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CLASSICAL POINTS OF LOW WEIGHT IN P-ADIC FAMILIES OF MODULAR FORMS

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To my family.

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ABSTRACT

We develop a new strategy for studying low weight specializations of p-adic families of ordinary modular forms. In the elliptic case, we give a new proof of a result of Ghate–Vatsal which states that a Hida family contains infinitely many classical eigenforms of weight one if and only if it has complex multiplication. Our strategy is designed to explicitly avoid use of the related facts that the Galois representation attached to a classical weight one eigenform has finite image, and that classical weight one eigenforms satisfy the Ramanujan conjecture. In the case of Hilbert modular forms, under some assumptions about partially ordinary families of modular forms, we prove a similar result. If F is a totally real field in which p splits completely with v a choice of prime dividing p in F, we prove that a 1-dimensional family of v-ordinary Hilbert modular forms contains infinitely many classical eigenforms of partial weight one if and only if it has complex multiplication, conditional on a geometric construction of these families. We also relate this result to the local splitting of the Galois representation attached to a p-adic family of p-ordinary Hilbert modular forms.

CHAPTER 1

INTRODUCTION

1.1 Summary

In this dissertation we develop a new strategy for studying points of low weight in p-adic families of ordinary modular forms. The theory of families of ordinary modular forms developed by Hida in [13] and [12] is an invaluable tool in number theory, and has inspired many of the major developments in number theory in the late 20th and early 21st centuries. The families constructed by Hida are families of p-adic modular forms which p-adically interpolate infinitely many classical modular forms. The families are finite over a weight space Λ which parameterizes the weight-character of the forms, a continuous character of the group $\mathbf{Z}_p^\times$. Hida's ordinary families have the benefit that any point on them in a classical weight $k \geq 2$ is guaranteed to be a classical, rather than just p-adic, modular form. By contrast, the weight $k = 1$ points of the family, which are those lying above the points of Λ corresponding to finite order characters, *may or may not be* classical modular forms. Our main result is a characterization of which such ordinary families admit many classical weight one points: a 1-dimensional family of ordinary modular forms contains infinitely many classical points of weight one if and only if the family has complex multiplication. In fact, we develop a new strategy for proving theorems of this flavour which has the feature that it explicitly avoids the use of the Ramanujan conjecture for points of low weights. While our main result is known in the elliptic case, our strategy has the benefit of more easily generalizing to the case of Hilbert modular forms of partial weight one.

For elliptic modular forms of weight one our main result was first proved by Ghate–Vatsal in [11]. Their proof relies crucially on the fact that the Galois representation attached to a classical weight one eigenform has *finite image*, shown by Deligne–Serre in [7]. This makes their proof quite elementary, but makes it unsuitable for generalization to the case of partial weight one Hilbert modular forms, where the attached Galois representations have infinite image. In the elliptic case, our proof avoids the use of the intimately related facts that the Galois representation attached to a classical weight one eigenform has finite image, and that such forms satisfy the Ramanujan conjecture, also established by Deligne–Serre in [7].

Compared to the case of elliptic modular forms of weight one, very little is known about Hilbert modular forms of partial weight one. It is expected that compared to Hilbert modular forms of regular weight (i.e. those with all weights at least 2) there are relatively few classical Hilbert modular forms of partial weight one except those having complex multiplication (hereafter abbreviated as CM). There are a few results known which establish the non-existence of non-CM classical partial weight one eigenforms in very specific settings, for example the work of Moy in [33] where it is proved that there are no non-CM classical Hilbert modular forms of weight $[2k+1, 1]$ and level $N = 1$ over $\mathbf{Q}(\sqrt{7})$. One may ask whether or not any non-CM classical Hilbert modular forms of partial weight one even exist! While they are rare, they do exist. There is only a single known example of a classical partial weight one which does not have complex multiplication, constructed by Moy–Specter in [32].

We now give a rough statement of our result in the case of partial weight one forms. Fix a totally real field F in which the prime p splits completely, and choose a place $v|p$. The weights of a Hilbert modular form can be thought of as being indexed by either the infinite places of F, or in this setting also by the primes above p (after fixing isomorphisms $\iota_p : \mathbf{C} \to \mathbf{C}_p$). We will consider families of modular forms where we allow only the weight-character at v to vary over a 1-dimensional weight space Λ and keep the other weight-characters fixed, and we require only that our forms are ordinary with respect to the U_v-operator with no ordinarity condition imposed at the other primes above p. If all of our fixed weights are at least 3 (really we want them to be at least 2 and have the same parity as 1) these families have the benefit of containing many points in regular weight (i.e. where all of the weights are at least 2) and also points of partial weight one (in this case where the weight at v is equal to 1). Conditional on a new construction of such families to adequately find all classical partial weight one forms within them our strategy produces the same result here as in the elliptic case: such a family contains infinitely many classical forms of partial weight one if and only if it has complex multiplication. This is the first general result establishing the finiteness in families of non-CM classical partial weight one eigenforms.

The key input to our results is that for forms of weight one there is no distinction between forms being *ordinary* and having *critical slope*. This is a rare case where it is in fact beneficial to work with forms of critical slope! For any modular form f (be it classical, p-adic, overconvergent) which is an eigenform for the U_p operator with eigenvalue a_p, the *slope* of f is the p-adic valuation $\mathrm{ord}_p(a_p)$. If f is a classical form of integer weight $k \geq 1$ with finite slope (in other words, its U_p-eigenvalue is non-zero) we always have that its slope falls in the range $[0, k-1]$. When $k = 1$ we thus have that a classical finite slope form is automatically both ordinary (slope 0) and critical slope (slope $k-1$).

Our strategy relies on a study of the growth of Hecke fields in ordinary families, building off of work of Hida characterizing CM ordinary families by the complexity of their Hecke fields [18]. Let's suppose that f is a classical eigenform. If σ is an element of the absolute Galois group of $\mathbf{Q}$, there is an eigenform f^σ whose Hecke eigenvalues are the Galois conjugates by σ of the Hecke eigenvalues of f. Starting with a p-ordinary eigenform f, in general its Galois conjugates f^σ need not be p-ordinary; the algebraic integer a_p may be a unit in some p-adic embeddings but not in others. The Hecke field of f has degree equal to the size of the orbit of f under the absolute Galois group of $\mathbf{Q}$; Hida's characterization of CM ordinary families essentially says that a family has CM exactly when these degrees don't vary too much as we look at forms across the family. If f has weight one, the fact that ordinary and critical slope coincide guarantees that if we start with a p-ordinary f then all the Galois conjugates f^σ will still be p-ordinary. Since all of these Galois conjugates are ordinary, and thus arise as specializations of the same finite set of ordinary families, we can parlay this "automatically ordinary" property into a uniform bound on the degrees of Hecke fields of classical p-ordinary weight one eigenforms with a fixed tame weight (more precisely, we control the degrees of the Hecke fields relative to the p-cyclotomic extension $\mathbf{Q}(\mu_{p^\infty})$ rather than simply over $\mathbf{Q}$ to account for variation of the p-part of the Nebentypus character over the family).

We would like to apply Hida's characterization of CM families, using as our input this fact that Hecke fields of weight one forms in an ordinary family are uniformly bounded. But there is an added wrinkle, in that the Ramanujan conjecture plays a key role in the proof of Hida's characterization of CM families, as he makes crucial use of the fact that the Frobenius eigenvalues of forms of regular weight are *Weil numbers.* Since we wish to avoid any appeal to the Ramanujan conjecture as it is still open for Hilbert modular forms of partial weight one, we cannot immediately apply Hida's characterization. To circumvent this issue we develop a new set of rigidity principles for p-adic power series, which we use to propagate information on the boundedness of degrees of Hecke fields from low weight into regular weight, where Hida's characterization does apply even in the Hilbert case.

1.2 Statement of Results: Elliptic Modular Forms

In the case of elliptic modular forms, our proof strategy recovers a result of Ghate–Vatsal. We state our result in the elliptic case here in the language of ordinary Λ-adic Hecke algebras. See section 3.2 for more details about these Hecke algebras.

Theorem 1.1 (Ghate–Vatsal, cf. proposition 14 of [11])**.** *Let* $\mathbf{I}$ *be a reduced irreducible component of the ordinary* Λ*-adic Hecke algebra* $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$*. Then* $\mathbf{I}$ *specializes to infinitely many classical eigenforms of weight one if and only if* $\mathbf{I}$ *has CM.*

Ghate–Vatsal go further than this statement, proving the equivalence of the following statements under some mild technical assumptions.

(i) The Galois representation attached to $\mathbf{I}$ splits as a direct sum of characters when restricted to the decomposition group at p.

(ii) $\mathbf{I}$ specializes to infinitely many classical eigenforms of weight one.

(iii) $\mathbf{I}$ has CM.

The bulk of the article [11] is devoted to the implication (ii) $\implies$ (iii), which Ghate–Vatsal prove through a counting argument with the finite image Galois representations attached to classical eigenforms of weight one. Statement (iii) implies the other two directly by the explicit nature of CM families, and the implication (i) $\implies$ (ii) is a consequence of a result of Buzzard in [3].

We provide a new proof of the implication (ii) $\implies$ (iii) which avoids the use of the fact that classical eigenforms of weight one satisfy the Ramanujan conjecture and the very closely related fact that their Galois representations have finite image. Since our strategy avoids using the Ramanujan conjecture it generalizes well to other settings where this conjecture is still open, for instance the case of Hilbert modular forms of partial weight one.

1.3 Statement of Results: Parallel or Partial Weight One Hilbert Modular Forms

The strategy that we develop in this dissertation is geared towards proving statements about one-dimensional families of Hilbert modular forms. Let $F/\mathbf{Q}$ be a totally real field of degree d, and let p be a prime number which splits completely in $F/\mathbf{Q}$. As before we think of the weights of Hilbert modular forms as being indexed by the primes $v|p$. For a prime $v|p$ we say that a Hilbert modular form for F is v-ordinary if it is an eigenvector for the U_v operator with eigenvalue a p-adic unit; we say that f is p-ordinary if it is v-ordinary for each $v|p$. There are several different types of p-adic families of Hilbert modular forms over F which appear in the literature.

The most commonly studied are the "cyclotomic" families. These are 1-dimensional families of p-ordinary forms, first introduced by Hida in [14]. The forms in these families are all p-ordinary, and the 1 dimension of the weight space corresponds to varying the weight in the "parallel" direction $(1, \ldots, 1)$. These families can parameterize forms of either parallel or non-parallel weight; however in a given family the weight of every form will be congruent mod $(1, \ldots, 1)$ as a vector in $\mathbf{Z}^d$. In the setting of a parallel weight cyclotomic family our strategy again recovers known results while avoiding the use of the Ramanujan conjecture for forms of parallel weight one; in this case the result is due to Balasubramanyam–Ghate–Vatsal ([1]) using the same techniques as Ghate–Vatsal do in the elliptic case in [11].

In the article [15], Hida unified all of the cyclotomic families of a given tame level into a single large family. We refer to these as "full" families. They live over a $d+1$-dimensional weight space, parameterizing the d weights of the form as well as the central character (which can be modified by twisting by Hecke characters), and again parameterize forms which are p-ordinary. As our method is adapted to work with 1-dimensional families we cannot prove anything directly about these families although we do discuss in chapter 7 how our results for 1-dimensional families may be parlayed into some new results for these full families. This result is stated in this introduction as theorem 1.3.

Of greatest interest to us is the case of "partially ordinary" families of Hilbert modular forms. These are p-adic families where we require that the forms being interpolated satisfy an ordinarity condition at only a subset of the primes above p. When considering partially ordinary families the natural weight space to consider is s-dimensional, tracking the s weights where we impose an ordinary condition, and all forms in the family share a fixed set of weights at the other $d-s$ primes. The reason that these families are the most suitable for our method when dealing with partial weight one forms is that they allow us to use the "automatically ordinary" property of forms of low weight: if f is an eigenform which is weight one at v and has non-zero U_v eigenvalue, not only is f v-ordinary but also any Galois conjugate f^σ of f is v-ordinary. Working with a partially ordinary family guarantees for us that all Galois conjugates of our low weight forms are contained in the same family, whereas if we worked with partial weight one forms in a cyclotomic family it need not be true that all the Galois conjugates of a low weight form live in the same family.

Fix a prime $v|p$. We wish to work with a 1-dimensional partially ordinary family parameterizing forms which are v-ordinary across a weight space varying the weight at v, and having a fixed weight character at the other $d-1$ primes. See section 3.3 for further discussion of partially ordinary families and what is known about them. In particular we encode as assumption 3.13 the properties of such families that are necessary for our method to apply, proving the following result.

Theorem 1.2. *Work under the conditions of assumption 3.13. Let* $\mathbf{I}$ *be a reduced irreducible component of the ordinary* Λ*-adic Hecke algebra* $\mathbf{H}^{v-\mathrm{ord}}$*. Then* $\mathbf{I}$ *specializes to infinitely many classical eigenforms which are weight one at* v *if and only if* $\mathbf{I}$ *has CM.*

Our result also has implications for the full ordinary families, through the connection that a suitable 1-dimensional subspace of such a family is in fact a component of a partially ordinary family which happens to consist of forms which are also ordinary at all primes above p. This result links the local splitting of the Galois representation attached to the full family to the presence of a dense set of classical partial weight one forms, and the family having CM. In addition to the conditions of assumption 3.13, this result requires a classicality statement for forms of partial weight one. While this classicality result is used in some references and may possibly be extracted from the literature, it is this author's opinion that theorem 1.3 should be viewed as conditional on that result (stated in chapter 7 as theorem 7.2).

Theorem 1.3. *Work under the conditions of assumption 3.13 and assume theorem 7.2. Let* $\mathbf{I}$ *be a component of a* $d+1$*-dimensional* p*-ordinary Hecke algebra* $\mathbf{H}^{p-\mathrm{ord}}$*. Assume that* $\overline{\rho}_{\mathbf{I}}|G_{F(\zeta_p)}$ *is absolutely irreducible and* $\overline{\rho}_{\mathbf{I}}$ *is residually distinguished. Then the following are equivalent.*

(i) There exists a prime $v|p$ *for which* $\rho_{\mathbf{I}}|G_{F_v}$ *is split.*

(ii) There exists a prime $v|p$ *for which* $\mathbf{I}$ *contains a Zariski dense set of classical eigenforms which are weight one at* v *and regular weight at all other primes above* p*.*

(iii) There exists a prime $v|p$ *for which* $\mathbf{I}$ *contains a Zariski dense set of classical CM eigenforms which are weight one at* v *and regular weight at all other primes above* p*.*

(iv) $\mathbf{I}$ *has CM.*

This theorem may be viewed as an analog of the full result of Ghate–Vatsal from [11], compare with Proposition 14 of [11] and Theorem 3 of [1].

1.4 Overview of the Argument

We give an overview of the argument used in proving theorem 1.1 and theorem 1.2.

Assumption 1.4. We make the following simplifying assumptions throughout section 1.4 so as to highlight the structure of the argument.

- We work solely in the elliptic case.
- We assume that the universal ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathbf{Z}_p)$ consists of a single reduced irreducible component $\mathbf{I}$ which is dimension 1 over Λ, i.e. $\mathbf{I}=\mathbf{H}^{\mathrm{ord}}(N;\mathbf{Z}_p)$ is isomorphic to $\Lambda=\mathbf{Z}_p[\![T]\!]$.

If we know that $\mathbf{I}$ has CM then it is immediate from the construction of CM families that all specializations in weight one will be classical. The other direction is the focus of our strategy.

Suppose we are given that $\mathbf{I}$ specializes to infinitely many classical forms of weight one, and we want to show that $\mathbf{I}$ has CM. Let's denote this set of classical weight one forms as $\{f_\epsilon\}$, where the forms are indexed by some infinite set of finite order characters ϵ. By our assumption that $\mathbf{I}$ is the only component, we have by lemma 3.7 that the Hecke field of f_ϵ is in fact simply the character field $\mathbf{Q}(\epsilon)$ (this is the "automatically ordinary" property alluded to in section 1.1).

The Rankin-Selberg bounds on Fourier coefficients in this case show that $|a_\ell(f_\epsilon)| < 2\ell^{1/2}$ for each finite order character ϵ (note that these bounds are weaker than those given by the Ramanujan conjecture, which amounts to $|a_\ell(f_\epsilon)| \le 2$). Note also that the $a_\ell(f_\epsilon)$ are cyclotomic integers. These two facts are combined in section 5.2 as the input to a result of Loxton (stated as theorem 5.4) to show that there is a bound, independent of ϵ, on the number of roots of unity necessary to write down the cyclotomic integers $a_\ell(f_\epsilon)$.

Translating these properties of the $a_\ell(f_\epsilon)$ back to our Λ-adic Hecke algebra, this construction thus shows that the element ℓ-th Hecke operator $T_\ell \in \Lambda$ is a power series for which $T_\ell(\zeta - 1)$ is a sum of a B roots of unity for some fixed integer B and infinitely many roots of unity ζ. Chapter 4 is devoted to proving rigidity principles for such power series. The key takeaway from chapter 4 is that as a power series in T, the Hecke operator T_ℓ must be a linear combination of power series of the form $(1+T)^e = \sum_{n=0}^{\infty} \binom{e}{n} T^n$.

Once we've established such exact formula for the T_ℓ, we can then look at their specializations in regular weight and derive consequences for the Hecke fields of the regular weight forms parameterized by $\mathbf{I}$. In particular we show that there are bounds C_ℓ such that for any classical f arising as a specialization of $\mathbf{I}$ we have the degree of $a_\ell(f)$ over the character field of f is at most C_ℓ. This is sufficient to apply a result of Hida characterizing CM families by their Hecke fields; we sketch a proof of this result and apply it to our setup in chapter 6 to deduce our main results.

Section 5.1 is devoted to dealing with the case when $\mathbf{H}^{\mathrm{ord}}(N;\mathbf{Z}_p)$ is larger than just a single copy of Λ. The issue is that $a_\ell(f_\epsilon)$ itself is no longer guaranteed be a cyclotomic integer, but we know from lemma 3.7 that it will live in a degree at most $\mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}}(N;\mathbf{Z}_p))$ extension of a cyclotomic field.

Rather than working with T_ℓ itself (which is the trace of Frob_ℓ under the Galois representation attached to $\mathbf{I}$) we construct a high-dimensional Galois representation as the sum of the Galois representations attached to several well-chosen components of $\mathbf{H}^{\mathrm{ord}}(N;\mathbf{Z}_p)$. This large Galois representation will have the property that when specialized at a weight one point ϵ the coefficients of the characteristic polynomial of Frob_ℓ will all be cyclotomic integers, essentially as they are traces from the Hecke field to the character field of the Hecke eigenvalues $a_\ell(f_\epsilon)$. Working with the coefficients of these characteristic polynomials of Frobenius rather than T_ℓ directly the strategy as outlined above goes through. We are able to use the rigidity results of chapter 4 to establish exact formulas for the coefficients of the characteristic polynomials as a sum of exponential power series $(1+T)^e$, and specializing these in regular weight provides enough information about $\mathbf{I}$ itself that we are able to establish the conditions of Hida's characterization of CM families and deduce that $\mathbf{I}$ has CM.

The strategy as detailed above adapts well to the Hilbert modular case, *as long as one works with the right choice of families of forms.* The key to picking the right families is that one wants to work with families for which the analog of lemma 3.7 holds; if the family contains a form of low weight it should also contain all Galois conjugates of that form over its character field. If we are working with parallel weight one Hilbert modular forms the natural setting is to work with *cyclotomic families*: these are families where the weight varies in the parallel direction and we require the forms to be ordinary with respect to the U_v operator for each $v|p$. The key is that these cyclotomic families will see any parallel weight one form and all of its Galois conjugates, but by varying the weight in the parallel direction they also contain forms of regular parallel weight (i.e. where all the weights are at least 2). If we instead want to deal with partial weight one Hilbert modular forms, the natural setting is families where we vary only a single weight and fix all the other weights to be regular, and where we impose an ordinarity condition only at the single place $v|p$ where the weight is allowed to vary. The issue here is that the Galois conjugates of a form which is weight one and ordinary at v are guaranteed to still be ordinary at v, but there is no guarantee that ordinarity at primes where the weight is bigger than one will be preserved by Galois conjugation.

1.5 Outline

In chapter 2 we recall the key facts about modular forms which are used in our arguments. We discuss Galois conjugation of modular forms and its relation to Hecke fields. Our arguments rely on two types of bounds on the Hecke eigenvalues of modular forms: archimedean bounds (as embodied by the Rankin-Selberg bounds) and non-archimedean bounds (as embodied by the slope bounds $0 \leq \mathrm{ord}_p(a_p(f)) \leq k-1$ for classical forms of weight k).

Chapter 3 covers the statements we need from the theory of ordinary families of p-adic modular forms. We cover what we need in the elliptic case, paying particular attention to the inclusion of forms of weight one into ordinary families, as the literature often works only with forms of weight $k \geq 2$. In the Hilbert case we state what our arguments need as a black-box and give some indication of how such families could be constructed, as the existing literature does not contain a reference which constructs the required partially ordinary families. While the techniques used in developing such families are not drastically different from others in the literature, some care is needed to get the definitions correct and include forms of partial weight one. The results in this dissertation for partial weight one Hilbert modular forms are thus conditional on the careful construction of such partially ordinary families. We hope to provide a detailed account of a construction of the relevant families in a future work.

Chapter 4 is where new results start appearing. This chapter focuses on rigidity principles for p-adic power series and integral extensions of power series rings. After recalling in section 4.1 some facts about Weierstrass preparation and Newton polygons to set the stage for how we approach thinking about elements of integral extension of power series rings, we prove our main rigidity result in section 4.2. This main result is that if an element of an integral extension of Λ specializes to a bounded number of roots of unity at infinitely many inputs of the form $\zeta - 1$ then it must be a linear combination of "exponential" power series $(1+T)^e = \sum_{n=0}^{\infty} \binom{e}{n} T^n$. We conclude this chapter with some tools for working with linear combinations of exponential power series, in particular for studying the fields of definition of their values.

Chapter 5 works throughout with a component $\mathbf{I}$ of the ordinary Λ-adic Hecke algebra, which is assumed to admit infinitely many classical weight one specializations. This chapter covers the construction of a high-dimensional Galois representation whose characteristic polynomials of Frobenius provide a way to propagate information about Hecke fields from low weight into regular weight. Section 5.1 carries out the actual construction, which is a careful selection of components of the Hecke algebra by an extended pigeonhole principle argument; the Galois representation we want is the direct sum of the representations attached to a well-chosen set of such components. In section 5.2 we show that the characteristic polynomials of Frobenius of our high-dimensional Galois representation satisfy the conditions necessary to apply the rigidity principles of chapter 4.

Chapter 6 sketches a proof of Hida's characterization of CM families in section 6.1 and then assembles the ingredients from chapter 5 in order to apply Hida's theorem to families containing infinitely many classical forms of low weight, proving our main theorem.

Chapter 7 deals with the application of our main result to the local splitting of Galois representations attached to Hida's full p-ordinary families of Hilbert modular forms. The results of this chapter are conditional on both the assumptions on partially ordinary families stated in section 3.3 and a classicality theorem for partial weight one forms which is only implicitly found in the literature. Ghate–Vatsal's main result in [11] and its generalization by Balasubramanyam–Ghate–Vatsal to families of Hilbert modular forms in [1] show the equivalence of an ordinary family having infinitely many classical points of (parallel) weight one, having complex multiplication, and having Galois representation which splits on a decomposition group for each prime above p. Using our result for 1-dimensional families, we prove under some mild technical assumptions the equivalence of a full p-ordinary family having infinitely many classical points of partial weight one, having complex multiplication, and having its Galois representation split on a decomposition group at a single prime above p.

1.6 Notation

For each prime p we fix an isomorphism $\iota_p : \mathbf{C} \to \mathbf{C}_p$. These isomorphisms are used in two places. First they are used in constructing the p-adic Hecke character associated to an Archimedean Hecke character. Second, they are used to allow us to index the weights of a

Hilbert modular form by both the infinite places and the p-adic places of a totally real field F in which the prime p splits completely.

We fix the p-adic valuation ord_p on $\mathbf{C}_p$, normalized so that $\mathrm{ord}_p(p) = 1$.

For any field F we let G_F denote the absolute Galois group of F, $\mathrm{Gal}(\overline{F}/F)$. In this thesis we will need this notion only for finite extensions of $\mathbf{Q}$ or $\mathbf{Q}_p$, so issues of separability do not come into play.

CHAPTER 2

MODULAR FORMS

In this chapter we collect some facts about elliptic modular eigenforms, their Hecke fields, and bounds on their Hecke eigenvalues. Everything in this chapter is either already known or easily deduced from known results, we are simply collecting the key facts for our arguments to have them in one place.

2.1 Galois conjugates

Acting on the space $S_k(N, \epsilon; \mathbf{C})$ of cuspidal modular forms of weight k, level $\Gamma_0(N)$, Nebentypus character ϵ, with complex coefficients is a commutative algebra $H_k(N, \epsilon; \mathbf{C})$, called the Hecke algebra. The generators of this algebra and their action on q-expansions are:

- for each prime $\ell \nmid N$ a Hecke operator T_ℓ acting by

$$T_\ell\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{n\ell} q^n + \epsilon(\ell)\ell^{k-1} \sum_{n=1}^{\infty} a_n q^{n\ell},$$

- for each prime $\ell \nmid N$ a diamond operator S_ℓ acting by

$$S_\ell\left(\sum_{n=1}^{\infty} a_n q^n\right) = \epsilon(\ell) \sum_{n=1}^{\infty} a_n q^n,$$

- for each prime $\ell | N$ an Atkin-Lehner operator U_ℓ acting by

$$U_\ell\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{n\ell} q^n.$$

Throughout this thesis we will mostly work with Hecke algebras rather than directly with spaces of forms. Fundamentally the two are equivalent thanks to the following proposition.

Proposition 2.1. *The pairing*

$$H_k(N,\epsilon;\mathbf{C}) \times S_k(N,\epsilon;\mathbf{C}) \to \mathbf{C}$$
$$(T,f) \mapsto a_1(T(f))$$

is a perfect pairing of complex vector spaces.

Working with the larger space of forms for the group $\Gamma_1(N)$, we can consider the $\mathbf{Q}$ or $\mathbf{Z}$-algebras generated by these same Hecke operators. As $H_k(\Gamma_1(N);\mathbf{C})$ is reduced we get that $H_k(\Gamma_1(N);\mathbf{Q})$ must be a product of finite extensions of $\mathbf{Q}$. Tracing back through the duality between Hecke algebras and spaces of forms, we see that $S_k(\Gamma_1(N);\mathbf{C})$ admits a basis of eigenvectors of the action of $H_k(\Gamma_1(N);\mathbf{C})$, each corresponding to a homomorphism $H_k(\Gamma_1(N);\mathbf{C})$ factoring through one of the number field components of the Hecke algebra. If we scale such a basis vector f so that $a_1(f) = 1$, we call f a normalized eigenform. Note that $S_k(N,\epsilon;\mathbf{C})$ is just the subspace of $S_k(\Gamma_1(N);\mathbf{C})$ where the quotient $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbf{Z}/N\mathbf{Z})^\times$ acts by ϵ.

With this structure in place we can define Galois conjugation on these spaces of modular forms. Given a normalized eigenform $f \in S_k(\Gamma_1(N);\mathbf{C})$ and an element σ of the absolute Galois group $G_\mathbf{Q}$ we define the Galois conjugate f^σ by letting $\psi_f : H_k(\Gamma_1(N);\mathbf{Z}) \to \mathbf{C}$ be the homomorphism corresponding to f by the duality between Hecke algebra and forms, and letting f^σ be the normalized eigenform corresponding to $\sigma \circ \psi_f : H_k(\Gamma_1(N);\mathbf{Z}) \to \mathbf{C}$. If our form has character ϵ the conjugated form f^σ has character ϵ^σ. Note that this is well-defined as we can either choose an extension of σ to all of $\mathbf{C}$ or observe that the image of ψ_f lands in a number field.

Moreover we can consider the $\mathbf{Q}$ or $\mathbf{Z}$-algebra generated by these same operators as a Hecke algebra $H_k(N,\epsilon;\mathbf{Q})$ or $H_k(N,\epsilon;\mathbf{Z})$. As $H_k(N,\epsilon;\mathbf{C})$ is reduced we get that $H_k(N,\epsilon;\mathbf{Q})$ must be a product of finite extensions of $\mathbf{Q}$. Tracing back through the duality between Hecke algebras and space of forms, we see that $S_k(N,\epsilon;\mathbf{C})$ admits a basis of eigenvectors for the action of $H_k(N, \epsilon; \mathbf{C})$, each corresponding to a homomorphism $H_k(N, \epsilon; \mathbf{Q}) \to \mathbf{C}$ factoring through one of the number field components of the Hecke algebra. If we scale such a basis vector f so that $a_1(f) = 1$, we call f a normalized eigenform.

With this structure in place we can define Galois conjugation on these spaces of modular forms. Given a normalized eigenform $f \in S_k(N, \epsilon; \mathbf{C})$ and an element σ of the absolute Galois group $G_{\mathbf{Q}}$ we define the Galois conjugate f^σ by letting $\psi_f : H_k(N, \epsilon; \mathbf{Z}) \to \mathbf{C}$ be the homomorphism corresponding to f by the duality between Hecke algebra and forms, and letting f^σ be the normalized eigenform corresponding to $\sigma \circ \psi_f : H_k(N, \epsilon^\sigma; \mathbf{Z}) \to \mathbf{C}$. Note that this is well-defined as we can either choose an extension of σ to all of $\mathbf{C}$ or observe that the image of ψ_f lands in a number field.

Remark 2.2. We could define the Galois action on modular forms directly on q-expansions by acting with any field automorphism of $\mathbf{C}/\mathbf{Q}$, but with that definition is it entirely unclear why the Galois conjugate of a modular form should still be a modular form! The given definition makes it clear that Galois conjugate of a Hecke eigenform is still an eigenform.

2.2 Hecke Fields

Definition 2.3. Given a normalized eigenform $f \in S_k(N, \epsilon; \mathbf{C})$ we define its character field

$$\mathbf{Q}(\epsilon) = \mathbf{Q}(\{\epsilon(x) : x \in (\mathbf{Z}/N\mathbf{Z})^\times\})$$

and its Hecke field

$$\mathbf{Q}(f) = \mathbf{Q}(\{a_n(f) : n \in \mathbf{N}\}).$$

Remark 2.4. We make several remarks about character and Hecke fields of modular forms. First off, the relation between Fourier coefficients

$$a_{\ell^2}(f) = a_\ell(f)^2 - \epsilon(\ell)\ell^{k-1}a_\ell(f)$$

for almost all primes ℓ shows that $\mathbf{Q}(f) \supseteq \mathbf{Q}(\epsilon)$.

Second, since we know that the Fourier coefficient $a_\ell(f)$ is equal to the eigenvalue of the operator T_ℓ acting on f (similarly $\epsilon(\ell)$ is the eigenvalue of S_ℓ acting on f) we have that $\mathbf{Q}(f)$ is the image of the homomorphism $H_k(N, \epsilon; \mathbf{Q}(\epsilon)) \to \mathbf{C}$ sending T_ℓ to $a_\ell(f)$. Since the Hecke algebra is finitely integrally generated we have that $\mathbf{Q}(f)$ is a finitely generated extension of $\mathbf{Q}$. Moreover the finiteness of the Hecke algebra shows that each $a_\ell(f)$ must be algebraic (integral even) and so we conclude that $\mathbf{Q}(f)$ is of finite degree over $\mathbf{Q}$.

We record here the important principle that the degree of the Hecke field tells us the number of Galois conjugates of a form.

Lemma 2.5. *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then the number of Galois conjugates of f over $\mathbf{Q}$ is equal to the degree $[\mathbf{Q}(f) : \mathbf{Q}]$ of the Hecke field of f over $\mathbf{Q}$.*

Proof. This is more generally just a fact about algebraic field extensions. The degree $[\mathbf{Q}(f) : \mathbf{Q}]$ is equal to the number of embeddings $\mathbf{Q}(f) \to \mathbf{C}$, which is equal to the size of the orbit of the generating set $\{a_n(f) : n \in \mathbf{N}\}$ under the absolute Galois group of $\mathbf{Q}$. Finally we have that the size of the orbit of the set $\{a_n(f) : n \in \mathbf{N}\}$ is equal to the number of Galois conjugates of f. This follows since for $\sigma, \tau \in G_{\mathbf{Q}}$ we have that $\sigma(a_n(f)) = \tau(a_n(g))$ for all n if and only if $f^\sigma = f^\tau$. □

2.3 Modular Forms with Complex Multiplication

Definition 2.6. Let E be an imaginary quadratic field. We say that a modular form f has complex multiplication (or CM for short) by E if $a_p(f) = 0$ whenever p is inert in the extension $E/\mathbf{Q}$. We say that a modular form has CM if it has CM by some imaginary quadratic field.

A good reference for basic facts about modular forms with complex multiplication is Sections 3 and 4 of [36]. Of interest to us is the fact that eigenforms with CM can be constructed using algebraic Hecke characters, which we recall here. Let E be an imaginary quadratic field with a chosen embedding $\sigma : E \to \mathbf{C}$, and let ψ be an algebraic Hecke character of infinity-type σ^{k-1} and conductor $\mathfrak{m}$. Given such a Hecke character one can construct a weight k eigenform given by the series

$$g = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{a})}$$

where the sum is over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_E$ which have $(\mathfrak{a}, \mathfrak{m}) = 1$. The modular form g is called the θ-series attached to ψ. The eigenform g thus constructed has level DM where D is the discriminant of $E/\mathbf{Q}$ and $M = \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{m})$, and character $\epsilon = \chi\eta$ where χ is the quadratic Dirichlet character attached to E and η is the "finite order" part of ψ on the integers, given by $\eta(n) = \psi((n))/\sigma(n)^{k-1}$ for $n \in \mathbf{Z}$. Note that it is immediate from this definition that g has CM by E; if p is inert in $E/\mathbf{Q}$ then there are no ideals of $\mathcal{O}_E$ having norm p, so $a_p(g) = \sum_{\mathfrak{a}, \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{a})=p} \psi(\mathfrak{a}) = 0$. This construction is studied in Section 3 of [36], and Section 4 of [36] deals with basic facts about Galois representations attached to CM eigenforms.

Of fundamental importance for our method is the characterization of CM families by the arithmetic complexity of their Hecke fields. This characterization due to Hida is recalled in chapter 6. The key philosophy is that Hecke fields attached to CM eigenforms are much simpler than those attached to non-CM eigenforms. We begin this study here with a description of the Hecke field of a CM eigenform.

Lemma 2.7. *Suppose that $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform with CM by the imaginary quadratic field E. Suppose that f is realized as a theta series by an algebraic Hecke character ψ of conductor $\mathfrak{m}$, and let $M = \mathrm{Norm}_{\mathbf{Q}}^{E}(\mathfrak{m})$. Let h be the class number of E. Then there are elements $a_1, \ldots, a_h$ in E such that*

$$\mathbf{Q}(f) \subseteq E(\mu_{h \cdot M}, a_1^{1/h}, \ldots, a_h^{1/h}).$$

In particular, the degree of the Hecke field over $\mathbf{Q}$ is bounded solely in terms of the CM field

E and the conductor $\mathfrak{m}$ of the character ψ (the degree is at most $2h^3M$ over $\mathbf{Q}$).

Proof. We know that our eigenform f has q-expansion given by

$$f = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{Norm}^E_{\mathbf{Q}}(\mathfrak{a})}$$

where the sum is over all integral ideals $\mathfrak{a}$ of $\mathcal{O}_E$ which have $(\mathfrak{a}, \mathfrak{m}) = 1$. We certainly have that $\mathbf{Q}(f) \subseteq \mathbf{Q}(\{\psi(\mathfrak{a})\})$, so it suffices to understand the field of definition of ψ.

If (a) is a principal ideal with generator $a \equiv 1 \mod \mathfrak{m}$, we have that $\psi((a)) = a^{k-1}$. If the generator a is not necessarily $\equiv 1 \mod \mathfrak{m}$, then we know that $a^{hM} \equiv 1 \mod \mathfrak{m}$ and hence $\frac{\psi((a))}{a^{k-1}} \in \mu_{hM}$.

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ be a complete set of representatives of the ideal class group of E. For each i we have that $\psi(\mathfrak{a}_i^h) \in E$ since $\mathfrak{a}_i^h$ is principal and ψ necessarily sends principal ideals to elements of E (using that the infinity-type of ψ is σ^{k-1} for $\sigma : E \to \mathbf{C}$). Let a_i be an element of E such that $\psi(\mathfrak{a}_i^h) = a_i$. We know that every ideal $\mathfrak{a}$ is equal to a principal ideal times one of our representatives $\mathfrak{a}_i$, hence we have that $\psi(\mathfrak{a})$ must be a root of unity of order hM times $a_i^{1/h}$. □

This idea of the uniformity of Hecke fields of CM forms will be studied further in section 3.2.3, where we show that there is a uniform description of the Hecke fields of all forms in an ordinary family which has CM.

2.4 Archimedean Bounds

In this section we give bounds on the (archimedean) absolute value of Hecke eigenvalues of modular forms.

Theorem 2.8 (The Ramanujan conjecture, due to Deligne [6]). *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then for all primes ℓ we have that*

$$|a_\ell(f)|_{\mathbf{C}} \leq 2\ell^{\frac{k-1}{2}}.$$

Remark 2.9. Let $\rho_{f,p}$ be the p-adic Galois representation attached to f. Let α_ℓ, β_ℓ be the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$; these are the roots of $x^2 - a_\ell(f)x + \epsilon(\ell)\ell^{k-1}$. The bound of theorem 2.8 then gives that α_ℓ, β_ℓ are ℓ-*Weil numbers*, in other words their complex absolute values are exactly $p^{\frac{k-1}{2}}$. This fact is key to the characterization of CM ordinary families due to Hida that we employ in chapter 6.

The analog of theorem 2.8 for Hilbert modular forms of regular weight also holds, due to Blasius in [2]. Theorem 2.8 also holds in weight one due to the construction of Galois representations attached to elliptic weight one eigenforms by Deligne–Serre in [7], and for Hilbert modular forms of parallel weight one it is due to Rogawski–Tunnell in [37]. However, we wish to avoid using it in low weight as the analog of this result is still open for Hilbert modular forms of partial weight one, and we wish to develop a method which will apply in that case. To get around this we will only use a weaker bound in the case $k = 1$, the analog of which is known for all Hilbert modular forms.

Theorem 2.10. *Let $f \in S_k(N, \epsilon; \mathbf{C})$ be a normalized eigenform. Then for all primes $\ell \nmid N$ we have that*

$$|a_\ell(f)|_{\mathbf{C}} \leq 2\ell^{\frac{k}{2}}.$$

This follows from the holomorphicity of a Rankin-Selberg L-function constructed from the automorphic representation π_f attached to f. This is worked out for unramified local factors of automorphic representations for GL_n over number fields by Jacquet–Shalika in [22]. All that our method requires is an upper bound on $|a_\ell(f)|_{\mathbf{C}}$ which is independent of f, so the result of Jacquet–Shalika is sufficient for our intended application to Hilbert modular forms of partial weight one.

2.5 Non-Archimedean Bounds

In this section we discuss non-archimedean bounds on the Hecke eigenvalues of modular forms. The main result is a classical bound on the U_p eigenvalue of classical forms: if f

has weight k, its U_p eigenvalue (if non-zero) has valuation bounded between 0 and $k-1$. This bound appears throughout the literature, and some cases can be easily proved using the Galois representations attached to eigenforms, but we provide a proof using the automorphic representations attached to eigenforms that covers all cases of interest. Following that we derive a key consequence for weight one forms.

Theorem 2.11. *Let $f \in S_k(N, \epsilon; \mathbf{C}_p)$ be a normalized eigenform. Suppose that p divides N, so we have that the U_p eigenvalue of f is $a_p(f)$. If $a_p(f) \neq 0$ then*

$$0 \leq \mathrm{ord}_p(a_p(f)) \leq k-1.$$

Proof. We know from the integrality of the Hecke algebra that $a_p(f)$ will be integral, i.e. $\mathrm{ord}_p(a_p(f)) \geq 0$. Let π_f be the automorphic representation attached to f. We choose to normalize π_f so that the U_p eigenvalue of the modular form f is $\sqrt{p}$ times the U_p eigenvalue of $\pi_{f,p}$. Thus we wish to show that the U_p eigenvalue of $\pi_{f,p}$ has valuation bounded above by $k - \frac{3}{2}$. Under the assumption that the U_p eigenvalue is non-zero, there are three possible cases for what $\pi_{f,p}$ can be: an irreducible principal series representation with both characters unramified, an irreducible principal series representation with one character unramified, or an unramified twist of the Steinberg representation. We treat each case separately to establish the upper bound.

Case 1: $\pi_{f,p}$ is an irreducible principal series representation $PS(\chi_1, \chi_2)$ where both characters χ_i are unramified. In this case we know that the U_p eigenvalue is either $\chi_1(p)$ or $\chi_2(p)$. Since the central character of π_f has weight $k-2$, we know that the product $\chi_1(p)\chi_2(p)$ has valuation $k-2$. Since we know that $\mathrm{ord}_p(\chi_i(p)) \geq -\frac{1}{2}$, we get that each must have valuation at most $k-2+\frac{1}{2} = k-\frac{3}{2}$. Thus the U_p eigenvalue of $\pi_{f,p}$ has valuation at most $k-\frac{3}{2}$.

Case 2: $\pi_{f,p}$ is an irreducible principal series representation $PS(\chi_1, \chi_2)$ where only χ_1 is unramified. Let $\alpha_1 = \chi_1(p)$, which is the U_p eigenvalue of $\pi_{f,p}$, hence it has valuation at least $-\frac{1}{2}$. Let χ be the character of $\prod_\ell \mathbf{Z}_\ell^\times$ which is equal to χ_2 on the p-component and trivial on all others; we can view this as a finite order Dirichlet character. Take g to be the eigenform $f \otimes \chi^{-1}$. Thus we have that $\pi_{g,p} = \pi_{f,p} \otimes \chi^{-1}|_{\mathbf{Z}_\ell^\times}$, which is the principal series representation $PS(\chi_1\chi^{-1}, \chi_2\chi^{-1})$. Note that our choice of χ means that $\chi_1\chi^{-1}$ is ramified and $\chi_2\chi^{-1}$ is unramified. Let $\alpha_2 = \chi_2(p)$, which is the U_p eigenvalue of g, hence it has valuation at least $-\frac{1}{2}$. Since we've only twisted f by a finite order character that weight of the central character remains unchanged. From this we conclude that $\mathrm{ord}_p(\alpha_1\alpha_2) \leq k-2$, and since each has valuation at least $-\frac{1}{2}$ we conclude that each has valuation at most $k - \frac{3}{2}$.

Case 3: $\pi_{f,p}$ is an unramified twist of the Steinberg representation $S(\chi)$. In this case the U_p eigenvalue of $\pi_{f,p}$ is $\chi(p)$. We know that the central character evaluation at p is equal to $p\chi(p)^2$. Since this must have valuation equal to $k-2$, we see that $\chi(p)$ has valuation equal to $\frac{k-3}{2}$ which is certainly less than $k - \frac{3}{2}$. Note in particular that this cannot occur when $k = 1$ since $\frac{k-3}{2} = -1$ is less than $-\frac{1}{2}$, which we already know to be a lower bound on the valuation.

Thus in all cases we have the desired bounds on the U_p eigenvalue of $pi_{f,p}$, which gives us the desired bounds on the U_p eigenvalue of f itself. □

Using this bound on the slope of an eigenform, we prove the "automatically ordinary" property for weight one eigenforms alluded to in section 1.1. This is simply a matter of applying the bound from theorem 2.11 in the case $k = 1$.

Corollary 2.12. *Let $f \in S_1(N, \epsilon; \mathbf{C}_p)$ be a normalized eigenform of weight one. If $a_p(f) \neq 0$ then*

$$\mathrm{ord}_p(a_p(f^\sigma) = 0$$

for all $\sigma \in G_{\mathbf{Q}}$.

Proof. A Galois conjugate f^σ of f will be a normalized eigenform in the space $S_1(N, \epsilon^\sigma; \mathbf{C}_p)$. In particular theorem 2.11 still applies to f^σ, since $a_p(f^\sigma) = \sigma(a_p(f)) \neq 0$. So we conclude that

$$0 \leq \mathrm{ord}_p(a_p(f^\sigma)) \leq 1 - 1 = 0.$$

$\square$

CHAPTER 3

ORDINARY FAMILIES OF MODULAR FORMS

3.1 Ordinary Parts of Modules

In this section we develop the theory of ordinary parts of modules with respect to an endomorphism. In subsequent sections these principles will be applied to spaces of modular forms, in order to split off of well-behaved subspaces (in the elliptic case we will consider subspaces ordinary with respect to the U_p operator). Throughout this section $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$.

Proposition 3.1. *Let A be an $\mathcal{O}$-algebra which is finite over $\mathcal{O}$, and $a \in A$. Then there exists a unique algebra decomposition*

$$A = A^{\mathrm{ord}} \oplus A^{\mathrm{nil}}$$

for which a is a unit in A^{ord} and a is topologically nilpotent in A^{nil}. In this decomposition, the idempotent corresponding to the factor A^{ord} is $e = \lim_{n\to\infty} a^{n!}$.

Proof. The uniqueness of such an ordinary decomposition is immediate if it exists.

A finite $\mathcal{O}$-algebra is necessarily a direct sum over its localizations at its finitely many maximal ideals, $A = \oplus_{\mathfrak{m}} A_{\mathfrak{m}}$ (Corollary 7.6 of [10]). We define A^{ord} to be the direct sum over those factors $A_{\mathfrak{m}}$ in which a is a unit, and A^{nil} to be the sum over the remaining factors. From this definition it is clear that a is a unit in A^{ord}.

We check that e converges on each factor $A_{\mathfrak{m}}$, to 1 if a is a unit in $A_{\mathfrak{m}}$ and to 0 otherwise. We use that each $A_{\mathfrak{m}}$ is finite over $\mathcal{O}$, and hence the quotient rings $A_{\mathfrak{m}}/\mathfrak{m}^k A_{\mathfrak{m}}$ for $k \geq 1$ have finite cardinality. If a is a unit in $A_{\mathfrak{m}}$, then in any quotient $A_{\mathfrak{m}}/\mathfrak{m}^k A_{\mathfrak{m}}$ a sufficiently large power of a is equal to 1, so e exists and is equal to 1 on such a factor $A_{\mathfrak{m}}$. If a is not a unit in $A_{\mathfrak{m}}$, then $a \in \mathfrak{m} A_{\mathfrak{m}}$ and so in any quotient $A_{\mathfrak{m}}/\mathfrak{m}^k A_{\mathfrak{m}}$ a sufficiently large power of a

is equal to 0. This shows both that a is topologically nilpotent in the factor A^{nil}, and that e converges to 0 on A^{nil}. □

Proposition 3.2. *Let M be a finitely generated $\mathcal{O}$-module, and $U \in \mathrm{End}_{\mathcal{O}}(M)$. Then there exists a unique $\mathcal{O}[U]$-module decomposition $M = M^{\mathrm{ord}} \oplus M^{\mathrm{nil}}$ which satisfies:*

1. *U acts invertibly on M^{ord} and topologically nilpotently on M^{nil}.*
2. *The operator $e = \lim_{n\to\infty} U^{n!}$ exists as an endomorphism of M, and is the idempotent sending M to M^{ord}.*
3. *Functoriality of ordinary and nilpotent parts: if $f : M \to N$ is a map of $\mathcal{O}[U]$-modules, then $f(M^{\mathrm{ord}}) \subseteq N^{\mathrm{ord}}$ and $f(M^{\mathrm{nil}}) \subseteq N^{\mathrm{nil}}$, and moreover e commutes with f.*
4. *Exactness of ordinary and nilpotent parts: the functors taking a finitely generated $\mathcal{O}[U]$-module M to its ordinary and nilpotent parts M^{ord} and M^{nil} are exact.*

Proof. The first uniqueness and first two properties follow immediately from the definition of M^{ord} and M^{nil}, which we give now. Let $A(M)$ be the image of $\mathcal{O}[U]$ in $\mathrm{End}_{\mathcal{O}}(M)$. $A(M)$ is a finite over $\mathcal{O}$ as M is finitely generated, and thus proposition 3.1 applies to $A(M)$. We define $M^{\mathrm{ord}} = M \otimes_{\mathcal{O}[U]} A(M)^{\mathrm{ord}}$ and $M^{\mathrm{nil}} = M \otimes_{\mathcal{O}[U]} A(M)^{\mathrm{nil}}$; equivalently we may define $M^{\mathrm{ord}} = e(M)$ and $M^{\mathrm{nil}} = (1-e)(M)$.

The functoriality and exactness of taking ordinary and nilpotent parts will follow from the fact that the ordinary idempotent e commutes with any $\mathcal{O}[U]$-module homomorphism f. Since f commutes with U, we have that f commutes with U^n for any n, and so also f commutes with the limit $e = \lim_{n\to\infty} U^{n!}$. Let us indicate why $f(M^{\mathrm{ord}}) \subseteq N^{\mathrm{ord}}$ for any $\mathcal{O}[U]$-module homomorphism $f : M \to N$. Suppose that $m \in M^{\mathrm{ord}}$, then $m = e(m)$ and so $f(m) = f(e(m)) = e(f(m))$, showing that $f(m) \in N^{\mathrm{ord}}$. Verification of the rest of functoriality and exactness is similar. □

3.2 Ordinary Families of Elliptic Modular Forms

In this section we summarize the elements of Hida's theory of ordinary families of elliptic modular forms which we will use in later chapters. The main idea is that for any space of forms with level divisible by p we have an action of the U_p operator. Hida's key realization was that the U_p-ordinary subspace of a space of modular forms has bounded dimension as we vary the weight and Nebentypus character. As a consequence of this the U_p-ordinary subspaces of forms in a fixed tame level can be interpolated into a single family, finite over a weight space Λ parameterizing the weight-character (viewed as a p-adic character of $\mathbf{Z}_p^\times$). Whenever we discuss ordinarity (of a space of modular forms, or Hecke algebra, etc.) from now we always mean ordinarity with respect to the U_p operator, so we will say "ordinary" rather than "U_p-ordinary". For elliptic forms all the statements we need can be found in Hida's original papers [13] and [12] together with Wiles' work on Galois representations attached to ordinary eigenforms [40].

We work with Λ-adic Hecke algebras as our main objects. We fix the following notation for use in this section.

- An odd prime number p.
- A positive integer N coprime to p, which will be the prime-to-p part of the level of our forms.
- K a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_K$.
- $\Lambda = \mathcal{O}_K[\![T]\!]$, the ring of formal power series in one variable over $\mathcal{O}_K$.
- For any positive integer k and p-power root of unity ζ, let $P_{k,\zeta}$ be the kernel of the homomorphism

$$\begin{aligned} \Lambda &\to \overline{\mathbf{Q}}_p \\ T &\mapsto \zeta(1+p)^{k-1} - 1. \end{aligned}$$

We view Λ as the $\mathcal{O}_K$ group ring of the torsion-free part of the Galois group

$$\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \mathbf{Z}_p^\times.$$

The torsion-free part is $(1+p)\mathbf{Z}_p$; the isomorphism with $\mathcal{O}_k[\![T]\!]$ is realized by sending $1+p$ to T. While we could view all of our Hecke algebras as living over the larger group ring $\mathcal{O}_K[\![\mathbf{Z}_p^\times]\!]$, the $(\mathbf{Z}/p\mathbf{Z})^\times$ part of the character plays no role in our arguments so we will work solely with Hecke algebras as Λ-modules. Given the above setup, Hida's theory asserts the existence of a "universal" ordinary Hecke algebra.

Theorem 3.3. *There exists a Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ with the following properties.*

1. *$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is a finitely generated free Λ-module.*

2. *$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is reduced.*

3. *(Base Change) If L is a finite extension of K, with ring of integers $\mathcal{O}_L$, we have that*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_L) \cong \mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_L.$$

4. *$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is generated as a Λ-module by a collection of elements T_ℓ, S_ℓ for each prime $\ell \nmid Np$, and elements U_ℓ for each prime $\ell | Np$.*

5. *(Control Theorem) Let $k \geq 2$ be an integer, and ζ a p^r-power root of unity for some $r \geq 0$. Suppose that K is large enough to contain ζ. Let $\epsilon : (1+p)\mathbf{Z}_p \to \mathcal{O}_K^\times$ be the character taking $1+p$ to ζ. Then the natural map*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/P_{k,\zeta}\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \overset{\cong}{\to} H_k^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_K)$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ on the left to the equivalently named Hecke operators on the right is an isomorphism of Λ-modules. In other words $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$

interpolates the ordinary subspaces of all spaces of forms with prime to p level $\Gamma_1(N)$ *and weight* $k \geq 2$.

Proof. The fact that $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$ is finite free over Λ is Theorem 3.1 of [13]. The control theorem is Theorem 1.2 of [12]. The other statements are all consequences of the definition of $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$, and can be found in [13]. □

There are two constructions of this universal ordinary Hecke algebra $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$. The first, using Katz's theory of geometric p-adic modular forms, appears in [13]. The second, based on Betti cohomology of modular curves and group cohomology of congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$, appears in [12]. We refer to these approaches as the *geometric* and *cohomological* approaches to Hida theory. Each approach has benefits and drawbacks, and both are necessary in order to develop all facets of the theory which we use in this work. The geometric approach is crucial to understanding how forms of weight one fit into $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$, a topic which we explore in section 3.2.2. If one wants freeness over Λ of $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$ rather than just torsion-freeness this is provided only by the geometric approach. A downside of the geometric approach is that, at least in Hida's original work, it only deals with the case $r = 0$ of the control theorem, i.e. forms with trivial Nebentypus character. While this is sufficient to uniquely determine $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$ it is not enough for our application, as we will need to specialize at infinitely many different Nebentypus characters in a single weight. The cohomological approach is comparatively simpler as it does not require the algebraic geometry machinery of the geometric approach. Proving the control theorem for all characters ϵ is much more straightforward under the cohomological framework than the geometric one. The downsides of the cohomological approach are that it only produces torsion-freeness of $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$ over Λ (as opposed to freeness) and that it gives no information about the weight one specializations of the Hecke algebra.

We will frequently talk about "components" of $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$, and we outline what we mean by that here. As $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K)$ is reduced, we know that $\mathbf{H}^{\text{ord}}(N;\mathcal{O}_K) \otimes_\Lambda \mathrm{Frac}(\Lambda)$ is a product of finite field extensions of Frac(Λ). Say that

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)\otimes_\Lambda \mathrm{Frac}(\Lambda)\cong\prod_{i=1}^n \mathrm{Frac}(\mathbf{I}_i)$$

where $\mathbf{I}_i$ is an integral extension of Λ. We then have that

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)\hookrightarrow\prod_{i=1}^n \mathbf{I}_i$$

where each projection map $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)\to\mathbf{I}_i$ is surjective, although the total map need not be surjective. The $\mathbf{I}_i$ are the "components" of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ (technically it is more accurate to say that Spec($\mathbf{I}_i$) is a component of Spec($\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$), though we won't use this point of view). Note that if $\mathbf{I}$ is a component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ we have that $\mathbf{I}=\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/\mathfrak{P}$ for some minimal prime ideal $\mathfrak{P}$ of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$.

Given a normalized eigenform $f\in S_k^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_L)$ for some weight $k\geq 2$, character ϵ of conductor r, and finite extension L of K, we say that f arises from $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ or f arises as a specialization of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. If f arises from $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ and $\mathbf{I}$ is a component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$, we say that f arises from $\mathbf{I}$ if the homomorphism $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)\to\mathcal{O}_K$ realizing the eigensystem of f factors through the surjective map $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)\to\mathbf{I}$. Thus far we have not said anything about what happens when $k=1$; treating the case of weight $k=1$ is the focus of section 3.2.2.

With this notion of forms arising from components, we can state an important uniqueness property of the Λ-adic Hecke algebra.

Theorem 3.4. *Let $f\in S_k^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_K)$ be a normalized eigenform. Then there is a unique component $\mathbf{I}$ of the Λ-adic Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ such that f arises from $\mathbf{I}$.*

Proof. This is Corollary 1.5 of [12]. □

3.2.1 Galois Representations Attached to Ordinary Families

Each of the normalized eigenforms which $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ interpolates has an attached 2-dimensional p-adic Galois representation. It should thus not be surprising that these Galois representations also interpolate into a single Λ-adic Galois representation. These Λ-adic representations were first studied by Hida in [12] and Wiles in [40]. We record here the minimal properties of these representations that we use in later chapters.

Theorem 3.5. *Suppose that $\mathbf{I}$ is a reduced, irreducible component of $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$. Then there exists a continuous 2-dimensional Galois representation*

$$\rho_{\mathbf{I}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}))$$

which has the following properties.

1. *$\rho_{\mathbf{I}}$ is absolutely irreducible.*

2. *$\rho_{\mathbf{I}}$ is unramified away from Np, and the characteristic polynomial of a Frobenius element at a prime $\ell \nmid Np$*
$$X^2 - T_\ell X - \ell S_\ell.$$

3. *When restricted to a decomposition group at p, $\rho_{\mathbf{I}}$ is of the form*
$$\rho_{\mathbf{I}}|_{G_{\mathbf{Q}_p}} \cong \begin{bmatrix} * & * \\ 0 & \lambda \end{bmatrix}$$
where $\lambda : G_{\mathbf{Q}_p} \to \mathbf{I}^\times$ is the unramified character sending Frob_p to U_p.

4. *For almost all primes P of $\mathbf{I}$, the representation $\rho_{\mathbf{I}}$ can be taken to have values in the localization $\mathbf{I}_P$. In particular for almost all normalized eigenforms f arising from $\mathbf{I}$ we have that the p-adic Galois representation $\rho_{f,p}$ attached to f is equal to the composition*

of $\rho_{\mathbf{I}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{I}_P)$ *combined with the quotient map* $\mathrm{GL}_2(I_P) \to \mathrm{GL}_2(I_P/PI_P)$ *for some prime* P *of* $\mathbf{I}$*.*

Proof. This is Theorem 2.1 of [12]. □

3.2.2 Weight One Forms in Ordinary Families

While Hida's articles are very precise about the specialization of ordinary families in weights $k \geq 2$, eigenforms of weight one are not discussed directly in these articles. It does follow from Hida's first construction of ordinary families, using geometric p-adic modular forms, that every classical p-ordinary weight one eigenform arises as the specialization of an ordinary family. As this is not obviously stated in the literature, we discuss this explicitly here, along with a key consequence for the Hecke fields of p-ordinary weight one eigenforms.

Proposition 3.6. *Given a character* $\epsilon : (1+p)\mathbf{Z}_p \to \mu_{p^r}$ *sending* $1+p$ *to a generator* ζ *of* μ_{p^r}*, there is a natural surjective homomorphism*

$$\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K) \twoheadrightarrow H_1^{\mathrm{ord}}(\Gamma_1(Np)\cap\Gamma_0(p^r),\epsilon;\mathcal{O}_K).$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ *on the left to the equivalently named Hecke operators on the right. Put differently, every* p*-ordinary weight one eigenform arises as the specialization of an ordinary family.*

Proof. In Hida's first article [13], the universal ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ is constructed as a limit of Hecke algebras acting on the spaces $S_k^{\mathrm{ord}}(\Gamma_1(N);K/\mathcal{O}_K)$. These spaces of forms (or a suitable direct sum of these spaces allowing for divided congruences) are dense in the space $\mathcal{S}$ of all ordinary geometric p-adic modular forms, and so $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O}_K)$ can also be viewed as the Hecke algebra acting on this single large space of p-adic modular forms. Any space of forms $S_k^{\mathrm{ord}}(\Gamma_1(Np^r);\mathcal{O}_K)$ can be viewed as a subspace of $\mathcal{S}$ by interpreting these classical forms of higher level as p-adic modular forms; in particular this holds for

$k = 1$. At the level of Hecke algebras, this means that we can realize the Hecke algebra of $S_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O}_K)$ as a quotient of the Hecke algebra on $\mathcal{S}$. Decomposing the Hecke algebra on $S_k^{\mathrm{ord}}(\Gamma_1(Np^r); \mathcal{O}_K)$ as a direct sum corresponding to the various possible Nebentypus characters, we get the desired result. □

Lemma 3.7. *Suppose that $f \in S_1(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$ is a classical eigenform of weight one arising as a specialization of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$. Recall the finite extensions of $\mathbf{Q}$ defined using the Hecke eigenvalues of f:*

$$\mathbf{Q}(\epsilon) = \text{ the character field of } f$$
$$\mathbf{Q}(f) = \text{ the Hecke field of } f.$$

Then we have that

$$[\mathbf{Q}(f) : \mathbf{Q}(\epsilon)] \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)).$$

Proof. The degree $[\mathbf{Q}(f) : \mathbf{Q}(\epsilon)]$ is equal to the number of distinct Galois conjugates of f by the absolute Galois group $G_{\mathbf{Q}(\epsilon)}$ of $\mathbf{Q}(\epsilon)$. Let us assume that our local coefficient field K is large enough to contain $\mathbf{Q}(f)$ and all of its Galois conjugates. Call these Galois conjugates $f_1 = f, f_2, \ldots, f_n$. Each f_i is a classical weight one eigenform of the same level and character as f, i.e. each $f_i \in S_1(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$.

Crucially, we know that f is ordinary since it is a specialization of an ordinary family $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$. As Galois conjugates of an eigenform with finite slope, each f_i necessarily has finite slope. But since the slope of a finite slope weight k eigenform must be between 0 and $k-1$ by theorem 2.11, we conclude that each f_i is in fact ordinary, since $k-1 = 0$ when $k = 1$. So each f_i is in the ordinary subspace $S_1^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$. This entire space is a quotient of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ for some height one prime ideal $P_{1,\zeta}$ of Λ by proposition 3.6, so in total we have that

$$[\mathbf{Q}(f) : \mathbf{Q}(\epsilon)] = \text{ the number of distinct Galois conjugates of } f \text{ by } G_{\mathbf{Q}(\epsilon)}$$

$$\leq \mathrm{rank}_{\mathcal{O}_K}(S_1^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon; \mathcal{O}_K)$$
$$\leq \mathrm{rank}_{\mathcal{O}_K}(\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)/P_{1,\zeta}\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K))$$
$$= \mathrm{rank}_{\Lambda}(\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)).$$

Note that the first inequality holds since distinct Galois conjugates of f are linearly independent, as they lie in distinct eigenspaces for the action of the Hecke algebra. □

Remark 3.8. If we are willing to use the Ramanujan conjecture for classical weight one eigenforms, then it is likely that lemma 3.7 already provides a sufficient input to prove our main result without appealing to the constructions of chapter 4 and chapter 5. The goal of chapter 4 and chapter 5 is to find a method by which the Hecke field bound of lemma 3.7 can be propagated into regular weight, where theorem 6.1 may be applied. We expect that theorem 6.1 can be adapted to require only that the forms in question satisfy the Ramanujan conjecture; see remark 6.3 for a discussion of adapting Hida's result to the weight one case.

Note that the Ramanujan conjecture *is* known for weight one forms, having been proved by Deligne–Serre as a consequence of their construction of the Galois representations attached to weight one forms in [7]); our interest in finding a method which avoids using the Ramanujan conjecture is so that this strategy also applies to the case of partial weight one Hilbert modular forms, where the Ramanujan conjecture is still open.

Remark 3.9. We remark that the principle encapsulated by lemma 3.7 is unique to weight one. For forms of weight $k \geq 2$ it is frequently the case that not all Galois conjugates of a given p-ordinary form are p-ordinary. Indeed, one may think of Hida's characterization of CM families theorem 6.1 as saying that for non-CM ordinary eigenforms, the proportion of Galois conjugates which are also ordinary goes to 0 as we increase the level.

3.2.3 Components with Complex Multiplication

In this section we recall properties of the CM components of the Λ-adic ordinary Hecke algebra. A good reference for these facts is Section 7 of [13].

We sketch the construction of CM components outlined by Hida in [13]. Let E be an imaginary quadratic field in which our fixed prime p splits as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Fix an integral ideal $\mathfrak{m}$ of E which is coprime to $\mathfrak{p}$; this $\mathfrak{m}$ will serve as the tame conductor of our CM components. Let W be the idèle class group of E of conductor $\mathfrak{m}\mathfrak{p}^\infty$, that is

$$W = \mathbf{A}_E^\times / \overline{U E_\infty^\times E^\times}$$

where $U = \prod_{\ell \neq \mathfrak{p}} U_\ell$, with U_ℓ the entire group of integral units of the completion E_ℓ if ℓ is coprime to $\mathfrak{m}$, and U_ℓ being those integral units which are congruent to 1 mod $\mathfrak{m}$ if ℓ divides $\mathfrak{m}$. The group $\Gamma = 1 + p\mathbf{Z}_p$ injects into $U_\mathfrak{p}$ which itself injects into W. Moreover since the idèle class group of conductor $\mathfrak{m}$ is finite we have that Γ has finite index in W.

Let us assume that our coefficient ring $\mathcal{O}_K$ is large enough to contains the values of all characters of the finite group W/Γ. Define $A = \mathcal{O}_K[\![W]\!]$ to be the $\mathcal{O}_K$ group ring of W. Then the inclusion $\Gamma \to W$ gives a map on group rings $\Lambda \to A$ which realizes A as a finite free Λ-module.

For an algebraic Hecke character ψ on E of conductor $\mathfrak{m}\mathfrak{p}^r$ for some $r \geq 0$, we can view the p-adic avatar ψ_p of ψ as a continuous p-adic character of W. Since our fixed prime p splits in E, any eigenform f with CM by E will necessarily be p-ordinary. If ψ is an algebraic Hecke character inducing the CM eigenform f_ψ, we have that the map $A \to \mathcal{O}_K$ corresponding to ψ_p realizes the Hecke eigensystem of f_ψ inside $\mathcal{O}_K$. Let $M = \mathrm{Norm}_{\mathbf{Q}}^K(\mathfrak{m})$, and let $-d$ be the discriminant of $E/\mathbf{Q}$. In particular we have that

$$a_\ell(f) = \begin{cases} 0 & \ell \text{ is inert in } E/\mathbf{Q} \\ \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}}) & \ell \text{ splits as } \mathfrak{l}\bar{\mathfrak{l}} \text{ in } E/\mathbf{Q} \end{cases}$$

for primes $\ell \nmid dMp$. Since these quantities vary continuously with the character ψ, we can patch them together into a single map with coefficients in A. Letting $\Psi : W \to A^\times$ be the tautological character, we have a map

$$\begin{aligned} \mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K) &\to A \\ T_\ell &\mapsto \begin{cases} 0 & \ell \text{ is inert in } E/\mathbf{Q} \\ \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}}) & \ell \text{ splits as } \mathfrak{l}\bar{\mathfrak{l}} \text{ in } E/\mathbf{Q} \end{cases}. \end{aligned}$$

This map is indeed a homomorphism of Λ-algebras since after composing with any of the Zariski dense specializations corresponding to characters $\psi : A \to \mathcal{O}_K$ it realizes the map $\mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K) \to \mathcal{O}_K$ coming from the eigenform f_ψ. The CM components of $\mathbf{H}^{\mathrm{ord}}(dM; \mathcal{O}_K)$ which have CM by E are those which are components of A. The full details of this construction are presented in Theorem 7.1 of [13].

Proposition 3.10. *Suppose that* $\mathbf{I}$ *is a reduced, irreducible component of* $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$*. If a CM eigenform of weight* $k \geq 2$ *arises as a specialization of* $\mathbf{I}$*, then* $\mathbf{I}$ *is a CM component, and in particular every specialization of* $\mathbf{I}$ *has CM by the same imaginary quadratic field.*

Proof. Let f be the CM eigenform arising from $\mathbf{I}$. We know by theorem 3.4 that there is a unique component of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ giving rise to f. By assumption this component is $\mathbf{I}$, however the construction above produces a CM component which specializes to any given CM eigenform. Thus we must have that $\mathbf{I}$ itself is one of the CM components constructed above. □

We return to the description of Hecke fields, building an explicit description of the Hecke fields of CM components. This is essentially a combination of lemma 2.7 with the explicit description of CM components given above.

Lemma 3.11. *Suppose that* $\mathbf{I}$ *is a reduced, irreducible component of* $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$*. Let* E *be the imaginary quadratic field by which* $\mathbf{I}$ *has CM, and let* h *be the class number of* E*. Fix* *a weight $k \geq 2$. There are elements $a_1, \dots, a_h$ of K such that any weight k specialization f of $\mathbf{I}$ has*

$$\mathbf{Q}(f) \subseteq E(\mu_{hNp^\infty}, a_1^{1/h}, \dots, a_h^{1/h}).$$

In particular the Hecke field of each weight k form arising from $\mathbf{I}$ has its Hecke field contained within a fixed finite extension of the p-th cyclotomic field $\mathbf{Q}(\mu_{p^\infty})$.

Proof. Suppose that f_1, f_2 are any two (CM) forms of weight k arising from $\mathbf{I}$, each as the theta series attached to an algebraic Hecke character ψ_1, ψ_2. We know by the construction of $\mathbf{I}$ that the character $\psi_1\psi_2^{-1}$ has finite p-power order (it is an algebraic Hecke character of trivial infinity-type). Pick $a_1, \dots, a_h$ as in lemma 2.7 as applied to the form f_1. Since the character $\psi_1\psi_2^{-1}$ is finite order and moreover has p-power order, we see that in the presence of all p-power roots of unity (and the required "tame" roots of unity of order hN) these same a_i generate over E a field containing the Hecke field of f_2. Since we could take f_2 to be any form of weight k arising from $\mathbf{I}$, we see that the field $E(\mu_{hNp^\infty}, a_1^{1/h}, \dots, a_h^{1/h})$ contains the Hecke field of any weight k specialization of $\mathbf{I}$. □

Remark 3.12. Hida's characterization of CM families, stated in this thesis as theorem 6.1, is essentially a converse of lemma 3.11. Lemma 3.11 shows that the Hecke fields of forms arising from a CM component are uniformly controlled. Hida's result theorem 6.1 shows that any component of $\mathbf{H}^{\mathrm{ord}}(N; \mathcal{O}_K)$ which has sufficiently controlled Hecke fields in a single weight must be a CM component. It is worth noting that theorem 6.1 is much weaker than requiring that a component has uniformly controlled Hecke fields; rather it only requires that for a density 1 set of primes ℓ, the degree of the "ℓ-Hecke field" $\mathbf{Q}(a_\ell(f))$ remains bounded over $\mathbf{Q}(\mu_{p^\infty})$ as one varies over forms f of a fixed weight which arise from $\mathbf{I}$.

Under the assumption that the family in question has infinitely many classical forms of low weight, we establish this boundedness of Hecke fields across the entire family using the special properties of Hecke fields in low weight as embodied by lemma 3.7, along with the rigidity principles of chapter 4 and construction of chapter 5 to extend from low weight to regular weight.

3.3 Partially Ordinary Families of Hilbert Modular Forms

Our strategy was developed with an eye towards proving finiteness results for Hilbert modular forms of partial weight one in families. We outline in this section what sort of p-adic families our method works for. The key is that we wish to work with families where we impose an ordinarity condition at only a single split prime above p and vary the weight only at that prime, as these will be the families where an analog of lemma 3.7 holds for partial weight one forms.

To state what we need from p-adic families of Hilbert modular forms we will work with a totally real field $F/\mathbf{Q}$. Our families will depend on a choice of prime above p in F and some "tame" data fixing the level structure away from that prime. We fix the following data:

- A totally real field $F/\mathbf{Q}$ with $[F : \mathbf{Q}] = d$.
- A prime p which splits completely in $F/\mathbf{Q}$.
- Supposing that the primes dividing p are $v_1, \dots, v_d$ we fix $v = v_1$ as the prime for which we will consider v-ordinary forms.
- We fix "tame weights" $k_2, \dots, k_d \in \mathbf{Z}$ such that $k_i \equiv k_j$ for $i \neq j$ and $k_i \geq 3$ for each i, and $w \in \mathbf{Z}$ having the same parity as the k_i. The forms that we consider will have weight $(k, k_2, \dots, k_d)$ where k (the weight at v) may vary but is chosen with the same parity as the k_i, and central character the $w-1$-st power of the cyclotomic character times a character of finite order. Denote the vector of tame weights by $\underline{k}^v$.
- Choose a "tame level" N which is prime to v, but not necessarily prime to p. The forms we consider will have level $\Gamma_0(Nv^r)$ for some r.
- We choose a "tame character" $\epsilon^v : \mathcal{O}_F^\times / N\mathcal{O}_F^\times \to \overline{\mathbf{Q}}_p^\times$ which will be the tame part of the Nebentypus character of our forms.

With all these choices made, we will work with spaces of cuspidal Hilbert modular forms of weight $((k, k_2, \ldots, k_d), w)$, level $\Gamma_0(Nv^r)$, and character $\epsilon_v \epsilon^v$ for some choice of local character at v ϵ_v. We now state what we need out of families of forms for our arguments to work.

Assumption 3.13. Given the data listed above there exists a $\Lambda = \mathcal{O}[\![T]\!]$ algebra $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K)$ which interpolates v-ordinary Hilbert modular forms with the tame weight, level, and character as above. More precisely we have that:

1. $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K)$ is a finitely generated free Λ-module.

2. $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K)$ is reduced.

3. (Base Change) If L is a finite extension of K, with ring of integers $\mathcal{O}_L$, we have that

$$\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_L) \cong \mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathcal{O}_L.$$

4. $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K)$ is generated as a Λ-module by a collection of elements T_ℓ, S_ℓ for each prime $\ell \nmid Np$, and elements U_ℓ for each prime $\ell | Np$.

5. (Control Theorem) Let $k \geq 3$ be an odd integer, and let $\underline{k} = (k, k_2, \ldots, k_d)$. Let ζ be a p^r-power root of unity for some $r \geq 0$ and suppose that K is large enough to contain ζ. Let $\epsilon_v : (1+p)\mathcal{O}_{F_v} \cong (1+p)\mathbf{Z}_p \to \mathcal{O}_K^\times$ be the character taking $1+p$ to ζ. Then the natural map

$$\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K)/P_{k,\zeta}\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N, \epsilon^v; \mathcal{O}_K) \overset{\cong}{\to} H_{(\underline{k},w)}^{v-\mathrm{ord}}(\Gamma_0(Nv^r), \epsilon_v\epsilon^v; \mathcal{O}_K)$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ on the left to the equivalently named Hecke operators on the right is an isomorphism of Λ-modules.

6. (Partial Control Theorem in Weight One) Keep the notation of the previous point, but

this time let the weight k at v equal 1. Then the natural map

$$\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N,\epsilon^v;\mathcal{O}_K)/P_{k,\zeta}\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N,\epsilon^v;\mathcal{O}_K) \twoheadrightarrow H_{(\underline{k},w)}^{v-\mathrm{ord}}(\Gamma_0(Nv^r),\epsilon_v\epsilon^v;\mathcal{O}_K)$$

sending the abstract elements T_ℓ, S_ℓ, U_ℓ on the left to the equivalently named Hecke operators on the right is a surjection of Λ-modules. In other words every classical partial weight one form which is weight one at v and v-ordinary arises as a specialization of $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N,\epsilon^v;\mathcal{O}_K)$.

7. (Galois Representation) To each reduced, irreducible component $\mathbf{I}$ of $\mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N,\epsilon^v;\mathcal{O}_K)$ there is attached a Galois representation

$$\rho_{\mathbf{I}} : G_F \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I})).$$

This representation satisfies similar properties to those discussed in theorem 3.5, with the main difference being that the structure of the representation is given only on the decomposition group at v and not at all primes above p.

3.3.1 State of the Literature on Partially Ordinary Families

We discuss here what is known about the desired families as laid out in assumption 3.13. The most comprehensive reference is the unpublished Ph.D. thesis of Brad Wilson [41]. In [41] a candidate for the algebra $\mathbf{H}^{v-\mathrm{ord}} = \mathbf{H}_{\underline{k}^v}^{v-\mathrm{ord}}(N,\epsilon^v;\mathcal{O}_K)$ is constructed using the cohomological construction: Wilson works with algebraic automorphic forms in the Betti cohomology of quaternionic Shimura varieties of dimension 0 or 1, depending on the parity of $d = [F : \mathbf{Q}]$. This construction produces an algebra $\mathbf{H}^{v-\mathrm{ord}}$ which is torsion-free over Λ, and for which the control theorem is known only up to a finite kernel.

Hida's article [19] also works with these families $\mathbf{H}^{v-\mathrm{ord}}$. Hida sketches some parts of their construction, but references [41] for many of their properties. In this article Hida states without proof the existence of the Galois representations attached to components of $\mathbf{H}^{v-\mathrm{ord}}$.

Finally we should mention that similar families have been constructed independently by Yamagami in [42] and by Johansson–Newton in [24]. Both Yamagami and Johansson–Newton work with more general v-finite-slope rather than v-ordinary families, though one can think of the $\mathrm{ord}_p(U_v) = 0$ locus in the v-finite-slope eigenvariety as being the v-ordinary families that we study.

All the existing constructions of these families rely on the cohomological construction, relating the Hilbert modular forms we wish to study to algebraic automorphic forms in the middle cohomology of a quaternionic Shimura variety, where the quaternion algebra which ramified at either all or all but one of the infinite places. This construction has the benefit of being relatively elementary in character, but from our point of view is insufficient as it provides no information about the partial weight one points of the family. Since we wish to prove results about the number of classical partial weight one forms living in families, we certainly need to know that all such forms can be put into a p-adic family of the right type! It is absolutely crucial for our method to know that *any* classical partial weight one eigenform with non-zero U_v-eigenvalue fits into such a family, as that is the key to bounding Hecke fields in low weight as in lemma 3.7.

In the elliptic case, it is possible to show that classical weight one eigenforms fit into cohomologically-constructed ordinary families using the fact that (powers of the) Hasse invariant can be lifted to characteristic 0. This allows one to establish congruences between a weight one form f and a sequence of regular weight forms f_n, with $f \equiv f_n \mod p^n$. This strategy is used by Wiles in [40] in studying cyclotomic families of Hilbert modular forms, showing that any classical eigenform of parallel weight one does fit into a 1-dimensional family of parallel weight p-ordinary Hilbert modular forms. In the non-parallel case, a similar approach was used by Kisin–Lai to construct 1-dimensional eigenvarieties in [28], and by Jarvis to construct the Galois representations attached to partial weight one eigenforms in [23]. We remark that the powers of the Hasse invariant used are all forms of parallel weight, and so it is natural that the families they establish are ones where the weight varies in the parallel direction. In the partially ordinary case this strategy will not work as one would want to multiply by partial Hasse invariants in order to establish congruences to forms with weight differing only at v; however the partial Hasse invariants do not admit lifts to characteristic 0. This makes it difficult to see how one can include forms of partial weight one in a cohomologically-constructed v-ordinary family through congruences. Given that the geometric construction as outlined axiomatically by Hida in [17] and [16] relies heavily on having lifts of Hasse invariants it is difficult to see how a geometric construction starting with the full d-dimensional Hilbert modular variety would construct the required families.

The most promising approach to constructing suitable families seems to be to develop the geometric construction in a suitable quaternionic setting. Working with a quaternion algebra which is split at the infinite place corresponding to v by the isomorphism ι_p and ramified at all other infinite places, the automorphic representations attached to a classical partial weight one eigenform does transfer through the Jacquet–Langlands correspondence. We will not be able to find these representations appearing in the middle Betti cohomology of the related quaternionic Shimura curve, but we expect they will appear as sections of a suitable automorphic vector bundle over this curve. This presents some difficulties as the Shimura curve thus constructed does not naturally have a moduli interpretation as the full Hilbert modular variety does, but this can be overcome by switching to related unitary Shimura curves by making an additional choice of imaginary quadratic extension E of F. This approach does not appear in the literature, but some similar work has been done for example by Kassaei in [25], which treats only modular forms over Shimura curves which correspond to Hilbert modular forms of weight $(k, 2, \ldots, 2)$. There is also work of Ding in [9] constructing p-adic modular forms over unitary Shimura curves; this work deals with the case of p inert and doesn't explicitly treat forms of low weight, but otherwise works in a very similar setting to the one we wish to employ in the future.

Given that the very families our main strategy applies to have yet to be carefully constructed, we have elected in the remainder of this thesis to deal solely with elliptic modular forms (with the exception being chapter 7), pointing out the few places in the arguments where modifications are needed to deal with the Hilbert case. Until such time as a detailed geometric construction of partially ordinary families of Hilbert modular forms appears, one should treat our strategy as only conditionally producing results in the partial weight one Hilbert case. We intend to return to this problem of constructing suitable partially ordinary families in future work.

CHAPTER 4

RIGIDITY PRINCIPLES FOR P-ADIC POWER SERIES

In this chapter we prove rigidity results for integral extensions of p-adic power series rings. These results will be used used to propagate the boundedness of Hecke fields in low weight to regular weight, where the Ramanujan conjecture is known and Hida's theorem (stated as theorem 6.1) relating boundedness of Hecke fields and complex multiplication may be applied. In particular the boundedness of Hecke fields in low weight is what motivates the conditions of theorem 4.12; see section 5.2 for the application of this theorem to the coefficients of the characteristic polynomial of Frobenius elements in a high-dimensional representation of the absolute Galois group of F.

We fix the following notation for use in this chapter.

- K is a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}$, uniformizer π, and residue field $\mathbf{F}$.
- $\mathbf{C}_p$ is the completion of an algebraic closure of K, $\mathcal{O}_{\mathbf{C}_p}$ is the integral closure of $\mathcal{O}$ within $\mathbf{C}_p$, and $\mathfrak{m}_{\mathbf{C}_p}$ is the maximal ideal of $\mathcal{O}_{\mathbf{C}_p}$.
- As in section 1.6 ord_p is the valuation on K and extensions thereof, normalized so that $\mathrm{ord}_p(p) = 1$.
- $\Lambda = \mathcal{O}[\![T]\!]$ is the ring of formal power series in one variable with coefficients in $\mathcal{O}$.
- M is the integral closure of Λ in some finite extension of $\mathrm{Frac}(\Lambda)$.

4.1 Weierstrass Preparation and Newton Polygons

The Weierstrass preparation theorem and the theory of Newton polygons will be the basic tools we use to describe the behaviour of elements of Λ and M. Recall that a distinguished polynomial $f(T) \in \mathcal{O}[T] \subset \Lambda$ is a monic polynomial such that every coefficient other than the leading one is divisible by the uniformizer π. With this notion we can state the p-adic Weierstrass preparation theorem.

Theorem 4.1 (p-adic Weierstrass preparation)**.** *If $F(T) \in \Lambda$ is non-zero there is a unique way to write it as*

$$F(T) = \pi^k f(T)u(T)$$

where $k \geq 0$ is an integer, $f(T)$ is a distinguished polynomial, and $u(T)$ is a unit in Λ (in other words, the constant term of u is an element of $\mathcal{O}^\times$).

For a proof of the p-adic Weierstrass preparation theorem, see [30], Chapter 5, Section 2, Theorem 2.2.

Lemma 4.2. *If $F(T) \in \Lambda$ is non-zero, then for any $t \in \mathfrak{m}_{\mathbf{C}_p}$ (i.e. $t \in \mathbf{C}_p$ and $\mathrm{ord}_p(t) > 0$) the series $F(t)$ converges in $\mathbf{C}_p$. Moreover such $F(T)$ have only finitely many roots $t \in \mathfrak{m}_{\mathbf{C}_p}$.*

Proof. Let $F(T) = \pi^k f(T)u(T)$ in Weierstrass preparation. If $\mathrm{ord}_p(t) > 0$, then $u(t)$ converges since $\mathrm{ord}_p(t^n) = n \cdot \mathrm{ord}_p(t)$ goes to infinity with n, and $\mathrm{ord}_p(u(t)) = 0$ since the unit constant term of $u(T)$ dominates the norm of any term involving t. Thus $F(t) = \pi^k u(t)f(t)$ converges since f is a polynomial and u converges at t. Finally we see that since $u(t)$ is always a unit, we have $F(t) = 0$ if and only if $f(t) = 0$, and f necessarily has finitely many roots in $\mathbf{C}_p$ as it is a polynomial. □

Lemma 4.3. *Let $F(T)$ have Weierstrass preparation $F(T) = \pi^k f(T)u(T)$ where $f(T)$ has degree d. If $t \in \mathfrak{m}_{\mathbf{C}_p}$ with $0 < \mathrm{ord}_p(t) < \frac{\mathrm{ord}_p(\pi)}{d}$, then*

$$\mathrm{ord}_p(F(t)) = k \cdot \mathrm{ord}_p(\pi) + d \cdot \mathrm{ord}_p(t).$$

Proof. We compute the valuation of $F(t)$ using its Weierstrass preparation

$$\mathrm{ord}_p(F(t)) = \mathrm{ord}_p(\pi^k) + \mathrm{ord}_p(u(t)) + \mathrm{ord}_p(f(t)).$$

We have that $\text{ord}_p(\pi^k) = k \cdot \text{ord}_p(\pi)$, and $\text{ord}_p(u(t)) = 0$ since the unit constant term dominates the norm. Finally we have that $\text{ord}_p(f(t)) = d \cdot \text{ord}_p(t)$ since the leading term t^d has smaller valuation than any of the other terms of $f(t)$, as $d \cdot \text{ord}_p(t) < \text{ord}_p(\pi)$ and every other term has valuation at least $\text{ord}_p(\pi)$ since $f(T)$ is a distinguished polynomial. □

We recall the construction of Newton polygons for polynomials over $\mathbf{C}_p$. Suppose that $f(X) = \sum_{i=0}^{d} a_i X^i$ in $\mathbf{C}_p[X]$. We plot the points $(d-i, \text{ord}_p(a_i))$ in the plane (allowing points "at ∞" if some of the coefficients a_i are 0 and hence have infinite valuation), and form their lower convex hull. The resulting set of line segments in the plane is called the *Newton polygon* of f. The usefulness of Newton polygons lies in the fact that this simple combinatorial construction gives us total knowledge of the valuations of the roots of f.

Theorem 4.4. *Suppose that the Newton polygon of f consists of n line segments, with the i-th segment having horizontal length ℓ_i and slope m_i. If there is a line segment of infinite slope it must occur at the end, and in that case we consider the length ℓ_i to be such that X^{ℓ_i} divides f exactly. Then for each i in the range $1 \leq i \leq n$ there are ℓ_i roots of f which have valuation equal to m_i.*

See Chapter 3, Section 3 of [29] for more information on the theory of Newton polygons. Note that Koblitz's convention for Newton polygons is slightly different from ours; his Newton polygons are vertical reflections of ours. Koblitz's convention has the benefit of also applying easily to power series, at the drawback that the slopes of the polygon correspond to inverses of the valuations of the roots. Our convention is chosen so that the slopes are the valuations, and we won't need to use Newton polygons for non-polynomial power series.

We will use Newton polygons to study specializations of elements of M, where M is the integral closure of Λ in a finite extension of $\text{Frac}(\Lambda)$. Suppose that we have a ring homomorphism $P : M \to \mathbf{C}_p$ which extends the ring homomorphism $P_t : \Lambda \to \mathbf{C}_p$ given by $T \mapsto t \in \mathfrak{m}_{\mathbf{C}_p}$. In a slight abuse of notation we call P a $\mathbf{C}_p$-valued point of M (rather than of $\text{Spec}(M)$). Given $F \in M$, we write $F(P)$ rather than $P(F)$, thinking of F as an "algebraic" analytic function, to align with how we think of elements of Λ as analytic functions. Note that if $F \in \Lambda$, $F(P_t)$ is simply the power series F evaluated at t.

Remark 4.5. Suppose that we're given $F \in M$, and P is a $\mathbf{C}_p$ point of M, extending the $\mathbf{C}_p$ point P_t of Λ. If $R(T, X)$ is a monic irreducible polynomial satisfied by F, we have that $R(T, F) = 0$ in M, and so also $R(t, F(P)) = 0$ in $\mathbf{C}_p$. By computing the Newton polygon of $R(t, X)$ we can obtain the valuation of $F(P)$; in particular for $t \in \mathfrak{m}_{\mathbf{C}_p}$ of sufficiently small valuation we get that each coefficient of $R(t, X)$ has valuation of the form $d_i \cdot \mathrm{ord}_p(t) + k_i \cdot \mathrm{ord}_p(\pi)$ as in lemma 4.3. We then have that for $\mathrm{ord}_p(t)$ sufficiently small, $\mathrm{ord}_p(F(P)) = a \cdot \mathrm{ord}_p(t) + b$ for some positive rational a, b. Of course since all of the valuations involved are rational there is always some choice of a and b making the above statement true; the point is that the Newton polygon produces such a choice for us, and those a and b can computed from the Weierstrass preparations of the coefficients of $R(T, X)$.

Lemma 4.6. *Let F be an element of M. Suppose that there is an infinite set S of $\mathbf{C}_p$ points of M, each extending points P_t of Λ, such that $F(P) = 0$ for all $P \in S$. Then $F = 0$.*

Proof. Suppose that $R(T, X) \in \Lambda[X]$ is a monic irreducible polynomial which F satisfies. Since $R(t, F(P)) = 0$ for any $P \in S$, the constant term $a_0(T)$ of $R(T, X)$ must satisfy $a_0(t) = 0$ for each t which one of the points of S lifts. Since each point P_t of Λ extends to at most finitely many points of M, there must be infinitely many such t. By lemma 4.2 since the constant term of $R(T, X)$ has infinitely many roots t in $\mathfrak{m}_{\mathbf{C}_p}$ it must be 0. Since $R(T, X)$ is irreducible by assumption, we must have that $R(T, X) = X$, and hence $F = 0$. $\square$

Lemma 4.7. *Let $R(T, X) \in \Lambda[X]$. For all $t \in \mathfrak{m}_{\mathbf{C}_p}$ with $\mathrm{ord}_p(t)$ sufficiently small, the vertices of the Newton polygon of $R(t, X)$ occur at the same indices.*

Proof. The Newton polygon of a monic degree d polynomial is completely determined by the set of valuations of the coefficients. Thinking about the set of valuations as living in $\mathbf{R}^d$, we have a stratification of $\mathbf{R}^d$ according to which vertices lie in the Newton polygon. The condition of an index contributing a vertex to the Newton polygon is given by a collection of linear inequalities; in other words the set of valuations having a given vertex in the Newton polygon is a finite intersection of half-spaces in $\mathbf{R}^d$. The boundary of these half-spaces correspond to multiple vertices lying on the same line segment of the Newton polygon. Note that even though the valuation map takes values in $\mathbf{Q} \cup \{\infty\}$ rather than $\mathbf{R}$ we are simply working with the defining inequalities over $\mathbf{R}$, and if needs be maybe we may replace any infinite valuations with sufficiently large non-infinite valuations without affecting any of the arguments.

We know that for $t \in \mathbf{C}_p$ with $\text{ord}_p(t)$ sufficiently small, the coefficients of $R(t, X)$ have valuation of the form $a \cdot \text{ord}_p(t) + b$ by lemma 4.3. Say that the i-th coefficient of $R(T, X)$ has valuation $a_i \cdot \text{ord}_p(t) + b_i$ for $\text{ord}_p(t)$ sufficiently small. We consider the curve in $\mathbf{R}^d$ given by $s \mapsto (a_1 s + b_1, \ldots, a_d s + b_d)$. Since the image of this curve is an affine line, we have that for a half-space in $\mathbf{R}^d$ the curve must satisfy one of the following three possibilities:

- the curve is contained entirely within the interior of either the half-space or its complement,
- the curve is contained entirely in the boundary of the half-space,
- the curve intersects the boundary of the half-space exactly once.

Since there are only finitely many affine conditions involved in defining the stratification, we see that the curve will intersect the boundaries of strata transversally only finitely many times. Therefore for $s \in (0, \epsilon)$ for a sufficiently small ϵ the image of the curve will be entirely contained within a single stratum (moving from $s = 0$ to $s > 0$ may change strata, but the curve cannot encounter a boundary within a sufficiently small interval above 0). Since the valuations of the coefficients of $R(t, X)$ for $\text{ord}_p(t)$ sufficiently small land on this curve, we see that the vertices of the Newton polygon occur at the same indices for any t with $0 < \text{ord}_p(t) < \epsilon$ and $\text{ord}_p(t)$ small enough for each coefficient of $R(T, X)$ to satisfy lemma 4.3. □

4.2 Bounded Sums of Roots of Unity

In this section we prove our main result on the rigidity of algebraic power series. By algebraic power series we mean elements of integral extensions M of Λ. Our main result (theorem 4.12) is the following: if an algebraic power series is a sum of at most B roots of unity when specialized at infinitely many points which extend points of the form $T \mapsto \zeta - 1$ for p-power roots of unity ζ, then it is a power series which is a linear combination of at most B terms of the form

$$(1+T)^e = \sum_{n=0}^{\infty} \binom{e}{n} T^n$$

where for $e \in \mathbf{Z}_p$, $\binom{e}{n}$ is the usual binomial coefficient $\binom{e}{n} = \frac{e(e-1)...(e-n+1)}{n!}$. We call power series of the form $(1+T)^e$ "exponential" power series. This result is inspired by the rigidity results used by Hida in his work on the relationship between Hecke fields and complex multiplication for ordinary families. For example see Lemma 5.1 and Proposition 5.2 of [18], see also [19], [20], and [21] for variations on these statements. In a different context, similar results are also used in [38] and [39] in studying p-adic families of automorphic forms over imaginary quadratic fields.

In Hida's work the need for these rigidity lemmas arises in the following way. Given infinitely many ℓ-Weil numbers of bounded degree over $\mathbf{Q}(\mu_{p^\infty})$ (ℓ a prime different from p), there are only finitely many up to equivalence (two Weil numbers are equivalent if their quotient is a root of unity, see corollary 2.2. of [20]). Hence if F is an algebraic power series specializing to ℓ-Weil numbers at roots of unity, it must be the case that after dividing out by some Weil number we have a power series which takes values in μ_{p^∞} infinitely often. The algebraic power series F in question are those interpolating Frobenius eigenvalues across an ordinary family of modular forms. Applying the rigidity statement allows us to produce forms in this family with controlled Hecke fields, and from there use those forms to establish that the family has complex multiplication.

the desired properties in order to establish complex multiplication of the family.

Since we want to work with partial weight one Hilbert modular forms, where the Ramanujan conjecture is not yet known, we cannot directly apply this result as we don't know that the Frobenius eigenvalues of such forms are Weil numbers. Instead we use what we do know about the boundedness of their Hecke fields to establish rigidity formulas for the Frobenius eigenvalues of suitable high-dimensional Galois representations in chapter 5, and use this rigidity to deduce boundedness of Hecke fields in regular weight in chapter 6 whence Hida's results can be applied.

The main difficulty in establishing rigidity statements for algebraic power series specializing to a bounded number of roots of unity, rather than a single root of unity, is that cancellation between different terms can interfere with precise control of valuations. The following facts about quotients of rings of integers are crucial to putting limits on the possible cancellations that can occur among sums of roots of unity.

Lemma 4.8. *Suppose that $\mathcal{O}/\pi^n$ has characteristic p, i.e. n is less than or equal to the ramification index $e = [\mathcal{O} : W(\mathbf{F})]$. Then $\mathbf{F}[x]/x^n \cong \mathcal{O}/\pi^n$, with the isomorphism given by $x \mapsto \pi$.*

Proof. Since $\mathcal{O}/\pi^n$ has characteristic p, the Teichmüller lift map $\mathbf{F} \to \mathcal{O}/\pi^n$ given by $a \mapsto \lim_{m\to\infty} \tilde{a}^{p^m}$, where $\tilde{a}$ is any lift of a, is an algebra homomorphism. Consider the map $\mathbf{F}[x] \to \mathcal{O}/\pi^n$ given by $x \mapsto \pi$. This map is surjective since $\mathcal{O}/\pi^n$ is generated by Teichmüller lifts and π. The kernel of this map is (x^n), and so we have the claimed isomorphism. □

Remark 4.9. Suppose that ζ is a primitive p^n-th root of unity, and $\mathcal{O} = W(\mathbf{F})$ is the ring of integers of an unramified extension of $\mathbf{Q}_p$. Then for $m < n$ we have that $(\zeta^{p^m} - 1) = (\zeta - 1)^{p^m}$ as ideals of $\mathcal{O}[\zeta]$ by comparing valuations. Since $\zeta^{p^m} - 1$ has positive valuation less than 1, the quotient $\mathcal{O}[\zeta]/(\zeta^{p^m} - 1)$ has characteristic p, and is a polynomial ring $\mathbf{F}[x]/x^{p^m}$ by lemma 4.8. Moreover, by changing variables to $y = x + 1$ we see that

$$\frac{\mathbf{F}[y]}{y^{p^m} - 1} \to \frac{\mathcal{O}[\zeta]}{\zeta^{p^m} - 1}$$

$$y \mapsto \zeta$$

is an isomorphism.

We begin by proving our main result in the special case of an algebraic power series which takes values in μ_{p^∞} infinitely often. This proof serves as a good introduction to the ideas in the proof of theorem 4.12 while being less technical. The first appearance of this result is in [18], where two proofs are given; our strategy builds off of the second proof in [18] which Hida credits to Kiran Kedlaya.

Theorem 4.10 (Lemma 5.1 in [18])**.** *Suppose that we are given the following data:*

- *an element F in an integral extension M of Λ*
- *an infinite set $S \subset \mu_{p^\infty}$*
- *for each $\zeta \in S$, a $\overline{\mathbf{Q}}_p$ point P_ζ of M which extends the point $T \mapsto \zeta - 1$ of Λ*

with the property that for each $\zeta \in S$, $F(P_\zeta)$ is a power of ζ. Then there is a root of unity ξ' and exponent $e \in \mathbf{Z}_p$ such that $F \in \Lambda[\xi']$ and $F = \xi'(1+T)^e$.

Remark 4.11. In Hida's formulation of this result (which is stated for power series only rather than elements of integral extensions of Λ), it is only required that $F(P_\zeta) \in \mu_{p^\infty}$ for infinitely many ζ. While this may seem more general in that $F(P_\zeta)$ could potentially be a p-th root of ζ for all ζ, the control of valuations as in remark 4.5 and lemma 4.7 is enough to show that if we have such an $F \in \Lambda$, then in fact $F(P_\zeta)$ is a power of ζ for all ζ of sufficiently large order.

Proof. Suppose that $F(P_\zeta) = \zeta^{e_\zeta}$ for some integer exponent e_ζ. Since $\mathbf{Z}_p$ is compact, the infinite set of e_ζ must have a limit point $e \in \mathbf{Z}_p$. We restrict S to a subset such that $e_\zeta \to e$ as the multiplicative order of ζ goes to ∞. Define $G(T) = (1+T)^e$. Define $H = F - G$, and let $R(T, X)$ be a monic irreducible polynomial in $\Lambda[X]$ which H satisfies.

On one hand we know from remark 4.5 and lemma 4.7 that there are positive rational numbers a, b such that for $\zeta \in S$ of sufficiently large order we have

$$\mathrm{ord}_p(H(P_\zeta)) = a \cdot \mathrm{ord}_p(\zeta - 1) + b.$$

On the other hand we can compute directly that if $\zeta \in S$ is of order p^n, and $e_\zeta \equiv e \bmod p^m$ for $n \geq m$, then

$$\begin{aligned}
\mathrm{ord}_p(H(P_\zeta)) &= \mathrm{ord}_p(\zeta^{e_\zeta} - \zeta^e) \\
&= \mathrm{ord}_p(\zeta^{e_\zeta - e} - 1) \\
&\geq \mathrm{ord}_p(\zeta^{p^m} - 1) \\
&= p^m \cdot \mathrm{ord}_p(\zeta - 1).
\end{aligned}$$

since $\zeta^{e_\zeta - e}$ has multiplicative order at most p^{n-m}. Choosing our ζ of large enough order so that $p^m > a$, we see by comparing our two expressions for $\mathrm{ord}_p(H(P_\zeta))$ that we must have $b > 0$.

Fix a k such that $b > \frac{1}{\varphi(p^k)}$ (note that $\frac{1}{\varphi(p^k)} = \mathrm{ord}_p(\zeta^{p^{n-k}} - 1)$ for ζ of order p^n). Then if $\zeta \in S$ is a primitive p^n-th root of unity for $n > k$ and e_ζ is sufficiently p-adically close to e, we have that $H(P_\zeta) = 0$ in the quotient ring $R_\zeta = \mathbf{Z}_p[\zeta]/(\zeta^{p^{n-k}} - 1)$; this follows by computing valuations since $\mathrm{ord}_p(H(P_\zeta)) \geq \mathrm{ord}_p(\zeta^{p^{n-k}} - 1)$ by the above choices. As in remark 4.9 we have that R_ζ is isomorphic to a truncated polynomial ring $\mathbf{F}_p[y]/(y^{p^{n-k}} - 1)$ where the isomorphism sends $y \mapsto \zeta$. In order for $y^{e_\zeta} - y^e = 0$ in R_ζ it must be the case that $e_\zeta \equiv e \bmod p^{n-k}$ for all such ζ. However there are only finitely many values that $\zeta^{e_\zeta - e}$ can take if $e_\zeta \equiv e \bmod p^{n-k}$ as this must be a p^k-th root of unity. So if we choose ξ' such that $\zeta^{e_\zeta - e} = \xi'$ for infinitely many ζ, we see that $F - \xi'(1+T)^e$ is 0 when specialized at infinitely many of the P_ζ. Therefore lemma 4.6 shows that $F = \xi'(1+T)^e$.

$\square$

We are now in place to prove the main result of this section. Before doing so we sketch the idea of the proof, which follows the same strategy as theorem 4.10. Given an F as in theorem 4.12, we use the density of the exponents appearing to produce a guess $G(T)$ for the form of F which is a linear combination of exponential power series. We can show that the difference $F - G$ is p-adically close to 0 under many specializations; the challenge is to show that this is because the terms of F match up with the terms of G to cancel out, rather than the terms of F cancelling out with each other. By working in the quotient ring by an appropriate power of $(\zeta - 1)$ as in remark 4.9 we are in a polynomial ring, where we can ensure that unexpected cancellations are limited. Some cancellation between terms may still occur, but we can classify such cancellations into groups of terms which are consistently close to each other p-adically. This grouping allows us to refine our guess G, possibly reducing the value B, and repeat until we've ruled out all possible unexpected cancellations.

Theorem 4.12. *Suppose that we are given the following data:*

- *an element F in an integral extension M of Λ*
- *a constant $B \in \mathbf{Z}_{\geq 0}$*
- *an infinite set $S \subset \mu_{p^\infty}$*
- *for each $\zeta \in S$, a $\overline{\mathbf{Q}}_p$ point P_ζ of M which extends the point $T \mapsto \zeta - 1$ of Λ*
- *a root of unity ξ*
- *coefficients $c_1, \ldots, c_B \in \mathbf{Z}[\xi]$*

with the property that for each $\zeta \in S$, $F(P_\zeta) \in \mathbf{Z}[\xi, \zeta]$ and $F(P_\zeta)$ can be written in the form

$$F(P_\zeta) = \sum_{i=1}^{B} c_i \zeta^{e_{\zeta,i}}$$

for some exponents $e_{\zeta,i}$*. Then there is a root of unity* ξ'*, coefficients* $d_i \in \mathbf{Z}[\xi']$*, and exponents* $e_i \in \mathbf{Z}_p$ *such that* $F \in \Lambda[\xi']$ *and*

$$F = \sum_{i=1}^{B} d_i (1+T)^{e_i}.$$

Proof. The proof proceeds by induction on B. If $B = 0$, we have that $F(P_\zeta) = 0$ for infinitely many points P_ζ, and lemma 4.6 allows us to conclude that $F = 0$, which is of the desired form. The bulk of the proof is therefore to show that given such an F as in the theorem statement, we may write F in the form $G + F_1$, where G is a power series of the desired form (a linear combination of terms of the form $(1+T)^e$) and F_1 satisfies the assumptions of the theorem with a smaller value of B than that of F.

The first step is to construct a candidate expression G, and then to show that the specializations of $H = F - G$ at many of the points P_ζ are p-adically close to 0. Considering the $e_{\zeta,i}$ as integers, we have an infinite set of points in $\mathbf{Z}_p^B$. Since $\mathbf{Z}_p^B$ is compact, the set of tuples $e_{\zeta,i}$ must have a limit point $(e_1, \ldots, e_B) \in \mathbf{Z}_p^B$. Define

$$H = F - \sum_{i=1}^{B} c_i (1+T)^e$$

as an element of $M[\xi]$, and let $R(T, X) \in \Lambda[X]$ be a monic irreducible polynomial satisfied by H.

Let us restrict ourselves to an infinite subset of S such that as the multiplicative order of $\zeta \in S$ goes to infinity we have that $e_{\zeta,i} \to e_i$ for each i. We know from lemma 4.7 that for ζ of sufficiently large multiplicative order the Newton polygon of $R(\zeta - 1, X)$ is stable, and hence the specialization $H(P_\zeta)$ for $\zeta \in S$ must have valuation determined by one of the slopes of this polygon. Passing to a further infinite subset of S we may assume that the specialization has valuation determined by a single line segment in the stable Newton polygon, and hence for ζ of large multiplicative order we must have that

$$\mathrm{ord}_p(H(P_\zeta)) = a \cdot \mathrm{ord}_p(\zeta - 1) + b$$

for some fixed rational a and b. Since we know that the $e_{\zeta,i} \to e_i$, let us restrict to ζ of sufficiently large multiplicative order so that $e_{\zeta,i} \equiv e_i \bmod p^m$ for m chosen large enough to ensure that $p^m > a$. Then we have that

$$\begin{aligned}
\mathrm{ord}_p(H(P_\zeta)) &\geq \min_i \left(\mathrm{ord}_p(c_i(\zeta^{e_{\zeta,i}} - \zeta^{e_i}))\right) \\
&= \min_i \left(\mathrm{ord}_p(c_i\zeta^{e_i}(\zeta^{e_{\zeta,i}-e_i} - 1))\right) \\
&\geq \min_i \left(\mathrm{ord}_p(c_i)\right) + \mathrm{ord}_p(\zeta^{p^m} - 1) \\
&\geq \min_i \left(\mathrm{ord}_p(c_i)\right) + p^m \cdot \mathrm{ord}_p(\zeta - 1)
\end{aligned}$$

using the construction of H and our choice of ζ large enough ensuring that $e_{\zeta,i} - e_\zeta$ is divisible by p^m. Combining these two perspectives on $\mathrm{ord}_p(H(P_\zeta))$ we get that

$$a \cdot \mathrm{ord}_p(\zeta - 1) + b \geq p^m \cdot \mathrm{ord}_p(\zeta - 1) + \min_i \mathrm{ord}_p(c_i).$$

Since $p^m > a$ by construction, we see that $b > \min_i \mathrm{ord}_p(c_i)$. In particular let c be one of the coefficients achieving the minimum valuation, then for every $\zeta \in S$ we have that $c^{-1}H(P_\zeta)$ has valuation at least $v > 0$ for some constant v. Note that $c^{-1}H(P_\zeta)$ is still an element of $\mathbf{Z}_p[\xi, \zeta]$ rather than $\mathbf{Q}_p(\xi, \zeta)$ since each coefficient c_i has valuation at least that of c.

Pick a k such that $v > \frac{1}{\varphi(p^k)}$. If $\zeta \in S$ is a primitive p^n-th root of unity for $n > k$, we know that $\zeta^{p^{n-k}}$ is a primitive p^k-th root of unity, and by valuations we have that $c^{-1}H(P_\zeta) = 0$ in the quotient ring $R_\zeta = \mathbf{Z}_p[\xi, \zeta]/(\zeta^{p^{n-k}} - 1)$. As in remark 4.9 we have that this quotient ring R_ζ is isomorphic to a truncated polynomial ring $\mathbf{F}[y]/(y^{p^{n-k}} - 1)$ where $\mathbf{F}$ is the residue field of $\mathbf{Z}_p[\xi, \zeta]$ and the isomorphism sends y to ζ. We restrict S to those ζ of large enough multiplicative order p^n so that

- if $e_i \neq e_j$, then $e_i \not\equiv e_j \bmod p^d$, and $n-k > d$,

- if $e_{\zeta,i} \equiv e_i \bmod p^m$ for each $i = 1, \ldots, B$, and $e_i \not\equiv e_j \bmod p^d$, then $n-k > m > d$. In particular this forces $e_{\zeta,i} \not\equiv e_{\zeta,j} \bmod p^{n-k}$ if $e_i \neq e_j$.

We know that $c^{-1}H(P_\zeta) = 0$ as an element of R_ζ; we also have by simply reducing the expression that

$$c^{-1}H(P_\zeta) = c^{-1}\sum_{i=1}^{B} c_i(y^{e_{\zeta,i}} - y^{e_i})$$

in R_ζ. Since this expression is equal to 0 in R_ζ and ζ is chosen large enough to ensure that the powers of y associated to $e_i \neq e_j$ cannot interact in R_ζ (as these powers are distinct mod p^{n-k}), there must be cancellation occurring among the terms corresponding to each of the values e_i. These cancellations must be some combination of the following three possibilities:

- the coefficients $c^{-1}c_i$ are 0 in R_ζ; this cannot happen for all coefficients, as at least one of these is equal to 1 since $c = c_i$ for some $i = 1, \ldots, B$.

- a set of the coefficients c_i sums to 0 in $\mathbf{F}$, and the corresponding terms $y^{e_{\zeta,i}}$ have exponents which are congruent $\bmod p^{n-k}$ (similarly the corresponding terms y^{e_i} have exponents congruent $\bmod p^{n-k}$ which in fact implies they are equal).

- $e_{\zeta,i} \equiv e_i \bmod p^{n-k}$.

There are finitely many patterns that such cancellations can occur in, so restrict to an infinite set of S such that the same cancellation pattern occurs for each ζ in the restricted S. If two terms ζ^x and ζ^y have exponents that agree $\bmod p^{n-k}$, then $\zeta^y = \zeta^x\zeta_0$ for a root of unity ζ_0 of order dividing p^k. Since there are finitely many such ζ_0, we restrict to an infinite subset of S where, after collapsing down terms in the cancellation pattern with exponents congruent $\mod p^{n-k}$, the pattern of p^k-th roots of unity appearing is the same. Note that this collapsing must occur at least once since not all of the coefficients $c^{-1}c_i$ are 0 in $\mathbf{F}$.

We are now in the situation where for an infinite subset of S we have that

$$F(P_\zeta) = \sum_{i=1}^{B} c'_i \zeta^{e'_{\zeta,i}}$$

where the c'_i are in $\mathbf{Z}[\xi, \zeta_0]$ for a primitive p^k-th root of unity ζ_0, and for at least some indices i the $e'_{\zeta,i}$ are either equal to $e_i \mod p^n$ for all ζ or there are several i for which the $e_{\zeta,i}$ are equal for all ζ. Thus through a combination of combining coefficients with equal $e'_{\zeta,i}$ or subtracting off a term of the form $c'_i(1+T)^{e_i}$ we have a new $F_1 \in M[\xi, \zeta_0]$ which satisfies the assumptions of the theorem (with an enlarged integral extension M and base ring $\mathbf{Z}[\xi]$ and) with a smaller value of B. □

4.3 Sums of Exponential Power Series

In this section we collect several results on power series of the form $F(T) = \sum_{i=1}^{n} d_i(1+T)^{e_i}$. Given such a power series F, define $\pi_i = (1+p)^{e_i}$. The use of π_i is that the specializations of F that we are interested in (namely at points extending $P_{k,\zeta} : T \mapsto \zeta(1+p)^{k-1} - 1$) can all be expressed using the π_i:

$$F(P_{k,\zeta}) = \sum_{i=1}^{n} d_i \zeta^{e_i} \pi_i^{k-1}.$$

In particular $F(P_{k,\zeta}) \in \mathbf{Q}(\zeta, d_i, \pi_i)$, so if we control the field of definition of the d_i and π_i, we have control of the field of definition of $F(P_{k,\zeta})$.

The F that we will use arise from families of (Hilbert) modular forms; in particular $F(P_{k,\zeta})$ will be related to Hecke eigenvalues of classical modular forms of weight k and character coming from ζ, and will be algebraic. Our goal in this section is to show that under assumptions of the algebraicity of d_i and $F(P_{k,\zeta})$ we have that the π_i are algebraic.

We begin with the following combinatorial lemma.

Lemma 4.13. *Let $d_1, \dots, d_n$ be non-zero algebraic numbers, and let $e_1, \dots, e_n$ be distinct p-adic integers. Then there are distinct p-power roots of unity $\zeta_1, \dots, \zeta_n$ such that the matrix with entries $x_{i,j} = d_i \zeta_j^{e_i}$ has non-zero determinant.*

Proof. We proceed by induction on n, with the base case $n = 1$ being satisfied by the choice of $\zeta = 1$ since $d_1 \neq 0$.

Assume by induction that we've chosen $\zeta_1, \dots, \zeta_m$ for some $m < n$ such that the $m \times m$ matrix with entries $x_{i,j} = d_i \zeta_j^{e_i}$ for $1 \leq i, j \leq m$ has non-zero determinant. Choose k large enough so that $e_1, \dots, e_{m+1}$ are distinct modulo p^k and such that $\zeta_1, \dots, \zeta_m$ are all in μ_{p^k}. Let $\tilde{e}_i$ be the unique integer which satisfies $0 \leq \tilde{e}_i < p^k$ and $e_i \equiv \tilde{e}_i \bmod p^k$. Consider the $(m+1) \times (m+1)$ matrix with entries

$$y_{i,j} = \begin{cases} x_{i,j} & j \leq m \\ d_i X^{\tilde{e}_i} & j = m+1 \end{cases}$$

where X is a formal variable. The determinant of this matrix is thus a polynomial in X which is necessarily non-zero as each matrix entry containing X appears with a different power of X so no cancellation can occur between them, and there is at least one term (the $X^{\tilde{e}_{m+1}}$ term) which appears with a non-zero coefficient, by the inductive hypothesis guaranteeing that the upper left minor of the matrix has non-zero determinant. The degree of this determinant polynomial is one of the $\tilde{e}_i$, hence it is strictly less than p^k. Since there are p^k roots of unity in μ_{p^k}, not every element of μ_{p^k} can be a root of this polynomial, hence there is a choice of $\zeta_{m+1} \in \mu_{p^k}$ which produces a non-zero determinant when we set $X = \zeta_{m+1}$. Note that $\zeta_{m+1} \neq \zeta_j$ for any $1 \leq j \leq m$, as that would cause two columns of the matrix to be equal (and the determinant to be 0).

Finally we conclude that this choice of ζ_{m+1} satisfies the original claim with e_i instead of $\tilde{e}_i$: since $e_i \equiv \tilde{e}_i \bmod p^k$, we have that $\zeta_{m+1}^{e_i} = \zeta_{m+1}^{\tilde{e}_i}$ for each i. □

This lemma is used in the following proposition to show that the π_i are algebraic, given algebraicity assumptions on the d_i and specializations of F. The strategy of the proof is to realize (powers of) the π_i as solutions to a system of linear equations in the d_i and specializations of F.

Proposition 4.14. *Suppose that $F(T) \in \mathcal{O}[\![T]\!]$ is of the form $\sum_{i=1}^{n} d_i(1+T)^{e_i}$ for some non-zero $d_i \in \mathcal{O}$ and distinct $e_i \in \mathbf{Z}_p$. Assume further that the d_i are algebraic, and that there is an integer $k \geq 2$ such that $F(P_{k,\zeta})$ is algebraic for almost all $\zeta \in \mu_{p^\infty}$. Then $\pi_i = (1+p)^{e_i}$ is algebraic for each i.*

Proof. Applying lemma 4.13 we see that there exists distinct p-power roots of unity $\zeta_1, \ldots, \zeta_n$ such that the system of equations

$$\begin{array}{ccccccc} F(P_{k,\zeta_1}) & = & d_1\zeta_1^{e_1}\pi_1^{k-1} & + & \ldots & + & d_n\zeta_1^{e_n}\pi_n^{k-1} \\ \vdots & = & \vdots & & \vdots & & \vdots \\ F(P_{k,\zeta_n}) & = & d_1\zeta_n^{e_1}\pi_1^{k-1} & + & \ldots & + & d_n\zeta_n^{e_n}\pi_n^{k-1} \end{array}$$

having "coefficients" $d_i\zeta_j^{e_i}$ has a unique solution (the matrix of these coefficients is invertible), the solution being the π_i^{k-1}. Note that we may choose our ζ_j of sufficiently large multiplicative order to guarantee that the $F(P_{k,\zeta})$ are algebraic.

By Cramer's rule the solutions π_i^{k-1} to the system of equations above have polynomial expressions in terms of the quantities $F(P_{k,\zeta})$ and $d_i\zeta_j^{e_i}$. As all of these quantities are algebraic, we conclude that the π_i^{k-1} are algebraic, and hence the π_i themselves are algebraic. □

CHAPTER 5

CONSTRUCTION OF LARGE GALOIS REPRESENTATIONS

In this chapter we perform the construction which will allow us to propagate information about the degrees of Hecke fields between different weights in our ordinary families.

The idea behind this construction is to essentially take the trace over a character field of the Galois representations attached to a component of the ordinary Hecke algebra which contains many weight one specializations. We do this to put ourselves in a situation where the characteristic polynomials of Frobenius will have cyclotomic integer coefficients at many weight one specializations, so that the results of section 4.2 apply. This will allow us to propagate the fact that we have bounded Hecke fields in weight one to higher weights, where the Ramanujan conjecture is known and we may apply results of Hida to deduce that our component of the ordinary Hecke algebra has complex multiplication. This link with higher weight will occur in the next chapter; in this chapter we content ourselves with performing this trace construction and showing that the results of section 4.2 apply to the resulting characteristic polynomials of Frobenius.

We set up the following notation for use in this chapter.

- Fix a prime p.
- For some large enough finite extension $\mathcal{O}$ of $\mathbf{Z}_p$, $\Lambda = \mathcal{O}[\![T]\!]$ is the weight space for p-ordinary Hecke algebras. We use the notation $P_{k,\zeta}$ for the map $\Lambda \to \overline{\mathbf{Q}}_p$ given by $T \mapsto \zeta(1+p)^{k-1} - 1$; where it will not cause confusion we also use $P_{k,\zeta}$ as notation for the kernel of this map.
- Fix a tame level $N \nmid p$.
- We let $\mathbf{H}^{\mathrm{ord}}$ be the Λ-adic ordinary Hecke algebra $\mathbf{H}^{\mathrm{ord}}(N;\mathcal{O})$ with tame level N.

5.1 Selecting Components

Our basic assumption will be that we have a component $\mathbf{I}$ of $\mathbf{H}^{\mathrm{ord}}$ which specializes to infinitely many classical weight one eigenforms. By an extended pigeonhole principle argument, we select several more components of $\mathbf{H}^{\mathrm{ord}}$ with the property that together these components see all Galois conjugates of the classical weight one forms arising from $\mathbf{I}$.

Theorem 5.1. *Suppose that $\mathbf{I} = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}$ is a reduced, irreducible component of $\mathbf{H}^{\mathrm{ord}}$, with the property that there are infinitely many classical weight one eigenforms arising as specializations of $\mathbf{H}^{\mathrm{ord}}$. Then there is an infinite set R of classical weight one eigenforms arising from $\mathbf{I}$ such that the following hold.*

(1) There is an integer $m \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ such that each $f \in R$ has exactly m external Galois conjugates over its character field C_f.

(2) There are reduced, irreducible components $\mathbf{I}_i = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}_i$ of $\mathbf{H}^{\mathrm{ord}}$ for $i = 1, \ldots, m$ and for each $f \in R$ there is a $\overline{\mathbf{Q}}_p$-point $P_{f,i}$ of $\mathbf{I}_i$ with the following property. In some ordering $f_1, \ldots, f_m$ of the m external Galois conjugates of f over its character field, the system of eigenvalues of f_i arises as the specialization of $\mathbf{I}_i$ at the point $P_{f,i}$. We may take $f_1 = f$ and $\mathbf{I}_1 = \mathbf{I}$.

(3) There exists a finite Galois extension $\mathrm{Frac}(M)$ of $\mathrm{Frac}(\Lambda)$ with M the integral closure of Λ, together with fixed embeddings $e_i : \mathbf{I}_i \to M$.

(4) For each $f \in F$ there is a $\overline{\mathbf{Q}}_p$-point P_f of M with the property that for each $i = 1, \ldots, m$, we have that

$$P_f|_{e_i(\mathbf{I}_i)} = P_{f,i}.$$

(5) Define $r : G_{\mathbf{Q}} \to \oplus_{i=1}^m \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i))$ to be the direct sum of the Galois representations attached to each of the $\mathbf{I}_i$. Using the embeddings e_i we can think of r as having image in $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$. Denote M_{P_f} the localization of M at the kernel of P_f; then for

each $f \in R$, the image of r lands in $\mathrm{GL}_{2m}(M_{P_f})$, and so can be pushed forward through P_f to obtain a representation $r_f : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$.

Proof. The theorem is an extended application of the pigeonhole principle. We show the conclusions of the theorem one at a time; at each stage we refine the results from the previous step while maintaining an infinite set of weight one forms.

We know from lemma 3.7 that any weight one form which is parameterized by $\mathbf{H}^{\mathrm{ord}}$ has at most $\mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ external Galois conjugates over its character field. Thus from our initial set of infinitely many classical weight one forms there must be an $m \leq \mathrm{rank}_\Lambda(\mathbf{H}^{\mathrm{ord}})$ which occurs infinitely often as the number of external Galois conjugates over the character field. Restrict to only those f having exactly m external Galois conjugates over their character field, and call this set R_1. This shows (1).

To show (2), we use that $\mathbf{H}^{\mathrm{ord}}$ has only finitely many irreducible components since it is finite over Λ, which is irreducible. For a given $f \in R_1$, its m external Galois conjugates over C_f arise from some set of m components of $\mathbf{H}^{\mathrm{ord}}$. Since there are only finitely many possible such sets of components, one must occur for infinitely many $f \in R_1$. Let R_2 be an infinite subset of R_1 for which all $f \in R_2$ have their Galois conjugates arising from the same set of m components. Pick this set of components and label them $\mathbf{I}_1, \ldots, \mathbf{I}_m$ such that (in some ordering) the conjugates $f_1, \ldots, f_m$ satisfy that f_i arises from $\mathbf{I}_i$. Without loss of generality we may assume that $\mathbf{I} = \mathbf{I}_1$ and $f = f_1$. Note that for each $f \in R_2$ we have a $\overline{\mathbf{Q}}_p$-point $P_{f,i}$ of $\mathbf{I}_i$ with the property that the specialization of $\mathbf{I}_i$ at $P_{f,i}$ is the system of Hecke eigenvalues of f_i.

The integral extension M/Λ as in (3) can be constructed as follows. Each $\mathbf{I}_i$ is an integral domain finite over Λ, so $\mathrm{Frac}\mathbf{I}_i$ is a finite extension of $\mathrm{Frac}(\Lambda)$. Take the Galois closure of the compositum of these fields $\mathrm{Frac}(\mathbf{I}_i)$; this is some finite extension of $\mathrm{Frac}(\Lambda)$, and we take M as the integral closure of Λ inside that field. If P is the point of $\mathbf{I}$ giving rise to f, choose any extension of P to a point P_f of M.

For a given $f \in R_2$, we have points $P_{f,i}$ of $\mathbf{I}_i$ for $i = 1, \ldots, m$. Since the Galois action is transitive on points in fibers of M/Λ, there is some embedding of $\mathbf{I}_i$ into M such that $P_{f,i}$ is the restriction of P_f to the image of $\mathbf{I}_i$. As there are only finitely many embeddings $\mathbf{I}_i \to M$ for each i, there are only finitely many possible choices total. Since for each $f \in R_2$ there is at least one choice of embeddings $\mathbf{I}_i \to M$ with the desired compatibility between points, and R_2 is infinite, there must be some choice of embeddings $e_i : \mathbf{I}_i \to M$ such that for an infinite subset $R_3 \subseteq R_2$ we have the desired compatibility of points. This shows (3) and (4) of the theorem.

As in the statement of the theorem we define r to be the representation

$$r : G_{\mathbf{Q}} \to \oplus_{i=1}^m \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i)) \overset{\oplus_{i=1}^m e_i}{\hookrightarrow} \mathrm{GL}_{2m}(\mathrm{Frac}(M))$$

obtained as the direct sum of the Galois representations attached to each $\mathbf{I}_i$, viewed as having coefficients in $\mathrm{Frac}(M)$. We would like to be able to specialize $r : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}(M))$ through the map $P_f : M \to \overline{\mathbf{Q}}_p$ in order to recover the representations attached to each f_i; however without knowing that M is a unique factorization domain it may not be possible to find a basis of $V = \mathrm{Frac}(M)^{2m}$ in which r takes values in $\mathrm{GL}_{2m}(M)$. For a given P_f we can always extend the point P_f to have domain M_{P_f} (localization of M at the kernel of P_f), so it will suffice to show that r has coordinates in M_{P_f} for an infinite subset of R_3.

The group $G_{\mathbf{Q}}$ is compact, so there is a lattice $\mathcal{L} \subset \mathrm{Frac}(M)^{2m}$ which is stable under the action of $G_{\mathbf{Q}}$ through r. Each element of r can thus be though of equally well as an element of $\mathrm{End}_M(\mathcal{L})$. We claim that $\mathrm{End}_M(\mathcal{L})$ is a finitely generated M-module. Let $a_1, \ldots, a_n$ be a set of generators for $\mathcal{L}$ as an M-module. Given an $n \times n$ matrix $X = \{x_{i,j}\}$ with coefficients in M we say that X descends to $\mathcal{L}$ if the map given by $a_i \mapsto \sum_{j=1}^n x_{i,j} a_j$ defines an endomorphism of $\mathcal{L}$; note that for any endomorphism of $\mathcal{L}$ there is at least one such matrix. The subset of $\mathrm{End}_M(M^n)$ of those matrices which descend to $\mathcal{L}$ is an M-submodule. Since M is noetherian (it is a finite extension of the noetherian ring Λ) we have that any submodule of the finitely generated $\mathrm{End}_M(M^n)$ is finitely generated. In particular one such submodule surjects onto $\mathrm{End}_M(\mathcal{L})$, proving that it is finitely generated.

Viewing each generator of $\mathrm{End}_M(\mathcal{L})$ as an element of $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$, we see that it has at most finitely many "poles", where by "pole" we mean a $\overline{\mathbf{Q}}_p$ point P of M such that the entries of that element of $\mathrm{GL}_{2m}(\mathrm{Frac}(M))$ are not in M_P. Since there are finitely many generators of $\mathrm{End}_M(\mathcal{L})$, each with finitely many poles, we see that there are at most finitely many $\overline{\mathbf{Q}}_p$ points of M which can arise as poles of an element $r(g)$. After removing finitely many of the points in R_3 in order to avoid these poles, we obtain a set R and have that the image of r lands in $\mathrm{GL}_{2m}(M_{P_f})$ for each $f \in R$. □

The representation r defined in theorem 5.1 is what we'll use to propagate control of Hecke fields from weight one into regular weight. Since our main focus will be on the specializations of r through the primes P_f of M, we set r_f to be that specialization

$$r_f : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}(M)) \xrightarrow{P_f} \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$$

which is well-defined by part (5) of theorem 5.1. We begin with some basic properties of the representations r and r_f.

Lemma 5.2. *The representation $r : G_{\mathbf{Q}} \to GL_{2m}(\mathrm{Frac}(M))$ satisfies the following properties.*

(1) For primes $\ell \nmid Np$, r is unramified at ℓ.

(2) For every $g \in G_{\mathbf{Q}}$, the characteristic polynomial of $r(g) \in \mathrm{GL}_{2m}(\mathrm{Frac}(M))$ has coefficients in M.

(3) For every $f \in R$, the representation $r_f : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$ is equal to the direct sum of the p-adic Galois representations attached to the conjugates $f_1, \dots, f_m$ of f.

Proof. (1) is immediate as r is constructed as a direct sum of representations which are unramified at primes $\ell \nmid Np$. Part (2) is a consequence of continuity of the representation as we now show. The characteristic polynomial map $G_{\mathbf{Q}} \to \mathrm{Frac}(M)[X]$ given by $g \mapsto \det(XI - r(g))$ is continuous since r itself is continuous and taking characteristic polynomials is continuous. For primes ℓ as above, each of the direct summands of r has the property that characteristic polynomials of Frob_ℓ land in $\mathbf{I}_i$; the trace and determinant are Hecke operators and so are in $\mathbf{I}_i$ rather than $\mathrm{Frac}(\mathbf{I}_i)$. Since the characteristic polynomial of the direct sum is simply the product of the characteristic polynomials, we see that the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ has coefficients in M. Finally since the Frob_ℓ are topologically dense in $G_{\mathbf{Q}}$ by the Cebotarev density theorem, we see that every element in the image of r must have characteristic polynomial in $M[X]$ since it is a closed subset of $\mathrm{Frac}(M)[X]$.

Part (3) of this lemma is a consequence of parts (3) and (4) of theorem 5.1. By our choice of embeddings we have that P_f restricted to $\mathbf{I}_i$ produces the system of Hecke eigenvalues of f_i; hence $r : G_{\mathbf{Q}} \to \oplus_{i=1}^{2m} \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_i))$ will specialize to the direct sum of the $\rho_{f_i,p}$. □

Since the representation r is unramified at primes $\ell \nmid Np$, we introduce notation for the characteristic polynomial of $r(\mathrm{Frob}_\ell)$. Let

$$A_\ell(X) = \sum_{j=0}^{2m} A_{\ell,j} X^j = \det(XI - r(\mathrm{Frob}_\ell)).$$

Note that the coefficients $A_{\ell,j}$ of $A_\ell(X)$ lie in M as per lemma 5.2. The key property of the representation r is contained in the following theorem, where we show how the choices in the construction of r lead to control over the field of definition of the specializations r_f. In particular studying the specializations r_f will amount to studying the specialized characteristic polynomials of Frobenius $A_\ell(P_f)(X) = \det(XI - r_f(\mathrm{Frob}_\ell))$.

Theorem 5.3. *Suppose that $f \in R$ and P_f is the corresponding $\overline{\mathbf{Q}}_p$-point of M as in theorem 5.1. Then for a prime $\ell \nmid Np$ we have that the characteristic polynomial $A_\ell(P_f)(X)$ of $r_f(\mathrm{Frob}_\ell)$ has coefficients in the character field C_f of f.*

Proof. Let $f_1, f_2, \ldots, f_m$ be the m external Galois conjugates of $f = f_1$. For each f_i, let α_i, β_i be the eigenvalues of $\rho_{f_i,p}(\mathrm{Frob}_\ell)$ (so that the T_ℓ-eigenvalue of f_i is equal to $\alpha_i + \beta_i$). We then have by lemma 5.2 that $A_\ell(P_f)(X)$ factors as

$$A_\ell(P_f)(X) = \prod_{i=1}^{m}(X - \alpha_i)(X - \beta_i)$$

since r_f is the direct sum of the representations $\rho_{f_i,p}$. The coefficients $A_{\ell,j}(P_f)$ of $A_\ell(P_f)(X)$ are thus (up to sign) the elementary symmetric polynomials of degree $2m$ evaluated at the α_i and β_i.

We know that α, β are in a degree at most 2 extension of K_f since they satisfy a degree 2 polynomial with coefficients in K_f (the characteristic polynomial of $\rho_{f,p}(\mathrm{Frob}_\ell)$). For any positive integer n, the expression $\alpha^n + \beta^n$ is invariant under switching these two roots, hence it must itself be in K_f. Therefore we have that the power sum $\sum_{i=1}^m \alpha_i^n + \beta_i^n$ is in C_f, since it is a field trace:

$$\sum_{i=1}^{m} \alpha_i^n + \beta_i^n = \mathrm{Trace}_{C_f}^{K_f}(\alpha^n + \beta^n).$$

Since both the power sums and the elementary symmetric polynomials are generating sets (over $\mathbf{Q}$ or extensions thereof) for the space of symmetric polynomials, all of the elementary symmetric polynomials can be expressed as polynomial combinations with rational coefficients of the power sums. We've shown that the power sums of the α_i, β_i all lie in the character field C_f, so the coefficients of $A_\ell(P_f)(X)$ will also lie in C_f. $\square$

5.2 Rigidity of Frobenius Characteristic Polynomials

The goal is this section is to show that the coefficients of the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ are constrained as in theorem 4.12. We've seen in the previous section that the specialized characteristic polynomials take values in cyclotomic fields (the characteristic polynomial of $r_f(\mathrm{Frob}_\ell)$ has coefficients in the character field of f). Since these characteristic polynomials are also necessarily integral, the coefficients must be *cyclotomic integers*, that is, elements of $\mathbf{Z}[\zeta]$ for some root of unity ζ. Following Cassels [5] and Loxton [31] we define the following quantities for a cyclotomic integer α:

- $N(\alpha)$ = the minimal natural number n such that α can be written as a sum of n roots of unity.
- $\lceil\alpha\rceil$ = the maximum of the complex absolute values of α. This is called the *house* of α.

The following theorem of Loxton relates these two measures of the size of α; this is the key link between the bounds we know on the sizes of Hecke eigenvalues and the rigidity results of section 4.2.

Theorem 5.4 (Loxton, Theorem 1 of [31])**.** *Choose a real number d with $d > \log(2)$. Then there is a positive constant c depending only on d such that if α is a cyclotomic integer with*

$$N(\alpha) = n$$

then

$$\lceil\alpha\rceil^2 \geq c \cdot n \cdot \exp(-d \log(n)/\log(\log(n))).$$

As explained in the introduction of [31], this theorem allows us to bound $N(\alpha)$ if we have a bound on $\lceil\alpha\rceil$. In particular if we know that

$$\lceil\alpha\rceil^2 < c \cdot n \cdot \exp(-d \log(n)/\log(\log(n)))$$

then it must be the case that $N(\alpha) < n$. Since the expression $c \cdot n \cdot \exp(-d \log(n)/\log(\log(n)))$ is increasing in n, if we have an absolute bound $\lceil\alpha\rceil^2$ for some collection of cyclotomic integers α, then it forces an absolute bound on the $N(\alpha)$.

Lemma 5.5. *There is a constant $C_{\ell,j}$, depending only on ℓ and j, such that for each $f \in R$ we have*

$$\lceil A_{\ell,j}(P_f)\rceil \leq C_{\ell,j}.$$

Proof. This follows from the fact that the Frobenius eigenvalues α and β of $\rho_{f,p}(\mathrm{Frob}_\ell)$ have house bounded by a polynomial in ℓ by theorem 2.10. The construction in the proof of theorem 5.3 shows that $A_{\ell,j}(P_f)$ is a polynomial expression with rational coefficients in α and β and their Galois conjugates, which all satisfy the same archimedean bound coming from theorem 2.10. Applying the triangle inequality liberally to the expression for $A_{\ell,j}(P_f)$ we obtain a bound on $\left|\overline{A_{\ell,j}(P_f)}\right|$ which is polynomial in the bound on $\left|\overline{\alpha}\right|, \left|\overline{\beta}\right|$. Since this polynomial expression in α and β and their conjugates is the same for all f in R , and the bound on $\left|\overline{\alpha}\right|, \left|\overline{\beta}\right|$ is the same for all f in R, we obtain a uniform (in f) upper bound on $\left|\overline{A_{\ell,j}(P_f)}\right|$. □

Lemma 5.6. *Fix a prime ℓ of F such that $\ell \nmid \mathfrak{N}p$. Then for each j in the range $0 \leq j \leq 2m$ the coefficient $A_{\ell,j}$ of the characteristic polynomial of $r(\mathrm{Frob}_\ell)$ satisfies the assumptions of theorem 4.12.*

Proof. For any $f \in R$, we have by theorem 5.3 that $A_{\ell,j}(P_f)$ is in the character field C_f which is a cyclotomic field. Since $A_{\ell,j}(P_f)$ is an elementary symmetric polynomial evaluated at integral inputs (the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$ are integral), it is integral, and hence is a cyclotomic integer.

Fix j, and let $C_{\ell,j}$ be the upper bound on all $\left|\overline{A_{\ell,j}(P_f)}\right|$ established in lemma 5.5. Choose n sufficiently large so that

$$C_{\ell,j}^2 < c \cdot n \cdot \exp(-\log(n)/\log(\log(n)))$$

where c is the constant associated to $d = 1 > \log(2)$ of theorem 5.4. Choosing such an n is possible since the function on the right is unbounded in n. Theorem 5.4 then guarantees for us that $N(A_{\ell,j}(P_f)) < n$, i.e. each $A_{\ell,j}(P_f)$ can be written as a sum of fewer than n roots of unity.

Let ξ be a root of unity that generates the tame (i.e. prime to p) part of the character fields C_f (changing f only changes the p-power roots of unity present). Among the finitely many combinations of $B < n$ and possible powers of ξ used to write an integral element of C_f as a sum of B roots of unity, we pick one that occurs infinitely often among the $A_{\ell,j}(P_f)$ for $f \in R$. For the subset R' of R where this combination occurs, we let B and $c_1, \ldots, c_B$ be the chosen values, and

$$S = \{\zeta \in \mu_{p^\infty} : \text{ there is } f \in R' \text{ such that } P_f \text{ extends the point } P_{1,\zeta} \text{ of } \Lambda\}.$$

With this data $A_{\ell,j}$ satisfies the assumptions of theorem 4.12 and hence is a linear combination of exponential power series. □

CHAPTER 6

BOUNDED HECKE FIELDS

Now that we have constructed our representation r and controlled the form of its characteristic polynomials of Frobenius using the rigidity results of chapter 4, we are in a good position to specialize in regular weight. The advantage of regular weight is that we can apply results of Hida on the complexity of Hecke fields associated to non-CM ordinary families in order to conclude that our family has CM. The flavour of Hida's results is that if the Hecke fields of the forms in an ordinary family are sufficiently complicated (as measured by their degree relative to the p-cyclotomic extension $\mathbf{Q}(\mu_{p^\infty})$), then the family cannot have complex multiplication. Put another way, if the Hecke fields of an ordinary family are sufficiently bounded then that family has CM.

Hida has published several variations on theorems of this flavour. In section 6.1 we sketch the proof of the version of this result that we use, in particular indicating how it will apply to the partially ordinary, non-parallel weight Hilbert case. We then assemble the results of chapter 5 together with Hida's Hecke field result to prove our main theorem in section 6.2.

We keep the notation introduced at the start of chapter 5. Most importantly $\mathbf{H}^{\mathrm{ord}}$ is a Λ-adic Hecke algebra parameterizing p-ordinary modular forms having a fixed tame level and character.

6.1 Hida's Results on Bounded Hecke Fields

Hida has proven several variations on theorems of this flavour (see [18], [19], [20], [21]). In this section we build off of [20] as it works with Hecke fields away from p (i.e. degrees of a_ℓ for $\ell \nmid Np$ rather than a_p). We found it to be more convenient to take this approach rather an approach that relies on bounding a_p, although in principle such a strategy could also work in our situation. The reason that regular weight is key is that the Frobenius eigenvalues of ρ_f in regular weight are Weil numbers by the Ramanujan conjecture, which is known for

Hilbert modular forms of regular weights due to work of Blasius ([2]). We offer a sketch of the proof of this theorem; the main idea is to use theorem 4.10 to find two eigenforms f and g whose p-adic Galois representations are "too similar" unless $\mathbf{I}$ is a CM family.

Of course we could also attempt to use this result directly for elliptic modular forms of weight one, as the Ramanujan conjecture is known for these forms! However, our goal is to access our main result without utilizing the Ramanujan conjecture in low weight, as the application of this strategy to partial weight one Hilbert modular forms cannot rely on the Ramanujan conjecture.

Theorem 6.1 (Hida, Theorem 3.1 of [20]). *Suppose that we are given the following data:*

1. *a set Σ of primes of $\mathbf{Q}$ of positive density*
2. *for each $\ell \in \Sigma$ a constant $C_\ell > 0$*
3. *an infinite set of p-power roots of unity S*
4. *a fixed integer $k \geq 2$*
5. *a reduced, irreducible component $\mathbf{I} = \mathbf{H}^{\mathrm{ord}}/\mathfrak{P}$ of $\mathbf{H}^{\mathrm{ord}}$*
6. *for each $\zeta \in S$, a point $P_{k,\zeta}$ of $\mathbf{I}$ extending the point $P_{k,\zeta}$ of Λ*

with the property that for each $\zeta \in S$, the specialization $\mathbf{I}(P_{k,\zeta})$ (which is a classical modular eigenform f_ζ) has its Hecke fields satisfying the following bounds

$$[\mathbf{Q}(\mu_{p^\infty}, a_\ell(f_\zeta)) : \mathbf{Q}(\mu_{p^\infty})] \leq C_\ell$$

for each $\ell \in \Sigma$. Then $\mathbf{I}$ has complex multiplication.

Proof Sketch. For $\ell \in \Sigma$ let α_ℓ be a choice of root of the characteristic polynomial of $\rho_{\mathbf{I}}(\mathrm{Frob}_\ell)$ where $\rho_{\mathbf{I}}$ is the p-adic Galois representation attached to $\mathbf{I}$. We assume that $\mathbf{I}$ is large enough to contain α_ℓ, extending it and the points $P_{k,\zeta}$ if necessary. Since $k \geq 2$ the Ramanujan conjecture is known for the specializations f_ζ of $\mathbf{I}$ under $P_{k,\zeta}$. As a consequence of the Ramanujan conjecture we have that $\alpha_\ell(P_{k,\zeta})$ is an ℓ-Weil number. Our condition bounding the degrees of Hecke fields yields that $\alpha_\ell(P_{k,\zeta})$ has degree at most $2C_\ell$ over $\mathbf{Q}(\mu_{p^\infty})$. As there are only finitely many such ℓ-Weil numbers up to equivalence (see corollary 2.2 of [20]), we pick π_ℓ which occurs infinitely often up to equivalence as the specialization $\alpha_\ell(P_{k,\zeta})$ for $\zeta \in S$. Thus we have that $\pi_\ell^{-1}\alpha_\ell$ specializes to a p-power root of unity infinitely often, hence by a generalization of theorem 4.10 which allows for p-power roots of $(1+T)$(see Proposition 4.1 of [20]), α_ℓ is of the form $\pi_l(1+T)^{e_\ell}$ with $e_\ell \neq 0 \in \mathbf{Q}_p$ for each $\ell \in \Sigma$.

Choose a p-power root of unity $\zeta \neq 1$, and consider the forms f and g which arise as specialization of $\mathbf{I}$ by $P_{k,1}$ and $P_{k,\zeta}$. We assume for a contradiction that neither f nor g has complex multiplication.

We let $\alpha_\ell(f) = \alpha_\ell(P_{k,1})$ be the Frobenius eigenvalue of f produced by α_ℓ, and similarly $\alpha_\ell(g) = \alpha_\ell(P_{k,\zeta})$. By our control of α_ℓ we know that $\alpha_\ell(g) = \zeta^{e_\ell}\alpha_\ell(f)$. Since the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ (with any choice of coefficients) has constant term a power of ℓ times a root of unity, we see that a similar relationship holds with the second eigenvalue of Frobenius of each form f, g. Choosing a prime q which splits completely in $\mathbf{Q}(f,g) = K_f K_g$ for convenience, we see that if $\zeta^m = 1$ that

$$\mathrm{Trace}(\rho_{f,q}(\mathrm{Frob}_\ell)^m) = \mathrm{Trace}(\rho_{g,q}(\mathrm{Frob}_\ell)^m)$$

for each $\ell \in \Sigma$. Using that $\mathrm{Trace}(\rho^m) = \mathrm{Trace}(\mathrm{Sym}^m(\rho)) - \mathrm{Trace}(\mathrm{Sym}^{m-2}(\rho) \otimes \det(\rho))$ for any 2-dimensional representation ρ, we obtain

$$\begin{aligned} &\mathrm{Trace}\left(\mathrm{Sym}^m(\rho_{f,q}) \oplus \left(\mathrm{Sym}^{m-2}(\rho_{g,q}) \otimes \det(\rho_{g,q})\right)\right) \\ =&\mathrm{Trace}\left(\mathrm{Sym}^m(\rho_{g,q}) \oplus \left(\mathrm{Sym}^{m-2}(\rho_{f,q}) \otimes \det(\rho_{f,q})\right)\right) \end{aligned}$$

when evaluated on any Frob_ℓ for $\ell \in \Sigma$.

Since f and g are not CM forms by assumption, we claim that for q sufficiently large the image of their q-adic Galois representations contains an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_q)$. The residual representations contain $\mathrm{SL}_2(\mathbf{F}_q)$ for large enough q (see Section 0.1 of [8]), and the classification of compact subgroups of $\mathrm{SL}_2(\mathbf{Z}_q)$ shows that we must therefore have an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_q)$ in the image. In particular since the representations in question are irreducible and Σ has positive density, a result of Rajan (see Theorem 2 of [35]) guarantees that we have an equality of representations

$$\mathrm{Sym}^m(\rho_{f,q}) \oplus \left(\mathrm{Sym}^{m-2}(\rho_{g,q}) \otimes \det(\rho_{g,q})\right) = \mathrm{Sym}^m(\rho_{g,q}) \oplus \left(\mathrm{Sym}^{m-2}(\rho_{f,q}) \otimes \det(\rho_{f,q})\right)$$

when restricted to a finite index subgroup G_K of $G_{\mathbf{Q}}$. We also have that $\mathrm{Sym}^m(\rho_{f,q}) = \mathrm{Sym}^m(\rho_{g,q}) \otimes \chi$ for some finite order character χ, using the same result of Rajan.

Since the representations $\rho_{f,q}$ and $\rho_{g,q}$ are members of compatible systems, so are their symmetric powers. Since one member of the compatible system $\mathrm{Sym}^m(\rho_f)$ agrees with one member of the compatible system $\mathrm{Sym}^m(\rho_g) \otimes \chi$, the whole systems agree; thus we conclude that for the prime p

$$\mathrm{Sym}^m(\rho_{f,p}) = \mathrm{Sym}^m(\rho_{g,p}) \otimes \chi.$$

We know that the p-adic Galois representation of a p-ordinary form is upper triangular when restricted to the decomposition group at p. In particular we know that $\rho_{f,p}|G_{\mathbf{Q}_p}$ has the form

$$\begin{bmatrix} \omega_p^{k-1}\psi_f & * \\ 0 & \lambda_f \end{bmatrix}$$

for some characters ψ_f, λ_f; similarly $\rho_{g,p}|G_{\mathbf{Q}_p}$ is of that form with characters ψ_g, λ_g. Since the symmetric powers agree up to twist, we have an equality of sets of characters

$$\{\omega_p^{i(k-1)}\psi_f^i\lambda_f^{m-i} : i = 0, \ldots, m\} = \{\omega_p^{i(k-1)}\psi_g^i\lambda_g^{m-i}\chi : i = 0, \ldots, m\}.$$

Note that λ_f, λ_g are unramified, and ψ_f, ψ_g have finite order on inertia with $\psi_f \neq \psi_g$ on inertia by the choice of $\zeta \neq 1$. By comparing powers of the cyclotomic character which appear in the above equality of sets of characters, we conclude that $\psi_f^i \lambda_f^{m-i} = \psi_g^i \lambda_g^{m-i} \chi$ for each $i = 0, \ldots, m$. Rearranging we get that

$$\frac{\psi_f^i}{\psi_g^i} = \frac{\lambda_g^{m-i}}{\lambda_f^{m-i}} \chi$$

for each $i = 0, \ldots, m$; in particular when restricted to inertia this yields $\frac{\psi_f^i}{\psi_g^i} = \chi$ for each i. Taking $i = 0$ we see that χ is unramified, but taking $i = 1$ shows that χ must be non-trivial on inertia. This is a contradiction.

The only assumption that we made outside of the original hypotheses was that neither f nor g has CM, in order to use the fact that Galois representations attached to non-CM forms have large image. Since we arrived at a contradiction it must be the case that at least one of them has CM, and hence the whole component **I** has CM by proposition 3.10.

□

Remark 6.2. While Hida's proof is written in the case of parallel weight $k \geq 2$ Hilbert modular forms, and we've only sketched the argument in the case of elliptic modular forms, the argument applies equally well to a partially ordinary family and fixed regular weight $(\underline{k}, w)$.

The only difference in the argument comes right at the end when extracting a contradiction from the equality of sets of characters. In the partially ordinary case it is natural to work with Galois representations having a fixed determinant, rather than determinant varying with the weight as is the common choice for elliptic modular forms. Since we're working with a fixed weight and varying the Nebentypus character, there's no obstruction to still matching up characters based on the power of the cyclotomic character that appears (i.e. based on their Hodge-Tate weights). From there an slightly modified argument from the elliptic provides a contradiction, so long as the chosen ζ has sufficiently large order.

Remark 6.3. One might ask if theorem 6.1 can be applied to a set of classical weight one eigenforms arising from **I** and having appropriately bounded Hecke fields. There are two places where the regular weight assumption is used in the proof. First, the fact that the Frobenius eigenvalues in the Galois representation attached to a regular weight form are Weil numbers, which is a consequence of the Ramanujan conjecture. Second, the fact that the ℓ-adic Galois representation attached to a non-CM eigenform of regular weight has large image for sufficiently large ℓ.

If we are willing to use the (known!) Ramanujan conjecture for weight one eigenforms then the first use of the regular weight assumption can be taken care of. The second presents more difficulty in generalizing directly to the weight one case. However, we expect that given the strong control over Frobenius eigenvalues across the entire component **I** provided by the Ramanujan conjecture it should be possible to start with bounded Hecke fields in weight one to establish rigidity of the Frobenius eigenvalues, and then carry out the rest of Hida's argument in regular weight.

We remark once again that the role of chapter 4 and chapter 5 is to provide a method by which bounds on Hecke fields may be propagated from low weight into regular weight. While this may be redundant for elliptic modular eigenforms, it is necessary for the case of Hilbert modular eigenforms of partial weight one, as the Ramanujan conjecture is not yet resolved for such forms so Hida's theorem 6.1 cannot use them as an input since it relies crucially on finiteness properties of Weil numbers.

6.2 Proof of the Main Theorem

We are finally in a position to assemble all the ingredients of the previous two chapters in order to prove our main theorem. We remind the reader that the main goal is to prove that a component **I** of $\mathbf{H}^{\mathrm{ord}}$ has CM if it admits infinitely many classical weight one specializations, which we do by propagating information about Hecke fields from low weight into regular weight so that we many apply Hida's result theorem 6.1.

Theorem 6.4. *A reduced irreducible component* $\mathbf{I}$ *of* $\mathbf{H}^{\mathrm{ord}}$ *has CM if and only if it admits infinitely many classical weight one specializations.*

Proof. If $\mathbf{I}$ has CM, then any specialization in a classical weight is a classical CM form; in particular there will be infinitely many classical weight one eigenforms arising as specializations of a CM component $\mathbf{I}$. Thus the real task in this proof is to show that if $\mathbf{I}$ admits infinitely many classical weight one specializations, then $\mathbf{I}$ has CM.

We show that the conditions of theorem 6.1 apply if we are given such an infinite set of classical weight one specializations, which is now just a matter of assembling the ingredients of chapter 4 and chapter 5. In fact we'll show a stronger statement than necessary to apply theorem 6.1. For each $\ell \nmid Np$ we produce a constant C_ℓ such that for almost all classical specializations f of $\mathbf{I}$ we have that

$$[\mathbf{Q}(\mu_{p^\infty}, a_\ell(f)) : \mathbf{Q}(\mu_{p^\infty})] \leq C_\ell.$$

We recall that the main result of chapter 5 was the construction of a Galois representation $r : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}M)$ for some integral extension M of Λ. The key property of r is that for some infinite set R of classical weight one specializations of $\mathbf{I}$ we have for $f \in R$ that

$$r : G_{\mathbf{Q}} \to \mathrm{GL}_{2m}(\mathrm{Frac}(M)) \stackrel{P_f}{\to} \mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$$

is well-defined, and equal to the direct sum of the p-adic Galois representations attached to the external Galois conjugates $f_1, \ldots, f_m$ of f over its character field. In lemma 5.6 we showed that each coefficient $A_{\ell,j}$ of the characteristic polynomial $A_\ell(X) = \sum_{j=0}^{2m} A_{\ell,j} X^j$ of $r(\mathrm{Frob}_\ell)$ is controlled by the rigidity results of chapter 4. Namely, for each $j = 0, \ldots, 2m$ we have an expression

$$A_{\ell,j} = \sum_{i=1}^{n_j} d_{i,j}(1+T)^{e_{i,j}}$$

for some algebraic $d_{i,j}$ and p-adic integers $e_{i,j}$.

A final key feature of the representation r is that it "sees" almost all classical specializations of $\mathbf{I}$. If f is a classical eigenform arising as the specialization of $\mathbf{I}$ through a $\overline{\mathbf{Q}}_p$ point P, denote by P again an extension of this point to M. Then for almost all P we have that the image of r lands in $\mathrm{GL}_{2m}(M_P)$ (as in part (5) of theorem 5.1), so we can push forward r through P to obtain a representation into $\mathrm{GL}_{2m}(\overline{\mathbf{Q}}_p)$. By the construction of r as a direct sum of representations attached to components of $\mathbf{H}^{\mathrm{ord}}$, we see that $\rho_{f,p}$ is a direct summand of this specialization of r. In particular this shows that the Frobenius eigenvalues α_f, β_f of $\rho_{f,p}(\mathrm{Frob}_\ell)$ are roots of $A_\ell(P)(X)$. Since the other direct summands of this specialization of r are the Galois representations attached to other classical forms, we may also conclude that all coefficients of $A_\ell(P)(X)$ are algebraic.

We now make use of our exact formula for the $A_{\ell,j}$, as explained in section 4.3. Define $\pi_{i,j} = (1+p)^{e_{i,j}}$. Since almost all specializations of $\mathbf{I}$ in weights $k \geq 2$ are classical and are witnessed by r, we have that almost all specializations $A_{\ell,j}(P_{k,\zeta})$ are algebraic, as these specializations are polynomial combinations of Frobenius eigenvalues of classical forms. Thus proposition 4.14 shows that the $\pi_{i,j}$ are all algebraic.

We conclude that for almost all points $P_{k,\zeta}$ of $\mathbf{I}$ for $k \geq 2$ satisfy that $A_\ell(P_{k,\zeta})(X)$ has coefficients in $L_\ell(\zeta)$, where

$$L_\ell = \mathbf{Q}(\{d_{i,j}\}, \{\pi_{i,j}\}).$$

Note that L_ℓ is a finite extension of $\mathbf{Q}$ since we've adjoined finitely many algebraic quantities to $\mathbf{Q}$. Therefore for almost all f arising as specializations of $\mathbf{I}$ we have that the eigenvalues of $\rho_{f,p}(\mathrm{Frob}_\ell)$, and hence also the T_ℓ-eigenvalue $a_\ell(f)$, lie in a degree at most $2m$ extension of $L_\ell(\zeta)$, since they are roots of a specialization of the degree $2m$ polynomial $A_\ell(X)$ which has coefficients in $L_\ell(\zeta)$. Adjoining all p-power roots of unity, we see that $a_\ell(f)$ has degree at most $2m[L_\ell : \mathbf{Q}]$ over $\mathbf{Q}(\mu_{p^\infty})$.

Define C_ℓ to be this constant

$$C_\ell = 2m[L_\ell : \mathbf{Q}]$$

which depends only on $\mathbf{I}$ and ℓ, and not on our choice of regular weight specialization. We now have that theorem 6.1 applies to $\mathbf{I}$ with any positive density subset of primes $\ell \nmid Np$, these choices of C_ℓ, any fixed choice of $k \geq 2$, and a choice of any infinite subset S of μ_{p^∞} which avoids a finite set of pairs (k, ζ) where the representation r has poles. Thus we conclude that $\mathbf{I}$ has CM! $\square$

CHAPTER 7

APPLICATIONS TO LOCAL SPLITTING OF GALOIS REPRESENTATIONS

Throughout this chapter we work under the notation set forth in section 3.3, and in particular we follow assumption 3.13 throughout the chapter. We present here an application of our main result to the local splitting of Galois representations attached to p-ordinary families of Hilbert modular forms. This application can be viewed as analogous to the results of Ghate–Vatsal ([11]) and Balasubramanyam–Ghate–Vatsal ([1]); where they prove that p-ordinary families are CM if and only if their Galois representation is split on a decomposition group at each prime above p, we prove that p- or v-ordinary families are CM if and only if their Galois representation is split on a decomposition group at v. We note that these results are conditional as they rely not only on the assumptions we make about 1-dimensional v-ordinary families, but also on classicality results in partial weight one which are not explicit in the literature.

We begin by showing that the Galois representation attached to a classical eigenform which is both weight one at v and v-ordinary is split on a decomposition group at v.

Proposition 7.1. *Let F be a totally real field in which the odd prime p splits completely, and let v be a prime of F dividing p. The Galois representation attached to a classical Hilbert modular eigenform of weight one at v and finite slope is locally split at v.*

Proof. An analog of theorem 2.11 for Hilbert modular forms shows that if f has finite slope at v and is weight one at v then in fact it is v-ordinary. The v component $\pi_{f,v}$ of the automorphic representation attached to f must therefore be an irreducible principal series representation $PS(\chi_1, \chi_2)$ with one or both characters unramified.

Case 1: both χ_1 and χ_2 are unramified. We suppose that the U_v eigenvalue of f is $\sqrt{p} \cdot \chi_1(\varpi_v)$. If V is the vector space underlying π_f, we let g the complementary eigenvector in $V^{U_0(v)}$; this g is a Hilbert modular eigenform with the same eigenvalues as f away from

v, and it has U_v eigenvalue $\sqrt{p}\cdot\chi_2(\varpi_v)$. In particular since their eigenvalues agree away from v we have that $\rho_{f,p}\cong\rho_{g,p}$. However, using that f and g are both v-ordinary we see by realizing their Galois representations as coming from a 1-dimensional v-ordinary Hecke algebra $\mathbf{H}$ that

$$\rho_{f,p}|_{G_{F_v}} = \begin{bmatrix} |\cdot|^{-1/2}\chi_2 & * \\ 0 & |\cdot|^{-1/2}\chi_1 \end{bmatrix}$$
$$\rho_{g,p}|_{G_{F_v}} = \begin{bmatrix} |\cdot|^{-1/2}\chi_1 & * \\ 0 & |\cdot|^{-1/2}\chi_2 \end{bmatrix}.$$

However using that $\rho_{f,p}\cong\rho_{g,p}$ the upper triangular decomposition of g implies that

$$\rho_{f,p}|_{G_{F_v}} = \begin{bmatrix} |\cdot|^{-1/2}\chi_2 & 0 \\ * & |\cdot|^{-1/2}\chi_1 \end{bmatrix}.$$

From this we conclude that $\rho_{f,p}|_{G_{F_v}}$ is split.

Case 2: χ_1 is unramified and χ_2 is ramified. We then have by realizing f as a member of a v-ordinary family that

$$\rho_{f,p}|_{G_{F_v}} = \begin{bmatrix} |\cdot|^{-1/2}\chi_2 & * \\ 0 & |\cdot|^{-1/2}\chi_1 \end{bmatrix}.$$

As in the proof of theorem 2.11, let $\chi : F^\times\backslash\mathbf{A}_F^\times\to\mathbf{C}^\times$ be a finite order Hecke character such that $\chi|_{\mathcal{O}_{F,v}^\times}=\chi_2|_{\mathcal{O}_{F,v}^\times}$, and let $g=f\otimes\chi^{-1}$. Then we have that $\rho_{g,p}\cong\rho_{f,p}\otimes\chi^{-1}$. Since g is v-ordinary with U_v eigenvalue $\sqrt{p}\cdot\chi^{-1}(\varpi_v)\chi_2(\varpi_v)$ and $\pi_{g,v}=PS(\chi^{-1}\chi_1,\chi^{-1}\chi_2)$, we have by realizing g as a member of a v-ordinary family (potentially larger than the one containing

f, as twisting by χ may increase the tame level) we have that

$$\rho_{g,p}|_{G_{F_v}} = \begin{bmatrix} |\cdot|^{-1/2}\chi^{-1}\chi_1 & * \\ 0 & |\cdot|^{-1/2}\chi^{-1}\chi_2 \end{bmatrix}.$$

Comparing these two upper triangular decompositions and using that $\rho_{g,p} \cong \rho_{f,p} \otimes \chi^{-1}$ we again conclude that $\rho_{f,p}|_{G_{F_v}}$ is split. $\square$

Work of Buzzard–Taylor in [4] and Buzzard in [3] shows that the main determining factor in whether overconvergent elliptic modular eigenforms of weight one are classical is whether or not their p-adic Galois representation is split on a decomposition group at p. These techniques have been extended to cover the case of Hilbert modular forms of parallel weight one by many authors in various stages. Using similar analytic continuation arguments, Kassaei has shown in [26] how to prove Coleman's classicality criterion for overconvergent modular forms of weight at least 2. Again, these results have been extended to the Hilbert case by many authors.

These analytic continuation techniques apply equally well in the case of non-parallel weights. In essence, one can apply the arguments prime by prime, allowing a combination of the strategies of Buzzard–Taylor and Kassaei to prove classicality for forms of partial weight one. The conditions for classicality in partial weight one are essentially also taken prime by prime: we require a split Galois representation at weight one primes, and a bound on slopes at the weight ≥ 2 primes. Since we will only apply this result to forms which are v-ordinary at all primes $v|p$ we take our slope condition to be v-ordinary (i.e. normalized slope 0) at primes of weight $k \geq 2$. As yet such a result does not appear explicitly in the literature; we record here what we expect these techniques to prove in this case.

Theorem 7.2. *Let F be a totally real field in which the odd prime p splits completely. Let f be an overconvergent Hilbert modular eigenform for F in a classical weight $(\underline{k}, w)$. Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be the p-adic Galois representation attached to f. Suppose that:*

- *For each $v|p$, f is v-ordinary.*
- *For each $v|p$, $\overline{\rho}|_{G_{F_v}}$ is distinguished (i.e. the characters appearing in the upper triangular decomposition are distinct).*
- *$\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.*
- *$\overline{\rho}$ is ordinarily modular, i.e. there is a classical p-ordinary parallel weight 2 form g such that $\overline{\rho}_g \cong \overline{\rho}$.*
- *For primes v dividing p at which $k_v = 1$, we have that $\rho|_{G_{F_v}}$ is a direct sum of distinct characters which are finite order when restricted to inertia.*

Then f is classical.

Remark 7.3. A theorem of this flavour is used implicitly in Newton's work [34] on local-global compatibility for the Galois representations attached to classical partial weight one eigenforms. Newton cites work of Kassaei [27] generalizing the method of Buzzard–Taylor to the case of parallel weight one Hilbert modular forms. While we certainly expect that such a method does in fact suffice for the case of partial weight one forms we have at hand, there are sufficiently many details to be checked and many different definitions of Hilbert modular forms in play that a reference treating this in detail is needed to be fully confident in these results.

Using this classicality result as an input, we prove the following, which can be seen a direct generalization of the results of [11] and [1] to the partially split case. Note that when we say that a form f has partial weight one at a prime $v|p$, we mean that the weight $k_v = 1$, and the other weights may or may not be equal to 1.

Theorem 7.4. *Work under the conditions of assumption 3.13 and assume theorem 7.2. Let* $\mathbf{I}$ *be a component of a $d+1$-dimensional p-ordinary Hecke algebra* $\mathbf{H}$*. Assume that $\overline{\rho}_{\mathbf{I}}|G_{F(\zeta_p)}$ is absolutely irreducible and $\overline{\rho}_{\mathbf{I}}$ is residually distinguished (the two characters appearing on the diagonal of $\overline{\rho}\mathbf{I}|_{G_{F_v}}$ are distinct for each $v|p$). Then the following are equivalent.*

(i) There exists a prime $v|p$ for which $\rho_{\mathbf{I}}|G_{F_v}$ is split.

(ii) There exists a prime $v|p$ for which $\mathbf{I}$ contains a Zariski dense set of classical eigenforms which are weight one at v and regular weight at all other primes above p.

(iii) There exists a prime $v|p$ for which $\mathbf{I}$ contains a Zariski dense set of classical CM eigenforms which are weight one at v and regular weight at all other primes above p.

(iv) $\mathbf{I}$ has CM.

Proof. The implications (4) $\implies$ (3) $\implies$ (2) are immediate; if $\mathbf{I}$ has CM, then any specialization at a $\overline{\mathbf{Q}}_p$ point has complex multiplication whether or not it is classical, and all weight one points are classical in a CM family. We will prove (2) $\implies$ (1) $\implies$ (4).

(2) $\implies$ (1): let $\rho_{\mathbf{I}}|_{G_{F_v}}$ be of the form $\sigma \mapsto \begin{bmatrix} a(\sigma) & b(\sigma) \\ 0 & d(\sigma) \end{bmatrix}$ where $a, d : G_{F_v} \to \mathbf{I}^\times$ are characters and $b : G_{F_v} \to \mathbf{I}$ is a cocycle (we know it is upper triangular by the analog of theorem 3.5 for the full p-ordinary family $\mathbf{H}$). For $P \in \mathrm{Spec}(\mathbf{I})(\overline{\mathbf{Q}}_p)$ which is in our Zariski dense set of partial weight one points, we know that the composition $b(P) : G_{F_v} \to \mathbf{I} \to \overline{\mathbf{Q}}_p$ is trivial by proposition 7.1. Since b vanishes on a Zariski dense set of points, we conclude that b is identically 0, and hence $\rho_{\mathbf{I}}|_{G_{F_v}}$ is split.

(1) $\implies$ (4): we may apply theorem 7.2 to any specializations $\mathbf{I}(P)$ for any $\overline{\mathbf{Q}}_p$ point P in a classical weight which is partial weight one at v and regular weight at all other primes. In particular let us choose a tame weight, level, and character as in section 5.1, and restrict $\mathbf{I}$ over weight space to those points having our fixed tame data. This restriction of $\mathbf{I}$ is then a component of the v-ordinary Hecke algebra considered in section 5.1, and we know by theorem 7.2 that all of its partial weight one points are classical. We may then apply theorem 6.4 to this restriction of $\mathbf{I}$ and conclude that it has complex multiplication. However this shows that $\mathbf{I}$ itself has specializations in regular weight which have complex multiplication, and hence by the uniqueness of ordinary families passing through forms of regular weight we have that $\mathbf{I}$ itself must have complex multiplication. $\square$

References

[1] Baskar Balasubramanyam, Eknath Ghate, and Vinayak Vatsal. On local Galois representations associated to ordinary Hilbert modular forms. *Manuscripta Math.*, 142(3-4):513–524, 2013.

[2] Don Blasius. Hilbert modular forms and the Ramanujan conjecture. In *Noncommutative geometry and number theory*, Aspects Math., E37, pages 35–56. Friedr. Vieweg, Wiesbaden, 2006.

[3] Kevin Buzzard. Analytic continuation of overconvergent eigenforms. *J. Amer. Math. Soc.*, 16(1):29–55, 2003.

[4] Kevin Buzzard and Richard Taylor. Companion forms and weight one forms. *Ann. of Math. (2)*, 149(3):905–919, 1999.

[5] J. W. S. Cassels. On a conjecture of R. M. Robinson about sums of roots of unity. *J. Reine Angew. Math.*, 238:112–131, 1969.

[6] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[7] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[8] Mladen Dimitrov. Galois representations modulo p and cohomology of Hilbert modular varieties. *Ann. Sci. École Norm. Sup. (4)*, 38(4):505–551, 2005.

[9] Yiwen Ding. Formes modulaires p-adiques sur les courbes de Shimura unitaires et compatibilité local-global. *Mém. Soc. Math. Fr. (N.S.)*, (155):viii+245, 2017.

[10] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[11] Eknath Ghate and Vinayak Vatsal. On the local behaviour of ordinary Λ-adic representations. *Ann. Inst. Fourier (Grenoble)*, 54(7):2143–2162 (2005), 2004.

[12] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[13] Haruzo Hida. Iwasawa modules attached to congruences of cusp forms. *Ann. Sci. École Norm. Sup. (4)*, 19(2):231–273, 1986.

[14] Haruzo Hida. On p-adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math. (2)*, 128(2):295–384, 1988.

[15] Haruzo Hida. On nearly ordinary Hecke algebras for GL(2) over totally real fields. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 139–169. Academic Press, Boston, MA, 1989.

[16] Haruzo Hida. Control theorems of coherent sheaves on Shimura varieties of PEL type. *J. Inst. Math. Jussieu*, 1(1):1–76, 2002.

[17] Haruzo Hida. *p-adic automorphic forms on Shimura varieties*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004.

[18] Haruzo Hida. Hecke fields of analytic families of modular forms. *J. Amer. Math. Soc.*, 24(1):51–80, 2011.

[19] Haruzo Hida. Hecke fields of Hilbert modular analytic families. In *Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro*, volume 614 of *Contemp. Math.*, pages 97–137. Amer. Math. Soc., Providence, RI, 2014.

[20] Haruzo Hida. Transcendence of Hecke operators in the big Hecke algebra. *Duke Math. J.*, 163(9):1655–1681, 2014.

[21] Haruzo Hida. Growth of Hecke fields along a p-adic analytic family of modular forms. In *Families of automorphic forms and the trace formula*, Simons Symp., pages 129–173. Springer, [Cham], 2016.

[22] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981.

[23] Frazer Jarvis. On Galois representations associated to Hilbert modular forms. *J. Reine Angew. Math.*, 491:199–216, 1997.

[24] Christian Johansson and James Newton. Parallel weight 2 points on Hilbert modular eigenvarieties and the parity conjecture. *Forum Math. Sigma*, 7:Paper No. e27, 36, 2019.

[25] Payman L. Kassaei. P-adic modular forms over Shimura curves over totally real fields. *Compos. Math.*, 140(2):359–395, 2004.

[26] Payman L. Kassaei. A gluing lemma and overconvergent modular forms. *Duke Math. J.*, 132(3):509–529, 2006.

[27] Payman L. Kassaei. Modularity lifting in parallel weight one. *J. Amer. Math. Soc.*, 26(1):199–225, 2013.

[28] Mark Kisin and King Fai Lai. Overconvergent Hilbert modular forms. *Amer. J. Math.*, 127(4):735–783, 2005.

[29] Neal Koblitz. *p-adic numbers, p-adic analysis, and zeta-functions*, volume 58 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1984.

[30] Serge Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.

[31] J. H. Loxton. On the maximum modulus of cyclotomic integers. *Acta Arith.*, 22:69–85, 1972.

[32] Richard A. Moy and Joel Specter. There exist non-CM Hilbert modular forms of partial weight 1. *Int. Math. Res. Not. IMRN*, (24):13047–13061, 2015.

[33] Richard Allen Moy. *Non-CM Hilbert Modular Forms of Partial Weight One*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–Northwestern University.

[34] James Newton. Towards local-global compatibility for Hilbert modular forms of low weight. *Algebra Number Theory*, 9(4):957–980, 2015.

[35] C. S. Rajan. On strong multiplicity one for l-adic representations. *International Mathematics Research Notices*, 1998(3):161–172, 01 1998.

[36] Kenneth A. Ribet. Galois representations attached to eigenforms with Nebentypus. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 17–51. Lecture Notes in Math., Vol. 601, 1977.

[37] J. D. Rogawski and J. B. Tunnell. On Artin L-functions associated to Hilbert modular forms of weight one. *Invent. Math.*, 74(1):1–42, 1983.

[38] Vlad Serban. An infinitesimal p-adic multiplicative Manin-Mumford conjecture. *J. Théor. Nombres Bordeaux*, 30(2):393–408, 2018.

[39] Vlad Serban. A finiteness result for p-adic families of bianchi modular forms. 2019.

[40] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

[41] Brad Lee Wilson. *p-adic Hecke algebras and L-functions*. ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of California, Los Angeles.

[42] Atsushi Yamagami. On p-adic families of Hilbert cusp forms of finite slope. *J. Number Theory*, 123(2):363–387, 2007.