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NONLINEAR VARIATIONAL PROBLEMS: CMC DOUBLINGS AND p-ENERGIES

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Dedicated to my parents

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ABSTRACT

This thesis studies a pair of nonlinear variational problems.

In Chapter 2, we give a variational construction of constant mean curvature doublings of certain minimal surfaces. More precisely, let Σ^n be a minimal surface in a Riemannian manifold (M^{n+1}, g) of dimension $3 \leq n+1 \leq 7$. Assume that Σ has index 0 or 1. We use the min-max theory developed by Zhou and Zhu to construct doublings of Σ with constant mean curvature ε for every small $\varepsilon > 0$. Constant mean curvature doublings of minimal surfaces had previously been constructed in ambient dimension $n + 1 \geq 4$ by Pacard and Sun using gluing techniques.

In Chapter 3, we prove a Weyl law for the variational spectrum of the p-laplacian. Let (M^n, g) be a Riemannian manifold and let $(\lambda_i)_{i=1}^{\infty}$ be the variational spectrum of Δ_p on M. We show that the associated counting function $N(\lambda) = \#\{i : \lambda_i < \lambda\}$ satisfies a Weyl law: there is a constant $C_{n,p}$ that depends only on n and p such that $\lambda^{-n/p} N(\lambda) \to C_{n,p} \operatorname{Vol}(M)$ as $\lambda \to \infty$. This proves a conjecture of Friedlander.

CHAPTER 1

INTRODUCTION

Nonlinear partial differential equations (PDEs) are PDEs that include nonlinear terms. One approach to studying nonlinear PDEs is the so-called variational method. In this approach, one first exhibits a given PDE as the Euler-Lagrange equation for some functional F, and then attempts to find solutions of the PDE by looking for critical points of F.

In certain cases, critical points of F can be found by looking for a minimizer. However, non-trivial minimizers do not always exist, and even when they do they may not give all the solutions of a given PDE. Thus one is often lead to look for saddle type critical points of F. Such critical points can be found using min-max theory. To give some intuition, we now describe one of the simplest cases of min-max theory: the mountain pass theorem for a smooth function $F\colon \mathbb{R}^2 \to \mathbb{R}$. While the application of min-max theory to nonlinear variational problems is often far more difficult and technical, much of the basic intuition remains the same as that suggested by the following example.

Theorem 1.0.1 (Finite Dimensional Mountain Pass Theorem). *Assume $F\colon \mathbb{R}^2 \to \mathbb{R}$ is smooth, proper, and Morse. Also assume that F has two distinct local minima $a, b \in \mathbb{R}^2$. Then F has a third critical point $c \in \mathbb{R}^2$ which is of saddle type.*

Proof. (Sketch) Let P be the collection of all smooth paths $\gamma\colon [0,1] \to \mathbb{R}^2$ with $\gamma(0) = a$ and $\gamma(1) = b$, and define the min-max number

$$W = \inf_{\gamma \in P} \left[\max_{t \in [0,1]} F(\gamma(t)) \right].$$

Note that any path connecting γ connecting a to b must pass through a point $p \in \mathbb{R}^2$ with

$F(p) \geq W$. Moreover, note that $W > \max\{F(a), F(b)\}$ since a and b are non-degenerate local minima.

For simplicity, assume there is an optimal path γ connecting a to b, i.e., assume there is a path $\gamma \in P$ with $\max_{t\in[0,1]} F(\gamma(t)) = W$. Again for simplicity, assume that there is a unique time t_0 where $F(\gamma(t_0)) = W$ and let $c = \gamma(t_0)$. We claim that c is the required saddle point of F. First note that c must be a critical point of F. Indeed, if c was not a critical point we could use the negative gradient flow of F in a neighborhood of c to deform γ into a new path $\tilde{\gamma}$ with

$$\max_{t\in[0,1]} F(\tilde{\gamma}(t)) < W.$$

Since $W > \max\{F(a), F(b)\}$, the new path $\tilde{\gamma}$ still has $\tilde{\gamma}(0) = a$ and $\tilde{\gamma}(1) = b$ and hence the dsiplayed inequality contradicts the definition of W.

It remains to see that c is a saddle point. Since F is Morse, the point c is either a local minimum, a saddle point, or a local maximum. It is clear that c cannot be a local minimum because then there would be other times t close to t_0 with $F(\gamma(t)) > F(\gamma(t_0)) = W$. It is also clear that c cannot be a local maximum. Indeed, if c was a local maximum, then the path γ would go over "the peak of a mountain" in the graph of F. Pushing the path γ slightly to one side of this peak would then yield a new path $\tilde{\gamma}$ with $\max_{t\in[0,1]} F(\tilde{\gamma}(t)) < W$ which is a contradiction for the same reason as before. Thus c must be a saddle point. $\square$

Broadly speaking, the main results of this thesis fit into the scheme outlined above. We study certain nonlinear PDEs from a variational point of view. Solutions to the PDEs are obtained by means of suitable min-max argument, and then these min-max characterizations are used to obtain further information about the solutions. Chapter 2 is concerned with the construction of constant mean curvature doublings of minimal surfaces. Chapter 3 is concerned with a Weyl law for the asymptotic growth rate for the eigenvalues of the p-laplacian. Next we give a brief outline of each chapter, explaining in more detail how the results fit in with the above methods.

1.1 OVERVIEW OF CHAPTER 2

Let Ω be a smooth domain in $\mathbb{R}^n$ and fix a constant $h \in \mathbb{R}$. Consider the following nonlinear PDE for a function $u\colon \Omega \to \mathbb{R}$:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h, \quad u\big|_{\partial\Omega} = 0. \tag{1.1}$$

If u is a solution to (1.1), then the graph of u has constant mean curvature h as a surface in $\mathbb{R}^3$. To approach the PDE (1.1) from a variational point of view, we need to rewrite the PDE as the Euler-Lagrange equation for a suitable functional.

To that end, define the functional

$$A^h(u) = \operatorname{Area}(u) + h \cdot \operatorname{Vol}(u) = \int_\Omega \sqrt{1+|\nabla u|^2} + h\int_\Omega u.$$

Here $\operatorname{Area}(u)$ denotes the area of the graph of u, and $\operatorname{Vol}(u)$ denotes the signed volume of the region between Ω and the graph of u. Now observe that

$$\frac{d}{dt}\bigg|_{t=0} A^h(u+tv) = \int_\Omega \frac{\langle \nabla u, \nabla v\rangle}{\sqrt{1+|\nabla u|^2}} + h\int_\Omega v.$$

Assuming that u and v vanish on $\partial\Omega$, we can integrate by parts to get

$$\frac{d}{dt}\bigg|_{t=0} A^h(u+tv) = \int_\Omega \left[h - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right] v.$$

Thus solutions to (1.1) are exactly the critical points of A^h. In the special case where $h=0$, solutions u to (1.1) are critical points of the area functional. Such critical points are called minimal graphs.

A similar story holds in Riemannian manifolds. A hypersurface Σ^n in a Riemannian manifold (M^{n+1}, g) has constant mean curvature h precisely when it is a critical point of a suitable A^h functional. As above, in the case $h=0$, critical points of the area functional are called minimal surfaces. One method for constructing constructing constant mean curvature surfaces is to look for critical points of the A^h functional.

Recently Zhou and Zhu [40] developed a min-max theory that applies to the A^h functional on a Riemannian manifold. Using this min-max theory, they were able to show that every closed Riemannian manifold (M^{n+1}, g) of dimension $3 \leq n+1 \leq 7$ contains a smooth, almost-embedded constant mean curvature hypersurface of mean curvature h. Zhou further refined this min-max theory in [38] and used it to prove the multiplicity one conjecture of Marques and Neves.

We can now state the main results of Chapter 2. Let Σ^n be a minimal surface of index 0 or 1 in a manifold (M^{n+1}, g) of dimension $3 \leq n+1 \leq 7$. We use Zhou and Zhu's min-max theory for the A^h functional to give a variational construction of constant mean curvature doublings of Σ. First we show that for every small $\varepsilon > 0$ there is a surface Λ^ε of constant mean curvature ε in M. See Theorem 2.3.1 for the construction in the stable case, and Theorem 2.4.1 for the construction in the index 1 case. These surfaces Λ^ε have the property that $\operatorname{Area}(\Lambda^\varepsilon) \to 2 \operatorname{Area}(\Sigma)$ as $\varepsilon \to 0$ which, for generic metrics, implies that $\Lambda^\varepsilon \to 2\Sigma$ as varifolds. This is the content of Theorem 2.4.2. Finally, when Σ is stable and $n+1 = 3$, we are able to adapt an argument of Chodosh, Ketover, and Maximo [3] to show that Λ^ε decomposes into two sheets over Σ that are joined by a small catenoidal neck. This is the content of Theorem 2.3.2.

As already mentioned, the main ingredient required for the construction is Zhou and Zhu's min-max theory for the A^h functional [40]. We also use ideas from the min-max arguments of De Lellis and Ramic [5], Marques and Neves [22], and Montezuma [28]. Constant mean curvature doublings of minimal surfaces were previously constructed using gluing methods by Pacard and Sun [29] in the case $n+1 \geq 4$.

1.2 OVERVIEW OF CHAPTER 3

Let Ω be a smooth domain in $\mathbb{R}^n$ and fix a constant $1 < p < \infty$. The p-laplacian of a smooth function $u\colon \Omega \to \mathbb{R}$ is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Consider the following PDE for a function $u\colon \Omega \to \mathbb{R}$ and a constant $\lambda \in \mathbb{R}$:

$$-\Delta_p u = \lambda |u|^{p-2} u, \quad u\big|_{\partial\Omega} = 0. \tag{1.2}$$

When $p = 2$, the p-laplacian reduces to the usual laplacian and the above PDE is a linear PDE for the eigenfunctions of Δ. However, when $p \neq 2$ the PDE (1.2) is nonlinear. In this case, a solution u to (1.2) is called a (nonlinear) eigenfunction of Δ_p and λ is called a (nonlinear) eigenvalue for Δ_p.

To recast (1.2) as a variational problem, define the p-energy

$$E_p(u) = \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

Now observe that

$$\frac{d}{dt}\bigg|_{t=0} E_p(u+tv) = \frac{p\|u\|_p^p \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla v\rangle - p\|\nabla u\|_p^p \int |u|^{p-2}uv}{\|u\|_p^{2p}}.$$

Assuming u and v vanish on $\partial\Omega$ we can integrate by parts in the first term of the numerator to get

$$\frac{d}{dt}\bigg|_{t=0} E_p(u+tv) = \frac{p}{\|u\|_p^p}\left[\int_\Omega -v\Delta_p u - E_p(u)|u|^{p-2}uv\right].$$

Thus a critical point u of E_p solves the PDE (1.2) with eigenvalue $\lambda = E_p(u)$.

In [2], Alonso and Azorero applied min-max theory to the p-energy (considered as a functional on the unit sphere in $W^{1,p}$) in order to produce an infinite sequence of solutions

u_i to (1.2) with eigenvalues λ_i. In the case $p = 2$, the construction reduces to the usual min-max formula

$$\lambda_i = \inf_{\substack{i+1 \text{ planes} \\ P \subset W^{1,2}}} \left[\max_{u \in P \setminus \{0\}} E_2(u) \right].$$

This sequence $(\lambda_i)_{i=1}^{\infty}$ is called the variational spectrum of the p-laplacian. When $p \neq 2$ it isn't known whether every nonlinear eigenvalue λ of Δ_p appears in the variational spectrum.

In [8], Friedlander proved bounds on asymptotic growth rate of the variational spectrum of Δ_p. More precisely, define the eigenvalue counting function

$$N_\Omega(\lambda) = \#\{i : \lambda_i < \lambda\}.$$

Friedlander showed that there are constants $0 < C_1 < C_2$ depending only on n and p such that

$$C_1 \operatorname{Vol}(\Omega)\lambda^{n/p} \leq N_\Omega(\lambda) \leq C_2 \operatorname{Vol}(\Omega)\lambda^{n/p}$$

for all sufficiently large λ. Moreover, he conjectured that the following Weyl law should hold.

Conjecture 1.2.1 (Friedlander)**.** *There is a constant $C_{n,p}$ depending only on n and p such that $\lambda^{-n/p} N_\Omega(\lambda) \to C_{n,p} \operatorname{Vol}(\Omega)$ as $n \to \infty$.*

In Chapter 3, we prove a Weyl law for the p-laplacian, confirming this conjecture of Friedlander. The proof is based on a general scheme for proving Weyl laws set forward by Gromov in [11]. It also uses techniques developed by Liokumovich, Marques, and Neves in their proof of the Weyl law for the volume spectrum [20].

CHAPTER 2

CMC DOUBLINGS

This work originally appeared in [27].

2.1 INTRODUCTION

Let M be a Riemannian manifold. A minimal hypersurface $\Sigma \subset M$ is a critical point of the area functional on M. A constant mean curvature (cmc) hypersurface is a critical point of the area functional subject to variations that preserve the enclosed volume. A fundamental problem in geometry is to construct minimal and cmc hypersurfaces in a given manifold.

Min-max methods have long proven to be a powerful tool for constructing minimal surfaces. In 1981, Pitts [31], building on work of Almgren [1], used min-max methods to show that every closed manifold M^{n+1} with $3 \leq n+1 \leq 6$ contains a smooth, embedded minimal hypersurface. Schoen and Simon [32] improved this to $3 \leq n+1 \leq 7$. In fact, the work of Schoen and Simon shows that every M^{n+1} with $n+1 \geq 3$ contains a minimal hypersurface which is smooth and embedded up to a set of codimension 7.

In 1982, Yau [37] conjectured that every closed manifold contains infinitely many minimal surfaces. Marques and Neves devised a program to prove Yau's conjecture by developing a detailed understanding of the Morse theory of the area functional on a manifold. This program has now been carried out to great success. In [13], Irie, Marques, and Neves showed that Yau's conjecture is true for a generic metric on M^{n+1} with $3 \leq n+1 \leq 7$. In fact, they proved more: generically the union of all minimal surfaces in M is dense in M. A crucial ingredient in the proof was the Weyl law for the volume spectrum proven by Liokumovich, Marques, and Neves [20].

Later Marques, Neves, and Song [25] improved the result in [13] by showing that, for a generic metric on M, some sequence of minimal surfaces becomes equidistributed in M. Gaspar and Guaraco [9] showed that the Weyl law and equidistribution results also hold in the Allen-Cahn setting. In the non-generic case, Song [34] has shown that a closed manifold of dimension $3 \leq n+1 \leq 7$ with an arbitrary metric contains infinitely many minimal hypersurfaces. Thus Yau's conjecture is fully resolved for these dimensions. In the higher dimensional case, Li [19] has shown that for a generic metric on M^{n+1} with $n+1 \geq 8$ there are infinitely many minimal hypersurfaces of optimal regularity.

Recently, Zhou [38] proved the multiplicity one conjecture of Marques and Neves [23]. Using this, Marques and Neves [23] [24] were able to prove the following: for a generic metric on M^{n+1} with $3 \leq n+1 \leq 7$ there is a smooth, embedded, two-sided, index p minimal hypersurface for every $p \in \mathbb{N}$. Moreover, the area of these surfaces grows with p according to the Weyl law for the volume spectrum [20].

Min-max methods for constructing constant mean curvature surfaces have only been developed more recently. Fix a number $h > 0$. Define a functional A^h on open sets in M with smooth boundary by setting

$$A^h(\Omega) = \operatorname{Area}(\partial\Omega) - h \operatorname{Vol}(\Omega).$$

It is known that the critical points of A^h are precisely those sets Ω whose boundary has constant mean curvature h with respect to the inward pointing normal vector. In [40], Zhou and Zhu developed a min-max theory for the A^h functional, and used this theory to show that every closed manifold M^{n+1} with $3 \leq n+1 \leq 7$ admits a smooth almost-embedded h-cmc hypersurface for every $h > 0$. In [39], Zhou and Zhu extended the theory to construct more general prescribed mean curvature hypersurfaces. Zhou [38] used this to give a proof of the multiplicity one conjecture of Marques and Neves [23]. Earlier work of Chodosh and Mantoulidis [4] had shown that the multiplicity one conjecture was true for dimension

$n + 1 = 3$ in the Allen-Cahn setting.

Another technique for constructing minimal and constant mean curvature hypersurfaces is the so-called gluing method. Starting from a collection of nearly minimal surfaces, one joins them together in a carefully chosen manner and then shows that the resulting surface can be perturbed to be minimal (or to have constant mean curvature). Kapouleas and Yang [17] used this technique to construct minimal doublings of the Clifford torus in S^3. Kapouleas has also used it to construct constant mean curvature surfaces of high genus in $\mathbb{R}^3$ [15], and to construct minimal doublings of the equator in S^3 [16]. In [29], Pacard and Sun used gluing methods to construct constant mean curvature doublings of minimal hypersurfaces. The following theorem is a special case of their results (see Theorem 2.1 and Corollary 2.1 in [29]).

Theorem 2.1.1 (Pacard and Sun)**.** *Let $n+1 \geq 4$. Let $\Sigma^n \subset M^{n+1}$ be an embedded minimal hypersurface. Assume that the Jacobi operator J for Σ is invertible, and that the unique solution ϕ to $J\phi = 1$ does not change sign, and that ϕ has a non-degenerate critical point. Then for every sufficiently small $\varepsilon > 0$, there is an embedded ε-cmc hypersurface which is a doubling of Σ.*

It is natural to ask whether surfaces produced by gluing methods can also be produced by variational techniques. In the case of the Clifford torus in S^3, Ketover, Marques, and Neves [18] proved the catenoid estimate and used it to give a min-max construction of the doublings of Kapouleas and Yang [17]. In this paper we show that, in certain circumstances, cmc doublings like those of Pacard and Sun can be constructed using min-max methods. Our results apply in the case $3 \leq n + 1 \leq 7$. In the remainder of the introduction, we give a heuristic explanation of the min-max construction of cmc doublings.

2.1.1 The Stable Case

Fix a dimension $3 \leq n + 1 \leq 7$. Suppose that $\Sigma^n \subset M^{n+1}$ is an embedded, two-sided, stable, minimal hypersurface. Assume that a neighborhood of Σ can be foliated by β-cmcs

Σ^β whose mean curvature vectors point towards Σ. Every strictly stable minimal surface admits such a neighborhood by the implicit function theorem and the maximum principle. A degenerate stable minimal surface may or may not admit such a neighborhood.

Let Ω^ε be the open set in between Σ^ε and $\Sigma^{-\varepsilon}$. Then Ω^ε is a critical point of A^ε. Moreover, using the second variation formula for A^ε, one can check that Ω^ε is strictly stable for A^ε. Thus Ω^ε is a strict local minimum for A^ε in the smooth topology. Now, by the isoperimetric inequality, the empty set is also a local minimum for A^ε. Thus one can attempt to do min-max for the A^ε functional over all 1-parameter families of open sets connecting the empty set to Ω^ε.

Theorem 2.3.1 and Theorem 2.3.2 are the main results of this paper in the stable case. In Theorem 2.3.1, we formalize the min-max argument outlined above to construct an ε-cmc doubling of Σ. The key tool in the proof is the min-max theory for the A^ε functional introduced by Zhou and Zhu in [40]. We also borrow ideas from previous mountain pass type arguments for minimal surfaces. See De Lellis and Ramic [5], Marques and Neves [22], and Montezuma [28]. In the case $n = 2$, we are further able to show that the ε-cmc doubling constructed in Theorem 2.3.1 consists of two parallel copies of Σ joined by a small catenoidal neck. This is the content of Theorem 2.3.2. The proof of this theorem is based on work of Chodosh, Ketover, and Maximo [3].

2.1.2 The Index 1 Case

Fix a dimension $3 \leq n+1 \leq 7$. Let $\Sigma^n \subset M^{n+1}$ be an embedded, two-sided, index 1, minimal hypersurface. Let L be the Jacobi operator on Σ and assume that L is non-degenerate and that the unique solution ϕ to $L\phi = 1$ is positive. The assumption that L is non-degenerate together with the fact that $\phi > 0$ implies that a neighborhood of Σ is foliated by cmc hypersurfaces. Again let Σ^β denote the β-cmc in this foliation and note that the mean curvature vector of Σ^β points away from Σ. Moreover, the surface Σ^β lies at a height on the order of β over Σ.

Now fix a small number $\varepsilon > 0$ and consider an ε-cmc doubling Λ^ε of Σ. If Λ^ε arises from the construction of Pacard and Sun, there is a decomposition

$$\Lambda^\varepsilon = \Lambda_+ \cup \Lambda_- \cup N$$

where N is a small neck, and Λ_+ and Λ_- are each diffeomorphic to Σ with a small ball removed. The sheet Λ_+ is the graph of a function of small norm over Σ^ε, and the sheet Λ_- is the graph of a function of small norm over $\Sigma^{-\varepsilon}$. From this structure, we expect that the index of Λ^ε is three, where the three deformations decreasing A^ε correspond to varying the height of Λ_+, varying the height of Λ_-, and pinching the neck. Thus Λ^ε should be the solution to a three parameter min-max problem.

Based on this, we construct a three parameter family of surfaces Φ parameterized by the cube

$$X = \left\{(x,y,t) : -\frac{\varepsilon}{2} \le x \le \frac{\varepsilon}{2},\ -\frac{\varepsilon}{2} \le y \le \frac{\varepsilon}{2},\ 0 \le t \le R\right\},$$

where $R \gg \varepsilon$ is a fixed small number. To define Φ, first let $\Phi(0,0,0) = \Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$. Think of this as a top sheet Σ^ε at height ε and a bottom sheet $\Sigma^{-\varepsilon}$ at height $-\varepsilon$. Then extend Φ to the rest of X as follows: changing the x-coordinate varies the height of the top sheet by up to $\pm\varepsilon/2$, changing the y-coordinate varies the height of the bottom sheet by up to $\pm\varepsilon/2$, and increasing the t-coordinate opens up a neck between the two sheets.

This family Φ has two important properties.

(i) The surface $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ is an index two critical point of A^ε and the bottom face of the cube X is a two parameter family of deformations that decreases A^ε.

(ii) The surface $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ maximizes A^ε over the boundary of X.

To see property (ii), first observe that $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ maximizes A^ε over the bottom face of the cube. Second, note that by opening a neck up to a fixed size $R \gg \varepsilon$, we can ensure that $A^\varepsilon(S) < A^\varepsilon(\Sigma^\varepsilon \cup \Sigma^{-\varepsilon})$ for every surface S in the top face of the cube. Finally, consider a surface T in the boundary of the bottom face of the cube. Since Σ is unstable, there is a uniform constant c such that

$$A^\varepsilon(T) < A^\varepsilon(\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}) - c\varepsilon^2.$$

On the other hand, by the catenoid estimate of Ketover, Marques, and Neves [18], it is possible to open a neck between the two sheets in T without ever increasing the area by more than $C\varepsilon^2/|\log \varepsilon|$. Therefore, we can ensure that $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ also maximizes A^ε over the side faces of the cube.

Theorem 2.4.1 and Theorem 2.4.2 are the main results of this paper in the index 1 case. In Theorem 2.4.1, we construct ε-cmc surfaces Λ^ε in M by doing min-max for the A^ε functional over all families of surfaces Ψ parameterized by the cube X with $\Psi = \Phi$ on ∂X. These surfaces Λ^ε have the property that $\operatorname{Area}(\Lambda^\varepsilon) \to 2\operatorname{Area}(\Sigma)$ as $\varepsilon \to 0$. In Theorem 2.4.2, we show that for a generic metric on M the surfaces Λ^ε of Theorem 2.4.1 are doublings of Σ.

2.1.3 Organization

The rest of this chapter is organized as follows. Section 2.2 reviews some concepts from geometric measure theory as well as some definitions and theorems from Zhou's min-max theory. Section 2.3 constructs cmc doublings in the stable case. Section 2.4 constructs cmc doublings in the index 1 case. Appendix 2.A contains a quantitative minimality theorem that is needed to check that the width of certain homotopy classes is non-trivial. Appendix 2.B proves that a certain class of metrics is generic.

2.2 NOTATION AND PRELIMINARIES

Let M^{n+1} be a smooth, closed Riemannian manifold. We begin by introducing some tools from geometric measure theory.

1. The set $\mathcal{I}_k(M, \mathbb{Z}_2)$ is the space of k-dimensional rectifiable flat chains mod 2 in M.

2. The flat norm on $\mathcal{I}_k(M, \mathbb{Z}_2)$ is denoted by $\mathcal{F}$, and the mass norm on $\mathcal{I}_k(M, \mathbb{Z}_2)$ is denoted by $\mathbf{M}$.
3. Given $T \in \mathcal{I}_k(M, \mathbb{Z}_2)$, the notation $|T|$ stands for the varifold induced by T.
4. The $\mathbf{F}$ metric on $\mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ is defined by

$$\mathbf{F}(\Omega_1, \Omega_2) = \mathcal{F}(\Omega_1, \Omega_2) + \mathbf{F}(|\partial\Omega_1|, |\partial\Omega_2|)$$

   where $\mathbf{F}$ on the right hand side is Pitts' $\mathbf{F}$-metric on varifolds.
5. Following Marques and Neves, an embedded minimal cycle in M is defined to be a varifold V of the form

$$V = a_1\Gamma_1 + \ldots + a_\ell\Gamma_\ell$$

   where the Γ_i are disjoint, smooth, embedded minimal surfaces in M and the a_i are positive integers.
6. Given $\varepsilon > 0$, define $A^\varepsilon \colon \mathcal{I}_{n+1}(M, \mathbb{Z}_2) \to \mathbb{R}$ by $A^\varepsilon(\Omega) = \operatorname{Area}(\partial\Omega) - \varepsilon \operatorname{Vol}(\Omega)$.

The following definitions are due to Zhou in [38]. Let X be a cubical complex and let Z be a subcomplex of X. Fix an $\mathbf{F}$-continuous map $\Phi \colon X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

Definition 2.2.1. *The (X, Z)-homotopy class of Φ consists of all sequences $\{\Psi_i\}_i$ with the following properties. First, each Ψ_i is an $\mathbf{F}$-continuous map $X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$. Second, for each i, there is a flat continuous homotopy $H_i \colon [0, 1] \times X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ such that*

(i) $H_i(0, x) = \Psi_i(x)$,

(ii) $H_i(1, x) = \Phi(x)$,

(iii) $\displaystyle\limsup_{i\to\infty} \left[\sup_{z \in Z,\ t \in [0,1]} \mathbf{F}(\Phi(z), H_i(t, z)) \right] = 0.$

Definition 2.2.2. *Let* Π *be the* (X, Z)*-homotopy class of* Φ*. Fix an* $\varepsilon > 0$*. Given a sequence* $\{\Psi_i\}_i$ *in* Π *we let*

$$L^\varepsilon(\{\Psi_i\}_i) = \limsup_{i\to\infty} \left[\max_{x\in X} A^\varepsilon(\Psi_i(x))\right].$$

The width of the homotopy class Π *is then defined by*

$$L^\varepsilon(\Pi) = \inf_{\{\Psi_i\}_i \in \Pi} L^\varepsilon(\{\Psi_i\}_i).$$

Definition 2.2.3. *Let* Γ *be a smooth, immersed, constant mean curvature hypersurface in* M*. Then* Γ *is said to be almost-embedded provided for every point* $p \in M$ *either*

- *(i)* Γ *is embedded in a neighborhood of* p*, or*
- *(ii)* Γ *decomposes into the union of two embedded pieces* Γ_1 *and* Γ_2 *in a neighborhood of* p *with* Γ_1 *on one side of* Γ_2*.*

The following min-max theorem for the A^ε functional is due to Zhou. See Theorem 1.7 and Theorem 3.1 in [38].

Theorem 2.2.4 (Zhou)**.** *Assume that the min-max width* Π *is non-trivial, i.e., that*

$$L^\varepsilon(\Pi) > \max_{z\in Z} A^\varepsilon(\Phi(z)).$$

Then there is a smooth, almost-embedded ε*-cmc hypersurface* Λ^ε *in* M*, and there is an open set* Θ^ε *in* M *with* $\partial\Theta^\varepsilon = \Lambda^\varepsilon$ *and* $A^\varepsilon(\Theta^\varepsilon) = L^\varepsilon(\Pi)$*. Moreover, the index of* Λ^ε *as a critical point of* A^ε *is at most the dimension of* X*.*

2.3 THE STABLE CASE

2.3.1 Statement of Results

We now formalize the assumptions outlined in the introduction. Fix a dimension $3 \le n+1 \le 7$. Let (M^{n+1}, g) be a closed Riemannian manifold and let $\Sigma^n \subset M^{n+1}$ be a closed, connected, two-sided, minimal hypersurface. Also assume the following.

- (S1) There is a neighborhood U of Σ and a smooth function f on U and a number $\alpha > 0$ such that $-\alpha < f < \alpha$ on U.
- (S2) The level set $\Sigma^\beta := f^{-1}(\beta)$ is a closed hypersurface diffeomorphic to Σ with constant mean curvature $|\beta|$ for $|\beta| < \alpha$. Moreover $\Sigma^0 = \Sigma$.
- (S3) The mean curvature vector of Σ^β points toward Σ for each $|\beta| < \alpha$.
- (S4) The gradient ∇f does not vanish anywhere on $U \setminus \Sigma$.

For future reference, we will refer to this collection of assumptions as (S). Let Ω^ε be the region contained between Σ^ε and $\Sigma^{-\varepsilon}$.

Our main theorems in the stable case are the following.

Theorem 2.3.1. *Fix (M^{n+1}, g) and Σ for which the assumptions (S) hold. Then there is a smooth, almost-embedded ε-cmc Λ^ε contained in Ω^ε. Moreover, there is an open set $\Theta^\varepsilon \subset \Omega^\varepsilon$ with $\Lambda^\varepsilon = \partial\Theta^\varepsilon$, and the index of $\Lambda^\varepsilon = \partial\Theta^\varepsilon$ as a critical point of A^ε is at most 1.*

Theorem 2.3.2. *Assume further that $n = 2$. Then the surface Λ^ε from the previous theorem admits a decomposition*

$$\Lambda^\varepsilon = \Lambda^\varepsilon_+ \cup \Lambda^\varepsilon_- \cup N$$

where each $\Lambda^\varepsilon_\pm$ is the graph of a function of small norm over Σ minus a ball and N is a catenoidal neck.

2.3.2 Sweepouts

We would like to use a mountain pass type argument to produce an ε-cmc. We now introduce the maps that will serve as sweepouts. Fix a number $0 < \varepsilon < \alpha$. For each $0 < \beta < \alpha$, let $\Omega^\beta = \{-\beta < f < \beta\}$ denote the open set between $\Sigma^{-\beta}$ and Σ^β. Also fix a small number $\eta > 0$ to be specified later and let $\Omega^* = \Omega^{\varepsilon+\eta}$.

Proposition 2.3.3. *There is an* $\mathbf{F}$*-continuous map* $\Phi\colon [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ *with* $\Phi(0) = \emptyset$ *and* $\Phi(1) = \Omega^\varepsilon$.

Proof. The map $\Psi\colon [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ given by $\Psi(t) = \Omega^{t\varepsilon}$ is continuous in the flat topology. By Lemma A.1 in Zhou and Zhu [40], it is possible to construct a sequence $\{\phi_i\}_i$ of better and better discrete approximations to Ψ. Applying Zhou's discrete to continuous interpolation theorem (Theorem 1.12 in [38]) produces the required map Φ from the sequence $\{\phi_i\}_i$. □

Definition 2.3.4. *Let* Φ *be the map constructed in the previous proposition. A sweepout is an* $\mathbf{F}$*-continuous map* $\Psi\colon [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ *with* $\Psi(0) = \emptyset$ *and* $\Psi(1) = \Omega^\varepsilon$ *that is flat homotopic to* Φ *relative to* $\partial[0,1]$*. More precisely, this last statement means that there is a flat continuous map* $H\colon [0,1] \times [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ *such that*

(i) $H(0,t) = \Phi(t)$,

(ii) $H(1,t) = \Psi(t)$,

(iii) $H(s,0) = \emptyset$,

(iv) $H(s,1) = \Omega^\varepsilon$,

for all s *and* t.

Remark 2.3.5. *Let* $X = [0,1]$ *and* $Z = \{0,1\}$*. Note that a sweepout* Ψ *is essentially an element of the* (X,Z)*-homotopy class of* Φ *as defined in Section 2.2. However, we require that* $\Psi(0)$ *exactly equals* $\Phi(0)$ *and that* $\Psi(1)$ *exactly equals* $\Phi(1)$*. Moreover, all sets in a sweepout* Ψ *are required to be contained in the set* Ω^*.

Definition 2.3.6. *The min-max width* W^ε *is defined by*

$$W^\varepsilon = \inf_{\text{sweepouts } \Psi} \left[\sup_{t \in [0,1]} A^\varepsilon(\Psi(t)) \right].$$

Definition 2.3.7. *A critical sequence is a sequence of sweepouts* $\{\Psi_i\}_i$ *with the property that*

$$\lim_{i\to\infty}\left[\sup_{t\in[0,1]} A^\varepsilon(\Psi_i(t))\right] = W^\varepsilon.$$

Definition 2.3.8. *Let* $\{\Psi_i\}_i$ *be a critical sequence. The associated critical set* $C(\{\Psi_i\}_i)$ *is the collection of all varifolds of the form*

$$V = \lim_{i\to\infty} |\partial\Psi_i(t_i)|$$

with $t_i \in [0,1]$ *and* $\lim_{i\to\infty} A^\varepsilon(\Psi_i(t_i)) = W^\varepsilon$*. Note that the critical set is always non-empty and compact.*

2.3.3 Non-trivial Width

Fix (M,g) and Σ satisfying the assumptions (S) and fix a number $0<\varepsilon<\alpha$. Recall that the notation Ω^β denotes the open set between $\Sigma^{-\beta}$ and Σ^β. Also $\eta>0$ is a fixed small number and $\Omega^* = \Omega^{\varepsilon+\eta}$. The number W^ε is the min-max width of the collection of all paths in $\mathcal{I}_{n+1}(M,\mathbb{Z}_2)$ joining $\emptyset$ to Ω^ε while staying inside Ω^*.

The goal of this section is to show that $W^\varepsilon > \max\{A^\varepsilon(\Omega^\varepsilon), 0\}$. The fact that $W^\varepsilon > A^\varepsilon(\emptyset) = 0$ is a consequence of a suitable isoperimetric inequality.

Proposition 2.3.9 (See Theorem 2.15 in [40])**.** *There are constants* C *and* V *such that*

$$\mathrm{Area}(\partial\Omega) \geq C\,\mathrm{Vol}(\Omega)^{n/(n+1)}$$

whenever $\Omega \in \mathcal{I}_{n+1}(M,\mathbb{Z}_2)$ *satisfies* $\mathrm{Vol}(\Omega) < V$*.*

Corollary 2.3.10. *The width* W^ε *is positive.*

Proof. Choose a small number $0 < v < \min\{V, \mathrm{Vol}(\Omega^\varepsilon)\}$. Let $\Psi\colon [0,1] \to \mathbf{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ be a sweepout. By continuity, there must be some Ω in the image of Ψ with $\mathrm{Vol}(\Omega) = v$. It follows that

$$\begin{aligned} A^\varepsilon(\Omega) &= \operatorname{Area}(\partial\Omega) - \varepsilon \operatorname{Vol}(\Omega) \\ &\geq C \operatorname{Vol}(\Omega)^{n/(n+1)} - \varepsilon \operatorname{Vol}(\Omega) = \operatorname{Vol}(\Omega)^{n/(n+1)} \left(C - \varepsilon \operatorname{Vol}(\Omega)^{1/(n+1)} \right). \end{aligned}$$

The number on the right hand side is positive provided v is taken sufficiently small. $\square$

It remains to show that $W^\varepsilon > A^\varepsilon(\Omega^\varepsilon)$. To begin, we first show that Ω^ε is a strictly stable critical point of A^ε.

Proposition 2.3.11. *Assume that (M, g) and Σ satisfy the assumptions (S). Then Ω^ε is a strictly stable critical point of A^ε.*

Proof. Let N be the outward pointing normal vector to $\partial\Omega^\varepsilon$. The second variation formula for A^ε says that

$$\delta^2 A^\varepsilon \Big|_{\Omega^\varepsilon} (uN) = -\int_{\Sigma^\varepsilon} u L_\varepsilon u - \int_{\Sigma^{-\varepsilon}} u L_{-\varepsilon} u,$$

where L_β is the Jacobi operator on Σ^β. Hence to prove the claim it suffices to show that the lowest eigenvalue of L_β is positive for $\beta = \pm\varepsilon$.

We will prove this for L_ε, the argument for $L_{-\varepsilon}$ being essentially identical. Let H be the mean curvature operator on Σ^ε (computed with respect to N). It is known that L_ε is the linearization of H. For γ close enough to ε, we can write Σ^γ as a normal graph of a function φ_γ over Σ^ε. Define

$$\psi = \frac{d}{d\gamma}\Big|_{\gamma=\varepsilon} (\varphi_\gamma)$$

and note that $\psi \geq 0$. Differentiating the equation $H(\varphi_\gamma) = -\gamma$ and evaluating at $\gamma = \varepsilon$ shows that $L_\varepsilon \psi = -1$.

The existence of a non-negative solution to this equation implies that the lowest eigenvalue of L_ε is positive. Indeed, let λ be the lowest eigenvalue of L_ε and let $\zeta > 0$ be the associated eigenfunction so that $L_\varepsilon\zeta + \lambda\zeta = 0$. Since

$$\int_{\Sigma^\varepsilon} \zeta = -\int_{\Sigma^\varepsilon} \zeta L_\varepsilon \psi = -\int_{\Sigma^\varepsilon} \psi L_\varepsilon \zeta = \lambda \int_{\Sigma^\varepsilon} \psi\zeta$$

it follows that λ must be positive. $\square$

The desired inequality for the width now follows from the quantitative minimality results in Appendix 2.A.

Proposition 2.3.12. *There are positive constants δ and C such that*

$$A^\varepsilon(\Omega) \geq A^\varepsilon(\Omega^\varepsilon) + C\mathcal{F}(\Omega, \Omega^\varepsilon)^2$$

for all $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ with $\mathcal{F}(\Omega, \Omega^\varepsilon) < \delta$.

Proof. Proposition 2.3.11 says that Ω^ε is strictly stable for A^ε. Hence the desired result follows from Corollary 2.A.7 in Appendix 2.A. $\square$

Corollary 2.3.13. *The width satisfies $W^\varepsilon > A^\varepsilon(\Omega^\varepsilon)$.*

Proof. Let δ and C be the constants from Proposition 2.3.12. Without loss we can assume that $\delta < \operatorname{Vol}(\Omega^\varepsilon)$. Let

$$\Psi\colon [0,1] \to \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$$

be a sweepout. By continuity there is some Ω in the image of Ψ with $\mathcal{F}(\Omega, \Omega^\varepsilon) = \delta/2$. But then Proposition 2.3.12 implies that

$$A^\varepsilon(\Omega) \geq A^\varepsilon(\Omega^\varepsilon) + \frac{C\delta^2}{4},$$

and the corollary follows. $\square$

2.3.4 A Deformation Lemma

The goal of this section is to prove a deformation lemma that will be used to show that the min-max surface lies in the interior of Ω^*. The proof closely follows an argument of Marques and Neves [23], and relies on the existence of a deformation that pushes currents away from $\partial\Omega^*$ while simultaneously decreasing A^ε.

Proposition 2.3.14. *It is possible to find an open set* Ω^{**} *with*

$$\Omega^\varepsilon \subset\subset \Omega^{**} \subset\subset \Omega^*$$

together with a Lipschitz vector field Z *supported on* $\Omega^* \setminus \Omega^\varepsilon$ *with flow* φ_t *such that the following properties hold.*

(i) $\operatorname{supp}((\varphi_1)_\# \Omega) \subset \Omega^{**}$, *for all* $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$

(ii) $A^\varepsilon((\varphi_1)_\# \Omega) \leq A^\varepsilon(\Omega)$, *for all* $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$

Proof. Recall that the cmc foliation near Σ is given by the level sets of a function f and that $\nabla f \neq 0$ on a neighborhood W of $\Sigma^{-\varepsilon} \cup \Sigma^\varepsilon$. By taking η small enough, we can assume that $\Omega^* \setminus \Omega^\varepsilon \subset W$. Define a vector field $X = \nabla f / |\nabla f|^2$ on W. Then define

$$Z = \begin{cases} -(f-\varepsilon)X, & \text{on } \Omega^* \setminus \Omega^\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

and note that Z is a Lipschitz vector field on Ω^*.

Fix some $\Omega \in \mathcal{I}_{n+1}(\Omega^*, \mathbb{Z}_2)$ and let ν be the outward pointing normal vector to $\partial\Omega$. According to the first variation formula,

$$\delta A^\varepsilon \Big|_{\Omega}(Z) = \int_{\partial\Omega} \operatorname{div}_\sigma Z - \varepsilon \int_{\partial\Omega} \langle Z, \nu \rangle \, d\mathcal{H}^n.$$

To understand the right hand side, we need to compute $\operatorname{div}_\sigma Z$.

Let ϕ_t denote the flow of X and let x denote a point in Σ^ε. Choose a point $y = \psi(x,t)$ and let $\sigma \subset T_yM$ be an n-plane. Let x_i be coordinates on a neighborhood of x in Σ^ε. Then the map $\psi(x,t) = \phi_t(x)$ gives coordinates on a neighborhood of y. Define $e_i = \partial\psi/\partial x_i$ and note that $\partial\psi/\partial t = X$. Let $N = \nabla f/|\nabla f|$ be the unit normal vector to the surfaces Σ^β and let A denote the second fundament form of the surfaces Σ^β. As in Marques and Neves [23], we compute

$$\begin{gathered}\langle \nabla_{e_i} Z, e_j\rangle = -\langle Z, A(e_i,e_j)\rangle,\\ \langle \nabla_N Z, N\rangle = \langle \nabla_N(-(f-\varepsilon)X), N\rangle = -1-(f-\varepsilon)\langle \nabla_N X, N\rangle.\end{gathered}$$

Also we have

$$\begin{gathered}\langle \nabla_{e_i} Z, N\rangle = (f-\varepsilon)\left\langle \frac{\partial\psi}{\partial x_i}, \nabla_{\frac{\partial\psi}{\partial t}} N\right\rangle = \frac{f-\varepsilon}{|\nabla f|}\langle e_i, \nabla_N N\rangle,\\ \langle e_i, -\nabla_N Z\rangle = \left\langle e_i, N\left(\frac{f-\varepsilon}{|\nabla f|}\right)N + \frac{f-\varepsilon}{|\nabla f|}\nabla_N N\right\rangle = \frac{f-\varepsilon}{|\nabla f|}\langle e_i, \nabla_N N\rangle,\end{gathered}$$

and so

$$\langle \nabla_{e_i} Z, N\rangle = -\langle e_i, \nabla_N Z\rangle.$$

Using this one can compute $\operatorname{div}_\sigma Z$ as follows.

Let $v_1,\dots,v_n$ be an orthonormal basis for σ. We can arrange that $v_1,\dots,v_{n-1}$ are tangent to $\Sigma^{\varepsilon+t}$ and that $v_n = (\cos\theta)u + (\sin\theta)N$ for some unit vector u which is tangent to $\Sigma^{\varepsilon+t}$ and orthogonal to $v_1,\dots,v_{n-1}$. Let H be the mean curvature vector for $\Sigma^{\varepsilon+t}$. Then from the above computations one finds

$$\begin{aligned}\operatorname{div}_\sigma Z &= \left(\langle \nabla_u Z, u\rangle + \sum_{i=1}^{n-1}\langle \nabla_{v_i} Z, v_i\rangle\right) + \langle \nabla_{v_n} Z, v_n\rangle - \langle \nabla_u Z, u\rangle\\ &= -\langle Z, H\rangle + (\cos^2\theta - 1)\langle \nabla_u Z, u\rangle + \sin^2\theta\langle \nabla_N Z, N\rangle\\ &= -\frac{\varepsilon(f-\varepsilon)}{|\nabla f|} - \sin^2\theta\Big(1 + (f-\varepsilon)\langle \nabla_N X, N\rangle + (f-\varepsilon)\langle X, A(u,u)\rangle\Big).\end{aligned}$$

Therefore, provided η is small enough, it follows that

$$\operatorname{div}_\sigma Z - \varepsilon\langle Z, \nu\rangle \leq -\frac{\varepsilon(f-\varepsilon)}{|\nabla f|} + \varepsilon|Z| = 0.$$

Hence following the flow of Z decreases A^ε. □

Corollary 2.3.15. *There exists an open set* Ω^{**} *with* $\Omega^\varepsilon \subset\subset \Omega^{**} \subset\subset \Omega^*$ *and a critical sequence* $\{\Psi_i\}_i$ *such that*

$$\operatorname{supp}(\Psi_i(x)) \subset \Omega^{**}$$

for all i *and all* $x \in [0,1]$.

Proof. Let $\{\Phi_i\}_i$ be a criticial sequence. Let φ_t denote the flow of Z. Define $\Psi_i(x) = (\varphi_1)_\# \Phi_i(x)$ for $x \in [0,1]$. By the previous proposition, $\{\Psi_i\}_i$ is as required. □

2.3.5 Constructing the Min-Max Surfaces

We can now perform a min-max argument to construct the doublings. The following min-max theorem essentially follows from Theorem 2.2.4. The theorem is not an immediate consequence of Theorem 2.2.4 because we require that the surfaces in a sweepout are contained in Ω^*. However, it is straightforward to modify the proof of Theorem 2.2.4 to handle our situation.

Theorem 2.3.16. *Assume that* $W^\varepsilon > \max\{0, A^\varepsilon(\Omega^\varepsilon)\}$. *Then for any critical sequence* $\{\Psi_i\}_i$ *there is a varifold* $V \in C(\{\Psi_i\}_i)$ *that is induced by a smooth, almost-embedded* ε*-cmc hypersurface* Λ^ε. *There is an open set* $\Theta^\varepsilon \subset \Omega^*$ *such that* $\partial\Theta^\varepsilon = \Lambda^\varepsilon$ *and* $A^\varepsilon(\Theta^\varepsilon) = W^\varepsilon$. *Moreover, there is a bound* $\operatorname{ind}(\Lambda^\varepsilon) \leq 1$.

Proof. We outline the necessary changes to the proof of Theorems 1.7 and 3.1 in [38]. Let $X = [0,1]$ and $Z = \{0,1\}$. Let Φ be the map from Proposition 2.3.3. Zhou defines the (X,Z)-homotopy class of Φ to consist of all sequences $\{\Psi_i\}_i$ such that each Ψ_i is flatly homotopic to Φ and

$$\lim_{i\to\infty} \max\{\mathbf{F}(\Psi_i(0), \emptyset), \mathbf{F}(\Psi_i(1), \Omega^\varepsilon)\} = 0.$$

However, because the domain X is one dimensional, the interpolation results of Zhou show that nothing changes if we instead insist that $\Psi_i(0) = \emptyset$ and $\Psi_i(1) = \Omega^\varepsilon$ for all i. This leads to the notion of homotopy in Definition 2.3.4.

Now let Ψ be a sweepout. Assume that Ψ' is obtained from Ψ by either the pulltight procedure, the combinatorial argument, or the deformations in the index estimates. Note that we can arrange so that the following property is true: if W is an open set and $\operatorname{supp}(\Psi(t)) \subset W$ for all $t \in [0,1]$ then $\operatorname{supp}(\Psi'(t)) \subset W'$ for all $t \in [0,1]$ where W' is a slightly larger open set containing W. Therefore, by Corollary 2.3.15, we can perform all the arguments of Zhou on a critical sequence $\{\Psi_i\}_i$ while always staying inside Ω^*. □

We can now prove the first main theorem.

Proof. (Theorem 2.3.1) Corollary 2.3.10 and Corollary 2.3.13 show that

$$W^\varepsilon > \max\{0, A^\varepsilon(\Omega^\varepsilon)\}.$$

Therefore Theorem 2.3.16 applies to produce Λ^ε and Θ^ε satisfying the conclusion of Theorem 2.3.1. □

2.3.6 Topology of the Min-Max Doubling

The goal of this section is to show that the min-max surfaces constructed above consist of two parallel copies of Σ joined by a small catenoidal neck. For this section only, we require that $n + 1 = 3$.

Choose a sequence $\varepsilon_j \to 0$. Let $\Lambda_j = \Lambda^{\varepsilon_j}$ be the ε_j-cmc given by Theorem 2.3.1. Note that Λ_j converges to Σ in the Hausdorff distance. Hence by the compactness theorem for cmcs with bounded area and index (Zhou [38]), there is a point $p \in \Sigma$ such that (up to a subsequence) Λ_j converges locally smoothly to Σ away from p.

Proposition 2.3.17. *The convergence $\Lambda_j \to \Sigma$ occurs with multiplicity two.*

Proof. First we show that the multiplicity is at most two. To prove this, it suffices to show that

$$\limsup_{\varepsilon \to 0} W^\varepsilon \leq 2\,\mathrm{Area}(\Sigma).$$

Fix some $\varepsilon > 0$. Since the map $\Phi : [0,1] \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ given by $\Phi(t) = \Omega^{t\varepsilon}$ can be interpolated to a sweepout, it follows that

$$W^\varepsilon \leq \max_{\beta \in [0,\varepsilon]} A^\varepsilon(\Omega^\beta) \leq \max_{\beta \in [0,\varepsilon]} \mathrm{Area}(\partial \Omega^\beta).$$

The quantity on the right hand side converges to $2\,\mathrm{Area}(\Sigma)$ as $\varepsilon \to 0$.

It remains to show that the multiplicity is at least 2. To prove this, it suffices to show that

$$\liminf_{\varepsilon \to 0} W^\varepsilon \geq 2\,\mathrm{Area}(\Sigma).$$

To see this, recall that

$$W^\varepsilon \geq A^\varepsilon(\Omega^\varepsilon) = \mathrm{Area}(\partial\Omega^\varepsilon) - \varepsilon\,\mathrm{Vol}(\Omega^\varepsilon).$$

Again the quantity on the right hand side converges to $2\,\mathrm{Area}(\Sigma)$ as $\varepsilon \to 0$. □

Proposition 2.3.18. *The surface Λ_j is connected.*

Proof. Otherwise there would be a component Λ_j' of Λ_j which is graphical over Σ. The maximum principle shows that such a surface Λ_j' cannot exist. □

Corollary 2.3.19. *The index of Λ_j is one.*

Proof. Suppose to the contrary that $\mathrm{ind}(\Lambda_j) = 0$. By the curvature estimates for stable cmcs (see Zhou [38]), the convergence $\Lambda_j \to \Sigma$ would consequently occur smoothly everywhere.

But, since Σ is two-sided, it is impossible for a connected surface Λ_j to converge smoothly to Σ with multiplicity two. □

We can now give the proof of Theorem 2.3.2.

Proof. (Theorem 2.3.2) The proof is based on results of Chodosh, Ketover, and Maximo [3]. Although the results in [3] are stated for minimal hypersurfaces, one can check that they continue to hold in our setting. For the sake of completeness, we sketch the details of the argument.

Let A_j denote the second fundamental form of Λ_j. Recall that stable cmcs have curvature estimates (see Zhou [38]). Therefore we must have

$$\lim_{j\to\infty} \max_{x\in\Lambda_j} |A_j(x)| = \infty$$

since the convergence $\Lambda_j \to \Sigma$ is not smooth near p. By a point picking argument together with the fact that $\operatorname{ind}(\Lambda_j) = 1$, it is possible to find a constant $C > 0$ and a sequence of points $p_j \in \Lambda_j$ with $|A_j(p_j)| \to \infty$ and such that

$$|A_j(x)| \operatorname{dist}_M(x, p_j) \le C$$

for all $x \in \Lambda_j$. Moreover, it is clear that $p_j \to p$.

Fix a small number $\sigma > 0$. Choose a sequence $\eta_j \to 0$ for which $\operatorname{dist}_M(p_j, p) < \eta_j$ and

$$\lim_{j\to\infty} \eta_j |A_j(p_j)| = \infty.$$

We claim that for j sufficiently large there is a bound

$$|A_j(x)| \operatorname{dist}_M(x, p_j) \le \frac{1}{4}$$

for all $x \in \Lambda_j \cap (B(p,\sigma) \setminus B(p_j, \eta_j))$. Suppose not. Then there would be points $x_j \in$

$\Lambda_j \cap (B(p,\sigma) \setminus B(p_j,\eta_j))$ with

$$|A_j(x_j)|\,\mathrm{dist}_M(x_j,p_j) > \frac{1}{4}.$$

Let Λ'_j be the surface Λ_j rescaled by a factor $\mathrm{dist}_M(x_j,p_j)^{-1}$ about the point p_j. Let A'_j denote the 2nd fundamental form of Λ'_j, and given a point $x \in \Lambda_j$ let x' denote the corresponding point in Λ'_j.

Notice that

$$|A'_j(x')| = |A_j(x)|\mathrm{dist}_M(x_j,p_j),$$

and hence the surfaces A'_j have uniform curvature bounds on compact sets that do not include the origin. Moreover,

$$|A'_j(0)| \geq |A_j(p_j)|\eta_j \to \infty$$

as $j \to \infty$. Therefore, (up to a subsequence) the surfaces Λ'_j converge locally smoothly away from the origin to a complete, embedded minimal surface Λ' with multiplicity two. Since the mean curvature vectors of the two sheets of Λ'_j point toward each other, it follows that Λ' must be stable. Hence Λ' is a plane. But this means that $|A'_j(x'_j)| \to 0$, and this contradicts the way the points x_j were chosen.

Next one combines the preceding curvature estimate with a Morse theory argument (Lemma 3.1 in [3]) to conclude that $\Lambda_j \cap B(p,\sigma)$ and $\Lambda_j \cap B(p_j,\eta_j)$ have the same topology. We are now reduced to showing that $\Lambda_j \cap B(p_j,\eta_j)$ is topologically a catenoid. Let Λ''_j be the surface Λ_j rescaled by a factor η_j^{-1} about the point p_j. It is equivalent to check that $\Lambda''_j \cap B(0,1)$ is a catenoid.

Let Λ'''_j be the surface Λ''_j rescaled by a factor $|A''_j(0)|$ about the origin. Then Λ'''_j has uniform curvature estimates everywhere. Thus (up to a subsequence) the surfaces Λ'''_j converge locally smoothly to a complete, embedded, two-sided, non-flat minimal hypersurface

$\Lambda''' \subset \mathbb{R}^3$. Moreover, we have $\operatorname{ind}(\Lambda''') \leq 1$. By the results in [7] and [21], it follows that Λ''' is a catenoid. Fix a radius $R > 0$ so that $|A'''(y)| \operatorname{dist}(y, 0) < 1/4$ for all $y \in \Lambda''' \setminus B(0, R)$.

We claim that for j sufficiently large there is a bound

$$|A_j''(y)| \operatorname{dist}(y, 0) \leq \frac{1}{4}$$

for all $y \in \Lambda_j'' \cap (B(0,2) \setminus B(0, R/|A_j''(0)|))$. Suppose not. Then there would be points $y_j \in \Lambda_j'' \cap (B(0,2) \setminus B(0, R/|A_j''(0)|))$ with

$$|A_j''(y_j)| \operatorname{dist}(y_j, 0) > \frac{1}{4}.$$

Let Λ_j'''' be the surface obtained by scaling Λ_j'' by a factor $\operatorname{dist}(y_j, 0)^{-1}$ about the origin.

We claim that $|A_j''''(0)| \to \infty$ as $j \to \infty$. Suppose this were not the case. Then since

$$|A_j''''(0)| = |A_j''(0)| \operatorname{dist}(y_j, 0),$$

it must be that

$$\frac{R}{|A_j''(0)|} \leq \operatorname{dist}(y_j, 0) \leq \frac{B}{|A_j''(0)|}$$

for some constant B. But then (up to a subsequence) Λ_j'''' must converge to a surface $\Lambda'''' = a\Lambda'''$ where

$$\frac{1}{B} \leq a \leq \frac{1}{R}.$$

Now observe that

$$\frac{1}{4} < |A''''(\operatorname{dist}(y_j, 0)^{-1} y_j)| = a^{-1} |A'''(a^{-1} \operatorname{dist}(y_j, 0)^{-1} y_j)|.$$

This contradicts the choice of R. Therefore it must be that $|A_j''''(0)| \to \infty$ as $j \to \infty$.

The surfaces Λ_j'''' have uniform curvature estimates on compact subsets that do not include the origin. Hence arguing as above, it follows that (up to a subsequence) the surfaces Λ_j''''

converge locally smoothly to a plane away from the origin. This contradicts the way the points y_j were chosen. Finally one repeats the Morse theory argument with this curvature estimate to deduce that $\Lambda_j'' \cap B(0,1)$ has the same topology as $\Lambda_j'' \cap B(0, R/|A_j''(0)|)$. Since the surface $\Lambda_j'' \cap B(0, R/|A_j''(0)|)$ has the same topology as $\Lambda_j''' \cap B(0,R)$, it follows that $\Lambda_j'' \cap B(0, R/|A_j''(0)|)$ is topologically a catenoid, as needed. This completes the proof of Theorem 2.3.2. $\square$

2.4 THE INDEX 1 CASE

2.4.1 Statement of Results

Now consider the index 1 case. Fix a dimension $3 \leq n+1 \leq 7$. Let (M^{n+1}, g) be a closed Riemannian manifold and let $\Sigma^n \subset M^{n+1}$ be a closed, connected, two-sided, minimal hypersurface. Also assume the following.

(U1) The hypersurface Σ has index 1 and the Jacobi operator L for Σ is non-degenerate. Moreover, the unique solution ϕ to $L\phi = 1$ is positive.

Note that by assumption (U1) and the implicit function theorem, there is a neighborhood of Σ that is foliated by constant mean curvature hypersurfaces whose mean curvature vectors point away from Σ. More precisely, we have the following.

(i) There is a neighborhood U of Σ and a smooth function $f\colon U \to (-\beta, \beta)$.

(ii) For each $\varepsilon \in (-\beta, \beta)$, the set $\Sigma^\varepsilon = f^{-1}(\varepsilon)$ is a smooth hypersurface with constant mean curvature $|\varepsilon|$. Moreover, $\Sigma^0 = \Sigma$.

(iii) For each $\varepsilon \in (-\beta, \beta)$, the mean curvature vector of Σ^ε points away from Σ.

The next theorem is the main result of the paper in the index 1 case.

Theorem 2.4.1. *Fix (M, g) and Σ for which the assumption (U1) holds. Then for each small $\varepsilon > 0$, there is a smooth, almost-embedded hypersurface Λ^ε of constant mean curvature ε in M. The index of Λ^ε is at most 3 and* $\operatorname{Area}(\Lambda^\varepsilon) \to 2\operatorname{Area}(\Sigma)$ *as* $\varepsilon \to 0$.

To ensure that Λ^ε is a doubling of Σ, we have to make an additional assumption. Namely, suppose the following additional property holds.

(U2) The varifold 2Σ is the only embedded minimal cycle in M with area $2\,\mathrm{Area}(\Sigma)$.

Then we have the following.

Theorem 2.4.2. *Fix* (M, g) *and* Σ *for which the assumptions (U1) and (U2) hold. Then the surfaces* Λ^ε *from Theorem 2.4.1 converge to* 2Σ *as varifolds as* $\varepsilon \to 0$.

Remark 2.4.3. *It is natural to ask whether hypothesis (U2) significantly restricts the applicability of Theorem 2.4.2. In Appendix 2.B we show that (U2) holds for a generic set of metrics on* M.

2.4.2 Construction of the Three Parameter Family

In this section, we formally construct the three parameter family Φ described in the introduction. Fix (M, g) and Σ satisfying the assumption (U1) and fix a small number $\varepsilon > 0$. For simplicity, we give the construction in the case where $n + 1 = 3$. The cases $4 \le n + 1 \le 7$ are similar but easier since one can use cylindrical necks rather than catenoidal ones.

Before constructing the three parameter family, we need to introduce some notation. Write Σ^β as the normal graph of a function ψ_β over Σ. Recall that ϕ is a positive function on Σ that solves $L\phi = 1$, and observe that $\psi_\beta/\beta \to \phi$ smoothly as $\beta \to 0$.

The following notation is taken from [18]. Fix a point $p \in \Sigma$ and for $x \in \Sigma$ let $r(x)$ be the distance from x to p. Fix a number $R > 0$ to be specified later. For each $0 \le t \le R$ define a function η_t on Σ by

$$\eta_t(x) = \begin{cases} 1, & \text{if } r(x) \ge t \\ (1/\log(t))(\log t^2 - \log r(x)), & \text{if } t^2 \le r(x) \le t \\ 0 & \text{if } r(x) \le t^2. \end{cases}$$

This function η_t will be used to construct the necks.

Definition 2.4.4. *Let* $X = [-\varepsilon/2, \varepsilon/2]^2 \times [0, R]$ *and define*

$$\Phi\colon X \to \mathcal{I}^{n+1}(M, \mathbb{Z}_2)$$

as follows. First, for each $(x, y, t) \in X$ *let* $S(x, y, t)$ *be the union of the graph of* $\eta_t\psi_{\varepsilon+x}$ *with the graph of* $\eta_t\psi_{-\varepsilon+y}$*. This is a piecewise smooth surface. Choose a point* $q \in \Sigma$ *with* $r(q) \gg R$*. Then let* $\Phi(x, y, t)$ *be the open set in* M *such that* $\partial\Phi(x, y, t) = S(x, y, t)$ *and* $q \notin \Phi(x, y, t)$*. The family* Φ *is continuous in the* **F** *topology.*

In the next sequence of propositions, we prove the two key properties of the family Φ outlined in the introduction.

Proposition 2.4.5. *The surface* $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ *is an index two critical point of* A^ε*. Moreover, there is a constant* $c > 0$ *that doesn't depend on* ε *such that*

$$A^\varepsilon(\Phi(x, y, 0)) \leq A^\varepsilon(\Phi(0, 0, 0)) - c(x^2 + y^2)$$

for all $(x, y, 0) \in X$*.*

Proof. Since Σ is an index one critical point of A^0, it follows that Σ^ε is an index one critical point of A^ε. Likewise $\Sigma^{-\varepsilon}$ is an index one critical point of A^ε and therefore the union $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$ is an index two critical point of A^ε. Next we study how $A^\varepsilon(\Sigma^t)$ depends on t. Let L_t be the Jacobi operator on Σ^t. Since the Jacobi operator on Σ is non-degenerate, L_t is also non-degenerate for all sufficiently small t. Moreover, the unique solution f_t to $L_t f_t = 1$ is uniformly positive for t small enough. Since

$$\frac{d}{dt} A^\varepsilon(\Sigma^t) = \int_{\Sigma_t} (t - \varepsilon) f_t \, dv_{\Sigma_t},$$

it follows that there is a constant $c > 0$ such that

$$A^\varepsilon(\Sigma^t) \le A^\varepsilon(\Sigma^\varepsilon) - c|t - \varepsilon|^2$$

for all $0 \le t \le 2\varepsilon$. The same reasoning applies to $\Sigma^{-\varepsilon}$ and this implies the proposition. $\square$

Lemma 2.4.6. *Let c be the constant from the previous proposition. Then for all ε sufficiently small and all $(x, y, t) \in X$ there is an inequality*

$$\mathrm{Area}(\partial\Phi(x, y, t)) \le \mathrm{Area}(\partial\Phi(x, y, 0)) + \frac{c\varepsilon^2}{2}.$$

Moreover, $\mathrm{Area}(\partial\Phi(x, y, R)) < \mathrm{Area}(\partial\Phi(x, y, 0))$ *for all choices of x and y.*

Proof. This essentially follows from the proof of Theorem 2.4 in [18]. We include the details for the sake of clarity. Let $\gamma = \varepsilon + x$ and let $g_{\gamma,t} = \psi_\gamma\eta_t/\gamma$. Note that there is a bound $\|g_{\gamma,t}\|_{L^\infty} \le C$ where C is a constant that does not depend on γ or t.

For a function f on $B(p, R) \subset \Sigma$, let S_f be the normal graph of f over $B(p, R)$. Proposition 2.5 in [18] gives the existence of an $h_0 > 0$ so that for $h \le h_0$ there is an expansion

$$\begin{aligned}\mathrm{Area}(S_{hg_{\gamma,t}}) \le \mathrm{Area}(B_t) &- \mathrm{Area}(B_{t^2}) \\ &+ \frac{h^2}{2}\int_{B_t\setminus B_{t^2}} (|\nabla g_{\gamma,t}|^2 - g_{\gamma,t}^2(|A|^2 + \mathrm{Ric}(N, N))) \\ &+ Ch^3 \int_{B_t\setminus B_{t^2}} (1 + |\nabla g_{\gamma,t}|^2).\end{aligned}$$

Moreover, the constants h_0 and C do not depend on ε or t.

In particular, for $\gamma < h_0$ we can set $h = \gamma$ in the above expansion to get

$$\begin{aligned}\operatorname{Area}(S_{\psi_\gamma \eta_t}) \leq \operatorname{Area}(B_t) - \operatorname{Area}(B_{t^2}) \\ + \frac{\gamma^2}{2}\int_{B_t\setminus B_{t^2}} (|\nabla g_{\gamma,t}|^2 - g_{\gamma,t}^2(|A|^2 + \operatorname{Ric}(N,N))) \\ + C\gamma^3 \int_{B_t\setminus B_{t^2}} (1 + |\nabla g_{\gamma,t}|^2).\end{aligned}$$

Recall that $\psi_\gamma/\gamma \to \phi$ smoothly as $\gamma \to 0$. Therefore, taking R small enough and ε small enough, we get that

$$\frac{\gamma^2}{2}\left|\int_{B_t\setminus B_{t^2}} g_\gamma^2(|A|^2 + \operatorname{Ric}(N,N))\right| \leq \frac{c\gamma^2}{128}.$$

Shrinking ε further to absorb the γ^3 terms, this implies that

$$\operatorname{Area}(S_{\psi_\gamma \eta_t}) \leq \operatorname{Area}(B_t) + \frac{c\gamma^2}{128} + \gamma^2 \int_{B_t\setminus B_{t^2}} |\nabla g_{\gamma,t}|^2.$$

Finally, using the logarithmic cutoff trick as in [18] together with the fact that $\psi_\gamma/\gamma \to \phi$ as $\gamma \to 0$, it follows that

$$\int_{B_t\setminus B_{t^2}} |\nabla g_{\gamma,t}|^2 \leq \frac{c}{128} + \frac{A}{|\log t|}$$

where A is a constant that does not depend on γ or t. For R small enough, this implies that

$$\operatorname{Area}(S_{\psi_\gamma \eta_t}) \leq \operatorname{Area}(B_R) + \frac{c\gamma^2}{32}$$

for all $t \in [0, R]$.

Therefore, letting $\Omega^+ = \{f > 0\} = \cup_{\beta>0}\Sigma^\beta$, it follows that

$$\begin{aligned}\operatorname{Area}(\partial\Phi(x,y,t)\cap\Omega^+) &- \operatorname{Area}(\partial\Phi(x,y,0)\cap\Omega^+)\\ &\leq \operatorname{Area}(S_{\psi_\gamma\eta_t}) - \operatorname{Area}(B_R)(1 - C\varepsilon^2)\\ &\leq C\varepsilon^2\operatorname{Area}(B_R) + \frac{c\gamma^2}{32}\\ &\leq \frac{c\varepsilon^2}{4}\end{aligned}$$

provided R is small enough. A similar argument shows that the above inequality is also true with Ω^+ replaced by $\Omega^- = \{f < 0\} = \cup_{\beta<0}\Sigma^\beta$. This proves the lemma. $\square$

Proposition 2.4.7. *For every $(x,y,t) \in \partial X$ it holds that*

$$A^\varepsilon(\Phi(x,y,t)) \leq A^\varepsilon(\Phi(0,0,0))$$

with equality if and only if $(x,y,t) = (0,0,0)$.

Proof. Fix a point $(x,y,t) \in \partial X$. The proposition is clearly true if $t = 0$, and the proposition is true if $t = R$ by the previous lemma. So assume that $0 < t < R$. The previous lemma implies that

$$\operatorname{Area}(\partial\Phi(x,y,t)) \leq \operatorname{Area}(\partial\Phi(x,y,0)) + \frac{c\varepsilon^2}{2}.$$

It follows that

$$\begin{aligned}A^\varepsilon(\Phi(x,y,t)) &= \operatorname{Area}(\partial\Phi(x,y,t)) - \varepsilon\operatorname{Vol}(\Phi(x,y,t))\\ &\leq A^\varepsilon(\Phi(x,y,0)) + \frac{c\varepsilon^2}{2}\\ &\leq A^\varepsilon(\Phi(0,0,0)) - \frac{c\varepsilon^2}{2}.\end{aligned}$$

This proves the proposition. $\square$

2.4.3 Non-trivial Width

Again fix (M^{n+1}, g) and Σ satisfying assumption (U1) and fix a small number $\varepsilon > 0$. Let Π be the $(X, \partial X)$-homotopy class of the map Φ constructed in the previous section. Let $\Omega^\varepsilon = \Phi(0,0,0)$ so that $\partial\Omega^\varepsilon = \Sigma^\varepsilon \cup \Sigma^{-\varepsilon}$. The goal of this section is to prove that the width of Π is non-trivial, i.e., to check that

$$L^\varepsilon(\Pi) > A^\varepsilon(\Omega^\varepsilon) = \max_{(x,y,t)\in\partial X} A^\varepsilon(\Phi(x,y,t)).$$

The proof is based on the quantitative minimality results in Appendix 2.A.

Proposition 2.4.8. *There are constants $\gamma > 0$ and $\eta > 0$ and $C > 0$ such that the following property holds. If $\Psi\colon X \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ is an $\mathbf{F}$-continuous map with*

$$\sup_{(x,y,t)\in\partial X} \mathbf{F}(\Psi(x,y,t), \Phi(x,y,t)) < \eta$$

then there is a point $(x_0, y_0, t_0) \in X$ such that

$$A^\varepsilon(\Psi(x_0,y_0,t_0)) \geq A^\varepsilon(\Omega^\varepsilon) + C\gamma^2.$$

Proof. Let $\delta > 0$ and $C > 0$ be the constants from Theorem 2.A.3 applied to $\Sigma^\varepsilon \cup \Sigma^{-\varepsilon} = \partial\Omega^\varepsilon$. Fix some $0 < \gamma < \delta/4$ and then choose a constant $\eta > 0$ to be specified later. Consider a map Ψ as in the statement of the proposition. If η is small enough, it is possible to find a piecewise linear surface $S \subset X$ such that the following properties hold.

- $\gamma < \mathcal{F}(\Psi(p), \Omega^\varepsilon) < 2\gamma$ for all $p \in S$
- ∂S is a connected curve in the bottom face of X that encloses $(0,0,0)$. Moreover, $\operatorname{dist}(\partial S, (0,0,0)) > d$ for some positive constant d that doesn't depend on Ψ.

This can be done, for example, by taking a suitable simplicial approximation to the function

$$(x, y, t) \in X \mapsto \mathcal{F}(\Psi(x, y, t), \Omega^\varepsilon).$$

Note that $A^\varepsilon(\Phi(p)) \leq A^\varepsilon(\Omega^\varepsilon) - d_1$ for all $p \in \partial S$. Here $d_1 > 0$ is a constant that does not depend on Ψ.

Fix a small number $\alpha > 0$. By Theorem 3.8 in [24], if η is small enough there exists an $\mathbf{F}$-continuous homotopy

$$H \colon \partial S \times [0, 1] \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$$

with the properties that

- $H(p, 0) = \Psi(p)$ for all $p \in \partial S$, and
- $H(p, 1) = \Phi(p)$ for all $p \in \partial S$, and
- $\mathbf{F}(H(p, s), \Phi(p)) < \alpha$ for all $p \in \partial S$ and all $s \in [0, 1]$.

For an appropriate choice of α, this ensures that

$$A^\varepsilon(H(p, s)) \leq A^\varepsilon(\Phi(p)) + \frac{d_1}{2} < A^\varepsilon(\Omega^\varepsilon) \tag{2.1}$$

for all $p \in \partial S$ and all $s \in [0, 1]$.

Now let $S_1 = S \cup_{\partial S} (\partial S \times [0, 1])$ and define a map $\Psi_1 \colon S_1 \to \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ by letting $\Psi_1 = \Psi$ on S and letting $\Psi_1 = H$ on $\partial S \times [0, 1]$. Note that part (ii) of Theorem 2.A.3 applies to the family Ψ_1 parameterized by S_1. Therefore, there is some point $q \in S_1$ such that

$$A^\varepsilon(\Psi_1(q)) \geq A^\varepsilon(\Omega^\varepsilon) + C\mathcal{F}(\Psi_1(q), \Omega^\varepsilon)^2.$$

By (2.1), the point $q = (x_0, y_0, t_0)$ must belong to S. Thus we have exhibited a point $(x_0, y_0, t_0) \in X$ with

$$A^\varepsilon(\Psi(x_0, y_0, t_0)) \geq A^\varepsilon(\Omega^\varepsilon) + C\gamma^2,$$

and the proposition follows. ☐

Corollary 2.4.9. *The width of* Π *satisfies* $L^\varepsilon(\Pi) > A^\varepsilon(\Omega^\varepsilon)$.

Proof. This is an immediate consequence of Proposition 2.4.8. ☐

2.4.4 Construction of the Doublings

Fix (M^{n+1}, g) and Σ satisfying assumption (U1). In this section ε will be allowed to vary, and so we write X^ε, Φ^ε, and Π^ε to emphasize the dependence of these objects on ε.

Proof. (Theorem 2.4.1) Corollary 2.4.9 shows that

$$L^\varepsilon(\Pi^\varepsilon) > \max_{(x,y,t)\in\partial X^\varepsilon} A^\varepsilon(\Phi^\varepsilon(x,y,t)),$$

and therefore Π^ε satisfies all the hypotheses of Theorem 2.2.4. Hence min-max produces an almost embedded ε-cmc hypersurface $\Lambda^\varepsilon = \partial\Theta^\varepsilon$ in M with $A^\varepsilon(\Theta^\varepsilon) = L^\varepsilon(\Pi^\varepsilon)$ and $\operatorname{ind}(\Lambda^\varepsilon) \leq 3$.

Observe that

$$A^\varepsilon(\Phi^\varepsilon(0,0,0)) \leq L^\varepsilon(\Pi^\varepsilon) \leq \max_{(x,y,t)\in X^\varepsilon} A^\varepsilon(\Phi^\varepsilon(x,y,t)),$$

and that both bounds for $L^\varepsilon(\Pi^\varepsilon)$ converge to $2\,\text{Area}(\Sigma)$ as $\varepsilon \to 0$. Therefore the area of Λ^ε converges to $2\,\text{Area}(\Sigma)$ as $\varepsilon \to 0$. ☐

Proof. (Theorem 2.4.2) Assume additionally that (U2) holds. By the compactness theorem for cmc surfaces with bounded area and index, there is an embedded minimal cycle V in M with $\|V\|(M) = 2\,\text{Area}(\Sigma)$ such that $\Lambda^\varepsilon \to V$ as $\varepsilon \to 0$ (up to a subsequence). Assumption (U2) implies that $V = 2\Sigma$. ☐

2.4.5 The Non-foliated Case

We close this section with some remarks on the non-foliated case. Assume that $\Sigma \subset M$ is an index one, non-degenerate minimal hypersurface. Let L be the Jacobi operator on Σ and let ϕ be the solution to $L\phi = 1$. One can show that ϕ has at most two nodal domains. In the case of exactly two nodal domains, ϕ changes sign and thus there is no cmc foliation of a neighborhood of Σ.

Nevertheless, it is still possible to foliate a neighborhood of Σ by surfaces whose mean curvature vectors point away from Σ. Let H be the mean curvature operator on Σ, and let $\zeta > 0$ be the first eigenfunction of L. Then by the implicit function theorem, for every small $\beta > 0$ there is a smooth function ψ_β on Σ with $H(\psi_\beta) = \beta\zeta$. The surfaces $\Sigma^\beta = \operatorname{graph}(\psi_\beta)$ foliate a neighborhood of Σ.

Let x be a system of coordinates on Σ and let (x, t) be Fermi coordinates on a tubular neighborhood of Σ. Let h be a smooth, positive function on M such that $h(x, t) = \zeta(x)$ on a tubular neighborhood of Σ. Fix some $\varepsilon > 0$ and note that Σ^ε is a critical point of the $A^{\varepsilon h}$ functional defined by

$$A^{\varepsilon h}(\Omega) = \operatorname{Area}(\partial\Omega) - \varepsilon \int_\Omega h.$$

Using the prescribed mean curvature (pmc) min-max theory of Zhou and Zhu [39] and the same arguments as above, one can show that there are εh-pmc surfaces Λ^ε with $\operatorname{Area}(\Lambda^\varepsilon) \to 2\operatorname{Area}(\Sigma)$. Generically these are doublings of Σ.

2.A QUANTITATIVE MINIMALITY

This appendix contains a quantitative minimality result for the A^ε functional. This result is needed to check that the widths of the min-max families in the paper are non-trivial. The result is based on the following theorem of Inauen and Marchese [12].

Theorem 2.A.1. *([12] Theorem 4.3) Let F be an elliptic parametric functional on M^{n+1}. Let $\Sigma^n \subset M^{n+1}$ be a smooth, closed, hypersurface which is a non-degenerate, index k critical* *point for F. Then there are constants $r > 0$, $c > 0$, $\delta > 0$, and $C > 0$ and a smooth k-parameter family of surfaces*

$$(\Sigma_v)_{v \in \overline{B}_r^k}$$

such that the following properties hold.

(i) For every $v \in \overline{B}_r^k$, the surface Σ_v is homologous to Σ and satisfies $\mathcal{F}(\Sigma_v, \Sigma) < \delta$ and $F(\Sigma_v) \leq F(\Sigma) - c|v|^2$.

(ii) Let S^k be an abstract k-manifold with $\partial S^k = \partial \overline{B}_r^k$. Then for any continuous family of integral currents

$$(\tilde{\Sigma}_v)_{v \in S},$$

each homologous to Σ with $\mathcal{F}(\tilde{\Sigma}_v, \Sigma) < \delta$ for all $v \in S$ and $\tilde{\Sigma}_v = \Sigma_v$ for $v \in \partial S$, it holds that

$$\sup_{v \in S} \left[F(\tilde{\Sigma}_v) - C\mathcal{F}(\tilde{\Sigma}_v, \Sigma)^2 \right] \geq F(\Sigma).$$

Remark 2.A.2. *Let $u_1, \ldots, u_k$ be the eigenfunctions for the second variation of F on Σ with negative eigenvalues. Let*

$$(\psi_v)_{v \in \overline{B}_r^k}$$

be a family of smooth functions on Σ for which the map

$$v \in \overline{B}_r^k \mapsto \left(\int_\Sigma \psi_v u_1, \ldots, \int_\Sigma \psi_v u_k \right) \in \mathbb{R}^k$$

is a diffeomorphism onto a neighborhood of 0. Then by inspecting the proof of Theorem 4.3 in [12] along with the proofs of Theorem 4 and Theorem 5 in [36], one sees that it is possible to take $\Sigma_v = $ graph(ψ_v) in the above theorem.

Unfortunately, Theorem 2.A.1 does not apply directly in our setting since the A^ε functional cannot be written globally as an elliptic parametric functional. Nevertheless, we have the following.

Theorem 2.A.3. *Let* $\Sigma = \partial\Omega$ *be a smooth, closed, hypersurface in* M *which is a non-degenerate, index* k *critical point for* A^ε*. Then there are constants* $r > 0$*,* $c > 0$*,* $\delta > 0$*, and* $C > 0$ *and a smooth* k*-parameter family of open sets*

$$(\Omega_v)_{v \in \overline{B}^k_r}$$

such that the following properties hold.

(i) *For every* $v \in \overline{B}^k_r$*, the set* Ω_v *satisfies* $\mathcal{F}(\Omega_v, \Omega) < \delta$ *and* $A^\varepsilon(\Omega_v) \leq A^\varepsilon(\Omega) - c|v|^2$.

(ii) *Let* S^k *be an abstract* k*-manifold with* $\partial S^k = \partial \overline{B}^k_r$*. Then for any* $\mathbf{F}$ *continuous family*

$$(\tilde{\Omega}_v)_{v \in S}$$

in $\mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ *with* $\mathcal{F}(\tilde{\Omega}_v, \Omega) < \delta$ *for all* $v \in S$ *and* $\tilde{\Omega}_v = \Omega_v$ *for* $v \in \partial S$*, there is a point* $v \in S$ *such that*

$$\sup_{v \in S} \left[A^\varepsilon(\tilde{\Omega}_v) - C\mathcal{F}(\tilde{\Omega}_v, \Omega)^2 \right] \geq A^\varepsilon(\Omega).$$

Moreover, the inequality is strict unless $\tilde{\Omega}_v = \Omega$.

Let $u_1, \ldots, u_k$ *be the eigenfunctions for the Jacobi operator on* Σ *with negative eigenvalues. Let*

$$(\psi_v)_{v \in \overline{B}^k_r}$$

be a family of smooth functions on Σ *for which the map*

$$v \in \overline{B}^k_r \mapsto \left(\int_\Sigma \psi_v u_1, \ldots, \int_\Sigma \psi_v u_k \right) \in \mathbb{R}^k$$

is a diffeomorphism onto a neighborhood of 0*. Then it is possible to choose* Ω_v *above so that* $\partial\Omega_v = $ *graph*(ψ_v)*.*

To prove Theorem 5.3, one essentially copies the arguments from [12] and observes that they continue to hold with F replaced by A^ε. We include the details for completeness.

Proof. Let $u_1, \dots, u_k$ be the eigenfunctions for the Jacobi operator on Σ with negative eigenvalues. Pick a smooth function $\vec{f}\colon M \to \mathbb{R}^k$ such that

$$\vec{f}(x) = 0, \quad \text{and} \quad \nabla \vec{f}(x) = (u_1(x), \dots, u_k(x))$$

for all $x \in \Sigma$. Let K be a very large constant and define

$$G(\Theta) = A^\varepsilon(\Theta) + K \left\| \int \vec{f}\, d\|\partial\Theta\| \right\|^2$$

for $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$. It follows from [36] that the functional G is lower-semicontinuous with respect to flat convergence, and $\Sigma = \partial\Omega$ is a strictly stable critical point of G.

Lemma 2.A.4. *There is some $\delta > 0$ such that $G(\Omega) < G(\Theta)$ for all $\Theta \neq \Omega$ with $\mathcal{F}(\Theta, \Omega) < \delta$.*

Proof. Suppose for contradiction that this is not the case. Then there are sets $\Omega_i \neq \Omega$ with $\mathcal{F}(\Omega_i, \Omega) \to 0$ and $G(\Omega_i) \leq G(\Omega)$. Define

$$G_i(\Theta) = G(\Theta) + \lambda|\mathcal{F}(\Theta, \Omega) - \mathcal{F}(\Omega_i, \Omega)|,$$

where $\lambda > 0$ is a constant to be specified later. Let Ω_i' be a minimizer of G_i. Passing to a subsequence, $\Omega_i' \to \Omega'$ in the flat topology. The proof of Lemma 3.3 in [12] applies verbatim to show that Ω' minimizes

$$G_0(\Theta) = G(\Theta) + \lambda|\mathcal{F}(\Theta, \Omega)|$$

over all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

Next one verifies the analog of Lemma 3.5 in [12].

Lemma 2.A.5. *There are constants $\delta > 0$ and $C > 0$ such that*

$$G(\Omega) - G(\Theta) \le C\mathcal{F}(\Omega, \Theta)$$

for all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

Proof. Note that

$$\begin{aligned} &G(\Omega) - G(\Theta) \\ &\qquad = [\text{Area}(\partial\Omega) - \text{Area}(\Theta)] - \varepsilon[\text{Vol}(\Omega) - \text{Vol}(\Theta)] - K\left\| \int \vec{f}\, d\|\partial\Theta\| \right\|^2 \\ &\qquad \le [\text{Area}(\partial\Omega) - \text{Area}(\Theta)] - \varepsilon[\text{Vol}(\Omega) - \text{Vol}(\Theta)] \\ &\qquad \le [\text{Area}(\partial\Omega) - \text{Area}(\Theta)] + \varepsilon\mathcal{F}(\Omega, \Theta). \end{aligned}$$

By Lemma 3.5 in [12], there is a constant C such that $\text{Area}(\partial\Omega) - \text{Area}(\partial\Theta) \le C\mathcal{F}(\Omega, \Theta)$, and the lemma follows. $\square$

The proof of Lemma 3.6 in [12] now applies verbatim to show that Ω is the only minimizer of G_0. Thus the minimizers Ω_i' converge to Ω in the flat topology. We claim that in fact $\Omega_i' \to \Omega$ in the $\mathbf{F}$-topology. Indeed, since Ω_i' minimizes G_i, there is an inequality

$$G(\Omega_i') + \lambda|\mathcal{F}(\Omega_i', \Omega) - \mathcal{F}(\Omega_i, \Omega)| \le G_i(\Omega_i) = G(\Omega_i) \le G(\Omega).$$

This implies that

$$\text{Area}(\partial\Omega_i') - \varepsilon\,\text{Vol}(\Omega_i') \le \text{Area}(\partial\Omega) - \varepsilon\,\text{Vol}(\Omega),$$

and it follows that

$$\limsup \text{Area}(\partial\Omega_i') \le \text{Area}(\partial\Omega)$$

since $\text{Vol}(\Omega_i') \to \text{Vol}(\Omega)$. This proves the $\mathbf{F}$-convergence.

Now observe that the varifolds $|\Omega_i'|$ have uniformly bounded first variation. This implies that they satisfy a monotonicity formula with uniform constants. Since $\Omega_i' \to \Omega$ in the $\mathbf{F}$-topology, it follows that $\partial\Omega_i'$ is eventually contained in a tubular neighborhood of Σ. According to White [36], this implies that $G(\Omega_i') > G(\Omega)$, and this is a contradiction. This establishes Lemma 2.A.4. □

Lemma 2.A.6. *There are constants $\delta > 0$ and $C > 0$ such that*

$$G(\Omega) \leq G(\Theta) + C\mathcal{F}(\Omega, \Theta)^2$$

for all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ with $\mathcal{F}(\Omega, \Theta) < \delta$.

Proof. Think of $C > 0$ as a fixed constant to be chosen later. Suppose for contradiction that the claim fails. Then there are sets $\Omega_i \neq \Omega$ with $\mathcal{F}(\Omega_i, \Omega) \to 0$ and

$$G(\Omega_i) + C\mathcal{F}(\Omega_i, \Omega)^2 \leq G(\Omega)$$

Define

$$H_i(\Theta) = G(\Theta) + \lambda[\mathcal{F}(\Theta, \Omega) - \mathcal{F}(\Omega_i, \Omega)]^2$$

where $\lambda > 0$ is a constant to be specified later. Let Ω_i' be a minimizer of H_i. Passing to a subsequence, $\Omega_i' \to \Omega'$ in the flat topology. The proof of Lemma 4.1 in [12] applies verbatim to show that Ω' minimizes

$$H_0(\Theta) = G(\Theta) + \lambda\mathcal{F}(\Theta, \Omega)^2$$

over all $\Theta \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$.

We claim that Ω is the unique minimizer of H_0 provided λ is large enough. Suppose for contradiction that there is some $\Omega_1 \neq \Omega$ with $H_0(\Omega_1) \leq H_0(\Omega)$. Then

$$G(\Omega_1) + \lambda\mathcal{F}(\Omega_1, \Omega)^2 \leq A^\varepsilon(\Omega)$$

which implies that

$$\mathcal{F}(\Omega_1, \Omega)^2 \leq \frac{A^\varepsilon(\Omega) - A^\varepsilon(\Omega_1)}{\lambda} \leq \frac{A^\varepsilon(\Omega) + \varepsilon \operatorname{Vol}(M)}{\lambda}.$$

In particular, if λ is large enough then Claim 2.A.4 applies to Ω_1 and so $G(\Omega_1) > G(\Omega)$. This is a contradiction.

Since Ω is the unique minimizer of H_0, it follows that $\Omega_i' \to \Omega$ in the flat topology. The same argument as above shows that this convergence is actually in the **F**-topology. Again the varifolds $|\Omega_i'|$ satisfy a monotonicity formula with uniform constants and hence are eventually contained in a tubular neighborhood of Σ. This contradicts Theorem 1.1 in [12] since the A^ε functional can locally be written as an elliptic parametric functional. (This is because the volume form ω on M is exact in a tubular neighborhood of Σ.) This establishes Lemma 2.A.6. $\square$

Finally Theorem 2.A.3 follows from Lemma 2.A.6 as explained in [36]. $\square$

Note that Theorem 2.A.3 has the following corollary.

Corollary 2.A.7. *Let* $\Sigma = \partial\Omega$ *be a smooth, closed,* ε*-cmc in* M *which is strictly stable for* A^ε*. Then there are constants* $\delta > 0$ *and* $C > 0$ *such that every* $\tilde{\Omega} \in \mathcal{I}_{n+1}(M, \mathbb{Z}_2)$ *with* $\mathcal{F}(\tilde{\Omega}, \Omega) < \delta$ *satisfies* $A^\varepsilon(\tilde{\Omega}) \geq A^\varepsilon(\Omega) + C\mathcal{F}(\tilde{\Omega}, \Omega)^2$.

2.B GENERIC METRICS

It is natural to ask whether assumption (U2) poses a significant restriction to the applicability of Theorem 2.4.2. The following proposition addresses this question. It shows that assumption (U2) holds for a generic set of metrics g on M.

Proposition 2.B.1. *Let* M *be a closed manifold. There is a (Baire) generic set* $\mathcal{G}$ *of smooth metrics on* M *with the following property: if* $g \in \mathcal{G}$ *then for any closed, connected, embedded* *minimal hypersurface* Σ *in* (M, g) *the varifold* 2Σ *is the only embedded minimal cycle in* (M, g) *with area* $2\,\mathrm{Area}(\Sigma)$*.*

Proposition 2.B.1 is a corollary of the following result of Marques and Neves [24]. Given a metric g on M and $C > 0$ and $I \in \mathbb{N}$, let $\mathcal{M}_{C,I}(g)$ denote the collection of all closed, connected, embedded minimal hypersurfaces in (M, g) with area at most C and index at most I.

Proposition 2.B.2. *([24] Proposition 8.6) Let* g *be a bumpy metric on* M*, and fix* $C > 0$ *and* $I \in \mathbb{N}$*. There exist metrics* $\tilde{g}$ *arbitrarily close to* g *in the smooth topology such that the following properties hold.*

- *(i) The set* $\mathcal{M}_{C,I}(\tilde{g}) = \{\Sigma_1, \ldots, \Sigma_N\}$ *is finite and every surface in* $\mathcal{M}_{C,I}(\tilde{g})$ *is non-degenerate.*
- *(ii) The areas* $\mathrm{Area}_{\tilde{g}}(\Sigma_1)$*, …,* $\mathrm{Area}_{\tilde{g}}(\Sigma_N)$ *are linearly independent over* $\mathbb{Q}$*.*

Remark 2.B.3. *Note that property (ii) above immediately implies the following weaker property.*

- *(iii) Let* $A = a_1\,\mathrm{Area}_{\tilde{g}}(\Sigma_1) + \ldots + a_N\,\mathrm{Area}_{\tilde{g}}(\Sigma_N)$ *for some integers* $a_i \geq 0$*. If* $A = 2\,\mathrm{Area}_{\tilde{g}}(\Sigma_i)$ *for some* i *then* $a_i = 2$ *and all the other* a_j*'s are zero.*

Proof. (Proposition 2.B.1) Given $C > 0$ and $I \in \mathbb{N}$, let $\mathcal{G}_{C,I}$ be the collection of all metrics g on M for which properties (i) and (iii) above hold (with g in place of $\tilde{g}$). We claim that $\mathcal{G}_{C,I}$ is open and dense in the set of all smooth metrics on M.

First we show that $\mathcal{G}_{C,I}$ is open. Fix some $g \in \mathcal{G}_{C,I}$ and write

$$\mathcal{M}_{C,I}(g) = \{\Sigma_1, \ldots, \Sigma_N\}.$$

Since every surface in $\mathcal{M}_{C,I}(g)$ is non-degenerate, there is a neighborhood U of g such that for any $\tilde{g} \in U$ and any $i = 1, \ldots, N$ there is a unique minimal surface $\Sigma_i(\tilde{g})$ in $(M, \tilde{g})$ that is smoothly close to Σ_i. Moreover, these surfaces $\Sigma_i(\tilde{g})$ are all non-degenerate. By Sharp's compactness theorem [33], it follows that there is a potentially smaller neighborhood U_1 of g such that

$$\mathcal{M}_{C,I}(\tilde{g}) \subseteq \{\Sigma_1(\tilde{g}), \dots, \Sigma_N(\tilde{g})\}$$

for all $\tilde{g} \in U_1$. Taking an even smaller neighborhood U_2 of g, it is then possible to ensure that condition (iii) holds for all $\tilde{g} \in U_2$.

Next we show that $\mathcal{G}_{C,I}$ is dense. Consider any metric g on M. Since bumpy metrics are dense, there is a bumpy metric g_1 on M arbitrarily close to g. Applying Proposition 3.2 to g_1 then yields $g_2 \in \mathcal{G}_{C,I}$ that is arbitrarily close to g_1. Thus there is a metric $g_2 \in \mathcal{G}_{C,I}$ arbitrarily close to g in the smooth topology.

To conclude the proof, take sequences $C_n \to \infty$ and $I_n \to \infty$ and define

$$\mathcal{G} = \bigcap_n \mathcal{G}_{C_n,I_n}.$$

Then $\mathcal{G}$ is Baire generic, and every metric $g \in \mathcal{G}$ satisfies the conclusion of Proposition 2.B.1. □

CHAPTER 3

A WEYL LAW FOR THE p-LAPLACIAN

This work originally appeared in [26].

3.1 INTRODUCTION

The classical Weyl law states that the Dirichlet eigenvalues λ_i of Δ on a domain $U \subset \mathbb{R}^n$ grow according to the asymptotics

$$\#\{i : \lambda_i < \lambda\} \sim c \operatorname{Vol}(U) \lambda^{n/2}$$

where c is a universal constant that depends only on n. Here the notation $f(\lambda) \sim g(\lambda)$ means that $f(\lambda)/g(\lambda) \to 1$ as $\lambda \to \infty$.

Briefly, the idea of the proof is to use the variational characterization

$$\lambda_i = \inf_{i\text{-planes } P \subset W_0^{1,2}(U)} \left(\sup_{u \in P \setminus \{0\}} \frac{\int_U |\nabla u|^2}{\int_U |u|^2} \right) \tag{3.1}$$

to relate the eigenvalues of Δ on U with the eigenvalues of Δ on a union of cubes that closely approximates U. The eigenvalues of Δ on a cube can be computed explicitly and the formula then follows. The Weyl law also holds for the spectrum of Δ on a closed Riemannian manifold. This can be proved by studying the asymptotics of the heat kernel.

In [2], the authors used variational methods to produce a sequence of eigenvalues λ_i for the p-Laplacian Δ_p. These eigenvalues are given by a min-max formula roughly similar to (3.1). In [8], Friedlander studied the asymptotic growth of these eigenvalues of Δ_p and proved growth bounds of the form

$$C_1 \operatorname{Vol}(U)\lambda^{n/p} \leq \#\{i : \lambda_i < \lambda\} \leq C_2 \operatorname{Vol}(U)\lambda^{n/p}$$

for some constants C_1 and C_2 that depend only on n and p. Moreover, Friedlander conjectured that a Weyl law should hold in this setting. In this paper, we prove the following theorem which confirms Friedlander's conjecture.

Theorem 3.1.1. *Let $1 < p < \infty$. Let (X^n, g) be a closed Riemannian manifold and let (λ_i) be the variational spectrum of Δ_p on X. Then*

$$\#\{i : \lambda_i < \lambda\} \sim c \operatorname{Vol}(X)\lambda^{n/p}$$

where c is a universal constant that depends only on n and p.

The proof is based on a general framework for studying Weyl laws proposed by Gromov in [11]. We also use ideas from the proof of the Weyl law for the volume spectrum due to Liokumovich, Marques, and Neves [20]. It is worth noting that the proof seems to be new even in the case $p = 2$ in the sense that it avoids the use of the heat kernel.

3.2 PRELIMINARIES: VARIATIONAL SPECTRUM OF Δ_p

Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. If u is a smooth function on U then the p-laplacian of u is defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

The Dirichlet eigenvalue problem for Δ_p asks for a function u satisfying

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } U \\ u = 0, & \text{on } \partial U. \end{cases} \tag{3.2}$$

A function $u \in W_0^{1,p}(U)$ is called a weak solution of (3.2) provided

$$\int_U |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = \lambda \int_U |u|^{p-2} u \phi$$

for every test function $\phi \in C_c^\infty(U)$. If u is a non-trivial weak solution of (3.2) then λ is called a Dirichlet eigenvalue for Δ_p on U and u is called an eigenfunction with eigenvalue λ.

It is possible to use variational methods to produce a sequence of eigenvalues for Δ_p on U. This was done in [2]. There is also a detailed treatment in the book [30]. For clarity we outline the argument from [30] below. Let $\mathcal{M}^0 = \mathcal{M}^0(U)$ be the set

$$\mathcal{M}^0 := \left\{ u \in W_0^{1,p}(U) : \|\nabla u\|_{L^p(U)} = 1 \right\}$$

equipped with the topology it inherits as a subspace of $W_0^{1,p}$ with the norm topology. This is a C^1 Banach manifold (see [30]). There is an energy functional $E : \mathcal{M}^0 \to \mathbb{R}$ given by

$$E(u) := \frac{\int_U |\nabla u|^p}{\int_U |u|^p}$$

and the eigenfunctions of (3.2) are precisely the critical points of E on $\mathcal{M}^0$. Moreover, E satisfies the Palais-Smale compactness condition ([30] Lemma 4.5).

Critical points of E can be produced using a min-max argument with the cohomological index. The following discussion of the cohomological index is based on Chapter 2 of [30]. Note that there is a natural $\mathbb{Z}_2$-action on $\mathcal{M}^0$ and that E respects this action, i.e., $E(u) = E(-u)$.

Definition 3.2.1. *A subset $A \subset \mathcal{M}^0$ is symmetric provided that $u \in A$ if and only if $-u \in A$*

for every $u \in \mathcal{M}^0$.

Definition 3.2.2. *A* $\mathbb{Z}_2$*-space is a Hausdorff, paracompact topological space equipped with a free* $\mathbb{Z}_2$ *action.*

Definition 3.2.3. *Let* A *and* B *be* $\mathbb{Z}_2$*-spaces. A map* $f : A \to B$ *is called odd provided* $f(-a) = -f(a)$ *for all* $a \in A$.

A symmetric set $A \subset \mathcal{M}^0$ is paracompact since $\mathcal{M}^0$ is a metric space and every metric space is paracompact. Moreover, A comes equipped with a free $\mathbb{Z}_2$-action $u \mapsto -u$. Thus A is a $\mathbb{Z}_2$-space. Now assume that A is any $\mathbb{Z}_2$-space and let $\overline{A}$ be the quotient $A/\mathbb{Z}_2$. Then

$$\pi : A \to \overline{A}$$

is a principal $\mathbb{Z}_2$-bundle over $\overline{A}$ and so there is a classifying map $f \colon \overline{A} \to \mathbb{R}\mathrm{P}^\infty$. Let H^*_{AS} denote Alexander-Spanier cohomology. The cohomology ring $H^*_{AS}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[\sigma]$ where σ is the non-zero element in $H^1_{AS}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ and the classifying map induces a map in cohomology

$$f^* : H^*_{AS}(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \to H^*_{AS}(\overline{A}; \mathbb{Z}_2).$$

The following definition is originally due to Fadell and Rabinowitz [6].

Definition 3.2.4. *Let* A *be a* $\mathbb{Z}_2$*-space and let* f *be a classifying map for the bundle* $\pi \colon A \to \overline{A}$*. Then the cohomological index of* A *is*

$$\mathrm{ind}_{AS}(A) = \sup\{k \geq 1 : \, f^*(\sigma^{k-1}) \neq 0 \ \textit{in} \ H^*_{AS}(\overline{A}; \mathbb{Z}_2)\}.$$

By convention $\mathrm{ind}_{AS}(\emptyset) = 0$.

The properties of the cohomological index will be discussed further in the next section of the paper.

It remains to perform the min-max argument. For complete details see Chapter 4 of [30]. Let $\mathcal{F}^0 = \mathcal{F}^0(U)$ be the collection of all symmetric subsets of $\mathcal{M}^0$. Define the classes

$$\mathcal{F}_k^0 = \mathcal{F}_k^0(U) = \{A \in \mathcal{F}^0 : \operatorname{ind}_{AS}(A) \geq k\}$$

and then consider the min-max values

$$\lambda_k = \lambda_k(U) = \inf_{A \in \mathcal{F}_k^0} \sup_{u \in A} E(u).$$

By [30] Theorem 4.6, the numbers λ_k are Dirichlet eigenvalues of Δ_p on U and $\lambda_k \to \infty$ as $k \to \infty$. Moreover, if we define the counting function

$$N_U^0(\lambda) = \#\{j : \lambda_j < \lambda\}$$

then N_U^0 satisfies $N_U^0(\lambda) = \operatorname{ind}_{AS}(E^{-1}[0,\lambda))$. The sequence (λ_k) is called the variational spectrum of Δ_p on U. It doesn't seem to be known (see [30] page 71) whether the variational spectrum contains every eigenvalue of Δ_p.

Remark 3.2.5. *Let A be a $\mathbb{Z}_2$-space with classifying map f. Let H^* denote singular cohomology and let σ be the non-zero element in $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$. Define*

$$\operatorname{ind}(A) = \sup\{k \geq 1 : f^*(\sigma^{k-1}) \neq 0 \text{ in } H^*(\overline{A}; \mathbb{Z}_2)\}.$$

If A is locally contractible then the Alexander-Spanier cohomology of A is isomorphic to the singular cohomology of A (see [35]) and thus $\operatorname{ind}_{AS}(A) = \operatorname{ind}(A)$. Suppose now that $A = E^{-1}[0,\lambda) \subset \mathcal{M}^0$. Note that $\mathcal{M}^0$ is locally contractible since it is a Banach manifold. Thus A is also locally contractible since it is an open subset of a locally contractible space. It follows that $\operatorname{ind}_{AS}(A) = \operatorname{ind}(A)$. In particular, the counting function satisfies $N_U^0(\lambda) = \operatorname{ind}(E^{-1}[0,\lambda))$.

3.3 PRELIMINARIES: THE NEUMANN PROBLEM

It will be useful to simultaneously investigate the corresponding Neumann eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u, & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial U. \end{cases}$$

To this end, define the set

$$\mathcal{M} = \mathcal{M}(U) := \left\{ u \in W^{1,p}(U) : \, \|u\|_{W^{1,p}(U)} = 1 \right\}.$$

Again there is an energy $E : \mathcal{M} \to \mathbb{R}$ given by

$$E(u) = \frac{\int_U |\nabla u|^p}{\int_U |u|^p}.$$

Define the counting function

$$N_U(\lambda) = \operatorname{ind}(E^{-1}[0, \lambda)).$$

Then $N_U^0(\lambda) \leq N_U(\lambda)$ for every λ. We will show that the Neumann counting function also satisfies a Weyl law $N_U(\lambda) \sim c \operatorname{Vol}(U) \lambda^{n/p}$.

Remark 3.3.1. *Throughout the paper objects associated with the Dirichlet problem will be decorated with a superscript zero while the corresponding objects in the Neumann problem will appear without decoration. Thus N_U^0 denotes the counting function for the Dirichlet problem while N_U denotes the counting function for the Neumann problem, and so on. This is consistent with the notation in [11].*

3.4 PRELIMINARIES: PROPERTIES OF THE COHOMOLOGICAL INDEX

This section collects a few properties of the cohomological index that will be needed later. Proofs of the following properties can be found in [6] or [30]. Since the proofs are relatively short, we reproduce them below for the convenience of the reader.

Proposition 3.4.1. *The index* ind *satisfies the following properties.*

(i) If A, B are $\mathbb{Z}_2$-spaces and there is an odd continuous map $f : A \to B$ then $\operatorname{ind}(A) \leq \operatorname{ind}(B)$.

(ii) If X is a $\mathbb{Z}_2$-space and $A, B \subset X$ are open, paracompact symmetric subsets with $X = A \cup B$ then $\operatorname{ind}(X) \leq \operatorname{ind}(A) + \operatorname{ind}(B)$.

Proof. (i) Let $\overline{f} : \overline{A} \to \overline{B}$ be the induced map and let $g : \overline{B} \to \mathbb{R}\mathrm{P}^\infty$ be a classifying map. Then $g\overline{f} : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ serves as a classifying map for $\overline{A}$. Notice that there are maps

$$H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) \xrightarrow{g^*} H^*(\overline{B}; \mathbb{Z}_2) \xrightarrow{\overline{f}^*} H^*(\overline{A}; \mathbb{Z}_2)$$

and thus $\operatorname{ind}(A) \leq \operatorname{ind}(B)$.

(ii) Let $f : \overline{X} \to \mathbb{R}\mathrm{P}^\infty$ be a classifying map and let

$$i_A : \overline{A} \to \overline{X}, \quad i_B : \overline{B} \to \overline{X}$$

be the inclusions. Then $f\iota_A$ is a classifying map for $\overline{A}$ and $f\iota_B$ is a classifying map for $\overline{B}$. Without loss, we may assume that $\operatorname{ind}(A) = i$ and $\operatorname{ind}(B) = j$ are finite. Put $\theta = f^*\sigma$. Then $\iota_A^*\theta^i = 0$ and $\iota_B^*\theta^j = 0$. There are exact sequences

$$H^i(\overline{X}, \overline{A}) \to H^i(\overline{X}) \to H^i(\overline{A}),$$

$$H^j(\overline{X}, \overline{B}) \to H^j(\overline{X}) \to H^i(\overline{B}),$$

and hence there are classes $\theta_A \in H^i(\overline{X}; \overline{A})$ and $\theta_B \in H^j(\overline{X}; \overline{B})$ that map to θ.

Since A, B are open in X there is a relative cup product

$$\smile : H^*(\overline{X}, \overline{A}) \times H^*(\overline{X}, \overline{B}) \to H^*(\overline{X}, \overline{A} \cup \overline{B}).$$

Moreover, since $\overline{X} = \overline{A} \cup \overline{B}$, it follows that $\theta_A \smile \theta_B = 0$ in $H^*(\overline{X}, \overline{A} \cup \overline{B})$. By naturality, θ^{i+j} is the image of $\theta_A \smile \theta_B$ under the map

$$H^*(\overline{X}, \overline{A} \cup \overline{B}) \to H^*(\overline{X}).$$

Therefore $\theta^{i+j} = 0$ and the result follows. □

One further property will be required. Assume that A and B are $\mathbb{Z}_2$-spaces. By definition, their join is the quotient

$$A * B := (A \times B \times [0,1]) / \sim$$

where $(a_1, b, 0) \sim (a_2, b, 0)$ and $(a, b_1, 1) \sim (a, b_2, 1)$. This is also $\mathbb{Z}_2$-space in a natural way.

Proposition 3.4.2. *Let A and B be $\mathbb{Z}_2$-spaces. Then*

$$\operatorname{ind}(A) + \operatorname{ind}(B) \leq \operatorname{ind}(A * B).$$

Remark 3.4.3. *The special case of this proposition where $B = S^0$ is proven in [6]. Presumably the general case is also known, but we could not find a reference in the literature and hence we provide a proof below.*

Our proof of Proposition 3.4.2 is based on a join operation in homology constructed in [10]. We summarize the construction below. Let X and Y be topological spaces. If Δ^m and Δ^n are the standard simplices, then there is a natural identification $\Delta^m * \Delta^n \cong \Delta^{m+n+1}$. Thus given singular simplices $\alpha : \Delta^m \to X$ and $\beta : \Delta^n \to Y$ one can form a new singular simplex

$$\alpha * \beta : \Delta^{m+n+1} \to X * Y.$$

In the following all groups have $\mathbb{Z}_2$-coefficients, even where this is not explicitly indicated in the notation. Extending linearly, there is a map on chains

$$* : C_m(X) \otimes C_n(Y) \to C_{m+n+1}(X * Y)$$

and since

$$\partial(\alpha * \beta) = (\partial\alpha) * \beta + \alpha * (\partial\beta),$$

this descends to a map in homology

$$* : H_m(X) \otimes H_n(Y) \to H_{m+n+1}(X * Y).$$

Using this, it is possible to construct an equivariant join operation on $\mathbb{Z}_2$-spaces.

Suppose now that X and Y are $\mathbb{Z}_2$-spaces and recall that $\overline{X} = X/\mathbb{Z}_2$ and $\overline{Y} = Y/\mathbb{Z}_2$. Let g be the antipodal map on some $\mathbb{Z}_2$-space Z. Then a chain c in Z is called $\mathbb{Z}_2$-equivariant provided $g_{\#}c = c$. Following [10], let $C_*(Z)^{\mathbb{Z}_2}$ denote the set of $\mathbb{Z}_2$-equivariant chains in Z. There is a natural identification

$$C_*(\overline{Z}) \leftrightarrow C_*(Z)^{\mathbb{Z}_2}$$

given by sending $\alpha : \Delta^m \to \overline{Z}$ to the sum of its two lifts to Z. Since the join operation respects equivariance, there is an induced operation

$$* : C_m(X)^{\mathbb{Z}_2} \otimes C_n(Y)^{\mathbb{Z}_2} \to C_{m+n+1}(X * Y)^{\mathbb{Z}_2}.$$

Using the above equivalence, this gives an operation

$$\star : C_m(\overline{X}) \otimes C_n(\overline{Y}) \to C_{m+n+1}(\overline{X * Y})$$

and again this descends to give an operation in homology

$$\star : H_m(\overline{X}) \otimes H_n(\overline{Y}) \to H_{m+n+1}(\overline{X * Y}).$$

It is possible to compute this map in the case where $X = S^\infty$ and $Y = S^\infty$.

Take $X = S^\infty$ and $Y = S^\infty$. Note that $\overline{X}$ and $\overline{Y}$ are both homeomorphic to $\mathbb{R}\mathrm{P}^\infty$ and hence $H_*(\overline{X}) \cong H_*(\overline{Y}) \cong \mathbb{Z}_2[x]$ in the sense of additive groups. Now think of S^∞ as the set of points $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^\infty$ such that

$$\sum_{i=1}^{\infty} x_i^2 = 1$$

and all but finitely many x_i are equal to 0. There is a homeomorphism $j : X * Y \to S^\infty$ given by

$$j((x_i),(y_i),t) = (x_1\sqrt{t}, y_1\sqrt{1-t}, x_2\sqrt{t}, y_2\sqrt{1-t}, \ldots).$$

This gives an identification of $X * Y$ with S^∞ and hence $H_*(\overline{X * Y})$ is also isomorphic to $\mathbb{Z}_2[x]$.

Fix a pair of non-negative integers m and n. Let $\mathcal{X}$ be the set of points (x_i) in X such that $x_i = 0$ for all $i > m+1$, and let $\mathcal{Y}$ be the set of points (y_i) in Y such that $y_i = 0$ for all $i > n+1$. Let

$$\overline{\mathcal{X}} \subset \overline{X}, \quad \overline{\mathcal{Y}} \subset \overline{Y}$$

be the quotients. Then $[\overline{\mathcal{X}}]$ is the non-trivial element of $H_m(\overline{X})$ and $[\overline{\mathcal{Y}}]$ is the non-trivial element of $H_n(\overline{Y})$. Let $\mathcal{Z}$ be the set of points (z_i) in S^∞ such that $z_{2i-1} = 0$ for all $i > m+1$ and $z_{2j} = 0$ for all $j > n+1$. Then from the definition of the operation $\star$, it follows that

$$j_*([\overline{\mathcal{X}}] \star [\overline{\mathcal{Y}}]) = [\overline{\mathcal{Z}}].$$

Since j_* is an isomorphism, this means that $[\overline{\mathcal{X}}]\star[\overline{\mathcal{Y}}]$ is the non-trivial class in $H_{m+n+1}(\overline{X * Y})$.

Proof. (Proposition 3.4.2) Let $\overline{A} = A/\mathbb{Z}_2$ and $\overline{B} = B/\mathbb{Z}_2$ and let

$$f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty, \quad g : \overline{B} \to \mathbb{R}\mathrm{P}^\infty$$

be classifying maps for the bundles $A \to \overline{A}$, $B \to \overline{B}$. Recall that there is a homeomorphism $j : S^\infty * S^\infty \to S^\infty$. Since f, g are classifying maps, we get the following commutative diagrams.

$$\begin{tikzcd}
A \arrow[r, "\widetilde{f}"] \arrow[d] & S^\infty \arrow[d] & B \arrow[r, "\widetilde{g}"] \arrow[d] & S^\infty \arrow[d] \\
\overline{A} \arrow[r, "f"'] & \mathbb{R}\mathrm{P}^\infty & \overline{B} \arrow[r, "g"'] & \mathbb{R}\mathrm{P}^\infty
\end{tikzcd}$$

Thus it is possible to define a map $\widetilde{h} = \widetilde{f} * \widetilde{g} : A * B \to S^\infty * S^\infty$. Moreover $\widetilde{h}$ is odd and so it induces a map

$$h : \overline{A * B} \to \overline{S^\infty * S^\infty} \cong \mathbb{R}\mathrm{P}^\infty.$$

This is a classifying map for the bundle $A * B \to \overline{A * B}$.

Since $\mathbb{Z}_2$ is a field, the universal coefficient theorem implies that

$$H^i(X; \mathbb{Z}_2) \cong \mathrm{Hom}(H_i(X; \mathbb{Z}_2), \mathbb{Z}_2).$$

Now assume $m+1 \leq \mathrm{ind}(A)$ and $n+1 \leq \mathrm{ind}(B)$. Then $f^*(\sigma^m)$ and $g^*(\sigma^n)$ are non-zero and hence there exist classes

$$\alpha \in H_m(\overline{A}; \mathbb{Z}_2), \quad \beta \in H_n(\overline{B}; \mathbb{Z}_2)$$

such that $f_*\alpha$ and $g_*\beta$ are non-zero. Define $\gamma = \alpha \star \beta \in H_{m+n+1}(\overline{A * B})$. By naturality of the above construction, it follows that

$$h_*(\gamma) = (f_*\alpha) \star (g_*\beta), \quad \text{in } H_{m+n+1}(\overline{S^\infty * S^\infty}).$$

But we know the class on the right hand side of the above equation is non-zero and therefore $h^*(\sigma^{m+n+1})$ is non-zero. The result follows. ☐

3.5 THE DIRICHLET DOMAIN MONOTONICITY INEQUALITY

Assume that V, W are disjoint open sets with $V, W \subseteq U$. Recall that when $p = 2$ the Dirichlet eigenvalues of the Laplacian satisfy a domain monotonicity inequality of the form

$$N_U^0(\lambda) \geq N_V^0(\lambda) + N_W^0(\lambda).$$

We want to show that a similar inequality holds for arbitrary p. The arguments in this section closely follow Gromov in [11]. The first step is to prove an inequality relating the energy of functions v on V and w on W with the energy of $v + w$ on U.

Lemma 3.5.1 (See [11] Lemma 3.2.A)**.** *Assume V, W are disjoint open sets with $V, W \subset U$. Let $v \in W_0^{1,p}(V)$ and $w \in W_0^{1,p}(W)$ not both zero and define $u := v + w \in W_0^{1,p}(U)$. Then $E(u) \leq \max\{E(v), E(w)\}$.*

Proof. This is obvious if either $v \equiv 0$ or $w \equiv 0$, so we may assume that v and w are not identically 0. Without loss of generality, we may also assume that $E(w) \leq E(v)$. Define the numbers

$$a := \int_V |\nabla v|^p, \quad b := \int_W |\nabla w|^p, \quad c := \int_V |v|^p, \quad d := \int_W |w|^p.$$

Elementary manipulations show that

$$\frac{a+b}{c+d} \leq \frac{a}{c} \iff \frac{b}{d} \leq \frac{a}{c},$$

and the inequality on the right holds since $E(w) \leq E(v)$. It follows that

$$E(u) = \frac{a+b}{c+d} \leq \frac{a}{c} = E(v).$$

and the lemma is proven. $\square$

The next step is to prove a monotonicity inequality.

Proposition 3.5.2 (See [11] 3.2.A$_1$)**.** *Assume V, W are disjoint open sets with $V, W \subset U$. Then we have $N^0_U(\lambda) \geq N^0_V(\lambda) + N^0_W(\lambda)$.*

Proof. Define the sets

$$A := \{v \in \mathcal{M}^0(V) : \, E(v) < \lambda\},$$
$$B := \{w \in \mathcal{M}^0(W) : \, E(w) < \lambda\},$$
$$C := \{u \in \mathcal{M}^0(U) : \, E(u) < \lambda\}.$$

Then $N^0_V(\lambda) = \operatorname{ind}(A)$, $N^0_W(\lambda) = \operatorname{ind}(B)$, $N^0_U(\lambda) = \operatorname{ind}(C)$ and hence we must verify that

$$\operatorname{ind}(A) + \operatorname{ind}(B) \leq \operatorname{ind}(C).$$

Consider the join $A * B$. The map $A * B \to \mathcal{M}^0(U)$ given by

$$(v, w, t) \mapsto tv + (1-t)w$$

is a homeomorphism onto its image. Hence $A * B$ can be viewed as a subset of $\mathcal{M}^0(U)$. Lemma 3.5.1 shows that actually $A * B \subset C$ and thus

$$\operatorname{ind}(A * B) \leq \operatorname{ind}(C)$$

by Proposition 3.4.1(i). But Proposition 3.4.2 says

$$\operatorname{ind}(A) + \operatorname{ind}(B) \leq \operatorname{ind}(A * B).$$

Combining these inequalities gives $\operatorname{ind}(A) + \operatorname{ind}(B) \leq \operatorname{ind}(C)$, as needed. $\square$

It is possible to be slightly more general. Given U and a positive real number a, let aU be the set $\{ax : x \in U\}$. Then the scaling properties of the energy lead to a relationship between $N^0_U(\lambda)$ and $N^0_{aU}(\lambda)$. This is the content of the following proposition.

Proposition 3.5.3. *Let a be a positive real number. Then $N^0_{aU}(\lambda) = N^0_U(a^p \lambda)$.*

Proof. Define a map $g : W^{1,p}_0(U) \setminus \{0\} \to W^{1,p}_0(aU) \setminus \{0\}$ by

$$(gu)(x) = u(x/a).$$

This map induces a homeomorphism

$$g : \mathcal{M}^0(U) \cong \mathcal{M}^0(aU).$$

Thus $\operatorname{ind}(A) = \operatorname{ind}(g(A))$ for every symmetric $A \subset \mathcal{M}^0(U)$. Moreover, a straightforward calculation shows that

$$E(gu) = a^{-p} E(u)$$

for every $u \in \mathcal{M}^0(U)$. The result follows. $\square$

Now suppose that $U, U_1, \ldots, U_m$ are open sets in $\mathbb{R}^n$ and that $a_1, \ldots, a_m$ are positive real numbers. Following Gromov [11], we will write

$$\sum_{i=1}^{m} a_i U_i \prec U$$

if there exist elements $b_1, \ldots, b_m \in \mathbb{R}^n$ such that the translates $a_i U_i + b_i$ are all disjoint and contained in U. Using Proposition 3.5.2 and Proposition 3.5.3 and induction shows that

$$\sum_{i=1}^{m} a_i U_i \prec U \implies N_U^0(\lambda) \geq \sum_{i=1}^{m} N_{U_i}^0(a_i^p \lambda). \tag{3.3}$$

We will refer to this as the Dirichlet domain monotonicity inequality.

3.6 THE WEYL LAW FOR DIRICHLET EIGENVALUES

In this section we prove the Dirichlet Weyl law for domains in $\mathbb{R}^n$. The first step is to prove the Weyl law for a cube. The argument is essentially the same as the proof of Lemma 3.3 in [20]. Also see the Trivial Lemma in Section 3.4 of [11].

Lemma 3.6.1. *Let C be the unit cube in $\mathbb{R}^n$ and define $f(\lambda) = \lambda^{-n/p} N_C^0(\lambda)$. Then f tends to a limit as $\lambda \to \infty$.*

Proof. Choose sequences (λ_j), (μ_k) so that

$$\limsup_{\lambda \to \infty} f(\lambda) = \lim_{j \to \infty} f(\lambda_j), \quad \liminf_{\lambda \to \infty} f(\lambda) = \lim_{k \to \infty} f(\mu_k).$$

Now fix some j and consider k large. Let M_k be the largest integer such that it is possible to pack M_k disjoint open cubes of volume $(\lambda_j/\mu_k)^{n/p}$ into C. Then

$$\sum_{\ell=1}^{M_k} (\lambda_j/\mu_k)^{1/p} C \prec C$$

and it follows by the domain monotonicity inequality (3.3) that

$$N_C^0(\lambda) \geq M_k N_C^0(\lambda_j \lambda/\mu_k).$$

Choose $\lambda = \mu_k$ and multiply both sides by $\mu_k^{-n/p}$ to get

$$\mu_k^{-n/p} N_C^0(\mu_k) \geq M_k \mu_k^{-n/p} N_C^0(\lambda_j).$$

Now let $k \to \infty$ and use the fact that $M_k \mu_k^{-n/p} \to \lambda_j^{-n/p}$ to get

$$\liminf_{\lambda\to\infty} f(\lambda) \geq \lambda_j^{-n/p} N_C^0(\lambda_j).$$

Finally let $j \to \infty$ to get

$$\liminf_{\lambda\to\infty} f(\lambda) \geq \limsup_{\lambda\to\infty} f(\lambda).$$

The result follows. $\square$

Define $c^0 = \lim_{\lambda\to\infty} f(\lambda)$. As mentioned in Section 1, estimates of Friedlander [8] imply that

$$C_1 \lambda^{n/p} \leq N_C^0(\lambda) \leq C_2 \lambda^{n/p}, \quad \text{as } \lambda \to \infty.$$

It follows at once that $0 < c^0 < \infty$. Thus the Weyl law holds on the unit cube, i.e., we have $N_C^0(\lambda) \sim c^0 \lambda^{n/p}$.

It is also possible to give a self-contained proof that $0 < c^0 < \infty$ which does not rely on the estimates of Friedlander. The next proposition shows that $c^0 > 0$. We will give the proof that $c^0 < \infty$ in a later section.

Proposition 3.6.2. *Put* $c^0 = \lim_{\lambda\to\infty} \lambda^{-n/p} N_C^0(\lambda)$. *Then* $c^0 > 0$.

Proof. As above let $f(\lambda) = \lambda^{-n/p} N_C^0(\lambda)$. Fix some $\lambda > 0$ and some positive integer k. It is possible to divide unit cube into k^n disjoint open cubes of volume k^{-n}. Hence

$$\sum_{\ell=1}^{k^n} k^{-1} C \prec C$$

and so the domain monotonicity inequality implies

$$N_C^0(\lambda) \geq k^n N_C^0(k^{-p}\lambda).$$

Multiplying by $\lambda^{-n/p}$ yields

$$f(\lambda) = \lambda^{-n/p} N_C^0(\lambda) \geq (k^{-p}\lambda)^{-n/p} N_C^0(k^{-p}\lambda) = f(k^{-p}\lambda).$$

Now there exists some λ_1 such that $N_C^0(\lambda) \geq 1$ for all $\lambda \geq \lambda_1$. Hence f has a positive minimum C_1 on the interval $[\lambda_1, 2^p\lambda_1]$. Since

$$(k+1)^p \leq (2k)^p, \quad \text{for all } k \geq 1$$

it follows that

$$[\lambda_1, \infty) = \bigcup_{k=1}^{\infty} [k^p\lambda_1, k^p 2^p \lambda_1].$$

Therefore, given an arbitrary $\lambda \geq \lambda_1$, there exists some positive integer k such that $k^{-p}\lambda \in [\lambda_1, 2^p\lambda_1]$. Using the above inequality, it follows that

$$f(\lambda) \geq f(k^{-p}\lambda) \geq C_1.$$

This proves that $c^0 \geq C_1 > 0$, as needed. □

Finally we derive the Weyl law for a general domain in $\mathbb{R}^n$.

Theorem 3.6.3. *Let $U \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then $N_U^0(\lambda) \sim c^0 \operatorname{Vol}(U)\lambda^{n/p}$ as $\lambda \to \infty$.*

Proof. Without loss of generality we can assume that U has unit volume. Define the numbers

$$\underline{\beta}(U) = \liminf_{\lambda\to\infty} \lambda^{-n/p} N_U^0(\lambda), \quad \overline{\beta}(U) = \limsup_{\lambda\to\infty} \lambda^{-n/p} N_U^0(\lambda).$$

Let $\varepsilon > 0$ and choose numbers $a_1, \ldots, a_m$ such that

$$\sum_{i=1}^{m} a_i C \prec U$$

and $\sum_{i=1}^{m} a_i^n \geq 1 - \varepsilon$. Applying the domain monotonicity inequality gives

$$N_U^0(\lambda) \geq \sum_{i=1}^{m} N_C^0(a_i^p \lambda).$$

But $N_C^0(a_i^p\lambda) \sim c^0 a_i^n \lambda^{n/p}$ as $\lambda \to \infty$ since the Weyl law holds for cubes. Hence multiplying both sides of the above inequality by $\lambda^{-n/p}$ and letting $\lambda \to \infty$ it follows that

$$\underline{\beta}(U) \geq c^0 \left(\sum_{i=1}^{m} a_i^n \right) \geq c(1 - \varepsilon).$$

Since ε was arbitrary this implies that $\underline{\beta}(U) \geq c^0$.

It remains to show that $\overline{\beta}(U) \leq c^0$. To this end, fix $a > 0$ so that some translate of aU is contained in C. Let $\varepsilon > 0$ and choose numbers $a_1, \ldots, a_m$ so that

$$aU + \sum_{i=1}^{m} a_i C \prec C$$

and $\sum_{i=1}^{m} a_i^n \geq 1 - a^n - \varepsilon$. Then

$$N_C^0(\lambda) \geq N_U^0(a^p\lambda) + \sum_{i=1}^{m} N_C^0(a_i^p \lambda).$$

Pick a sequence $\lambda_j \to \infty$ so that

$$a^{-n}\lambda_j^{-n/p} N_U^0(a^p \lambda_j) \to \overline{\beta}(U).$$

Taking $\lambda = \lambda_j$ in the above equation and multiplying both sides by $a^{-n}\lambda_j^{-n/p}$ gives

$$a^{-n}\lambda_j^{-n/p} N_C^0(\lambda_j) \geq a^{-n}\lambda_j^{-n/p} N_U^0(a^p\lambda_j) + \sum_{i=1}^{m} a^{-n}\lambda_j^{-n/p} N_C^0(a_i^p \lambda_j).$$

Letting $j \to \infty$ now yields

$$\begin{aligned} a^{-n}c^0 &\geq \overline{\beta}(U) + a^{-n}\sum_{i=1}^{m} a_i^n c^0 \\ &\geq \overline{\beta}(U) + a^{-n}(1 - a^n - \varepsilon)c^0. \end{aligned}$$

Therefore

$$\overline{\beta}(U) \leq (1 + a^{-n}\varepsilon)c^0$$

and hence $\overline{\beta}(U) \leq c^0$ since ε was arbitrary. This proves that $\overline{\beta}(U) = \underline{\beta}(U) = c^0$ and so the Weyl law $N_U^0(\lambda) \sim c^0\lambda^{n/p}$ holds for U. $\square$

3.7 THE NEUMANN MONOTONICITY INEQUALITY

Assume that V, W are open subsets of U with $\overline{U} = \overline{V} \cup \overline{W}$ and that $\overline{V} \cap \overline{W}$ has measure 0. Recall that when $p = 2$, the Neumann eigenvalues of the Laplacian satisfy a domain monotonicity inequality of the form

$$N_U(\lambda) \leq N_V(\lambda) + N_W(\lambda).$$

We want to show that a similar inequality holds for arbitrary p. Again the first step is to prove an energy inequality.

Lemma 3.7.1 (See [11] Lemma 3.2.A)**.** *Assume V, W are open subsets of U with $\overline{U} = \overline{V} \cup \overline{W}$ and that $\overline{V} \cap \overline{W}$ has measure 0. Let $u \in W^{1,p}(U)$ and then define $v = u|_V \in W^{1,p}(V)$ and $w = u|_W \in W^{1,p}(W)$ so that $u = v + w$. Assume neither v nor w is identically zero. Then $E(u) \geq \min\{E(v), E(w)\}$.*

Proof. Without loss of generality we can assume that $E(v) \leq E(w)$. Define the numbers

$$a := \int_V |\nabla v|^p, \quad b := \int_W |\nabla w|^p, \quad c := \int_V |v|^p, \quad d := \int_W |w|^p.$$

Elementary manipulations show that

$$\frac{a+b}{c+d} \geq \frac{a}{c} \iff \frac{a}{c} \leq \frac{b}{d}$$

and the inequality on the right holds since $E(w) \geq E(v)$. It follows that

$$E(u) = \frac{a+b}{c+d} \geq \frac{a}{c} = E(v),$$

as needed. □

The next step is a monotonicity inequality.

Proposition 3.7.2 (See [11] 3.2.A$_2$)**.** *Assume V, W are open subsets of U with $\overline{U} = \overline{V} \cup \overline{W}$ and that $\overline{V} \cap \overline{W}$ has measure 0. Then $N_U(\lambda) \leq N_V(\lambda) + N_W(\lambda)$.*

Proof. Define the sets

$$A := \{v \in \mathcal{M}(V) : E(v) < \lambda\},$$

$$B := \{w \in \mathcal{M}(W) : E(w) < \lambda\},$$

$$C := \{u \in \mathcal{M}(U) : E(u) < \lambda\}.$$

Then $N_U(\lambda) = \operatorname{ind}(C)$ and $N_V(\lambda) + N_W(\lambda) = \operatorname{ind}(A) + \operatorname{ind}(B)$. Hence it is enough to show that

$$\operatorname{ind}(C) \leq \operatorname{ind}(A) + \operatorname{ind}(B).$$

To see this, define the sets

$$C_A := \{u \in C : u|_V \not\equiv 0 \text{ and } E(u|_V) < \lambda\},$$

$$C_B := \{u \in C : u|_W \not\equiv 0 \text{ and } E(u|_W) < \lambda\}.$$

Lemma 3.7.1 shows that $C \subset C_A \cup C_B$. Moreover, the sets C_A and C_B are open in C and hence $\operatorname{ind}(C) \leq \operatorname{ind}(C_A) + \operatorname{ind}(C_B)$ by Proposition 3.4.1(ii). But there is an odd continuous map

$$C_A \to A, \quad u \mapsto \frac{u|_V}{\|u|_V\|}$$

and therefore $\operatorname{ind}(C_A) \leq \operatorname{ind}(A)$ by Proposition 3.4.1(i). The same argument shows that $\operatorname{ind}(C_B) \leq \operatorname{ind}(B)$ and thus $\operatorname{ind}(C) \leq \operatorname{ind}(A) + \operatorname{ind}(B)$. The result follows. □

As in the Dirichlet case, this can be combined with the scaling properties of the energy to give a more general monotonicity inequality. Suppose $U, U_1, \ldots, U_n$ are open in $\mathbb{R}^n$ and $a_1, \ldots, a_m$ are positive real numbers. Then we will write

$$\sum_{i=1}^{m} a_i U_i \approx U$$

if there exist elements $b_1, \ldots, b_n \in \mathbb{R}^n$ such that

$$\overline{U} = \bigcup_{i=1}^{m} (\overline{a_i U + b_i})$$

and each intersection $(\overline{a_i U_i + b_i}) \cap (\overline{a_j U_j + b_j})$ has measure zero. Using Proposition 3.4, the scaling properties of the energy, and induction shows that

$$\sum_{i=1}^{m} a_i U_i \approx U \implies N_U(\lambda) \leq \sum_{i=1}^{m} N_{U_i}(a_i^p \lambda). \tag{3.4}$$

We will refer to this as the Neumann domain monotonicity inequality.

3.8 THE WEYL LAW FOR NEUMANN EIGENVALUES

In this section we prove the Neumann Weyl law for domains in $\mathbb{R}^n$. The first step is to prove the Weyl law on a cube.

Lemma 3.8.1. *Let C be the unit cube in $\mathbb{R}^n$ and define $g(\lambda) = \lambda^{-n/p} N_C(\lambda)$. Then g tends to a limit as $\lambda \to \infty$.*

Proof. Choose sequences (λ_j), (μ_k) so that

$$\limsup_{\lambda\to\infty} g(\lambda) = \lim_{j\to\infty} g(\lambda_j), \quad \liminf_{\lambda\to\infty} g(\lambda) = \lim_{k\to\infty} g(\mu_k).$$

Now fix some k and consider j large. Let $\varepsilon_j \geq 0$ be the smallest number such that $(\mu_k/\lambda_j + \varepsilon_j)^{1/p}$ is the reciprocal of an integer. Then cubes of volume $(\mu_k/\lambda_j + \varepsilon_j)^{n/p}$ partition the cube C. Let M_j be the number of cubes in such a partition.

We now estimate ε_j and M_j. Define $t_j := \mu_k/\lambda_j$ and then let ℓ_j be the largest integer such that

$$\ell_j \leq t_j^{-1/p}.$$

Then $\varepsilon_j = \ell_j^{-p} - t_j \geq 0$. Moreover $t_j^{-1/p} - 1 \leq \ell_j \leq t_j^{-1/p}$ and thus

$$\varepsilon_j \lambda_j \leq \left((t_j^{-1/p} - 1)^{-p} - t_j \right) \frac{\mu_k}{t_j} \to 0, \quad \text{as } j \to \infty.$$

Also notice that $M_j = \ell_j^n$ and therefore

$$(t_j^{-1/p} - 1)^n \left(\frac{\mu_k}{t_j} \right)^{-n/p} \leq M_j \lambda_j^{-n/p} \leq \mu_k^{-n/p}$$

so that $M_j \lambda_j^{-n/p} \to \mu_k^{-n/p}$ as $j \to \infty$.

Given these estimates, the result can be obtained as follows. Observe that

$$\sum_{\ell=1}^{M_j} (\mu_k/\lambda_j + \varepsilon_j)^{1/p} C \approx C$$

and hence the domain monotonicity inequality gives

$$N_C(\lambda) \leq M_j N_C(\mu_k \lambda/\lambda_j + \varepsilon_j \lambda).$$

Choosing $\lambda = \lambda_j$ and multiplying both sides by $\lambda_j^{-n/p}$ yields

$$\lambda_j^{-n/p} N_C(\lambda_j) \leq M_j \lambda_j^{-n/p} N_C(\mu_k + \varepsilon_j \lambda_j).$$

Letting $j \to \infty$ and using the fact that $M_j \lambda_j^{-n/p} \to \mu_k^{-n/p}$ and $\varepsilon_j \lambda_j \to 0$ this gives

$$\limsup_{\lambda \to \infty} g(\lambda) \leq \mu_k^{-n/p} N_C(\mu_k).$$

Finally let $k \to \infty$ to get

$$\limsup_{\lambda \to \infty} g(\lambda) \leq \liminf_{\lambda \to \infty} g(\lambda),$$

as needed. $\square$

Define $c = \lim_{\lambda \to \infty} g(\lambda)$. It is obvious that $c^0 \leq c$ and hence $c > 0$. The next proposition shows that $c < \infty$. It will be shown in a later section that actually $c^0 = c$.

Proposition 3.8.2. *Set* $c = \lim_{\lambda \to \infty} \lambda^{-n/p} N_C(\lambda)$. *Then* $c < \infty$.

Proof. As above, let $g(\lambda) = \lambda^{-n/p} N_C(\lambda)$. Fix some $\lambda > 0$ and some positive integer k. The unit cube can be partitioned into k^n cubes of volume k^{-n}. Hence

$$\sum_{\ell=1}^{k^n} k^{-1} C \approx C$$

and so the domain monotonicity inequality implies

$$N_C(\lambda) \leq k^n N_C(k^{-p} \lambda).$$

Multiplying by $\lambda^{-n/p}$ gives

$$g(\lambda) = \lambda^{-n/p} N_C(\lambda) \leq (k^{-p} \lambda)^{-n/p} N_C(k^{-p} \lambda) = g(k^{-p} \lambda).$$

Now g has some finite maximum C_2 on the interval $[1, 2^p]$. Since

$$(k+1)^p \leq (2k)^p, \quad \text{for all } k \geq 1$$

it follows that

$$[1, \infty) = \bigcup_{k=1}^{\infty} [k^p, k^p 2^p].$$

Therefore, given an arbitrary $\lambda \geq 1$, there exists some positive integer k such that $k^{-p}\lambda \in [1, 2^p]$. Using the above inequality, it follows that

$$g(\lambda) \leq g(k^{-p}\lambda) \leq C_2.$$

This proves that $c \leq C_2 < \infty$, as needed. □

The proof of the following proposition is somewhat technical so we delay it until Section 3.11.

Proposition 3.8.3. *Let $W \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then*

$$N_W(\lambda) \leq C_2 \operatorname{Vol}(W) \lambda^{n/p},$$

as $\lambda \to \infty$. Here C_2 is a constant that depends on n, p, and the Lipschitz constant of W.

We can now prove the Neumann Weyl law for a general domain in $\mathbb{R}^n$.

Theorem 3.8.4. *Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. Then we have $N_V(\lambda) \sim c \operatorname{Vol}(V) \lambda^{n/p}$ as $\lambda \to \infty$.*

Proof. Without loss of generality we can assume that U has unit volume. Define the numbers

$$\underline{\gamma}(U) = \liminf_{\lambda \to \infty} \lambda^{-n/p} N_U(\lambda), \quad \overline{\gamma}(U) = \limsup_{\lambda \to \infty} \lambda^{-n/p} N_U(\lambda).$$

Given $\varepsilon > 0$, there exist numbers $a_1, \dots, a_m$ and a set V with Lipschitz boundary such that $\mathrm{Vol}(V) \le \varepsilon$ and

$$V + \sum_{i=1}^{m} a_i C \approx U.$$

Moreover, it is possible to choose V so that $\mathrm{lip}(V) \le K$ for some constant K that depends only on $\mathrm{lip}(U)$ and the dimension n. In particular, this means that $N_V(\lambda)$ satisfies a growth bound

$$N_V(\lambda) \le C_2 \,\mathrm{Vol}(V) \lambda^{n/p} \le C_2 \varepsilon \lambda^{n/p}$$

as $\lambda \to \infty$ where the constant C_2 is independent of ε.

Applying the domain monotonicity inequality shows that

$$N_U(\lambda) \le N_V(\lambda) + \sum_{i=1}^{m} N_C(a_i^p \lambda).$$

But $N_C(a_i^p \lambda) \sim c a_i^n \lambda^{n/p}$ as $\lambda \to \infty$ by the Weyl law for cubes. Hence multiplying both sides of the above inequality by $\lambda^{-n/p}$ and letting $\lambda \to \infty$ it follows that

$$\overline{\gamma}(U) \le c \left(\sum_{i=1}^{m} a_i^n \right) + C_2 \varepsilon = c(1 - \varepsilon) + C_2 \varepsilon.$$

Since ε was arbitrary, this implies $\overline{\gamma}(U) \le c$.

It remains to show that $\underline{\gamma}(U) \ge c$. To this end, fix $a > 0$ so that some translate of aU is contained in C. Let $\varepsilon > 0$ and choose numbers $a_1, \dots, a_m$ and a set V with Lipschitz boundary such that $\mathrm{Vol}(V) \le \varepsilon$ and

$$C \approx V + aU + \sum_{i=1}^{m} a_i C.$$

It is possible to pick V so that $\mathrm{lip}(V) \le K$ where K is some constant depending only on $\mathrm{lip}(U)$ and n. Notice then that $\sum_{i=1}^{m} a_i^n = 1 - a^n - \mathrm{Vol}(V)$. The domain monotonicity inequality gives

$$N_C(\lambda) \le N_V(\lambda) + N_U(a^p\lambda) + \sum_{i=1}^{m} N_C(a_i^p\lambda).$$

Pick a sequence $\lambda_j \to \infty$ so that

$$a^{-n}\lambda_j^{-n/p} N_U(a^p\lambda_j) \to \underline{\gamma}(U).$$

Taking $\lambda = \lambda_j$ in the above equation and multiplying both sides by $a^{-n}\lambda_j^{-n/p}$ yields

$$a^{-n}\lambda_j^{-n/p} N_C(\lambda_j) \le a^{-n}\lambda_j^{-n/p}\left(N_V(\lambda_j) + N_U(a^p\lambda_j) + \sum_{i=1}^{m} N_C(a_i^p\lambda_j)\right)$$

Letting $j \to \infty$ and using the fact that $\lambda_j^{-n/p} N_V(\lambda_j) \le C_2 \operatorname{Vol}(V) \le C_2\varepsilon$ for all large j, it follows that that

$$a^{-n}c \le a^{-n}C_2\varepsilon + \underline{\gamma}(U) + a^{-n}\sum_{i=1}^{m} a_i^n c.$$

Thus

$$\begin{aligned}\underline{\gamma}(U) &\ge a^{-n}\left(c - \sum_{i=1}^{m} a_i^n c - C_2\varepsilon\right)\\ &= a^{-n}c - a^{-n}(1 - a^n - \operatorname{Vol}(V))c - a^n C_2\varepsilon \ge c - a^n C_2\varepsilon.\end{aligned}$$

But ε is arbitrary and C_2 is independent of ε and hence $\underline{\gamma}(U) \ge c$. This proves that $\overline{\gamma}(U) = \underline{\gamma}(U) = c$ and so the Weyl law $N_U(\lambda) \sim c\lambda^{n/p}$ holds for U. $\square$

3.9 EQUALITY OF DIRICHLET AND NEUMANN CONSTANTS

In this section we prove that the constant c^0 in the Dirichlet Weyl law is equal to the constant c in the Neumann Weyl law. The following lemma of Gromov ([11] Lemma 3.2.E$_1$) is the key ingredient in the proof.

Lemma 3.9.1. *Let $U \subset \mathbb{R}^n$ with smooth boundary. Let $\varepsilon > 0$ and define the set*

$$U_\varepsilon := \{x \in U : 0 < dist(x, \partial U) < \varepsilon\}.$$

Let $\lambda', \lambda'' > 0$ and set

$$\lambda = \frac{\lambda'\lambda''}{\lambda' + \lambda'' + \varepsilon^{-1}}.$$

Then there is an inequality

$$N_U(\lambda^p) \leq N_U^0((\lambda')^p) + N_{U_\varepsilon}((\lambda'')^p).$$

Proof. Let ε, λ, λ', λ'' be as in the statement of the lemma. Define the sets

$$A := \{v \in \mathcal{M}^0(U) : E(v) < (\lambda')^p\},$$

$$B := \{w \in \mathcal{M}(U_\varepsilon) : E(w) < (\lambda'')^p\},$$

$$C := \{u \in \mathcal{M}(U) : E(u) < \lambda^p\}.$$

Define $\varphi : U \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \varepsilon^{-1}\text{dist}(x, \partial U), & \text{if } x \in U_\varepsilon \\ 1, & \text{otherwise} \end{cases}$$

and then set

$$C_A := \{u \in C : E(\varphi u) < (\lambda')^p\}.$$

Also define

$$C_B := \{u \in C : u|_{U_\varepsilon} \not\equiv 0 \text{ and } E(u|_{U_\varepsilon}) < (\lambda'')^p\}.$$

We claim that $C \subset C_A \cup C_B$. Given this the result follows. Indeed C_A and C_B are open in

C and so Proposition 3.4.1(ii) gives

$$\operatorname{ind}(C) \leq \operatorname{ind}(C_A) + \operatorname{ind}(C_B).$$

Moreover, there are odd continuous maps

$$\begin{aligned} C_A \to A, &\quad u \mapsto \frac{\varphi u}{\|\varphi u\|} \\ C_B \to B, &\quad u \mapsto \frac{u|_{U_\varepsilon}}{\|u|_{U_\varepsilon}\|}. \end{aligned}$$

Hence Proposition 3.4.1(i) gives

$$\operatorname{ind}(C_A) \leq \operatorname{ind}(A), \quad \operatorname{ind}(C_B) \leq \operatorname{ind}(B)$$

and it follows that $\operatorname{ind}(C) \leq \operatorname{ind}(A) + \operatorname{ind}(B)$, as needed.

It remains to show the claim. So suppose that $u \in C$. If $u|_{U_\varepsilon} = 0$, then $\varphi u = u$ and so $E(\varphi u) = E(u) < \lambda^p < (\lambda')^p$. Thus $u \in C_A$. Hence we may assume that $u|_{U_\varepsilon} \neq 0$. Suppose for contradiction that $u \notin C_A \cup C_B$. Then

$$\begin{gathered} \|\nabla(\varphi u)\|_{L^p(U)} \geq \lambda' \|\varphi u\|_{L^p(U)}, \\ \|\nabla u\|_{L^p(U)} \geq \|\nabla u\|_{L^p(U_\varepsilon)} \geq \lambda'' \|u\|_{L^p(U_\varepsilon)}. \end{gathered}$$

Moreover, there are inequalities

$$\begin{gathered} \|\nabla(\varphi u)\|_{L^p(U)} \leq \|\nabla u\|_{L^p(U)} + \varepsilon^{-1} \|u\|_{L^p(U_\varepsilon)}, \\ \|u\|_{L^p(U)} \leq \|\varphi u\|_{L^p(U)} + \|u\|_{L^p(U_\varepsilon)}. \end{gathered}$$

Therefore

$$\begin{aligned}
\|\nabla u\|_{L^p(U)} &\geq \lambda''\|u\|_{L^p(U_\varepsilon)} \\
&= \lambda''\left(\|u\|_{L^p(U_\varepsilon)} + \|\varphi u\|_{L^p(U)}\right) - \lambda''\|\varphi u\|_{L^p(U)} \\
&\geq \lambda''\|u\|_{L^p(U)} - \frac{\lambda''}{\lambda'}\|\nabla(\varphi u)\|_{L^p(U)} \\
&\geq \lambda''\|u\|_{L^p(U)} - \frac{\lambda''}{\lambda'}\left(\|\nabla u\|_{L^p(U)} + \varepsilon^{-1}\|u\|_{L^p(U_\varepsilon)}\right) \\
&\geq \lambda''\|u\|_{L^p(U)} - \left(\frac{\lambda''}{\lambda'} + \frac{1}{\varepsilon\lambda'}\right)\|\nabla u\|_{L^p(U)}.
\end{aligned}$$

Rearranging and using the definition of λ gives

$$\|\nabla u\|_{L^p(U)} \geq \lambda\|u\|_{L^p(U)}.$$

This is a contradiction, and the claim follows. □

Theorem 3.9.2. *Let c^0 be the constant appearing in the Dirichlet Weyl law and let c be the constant appearing the Neumann Weyl law. Then $c^0 = c$.*

Proof. Let B be the unit ball in $\mathbb{R}^n$. Fix some small $\eta > 0$. Let $\varepsilon > 0$ and for each $\lambda > 0$ put

$$\lambda' = (1+\eta)\lambda, \quad \lambda'' = \frac{\lambda + \varepsilon^{-1}}{\eta}$$

so that

$$\lambda = \frac{\lambda'\lambda''}{\lambda' + \lambda'' + \varepsilon^{-1}}.$$

Notice that for λ large enough there is an inequality $\lambda'' \leq 2\lambda/\eta$. Using this inequality in conjunction with Lemma 3.9.1 shows that

$$\begin{aligned}
\lambda^{-n}N_B(\lambda^p) &\leq \lambda^{-n}N_B^0((1+\eta)^p\lambda^p) + \lambda^{-n}N_{B_\varepsilon}(2^p\lambda^p/\eta^p) \\
&\leq \lambda^{-n}N_B^0((1+\eta)^p\lambda^p) + C_2\operatorname{Vol}(B_\varepsilon)(2/\eta)^n
\end{aligned}$$

for all large λ. Letting $\lambda \to \infty$, this implies that

$$c \leq c^0(1+\eta)^n + C_2 \operatorname{Vol}(B_\varepsilon)(2/\eta)^n.$$

Taking $\varepsilon \to 0$ this gives $c \leq c^0(1+\eta)^n$, and then letting $\eta \to 0$ yields $c \leq c^0$. The opposite inequality $c^0 \leq c$ is clear, and the result follows. □

3.10 THE WEYL LAW ON CLOSED MANIFOLDS

Let (X^n, g) be a closed Riemannian manifold. One defines the p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

and the space

$$\mathcal{M}(X) = \left\{u \in W^{1,p}(X) : \|u\|_{W^{1,p}} = 1\right\}.$$

The variational spectrum of Δ_p on X and the counting function $N_X(\lambda)$ are then defined as before via a min-max procedure involving the cohomological index. The goal of this section is to show that a Weyl law $N_X(\lambda) \sim c \operatorname{Vol}(X)\lambda^{n/p}$ holds on X, thus proving Theorem 3.1.1.

As a first step consider some $U \subset X$ with Lipschitz boundary. Suppose U is small enough that we can find a chart $\varphi : V \to U$ where the metric $g = (g_{ij})$ on U satisfies

$$(1-\varepsilon)^2(\delta_{ij}) \leq (g_{ij}) \leq (1+\varepsilon)^2(\delta_{ij}).$$

We will call such a set ε-admissible. If U is ε-admissible then energy function on U is given by

$$E_U(u) = \frac{\int_V \left|g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right|^{p/2}\sqrt{g}\,dx}{\int_V |u|^p\sqrt{g}\,dx}$$

and hence there are inequalities

$$K_1(\varepsilon)E_V(u) \leq E_U(u) \leq K_2(\varepsilon)E_V(u)$$

where

$$K_1(\varepsilon), K_2(\varepsilon) \to 1, \quad \text{as } \varepsilon \to 0.$$

It follows that there are comparisons

$$N_V^0(K_2(\varepsilon)^{-1}\lambda) \leq N_U^0(\lambda) \leq N_U(\lambda) \leq N_V(K_1(\varepsilon)^{-1}\lambda)$$

for every $\lambda > 0$. There is also a volume comparison

$$(1-\varepsilon)^n \operatorname{Vol}(V) \leq \operatorname{Vol}(U) \leq (1+\varepsilon)^n \operatorname{Vol}(V).$$

It is now possible to prove a Weyl law for Δ_p on X.

Theorem 3.10.1. *Let (X^n, g) be a closed Riemannian manifold. Then there is a Weyl law $N_X(\lambda) \sim c \operatorname{Vol}(X)\lambda^{n/p}$.*

Proof. Without loss of generality we may assume that $\operatorname{Vol}(X) = 1$. Define the quantities

$$\underline{\gamma}(X) = \liminf_{\lambda\to\infty} \lambda^{-n/p} N_X(\lambda), \quad \overline{\gamma}(X) = \limsup_{\lambda\to\infty} \lambda^{-n/p} N_X(\lambda).$$

Pick some $\varepsilon > 0$. Choose disjoint ε-admissible open sets $U_1, \ldots, U_m$ in M such that

$$\sum_{i=1}^{m} \operatorname{Vol}(U_i) \geq 1 - \varepsilon.$$

For each i, let $\varphi_i : V_i \to U_i$ be a chart where the metric is almost Euclidean. Arguing as in

Section 3.5 there is an inequality

$$N_M(\lambda) \geq \sum_{i=1}^{m} N^0_{U_i}(\lambda) \geq \sum_{i=1}^{m} N^0_{V_i}(K_2(\varepsilon)^{-1}\lambda).$$

Multiplying by $\lambda^{-n/p}$ and then letting $\lambda \to \infty$ and using the Weyl law for domains in Euclidean space, this implies

$$\underline{\gamma}(M) \geq \sum_{i=1}^{m} c \operatorname{Vol}(V_i) K_2(\varepsilon)^{-n/p} \geq c(1+\varepsilon)^{-n}(1-\varepsilon)K_2(\varepsilon)^{-n/p}.$$

Letting $\varepsilon \to 0$ it follows that $\underline{\gamma}(M) \geq c$.

It remains to show that $\overline{\gamma}(M) \leq c$. To this end, let $\varepsilon > 0$ and then choose ε-admissible sets $U_1, \ldots, U_m$ so that

$$M = \overline{U}_1 \cup \ldots \cup \overline{U}_m,$$

and each intersection $\overline{U}_i \cap \overline{U}_j$ has measure 0. Such sets can be constructed using the argument in the proof of 4.2 in [20]. For each i, let $\varphi_i : V_i \to U_i$ be a chart where the metric is almost Euclidean. Arguing as in Section 3.7 there is an inequality

$$N_M(\lambda) \leq \sum_{i=1}^{m} N_{U_i}(\lambda) \leq \sum_{i=1}^{m} N_{V_i}(K_1(\varepsilon)^{-1}\lambda).$$

Multiplying by $\lambda^{-n/p}$ and letting $\lambda \to \infty$ and using the Weyl law for domains in Euclidean space, this implies

$$\overline{\gamma}(M) \leq \sum_{i=1}^{m} c \operatorname{Vol}(V_i) K_1(\varepsilon)^{-n/p} \leq c(1-\varepsilon)^{-n} K_1(\varepsilon)^{-n/p}.$$

Letting $\varepsilon \to 0$ it follows that $\overline{\gamma}(M) \leq c$. This proves the result. $\square$

3.11 GROWTH BOUND FOR THE NEUMANN COUNTING FUNCTION

The goal of this final section is to prove the growth bound in Proposition 3.8.3. The argument uses Sobolev extension operators.

Definition 3.11.1. *Let $U \subset \mathbb{R}^n$ be a bounded open set. Then U is L-Lipschitz provided there is a covering of ∂U by balls (B_i) with the following property: for each i there is an L-Lipschitz function $f_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, up to translation and rotation, $U \cap B_i$ coincides with the set*

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \, x_n < f(x_1, \ldots, x_{n-1})\} \cap B_i.$$

Jones introduced the following more general class of domains in [14].

Definition 3.11.2. *Let U be an open set in $\mathbb{R}^n$ and let $\varepsilon, \delta > 0$. Then U is called an (ε, δ)-domain provided for every $x, y \in U$ with $|x-y| < \delta$ there is a rectifiable arc γ joining x to y with*

(i) $\text{Length}(\gamma) \leq \dfrac{|x-y|}{\varepsilon}$,

(ii) $\operatorname{dist}(z, \partial U) \geq \dfrac{\varepsilon |x-z||y-z|}{|x-y|}$ *for all* $z \in \gamma$.

Every L-Lipschitz domain U is an (ε, δ) domain for some choice of ε and δ. Moreover, ε can be taken to depend only on L and n. The following result is due to Jones [14].

Theorem 3.11.3. *Let $U \subset \mathbb{R}^n$ be an (ε, δ)-domain. Then there exists a continuous linear map*

$$\mathcal{E} : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

with the property that $(\mathcal{E}u)|_U = u$. Moreover, the norm of $\mathcal{E}$ depends only on n, p, ε, and δ.

It is now possible to prove the growth bound.

Proof. (Proposition 3.8.3) Let $U \subset \mathbb{R}^n$ be a bounded open set and assume that U is L-Lipschitz. We need to check that

$$N_U(\lambda) \leq C\,\mathrm{Vol}(U)\lambda^{n/p}, \quad \text{as } \lambda \to \infty$$

for some constant C that depends only on n, p, and L. Notice that if U is L-Lipschitz then so is the scaled copy aU for any $a > 0$. Since the above inequality is scale invariant, it suffices to show that

$$N_{aU}(\lambda) \leq C\,\mathrm{Vol}(aU)\lambda^{n/p}, \quad \text{as } \lambda \to \infty$$

for some $a > 0$.

Since U is L-Lipschitz, it is an (ε, δ)-domain for some choice of ε and δ. Hence for a large enough, aU will be an $(\varepsilon, 1)$-domain. Choose an open set V containing the closure of U with $\mathrm{Vol}(V) \leq 2\,\mathrm{Vol}(U)$. Then it is still true that $\mathrm{Vol}(aV) \leq 2\,\mathrm{Vol}(aU)$. Moreover, for a large enough aV will contain the 1-neighborhood of aU. For notational convenience put

$$\widetilde{U} = aU \quad \text{and} \quad \widetilde{V} = aV.$$

By Theorem 3.11.3 there is an extension operator

$$\mathcal{E} : W^{1,p}(\widetilde{U}) \to W^{1,p}(\mathbb{R}^n).$$

Since $\delta = 1$ and ε depends only on L and n, it follows that the norm of $\mathcal{E}$ depends only on n, p, and L.

Let ζ be a cutoff function with $\zeta \equiv 1$ on $\widetilde{U}$ and $\zeta \equiv 0$ outside of $\widetilde{V}$. It is possible to choose ζ so that $|\nabla\zeta| \le 2$ everywhere. Define an odd continuous mapping

$$G : W^{1,p}(\widetilde{U}) \to W_0^{1,p}(\widetilde{V}),$$
$$u \mapsto \zeta \cdot \mathcal{E}u.$$

There is an estimate

$$\begin{aligned} E(Gu) = \frac{\int_{\widetilde{V}} |\nabla(\zeta \cdot \mathcal{E}u)|^p}{\int_{\widetilde{V}} |\zeta \cdot \mathcal{E}u|^p} &\le C\left(\frac{\int_{\widetilde{V}} |\nabla(\mathcal{E}u)|^p|\zeta|^p + |\mathcal{E}u|^p|\nabla\zeta|^p}{\int_{\widetilde{U}} |u|^p}\right) \\ &\le C\left(\frac{\int_{\widetilde{V}} |\nabla(\mathcal{E}u)|^p + |\mathcal{E}u|^p}{\int_{\widetilde{U}} |u|^p}\right) \\ &\le C\left(\frac{\int_{\widetilde{U}} |\nabla u|^p + |u|^p}{\int_{\widetilde{U}} |u|^p}\right) \\ &\le C(E(u)+1). \end{aligned}$$

Now let $A = \{u \in W^{1,p}(\widetilde{U}) :\ E(u) < \lambda\}$. Then the above estimate implies that

$$E(Gu) \le C(\lambda + 1)$$

for every $u \in A$. By Proposition 3.4.1(i) this implies

$$N_{\widetilde{U}}(\lambda) \le N_{\widetilde{V}}^0\left(C(\lambda+1)\right) \le C\,\mathrm{Vol}(\widetilde{V})(\lambda+1)^{n/p}.$$

Therefore

$$N_{\widetilde{U}}(\lambda) \le C\,\mathrm{Vol}(\widetilde{U})\lambda^{n/p}$$

for all sufficiently large λ, as needed. $\square$

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