3.2 Reduction of $(1,1)$ *to* $(0,1)$

Any theory with $(1,1)$ global supersymmetry can also be viewed as a theory with $(0,1)$ global supersymmetry. Up to possible field redefinitions, we should therefore be able to relate the $(1,1)$ theory deformed by the supercurrent-squared deformation defined in (3.1.5) to the $(0,1)$ of (3.3.4), which we would have used if we had simply restricted to $(0,1)$ supersymmetry.

To be more precise, consider a $(1,1)$ theory whose physical Lagrangian $\mathcal{L}$ is given by the integral of a superspace Lagrangian $\mathcal{A}^{(1,1)}$ over $(1,1)$ superspace. We can also view this as a $(0,1)$ theory,

$$\mathcal{L} = \int d^2\theta \, \mathcal{A}^{(1,1)} = \int d\theta^+ \, \mathcal{A}^{(0,1)}. \tag{3.3.8}$$

The flow equation defining the supercurrent-squared deformation of $\mathcal{A}^{(1,1)}$ is $\partial_t \mathcal{A}^{(1,1)} = \mathcal{T}_{+++}\mathcal{T}_{---} - \mathcal{T}_{--+}\mathcal{T}_{++-}$. By performing the integral over θ^-, this induces a flow for $\mathcal{A}^{(0,1)}$ due to (3.3.8), namely

$$\frac{\partial}{\partial t}\mathcal{A}^{(0,1)} = \int d\theta^+ \, \left(\mathcal{T}_{+++}\mathcal{T}_{---} - \mathcal{T}_{--+}\mathcal{T}_{++-}\right). \tag{3.3.9}$$

For instance, let us consider the deformation of the free theory $\mathcal{A}^{(1,1)} = D_+\Phi^{(1,1)}D_-\Phi^{(1,1)}$. This can be written as an integral over $(0,1)$ superspace as

$$\begin{aligned}\int d^2\theta \, D_+\Phi^{(1,1)}D_-\Phi^{(1,1)} &= \int d\theta^+ \Big(- i\psi_+\partial_{--}\phi - i\psi_- f - \theta^+\big(f^2 + \partial_{++}\phi\partial_{--}\phi \\ &\qquad + \psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\big)\Big) \\ &= -\int d\theta^+ \, \Big(D_+\Phi^{(0,1)}\partial_{--}\Phi^{(0,1)} + \Psi_- D_+\Psi_-\Big). \end{aligned} \tag{3.3.10}$$

Here we have written the integrand on the right side of (3.3.10) as a superspace Lagrangian $\mathcal{A}^{(0,1)}\left(\Phi^{(0,1)}, \Psi_-\right)$ for a superfield $\Phi^{(0,1)} = \phi + i\theta^+\psi_+$ of the form discussed above, along with an extra Fermi superfield $\Psi_- = i\psi_- + \theta^+ f$:

$$\mathcal{A}^{(0,1)}\left(\Phi^{(0,1)}, \Psi_-\right) = D_+\Phi^{(0,1)}\partial_{--}\Phi^{(0,1)} + \Psi_- D_+\Psi_-. \tag{3.3.11}$$

For comparison, we compute the supercurrent-squared deformation to leading order in t; that is, we compute the tangent vector $\frac{\partial \mathcal{A}^{(1,1)}}{\partial t}|_{t=0}$ to the free theory along the flow and compare it to that of the free $(0,1)$ theory with an extra fermion.

The components of the supercurrent superfield associated with the free theory, after integrating out the auxiliary using $f = 0$, are given in equation (3.1.13). Using these and performing the integral over θ^-, the reduced flow equation (3.3.9) at $t = 0$ becomes

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}^{(0,1)}|_{t=0} = {}& i\left(\psi_+\partial_{++}\phi T_{----} + \psi_+\psi_-\partial_{++}\psi_-\partial_{--}\phi\right) + \theta^+\Big(T_{++++}T_{----} \\ & + 2\partial_{++}\phi\partial_{--}\phi\left(\psi_+\partial_{--}\psi_+ + \psi_-\partial_{++}\psi_-\right) + \psi_-\partial_{++}\psi_-\psi_+\partial_{--}\psi_+\Big),\end{aligned} \tag{3.3.12}$$

where we have used $T_{\pm\pm\pm\pm} = \left(\partial_{\pm\pm}\phi\right)^2 + \psi_\pm\partial_{\pm\pm}\psi_\pm$.

We know that the solution to (3.3.12) must represent a solvable deformation of the original $(0,1)$ theory because it descends from a solvable deformation in the parent $(1,1)$ theory. On the other hand, one can construct the flow equation (3.3.4) directly in the $(0,1)$ theory. This must also yield a solvable deformation since it is built out of currents which satisfy a superspace conservation equation of the same form as in the $(1,1)$ supercurrent multiplet. One might suspect that these two deformations should be the same, up to field redefinitions which do not affect the spectrum.

To check this, let us compare the leading-order deformations for these two cases in components. After including the contributions $\partial_{\pm\pm}\Psi_-\frac{\delta\mathcal{A}}{\delta D_\pm\Psi_-}$ to $\mathcal{T}_{\pm\pm--}$ due to the fermion

Ψ_-, the currents (3.3.3) for this theory are

$$\begin{aligned}
\mathcal{S}_{+++} &= D_+\Phi\partial_{++}\Phi \\
&= i\psi_+\partial_{++}\phi + \theta^+\left(\psi_+\partial_{++}\psi_+ + (\partial_{++}\phi)^2\right), \\
\mathcal{S}_{--+} &= -\Psi_- D_+\Psi_- \\
&= -i\psi_- f - \theta^+\left(f^2 + \psi_-\partial_{++}\psi_-\right), \\
\mathcal{T}_{++--} &= \partial_{--}\Phi\partial_{++}\Phi + \Psi_-\partial_{++}\Psi_- - D_+\left(D_+\Phi\partial_{--}\Phi + \Psi_- D_+\Psi_-\right) \\
&= -\psi_+\partial_{--}\psi_+ - f^2 + i\theta^+\left(\partial_{++}\phi\partial_{--}\psi_+ - \psi_+\partial_{++}\partial_{--}\phi\right), \\
\mathcal{T}_{----} &= (\partial_{--}\Phi)^2 - \Psi_-\partial_{--}\Psi_- \\
&= (\partial_{--}\phi)^2 + \psi_-\partial_{--}\psi_- + i\theta^+\left(-f\partial_{--}\psi_- + \psi_-\partial_{--}f + 2\partial_{--}\phi\partial_{--}\psi_+\right).
\end{aligned} \tag{3.3.13}$$

The bilinears appearing in the $(0,1)$ deformation are

$$\begin{aligned}
\mathcal{S}_{+++}\mathcal{T}_{----} &= i\psi_+\partial_{++}\phi T_{----} + \theta^+\big(T_{++++}T_{----} \\
&\qquad + \psi_+\partial_{++}\phi\left(\psi_-\partial_{--}f - f\partial_{--}\psi_- + 2\partial_{--}\phi\partial_{--}\psi_+\right)\big), \\
\mathcal{S}_{--+}\mathcal{T}_{++--} &= i\psi_- f\left(\psi_+\partial_{--}\psi_+ + f^2\right) + \theta^+\big(\left(f^2 + \psi_-\partial_{++}\psi_-\right)\left(f^2 + \psi_+\partial_{--}\psi_+\right) \\
&\qquad + \psi_- f\left(\psi_+\partial_{++}\partial_{--}\phi - \partial_{++}\phi\partial_{--}\psi_+\right)\big),
\end{aligned} \tag{3.3.14}$$

and thus the $(0,1)$ flow equation at $t=0$ is given by

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{A}^{(0,1)}|_{t=0} &= \mathcal{S}_{+++}\mathcal{T}_{----} - \mathcal{S}_{--+}\mathcal{T}_{++--} \\
&= i\psi_+\partial_{++}\phi T_{----} - i\psi_- f\left(\psi_+\partial_{--}\psi_+ + f^2\right) \\
&+ \theta^+\Big(T_{++++}T_{----} + \psi_+\partial_{++}\phi\left(\psi_-\partial_{--}f - f\partial_{--}\psi_- + 2\partial_{--}\phi\partial_{--}\psi_+\right) \\
&- \left(f^2 + \psi_-\partial_{++}\psi_-\right)\left(f^2 + \psi_+\partial_{--}\psi_+\right) - \psi_- f\left(\psi_+\partial_{++}\partial_{--}\phi - \partial_{++}\phi\partial_{--}\psi_+\right)\Big).
\end{aligned} \tag{3.3.15}$$

The deformations (3.3.12) and (3.3.15) agree up to terms proportional to the equations of motion $f = 0$ and $\partial_{++}\psi_- = 0$. At this order in t, such terms can be removed by making a field redefinition involving f and ψ_-. If this can be repeated order-by-order in t, as we suspect should be the case, then the two flows are genuinely equivalent and give rise to deformed theories with the same energies.

CHAPTER 4
$(2,2)$ SUPERCURRENT-SQUARED DEFORMATIONS

In this chapter, we will extend the supercurrent-squared deformation of Chapter 3 to theories with more supersymmetry, focusing on the case of $(2,2)$ theories. Our discussion is based on the paper "$T\overline{T}$ Flows and $(2,2)$ Supersymmetry" [6].

4.1 $D = 2$ $\mathcal{N} = (2,2)$ Supercurrent Multiplets

Our manifestly supersymmetric modification of $T\overline{T}$ is built from bilinears in fields of the supercurrent multiplet. In this section we review the structure of such multiplets in $D = 2$ $\mathcal{N} = (2,2)$ theories.

4.1.1 The $\mathcal{S}$-multiplet

For Lorentz invariant supersymmetric theories, there is an essentially unique supermultiplet which contains the stress-energy tensor $T_{\mu\nu}$, the supercurrent $S_{\mu\alpha}$, and no other operators with spin larger than one, under the assumption that the multiplet, though in general reducible, cannot be separated into decoupled supersymmetry multiplets; namely that it is indecomposable [17]. This $\mathcal{S}$-multiplet can be defined in any theory with $D = 2$ $\mathcal{N} = (2,2)$ supersymmetry. By "essentially unique," we mean that the $\mathcal{S}$-multiplet is unique up to improvement terms which preserve the superspace constraint equations.

For two-dimensional theories with $(2,2)$ supersymmetry, the $\mathcal{S}$-multiplet consists of superfields $\mathcal{S}_{\pm\pm}$, $\chi_\pm$, and $\mathcal{Y}_\pm$ which satisfy the constraints:

$$\overline{D}_\pm \mathcal{S}_{\mp\mp} = \pm\left(\chi_\mp + \mathcal{Y}_\mp\right) \ , \tag{4.1.1a}$$

$$\overline{D}_\pm \chi_\pm = 0 \ , \qquad \overline{D}_\pm \chi_\mp = \pm C^{(\pm)} \ , \qquad D_+\chi_- - \overline{D}_-\overline{\chi}_+ = k \ , \tag{4.1.1b}$$

$$D_\pm \mathcal{Y}_\pm = 0 \ , \qquad \overline{D}_\pm \mathcal{Y}_\mp = \mp C^{(\pm)} \ , \qquad D_+\mathcal{Y}_- + D_-\mathcal{Y}_+ = k' \ . \tag{4.1.1c}$$

Here k and k' are real constants and $C^{(\pm)}$ is a complex constant. The $\mathcal{S}$-multiplet contains 8+ 8 independent real component operators and the constants $k, k', C^{(\pm)}$ [17]. The expansion in components of $\mathcal{S}_{\pm\pm}$, $\chi_\pm$, and $\mathcal{Y}_\pm$ are given for convenience in Appendix A.3.

Among the various component fields it is important to single out the complex supersymmetry current $S_{\alpha\mu}$ and the energy-momentum tensor $T_{\mu\nu}$. The complex supersymmetry current, associated to $S_{+\pm\pm}$ and $S_{-\pm\pm}$, is conserved: $\partial^\mu S_{\alpha\mu} = 0$. The energy-momentum tensor, associated with $T_{\pm\pm\pm\pm}$ and $T_{++--} = T_{--++}$, is real, conserved ($\partial^\mu T_{\mu\nu} = 0$), and symmetric ($T_{\mu\nu} = T_{\nu\mu}$). In light-cone notation the conservation equations are given by

$$\begin{aligned}
\partial_{++}S_{+--}(x) &= -\partial_{--}S_{+++}(x)\,, && \text{(4.1.2a)}\\
\partial_{++}\bar{S}_{+--}(x) &= -\partial_{--}\bar{S}_{+++}(x)\,, && \text{(4.1.2b)}\\
\partial_{++}T_{----}(x) &= -\partial_{--}\Theta(x)\,, && \text{(4.1.2c)}\\
\partial_{++}\Theta(x) &= -\partial_{--}T_{++++}(x)\,, && \text{(4.1.2d)}
\end{aligned}$$

where we have defined as usual

$$\Theta(x) := T_{++--}(x) = T_{--++}(x)\ . \qquad (4.1.3)$$

To conclude this subsection, let us describe the ambiguity in the form of the $\mathcal{S}$-multiplet which is parametrized by a choice of improvement term. If $\mathcal{U}$ is a real superfield, we are free to modify the $\mathcal{S}$-multiplet superfields as follows

$$\begin{aligned}
\mathcal{S}_{\pm\pm} &\to \mathcal{S}_{\pm\pm} + [D_\pm, \overline{D}_\pm]\mathcal{U}\ , && \text{(4.1.4a)}\\
\chi_\pm &\to \chi_\pm - \overline{D}_+\overline{D}_-D_\pm\mathcal{U}\ , && \text{(4.1.4b)}\\
\mathcal{Y}_\pm &\to \mathcal{Y}_\pm - D_\pm\overline{D}_+\overline{D}_-\mathcal{U}\ , && \text{(4.1.4c)}
\end{aligned}$$

which keeps invariant the conservation equations (4.1.1). In general the $\mathcal{S}$-multiplet is a reducible representation of supersymmetry and some of its component can consistently be set to zero by a choice of improvement. The reduced Ferrara-Zumino supercurrent multiplet, which plays a central role in this chapter, is described next.

4.1.2 The Ferrara-Zumino (FZ) multiplet and old-minimal supergravity

If there exists a well-defined superfield $\mathcal{U}$ such that $\chi_\pm = \overline{D}_+\overline{D}_-D_\pm\mathcal{U}$, then we may use the transformation (4.1.4) to set $\chi_\pm = 0$ in the $\mathcal{S}$-multiplet. If in addition $k = C^{(\pm)} = 0$, then we will rename $\mathcal{S}_{\pm\pm}$ to $\mathcal{J}_{\pm\pm}$; then the fields $\mathcal{J}_{\pm\pm}$ and $\mathcal{Y}_\pm$ satisfy the constraints

$$\begin{aligned}
\overline{D}_\pm\mathcal{J}_{\mp\mp} &= \pm\mathcal{Y}_\mp\ , && \text{(4.1.5a)}\\
D_\pm\mathcal{Y}_\pm &= 0\ , && \text{(4.1.5b)}\\
\overline{D}_\pm\mathcal{Y}_\mp &= 0\ , && \text{(4.1.5c)}\\
D_+\mathcal{Y}_- + D_-\mathcal{Y}_+ &= k'\ . && \text{(4.1.5d)}
\end{aligned}$$

These are the defining equations for the Ferrara-Zumino (FZ) multiplet $(\mathcal{J}_{\pm\pm}, \mathcal{Y}_\pm)$. In this case, $\mathcal{J}_{\pm\pm}$ turns out to be associated to the axial $U(1)_A$ R-symmetry current and satisfies the conservation equation

$$\partial_{--}\mathcal{J}_{++} - \partial_{++}\mathcal{J}_{--} = 0 \ . \tag{4.1.6}$$

This multiplet, which has $4+4$ real components, is the dimensionally-reduced version of the $D = 4$ $\mathcal{N} = 1$ FZ-multiplet [13]; see Appendix A.3 for more details. All of the models we consider in section 4.3 have the property that $\chi_\pm$ can be improved to zero; that is, they all have a well-defined FZ-multiplet.

In Chapter 3, we obtained the components of the supercurrent superfield by finding the Noether currents associated with translations in superspace. One could use a similar Noether procedure in the case $(2,2)$ superspace, as is done for $4D\,\mathcal{N} = 1$ in [18]. However, we will find it more convenient to avoid the Noether procedure in this chapter and instead use the supersymmetrized version of the Hilbert definition of the stress tensor. Just as the bosonic Hilbert stress tensor $T_{\mu\nu}$ represents the response function of the Lagrangian to a linearized perturbation $h_{\mu\nu}$ of the metric, the supercurrent multiplets correspond to linearized couplings to supergravity. Therefore, we can compute the components of the supercurrent by coupling our $(2,2)$ theories to one of the formulations of $2D$ supergravity, which we now briefly review.

Different formulations of off-shell supergravity couple to different supercurrent multiplets. If a theory has a well-defined FZ-multiplet, as is the case for all the examples found in section 4.3, then the theory can be consistently coupled to the old-minimal supergravity prepotentials $H^{\pm\pm}$ and σ. The nomenclature "old-minimal" is again inherited from $D = 4$ $\mathcal{N} = 1$ supergravity; see [19, 20] for pedagogical reviews and references. Here $H^{\pm\pm}$ is the conformal supergravity prepotential—the analogue of the traceless part of the metric—and σ is a chiral conformal compensator.

We refer the reader to [21–25] and references therein for an exhaustive description of $D = 2$ $\mathcal{N} = (2,2)$ off-shell supergravity in superspace, which we will use in our analysis; see also Appendix A.4. For the scope of this work, it will be enough to know the structure of linearized old-minimal supergravity. For instance, at the linearized level the gauge symmetry of the supergravity prepotentials $H^{\pm\pm}$, σ and $\bar{\sigma}$, can be parameterized as follows

$$\delta H^{++} = \frac{i}{2}\left(\overline{D}_- L^+ - D_-\overline{L}^+\right) \ , \tag{4.1.7a}$$

$$\delta H^{--} = \frac{i}{2}\left(\overline{D}_+ L^- - D_+\overline{L}^-\right) \ , \tag{4.1.7b}$$

$$\delta\sigma = -\frac{i}{2}\overline{D}_+\overline{D}_-\left(D_+ L^+ - D_- L^-\right) \ , \tag{4.1.7c}$$

$$\delta\bar{\sigma} = -\frac{i}{2}D_- D_+\left(\overline{D}_+\overline{L}^+ - \overline{D}_-\overline{L}^-\right) \ , \tag{4.1.7d}$$

in terms of unconstrained spinor superfields $L^\pm$ and their complex conjugates.

The conservation law (4.1.5) for the FZ-multiplet can be derived by using the previous gauge transformations. The linearized supergravity couplings for a given model are written as

$$\mathcal{L}_{\text{linear}} = \int d^4\theta \, \left(H^{++}\mathcal{J}_{++} + H^{--}\mathcal{J}_{--}\right) - \int d^2\theta \, \sigma \, \mathcal{V} - \int d^2\bar{\theta} \, \bar{\sigma} \, \overline{\mathcal{V}} \, , \tag{4.1.8}$$

with $\mathcal{V}$ a chiral superfield and $\overline{\mathcal{V}}$ its complex conjugate. Assuming the matter superfields satisfy their equations of motion, the change in the Lagrangian (4.1.8) under the gauge transformation (4.1.7) is

$$\begin{aligned}
\delta\mathcal{L}_{\text{linear}} &= \int d^4\theta \, \left(\delta H^{++}\mathcal{J}_{++} + \delta H^{--}\mathcal{J}_{--}\right) - \int d^2\theta \, \delta\sigma \, \mathcal{V} - \int d^2\bar{\theta} \, \delta\bar{\sigma} \, \overline{\mathcal{V}} \\
&= \frac{i}{2} \int d^4\theta \Big\{ \left(\overline{D}_- L^+ - D_- \overline{L}^+\right) \mathcal{J}_{++} + \left(\overline{D}_+ L^- - D_+ \overline{L}^-\right) \mathcal{J}_{--} \\
&\qquad - \left(D_+ L^+ - D_- L^-\right) \mathcal{V} - \left(\overline{D}_+ \overline{L}^+ - \overline{D}_- \overline{L}^-\right) \overline{\mathcal{V}} \Big\} \\
&= \frac{i}{2} \int d^4\theta \Big\{ L^+ \left(\overline{D}_- \mathcal{J}_{++} + D_+ \mathcal{V}\right) + L^- \left(\overline{D}_+ \mathcal{J}_{--} - D_- \mathcal{V}\right) + \text{c.c.} \Big\} \, ,
\end{aligned} \tag{4.1.9}$$

where we have integrated by parts. Demanding that the variation vanishes for any gauge parameter $L^\pm$ gives

$$\overline{D}_- \mathcal{J}_{++} + D_+ \mathcal{V} = 0 \, , \qquad \overline{D}_+ \mathcal{J}_{--} - D_- \mathcal{V} = 0 \, . \tag{4.1.10}$$

This matches the constraints (4.1.5) for the FZ-multiplet if we identify

$$\mathcal{Y}_\pm = D_\pm \mathcal{V} \, , \tag{4.1.11}$$

and set $k' = 0$.

As we will soon see, studying $T\overline{T}$ deformations requires consideration of a composite operator constructed out of the square of the supercurrent multiplet. Hence to solve the $T\overline{T}$ flow equations we need to be able to calculate the supercurrent multiplet explicitly. The coupling to supergravity provides a straightforward prescription for computing the FZ-multiplet for matter models that can be coupled to old-minimal supergravity.[1] In particular, for a given $\mathcal{N} = (2, 2)$ matter theory we will:

1. Begin with an undeformed superspace Lagrangian $\mathcal{L}$ in flat $\mathcal{N} = (2, 2)$ superspace.

1. Though we will not need it in this thesis, it is worth mentioning that the non-minimal supergravity results of [21–25] allow the computation of the supercurrent multiplet for more general classes of models.

2. Minimally couple $\mathcal{L}$ to the supergravity superfield prepotentials $H^{\pm\pm}$, σ and $\bar{\sigma}$.
3. Extract the superfields $\mathcal{J}^{\pm\pm}$, $\mathcal{V}$ and $\overline{\mathcal{V}}$ which couple linearly to $H^{\pm\pm}$, σ and $\bar{\sigma}$, respectively, in the D- and F-terms of (4.1.8).

Thanks to the analysis given above, the superfields $\mathcal{J}^{\pm\pm}$, $\mathcal{V}$ and $\overline{\mathcal{V}}$ will automatically satisfy the FZ-multiplet constraints (4.1.10). A detailed description of the computation of the FZ-multiplet for the models relevant for our discussion is given in Appendix A.4.

4.2 The $T\overline{T}$ Operator and $\mathcal{N} = (2,2)$ Supersymmetry

After having reviewed in the previous section the structure of the $\mathcal{S}$-multiplet, we are ready to describe $\mathcal{N} = (2,2)$ $T\overline{T}$ deformations.

4.2.1 The $\mathcal{T}\overline{\mathcal{T}}$ operator

Given a $D = 2$ $\mathcal{N} = (2,2)$ supersymmetric theory with an $\mathcal{S}$-multiplet, we define the supercurrent-squared deformation of this theory, denoted $\mathcal{T}\overline{\mathcal{T}}$ in analogy with $T\overline{T}$, by the flow equation

$$\partial_\lambda \mathcal{L} = -\frac{1}{8}\mathcal{T}\overline{\mathcal{T}} \ , \tag{4.2.1}$$

where $\mathcal{T}\overline{\mathcal{T}}$ is constructed from current bilinears with

$$\mathcal{T}\overline{\mathcal{T}} \equiv -\int d^4\theta\, \mathcal{S}_{++}\mathcal{S}_{--} - \left(\int d\theta^- d\theta^+\, \chi_+\chi_- + \int d\bar{\theta}^- d\theta^+\, \overline{\mathcal{Y}}_+\mathcal{Y}_- + \text{c.c.}\right) \ , \tag{4.2.2}$$

and where the factor of $\frac{1}{8}$ is chosen for later convenience. This deformation generalizes the results we recently obtained for $D = 2$ theories possessing $\mathcal{N} = (0,1)$, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (0,2)$ supersymmetry [5, 26, 27] to theories with $\mathcal{N} = (2,2)$ supersymmetry.

Let us recall the form of the $T\overline{T}$ composite operator [1], which we denote

$$T\overline{T}(x) = T_{++++}(x)\, T_{----}(x) - \big[\Theta(x)\big]^2 \ . \tag{4.2.3}$$

An important property of the $\mathcal{N} = (0,1)$, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (0,2)$ cases is that the $T\overline{T}$ operator turns out to be the bottom component of a long supersymmetric multiplet. This is true up to both total vector derivatives (∂_{++} and ∂_{--}), and terms that vanish upon using the supercurrent conservation equations (Ward identities). For this reason, in the supersymmetric cases studied previously, the original $T\overline{T}$ deformation of [1] is manifestly supersymmetric and equivalent to the deformations constructed in terms of the full superspace integrals of primary supercurrent-squared composite operators [5, 26, 27].

Remarkably, despite the much more involved structure of the $(2,2)$ $\mathcal{S}$-multiplet compared to theories with fewer supersymmetries, it is possible to prove that the following relation holds:

$$\mathcal{T}\overline{\mathcal{T}}(x) = T\overline{T}(x) + \text{EOM}'\text{s} + \partial_{++}(\cdots) + \partial_{--}(\cdots) \ . \tag{4.2.4}$$

In (4.2.4), we use EOM′s to denote terms that are identically zero when (4.1.1) are used. Showing (4.2.4) requires using (A.3.1)–(A.3.3), along with several cancellations, integration by parts and the use of the $(2,2)$ $\mathcal{S}$-multiplet conservation equations (4.1.1).

In fact, the specific combination of current superfields given in (4.2.2) was chosen precisely for (4.2.4) to hold. The combination (4.2.4) is also singled out by being invariant under the improvement transformation (4.1.4). The important implication of (4.2.4) is that the $T\overline{T}$ deformation for an $\mathcal{N} = (2,2)$ supersymmetric quantum field theory is manifestly supersymmetric and equivalent to the $\mathcal{T}\overline{\mathcal{T}}$ deformation of eq. (4.2.2).

Note that in the $\mathcal{N} = (2,2)$ case the deformation we have introduced in (4.2.2) is conceptually different from the cases with less supersymmetry. Specifically, the deformation is not given by the descendant of a single composite superfield. On the other hand, suppose the $\mathcal{S}$-multiplet is such that $C^{(\pm)} = k = k' = 0$ and it is possible to improve the superfields $\chi_\pm$ and $\mathcal{Y}_\pm$ to a case where

$$\mathcal{Y}_\pm = D_\pm \mathcal{V} \ , \quad \overline{\mathcal{Y}}_\pm = \overline{D}_\pm \overline{\mathcal{V}} \ , \tag{4.2.5a}$$

$$\chi_+ = i\overline{D}_+\overline{\mathcal{B}} \ , \quad \chi_- = i\overline{D}_-\mathcal{B} \ , \quad \bar{\chi}_+ = -iD_+\mathcal{B} \ , \quad \bar{\chi}_- = -iD_-\overline{\mathcal{B}} \ , \tag{4.2.5b}$$

with $\mathcal{V}$ chiral and $\mathcal{B}$ twisted-chiral:

$$\overline{D}_\pm \mathcal{V} = 0 \ , \quad D_\pm \overline{\mathcal{V}} = 0 \ , \tag{4.2.6a}$$

$$\overline{D}_+\mathcal{B} = D_-\mathcal{B} = 0 \ , \quad D_+\overline{\mathcal{B}} = \overline{D}_-\overline{\mathcal{B}} = 0 \ . \tag{4.2.6b}$$

In this case (4.2.2) simplifies to

$$\begin{aligned} \mathcal{T}\overline{\mathcal{T}} &= -\int d^4\theta\, \mathcal{S}_{++}\mathcal{S}_{--} + \left(\int d\theta^- d\theta^+ \, \overline{D}_+\overline{\mathcal{B}}\overline{D}_-\mathcal{B} - \int d\bar{\theta}^- \, d\theta^+ \, \overline{D}_+\overline{\mathcal{V}}D_-\mathcal{V} + \text{c.c.} \right) \\ &= -\int d^4\theta \, \left(\mathcal{S}_{++}\mathcal{S}_{--} - 2\mathcal{B}\overline{\mathcal{B}} - 2\mathcal{V}\overline{\mathcal{V}} \right) \ , \end{aligned} \tag{4.2.7}$$

and we see that, up to EOM's, $T\overline{T}(x)$ is the bottom component of a long supersymmetric multiplet. In this situation, once we define the composite superfield

$$\mathcal{O}(\zeta) := -\mathcal{S}_{++}(\zeta)\mathcal{S}_{--}(\zeta) + 2\mathcal{B}(\zeta)\overline{\mathcal{B}}(\zeta) + 2\mathcal{V}(\zeta)\overline{\mathcal{V}}(\zeta) \ , \qquad (4.2.8)$$

eq. (4.2.4) turns into the equivalent result[2]

$$\int d^4\theta\, \mathcal{O}(\zeta) = D_-D_+\overline{D}_+\overline{D}_-\mathcal{O}(\zeta)|_{\theta=0} = T\overline{T}(x) + \mathrm{EOM's} + \partial_{++}(\cdots) + \partial_{--}(\cdots) \ , \quad (4.2.9)$$

stating that the D-term of the operator $\mathcal{O}(\zeta)$ is equivalent to the standard $T\overline{T}(x)$ operator.

For a matter theory that can be coupled to old-minimal supergravity, leading to the FZ-multiplet described by (4.1.10), the operator $\mathcal{O}(\zeta)$ further simplifies thanks to the fact that the twisted-(anti-)chiral operators $\mathcal{B}$ and $\overline{\mathcal{B}}$ disappear. For these cases, the $T\overline{T}$ flow turns into the following equation

$$\partial_\lambda \mathcal{L} = \frac{1}{8}\int d^4\theta \, \left(\mathcal{J}_{++}\mathcal{J}_{--} - 2\mathcal{V}\overline{\mathcal{V}}\right) \ . \qquad (4.2.10)$$

This will be our starting point in analyzing $\mathcal{N} = (2, 2)$ deformed models in section 4.3.

4.2.2 Point-splitting and well-definedness

The $T\overline{T}(x)$ operator (4.2.3) is quite magical because it is a well-defined irrelevant composite local operator, free of short distance divergences [1]. In fact, this property generalizes to the larger class of operators

$$[A_s(x)\, A'_{s'}(x) - B_{s+2}(x)\, B_{s'-2}(x)] \qquad (4.2.11)$$

where (A_s, B_{s+2}) and $(A'_{s'}, B'_{s'-2})$ are two pairs of conserved currents with spins s and s'. The operator $T\overline{T}(x)$ is a particular example with $s = s' = 0$. As proven in [28], these composite operators of "Smirnov-Zamolodchikov"-type have a well-defined point splitting which is free of short-distance divergences. In the case of $\mathcal{N} = (0, 1)$ and $\mathcal{N} = (1, 1)$ supersymmetric $T\overline{T}$ deformations, the entire supermultiplet whose bottom component is $T\overline{T}(x)$ is comprised of well-defined Smirnov-Zamolodchikov-type operators [5, 26]. In the $\mathcal{N} = (0, 2)$ case, the primary[3] operator whose bottom component is $T\overline{T}(x)$ is not of Smirnov-Zamolodchikov-type. Nevertheless, also in this case it was recently shown that, thanks to

2. In the subsequent discussion by $\theta = 0$ we will always mean $\theta^\pm = \overline{\theta}^\pm = 0$.

3. We denote as primary operator the top component of a supersymmetric multiplet even when the theory is not superconformal.

supersymmetry, the whole multiplet is well-defined [27].

In the $\mathcal{N} = (2, 2)$ case it is clear that the situation is more complicated than any of the cases mentioned above. First, in the general situation, according to (4.2.2), the $\mathcal{T}\overline{\mathcal{T}}$ deformation is a linear combination of a D-term together with chiral and twisted-chiral F-terms contributions. Though the F-terms might be protected by standard perturbative non-renormalization theorems (see, for example, [19, 20] for the $D = 4$ $\mathcal{N} = 1$ case which dimensionally reduces to $D = 2$ $\mathcal{N} = (2, 2)$), the D-term associated to the $\mathcal{S}_{++}\mathcal{S}_{--}$ operator has no clear reason to be protected in general from short-distance divergences in point-splitting regularization, and hence has no obvious reason to be well-defined. This indicates that there might be a clash between supersymmetry and a point-splitting procedure in the general setting.

We will not attempt to analyze this issue in full generality in the current work; instead our aim is to describe a subclass of models for which the $\mathcal{T}\overline{\mathcal{T}}$ deformation turns out to be well-defined. A natural restriction to impose is that the $\mathcal{S}$-multiplet is constrained by (4.2.5) and the $\mathcal{T}\overline{\mathcal{T}}$ deformation is therefore described by the D-term (4.2.7). By trivially extending the arguments used in [27] for the $\mathcal{N} = (0, 2)$ case, it is not difficult to show that these restrictions are sufficient to imply that the multiplet described by the $\mathcal{N} = (2, 2)$ primary operator $\mathcal{O}(\zeta)$ of (4.2.8) is indeed well-defined despite not being of Smirnov-Zamolodchikov-type. As in the $\mathcal{N} = (0, 2)$, unbroken $\mathcal{N} = (2, 2)$ supersymmetry turns out to be the reason for this to happen.

Let us quickly explain how this works for the FZ-multiplet and the deformation (4.2.10), which are the main players in this chapter. Note, however, that the same argument extends to more general cases where both chiral and twisted-chiral current superfields, $\chi_\pm$ and $\mathcal{Y}_\pm$, satisfying (4.2.5) are turned on. We also refer to [27] for details that we will skip in the following discussion, which are trivial extensions from the $(0, 2)$ to the $(2, 2)$ case.

A first indication of the well-definedness of the multiplet associated to $\mathcal{O}(\zeta)$ comes by looking at the vacuum expectation value of its lowest component. Define the primary composite operator

$$O(x) := -j_{--}(x)j_{++}(x) + 2v(x)\bar{v}(x) = \mathcal{O}(\zeta)|_{\theta=0} \ , \tag{4.2.12}$$

and its point-split version

$$O(x, x') := -j_{--}(x)j_{++}(x') + v(x)\bar{v}(x') + \bar{v}(x)v(x') \ , \tag{4.2.13}$$

where

$$j_{\pm\pm}(x) := \mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} \ , \quad v(x) := \mathcal{V}(\zeta)|_{\theta=0} \ , \quad \bar{v}(x) := \overline{\mathcal{V}}(\zeta)|_{\theta=0} \ . \tag{4.2.14}$$

Note that equation (4.1.10) implies the following relation among the component operators

$$\big[\overline{Q}_{\pm}, j_{\mp\mp}(x)\big] = \pm\big[Q_{\mp}, v(x)\big] \ , \quad \big[Q_{\pm}, j_{\mp\mp}(x)\big] = \pm\big[\overline{Q}_{\mp}, \bar{v}(x)\big] \ , \tag{4.2.15}$$

with $Q_{\pm}$ and $\overline{Q}_{\pm}$ denoting the $\mathcal{N} = (2,2)$ supercharges.[4] By then using $\partial_{\pm\pm} = i\{Q_{\pm}, \overline{Q}_{\pm}\}$, $\{Q_+, Q_-\} = \{\overline{Q}_+, \overline{Q}_-\} = 0$, $[\overline{Q}_{\pm}, v(x)] = [Q_{\pm}, \bar{v}(x)] = 0$, super-Jacobi identities, together with the conservation equations (4.2.15), and the assumption that the vacuum is invariant under supersymmetry, it is straightforward to show that vacuum expectation value of $O(x, x')$ satisfies

$$\begin{aligned}
\partial_{++}\big\langle j_{--}(x)j_{++}(x')\big\rangle &= i\big\langle\big[\{Q_+, \overline{Q}_+\}, j_{--}(x)\big]\, j_{++}(x')\big\rangle \\
&= i\big\langle\{Q_+, [Q_-, v(x)]\}\, j_{++}(x') + \big\{\overline{Q}_+, [\overline{Q}_-, \bar{v}(x)]\big\}\, j_{++}(x')\big\rangle \\
&= -i\big\langle[Q_+, v(x)][Q_-, j_{++}(x')] + [\overline{Q}_+, \bar{v}(x)][\overline{Q}_-, j_{++}(x')]\big\rangle \\
&= i\big\langle[Q_+, v(x)][\overline{Q}_+, \bar{v}(x')] + [\overline{Q}_+, \bar{v}(x)][Q_+, v(x')]\big\rangle \\
&= \big\langle\big[i\{\overline{Q}_+, Q_+\}, v(x)\big\}\, \bar{v}(x') + \big[i\{Q_+, \overline{Q}_+\}, v(x)\big\}\, \bar{v}(x')\big\rangle \\
&= \partial_{++}\big\langle v(x)\bar{v}(x') + v(x)\bar{v}(x')\big\rangle \ ,
\end{aligned} \tag{4.2.16}$$

and, after performing a similar calculation for $\big\langle\partial_{--}j_{--}(x)j_{++}(x')\big\rangle = -\big\langle j_{--}(x)\partial'_{--}j_{++}(x')\big\rangle$, it is clear that the relation

$$\partial_{\pm\pm}\big\langle O(x, x')\big\rangle = 0 \tag{4.2.17}$$

holds. Therefore, $\big\langle O(x, x')\big\rangle$ is independent of the positions and free of short distance divergences. It is worth noting that similarly to the argument showing that the two point function of two chiral or twisted-chiral operators is independent of the positions x and x', the previous analysis for $\big\langle O(x, x')\big\rangle$ necessarily relies on unbroken $\mathcal{N} = (2,2)$ supersymmetry.

The argument given above can be generalized to a statement about operators in superspace in complete analogy to the $\mathcal{N} = (0,2)$ case of [27]. Let us investigate the short distance singularities in the bosonic coordinates by defining a point-split version of the $\mathcal{N} = (2,2)$

4. Given an operator $F(x)$ defined as the $\theta = 0$ component of the superfield $\mathcal{F}(\zeta)$, $F(x) := \mathcal{F}(\zeta)|_{\theta=0}$, then its supersymmetry transformations are such that $\big[Q_{\pm}, F(x)\big\} = \mathcal{Q}_{\pm}\mathcal{F}(\zeta)|_{\theta=0} = D_{\pm}\mathcal{F}(\zeta)|_{\theta=0}$ and $\big[\overline{Q}_{\pm}, F(x)\big\} = \overline{\mathcal{Q}}_{\pm}\mathcal{F}(\zeta)|_{\theta=0} = \overline{D}_{\pm}\mathcal{F}(\zeta)|_{\theta=0}$.

primary $\mathcal{T}\overline{\mathcal{T}}$ operator,

$$\mathcal{O}(x,x',\theta) := -\mathcal{J}_{--}(x,\theta)\,\mathcal{J}_{++}(x',\theta) + \mathcal{V}(x,\theta)\,\overline{\mathcal{V}}(x',\theta) + \overline{\mathcal{V}}(x,\theta)\,\mathcal{V}(x',\theta)\ . \qquad (4.2.18)$$

We want to show that the preceding bilocal superfield is free of short distance divergences in the limit $x \to x'$. A straightforward calculation shows that

$$\begin{aligned}\partial_{++}\mathcal{O}(x,x',\theta) \;=\; & -\Big\{iD_+\mathcal{V}(\zeta)\left[D'_-\mathcal{J}_{++}(\zeta') + \overline{D}'_+\overline{\mathcal{V}}(\zeta')\right] + i\overline{D}_+\overline{\mathcal{V}}(\zeta)\left[\overline{D}'_-\mathcal{J}_{++}(\zeta') + D'_+\mathcal{V}(\zeta')\right] \\ & + i(\mathcal{Q}_+ + \mathcal{Q}'_+)\left[(\overline{D}_+\overline{\mathcal{V}}(\zeta))\mathcal{V}(\zeta')\right] + i(\overline{\mathcal{Q}}_+ + \overline{\mathcal{Q}}'_+)\left[(D_+\mathcal{V}(\zeta))\overline{\mathcal{V}}(\zeta')\right] \\ & + i(\mathcal{Q}_- + \mathcal{Q}'_-)\left[(D_+\mathcal{V}(\zeta))\mathcal{J}_{++}(\zeta')\right] + i(\overline{\mathcal{Q}}_- + \overline{\mathcal{Q}}'_-)\left[(\overline{D}_+\overline{\mathcal{V}}(\zeta))\mathcal{J}_{++}(\zeta')\right] \\ & + (\partial_{++} + \partial'_{++})\Big[\bar{\theta}^+(\overline{D}_+\overline{\mathcal{V}}(\zeta))\mathcal{V}(\zeta') + \bar{\theta}^-(D_+\mathcal{V}(\zeta))\mathcal{J}_{++}(\zeta')\Big] \\ & - (\partial_{++} + \partial'_{++})\left[\theta^+(D_+\mathcal{V}(\zeta))\overline{\mathcal{V}}(\zeta') + \theta^-(\overline{D}_+\overline{\mathcal{V}}(\zeta))\mathcal{J}_{++}(\zeta')\right]\Big\}\Big|_{\theta=\theta'}\ . \end{aligned} \qquad (4.2.19)$$

Note that the first line in the preceding expression is zero because of the FZ conservation equations (4.1.10), which hold up to contact terms in correlation functions. The other lines are either total vector derivatives or supersymmetry transformations of bilocal operators. A similar equation holds for $\partial_{--}\mathcal{O}(x,x',\theta)$ showing that the operator $\mathcal{O}(x,x',\theta)$ satisfies

$$\partial_{\pm\pm}\mathcal{O}(x,x',\theta) \;=\; 0 + \text{EOM's} + [P,\cdots] + [Q,\cdots]\ , \qquad (4.2.20)$$

where $[P,\cdots]$ and $[Q,\cdots]$ schematically indicate a translation and supersymmetry transformation of some bilocal superfield operator.[5] To conclude, by employing an OPE argument completely analogous to the one originally given by Zamolodchikov in [1] and extended to the $\mathcal{N}=(0,2)$ supersymmetric case in [27], one can show that eq. (4.2.20) implies

$$\mathcal{O}(x,x',\theta) = \mathcal{O}(\zeta) + \text{derivative terms}\ . \qquad (4.2.21)$$

Here "derivative terms" indicate superspace covariant derivatives $D_A = (\partial_{\pm\pm}, D_\pm, \overline{D}_\pm)$ acting on local superfield operators while $\mathcal{O}(\zeta)$ arises from the regular, non-derivative part of the OPE of $\mathcal{O}(x,x',\theta)$. As a result the integrated operator

$$S_{\mathcal{O}} = \int d^2x\, d^4\theta \lim_{\varepsilon\to 0}\mathcal{O}(x,x+\varepsilon,\theta) = \int d^2x\, d^4\theta :\mathcal{O}(x,x,\theta):\ , \qquad (4.2.22)$$

5. See Appendix A of [27] for the relation between the operators $(\mathcal{Q}_\pm + \mathcal{Q}'_\pm)$, $(\overline{\mathcal{Q}}_\pm + \overline{\mathcal{Q}}'_\pm)$ and the generators of supersymmetry transformations on bilocal superfields such as $\mathcal{O}(x,x',\theta)$. The extension of that analysis from $\mathcal{N}=(0,2)$ to $\mathcal{N}=(2,2)$ is straightforward.

which can be considered as a definition of the integrated $\mathcal{T}\overline{\mathcal{T}}(x)$ operator,[6] is free of short distance divergences and well-defined in complete analogy to the non-supersymmetric case [1] and the $\mathcal{N} = (0, 1)$, $\mathcal{N} = (1, 1)$, and $\mathcal{N} = (0, 2)$ cases [5, 26, 27].

4.3 Deformed (2, 2) Models

In this section, we will apply our supercurrent-squared deformation (4.2.10) to a few examples of $\mathcal{N} = (2, 2)$ supersymmetric theories for a chiral multiplet Φ. The superfield Φ can be written in components as

$$\begin{aligned}\Phi \;=\;& \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F - i\theta^+\overline{\theta}^+\partial_{++}\phi - i\theta^-\overline{\theta}^-\partial_{--}\phi \\ & -i\theta^+\theta^-\overline{\theta}^-\partial_{--}\psi_+ - i\theta^-\theta^+\overline{\theta}^+\partial_{++}\psi_- - \theta^+\theta^-\overline{\theta}^-\overline{\theta}^+\partial_{++}\partial_{--}\phi \ ,\end{aligned} \tag{4.3.1}$$

where ϕ is a complex scalar field, $\psi_\pm$ are Dirac fermions, and F is a complex auxiliary field. The multiplet Φ satisfies the chirality constraint $\overline{D}_\pm\Phi = 0$.

We denote the physical Lagrangian by $\mathcal{L}$ and the superspace D-term Lagrangian by $\mathcal{A}$, so that

$$S = \int d^2x\, \mathcal{L} = \int d^2x\, d^4\theta\, \mathcal{A} \ . \tag{4.3.2}$$

A broad class of two-derivative theories for a chiral superfield can be described by superspace Lagrangians of the form

$$\mathcal{L} = \int d^4\theta\, K(\Phi, \overline{\Phi}) + \int d^2\theta\, W(\Phi) + \int d^2\overline{\theta}\overline{W}(\overline{\Phi}) \ , \tag{4.3.3}$$

where $K(\Phi, \overline{\Phi})$ is a real function called the Kähler potential and $W(\Phi)$ is a holomorphic function called the superpotential. These are $\mathcal{N} = (2, 2)$ Landau-Ginzburg models. In order for the kinetic terms of the component fields of Φ to have the correct sign, we will assume that $K_{\Phi\overline{\Phi}} = \frac{\partial^2 K}{\partial\Phi\partial\overline{\Phi}}$ is positive.

Although we will not expand on this point in detail, all the results found in this section can be derived almost identically for the case of a generic model of a single scalar twisted-chiral superfield $\mathcal{Y}$, $\overline{D}_+\mathcal{Y} = D_-\mathcal{Y} = 0$, and its conjugate. This is not surprising since theories

6. Note that, consistently, one can show that

$$\{Q_+, [\overline{Q}_+, \{Q_-, [\overline{Q}_-, O(x, x')]\}]\} = T_{----}(x)T_{++++}(x') - \Theta(x)\Theta(x') + \text{EOM's} + [P, \cdots] \tag{4.2.23}$$

implying that the descendant of the point-split primary operator $O(x)$ is equivalent, up to Ward identities and total vector derivatives ($\partial_{\pm\pm}$), to the point-split version of the descendant $T\overline{T}(x)$ operator.

containing only chiral superfields are physically equivalent to theories formulated in terms of twisted-chiral superfields; see, for example, [21–25] for a discussion of this equivalence in models with global and local supersymmetry. There are also many more involved $(2,2)$ theories that one might also want to study involving chiral, twisted-chiral and semi-chiral superfields; see, for example, [29] for a recent discussion and references. For this analysis, we have chosen to consider only models based on a single chiral multiplet.

4.3.1 Kähler potential

First we will set the superpotential W to zero and begin with an undeformed superspace Lagrangian of the form

$$\mathcal{L} = \int d^4\theta \, K(\Phi, \overline{\Phi}) \tag{4.3.4}$$

for some Kähler potential K. To leading order around this undeformed theory, the FZ supercurrents are

$$\mathcal{J}_{\pm\pm} = 2K_{\Phi\overline{\Phi}} D_\pm \Phi \overline{D}_\pm \overline{\Phi} \ , \tag{4.3.5a}$$

$$\mathcal{V} = 0 \ , \tag{4.3.5b}$$

where $K_\Phi = \frac{\partial K}{\partial \Phi}, K_{\Phi\overline{\Phi}} = \frac{\partial^2 K}{\partial\Phi\partial\overline{\Phi}}$, etc. Therefore, at first order the supercurrent-squared deformation driven by $\mathcal{O} = \left(-\mathcal{S}_{++}\mathcal{S}_{--} + 2\mathcal{V}\overline{\mathcal{V}}\right)$ will source a four-fermion contribution in the D-term, giving

$$\mathcal{L}^{(1)} = \mathcal{L}^{(0)} + \frac{1}{2}\lambda K_{\Phi\overline{\Phi}}^2 D_+\Phi\overline{D}_+\overline{\Phi} D_-\Phi\overline{D}_-\overline{\Phi} \ . \tag{4.3.6}$$

Next, we would like to find the all-orders solution for the deformed theory. We make the ansatz that, at finite deformation parameter λ, the Lagrangian takes the form

$$\mathcal{L}_\lambda = \int d^4\theta \Big\{ K(\Phi, \overline{\Phi}) + f(\lambda, x, \overline{x}, y) K_{\Phi\overline{\Phi}}^2 D_+\Phi\overline{D}_+\overline{\Phi} D_-\Phi\overline{D}_-\overline{\Phi} \Big\} \ , \tag{4.3.7}$$

where we define the combinations

$$x = K_{\Phi\overline{\Phi}} \partial_{++}\Phi \partial_{--}\overline{\Phi} \ , \qquad y = K_{\Phi\overline{\Phi}} \left(D_+ D_- \Phi\right)\left(\overline{D}_+ \overline{D}_- \overline{\Phi}\right) \ . \tag{4.3.8}$$

Using the results in Appendix A.4, one finds that the superfields $\mathcal{J}_{\pm\pm}$ and $\mathcal{V}$ appearing in our supercurrent-squared deformation, computed for the Lagrangian (4.3.7), are given by

$$\begin{aligned}
\mathcal{J}_{++} = {} & 2K_{\Phi\overline{\Phi}} D_+\Phi\overline{D}_+\overline{\Phi}\left[1 + f(x+\overline{x}-3y) + x\frac{\partial f}{\partial x}(\overline{x}-y) + \overline{x}\frac{\partial f}{\partial \overline{x}}(x-y) + y\frac{\partial f}{\partial y}(x+\overline{x}-2y)\right] \\
& + 2K^2_{\Phi\overline{\Phi}} D_-\Phi\overline{D}_-\overline{\Phi}\partial_{++}\Phi\partial_{++}\overline{\Phi}\left[-f - x\frac{\partial f}{\partial x} - \overline{x}\frac{\partial f}{\partial \overline{x}} + y\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \overline{x}}\right)\right] \\
& - 2iK^2_{\Phi\overline{\Phi}} D_+\Phi D_-\Phi\partial_{++}\overline{\Phi}\overline{D}_+\overline{D}_-\overline{\Phi}\left[-f + (x-\overline{x})\frac{\partial f}{\partial \overline{x}} + (x-y)\frac{\partial f}{\partial y}\right] \\
& - 2iK^2_{\Phi\overline{\Phi}} \overline{D}_+\overline{\Phi}\overline{D}_-\overline{\Phi}\partial_{++}\Phi D_+D_-\Phi\left[f + (x-\overline{x})\frac{\partial f}{\partial x} + (y-\overline{x})\frac{\partial f}{\partial y}\right] ,
\end{aligned} \tag{4.3.9}$$

and

$$\begin{aligned}
\mathcal{J}_{--} = {} & 2K_{\Phi\overline{\Phi}} D_-\Phi\overline{D}_-\overline{\Phi}\left[1 + f(x+\overline{x}-3y) + x\frac{\partial f}{\partial x}(\overline{x}-y) + \overline{x}\frac{\partial f}{\partial \overline{x}}(x-y) + y\frac{\partial f}{\partial y}(x+\overline{x}-2y)\right] \\
& + 2K^2_{\Phi\overline{\Phi}} D_+\Phi\overline{D}_+\overline{\Phi}\partial_{--}\Phi\partial_{--}\overline{\Phi}\left[-f - x\frac{\partial f}{\partial x} - \overline{x}\frac{\partial f}{\partial \overline{x}} + y\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \overline{x}}\right)\right] \\
& - 2iK^2_{\Phi\overline{\Phi}} D_+\Phi D_-\Phi\partial_{--}\overline{\Phi}\overline{D}_+\overline{D}_-\overline{\Phi}\left[-f + (\overline{x}-x)\frac{\partial f}{\partial x} + (\overline{x}-y)\frac{\partial f}{\partial y}\right] \\
& - 2iK^2_{\Phi\overline{\Phi}} \overline{D}_+\overline{\Phi}\overline{D}_-\overline{\Phi}\partial_{--}\Phi D_+D_-\Phi\left[f + (\overline{x}-x)\frac{\partial f}{\partial \overline{x}} + (y-x)\frac{\partial f}{\partial y}\right] ,
\end{aligned} \tag{4.3.10}$$

and

$$\begin{aligned}
\mathcal{V} = {} & 2K^2_{\Phi\overline{\Phi}}\left(f + y\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x} + \overline{x}\frac{\partial f}{\partial \overline{x}}\right)\Big[- i\partial_{++}\overline{\Phi}(D_+D_-\Phi)D_-\Phi\overline{D}_-\overline{\Phi} + \partial_{++}\overline{\Phi}\partial_{--}\overline{\Phi}D_-\Phi D_+\Phi \\
& - \overline{D}_-\overline{\Phi}\overline{D}_+\overline{\Phi}\,(D_+D_-\Phi)^2 - i\partial_{--}\overline{\Phi}(D_+D_-\Phi)D_+\Phi\overline{D}_+\overline{\Phi}\Big] .
\end{aligned} \tag{4.3.11}$$

The supercurrent-squared flow then induces a differential equation for the superspace Lagrangian $\mathcal{A}_\lambda$ (where, again, $\mathcal{L}_\lambda = \int d^4\theta\,\mathcal{A}_\lambda$) given by

$$\frac{d}{d\lambda}\mathcal{A}_\lambda = -\frac{1}{8}\mathcal{O} = \frac{1}{8}\left(\mathcal{J}_{++}\mathcal{J}_{--} - 2\mathcal{V}\overline{\mathcal{V}}\right) . \tag{4.3.12}$$

Given our ansatz (4.3.7), we see that

$$\frac{d\mathcal{A}_\lambda}{d\lambda} = \frac{df}{d\lambda}K^2_{\Phi\overline{\Phi}}\,D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi} . \tag{4.3.13}$$

On the other hand, plugging in our expressions (4.3.9), (4.3.10), and (4.3.11) for the supercurrents into the right hand side of (4.3.12) also gives a result proportional to $K^2_{\Phi\overline{\Phi}}D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}$.

Equating the coefficients, we find a differential equation for f:

$$\frac{df}{d\lambda} = \frac{1}{2}\Bigg\{ -\overline{x}y\left[f + (\overline{x}-x)\frac{\partial f}{\partial \overline{x}} + (y-x)\frac{\partial f}{\partial y}\right]^2 - xy\left[f + (x-\overline{x})\frac{\partial f}{\partial x} + (y-\overline{x})\frac{\partial f}{\partial y}\right]^2$$
$$+ 2(x-y)(y-\overline{x})\left[f + y\frac{\partial f}{\partial y} + \overline{x}\frac{\partial f}{\partial \overline{x}} + x\frac{\partial f}{\partial x}\right]^2 + x\overline{x}\left[f + (\overline{x}-y)\frac{\partial f}{\partial \overline{x}} + (x-y)\frac{\partial f}{\partial x}\right]^2$$
$$+ \left[1 + (x+\overline{x}-3y)f + (x+\overline{x}-2y)y\frac{\partial f}{\partial y} + \overline{x}(x-y)\frac{\partial f}{\partial \overline{x}} + x(\overline{x}-y)\frac{\partial f}{\partial x}\right]^2\Bigg\} . \qquad (4.3.14)$$

In particular, this shows that our ansatz (4.3.7) for the finite-λ superspace action is consistent: the supercurrent-squared deformation closes on an action of this form. It could have been otherwise: the flow equation might have sourced additional terms proportional, say, to two-fermion combinations $D_+\Phi\overline{D}_+\overline{\Phi}$, or required dependence on other dimensionless variables like $\lambda(D_+D_-\Phi)^2$, but these complications do not arise in the case where the undeformed theory only has a Kähler potential.

On dimensional grounds, f must be proportional to λ times a function of the dimensionless combinations λx and λy. Thus, although the differential equation for f determined by (4.3.14) is complicated, one can solve order-by-order in λ. The solution to $\mathcal{O}(\lambda^3)$ is

$$f(\lambda, x, \overline{x}, y) = \frac{\lambda}{2} + \lambda^2\left(\frac{x+\overline{x}}{4} - \frac{3}{4}y\right)$$
$$+ \lambda^3\left(\frac{x^2+\overline{x}^2+3x\overline{x}}{8} + \frac{37}{24}y^2 - \frac{25}{24}(x+\overline{x})\,y\right) + \cdots . \qquad (4.3.15)$$

We were unable to find a closed-form expression for f to all orders in λ. However, the differential equation simplifies dramatically when we impose the equations of motion for the theory, and in this case one can write down an exact formula. This is similar to the $T\overline{T}$ flow of the free action for a real $\mathcal{N} = (1, 1)$ scalar multiplet that was analyzed in [5, 26].

We claim that, on-shell, one may drop any terms where $y \sim (D_+D_-\Phi)(\overline{D}_+\overline{D}_-\overline{\Phi})$ multiplies the four-fermion term $|D\Phi|^4 \equiv D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}$. This is shown explicitly in Appendix A.5 and follows directly from the superspace equation of motion and nilpotency of the fermionic terms $D_\pm\Phi$ and $\overline{D}_\pm\overline{\Phi}$. It is also an intuitive statement associated to the fact that for these models, on-shell, $\mathcal{N} = (2, 2)$ supersymmetry is not broken. In fact, note that the superfields $(D_+D_-\Phi)$ and $(\overline{D}_+\overline{D}_-\overline{\Phi})$ have as their lowest components the auxiliary fields F and $\overline{F}$. If supersymmetry is not broken, the vev of F has to be zero, $\langle F\rangle = 0$, which implies that the auxiliary field F is on-shell at least quadratic in fermions and, more precisely, can be proven to be at least linear in $\psi_\pm = D_\pm\Phi|_{\theta=0}$ and $\bar{\psi}_\pm = \overline{D}_\pm\overline{\Phi}|_{\theta=0}$. From this argument it follows that on-shell $(D_+D_-\Phi)$ is at least linear in $D_\pm\Phi$ and $\overline{D}_\pm\overline{\Phi}$, and then the two conditions $(D_+D_-\Phi)|D\Phi|^4 = 0$ and $y|D\Phi|^4 = 0$ follow.

After removing from (4.3.14) the y-dependent terms which vanish on-shell, we find a simpler differential equation for the function f,

$$\frac{df}{d\lambda} = \frac{1}{2}\left\{-x\overline{x}\left[f + x\frac{\partial f}{\partial x} + \overline{x}\frac{\partial f}{\partial \overline{x}}\right]^2 + \left[1 + (x+\overline{x})f + x\overline{x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \overline{x}}\right)\right]^2\right\} , \qquad (4.3.16)$$

whose solution is

$$f(\lambda, x, \overline{x}, y = 0) = \frac{\lambda}{1 - \frac{\lambda}{2}(x+\overline{x}) + \sqrt{1 - \lambda(x+\overline{x}) + \frac{\lambda^2}{4}(x-\overline{x})^2}} . \qquad (4.3.17)$$

Thus we have shown that the supercurrent-squared deformed Lagrangian at finite λ is equivalent on-shell to the following superspace Lagrangian

$$\mathcal{L}_\lambda = \int d^4\theta \left(K(\Phi,\overline{\Phi}) + \frac{\lambda K^2_{\Phi\overline{\Phi}} D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}}{1 - \frac{1}{2}\lambda K^2_{\Phi\overline{\Phi}}A + \sqrt{1 - \lambda K^2_{\Phi\overline{\Phi}}A + \frac{1}{4}\lambda^2 K^4_{\Phi\overline{\Phi}}B^2}}\right) , \qquad (4.3.18)$$

where

$$A = \partial_{++}\Phi\partial_{--}\overline{\Phi} + \partial_{++}\overline{\Phi}\partial_{--}\Phi , \qquad B = \partial_{++}\Phi\partial_{--}\overline{\Phi} - \partial_{++}\overline{\Phi}\partial_{--}\Phi . \qquad (4.3.19)$$

When $K(\Phi,\overline{\Phi}) = \overline{\Phi}\Phi$, it is simple to show that this model represents an $\mathcal{N} = (2,2)$ off-shell supersymmetric extension of the $D = 4$ Nambu-Goto string in an appropriate gauge—often referred to as a static gauge in presence of a B field, though it can be more naturally described as uniform light-cone gauge [30, 31] (see refs. [32, 33] for a discussion of this point). In particular, by setting various component fields to zero and performing the superspace integrals, one can show that (4.3.18) matches the expected answer for $T\overline{T}$ deformations in previously known non-supersymmetric cases. For instance, setting the fermions to zero and integrating out the auxiliary fields F and $\overline{F}$ gives the $T\overline{T}$ deformation of the complex free boson ϕ, whose Lagrangian is

$$\mathcal{L}_{\lambda,\mathrm{bos}} = \frac{\sqrt{1 + 2\lambda a + \lambda^2 b^2} - 1}{4\lambda} = \frac{a}{4} - \lambda\frac{\partial_{++}\phi\partial_{--}\phi\partial_{++}\bar{\phi}\partial_{--}\bar{\phi}}{1 + \lambda a + \sqrt{1 + 2\lambda a + \lambda^2 b^2}} , \qquad (4.3.20)$$

where

$$a = \partial_{++}\phi\partial_{--}\bar{\phi} + \partial_{++}\bar{\phi}\partial_{--}\phi , \qquad b = \partial_{++}\phi\partial_{--}\bar{\phi} - \partial_{++}\bar{\phi}\partial_{--}\phi . \qquad (4.3.21)$$

The Lagrangian (4.3.20) indeed describes the $D = 4$ light-cone gauge-fixed Nambu-Goto string model.

Alternatively, setting all the bosons to zero in (4.3.18) can be shown to give the $T\overline{T}$ deformation of a complex free fermion. These calculations are similar to those in the case of the $(0, 2)$ supercurrent-squared action, which are presented in [27]. In fact, it can even be easily shown that an $\mathcal{N} = (0, 2)$ truncation of (4.3.18) gives precisely the $T\overline{T}$ deformation of a free $\mathcal{N} = (0, 2)$ chiral multiplet that was derived in [27].

It is worth highlighting that, unlike the $\mathcal{N} = (2, 2)$ case, an off-shell $(0, 2)$ chiral scalar multiplet contains only physical degrees of freedom and no auxiliary fields. Interestingly, related to this fact, it turns out that (up to integration by parts and total derivatives) the $\mathcal{N} = (0, 2)$ off-shell supersymmetric extension of the $D = 4$ Nambu-Goto string action in light-cone gauge is unique and precisely matches the off-shell $T\overline{T}$ deformation of a free $\mathcal{N} = (0, 2)$ chiral multiplet action [27].

In the $\mathcal{N} = (2, 2)$ case, because of the presence of the auxiliary field F in the chiral multiplet Φ, there are an infinite set of inequivalent $\mathcal{N} = (2, 2)$ off-shell extensions of the Lagrangian (4.3.20) that are all equivalent on-shell. A representative of these equivalent actions is described by (4.3.18) when $K(\Phi, \overline{\Phi}) = \overline{\Phi}\Phi$.

The non-uniqueness of dynamical systems described by actions of the form (4.3.18) can also be understood by noticing that, for example, it is possible to perform a class of redefinitions that leaves the action (4.3.18) invariant on-shell. As a (very particular) example, note that we are free to perform a shift of the form

$$D_+\overline{D}_- \left(\overline{D}_+\overline{\Phi}D_-\Phi\right) \longrightarrow D_+\overline{D}_- \left(\overline{D}_+\overline{\Phi}D_-\Phi\right) + a\left(D_+D_-\Phi + \overline{D}_+\overline{D}_-\overline{\Phi}\right)^2 \tag{4.3.22}$$

for any real number a. In terms of A and B, (4.3.22) implements the shifts

$$A \longrightarrow A + a\left((D_+D_-\Phi)^2 + 2y + \left(\overline{D}_+\overline{D}_-\overline{\Phi}\right)^2\right) , \qquad B \longrightarrow B \tag{4.3.23}$$

in (4.3.18). The resulting Lagrangian would enjoy the same on-shell simplifications described in Appendix A.5 and would turn out to be on-shell equivalent to the Lagrangian (4.3.18). In this infinite set of on-shell equivalent actions, a particular choice would represent an exact solution of the $T\overline{T}$ flow equation (4.3.12)–(4.3.14), whose leading terms in a λ series expansion are given in (4.3.15). Another representative in this on-shell equivalence class is the simplified model described by (4.3.18).

These types of redefinition and on-shell equivalentness are not a surprise, nor really new. In fact, they are of the same nature as redefinitions that have been studied in detail in

[34] (see also [35] for a description of these types of "trivial symmetries") in the context of $D = 4\,\mathcal{N} = 1$ chiral and linear superfield models possessing a non-linearly realised additional supersymmetry [34, 36]. As in (4.3.23), the field redefinition in this context does not affect the dynamics of the physical fields— it basically corresponds only to an arbitrariness in the definition of the auxiliary fields that always appear quadratically in the action and then are set to zero (up to fermion terms that will not contribute due to nilpotency in the action) on-shell. Although here we only focused on discussing the on-shell ambiguity of the solution of the $\mathcal{N} = (2,2)$ $T\overline{T}$ flow, we expect that the exact solution of the flow equations with y nonzero (4.3.12)–(4.3.14) can be found by a field redefinition of the kind we made in the action (4.3.18).

It is also interesting to note that similar freedoms and field redefinitions are also described in the construction of $D = 4$ $\mathcal{N} = 1$ supersymmetric Born-Infeld actions; see, for example, [37]. In fact, as will be analyzed in more detail elsewhere [38], it can be shown that the Lagrangian (4.3.18) is structurally of the type described by Bagger and Galperin for the $D = 4$ $\mathcal{N} = 1$ supersymmetric Born-Infeld action [37]. The equivalence can be formally shown by identifying $W_+ = \overline{D}_+\overline{\Phi}$, $W_- = D_-\Phi$, $W^2 = \overline{D}_+\overline{\Phi}D_-\Phi$, and $D^\alpha W_\alpha = D_+D_-\Phi + \overline{D}_-\overline{D}_+\overline{\Phi}$ to match their conventions. As a consequence, we can show that our solution for the $T\overline{T}$ flow possesses a second non-linearly realised $\mathcal{N} = (2,2)$ supersymmetry, besides the $(2,2)$ supersymmetry which is made manifest by the superspace construction, which is discussed in much greater detail in Chapter 5. We note that the presence of a second supersymmetry is analogous to what happens in the $\mathcal{N} = (0,2)$ case [27].

4.3.2 Adding a superpotential

Now suppose we begin with an undeformed theory that has a superpotential $W(\Phi)$,

$$\mathcal{L}^{(0)} = \int d^4\theta\, K(\Phi,\overline{\Phi}) + \left(\int d^2\theta\, W(\Phi)\right) + \left(\int d^2\overline{\theta}\, \overline{W}(\overline{\Phi})\right) . \tag{4.3.24}$$

As shown in Appendix A.4, the superpotential F-term gives a contribution $\delta\mathcal{V} = 2W(\Phi)$ to the field $\mathcal{V}$ which appears in supercurrent-squared. To leading order in the deformation parameter, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}^{(0)} &\to \mathcal{L}^{(0)} + \mathcal{L}^{(1)} \\ &= \mathcal{L}^{(0)} + \lambda \int d^4\theta \left(\frac{1}{2}K^2_{\Phi\overline{\Phi}} D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi} + W(\Phi)\overline{W}(\overline{\Phi})\right) .\end{aligned} \tag{4.3.25}$$

In addition to the four-fermion term which we saw in section 4.3.1, we see that the deformation modifies the Kähler potential, adding a term proportional to $|W(\Phi)|^2$.

Next consider the second order term in λ. For convenience, we use the combination $|D\Phi|^4 = D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}$, which is the four-fermion combination that appeared at first order. Then

$$\mathcal{L}^{(2)} = \frac{\lambda^2}{4}\int d^4\theta \left(x + \overline{x} - 3y - 2|W'(\Phi)|^2 + WD_-D_+ + \overline{W}\overline{D}_+\overline{D}_-\right)|D\Phi|^4 \ . \qquad (4.3.26)$$

The new terms involving supercovariant derivatives of $|D\Phi|^4$ will generate contributions with two fermions in the D-term.

As we continue perturbing to higher orders, the form of the superspace Lagrangian becomes more complicated. It is no longer true that the supercurrent-squared flow closes on a simple ansatz with one undetermined function, as it did in the case with only a Kähler potential. Indeed, the finite-λ deformed superspace Lagrangian in the case with a superpotential will depend not only on the variables x, $\overline{x}$, and y as in section 4.3.1, but also, for example, on combinations like $\partial_{++}\Phi\overline{D}_+\overline{D}_-\overline{\Phi}$, which can appear multiplying the two-fermion term $D_-\Phi\overline{D}_-\overline{\Phi}$ in the superspace Lagrangian. To find the full solution, one would need to determine several functions contributing to the D-term—one multiplying the four-fermion term $|D\Phi|^4$ as in the Kähler case; one for the deformed Kähler potential which may now depend on x, y, and other combinations; and four functions multiplying the two-fermion terms $D_+\Phi D_-\Phi$, $D_+\Phi\overline{D}_-\overline{\Phi}$, etc. Each function can depend on several dimensionless combinations.

In the presence of a superpotential, the situation might further be complicated by the fact that supersymmetry can be spontaneously broken. This would make it impossible, for example, to use on-shell simplifications like $y|D\Phi|^4 = 0$ that we employed in the section 4.3.1, where supersymmetry is never spontaneously broken.

It should be clear that the case with a superpotential is significantly more involved and rich than just a pure Kähler potential. In this case, we have not attempted to find a solution of the $T\overline{T}$ flow equation in closed form. However, it is evident from the form of supercurrent-squared eq. (4.3.12)—which is always written as a D-term integral of current bilinears—that this deformation will only affect the D term and not the $\mathcal{N} = (2,2)$ superpotential W appearing in the chiral integral. Therefore the superpotential, besides being protected from perturbative quantum corrections, is also protected from corrections along the supercurrent-squared flow.

4.3.3 The physical classical potential

In view of the difficulty of finding the all-orders deformed superspace action for a theory with a superpotential, we now consider the simpler problem of finding the local-potential approximation (or zero-momentum potential) for the bosonic complex scalar ϕ contained in the superfield Φ. We stress that our analysis here is purely classical and we will make a couple of comments about possible quantum effects later in this section. For simplicity, we will also restrict to the case in which the Kähler potential is flat, $K(\Phi, \overline{\Phi}) = \overline{\Phi}\Phi$. By "zero-momentum potential" we mean the physical potential $V(\phi)$ which appears in the Lagrangian after performing the superspace integral in the deformed theory and then setting $\partial_{\pm\pm}\phi = 0$. For instance, consider the undeformed Lagrangian

$$\mathcal{L}^{(0)} = \int d^4\theta \, \overline{\Phi}\Phi + \int d^2\theta \, W(\Phi) + \int d^2\overline{\theta} \, \overline{W}(\overline{\Phi}) \ . \tag{4.3.27}$$

When we ignore all terms involving derivatives and the fermions $\psi_{\pm}$, the only contributions to the physical Lagrangian (after performing the superspace integral) come from an $|F|^2$ term from the kinetic term, plus the term $W(\Phi) = W(\phi) + W'(\phi)\theta^+\theta^- F$. This gives us the zero-momentum, zero-fermion component action

$$S = \int d^2x \, \left(|F|^2 + W'(\phi)F + \overline{W}'(\bar{\phi})\overline{F} \right) \ . \tag{4.3.28}$$

We may integrate out the auxiliary field F using its equation of motion $\overline{F} = -W'(\phi)$, which yields

$$S = \int d^2x \, \left(-|W'(\phi)|^2 \right) \ , \tag{4.3.29}$$

so the zero-momentum potential for ϕ is $V = |W'(\phi)|^2$, as expected. Note that the previous potential might have extrema that breaks $\mathcal{N} = (2, 2)$ supersymmetry while supersymmetric vacua will always set $\langle F \rangle = \langle W'(\phi) \rangle = 0$. We will assume supersymmetry of the undeformed theory not to be spontaneously broken in our discussion.

Now suppose we deform by the supercurrent-squared operator to second order in λ, which gives the superspace expression (4.3.26). If we again perform the superspace integral and discard any terms involving derivatives or fermions, we now find the physical Lagrangian

$$\begin{aligned} \mathcal{L}\big|_{\partial_{\pm\pm}\phi=0} \ &= \ |F|^2 + FW' + \overline{F}\overline{W}' + \lambda \left(\frac{1}{2}|F|^4 - |F|^2|W'|^2 \right) + \frac{1}{4}\lambda^2|F|^4 \left(W'F + \overline{W}'\overline{F} \right) \\ &\quad -\frac{1}{2}\lambda^2|W'|^2|F|^4 + \frac{3}{4}\lambda^2|F|^6 \ . \end{aligned} \tag{4.3.30}$$

Remarkably, the equations of motion for the auxiliary F in (4.3.30) admit the solution $F = -\overline{W}'(\bar{\phi})$, $\overline{F} = -W'(\phi)$, which is the same as the unperturbed solution. This for instance implies that if we start from a supersymmetric vacua in the undeformed theory we will remain supersymmetric along the $T\overline{T}$ flow. On the one hand, this is not a surprise considering that we know the $T\overline{T}$ flow preserves the structure of the spectrum, and in particular should leave a zero-energy supersymmetric vacuum unperturbed. On the other hand, it is a reassuring check to see this property explicitly appearing in our analysis.

Returning to (4.3.30) and integrating out the auxiliary fields gives

$$\mathcal{L}\big|_{\partial_{\pm\pm}\phi=0} = -|W'(\phi)|^2 - \frac{1}{2}\lambda|W'(\phi)|^4 - \frac{1}{4}\lambda^2|W'(\phi)|^6 \ . \tag{4.3.31}$$

These are the leading terms in the geometric series $\frac{-|W'|^2}{1-\frac{1}{2}\lambda|W'|^2}$. In fact, up to conventions for the scaling of λ, one could have predicted this outcome from the form of the supercurrent-squared operator and the known results for $T\overline{T}$ deformations of a bosonic theory with a potential [3]. We know that, up to terms which vanish on-shell, the effect of adding supercurrent-squared to the physical Lagrangian is to deform by the usual $T\overline{T}$ operator. However, in the zero-momentum sector, we see that the $T\overline{T}$ deformation reduces to deforming by the square of the potential:

$$T\overline{T}\Big|_{\partial_{\pm\pm}\phi=0} = \mathcal{L}^2\Big|_{\partial_{\pm\pm}\phi=0} = V^2 \ . \tag{4.3.32}$$

Therefore, it is easy to solve for the deformed potential if we deform a physical Lagrangian $\mathcal{L} = f(\lambda, \partial_{\pm\pm}\phi) + V(\lambda, \phi)$ by $T\overline{T}$, since the flow equation for the potential term is simply

$$\partial_\lambda \mathcal{L} = \frac{\partial V}{\partial \lambda} = V^2 \ , \tag{4.3.33}$$

which admits the solution

$$V(\lambda, \phi) = \frac{V(0,\phi)}{1 - \lambda V(0,\phi)} \ . \tag{4.3.34}$$

We can apply this result to the Lagrangian (4.3.28), treating the entire expression involving the auxiliary field F as a potential (since it is independent of derivatives). The deformed theory has a zero-momentum piece which is therefore equivalent to

$$S(\lambda)\Big|_{\partial_{\pm\pm}\phi=0} = \int d^2x \, \frac{\left(|F|^2 + W'(\phi)F + \overline{W}'(\bar{\phi})\overline{F}\right)}{1 - \lambda\left(|F|^2 + W'(\phi)F + \overline{W}'(\bar{\phi})\overline{F}\right)} \ , \tag{4.3.35}$$

at least on-shell. Integrating out the auxiliary now gives

$$S(\lambda)\Big|_{\partial_{\pm\pm}\phi=0} = \int d^2x \, \frac{-|W'(\phi)|^2}{1-\lambda|W'(\phi)|^2} \tag{4.3.36}$$

as the deformed physical potential. This matches the first few terms of (4.3.31), up to a convention-dependent factor of $\frac{1}{2}$ in the scaling of λ.

Now one might ask what superspace Lagrangian would yield the physical action (4.3.36) after performing the $d\theta$ integrals. One candidate is

$$\mathcal{L}(\lambda)\Big|_{\partial_{\pm\pm}\phi=0} \sim \int d^4\theta \, \left(\overline{\Phi}\Phi - \lambda|W(\Phi)|^2\right) + \int d^2\theta \, W(\Phi) + \int d^2\overline{\theta} \, \overline{W}(\overline{\Phi}) \ , \tag{4.3.37}$$

where here $\sim$ means "this superspace Lagrangian gives an equivalent zero-momentum physical potential for the boson ϕ on-shell."

It is important to note that (4.3.37) is not the true solution for the deformed superspace Lagrangian using supercurrent-squared. The genuine solution involves a four-fermion term, all possible two-fermion terms, and more complicated dependence on the variable $y = \lambda(D_+D_-\Phi)(\overline{D}_+\overline{D}_-\overline{\Phi})$ in the zero-fermion term. However, if one were to perform the superspace integral in the true solution and then integrate out the auxiliary field F using its equation of motion, one would obtain the same zero-momentum potential for ϕ as we find by performing the superspace integral in (4.3.37) and integrating out F.

The form (4.3.37) is interesting because it shows that the effect of supercurrent-squared on the physical potential for ϕ can be interpreted as a change in the Kähler metric, which for this Lagrangian is

$$K_{\Phi\overline{\Phi}} = 1 - \lambda|W'(\Phi)|^2 \ . \tag{4.3.38}$$

When one performs the superspace integrals in (4.3.37), the result is

$$\mathcal{L}\Big|_{\partial_{\pm\pm}\phi=0} = K_{\Phi\overline{\Phi}}|F|^2 + W'(\phi)F + \overline{W}'(\bar{\phi})\overline{F} \ , \tag{4.3.39}$$

which admits the solution $F = -\frac{\overline{W}'(\bar{\phi})}{K_{\Phi\overline{\Phi}}}$. Substituting this solution gives

$$\mathcal{L}\Big|_{\partial_{\pm\pm}\phi=0} = \frac{-|W'(\phi)|^2}{K_{\Phi\overline{\Phi}}} = \frac{-|W'(\phi)|^2}{1-\lambda|W'(\phi)|^2} \ , \tag{4.3.40}$$

which agrees with (4.3.36).

As already mentioned, supersymmetric vacua of the original, undeformed, theory are associated with critical points of the superpotential $W(\phi)$. Any vacuum of the undeformed theory will persist in the deformed theory: near a point where $W'(\phi) = 0$, we see that the physical potential $V(\phi) = \frac{|W'|^2}{1-\lambda|W'|^2}$ also vanishes (away from the pole $|W'|^2 = \frac{1}{\lambda}$, the deformed potential is a monotonically increasing function of $|W'|^2$). Further, the auxiliary field F does not acquire a vacuum expectation value because $F = -\overline{W}'(\bar{\phi})$ remains a solution to its equations of motion in the deformed theory. Once more, this indicates that supersymmetry is unbroken along the whole $T\overline{T}$ flow if is in the undeformed theory.

However, this classical analysis suggests that the soliton spectrum of the theory has changed dramatically at any finite deformation parameter λ. There are now generically poles in the physical potential $V(\phi)$ at points where $|W'|^2 = \frac{1}{\lambda}$ which might separate distinct supersymmetric vacua of the theory. For instance, if the original theory had a double-well superpotential with two critical points ϕ_1, ϕ_2 where $W'(\phi_i) = 0$, then this undeformed theory supports BPS soliton solutions which interpolate between these two vacua. But if the superpotential W reaches a value of order $\frac{1}{\lambda}$ at some point between ϕ_1 and ϕ_2, then this soliton solution appears naively forbidden in the deformed theory because it requires crossing an infinite potential barrier. Another way of seeing this is by considering the effective Kähler potential (4.3.38), which would change sign at some point between the two supersymmetric vacua in the deformed theory and thus give rise to a negative-definite Kähler metric.

Our discussion has been purely classical. As we emphasised in the introduction, a fully quantum analysis of this problem is desirable, though subtle because of the non-local nature of the $T\overline{T}$ deformation. The advantage of performing such an analysis in models with extended supersymmetry is that holomorphy and associated non-renormalization theorems provide control over the form of any possible quantum corrections. For example, the superpotential for the models studied in this work is not renormalized perturbatively along the flow. It would be interesting to examine the structure of perturbative quantum corrections along the lines of [39], but in superspace with manifest supersymmetry. It should be possible to absorb any quantum corrections visible in perturbation theory by a change in the D-term Kähler potential meaning that at least the structure of the supersymmetric vacua would be preserved.

CHAPTER 5
NON-LINEARLY REALIZED SYMMETRIES

In this chapter, we will investigate certain additional symmetries which are present in some $T\overline{T}$ and supercurrent-squared deformed models. The treatment of this chapter follows the paper "Non-Linear Supersymmetry and $T\overline{T}$-like Flows" [7].

It was briefly mentioned in Chapter 4 that our solution (4.3.18) for the supercurrent-squared deformation of a chiral multiplet possesses a second non-linearly realized supersymmetry – which we will demonstrate in more detail shortly – but there are already examples of such additional symmetries in the non-supersymmetric context. For instance, we reviewed in Section 2.1.2 that the $T\overline{T}$ deformation of a free boson is given by

$$\mathcal{L}_\lambda = \frac{1}{2\lambda}\left(\sqrt{1 + 2\lambda \partial_\mu \phi \partial^\mu \phi + 1} - 1\right) . \tag{5.0.1}$$

Ignoring the constant $-\frac{1}{2\lambda}$, this is of the same form as the Dirac Lagrangian which describes the transverse fluctuations ϕ on a brane. The Dirac action for a p-brane is usually written

$$S_{\text{Dirac}} = -T_p \int d^{p+1}\xi \, \sqrt{-\det(\gamma)} \, , \tag{5.0.2}$$

where T_p is the tension of the Dp-brane and

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu} \tag{5.0.3}$$

is the pullback of the target space metric $\eta_{\mu\nu}$ onto the brane's worldvolume. We can choose static gauge which identifies the $p+1$ worldvolume coordinates ξ^a with the target space coordinates X^a, and the remaining worldvolume coordinates are transverse oscillations:

$$\begin{aligned} X^a &= \xi^a \, , & a &= 0, \cdots, p \, , \\ X^I &= 2\pi\alpha' \phi^I(\xi) \, , & I &= p+1, \cdots D-1 \, . \end{aligned} \tag{5.0.4}$$

Here D is the total dimension of the target spacetime and α' is a parameter which is proportional to the square of the string length. In the case of a $D1$-string embedded in a three dimensional spacetime, so that $p = 1$ and $D = 3$, the Dirac action can be written as

$$S_{\text{Dirac}} = -T_p \int d^2\xi \sqrt{1 + 2\pi\alpha' \partial^a \phi \partial_a \phi} \, , \tag{5.0.5}$$

which is the same form as the $T\overline{T}$ deformed Lagrangian (5.0.1). Here α' plays the role of the $T\overline{T}$ parameter λ.

It is well-known that the Dp-brane Lagrangian (5.0.2) possesses a non-linearly realized symmetry for any p. We will describe this symmetry in the case (5.0.5) of a $D1$-string. In this case, for either fixed index $a \in \{0, 1\}$, the action (5.0.5) is invariant under the transformation

$$\delta\phi = \xi^a + \phi\partial^a\phi \, . \tag{5.0.6}$$

The interpretation of this symmetry is that the presence of the brane as an embedded hypersurface spontaneously breaks part of the Poincaré symmetry of the ambient spacetime. The scalars ϕ are then the Nambu-Goldstone bosons of this spontaneously broken symmetry, which is then non-linearly realized. One can view the transformation (5.0.6) as a rotation which rotates the transverse direction ϕ into the worldvolume direction ξ^a, breaking static gauge, along with a compensating worldvolume diffeomorphism which then restores static gauge. This symmetry, and its extensions to the Dirac-Born-Infeld action which includes a gauge field, are nicely discussed in [40–42].

5.1 $D = 2$ $\mathcal{N} = (2, 2)$ Flows and Non-Linear $\mathcal{N} = (2, 2)$ Supersymmetry

In this section, we are going to explore in detail how the non-linear supersymmetries – which we briefly alluded to in Chapter 4 – arise for the simplest $\mathcal{N} = (2, 2)$ $T\overline{T}$ flows. The undeformed models are supersymmetrized theories of free scalars, while the deformed models are $\mathcal{N} = (2, 2)$ supersymmetric extensions of the $D = 4$ gauge-fixed Nambu-Goto string.

5.1.1 $T\overline{T}$ deformations with $\mathcal{N} = (2, 2)$ supersymmetry

In Chapter 4, supersymmetric flows for various theories were studied; the conclusion of Section 4.2.2 was that $T\overline{T}(x)$ operator of a supersymmetric theory is related to a supersymmetric descendant operator $\mathcal{T}\overline{\mathcal{T}}(x)$,

$$\mathcal{T}\overline{\mathcal{T}}(x) = T\overline{T}(x) + \text{EOM} + \partial_{++}(\cdots) + \partial_{--}(\cdots) \, . \tag{5.1.1}$$

The previous equation states the equivalence of $T\overline{T}(x)$ and $\mathcal{T}\overline{\mathcal{T}}(x)$ up to total derivatives and terms that vanish on-shell, which we have indicated with "EOM".

The simplest cases of deformed models, on which we will focus in this section, are $T\overline{T}$-deformed theories of free scalars, fermions and auxiliary fields. In the case of $D = 2$ $\mathcal{N} = (2, 2)$ supersymmetry, a scalar multiplet can have several different off-shell representations

[43–46]. The two cases we will consider here are chiral and twisted-chiral supermultiplets, which are the most commonly studied cases.

In $\mathcal{N}=(2,2)$ superspace, parametrized by coordinates $\zeta^M=(x^{\pm\pm},\theta^\pm,\bar\theta^\pm)$, let the complex superfields $X(x,\theta)$ and $Y(x,\theta)$ satisfy chiral and twisted-chiral constraints, respectively,

$$\bar D_\pm X=0\ ,\quad \bar D_+ Y=D_- Y=0\ . \tag{5.1.2}$$

In this chapter, we use a slightly different conventions for the supercovariant derivatives than in Chapter 4. These were referred to as $D'_\pm$ and $\overline{D}'_\pm$ in equation (2.2.13), but we will drop the primes in the following discussion. For convenience, the conventions are repeated here:

$$D_\pm = \frac{\partial}{\partial\theta^\pm}+i\bar\theta^\pm\partial_{\pm\pm}\ ,\quad \bar D_\pm=-\frac{\partial}{\partial\bar\theta^\pm}-i\theta^\pm\partial_{\pm\pm}\ , \tag{5.1.3a}$$

$$Q_\pm = i\frac{\partial}{\partial\theta^\pm}+\bar\theta^\pm\partial_{\pm\pm}\ ,\quad \bar Q_\pm=-i\frac{\partial}{\partial\bar\theta^\pm}-\theta^\pm\partial_{\pm\pm}\ . \tag{5.1.3b}$$

These satisfy

$$D_\pm^2 = \bar D_\pm^2=0\ ,\qquad \{D_\pm,\bar D_\pm\}=-2i\partial_{\pm\pm}\ ,\qquad [D_\pm,\partial_{\pm\pm}]=[\bar D_\pm,\partial_{\pm\pm}]=0\ , \tag{5.1.4a}$$

$$Q_\pm^2 = \bar Q_\pm^2=0\ ,\qquad \{Q_\pm,\bar Q_\pm\}=-2i\partial_{\pm\pm}\ ,\qquad [Q_\pm,\partial_{\pm\pm}]=[\bar Q_\pm,\partial_{\pm\pm}]=0\ . \tag{5.1.4b}$$

There is one more caveat worth mentioning: in much of the $\mathcal{N}=(2,2)$ literature, twisted-chiral multiplets, often denoted Σ in this context, naturally arise as field strengths for $\mathcal{N}=(2,2)$ vector superfields V. The lowest component of such a superfield is a complex scalar, but the top component proportional to $\bar\theta^-\theta^+$ encodes the gauge-field strength along with a real auxiliary field. On the other hand, there are twisted chiral superfields denoted Y whose bottom component is a complex scalar and whose top component is just a complex auxiliary field. It is to this latter case that we restrict. The free Lagrangians for these supermultiplets are given by

$$\mathcal{L}_0^{\rm c}=\int d^4\theta\, X\bar X\ ,\qquad \mathcal{L}_0^{\rm tc}=-\int d^4\theta\, Y\bar Y\ . \tag{5.1.5}$$

In [6] it was shown that the following Lagrangian

$$\mathcal{L}_\lambda^{\rm c} = \int d^4\theta\left(X\bar X+\frac{\lambda D_+X\overline{D}_+\bar X D_-X\overline{D}_-\bar X}{1-\frac12\lambda A+\sqrt{1-\lambda A+\frac14\lambda^2B^2}}\right)\ , \tag{5.1.6a}$$

with

$$A=\partial_{++}X\partial_{--}\bar{X}+\partial_{++}\bar{X}\partial_{--}X\ ,\quad B=\partial_{++}X\partial_{--}\bar{X}-\partial_{++}\bar{X}\partial_{--}X\ ,\qquad (5.1.6b)$$

is a solution of the flow equation (4.2.10) on-shell, and hence describes the $T\overline{T}$-deformation of the free chiral supermultiplet Lagrangian (5.1.5).

A simple way to generate the $T\overline{T}$-deformation of the free twisted-chiral theory is to remember that a twisted-chiral multiplet can be obtained from a chiral one by acting with a $\mathbb{Z}_2$ automorphism on the Grassmann coordinates of $\mathcal{N}=(2,2)$ superspace:

$$\theta^{+}\leftrightarrow\theta^{+}\ ,\quad \theta^{-}\leftrightarrow-\bar{\theta}^{-}\ .\qquad (5.1.7)$$

This leaves the D_+ and $\bar{D}_+$ derivatives invariant while it exchanges D_- with $\bar{D}_-$. As a result, the chiral and twisted-chiral differential constraints (5.1.2) are mapped into each others under the automorphism (5.1.7).[1]

Under the $\mathbb{Z}_2$ automorphism (5.1.7), the Lagrangian (5.1.6a) turns into the following twisted-chiral Lagrangian

$$\mathcal{L}^{\rm tc}_{\lambda}\ =\ -\int d^4\theta\left(Y\bar{Y}+\frac{\lambda D_+Y\overline{D}_+\bar{Y}\overline{D}_-YD_-\bar{Y}}{1-\frac{1}{2}\lambda A+\sqrt{1-\lambda A+\frac{1}{4}\lambda^2B^2}}\right)\ ,\qquad (5.1.8a)$$

where

$$A=\partial_{++}Y\partial_{--}\bar{Y}+\partial_{++}\bar{Y}\partial_{--}Y\ ,\quad B=\partial_{++}Y\partial_{--}\bar{Y}-\partial_{++}\bar{Y}\partial_{--}Y\ .\qquad (5.1.8b)$$

Thanks to the map (5.1.7), by construction the Lagrangian (5.1.8a) is a $T\overline{T}$-deformation and its superspace Lagrangian $\mathcal{A}^{\rm tc}_{\lambda}$, $\mathcal{L}^{\rm tc}_{\lambda}=\int d^4\theta\,\mathcal{A}^{\rm tc}_{\lambda}$, is an on-shell solution of the following flow equation

$$\frac{d}{d\lambda}\mathcal{A}^{\rm tc}_{\lambda}=\frac{1}{8}\left(\mathcal{R}_{++}\mathcal{R}_{--}-2\mathcal{B}\bar{\mathcal{B}}\right)\ .\qquad (5.1.9)$$

Here $\mathcal{R}_{\pm\pm}(x,\theta)$, $\mathcal{B}(x,\theta)$ and its complex conjugate $\bar{\mathcal{B}}(x,\theta)$ are the local operators describing the $\mathcal{R}$-multiplet of currents for $D=2$ $\mathcal{N}=(2,2)$ supersymmetry that arise by applying (5.1.7) to the FZ multiplet of the chiral theory (5.1.6a) [6]. They satisfy the conservation

1. In the literature this $\mathbb{Z}_2$ automorphism (5.1.7) is often called the “mirror-map” or “mirror-image" because it exchanges the vector and axial $U(1)$ R-symmetries.

equations,

$$\overline{D}_+\mathcal{R}_{--} = i\overline{D}_-\mathcal{B} \ , \qquad D_-\mathcal{R}_{++} = iD_+\mathcal{B} \ , \qquad \overline{D}_+\mathcal{B} = D_-\mathcal{B} = 0 \ , \tag{5.1.10}$$

together with their complex conjugates. Like the case of the FZ-multiplet, the supercurrent-squared operator

$$\mathcal{T}\overline{\mathcal{T}}(x) = \int d^4\theta \, \mathcal{O}^{\mathcal{R}}(x,\theta) \ , \quad \mathcal{O}^{\mathcal{R}}(x,\theta) := -\mathcal{R}_{++}(x,\theta)\mathcal{R}_{--}(x,\theta) + 2\mathcal{B}(x,\theta)\bar{\mathcal{B}}(x,\theta) \ , \tag{5.1.11}$$

satisfies (5.1.1); namely, $\mathcal{T}\overline{\mathcal{T}}(x)$ is equivalent to $T\bar{T}(x)$ up to total derivatives and EOM, as we showed in Chapter 4.

Note that the bosonic truncation of both (5.1.6a) and (5.1.8a) give the Lagrangian

$$\mathcal{L}_{\lambda,\text{bos}} = \frac{\sqrt{1+2\lambda a+\lambda^2 b^2}-1}{4\lambda} = \frac{a}{4} - \lambda\frac{\partial_{++}\phi\partial_{--}\phi\partial_{++}\bar{\phi}\partial_{--}\bar{\phi}}{1+\lambda a+\sqrt{1+2\lambda a+\lambda^2 b^2}} \ , \tag{5.1.12}$$

where

$$a = \partial_{++}\phi\partial_{--}\bar{\phi} + \partial_{++}\bar{\phi}\partial_{--}\phi \ , \qquad b = \partial_{++}\phi\partial_{--}\bar{\phi} - \partial_{++}\bar{\phi}\partial_{--}\phi \ , \tag{5.1.13}$$

and ϕ is either $\phi = X|_{\theta=0}$ or $\phi = Y|_{\theta=0}$. This is the Lagrangian for the gauge-fixed Nambu-Goto string in four dimensions [3].

The aim of the remainder of this section is to show that the Lagrangians (5.1.6a) and (5.1.8a) are structurally identical to the Bagger–Galperin action for the $D = 4$ $\mathcal{N} = 1$ supersymmetric Born-Infeld theory [37], which we will analyse in detail in section 5.4. Since the Bagger–Galperin action possesses a second non-linearly realized $D = 4$ $\mathcal{N} = 1$ supersymmetry, we will show that the theories described by (5.1.6a) and (5.1.8a) also possess an extra set of non-linearly realized $\mathcal{N} = (2, 2)$ supersymmetries.

5.1.2 The $T\overline{T}$-deformed twisted-chiral model and partial-breaking

Let us start with the twisted-chiral Lagrangian (5.1.8a) which, as we will show, is the one more directly related to the $D = 4$ Bagger-Galperin action. In complete analogy to the $D = 4$ case, we are going to show that (5.1.8a) is a model for a Nambu-Goldstone multiplet of $D = 2$ $\mathcal{N} = (4, 4) \to \mathcal{N} = (2, 2)$ partial supersymmetry breaking. The analysis is similar in spirit to the $D = 4$ construction of the Bagger-Galperin action using $D = 4$ $\mathcal{N} = 2$ superspace proposed by Roček and Tseytlin [47]; see also [48–50] for more recent analysis.

To describe manifest $\mathcal{N} = (4, 4)$ supersymmetry we can use $\mathcal{N} = (4, 4)$ superspace which augments the $\mathcal{N}=(2,2)$ superspace coordinates $\zeta^M=(x^{\pm\pm},\theta^\pm,\bar\theta^\pm)$ of the previous section with the following additional complex Grassmann coordinates $(\eta^\pm,\bar\eta^\pm)$. The extra supercovariant derivatives and supercharges are given by

$$\mathcal{D}_+ = \frac{\partial}{\partial\eta^+}+i\bar\eta^+\partial_{++}\ , \quad \bar{\mathcal{D}}_+=-\frac{\partial}{\partial\bar\eta^+}-i\eta^+\partial_{++}\ , \tag{5.1.14a}$$

$$\mathcal{Q}_+ = i\frac{\partial}{\partial\eta^+}+\bar\eta^+\partial_{++}\ , \quad \bar{\mathcal{Q}}_+=-i\frac{\partial}{\partial\bar\eta^+}-\eta^+\partial_{++}\ , \tag{5.1.14b}$$

with similar expressions for $\mathcal{D}_-$ and $\mathcal{Q}_-$. They satisfy

$$\mathcal{D}_\pm^2 = \bar{\mathcal{D}}_\pm^2=0\ , \qquad \{\mathcal{D}_\pm,\bar{\mathcal{D}}_\pm\}=-2i\partial_{\pm\pm}\ , \qquad [\mathcal{D}_\pm,\partial_{\pm\pm}]=[\bar{\mathcal{D}}_\pm,\partial_{\pm\pm}]=0\ , \tag{5.1.15a}$$

$$\mathcal{Q}_\pm^2 = \bar{\mathcal{Q}}_\pm^2=0\ , \qquad \{\mathcal{Q}_\pm,\bar{\mathcal{Q}}_\pm\}=-2i\partial_{\pm\pm}\ , \qquad [\mathcal{Q}_\pm,\partial_{\pm\pm}]=[\bar{\mathcal{Q}}_\pm,\partial_{\pm\pm}]=0\ , \tag{5.1.15b}$$

while they (anti-)commute with all the usual $D_\pm$ and $Q_\pm$ operators.

Two-dimensional $\mathcal{N}=(4,4)$ supersymmetry can also be usefully described in the language of $\mathcal{N}=(2,2)$ superspace. In this section, we will largely refer to [51] for such a description. In this approach from the full $(4,4)$ supersymmetry, one copy of $(2,2)$ is manifest while a second $(2,2)$ is hidden. For our goal of describing a model of partial supersymmetry breaking, we view the hidden $(2,2)$ supersymmetry as broken and non-linearly realized. We will derive such a description starting from $\mathcal{N}=(4,4)$ superspace and describe the broken/hidden supersymmetry using the $\eta^\pm$ directions.

The hidden supersymmetry transformation of a generic $D=2$ $\mathcal{N}=(4,4)$ superfield $U=U(x^{\pm\pm},\theta^\pm,\bar\theta^\pm,\eta^\pm,\bar\eta^\pm)$ under the hidden $(2,2)$ supersymmetry is

$$\delta U=i(\epsilon^+\mathcal{Q}_++\epsilon^-\mathcal{Q}_--\bar\epsilon^+\bar{\mathcal{Q}}_+-\bar\epsilon^-\bar{\mathcal{Q}}_-)U\ . \tag{5.1.16}$$

The $(2,2)$ supersymmetry, generated by the $Q_\pm$ and $\bar Q_\pm$ operators, will always be manifest and preserved, so we will not bother to discuss it in detail. For convenience, we also introduce the chiral coordinate $y^{\pm\pm}=x^{\pm\pm}+i\eta^\pm\bar\eta^\pm$. Using this coordinate, the spinor covariant derivatives and supercharges take the form

$$\mathcal{D}_\pm = \frac{\partial}{\partial\eta^\pm}+2i\bar\eta^\pm\frac{\partial}{\partial y^{\pm\pm}}\ , \quad \bar{\mathcal{D}}_\pm=-\frac{\partial}{\partial\bar\eta^\pm}\ , \tag{5.1.17a}$$

$$\mathcal{Q}_\pm = i\frac{\partial}{\partial\eta^\pm}\ , \quad \bar{\mathcal{Q}}_\pm=-i\frac{\partial}{\partial\bar\eta^\pm}-2\eta^\pm\frac{\partial}{\partial y^{\pm\pm}}\ . \tag{5.1.17b}$$

After this technical introduction, let us turn to our main construction. Consider a $(4,4)$

superfield which is chiral under the hidden $(2,2)$ supersymmetry:

$$\bar{\mathcal{D}}_\pm \boldsymbol{\mathcal{X}} = 0 \ . \tag{5.1.18}$$

We can expand it in terms of hidden fermionic coordinates,

$$\boldsymbol{\mathcal{X}} = X + \eta^+ X_+ + \eta^- X_- + \eta^+ \eta^- F \ , \tag{5.1.19}$$

where $X = X(y^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$, $X_\pm = X_\pm(y^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$ and $F = F(y^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$ are themselves $(2,2)$ superfields. In the following discussion, we will keep the $\theta^\pm$, $\bar{\theta}^\pm$ dependence implicit. The hidden $(2,2)$ supersymmetry transformation rules can then be straightforwardly computed using (5.1.16) and (5.1.19). They take the form

$$\delta X = -\epsilon^+ X_+ - \epsilon^- X_- \ , \tag{5.1.20a}$$

$$\delta X_\pm = \mp \epsilon^\mp F - 2i\bar{\epsilon}^\pm \partial_{\pm\pm} X, \tag{5.1.20b}$$

$$\delta F = -2i\bar{\epsilon}^- \partial_{--} X_+ + 2i\bar{\epsilon}^+ \partial_{++} X_- \ . \tag{5.1.20c}$$

The $\boldsymbol{\mathcal{X}}$ superfield is still reducible under $\mathcal{N} = (4,4)$ supersymmetry so we can put additional constraints on the $(2,2)$ superfields X, $X_\pm$ and F. Here we will consider $(4,4)$ twisted multiplets, and refer the reader to [43, 52–57] for a more detailed analysis. For this discussion, we will follow the $\mathcal{N} = (2,2)$ superspace description of [51]. One type of twisted multiplet with $(4,4)$ supersymmetry can be defined by setting

$$X_+ = \bar{D}_+ \bar{Y}, \qquad X_- = -\bar{D}_- Y \ , \tag{5.1.21}$$

where X and Y are chiral and twisted-chiral, respectively, under the manifest $(2,2)$ supersymmetry:

$$\bar{D}_+ X = \bar{D}_- X = \bar{D}_+ Y = D_- Y = 0 \ , \qquad D_+ \bar{X} = D_- \bar{X} = D_+ \bar{Y} = \bar{D}_- \bar{Y} = 0 \ . \tag{5.1.22}$$

The superfield (5.1.19) becomes

$$\boldsymbol{\mathcal{X}} = X + \eta^+ \overline{D}_+ \bar{Y} - \eta^- \bar{D}_- Y + \eta^+ \eta^- F \ . \tag{5.1.23}$$

The supersymmetry transformation rules then become

$$\delta X = -\epsilon^+ \bar{D}_+ \bar{Y} + \epsilon^- \bar{D}_- Y \ , \tag{5.1.24a}$$

$$\delta F = -2i\bar{\epsilon}^- \partial_{--} \bar{D}_+ \bar{Y} - 2i\bar{\epsilon}^+ \partial_{++} \bar{D}_- Y \ , \tag{5.1.24b}$$

while $\delta X_\pm$ remains the same as (5.1.20b). By using the conjugation property for two fermions, $\overline{\chi\xi} = \bar{\xi}\bar{\chi} = -\bar{\chi}\bar{\xi}$, and the conjugation property $\overline{D_+ A} = \bar{D}_+ \bar{A}$ for a bosonic superfield A, it follows that

$$\delta \bar{X} = \bar{\epsilon}^+ D_+ Y - \bar{\epsilon}^- D_- \bar{Y} \ . \tag{5.1.25}$$

One can check that

$$\delta \bar{D}^2 \bar{X} = \bar{D}^2 \delta \bar{X} = \bar{D}^2 \Big(\bar{\epsilon}^+ D_+ Y - \bar{\epsilon}^- D_- \bar{Y} \Big) = 2i\bar{\epsilon}^+ \partial_{++} \bar{D}_- Y + 2i\bar{\epsilon}^- \partial_{--} \bar{D}_+ Y \ , \tag{5.1.26}$$

where $\bar{D}^2 = \bar{D}_+ \bar{D}_-$. Note that in the first equality we made use of the fact that the manifest and hidden $(2,2)$ supersymmetries are independent. The supersymmetry transformation rule for $-\bar{D}^2 X$ is then exactly that of the auxiliary field F. Thus we can consistently set

$$F = -\bar{D}^2 \bar{X} \ , \tag{5.1.27}$$

which is the last constraint necessary to describe a version of the $(4,4)$ twisted multiplet in terms of a chiral and twisted-chiral $\mathcal{N} = (2,2)$ superfields. The resulting $(4,4)$ superfield $\boldsymbol{\mathcal{X}}$, expanded in terms of the hidden $(2,2)$ fermionic coordinates, takes the form

$$\boldsymbol{\mathcal{X}} = X + \eta^+ \overline{D}_+ \bar{Y} - \eta^- \bar{D}_- Y - \eta^+ \eta^- \bar{D}^2 \bar{X} \ , \tag{5.1.28}$$

which closely resembles the expansion of a $D = 4$ $\mathcal{N} = 2$ vector multiplet when one identifies the analogue of the $D = 4$ $\mathcal{N} = 1$ chiral vector multiplet field strength W_α with the $(2,2)$ chiral superfields $\overline{D}_+ \bar{Y}$ and $\overline{D}_- Y$. Note in particular that $\boldsymbol{\mathcal{X}}$ turns to be $(4,4)$ chiral:

$$\bar{D}_\pm \boldsymbol{\mathcal{X}} = 0 \ , \quad \bar{\mathcal{D}}_\pm \boldsymbol{\mathcal{X}} = 0 \ . \tag{5.1.29}$$

To summarize: the entire $(4,4)$ off-shell twisted multiplet is described in terms of one chiral and one twisted-chiral $(2,2)$ superfield, which possess the following hidden $(2,2)$ supersymmetry transformations:

$$\delta X = -\epsilon^+ \bar{D}_+ \bar{Y} + \epsilon^- \bar{D}_- Y \ , \tag{5.1.30a}$$

$$\delta Y = \bar{\epsilon}^- D_- X + \epsilon^+ \bar{D}_+ \bar{X} \ . \tag{5.1.30b}$$

Let us now introduce the action for a free $\mathcal{N} = (4,4)$ twisted multiplet. Taking the square of $\boldsymbol{\mathcal{X}}$ in (5.1.28) we obtain

$$\boldsymbol{\mathcal{X}}^2 = \eta^+\eta^-\Big(- 2X\bar{D}^2\bar{X} + 2\bar{D}_+\bar{Y}\bar{D}_-Y\Big) + \dots \ , \tag{5.1.31}$$

where the ellipses denote terms that are not important for our analysis. Since $\boldsymbol{\mathcal{X}}$ and therefore $\boldsymbol{\mathcal{X}}^2$ are chiral superfields, we can consider the chiral integral in the hidden direction

$$\int d\eta^+ d\eta^- \boldsymbol{\mathcal{X}}^2 = 2X\bar{D}^2\bar{X} - 2\bar{D}_+\bar{Y}\cdot\bar{D}_-Y \ . \tag{5.1.32}$$

Note also that, since X and Y are chiral and twisted-chiral under the manifest supersymmetry (5.1.22), it follows that

$$\begin{aligned}\int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) &= \int d^2x\, d\theta^+ d\theta^- \bar{D}_+\bar{D}_-(X\bar{X} - Y\bar{Y}) \ , \\ &= \int d^2x\, d\theta^+ d\theta^- \Big(X\bar{D}_+\bar{D}_-\bar{X} - \bar{D}_+\bar{Y}\cdot\bar{D}_-Y\Big) , \end{aligned} \tag{5.1.33}$$

which can also be rewritten as

$$\int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) = \int d^2x\, d\bar{\theta}^+ d\bar{\theta}^- \Big(\bar{X}D_+D_-X - D_+Y\cdot D_-\bar{Y}\Big) \ . \tag{5.1.34}$$

The sum of the two equations above yields

$$4\int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) = \int d^2x\, d\theta^+ d\theta^- d\eta^+ d\eta^- \boldsymbol{\mathcal{X}}^2 + c.c. \ . \tag{5.1.35}$$

The left-hand side has an enhanced $\mathcal{N} = (4,4)$ supersymmetry as discussed in [51]. This becomes manifest from our $(4,4)$ superspace construction on the right-hand side.

To describe $\mathcal{N} = (4,4) \to \mathcal{N} = (2,2)$ supersymmetry breaking we can appropriately deform the $(4,4)$ twisted multiplet. Analogous to the case of a $D = 4$ $\mathcal{N} = 2$ vector multiplet deformed by a magnetic Fayet-Iliopoulos term [58] (see also [47–50, 59]), we add a deformation parameter to the auxiliary field F of $\boldsymbol{\mathcal{X}}$, which is deformed to

$$\boldsymbol{\mathcal{X}}_{\rm def} = X + \eta^+ D_+\bar{Y} - \eta^-\bar{D}_-Y - \eta^+\eta^-\Big(\bar{D}^2\bar{X} + \kappa\Big) \ . \tag{5.1.36}$$

Assuming that the auxiliary field F gets a VEV, $\langle F\rangle = \kappa$ or equivalently $\langle \overline{D}^2\bar{X}\rangle = 0$, then by looking at the supersymmetry transformations of $X_\pm$ for the deformed multiplet

$$\delta X_\pm \;=\; \pm\epsilon^\mp\Big(\bar{D}^2\bar{X}+\kappa\Big)-2i\bar{\epsilon}^\pm\partial_{\pm\pm}X\ , \tag{5.1.37}$$

we can see the $\mathcal{N}=(4,4)\to\mathcal{N}=(2,2)$ supersymmetry breaking pattern arises; specifically, the hidden $\mathcal{N}=(2,2)$ is spontaneously broken and non-linearly realized. For later use, it is important to stress that, though the hidden transformations of $\delta X_\pm$ are modified by the non-linear term proportional to κ, the hidden transformation of X remains the same as in the undeformed case given in eq. (5.1.24a).

In analogy to the $D=4$ case of [47–49], to describe the Goldstone multiplet associated to partial supersymmetry breaking we impose the following nilpotent constraint on the deformed $(4,4)$ twisted superfield:

$$\boldsymbol{\mathcal{X}}^2_{\text{def}}=0=-2\eta^+\eta^-\Big(X(\kappa+\bar{D}^2\bar{X})-\bar{D}_+\bar{Y}\cdot\bar{D}_-Y\Big)+\dots\ . \tag{5.1.38}$$

This implies the constraint

$$X\Big(\kappa+\bar{D}^2\bar{X}\Big)-\bar{D}_+\bar{Y}\cdot\bar{D}_-Y=0\ , \tag{5.1.39}$$

which requires

$$X=\frac{\bar{D}_+\bar{Y}\cdot\bar{D}_-Y}{\kappa+\bar{D}^2\bar{X}}=\frac{W^2}{\kappa+\bar{D}^2\bar{X}}\ , \tag{5.1.40}$$

and its conjugate

$$\bar{X}=-\frac{D_+Y\cdot D_-\bar{Y}}{\kappa+D^2X}=-\frac{\bar{W}^2}{\kappa+D^2X}\ . \tag{5.1.41}$$

Here $\bar{D}^2=\bar{D}_+\bar{D}_-$, $D^2=-D_+D_-$ and we have introduced the superfields:

$$W^2 \;=\; -X_+X_-=\bar{D}_+\bar{Y}\cdot\bar{D}_-Y=\bar{D}_+\bar{D}_-(Y\bar{Y})=\bar{D}^2(Y\bar{Y})\ , \tag{5.1.42a}$$

$$\bar{W}^2 \;=\; \bar{X}_+\bar{X}_-=-D_+Y\cdot D_-\bar{Y}=-D_+D_-(Y\bar{Y})=D^2(Y\bar{Y})\ . \tag{5.1.42b}$$

The constraint (5.1.39) is the $D=2$ analogue of the Bagger-Galperin constraint for a Maxwell-Goldstone multiplet for $D=4$ $\mathcal{N}=2\to\mathcal{N}=1$ supersymmetry breaking [37]. Combining (5.1.40) and (5.1.41) gives

$$\kappa X=\bar{D}^2(Y\bar{Y}-X\bar{X})=\bar{D}^2\Big[Y\bar{Y}-\frac{\bar{D}_+\bar{Y}\cdot\bar{D}_-Y\cdot D_-\bar{Y}\cdot D_+Y}{(\kappa+D^2X)(\kappa+\bar{D}^2\bar{X})}\Big]\ , \tag{5.1.43}$$

which is consistent thanks to the κ terms in the denominator. Because of the four fermion coupling in the numerator of the last term, no fermionic terms can appear in the denominator. So effectively we have the equation

$$(\kappa + D^2 X)_{\rm eff} = \Big(\kappa + D^2 \frac{W^2}{\kappa + \bar{D}^2 \bar{X}}\Big)_{\rm eff} = \kappa + \frac{D^2 W^2}{\kappa + (\bar{D}^2 \bar{X})_{\rm eff}} \ , \tag{5.1.44}$$

and its conjugate

$$(\kappa + \bar{D}^2 \bar{X})_{\rm eff} = \kappa + \frac{\bar{D}^2 \bar{W}^2}{\kappa + (D^2 X)_{\rm eff}} \ . \tag{5.1.45}$$

Solving them we get

$$(D^2 X)_{\rm eff} = \frac{B - \kappa^2 + \sqrt{B^2 + 2\kappa^2 A + \kappa^4}}{2\kappa} \ , \tag{5.1.46a}$$

$$(\bar{D}^2 \bar{X})_{\rm eff} = \frac{-B - \kappa^2 + \sqrt{B^2 + 2\kappa^2 A + \kappa^4}}{2\kappa} \ . \tag{5.1.46b}$$

Substituting these expressions into (5.1.43) gives

$$X = \frac{1}{\kappa} \bar{D}^2 \Upsilon \ , \qquad \bar{X} = \frac{1}{\kappa} D^2 \Upsilon \ , \qquad \Upsilon = \bar{\Upsilon} = Y\bar{Y} - \frac{2W^2\bar{W}^2}{A + \kappa^2 + \sqrt{B^2 + 2\kappa^2 A + \kappa^4}} \ , \tag{5.1.47}$$

where

$$A = D^2 W^2 + \bar{D}^2 \bar{W}^2 = \{D^2, \bar{D}^2\}(Y\bar{Y}) = \partial_{++} Y \partial_{--} \bar{Y} + \partial_{++} \bar{Y} \partial_{--} Y \ , \tag{5.1.48a}$$

$$B = D^2 W^2 - \bar{D}^2 \bar{W}^2 = [D^2, \bar{D}^2](Y\bar{Y}) = \partial_{++} Y \partial_{--} \bar{Y} - \partial_{++} \bar{Y} \partial_{--} Y \ . \tag{5.1.48b}$$

The result is that the $\mathcal{N} = (2,2)$ chiral part X of the $\mathcal{N} = (4,4)$ twisted multiplet is expressed in terms of the $(2,2)$ twisted-chiral superfield Y. Thanks to the linearly realized construction in terms of $(4,4)$ superfields, it is straightforward to obtain the non-linearly realized $\mathcal{N} = (2,2)$ supersymmetry transformations for Y. In particular, it suffices to look at the transformations of $D_+ Y$ and $\overline{D}_- Y$ that can be obtained by substituting back the composite expression for $X = X[Y]$ into the transformations (5.1.37). By construction, these expressions ensure that δX transforms according to (5.1.24a).

Since X is chiral under the manifest $(2,2)$ supersymmetry (5.1.22), we can consider the chiral integral

$$S_{\kappa^2} = -\frac{1}{2}\kappa \int d^2x \, d\theta^+ d\theta^- X + c.c = -\frac{1}{2} \int d^2x \, d\theta^+ d\theta^- \bar{D}^2 \Upsilon + c.c.$$

$$= - \int d^2x \, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \Upsilon \ . \tag{5.1.49}$$

A remarkable property of this action is that it is invariant under the hidden non-linearly realized supersymmetry. Using (5.1.24a), we see that

$$\delta S_{\kappa^2} = -\frac{1}{2}\kappa \int d^2x D_+ D_- \delta X\Big|_{\theta=\bar{\theta}=0} + c.c. \ , \tag{5.1.50a}$$

$$= -\frac{1}{2}\kappa \int d^2x\Big(-2i\epsilon^- \partial_{--} D_+ Y - 2i\epsilon^+ \partial_{++} D_- \bar{Y}\Big)\Big|_{\theta=\bar{\theta}=0} + c.c. = 0 \ , \tag{5.1.50b}$$

where we used the fact that Y is a twisted-chiral superfield (5.1.22).

Explicitly, the action reads

$$S_{\kappa^2} = -\int d^2x \, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \left(Y\bar{Y} - \frac{2W^2\bar{W}^2}{\kappa^2 + A + \sqrt{\kappa^4 + 2\kappa^2 A + B^2}} \right) , \tag{5.1.51}$$

which precisely matches the model of eq. (5.1.8a) if we identify the coupling constants:

$$\lambda = -\frac{2}{\kappa^2} \ . \tag{5.1.52}$$

This shows explicitly that the $T\overline{T}$-deformation of the free twisted-chiral action possesses a non-linearly realized $\mathcal{N} = (2,2)$ hidden supersymmetry.

5.1.3 The $T\overline{T}$-deformed chiral model and partial-breaking

Let us now turn to the $T\overline{T}$ deformation of the free chiral model of eq. (5.1.6a). The construction follows the previous subsection with the difference that we will start with a different formulation of the (4,4) twisted multiplet described in terms of (2,2) superfields. Consider again an $\mathcal{N} = (4,4)$ superfield which is chiral under the hidden (2,2) supersymmetry:

$$\bar{\mathcal{D}}_+ \boldsymbol{\mathcal{Y}} = \bar{\mathcal{D}}_- \boldsymbol{\mathcal{Y}} = 0 \ . \tag{5.1.53}$$

Its expansion in hidden superspace variables is

$$\boldsymbol{\mathcal{Y}} = Y + \eta^+ Y_+ + \eta^- Y_- + \eta^+ \eta^- G \ , \tag{5.1.54}$$

where $Y = Y(y^{\pm\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$, $Y_\pm = Y_\pm(y^{\pm\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$ and $G = G(y^{\pm\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$ are themselves superfields with manifest (2,2) supersymmetry. The hidden (2,2) supersymmetry transformation rules of the components are

$$\begin{aligned} \delta Y &= -\epsilon^{+} Y_{+} - \epsilon^{-} Y_{-} \ , && (5.1.55a) \\ \delta Y_{\pm} &= \mp \epsilon^{\mp} G - 2i\bar{\epsilon}^{\pm} \partial_{\pm\pm} Y \ , && (5.1.55b) \\ \delta G &= -2i\bar{\epsilon}^{-} \partial_{--} Y_{+} + 2i\bar{\epsilon}^{+} \partial_{++} Y_{-} \ . && (5.1.55c) \end{aligned}$$

This representation of $(4,4)$ off-shell supersymmetry is again reducible so we can impose constraints. As in the construction of the previous section, we impose

$$Y_{+} = \bar{D}_{+} \bar{X} \ , \qquad Y_{-} = D_{-} X \ , \tag{5.1.56}$$

then

$$\boldsymbol{\mathcal{Y}} = Y + \eta^{+} \bar{D}_{+} \bar{X} + \eta^{-} D_{-} X + \eta^{+} \eta^{-} G \ . \tag{5.1.57}$$

Here X and Y are consistently chosen to be chiral and twisted-chiral under the manifest $(2,2)$ supersymmetry:

$$\bar{D}_{+} X = \bar{D}_{-} X = \bar{D}_{+} Y = D_{-} Y = 0 \ , \qquad D_{+} \bar{X} = D_{-} \bar{X} = D_{+} \bar{Y} = \bar{D}_{-} \bar{Y} = 0 \ . \tag{5.1.58}$$

Then we have

$$\delta Y = -\epsilon^{+} \bar{D}_{+} \bar{X} - \epsilon^{-} D_{-} X \ , \tag{5.1.59}$$

as well as its conjugate

$$\delta \bar{Y} = \bar{\epsilon}^{+} D_{+} X + \bar{\epsilon}^{-} \bar{D}_{-} \bar{X} \ . \tag{5.1.60}$$

Hence it follows that

$$\delta(\bar{D}_{+} D_{-} \bar{Y}) = \bar{D}_{+} D_{-} \delta \bar{Y} = 2i\bar{\epsilon}^{+} \partial_{++} D_{-} X - 2i\bar{\epsilon}^{-} \partial_{--} \bar{D}_{+} \bar{X} \ . \tag{5.1.61}$$

This should be compared with

$$\delta G = 2i\bar{\epsilon}^{+} \partial_{++} D_{-} X - 2i\bar{\epsilon}^{-} \partial_{--} \bar{D}_{+} \bar{X} \ , \tag{5.1.62}$$

showing that $\bar{D}_{+} D_{-} \bar{Y}$ transforms exactly like the auxiliary field G. This enables us to further constrain the $(4,4)$ multiplet by setting

$$G = \bar{D}_{+} D_{-} \bar{Y} \ . \tag{5.1.63}$$

Imposing these conditions gives a $(4,4)$ twisted superfield

$$\boldsymbol{\mathcal{Y}} = Y + \eta^+ \bar{D}_+ \bar{X} + \eta^- D_- X + \eta^+\eta^- \bar{D}_+ D_- \bar{Y} \ , \tag{5.1.64}$$

which by construction is twisted-chiral and chiral with respect to the manifest and hidden $(2,2)$ supersymmetries, respectively:

$$\bar{D}_+ \boldsymbol{\mathcal{Y}} = D_- \boldsymbol{\mathcal{Y}} = 0 \ , \quad \bar{\mathcal{D}}_\pm \boldsymbol{\mathcal{Y}} = 0 \ . \tag{5.1.65}$$

Its free dynamical action can be easily constructed by considering its square

$$\boldsymbol{\mathcal{Y}}^2 = 2\eta^+\eta^- \Big(Y \bar{D}_+ D_- \bar{Y} - \bar{D}_+ \bar{X} \cdot D_- X \Big) + \dots \ . \tag{5.1.66}$$

In fact, the following relations hold:

$$\begin{aligned} \int d^2x \, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) &= \int d^2x \, d\theta^+ d\bar{\theta}^- \bar{D}_+ D_- (X\bar{X} - Y\bar{Y}) \ , \\ &= \int d^2x \, d\theta^+ d\bar{\theta}^- \Big(D_+ X \cdot \bar{D}_- \bar{X} - \bar{Y} D_+ \bar{D}_- Y \Big) . \end{aligned} \tag{5.1.67}$$

Alternatively,

$$\begin{aligned} \int d^2x \, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) &= \int d^2x \, d\bar{\theta}^+ d\theta^- D_+ \bar{D}_- (X\bar{X} - Y\bar{Y}) \ , \\ &= \int d^2x \, d\bar{\theta}^+ d\theta^- \Big(\bar{D}_+ \bar{X} \cdot D_- X - Y \bar{D}_+ D_- \bar{Y} \Big) . \end{aligned} \tag{5.1.68}$$

These relations imply

$$4 \int d^2x \, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- (X\bar{X} - Y\bar{Y}) = \int d^2x \, d\theta^+ d\bar{\theta}^- d\eta^+ d\eta^- \ \boldsymbol{\mathcal{Y}}^2 + c.c. \ . \tag{5.1.69}$$

Once again the $(4,4)$ supersymmetry of the left hand side becomes manifest on the right hand side.

As in the $(4,4)$ twisted multiplet considered in the previous subsection, we can deform this representation to induce the partial breaking. The deformed multiplet is described by the following $(4,4)$ superfield:

$$\boldsymbol{\mathcal{Y}}_{\rm def} = Y + \eta^+ \bar{D}_+ \bar{X} + \eta^- D_- X + \eta^+\eta^- \Big(\bar{D}_+ D_- \bar{Y} + \kappa \Big) \ . \tag{5.1.70}$$

The hidden supersymmetry transformations of the component $(2,2)$ superfields can be straightforwardly computed using the arguments of the previous subsection. For the goal of this section, it is enough to mention that δY is the same as the undeformed case of eq. (5.1.59).

To eliminate half of the degrees of freedom of $\boldsymbol{\mathcal{Y}}_{\text{def}}$ and describe a Goldstone multiplet for $\mathcal{N}=(4,4)\to\mathcal{N}=(2,2)$ partial supersymmetry breaking, we again impose the nilpotent constraint

$$\boldsymbol{\mathcal{Y}}_{\text{def}}^2=0=2\eta^+\eta^-\Big(Y(\kappa+\bar{D}_+D_-\bar{Y})-\bar{D}_+\bar{X}\cdot D_-X\Big)+\dots\ . \tag{5.1.71}$$

This yields the following constraint for the $(2,2)$ superfields

$$Y(\kappa+\bar{D}_+D_-\bar{Y})-\bar{D}_+\bar{X}\cdot D_-X=0\ , \tag{5.1.72}$$

which is equivalent to

$$Y=\frac{\bar{D}_+\bar{X}\cdot D_-X}{\kappa+\bar{D}_+D_-\bar{Y}}=\frac{\widetilde{W}^2}{\kappa+\bar{\widetilde{D}}^2\bar{Y}}\ ,\qquad \bar{Y}=\frac{\bar{D}_-\bar{X}\cdot D_+X}{\kappa+\bar{D}_+D_-\bar{Y}}=\frac{\bar{\widetilde{W}}^2}{\kappa+\widetilde{D}^2\bar{Y}}\ . \tag{5.1.73}$$

Here $\bar{\widetilde{D}}^2=\bar{D}_+D_-$, $\widetilde{D}^2=-D_+\bar{D}_-$ and we have introduced the following bilinears:

$$\widetilde{W}^2\equiv\bar{D}_+\bar{X}\cdot D_-X=\bar{\widetilde{D}}^2(X\bar{X}),\qquad \bar{\widetilde{W}}^2\equiv\bar{D}_-\bar{X}\cdot D_+X=\widetilde{D}^2(X\bar{X})\ . \tag{5.1.74}$$

Using exactly the same tricks as before and inspired by the $D=4$ Bagger-Galperin model, we can solve the constraints (5.1.72) to find

$$Y=\frac{1}{\kappa}\bar{\widetilde{D}}^2\widetilde{\Upsilon}\ ,\quad \bar{Y}=\frac{1}{\kappa}\widetilde{D}^2\widetilde{\Upsilon}\ ,\qquad \widetilde{\Upsilon}=\bar{\widetilde{\Upsilon}}=X\bar{X}-\frac{2\widetilde{W}^2\bar{\widetilde{W}}^2}{\widetilde{A}+\kappa^2+\sqrt{\widetilde{B}^2+2\kappa^2\widetilde{A}+\kappa^4}}\ , \tag{5.1.75}$$

where

$$\widetilde{A}\ =\ \widetilde{D}^2\widetilde{W}^2+\bar{\widetilde{D}}^2\bar{\widetilde{W}}^2=\{\widetilde{D}^2,\bar{\widetilde{D}}^2\}(X\bar{X})=\partial_{++}X\partial_{--}\bar{X}+\partial_{++}\bar{X}\partial_{--}X\ , \tag{5.1.76a}$$
$$\widetilde{B}\ =\ \widetilde{D}^2\widetilde{W}^2-\bar{\widetilde{D}}^2\bar{\widetilde{W}}^2=[\widetilde{D}^2,\bar{\widetilde{D}}^2](X\bar{X})=\partial_{++}X\partial_{--}\bar{X}-\partial_{++}\bar{X}\partial_{--}X\ . \tag{5.1.76b}$$

Since Y is twisted-chiral under the manifest $(2,2)$ supersymmetry (5.1.58), we can consider the twisted-chiral integral

$$S_{\kappa^2}=\frac{1}{2}\kappa\int d^2x\,d\theta^+d\bar{\theta}^-Y+c.c=\frac{1}{2}\int d^2x\,d\theta^+d\bar{\theta}^-\bar{\widetilde{D}}^2\widetilde{\Upsilon}+c.c.=\int d^2x\,d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-\widetilde{\Upsilon}\ . \tag{5.1.77}$$

By using arguments analogous to those around eqs. (5.1.50) of the previous subsection, the action (5.1.77) proves to be $\mathcal{N} = (4,4)$ supersymmetric.

Explicitly, the action reads

$$S_{\kappa^2} = \int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \left(X\bar{X} - \frac{2\widetilde{W}^2 \bar{\widetilde{W}}^2}{\kappa^2 + \widetilde{A} + \sqrt{\kappa^4 + 2\kappa^2 \widetilde{A} + \widetilde{B}^2}} \right) , \qquad (5.1.78)$$

which precisely matches the model of eq. (5.1.6a) if we identify the coupling constants:

$$\lambda = -\frac{2}{\kappa^2} . \qquad (5.1.79)$$

This shows explicitly that the $T\overline{T}$ deformation of the free chiral action possesses a non-linearly realized $\mathcal{N} = (2,2)$ supersymmetry.

5.2 T^2 Deformations in $D = 4$ and Supersymmetric Extensions

In section (5.1) we exhibited the non-linear supersymmetry possessed by two $D = 2$ $\mathcal{N} = (2,2)$ models constructed in [6] from the $T\overline{T}$ deformation of free actions. The striking relationship with the $D = 4$ supersymmetric Born-Infeld (BI) theory naturally makes one wonder whether some kind of $T\overline{T}$ flow equation is satisfied by supersymmetric $D = 4$ BI, and related actions. We will spend the rest of the chapter exploring this possibility. In this section, we start with a few general observations on T^2 or supercurrent-squared operators in $D > 2$.

5.2.1 Comments on the T^2 operator in $D = 4$

In two dimensions, by $T\bar{T}$ we mean the operator $T_{\mu\nu}T^{\mu\nu} - (T^{\mu}_{\mu})^2$, which is proportional to $\det\big[T_{\mu\nu}\big]$ [1, 3, 28]. One can attempt to generalize this structure to $D > 2$. In general, one could consider the following stress-tensor squared operator

$$O^{[r]}_{T^2} = T^{\mu\nu}T_{\mu\nu} - r\,\Theta^2 , \qquad \Theta \equiv T^{\mu}{}_{\mu} , \qquad (5.2.1)$$

with r a real constant parameter. In two dimensions, the unique choice $r = 1$ yields a well defined operator which is free of short distance singularities [1, 28]. However, to the best of our knowledge, there is no analogous argument in higher dimensions that guarantees a well-defined irrelevant operator $O^{[r]}_{T^2}$ at the quantum level. Nevertheless, in a D-dimensional space-time, one possible extension is given by $O^{[r]}_{T^2}$ with $r = 1/(D-1)$, which reduces to the

$T\bar{T}$ operator in two dimensions.

This operator has received some attention recently since it is motivated by a particular holographic picture in $D > 2$ [60, 61]. We will not enter into a detailed discussion of the physical properties enjoyed by $O_{T^2}^{[1/(D-1)]}$, but simply comment that this combination is invariant under a set of improvement transformations of the stress-energy tensor. Indeed it is easy to show that such a T^2 operator transforms by,

$$O_{T^2}^{[1/(D-1)]} \to O_{T^2}^{[1/(D-1)]} + \text{total derivatives} \ , \tag{5.2.2}$$

if the (symmetric) stress-energy tensor shifts by the following improvement transformation,

$$T_{\mu\nu} \to T_{\mu\nu} + \left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2\right) u \ , \tag{5.2.3}$$

for an arbitrary scalar field u.

In four dimensions, there is another choice of interest, specifically $r = 1/2$. In fact, it was shown in [62] that the bosonic Born-Infeld action can be obtained by deforming the free Maxwell theory with the operator $O_{T^2}^{[1/2]}$.[2] In this work, we are going to use $O_{T^2}^{[1/2]}$ as our deforming operator. Once generalized to the supersymmetric case, we will see that this operator plays a central role for various models possessing non-linearly realized symmetries.

One interesting property enjoyed by $O_{T^2}^{[1/2]}$ is its invariance under a shift of the Lagrangian density of the theory, or equivalently a shift of the zero point energy. This can serve as motivation for this particular combination. Under a constant shift of the Lagrangian density $\mathcal{L}$, and correspondingly its stress-energy tensor,

$$\mathcal{L} \to \mathcal{L} + c \ , \qquad T^{\mu\nu} \to T^{\mu\nu} - c\,\eta^{\mu\nu} \ , \tag{5.2.4}$$

the composite operator $O_{T^2}^{[r]}$ transforms in the following way:

$$O_{T^2}^{[r]} \to O_{T^2}^{[r]} + 2c(2r-1)\Theta + 4c^2(1-r) \ . \tag{5.2.5}$$

When the theory is not conformal, which is the general situation at an arbitrary point in the flow since the deformation introduces a scale, and $r \neq 1/2$, the operator $O_{T^2}^{[r]}$ always transforms in a non-trivial way because of the extra trace term. This implies that under a constant shift in the Lagrangian, the dynamics is modified which is certainly peculiar since

2. It is worth mentioning that another type of higher-dimensional generalization of $T\overline{T}$-deformations, specifically the operator $|\det T|^{1/(D-1)}$, was studied in [10, 63].

the shift is trivial in the undeformed theory.[3]

However if $r = \frac{1}{2}$, $O^{[r]}_{T^2}$ is unaffected up to an honest field-independent cosmological constant term. The shift of the vacuum energy does not affect the dynamics of the theory, as long as the theory is not coupled to gravity. This property is especially interesting, since the $D = 4$ $\mathcal{N} = 1$ Goldstino action, which is also given by a $T\overline{T}$ flow, is the low-energy description of supersymmetry breaking which can generate a cosmological constant. For these reasons, we will study the particular operator quadratic in stress-energy tensors given by

$$O_{T^2} \equiv T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}\Theta^2 , \tag{5.2.6}$$

in the remainder of this chapter.

5.2.2 $D = 4$ $\mathcal{N} = 1$ *supercurrent-squared operator*

We would like to find the $\mathcal{N} = 1$ supersymmetric extension of the O_{T^2} operator in four dimensions. As reviewed in section 5.1, in two dimensions the manifestly supersymmetric $T\bar{T}$ deformation is roughly given by the square of the supercurrent superfields. One might suspect that a similar construction holds in four dimensions.

For the remainder of this work, we will assume that the $D = 4$ $\mathcal{N} = 1$ supersymmetric theories under our consideration admit a Ferrara-Zumino (FZ) multiplet of currents [13]. Generalizations of this case involving the supercurrent multiplets described in [17, 19, 64–68] might be possible, but merit separate investigation. The operator content of the FZ multiplet, which has 12+12 component fields, includes the conserved supersymmetry current $S_{\mu\alpha}$, its conjugate $\bar{S}_\mu{}^{\dot{\alpha}}$ and the conserved symmetric energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = T_{\nu\mu} , \qquad \partial^\mu T_{\mu\nu} = 0 , \qquad \partial^\mu S_\mu = \partial^\mu \bar{S}_\mu = 0 . \tag{5.2.7}$$

The FZ multiplet also includes a complex scalar field x, as well as the R-current vector field j_μ, which is not necessarily conserved [13].

In $D = 4$ $\mathcal{N} = 1$ superspace, the FZ multiplet is described by a vector superfield $\mathcal{J}_\mu$ and a complex scalar scalar superfield $\mathcal{X}$ satisfying the following constraints:

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha\mathcal{X} , \qquad \bar{D}_{\dot{\alpha}}\mathcal{X} = 0 . \tag{5.2.8}$$

The constraints can be solved, and the FZ supercurrents expressed in terms of its $12 + 12$

3. It is worth noting that $T\overline{T}$ in $D = 2$ shares this peculiarity.

independent components read[4]

$$
\begin{aligned}
\mathcal{J}_\mu(x) &= j_\mu + \theta\Big(S_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\chi}\Big) + \bar{\theta}\Big(\bar{S}_\mu + \frac{1}{\sqrt{2}}\bar{\sigma}_\mu\chi\Big) + \frac{i}{2}\theta^2\partial_\mu\bar{\mathsf{x}} - \frac{i}{2}\bar{\theta}^2\partial_\mu\mathsf{x} \\
&\quad +\theta\sigma^\nu\bar{\theta}\Big(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}\Theta - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\Big) \\
&\quad -\frac{i}{2}\theta^2\bar{\theta}\Big(\bar{\partial\!\!\!/}S_\mu + \frac{1}{\sqrt{2}}\bar{\sigma}_\mu\partial\!\!\!/\bar{\chi}\Big) - \frac{i}{2}\bar{\theta}^2\theta\Big(\partial\!\!\!/\bar{S}_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\partial\!\!\!/}\chi\Big) \\
&\quad +\frac{1}{2}\theta^2\bar{\theta}^2\Big(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\Big) \ , \qquad (5.2.9)
\end{aligned}
$$

and

$$
\mathcal{X}(y) = \mathsf{x}(y) + \sqrt{2}\theta\chi(y) + \theta^2\mathsf{F}(y) \ , \qquad (5.2.10a)
$$

$$
\chi_\alpha = \frac{\sqrt{2}}{3}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{S}^{\dot{\alpha}}_\mu \ , \qquad \mathsf{F} = \frac{2}{3}\Theta + i\partial_\mu j^\mu \ , \qquad (5.2.10b)
$$

where the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, and we used $\partial\!\!\!/ = \sigma^\mu\partial_\mu$, $\bar{\partial\!\!\!/} = \bar{\sigma}^\mu\partial_\mu$.

If we seek a manifestly supersymmetric completion of the operator (5.2.6) by using combinations of the supercurrent superfields with dimension 4, it is clear that the only possibility is the full superspace integral of a linear combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$. Up to total derivatives and terms that vanish by using the supercurrent conservation equations, or equivalently that vanish on-shell, the D-terms of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ are given by[5]

$$
\begin{aligned}
\mathcal{J}^2|_{\theta^2\bar{\theta}^2} \equiv \eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu|_{\theta^2\bar{\theta}^2} &= -\frac{1}{2}\Big(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}\Theta - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\Big)^2 + j^\mu\Big(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\Big) \\
&\quad +\frac{1}{2}\partial_\mu\mathsf{x}\partial^\mu\bar{\mathsf{x}} + \frac{i}{2}\Big(A - \bar{A}\Big) \qquad (5.2.12a) \\
&= -2(T_{\mu\nu})^2 + \frac{4}{9}\Theta^2 - \frac{5}{4}\Big(\partial_\mu j^\mu\Big)^2 - \frac{3}{4}j_\mu\partial^2 j^\mu + \frac{1}{2}\partial_\mu\bar{\mathsf{x}}\partial^\mu\mathsf{x} \\
&\quad +i\Big(S_\mu\partial\!\!\!/\bar{S}^\mu - \bar{\chi}\bar{\partial\!\!\!/}\chi\Big) + \text{total derivatives} + \text{EOM} \ , \qquad (5.2.12b)
\end{aligned}
$$

4. For convenience, we have rescaled the supersymmetry current compared to [17]: $S_\mu^{\text{here}} = -iS_\mu^{\text{there}}$.

5. The composite A (and analogously its conjugate $\bar{A}$) is given by

$$
A = \Big(S_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\chi}\Big)\Big(\partial\!\!\!/\bar{S}^\mu - \frac{1}{\sqrt{2}}\sigma^\mu\bar{\partial\!\!\!/}\chi\Big) = S_\mu\partial\!\!\!/\bar{S}^\mu - \bar{\chi}\bar{\partial\!\!\!/}\chi + \sqrt{2}\bar{S}^\mu\partial_\mu\bar{\chi} + \text{total derivatives} \ . \qquad (5.2.11)
$$

The equality can be obtained with some algebra. Note that the last term drops after integration by parts because of the conservation equation for S_μ.

and

$$\mathcal{X}\bar{\mathcal{X}}|_{\theta^2\bar{\theta}^2} = \mathsf{F}\bar{\mathsf{F}} - \partial_\mu \mathsf{x} \partial^\mu \bar{\mathsf{x}} - i\bar{\chi}\bar{\partial\!\!\!/}\chi + \text{total derivatives} \tag{5.2.13a}$$

$$= \frac{4}{9}\Theta^2 + (\partial_\mu j^\mu)^2 - \partial_\mu \mathsf{x} \partial^\mu \bar{\mathsf{x}} - i\bar{\chi}\bar{\partial\!\!\!/}\chi + \text{total derivatives} \ . \tag{5.2.13b}$$

To get a manifestly supersymmetric extension of $O_{T^2} = T^2 - \frac{1}{2}\Theta^2$, we have to consider the following linear combination

$$\mathcal{O}_{T^2} = -\frac{1}{2}\Big(\eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu + \frac{5}{4}\mathcal{X}\bar{\mathcal{X}}\Big) = \frac{1}{16}\mathcal{J}^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} - \frac{5}{8}\mathcal{X}\bar{\mathcal{X}} \ . \tag{5.2.14}$$

In fact, the supersymmetric descendant of the supercurrent-squared operator $\mathcal{O}_{T^2}$ is

$$\mathsf{O}_{T^2} = \int d^4\theta \, \mathcal{O}_{T^2} \tag{5.2.15a}$$

$$= T^2 - \frac{1}{2}\Theta^2 + \frac{3}{8}j_\mu\partial^2 j^\mu + \frac{3}{8}\partial_\mu \mathsf{x} \partial^\mu \bar{\mathsf{x}} - \frac{i}{2}\Big(S_\mu \partial\!\!\!/ \bar{S}^\mu - \frac{9}{4}\bar{\chi}\bar{\partial\!\!\!/}\chi\Big)$$
$$+\text{total derivatives} + \text{EOM} \ . \tag{5.2.15b}$$

This result shows that O_{T^2} is the natural supersymmetric extension of O_{T^2}. However, it is worth emphasizing that in the $D = 4$ case, the supersymmetric descendent O_{T^2} of $\mathcal{O}_{T^2}$ has extra non-trivial contributions from other currents. This should be contrasted with the $D = 2$ case where $\mathsf{O}_{T^2} = O_{T^2}$ up to EOM and total derivatives, see eq. (5.1.1).

It actually does not seem possible to find a linear combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ such that an analogue of eq. (5.1.1) holds in $D = 4$. This suggests that, in contrast with the $D = 2$ case, deformations of a Lagrangian triggered by the operators O_{T^2} and O_{T^2} will in general lead to different flows: one manifestly supersymmetric, while the other not.

5.3 Bosonic Born-Infeld as a T^2 Flow

It was shown in [62] that the $D = 4$ Born-Infeld action arises from a $D > 2$ generalization of the $T\bar{T}$ deformation. Specifically, the operator driving the flow equation was shown to be the O_{T^2} defined in eq. (5.2.6) of the preceding section. In this section we review this result in detail as it is a primary inspiration for our supersymmetric extensions.

The $D = 4$ bosonic BI action on a flat background is given by

$$\begin{aligned} S_{\rm BI} &= \frac{1}{\alpha^2}\int d^4x \left[1 - \sqrt{-\det\big(\eta_{\mu\nu} + \alpha F_{\mu\nu}\big)}\right] , \\ &= \frac{1}{\alpha^2}\int d^4x \left[1 - \sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2}\right] , \\ &= -\frac{1}{4}\int d^4x\, F^2 + \text{higher derivative terms} , \end{aligned} \tag{5.3.1}$$

where $F_{\mu\nu} = (\partial_\mu v_\nu - \partial_\nu v_\mu)$ is the field strength for an Abelian gauge field v_μ, and

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \qquad F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} . \tag{5.3.2}$$

The Hilbert stress-energy tensor for the BI action can be computed straightforwardly and it reads [69]

$$T^{\mu\nu} = -\frac{F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{\alpha^2}\Big(\sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2} - 1 - \frac{\alpha^2}{2}F^2\Big)\eta^{\mu\nu}}{\sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2}} . \tag{5.3.3}$$

This can be written in the following useful form

$$T^{\mu\nu} = \frac{T^{\mu\nu}_{\rm Maxwell}}{\sqrt{1+2A+B^2}} + \frac{\eta^{\mu\nu}}{\alpha^2\sqrt{1+2A+B^2}}\frac{A^2 - B^2}{1 + A + \sqrt{1+2A+B^2}} , \tag{5.3.4}$$

where we used the stress-energy tensor for the Maxwell theory

$$T^{\mu\nu}_{\rm Maxwell} = -F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4}F^2\eta^{\mu\nu} , \tag{5.3.5}$$

while A and B are defined by

$$A = \frac{1}{4}\alpha^2 F^2 , \qquad B = \frac{i}{4}\alpha^2 F\tilde{F} . \tag{5.3.6}$$

It is easy to compute the trace of the stress-energy tensor

$$\Theta = T^{\mu\nu}\eta_{\mu\nu} = \frac{4}{\alpha^2\sqrt{1+2A+B^2}}\frac{A^2 - B^2}{1 + A + \sqrt{1+2A+B^2}} , \tag{5.3.7}$$

where, interestingly, the combination $(A^2 - B^2)$ proves to be related to the square of $T^{\mu\nu}_{\rm Maxwell}$.

Using the identity

$$(F\tilde{F})^2 = \frac{1}{4}(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma})^2 = 4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - 2(F^2)^2 \ , \tag{5.3.8}$$

we see that

$$T^2_{\text{Maxwell}} = F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}(F^2)^2 = \frac{1}{4}\Big((F^2)^2 + (F\tilde{F})^2\Big) = \frac{4}{\alpha^4}(A^2 - B^2) \ . \tag{5.3.9}$$

Using tracelessness of the free Maxwell stress-energy tensor, the O_{T^2} operator can be easily computed:

$$\begin{aligned} O_{T^2} = T^2 - \frac{1}{2}\Theta^2 \quad &= \quad \frac{4(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}^2}\Big(1 - \frac{A^2-B^2}{(1+A+\sqrt{1+2A+B^2})^2}\Big) \ , && (5.3.10a)\\ &= \quad \frac{4(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}^2}\Big(1 - \frac{1+A-\sqrt{1+2A+B^2}}{1+A+\sqrt{1+2A+B^2}}\Big) \ , && (5.3.10b)\\ &= \quad \frac{8(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}}\frac{1}{1+A+\sqrt{1+2A+B^2}} \ , && (5.3.10c)\\ &= \quad \frac{8(1+A-\sqrt{1+2A+B^2})}{\alpha^2\sqrt{1+2A+B^2}} \ . && (5.3.10d) \end{aligned}$$

The variation of the BI Lagrangian with respect to the parameter α^2 can be readily computed, and it is given by

$$\frac{\partial \mathcal{L}_\alpha}{\partial \alpha^2} = \frac{1 + \frac{1}{4}\alpha^2 F^2 - \sqrt{1 + \frac{1}{2}\alpha^2 F^2 - \frac{1}{16}\alpha^4 (F\tilde{F})^2}}{\alpha^2\sqrt{1 + \frac{1}{2}\alpha^4 F^2 - \frac{1}{16}\alpha^4 (F\tilde{F})^2}} \ . \tag{5.3.11}$$

Once we use (5.3.6) it is clear that (5.3.10a) and (5.3.11) have exactly the same structure and satisfy the following equivalence equation

$$\frac{\partial \mathcal{L}_\alpha}{\partial \alpha^2} = \frac{1}{8}O_{T^2} \ , \tag{5.3.12}$$

showing that the BI Lagrangian satisfies a T^2-flow driven by the operator O_{T^2}.

Before turning to $D = 4$ supersymmetric analysis, it is worth mentioning that the structure of the computation relating the O_{T^2} operator to the bosonic BI theory, which we just reviewed, is quite similar to what we saw in section 5.1 for the $D = 2$ $\mathcal{N} = (2, 2)$ supersymmetric $T\bar{T}$ flows. For example, in the deformation of the free twisted-chiral multiplet action, the analogue of the A and B combinations of (5.3.6) is given by (5.1.48), but the square root structure of the actions is completely analogous. This fact, together with the non-linearly realized supersymmetry we investigated in section 5.1, naturally lead to the guess that the $D = 4$ $\mathcal{N} = 1$ supersymmetric Born-Infeld (BI) theory may also satisfy a T^2 flow. The next section is devoted to explaining how this is the case.

5.4 Supersymmetric Born-Infeld from Supercurrent-Squared Deformation

In Section 5.1 we proved, by analogy and extension of the $D = 4$ results of [37], that two $D = 2$ supercurrent-squared flows possess additional non-linearly realized supersymmetry. In this section we reverse the logic. We will look at a well-studied model, namely the Bagger-Galperin construction [37] of $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory [70, 71], and show that it satisfies a supercurrent-squared flow equation.

5.4.1 $D = 4$ $\mathcal{N} = 1$ supersymmetric BI and non-linear supersymmetry

Let us review some well known results about the $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory [71], the Bagger-Galperin action [37], the non-linearly realized second supersymmetry, and its precise $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking pattern. For more detail, we refer to the following references on the subject [37, 47–50, 71].

We start with the following $\mathcal{N} = 2$ superfield,

$$\mathcal{W}(y, \theta, \tilde{\theta}) = X(y, \theta) + \sqrt{2} i \tilde{\theta} W(y, \theta) - \tilde{\theta}^2 G(y, \theta) \ , \qquad y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}} \ , \quad (5.4.1)$$

which is chiral with respect to both supersymmetries:

$$\bar{D}_{\dot{\alpha}}\mathcal{W} = \bar{\tilde{D}}_{\dot{\alpha}}\mathcal{W} = 0 \ . \qquad (5.4.2)$$

Since we are ultimately interested in partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking, we will mostly use $\mathcal{N} = 1$ superfields associated to the θ Grassmann variables to describe manifest supersymmetry, while we use the $\tilde{\theta}$ variable for the hidden non-linearly realized supersymmetry. The $\mathcal{N} = 1$ superfields X, W_α, and G of eq. (5.4.1) are chiral under the manifest $\mathcal{N} = 1$ supersymmetry. Under the additional hidden $\mathcal{N} = 1$ supersymmetry, they transform as follows:

$$\tilde{\delta}X = \sqrt{2}i\epsilon W \ , \tag{5.4.3a}$$

$$\tilde{\delta}W = \sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu X + \sqrt{2}i\epsilon G \ , \tag{5.4.3b}$$

$$\tilde{\delta}G = -\sqrt{2}\partial_\mu W\sigma^\mu\bar{\epsilon} \ . \tag{5.4.3c}$$

The superfield (5.4.1) has 16+16 independent off-shell components and is reducible. It contains the degrees of freedom of an $\mathcal{N} = 2$ vector and tensor multiplet. To reduce the degrees of freedom and describe an irreducible $\mathcal{N} = 2$ off-shell vector multiplet, we impose the following conditions on the $\mathcal{N} = 1$ components of $\mathcal{W}$:

(i) First that W_α is the field-strength superfield of an $\mathcal{N} = 1$ vector multiplet satisfying,

$$D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = 0 \ , \tag{5.4.4}$$

(ii) and that

$$G = \frac{1}{4}\bar{D}^2\bar{X} \ . \tag{5.4.5}$$

The latter condition can easily be seen to be consistent since it is straightforward to verify that $\frac{1}{4}\bar{D}^2\bar{X}$ transforms in the same way as G given in (5.4.3c). Therefore we can impose (5.4.5) without violating $\mathcal{N} = 2$ supersymmetry.

Since $\mathcal{W}$ is chiral with respect to both sets of supersymmetries, we can consider the following Lagrangian,

$$\mathcal{L}^{\mathcal{N}=2}_{\mathcal{W}^2} = \frac{1}{4}\int d^2\theta d^2\tilde{\theta}\,\mathcal{W}^2 + c.c. = \frac{1}{4}\int d^2\theta\Big(W^2 - \frac{1}{2}X\bar{D}^2\bar{X}\Big) + c.c. \ . \tag{5.4.6}$$

On the other hand, the $\mathcal{N} = 2$ Maxwell theory written in terms of the $\mathcal{N} = 1$ chiral superfields X and W_α is given by

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=2}_{\text{Maxwell}} &= \int d^2\theta d^2\bar{\theta}\,\bar{X}X + \frac{1}{4}\int d^2\theta\,W^2 + \frac{1}{4}\int d^2\theta\,\bar{W}^2 \ , \\ &= \frac{1}{4}\int d^2\theta\Big(W^2 - \frac{1}{2}X\bar{D}^2\bar{X}\Big) + c.c. + \text{total derivative} \ . \end{aligned}\tag{5.4.7}$$

We see that these two Lagrangians are the same, confirming that the extra constraint imposed on $\mathcal{W}$ is correct. The off-shell $\mathcal{N} = 2$ vector multiplet can therefore be described in term of the following $\mathcal{N} = 2$ superfield

$$\mathcal{W}(y,\theta,\tilde{\theta}) = X(y,\theta) + \sqrt{2}i\tilde{\theta}W(y,\theta) - \frac{1}{4}\tilde{\theta}^2\bar{D}^2\bar{X}(y,\theta) \ , \tag{5.4.8}$$

where X and W_α are $\mathcal{N}=1$ chiral and vector multiplets, respectively. Their component expansion reads:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha \mathsf{D} - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha \ , \tag{5.4.9a}$$
$$X = x + \sqrt{2}\theta\chi - \theta^2\,\mathsf{F} \ . \tag{5.4.9b}$$

Following [47] (see also [48–50]), we break $\mathcal{N}=2$ supersymmetry by considering a Lorentz and $\mathcal{N}=1$ invariant condensate with a non-trivial dependence on the hidden Grassmann variables $\langle\mathcal{W}\rangle = \mathcal{W}_{\text{def}} \propto \tilde{\theta}^2 \neq 0$, such that

$$\mathcal{W} \rightarrow \mathcal{W}_{\text{new}} = \langle\mathcal{W}\rangle + \mathcal{W} = \mathcal{W} + \mathcal{W}_{\text{def}} \ , \tag{5.4.10a}$$
$$\mathcal{W}_{\text{new}} = X + \sqrt{2}i\tilde{\theta}W - \frac{1}{4}\tilde{\theta}^2\Big(\bar{D}^2\bar{X} + \frac{2}{\kappa}\Big) \ . \tag{5.4.10b}$$

The hidden supersymmetry transformations of the $\mathcal{N}=1$ components of the deformed $\mathcal{N}=2$ vector multiplet turn out to be

$$\tilde{\delta}X = \sqrt{2}i\epsilon W \ , \tag{5.4.11a}$$
$$\tilde{\delta}W = \frac{i}{\sqrt{2}\kappa}\epsilon + \frac{i}{2\sqrt{2}}\epsilon\overline{D}^2\bar{X} + \sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu X \ . \tag{5.4.11b}$$

Assuming the model under consideration preserves the manifest $\mathcal{N}=1$ supersymmetry, which implies $\langle\overline{D}^2X\rangle = 0$, the explicit non-linear κ-dependent term in the transformation of the fermionic W_α signals the spontaneous partial breaking $\mathcal{N}=2\rightarrow\mathcal{N}=1$ of the hidden supersymmetry.

To describe the Maxwell-Goldstone multiplet for the partial breaking $\mathcal{N}=2\rightarrow\mathcal{N}=1$, we can impose the following nilpotent constraint on the deformed $\mathcal{N}=2$ superfield strength $\mathcal{W}_{\text{new}}$ [47]

$$(\mathcal{W}_{\text{new}})^2 = 0 \ . \tag{5.4.12}$$

Once reduced to $\mathcal{N}=1$ superfields, following the expansion (5.4.10b), this constraint implies the Bagger-Galperin constraint [37]

$$\frac{1}{\kappa}X = W^2 - \frac{1}{2}X\bar{D}^2\bar{X} \ , \tag{5.4.13}$$

which can be solved to eliminate X in terms of $W^2 = W^\alpha W_\alpha$ and its complex conjugate $\bar{W}^2 = \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$:

$$X = \kappa W^2 - \kappa^3\bar{D}^2\Big[\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\Big] \ , \tag{5.4.14}$$

where we have introduced:

$$\mathcal{A} = \frac{\kappa^2}{2}(D^2W^2 + \bar{D}^2\bar{W}^2) = \overline{\mathcal{A}} \; , \qquad \mathcal{B} = \frac{\kappa^2}{2}(D^2W^2 - \bar{D}^2\bar{W}^2) = -\overline{\mathcal{B}} \; . \tag{5.4.15}$$

For later use we denote the lowest components of the composite superfields $\mathcal{A}$ and $\mathcal{B}$

$$A = \mathcal{A}|_{\theta=0} \; , \qquad B = \mathcal{B}|_{\theta=0} \; . \tag{5.4.16}$$

We will not repeat the derivation of (5.4.14) which can be found in the original paper [37], and was reviewed and slightly modified in section 5.1 for our analysis in two dimensions.

The $\mathcal{N} = 1$ supersymmetric BI action can be constructed using the following $\mathcal{N} = 1$ (anti-)chiral Lagrangian linear in X:

$$\mathcal{L}_\kappa = \frac{1}{4\kappa}\Big(\int d^2\theta X + \int d^2\bar{\theta}\bar{X}\Big) \; . \tag{5.4.17}$$

The second hidden supersymmetry eq. (5.4.11a) written in terms of the unconstrained real vector multiplet V, where $W_\alpha = -1/4\overline{D}^2 D_\alpha V$, takes the form:

$$\tilde{\delta}X = -\frac{1}{4}\sqrt{2}i\epsilon^\alpha \bar{D}^2 D_\alpha V \; . \tag{5.4.18}$$

Using the fact that $D^2\overline{D}^2 D_\alpha \propto \partial_{\alpha\dot{\alpha}} D^2\overline{D}^{\dot{\alpha}}$, one can immediately see that the supersymmetry variation of $\mathcal{L}_\kappa$ in (5.4.17) is a total derivative. Therefore this supersymmetric BI action is invariant under the second hidden non-linear supersymmetry.

Using the solution (5.4.14), the supersymmetric BI Lagrangian takes the explicit form

$$\begin{aligned} \mathcal{L}_\kappa &= \frac{1}{4\kappa}\int d^2\theta\Big(\kappa W^2 - \kappa^3\bar{D}^2\Big[\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\Big]\Big) + c.c. \; , \\ &= \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar{\theta}\,\bar{W}^2 + 2\kappa^2\int d^2\theta d^2\bar{\theta}\,\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}} \; , \end{aligned} \tag{5.4.19}$$

which makes it clear that the supersymmetric BI is a non-linear deformation of the free $\mathcal{N} = 1$ Maxwell theory. This supersymmetric extension of BI was first constructed by Bagger and Galperin in [37]. In this work when we refer to the supersymmetric BI theory, we will always mean the Bagger-Galperin action.

We can easily calculate the flow under the κ^2 coupling constant,

$$\frac{\partial\mathcal{L}_\kappa}{\partial\kappa^2} = 2\int d^2\theta d^2\bar{\theta}\,\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\frac{1}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}} \; . \tag{5.4.20}$$

Our goal is now to show that the right hand side of this flow equation on-shell is the specific supercurrent bilinear (5.2.14) that we introduced earlier. This will establish a supercurrent-squared flow for the supersymmetric BI action.

Before turning to the core of this analysis let us recall that at the leading order in κ^2, the fact that $D = 4$ $\mathcal{N} = 1$ BI satisfies a supercurrent-squared flow was already noticed in [71]. This result was also highlighted recently in the introduction of [5]. In fact, note that in the free limit $\alpha = \kappa = 0$, the Lagrangian (5.4.19) becomes the $\mathcal{N} = 1$ supersymmetric Maxwell theory. Its supercurrent multiplet is

$$\mathcal{J}_{\alpha\dot{\alpha}} = -4W_\alpha \bar{W}_{\dot{\alpha}} \ , \qquad \mathcal{X} = 0 \ , \tag{5.4.21}$$

where $\mathcal{X} = 0$ because super-Maxwell theory is scale invariant. The supersymmetric T^2 deformation operator (5.2.14) is then simply given by

$$\mathcal{O}_{T^2} = \frac{1}{16}\mathcal{J}_{\alpha\dot{\alpha}}\mathcal{J}^{\alpha\dot{\alpha}} - \frac{5}{8}\mathcal{X}\bar{\mathcal{X}} = W^2\bar{W}^2 \ , \tag{5.4.22}$$

and to leading order (5.4.20) turns into [71]

$$\frac{\partial \mathcal{L}_\kappa}{\partial \kappa^2} = \int d^2\theta d^2\bar{\theta}\, W^2\bar{W}^2 + \mathcal{O}(\kappa^2) = \int d^2\theta d^2\bar{\theta}\, \mathcal{O}_{T^2} + \mathcal{O}(\kappa^2) \ . \tag{5.4.23}$$

This shows that the supercurrent-squared flow equation is satisfied at this order. The rest of this section is devoted to demonstrating the full non-linear extension of this result. First, we are going to look at the bosonic truncation of (5.4.19) and (5.4.20).

5.4.2 Bosonic truncation

In the pure bosonic case the gauginos are set to zero in (5.4.9a), $\lambda = \bar{\lambda} = 0$, and $W^2, \bar{W}^2$ only have $\theta^2, \bar{\theta}^2$ components, so $\mathcal{A}, \mathcal{B}$ can only contribute the lowest components:

$$A = \mathcal{A}|_{\theta=0} = 2\kappa^2\Big(F^2 - 2\mathsf{D}^2\Big) \ , \qquad B = \mathcal{B}|_{\theta=0} = 2\kappa^2 iF\tilde{F} \ . \tag{5.4.24}$$

Therefore the supersymmetric BI Lagrangian reduces to

$$\mathcal{L} = \frac{1}{8\kappa^2}\left[1 - \sqrt{1 + 4\kappa^2\Big(F^2 - 2\mathsf{D}^2\Big) - 4\kappa^4\Big(F\tilde{F}\Big)^2}\,\right] \ . \tag{5.4.25}$$

The auxiliary field $\mathsf{D} = 0$ after using its EOM, and the Lagrangian is equivalent to the bosonic BI Lagrangian (5.3.1) with the identification $\alpha^2 = 8\kappa^2$. This immediately implies that on-shell the bosonic truncation of the supersymmetric BI satisfies a T^2 flow equation driven by the O_{T^2} operator (5.2.6), as we discussed in (5.3.10a). A similar conclusion holds for the complete supersymmetric model of (5.4.19) and (5.4.20).

5.4.3 Supersymmetric Born-Infeld as a supercurrent-squared flow

The supercurrent for the supersymmetric BI action (5.4.19) was computed in [72] for $\kappa^2 = \frac{1}{2}$. To simplify notation, we will also consider the special case $\kappa^2 = \frac{1}{2}$ in our intermediate computations. The κ-dependence can be restored easily and will appear in the final formulae.

We can straightforwardly use the results of [72] for our supercurrent-squared flow analysis. The FZ multiplet was computed for a class of models described by the following Lagrangian,

$$\mathcal{L} = \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar\theta\, \bar W^2 + \frac{1}{4}\int d^2\theta d^2\bar\theta\, W^2 \bar W^2 \Lambda(u,\bar u) \ , \tag{5.4.26}$$

where

$$u = \frac{1}{8}D^2 W^2 \ , \qquad \bar u = \frac{1}{8}\bar D^2 \bar W^2 \ . \tag{5.4.27}$$

The action (5.4.19) turns out to be given by the following choice of $\Lambda(u,\bar u)$

$$\Lambda(u,\bar u) = \frac{4}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}} \ , \tag{5.4.28}$$

where

$$\mathcal{A} = 2(u+\bar u) \ , \qquad \mathcal{B} = 2(u-\bar u) \ . \tag{5.4.29}$$

Following [72], we also introduce the composite superfields

$$\Gamma(u,\bar u) = \frac{\partial(u\Lambda)}{\partial u} \ , \qquad \bar\Gamma(u,\bar u) = \frac{\partial(\bar u\Lambda)}{\partial \bar u} \ , \tag{5.4.30}$$

which, in the case of interest to us where (5.4.28) holds, satisfy

$$\Gamma + \bar\Gamma - \Lambda = \frac{4}{\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)\sqrt{1+2\mathcal{A}+\mathcal{B}^2}} \ , \tag{5.4.31a}$$

$$\bar u\Gamma + u\bar\Gamma = 1 - \frac{1}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}} \ . \tag{5.4.31b}$$

The supercurrents will also be functionals of the following composite

$$iM_\alpha = W_\alpha\Big[1 - \frac{1}{4}\bar{D}^2\Big(\bar{W}^2\Big(\Lambda + \frac{1}{8}D^2(W^2\frac{\partial\Lambda}{\partial u})\Big)\Big)\Big] , \tag{5.4.32a}$$

$$= W_\alpha\Big(1 - 2\bar{u}\Gamma\Big) + W\bar{W}(\cdots) + W^2(\cdots) , \tag{5.4.32b}$$

where $W\bar{W}(\cdots)$ denotes terms which are proportional to $W_\alpha\bar{W}_{\dot{\alpha}}$, while $W^2(\cdots)$ denotes terms proportional to W^2. We will use similar notation with ellipses denoting quantities with bare fermionic terms that will not contribute to the calculation because of nilpotency conditions.

With the ingredients introduced above, the FZ multiplet for the supersymmetric BI action is given by [72]

$$\mathcal{X} = \frac{1}{6}W^2\bar{D}^2\Big(\bar{W}^2(\Gamma + \bar{\Gamma} - \Lambda)\Big) , \tag{5.4.33a}$$

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}} &= -2iM_\alpha\bar{W}_{\dot{\alpha}} + 2iW_\alpha\bar{M}_{\dot{\alpha}} + \frac{1}{12}[D_\alpha, \bar{D}_{\dot{\alpha}}]\Big(W^2\bar{W}^2\Big)\cdot\Big(\Gamma + \bar{\Gamma} - \Lambda\Big) \\ &\quad + W^2\bar{W}(\cdots) + \bar{W}^2W(\cdots) .\end{aligned} \tag{5.4.33b}$$

For our purposes, the superfields X and $\mathcal{J}_{\alpha\dot{\alpha}}$ can be further simplified as follows:

$$\mathcal{X} = \frac{1}{6}W^2\bar{D}^2\bar{W}^2\cdot\Big(\Gamma + \bar{\Gamma} - \Lambda\Big) + W^2\bar{W}(\cdots) , \tag{5.4.34a}$$

$$= \frac{2W^2\bar{D}^2\bar{W}^2}{3\Big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\Big)} + W^2\bar{W}(\cdots) , \tag{5.4.34b}$$

and

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}} &= -4W_\alpha\bar{W}_{\dot{\alpha}}(1 - \bar{u}\Gamma - u\bar{\Gamma}) + \frac{1}{12}[D_\alpha, \bar{D}_{\dot{\alpha}}]\Big(W^2\bar{W}^2\Big)\cdot\Big(\Gamma + \bar{\Gamma} - \Lambda\Big) \\ &\quad + W^2\bar{W}(\cdots) + \bar{W}^2W(\cdots) ,\end{aligned} \tag{5.4.35a}$$

$$\begin{aligned}&= -\frac{4W_\alpha\bar{W}_{\dot{\alpha}}}{\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}} + \frac{2D_\alpha W^2\cdot\bar{D}_{\dot{\alpha}}\bar{W}^2}{3\Big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\Big)\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}} \\ &\quad + W^2\bar{W}(\cdots) + \bar{W}^2W(\cdots) ,\end{aligned} \tag{5.4.35b}$$

where we used (5.4.31a).

The computation of $\mathcal{X}\bar{\mathcal{X}}$ is trivial and receives contributions only from the square of the first term in (5.4.34b). The computation of $\mathcal{J}^2$ is less trivial. It is obvious that the last two complicated terms in the second line of (5.4.35b) make no contribution since all the terms are proportional to $W\bar{W}$, and we have the nilpotency property $W_\alpha W_\beta W_\gamma = 0$. The square of the first term is easy to compute, and it is proportional to $W^2\bar{W}^2$. Next we consider the cross product between the first and second term in (5.4.35b) which leads to the relation:

$$W_\alpha \bar{W}_{\dot\alpha} \cdot D^\alpha W^2 \cdot \bar{D}^{\dot\alpha}\bar{W}^2 = W^2(DW)\cdot \bar{W}^2(\bar{D}\bar{W}) = 0\ . \tag{5.4.36}$$

Remarkably, this cross term vanishes since, as shown in Appendix A.6. Therefore, we have the on-shell relation that

$$W^2\overline{W}^2 DW = 0\ . \tag{5.4.37}$$

A simple physical interpretation of this condition is that the manifest supersymmetry is preserved on-shell, implying that the auxiliary field $\mathsf{D} \propto D^\alpha W_\alpha|_{\theta=0}$ has no vev, and is at least linear in gaugino fields $\lambda_\alpha \propto W_\alpha|_{\theta=0}$. The vanishing of this cross term can be compared with the pure bosonic case where the cross terms in T^2 vanish because of the tracelessness property of the free Maxwell stress tensor; see section 5.3. Finally, we compute the square of the second term in (5.4.35b) which includes the following structure:

$$D^\alpha W^2 \cdot \bar{D}^{\dot\alpha}\bar{W}^2 \cdot D_\alpha W^2 \cdot \bar{D}_{\dot\alpha}\bar{W}^2 = W^2\bar{W}^2 D^2W^2\bar{D}^2\bar{W}^2\ . \tag{5.4.38}$$

Here we have used $(D_\alpha W_\beta)(D^\alpha W^\beta) = -\frac{1}{2}D^2W^2 + W^\beta D^2 W_\beta$ to simplify the result.

In summary, on-shell the contributions to the supercurrent-squared operator $\mathcal{O}_{T^2}$ defined in eq. (5.2.14) are given by

$$\mathcal{J}^2 = -\frac{1}{8}\left\{\frac{16W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2} + \frac{4W^2\bar{W}^2D^2W^2\bar{D}^2\bar{W}^2}{9\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)^2}\right\}, \tag{5.4.39a}$$

$$\mathcal{X}\bar{\mathcal{X}} = \frac{4}{9}\frac{W^2\bar{W}^2D^2W^2\bar{D}^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)^2}\ . \tag{5.4.39b}$$

Adding these results gives the supersymmetric T^2 primary operator $\mathcal{O}_{T^2}$:

$$\mathcal{O}_{T^2} = -\frac{1}{2}\left(\mathcal{J}^2 + \frac{5}{4}\mathcal{X}\bar{\mathcal{X}}\right) = \frac{W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2}\left(1 - \frac{D^2W^2\bar{D}^2\bar{W}^2}{4\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)^2}\right) \tag{5.4.40a}$$

$$= \frac{W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2}\left(1 - \frac{\mathcal{A}^2-\mathcal{B}^2}{(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2})^2}\right), \tag{5.4.40b}$$

$$= \frac{2W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)}\ . \tag{5.4.40c}$$

It is worth noting that the simplifications occurring in constructing $\mathcal{O}_{T^2}$ from the supercurrents are very similar to the bosonic case of (5.3.10a).

Comparing with (5.4.20), we see that eq. (5.4.40c) proves that the supersymmetric BI action (5.4.19) is an on-shell solution of the flow equation

$$\frac{\partial \mathcal{L}_\kappa}{\partial \kappa^2} = \int d^2\theta d^2\bar{\theta}^2 \, \frac{2W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\Big(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\Big)} \, , \tag{5.4.41a}$$

$$= \int d^2\theta d^2\bar{\theta}^2 \, \mathcal{O}_{T^2} + \text{total derivatives} + \text{EOM} \, . \tag{5.4.41b}$$

It therefore describes a supercurrent-squared deformation of the $\mathcal{N} = 1$ free Maxwell Lagrangian. This result establishes a relationship between non-linearly realized supersymmetry and supercurrent-squared flow equations in $D = 4$.

Before closing this section, we should make a few comments regarding the on-shell condition (5.4.37) used in establishing the supercurrent-squared flow equation for the $D = 4$ $\mathcal{N} = 1$ BI action. First it is important to stress that the flow equation is not satisfied by the supersymmetric BI action off-shell. Second, we note that the specific combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ studied is the unique choice for which (5.4.19) satisfies a supercurrent-squared flow equation, even if only on-shell.

Such a non-trivial condition satisfied by the on-shell supersymmetric BI action is intriguing and hints at the existence of appropriate (super)field redefinitions which might lead to a different supersymmetric extension of BI that satisfies the flow equation off-shell. For example, it is know that the dependence of the off-shell extension on the auxiliary field D can be modified by appropriate (super)field redefinitions, as well as redefinitions of the full superspace Lagrangian. We refer to [34–36, 73] for a list of relevant papers on this subject. Under field redefinitions, the hidden supersymmetry will be modified but will remain a non-linearly realized symmetry of the theory. The existence of an off-shell solution of the supercurrent-squared flow is an interesting question for future research.

5.5 Higher Dimensions and Connections to Amplitudes

In this thesis, we have primarily focused on two-dimensional quantum field theories, where $T\overline{T}$ and related supercurrent-squared operators can be unambiguously defined at the quantum level. For $D > 2$, there appears to be no complete argument showing that any of the proposed operators $O_{T^2}^{[r]}$ of eq. (5.2.1), including the holographic operator of [60, 61], possess any particularly nice quantum properties.

However, despite the absence of a well-defined quantum operator in higher dimensions, the connection between non-linearly realized symmetries and our $D = 4$, $\mathcal{N} = 1$ example suggests that operators of this form still have some special properties (at least at the classical level). It would be very interesting to understand whether any supersymmetric completion of our descendant operator O_{T^2} of (5.2.15) could provide a well-defined deformation at the quantum level. Such an extension would necessarily involve contributions from other current-squared type operators which do not vanish on-shell. This possibility seems most promising in theories with extended, and perhaps even maximal, supersymmetry.

One interesting direction for future investigation concerns the relationship between $T\overline{T}$ deformations and amplitudes. In two dimensions, $T\overline{T}$ simply modifies the S-matrix of the undeformed theory by a CDD factor [74], but one might wonder about the S-matrices of higher-dimensional theories deformed by generalizations of $T\overline{T}$. One hint is that theories with non-linearly realized symmetries exhibit enhanced soft behavior – indeed, in the case of non-linearly realized *super*symmetry, there is a proof that such symmetries generically lead to constraints on the soft behavior of the S-matrix [75]. This fact which has been applied to the Volkov-Akulov action [76], which also satisfies a $T\overline{T}$-like flow.

There are also examples involving purely bosonic theories. For instance, in four dimensions, the Dirac action is the unique Lorentz-invariant Lagrangian for a single scalar which is consistent with factorization, has one derivative per field, and exhibits soft degree $\sigma = 2$ for its scattering amplitudes [77]. Similarly, it has been shown that the Born-Infeld action for a vector can be fixed by demanding enhanced soft behavior in a particular multi-soft limit [78], which can be understood in the context of T-duality and dimensional reduction [79]. Given the hints of a deeper relationship between supercurrent-squared deformations, non-linearly realized symmetries, and actions of Dirac or Born-Infeld type, it is natural to ask whether such deformations enhance the soft behavior of scattering amplitudes in a more general context.

CHAPTER 6
$T\overline{T}$ AND NON-ABELIAN GAUGE THEORY

In this chapter, we will define a theory for a non-abelian gauge field in two dimensions using the $T\overline{T}$ operator, following the discussion in “A Non-Abelian Analogue of DBI from $T\overline{T}$” [8].

Although we focus on the bosonic formulation of $T\overline{T}$ rather than the supercurrent-squared deformations developed in the preceding chapters, the gauge theory which we define here is compatible with maximal supersymmetry. This deformed theory shares some properties with the Dirac-Born-Infeld theory which describes gauge fields living on a brane, although it does not reduce to DBI even in the abelian case. We begin by reviewing some facts and motivation about brane physics.

6.1 Background on Brane Physics

Imagine a p-brane embedded in an ambient $(D+1)$-dimensional Minkowski space-time. By definition, any such brane spontaneously breaks the Poincaré symmetry of the ambient space-time:

$$ISO(D,1) \to ISO(p,1). \tag{6.1.1}$$

In particular, the breaking of translational symmetry guarantees the existence of $D-p$ universal scalar fields on the brane world-volume, collectively denoted ϕ, which are Nambu-Goldstone (NG) bosons for the broken translations. The physics of these modes is governed by the Dirac action,

$$S_{\text{Dirac}} = -T_p \int d^{p+1}\sigma \sqrt{-\det\left(\eta_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi\right)}, \tag{6.1.2}$$

with brane world-volume coordinates σ and a single dimensionful parameter T_p. The form of this action is fixed by the broken Lorentz symmetries, which are non-linearly realized. There might also be a function of any additional non-universal scalar fields multiplying this form, which we will not consider here. This action can also equivalently be viewed as the Nambu-Goto action for the brane in static gauge.

The Born-Infeld action, on the other hand, defines a non-linear interacting extension of Maxwell theory with action:

$$S_{\text{BI}} = -T_p \int d^{p+1}\sigma \sqrt{-\det\left(\eta_{\mu\nu} + \alpha F_{\mu\nu}\right)}. \tag{6.1.3}$$

One of the striking physical differences between Born-Infeld theory and Maxwell theory is the existence of a critical electric field determined by the dimensionful parameter α. In string theory, Born-Infeld theory describes the leading interactions for the gauge-field supported on a D-brane [80]. In that context both T_p and α are fixed in terms of the fundamental string tension α' with $\alpha = 2\pi\alpha'$.

The combined Dirac-Born-Infeld (DBI) action is a complete description of the physics of a single D-brane at leading order in string perturbation theory, and under the assumption that acceleration terms like ∂F or $\partial^2\phi$ are negligible:

$$S_{\rm DBI} = -T_p \int d^{p+1}\sigma \sqrt{-\det\left(\eta_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi + \alpha F_{\mu\nu}\right)}. \tag{6.1.4}$$

For multiple coincident branes, the abelian gauge symmetry is replaced by a non-abelian symmetry, and the fields (ϕ, F) typically take values in the adjoint representation of the gauge group. We will not assume any particular representation for the scalar fields in this discussion. It is natural to pose the following long considered question: what might replace (6.1.4) in the non-abelian theory? For the scalar fields appearing in the induced metric of the Dirac action (6.1.2), one could easily imagine making the replacement

$$\partial_\mu\phi\partial_\nu\phi \;\rightarrow\; {\rm Tr}\left(D_\mu\phi D_\nu\phi\right), \tag{6.1.5}$$

where ϕ is now matrix-valued and D is an appropriate covariant derivative. For the Born-Infeld action of (6.1.3), however, an interesting gauge-invariant replacement of this sort is not possible. In (6.1.5) Tr denotes the trace over gauge indices. When needed, we will use tr to denote the trace over Lorentz indices so that,

$${\rm tr}\Big(F^2\Big) = F_{\mu\nu}F^{\mu\nu}. \tag{6.1.6}$$

Indeed what one means by the Born-Infeld approximation, namely neglecting acceleration terms like DF, is ambiguous. Unlike the abelian case,

$$\big[D_\mu, D_\nu\big] = -iF_{\mu\nu}, \tag{6.1.7}$$

so there is no clear cut way of truncating the full brane effective action by throwing out acceleration terms.

With considerable hard work there is, however, some data known about brane couplings beyond the two derivative non-abelian kinetic terms ${\rm Tr}\big(F_{\mu\nu}F^{\mu\nu}\big)$ at leading order in string perturbation theory. This information is very nicely summarized in the thesis [81] to which we refer for a more complete discussion. For comparative purposes with our analysis, we note that the known F^4 terms are correctly captured by a symmetrized trace prescription [82, 83]. Up to overall scaling, they are given by

$$\mathrm{STr}\left(\mathrm{tr}\, F^4 - \frac{1}{4}(\mathrm{tr}\, F^2)^2\right), \tag{6.1.8}$$

with

$$\mathrm{STr}\,(T_1 T_2 \ldots T_n) = \frac{1}{n!}\sum_{\sigma\in \mathrm{S_n}} \mathrm{Tr}\Big(T_{\sigma(1)}T_{\sigma(2)}\ldots T_{\sigma(n)}\Big). \tag{6.1.9}$$

This prescription is known to fail for higher derivative terms. What is important for us is that (6.1.8) defines a single trace operator.

To define a non-abelian analogue of the abelian DBI theory (6.1.4), our approach will be to $T\overline{T}$ deform a non-abelian gauge theory with scalar matter in two dimensions. In contrast to the preceding parts of this thesis, in this chapter we will restrict to bosonic theories for simplicity. A priori this approach has no connection to either brane physics or string theory. We will do this in steps by first recalling known results about deforming free scalars and Maxwell fields [3, 10, 62], and then extending to charged matter and non-abelian gauge theories. Other than also involving an infinite collection of irrelevant operators, the reason this approach should be viewed as giving a non-abelian analogue of DBI is that the $T\overline{T}$ deformation of a free scalar with parameter λ already gives the Dirac action, as we reviewed in Chapter 2:

$$S_\lambda = \int d^2\sigma \frac{1}{2\lambda}\left(\sqrt{1+4\lambda|\partial\phi|^2}-1\right). \tag{6.1.10}$$

This direct connection with brane physics is reason enough to suspect that $T\overline{T}$ applied to gauge-theory will give further insight into brane physics.

The $T\overline{T}$ deformation of Maxwell theory, however, is already different from the Born-Infeld theory of (6.1.3). This is the reason we call the $T\bar{T}$-deformed theory an analogue rather than a generalization of DBI; it does not reduce to DBI even in the case of abelian gauge theory. The couplings are not given by the square-root structure of a relativistic particle but rather by a hypergeometric function [62]. This might not seem very exciting in two dimensions where pure gauge-fields have no propagating degrees of freedom, but that is no longer the case when we add scalar fields, even in the abelian setting. For interesting recent discussions of $T\overline{T}$-deformed gauge theories, see [84, 85].

For the non-abelian theory defined using $T\overline{T}$, the $O(F^4)$ terms are already very different from what is known about non-abelian BI theory. Rather than involving a single trace operator like (6.1.8), they involve double trace operators. The $T\overline{T}$-deformed theory is quite remarkable because it has the following properties:

- The theory is compatible with supersymmetry [5–7, 26, 27, 86, 87]. Indeed, we have discussed at length in Chapters 3 and 4 that $T\overline{T}$ *always* preserves degeneracies in energy levels like those implied by supersymmetry, and that one can often make the supersymmetry manifest via a superspace flow construction. In particular, if one $T\overline{T}$ deforms a maximally supersymmetric starting theory then this supersymmetry is preserved!

- The theory is believed to exist at the quantum level, unlike DBI which is an effective theory with our present level of understanding.

- The theory has a critical electric field like BI.

This is already quite surprising in the abelian case with uncharged matter. Folklore suggests that some of these properties, like compatibility with maximal supersymmetry, should only have been true for DBI. Indeed there are no obvious reasons that the structures seen here should not emerge from string theory, either in a closed or open string setting. In fact, the $D = 10$ space-time effective action for the type I/heterotic strings does contain a double trace F^4 term, which is required for anomaly cancelation in $D = 10$, or more generally required by supersymmetry [88]. In the heterotic string, the term arises at tree-level and takes the schematic form:

$$S_{\text{het}} \sim \int d^{10}x\sqrt{g}e^{-2\phi}\left(\operatorname{Tr} F^2\right)^2 + \dots . \tag{6.1.11}$$

In the dual type I frame, relevant for a brane picture, the same coupling arises from diagrams with Euler characteristic -1 [89, 90]. This leads us to suspect that the $T\overline{T}$ flow equation is connected with corrections to two-dimensional beta functions from higher orders in string perturbation theory.

6.2 Deforming Pure Gauge Theory

Before we go on to solve the $T\overline{T}$ equation to find a non-abelian analogue of the DBI action, we will first illustrate the above techniques by solving for the $T\overline{T}$-deformed Yang-Mills theory. That is, we begin with an undeformed Lagrangian of the form

$$k\mathcal{L}_0 = F^a_{\mu\nu}F_a^{\mu\nu} = \operatorname{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{6.2.1}$$

where we will often suppress the trace for convenience and simply write F^2 for $\mathrm{Tr}(F^2)$, so that

$$\mathcal{L}_0 = \frac{1}{k} F^2. \tag{6.2.2}$$

We retain an overall dimensionless constant k in the Lagrangian; in most of the calculations that follow, which we will suppress factors of k by setting $k = 1$. The value of k does not affect the equations of motion associated with the action (6.2.2), but the sign of k will be important for determining critical value of the field strength F^2. To see the maximum allowed electric field for the deformed theory in Minkowski signature, we will find that we must take $k < 0$ so that the undeformed action is positive (however, all of our other results are valid in either Minkowski or Euclidean signature). We will restore factors of k, replacing $F^2 \to \frac{1}{k} F^2$, when the sign is relevant.

Our goal is to find the deformed Lagrangian $\mathcal{L}(\lambda) = f(\lambda, F^2)$ which solves the flow equation

$$\partial_\lambda \mathcal{L}_\lambda = \det T_{\mu\nu} \ , \tag{6.2.3}$$

with initial condition $\mathcal{L}(0) = \mathcal{L}_0$.

It is first convenient to rewrite this flow equation using the fact that 2×2 matrices M satisfy

$$\det(M) = \frac{1}{2} \left((\mathrm{tr}\, M)^2 - \mathrm{tr} \left(M^2 \right) \right) . \tag{6.2.4}$$

Thus we can write the $T\overline{T}$ flow equation in a manifestly Lorentz-invariant way as

$$\partial_\lambda \mathcal{L}_\lambda = \frac{1}{2} \left(\left(T^\mu{}_\mu \right)^2 - T^{\mu\nu} T_{\mu\nu} \right) . \tag{6.2.5}$$

Notice that the stress-energy tensor is a single-trace operator in the undeformed theory. In the λ expansion, the leading order deformation of the Lagrangian is therefore automatically a double trace operator. Including further corrections in λ will only generate higher order multi-trace operators. In particular – as we show in Appendix A.7 – a single-trace deformation like the leading F^4 terms of non-abelian DBI in (6.1.8) will never be generated from a $T\overline{T}$ flow beginning from an undeformed Lagrangian which is only a function of F^2.

The details of the calculation of the stress tensor components $T^{(\lambda)}_{\mu\nu}$ for the Lagrangian $\mathcal{L}(\lambda, F^2)$ are presented in Appendix A.7, where we find the $T\overline{T}$ operator for an arbitrary Lagrangian depending on a field strength $F_{\mu\nu}$ and a complex scalar ϕ. We can set the scalar ϕ to zero in the result of that Appendix to find the flow equation for a pure gauge field

Lagrangian. The result, using the shorthand notation $x = F^2$, is

$$\frac{df}{d\lambda} = f(x)^2 - 4f(x)x\frac{\partial f}{\partial x} + 4x^2\left(\frac{\partial f}{\partial x}\right)^2 . \tag{6.2.6}$$

Next we will present several methods for solving (6.2.6):

1. by directly solving the flow equation (6.2.5) in a series expansion;

2. by writing the solution implicitly in terms of a complete integral;

3. and by dualizing the field strength F^2 to a scalar, deforming, and dualizing back.

The same three methods will also be applied to the case of a gauge field coupled to scalars in Section 6.3.

6.2.1 Series solution of flow equation

The differential equation (6.2.6) derived in the preceding subsection can be brought into a simpler form by refining our ansatz to

$$f(\lambda, F^2) = F^2 g(\lambda F^2) \tag{6.2.7}$$

for some new function g. For convenience, we define the dimensionless variable $\chi = \lambda F^2 = \lambda x$. Then the function g satisfies the differential equation

$$\frac{\partial g}{\partial \chi} = \left(g(\chi) + 2\chi g'(\chi)\right)^2 . \tag{6.2.8}$$

One can solve this differential equation by making a series ansatz of the form $g(\chi) = \sum_{n=0}^{\infty} c_n \chi^n$, determining the first several coefficients c_n. To order χ^6, the function $g(\chi)$ is given by

$$g(\chi) = 1 + \chi + 3\chi^2 + 13\chi^3 + 68\chi^4 + 399\chi^5 + 2530\chi^6 + \mathcal{O}(\chi^7). \tag{6.2.9}$$

To determine the generating function, one can refer to an encyclopedia of integer sequences [91] to find that g can be written as a generalized hypergeometric function,

$$g(\chi) = {}_4F_3\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27}\cdot\chi\right) . \tag{6.2.10}$$

Thus the full solution for the deformed Lagrangian can be written as

$$\begin{aligned}\mathcal{L}(\lambda) &= F^2 \cdot {}_4F_3\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27} \cdot \lambda F^2\right) \\ &= \frac{3}{4\lambda}\left({}_3F_2\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}; \frac{1}{3}, \frac{2}{3}; \frac{256}{27} \cdot \lambda F^2\right) - 1\right). \end{aligned} \tag{6.2.11}$$

The functions on the first and second lines of (6.2.11) are equivalent because of a hypergeometric functional identity. We will use the expression in the first line, written in terms of ${}_4F_3$ rather than ${}_3F_2$, but we include the second expression to make contact with the work of [62], where this expression was first derived. We also note that the function (6.2.11) has appeared in an analogue of $T\overline{T}$ defined for $(0+1)$ dimensional theories [92].

6.2.2 Implicit solution

We will also find later that it will be useful to solve the $T\overline{T}$ flow equation using a different method. We begin with the differential equation (6.2.6), but this time we make the ansatz

$$f(\lambda, F^2) = \frac{1}{\lambda} h(\lambda F^2). \tag{6.2.12}$$

As before, we define the dimensionless variable $\chi = \lambda F^2$. In terms of h, the differential equation becomes

$$4\chi^2 \left(h'(\chi)\right)^2 - 4\chi h(\chi) h'(\chi) - \chi h'(\chi) + h(\chi)^2 + h(\chi) = 0. \tag{6.2.13}$$

Equation (6.2.13) is quadratic in $h'(\chi)$, so we can solve to find

$$\frac{dh}{d\chi} = \frac{1 + 4h(\chi) - \sqrt{1 - 8h(\chi)}}{8\chi}, \tag{6.2.14}$$

where we have chosen the root which makes $h'(\chi)$ finite as $\chi \to 0$, assuming $\lim_{\chi \to 0} h(\chi) = 0$.

We may separate variables in (6.2.14) to write

$$\int \frac{8\, dh}{1 + 4h(\chi) - \sqrt{1 - 8h(\chi)}} = \int \frac{d\chi}{\chi}. \tag{6.2.15}$$

The integrals can be evaluated in terms of logarithms; exponentiating both sides then yields

$$\chi = C\left(1 - \sqrt{1 - 8h}\right)\left(3 + \sqrt{1 - 8h}\right)^3. \tag{6.2.16}$$

Equation (6.2.16) implicitly defines the solution $h(\chi)$ to the $T\overline{T}$ flow equation via the roots of an algebraic equation.

We note that, choosing $C = \frac{1}{256}$, equation (6.2.16) is consistent with the solution derived in the previous section. Recall that the two ansatzes we made here and in subsection (6.2.1) are related by

$$f(\lambda, F^2) = F^2 f(\chi) = \frac{1}{\lambda} h(\chi), \tag{6.2.17}$$

so $h(\chi) = \chi f(\chi)$. Indeed, one can check that the function

$$h(\chi) = \chi \cdot {}_4F_3 \left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27} \cdot \chi \right), \tag{6.2.18}$$

satisfies the functional identity

$$\chi = \frac{1}{256} \left(1 - \sqrt{1 - 8h(\chi)} \right) \left(3 + \sqrt{1 - 8h(\chi)} \right)^3 . \tag{6.2.19}$$

We therefore see that the hypergeometric (6.2.11) obtained earlier is, in fact, an algebraic function that can be defined as a root of (6.2.19).[1]

6.2.3 Solution via dualization

The above result can also be derived in a different way. The details of this procedure do not depend on the sign of our constant k nor the signature, so we will set $k = 1$ and take Minkowski signature for concreteness. The undeformed Lagrangian (6.2.2) is then

$$\mathcal{L}_0 = \frac{1}{k} F_{\mu\nu} F^{\mu\nu} = -2F_{01}^2, \tag{6.2.20}$$

which can be equivalently expressed by dualizing the field strength to a scalar, as in

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\phi^2 + \phi\, \epsilon^{\mu\nu} F_{\mu\nu} \\ &= \frac{1}{2}\phi^2 + 2\phi F_{01}. \end{aligned} \tag{6.2.21}$$

1. Other examples of hypergeometric functions which can be expressed algebraically include those on Schwarz's list [93], which is summarized on Wikipedia.

The equation of motion for ϕ arising from (6.2.21) is

$$\frac{\delta \mathcal{L}_0}{\delta \phi} = \phi + 2F_{01} = 0, \tag{6.2.22}$$

so $\phi = -2F_{01}$, and then replacing ϕ with its equation of motion yields

$$\mathcal{L}_0 = \frac{1}{2}\left(-2F_{01}\right)^2 + 2\left(-2F_{01}\right)F_{01} = -2F_{01}^2, \tag{6.2.23}$$

which matches (6.2.20). On the other hand, (6.2.21) is easy to $T\overline{T}$ deform. After coupling to gravity, one has

$$S_0[g] = \left(\frac{1}{2}\int \sqrt{-g}\,\phi^2\, d^2x\right) + \left(\int \phi\,\epsilon^{\mu\nu} F_{\mu\nu}\, d^2x\right), \tag{6.2.24}$$

where the second term is purely topological and thus independent of the metric. The undeformed Lagrangian, then, is a pure potential term $V(\phi) = \phi^2$ for the boson ϕ. The solution to the $T\overline{T}$ flow equation at finite λ for a general potential is well-known [3, 10]; in this case, one finds

$$\begin{aligned}\mathcal{L}(\lambda) &= \frac{\frac{1}{2}\phi^2}{1-\frac{\lambda}{2}\phi^2} + \phi\,\epsilon^{\mu\nu}F_{\mu\nu} \\ &= \frac{\phi^2}{2-\lambda\phi^2} + 2\phi F_{01}.\end{aligned} \tag{6.2.25}$$

We now integrate out ϕ. The equation of motion for ϕ arising from (6.2.25) is

$$\begin{aligned}0 &= \frac{\delta\mathcal{L}(\lambda)}{\delta\phi} \\ &= 2\phi + F_{01}\left(2-\lambda\phi^2\right)^2,\end{aligned} \tag{6.2.26}$$

or

$$F_{01} = -\frac{2\phi}{(2-\lambda\phi^2)^2}. \tag{6.2.27}$$

To proceed, we series expand (6.2.27) in ϕ to find

$$F_{01} = -\frac{\phi}{2} - \frac{\lambda\phi^3}{2} - \frac{3\lambda^2\phi^5}{8} - \frac{\lambda^3\phi^7}{4} - \frac{5\lambda^4\phi^9}{32} + \mathcal{O}\left(\phi^{11}\right), \tag{6.2.28}$$

and then apply the Lagrange inversion theorem to find a series expansion for ϕ in terms of F_{01}, yielding

$$\phi = -2F_{01} + 8\lambda F_{01}^2 - 72\lambda^2 F_{01}^5 + 832\lambda^3 F_{01}^7 - 10880\lambda^4 F_{01}^9 + \mathcal{O}\left(F_{01}^{11}\right). \tag{6.2.29}$$

Substituting the expansion (6.2.29) into the action (6.2.25), and expressing the result in terms of $F^2 = F_{\mu\nu}F^{\mu\nu} = -2F_{01}^2$, gives

$$\mathcal{L}(\lambda) = F^2 \left(1 + \lambda F^2 + 3\lambda F^4 + 13\lambda^3 F^6 + 68\lambda^4 F^8 + \cdots\right). \tag{6.2.30}$$

The Taylor coefficients appearing in (6.2.30) are precisely those of the hypergeometric (6.2.10). The procedure of iteratively solving (6.2.27) for ϕ and substituting into (6.2.25), therefore, reproduces

$$\mathcal{L}(\lambda) = F^2 \cdot {}_4F_3\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27} \cdot \lambda F^2\right) \tag{6.2.31}$$

which matches the solution which we derived by different methods above.

This procedure – dualizing the field strength F^2 to a scalar ϕ, $T\overline{T}$ deforming the scalar action, and then dualizing back – is closely related to an observation made in [62]. There the authors noted that, although the deformed Lagrangian (6.2.11) is quite complicated, the corresponding Hamiltonian satisfies the simple relation

$$\mathcal{H}_\lambda = \frac{\mathcal{H}_0}{1 - \lambda \mathcal{H}_0}, \tag{6.2.32}$$

where the Hamiltonian is a function of the conjugate momentum

$$\Pi^1 = \frac{\partial \mathcal{L}_\lambda}{\partial \dot{A}_1}. \tag{6.2.33}$$

The Legendre transform which converts the Lagrangian $\mathcal{L}_\lambda$ to the Hamiltonian $\mathcal{H}_\lambda$ is mathematically equivalent to the process of dualizing the field strength F_{01} to a scalar ϕ.

6.2.4 Comparison to Born-Infeld

For the moment, we specialize to the abelian case where the Born-Infeld action can be unambiguously defined. The Lagrangian (6.2.11) differs from the Born-Infeld action in two dimensions. To order λ^4, our solution has the series expansion

$$\mathcal{L}(\lambda) = F^2 + \lambda F^4 + 3\lambda^2 F^6 + 13\lambda^3 F^8 + 68\lambda^4 F^{10} + \mathcal{O}(\lambda^5). \tag{6.2.34}$$

On the other hand, the Born-Infeld action (after normalizing the coefficient of F^2 to match (6.2.34) at order F^2) has the expansion

$$\frac{1}{2\lambda}\sqrt{1+4\lambda F^2} = \frac{1}{2\lambda} + F^2 - \lambda F^4 + 2\lambda^2 F^6 - 5\lambda^3 F^8 + 14\lambda^4 F^{10} + \mathcal{O}(\lambda^5). \tag{6.2.35}$$

Although the Taylor coefficients for the Born-Infeld action and the "hypergeometric action" differ, both exhibit a critical value for the electric field. In the case of Born-Infeld, this is obvious; replacing $F^2 = -2F_{01}^2$, we see that the action

$$\frac{1}{2\lambda}\sqrt{1 - 8\lambda F_{01}^2} \tag{6.2.36}$$

is only real for

$$F_{01} < \frac{1}{\sqrt{8\lambda}}. \tag{6.2.37}$$

To see the critical electric field for the action $\mathcal{L}(\lambda)$ defined in (6.2.10), it is most convenient to use the implicit form (6.2.19):

$$\lambda F^2 = \frac{1}{256}\left(1 - \sqrt{1 - 8\lambda\mathcal{L}(\lambda)}\right)\left(3 + \sqrt{1 - 8\lambda\mathcal{L}(\lambda)}\right)^3. \tag{6.2.38}$$

The right side is maximized when $\mathcal{L}(\lambda) = \frac{1}{8\lambda}$, where it takes the value $\frac{27}{256}$, which means that

$$F^2 < \frac{27}{256\lambda}. \tag{6.2.39}$$

Recall that our Lagrangian (6.2.2) contained an overall constant to track signs; to restore factors of k, we replace $F^2 \to \frac{1}{k}F^2$. In Minkowski signature, we should take $k < 0$ so that $\mathcal{L}_0 = -\frac{1}{k}F_{\mu\nu}F^{\mu\nu} = -\frac{2}{k}F_{01}^2$ is positive. Letting $k = -1$, we find

$$F_{01} < \sqrt{\frac{27}{512\lambda}}, \tag{6.2.40}$$

which is a different critical value for the electric field than (6.2.37).

However, pure Yang-Mills theory in two dimensions has no propagating degrees of freedom, so the difference between the expansions (6.2.34) and (6.2.35) does not have much physical effect (at least in infinite volume). To detect the difference between these theories, we should couple the gauge field to matter, as we do in section 6.3.

6.3 Non-Abelian Analogue of DBI

In this section, we will consider an action for a Yang-Mills gauge field $F^a_{\mu\nu}$ coupled to a scalar ϕ in some representation of the gauge group. The undeformed Lagrangian is taken to be

$$\begin{aligned}\mathcal{L}_0 &= F^a_{\mu\nu} F_a^{\mu\nu} + |D\phi|^2 \\ &\equiv F^2 + |D\phi|^2,\end{aligned} \tag{6.3.1}$$

where we again use the shorthand $F^2 = \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$. We have set the overall constant k, which appears in (6.2.2), equal to 1 because we will not analyze critical field strengths in the deformed coupled model. If one were to carry out this analysis, however, one would need an overall minus sign in (6.3.1) in Minkowski signature.

In what follows, we will also define $x = F^2$ and $y = |D\phi|^2$ for convenience; here

$$\begin{aligned}|D\phi|^2 &= \left(D_\mu \phi\right)\left(D^\mu \phi\right)^*, \\ D_\mu &= \partial_\mu - iA_\mu,\end{aligned} \tag{6.3.2}$$

and gauge group indices will be suppressed.

At finite λ, we take a general ansatz of the form

$$\mathcal{L}_\lambda = f(\lambda, x = F^2, y = |D\phi|^2). \tag{6.3.3}$$

The stress tensor components $T^{(\lambda)}_{\mu\nu}$ for the Lagrangian (6.3.3), and the differential equation arising from (6.2.5), are worked out in Appendix A.7. The resulting partial differential equation, equation (A.7.7), is copied here for convenience:

$$\frac{df}{d\lambda} = f^2 - 4fx\frac{\partial f}{\partial x} - 2fy\frac{\partial f}{\partial y} + 4x^2\left(\frac{\partial f}{\partial x}\right)^2 + 4xy\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}. \tag{6.3.4}$$

Our goal in the following subsections will be to solve (A.7.7) by several methods, just as we did in the case of pure gauge theory.

6.3.1 Series solution of flow equation

We know that (A.7.7) reduces to the Dirac action, (A.7.9), when the gauge field is set to zero, and that it reduces to the hypergeometric action of Section 6.2, (A.7.11), when the scalar is set to zero. Therefore, in the coupled case it is natural to make an ansatz of the form

$$f(\lambda, x, y) = \frac{1}{2\lambda}\left(\sqrt{1+4\lambda y}-1\right) + {}_3F_4\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27}\cdot\lambda x\right)\cdot x + \sum_{n=3}^{\infty}\sum_{k=1}^{n} c_{n,k}\lambda^{n-1}x^k y^{n-k}. \tag{6.3.5}$$

The sum on the final line of (6.3.5) allows for all possible couplings between F^2 and $|D\phi|^2$, with the appropriate power of λ required by dimensional analysis. One can then determine the coefficients $c_{n,k}$ by plugging the ansatz (6.3.5) into (A.7.7) and solving order-by-order in λ. The result, up to coupled terms of order λ^8, is

$$\begin{aligned} f(\lambda, x, y) = {} & \frac{1}{4\lambda}\left(\sqrt{1+16\lambda y}-1\right) + {}_3F_4\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27}\cdot\lambda x\right)\cdot x \\ & - \lambda^2 xy^2 + \lambda^3\left(4\,xy^3 - 4\,x^2y^2\right) + \lambda^4\left(18\,x^2y^3 - 22\,x^3y^2 - 14\,xy^4\right) \\ & + \lambda^5\left(-140\,x^4y^2 + 104\,x^3y^3 - 65\,x^2y^4 + 48\,xy^5\right) \\ & + \lambda^6\left(-165\,xy^6 + 220\,x^2y^5 - 364\,x^3y^4 + 680\,x^4y^3 - 969\,x^5y^2\right) \\ & + \lambda^7\left(572\,xy^7 - 726\,x^2y^6 + 1120\,x^3y^5 - 2244\,x^4y^4 + 4788\,x^5y^3 - 7084\,x^6y^2\right) \\ & + \lambda^8\Big(-2002\,xy^8 + 2392\,x^2y^7 - 3160\,x^3y^6 + 5814\,x^4y^5 - 14630\,x^5y^4 + 35420\,x^6y^3 \\ & \quad - 53820\,x^7y^2\Big). \end{aligned} \tag{6.3.6}$$

We were unable to find an closed-form expression for the function which generates the couplings (6.3.6). However, it is interesting to study the corrections in various approximations.

For instance, consider the coupled terms between F^2 and $|D\phi|^2$ to leading order in the variable $y = |D\phi|^2$. Retaining only the couplings in (6.3.6) proportional to y^2, one finds

$$\begin{aligned} f(\lambda, x, y) = {} & \frac{1}{4\lambda}\left(\sqrt{1+16\lambda y}-1\right) + {}_3F_4\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27}\cdot\lambda x\right)\cdot x - \lambda^2 xy^2 - 4\lambda^3\,x^2y^2 \\ & - 2\lambda^4\,x^3y^2 - 140\lambda^5\,x^4y^2 - 969\lambda^6\,x^5y^2 - 7084\lambda^7\,x^6y^2 - 53820\lambda^8\,x^7y^2 \\ & + \mathcal{O}\left(\lambda^9, \lambda^3 xy^3\right). \end{aligned} \tag{6.3.7}$$

These series coefficients resum into another hypergeometric function [94],

$$
\begin{aligned}
f(\lambda, x, y) = \frac{1}{4\lambda}\left(\sqrt{1+16\lambda y}-1\right) + {}_3F_4\left(\frac{1}{2},\frac{3}{4},1,\frac{5}{4};\frac{4}{3},\frac{5}{3},2;\frac{256}{27}\cdot\lambda x\right)\cdot x \\
-\lambda^2 x y^2\, {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{2}{3},\frac{4}{3};\frac{256}{27}\cdot\lambda x\right) + \mathcal{O}\left(\lambda^3 x y^3\right).
\end{aligned}
\tag{6.3.8}
$$

Defining the hypergeometric appearing in the correction term as

$$
g(\chi) = {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{2}{3},\frac{4}{3};\frac{256}{27}\cdot\chi\right), \tag{6.3.9}
$$

one can show that g satisfies the functional relation

$$
\chi = \frac{g(\chi)-1}{g(\chi)^4}. \tag{6.3.10}
$$

The maximum of the function $\frac{g-1}{g^4}$ occurs when $g = \frac{4}{3}$, at which this function takes the maximal value of $\frac{27}{256}$. Therefore, the maximum value of χ for which the function (6.3.10) is defined is $\chi = \frac{27}{256}$, giving a critical field strength

$$
F^2 < \frac{27}{256\lambda}. \tag{6.3.11}
$$

We note that this is the same value of the critical electric field as that in the uncoupled term involving ${}_3F_4$ in (6.3.8), which we saw in (6.2.40) for the case of pure gauge theory. It is reasonable to expect that the value of the critical electric field is modified if one includes corrections to all orders in $|D\phi|^2$, as is the case for the Dirac-Born-Infeld action.

6.3.2 Implicit solution

We can instead solve (A.7.7) in terms of a complete integral. First, we refine our ansatz to

$$
f(\lambda) = \frac{1}{\lambda} g(\chi,\eta) \quad , \quad \chi = \lambda x\ ,\ \eta = \lambda y\ . \tag{6.3.12}
$$

After doing this, the differential equation becomes

$$
0 = 4\chi^2\left(\frac{\partial g}{\partial\chi}\right)^2 + 4\eta\chi\frac{\partial g}{\partial\chi}\frac{\partial g}{\partial\eta} - 4\chi g\frac{\partial g}{\partial\chi} - \chi\frac{\partial g}{\partial\chi} - 2\eta g\frac{\partial g}{\partial\eta} - \eta\frac{\partial g}{\partial\eta} + g^2 + g. \tag{6.3.13}
$$

Making a change of variables to

$$p = \log(\chi) \quad , \quad q = \log(\eta) \ , \tag{6.3.14}$$

and writing $g(\chi, \eta) = w(p, q)$, the differential equation for h becomes

$$0 = 4\left(\frac{\partial w}{\partial p}\right)^2 + 4\frac{\partial w}{\partial p}\frac{\partial w}{\partial q} - (4w+1)\frac{\partial w}{\partial p} - (2w+1)\frac{\partial w}{\partial q} + w^2 + w \ . \tag{6.3.15}$$

This can be solved by consulting a handbook of partial differential equations (see, for instance, equation 15 in section 2.2.6 of [95]). For any partial differential equation of the form

$$0 = f_1(w)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(w)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + f_3(w)\left(\frac{\partial w}{\partial y}\right)^2 + g_1(w)\frac{\partial w}{\partial x} + g_2(w)\frac{\partial w}{\partial y} + h(w), \tag{6.3.16}$$

the solution $w(x, y)$ is given implicitly by the following complete integral:

$$\begin{aligned} C_3 &= C_1 x + C_2 y + \int \frac{2F(w)\, dw}{G(w) \pm \sqrt{G(w)^2 - 4F(w)h(w)}}, \\ F(w) &= C_1^2 f_1(w) + C_1 C_2 f_2(w) + C_2^2 f_3(w), \\ G(w) &= C_1 g_1(w) + C_2 g_2(w). \end{aligned} \tag{6.3.17}$$

Our equation (6.3.15) is precisely of the form (6.3.16), after identifying the independent variables $p \sim x$, $q \sim y$, and with the following functions:

$$\begin{aligned} &f_1 = f_2 = 4 \quad , \quad f_3 = 0 \quad , \quad g_1 = -4w - 1 \ , \\ &g_2 = -2w - 1 \quad , \quad h(w) = w^2 + w. \end{aligned} \tag{6.3.18}$$

Therefore, the functions F and G are

$$\begin{aligned} F(w) &= 4C_1^2 + 4C_1 C_2, \\ G(w) &= C_1(-4w - 1) + C_2(-2w - 1) \\ &= (-4C_1 - 2C_2)\, w - (C_1 + C_2)\, . \end{aligned} \tag{6.3.19}$$

Our solution, then, is

$$C_3 = C_1 p + C_2 q \tag{6.3.20}$$
$$+ \int \frac{8\left(C_1^2 + C_1 C_2\right)\, dw}{-\left(4C_1 + 2C_2\right) w - \left(C_1 + C_2\right) - \sqrt{\left(\left(4C_1 + 2C_2\right) w + \left(C_1 + C_2\right)\right)^2 - 16\left(C_1^2 + C_1 C_2\right)\left(w^2 + w\right)}} \, .$$

where we have taken the negative root in the denominator, appropriate if $C_1 + C_2 < 0$.

Choosing values of the constants C_1, C_2, C_3 in (6.3.21) gives an implicit relation for the function $w(p, q)$ which solves the $T\overline{T}$ flow equation. For instance, if we set $C_2 = 0$ and $C_1 = -1$, equation (6.3.21) becomes

$$C_3 + \log(\chi) = \int \frac{8\, dw}{4w + 1 - \sqrt{1 - 8w}}, \tag{6.3.21}$$

which reproduces the result (6.2.15) which we found in the case of pure gauge theory. In this sense, our implicit solution is a generalization of the technique of section 6.2.2 to the case where $|D\phi|^2 \neq 0$.

The integral appearing in (6.3.21) can be computed explicitly in terms of logarithms (or, equivalently, inverse hyperbolic tangents). The integral is of the form

$$I(w) = \int \frac{\alpha\, dw}{-\beta w - \gamma - \sqrt{(\beta w + \gamma)^2 - 2\alpha\left(w^2 + w\right)}}, \tag{6.3.22}$$

where

$$\begin{aligned} \alpha &= 8(C_1^2 + C_1 C_2), \\ \beta &= 4C_1 + 2C_2, \\ \gamma &= C_1 + C_2. \end{aligned} \tag{6.3.23}$$

The result can be written as

$$
\begin{aligned}
I(w) = \frac{1}{2}\Bigg(& (\gamma - \beta)\tanh^{-1}\left(\frac{\alpha(w+1) + (\beta-\gamma)(\gamma+\beta w)}{(\beta-\gamma)\sqrt{\gamma^2 + w^2\left(2\alpha+\beta^2\right) + 2w(\alpha+\beta\gamma)}}\right) \\
& - \gamma\tanh^{-1}\left(\frac{\gamma^2 + w(\alpha+\beta\gamma)}{\gamma\sqrt{\gamma^2 + w^2\left(2\alpha+\beta^2\right) + 2w(\alpha+\beta\gamma)}}\right) \\
& + \sqrt{2\alpha+\beta^2}\tanh^{-1}\left(\frac{\alpha + 2\alpha w + \beta(\gamma+\beta w)}{\sqrt{2\alpha+\beta^2}\sqrt{\gamma^2 + w^2\left(2\alpha+\beta^2\right) + 2w(\alpha+\beta\gamma)}}\right) \\
& + (\beta-\gamma)\log(w+1) + \gamma\log(w)\Bigg) .
\end{aligned} \tag{6.3.24}
$$

Exponentiating both sides then gives

$$
\exp\left(C_3 - C_1 p - C_2 q\right) = \exp\left(I(w)\right) . \tag{6.3.25}
$$

After simplifying the exponentials of the inverse hyperbolic tangents in (6.3.25), the right side only involves rational functions and radicals. This relation, therefore, gives an algebraic equation in w whose roots are the solution to the $T\overline{T}$ flow.

By construction, a function $w(p, q)$ which satisfies (6.3.25) solves the differential equation (6.3.15). However, this technique is more unwieldy than the direct series solution for generating Taylor coefficients. The main utility of this strategy is the conceptual result that the solution $w(p, q)$ is, in principle, defined by the root of an equation which involves only radicals and quotients, as we saw for the pure gauge theory case in (6.2.19).

6.3.3 Solution via dualization

One can also apply the dualization technique of section (6.2) to the coupled action. Begin with the undeformed action

$$
\mathcal{L}_0 = |D\phi|^2 + F^2 , \tag{6.3.26}
$$

where we put $k = 1$ since the sign will not affect this calculation. Exactly as before, this action is equivalent to

$$
\mathcal{L}_0 = |D\phi|^2 + \frac{1}{2}\chi^2 + \chi\epsilon^{\mu\nu}F_{\mu\nu} , \tag{6.3.27}
$$

after integrating out the field χ, although this form of the Lagrangian hides some complexity because the covariant derivative D is now non-local in χ. Ignoring this for the moment, we again note that the action coupled to a background metric is of the form

$$S_0 = \left(\int \sqrt{-g}\, d^2x \left(|D\phi|^2 + \frac{1}{2}\chi^2 \right) \right) + \left(\int d^2x\, \chi \epsilon^{\mu\nu} F_{\mu\nu} \right) . \tag{6.3.28}$$

As far as the $T\overline{T}$ deformation is concerned, (6.3.28) is simply the action of a complex scalar ϕ with a constant potential $V = \frac{1}{2}\chi^2$. The solution to the flow equation at finite λ is [3, 10]

$$\mathcal{L}(\lambda) = \frac{1}{2\lambda} \sqrt{ \frac{\left(1 - \lambda\chi^2\right)^2}{\left(1 - \frac{1}{2}\lambda\chi^2\right)^2} + 2\lambda \frac{2|D\phi|^2 + \chi^2}{1 - \frac{1}{2}\lambda\chi^2} } - \frac{1}{2\lambda} \frac{1 - \lambda\chi^2}{1 - \frac{1}{2}\lambda\chi^2} + \chi\epsilon^{\mu\nu} F_{\mu\nu} . \tag{6.3.29}$$

As in section (6.2), one might hope to iteratively integrate out the auxiliary field χ in (6.3.29) in order to express the result in terms of F^2. The equation of motion for χ resulting from (6.3.29), after solving for F_{01} (and assuming that $\lambda\chi^2 < 2$), is

$$F_{01} = \frac{\chi \left(-|D\phi|^2 \lambda \left(2 - \lambda\chi^2\right) - \sqrt{2|D\phi|^2 \lambda \left(2 - \lambda\chi^2\right) + 1} - 1 \right)}{\left(2 - \lambda\chi^2\right)^2 \sqrt{2|D\phi|^2 \lambda \left(2 - \lambda\chi^2\right) + 1}} . \tag{6.3.30}$$

Solving (6.3.30) by series inversion to give χ as a function of F_{01}, then substituting back into (6.3.29), then determines the full $T\overline{T}$ deformed action. The result, up to order F^8 and using the shorthand $x = F^2, y = |D\phi|^2$, is

$$\begin{aligned}
\mathcal{L}(\lambda) = {} & \frac{\sqrt{1 + 4\lambda y} - 1}{2\lambda} + \frac{2x \left(\sqrt{1 + 4\lambda y} + 2\lambda y \left(\sqrt{1 + 4\lambda y} + 2 \right) + 1 \right)}{\left(2\lambda y + \sqrt{1 + 4\lambda y} + 1 \right)^2} \\
& + \frac{16\lambda x^2}{\left(2\lambda y + \sqrt{1 + 4\lambda y} + 1 \right)^5} \cdot \Big[2\lambda^3 y^3 \left(3\sqrt{1 + 4\lambda y} + 14 \right) + \lambda^2 y^2 \left(17\sqrt{1 + 4\lambda y} + 31 \right) \\
& \qquad + 2\lambda y \left(4\sqrt{1 + 4\lambda y} + 5 \right) + \sqrt{1 + 4\lambda y} + 1 \Big] \\
& + \frac{128\lambda^2 x^3}{\left(2\lambda y + \sqrt{1 + 4\lambda y} + 1 \right)^8} \cdot \Big[4\lambda^5 y^5 \left(13\sqrt{1 + 4\lambda y} + 96 \right) + 2\lambda^4 y^4 \left(183\sqrt{1 + 4\lambda y} + 496 \right) \\
& \qquad + 8\lambda^3 y^3 \left(57\sqrt{1 + 4\lambda y} + 101 \right) + 2\lambda^2 y^2 \left(106\sqrt{1 + 4\lambda y} + 145 \right) \\
& \qquad + 6\lambda y \left(7\sqrt{1 + 4\lambda y} + 8 \right) + 3 \left(\sqrt{1 + 4\lambda y} + 1 \right) \Big]
\end{aligned}$$

$$+\frac{1024\lambda^3 x^4}{\left(2\lambda y+\sqrt{1+4\lambda y}+1\right)^{11}}\cdot\Big[2\lambda^7 y^7\left(323\sqrt{1+4\lambda y}+3266\right)+\lambda^6 y^6\left(8471\sqrt{1+4\lambda y}+30585\right)$$
$$+2\lambda^5 y^5\left(9879\sqrt{1+4\lambda y}+22795\right)+4\lambda^4 y^4\left(4589\sqrt{1+4\lambda y}+8019\right)$$
$$+2\lambda^3 y^3\left(4244\sqrt{1+4\lambda y}+6093\right)+\lambda^2 y^2\left(2083\sqrt{1+4\lambda y}+2577\right)$$
$$+26\lambda y\left(10\sqrt{1+4\lambda y}+11\right)+13\left(\sqrt{1+4\lambda y}+1\right)\Big]. \qquad (6.3.31)$$

We have checked by explicit computation that the series expansion (6.3.31) solves the flow equation (A.7.7) to order x^4.

APPENDIX A
DETAILS OF SUPERCURRENT AND FLOW EQUATION COMPUTATIONS

A.1 Derivation of $(1,1)$ Flow Equation

Here we will show some steps of the calculation which leads to the partial differential equation (3.2.15) defining the supercurrent-squared deformation of a $(1,1)$ free theory with a potential. By setting the superpotential h to zero, this calculation also reproduces the PDE (3.2.3) which describes deformations of the free theory.

We would like to consider what happens when we deform the superspace Lagrangian $\mathcal{A}^{(0)} = D_+\Phi D_-\Phi + h(\Phi)$, according to the flow equation (3.1.7),

$$\frac{\partial}{\partial t}\mathcal{A}^{(t)} = \mathcal{T}^{(t)}_{+++}\mathcal{T}^{(t)}_{---} - \mathcal{T}^{(t)}_{--+}\mathcal{T}^{(t)}_{++-}.$$

It will help to introduce some shorthand: we define $A = D_+\Phi D_-\Phi$ so that $\mathcal{A}^{(0)} = A$, and let $x = t\partial_{++}\Phi\partial_{--}\Phi$ and $y = t\left(D_+D_-\Phi\right)^2$ as before. Also define the dimensionful combinations

$$X = \partial_{++}\Phi\partial_{--}\Phi = \frac{x}{t}, \qquad Y = \left(D_+D_-\Phi\right)^2 = \frac{y}{t}. \tag{A.1.1}$$

Our ansatz for the superspace Lagrangian at finite t will be $\mathcal{A}^{(t)} = F(x,y)A + h(\Phi)$.

With this ansatz, some of the terms in (3.1.5) will not contribute to the right side of (3.1.7). For instance, the terms $\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}D_\pm\partial_\pm\Phi$ will be proportional to $D_+\Phi D_-\Phi = A$. However, every term in the superspace supercurrent is proportional to $D_+\Phi$, $D_-\Phi$, or $D_+\Phi D_-\Phi$. Therefore, when we construct a bilinear in $\mathcal{T}$, any term containing $D_+\Phi D_-\Phi$ will not contribute because it can only appear multiplying another term which contains at least one of $D_\pm\Phi$, which vanishes because $\left(D_\pm\Phi\right)^2 = 0$.

For our special ansatz, we will re-write the components of $\mathcal{T}$ keeping only terms which contribute to bilinears,

$$
\begin{aligned}
\mathcal{T}_{++-} &\sim \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + \partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) - \frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_+\mathcal{A},\\
\mathcal{T}_{+++} &\sim \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right),\\
\mathcal{T}_{---} &\sim \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + \partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) - \frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right),\\
\mathcal{T}_{--+} &\sim \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} + \partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) + \frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) - D_-\mathcal{A}.
\end{aligned}
\tag{A.1.2}
$$

The terms are

$$
\begin{aligned}
D_+\mathcal{A} &\sim FD_+A + h'(\Phi)D_+\Phi,\\
D_-\mathcal{A} &\sim FD_-A + h'(\Phi)D_-\Phi,\\
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim F\partial_{++}\Phi D_-\Phi,\\
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} &\sim -F\partial_{++}\Phi D_+\Phi,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim F\partial_{--}\Phi D_-\Phi,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_-\Phi} &\sim -F\partial_{--}\Phi D_+\Phi,\\
\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim X\frac{\partial F}{\partial X}D_+A,\\
\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) &\sim (\partial_{++}\Phi)^2\frac{\partial F}{\partial X}D_-A,\\
\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim (\partial_{--}\Phi)^2\frac{\partial F}{\partial X}D_+A,\\
\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi}\right) &\sim X\frac{\partial F}{\partial X}D_-A,\\
\frac{1}{2}\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim \sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}\cdot D_+A,\\
\frac{1}{2}\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim \sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}\cdot D_+A,\\
-\frac{1}{2}\partial_{++}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim -\sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y}\cdot D_-A,\\
-\frac{1}{2}\partial_{--}\Phi D_-\left(\frac{\delta\mathcal{A}}{\delta D_+D_-\Phi}\right) &\sim -\sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y}\cdot D_-A,
\end{aligned}
$$

where $\sim$ means "equal modulo terms which are proportional to $D_+\Phi D_-\Phi$," since any products involving these terms will contain two nilpotent factors and thus vanish.

The first piece of supercurrent-squared is

$$\begin{aligned}\mathcal{T}_{++|+}\mathcal{T}_{--|-} &= \left(-F\partial_{++}\Phi D_+\Phi + (\partial_{++}\Phi)^2 \frac{\partial F}{\partial X} D_- A + \sqrt{Y}\partial_{++}\Phi \frac{\partial F}{\partial Y} \cdot D_+ A\right) \\ &\times \left(F\partial_{--}\Phi D_-\Phi + (\partial_{--}\Phi)^2 \frac{\partial F}{\partial X} D_+ A - \sqrt{Y}\partial_{--}\Phi \frac{\partial F}{\partial Y} D_- A\right), \\ &= -F^2 XA - FX\frac{\partial F}{\partial X}\partial_{--}\Phi D_+\Phi D_+ A + FX\frac{\partial F}{\partial X}\partial_{++}\Phi D_- A D_-\Phi \\ &+ X^2\left(\frac{\partial F}{\partial X}\right)^2 D_- A D_+ A + F\frac{\partial F}{\partial Y}\sqrt{Y} X D_+ A D_-\Phi \\ &+ FX\sqrt{Y}\frac{\partial F}{\partial Y} D_+\Phi D_- A - YX\left(\frac{\partial F}{\partial Y}\right)^2 D_+ A D_- A.\end{aligned} \tag{A.1.3}$$

The second piece is

$$\begin{aligned}\mathcal{T}_{++|-}\mathcal{T}_{--|+} &= \left(F\partial_{++}\Phi D_-\Phi + \left(X\frac{\partial F}{\partial X} - F\right) D_+ A - G' D_+\Phi - \sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y} D_- A\right) \\ &\times \left(-F\partial_{--}\Phi D_+\Phi + \left(X\frac{\partial F}{\partial X} - F\right) D_- A - G' D_-\Phi + \sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y} \cdot D_+ A\right), \\ &= F^2 XA + F\left(X\frac{\partial F}{\partial X} - F\right)\partial_{++}\Phi D_-\Phi D_- A + FX\sqrt{Y}\frac{\partial F}{\partial Y} D_-\Phi D_+ A \\ &+ FX\sqrt{Y}\frac{\partial F}{\partial Y} D_- A D_+\Phi - F\left(X\frac{\partial F}{\partial X} - F\right)\partial_{--}\Phi D_+ A D_+\Phi \\ &+ \left(X\frac{\partial F}{\partial X} - F\right)^2 D_+ A D_- A - YX\left(\frac{\partial F}{\partial Y}\right)^2 D_- A D_+ A \\ &- G'\left(X\frac{\partial F}{\partial X} - F\right) D_+\Phi D_- A + \left(G'\right)^2 D_+\Phi D_-\Phi - G'\sqrt{Y}\partial_{--}\Phi\frac{\partial F}{\partial Y} D_+\Phi D_+ A \\ &- G'\left(X\frac{\partial F}{\partial X} - F\right) D_+ A D_-\Phi + G'\sqrt{Y}\partial_{++}\Phi\frac{\partial F}{\partial Y} D_- A D_-\Phi.\end{aligned} \tag{A.1.4}$$

Using the definitions $A = D_+\Phi D_-\Phi$, $X = \partial_{++}\Phi\partial_{--}\Phi$, and $\sqrt{Y} = D_+ D_-\Phi$, we see that the products appearing in the above bilinears can be simplified as follows:

$$
\begin{aligned}
D_+\Phi D_+ A &= D_+\Phi D_+ \left(D_+\Phi D_-\Phi\right) = D_+\Phi D_+ D_+\Phi D_-\Phi = A\partial_{++}\Phi, \\
D_+\Phi D_- A &= D_+\Phi D_- \left(D_+\Phi D_-\Phi\right) = D_+\Phi D_- D_+\Phi D_-\Phi = -A\sqrt{Y}, \\
D_-\Phi D_+ A &= D_-\Phi D_+ \left(D_+\Phi D_-\Phi\right) = -D_-\Phi D_+\Phi D_+ D_-\Phi = A\sqrt{Y}, \\
D_-\Phi D_- A &= D_-\Phi D_- \left(D_+\Phi D_-\Phi\right) = -D_-\Phi D_+\Phi D_- D_-\Phi = A\partial_{--}\Phi, \\
D_+ A D_- A &= \left(\partial_{++}\Phi D_-\Phi - D_+\Phi\sqrt{Y}\right)\left(-\sqrt{Y}D_-\Phi - \partial_{--}\Phi D_+\Phi\right) = (X+Y)A.
\end{aligned}
\tag{A.1.5}
$$

So after simplifying,

$$
\begin{aligned}
\mathcal{T}_{++|+}\mathcal{T}_{--|-} &= -F^2XA - 2FX^2\frac{\partial F}{\partial X}A - X^2\left(\frac{\partial F}{\partial X}\right)^2 A(X+Y) - 2F\frac{\partial F}{\partial Y}YXA, \\
\mathcal{T}_{++|-}\mathcal{T}_{--|+} &= F^2XA + 2FX\left(X\frac{\partial F}{\partial X} - F\right)A + 2FXY\frac{\partial F}{\partial Y}A + \left(X\frac{\partial F}{\partial X} - F\right)^2 (X+Y)A \\
&\quad + \left(\left(h'\right)^2 + 2h'\sqrt{Y}\left(X\frac{\partial F}{\partial X} - F\right) - 2\sqrt{Y}Xh'\frac{\partial F}{\partial Y}\right)A.
\end{aligned}
\tag{A.1.6}
$$

In particular, we see that every term appearing in (A.1.6) is proportional to $A = D_+\Phi D_-\Phi$. This means that the deformation only generates a change in the first term of our ansatz $\mathcal{A}^{(t)} = FD_+\Phi D_-\Phi + h(\Phi)$, but it does not source any change in the potential. This justifies our choice of ansatz which leaves the potential as $h(\Phi)$ rather than allowing a more general function $G(t, \Phi)$ with $G(0, \Phi) = h(\Phi)$.

Adding the contributions gives,

$$
\begin{aligned}
\mathcal{T}_{++|+}\mathcal{T}_{--|-} + \mathcal{T}_{++|-}\mathcal{T}_{--|+} = \Big[&(Y-X)\,F^2 - 2FX(X+Y)\frac{\partial F}{\partial X} + 2h'\sqrt{Y}\left(X\frac{\partial F}{\partial X} - F\right) \\
&- 2\sqrt{Y}Xh'\frac{\partial F}{\partial Y} + \left(h'\right)^2\Big]A.
\end{aligned}
\tag{A.1.7}
$$

Setting this deformation equal to $\frac{\partial}{\partial t}\mathcal{A}^{(t)}$, and multiplying both sides by t to convert dimensionlful variables X and Y into their dimensionless counterparts x and y, gives our final result (3.2.15),

$$x\frac{\partial}{\partial x}F + y\frac{\partial}{\partial y}F = (y-x)\,F^2 - 2Fx(x+y)\frac{\partial F}{\partial x} + \left(h'\right)^2 + 2h'\sqrt{y}\left(x\frac{\partial F}{\partial x} - F\right)$$
$$- 2\sqrt{y}xh'\frac{\partial F}{\partial y}. \tag{A.1.8}$$

We were unable to find a closed-form solution to (3.2.15) in the general case. However, we can find the solution in a few special cases. If $y = 0$, (A.1.8) reduces to

$$xF'(x) = -x\left(F(x)^2 + 2F(x)F'(x)x\right), \tag{A.1.9}$$

which is solved by the Dirac-type ansatz $F(x) = \frac{\sqrt{1+4x}-1}{2x}$. If $x = 0$, equation (A.1.8) is solved by $F(y) = \frac{1}{1-y}$. If $y = -x$, the second term on the right side of (A.1.8) drops out and the solution is $F(x) = \frac{1}{1+2x}$.

A.2 Derivation of $(0,1)$ Flow Equation

Next we would like to write down a partial differential equation, similar to (3.2.3) in the $(1,1)$ case, which determines the Lagrangian deformed by the $(0,1)$ supercurrent-squared operator at finite t.

Define the three combinations of fields

$$x = t\partial_{++}\Phi\partial_{--}\Phi, \qquad y = t\left(D_+\Psi_-\right)^2, \qquad z = tD_+\Phi D_+\partial_{--}\Phi, \tag{A.2.1}$$

and their dimensionful counterparts $X = \frac{x}{t}$, $Y = \frac{y}{t}$, $Z = \frac{z}{t}$. Our ansatz for the Lagrangian at finite t is

$$\mathcal{A}^{(t)} = F(x,y,z)\left(D_+\Phi\partial_{--}\Phi + \Psi_- D_+\Psi_-\right) + F_{2,-}(x,y,z)\left(\Psi_- D_+\Phi\right). \tag{A.2.2}$$

Since the function $F_{2,-}$ is fermionic, it actually contains several different functions since we may combine the fields Φ, Ψ_- and derivatives in a few independent ways to obtain a fermionic function. We will expand $F_{2,-}$ as follows:

$$F_{2,-} = G(x,y,z)D_+\Psi_- D_+\partial_{--}\Phi + H(x,y,z)\partial_{++}\Phi\partial_{--}\Psi_- + J(x,y,z)\partial_{--}\Phi\partial_{++}\Psi_-. \tag{A.2.3}$$

Altogether our ansatz for the deformed Lagrangian is,

$$\begin{aligned}\mathcal{A}^{(t)} &= F\left(D_+\Phi\partial_{--}\Phi + \Psi_- D_+\Psi_-\right) \\ &\quad + \left(G D_+\Psi_- D_+\partial_{--}\Phi + H\partial_{++}\Phi\partial_{--}\Psi_- + J\partial_{--}\Phi\partial_{++}\Psi_-\right)\left(\Psi_- D_+\Phi\right). \end{aligned} \tag{A.2.4}$$

This is a $(0,1)$ superspace Lagrangian with the functional dependence

$$\mathcal{A} = \mathcal{A}\left(\Phi, \Psi_-, D_+\Phi, D_+\Psi_-, \partial_{\pm\pm}\Phi, \partial_{\pm\pm}\Psi_-, D_+\partial_{--}\Phi\right). \tag{A.2.5}$$

Following the procedure of section (3.1.1), we can consider a transformation $x^{\pm\pm} \to x^{\pm\pm} + a^{\pm\pm}$ and extract the components of conserved currents. In this case, they are

$$\begin{aligned}\mathcal{T}_{++++} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi} + \partial_{++}\Psi_-\frac{\delta\mathcal{A}}{\delta\partial_{++}\Psi_-} - \partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}\right), \\ \mathcal{T}_{++--} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi} + \partial_{--}\Psi_-\frac{\delta\mathcal{A}}{\delta\partial_{--}\Psi_-} - \partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}\right) - \mathcal{A}, \\ \mathcal{S}_{++-} &= \partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi} - \mathcal{A}\right) + \partial_{++}\Psi_-\frac{\delta\mathcal{A}}{\delta D_+\Psi_-} \\ &\quad + \left(\partial_{--}\partial_{++}\Phi\right)\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}, \\ \mathcal{S}_{---} &= \partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} + D_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) + \partial_{--}\Psi_-\frac{\delta\mathcal{A}}{\delta D_+\Psi_-} + \partial_{--}^2\Phi\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}. \end{aligned} \tag{A.2.6}$$

We compute each of these contributions. As before, we will drop terms which are proportional to $\Psi_- D_+\Phi$, since every term in $\mathcal{S}$ and $\mathcal{T}$ is proportional to either Ψ_- or to $D_+\Phi$, so any terms involving both of these nilpotent factors will not contribute to bilinears. We will also introduce the shorthand $A = D_+\Phi\partial_{--}\Phi$ and $B = \Psi_- D_+\Psi_-$.

Doing this, we see that:

$$
\begin{aligned}
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi} &\sim FD_+\Phi\partial_{++}\Phi + \frac{\partial F}{\partial x}\left(\partial_{++}\Phi\right)^2(A+B)\,,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{--}\Phi} &\sim FA + x\frac{\partial F}{\partial x}(A+B)\,,\\
\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim \partial_{++}\Phi\frac{\partial F}{\partial z}\left(D_+\partial_{--}\Phi\right)(A+B) + Fx\\
&\quad - \partial_{++}\Phi\left(GD_+\Psi_-D_+\partial_{--}\Phi + H\partial_{++}\Phi\partial_{--}\Psi_- + J\partial_{--}\Phi\partial_{++}\Psi_-\right)\Psi_-,\\
\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta D_+\Phi} &\sim \partial_{--}\Phi\frac{\partial F}{\partial z}\left(D_+\partial_{--}\Phi\right) + F\left(\partial_{--}\Phi\right)^2\\
&\quad - \partial_{--}\Phi\left(GD_+\Psi_-D_+\partial_{--}\Phi + H\partial_{++}\Phi\partial_{--}\Psi_- + J\partial_{--}\Phi\partial_{++}\Psi_-\right)\Psi_-,\\
\partial_{++}\Psi_-\frac{\delta\mathcal{A}}{\delta\partial_{++}\Psi_-} &\sim 0,\\
\partial_{--}\Psi_-\frac{\delta\mathcal{A}}{\delta\partial_{--}\Psi_-} &\sim 0,\\
D_+\left(\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim D_+\left(x\frac{\partial F}{\partial x}(A+B)\right) + \Bigg(x\frac{\partial G}{\partial x}D_+\Psi_-D_+\partial_{--}\Phi + x\frac{\partial H}{\partial x}\partial_{++}\Phi\partial_{--}\Psi_-\\
&\quad + x\frac{\partial J}{\partial x}\partial_{--}\Phi\partial_{++}\Psi_-\Bigg)\cdot\Big(D_+\Psi_-D_+\Phi - \Psi_-\partial_{++}\Phi\Big),\\
D_+\left(\partial_{--}\Phi\frac{\delta\mathcal{A}}{\delta\partial_{++}\Phi}\right) &\sim D_+\left(\left(\partial_{--}\Phi\right)^2\frac{\partial F}{\partial x}(A+B)\right) + \Bigg(\frac{\partial G}{\partial x}\left(\partial_{--}\Phi\right)^2D_+\partial_{--}\Phi\\
&\quad + \frac{\partial H}{\partial x}\partial_{--}\Phi\partial_{--}\Psi_- + \frac{\partial J}{\partial x}\left(\partial_{--}\Phi\right)^3\partial_{++}\Psi_-\Bigg)\Big(D_+\Psi_-D_+\Phi - \Psi_-\partial_{++}\Phi\Big),\\
\partial_{++}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}\right) &\sim \partial_{++}\Phi\frac{\partial F}{\partial z}\left(\partial_{++}\Phi\Psi_-D_+\Psi_- - D_+\Phi\left(D_+\Psi_-\right)^2\right),\\
\partial_{--}\Phi D_+\left(\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi}\right) &\sim \partial_{--}\Phi\frac{\partial F}{\partial z}\left(\partial_{++}\Phi\Psi_-D_+\Psi_- - D_+\Phi\left(D_+\Psi_-\right)^2\right),\\
\partial_{++}\Psi_-\frac{\delta\mathcal{A}}{\delta D_+\Psi_-} &\sim \partial_{++}\Psi_-\left(\frac{\partial F}{\partial y}(A+B) + F\Psi_-\right),\\
\partial_{--}\Psi_-\frac{\delta\mathcal{A}}{\delta D_+\Psi_-} &\sim \partial_{--}\Psi_-\left(\frac{\partial F}{\partial y}(A+B) + F\Psi_-\right),\\
\partial_{--}\partial_{++}\Phi\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi} &\sim \partial_{--}\partial_{++}\Phi\left(\frac{\partial F}{\partial z}D_+\Phi(A+B)\right) \sim 0,\\
\partial_{--}^2\Phi\frac{\delta\mathcal{A}}{\delta D_+\partial_{--}\Phi} &\sim \partial_{--}^2\Phi\left(\frac{\partial F}{\partial z}D_+\Phi(A+B)\right) \sim 0,\\
D_+\mathcal{A} &\sim \left(\frac{\partial F}{\partial x}D_+x + \frac{\partial F}{\partial y}D_+y + \frac{\partial F}{\partial z}D_+z\right)(A+B) + F\left(D_+A + D_+B\right)\\
&\quad + \left(GD_+\Psi_-D_+\partial_{--}\Phi + H\partial_{++}\Phi\partial_{--}\Psi_- + J\partial_{--}\Phi\partial_{++}\Psi_-\right)\\
&\quad \times\left(D_+\Psi_-D_+\Phi - \Psi_-\partial_{++}\Phi\right).
\end{aligned}
\tag{A.2.7}
$$

We will argue that the coupled differential equations for F, G, H, and J resulting from (A.2.7) are consistent. This will be the case if they do not source any additional combinations of fields that do not appear in the ansatz (A.2.4).

The only thing that could spoil consistency is a $D_+ x$ term, since

$$D_+ x = (D_+ \partial_{++} \Phi)\, \partial_{--} \Phi + \partial_{++} \Phi D_+ \partial_{--} \Phi. \tag{A.2.8}$$

We have already allowed for dependence on $D_+ \partial_{--} \Phi$ in our Lagrangian, but terms proportional to $D_+ \partial_{++} \Phi$ are forbidden. We will show that, in the $\mathcal{S} \cdot \mathcal{T}$ deformation resulting from (A.2.7), all $D_+ x$ terms drop out.

Tracking only the $D_+ x$ terms in bilinears, the supercurrent components are

$$\begin{aligned}
\mathcal{S}_{++-} &\sim x D_+ \left(\frac{\partial F}{\partial x} \right) (A + B) + \ldots, \\
\mathcal{T}_{++--} &\sim x \frac{\partial F}{\partial x} (A + B) - FB - \partial_{--} \Phi \frac{\partial F}{\partial z} D_+ \left(D_+ \Phi \Psi_- D_+ \Psi_- \right), \\
\mathcal{S}_{---} &\sim \left(\partial_{--} \phi \right)^2 D_+ \left(\frac{\partial F}{\partial x} \right) (A + B) + \ldots, \\
\mathcal{T}_{++++} &\sim F D_+ \Phi \partial_{++} \Phi + \frac{\partial F}{\partial x} \left(\partial_{++} \Phi \right)^2 (A + B) - \partial_{++} \Phi \frac{\partial F}{\partial z} D_+ \left(D_+ \Phi \Psi_- D_+ \Psi_- \right),
\end{aligned} \tag{A.2.9}$$

where $\ldots$ indicates terms that are not proportional to $D_+ x$ or $D_+ \left(\frac{\partial F}{\partial x} \right)$.

The relevant contributions in the deformation are

$$\begin{aligned}
\mathcal{T}_{++++} \mathcal{S}_{---} - \mathcal{T}_{++--} \mathcal{S}_{++-} &\sim -x D_+ \left(\frac{\partial F}{\partial x} \right) (A + B) \cdot \left(x \frac{\partial F}{\partial x} (A + B) - FB \right) \\
&\quad + \left(\partial_{--} \Phi \right)^2 D_+ \left(\frac{\partial F}{\partial x} \right) (A + B) \cdot \left(F D_+ \Phi \partial_{++} \Phi + \frac{\partial F}{\partial x} \left(\partial_{++} \Phi \right)^2 (A + B) \right) + \ldots, \\
&= \left(\partial_{--} \Phi \right)^2 D_+ \left(\frac{\partial F}{\partial x} \right) \left(FB D_+ \Phi \partial_{++} \Phi \right) - x D_+ \left(\frac{\partial F}{\partial x} \right) (-FAB) + \ldots,
\end{aligned} \tag{A.2.10}$$

where we have used the fermionic nature of A and B so $A^2 = B^2 = (A + B)^2 = 0$. However in the last line, we recognize that $\left(\partial_{--} \Phi \right)^2 D_+ \Phi \partial_{++} \Phi = xA$, since $x = \partial_{++} \Phi \partial_{--} \Phi$ and $A = D_+ \Phi \partial_{--} \Phi$, so

$$\mathcal{T}_{++++} \mathcal{S}_{---} - \mathcal{T}_{++--} \mathcal{S}_{++-} \sim x D_+ \left(\frac{\partial F}{\partial x} \right) FBA + x D_+ \left(\frac{\partial F}{\partial x} \right) FAB = 0, \tag{A.2.11}$$

and thus the problematic $D_+\left(\frac{\partial F}{\partial x}\right)$ terms do not contribute.

A.3 The $\mathcal{S}$-multiplet in Components

In this appendix, we provide the component expansion of the superfields of the $\mathcal{S}$-multiplet introduced in section 4.1.1. The results presented below are equivalent to the results first obtained in [17] up to differences in notation.

The constraints (4.1.1) are solved in terms of component fields by,

$$\begin{aligned}
\mathcal{S}_{\pm\pm} &= j_{\pm\pm} - i\theta^{\pm}S_{\pm\pm\pm} - i\theta^{\mp}\left(S_{\mp\pm\pm} \mp 2\sqrt{2}i\bar{\rho}_{\pm}\right) - i\bar{\theta}^{\pm}\bar{S}_{\pm\pm\pm} \\
&\quad - i\bar{\theta}^{\mp}\left(\bar{S}_{\mp\pm\pm} \pm 2\sqrt{2}i\rho_{\pm}\right) - \theta^{\pm}\bar{\theta}^{\pm}T_{\pm\pm\pm\pm} + \theta^{\mp}\bar{\theta}^{\mp}\left(A \mp \frac{k+k'}{2}\right) \\
&\quad + i\theta^{+}\theta^{-}\bar{Y}_{\pm\pm} + i\bar{\theta}^{+}\bar{\theta}^{-}Y_{\pm\pm} \pm i\theta^{+}\bar{\theta}^{-}\bar{G}_{\pm\pm} \mp i\theta^{-}\bar{\theta}^{+}G_{\pm\pm} \\
&\quad \mp \frac{1}{2}\theta^{+}\theta^{-}\bar{\theta}^{\pm}\partial_{\pm\pm}S_{\mp\pm\pm} \mp \frac{1}{2}\theta^{+}\theta^{-}\bar{\theta}^{\mp}\partial_{\pm\pm}\left(S_{\pm\mp\mp} \pm 2\sqrt{2}i\bar{\rho}_{\mp}\right) \\
&\quad \mp \frac{1}{2}\bar{\theta}^{+}\bar{\theta}^{-}\theta^{\pm}\partial_{\pm\pm}\bar{S}_{\mp\pm\pm} \mp \frac{1}{2}\bar{\theta}^{+}\bar{\theta}^{-}\theta^{\mp}\partial_{\pm\pm}\left(\bar{S}_{\pm\mp\mp} \mp 2\sqrt{2}i\rho_{\mp}\right) \\
&\quad + \frac{1}{4}\theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{\pm\pm}^{2}j_{\mp\mp}\ .
\end{aligned} \tag{A.3.1}$$

Let us introduce the usual useful combinations: $y^{\pm\pm} = x^{\pm\pm} - \frac{i}{2}\theta^{\pm}\bar{\theta}^{\pm}$ and $\tilde{y}^{\pm\pm} = x^{\pm\pm} \mp \frac{i}{2}\theta^{\pm}\bar{\theta}^{\pm}$. The chiral superfields $\chi_{\pm}$ are

$$\chi_{+} = -i\lambda_{+}(y) - i\theta^{+}\bar{G}_{++}(y) + \theta^{-}\left(E(y) + \frac{k}{2}\right) + \bar{\theta}^{-}C^{(-)} + \theta^{+}\theta^{-}\partial_{++}\bar{\lambda}_{-}(y)\ , \tag{A.3.2a}$$

$$\chi_{-} = -i\lambda_{-}(y) - \theta^{+}\left(\bar{E}(y) - \frac{k}{2}\right) + i\theta^{-}G_{--}(y) - \bar{\theta}^{+}C^{(+)} - \theta^{+}\theta^{-}\partial_{--}\bar{\lambda}_{+}(y)\ , \tag{A.3.2b}$$

$$\lambda_{\pm} = \pm\bar{S}_{\mp\pm\pm} + \sqrt{2}i\rho_{\pm}\ , \tag{A.3.2c}$$

$$E = \frac{1}{2}(\Theta - A) + \frac{i}{4}(\partial_{++}j_{--} - \partial_{--}j_{++})\ , \tag{A.3.2d}$$

$$0 = \partial_{++}G_{--} - \partial_{--}G_{++}\ , \tag{A.3.2e}$$

and the twisted-(anti-)chiral superfields $\mathcal{Y}_{\pm}$ are given by

$$\mathcal{Y}_{+} = \sqrt{2}\rho_{+}(\bar{\tilde{y}}) + \theta^{-}\left(F(\bar{\tilde{y}}) + \frac{k'}{2}\right) - i\bar{\theta}^{+}Y_{++}(\bar{\tilde{y}}) - \bar{\theta}^{-}C^{(-)} + \sqrt{2}i\theta^{-}\bar{\theta}^{+}\partial_{++}\rho_{-}(\bar{\tilde{y}})\ , \tag{A.3.3a}$$

$$\mathcal{Y}_{-} = \sqrt{2}\rho_{-}(\tilde{y}) - \theta^{+}\left(F(\tilde{y}) - \frac{k'}{2}\right) + \bar{\theta}^{+}C^{(+)} - i\bar{\theta}^{-}Y_{--}(\tilde{y}) + \sqrt{2}i\theta^{+}\bar{\theta}^{-}\partial_{--}\rho_{+}(\tilde{y})\ , \tag{A.3.3b}$$

$$F = -\frac{1}{2}(\Theta + A) - \frac{i}{4}(\partial_{++}j_{--} + \partial_{--}j_{++})\ , \tag{A.3.3c}$$

$$0 = \partial_{++}Y_{--} - \partial_{--}Y_{++}\ . \tag{A.3.3d}$$

For the FZ-multiplet defined by the constraints (4.1.10), the $\mathcal{S}$-multiplet reduces to a set of $4+4$ real independent component fields described by the $j_{\pm\pm}$ $U(1)_A$ axial conserved R-symmetry current ($\partial_{++}j_{--} - \partial_{--}j_{++} = 0$). In addition, there is a complex scalar field $v(x)$, see eq. (4.2.14), together with the independent supersymmetry current and energy momentum tensor:

$$
\begin{aligned}
S_{\pm\pm\pm}(x) &:= \quad iD_\pm \mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} \ , && \text{(A.3.4a)}\\
\bar{S}_{\pm\pm\pm}(x) &:= -i\overline{D}_\pm \mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} \ , && \text{(A.3.4b)}\\
S_{\mp\pm\pm}(x) &:= -iD_\mp \mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} = \pm i\overline{D}_\pm \overline{\mathcal{V}}(\zeta)|_{\theta=0} \ , && \text{(A.3.4c)}\\
\bar{S}_{\mp\pm\pm}(x) &:= \quad i\overline{D}_\mp \mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} = \mp iD_\pm \mathcal{V}(\zeta)|_{\theta=0} \ , && \text{(A.3.4d)}\\
T_{\pm\pm\pm\pm}(x) &:= \quad \frac{1}{2}[D_\pm, \overline{D}_\pm]\mathcal{J}_{\pm\pm}(\zeta)|_{\theta=0} \ , && \text{(A.3.4e)}\\
\Theta(x) &:= -\frac{1}{2}[D_+, \overline{D}_+]\mathcal{J}_{--}(\zeta)|_{\theta=0} = -\frac{1}{2}[D_-, \overline{D}_-]\mathcal{J}_{++}(\zeta)|_{\theta=0} \\
&= -\frac{1}{2}D_+D_-\mathcal{V}(\zeta)|_{\theta=0} + \frac{1}{2}\overline{D}_+\overline{D}_-\overline{\mathcal{V}}(\zeta)|_{\theta=0} \ . && \text{(A.3.4f)}
\end{aligned}
$$

For the FZ-multiplet, the following relation holds:

$$
\begin{aligned}
\mathcal{J}_{\pm\pm} = \; & j_{\pm\pm} - i\theta^\pm S_{\pm\pm\pm} - i\bar{\theta}^\pm \bar{S}_{\pm\pm\pm} + i\theta^\mp S_{\mp\pm\pm} + i\bar{\theta}^\mp \bar{S}_{\mp\pm\pm} \\
& -\theta^\pm\bar{\theta}^\pm T_{\pm\pm\pm\pm} + \theta^\mp\bar{\theta}^\mp\Theta + i\theta^+\theta^-\partial_{\pm\pm}\bar{v} + i\bar{\theta}^+\bar{\theta}^-\partial_{\pm\pm}v \\
& \mp\frac{1}{2}\theta^+\theta^-\bar{\theta}^\pm\partial_{\pm\pm}S_{\mp\pm\pm} \pm \frac{1}{2}\theta^+\theta^-\bar{\theta}^\mp\partial_{\pm\pm}S_{\pm\mp\mp} \\
& \mp\frac{1}{2}\bar{\theta}^+\bar{\theta}^-\theta^\pm\partial_{\pm\pm}\bar{S}_{\mp\pm\pm} \pm \frac{1}{2}\bar{\theta}^+\bar{\theta}^-\theta^\mp\partial_{\pm\pm}\bar{S}_{\pm\mp\mp} \\
& +\frac{1}{4}\theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\partial^2_{\pm\pm}j_{\mp\mp} \ . \qquad \text{(A.3.5)}
\end{aligned}
$$

Moreover, the chiral superfields $\chi_\pm$ are set to zero and the twisted-(anti-)chiral superfields $\mathcal{Y}_\pm = D_\pm\mathcal{V}$ are given by

$$
\begin{aligned}
\mathcal{Y}_+ &= i\bar{S}_{-++}(\bar{\tilde{y}}) + \theta^- G(\bar{\tilde{y}}) - i\bar{\theta}^+\partial_{++}v(\bar{\tilde{y}}) + \theta^-\bar{\theta}^+\partial_{++}\bar{S}_{+--}(\bar{\tilde{y}}) \ , && \text{(A.3.6a)}\\
\mathcal{Y}_- &= -i\bar{S}_{+--}(\tilde{y}) - \theta^+ G(\tilde{y}) - i\bar{\theta}^-\partial_{--}v(\tilde{y}) + \theta^+\bar{\theta}^-\partial_{++}\bar{S}_{---}(\tilde{y}) \ , && \text{(A.3.6b)}\\
G &= -\Theta - \frac{i}{2}\partial_{++}j_{--} \ . && \text{(A.3.6c)}
\end{aligned}
$$

A.4 Details of the (2, 2) FZ Multiplet Calculation

In this appendix, we compute the fields $\mathcal{J}_{\pm\pm}$ and σ appearing in the FZ-multiplet for Lagrangians of a chiral superfield Φ with the general form

$$\mathcal{L}_0 = \left(\int d^4\theta \, \mathcal{A}(\Phi, D_\pm \Phi, D_+ D_- \Phi, \partial_{\pm\pm}\Phi, \text{c.c.}) \right) + \left(\int d^2\theta \, W(\Phi) \right) + \left(\int d^2\overline{\theta} \, \overline{W}(\overline{\Phi}) \right) \,, \tag{A.4.1}$$

where "c.c." indicates dependence on the conjugates $\overline{\Phi}, \overline{D}_\pm \overline{\Phi}, \overline{D}_+ \overline{D}_- \overline{\Phi}$, and $\partial_{\pm\pm}\overline{\Phi}$. To do this, we will minimally couple the theory to supergravity using the old-minimal supergravity formulation and extract the currents which couple to the metric superfield $H^{\pm\pm}$ and the chiral compensator σ. The minimal coupling prescription involves promoting $\mathcal{L}_0$ to[1]

$$\begin{aligned} \mathcal{L}_0 \longrightarrow \mathcal{L}_{\text{SUGRA}} = & \left(\int d^4\theta \, E^{-1} \, \mathcal{A}(\boldsymbol{\Phi}, \nabla_\pm \boldsymbol{\Phi}, \nabla_+ \nabla_- \boldsymbol{\Phi}, \nabla_{\pm\pm}\boldsymbol{\Phi}, \text{c.c.}) \right) \\ & + \left(\int d^2\theta \, \mathcal{E}^{-1} \, W(\boldsymbol{\Phi}) \right) + \left(\int d^2\overline{\theta} \, \overline{\mathcal{E}}^{-1} \, \overline{W}(\overline{\boldsymbol{\Phi}}) \right) \,. \end{aligned} \tag{A.4.2}$$

Here $\nabla_\pm$ is the derivative which is covariant with respect to the full local supergravity gauge group, E^{-1} is the full superspace measure, $\mathcal{E}^{-1}$ is the chiral measure, and $\boldsymbol{\Phi}$ is the covariantly chiral version of the chiral superfield Φ—that is, $\overline{\nabla}_\pm \boldsymbol{\Phi} = 0$ whereas $\overline{D}_\pm \Phi = 0$.

Expressions for these supercovariant derivatives and measures have been worked out in a series of papers [21–25] from which we will import the results that we need for our analysis. To leading order in H^m, the linearized inverse superdeterminant of the supervielbein is

$$E^{-1} = 1 - \left[\overline{D}_+, D_+ \right] H^{++} - \left[\overline{D}_-, D_- \right] H^{--} \tag{A.4.3}$$

while the chiral measure is given by

$$\mathcal{E}^{-1} = e^{-2\sigma} \left(1 \cdot e^{iH^m \overleftarrow{\partial}_m} \right) = 1 - 2\sigma + i(\partial_m H^m) + \cdots \,, \tag{A.4.4}$$

where the ellipsis are terms of higher-order in H^m and σ. The covariantly chiral superfield $\boldsymbol{\Phi}$ is related to the ordinary chiral superfield Φ by

$$\boldsymbol{\Phi} = e^{iH^m \partial_m} \Phi = \Phi + i \left(H^{++}\partial_{++} + H^{--}\partial_{--} \right) \Phi + \mathcal{O}(H^2) \,. \tag{A.4.5}$$

1. Conforming to notation of [21–25], in this section we will sometimes use the index notations $\alpha = +, -$ and $m = ++, --$.

The spinor supercovariant derivatives $\nabla_\pm$ are

$$\nabla_\alpha = E_\alpha + \Omega_\alpha M + \Gamma_\alpha \overline{M} + \Sigma_\alpha N \ , \tag{A.4.6}$$

where M and N are linear combinations of the Lorentz, $U(1)_V$, and $U(1)_A$ generators which act on spinors as

$$[M, \psi_\pm] = \pm\frac{1}{2}\psi_\pm \ , \qquad [M, \overline{\psi}_\pm] = 0 \ , \tag{A.4.7a}$$
$$[\overline{M}, \overline{\psi}_\pm] = \pm\frac{1}{2}\overline{\psi}_\pm \ , \qquad [\overline{M}, \psi] = 0 \ , \tag{A.4.7b}$$
$$[N, \psi_\pm] = -\frac{i}{2}\psi_\pm \ , \qquad [N, \overline{\psi}_\pm] = +\frac{i}{2}\overline{\psi}_\pm \ . \tag{A.4.7c}$$

The spinor inverse of the supervielbein $E_\alpha = E_\alpha{}^M \partial_M$, and the structure group connections Ω_α, Γ_α, and Σ_α can be expressed to linear order in terms of the metric superfield $H^{\pm\pm}$ and an unconstrained complex scalar compensator S. In the case of old-minimal supergravity, the unconstrained superfield S is related to the chiral compensator σ by

$$S = \sigma - \frac{i}{2}\partial_m H^m - \frac{1}{2}\left[\overline{D}_+, D_+\right] H^{++} - \frac{1}{2}\left[\overline{D}_-, D_-\right] H^{--} \ , \tag{A.4.8}$$

to linear order. In the following analysis we will first obtain expressions for the supercovariant derivatives in terms of $S = S(H^m, \sigma)$, and use (A.4.8) to give them in terms of H^m and σ.

The spinorial inverse of the supervielbein is given at first order in the prepotentials by

$$E_\pm = (1 + \overline{S})D_\pm + i(D_\pm H^m)\partial_m - 2\left(\overline{D}_\mp D_\pm H^{\mp\mp}\right) D_\mp \ , \tag{A.4.9}$$

together with their complex conjugates. Meanwhile, the connections Ω_α, Γ_α, and Σ_α can be written to leading order as

$$\Gamma_\pm = \pm 2D_\pm\left(S + \overline{S}\right) \mp 2D_\mp\overline{D}_\mp D_\pm H^{\mp\mp} \ , \tag{A.4.10a}$$
$$\Sigma_\pm = -2iD_\pm\overline{S} + 2iD_\mp\overline{D}_\mp D_\pm H^{\mp\mp} \ , \tag{A.4.10b}$$
$$\Omega_\pm = \mp 2D_\mp\overline{D}_\mp D_\pm H^{\mp\mp} \ . \tag{A.4.10c}$$

Using (A.4.6), the vielbeins (A.4.9), and the expression (A.4.5) for $\boldsymbol{\Phi}$, we find the supercovariant derivatives

$$\nabla_\pm\boldsymbol{\Phi} = \left(1 + \overline{S}\right) D_\pm\Phi + 2i(D_\pm H^m)\partial_m\Phi + iH^m(D_\pm\partial_m\Phi) - 2\left(\overline{D}_\mp D_\pm H^{\mp\mp}\right) D_\mp\Phi \ , \tag{A.4.11a}$$
$$\overline{\nabla}_\pm\overline{\boldsymbol{\Phi}} = (1 + S)\,\overline{D}_\pm\overline{\Phi} - 2i\left(\overline{D}_\pm H^m\right)\partial_m\overline{\Phi} - iH^m(\overline{D}_\pm\partial_m\overline{\Phi}) - 2\left(\overline{D}_\pm D_\mp H^{\mp\mp}\right)\overline{D}_\mp\overline{\Phi} \tag{A.4.11b}$$

To compute the second supercovariant derivatives acting on $\mathbf{\Phi}$ and $\overline{\mathbf{\Phi}}$, we must include the contributions from Ω_α, Γ_α, Σ_α, and their conjugates. One finds

$$\begin{aligned}
\overline{\nabla}_+\nabla_+\mathbf{\Phi} &= i(1+S+\overline{S})\partial_{++}\Phi - 2(\overline{D}_+\overline{D}_-D_+H^{--})D_-\Phi + 2i(\overline{D}_+D_+H^m)\partial_m\Phi \\
&\quad - H^m\partial_{++}\partial_m\Phi + 2\left(\overline{D}_+(S+\overline{S}) + \overline{D}_-D_-\overline{D}_+H^{--}\right)D_+\Phi \ , && \text{(A.4.12a)} \\
\nabla_+\nabla_-\mathbf{\Phi} &= (1+2\overline{S})D_+D_-\Phi + 2i(D_+D_-H^m)\partial_m\Phi - 2i(D_-H^m)D_+\partial_m\Phi \\
&\quad + 2i(D_+H^m)D_-\partial_m\Phi + iH^mD_+D_-\partial_m\Phi - 2(D_+\overline{D}_+D_-H^{++})D_+\Phi \\
&\quad + 2\left(D_-\overline{D}_-D_+H^{--}\right)D_-\Phi \ , && \text{(A.4.12b)} \\
\overline{\nabla}_-\nabla_-\mathbf{\Phi} &= i(1+S+\overline{S})\partial_{--}\Phi - 2(\overline{D}_-\overline{D}_+D_-H^{++})D_+\Phi + 2i(\overline{D}_-D_-H^m)\partial_m\Phi \\
&\quad - H^m\partial_{--}\partial_m\Phi + 2\left(\overline{D}_-(S+\overline{S}) + \overline{D}_+D_+\overline{D}_-H^{++}\right)D_-\Phi \ , && \text{(A.4.12c)}
\end{aligned}$$

together with their complex conjugates. Armed with these expressions, we can linearize the supergravity couplings in (A.4.2). First let us consider the contribution from the D-term. We would like to extract the terms proportional to $H^{\pm\pm}$ and σ in

$$\begin{aligned}
\mathcal{L} &= \int d^4\theta\, E^{-1}\,\mathcal{A}(\mathbf{\Phi},\nabla_\pm\mathbf{\Phi},\nabla_+\nabla_-\mathbf{\Phi},\nabla_{\pm\pm}\mathbf{\Phi},\text{c.c.}) \ , \\
&\sim \int d^4\theta\Big(H^{\alpha\dot{\alpha}}[D_\alpha,\overline{D}_{\dot{\alpha}}]\mathcal{A} + i\frac{\partial\mathcal{A}}{\partial\Phi}H^m\partial_m\Phi + (\nabla_\alpha\mathbf{\Phi} - D_\alpha\Phi)\,\frac{\partial\mathcal{A}}{\partial\nabla_\alpha\Phi} \\
&\quad + \frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}\,(\nabla_+\nabla_-\mathbf{\Phi} - D_+D_-\Phi) + \frac{\partial\mathcal{A}}{\partial\nabla_m\Phi}\,(\nabla_m\mathbf{\Phi} - \partial_m\Phi) + \text{c.c.}\Big) \ , \qquad \text{(A.4.13)}
\end{aligned}$$

where $\nabla_{\pm\pm} = -i\left\{\nabla_\pm, \overline{\nabla}_\pm\right\}$. Doing so, we see that the currents which couple to $H^{\pm\pm}$ are

$$
\begin{aligned}
\mathcal{J}_{++} = [D_+, \overline{D}_+]\Bigg[& \frac{1}{2}\mathcal{A} - \frac{1}{2}\frac{\partial\mathcal{A}}{\partial\nabla_-\Phi}D_-\Phi - \frac{1}{2}\frac{\partial\mathcal{A}}{\partial\nabla_+\Phi}D_+\Phi + \frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_+D_-\Phi + \frac{\partial\mathcal{A}}{\partial\nabla_{++}\Phi}\partial_{++}\Phi \\
& + 2i\overline{D}_-\left(\frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}D_-\Phi\right) + 2i\overline{D}_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_{++}\Phi}D_+\Phi\right) + \frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}\partial_{--}\Phi\Bigg] \\
+ i\Bigg[& \frac{\partial\mathcal{A}}{\partial\Phi}\partial_{++}\Phi + \frac{1}{2}\partial_{++}\left(\frac{\partial\mathcal{A}}{\partial\nabla_-\Phi}D_-\Phi\right) - \partial_{++}\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_+D_-\Phi\right) - \frac{\partial\mathcal{A}}{\partial\nabla_-\Phi}D_-\partial_{++}\Phi \\
& - 2D_-\left(\frac{\partial\mathcal{A}}{\partial\nabla_-\Phi}\partial_{++}\Phi\right) + \frac{1}{2}\partial_{++}\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\Phi}D_+\Phi\right) - \frac{\partial\mathcal{A}}{\partial\nabla_+\Phi}D_+\partial_{++}\Phi - 2D_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\Phi}\partial_{++}\Phi\right) \\
& - 2D_-D_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}\partial_{++}\Phi\right) + 2D_-\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_+\partial_{++}\Phi\right) \\
& - 2D_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_-\partial_{++}\Phi\right) + \frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_+D_-\partial_{++}\Phi + 2iD_+\overline{D}_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_{++}\Phi}\partial_{++}\Phi\right) \\
& + 2iD_-\overline{D}_-\left(\frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}\partial_{++}\Phi\right) + \frac{\partial\mathcal{A}}{\partial\nabla_{++}\Phi}\partial_{++}^2\Phi + \frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}\partial_{--}\partial_{++}\Phi\Bigg] \\
+ 2\Bigg[& - D_-\overline{D}_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_-\Phi}D_+\Phi\right) - D_-\overline{D}_+D_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_+\nabla_-\Phi}D_+\Phi\right) \\
& + iD_-\overline{D}_+\overline{D}_-\left(\frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}D_+\Phi\right) - i\overline{D}_-D_+\overline{D}_+\left(\frac{\partial\mathcal{A}}{\partial\nabla_{--}\Phi}D_-\Phi\right)\Bigg] \\
& + \text{ c.c. } ,
\end{aligned}
\tag{A.4.14}
$$

and

$$
\begin{aligned}
\mathcal{J}_{--} = [D_-, \overline{D}_-] \Bigg[& \frac{1}{2}\mathcal{A} - \frac{1}{2}\frac{\partial \mathcal{A}}{\partial \nabla_- \Phi} D_-\Phi - \frac{1}{2}\frac{\partial \mathcal{A}}{\partial \nabla_+ \Phi} D_+\Phi + \frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_+ D_- \Phi + \frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi}\partial_{++}\Phi \\
& + 2i\overline{D}_- \left(\frac{\partial \mathcal{A}}{\partial \nabla_{--}\Phi} D_-\Phi \right) + 2i\overline{D}_+ \left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi} D_+\Phi \right) + \frac{\partial \mathcal{A}}{\partial \nabla_{--}\Phi}\partial_{--}\Phi \Bigg] \\
+ i \Bigg[& \frac{\partial \mathcal{A}}{\partial \Phi}\partial_{--}\Phi + \frac{1}{2}\partial_{--}\left(\frac{\partial \mathcal{A}}{\partial \nabla_- \Phi} D_-\Phi \right) - \partial_{--}\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_+ D_- \Phi \right) - \frac{\partial \mathcal{A}}{\partial \nabla_- \Phi} D_- \partial_{--}\Phi \\
& - 2D_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_- \Phi}\partial_{--}\Phi \right) + \frac{1}{2}\partial_{--}\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \Phi} D_+\Phi \right) - \frac{\partial \mathcal{A}}{\partial \nabla_+ \Phi} D_+ \partial_{--}\Phi - 2D_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \Phi}\partial_{--}\Phi \right) \\
& - 2D_- D_+ \left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi}\partial_{--}\Phi \right) + 2D_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_+ \partial_{--}\Phi \right) \\
& - 2D_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_- \partial_{--}\Phi \right) + \frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_+ D_- \partial_{--}\Phi + 2iD_+\overline{D}_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi}\partial_{--}\Phi \right) \\
& + 2iD_-\overline{D}_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_{--}\Phi}\partial_{--}\Phi \right) + \frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi}\partial_{++}\partial_{--}\Phi + \frac{\partial \mathcal{A}}{\partial \nabla_{--}\Phi}\partial^2_{--}\Phi \Bigg] \\
+ 2 \Bigg[& - D_+\overline{D}_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \Phi} D_-\Phi \right) + D_+\overline{D}_- D_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_-\Phi \right) \\
& + iD_+\overline{D}_-\overline{D}_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi} D_-\Phi \right) - i\overline{D}_+ D_- \overline{D}_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi} D_+\Phi \right) \Bigg] \\
& + \text{ c.c. } \quad ,
\end{aligned}
\tag{A.4.15}
$$

where +c.c. means to add the complex conjugates of <u>all</u> preceding terms (including the real quantity $\frac{1}{2}[D_\pm, \overline{D}_\pm]\mathcal{A}$ for which the complex conjugate merely removes the factor of $\frac{1}{2}$).

The field $\mathcal{V}$ which appears in our deformation (4.2.10) receives two contributions, one from the D-term coupling which depends only on $\mathcal{A}$, and one from the F-term coupling which depends only on the superpotential W. Adding them, we find

$$
\begin{aligned}
\mathcal{V} \quad = \quad \overline{D}_+\overline{D}_- \Bigg[& - \frac{\partial \mathcal{A}}{\partial \nabla_\alpha \Phi} D_\alpha \Phi + 2\frac{\partial \mathcal{A}}{\partial \nabla_+ \nabla_- \Phi} D_+ D_- \Phi + \frac{\partial \mathcal{A}}{\partial \nabla_m \Phi}\partial_m \Phi + \frac{\partial \mathcal{A}}{\partial \nabla_m \overline{\Phi}}\partial_m \overline{\Phi} \\
& + 2i\overline{D}_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\Phi} D_+\Phi \right) + 2iD_+\left(\frac{\partial \mathcal{A}}{\partial \nabla_{++}\overline{\Phi}} \overline{D}_+\overline{\Phi} \right) \\
& + 2i\overline{D}_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_{--}\Phi} D_-\Phi \right) + 2iD_-\left(\frac{\partial \mathcal{A}}{\partial \nabla_{--}\overline{\Phi}} \overline{D}_-\overline{\Phi} \right) \Bigg] \\
& + 2W(\Phi) \ .
\end{aligned}
\tag{A.4.16}
$$

A.5 On-Shell Simplification of Chiral Scalar Theories

In this appendix, we prove the following claim which applies to $(2,2)$ theories of a chiral superfield of the form considered in Chapter 4: one can drop all terms which involve products of $(D_+D_-\Phi)$ or $(\overline{D}_+\overline{D}_-\overline{\Phi})$ and the four-fermion term $|D\Phi|^4 = D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}$ when the equations of motion are satisfied.

To see this for the models we consider, it suffices to consider a superspace Lagrangian of the form

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\, \mathcal{A}\left(\Phi, D_\pm\Phi, D_+D_-\Phi, \partial_{\pm\pm}\Phi, \text{c.c.}\right)\\ &= \int d^4\theta\, \left(K(\Phi,\overline{\Phi}) + f(x,\overline{x},y)|D\Phi|^4\right) \ , \end{aligned} \tag{A.5.1}$$

which has the superspace equation of motion

$$\begin{aligned}\overline{D}_+\overline{D}_-K_\Phi = \overline{D}_+\overline{D}_-\Bigg\{ D_\alpha\left[\frac{\partial(f|D\Phi|^4)}{\partial D_\alpha\Phi}\right] - D_+D_-\left[\frac{\partial(f|D\Phi|^4)}{\partial D_+D_-\Phi}\right]\\ - \partial_m\left[\frac{\partial(f|D\Phi|^4)}{\partial(\partial_m\Phi)}\right]\Bigg\}\end{aligned} \tag{A.5.2}$$

for Φ, and the conjugate equation of motion for $\overline{\Phi}$. If we multiply (A.5.2) on both sides by the four-fermion term $|D\Phi|^4 = D_+\Phi\overline{D}_+\overline{\Phi}D_-\Phi\overline{D}_-\overline{\Phi}$ then any term containing $(D_\pm\Phi)$ and $(\overline{D}_\pm\overline{\Phi})$ fermions in (A.5.2) will vanish by nilpotency. On the left, the only surviving term is $K_{\Phi\overline{\Phi}}\overline{D}_+\overline{D}_-\overline{\Phi}$, while on the right we get contributions from the first and second terms:

$$\begin{aligned}K_{\Phi\overline{\Phi}}\left(\overline{D}_+\overline{D}_-\overline{\Phi}\right)|D\Phi|^4 = \left(\overline{D}_+\overline{D}_-\overline{\Phi}\right)|D\Phi|^4\Bigg\{\lambda\overline{D}_+\overline{D}_-\left[\frac{\partial f}{\partial y}(\partial_{--}\overline{\Phi})(\partial_{++}\overline{\Phi})\right]\\ - \left(\frac{x+\overline{x}}{\lambda}\right)f\Bigg\} \ .\end{aligned} \tag{A.5.3}$$

On collecting terms, the previous equation turns into

$$\left(\overline{D}_+\overline{D}_-\overline{\Phi}\right)|D\Phi|^4\left\{K_{\Phi\overline{\Phi}} + \left(\frac{x+\overline{x}}{\lambda}\right)f - \lambda\overline{D}_+\overline{D}_-\left[\frac{\partial f}{\partial y}(\partial_{--}\overline{\Phi})(\partial_{++}\overline{\Phi})\right]\right\} = 0 \ . \tag{A.5.4}$$

The parenthesis multiplying $(\overline{D}_+\overline{D}_-\overline{\Phi})|D\Phi|^4$ in the previous expression does not vanish in general, at least for λ small enough. Then for (A.5.4) to be satisfied, the equation

$$\left(\overline{D}_+\overline{D}_-\overline{\Phi}\right)|D\Phi|^4 = 0 \tag{A.5.5}$$

has to hold when the equations of motion are satisfied. This justifies our claim in section 4.3.1 that we may drop all terms involving the product $y|D\Phi|^4$ in the deformation, assuming we restrict to on-shell configurations.

A.6 On-Shell Simplification of Born-Infeld-Type Theories

This appendix is devoted to deriving the on-shell relation (5.4.37). We are going to prove this holds for an action of the form (5.4.26). Let us start by considering the following Lagrangian

$$\mathcal{L} \;=\; \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar{\theta}\,\overline{W}^2 + \int d^2\theta d^2\bar{\theta}\, W^2\overline{W}^2\,\Omega\left[D^2W^2, \overline{D}^2\overline{W}^2\right] . \tag{A.6.1}$$

We recall that W_α and $\overline{W}_{\dot{\alpha}}$ satisfy the Bianchi identity $D^\alpha W_\alpha = \overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}$, whose solution is given in terms of a real but otherwise unconstrained scalar prepotential superfield V: $W_\alpha = -1/4\,\overline{D}^2 D_\alpha V$ and $\overline{W}_{\dot{\alpha}} = -1/4\, D^2\overline{D}_{\dot{\alpha}} V$. It is a straightforward calculation to derive the EOM by varying the action (A.6.1) with respect to the prepotential V. The EOM reads

$$\begin{aligned} 0 \;=\; & -D^\alpha W_\alpha + \frac{1}{2}D^\alpha\overline{D}^2\left(W_\alpha\overline{W}^2\Omega\right) + \frac{1}{2}\overline{D}_{\dot{\alpha}}D^2\left(W^2\overline{W}^{\dot{\alpha}}\Omega\right) \\ & + \frac{1}{2}D^\alpha\left[W_\alpha\overline{D}^2D^2\left(W^2\overline{W}^2\frac{\partial\Omega}{\partial(D^2W^2)}\right)\right] + \frac{1}{2}\overline{D}_{\dot{\alpha}}\left[\overline{W}^{\dot{\alpha}}\left(D^2\overline{D}^2W^2\overline{W}^2\frac{\partial\Omega}{\partial(\overline{D}^2\overline{W}^2)}\right)\right] \end{aligned} \tag{A.6.2}$$

Because of the constraint that $W_\alpha W_\beta W_\gamma = 0$ and its complex conjugate, multiplying eq. (A.6.2) by $W^2\overline{W}^2$ and using the EOM gives the following condition

$$W^2\overline{W}^2(D^\alpha W_\alpha)\Big(1 + f(\Omega)\Big) = 0 , \tag{A.6.3}$$

where the functional $f(\Omega)$ is given by

$$\begin{aligned} f(\Omega) \;:=\; & -\frac{1}{2}(\overline{D}^2\overline{W}^2 + D^2W^2)\Omega \\ & -\frac{1}{2}\left[(D^2W^2)(\overline{D}^2\overline{W}^2)\frac{\partial\Omega}{\partial(D^2W^2)} + (D^2W^2)(\overline{D}^2\overline{W}^2)\frac{\partial\Omega}{\partial(\overline{D}^2\overline{W}^2)}\right] . \end{aligned} \tag{A.6.4}$$

This implies

$$W^2\overline{W}^2(D^\alpha W_\alpha) = 0 \ , \tag{A.6.5}$$

which is precisely condition (5.4.37).

A.7 Derivation of General Flow Equation for Gauge Field and Scalars

In this appendix, we will obtain the flow equation of a sufficiently general Lagrangian for all cases involving gauge fields and scalars that are of interest in Chapter 6.

Consider a general λ-dependent Lagrangian for a complex scalar ϕ and field strength F:

$$\mathcal{L} = f(\lambda, F^2, |D\phi|^2), \tag{A.7.1}$$

For convenience, we will also define $x = F^2$ and $y = |D\phi|^2$. As in the main body of Chapter 6, D is the gauge-covariant derivative and the field strength F need not be abelian; we use the shorthand

$$F^2 = F^a_{\mu\nu} F^{\mu\nu}_a = \text{Tr}\left(F^2\right) , \tag{A.7.2}$$

and we will suppress gauge group indices in what follows.

We can now compute the stress-energy tensor by coupling to a background metric and varying with respect to the metric, which gives

$$\begin{aligned} T^{(\lambda)}_{\mu\nu} &= \eta_{\mu\nu}\, f - 4\frac{\partial f}{\partial x} F_\mu{}^\sigma F_{\sigma\nu} - 2\frac{\partial f}{\partial y} D_\mu\phi D_\nu\bar{\phi} \\ &= \eta_{\mu\nu}\, f + 2\frac{\partial f}{\partial x}\eta_{\mu\nu} F^2 - 2\frac{\partial f}{\partial y} D_\mu\phi D_\nu\bar{\phi} \ , \end{aligned} \tag{A.7.3}$$

where we have used that $F_\mu{}^\sigma F_{\sigma\nu} = -\frac{1}{2}\eta_{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right)$ in two dimensions.

The determinant of T is then expressed in terms of the combinations

$$\begin{aligned} T^{\mu\nu}T_{\mu\nu} &= \left(\eta^{\mu\nu}\, f + 2\eta^{\mu\nu}F^2\frac{\partial f}{\partial x} - 2D^\mu\phi D^\nu\bar{\phi}\frac{\partial f}{\partial y}\right)\left(\eta_{\mu\nu}\, f + 2\eta_{\mu\nu}F^2\frac{\partial f}{\partial x} - 2D_\mu\phi D_\nu\bar{\phi}\frac{\partial f}{\partial y}\right) \\ &= 2f^2 - 8F^2 f\frac{\partial f}{\partial x} - 4|D\phi|^2 f\frac{\partial f}{\partial y} + 8F^4\left(\frac{\partial f}{\partial x}\right)^2 + 8F^2|D\phi|^2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 4|D\phi|^4\left(\frac{\partial f}{\partial y}\right)^2 \\ &= 2f^2 - 8xf\frac{\partial f}{\partial x} - 4yf\frac{\partial f}{\partial y} + 8x^2\left(\frac{\partial f}{\partial x}\right)^2 + 8xy\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 4y^2\left(\frac{\partial f}{\partial y}\right)^2 \ , \end{aligned} \tag{A.7.4}$$

and

$$
\begin{aligned}
\left(T^{\mu}{}_{\mu}\right)^2 &= \left(2f - 4F^2\frac{\partial f}{\partial x} - 2|D\phi|^2\frac{\partial f}{\partial y}\right)^2 \\
&= 4f^2 - 16F^2 f\frac{\partial f}{\partial x} - 8|D\phi|^2 f\frac{\partial f}{\partial y} + 16F^4\left(\frac{\partial f}{\partial x}\right)^2 + 4|D\phi|^4\left(\frac{\partial f}{\partial y}\right)^2 + 16F^2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}|D\phi|^2 \\
&= 4f^2 - 16xf\frac{\partial f}{\partial x} - 8yf\frac{\partial f}{\partial y} + 16x^2\left(\frac{\partial f}{\partial x}\right)^2 + 16xy\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 4y^2\left(\frac{\partial f}{\partial y}\right)^2 . \qquad \text{(A.7.5)}
\end{aligned}
$$

Using these, we can write the $T\overline{T}$ operator as

$$
\begin{aligned}
\det(T) &= \frac{1}{2}\left(\left(T^{\mu}{}_{\mu}\right)^2 - T^{\mu\nu}T_{\mu\nu}\right) \\
&= f^2 - 4fx\frac{\partial f}{\partial x} - 2fy\frac{\partial f}{\partial y} + 4x^2\left(\frac{\partial f}{\partial x}\right)^2 + 4xy\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} , \qquad \text{(A.7.6)}
\end{aligned}
$$

and hence the $T\overline{T}$-flow equation as

$$
\frac{df}{d\lambda} = f^2 - 4fx\frac{\partial f}{\partial x} - 2fy\frac{\partial f}{\partial y} + 4x^2\left(\frac{\partial f}{\partial x}\right)^2 + 4xy\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} . \qquad \text{(A.7.7)}
$$

This is the main differential equation of interest which we will study in Chapter 6. Although we have used $\eta_{\mu\nu}$ for the metric, these results are valid either in Minkowski signature or in Euclidean signature – replacing $\eta_{\mu\nu}$ with $\delta_{\mu\nu}$ in the intermediate steps of these calculations does not affect our final result (A.7.7).

In the case where we turn off the field strength, setting $x = 0$, this differential equation becomes

$$
\frac{df}{d\lambda} = f^2 - 2fy\frac{\partial f}{\partial y} . \qquad \text{(A.7.8)}
$$

Imposing the boundary condition that $f(\lambda = 0) = |D\phi|^2$, we find

$$
f(\lambda, |D\phi|^2) = \frac{1}{2\lambda}\left(\sqrt{1 + 4\lambda|D\phi|^2} - 1\right) . \qquad \text{(A.7.9)}
$$

On the other hand, in the case where we turn off the scalars (setting $|D\phi|^2 = 0$), the differential equation (A.7.7) becomes

$$
\frac{df}{d\lambda} = f^2 - 4fx\frac{\partial f}{\partial x} + 4x^2\left(\frac{\partial f}{\partial x}\right)^2 . \qquad \text{(A.7.10)}
$$

which has the solution

$$f(\lambda, F^2) = F^2 \cdot {}_4F_3\left(\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}, 2; \frac{256}{27} \cdot \lambda F^2\right) \ . \tag{A.7.11}$$

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