THE UNIVERSITY OF CHICAGO

FROM POINT-PICKING TO SECTIONS OF SURFACE BUNDLES

A DISSERTATION SUBMITTED TO

THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES

IN CANDIDACY FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

LEI CHEN

CHICAGO, ILLINOIS

JUNE 2018

Dedicated to my parents and Benson Farb

“I am a wild animal, not an animal trainer.
I am going to chase you deep into the mountains and forests
and you are going to have to do your damnedest to run ahead of me.
The faster you run, the faster I will chase you.
If you run slowly, I will chase you slowly.
Whatever happens, you must run,
you must never stop, whatever difficulties you face.
The day that you stop running, our relationship is over
The day that you run deep into the woods and disappear from sight,
our relationship is also over.
In the first instance, I have given up on you;
in the second, you have set yourself free.”

–By Mai Jia, told by Benson Farb

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

First of all, I would like to thank my advisor Benson Farb, for his time, his wisdom and his support. I am very grateful for all math he shared with me that opened my mind and enriched my world. More importantly, I am super grateful for his encouragement and empathy for me when I doubted myself deeply.

I also want to thank Nick Salter, who helped me a lot during graduate school. I would like to thank the following mathematicians for sharing their thoughts with: Paul Apisa, Joan Birman, Maxime Bergeron, Tara Brendle, Kevin Casto, Danny Calegari, Lvzhou Chen, Nir Gadish, Kathryn Mann, Dan Margalit, Vlad Markovic, Howard Masur, Peter May, Andy Putman, Bena Tshishiku and Amie Wilkinson.

Lastly, I need to thank my parents, for their patience and love. Thanks my mother for listening to my crying many times through the 5 years. And thanks my dad for sharing his experience with me in struggling with career crisis.

ABSTRACT

Given any n points on a manifold, how can we systematically and continuously find a new point? What if we ask them to be distinct? In this thesis, I will try to answer these question on surfaces. Then I will connect these questions to sections of some universal surface bundles. In the end, I will classify automorphisms of n-strand surface braid group. The slogan is "there is no center of mass on closed hyperbolic surfaces".

CHAPTER 1

INTRODUCTION

1.1 Point-picking

Let S be a surface and let $\mathrm{PConf}_n(S)$, *the pure configuration space*, be the space of ordered n-tuple of distinct points on S. Let $f_n(S) : \mathrm{PConf}_{n+1}(S) \to \mathrm{PConf}_n(S)$ be the map given by $f_n(x_0, x_1, \ldots, x_n) := (x_1, \ldots, x_n)$; the fiber bundle $f_n(S)$ is called the Fadell-Neuwirth fibration. Given n distinct points on S, how can we continuously associate a new point on S that is distinct from the other n points? The sections of $f_n(S)$ gives the answer in the situation of n ordered points. Let S_g be a surface of genus g and S^2 a 2-sphere. In this paper, we classify the sections of the fiber bundle $f_n(S)$ for 3 cases: $\mathbb{R}^2$, S^2 and S_g when $g > 1$. Here by *section* we mean continuous section.

We call a section s of $f_n(\mathbb{R}^2)$ (resp. $f_n(S^2)$) *"adding a point near x_k"* if s is homotopic to an element in the collection of sections $\mathrm{Add}_{n,k}(\mathbb{R}^2)$ (resp. $\mathrm{Add}_{n,k}(S^2)$). Informally, we assign x_0 at a sufficiently small distance to x_k along some nonvanishing vector field. See Figure 1.1.1 for a demonstration of "adding a point near x_k". Notice that there are infinitely many homotopy classes of sections in $\mathrm{Add}_{n,k}(\mathbb{R}^2)$ and $\mathrm{Add}_{n,k}(S^2)$ and they are classified by a kind of twists or sections of a circle bundle. See Section 2.1 and Section 3.2 for formal definitions of $\mathrm{Add}_{n,k}(\mathbb{R}^2)$ and $\mathrm{Add}_{n,k}(S^2)$ respectively.

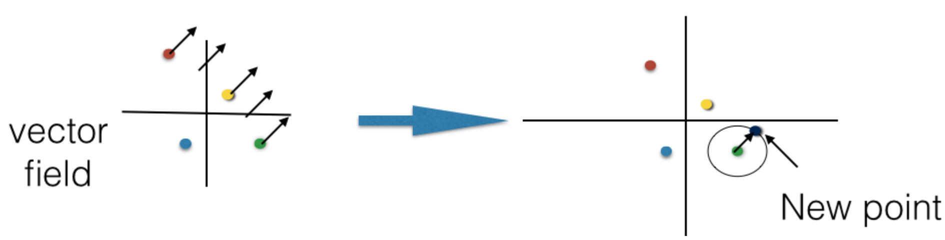


Figure 1.1.1: "adding a point near x_k"

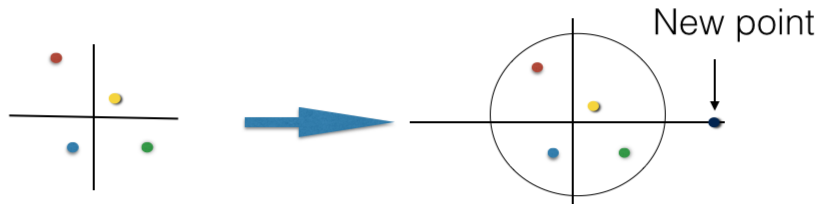


Figure 1.1.2: "adding a point at infinity"

We call a section s of $f_n(\mathbb{R}^2)$ *"adding a point at infinity"* if s is homotopic to an element in the collection of sections $\mathrm{Add}_{n,\infty}(\mathbb{R}^2)$; see Figure 1.1.2. Informally, we consider $\mathbb{R}^2$ as S^2 missing a point ∞, we can assign x_0 at a sufficiently small distance to ∞ along some nonvanishing vector field. See Section 2.1 for a formal definition of $\mathrm{Add}_{n,\infty}(\mathbb{R}^2)$. Whether we can define "adding a point near x_k" or "adding a point near x_k" depends in a delicate way on properties of S.

Theorem 1.1 (Classification of sections for ordered configurations). *The following holds:*

(1) If $S = \mathbb{R}^2$ and $n > 3$, any section of $f_n(S)$ is either "adding a point at infinity" or "adding a point near x_k" for some $1 \leq k \leq n$.

(2) If $S = S^2$ and $n = 2$, the bundle $f_n(S)$ does not have a section. If $S = S^2$ and $n > 4$, any section of $f_n(S)$ is "adding a point near x_k" for some $1 \leq k \leq n$.

(3) If $S = S_g$ a surface of genus $g > 1$ and for $n > 1$, the bundle $f_n(S)$ does not have a section.

There is a natural action of permutation group Σ_n on $\mathrm{PConf}_n(S)$ by permuting the n points. Thus the quotient space is the space of unordered n-tuples of distinct points in S. Permutation group Σ_n acts on the fiber bundle $f_n(S)$ as well. Let $F_n(S) : \mathrm{PConf}_{n+1}(S)/\Sigma_n \to \mathrm{PConf}_n(S)/\Sigma_n$ be the map given by $F_n(x_0, \{x_1, \ldots, x_n\}) := \{x_1, \ldots, x_n\}$.

Corollary 1.2 (Classification of sections for unordered configurations). *The following holds:*

(1) If $S = \mathbb{R}^2$ and $n > 3$, any section of $F_n(S)$ is "adding a point at infinity";

(2) If $S = S^2$ and $n > 4$ or $n = 2, 3$, the bundle $F_n(S)$ does not have a section.

We will discuss the exceptional cases when $n = 3$ for $S = S^2$ in Section 3.5. Our method does not work for the case $n = 4$ but in [19, Theorem 2], they proved that $F_4(S^2)$ does not have sections. We do not do the $g = 1$ case because our methods do not apply to it. But the construction "adding a point near x_k" works for the torus as well; see Section 2.1.

It is classical that $f_n(\mathbb{R}^2)$ admits a section. In [14, Theorem 3.1], Fadell showed that when $n > 2$, the bundle $f_n(S^2)$ admits a section. The unordered case for $S = \mathbb{R}^2$, i.e. (1) of Corollary 1.2 has been proved by [4, Main Theorem 2] and [7, Theorem 4]. In [19, Theorem 2], they prove the case (2) of Corollary 1.2, and even stronger, they deal with the multi-section problems. All the previous proofs make use of the braid relation and the presentations of braid groups and do not imply (1) and (2) in Theorem 1.1. Our main tool is the canonical reduction system for a mapping class, which in turn uses the Thurston classification of isotopy classes of diffeomorphisms of surfaces. This idea originated from [5].

The ordered case for $S = S_g$ of $g > 1$, i.e. (3) of Theorem 1.1 has been proved by [18, Theorem 2]. Their proof makes heavily use of the presentations of surface braid group. We give a simple proof using the cohomology of surface braid group and a classification theorem in [8, Theorem 5].

1.2 Sections of n-pointed surface bundles

Let $\mathrm{Diff}(S_{g,n})$ be the orientation-preserving diffeomorphism group of a surface S_g of genus $g > 1$ fixing n distinct points $\{x_1, x_2, ..., x_n\} \subset S_g$ pointwise. There is a fiber bundle

$$S_g \to \mathrm{UDiff}(S_{g,n}) \xrightarrow{u_{g,n}} \mathrm{BDiff}(S_{g,n}), \tag{1.2.1}$$

which is universal in the sense that any S_g-bundle endowed with n disjoint sections is a pullback of this bundle. Since $\mathrm{Diff}(S_{g,n})$ fixes the n points $x_1, x_2, ..., x_n$, we associate n points on each fiber, i.e. n disjoint sections of (1.2.1) which are denoted by $s_1, s_2, ..., s_n$. A natural question is: are there more sections?

R. Hain conjectured that every section of (1.2.1) is homotopic to one of these n sections. This is the main theorem of this paper.

Theorem 1.3 (The classification of sections for ordered case). *For $n \geq 0$ and $g > 2$, every section of the universal bundle (1.2.1) is homotopic to s_i for some $i \in \{1, 2, ..., n\}$. For $g = 2$, there are precisely $2n$ homotopy classes of sections of the universal bundle (1.2.1).*

Since each section s_i has nontrivial self-intersection, we have the following corollary.

Corollary 1.4. *The universal bundle (1.2.1) does not admit $n+1$ disjoint sections.*

What if we only fix the n points as a set? More precisely, let $\mathrm{Diff}(S_{g,\overline{n}})$ denote the orientation-preserving diffeomorphism group of a surface S_g of genus $g > 1$ fixing n points $\{x_1, x_2, ..., x_n\} \subset S_g$ as a set. There is a fiber bundle

$$S_g \to \mathrm{UDiff}(S_{g,\overline{n}}) \xrightarrow{u'_{g,n}} \mathrm{BDiff}(S_{g,\overline{n}}). \tag{1.2.2}$$

We also have the following result.

Theorem 1.5 (No sections for unordered case). *For $n > 1$ and $g > 1$, surface bundle (1.2.2) has no sections.*

We see below that Hain's conjecture can be interpreted both in terms of mapping class groups and also in terms of moduli spaces. Let $\mathcal{M}_{g,m,n}$ be the moduli space of smooth Riemann surfaces of genus g with $m + n$ distinct points, m labelled and n unlabelled. Earle-Kra [11, Theorem 2.2] proved that the only holomorphic section of the forgetful map

$f : \mathcal{M}_{g,m,n} \to \mathcal{M}_{g,m,0}$ occurs when $g = 2$ and $n = 6$. This section is constructed by marking all six Weierstrass points.

Corollary 1.4 and Theorem 1.5 give a topological proof of the fact that there is no continuous section of $\mathcal{M}_{g,m+1,0} \to \mathcal{M}_{g,m,0}$ for $m \geq 0$ and there is no continuous section of $\mathcal{M}_{g,1,n} \to \mathcal{M}_{g,0,n}$ for $n > 1$. Recently, we found out that Theorem 1.8 can be deduced from [2, Theorem 1.1]. Their proof substantially uses the tool of canonical reduction system. We provide a more elementary proof of this result.

When we talk about fundamental group in this paper, we omit the base point and that brings no ambiguity.

1.2.1 *The strategy of proof*

Let $\mathrm{PConf}_n(S_g)$ be the space of ordered n-tuple of distinct points on S_g and let $PB_n(S_g) = \pi_1(\mathrm{PConf}_n(S_g))$. Let $\mathrm{Mod}_{g,n}$ (resp. $\mathrm{PMod}_{g,n}$) be the *mapping class group* of $S_{g,n}$, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing n punctures as a set (resp. pointwise). We omit n when $n = 0$.

We first translate the problem into a group-theoretical problem of determining a homomorphism p satisfying the following diagram, where the horizontal exact sequences are the *Birman exact sequences.*

$$\begin{array}{ccccccccc} 1 \to & PB_n(S_g) & \longrightarrow & \mathrm{PMod}_{g,n} & \xrightarrow{\pi_{g,n}} & \mathrm{Mod}_g & \longrightarrow & 1 \\ & \downarrow R & & \downarrow p & & \downarrow = & & \\ 1 \to & \pi_1(S_g) & \longrightarrow & \mathrm{Mod}_{g,1} & \xrightarrow{\pi_1} & \mathrm{Mod}_g & \longrightarrow & 1. \end{array} \qquad (1.2.3)$$

The analysis of p is decomposed into two parts: first classifying R and then trying to extend R to p. In the second part, we use the commutativity of diagram (1.2.3) and the action of Mod_g on $\pi_1(S_g)$. In classifying R, we have the following key ingredient.

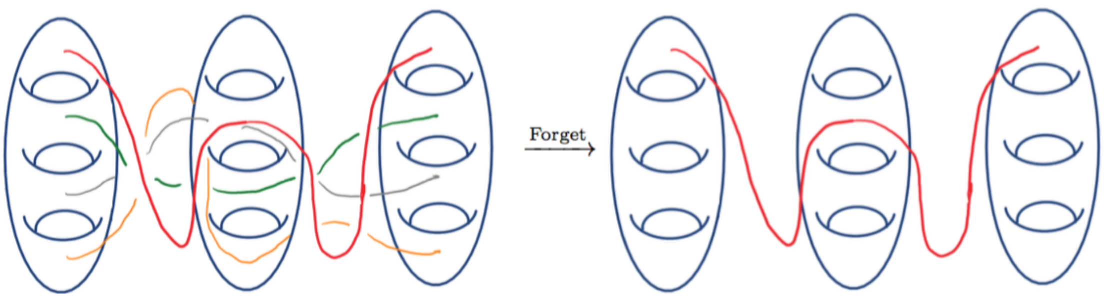
1.2.2 The key ingredient

The key ingredient is the following question.

Question 1.6. *How many homotopy classes of maps are there from $PConf_n(S_g)$ to S_g?*

Let $p_i : \mathrm{PConf}_n(S_g) \to S_g$ be the projection onto the ith component. Let $p_{i*} : PB_n(S_g) \to \pi_1(S_g)$ be the map on the fundamental groups of p_i. Since p_i does not fix a basepoint, the map p_{i*} is only defined up to conjugacy. Do we have more maps?

Remark. The following figure is a cartoon version of what the following theorem talks about.



We answer Question 1.6 by the following classification theorem.

Theorem 1.7 (The classification of homomorphisms $PB_n(S_g) \to \pi_1(S_g)$). *Let $g > 1$ and $n > 0$. Let $R : PB_n(S_g) \to \pi_1(S_g)$ be a homomorphism. The followings hold:*

1)If R is surjective, then $R = A \circ p_{i*}$ for some i and A an automorphism of $\pi_1(S_g)$.

2)If Image(R) is not a cyclic group, the homomorphism $PB_n(S_g) \to \pi_1(S_g)$ factors through p_{i*} for some i.

In our next paper [9], we classify the surjective homomorphisms between $PB_n(S_g)$ and $PB_m(S_g)$ for any n and m. We also give a new proof of the result in [13, Theorem 1] about the automorphism group of $PB_n(S_g)$.

1.2.3 Other geometric applications

It is a basic question to understand the classification of sections of a surface bundle. Theorem 1.7 has many geometric applications regarding the section problems. In this paper, we also deal with the case of the universal hyperelliptic surface bundle. This result is recently proved in [36, Theorem 1] as well. The genus 2 case in Theorem 1.8 is also part of the hyperelliptic case.

1.3 Autormophism of surface braid groups

Given a surface S and a positive number n, we denote the *pure configuration space* of S by

$$\mathrm{PConf}_n(S) = \{(x_1, ..., x_n) \in S^n : x_i \neq x_j \text{ for } i \neq j\}.$$

There is a natural free action of the permutation group Σ_n on $\mathrm{PConf}_n(S)$ given by permuting the coordinates and we refer to the corresponding quotient $\mathrm{Conf}_n(S) := \mathrm{PConf}_n(S)/\Sigma_n$ as the *configuration space* of S. Lastly, let

$$PB_n(S) := \pi_1(\mathrm{PConf}_n(S)) \text{ and } B_n(S) := \pi_1(\mathrm{Conf}_n(S))$$

denote the *n-strand pure braid group* and the *n-strand braid group* of S. Our goal in this paper is to understand surjective homomorphisms between surface braid groups on different numbers of strands. For example, when $n \geq m > 0$, there are natural maps $\mathrm{PConf}_n(S) \to \mathrm{PConf}_m(S)$ forgetting $n-m$ coordinates, which induce *forgetful homomorphisms* $PB_n(S) \to PB_m(S)$. In fact, up to automorphisms, these are the only surjections that arise:

Theorem 1.8 (Surjections $PB_n(S) \to PB_m(S)$)**.** *Let S be a (possibly noncompact) hyperbolic surface of genus at least two. For $m, n > 0$, every surjective homomorphism $F : PB_n(S) \to PB_m(S)$ factors through some forgetful homomorphism, possibly post-*

composed with an automorphism of $PB_m(S)$.

In particular, when $m > n$, there is no surjection $F : PB_n(S) \to PB_m(S)$. Moreover, when $n = m$, every surjection $F : PB_n(S) \to PB_n(S)$ is an isomorphism. This gives a new proof of the fact that $PB_n(S)$ is Hopfian. Another consequence of our theorem is the following:

Corollary 1.9 (Surjections $B_n(S) \to PB_m(S)$)**.** *Let S be a (possibly noncompact) hyperbolic surface of genus at least two. For $m > 0, n > 1$, there is no surjective homomorphisms*

$$F : B_n(S) \to PB_m(S).$$

Historically, the first nontrivial surjective homomorphism between braid groups arose from a classical miracle: "resolving the quartic". Indeed, let $RQ : \mathrm{Conf}_4(\mathbb{C}) \to \mathrm{Conf}_3(\mathbb{C})$ be the map given by

$$RQ(a, b, c, d) = (ab + cd, ac + bd, ad + bc).$$

By computation, the induced homomorphism on the fundamental groups $RQ_* : B_4(\mathbb{C}) \to B_3(\mathbb{C})$ is surjective. Theorem 1.8 says that there is no such miracle map between pure surface braid groups.

The readers may be wondering why we refer to RQ_* as a miracle. One of the reasons behind this terminology is a result of Lin [28] proving that there is no surjective homomorphism $B_n(\mathbb{C}) \to B_m(\mathbb{C})$ when $n > m$ except the RQ_* map. In proving this, Lin extended Artin [3] classification of all homomorphisms $B_n(\mathbb{C}) \to \Sigma_n$ to the case $B_n(\mathbb{C}) \to \Sigma_m$. To get a similar result of surface braid group, we would need to classify homomorphisms $B_n(S) \to \Sigma_m$, extending the result of Ivanov [23, Theorem 1] classifying surjections $B_n(S) \to \Sigma_n$. Based on the above discussion, we have the following conjecture:

Conjecture 1. Let S be a (possibly noncompact) hyperbolic surface of genus at least two and $m \neq n > 0$. There is no surjective homomorphism

$$F : B_n(S) \to B_m(S)$$

In light of our first theorem, in order to further understand all surjections between pure surface braid groups, we need to study the automorphism groups of braid groups: $\mathrm{Aut}(B_n(S))$ and $\mathrm{Aut}(P_n(S))$. To this end, for $g > 1$ let $\mathrm{Diff}^{\pm}(S_{g,p,n})$ be the group of diffeomorphisms of S_g fixing two sets of punctures, one with p points and the other with n points including both orientation-preserving and orientation-reversing maps. Let

$$\mathrm{Mod}^{\pm}(S_{g,p,n}) := \pi_0(\mathrm{Diff}^{\pm}(S_{g,p,n}))$$

called *extended mapping class group* of $S_{g,n,p}$. The following theorem computes $\mathrm{Aut}(PB_n(S_{g,p}))$ except when $n = 1, p > 0$.

Theorem 1.10 (Automorphism group of $PB_n(S_{g,p})$ and $B_n(S_{g,p})$)**.** *When $g > 1$, either $n > 1, p \geq 0$ or $n \geq 1, p = 0$,*

$$\mathrm{Mod}^{\pm}(S_{g,p,n}) \cong \mathrm{Aut}(PB_n(S_{g,p})) \cong \mathrm{Aut}(B_n(S_{g,p})).$$

Remark. When $n = 1, p > 0$, the statement of Theorem 1.10 is simply false. The group $PB_1(S_{g,p})$ is a free group and there are many isomorphisms of $PB_1(S_{g,p})$ that are not induced from diffeomorphisms. But as Theorem 1.10 shows, as long as we have more strands in the braid group, every automorphism is induced from a diffeomorphism again.

It should be mentioned at this point that Theorem 1.10 had a predecessor: Irmak–Ivanov–McCarthy [22, Theorem 1] first computed the automorphism group of $PB_n(S_g)$ and showed that every element is geometric in the sense that it comes from a diffeomorphism of S_g. After this work was completed, we also found out that An [1] obtained Theorem 1.10 through similar method as [22, Theorem 1]. Moreover, Kida–Yamagata [25] [26] showed that all injective homomorphism from a finite index subgroup of $PB_n(S_g)$ to itself is induced by a diffeomorphism of surface. Nevertheless our method is new and appears to be much simpler than all of the above. In particular, we do not rely on canonical reduction systems but we use cohomology obstruction theory.

CHAPTER 2

POINT-PICKING

2.1 The case when $S = \mathbb{R}^2$

Let S be a surface and let $\mathrm{PConf}_n(S)$ *the pure configuration space* be the space of ordered n-tuple of distinct points on S. The natural embedding $\mathrm{PConf}_n(S) \subset S^n$ gives the topology on $\mathrm{PConf}_n(S)$. Let $f_n(S) : \mathrm{PConf}_{n+1}(S) \to \mathrm{PConf}_n(S)$ be the map given by $f_n(x_0, x_1, \ldots, x_n) := (x_1, \ldots, x_n)$.

There is a natural action of permutation group Σ_n on $\mathrm{PConf}_n(S)$ by permuting the n points. Thus the quotient space is the space of unordered n-tuples of distinct points in S. Permutation group Σ_n acts on the fiber bundle $f_n(S)$ as well. Let $F_n(S) : \mathrm{PConf}_{n+1}(S)/\Sigma_n \to \mathrm{PConf}_n(S)/\Sigma_n$ be the map given by $F_n(x_0, \{x_1, \ldots, x_n\}) := \{x_1, \ldots, x_n\}$. The subject of this section is to classify the sections of the fiber bundles $f_n(\mathbb{R}^2)$ and $F_n(\mathbb{R}^2)$. We omit the base point throughout the whole paper when we use fundamental groups and that does not bring ambiguity in our discussions.

2.1.1 Constructing sections

In this subsection we give constructions of sections of the fiber bundle $f_n(\mathbb{R}^2)$. There are two cases: "adding a point near x_k" and "adding a point at infinity". The construction originated from [12].

Case 1: adding a point near x_k. Let

$$\mathrm{PConf}_{n,k}(\mathbb{R}^2) = \{(v_k, x_1, ..., x_n) | x_1, ..., x_n \text{ be } n \text{ points on } \mathbb{R}^2 \text{ and } v_k \text{ be a unit vector at } x_k\}.$$

This is the total space of a circle bundle by forgetting the vector v_k

$$S^1 \to \mathrm{PConf}_{n,k}(\mathbb{R}^2) \to \mathrm{PConf}_n(\mathbb{R}^2). \tag{2.1.1}$$

Equip $\mathbb{R}^2$ with the Euclidean metric. Let

$$\epsilon(x_1, ..., x_n) = \frac{1}{2}\mathrm{min}_{1\leq i\neq j\leq n}\{d(x_i, x_j)\}.$$

By the definition of $\epsilon(x_1, ..., x_n)$, setting x_0 to be the image of the v_k-flow at time $\epsilon(x_1, ..., x_n)$ from x_k gives a map:

$$em_{n,k}(\mathbb{R}^2) : \mathrm{PConf}_{n,k}(\mathbb{R}^2) \hookrightarrow \mathrm{PConf}_{n+1}(\mathbb{R}^2).$$

Composing a continuous section $s : \mathrm{PConf}_n(\mathbb{R}^2) \to \mathrm{PConf}_{n,k}(\mathbb{R}^2)$ of the fiber bundle (2.1.1) with $em_{n,k}(\mathbb{R}^2)$ gives a section of the fiber bundle $f_n(\mathbb{R}^2)$.

Definition 2.1 (**Adding a point near $\boldsymbol{x_k}$).** We denote by $\mathrm{Add}_{n,k}(\mathbb{R}^2)$ the collection of sections of $f_n(\mathbb{R}^2)$ consisting of compositions of a section of (2.1.1) with $em_{n,k}(\mathbb{R}^2)$.

Notice that there are infinitely many homotopy classes of sections in $\mathrm{Add}_{n,k}(\mathbb{R}^2)$ and they are in one-to-one correspondence with the homotopy classes of sections of (2.1.1).

Case 2: adding a point at infinity. Let us call the north pole of a 2-sphere the point at infinity ∞. Then $\mathbb{R}^2 \cong S^2 - \infty$ through the stereographic projection. Let

$$\mathrm{PConf}_{n,\infty}(\mathbb{R}^2) = \{(v_\infty, x_1, ..., x_n)|x_1, ..., x_n \text{ be } n \text{ points on } \mathbb{R} \text{ and } v_\infty \text{ be a unit vector at } \infty\}.$$

This is the total space of a circle bundle by forgetting the vector

$$S^1 \to \mathrm{PConf}_{n,\infty}(\mathbb{R}^2) \to \mathrm{PConf}_n(\mathbb{R}^2). \tag{2.1.2}$$

Equip S^2 with the spherical metric; i.e. the metric that is induced from the standard embedding $S^2 \subset \mathbb{R}^3$. Let

$$\epsilon(x_1, ..., x_n) = \frac{1}{2}\min_{1 \le i \le n}\{d(x_i, \infty)\}.$$

By the definition of $\epsilon(x_1, ..., x_n)$, setting x_0 to be the image of the v_∞-flow at time ϵ from ∞ gives a map:

$$em_{n,\infty}(\mathbb{R}^2) : \mathrm{PConf}_{n,\infty}(\mathbb{R}^2) \hookrightarrow \mathrm{PConf}_{n+1}(\mathbb{R}^2).$$

Composing a continuous section $s : \mathrm{PConf}_n(\mathbb{R}^2) \to \mathrm{PConf}_{n,\infty}(\mathbb{R}^2)$ of the fiber bundle (2.1.2) with $em_{n,\infty}(\mathbb{R}^2)$ gives a section of the fiber bundle $f_n(\mathbb{R}^2)$.

Definition 2.2 (**Adding a point at infinity**)**.** We denote by $\mathrm{Add}_{n,\infty}(\mathbb{R}^2)$ the collection of sections of $f_n(\mathbb{R}^2)$ consisting of compositions of a section of (2.1.2) with $em_{n,\infty}(\mathbb{R}^2)$.

Notice that there are infinitely many homotopy classes of sections in $\mathrm{Add}_{n,\infty}(\mathbb{R}^2)$ and they are in one-to-one correspondence with the homotopy classes of sections of (2.1.2).

2.1.2 Background

In this subsection we discuss some properties of canonical reduction systems and the lantern relation. Let $S = S^b_{g,p}$ be a surface with b boundary components and p punctures. Let $\mathrm{Mod}(S)$ (reps. $\mathrm{PMod}(S)$) be the *mapping class group* (resp. *pure mapping class group*) of S, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of S fixing the boundary components pointwise and punctures as a set (resp. pointwise). By "simple closed curves", we often mean isotopy class of simple closed curves, e.g. by "preserve a simple closed curve", we mean preserve the isotopy class of a curve.

Thurston's classification of elements of $\mathrm{Mod}(S)$ is a very powerful tool to study mapping class groups. We call a mapping class $f \in \mathrm{Mod}(S)$ *reducible* if a power of f fixes a nonperipheral simple closed curve. Each nontrivial element $f \in \mathrm{Mod}(S)$ is of exactly one of the following types: periodic, reducible, pseudo-Anosov. See [16, Chapter 13] and [17] for more details. We now give the definition of canonical reduction system.

Definition 2.3 (**Reduction systems).** A *reduction system* of a reducible mapping class h in $\text{Mod}(S)$ is a set of disjoint nonperipheral curves that h fixes as a set up to isotopy. A reduction system is *maximal* if it is maximal with respect to inclusion of reduction systems for h. The *canonical reduction system* $\text{CRS}(h)$ is the intersection of all maximal reduction systems of h.

For a reducible element f, there exists n such that f^n fixes each element in $\text{CRS}(f)$ and after cutting out $\text{CRS}(f)$, the restriction of f^n on each component is either periodic or pseudo-Anosov. See [16, Corollary 13.3]. Now we mention three properties of the canonical reduction systems that will be used later.

Proposition 2.4. $\text{CRS}(h^n)$=$\text{CRS}(h)$ *for any* n*.*

Proof. This is classical; see [16, Chapter 13]. □

For a curve a on a surface S, denote by T_a the Dehn twist about a. For two curves a, b on a surface S, let $i(a, b)$ be the geometric intersection number of a and b. For two sets of curves P and T, we say that S and T *intersect* if there exist $a \in P$ and $b \in T$ such that $i(a, b) \neq 0$. Notice that two sets of curves intersecting does not mean that they have a common element.

Proposition 2.5. *Let* h *be a reducible mapping class in* $\text{Mod}(S)$*. If* $\{\gamma\}$ *and* $\text{CRS}(h)$ *intersect, then no power of* h *fixes* γ*.*

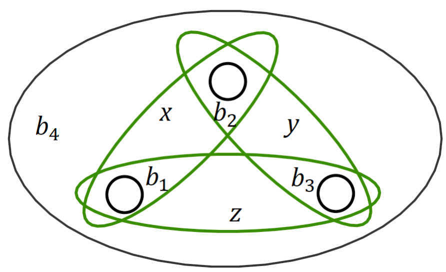
Proof. Suppose that h^n fixes γ. Therefore γ belongs to a maximal reduction system M. By definition, $\text{CRS}(h) \subset M$. However γ intersects some curve in $\text{CRS}(f)$; this contradicts the fact that M is a set of disjoint curves. □

Proposition 2.6. *Suppose that $h, f \in \mathrm{Mod}(S)$ and $fh = hf$. Then $\mathrm{CRS}(h)$ and $\mathrm{CRS}(f)$ do not intersect.*

Proof. By conjugation, we have that $\mathrm{CRS}(hfh^{-1}) = h(\mathrm{CRS}(f))$. Since $hfh^{-1} = f$, we get that $\mathrm{CRS}(f) = h(\mathrm{CRS}(f))$. Therefore h fixes the whole set $\mathrm{CRS}(f)$. A power of h fixes all curves in $\mathrm{CRS}(f)$. By Proposition 2.5, curves in $\mathrm{CRS}(h)$ do not intersect curves in $\mathrm{CRS}(f)$. □

Now, we introduce a remarkable relation for $\mathrm{Mod}(S)$ that will be used in the proof.

Proposition 2.7 (The lantern relation). *There is an orientation-preserving embedding of $S_{0,4} \subset S$ and let $x, y, z, b_1, b_2, b_3, b_4$ be simple closed curves in $S_{0,4}$ that are arranged as the curves shown in the following figure.*



In $\mathrm{Mod}(S)$ we have the relation

$$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}.$$

Proof. This is classical; see [16, Chapter 5.1]. □

2.1.3 An algebraic result and how it implies (1) of Theorem 1.1

In this subsection we give an algebraic result about the braid groups and prove how it implies (1) of Theorem 1.1.

$\mathrm{PConf}_n(\mathbb{R}^2)$ and $\mathrm{PConf}_{n+1}(\mathbb{R}^2)$ are both $K(\pi, 1)$ spaces. This can be seen by induction on n and taking the long exact sequence of homotopy groups of the fiber bundle

$f_n(\mathbb{R}^2)$. Therefore, the homotopy classes of sections of $f_n(\mathbb{R}^2)$ only depend on the homomorphisms of the fundamental groups. Let $PB_n = \pi_1(\mathrm{PConf}_n(\mathbb{R}^2))$ and let F_n be a free group of n generators. The fundamental groups of the fiber bundle $f_n(S)$ gives us the following short exact sequence, i.e. the Fadell-Neuwirth short exact sequence:

$$1 \to F_n \to PB_{n+1} \xrightarrow{f_n(\mathbb{R}^2)_*} PB_n \to 1. \tag{2.1.3}$$

Let D_n be the disk with n punctures $\{x_1, ..., x_n\}$ and D_{n+1} be the disk with $n+1$ punctures $\{x_0, x_1, ..., x_n\}$ and the forget map forgets the point x_0. We view PB_n and PB_{n+1} as mapping class groups as the following:

$$PB_n = \mathrm{PMod}(D_n) \text{ and } PB_{n+1} = \mathrm{PMod}(D_{n+1}).$$

A simple closed curve a on D_n separates D_n into two parts: the *outside of* a, i.e. the component containing the boundary of D_n and the *inside of* a, i.e. the one not containing the boundary of D_n. We say that a *surrounds* x_k if $x_k \in$ the inside of a. The following algebraic result on the splittings of the exact sequence (2.1.3) is a key ingredient in the proof of Theorem 1.1.

Theorem 2.8. *Suppose that we have a section $s : PB_n \to PB_{n+1}$. Then the image $s(PB_n)$ either preserves a simple closed curve c surrounding points $\{x_1, ..., x_n\}$, or preserves a simple closed curve c surrounding $\{x_i, x_0\}$ for some $i \in \{1, 2, ..., n\}$.*

The rest of this subsection focuses on how Theorem 2.8 implies part (1) of Theorem 1.1. Let c be a curve inside D_{n+1} surrounding k points. Let D_k^l be a disk with k punctures and l open disks removed. We call the boundary of the l disks the *small boundary components* and the original boundary of D *the big boundary component*. See the following figure for a geometric explanation.

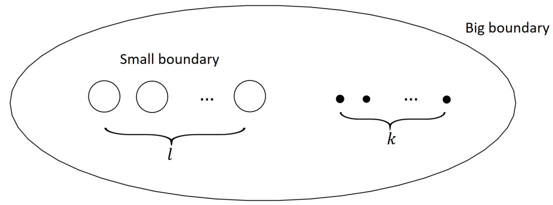


Figure 2.1.1: D_k^l where small boundaries are the l small circles and big boundary is the outside circle.

Let

$$PB_{k,l} := \mathrm{PMod}(D_k^l)$$

be the pure mapping class group of D_k^l. The difference between punctures and boundary components is that the Dehn twist about a puncture is trivial but the Dehn twist about a boundary component is nontrivial. The following proposition describes the centralizer of T_c. Denote the centralizer of T_c by $C_{PB_{n+1}}(T_c)$.

Proposition 2.9 (Centralizer of $\boldsymbol{T_c}$). *$C_{PB_{n+1}}(T_c)$ satisfies the following exact sequence*

$$1 \to \mathbb{Z} \xrightarrow{(T_c, T_c^{-1})} PB_k \times PB_{n+1-k,1} \to C_{PB_{n+1}}(T_c) \to 1.$$

Proof. This is classical. The centralizer of T_c is the subgroup of $\mathrm{Mod}(D_n)$ that fixes c. The curve c separates D_n into two components: C_1 that contains the boundary and C_2 that does not contain the boundary. Since C_1 and C_2 are not homeomorphic, we have that $C_{PB_{n+1}}(T_c)$ only contains elements that preserve C_1 and C_2. Therefore, our statement holds. □

Now we are ready to prove (1) of Theorem 1.1.

Proof of (1) of Theorem 1.1 assuming Theorem 2.8. Let $g : \mathrm{PConf}_n(\mathbb{R}^2) \to \mathrm{PConf}_{n+1}(\mathbb{R}^2)$ be a section of the fiber bundle $f_n(\mathbb{R}^2)$. Let $s = g_* : PB_n \to PB_{n+1}$ be the induced map on the fundamental groups of g. By Theorem 2.8, the image $s(PB_n)$ preserves a curve c

that either surrounds 2 points or n points. Therefore, $s(PB_n)$ is in the centralizer of T_c in PB_{n+1} by the fact that $fT_cf^{-1} = T_{f(c)}$.

Case 1: when c surrounds $\{x_0, x_k\}$. $PB_2 \cong \mathbb{Z}$, which is generated by the Dehn twist about the boundary component. From Proposition 2.9 we have

$$1 \to \mathbb{Z} \xrightarrow{(T_c, T_c^{-1})} \mathbb{Z} \times PB_{n-1,1} \to C_{PB_{n+1}}(T_c) \to 1.$$

Therefore $C_{PB_{n+1}}(T_c) \cong PB_{n-1,1}$. The inclusion $PB_{n-1,1} \hookrightarrow PB_{n+1}$ is induced by gluing a 2-punctured disk inside the small boundary of D^1_{n-1}.

On the other hand, we have that

$$\pi_1(\mathrm{PConf}_{n,k}(\mathbb{R}^2)) = PB_{n-1,1}.$$

The fundamental groups of the fiber bundle (2.1.1) is the following exact sequence:

$$1 \to \mathbb{Z} \xrightarrow{T_d} PB_{n-1,1} \to PB_n \to 1. \tag{2.1.4}$$

Here T_d is the Dehn twist about the small boundary component. The embedding $em_{n,k}$: $\mathrm{PConf}_{n,k}(\mathbb{R}^2) \hookrightarrow \mathrm{PConf}_{n+1}(\mathbb{R}^2)$ induces a homomorphism on the fundamental group $em_{n,k*} : PB_{n-1,1} \to PB_{n+1}$. On the mapping class group level, since T_d in $PB_{n-1,1}$ is mapped to the Dehn twist about a curve surrounding $\{x_0, x_k\}$, we know that $em_{n,k*}$ is also induced by gluing a 2-punctured disk inside the small boundary of D^1_{n-1}. The theorem holds.

Case 2: when c surrounds $\{x_1, ..., x_n\}$. Since $PB_{1,1} \cong \mathbb{Z}\times\mathbb{Z}$, which is generated by the Dehn twists about the two boundaries, we have the following exact sequence:

$$1 \to \mathbb{Z} \xrightarrow{(0, T_c, T_c^{-1})} \mathbb{Z} \times \mathbb{Z} \times PB_n \to C_{PB_{n+1}}(T_c) \to 1.$$

On the mapping class group level, $PB_n \times PB_{1,1} \to C_{PB_{n+1}}(T_c) \to PB_{n+1}$ is induced by gluing D_1^1 outside the big boundary component of D_n. Therefore $\mathbb{Z} \times PB_n \cong C_{PB_{n+1}}(T_c)$ and the generator of $\mathbb{Z}$ is mapped to $T_c T_b^{-1}$ where b is the big boundary of D_{n+1}.

On the other hand, we have that

$$\pi_1(\mathrm{PConf}_{n,\infty}(\mathbb{R}^2)) = \mathbb{Z} \times PB_n.$$

The embedding $em_{n,\infty} : \mathrm{PConf}_{n,\infty}(\mathbb{R}^2) \hookrightarrow \mathrm{PConf}_{n+1}(\mathbb{R}^2)$ induces $em_{n,\infty*} : \mathbb{Z} \times PB_n \to PB_{n+1}$ on the fundamental groups. On the level of mapping class groups, since $\mathbb{Z}$ maps to $T_c T_b^{-1}$, we know that $em_{n,\infty*}$ is induced by the embedding of D_n in D_{n+1} and maps the generator of $\mathbb{Z}$ to $T_c T_b^{-1}$. Therefore, $em_{n,\infty*}$ is induced by gluing D_1^1 outside the big boundary component of D_n as well. Our theorem holds.

□

The classification of the sections of the fiber bundle $f_n(S)$ is not entirely the same as the classification of the splittings of the exact sequence (2.1.3). There is an subtlety coming from the choice of base point in the fundamental groups. Therefore, we classify the splittings of the exact sequence (2.1.3) up to conjugacy. In Theorem 2.8, all the choices of c is coming from a conjugacy by an element F_n; thus they decide the same sections.

2.1.4 The proof of Theorem 2.8

Throughout the subsection we will prove Theorem 2.8. We assume that there exists a section $s : PB_n \to PB_{n+1}$, i.e. $f_n(\mathbb{R}^2)_* \circ s = id$. The strategy is that we first determine $s(T_a)$ for any simple closed curve a on D_n. We will prove that the lift $s(T_a)$ is always a multicurve about at most two curves on D_{n+1}; these two curves or one curve are either trivial or isotopic to a after forgetting the point x_0. Then we decompose the discussion into cases depending on whether $s(T_a)$ is a multicurve on two curves or a single twist. We use the lantern relation to simplify each cases and derive a contradiction. The following lemma determines $s(T_a)$ for any simple closed curve a on D_n.

Lemma 2.10 (The lift of a Dehn twist). *Let a be a simple closed curve on D_n, then $s(T_a)$ can only be one of the following three cases:*

(1) It can be a Dehn twist $T_{a'}$ about a curve a' on D_{n+1} such that after forgetting x_0, we have $a' = a$.

(2) It can be a multitwist $T_{a'}T_c^m$ (i.e. a product of twists on disjoint curves) about two curves a' and c on D_{n+1} for $m \in \mathbb{Z}$, where c surrounds 2 points $\{x_0, x_k\}$ and after forgetting x_0, we have that $a' = a$.

(3) It can be $T_{a'}(T_{a'}T_{a''}^{-1})^n$, where a' and a'' are disjoint on D_{n+1} such that after forgetting x_0, we have that $a' = a'' = a$.

Proof. We break our proof into two steps.

Step 1: The proof of the fact that "after forgetting x_0, any element of CRS($s(T_a)$) is either a or surrounding one puncture (trivial)."

Firstly by [31, Theorem 1], the centralizer of a pseudo-Anosov element is virtually cyclic. However the centralizer of $s(T_a)$ at least contains a copy of $\mathbb{Z}^2$ when $n > 3$. Therefore we know that $s(T_a)$ is not pseudo-Anosov. Since $s(T_a)$ is not torsion element, we have that $s(T_a)$ is reducible. Assume there exists $b' \in \text{CRS}(s(T_a))$ such that after forgetting x_0, we have that b is not trivial and $b \neq a$. Since $b' \in \text{CRS}(s(T_a))$, we have that a power of $s(T_a)$ fixes b'. Also a power of any mapping class that commutes with $s(T_a)$ fixes b as well. We break our discussion into the following two cases.

Case 1: If $i(a, b) \neq 0$, then no power of T_a fixes b. However we also have some power of $s(T_a)$ fixes b'. This is a contradiction.

Case 2: If $i(a, b) = 0$ but $b \neq a$, then there exists a curve c such that $i(c, b) \neq 0$ but $i(c, a) = 0$. Since $s(T_c)$ commutes with $s(T_a)$, we know that $s(T_c)$ preserves $\text{CRS}(s(T_a))$. However $i(b, c) \neq 0$, which shows that no power of T_c preserve b. This contradicts the fact that a power $s(T_c)$ preserves b.

By the disjointness of curves in canonical reduction system, we have that $\mathrm{CRS}(s(T_a))$ contains at most 2 curves.

Step 2: Finishing the proof of the lemma. We break the proof into 3 cases.

Case 1: $\mathbf{CRS}(\boldsymbol{T_a})$ only contains one curve $\boldsymbol{a'}$. It depends on the location of x_0, only one side of a' will contain x_0. Assume without loss of generality that the inside of a contains x_0, then $s(T_a)$ is the identity on the outside. If the inside contains more than 2 points, there is a curve b inside of a containing 2 points, therefore $s(T_a)$ fixes $\mathrm{CRS}(s(T_b))$, which means $s(T_a)$ does not acts as pseudo-Anosov inside a'. This proves that a power of $s(T_a)$ is the identity on the inside. Therefore a power of $s(T_a)$ is a power of the $T_{a'}$. Since $s(T_a)$ is a lift of T_a, we have that a power of $s(T_a)T_{a'}^{-1}$ is the identity. Therefore $s(T_a) = T_{a'}$.

Case 2: $\mathbf{CRS}(\boldsymbol{T_a})$ only contains two curves $\boldsymbol{a'}$ and $\boldsymbol{c}$ such that $\boldsymbol{c}$ surrounds 2 points $\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$. By the same argument as Case 1, we show that $s(T_a) = T_a T_c^m$.

Case 3: $\mathbf{CRS}(\boldsymbol{T_a})$ only contains two curves $\boldsymbol{a'}$ and $\boldsymbol{a''}$ such that after forgetting $\boldsymbol{x_0}$, both curves $\boldsymbol{a'}$ and $\boldsymbol{a''}$ become $\boldsymbol{a}$. By the same argument as Case 1, we show that $s(T_a) = T_{a'}(T_{a'}T_{a''}^{-1})^n$.

□

Notation. In the following argument, we will use small letters like $a, b, c, ...$ to represent simple closed curves on D_n and small letters with a prime or double primes like $a', a'', b', ...$ to represent the canonical reduction systems of $s(T_a), s(T_b), ...$ If we have two curves in $\mathrm{CRS}(s(T_a))$, we use a' and a''.

The following proposition characterizes intersection number 2 of two curves.

Proposition 2.11. *Let $i(a,b) \neq 0$. Then T_aT_b is a multitwist if and only if $i(a,b) = 2$.*

Proof. This result was previously obtained by Margalit [29] and Hamidi-Tehrani [20]. We give a different proof using Thurston's construction; see e.g. [16, Theorem 14.1]. There is a subspace T of S that a, b fills, i.e. the tubular neighborhood of $a \cup b$. Let $\langle T_a, T_b \rangle$ be the group generated by T_a and T_b in $\text{Mod}(T)$. Thurston's theorem says that when a, b fill, there is a representation $\rho : \langle T_a, T_b \rangle \to \text{PSL}(2, \mathbb{R})$ such that

$$T_a \to \begin{bmatrix} 1 & -i(a,b) \\ 0 & 1 \end{bmatrix} \text{ and } T_b \to \begin{bmatrix} 1 & 0 \\ i(a,b) & 1 \end{bmatrix}.$$

$\rho(h)$ is parabolic if and only if h is reducible on T. We know that

$$\rho(T_a T_b) = \begin{bmatrix} 1 & -i(a,b) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i(a,b) & 1 \end{bmatrix} = \begin{bmatrix} 1 - i(a,b)^2 & -i(a,b) \\ i(a,b) & 1 \end{bmatrix}$$

Since $\text{Trace}(\rho(T_a T_b)) = 2 - i(a,b)^2$, we know that $T_a T_b$ is reducible on T if and only if $i(a,b) = 2$. By the lantern relation, we know that $T_a T_b$ is a multitwist when $i(a,b) = 2$. □

On D_n, we call a simple closed curve surrounding 2 points by *a basic simple closed curve*. The following lemma gives one condition for $s(PB_n)$ to preserve a simple closed curve surrounding $\{x_1, ..., x_n\}$.

Lemma 2.12. *If the canonical reduction system of any basic simple closed curve does not contain a curve surrounding x_0, then $s(PB_n)$ preserves a simple closed curve surrounding $\{x_1, ..., x_n\}$.*

Proof. Suppose that there exists a simple closed curve a such that $\text{CRS}(a)$ contains a curve surrounding x_0. We call a the *innermost* if a surrounds k points and the canonical reduction systems of all curves surrounding $k - 1$ points does not contains a curve surrounding x_0. Take an innermost curve a such that a surrounds k points in D_n. By the assumption of Lemma 2.12, we have that $k > 2$. There are three cases according to Lemma 2.10.

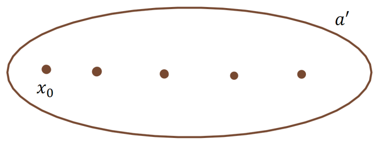


Figure 2.1.2: Case 1: CRS$(a) = \{a'\}$.

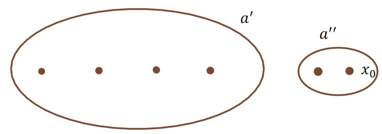


Figure 2.1.3: Case 2: CRS$(a) = \{a', a''\}$ and a'' can possibly be inside a'.

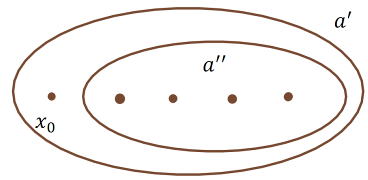
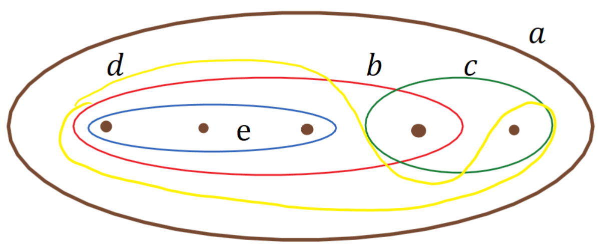


Figure 2.1.4: Case 3: CRS$(a) = \{a', a''\}$.

Case 1: CRS**$(a) = \{a'\}$ such that after forgetting x_0, we have $a' = a$.** We take b and c inside D_n like the following figure, we have the lantern relation $T_bT_c = T_eT_aT_d^{-1}$.



Because b, c, d, e surround less points than k and a is the innermost curve, we know that CRS$(s(T_b))$, CRS$(s(T_c))$, CRS$(s(T_d))$ and CRS$(s(T_e))$ each only contains one curve not surrounding x_0 and we denote them by b', c', d', e'. Since T_e, T_a and T_d commute with each other, their canonical reduction systems would be disjoint. By Lemma 2.10, we know that $s(T_eT_aT_d^{-1})$ is also a multitwist. Therefore by Lemma 2.11, we know that $i(b', c') = 2$. However CRS$(T_{b'}T_{c'})$ does not contain a'. This is a contradiction.

We have two more cases:

Case 2: CRS$(a) = \{a', a''\}$ such that a'' surrounds 2 points $\{x_0, x_k\}$ and after forgetting x_0, we have that $a' = a$.

Case 3: CRS$(a) = \{a', a''\}$ such that after forgetting x_0, we have that $a' = a'' = a$.

Case 2 and 3 can be proved using the same argument as Case 1. We construct the same lantern relation and use the fact that a is the innermost curve to read a contradiction. Therefore if canonical reduction systems of all basic simple closed curves do not contain a curve surrounding x_0, then the canonical reduction systems of any curve does not surround x_0. This is true for the center element of PB_n as well. Let c be the boundary curve of D_n. Then CRS$(c) = \{c'\}$ does not contain x_0. However, all Dehn twists commute with T_c which preserves c'. □

Now we need a generating set of PB_n. Let Figure 2.1.4 be an n-punctured disk D_n. Let L be a segment below all the other points. Let $L_1, ..., L_n$ be segments connecting $x_1, ..., x_n$ to the segment L. Figure 2.1.6 is the corresponding figure for D_{n+1}.

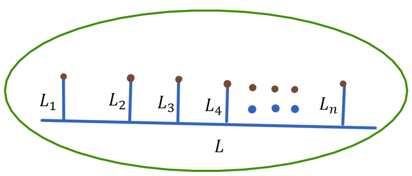


Figure 2.1.5: D_n.

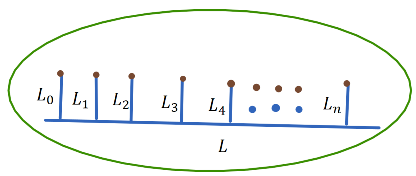


Figure 2.1.6: D_{n+1}.

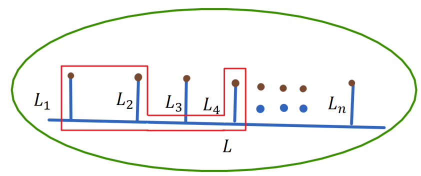


Figure 2.1.7: An example of Notation 2.1.4 for a_{124}.

Notation. For $\{i_1, ..., i_k\}$ a subset of $\{1, ..., n\}$, let $a_{i_1 i_2 ... i_k}$ be the boundary curve of the tubular neighborhood of $L \cup \cup_{m=1}^{k} L_{i_k}$. Denote by $A_{i_1 i_2 ... i_k}$ the Dehn twist about $a_{i_1 i_2 ... i_k}$. For $\{i_1, ..., i_k\}$ a subset of $\{0, 1, ..., n\}$, let $b_{i_1 i_2 ... i_k}$ be the boundary curve of the tubular neighborhood of $L \cup \cup_{m=1}^{k} L_{i_k}$. Denote by $B_{i_1 i_2 ... i_k}$ the Dehn twist about $b_{i_1 i_2 ... i_k}$. See Figure 2.1.7 for an example representing a_{124}.

The following proposition describes a generating set of the group PB_n.

Proposition 2.13. *There is a generating set of PB_n consisting of all the Dehn twists about the basic curves a_{ij} for $1 \leq i < j \leq n$.*

Proof. This is classical and can be prove it by induction on the exact sequence

$$1 \to F_k \to PB_{k+1} \to PB_k \to 1.$$

This generating set is given by Artin; e.g. see [30, Theorem 2.3] □

Now, we are ready to prove Theorem 2.8. We break the proof into several cases.

The proof of Theorem 2.8. By Lemma 2.12, we only need to consider the case that there exists at least one basic simple closed curve a such that some element of CRS(a) surrounds x_0. The curves in following four figures represent CRS(a). We will break our discussion into the following four cases.

Proof of Case 1: The canonical reduction systems of all basic curves only contain one curve and a' surrounds x_0. By the assumption of Case 1, for any basic curve c, there exists c' such that $s(T_c) = T_{c'}$. Let a, b, c, d be the curves in Figure 2.1.8. We have the lantern relation $T_a T_b T_c = T_d$.

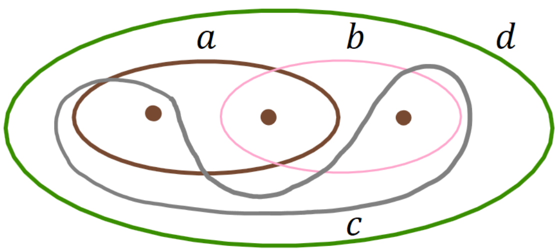


Figure 2.1.8: D_n: $a = a_{12}$, $b = a_{23}$, $c = a_{13}$ and $d = a_{123}$.

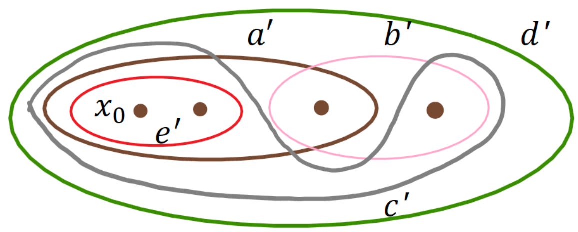


Figure 2.1.9: D_{n+1}: $a' = b_{012}$, $b' = a_{23}$, $c = a_{013}$, $e' = b_{01}$ and $d' = b_{0123}$.

Since T_c and T_d commute, $s(T_c) = T_{c'}$ and $s(T_d) = T_{d'}$ also commute. Therefore $s(T_a)s(T_b) = T_{c'}^{-1}T_{d'}$ is a multitwist by Lemma 2.10. By Lemma 2.11, we know that $i(b', a') = 2$ as in Figure 2.1.9. Suppose that b' does not surround x_0. By the lantern relation, $T_{a'}T_{b'} = T_{d'}T_{e'}T_{c'}^{-1}$. Since $s(T_d)$ and $s(T_c)$ are commuting multicurves, $s(T_d)s(T_c)^{-1}$ is multicurve as well. Since a, b, e are basic curves, we know that $s(T_d) = T_{d'}T_{e'}$ and $s(T_c) = T_{c'}$. By the same reason, we have that $s(T_f) = T_{f'}$ in Figure 2.1.10 and 2.1.11.

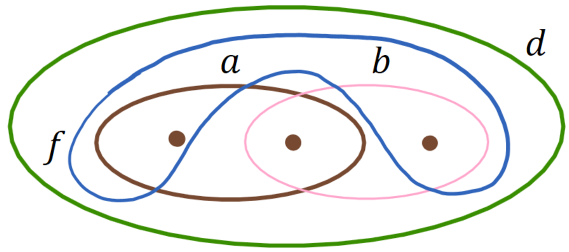


Figure 2.1.10: Lantern relation $T_f T_a T_b = T_d$.

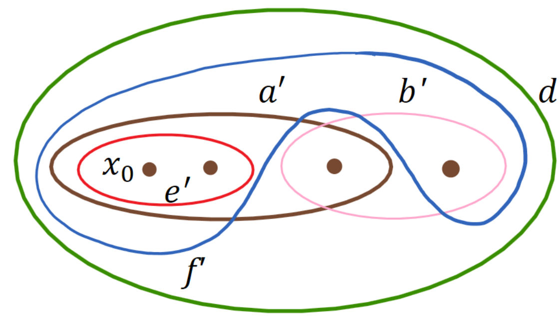


Figure 2.1.11: Lantern relation $T_{f'} T_{a'} T_{b'} = T_{e'} T_{d'}$.

In the following, we will prove that $s(PB_n)$ preserves b_{01}. Under Notation 2.1.4 for PB_n, we have $a = a_{12}$, $b = a_{23}$, $c = a_{13}$ and $d = a_{123}$. We also have that $s(A_{12}) = B_{012}$, $s(A_{23}) = B_{23}$ and $s(A_{13}) = B_{013}$. Since A_{ij} generates PB_n, all we need to show is that $s(A_{ij})$ preserves b_{01}. Since CRS(d) contains b_{01}, any curve disjoint from d preserves b_{01}. We only need to consider the curves that intersect with d. Without loss of generality, we only need to show that $s(A_{14})$, $s(A_{24})$ and $s(A_{34})$ preserve b_{01}. By the assumption of Case 1, we only need to show that the CRS(a_{14}), CRS(a_{24}) and CRS(a_{34}) are disjoint from b_{01}.

Since $i(a_{12}, a_{34}) = 0$, we have that CRS(a_{12}) is disjoint from CRS(a_{34}), which means that CRS(a_{34}) is disjoint from b_{01}. Since $s(T_f) = T_{f'}$ in Figure 2.1.10 and 2.1.11, CRS(a_{24}) is also disjoint from b_{01}. We have the following lantern relation:

$$A_{13}A_{34}A_{14} = A_{134}.$$

The image of relation under lift s is:

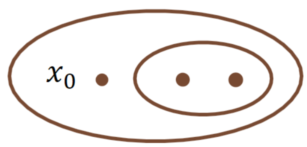
$$B_{013}B_{34}s(A_{14}) = s(A_{134}).$$

A_{134} commutes with A_{13} and A_{34}, thus CRS(a_{134}) is disjoint from b_{013} and b_{34}. The only possible curves are b_{01} and b_{0134}. If $s(A_{134}) = B_{0134}$, we have another lantern relation in D_{n+1}:

$$B_{013}B_{34}B_{014} = B_{0134}B_{01}.$$

This proves that $s(A_{14}) = B_{014}B_{01}^{-1}$ preserving $b_{01} = e'$. If CRS(a_{134}) contains b_{01}, we also have that $s(A_{14})$ preserves $b_{01} = e'$. The case when b' surrounds x_0 follows from the same argument.

Proof of Case 2: CRS(a) has two curves and both are isotopic to a after forgetting x_0.



Let b, c, d, e be curves in Figure 2.1.12. We have the lantern relation $T_bT_cT_d = T_eT_a$.

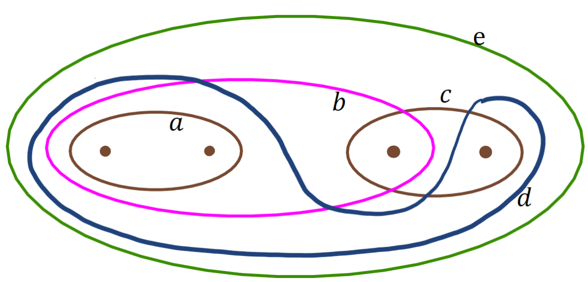


Figure 2.1.12: Lantern relation $T_bT_cT_d = T_aT_e$.

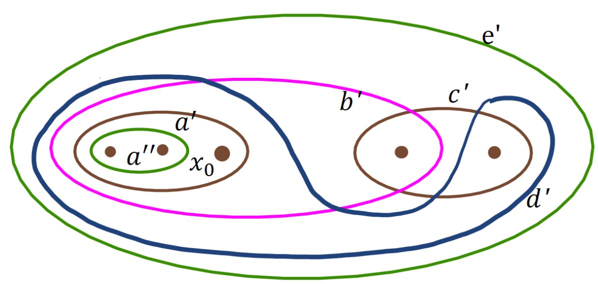
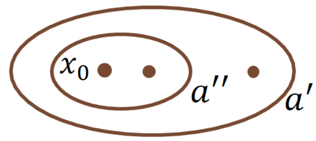


Figure 2.1.13: Lantern relation $T_{b'}T_{c'}T_{d'} = T_{a'}T_e$.

Since b, c, d, e are disjoint from a, we have that CRS(b), CRS(c), CRS(d) and CRS(e) are disjoint from $\{a', a''\}$. Therefore $s(T_b) = T_{b'}$, $s(T_c) = T_{c'}$, $s(T_d) = T_{d'}$ and $s(T_e) = T_{e'}$ as in Figure 2.1.13. But we also have the lantern relation $T_{b'}T_{c'}T_{d'} = T_{e'}T_{a'}$. Thus $s(T_a) = T_{a'}$. This contradicts the assumption of Case 2.

Proof of Case 3: CRS(a) has two curves a', a'' such that a' is isotopic to a and a'' is trivial after forgetting x_0, and a' surrounds a''.



In this case, we have $s(A_{12}) = B_{012}B_{01}^k$ for $k \neq 0$ by Lemma 2.10. Without loss of generality, we only need to show that $\mathrm{CRS}(a_{13})$ and $\mathrm{CRS}(a_{23})$ are disjoint from b_{01}. First of all, we have the following lantern relation:

$$A_{123}A_{34}A_{124} = A_{12}A_{1234}.$$

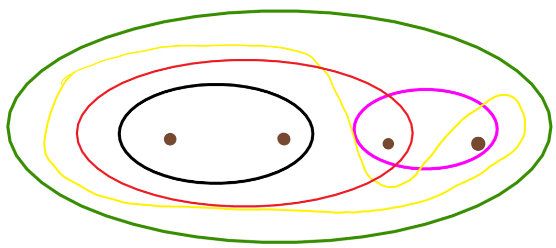


Figure 2.1.14: Lantern relation $A_{123}A_{34}A_{124} = A_{12}A_{1234}$.

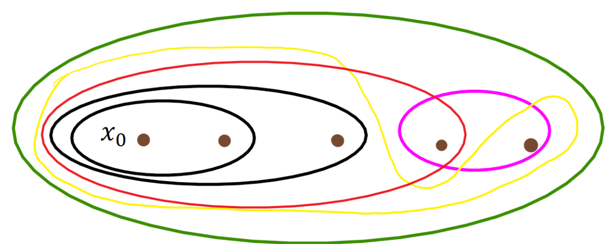


Figure 2.1.15: Lantern relation $B_{0123}B_{34}B_{0124} = B_{012}B_{01234}$.

Since all of the curves above are disjoint from a_{12}, their canonical reduction systems are disjoint from $a_{12'} = b_{012}$ and $a''_{12} = b_{01}$. We have the lantern relation:

$$B_{0123}B_{34}B_{0124} = B_{012}B_{01234}.$$

Since $s(A_{12}) = B_{012}B_{01}^k$, there exists at least one other curve in a_{123}, a_{34}, a_{124}, a_{1234}, whose canonical reduction system contains b_{01}. We break our discussion into the following four subcases depending on whether b_{01} is an element in $\mathrm{CRS}(A_{1234})$, $\mathrm{CRS}(A_{123})$, $\mathrm{CRS}(A_{124})$ or $\mathrm{CRS}(A_{34})$, respectively.

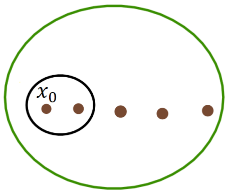


Figure 2.1.16: Subcase 1: $b_{01} \in$ CRS(A_{1234}).

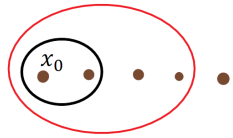


Figure 2.1.17: Subcase 2: $b_{01} \in$ CRS(A_{123}).

Figure 2.1.18: Subcase 3: $b_{01} \in$ CRS(A_{124}).

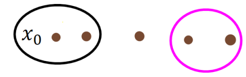


Figure 2.1.19: Subcase 4: $b_{01} \in$ CRS(A_{34}).

Subcase 1 and 2: In the first two cases, it is clear that CRS(a_{13}) and CRS(a_{23}) are disjoint from b_{01} because a_{13} and a_{23} are disjoint from a_{123} and a_{1234}.

Subcase 3: By $i(a_{14}, a_{124}) = 0$, we have that $b_{014} \in$ CRS(a_{14}) and b_{01} does not intersect CRS(a_{14}). Since $i(a_{23}, a_{14}) = 0$, we have that b_{014} does not intersect CRS(a_{23}). Suppose CRS(a_{23}) contains another curve z that is trivial after forgetting x_0. Since a_{23} is disjoint from a_{123} and a_{14}, we have that z has to be disjoint from b_{014} and b_{0123}. The only possibility is that $z = b_{01}$.

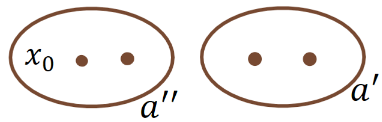
Because of the disjointness of a_{123} and a_{12}, we have that $s(A_{123})$ preserves CRS(a_{12}). This shows that $s(A_{123})$ preserves b_{01}. We have a lantern relation

$$A_{12}A_{23}A_{13} = A_{123}$$

After applying the homomorphism s to the above relation, all of the above element except $s(A_{13})$ preserves b_{01}. Therefore $s(A_{13})$ fixes b_{01}.

Subcase 4: Since $i(a_{234}, a_{34}) = 0$, we have that $b_{234} \in$ CRS(a_{234}). Since $i(a_{123}, a_{12}) = 0$, we have that $b_{0123} \in$ CRS(a_{123}). Therefore $b_{23} \in$ CRS(a_{23}) and CRS(a_{23}) may contain another curve z that is trivial after forgetting x_0. However a_{23} is disjoint from a_{123} and a_{234}, which implies that z is disjoint from b_{0123} and b_{234}. Therefore z can only be b_{01}. By the same argument as Subcase 3, we know that A_{13} also fixes b_{01}.

Proof of Case 4: CRS(a) has two curves a', a'' such that a' is isotopic to a and a'' is trivial after forgetting x_0, and a' does not surround a''.



Let a', a'' be positioned into the following Figure 2.1.20 such that $a' = b_{34}$ and $a'' = b_{01}$. If a curve c is disjoint from a_{34}, then $s(T_c)$ preserves b_{01}. Therefore without loss of generality, we only need to show that $s(A_{23})$ and $s(A_{13})$ preserve b_{01}.

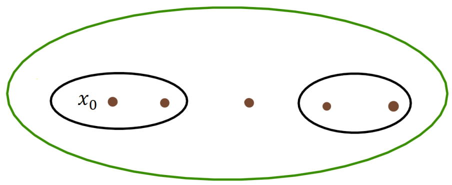


Figure 2.1.20: $a' = b_{34}$ and $a'' = b_{01}$.

Since $i(a_{12}, a_{34}) = 0$, we have that $b_{012} \in \text{CRS}(a_{12})$. Since $i(a_{124}, a_{12}) = 0$, we have that $b_{0124} \in \text{CRS}(a_{124})$. Possibly CRS($a_{124}$) contains another curve z that is trivial after forgetting x_0. However $b_{24} \in \text{CRS}(a_{24})$ because $b_{234} \in \text{CRS}(a_{234})$ and $i(a_{234}, a_{24}) = 0$. Therefore, z is disjoint from b_{24} and b_{012}, which means $z = b_{01}$. By the same reason, we can prove that $b_{0123} \in \text{CRS}(a_{123})$ and $s(A_{123})$ preserves b_{01}. We have the following lantern relation.

$$A_{123}A_{34}A_{124} = A_{12}A_{1234}. \tag{2.1.5}$$

Since $A_{34} = B_{34}B_{01}^k$ for nonzero k, therefore the canonical reduction system of one of the curves in the relation (2.1.5) contains b_{01}. The rest of the discussion is similar to Case 3 by doing a case study.

□

2.1.5 *The proof of (1) of Corollary 1.2*

Proof of (1) Corollary 1.2. Let $B_n = \pi_1(\mathrm{PConf}_n(\mathbb{R}^2)/\Sigma_n)$ and $B_{n,1} = \pi_1(\mathrm{PConf}_{n+1}(\mathbb{R}^2)/\Sigma_n)$. The fiber bundle $F_n(S)$ gives the first line of the following commutative diagram:

$$\begin{CD}
1 @>>> PB_{n+1} @>>> B_{n,1} @>>> \Sigma_n @>>> 1\\
@. @VV{f_n(\mathbb{R}^2)_*}V @VV{F_n(\mathbb{R}^2)_*}V @VV{=}V @.\\
1 @>>> PB_n @>>> B_n @>>> \Sigma_n @>>> 1.
\end{CD}$$

Every splitting of $F_n(\mathbb{R}^2)_*$ induces a splitting of $f_n(\mathbb{R}^2)_*$. Therefore, we only need to study the extension of a splitting of $f_n(\mathbb{R}^2)_*$ to a splitting of $F_n(\mathbb{R}^2)_*$. Let $\phi : B_n \to B_{n,1}$ be a splitting of $F_n(\mathbb{R}^2)_*$. Let $x \in PB_n$ and $e \in B_n$. We have that

$$\phi(exe^{-1}) = \phi(e)\phi(x)\phi(e)^{-1}.$$

Denote by C_e the conjugation of e on PB_n. Therefore, we have the following diagram:

$$\begin{CD}
PB_n @>{C_e}>> PB_n\\
@VV{\phi|_{PB_n}}V @VV{\phi|_{PB_n}}V\\
PB_{n+1} @>{C_{\phi(e)}}>> PB_{n+1}.
\end{CD} \tag{2.1.6}$$

By Theorem 2.8, there are two possibilities of $\phi|_{PB_n}$:

(1) ϕ fixes a simple closed curve c surrounding $\{x_k, x_0\}$

(2) ϕ fixes a simple closed curve c surroundding $\{x_1, ..., x_n\}$.

We claim that $\phi|_{PB_n}$ fixes a simple closed curve surroundding $\{x_1, ..., x_n\}$. To prove this claim, we assume the opposite that $\phi|_{PB_n}$ fixes a simple closed curve c surrounding $\{x_k, x_0\}$. There exists an element $e \in B_n$ such that e permutes punctures k and $j \neq k$. Since $\phi(PB_n)$ fixes c, we have that $\phi(C_e(PB_n)) = PB_n$ fixes c uniquely. However $C_{\phi(e)}(\phi(PB_n))$ fixes $\phi(e)(c)$ surrounding $\{x_j, x_0\}$ uniquely. This contradicts the diagram

(2.1.6). Therefore, $\phi|_{PB_n}$ fixes a simple closed curve surrounds $\{x_1, ..., x_n\}$. In this case, the section is the section that adds a point at infinity.

□

2.2 The case when S is the 2-sphere S^2

In this subsection we give a construction of sections of the fiber bundle $f_n(S^2)$.

2.2.1 Nonexistence of a continuous section for $n = 2$

We prove a more general result on the sections of the fiber bundle $f_n(S^2)$ for $n = 2$. Let S^{2k} be $2k$-dimensional sphere for $k > 0$ integer. Let x_1, x_2 be two distinct points in S^{2k}. The following is classical; see [15, Chapter 3].

Proposition 2.14. *The following fiber bundle*

$$S^{2k} - \{x_1, x_2\} \to \mathrm{PConf}_3(S^{2k}) \xrightarrow{f_2(S^{2k})} \mathrm{PConf}_2(S^{2k})$$

does not have a continuous section.

Proof. Suppose that there is a continuous map $s : \mathrm{PConf}_2(S^{2k}) \to \mathrm{PConf}_3(S^{2k})$ such that $f_2(S^{2k}) \circ s =$ identity. Then after post-composing with a forgetful map to the last coordinate, we get a map $f : \mathrm{PConf}_2(S^{2k}) \to S^{2k}$. We denote by $p_i : \mathrm{PConf}_2(S^{2k}) \to S^{2k}$ the projection to the ith component. Let

$$g_i : \mathrm{PConf}_2(S^{2k}) \xrightarrow{(f,p_i)} \mathrm{PConf}_2(S^{2k}) \subset S^{2k} \times S^{2k}.$$

Let $\triangle \subset S^{2k} \times S^{2k}$ be the diagonal subspace in the product. Let $[\triangle] \in H^{2k}(S^{2k} \times S^{2k}, \mathbb{Q})$ be the Poincaré dual of $\triangle$. By the Thom isomorphism, there is an exact sequence for the computation of cohomology:

$$0 \to \mathbb{Q} \xrightarrow{\text{diagonal}} H^{2k}(S^{2k} \times S^{2k}, \mathbb{Q}) \to H^{2k}(S^{2k} \times S^{2k} - \triangle; \mathbb{Q}) \to 0.$$

Let $c \in H^{2k}(S^{2k}; \mathbb{Q})$ be the fundamental class and $c_i = p_i^*(c)$. The image of diagonal is the Thom class $c_1 + c_2$. Therefore in $H^{2k}(S^{2k} \times S^{2k} - \triangle; \mathbb{Q})$, we have $c_1 + c_2 = 0$. This means that

$$H^{2k}(S^{2k} \times S^{2k} - \triangle; \mathbb{Q}) = \mathbb{Z}c_1.$$

Suppose that $f^*(x) = kc_1$ for k an integer. Therefore we will have $g_i^*([\triangle]) = kc_1 + c_i$. Since the image of g_i misses the diagonal $\triangle$, we have that $g_1^*([\triangle]) = kc_1 + c_1 = 0$ and $g_2^*([\triangle]) = kc_1 + c_2 = kc_1 - c_1 = 0$. Since c_1 is a generator of $H^{2k}(S^{2k} \times S^{2k} - \triangle; \mathbb{Q})$, we have that $k + 1 = 0$ and $k - 1 = 0$. These two formulas cannot happen at the same time.

□

2.2.2 Constructing sections when $n > 2$

Let

$$\mathrm{PConf}_{n,k}(S^2) = \{(v_1, x_1, ..., x_n) | x_1, ..., x_n \text{ be } n \text{ points in } S^2 \text{ and } v_k \text{ be a unit vector at } x_k\}.$$

This is the total space of a circle bundle by forgetting the vector:

$$S^1 \to \mathrm{PConf}_{n,k}(S^2) \to \mathrm{PConf}_n(S^2). \tag{2.2.1}$$

Proposition 2.15. *For $n > 2$, the fiber bundle (2.2.1) is a trivial bundle.*

Proof. Any S^1 bundle is classified by Euler class, i.e. a second cohomology class of the base. We investigate $H^2(\mathrm{PConf}_n(S^2); \mathbb{Z})$ first. There is a graded-commutative $\mathbb{Q}$-algebra $[G_{ij}]$ defined in [35, Theorem 1], where the degree of the generators G_{ij} is 1. By Totaro

[35, Theorem 1], there is a spectral sequence $E_2^{p,q} = H^p((S^2)^n;\mathbb{Q})[G_{ij}]^q$ converging to $H^*(\mathrm{PConf}_n(S^2);\mathbb{Q})$. Since we only compute H^2, the differential involved is $d_2 : E_2^{0,1} = H^0(S^2;\mathbb{Q})[G_{ij}] \to E_2^{2,0} = H^2(S^2;\mathbb{Q})$. Let $[\triangle_{ij}] \in H^2(S^2;\mathbb{Q})$ be the Poincaré dual of $\triangle_{ij} \subset S^2$. By [35, Theorem 2], the differential $d_2(G_{ij}) = [\triangle_{ij}]$. Let $p_i : (S^2)^n \to S^2$ be the projection to the ith coordinate and $[S^2] \in H^2(S^2;\mathbb{Z})$ be the generator of $H^2(S^2;\mathbb{Z})$. Therefore we have that

$$H^2(\mathrm{PConf}_n(S^2);\mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}p_i^*[S^2]/(p_i^*[S^2]+p_j^*[S^2]) \cong \mathbb{Z}/2,$$

which is generated by $p_k^*[S^2]$ and we have that $2p_k^*[S^2] = 0$. The circle bundle (2.2.1) is induced from the circle bundle

$$S^1 \to \mathrm{PConf}_{1,1}(S^2) \to S^2 \tag{2.2.2}$$

by the projection to the kth coordinate. The bundle (2.2.2) is the unit tangent bundle over S^2. Since the Euler characteristic of S^2 is 2, the Euler class of (2.2.2) is $eu = 2[S^2] \in H^2(S^2;\mathbb{Z})$. Therefore the Euler class of (2.2.1) is $p_k^*[eu] = 2p_k^*[S^2] = 0 \in H^2(\mathrm{PConf}_n(S^2);\mathbb{Z})$.

□

Equip S^2 with the spherical metric; i.e. the metric that is induced from the standard embedding $S^2 \subset \mathbb{R}^3$. Let

$$\epsilon(x_1,...,x_n) = \frac{1}{2}\min_{1\le i<j\le n}\{d(x_i,x_j)\}.$$

Set x_0 to be the image of the v_k-flow at time ϵ from x_k; that is

$$em_{n,k}(S^2) : \mathrm{PConf}_{n,k}(S^2) \hookrightarrow \mathrm{PConf}_{n+1}(S^2)$$

Composing a continuous section $s : \mathrm{PConf}_n(S^2) \to \mathrm{PConf}_{n,k}(S^2)$ of the fiber bundle (2.2.1) with $em_{n,k}(S^2)$ gives a section of the fiber bundle $f_n(S^2)$.

Definition 2.16 (Adding a point near x_k). We denote by $\mathrm{Add}_{n,k}(S^2)$ the collection of sections of $f_n(S^2)$ consisting of compositions of a section of (2.2.1) with $em_{n,k}(S^2)$.

Notice that there are infinitely many homotopy classes of sections in $\mathrm{Add}_{n,k}(S^2)$ and they are classified by sections of (2.2.1).

A special section for $n = 3$. Since there is a unique Mobius transformation $\phi(x_1, x_2, x_3)$ that transforms $(0, 1, \infty)$ to any ordered three points (x_1, x_2, x_3). we have that

$$\mathrm{PConf}_3(S^2) \xrightarrow[\approx]{\phi} \mathrm{PSL}(2, \mathbb{C}).$$

We can assign any new point $x_0 = \phi(x_1, x_2, x_3))(a)$ such that $a \neq 0, 1, \infty$.

2.2.3 The proof of (2) of Theorem 1.1

In this subsection we prove (2) of Theorem 1.1. Let $S_{0,n}$ a sphere with n punctures. Let $\mathrm{Diff}(S_{0,n})$ be the orientation-preserving diffeomorphism group of $S_{0,n}$ fixing the n punctures pointwise. While the following is surely known to experts, we could not find this statement or a proof in the literature. I am thus incluing it for completeness. We believe that it follows from Earle-Eells [10, Theorem 1] in the punctured case.

Proposition 2.17. *For $n > 2$, we have that*

$$\mathrm{BDiff}(S_{0,n}) \cong K(\mathrm{PMod}(S_{0,n}), 1)$$

Proof. We only need to prove that the homotopy group $\pi_k(\mathrm{Diff}(S_{0,n})) = 0$ for $k > 0$. For

$n = 0$, by Smale [34, Theorem A], $\mathrm{Diff}(S^2) \simeq \mathrm{SO}(3)$. By fiber bundle

$$\mathrm{Diff}(S_{0,n+1}) \to \mathrm{Diff}(S_{0,n}) \to S_{0,n}, \tag{2.2.3}$$

we deduce that $\mathrm{Diff}(S_{0,1}) \simeq \mathrm{SO}(2)$ and $\mathrm{Diff}(S_{0,2}) \simeq \mathrm{SO}(2)$. The long exact sequence of homotopy groups of the fiber bundle (2.2.3) is

$$1 \to \pi_1(\mathrm{Diff}(S_{0,3})) \to \pi_1(\mathrm{Diff}(S_{0,2})) \to \pi_1(S_{0,2}) \to \mathrm{PMod}_{0,3} \to \mathrm{PMod}_{0,2} \to 1.$$

However we know that $\mathrm{PMod}_{0,3} = 1$ (see [16, Proposition 2.3]), we get that $\pi_1(\mathrm{Diff}(S_{0,3})) = 0$. The other cases are the same. □

Let $PB_n(S^2) = \pi_1(\mathrm{PConf}_n(S^2))$. Now we are ready to prove (2) of Theorem 1.1

Proof of (2) of Theorem 1.1. Let

$$S_{0,n+1} \to \mathrm{UDiff}(S_{0,n+1}) \xrightarrow{u_{n+1}} \mathrm{BDiff}(S_{0,n+1})$$

be the universal $S_{0,n+1}$-bundle in the sense that any S^2 bundle with $n+1$ sections

$$S_{0,n+1} \to E \to B$$

is pullback from u_{n+1} by a continuous map $f : B \to \mathrm{BDiff}(S_{0,n+1})$. By Proposition 2.17, $\mathrm{BDiff}(S_{0,n+1}) \cong K(\mathrm{PMod}(S_{0,n+1}), 1)$. This means that $\mathrm{UDiff}(S_{0,n+1})$ is also a $K(\pi, 1)$-space. Therefore $S_{0,n+1}$-bundles are determined by their monodromy representations and the sections of an $S_{0,n+1}$-bundle are also determined by the maps on fundamental groups. A splitting of the following exact sequence will gives us a section of the fiber bundle $f_n(S^2)$.

$$1 \to F_n \to PB_{n+2}(S^2) \xrightarrow{f_n(S^2)_*} PB_{n+1}(S^2) \to 1.$$

We have the following diagram:

$$
\begin{array}{ccccc}
S_{0,n+1} & \longrightarrow & \mathrm{PConf}_{n+1}(S_{0,1}) & \xrightarrow{f_n(S_{0,1})} & \mathrm{PConf}_n(S_{0,1}) \\
\downarrow & & \downarrow & & \downarrow \\
S_{0,n+1} & \longrightarrow & \mathrm{PConf}_{n+2}(S^2) & \xrightarrow{f_{n+1}(S^2)} & \mathrm{PConf}_{n+1}(S_0) \\
 & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle p_{n+1}} \\
 & & S^2 & \longrightarrow & S^2.
\end{array}
$$

By the long exact sequence of homotopy groups of the fiber bundle

$$\mathrm{PConf}_n(S_{0,1}) \to \mathrm{PConf}_{n+1}(S^2) \to S^2,$$

we have that $PB_{n+1}(S^2) = PB_n/Z$ where Z denotes the center of PB_n and is generated by the Dehn twist about the boundary of D_n; see [16, Page 247]. Therefore a section of $f_n(S_{0,1})$ induced from a section of $f_{n+1}(S^2)$ satisfies that $f_n(S_{0,1})_*$ maps the center to the center.

Since $S_{0,1} \approx \mathbb{R}^2$, the section problem for $f_n(S_{0,1})$ has been fully discussed in Section 2. Every section of $f_{n+1}(S^2)$ induces a section of $f_n(\mathbb{R}^2)$, thus we could use the classification of sections of $f_n(\mathbb{R}^2)$ to study the sections of $f_{n+1}(S^2)$. Let $s : PB_{n+1}(S^2) \to PB_{n+2}(S^2)$ be a splitting of $f_{n+1}(S^2)_*$ such that $f_{n+1}(S^2)_* \circ s = id$. By (1) of Theorem 1.1, we break the discussion into the following two cases according to the sections of $f_n(S_{0,1})$.

Case 1: the section of $f_n(S_{0,1})$ is adding a point near x_k. In this case, $s(PB_{n+1}(S^2))$ fixes a curve c around $\{x_0, x_k\}$. Then the image lies in the stabilizer of c. The stabilizer of c in $\mathrm{PMod}(S_{0,n+2})$ is $\mathrm{PMod}(D_n) \cong PB_n$. The boundary of D_n is c surrounding $\{x_1, ..., x_{k-1}, x_{k+1}, ..., x_n$
On the other hand by Proposition 2.15 the circle bunlde

$$S^1 \to \mathrm{PConf}_{n,k}(S^2) \to \mathrm{PConf}_n(S^2)$$

is trivial, we have that

$$\pi_1(\mathrm{PConf}_{n+1,k}(S^2)) \cong \mathbb{Z} \times \pi_1(\mathrm{PConf}_{n+1}(S^2)) \cong \mathbb{Z} \times PB_{n+1}(S^2) \cong PB_n.$$

The last isomorphism is coming from the splitting of the following exact sequence; see [16, Page 252].

$$1 \to Z \to PB_n \to PB_n/Z \to 1.$$

Since the $\mathbb{Z}$ component of $\pi_1(\mathrm{PConf}_{n+1,k}(S^2))$ is mapped to the Dehn twist about a curve d surrounding $\{x_0, x_k\}$ thus d also surrounds $\{x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}\}$. Therefore we have that $f_{n+1}(S^2)$ is adding a point near x_k.

Case 2: the section of $f_n(S_{0,1})$ is adding a point near ∞. In this case, $s(PB_{n+1}(S^2))$ fixes a curve c around $\{x_1, ..., x_n\}$. Then the image lies in the stabilizer of c. The stabilizer of c in $\mathrm{Mod}(S_{0,n+2})$ is $\mathrm{PMod}(D_n) \cong PB_n$. The boundary of D_n is c surrounding $\{x_1, ..., x_n\}$. On the other hand by Proposition 2.15 the circle bunlde

$$S^1 \to \mathrm{PConf}_{n+1,n+1}(S^2) \to \mathrm{PConf}_{n+1}(S^2).$$

is trivial, we have that

$$\pi_1(\mathrm{PConf}_{n+1,n+1}(S^2)) \cong \mathbb{Z} \times \pi_1(\mathrm{PConf}_{n+1}(S^2)) \cong \mathbb{Z} \times PB_{n+1}(S^2) \cong PB_n.$$

Since the $\mathbb{Z}$ component of $\pi_1(\mathrm{PConf}_{n+1,k}(S^2))$ is mapped to the Dehn twist about a curve d surrounding $\{x_0, x_{n+1}\}$, thus d also surrounds $\{x_1, ..., x_n\}$. Therefore we have that $f_{n+1}(S^2)$ is adding a point near x_{n+1}.

□

2.2.4 The unordered case

Proof of (2) of Corollary 1.2. By the same argument as the proof of (1) of Corollary 1.2, we show that none of the sections of

$$f_n(S^2) : \mathrm{PConf}_{n+1}(S^2) \to \mathrm{PConf}_n(S^2)$$

can be extend to a section of

$$F_n(S^2) : \mathrm{PConf}_{n+1}(S^2)/\Sigma_n \to \mathrm{PConf}_n(S^2)/\Sigma_n.$$

$\square$

2.2.5 The exceptional cases

For the special cases $n = 3$, we have the following classification.

Theorem 2.18 (Classification of sections of $f_3(S^2)$ and $F_3(S^2)$). *There is a unique section for the fiber bundle $f_3(S^2)$ up to homotopy. There is no section for the bundle $F_3(S^2)$.*

Proof. By Proposition 2.17, we have that $\mathrm{BDiff}(S_{0,3}) \cong K(\mathrm{PMod}(S_{0,3}), 1)$. Since $\mathrm{PMod}(S_{0,3}) = 1$, the classifying space $\mathrm{BDiff}(S_{0,3})$ is contractible. Therefore every S^2-bundle with 3 sections is a trivial bundle. Thus $f_3(S^2)$ is a trivial bundle. Therefore, a section of $f_3(S^2)$ is determined by a map $\mathrm{PConf}_3(S^2) \to S_{0,3}$. Since $S_{0,3} \cong K(F_2, 1)$, a map $\mathrm{PConf}_3(S^2) \to S_{0,3}$ up to homotopy is determined by $\mathrm{Hom}(PB_3(S^2), F_2)$ up to conjugacy. However $PB_3(S^2) = PB_2/Z = 1$ implying that $\mathrm{Hom}(PB_3(S^2), F_2) = 1$. Therefore, there is a unique section up to homotopy. For the unordered case $F_3(S^2)$, let $\mathrm{Mod}(S_{0,3,1})$ be the mapping class group of S^2 fixing a set of 3 points and a set of 1 point. There is an exact sequence

$$1 \to \mathrm{PMod}(S_{0,4}) \to \mathrm{Mod}(S_{0,3,1}) \to \Sigma_3 \to 1. \tag{2.2.4}$$

Since $\pi_1(\mathrm{PConf}_3(S^2)/\Sigma_3) \cong \Sigma_3$ and $\pi_1(S_{0,3}) \cong \mathrm{PMod}(S_{0,4})$, we have that $\mathrm{Mod}(S_{0,3,1}) = \pi_1(\mathrm{PConf}_4(S^2)/\Sigma_3)$. Therefore, the section of $F_3(S^2)$ is the determined by the splittings of the exact sequence (2.2.4).

Let $\overline{\mathrm{Diff}}(S_{0,3,1})$ be the orientation-preserving diffeomorphism group of S^2 fixing a set of 3 points and a set of 1 point. By definition there is a map $\rho : \overline{\mathrm{Diff}}(S_{0,3,1}) \to \mathrm{Mod}(S_{0,3,1})$ which induces isomorphism on π_0. A version of the Nielsen Realisation Theorem (e.g. [16, Theorem 7.2] and [37]) tells us that a finite subgroup of $\mathrm{Mod}(S_{0,3,1})$ has a lift to $\overline{\mathrm{Diff}}(S_{0,3,1})$. However every finite subgroup of $\overline{\mathrm{Diff}}(S_{0,3,1})$ is cyclic because $\overline{\mathrm{Diff}}(S_{0,3,1})$ fixes a point. Therefore every finite subgroup of $\mathrm{Mod}(S_{0,3,1})$ is cyclic. Since Σ_3 is noncyclic, (2.2.4) does not split. □

For the special cases $n = 4$, we have the following classification.

Theorem 2.19 (Classification of sections of $f_4(S^2)$). *The sections of fiber bundle $f_4(S^2)$ correspond to the splittings of the exact sequence*

$$1 \to F_3 \to PB_5(S^2) \xrightarrow{f_4(S^2)_*} F_2 \to 1$$

up to conjugacy.

Proof. We have the following Birman exact sequence; see [16, Theorem 4.6].

$$1 \to \pi_1(S_{0,3}) \to PB_4(S^2) \xrightarrow{f_3(S^2)_*} PB_3(S^2) \to 1.$$

Since $PB_3(S^2) = 1$, we have that $PB_4(S^2) = \pi_1(S_{0,3}) \cong F_2$. By Proposition 2.17, the sections of $f_4(S^2)$ is determined by the splittings of the following Birman exact sequence up to conjugacy.

$$1 \to \pi_1(S_{0,4}) \to PB_5(S^2) \xrightarrow{f_4(S^2)_*} PB_4(S^2) \to 1. \tag{2.2.5}$$

□

2.3 The case when $S = S_g$ a closed surface of genus $g > 1$

In this subsection we give a construction of sections of the fiber bundle $f_n(S_g)$.

Let S_g^n be the product of n copies of S_g. There is a natural embedding $\mathrm{PConf}_n(S_g) \subset S_g^n$. Let $p_i : \mathrm{PConf}_n(S_g) \to S_g$ be the projection onto the ith component. Denote by $\triangle_{ij} \approx S_g^{n-1} \subset S_g^n$ the ijth diagonal subspace of S_g^n, i.e. $\triangle_{ij}$ consists of points in S_g^n such that the ith and jth coordinates are equal. Let $H_i := p_i^* H^1(S_g;\mathbb{Q})$ and let $[S_g]$ be the fundamental class in $H^2(S_g;\mathbb{Q})$. Now, we display the computation of $H^*(\mathrm{PConf}_n(S_g);\mathbb{Q})$ from [8].

Lemma 2.20. *(1) For $g > 1$ and $n > 0$,*

$$H^1(\mathrm{PConf}_n(S_g);\mathbb{Q}) \cong H^1(S_g^n;\mathbb{Q}) \cong \bigoplus_{i=1}^n H_i. \tag{2.3.1}$$

(2)We have an exact sequence

$$1 \to \oplus_{1\le i<j\le n}\mathbb{Q}[G_{ij}] \xrightarrow{\phi} H^2(S_g^n;\mathbb{Q}) \cong \bigoplus_{i=1}^n \mathbb{Q}p_i^*[S_g] \oplus \bigoplus_{i\neq j} H_i \otimes H_j \xrightarrow{Pr} H^2(PConf_n(S_g);\mathbb{Q}), \tag{2.3.2}$$

where $\phi(G_{ij}) = [\triangle_{ij}] \in H^2(S_g^n;\mathbb{Q})$ is the Poincaré dual of the diagonal $\triangle_{ij}$.

Proof. See [8, Lemma 3.1]. □

Let $\{a_k, b_k\}_{k=1}^g$ be a symplectic basis for $H^1(S_g;\mathbb{Q})$. For $1 \le i,j \le m$, we denote

$$M_{i,j} = \sum_{k=1}^n p_i^* a_k \otimes p_j^* b_k - p_i^* b_k \otimes p_j^* a_k.$$

Lemma 2.21. *The diagonal element* $[\triangle_{ij}] = p_i^*[S_g] + p_j^*[S_g] + M_{ij} \in \bigoplus_{i=1}^n \mathbb{Q}p_i^*[S_g] \oplus \bigoplus_{i\neq j} H_i \otimes H_j \cong H^2(S_g^n;\mathbb{Q})$.

Proof. See [8, Lemma 3.2]. □

The following lemma is the classification of homomorphisms $\pi_1(\mathrm{PConf}_n(S_g)) \to \pi_1(S_g)$ from [8].

Theorem 2.22 (The classification of homomorphisms $\pi_1(\mathrm{PConf}_n(S_g)) \to \pi_1(S_g)$). *Let $g > 1$ and $n > 0$. Let $R : \pi_1(PConf_n(S_g)) \to \pi_1(S_g)$ be a homomorphism. The followings hold:*

(1)If R is surjective, then $R = A \circ p_{i*}$ for some i and A an automorphism of $\pi_1(S_g)$.

(2)If Image(R) is not a cyclic group, the homomorphism $\pi_1(PConf_n(S_g)) \to \pi_1(S_g)$ factors through p_{i*} for some i.

Proof. See [8, Theorem 1.5]. □

Now, we are ready to prove (3) of Theorem 1.1.

Proof of (3) of Theorem 1.1. Suppose that there is a map $s : \mathrm{PConf}_n(S_g) \to \mathrm{PConf}_{n+1}(S_g)$ such that $f_n(S_g) \circ s =$ identity. Then after post-composing with a forgetful map to the last coordinate, we get a map $f : \mathrm{PConf}_n(S_g) \to S_g$. We denote

$$g_i : \mathrm{PConf}_n(S_g) \xrightarrow{(f,p_i)} \mathrm{PConf}_2(S_g) \subset S_g \times S_g$$

Let $\triangle \subset S_g \times S_g$ be the diagonal subspace. Let $[\triangle] \in H^2(S_g \times S_g;\mathbb{Q})$ be the Poincaré dual of $\triangle$. Let $f^* : H^1(S_g) \to H^1(\mathrm{PConf}_n(S_g))$. Let $f_* : \pi_1(\mathrm{PConf}_n(S_g)) \to \pi_1(S_g)$ be the induced map on the fundamental groups. By Lemma 2.22, either f_* factors though a forgetful map p_{i*} or Image(f_*) $\cong \mathbb{Z}$. We break the proof into two cases according to the image of f_*.

Case 1: Image(f_*) $\cong \mathbb{Z}$. There are two subcases:

(1) If $f^*=0$, then $g_i^*([\triangle])=p_i^*[S_g]\neq 0$. This contradicts the fact that the image of g_i misses $\triangle$.

(2) If $f^*\neq 0$, then $\mathrm{Im}f^*\cong\mathbb{Z}$. We can assume that there exists a symplectic basis $\{a_k,b_k\}_{k=1}^g$ for $H^1(S_g;\mathbb{Q})$ such that $f^*(a_i)=0$ for any $i\neq 1$ and $f^*(b_i)=0$ for any i. Let $f^*(a_1)=(x_1,x_2,...,x_n)\neq 0\in\bigoplus_{i=1}^n H_i\cong H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$. Assume without loss of generality that $x_1\neq 0$. Therefore for $k\neq 1$ by Lemma 2.21, we have that

$$g_k^*([\triangle])=p_k^*[S_g]+x_k\smile p_k^*b_1+\sum_{i=1,i\neq k}^n x_i\otimes p_k^*b_1\in\bigoplus_{i=1}^n\mathbb{Q}p_i^*[S_g]\oplus\bigoplus_{i\neq j}H_i\otimes H_j\cong H^2(S_g^n;\mathbb{Q}).$$

The coordinate $x_1\otimes p_k^*b_1$ is not zero, therefore $g_k^*([\triangle])\neq 0$. This contradicts the fact that the image of g_i misses $\triangle$.

Case 2: f_* factors though the forgetful map p_{1*}. We have that

$$g_2^*([\triangle])=f^*[S_g]+p_2^*[S_g]+\sum_k f^*a_k\smile p_2^*b_k-f^*b_k\smile p_2^*a_k.$$

Since $\mathrm{Image}(f^*)\subset\mathrm{Image}(p_1^*)$, we have that $g_2^*([\triangle])$ only has the terms in $\mathbb{Q}G_{12}\oplus H_1\otimes H_2$. The fact that g_2 misses $\triangle$ implies

$$f^*[S_g]+p_2^*[S_g]+\sum_k f^*a_k\otimes p_2^*b_k-f^*b_k\otimes p_2^*a_k=\lambda([\triangle_{12}])\in\mathbb{Q}p_1^*[S_g]\oplus\mathbb{Q}p_2^*[S_g]\oplus H_1\otimes H_2.$$

The coefficient of $p_2^*[S_g]$ tells us that $\lambda=1$. Therefore we have that $f^*[S_g]=p_1^*[S_g]$ and

$$\sum_k(f^*a_k-p_1^*a_k)\otimes p_2^*b_k-(f^*b_k-p_1^*b_k)\otimes p_2^*a_k=0\in H_1\otimes H_2$$

By the property of tensor product, we know that $f^*a_k-p_1^*a_k=0$ and $f^*b_k-p_1^*b_k=0$.

However in this case, if we look at the map $g_1 : \mathrm{PConf}_n(S_g) \xrightarrow{(f,p_1)} S_g \times S_g$. We have that

$$g_1^*([\triangle]) = f^*[S_g]+p_1^*[S_g]+\sum_k f^*a_k \smile p_1^*b_k - f^*b_k \smile p_1^*a_k = 2p_1^*[S_g]-2gp_1^*[S_g] = (2-2g)p_1^*[S_g] \neq 0.$$

This contradicts the fact that the image of g_1 misses $\triangle$

□

2.4 Further questions

In this section we list a few further questions. Let m, n be two positive integers. Let $(x_1, ..., x_n) \in \mathrm{PConf}_n(S)$ for any manifold S. Let the permutation group Σ_m acts on $\mathrm{PConf}_{n+m}(S)$ by permuting the last m points. We have the following fiber bundle:

$$\mathrm{PConf}_m(S - \{x_1, ..., x_n\})/\Sigma_m \to \mathrm{PConf}_{n+m}(S)/\Sigma_m \xrightarrow{f_{n+m,n}(S)} \mathrm{PConf}_n(S). \qquad (2.4.1)$$

Here denote by $f_{n+m,n}(S)$ the forgetful map that forgets the first n points. A section of the fiber bundle (2.4.1) is called a multi-section.

Main Problem. Classify the continuous sections of the fiber bundle (2.4.1) up to homotopy for S a surface.

Main Problem. Classify the continuous sections of the fiber bundle (2.4.1) up to homotopy for any manifold S.

CHAPTER 3

SECTIONS OF N-POINTED SURFACE BUNDLES

3.1 The translation of the problem into a group-theoretical problem

In this section, we translate the problem of finding a section of the universal surface bundle into a purely group-theoretical problem of finding homomophisms of mapping class groups.

3.1.1 The translation of the section problem

In this subsection, we will translate the problem of finding a section of a surface bundle into a purely group-theoretical problem of finding homomophisms of groups.

Let $\mathrm{Diff}(S_g)$ denote the orientation-preserving diffeomorphism group of a surface S_g of genus $g > 1$. We have the universal $\mathrm{Diff}(S_g)$ principal bundle

$$\mathrm{Diff}(S_g) \to \mathrm{EDiff}(S_g) \to \mathrm{BDiff}(S_g).$$

Here $\mathrm{EDiff}(S_g)$ is the total space of the universal $\mathrm{Diff}(S_g)$ bundle, i.e. a contractible principal $\mathrm{Diff}(S_g)$ bundle. Let $\mathrm{UDiff}(S_g) = \mathrm{EDiff}(S_g) \times_{\mathrm{Diff}(S_g)} S_g$ be the universal surface bundle

$$S_g \to \mathrm{UDiff}(S_g) \xrightarrow{u_g} \mathrm{Diff}(S_g).$$

$\mathrm{BDiff}(S_g)$ classifies surface bundles, which means that any surface bundle $S_g \to E \to B$ is the pullback of u_g via a continuous map $f_C : B \to \mathrm{BDiff}(S_g)$. Let $\mathrm{Mod}_{g,n}$ (resp. $\mathrm{PMod}_{g,n}$) be the *mapping class group* (resp. *pure mapping class group*) of $S_{g,n}$, i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of S_g fixing n punctures as a set (resp. pointwise). We omit n when $n = 0$.

Earle and Eells [10, Theorem 1] says that $\mathrm{Diff}_0(S_g)$, i.e. the identity component of $\mathrm{Diff}(S_g)$, is contractible for $g > 1$. Therefore we have $\mathrm{BDiff}(S_g) = K(\mathrm{Mod}_g, 1)$. By the property of $K(\pi, 1)$ space, $f : B \to \mathrm{BDiff}(S_g)$ is determined by the monodromy representation

$$f_* : \pi_1(B) \to \mathrm{Mod}_g.$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{Conjugacy classes of} \\ \text{representations} \\ f : \pi_1(B) \to \mathrm{Mod}_g \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes} \\ \text{of oriented} \\ S_g\text{-bundles over } B \end{array} \right\}. \tag{3.1.1}$$

Let $\mathrm{Diff}(S_{g,1})$ be the orientation-preserving diffeomorphism group of a surface S_g of genus $g > 1$ fixing one point. There is a natural inclusion $\mathrm{Diff}(S_{g,1}) \hookrightarrow \mathrm{Diff}(S_g)$.

Proposition 3.1. *For $g > 1$,*

$$UDiff(S_g) = BDiff(S_{g,1}).$$

Proof.

$$\begin{aligned} \mathrm{UDiff}(S_g) &= \mathrm{EDiff}(S_g) \times_{\mathrm{Diff}(S_g)} S_g && \text{By definition} \\ &= \mathrm{EDiff}(S_g) \times_{\mathrm{Diff}(S_g)} \mathrm{Diff}(S_g)/\mathrm{Diff}(S_{g,1}) && \text{Because } S_g = \mathrm{Diff}(S_g)/\mathrm{Diff}(S_{g,1}) \\ &= \mathrm{EDiff}(S_g)/\mathrm{Diff}(S_{g,1}) && \mathrm{Diff}(S_{g,1}) \text{ is a subgroup of } \mathrm{Diff}(S_g) \\ &= \mathrm{BDiff}(S_{g,1}). && \mathrm{EDiff}(S_g) \text{ is contractible} \end{aligned}$$

□

Proposition 3.1 implies that the universal surface bundle is

$$S_g \to K(\mathrm{Mod}_{g,1}, 1) \to K(\mathrm{Mod}_g, 1). \tag{3.1.2}$$

The fundamental groups of surface bundle (3.1.2) gives the following short exact sequence.

$$1 \to \pi_1(S_g) \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g \to 1. \tag{3.1.3}$$

Question 3.2 (Section problems). *For a surface bundle $S_g \to E \xrightarrow{f} B$, denote by $\rho : \pi_1(B) \to \mathrm{Mod}_g$ the monodromy representation of f. The fundamental groups of surface bundle f gives the following short exact sequence.*

$$1 \to \pi_1(S_g) \to \pi_1(E) \xrightarrow{f_*} \pi_1(B) \to 1. \tag{3.1.4}$$

How many splittings are there of exact sequence (3.1.4)?

It is well-known that exact sequence (3.1.3) has no splittings. This is $n = 0$ case of Theorem 1.8. The answer is no because of torsion, e.g. [16, Corollary 5.11]. The key fact is that there are noncyclic finite subgroups in Mod_g but there does not exist noncyclic finite subgroups in $\mathrm{Mod}_{g,1}$.

By the property of the pullback diagram, finding a splitting of f_* is the same as finding a homomorphism p making the following diagram commute, i.e. $\pi_1 \circ p = \rho$.

$$\begin{array}{ccc} \pi_1(E) & \longrightarrow & \mathrm{Mod}_{g,1} \\ \big\downarrow f_* & \overset{p}{\nearrow} & \big\downarrow \pi_{g,1} \\ \pi_1(B) & \xrightarrow{\rho} & \mathrm{Mod}_g. \end{array} \tag{3.1.5}$$

For a surface bundle $S_g \to E \xrightarrow{f} B$, we have the following correspondence:

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{continuous sections of} \\ S_g \to E \xrightarrow{f} B \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Homomorphisms } p \text{ satisfying diagram} \\ \text{(3.1.5) up to an conjugacy by an element} \\ \text{in } \mathrm{Ker}(\pi_{g,1}) \cong \pi_1(S_g) \end{array} \right\}. \tag{3.1.6}$$

Remark 3.3. *The conjugation is needed here for the lack of base points on the spaces. The classification of homomorphisms of fundamental groups classifies continuous maps fixing a point.*

3.1.2 The translation of Theorem 1.8 and 1.5

In this subsection, we translate Theorem 1.8 and 1.5 into group-theoretic theorems. We also study the section problem for the universal hyperelliptic surface bundle.

The mapping class groups

In this subsection, we translate Theorem 1.8 and 1.5 into group-theoretical theorems.

Let $\mathrm{Diff}(S_{g,n})$ denote the orientation-preserving diffeomorphism group of a surface S_g of genus $g > 1$ fixing n distinct points $\{x_1, x_2, ..., x_n\} \subset S_g$ pointwise. There is a fiber bundle

$$S_g \to \mathrm{UDiff}(S_{g,n}) \xrightarrow{u_{g,n}} \mathrm{BDiff}(S_{g,n}), \tag{3.1.7}$$

which is universal in the sense that any S_g-bundle endowed with n disjoint sections is a pullback of this bundle. Since $\mathrm{Diff}(S_{g,n})$ fixes the n points $x_1, x_2, ..., x_n$, we associate n points on each fiber, i.e. n disjoint sections of (3.1.7) which are denoted by $s_1, s_2, ..., s_n$.

Let $\mathrm{Diff}(S_{g,\overline{n}})$ denote the orientation-preserving diffeomorphism group of a surface S_g

of genus $g > 1$ fixing n points $\{x_1, x_2, ..., x_n\} \subset S_g$ as a set. There is a fiber bundle

$$S_g \to \mathrm{UDiff}(S_{g,\overline{n}}) \xrightarrow{u'_{g,n}} \mathrm{BDiff}(S_{g,\overline{n}}). \tag{3.1.8}$$

Since $\mathrm{Diff}_0(S_{g,n})$ and $\mathrm{Diff}_0(S_{g,\overline{n}})$ are contractible by Earle and Eells [10, Theorem 1], we have that $\mathrm{BDiff}(S_{g,n}) = K(\mathrm{PMod}_{g,n}, 1)$ and $\mathrm{BDiff}(S_{g,\overline{n}}) = K(\mathrm{Mod}_{g,n}, 1)$.

Let $\mathrm{PConf}_n(S_g)$ be the space of ordered n-tuple of distinct points on S_g. There is a natural permutation group Σ_n-free action on $\mathrm{PConf}_n(S_g)$. Let $\mathrm{Conf}_n(S_g) := \mathrm{PConf}_n(S_g)/\Sigma_n$ be the ordered n-tuple of distinct points on S_g. Let $PB_n(S_g) := \pi_1(\mathrm{PConf}_n(S_g))$ and $B_n(S_g) := \pi_1(\mathrm{Conf}_n(S_g))$ be the n-strand ordered and unordered *surface braid groups*. We have the following *Birman exact sequences* describing the monodromy representations of fiber bundle (3.1.7) and (3.1.8).

$$1 \to PB_n(S_g) \xrightarrow{\text{point pushing}} \mathrm{PMod}_{g,n} \xrightarrow{\pi_{g,n}} \mathrm{Mod}_g \to 1 \tag{3.1.9}$$

and

$$1 \to B_n(S_g) \xrightarrow{\text{point pushing}} \mathrm{Mod}_{g,n} \xrightarrow{\pi'_{g,n}} \mathrm{Mod}_g \to 1. \tag{3.1.10}$$

Because of correspondence (3.1.6), the classification of continuous sections of fiber bundle (3.1.7) and (3.1.8) is the same as the classification of homomorphisms p and p' up to conjugacy that make the following diagrams (3.1.11) and (3.1.12) commute.

$$\begin{array}{ccccccc}
1 \to PB_n(S_g) & \longrightarrow & \mathrm{PMod}_{g,n} & \xrightarrow{\pi_{g,n}} & \mathrm{Mod}_g & \longrightarrow & 1 \\
\Big\downarrow R & & \Big\downarrow p & & \Big\downarrow = & & \\
1 \to \pi_1(S_g) & \longrightarrow & \mathrm{Mod}_{g,1} & \xrightarrow{\pi_{g,1}} & \mathrm{Mod}_g & \longrightarrow & 1
\end{array} \tag{3.1.11}$$

and

$$\begin{array}{ccccccccc} 1 \to & B_n(S_g) & \longrightarrow & \mathrm{Mod}_{g,n} & \xrightarrow{\pi'_{g,n}} & \mathrm{Mod}_g & \longrightarrow & 1 \\ & \Big\downarrow R' & & \Big\downarrow p' & & \Big\downarrow = & & \\ 1 \to & \pi_1(S_g) & \longrightarrow & \mathrm{Mod}_{g,1} & \xrightarrow{\pi_{g,1}} & \mathrm{Mod}_g & \longrightarrow & 1. \end{array} \tag{3.1.12}$$

For p and p' satisfying diagrams (3.1.11) and (3.1.12), denote by R and R' the restrictions of p and p' on the subgroups $PB_n(S_g)$ and $B_n(S_g)$. Let $\mathrm{PMod}_{g,n} \xrightarrow{p_{g,n,i}} \mathrm{Mod}_{g,1}$ be the forgetful homomorphism that forgets the fixed points $\{x_1, ..., \hat{x_i}, ..., x_n\}$. Theorem 1.8 is thus equivalent to the following Theorem.

Theorem 3.4. *For $g > 2$ and $n \geq 0$, every homomorphism p satisfying diagram (3.1.11) is conjugate to $p_{g,n,i}$ for some i by an element A in $\pi_1(S_g)$.*

Theorem 1.5 is thus equivalent to the following.

Theorem 3.5. *For $g > 1$ and $n > 1$, there is no homomorphism p' satisfying diagram (3.1.12).*

The hyperelliptic mapping class groups

In this subsection, we translate the section problem of the hyperelliptic surface bundle into a group-theoretical statement. Let τ be the involution as in the following figure.

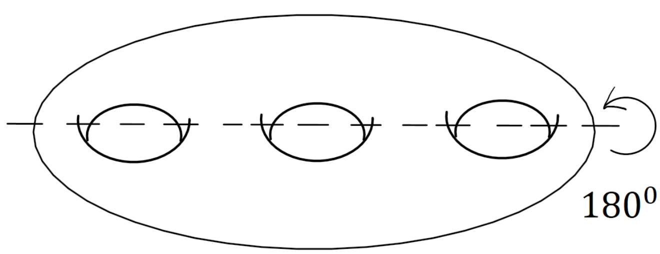


Figure 3.1.1: Hyperelliptic involution τ for $g = 3$ case

Let $\mathcal{H}_g$ be the *hyperelliptic mapping class group*, i.e. the subgroup of Mod_g consisting of all the mapping classes that are commutative with τ. Denote by $\mathcal{H}_{g,n}$ (resp. $\mathcal{PH}_{g,n}$)

the hyperelliptic mapping class group fixing n points as a set (resp. pointwise), i.e. they satisfy the following pullback diagrams.

$$\begin{array}{ccc} \mathcal{H}_{g,n} & \longrightarrow & \mathcal{H}_g \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Mod}_{g,n} & \longrightarrow & \mathrm{Mod}_g \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{PH}_{g,n} & \longrightarrow & \mathcal{H}_g \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{PMod}_{g,n} & \longrightarrow & \mathrm{Mod}_g. \end{array}$$

Let $\mathcal{BPH}_{g,n} = K(\mathcal{PH}_{g,n}, 1)$ be the *pure universal hyperelliptic space* fixing n punctures pointwise and let

$$S_g \to \mathcal{UPH}_{g,n} \xrightarrow{Hu_{g,n}} \mathcal{BPH}_{g,n} \tag{3.1.13}$$

be the *pure universal hyperelliptic bundle*, i.e. the bundle that corresponds to the monodromy $\rho_{H,g,n} : \mathcal{PH}_{g,n} \to \mathrm{PMod}_{g,n}$. Surface bundle (3.1.13) classifies smooth S_g-bundle equipped with a τ-action and n unordered points on each fiber. For any section s, we could generate another section $t = \tau(s)$. Denote by Hs_i the pullback of s_i as a section of bundle (3.1.13) and denote by Ht_i their hyperelliptic conjugates.

Let $\mathcal{BH}_{g,n} = K(\mathcal{H}_{g,n}, 1)$ be the *universal hyperelliptic space* fixing n punctures as a set and let

$$S_g \to \mathcal{UH}_{g,n} \xrightarrow{Hu'_{g,n}} \mathcal{BH}_{g,n} \tag{3.1.14}$$

be the *universal hyperelliptic bundle*, i.e. the bundle that corresponds to the monodromy $\rho'_{H,g,n} : \mathcal{H}_{g,n} \to \mathrm{PMod}_{g,n}$. Surface bundle (3.1.14) classifies smooth S_g-bundle equipped with a τ-action and n unordered points on each fiber. We have the following classification of sections for bundles (3.1.13) and (3.1.14).

Theorem 3.6 (Hyperelliptic case). *1) For $n \geq 0$ and $g > 1$, every section of the universal hyperelliptic undle (3.1.13) is homotopic to Hs_i or Ht_i for some $i \in \{1, 2, ..., n\}$.*

2) For $n > 1$ and $g > 1$, the universal hyperelliptic bundle (3.1.14) has no sections.

By correspondence (3.1.6), we can translate Theorem 3.6 into the following group-theoretical statement. Let $\mathcal{PH}_{g,n} \xrightarrow{H\pi_{g,n}} \mathcal{H}_g$ and $\mathcal{H}_{g,n} \xrightarrow{H\pi'_{g,n}} \mathcal{H}_g$ be the forgetful maps forgetting the punctures. Let $\mathcal{H}_{g,n} \xrightarrow{Hp_{g,n,i}} \mathrm{Mod}_{g,1}$ be the forgetful homomorphism forgetting the fixed points $\{x_1, ..., \hat{x_i}, ..., x_n\}$.

Proposition 3.7. *1) Every homomorphism p satisfying the following diagram is either conjugate to the forgetful homomorphism $Hp_{g,n,i}$ by an element in $\mathcal{PH}_{g,n}$ or factors through $H\pi_{g,n}$, i.e. there exists f such that $p = f \circ H\pi_{g,n}$.*

$$\begin{array}{ccccccc} 1 \to PB_n(S_g) & \longrightarrow & \mathcal{PH}_{g,n} & \xrightarrow{H\pi_{g,n}} & \mathcal{H}_g & \longrightarrow & 1 \\ \big\wr R & & \big\wr p & & \big\downarrow \rho_{H,g} & & \\ 1 \to \pi_1(S_g) & \longrightarrow & \mathrm{Mod}_{g,1} & \xrightarrow{H\pi_{g,1}} & \mathrm{Mod}_g & \longrightarrow & 1. \end{array} \tag{3.1.15}$$

2) For $n > 1$, every homomorphism p' satisfying the following diagram factors through $H\pi'_{g,n}$, i.e. there exists f' such that $p' = f' \circ H\pi'_{g,n}$

$$\begin{array}{ccccccc} 1 \to B_n(S_g) & \longrightarrow & \mathcal{H}_{g,n} & \xrightarrow{H\pi'_{g,n}} & \mathcal{H}_g & \longrightarrow & 1 \\ \big\wr R' & & \big\wr p' & & \big\downarrow \rho_{H,g} & & \\ 1 \to \pi_1(S_g) & \longrightarrow & \mathrm{Mod}_{g,1} & \xrightarrow{H\pi_{g,1}} & \mathrm{Mod}_g & \longrightarrow & 1. \end{array} \tag{3.1.16}$$

Proof of Theorem 3.6 assuming Proposition 3.7. By Proposition 3.7, p has the following two cases.

Case 1: p is conjugate to the forgetful homomorphism $Hp_{g,n,i}$ by an element $A \in \mathcal{PH}_{g,1}$.

By the commutativity of diagram (3.1.15), the mapping class $H\pi_{g,n}(A)$ is in the center of $\mathcal{H}_g$. Since $\mathrm{Center}(\mathcal{H}_g) = \langle \tau \rangle$, e.g. see [16, Section 3.4 and Section 9.4], we have that $\mathcal{H}\pi_{g,n}(A) = 1$ or τ, which represent section Hs_i and Ht_i.

Case 2: p factors throught $H\pi_{g,n}$.

To prove the result, we only need to show that the exact sequence

$$1 \to \pi_1(S_g) \to \mathcal{H}_{g,1} \to \mathcal{H}_g \to 1 \tag{3.1.17}$$

does not split. The following finite order mapping class σ is commutative with τ.

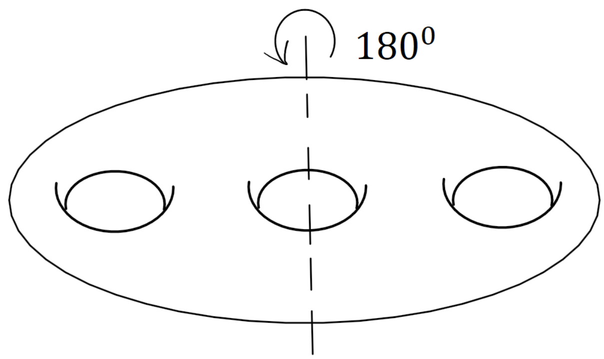


Figure 3.1.2: Torsion mapping class σ for $g = 3$ case

In $\mathcal{H}_g$, mapping classes τ and σ generate a $\mathbb{Z}/2 \times \mathbb{Z}/2$ subgroup; this contradicts the fact that every finite subgroup of $\mathrm{Mod}_{g,1}$ is cyclic. Therefore exact sequence (3.1.17) does not split.

□

3.2 The classification of homomorphisms $PB_n(S_g) \xrightarrow{R} \pi_1(S_g)$

This section is divided into three parts. We first compute $H^*(\mathrm{PConf}_n(S_g)); \mathbb{Q})$, then study an algebraic property of $H^*(\mathrm{PConf}_n(S_g)); \mathbb{Q})$, Finally we use the computation and the property to prove Theorem 1.7. The key idea is an argument of [24] that there is some cohomological constraint on the existence of homomorphisms $PB_n(S_g) \xrightarrow{R} \pi_1(S_g)$. We assume throughout that $g > 1$ and $n > 0$.

3.2.1 The computation of $H^*(PConf_n(S_g);\mathbb{Q})$

In this subsection, we compute $H^*(\mathrm{PConf}_n(S_{g,p});\mathbb{Q})$. Let S_g^n be the product of n copies of S_g. There is a natural embeddings $\mathrm{PConf}_n(S_g) \subset S_g^n$. Let $p_i : \mathrm{PConf}_n(S_g) \to S_g$ be the projection onto the ith component. Denote by $\triangle_{ij} \approx S_g^{n-1} \subset S_g^n$ be the ijth diagonal subspace of S_g^n, i.e. $\triangle_{ij}$ consists of points in S_g^n such that the ith and jth coordinates are equal.. Let $H_i := p_i^* H^1(S_g;\mathbb{Q})$ and let $[S_g]$ be the fundamental class in $H^2(S_g;\mathbb{Q})$.

Lemma 3.8. *1) For $g > 1$ and $n > 0$,*

$$H^1(\mathrm{PConf}_n(S_g);\mathbb{Q}) \cong H^1(S_g^n;\mathbb{Q}) \cong \bigoplus_{i=1}^{n} H_i. \tag{3.2.1}$$

2)We have an exact sequence

$$1 \to \oplus_{1\le i<j\le n}\mathbb{Q}[G_{ij}] \xrightarrow{\phi} H^2(S_g^n;\mathbb{Q}) \cong \bigoplus_{i=1}^{n} \mathbb{Q}p_i^*[S_g] \oplus \bigoplus_{i\neq j} H_i \otimes H_j \xrightarrow{Pr} H^2(PConf_n(S_g);\mathbb{Q}), \tag{3.2.2}$$

where $\phi(G_{ij}) = [\triangle_{ij}] \in H^2(S_g^n;\mathbb{Q})$ is the Poincaré dual of the diagonal $\triangle_{ij} \subset S_g^n$.

Proof. There is a graded-commutative $\mathbb{Q}$-algebra $[G_{ij}]$ defined in [35, Theorem 1], where the degree of the generators G_{ij} is 1. By Totaro [35, Theorem 1], there is a spectral sequence $E_2^{p,q} = H^p(S_g^n;\mathbb{Q})[G_{ij}]^q$ converging to $H^*(\mathrm{PConf}_n(S_g);\mathbb{Q})$. Since we only compute H^1 and H^2, the differential involved is $d_2 : E_2^{0,1} = H^0(S_g^n;\mathbb{Q})[G_{ij}] \to E_2^{2,0} = H^2(S_g^n;\mathbb{Q})$. Let $[\triangle_{ij}] \in H^2(S_g^n;\mathbb{Q})$ be the Poincaré dual of $\triangle_{ij} \subset S_g^n$. By [35, Theorem 2], the differential $d_2(G_{ij}) = [\triangle_{ij}]$. All the isomorphisms in the lemma are coming from the Künneth formula. $\square$

Let $\{a_k, b_k\}_{k=1}^{g}$ be a symplectic basis for $H^1(S_g;\mathbb{Q})$. For $1 \le i, j \le m$, we denote

$$M_{i,j} = \sum_{k=1}^{n} p_i^* a_k \otimes p_j^* b_k - p_i^* b_k \otimes p_j^* a_k.$$

The following lemma describes $[\triangle_{ij}] \in H^2(S_g^n;\mathbb{Q}) \cong \bigoplus_{i=1}^{n} \mathbb{Q} p_i^*[S_g] \oplus \bigoplus_{i \ne j} H_i \otimes H_j$.

Lemma 3.9. *The diagonal element* $[\triangle_{ij}] = p_i^*[S_g] + p_j^*[S_g] + M_{ij} \in \bigoplus_{i=1}^{n} \mathbb{Q} p_i^*[S_g] \oplus \bigoplus_{i \ne j} H_i \otimes H_j \cong H^2(S_g^n;\mathbb{Q})$.

Proof. This is classical. See [32, Section 11]. □

3.2.2 A property of the cup product structure of $H^*(PConf_n(S_g);\mathbb{Q})$

In this subsection, we talk about a property of the cup product $H^1 \otimes H^1 \to H^2$ for $\mathrm{PConf}_n(S_g)$.

Definition 3.10. We call an element $x = (x^1, ..., x^n) \in \bigoplus_{i=1}^{n} H_i = H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$ a *crossing element* if $\#\{i : x^i \ne 0\} > 1$, i.e. $x \notin H_i$ for any i.

Lemma 3.11. *Let* $x = (x^1, ..., x^n)$ *and* $y = (y^1, ..., y^n)$ *be two elements in* $H^1(PConf_n(S_g);\mathbb{Q})$. *Suppose that* x *or* y *is a crossing element. If* $x \smile y = 0 \in H^2(PConf_n(S_g);\mathbb{Q})$, *then* x *and* y *are proportional, i.e.* $\lambda x = \mu y$ *for some constants* $\lambda \in \mathbb{Q}$ *and* $\mu \in \mathbb{Q}$.

Proof. The multiplication of x and y is the following:

$$x \smile y = x^1 \smile y^1 + ... + x^n \smile y^n + \sum_{i \ne j} (x^i \otimes y^j - y^i \otimes x^j) \in \bigoplus_{i=1}^{n} \mathbb{Q} p_i^*[S_g] \oplus \bigoplus_{i \ne j} H_i \otimes H_j.$$

By $x \smile y = 0 \in H^2(\mathrm{PConf}_n(S_g);\mathbb{Q})$ and exact sequence (3.2.2), we have the following equality in $\bigoplus_{i=1}^{n} \mathbb{Q} p_i^*[S_g] \oplus \bigoplus_{i \ne j} H_i \otimes H_j$:

$$x^1 \smile y^1 + ... + x^n \smile y^n + \sum_{i \ne j} (x^i \otimes y^j - y^i \otimes x^j) = \sum k_{i,j}[\triangle_{i,j}] = \sum k_{i,j}(p_i[S_g] + p_j[S_g] + M_{i,j}).$$

By the independence of all the terms in $\mathbb{Q}^n \oplus \bigoplus_{i \neq j} H_i \otimes H_j$, we have

$$x^i \otimes y^j - y^i \otimes x^j = k_{i,j} M_{i,j} \text{ for all } i, j.$$

If x^i and y^i are proportional in H_i, since $g > 1$, we have at least 4 terms in $M_{i,j}$, we don't have enough basis to span our $M_{i,j}$. If x^i and y^i are independent in H_i, since $g > 1$, we have $x^i \otimes y^j - y^i \otimes x^j \neq M_{i,j}$. Therefore $k_{i,j} = 0$ and $x^i \otimes y^j - y^i \otimes x^j = 0 \in H_i \otimes H_j$. Assume without loss of generality that x a crossing element and $x_1 \neq 0$ and $x_2 \neq 0$. We break the proof into the following cases.

Case 1) $y^1 \neq 0$ and y^1 is not proportional to x^1

$x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $y^j = 0$ and $x^j = 0$ for j. However $x_2 \neq 0$. Therefore this case is invalid.

Case 2) $y^1 \neq 0$ and $\lambda x^1 = \mu y^1$

$x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $\lambda x^j = \mu y^j$ for all j, which verifies oi=1ur lemma that x and y are proportional.

Case 3) $y^1 = 0$

$x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $y^j = 0$ for all j. This means $y = 0$ therefore x and y are also proportional. □

3.2.3 The proof of Theorem 1.7

In this subsection, we use the computation of $H^*(\mathrm{PConf}_n(S_g); \mathbb{Q})$ and Lemma 3.11 to prove Theorem 1.7. Let $p_{i*} : PB_n(S_g) \to \pi_1(S_g)$ be the induced map on the fundamental groups of $p_i : \mathrm{PConf}_n(S_g) \to S_g$.

Lemma 3.12. *Let F_h be a free group of h generators and let S_r be a surface of genus r. If we have a surjective homomorphism $PB_n(S_g) \xrightarrow{S} \Gamma$ when $\Gamma = F_h$ with $h > 1$ or $\Gamma = \pi_1(S_r)$ with $r > 1$, and we also have $p_i^*(H^1(S_g; \mathbb{Q})) \cap S^*(H^1(\Gamma; \mathbb{Q})) \neq \{0\}$, then S factors* *through p_{i*} for some i.*

Proof. The proof of this lemma uses the same idea as [24]. The method can also be found in [33, Lemma 3.3 and 3.4]. If there is a common nonzero cohomology element $S^*(x) = p_{i*}^*(y)$ for $x \in H^1(F_h;\mathbb{Q})$ and $y \in H^1(\pi_1(S_g);\mathbb{Q})$, we have the following commutative diagram by the identification $H^1(\_\_;\mathbb{Q}) \cong \mathrm{Hom}(\_\_,\mathbb{Q})$.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{S} & F_h \\ \big\downarrow{\scriptstyle p_{i*}} & & \big\downarrow{\scriptstyle x} \\ \pi_1(S_g) & \xrightarrow{y} & \mathbb{Q} \end{array}$$

Let K be the kernel of p_{i*}, which is a finitely generated normal subgroup of $PB_n(S_g)$. The image of $S(K)$ is also a finite generated normal subgroup of $\pi_1(F_h)$. However every finitely generated normal subgroup of F_h either is finite index or is trivial. For a surface group of genus r case, any nontrivial finitely-generated normal subgroup of $\pi_1(S_r)$ has finite index; see Property (D6) in [24]. If $S(K) \subset F_h$ has finite index, then after composing with x, the image $x \circ S(K)$ won't be trivial in $\mathbb{Q}$; however K is the kernel of p_{i*} so $x \circ S(K) = y \circ p_{i*}(K) = 1$.

If the $S(K) = 1$, then S factors through p_{i*}. □

To prove Theorem 1.7, we have to include a lemma talking about the possible image of the homomorphism.

Lemma 3.13. *Every finitely generated subgroup of $\pi_1(S_g)$ is either finitely generated free group F_h or surface group $\pi_1(S_r)$ with $r \geq g$. When $r = g$, the subgroup is the whole group $\pi_1(S_g)$.*

Proof. A subgroup G of $\pi_1(S_g)$ corresponds to a cover S of S_g such that $G = \pi_1(S)$. If S is noncompact, then $\pi_1(S)$ is free group. If S is compact, it is a finite cover. Therefore $\pi_1(S) = S_r$ for some r. If S is a k-cover. The Euler characteristic is multiplicative under cover, thus $\chi(S_r) = k\chi(S_g)$. If $g > 1$ and $k > 1$, we have $r > g$. If $n = 1$, this is trivial cover. □

Proof of Theorem 1.7. Let $R : PB_n(S_g) \to \pi_1(S_g)$ be a homomorphism. By Lemma 3.13, if $\text{Im}(R) \cong \mathbb{Z}$, the image has to be F_h with $h > 1$ or $\pi_1(S_r)$ with $r \geq g$. Furthermore, if S does not factor through p_{i*} for some i, then by Lemma 3.12, $S^*(H^1(\text{Im}(R);\mathbb{Q}))$ does not intersect nontrivially with any H_i. This means that all nonzero elements of $S^*(H^1(\text{Im}(R);\mathbb{Q}))$ are crossing elements. However $r \geq g > 1$ and $h > 1$ mean that there are two crossing elements x and y in $S^*(H^1(\text{Im}(R);\mathbb{Q}))$ that are independent and their cup product is zero. Lemma 3.11 tells us that this is impossible, which successfully proves 2) of Theorem 1.7.

Now to prove 1), we have a surjection homomorphism $PB_n(S_g) \xrightarrow{p_{i*}} \pi_1(S_g) \xrightarrow{A} \pi_1(S_g)$. However surface groups are Hopfian which means that a surjective self homomorphism between the surface group $\pi_1(S_g)$ must be an automorphism. Therefore A is an automorphism, which concludes the proof of 1) in Theorem 1.7.

□

3.3 Applications of Theorem 1.7

In this section, we apply Theorem 1.7 to the study of section problems of universal surface bundles.

3.3.1 The proof of Theorem 3.4

Since we already established all possible homomorphisms R in Theorem 1.7, the key idea of extending the homomorphism to $\text{PMod}_{g,n}$ is that it has to be equivariant with the action of Mod_g. We then use homology to rule out other possibilities.

Definition 3.14. Let a subspace $H \subset H^1(\text{PConf}_n(S_g);\mathbb{Q}) \cong \bigoplus_{i=1}^n H_i$ be an *isotropic subspace* if any $a, b \in H$, we have $a \smile b = 0 \in H^2(\text{PConf}_n(S_g);\mathbb{Q})$.

The following lemma is needed in the proof.

Lemma 3.15. *Mod_g does not fix any isotropic subspace of $H^1(PConf_n(S_g);\mathbb{Q})$.*

Proof. If there exists a crossing element $x \in H$, because of Lemma 3.11, we know that $x \smile y = 0$ if and only if y is proportional to x. Therefore if H is isotropic, $H = \mathbb{Q}x \subset H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$.

If $\dim(H) > 1$, then H does not contain crossing elements by Lemma 3.11. In this case, if there exist $x, y \in H$ and $i \neq j \in \{1, 2, ..., n\}$ such that $x \neq 0 \in H_i$ and $y \neq 0 \in H_j$, we would have $x + y$ a crossing element. Therefore there exists i such that $H \subset H_i$.

Mod_g acts on $H^1(\mathrm{PConf}_n(S_g);\mathbb{Q}) \cong \bigoplus_{i=1}^n H_i$ by acting on each component. We know that the action of Mod_g on $H^1(S_g;\mathbb{Q})$ does not fix any isotropic subspace, therefore if $H \subset H_i$, Mod_g does not fix H. If $\dim(H) = 1$, Mod_g also does not fix it. □

Now we finish the proof of Theorem 3.4.

Proof of Theorem 3.4. If we can extend R, then for $e \in \mathrm{PMod}_{g,n}$ and $f \in PB_n(S_g)$ we have $R(efe^{-1}) = p(e)R(f)p(e)^{-1}$. The action of $\mathrm{PMod}_{g,n}$ and $\mathrm{Mod}_{g,n}$ on $PB_n(S_g)$ and $B_n(S_g)$ are given by conjugation in the exact sequence (3.1.9) and (3.1.10), respectively. Therefore we have a commutative diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \pi_1(S_g) & \xrightarrow{p(e)} & \pi_1(S_g). \end{array}$$

Since both e and $p(e)$ are isomorphisms of groups, we have that $\mathrm{Im}(R) = \mathrm{Im}(R \circ e)$. This gives us the following diagram:

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ \mathrm{Im}(R) & \xrightarrow{p(e)} & \mathrm{Im}(R). \end{array} \tag{3.3.1}$$

Because of Lemma 3.13, we know that we have 4 possibilities for $\mathrm{Im}(R)$: F_h for $h = 0$, $h > 0$ and $\pi_1(S_r)$ for $r = g$ or $r > g$. Now, we go over all possibilities.

Case 1) Im$(R) = 1$

In this case, we have a homomorphism $\mathrm{PMod}_{g,n}/PB_n(S_g) = \mathrm{Mod}_g \to \mathrm{Mod}_{g,1}$. However, the $n = 0$ case has already been proved, for example in [16, Corollary 5.11].

Case 2) Im$(R) = F_h$, while $h > 0$

We have the following diagram.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{e} & PB_n(S_g) \\ \downarrow R & & \downarrow R \\ F_h & \xrightarrow{p(e)} & F_h \end{array} \tag{3.3.2}$$

For every $e \in \mathrm{PMod}_{g,n}$, diagram (3.3.2) means that $R^*(H^1(F_h;\mathbb{Q})) \subset H^1(S_g;\mathbb{Q})$ has to be fixed under the action of Mod_g. This is impossible because $R^*(H^1(F_h;\mathbb{Q})) \subset H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$ is an isotropic subspace of $H^1(\mathrm{PCon}f_n(S_g);\mathbb{Q})$, but Mod_g does not fix any isotropic subspace.

Case 3) Im$(R) = \pi_1(S_g)$

If R is one of the forgetful homomorphism p_i, then for $e \in \mathrm{PMod}_{g,n}$ and $f \in PB_n(S_g)$,

$$p_i(efe^{-1}) = p(e)p_i(f)p(e)^{-1}.$$

We get that $p_i(e)p_i(f)p_i(e)^{-1} = p(e)p_i(f)p(e)^{-1}$. Therefore $p(e)^{-1}p_i(e)$ commutes with $p_i(f)$ for any $f \in PB_n(S_g)$. The image of p_i on $PB_n(S_g)$ is the whole group $\pi_1(S_g)$. Therefore, $p(e)^{-1}p_i(e) \in \mathrm{Mod}_{g,1}$ commutes with the subgroup $\pi_1(S_g)$. However, the centralizer of $\pi_1(S_g) < Mod_{g,1}$ is 1, so we get that $p(e)^{-1}p_i(e) = 1 \in \mathrm{Mod}_{g,1}$. This tells us that $p = p_i$.

If R is one of the forgetful homomorphism p_i post-composing with an automorphism A, with a similar argument as above, we get that $p(e) = Ap_i(e)A^{-1}$. Considering that the images of $Ap_i(e)A^{-1}$ and $p_i(e)$ have to be equal in Mod_g for any e, we have $Ap_i(e) = p_i(e)A$ for any $e \in \mathrm{Mod}_g$. Therefore, we have $A \in \mathrm{Center}(\mathrm{Mod}_g)$. For $g > 2$, $\mathrm{Center}(\mathrm{Mod}_g) = 1$, therefore we have $A \in \pi_1(S_g)$. For $g = 2$, we could have $A = \tau$.

Case 4) $\mathbf{Im}(R) = \pi_1(S_r)$ while $r > g$

Because of Lemma 1.7, R factors through p_i. However there is no surjective homomorphism from $\pi_1(S_g) \to \pi_1(S_r)$ since $\mathrm{Rank}(H^1(S_r;\mathbb{Q})) > \mathrm{Rank}(H^1(S_g;\mathbb{Q}))$.

□

3.3.2 The proof of Theorem 3.5

In this section, we begin with the proof of Theorem 3.5

Lemma 3.16.

$$H^1(Conf_n(S_g);\mathbb{Q}) \cong H^1(S_g;\mathbb{Q})$$

and the image of $H^1(Conf_n(S_g);\mathbb{Q}) \to H^1(PConf_n(S_g);\mathbb{Q})$ is equal to the image of the diagonal map, i.e. $x \to (x, x, ..., x) \in H^1(PConf_n(S_g);\mathbb{Q}) \cong \bigoplus_{i=1}^{n} H_i$.

Proof. Since $\mathrm{Conf}_n(S_g) = \mathrm{PConf}_n(S_g)/\Sigma_n$, we can use the transfer map to get

$$H^1(\mathrm{Conf}_n(S_g);\mathbb{Q}) = H^1(\mathrm{PCon}f_n(S_g);\mathbb{Q})^{\Sigma_n}.$$

It is not hard to see that $H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})^{\Sigma_n}$ is the diagonal subspace. □

Proof of Theorem 3.5. If we have a homomorphism p' in the diagram in Theorem 3.5, after composing an injection $\mathrm{PMod}_{g,n} \xrightarrow{i} \mathrm{Mod}_{g,n}$, we get a homomorphism p as in Theorem 3.4. Let C_A denote the conjugate by $A \in \pi_1(S_g)$. We have already proved Theorem 3.4 that $p' \circ i = C_A \circ p_i \in \pi_1(S_g)$ for some i and $A \in \pi_1(S_g)$. Restricting to the kernel of

$\pi_{g,n}$ and $\pi'_{g,n}$, the following diagram holds.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{i} & B_n(S_g) \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle R'} \\ \pi_1(S_g) & \xrightarrow{C_A} & \pi_1(S_g) \end{array}$$

The image of $H^1(S_g;\mathbb{Q}) \xrightarrow{(C_A \circ p_i)^*} H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$ is H_i; however the image of $H^1(\mathrm{Conf}_n(S_g);\mathbb{Q}) \to H^1(\mathrm{PConf}_n(S_g);\mathbb{Q})$ as described in the previous lemma is the diagonal. Thus this is a contradiction. □

3.3.3 The hyperelliptic case

In this subsection, we prove Proposition 3.7. The proof follows the same argument as the proof of Theorem 3.4. The following lemma is a key ingredient in the proof.

Lemma 3.17. *The action of $\mathcal{H}_g$ on $H^1(S_g;\mathbb{Q})$ does not preserve any isotropic subspace.*

Proof. Let $\mathrm{Sp}_{2g}(\mathbb{Z})[m]$ be the kernel of the map $\mathrm{Sp}_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/m)$. By [6, Theorem 3.3], the image of the monodromy representation $\rho_s : \mathcal{H}_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ contains $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$. Since $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ is a finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$, we only need to show that $\mathrm{Sp}_{2g}(\mathbb{Z})[2]$ does not preserve any isotropic subspace. We prove a stronger result that the stabilizer of any isotropic subspace of $H^1(S_g;\mathbb{Q})$ has infinite index in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

For an isotropic subspace $H \subset H^1(S_g;\mathbb{Q})$, let $\mathrm{Stab}_H(\mathrm{Sp}_{2g}(\mathbb{Z}))$ be the stabilizer of H in $\mathrm{Sp}_{2g}(\mathbb{Z})$ and let $\mathrm{Orb}_H(\mathrm{Sp}_{2g}(\mathbb{Z}))$ be the orbit of H under the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. We have the following equation:

$$[\mathrm{Sp}_{2g}(\mathbb{Z}) : \mathrm{Stab}_H(\mathrm{Sp}_{2g}(\mathbb{Z}))] \cong \mathrm{Orb}_H(\mathrm{Sp}_{2g}(\mathbb{Z})).$$

Since the order of the $\mathrm{Orb}_H(\mathrm{Sp}_{2g}(\mathbb{Z}))$ is infinite, we have that $\mathrm{Stab}_H(\mathrm{Sp}_{2g}(\mathbb{Z}))$ has infinite index in $\mathrm{Sp}_{2g}(\mathbb{Z})$. This concludes the proof since the $\rho_s(\mathcal{H}_g) \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ has finite index.

□

The proof is identical to the proof of Theorem 3.4 with the help of Lemma 3.17.

The proof of Proposition 3.7. The proof of Case 1), 3), and 4) are the same. Case 2) needs the fact that $\mathcal{PH}\pi_{g,n}$ does not preserve any isotropic subspace in $H^1(S_g;\mathbb{Q})$ which can be deduced by Lemma 3.17 that $\mathcal{H}\pi_g$ does not preserve any isotropic subspace in $H^1(S_g;\mathbb{Q})$. □

CHAPTER 4

AUTOMORPHISM OF SURFACE BRAID GROUPS

4.1 Proof of the classification of surjections

Let $S = S_{g,p}$ be a surface of genus g with p punctures and let

$$\mathrm{PConf}_m(S) = \{(x_1, ..., x_m) \in S^m : x_i \neq x_j \text{ for } i \neq j\}.$$

In this section, we will compute $H^*(\mathrm{PConf}_m(S_{g,p}); \mathbb{Q})$ and use this to prove Theorem 1.8. Most computations here are similar to the computations in [8]. Let

$$\triangle_{ij} = \{(x_1, ..., x_m) \in S_{g,p}^m : x_i = x_j\} \subset S_{g,p}^m,$$

which is a real codimensional two subspace of $S_{g,p}^m$. By Poincaré, subspace $\triangle_{ij}$ determines a class $[\triangle_{ij}] \in H^2(S_{g,p}^m; \mathbb{Q})$.

Lemma 4.1. *(1) Let $g > 1$ and $p, m > 0$ be integers. We have the following*

$$H^1(\mathrm{PConf}_m(S_{g,p}); \mathbb{Q}) \cong H^1(S_{g,p}^m; \mathbb{Q}) \cong H^1(S_{g,p}; \mathbb{Q})^{\oplus m}.$$

(2)The following sequence is exact:

$$0 \to \oplus_{1 \leq i < j \leq m} \mathbb{Q}[\triangle_{ij}] \to H^2(S_{g,p}^m; \mathbb{Q}) \to H^2(\mathrm{PConf}_m(S_{g,p}); \mathbb{Q}).$$

Proof. There is a graded-commutative $\mathbb{Q}$-algebra $[G_{ij}]$ defined in [35, Theorem 1], where the degree of the generators G_{ij} is 1. By Totaro [35, Theorem 1], there is a spectral sequence $E_2^{r,q} = H^r(S_{g,p}^n; \mathbb{Q})[G_{ij}]^q$ converging to $H^*(\mathrm{PConf}_n(S_{g,p}); \mathbb{Q})$. Since we only compute H^1 and H^2, the differential involved is $d_2 : E_2^{0,1} = H^0(S_{g,p}^n; \mathbb{Q})[G_{ij}] \to E_2^{2,0} =$

$H^2(S^n_{g,p};\mathbb{Q})$. Let $[\triangle_{ij}] \in H^2(S^n_{g,p};\mathbb{Q})$ be the Poincaré dual of $\triangle_{ij} \subset S^n_g$. By [35, Theorem 2], the differential $d_2(G_{ij}) = [\triangle_{ij}]$. □

Let $\{a_k, b_k\}_{k=1}^g$ be a symplectic basis for $H^1(S_g;\mathbb{Q})$. The natural embedding $S_{g,p} \xrightarrow{i} S_g$ gives us the pullback map $i^* : H^1(S_g;\mathbb{Q}) \to H^1(S_{g,p};\mathbb{Q})$. We denote the pull-backs of $\{a_k, b_k\}$ in $H^1(S_{g,p};\mathbb{Q})$ as the same notations. Let $p_i : S^m_{g,p} \to S_{g,p}$ be the projection to the ith component. For $1 \le i, j \le m$, we denote

$$M_{i,j} = \sum_k p_i^* a_k \otimes p_j^* b_k - p_i^* b_k \otimes p_j^* a_k.$$

Lemma 4.2. *The diagonal element* $[\triangle_{ij}] = p_i^*[S_{g,p}] + p_j^*[S_{g,p}] + i^* M_{ij} \in H^2(S^m_{g,p};\mathbb{Q})$. *When* $p > 0$*, we have* $\triangle_{ij} = i^* M_{ij} \in H^2(S^m_{g,p};\mathbb{Q})$.

Proof. This is classical. See Milnor–Stasheff [32, Section 11]. □

Denote by $H_i := p_i^*(H^1(S_{g,p};\mathbb{Q})) \subset H^1(\mathrm{PConf}_m(S_{g,p});\mathbb{Q})$. We have the following property of the cup product structure of $H^*(\mathrm{PConf}_m(S_{g,p}))$.

Lemma 4.3. *Let* $x = x^1 + ... + x^m, y = y^1 + ... + y^m \in \bigoplus_{i=1}^m H_i = H^1(\mathrm{PConf}_m(S_{g,p});\mathbb{Q})$. *If* x, y *independent and* $x \smile y = 0$*, then there exists* i *such that* $x \in H_i$ *or* $y \in H_i$.

Proof. The multiplication of x and y is the following:

$$x \smile y = x^1 \smile y^1 + ... + x^n \smile y^n + \sum_{i \neq j} (x^i \otimes y^j - y^i \otimes x^j) \in \bigoplus_{i=1}^n \mathbb{Q} p_i^*[S_{g,p}] \oplus \bigoplus_{i \neq j} H_i \otimes H_j$$

By $x \smile y = 0 \in H^2(\mathrm{PConf}_n(S_{g,p});\mathbb{Q})$ and (2) of Lemma 4.1, we have the following equality

$$x^1 \smile y^1 + ... + x^n \smile y^n + \sum_{i \neq j} (x^i \otimes y^j - y^i \otimes x^j) = \sum k_{i,j} [\triangle_{i,j}] = \sum k_{i,j} (p_i[S_{g,p}] + p_j[S_{g,p}] + M_{i,j})$$

in $\bigoplus_{i=1}^n \mathbb{Q}p_i^*[S_{g,p}] \oplus \bigoplus_{i\neq j} H_i \otimes H_j$. By the independence of all the terms in $\mathbb{Q}^n \oplus \bigoplus_{i\neq j} H_i \otimes H_j$, we have

$$x^i \otimes y^j - y^i \otimes x^j = k_{i,j} M_{i,j} \text{ for all } i, j.$$

If x^i and y^i are proportional in H_i, since $g > 1$, we have at least 4 terms in $M_{i,j}$, we don't have enough basis to span our $M_{i,j}$. If x^i and y^i are independent in H_i, since $g > 1$, we have $x^i \otimes y^j - y^i \otimes x^j \neq M_{i,j}$. Therefore $k_{i,j} = 0$ and $x^i \otimes y^j - y^i \otimes x^j = 0 \in H_i \otimes H_j$. Assume without loss of generality that $x \notin H_i$ for any i and that $x_1 \neq 0$ and $x_2 \neq 0$. We break the proof into the following cases.

- Case 1) $y^1 \neq 0$ and y^1 is not proportional to x^1:

  $x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $y^j = 0$ and $x^j = 0$ for j. However $x_2 \neq 0$. Therefore this case is invalid.

- Case 2) $y^1 \neq 0$ and $\lambda x^1 = \mu y^1$:

  $x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $\lambda x^j = \mu y^j$ for all j, which verifies our lemma that x and y are proportional.

- Case 3) $y^1 = 0$:

  $x^1 \otimes y^j = y^1 \otimes x^j \in H_1 \otimes H_j$ implies that $y^j = 0$ for all j. This means $y = 0$ therefore x and y are also proportional.

□

Lemma 4.4. *For $g > 1, p, n \geq 0$, an homomorphism*

$$R : PB_n(S_{g,p}) \to \pi_1(S_{g,p})$$

either factors through some forgetful map or Image$(R) \cong \mathbb{Z}$.

Proof. The proof of this lemma uses the same idea as [24]. The method can also be found in [33, Lemma 3.3 and 3.4]. We use group cohomology in what follows. By the classification of subgroups of $\pi_1(S_{g,p})$, if Image$(R) \not\cong \mathbb{Z}$, then Image(R) is either a free group F_k with $k > 1$ or a surface group $\pi_1(S_h)$ such that $h \geq g$. In both cases, there are independent elements $x, y \in H^1(\text{Image}(R); \mathbb{Q})$ such that $x \smile y = 0$. Denote by $S : PB_n(S_{g,p}) \to \text{Image}(R)$ the map to the image of R, which is surjective by definition. Then $S^*(x), S^*(y) \in H^1(PB_n(S_{g,p}); \mathbb{Q})$ are independent and $S^*(x) \smile S^*(y) = 0$. By Lemma 4.3, we have $S^*(x)$ or $S^*(y) \in H_i$ for some i. Without loss of generality, assume that $S^*(x) = p_i^*(x')$. We have the following commutative diagram by the identification $H^1(\_\_; \mathbb{Q}) \cong \text{Hom}(\_\_, \mathbb{Q})$.

$$\begin{array}{ccc} PB_n(S_g) & \xrightarrow{S} & \text{Image}(R) \\ \downarrow{\scriptstyle p_{i*}} & & \downarrow{\scriptstyle x} \\ \pi_1(S_g) & \xrightarrow{x'} & \mathbb{Q} \end{array}$$

Let K be the kernel of p_{i*}, which is a finitely generated normal subgroup of $PB_n(S_g)$. The image $S(K)$ is also a finite generated normal subgroup of Image(R). However every finitely generated normal subgroup of Image(R) is either finite index or trivial; see Property (D6) in [24]. If $S(K) <$ Image(R) has finite index, then after composing with x, the image $x \circ S(K)$ won't be trivial in $\mathbb{Q}$. This is a contradiction because K is the kernel of p_{i*}; i.e. $x \circ S(K) = x' \circ p_{i*}(K) = \{1\}$. If the $S(K) = 1$, then S factors through p_{i*}. □

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8. We will prove that every surjective homomorphism $F : PB_n(S) \to PB_m(S)$ factors through some forgetful homomorphism by induction on m. The case $m = 1$ is Lemma 4.4. We assume that when $m < k$, Theorem 1.8 is true. For $m = k$, we have a surjection $f : PB_n(S_{g,p}) \to PB_k(S_{g,p})$. By post-composing with a projection $p_k : PB_k(S_{g,p}) \to PB_{k-1}(S_{g,p})$, we have a new surjection $p_k \circ f : PB_n(S_{g,p}) \to PB_{k-1}(S_{g,p})$. By the inductive hypothesis, $p_k \circ f$ must factor through some forgetful map. We have the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & PB_{n-k+1}(S_{g,p+k-1}) & \longrightarrow & PB_n(S_{g,p}) & \xrightarrow{\text{forget}} & PB_{k-1}(S_{g,p}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & PB_1(S_{g,p+k-1}) & \longrightarrow & PB_k(S_{g,p}) & \xrightarrow{\text{forget}} & PB_{k-1}(S_{g,p}) & \longrightarrow & 1
\end{array}$$

Using the fact that $PB_{n-k+1}(S_{g,p+k-1}) \to PB_1(S_{g,p+k-1})$ factors through a forgetful homomorphism, we get our result that $PB_n(S_{g,p}) \to PB_k(S_{g,p})$ factors through a forgetful homomorphism.

□

Now we are ready to prove Corollary 1.9.

Proof of Corollary 1.9. We only need to prove that there does not exist a surjective homomorphism $F : B_n(S) \to \pi_1(S)$ when $n > 1$. Assume that we have a surjection $F : B_n(S) \to \pi_1(S)$. Restricting to the index $n!$ subgroup $PB_n(S)$, we have a map $F|_{PB_n(S)} : PB_n(S) \to \text{Image}(F|_{PB_n(S)})$. As a finite index subgroup of $\pi_1(S)$, the group $\text{Image}(F|_{PB_n(S)})$ is not cyclic. Then Lemma 4.4 says that $F|_{PB_n(S)} = Q \circ \text{forget}$ for some $Q : \pi_1(S) \to \text{Image}(F|_{PB_n(S)})$. However Q cannot be surjective because $\text{Image}(F|_{PB_n(S)})$ has higher first betti number than $\pi_1(S)$. □

4.2 Automorphism group of $\mathbf{PB}_n(S_{g,p})$

In this section, we will compute the automorphism group of $\text{PB}_n(S_{g,p})$. The main idea is to use the existence of a pseudo-Anosov element in the point-pushing subgroup. Before the proof of the result, we will introduce 3 classical results we will use in the proof. Firstly, we have the following result of Handel–Thurston [21, Lemma 2.2].

Theorem 4.5 ([21])**.** *A pseudo-Anosov element of mapping class group does not fix any nonperipheral isotopy class of curves (including nonsimple curves).*

Another ingredient is Kra's construction [27]. Let S be a surface possibly with punctures. The *extended mapping class group* $\mathrm{Mod}^{\pm}(S)$ is defined to be the group of isotopy classes of diffeomorphisms of S fixing punctures as a set. Later, we will define other types of extended mapping class groups by specifying exactly how they preserve punctures. Let $\mathrm{Mod}^{\pm}(S,b)$ be the extended mapping class group of S fixing a point b ($\mathrm{Mod}(S,b)$ also fixes punctures of S as a set). The following is the Birman exact sequence for S (see Farb–Margalit [16, Section 4.2]):

$$1 \to \pi_1(S,b) \xrightarrow{Push} \mathrm{Mod}^{\pm}(S,b) \to \mathrm{Mod}^{\pm}(S) \to 1.$$

We say that an element γ of $\pi_1(S,n)$ *fills* S if a curve representing of γ intersects every essential simple closed curve in S.

Theorem 4.6 (Kra's construction [27])**.** *Let $S = S_{g,n}$ and assume that S is hyperbolic. Let $\gamma \in \pi_1(S,b)$. The mapping class $Push(\gamma) \in \mathrm{Mod}^{\pm}(S,b)$ is pseudo-Anosov if and only if γ fills S.*

The third ingredient is the following punctured Dehn–Nielsen–Baer theorem; e.g. see [16, Theorem 8.8]. For a group G, denote by $\mathrm{Out}(G)$ the outer automorphism group of G; i.e. $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ denotes the group of conjugaction automorphisms.

Theorem 4.7 (Punctured Dehn–Nielsen–Baer Theorem)**.** *Let $S = S_{g,n}$ be a hyperbolic surface of genus g with n punctures. Let $\mathrm{Out}^*(\pi_1(S))$ (resp. $\mathrm{Aut}^*(\pi_1(S))$) be the subgroup of $\mathrm{Out}(\pi_1(S))$ (resp. $\mathrm{Aut}(\pi_1(S))$) consisting of elements that leave invariant the set of conjugacy classes in $\pi_1(S)$ of the simple closed curves surrounding individual punctures. Then the natural map*

$$\mathrm{Mod}^{\pm}(S) \to \mathrm{Out}^*(\pi_1(S)) \textit{ and } \mathrm{Mod}^{\pm}(S,b) \to \mathrm{Aut}^*(\pi_1(S))$$

are isomorphisms.

Another result of this paper is described as the following, which is the main ingredient in proving Theorem 1.10. Given any element in $\mathrm{Mod}^{\pm}(S_{g,p+n})$, there is an induced action on the fundamental group $\pi_1(S_{g,p+n})$. Since $\mathrm{Mod}^{\pm}(S_{g,p+n})$ does not fix any base point, the action is only defined up to conjugation. Therefore we have an injective homomorphism $\mathrm{Mod}^{\pm}(S_{g,p+n}) \rightarrow \mathrm{Out}(\pi_1(S_{g,p+n}))$. The braid point-pushing subgroup $PB_n(S_{g,p}) < \mathrm{Mod}^{\pm}(S_{g,p+n})$ is a normal subgroup; in other words, the normalizer of $PB_n(S_{g,p}) < \mathrm{Out}(\pi_1(S_{g,p+n}))$ contains $\mathrm{Mod}^{\pm}(S_{g,p+n})$. Let $\mathrm{Mod}^{\pm}(S_{g,p,n})$ be the extended mapping class group that fixes two sets of punctures, one with p points and the other with n points. The following proposition computes the normalizer of $PB_n(S_{g,p}) < \mathrm{Out}(\pi_1(S_{g,p+n}))$.

Proposition 4.8. *For $n > 0, g > 1, p \geq 0$, the normalizer of $PB_n(S_{g,p}) < Out(\pi_1(S_{g,p+n}))$ is* $\mathrm{Mod}^{\pm}(S_{g,p,n})$.

Proof. Let R be an element of the normalizer of $PB_n(S_{g,p})$ in $\mathrm{Out}(\pi_1(S_{g,n+p}))$. Therefore R acts as an automorphism A on $PB_n(S_{g,p})$ by conjugation. That is $R \circ e \circ R^{-1} = A(e)$, which gives the following equation.

$$R \circ e = A(e) \circ R \in \mathrm{Out}(\pi_1(S_{g,p+n}))$$

By Theorem 4.7, $R \in \mathrm{Mod}^{\pm}(S_{g,p+n})$ if and only if R fixes the set of conjugacy classes of the punctures. We plan to prove $R \in \mathrm{Mod}^{\pm}(S_{g,p+n})$ by contradiction. Let $\{c_i\}$ be the set of homotopy classes of closed curves surrounding punctures. Assume that R does not fix the set of punctures. Without loss of generality, we can assume $R(c_1) = a$ where a is nonperipheral. Since $PB_n(S_{g,p})$ fixes all punctures, we have $R \circ e(c_1) = R(c_1) = a$. Therefore $A(e)(a) = A(e)(R(c_1)) = R(c_1) = a$ which is true for any $e \in PB_n(S_{g,p})$.

However, because of Theorem 4.6, we know that that there is an element in $PB_n(S_{g,p})$

that is pseudo-Anosov. By Theorem 4.5, we know that the pseudo-Anosov element does not fix any nonperipheral isotopy class of curves a. This is a contradiction. Therefore we have that

$$R \in \mathrm{Out}^*(\pi_1(S_{g,p+n})) \cong \mathrm{Mod}^{\pm}(S_{g,p+n}).$$

We just established that $R \in \mathrm{Mod}^{\pm}(S_{g,p+n})$. What remains to be proven is that $R \in \mathrm{Mod}^{\pm}(S_{g,p,n})$. An element $R \in \mathrm{Mod}^{\pm}(S_{g,p+n})$ induces an automorphism of $PB_n(S_{g,p})$ if and only if R acts on all terms of the following exact sequence

$$1 \to PB_n(S_{g,p}) \to PB_{n+p}(S_g) \to PB_p(S_g) \to 1.$$

As a result, R fixes the p punctures as a set, which implies that R should also fixes the other n punctures as well. Therefore $R \in \mathrm{Mod}^{\pm}(S_{g,p,n})$. □

Let $\mathrm{Mod}^{\pm}(S_{g,p,n-1,1})$ be the extended mapping class group that fixes three sets of punctures, one with p points, one with $n-1$ points and one with one point. Let

$$PB_n(S_{g,p}) < \mathrm{Mod}^{\pm}(S_{g,p+n-1,1}) \to \mathrm{Aut}(\pi_1(S_{g,p+n-1}))$$

be the natural embedding as before. By a similar argument, we have the following:

Proposition 4.9. *For $g > 1, n > 1, p \geq 0$, the normalizer of*

$$PB_n(S_{g,p}) < \mathrm{Aut}(\pi_1(S_{g,p+n-1})) \textit{ is } \mathrm{Mod}^{\pm}(S_{g,p,n-1,1}).$$

We are now ready to prove Theorem 1.10; that is $\mathrm{Mod}^{\pm}(S_{g,p,n}) \cong \mathrm{Aut}(PB_n(S_{g,p}))$.

Proof of Theorem 1.10. First of all, as $PB_n(S_{g,p})$ is a normal subgroup in $\mathrm{Mod}^{\pm}(S_{g,p,n})$, there is a map

$$C : \mathrm{Mod}^{\pm}(S_{g,p,n}) \to \mathrm{Aut}(PB_n(S_{g,p}))$$

given by conjugation. Let $F_i : PB_n(S_{g,p}) \to PB_{n-1}(S_{g,p})$ be the forgetful map forgetting the ith coordinate. By Theorem 1.8, given any automorphism $A : PB_n(S_{g,p}) \to PB_n(S_{g,p})$, we have that $F_i \circ A$ factors through F_j for some j. Denote by $\mathrm{Aut}^0(PB_n(S_g))$ the subgroup of $\mathrm{Aut}(PB_n(S_g))$ such that $F_i \circ A$ factors through F_i for all i. Therefore we have the following exact sequence

$$1 \to \mathrm{Aut}^0(PB_n(S_g)) \to \mathrm{Aut}(PB_n(S_g)) \xrightarrow{S} \Sigma_n \to 1,$$

where for $A \in \mathrm{Aut}(PB_n(S_g))$, the image $S(A)$ satisfies that $F_i \circ A$ factors through $F_{S(A)(i)}$. Let $\mathrm{Mod}^{\pm}(S_{g,p,\overline{n}})$ be the extended mapping class group of S_g that fixes p points as a set and n points individually. By Five lemma, the following commutative diagram implies that to show that C is an isomorphism, it suffices to show that C' is an isomorphism.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Mod}^{\pm}(S_{g,p,\overline{n}}) & \longrightarrow & \mathrm{Mod}^{\pm}(S_{g,p,n}) & \longrightarrow & \Sigma_n & \longrightarrow & 1 \\ & & \downarrow C' & & \downarrow C & & \downarrow = & & \\ 1 & \longrightarrow & \mathrm{Aut}^0(PB_n(S_g)) & \longrightarrow & \mathrm{Aut}(PB_n(S_g)) & \longrightarrow & \Sigma_n & \longrightarrow & 1. \end{array}$$

Now we want to find the inverse of C'. Let $A \in \mathrm{Aut}^0(PB_n(S_{g,p}))$. Then $F_n \circ A$ factors through F_n, which gives the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(S_{g,p+n-1}) & \longrightarrow & PB_n(S_{g,p}) & \xrightarrow{F_n} & PB_{n-1}(S_{g,p}) & \longrightarrow & 1 \\ & & \downarrow R & & \downarrow A & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(S_{g,p+n-1}) & \longrightarrow & PB_n(S_{g,p}) & \xrightarrow{F_n} & PB_{n-1}(S_{g,p}) & \longrightarrow & 1 \end{array}$$

The above diagram gives us the following map P such that $P(A) = R$.

$$P : \mathrm{Aut}^0(PB_n(S_{g,p})) \to \mathrm{Aut}(\pi_1(S_{g,p+n-1}))$$

Now we want to show that C' is an isomorphism and P is the inverse of C'. This will conclude the proof of the result for $PB_n(S_{g,p})$.

- Step 1: P is injective.

  If $R = id$, then for any $e \in PB_n(S_{g,p}), x \in \pi_1(S_{g,p+n-1})$, we have that

$$R(exe^{-1}) = A(e)R(x)A(e)^{-1} \Longrightarrow e^{-1}A(e) \text{ commutes with } x \text{ for } x \in \pi_1(S_{g,p+n-1})$$

  However, the map $PB_n(S_{g,p}) \to \mathrm{Aut}(\pi_1(S_{g,p}))$ given by the conjugation action is injective so $e^{-1}A(e) = id$; i.e. $A(e) = e \Longrightarrow A = id$.

- Step 2: $\mathrm{Image}(P) = \mathrm{Mod}^{\pm}(S_{g,p,\overline{n}})$.

  For any $e \in PB_n(S_{g,p})$ and $x \in \pi_1(S_{g,p+n-1})$, we have

$$R(exe^{-1}) = A(e)R(x)A(e)^{-1}.$$

  That is to say

$$R \circ e = A(e) \circ R \Longrightarrow R \circ e \circ R^{-1} = A(e) \in \mathrm{Aut}(\pi_1(S_{g,p+n-1})),$$

  where $e, A(e) \in PB_n(S_{g,p})$ acts on $\pi_1(S_{g,p+n-1})$ by conjugation. By Proposition 4.9, we have that $R \in \mathrm{Mod}^{\pm}(S_{g,p,n-1,1})$. Let $\{c_i\}$ be the set of closed curves surrounding punctures. By observation, $F_j(c_i) = 1$ if and only if $j = i$. However for every i, we have the following commutative diagram (not exact sequence):

$$\begin{array}{ccccc}
\pi_1(S_{g,p+n-1}) & \longrightarrow & PB_n(S_{g,p}) & \xrightarrow{F_i} & PB_{n-1}(S_{g,p}) \\
\downarrow{\scriptstyle R} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A_i} \\
\pi_1(S_{g,p+n-1}) & \longrightarrow & PB_n(S_{g,p}) & \xrightarrow{F_i} & PB_{n-1}(S_{g,p}).
\end{array}$$

  If $R(c_i) = c_j$ where $i \neq j$, then on the one hand, we should have that $F_i \circ R(c_i) =$

$F_i(c_j) \neq 1$; on the other hand, $A_i \circ F_i(c_i) = A_i(1) = 1$. This is a contradiction. Therefore, we have that $R \in \mathrm{Mod}^{\pm}(S_{g,p,\overline{n}})$.

- Step 3: by definition $P \circ C' = id$.

The result for $B_n(S)$ follows from Ivanov [23, Theorem 2] that $PB_n(S)$ is a characteristic subgroup of $B_n(S)$. □

REFERENCES

[1] B. An. Automorphisms of braid groups on orientable surfaces. *J. Knot Theory Ramifications*, 25(5):1650022, 32, 2016.

[2] J. Aramayona and J. Souto. Homomorphisms between mapping class groups. *Geom. Topol.*, 16(4):2285–2341, 2012.

[3] E. Artin. Braids and permutations. *Annals of Mathematics*, pages 643–649, 1947.

[4] R. Bell and D. Margalit. Braid groups and the co-Hopfian property. *J. Algebra*, 303(1):275–294, 2006.

[5] J. Birman, A. Lubotzky, and J. McCarthy. Abelian and solvable subgroups of the mapping class groups. *Duke Math. J.*, 50(4):1107–1120, 1983.

[6] T. Brendle and D. Margalit. The level four braid group. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015.

[7] F. Castel. Geometric representations of the braid groups. *Astérisque*, (378):vi+175, 2016.

[8] L. Chen. The universal n-pointed surface bundle only has n sections. Pre-print, https://arxiv.org/abs/1611.04624, 2016.

[9] L. Chen. Surjective homomorphisms between surface braid groups. Pre-print, https://arxiv.org/abs/1704.05142, 2017.

[10] C. J. Earle and J. Eells. The diffeomorphism group of a compact Riemann surface. *Bull. Amer. Math. Soc.*, 73:557–559, 1967.

[11] C. J. Earle and I. Kra. On sections of some holomorphic families of closed Riemann surfaces. *Acta Math.*, 137(1-2):49–79, 1976.

[12] J. Ellenberg and J. Wiltshire-Gordon. Algebraic structures on cohomology of configuration spaces of manifolds with flows. Pre-print, https://arxiv.org/pdf/1508.02430.pdf, 2015.

[13] J. Elmas, I. McCarthy and N. Ivanov. Automorphisms of surface braid groups. Preprint, https://arxiv.org/abs/math/0306069, 2003.

[14] E. Fadell. Homotopy groups of configuration spaces and the string problem of Dirac. *Duke Math. J.*, 29:231–242, 1962.

[15] E. Fadell and S. Husseini. *Geometry and topology of configuration spaces.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.

[16] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 2012.

[17] A. Fathi, F. Laudenbach, and V. Poénaru. *Thurston's work on surfaces*, volume 48 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit.

[18] D. L. Gonçalves and J. Guaschi. On the structure of surface pure braid groups. *J. Pure Appl. Algebra*, 182(1):33–64, 2003.

[19] D. L. Gonçalves and J. Guaschi. The braid group $B_{n,m}(\mathbb{S}^2)$ and a generalisation of the Fadell-Neuwirth short exact sequence. *J. Knot Theory Ramifications*, 14(3):375–403, 2005.

[20] H. Hamidi-Tehrani. Groups generated by positive multi-twists and the fake lantern problem. *Algebr. Geom. Topol.*, 2:1155–1178, 2002.

[21] M. Handel and W. Thurston. New proofs of some results of nielsen. *Advances in Mathematics*, 56(2):173–191, 1985.

[22] E. Irmak, N. Ivanov, and J. McCarthy. Automorphisms of surface braid groups, 2003.

[23] N. Ivanov. Permutation representations of braid groups of surfaces. *Matematicheskii Sbornik*, 181(11):1464–1474, 1990.

[24] F. E. A. Johnson. A rigidity theorem for group extensions. *Arch. Math. (Basel)*, 73(2):81–89, 1999.

[25] Y. Kida and S. Yamagata. Commensurators of surface braid groups. *J. Math. Soc. Japan*, 63(4):1391–1435, 2011.

[26] Y. Kida and S. Yamagata. The co-Hopfian property of surface braid groups. *J. Knot Theory Ramifications*, 22(10):1350055, 46, 2013.

[27] I. Kra. On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces. *Acta Math.*, 146(3-4):231–270, 1981.

[28] V. Lin. Braids and permutations. Pre-print: https://arxiv.org/abs/math/0404528, 2004.

[29] D. Margalit. A lantern lemma. *Algebr. Geom. Topol.*, 2:1179–1195, 2002.

[30] D. Margalit and J. McCammond. Geometric presentations for the pure braid group. *J. Knot Theory Ramifications*, 18(1):1–20, 2009.

[31] J. McCarthy. Normalizers and centralizers of pseudo-anosov mapping classes. Pre-print: http://users.math.msu.edu/users/mccarthy/publications/normcent.pdf, 1982.

[32] J Milnor and J Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[33] N. Salter. Cup products, the Johnson homomorphism and surface bundles over surfaces with multiple fiberings. *Algebr. Geom. Topol.*, 15(6):3613–3652, 2015.

[34] S. Smale. Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.*, 10:621–626, 1959.

[35] B. Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.

[36] T. Watanabe. On the sections of universal hyperelliptic curves. Preprint https://arxiv.org/abs/1609.04124, 2016.

[37] S. Wolpert. Geodesic length functions and the Nielsen problem. *J. Differential Geom.*, 25(2):275–296, 1987.