THE UNIVERSITY OF CHICAGO

MEASURE RIGIDITY, EQUIDISTRIBUTION AND ORBIT CLOSURE CLASSIFICATION OF RANDOM WALKS ON SURFACES AND HOMOGENEOUS SPACES

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This thesis is dedicated to my family and friends.

Table of Contents

List of Figures

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ABSTRACT

In this thesis, we study the problem of stationary measure classification, equidistribution and orbit closure classification in three different settings. We use tools from homogeneous dynamics, smooth dynamics and random product of matrices to make progress in each setting.

In Chapter 2, we study the problem of classifying stationary measures and orbit closures for non-abelian action on a surface with a given smooth invariant measure. Using a result of Brown and Rodriguez Hertz, we show that under a certain finite verifiable average growth condition, the only nonatomic stationary measure is the given smooth invariant measure, and every orbit closure is either finite or dense. Moreover, every point with infinite orbit equidistributes on the surface with respect to the smooth invariant measure. This is analogous to the results of Benoist-Quint and Eskin-Lindenstrauss in the homogeneous setting, and the result of Eskin-Mirzakhani in the setting of moduli spaces of translation surfaces. We then apply this result to two concrete settings, namely discrete perturbation of the standard map and $\mathrm{Out}(F_2)$-action on a certain character variety. We verify the growth condition analytically in the former setting, and verify numerically in the latter setting.

In Chapter 3, we provide a self-contained proof of the classification of stationary measures for linear actions on vector spaces. This will be a major input of the result in the next chapter.

In Chapter 4, we study the problem of classifying stationary measures on homogeneous spaces of the form G/H, where G is a connected real Lie group, and H is a closed unimodular subgroup of G. Under an assumption of relative uniform expansion, we show that the stationary measures can be decomposed into homogeneous parts and generalized Bernoulli convolutions.

The main tools used are a relative version of the technique of Eskin-Lindenstrauss, and the measure classification result of linear action on real vector spaces from Chapter 3.

CHAPTER 1

INTRODUCTION

1.1 Background

One of the central themes in dynamical systems is to describe the orbits and invariant measures of the system. One version of this question can be described as follows: given a topological space M (or manifold, algebraic variety, $\{0,1\}^{\mathbb{N}}$ etc.), and a set $\mathcal{S}$ of self-maps $f_1, \ldots, f_k : M \to M$ with suitable regularity, let the *orbit* of a point $x_0 \in M$ under $\mathcal{S}$ be the set of all elements in M obtained by applying compositions of finitely many maps (possibly with repetition) in $\mathcal{S}$ to x_0. For instance the orbit of x_0 under $\mathcal{S} = \{f\}$ and $\{f, g\}$ are, respectively,

$$\mathrm{Orbit}(\{f\}, x_0) := \{x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots\},$$
$$\mathrm{Orbit}(\{f, g\}, x_0) := \{x_0, f(x_0), g(x_0), f(g(x_0)), g(f(x_0)), f(f(x_0)), f(g(f(x_0))), \ldots\}.$$

It is often natural to consider the (topological) *closures* of orbits to capture the topological limiting behaviors of such actions. We consider these topological and measure-theoretic questions:

Question 1. What are the possible (closures of) orbits of a given element $x_0 \in M$ under $\mathcal{S}$?

Question 2. Are there any Borel probability measures on M invariant under $\mathcal{S}$? If so can they be classified?

A "random walk" variant of such dynamical system is often considered if a probability distribution μ on the finite set $\mathcal{S}$ is given: start with a point $x_0 \in M$, at each stage take an element f_i in $\mathcal{S}$ according to the law μ and act on the current point x_n to get a new point $x_{n+1} := f_i x_n$. Iterating this indefinitely, one obtains a countable set $\{x_n\}$ that is called the *random walk orbit* of x_0 under μ (which is a set-valued random variable). One may ask the analogous topological question:

Question 3. Given $x_0 \in M$, what do the random walk orbits $\{x_n\}$ of x_0 under μ look like?

In the random walk setting, a natural generalization of invariant measures is the so-called *μ-stationary measures.* A measure ν on the space M is *μ-stationary* if $\nu = \mu * \nu := \int g_* \nu \, d\mu(g)$ - in other words, while the measure ν may not necessarily be invariant under any individual element g in the support of μ, it is "invariant on average" if in each step of the action, a random acting element is chosen according to the law given by μ. The corresponding measure-theoretic question in the random walk setting is:

Question 4. What are the possible μ-stationary measures on M?

Since $\mathcal{S}$-invariant measures are, in particular, μ-stationary, a classification of stationary measures automatically yields a classification of invariant measures (thus **Question 4** subsumes **Question 2**).

As an example, consider the 2-torus $M = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. Any unimodular 2×2 integer matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ acts naturally on M by left multiplication. One can ask, what are the orbit closures under the self-maps given by, say,

$$A_1 := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}?$$

We first notice that if x_0 is a rational point of the form $x_0 = (p_1/q, p_2/q) \in \mathbb{R}^2/\mathbb{Z}^2$, its orbit is finite since the denominator does not increase (hence the orbit is contained in the finite set $(\frac{1}{q}\mathbb{Z}/\mathbb{Z})^2$). On the other hand, if the starting point x_0 is an irrational point (at least one coordinate is irrational), the orbit is infinite. In fact, it can be shown using Birkhoff's pointwise ergodic theorem that *almost every* point on $\mathbb{T}^2$ has *dense* orbit. In other words, if a point on the torus $\mathbb{T}^2$ is chosen at random (with respect to the uniform probability measure), then with probability 1, the orbit closure of that point is the entire space $\mathbb{T}^2$. However Birkhoff's theorem tells us nothing about any specific $x_0 \in \mathbb{T}^2$. For instance, is the orbit of $(\pi/5, \sqrt{2}/4)$ dense?

It turns out that if only the first matrix A_1 is used, then there are orbit closures that are neither finite, nor all of $\mathbb{T}^2$. In fact, there are orbit closures of arbitrary fractional Hausdorff dimensions in $(0,2)$! But if both matrices A_1 and A_2 are used, then the situation is more rigid, in the sense that *every* orbit is either finite or dense. This is a special case of the following recent breakthrough theorem, answering a conjecture of Furstenberg.

Theorem 1.1.1 (Bourgain-Furman-Lindenstrauss-Mozes [BFLM11], Benoist-Quint [BQ11])**.** Let μ be a compactly supported probability measure on $\mathrm{SL}_d(\mathbb{Z})$. If $\mathcal{S} = \operatorname{supp} \mu$ generates a Zariski dense subsemigroup of $\mathrm{SL}_d(\mathbb{R})$, then

- for all $x \in \mathbb{T}^d$, $\mathrm{Orbit}(\mathcal{S}, x)$ is either finite or dense.
- Every μ-stationary probability measure ν on $\mathbb{T}^d$ is a convex combination of the Lebesgue measure on $\mathbb{T}^d$ and invariant probability measures supported on finite orbits.
- For every $x \in \mathbb{T}^d$ with infinite $\mathrm{Orbit}(\mathcal{S}, x)$, the random walk orbit equidistributes on $\mathbb{T}^d$ almost surely.

Theorem 1.1.1 can then be applied to the previous example, by taking $\mu = \frac{1}{2}(\delta_{A_1} + \delta_{A_2})$, an atomic measure on $\mathrm{SL}_2(\mathbb{Z})$, to show that the orbit of *every* irrational point equidistributes on $\mathbb{T}^d$, thus is, in particular, dense.

The ability to promote the almost-sure statement from Birkhoff's theorem to the everywhere statement of Benoist-Quint is crucial for applications - in practice one often concerns the behavior of a specific orbit, which Birkhoff's theorem says nothing about.

In fact, the result of Benoist-Quint applies not just to the torus $\mathbb{T}^d$, but also to more general homogeneous spaces (will be described in Theorem 1.4.1 of Section 1.4).

Since the breakthrough work of Benoist-Quint, there has been a long list of work trying to answer the following meta-theorem in different settings:

Main Question. Let M be a manifold, Γ be a semigroup acting on M, μ be a probability measure on Γ and $\mathcal{S} := \operatorname{supp} \mu$, the support of μ. Under what conditions on M and μ can we

- (**Orbit closure classification**) classify all the orbit closures under $\mathcal{S}$?
- (**Measure rigidity**) classify all the μ-stationary measures?

- (**Equidistribution**) obtain equidistribution of typical random walk orbits of every point?

One theme of this circle of ideas is that even special cases of such theorems can have strong applications in other areas of mathematics. For instance, the celebrated Ratner's theorem [Rat91] resolved these questions in the setting of the (deterministic) action of Lie groups generated by unipotent elements on homogeneous spaces, which in particular implies the half-century old Oppenheim conjecture (proved by Margulis [Mar87]) when applied to the special case of the $\mathrm{SO}(2,1)$-action on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. Eskin-Mirzakhani [EM18] and Eskin-Mirzakhani-Mohammadi [EMM15] proved these results for the $\mathrm{SL}_2(\mathbb{R})$-action on moduli spaces of flat surfaces, resolving a longstanding conjecture in the field of translation surfaces. In the case of abelian actions, the work of Einsiedler-Katok-Lindenstrauss [EKL06] about positive diagonal actions on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ made significant progress towards the century-old Littlewood conjecture, and the work of Lindenstrauss [Lin06] proved the Arithmetic Quantum Unique Ergodicity Conjecture of Rudnick-Sarnak [RS94]. In other settings, Simmons-Weiss [SW19] considered a special class of locally homogeneous spaces and proved implications about diophantine approximation on fractals. Sargent-Shapira [SS19] proved such results for a specific kind of projective bundle and made progress towards a conjecture of Furstenberg about cubic irrational numbers.

In the following, we describe three settings where we made progress towards the **Main Question** in this thesis, namely the case of volume-preserving C^2-actions on closed Riemannian surfaces (Section 1.2), linear actions on vector spaces (Section 1.3), and homogeneous actions on locally homogeneous spaces assuming relative uniform expansion (Section 1.4).

1.2 Random walks on surfaces

As a start, we generalize Theorem 1.1.1 in the two-dimensional case to general Riemannian manifolds when the action preserves a volume measure.

Theorem 1.2.1 (Proposition 2.3.1, 2.4.1 and 2.4.2 of this thesis)**.** Let M be a closed surface (compact connected two-dimensional Riemannian manifold without boundary) with volume measure vol induced by the Riemannian metric. Let μ be a compactly supported probability measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ that is *uniformly expanding*, and $\mathcal{S} := \mathrm{supp}\,\mu$ be the support of μ. Then

- for all $x \in M$, $\mathrm{Orbit}(\mathcal{S}, x)$ is either finite or dense.
- Every ergodic μ-stationary probability measure ν on M is either finitely supported or vol.
- For every $x \in M$ with infinite $\mathrm{Orbit}(\mathcal{S}, x)$, the random walk orbit equidistributes on M almost surely.

Moreover, we proposed a finite algorithm to verify uniform expansion in explicit settings, and verified it in two examples.

Here $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ is the set of C^2 diffeomorphisms on M that preserve the volume measure vol. Note that it is a general theorem in ergodic theory that any μ-stationary measure is a convex combination of ergodic stationary measures, hence it suffices to classify *ergodic* ones. *Uniformly expanding* is an assumption that generalizes the Zariski density assumption of the homogeneous setting considered in Theorem 1.1.1. In particular, Theorem 1.1.1 in the case when $d = 2$ is a special case of Theorem 1.2.1. We remark that the method of Benoist-Quint for Theorem 1.1.1 needs substantial modification to be adapted in this non-homogeneous setting. Our main inputs are the deep work of Brown-Rodriguez Hertz [BRH17], ideas of Dolgopyat-Krikorian [DK07] to prove ergodicity, and ideas of Margulis functions originated from Eskin-Margulis [EM04].

1.3 Random walks for linear actions on vector spaces

In this section, we consider the linear action of $\mathrm{GL}(V)$ on a finite dimensional real vector space V driven by a finitely supported probability measure μ on $\mathrm{GL}(V)$. The reason for considering this setting is twofold: on one hand, linear actions on vector space are special cases of homogeneous actions on homogeneous spaces where **Main Question** can be answered completely. On the other hand, this result is used critically in the result described in the next section.

Theorem 1.3.1 (Theorem 3.1.1 and 3.1.2 of this thesis)**.** Let V be a nonzero finite dimensional real vector space, μ be a finitely supported probability measure on $\mathrm{GL}(V)$ and Γ_μ be the closed subsemigroup of $\mathrm{GL}(V)$ generated by the support of μ. Then there exist Γ_μ-invariant vector subspaces $W' \subsetneq W \subset V$ such that

1. every μ-stationary probability measure on V is supported in W,
2. the map $\nu \mapsto \operatorname{supp} \pi_*\nu$ gives a one-to-one correspondence between

$$\{\text{ergodic } \mu\text{-stationary measure on } V\} \qquad \leftrightarrow \qquad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

   where $\pi : W \to W/W'$ is the quotient map,

3. every ergodic μ-stationary probability measure on V is the convolution of a compactly supported Γ_μ-invariant probability measure on W/W' and a "Bernoulli convolution".

Here "Bernoulli convolution" is a generalization of a class of measures well studied in the literature (see e.g. a survey by Peres-Schlag-Solomyak [PSS00]). In our context, the resulting Bernoulli convolution can be computed explicitly using the measure μ. Rather than giving the precise definition (the precise definition is in Chpater 4 Definition 4.3.9), we consider the following example which illustrates the theorem. Consider $G = \mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$ by left multiplication, and μ the probability measure on G that gives the following two elements equal probability 1/2:

$$\begin{pmatrix} \lambda & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & -1 \\ 0 & 1 \end{pmatrix},$$

where λ is any real number in $(0, 1)$. Then for any real number c, the random series

$$c \begin{pmatrix} \sum_{n=0}^{\infty} \pm \lambda^n \\ 1 \end{pmatrix}$$

is a μ-stationary probability measure on $\mathbb{R}^2$, and is an example of a Bernoulli convolution.

At first sight the existence of such stationary measure may suggest that the situation is not rigid - after all, these measures have fractional dimensions, and there are uncountably many of them (one for each $c \in \mathbb{R}$). However, the theorem states that these measures are *unique* up to the scaling factor c (in particular they all have the same Hausdorff dimension). Using the notations in Theorem 1.3.1, we take $V = W = \mathbb{R}^2$ and W' to be the x-axis. Then clearly Γ_μ acts as the identity on W/W', and the ergodic μ-stationary probability measures on W/W' are precisely the delta masses at each coset $\begin{pmatrix} 0 \\ c \end{pmatrix} + W' \in W/W'$. Now each such probability measure corresponds to one of the Bernoulli convolutions described above. The theorem states that these are the only ergodic μ-stationary probability measures on V. Corresponding equidistribution and orbit closure classification theorems can also be appropriately stated.

1.4 Random walks on homogeneous spaces with nondiscrete quotients

In a later work, Benoist-Quint generalized Theorem 1.1.1 to the following setting of a locally homogeneous space.

Theorem 1.4.1 (Benoist-Quint [BQ13a]). Let G be a connected real Lie group, Λ be a lattice in G, μ be a compactly supported probability measure on G. If $\mathcal{S} = \text{supp}\,\mu$ generates a subsemigroup Γ_μ of G whose Zariski closure is semisimple and Zariski connected with no compact factors, then

- for all $x \in G/\Lambda$, the orbit closure $\overline{\text{Orbit}(\mathcal{S}, x)} \subset G/\Lambda$ is homogeneous.
- Every ergodic μ-stationary probability measure ν on G/Λ is Γ_μ-invariant and homogeneous.
- For every $x \in G/\Lambda$, the random walk orbit equidistributes on $\overline{\text{Orbit}(\mathcal{S}, x)}$ almost surely.

Here a orbit closure is *homogeneous* if $\overline{\text{Orbit}(\mathcal{S}, x)} = Hx$ for some *closed* subgroup $H \subset G$. A probability measure ν on G/Λ is *homogeneous* if the support of ν is Hx for some closed subgroup $H \subset G$ and some $x \in G$, and ν is the unique H-invariant probability measure on Hx. Note that if the acting group is isomorphic to $\mathbb{R}$ or $\mathbb{Z}$ (corresponding to a flow and a single invertible transformation), these notions coincide with the usual notion of a *periodic orbit* and the uniform probability measure on the periodic orbit. Hence these can be considered as natural generalization of periodicity.

The assumption that $\mathcal{S}$ generates a semigroup with semisimple Zariski closure is necessary to guarantee that all the ergodic stationary measures are homogeneous. In Eskin-Lindenstrauss, where they relaxed the assumptions to *uniform expansion* on G, they demonstrated examples of non-homogeneous stationary measures when the Zariski closure is not semisimple (but uniform expansion still holds). In fact their main result is that assuming uniform expansion, any stationary measure is the convolution of an H-homogeneous measure on G/Λ and a stationary probability measure on G/H for some nondiscrete closed unimodular subgroup $H \subset G$. Since their statement focuses on homogeneous spaces of the form G/Λ with discrete Λ, their results do not apply directly to G/H. This motivates the following question.

Question. Under what conditions can we classify all the stationary measures on G/H, where H is a closed unimodular subgroup of the Lie group G?

It turns out that under suitable assumptions, we can study this question using the results from both the vector space case as studied in Chapter 3 and the semisimple case studied by Benoist-Quint (and the more general case by Eskin-Lindenstrauss [ELa]). To do so, we consider $L = N_G^1(H^\circ) := \{g \in N_G(H^\circ) \mid \mathrm{Ad}(g) \text{ preserves the Haar measure of } H^\circ\}$, where H° is the connected component of identity of H, $N_G(H^\circ)$ is the normalizer of H° in G.

Such L is one way to construct the so-called H-*envelope* (see Section 1 of Eskin-Lindenstrauss and Chapter 4 of this thesis for the precise definition and other constructions). The key properties of an H-envelope L are:

1. $L/H = (L/H^\circ)/(H/H^\circ)$ is the quotient of a real Lie group L/H° by a discrete subgroup H/H° (hence in the setting of Eskin-Lindenstrauss).

2. there is a G-equivariant continuous injection $G/L \to V$ into a vector space V, thus any μ-stationary measure on G/L is a μ-stationary measure on the vector space V.

Now we consider G/H as the total space of the fiber bundle

$$\begin{array}{ccccc} (L/H^\circ)/(H/H^\circ) = L/H & \longrightarrow & G/H & & \\ & & \downarrow & & \\ & & G/L & \hookrightarrow & V \end{array}$$

We then apply the technique of Eskin-Lindenstrauss for G/Λ to obtain extra invariance in the fiber direction. The main assumption is a "relative" version of their uniform expansion assumption. The following result summarizes the conclusion.

Theorem 1.4.2 (Theorem 4.1.1 of this thesis)**.** Let G be a real linear algebraic group, and μ be a Borel probability measure on G with finite first moment. Let Γ_μ be the (topological) closure of the subsemigroup generated by the support of μ in G, and $\overline{\Gamma}_\mu^Z$ be the Zariski closure of Γ_μ.

Let $H \subset G$ be a closed unimodular subgroup, and H° be the connected component of the identity in H. Suppose there exists an H-envelope L and $x_0 \in G/L$ such that μ is uniformly expanding on L/H at x_0.

Let $\nu_{G/H}$ be an ergodic μ-stationary probability measure on $\overline{\Gamma}_\mu^Z x_0 L/H$. We also assume an algebraic condition (†) (see Chapter 4). Then one of the following holds:

(I) there exist a Lie subgroup $H' \subset G$ with $H^\circ \subset H' \subset L \subset G$ and $\dim(H'/H^\circ) > 0$, an H'-homogeneous probability measure $\nu_{L/H}$ on L/H and finite μ-stationary measure $\nu_{G/H'}$ on $\overline{\Gamma}_\mu^Z xL/H'$ such that

$$\nu_{G/H} = \nu_{G/H'} * \nu_{L/H} := \int_{G/H'} g_* \nu_{L/H} \, d\nu_{G/H'}(g).$$

(II) the stationary measure $\nu_{G/H}$ can be written as

$$\nu_{G/H} = \int_{G/L} \nu_x \, d\overline{\nu}(x),$$

where

(a) $\overline{\nu}$ is a generalized μ-Bernoulli measure (see Definition 4.3.9) supported on $\overline{\Gamma}_\mu^Z x_0 L/L$,

(b) there exists a positive integer k such that for $\overline{\nu}$-almost every $x \in G/L$, ν_x is the uniform measure on k points in $\pi^{-1}(x) = xL/H$, where $\pi : G/H \to G/L$ is the natural quotient map,

(c) there exists a Γ_μ-invariant locally Zariski closed subset $\mathcal{F}$ such that $\operatorname{supp} \nu_{G/H} \subset \mathcal{F}$, and $\mathcal{F}$ has finite intersection with xL/H for all $x \in \overline{\Gamma}_\mu^Z x_0 L/L$ (the set $\mathcal{F}$ is defined dynamically and can be made more explicit and computable - see Theorem 4.4.9).

We remark that if H is a discrete subgroup of G, this statement recovers [ELa, Thm. 1.7] for trivial Z (in this case (†) is always satisfied).

Theorem 1.4.2 together with Theorem 1.7 of Eskin-Lindenstrauss form one step of an induction, which allows us to say more about measure rigidity even in the cases considered in Eskin-Lindenstrauss (with extra assumptions in the form of relative uniform expansion). See Section 4.2 for one such example (and it will be clear how to generalize the example to a family of such) in which all the ergodic stationary measures can be classified.

CHAPTER 2

STATIONARY MEASURES AND ORBIT CLOSURES OF UNIFORMLY EXPANDING RANDOM DYNAMICAL SYSTEMS ON SURFACES

2.1 Introduction

Given a Riemannian manifold M and an acting semigroup Γ, the closure of the Γ-orbit of some points of M may exhibit fractal-like structure. For instance in the case when M is a compact manifold and Γ is generated by a single Anosov diffeomorphism, there are orbit closures with fractional Hausdorff dimension. A one-dimensional example is the action of $\mathbb{N}$ on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ generated by

$$x \mapsto 3x \bmod 1.$$

By the Birkhoff ergodic theorem, we know that (Lebesgue-)almost every point on the circle has dense orbit. Nonetheless the orbit of every rational number is clearly finite, and one can get orbit closures that are neither finite nor the whole circle, for instance the standard Cantor middle third set.

It turns out that if one consider instead the action of a larger group, the situation becomes more rigid. Furstenberg [Fur67] showed that the orbits of the action of $\mathbb{N}^2$ generated by

$$x \mapsto 2x \bmod 1 \qquad \text{and} \qquad x \mapsto 3x \bmod 1,$$

are either finite or dense. Moreover, he famously asked whether all the ergodic invariant Borel probability measures are either finitely supported or the Lebesgue measure on S^1. Major progress on this conjecture was made by Rudolph [Rud90], who showed that the ergodic invariant measures either have zero-entropy for the action of every one-parameter subgroup or is the Lebesgue measure on S^1.

In two or higher dimensions, similar phenomena have been observed. For example, the action of $\mathbb{Z}$ on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ generated by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

has orbits that are neither finite nor dense. In fact using the theory of Markov partitions [Bow75], one can conjugate this system to a subshift of finite type to obtain orbit closures of any Hausdorff dimension between 0 and 2. If one consider instead the nonabelian action on $\mathbb{T}^2$ generated by, say,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

then it follows from a result of Bourgain, Furman, Lindenstrauss and Mozes [BFLM11] that the orbits are either finite or dense (for this particular example, the orbit closure classification statement was already shown in [GS04] and [Muc05], in [BFLM11] measure rigidity, orbit closure classification and quantitative equidistribution were shown). In fact Benoist and Quint has proved in a series of papers [BQ11, BQ13a, BQ13b] a number of such orbit closure classifications and the corresponding measure rigidity results. A special case of their result is the following: let μ be a finitely supported measure on $\mathrm{SL}(n, \mathbb{Z})$ and let $\Gamma_\mu \subset \mathrm{SL}(n, \mathbb{Z})$ be the closed subgroup generated by the support of μ. If Γ_μ is "large enough", in this case this means that every finite-index subgroup of Γ_μ acts irreducibly on $\mathbb{R}^n$, then every ergodic μ-stationary probability measure on $\mathbb{T}^n$ is either finitely supported or the Haar measure on $\mathbb{T}^n$. In particular every μ-stationary probability measure is $\mathrm{SL}(n, \mathbb{Z})$-invariant. They used this measure rigidity result to show that every orbit closure is either finite or dense, by first showing a stronger equidistribution result.

The results of Benoist and Quint are in the setting of homogeneous dynamics, where one considers the natural action of a Lie group G acting on a homogeneous space G/Λ. In [BQ11], it was proved that if μ is a compactly supported measure on a simple real Lie group G, and the subgroup $\Gamma \subset G$ generated by the support of μ is Zariski dense in G, then every Γ-orbit is either finite or dense. Moreover, the corresponding μ-stationary probability measures are either finitely supported or the Haar measure on G/Λ, hence in particular are Γ-invariant. The result was extended to a general real Lie group G in [BQ13a], where they showed that assuming the Zariski closure of Γ is semisimple, Zariski connected with no compact factor, any μ-stationary measure is homogeneous. This result was further generalized by Eskin-Lindenstrauss [ELa] where they relaxed the assumption on Γ to the "uniform expansion" assumption to include many cases where the Zariski closure of Γ is not semisimple. In contrast with the case of abelian actions (for instance Rudolph's theorem mentioned above), the measure classification has no entropy assumption, and the orbit closure classification follows as a corollary of the measure rigidity theorem.

In this paper, we study the question of measure rigidity and orbit closure classification in the setting of smooth dynamics, i.e. the action of a subgroup of diffeomorphisms on a manifold M. In particular, we shall prove positivity of Lyapunov exponent, measure rigidity and orbit closure classification theorems in the following two settings.

- Discrete random perturbation of the standard map.
- Outer automorphism group action on the character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2)$.

The first setting was studied by Blumenthal, Xue and Young [BXY17], where they considered a "continuous" random perturbation of the standard map and obtained positivity of Lyapunov exponent, even though positivity of exponent for the standard map is notoriously hard. Their method, however, does not apply to discrete perturbations that we consider in this paper, as it is no longer clear that any stationary measure is absolutely continuous with respect to Lebesgue. This will be explained in Section 2.6.

The second setting was studied by Goldman [Gol07], which is based on his earlier work [Gol97]. In [Gol07], the ergodic decomposition of the $\mathrm{Out}(F_2)$-action on the character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2))/\mathrm{SU}(2)$ is given. The topological dynamics was studied by Previte and Xia [PX00], who proved that on each ergodic component, every $\mathrm{Out}(F_2)$-orbit is either finite or dense. Their method uses crucially the fact that $\mathrm{Out}(F_2)$ is generated by Dehn twists. In this paper, we shall prove that for some finite set of generators $\mathcal{S}$ of $\Gamma := \mathrm{Out}(F_2)$ that does not contain any nontrivial powers of Dehn twist, every Γ-orbit on each ergodic component is either finite or dense. This will be explained in Section 2.8.

Both results are part of a more general theorem concerning the volume-preserving action of a group $\Gamma \subset \mathrm{Diff}^2(M)$ on a closed surface M. The measure rigidity problem in this setting was studied by Brown and Rodriguez-Hertz [BRH17]. Based on the "exponential drift" technique first introduced in [BQ11] and some ideas in [EM18], they proved that in this setting, if "the stable distribution is not nonrandom" (see Section 2.3 for the precise definition), then the stationary measures are either finitely supported, or the restriction of the volume on a positive volume subset. In this paper, we will build on the work of [BRH17] to give a more verifiable (but stronger) criterion on the acting group Γ so that the stationary measures and orbit closures can be classified. Such a criterion should, on one hand, be strict enough to rule out the case of a one-parameter acting group (in which case we can see from above that there can be measures of arbitrary Hausdorff dimension in general), and, on the other hand, be flexible enough to include many larger group Γ. We will then verify this criterion in both of the aforementioned settings.

Our measure rigidity result relies heavily on the result of Brown and Rodriguez-Hertz [BRH17], hence only works in the two-dimensional case. The assumption we introduce will be stronger than that of [BRH17], in order to give us the proof of the orbit closure classification. Nonetheless, such an assumption is a finite criterion and hence can be checked, at least in principle, in concrete settings.

2.1.1 Main results

In this paper, we shall prove positivity of Lyapunov exponent, measure rigidity and orbit closure classification in the following two settings.

1. Discrete random perturbation of the standard map

Theorem A. Let $\mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2$ be the 2-torus. For $L > 0$, $\varepsilon > 0$ and positive integer r, let

- $F_L : \mathbb{T}^2 \to \mathbb{T}^2$ be the standard map $F_L(x, y) = (L \sin x + 2x - y, x)$,
- $F_{L,\omega} : \mathbb{T}^2 \to \mathbb{T}^2$ be the perturbation $F_{L,\omega}(x, y) := F_L(x + \omega, y)$ by $\omega \in \Omega := \{k\varepsilon : k = 0, \pm 1, \pm 2, \ldots, \pm r\}$,

Let $\delta \in (0, 1)$. There exists an integer $r_0 = r_0(\delta) > 0$ such that if $r \geq r_0$ and $\varepsilon \in [L^{-1+\delta}, 1/(2r + 1))$, then for all large enough L,

(a) the random dynamical system defined by $F_{L,\Omega} := \{F_{L,\omega} : \omega \in \Omega\} \subset \mathrm{Diff}^2(\mathbb{T}^2)$ has positive Lyapunov exponent with respect to the Lebesgue measure on $\mathbb{T}^2$,

(b) every orbit of the system defind by $F_{L,\Omega}$ is either finite or dense.

2. Outer automorphism group action on character variety

Theorem B. Let $\mathfrak{X}_s := \mathrm{Hom}_s(F_2, \mathrm{SU}(2))//\mathrm{SU}(2)$ be the relative character variety corresponding to the boundary conjugacy class $s \in [-2, 2]$. Each $\mathfrak{X}_s$ has a natural finite measure λ_s inherited from the natural measure on $\mathrm{Hom}(F_2, \mathrm{SU}(2))$ that is invariant under the natural action of $\mathrm{Out}(F_2)$ (see Section 2.8 for the precise definitions and motivations).

There exists a finite set $\mathcal{S} \subset \mathrm{Out}(F_2)$ without any nontrivial powers of Dehn twists such that for the semigroup Γ generated by $\mathcal{S}$, and for $s = 1.99$,

(a) the only Γ-invariant measure ν on $\mathfrak{X}_s$ that is not finitely supported is the natural finite measure λ_s.

(b) Every orbit of Γ on $\mathfrak{X}_s$ is either finite or dense,

(c) Each dense Γ-orbit equidistributes (with respect to $\mathcal{S}$) on $\mathfrak{X}_s$ (in the precise sense defined in Proposition 2.4.1).

In [BXY17], Theorem A(a) was proved when $\Omega = [-\varepsilon, \varepsilon]$, and $\varepsilon > e^{-L^{2-\delta}}$. However, in this paper, we shall prove a stronger condition (called *uniform expansion*), and we are only able to show this for $\varepsilon > L^{-1+\delta}$. In fact, in a subsequence paper [BXY18], the same authors essentially showed uniform expansion in the case when $\Omega = [-\varepsilon, \varepsilon]$ and $\varepsilon > L^{-1+\delta}$ [BXY18, Prop. 9]. Their method, however, does not apply in this discrete setting, since their approach relies heavily on the fact that any stationary measure is absolutely continuous with respect to Lebesgue measure (see [BXY17, Lem. 5] and [BXY18, Lem. 8]), which is not necessarily true in the discrete setting.

In [PX00], the orbit closure classification in Theorem B was proved for $\Gamma = \mathrm{Out}(F_2)$ without going through a measure rigidity result. Instead, the topological dynamics was analyzed directly using critically the fact that $\mathrm{Out}(F_2)$ is generated by two Dehn twists. These Dehn twists take a particularly simple form on the space, which allow an explicit analysis of the orbits generated by them. In this paper, we shall prove uniform expansion for generators $\mathcal{S}$ of $\mathrm{Out}(F_2)$ that does not have any nontrivial powers of Dehn twists, hence does not admit such explicit analysis. The difference between these two results is analogous to the classical setting of the action on the 2-torus $\mathbb{T}^2$ generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where the action by each individual generator is rotation on a circle, versus the action generated by hyperbolic elements in $\mathrm{SL}(2,\mathbb{Z})$ that generate a subgroup Zariski dense in $\mathrm{SL}(2,\mathbb{R})$, where the generic orbit (though certainly not all orbit) of each individual generator is dense in $\mathbb{T}^2$.

Our method in the proof of Theorem B goes through a numerical verification using a computer program. We demonstrate such verification on one particular shell $s = 1.99$ and for one particular set of generators $\mathcal{S}$, though just by some derivative bounds (to be made explicit in Section 2.8) the same result can be extended to nearby shells. Such verification is faster for s close to 2, though there is no theoretical obstruction in applying the same verification to any shells $\mathfrak{X}_s$ with $s \in (-2, 2)$ (just the computation time grows as $s \to -2$). There is also no theoretical obstruction in applying it to other finite subsets $\mathcal{S}$ that generate a non-elementary subgroup $\Gamma \subset \mathrm{Out}(F_2)$.

2.1.2 *Uniform expansion*

As mentioned in the introduction, both theorems are special cases of a more general result. In this section, we shall introduce a general criterion called *uniform expansion*, and state that this criterion implies positivity of Lyapunov exponents, measure rigidity and orbit closure classification.

Given a Riemannian manifold M, let $\mathrm{Diff}^k(M)$ be the group of C^k diffeomorphisms on M. Given a measure m on M, let $\mathrm{Diff}^k_m(M)$ be the group of C^k diffeomorphisms on M that preserve m, i.e.

$$\mathrm{Diff}^k_m(M) := \{f \in \mathrm{Diff}^k(M) : f_*m = m\}.$$

Throughout this paper, any measure is assumed to be a Borel probability measure on the corresponding topological space.

Definition. A probability measure ν on M is called *μ-stationary* if

$$\mu * \nu = \nu, \qquad \text{where} \qquad \mu * \nu = \int_{\mathrm{Diff}^2(M)} f_*\nu \, d\mu(f).$$

Definition. Let M be a Riemannian manifold, μ be a measure on $\mathrm{Diff}^2(M)$. We say that μ is *uniformly expanding* if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and $v \in T_xM$,

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_xf(v)\|}{\|v\|} d\mu^{(N)}(f) > C.$$

Here $\mu^{(N)} := \mu * \mu * \cdots * \mu$ is the N-th convolution power of μ. We remark that if M is compact, this is equivalent to the weaker formulation where we allow C and N to depend on x and v (see e.g. [LX, Lem. 4.3.1], where such weaker criterion is called "weakly expanding").

Sometimes we say that a finite subset $\mathcal{S} \subset \mathrm{Diff}^2(M)$ is *uniformly expanding* if the uniform measure supported on $\mathcal{S}$ is uniformly expanding in the above sense. Note that in this case the integral in the uniform expansion condition reduces to a finite sum.

The goal of the first half of the paper is to classify μ-stationary measures on a closed surface M and the corresponding orbit closures if μ is uniformly expanding and supported on $\mathrm{Diff}^2_m M$ for some *smooth* measure m on M, i.e. a Borel probability measure m equivalent to the Riemannian volume on M.

Theorem C. Let M be a closed surface (compact connected two-dimensional C^∞ Riemannian manifold without boundary) and m be a smooth measure on M. Let μ be a uniformly expanding probability measure on $\mathrm{Diff}^2_m(M)$ with

$$\int_{\mathrm{Diff}^2_m(M)} \log^+(|f|_{C^2}) + \log^+(|f^{-1}|_{C^2})\, d\mu(f) < \infty. \tag{*}$$

Let ν be an ergodic, μ-stationary Borel probability measure on M. Then

(a) ν has positive Lyapunov exponent;

(b) either ν is finitely supported, or $\nu = m$.

This result was proved in [LX, Thm. 4.1.4], where they used this statement to prove a large deviation result. We shall recall the proof in Section 2.2 and 2.3 for completeness.

Here we are more concerned with the following orbit closure classification which follows from Theorem C, and its applications in concrete settings.

Theorem D. Let M be a closed surface, m be a smooth measure on M, and $\mathcal{S} \subset \mathrm{Diff}^2_m(M)$ be a finite subset of diffeomorphisms that preserve m. Let $\Gamma \subset \mathrm{Diff}^2_m(M)$ be the subsemigroup generated by $\mathcal{S}$. If $\mathcal{S}$ is uniformly expanding, then

(a) every orbit of Γ is either finite or dense,

(b) every dense Γ-orbit equidistributes on M (in the precise sense defined in Proposition 2.4.1).

Note that we could have replaced the word "subsemigroup" with "subgroup" to get a weaker statement. Also if $\mathcal{S}$ is uniformly expanding, then Γ cannot be cyclic (see Lemma 2.3.3 below). An analogous statement has been proved in greater generality in the homogeneous setting by Eskin and Lindenstrauss [ELa].

In the setting of homogeneous dynamics, uniform expansion has been verified in some cases. For instance, let G be a real semisimple Lie group with no compact factors and Λ be a discrete subgroup of G. Let μ be a countably supported probability measure on G whose support generates a Zariski dense subgroup of G. Then μ is uniformly expanding, see e.g. [EM04, Lem. 4.1], the idea of which goes back to Furstenberg [Fur63]. As a second example, one may consider the case of the $\mathrm{SL}(n,\mathbb{Z})$-action on the n-torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Let μ be a finitely supported probability measure on $\mathrm{SL}(n,\mathbb{Z})$ such that the support of μ generates a Zariski dense subgroup of $\mathrm{SL}(n,\mathbb{R})$. Using the classical theory of product of random matrices (for instance in Goldsheid and Margulis [GM89]), one can show that μ is uniformly expanding (see e.g. the proof of Theorem 4.1.3 in [LX] for the precise argument). Clearly uniform expansion is a C^1-open property, therefore any small C^1-perturbations of these examples also support uniformly expanding measures.

2.1.3 Verification of Uniform Expansion

Theorem A and B are both proved by verifying uniform expansion and then applying Theorem C and D. Theorem A will be proved in Section 2.6 by verifying uniform expansion analytically. Theorem B will be proved in Section 2.8 by verifying uniform expansion numerically, using an algorithm described in Section 2.7. The context and motivation will be provided in the respective sections.

Other than the fact that these examples are interesting in their own right, they are also chosen to illustrate how to overcome two difficulties in the verification of uniform expansion.

First of all, as we saw in Theorem C, uniform expansion is a stronger criterion than positivity of Lyapunov exponent, and the latter is notoriously difficult to verify for one-parameter group actions without some sort of uniform hyperbolicity. The reason is that even strong expansion in the early stages of the dynamics can be cancelled out by strong contraction in the future, for instance when the dynamics hit a region where it behaves like a rotation, such "backtracking" phenomenon may occur. In our examples, there are small rotation regions for each individual map. Nonetheless we show that as long as the random dynamics enter these rotation regions with small enough probability, the overall dynamics is expanding on average.

Secondly, it is clear that if the dynamics is generated by a single volume-preserving hyperbolic diffeomorphism, then uniform expansion never holds, since the stable direction is contracted by the dynamics. For higher rank actions, it is still possible that the contracting directions of the maps may overlap for some subset of points but not all. Note that this does not happen in the homogeneous setting, in the sense that if the contracting directions are separated at one point, then by homogeneity, they are separated at every point of the space. In our examples, the contracting directions may overlap in a codimension one subset, and again we show that uniform expansion occurs as long as the random dynamics enter a neighborhood of such subset with small enough probability. Proposition 2.5.4 illustrates that rotation regions and overlapping contracting directions are essentially the only two obstructions to uniform expansion.

For Theorem A, we are able to verify uniform expansion directly since at each point, with high probability, the map has strong expansion in the same (horizontal) direction. Moreover, one can compute with high accuracy the separation of the contracting directions of the maps. These allow us to understand exactly where the rotation regions and overlapping contracting directions occur. In particular, for each point and each direction, we can obtain an upper bound on the probability that the map contracts in that direction after n steps. Depending on how small the separation of the contracting directions is, one can then choose a suitable N so that uniform expansion occurs.

For Theorem B, however, the contracting directions of each map vary for different points on the space. In particular, we can no longer prove explicitly that backtracking occur with low probability (though we expect so). Therefore we can only check unifom expansion at each point on a fine enough grid, and then show that such expansion still occur at neighboring points using a C^2-bound.

The paper is structured as follows:

- In Section 2.2, positivity of Lyapunov exponents for uniformly expanding systems (Theorem C (a)) is proved (Proposition 2.2.2).

- In Section 2.3, classification of stationary measures of uniformly expanding systems (Theorem C (b)) is proved using a result of Brown and Rodriguez-Hertz [BRH17] (Proposition 2.3.1).
- In Section 2.4, using the measure rigidity result in Section 2.3, an equidistribution result (Proposition 2.4.1) will be proved. The orbit closure classification (Theorem D) is then obtained as a corollary (Proposition 2.4.2).
- In Section 2.5, we introduce a geometric way to view uniform expansion and prove a general criterion for uniform expansion (Proposition 2.5.4).
- In Section 2.6, the setting of perturbation of the standard map is introduced, and uniform expansion is verified analytically in this setting (Proposition 2.6.1). This proves Theorem A.
- In Section 2.7, an algorithm to check uniform expansion is presented.
- In Section 2.8, the setting of the $\mathrm{Out}(F_2)$ action on character variety is introduced, and uniform expansion is verified using the algorithm introduced in Section 2.7. This proves Theorem B.

2.2 Positive exponent

We first recall the celebrated Oseledets theorem in the setting of random dynamical systems. Here we adopt the notation in [BRH17] and define $f_\omega^n := \omega_{n-1} \circ \omega_{n-2} \circ \cdots \circ \omega_1 \circ \omega_0$ for $\omega = (\omega_0, \omega_1, \omega_2, \ldots) \in \mathrm{Diff}^2(M)^{\mathbb{N}}$. Let $\sigma : \mathrm{Diff}^2(M)^{\mathbb{N}} \to \mathrm{Diff}^2(M)^{\mathbb{N}}$ be the left shift map given by $(\omega_0, \omega_1, \omega_2, \ldots) \mapsto (\omega_1, \omega_2, \omega_3, \ldots)$.

Proposition 2.2.1 (Random Oseledets multiplicative ergodic theorem)**.** Let M be a closed smooth Riemannian manifold, μ be a measure on $\mathrm{Diff}^2(M)$ satisfying the moment condition (*). Let ν be an ergodic, μ-stationary Borel probability measure. Then there are numbers

$\lambda_1(\nu) > \lambda_2(\nu) > \cdots > \lambda_\ell(\nu)$ such that for $\mu^{\mathbb{N}}$-almost every sequence $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$ and ν-almost every $x \in M$, there is a filtration

$$T_x M = V_\omega^1(x) \supsetneqq V_\omega^2(x) \supsetneqq \cdots \supsetneqq V_\omega^\ell(x) \supsetneqq V_\omega^{\ell+1} = 0$$

such that for $v \in V_\omega^k(x) \setminus V_\omega^{k+1}(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|D_x f_\omega^n(v)\|}{\|v\|} = \lambda_k(\nu).$$

The subspaces $V_\omega^i(x)$ are invariant in the sense that

$$D_x f_\omega V_\omega^k(x) = V_{\sigma(\omega)}^k(f_\omega(x)).$$

For a proof of the theorem, see e.g. [LQ95, Prop. I.3.1].

Proposition 2.2.2 (Uniform positive exponent)**.** Let M be a closed surface, μ be a uniformly expanding probability measure on $\mathrm{Diff}^2(M)$ satisfying (*). Then there exists a uniform constant $\lambda_\mu > 0$, depending only on μ, such that for *all* $x \in M$, and $\mu^{\mathbb{N}}$-almost every $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, there exists $\lambda(\omega, x) \geq \lambda_\mu$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|D_x f_\omega^n\| = \lambda(\omega, x).$$

In particular for all ergodic, μ-stationary probability measure ν, for ν-almost every $x \in M$ and $\mu^{\mathbb{N}}$-almost every ω, the top Lyapunov exponent $\lambda_1(\nu) = \lambda(\omega, x) \geq \lambda_\mu > 0$.

Sketch of Proof. The point of this proposition is that assuming uniform expansion, Oseledets theorem holds for *every* point $x \in M$ and almost every sequence $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, and the top exponent is positive. See Lemma 4.3.5 of [LX] for a more refined version of this proposition, where it is shown that there is an Oseledets splitting for every point. Here we only need positivity of exponent. We include a sketch of the proof here for completeness.

Let T^1M be the unit tangent bundle of M. By definition of uniform expansion, there exists

$C > 0$ and $N \in \mathbb{N}$ such that for all $(x, v) \in T^1M$,

$$\int \log \|D_x f(v)\| d\mu^{(N)}(f) > C.$$

Let $(x, v_0) \in T^1M$. For each $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$ and $n \in \mathbb{N}$, let

$$(x_n, v_n) = (x_n(\omega), v_n(\omega)) := \left(f_\omega^n(x), \frac{D_x f_\omega^n(v_0)}{\|D_x f_\omega^n(v_0)\|} \right)$$

be the image of (x, v_0) in T^1M after n steps of the random dynamics following the sequence ω. For $k \geq 1$, consider the event

$$X_k(\omega) := \log \|D_{x_{(k-1)N}} f^N_{\sigma^{(k-1)N}\omega}(v_{(k-1)N})\| - \int \log \|D_{x_{(k-1)N}} f(v_{(k-1)N})\| d\mu^{(N)}(f).$$

Notice that

$$X_k(\omega) = \log \frac{\|D_x f_\omega^{kN}(v_0)\|}{\|D_x f_\omega^{(k-1)N}(v_0)\|} - \int \log \|D_{x_{(k-1)N}} f(v_{(k-1)N})\| d\mu^{(N)}(f).$$

Let $S_j = \sum_{k=1}^n X_k$. Then

$$S_j(\omega) = \log \|D_x f_\omega^{jN}(v_0)\| - \sum_{k=1}^{j} \int \log \|D_{x_{(k-1)N}} f(v_{(k-1)N})\| d\mu^{(N)}(f).$$

Thus by uniform expansion,

$$\log \|D_x f_\omega^{jN}(v_0)\| = S_j(\omega) + \sum_{k=1}^{j} \int \log \|D_{x_{(k-1)N}} f(v_{(k-1)N})\| d\mu^{(N)}(f) \geq S_j(\omega) + jC.$$

The main observation is that the family $\{S_n\}_{n\in\mathbb{N}}$ form a square integrable martingale. Then by the strong law of large numbers for square integrable martingales, for $\mu^{\mathbb{N}}$-almost every $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, we have the limit

$$\lim_{n\to\infty} \frac{S_n}{n} = 0.$$

Thus if we write $j = \lfloor n/N \rfloor$, then $\lim_{n\to\infty} j/n = 1/N$, and we have for almost every ω,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \|D_x f_\omega^n(v_0)\| &= \liminf_{n\to\infty} \frac{1}{n} \log \|D_{x_{jN}} f_{\sigma^{jN}\omega}^{n-jN}(v_{jN})\| + \liminf_{n\to\infty} \frac{1}{n} \log \|D_x f_\omega^{jN}(v_0)\| \\ &\geq \liminf_{n\to\infty} \left(0 + \frac{S_j(\omega)}{n} + \frac{jC}{n}\right) \geq \frac{C}{N} > 0.\end{aligned}$$

Hence we can take $\lambda_\mu := C/N$. □

2.3 Measure rigidity

We prove the measure rigidity result in this section. The precise statement was already proved in [LX, Thm. 4.1.4]. We include the proof here for completeness.

The main input of the proof is a result of Brown and Rodriguez-Hertz [BRH17, Thm. 3.4]. This result provides a trichotomy for the ergodic μ-stationary Borel probability measures ν: either the stable distribution is non-random, ν is finitely supported or ν is an ergodic component of the volume on M. The uniform expansion condition eliminates the possibility that the stable distribution is non-random. The same condition also implies that the volume is Γ-ergodic using a refined version of the classical Hopf argument inspired by [DK07, Sect. 10], as detailed in [LX, Prop. 4.4.1].

2.3.1 Main statement

Proposition 2.3.1 (Measure Rigidity)**.** Let M be a closed surface, $\Gamma \subset \mathrm{Diff}^2(M)$ be a subgroup that preserve a smooth measure m on M. Let μ be a uniformly expanding probability measure on $\mathrm{Diff}^2(M)$ with $\mu(\Gamma) = 1$ satisfying (*). Let ν be an ergodic, μ-stationary Borel probability measure on M. Then either ν is finitely supported or $\nu = m$.

Following [BRH17], we write

$$E_\omega^s(x) := \bigcup_{\lambda_j<0} V_\omega^j(x) = \left\{ v \in T_x M : \limsup_{n\to\infty} \frac{1}{n} \log \frac{\|D_x f_\omega^n(v)\|}{\|v\|} < 0 \right\}.$$

for the *stable Lyapunov subspace* for the word ω at the point $x \in M$. We say that the stable distribution is *non-random* if there exists μ-almost surely invariant ν-measurable subbundle $\hat{V} \subset TM$ such that $\hat{V}(x) = E^s_\omega(x)$ for $(\mu^{\mathbb{N}} \times \nu)$-almost every (ω, x), i.e. $Df(E^s_\omega(x)) = E^s_\omega(f(x))$ for μ-a.e. $f \in \mathrm{Diff}^2(M)$ and ν-a.e. $x \in M$.

Given a smooth probability measure m on M, let $\mathrm{Diff}^2_m(M) := \{f \in \mathrm{Diff}^2(M) \mid f_* m = m\}$. We recall the theorem of Brown and Rodriguez Hertz.

Theorem 2.3.2. [BRH17, Thm. 3.4] Let M be a closed surface, $\Gamma \subset \mathrm{Diff}^2(M)$ be a subgroup that preserve a smooth measure m on M. Let μ be a uniformly expanding probability measure on $\mathrm{Diff}^2_m(M)$ with $\mu(\Gamma) = 1$ satisfying (*). Let ν be an ergodic, hyperbolic μ-stationary Borel probability measure on M. Then either

(1) ν has finite support,

(2) the stable distribution $E^s_\omega(x)$ is non-random, or

(3) ν is - up to normalization - the restriction of m to a positive volume subset.

It remains to refine the conclusion of this theorem using the condition of uniform expansion. We will eliminate the second possibility in the next lemma, and refine the third possibility in the next subsection.

Lemma 2.3.3. If μ is uniformly expanding, then the stable distribution is not non-random.

Proof. Assume that the stable distribution $E^s_\omega(x)$ is non-random, i.e. there is a μ-almost surely invariant subbundle $\hat{V} \subset TM$ with $\hat{V}(x) = E^s_\omega(x)$ for $(\mu^{\mathbb{N}} \times \nu)$-a.e. (ω, x). By definition of the stable distribution, for ν-almost every $x \in M$, for all large enough n, we have $\log(\|D_x f^n_w(v)\|/\|v\|) < 0$ for all nonzero $v \in E^s_\omega(x)$. Hence by taking average, for ν-almost all $x \in M$, and for all nonzero $v \in \hat{V}(x)$, we have

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(n)}(f) < 0$$

for all large enough n. However, this contradicts the uniform expansion property of μ, as it is striaghtforward from definition that there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$, nonzero $v \in T_xM$ and $k \in \mathbb{N}$,

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(kN)}(f) > kC.$$

□

2.3.2 *Ergodicity*

The main theorem of [BRH17, Thm. 3.1] did not assume the existence of a smooth invariant measure, in which case the third possibility is that the stationary measure is SRB (see [BRH17, Def. 6.8] for a precise definition). The existence of a smooth invariant measure m allows the authors to refine the third possibility to being a restriction of m to a positive volume subset using a local ergodicity argument (see [BRH17, Ch. 13]), as stated above.

In this section, using uniform expansion, we further refine the third possibility to show that the stationary measure has to be the smooth invariant measure m.

Proposition 2.3.4. Let M be a closed (connected) surface, μ be a Borel probability measure on $\mathrm{Diff}^2_m(M)$. Suppose there exists a positive volume subset $A \subset M$ such that $\nu := \frac{1}{m(A)} m|_A$ is an ergodic μ-stationary Borel probability measure. If μ is uniformly expanding, then in fact $\nu = m$.

This is proved in [LX, Prop. 4.4.1] based on ideas from [DK07, Sect. 10]. For completeness we give a detailed outline of the proof.

The main idea of the proof is to perform a version of the classical Hopf argument. Rather than transversing along the stable and unstable leaves as in the setting of Anosov systems, the argument goes by transversing along the stable leaves $W^s_\omega(x)$ and $W^s_{\omega'}(x)$ of two distinct words ω, ω' with suitable geometric and dynamical properties.

Classical facts about the stable manifolds of a random system

We first collect some standard facts about stable manifolds of a random dynamical system.

Given $x \in M$ and $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, let

$$W^s_\omega(x) := \left\{ y \in M \mid \limsup_{n\to\infty} \frac{1}{n} \log d(f^n_\omega(x), f^n_\omega(y)) < 0 \right\}.$$

There exists a $(\mu^{\mathbb{N}} \times \mathrm{vol})$-co-null set $\Lambda \subset \mathrm{Diff}^2(M)^{\mathbb{N}} \times M$ such that $W^s_\omega(x)$ is a C^2-embedded curve in M for all $(\omega, x) \in \Lambda$. We call $W^s_\omega(x)$ the *global stable manifold* at x for ω.

We define local stable manifolds using the classical stable manifold theorem (we only list properties needed for our purpose).

Theorem 2.3.5 (Local stable manifold theorem)**.** Let $\lambda_\mu > 0$ be the constant from Proposition 2.2.2. For every $0 < \varepsilon < \lambda_\mu/200$, for $\mu^{\mathbb{N}}$-almost every word $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, there exists a full volume set $\Lambda_\omega \subset M$ and a measurable family of *local stable manifolds* $\{W^s_{\omega,\mathrm{loc}}(x)\}_{x\in\Lambda_\omega}$ with the following properties:

(a) $W^s_{\omega,\mathrm{loc}}(x)$ is a C^2 embedded curve, i.e. the image of a C^2 embedding $\psi : (-1, 1) \to M$.

(b) $T_x W^s_{\omega,\mathrm{loc}}(x) = E^s_\omega(x)$.

(c) for $n \geq 0$, $f^n_\omega(W^s_{\omega,\mathrm{loc}}(x)) \subset W^s_{\omega,\mathrm{loc}}(f^n_\omega(x))$.

(d) for $y, z \in W^s_{\omega,\mathrm{loc}}(x)$ and $n \geq 0$,

$$d(f^n_\omega(y), f^n_\omega(z)) \leq L(\omega, x)e^{(-\lambda_\mu+\varepsilon)n} d(y, z),$$

where $L : \mathrm{Diff}^2(M)^{\mathbb{N}} \times M \to [1, \infty)$ is a Borel measurable function such that for all $x \in \Lambda_\omega$ and $n \geq 0$,

$$L(\sigma^n(\omega), f^n_\omega(x)) \leq e^{n\varepsilon} L(\omega, x).$$

Here $\sigma : \mathrm{Diff}^2(M)^{\mathbb{N}} \to \mathrm{Diff}^2(M)^{\mathbb{N}}$ is the left shift given by $\sigma(\omega)_n := \omega_{n+1}$.

(e) $W^s_\omega(x) = \bigcup_{n\geq 0} (f^n_\omega)^{-1}(W^s_{\sigma^n(\omega),\mathrm{loc}}(f^n_\omega(x)))$.

We refer to [BP13, Ch. 7] for a treatment in the deterministic setting, and [LQ95, Ch. III.3] in the random setting.

Definition (Measures on stable leaves)**.** We recall the following notions related to the induced volume measure on the local stable manifolds.

1. Given $r > 0$ and $(\omega, x) \in \Lambda$, let $W^s_{\omega,r}(x) := \{y \in W^s_\omega(x) \mid d_{W^s}(x, y) < r\}$, where d_{W^s} is the Riemannian distance along the C^2-curve $W^s_\omega(x)$.

2. Given a C^1-curve γ on M, there is a natural measure on γ induced by the restriction of the Riemannian metric on M to γ. We call this measure the *leaf-volume* of γ, denoted vol_γ.

3. Given a measurable subset $T \subset W^s_\omega(x)$ for some word ω and point $x \in M$, we write

$$\mathrm{vol}_{W^s}(T) := \mathrm{vol}_{W^s_\omega(x)}(T),$$

as the dependence on ω and x is clear from the definition of T.

4. Unless otherwise specified, "almost every" point on γ means almost every point with respect to the leaf-volume.

We will also need the standard fact that for $(\mu^{\mathbb{N}} \times \mathrm{vol})$-almost every (ω, x), the stable manifold $W^s_\omega(x)$ satisfies two versions of absolute continuity that we will describe in the next lemma.

By Lusin theorem and Theorem 2.3.5 (a), for all $\delta > 0$, there exists a measurable subset $Q \subset M$ with $\mathrm{vol}(Q) > 1 - \delta$ such that $W^s_{\omega,\mathrm{loc}}(y)$ varies continuously in $y \in Q$ in the C^2 topology.

Lemma 2.3.6 (Absolute Continuity)**.** For $(\mu^{\mathbb{N}} \times \mathrm{vol})$-almost every $(\omega, x) \in \mathrm{Diff}^2(M)^{\mathbb{N}} \times M$, for sufficiently small $R > 0$, the family of local stable manifolds $\mathcal{F} := \{W^s_{\omega,\mathrm{loc}}(y)\}_{y \in Q\cap B(x,R)}$ satisfies the following properties:

1. For all $y \in Q \cap B(x,R)$, $W^s_{\omega,\mathrm{loc}}(y)$ intersects $\partial B(x,R)$ at two points.

2. For $y, y' \in Q \cap B(x,R)$, if $y' \in W^s_{\omega,\mathrm{loc}}(y)$, then $W^s_{\omega,\mathrm{loc}}(y) \cap B(x,R) = W^s_{\omega,\mathrm{loc}}(y') \cap B(x,R)$.

Then the following two versions of absolute continuity hold (we write $\mathcal{F}(y)$ for the element in $\mathcal{F}$ containing the point y).

(AC1) Let γ_1 and γ_2 be two C^1-curves in $B(x,R)$ everywhere uniformly transverse to $\mathcal{F}$. Let

$$T_1 := \gamma_1 \cap \bigcup_{y \in \gamma_2} \mathcal{F}(y), \qquad \text{and} \qquad T_2 := \gamma_2 \cap \bigcup_{y \in T_1} \mathcal{F}(y).$$

Define the holonomy map $h_{\mathcal{F}} : T_1 \to T_2$ given by "sliding" along the leaves in $\mathcal{F}$, i.e. $h_{\mathcal{F}}(y)$ is the only point in $\gamma_2 \cap \mathcal{F}(y)$ for all $y \in T_1$.

Then on T_2, we have

$$\mathrm{vol}_{\gamma_2} \ll (h_{\mathcal{F}})_* \mathrm{vol}_{\gamma_1}.$$

(AC2) For any Borel subset $A \subset M$, we have

$$\mathrm{vol}(A) = 0 \qquad \Leftrightarrow \qquad \mathrm{vol}_{W^s_\omega(y)}(A \cap W^s_\omega(y)) = 0 \quad \text{for} \quad \text{vol-a.e. } y \in M.$$

See [BP13, Ch. 8] for a statement in the case of deterministic systems, [LY88, Sect. 4.2] or [LQ95, Sect. III.5] for a statement in the case of random systems.

Implications of uniform expansion

One consequence of uniform expansion is uniform control on the angles between stable directions of different words.

Lemma 2.3.7 (Uniform avoidance of the stable direction)**.** [LX, Prop. 4.4.4] [Zha19, Prop. 3] If μ is uniformly expanding, then there exists $\alpha > 0$ with the following property: for any $(x,v) \in T^1M$, there exists a subset $\Gamma_{x,v} \subset \mathrm{Diff}^2(M)^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\Gamma_{x,v}) > 0.99$ such that, for any $\omega \in \Gamma_{x,v}$,

$$\measuredangle(E^s_\omega(x), v) > \alpha.$$

Another property of uniformly expanding systems is that for every point on the surface, the dynamics exhibit uniform hyperbolicity for a large proportion of words. This implies uniform control on the lengths and curvatures of the local stable manifolds.

Lemma 2.3.8 (Uniform control of the local stable manifolds)**.** [LX, Prop. 4.4.9] [Zha19, Prop. 3] If μ is uniformly expanding, then there exist a constant $\ell = \ell(\mu) > 0$ with the following properties:

for any $x \in M$, there exists a subset $\Lambda_x \subset \mathrm{Diff}^2(M)^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\Lambda_x) > 0.99$ such that for all $\omega \in \Lambda_x$,

(i) $W^s_{\omega,\ell}(x) \subset\subset W^s_{\omega,\mathrm{loc}}(x)$,

(ii) the *angle change* of the curve $\exp_x^{-1}(W^s_{\omega,\ell}(x))$ is less than $\alpha/100$.

Here α is as in Lemma 2.3.7, and for a C^1-curve $\gamma : [a,b] \to \mathbb{R}^2$, the *angle change* of γ is

$$\max_{t,s\in[a,b]} \measuredangle(\gamma'(t), \gamma'(s)).$$

The notation $A \subset\subset B$ in (i) means A is compactly contained in B, i.e. the closure of A is compact and is contained in B. Note that (i) implies that the leaf-volume of $W^s_{\omega,\ell}(x)$ is at least 2ℓ since the condition implies, in particular, that $W^s_{\omega,\ell}(x) \subsetneq W^s_\omega(x)$.

We say that $W^s_{\omega,\mathrm{loc}}(x)$ is a *nice curve* if $\omega \in \Lambda_x$.

We shall use these constants α and ℓ, which depend only on μ, later in the proof. The set Λ_x of words in Lemma 2.3.8 will also appear a few times in the proof.

Basin of ν

We will consider the classical notion of a basin of ν in this random setting, and remark that to show that $\nu = m$, it suffices to show that the basin $B(\nu)$ has full volume. This will be used in

Step 1 below.

Definition. Given $x \in M$, $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$ and a continuous function $\varphi : M \to \mathbb{R}$, define the *$\omega$-Birkhoff average* of φ at x as

$$S_\omega(\varphi)(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\omega^j(x))$$

if the limit on the right exists.

Definition. Given an ergodic μ-stationary measure ν on M, define the *basin* of ν, denoted $B(\nu) \subset M$, as the set of points $x \in M$ such that for any continuous function $\varphi : M \to \mathbb{R}$ and $\mu^{\mathbb{N}}$-almost every $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$,

$$S_\omega(\varphi)(x) = \int_M \varphi \, d\nu.$$

Lemma 2.3.9. If $\mathrm{vol}(B(\nu)) = 1$, then $\nu = m$.

Proof. Assume that $\mathrm{vol}(B(\nu)) = 1$. Then $m(B(\nu)) = 1$. Let $\varphi \in C^0(M)$. By the pointwise ergodic theorem (and the argument in the proof of Lemma 2.3.12), there exists a function $\overline{\varphi}(x)$ such that for $(\mu^{\mathbb{N}} \times m)$-a.e. (ω, x),

$$S_\omega(\varphi)(x) = \overline{\varphi}(x) \qquad \text{and} \qquad \int \overline{\varphi}(x)\, dm(x) = \int \varphi(x)\, dm(x).$$

On the other hand, by definition of the basin $B(\nu)$, for all $x \in B(\nu)$, we have

$$S_\omega(\varphi)(x) = \int \varphi \, d\nu.$$

Therefore $\overline{\varphi}(x) = \int \varphi \, d\nu$ for all $x \in B(\nu)$. But since $m(B(\nu)) = \mathrm{vol}(B(\nu)) = 1$, we have

$$\int \varphi(x)\, dm(x) = \int \overline{\varphi}(x)\, dm(x) = \int_{B(\nu)} \overline{\varphi}(x)\, dm(x) = \int \varphi(x)\, d\nu(x).$$

Since φ is arbitrary, we have $\nu = m$. $\square$

Reduction to a local argument via Lebesgue density theorem

By Lemma 2.3.9, it suffices to argue that the basin has full volume. In this section, we argue that it suffices to show that in every (uniformly) small enough ball, the basin either has zero density or has density bounded from below by a positive uniform constant. This allows us to reduce the problem to a local argument in a small ball. This will be used in **Step 2** below.

Definition (Density)**.** Given a Borel measurable subset $U \subset M$, a point $x \in M$ and $r > 0$, define the *density* of U in the ball $B(x, r)$ as

$$\mathrm{vol}(U : B(x,r)) := \frac{\mathrm{vol}(U \cap B(x,r))}{\mathrm{vol}(B(x,r))}.$$

Lemma 2.3.10. Assume that a measurable subset $U \subset M$ satisfies the following: there exist $c > 0$ and $R_0 > 0$ such that for all $x \in M$ and positive $r < R_0$, either

$$\mathrm{vol}(U : B(x,r)) = 0 \qquad \text{or} \qquad \mathrm{vol}(U : B(x,r)) > c.$$

Then $\mathrm{vol}(U) = 0$ or 1.

Proof. Assume the contrary that $\mathrm{vol}(U) \in (0, 1)$. Clearly the assumption continues to hold if we decrease c. Thus without loss of generality assume that $0 < c < 1/2$.

Since $\mathrm{vol}(U)$ and $\mathrm{vol}(U^c)$ are both positive by assumption, by Lebesgue density theorem, there exist $y \in U$, $z \in U^c$ and $r \in (0, R_0)$ such that

$$\mathrm{vol}(U : B(y,r)) > 1 - c \qquad \text{and} \qquad \mathrm{vol}(U : B(z,r)) < c/4.$$

Now observe that the function $x \mapsto \mathrm{vol}(U : B(x,r))$ is continuous in $x \in M$ for fixed $U \subset M$ and $r > 0$. Since M is connected, there exists $x \in M$ such that $\mathrm{vol}(U : B(x,r)) = c/2$. This yields a contradiction. □

In the rest of this section, we shall find uniform constants $c > 0$ and $R_0 > 0$ so that the assumptions of Lemma 2.3.10 hold for the basin $U = B(\nu)$.

Regular points

Similar to the proof of ergodicity in [DK07, Sect. 10], we define a notion of regular points, and show that almost every point on M is regular. This will be used in **Step 3** of the main argument.

Informally, the notions of regular points can be summarized as follows: for $x \in M$ and $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$,

1. x is ω-*regular* if the ω-Birkhoff averages at x agree with the ω'-Birkhoff averages at x for $\mu^{\mathbb{N}}$-a.e. ω'.
2. x is *regular* if for $\mu^{\mathbb{N}}$-a.e. ω, x is ω-regular and almost every $y \in W^s_\omega(x)$ is ω-regular.

Definition. For $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, a point $x \in M$ is called ω-*regular* if for $\mu^{\mathbb{N}}$-almost every $\omega' \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, for any continuous function $\varphi : M \to \mathbb{R}$, we have

$$S_\omega(\varphi)(x) = S_{\omega'}(\varphi)(x)$$

(in particular the Birkhoff averages exist).

Remark 2.3.11. Note that if x is ω-regular, then for $\mu^{\mathbb{N}}$-almost every $\omega' \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, x is ω'-regular.

Lemma 2.3.12. [Kif86, Cor. I.2.2, Page 24] For $\mu^{\mathbb{N}} \times \mathrm{vol}$-almost every $(\omega, x) \in \mathrm{Diff}^2(M)^{\mathbb{N}} \times M$, x is ω-regular.

Definition. A point $x \in M$ is called *regular* if for $\mu^{\mathbb{N}}$-almost every word $\omega \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, x is ω-regular and almost every point $y \in W^s_\omega(x)$ is ω-regular.

It can be shown using Lemma 2.3.12 and absolute continuity of the stable manifolds that almost every point on M is regular.

Lemma 2.3.13. [LX, Lem. 4.4.18] vol-almost every point $x \in M$ is regular.

Proof. We need to show that the set

$$B_1 = \{(\omega, x) \in \mathrm{Diff}^2(M)^{\mathbb{N}} \times M \mid \mathrm{vol}_{W^s}(\{y \in W^s_\omega(x) \mid y \text{ is not } \omega\text{-regular}\}) > 0\}$$

has $\mu^{\mathbb{N}} \times \mathrm{vol}$-measure zero. By Lemma 2.3.12, for $\mu^{\mathbb{N}}$-almost every word ω, we have

$$\mathrm{vol}(\{y \in M \mid y \text{ is not } \omega\text{-regular}\}) = 0.$$

By absolute continuity of the foliation W^s_ω (Lemma 2.3.6 (AC2), ignore a null set of words ω if necessary), for vol-almost every point $x \in M$, we have

$$\mathrm{vol}_{W^s}(\{y \in W^s_\omega(x) \mid y \text{ is not } \omega\text{-regular}\}) = 0.$$

This is enough to show that B_1 has measure zero. □

The following lemma is a direct consequence of the definitions, and will be used repeatedly in **Step 6**.

Lemma 2.3.14. [LX, Lem. 4.4.19] Given an ergodic μ-stationary measure ν on M and $\omega \in \mathrm{Diff}^2_m(M)^{\mathbb{N}}$, if $x, y \in M$ are both ω-regular and $y \in W^s_\omega(x)$, then $x \in B(\nu)$ if and only if $y \in B(\nu)$.

Proof. For any continuous function $\varphi : M \to \mathbb{R}$, and for $\mu^{\mathbb{N}}$-almost every $\omega' \in \mathrm{Diff}^2(M)^{\mathbb{N}}$, we have

$$S_{\omega'}(\varphi)(x) = S_\omega(\varphi)(x) = S_\omega(\varphi)(y) = S_{\omega'}(\varphi)(y),$$

where the second equality uses the fact that $y \in W^s_\omega(x)$, and M is compact so that φ is uniformly continuous. The first and third equalities use the fact that x and y are ω-regular. Therefore the leftmost term equals $\int \varphi \, d\nu$ if and only if the rightmost term equals $\int \varphi \, d\nu$. □

Basic setup of the Hopf argument

Using Lemma 2.3.7 and 2.3.8, we can set up the Hopf argument in a small local ball $B(x_0, r)$ containing a regular point x by finding two words $\omega, \omega' \in \mathrm{Diff}^2(M)^{\mathbb{N}}$ whose local stable manifolds through x have nice geometric and dynamical properties. Throughout this subsection we shall fix $x_0 \in M$ and $r > 0$. We first give an outline of the main argument (see Figure 2.1 for an illustration).

Step 1: By Lemma 2.3.9, to show that $\nu = m$, it suffices to show that $\mathrm{vol}(B(\nu)) = 1$.

Step 2: By Lemma 2.3.10, to show that $\mathrm{vol}(B(\nu)) = 1$, it suffices to show that for some uniform constants $R_0 > 0$ and $c > 0$, for all $x_0 \in M$ and $r < R_0$, either $\mathrm{vol}(B(\nu) : B(x_0, r)) = 0$ or $\mathrm{vol}(B(\nu) : B(x_0, r)) > c$. We will choose R_0 in subsection 2.3.2. We fix $x_0 \in M$ and $r < R_0$ in the rest of the outline.

Step 3: Assume that $\mathrm{vol}(B(\nu) : B(x_0, r)) > 0$. Choose a regular point x in $B(\nu) \cap B(x_0, r)$.

Step 4: Choose words ω, ω' and a subset $T \subset W^s_{\omega,\mathrm{loc}}(x) \cap B(x_0, r)$ with positive leaf-volume such that for all $y \in T$,

(i) $W^s_{\omega,\mathrm{loc}}(x)$ and $W^s_{\omega',\mathrm{loc}}(y)$ are nice curves (in the sense of Lemma 2.3.8) and uniformly transverse;

(ii) x and y are ω-regular;

(iii) y and almost every $z \in W^s_{\omega',\mathrm{loc}}(y)$ are ω'-regular.

We will choose ω, ω' and T in subsection 2.3.2.

Step 5: Construct a good set $U' \subset B(x_0, r)$ with (uniformly) positive density in $B(x_0, r)$, a word ω'', and a subset $T' \subset T$ with positive leaf-volume such that for all $p \in U'$,

(i) $W^s_{\omega'',\mathrm{loc}}(p)$ is a nice curve, and is uniformly transverse to the family $\{W^s_{\omega',\mathrm{loc}}(y)\}_{y \in T'}$.

(ii) p is ω''-regular,

(iii) the set of intersection points between $W^s_{\omega'',\mathrm{loc}}(p)$ and $\{W^s_{\omega',\mathrm{loc}}(y)\}_{y\in T'}$ that are both ω'-regular and ω''-regular has positive leaf-volume in $W^s_{\omega'',\mathrm{loc}}(p)$.

We will choose T' in subsection 2.3.2. We will choose the word ω'', the set U' and the uniform positive lower bound c_3 on the density of U' in subsection 2.3.2.

Step 6: Apply Lemma 2.3.14 to show that $U' \subset B(x_0, r)$ is contained in the basin $B(\nu)$. In fact, for $p \in U'$,

(i) $x \in B(\nu)$ by **Step 3**.

(ii) Let $y \in T' \subset T \subset W^s_{\omega,\mathrm{loc}}(x)$. Both x and y are ω-regular, so by (i) and Lemma 2.3.14, $y \in B(\nu)$.

(iii) Let $z \in W^s_{\omega',\mathrm{loc}}(y)$ for some $y \in T'$. Suppose that z is both ω'-regular and ω''-regular. By (ii), $y \in B(\nu)$. Since y is ω'-regular, by Lemma 2.3.14, $z \in B(\nu)$.

(iv) By **Step 5**, a positive leaf-volume set of points z in $W^s_{\omega'',\mathrm{loc}}(p)$ are in $W^s_{\omega',\mathrm{loc}}(y)$ for some $y \in T'$, and are ω'-regular and ω''-regular. By (iii), $z \in B(\nu)$. Since p is ω''-regular, by Lemma 2.3.14, $p \in B(\nu)$.

This concludes the argument, since $U' \subset B(x_0, r) \cap B(\nu)$ and has (uniformly) positive density in $B(x_0, r)$.

In the rest of this section, we shall make **Step 4-6** precise by choosing the appropriate parameters.

Choice of the radius R_0

We choose $R_0 = R_0(\alpha, \ell) > 0$ with the following properties: for positive $r < R_0$ and $y \in B(x_0, r)$,

1. (Angle between off center tangent vectors)

   - for $v, w \in T_yM$, if $\measuredangle(v, w) > \alpha$, then $\measuredangle(D_y \exp_{x_0}^{-1} v, D_y \exp_{x_0}^{-1} w) > \alpha/2$.

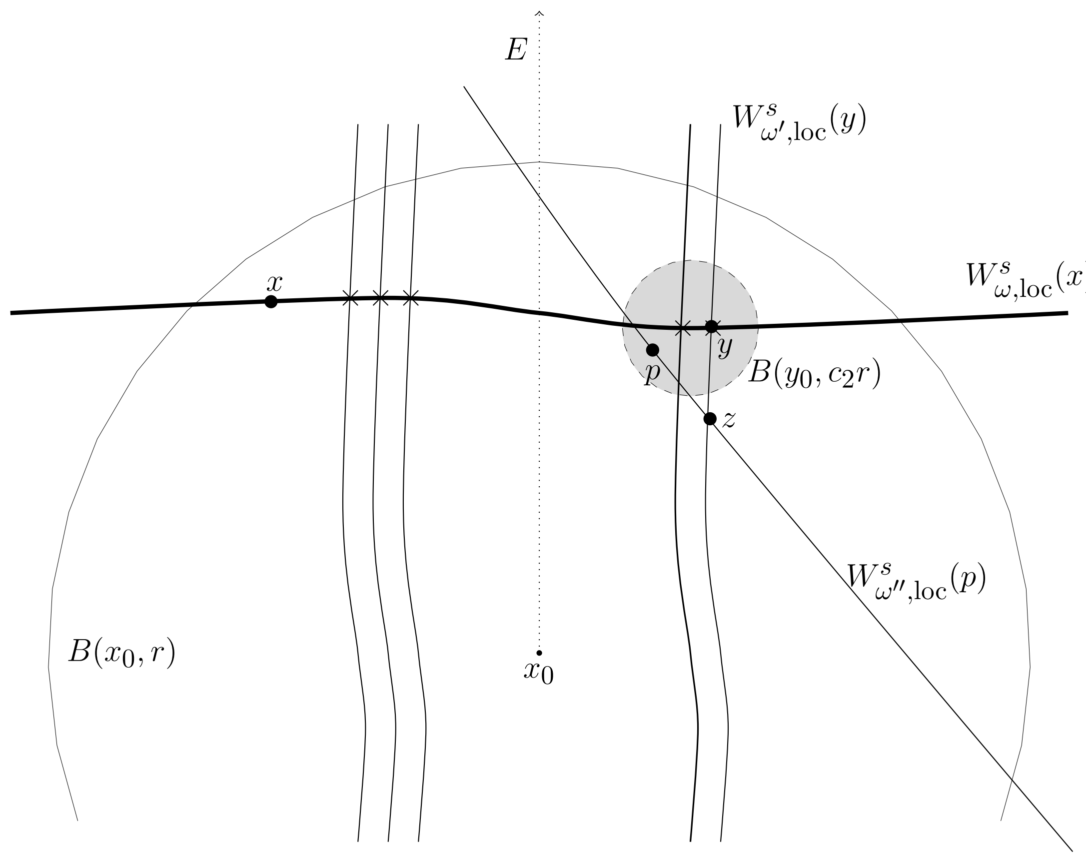


Figure 2.1: Illustration of the main argument (U' is a subset of the shaded region $B(y_0, c_2 r)$ with positive density; $\times \in T \subset W^s_{\omega,\mathrm{loc}}(x)$)

- for $v, w \in T_y M$, if $\measuredangle(v, w) > \alpha/4$, then $\measuredangle(D_y \exp_{x_0}^{-1} v, D_y \exp_{x_0}^{-1} w) > \alpha/8$.

2. (Angle change of curves) given a C^2-curve $\gamma_1 \subset B(x_0, r)$ through y, if $\exp_y^{-1} \gamma_1$ has angle change less than $\alpha/100$, then $\exp_{x_0}^{-1} \gamma_1$ has angle change less than $\alpha/99$.

3. Also choose $R_0 < \ell/10$, where $\ell = \ell(\mu)$ is the constant from Lemma 2.3.8.

Such conditions hold for small enough r such that for $y \in B(x_0, r)$, the map $D_y \exp_{x_0}^{-1} : T_y M \to TT_{x_0} M$ is close enough to the identity (using the C^2 assumption and compactness of the manifold, the appropriate constants depend only on α, ℓ and the geometry of the smooth Riemannian manifold M, in particular R_0 can be taken independent of x_0).

Choice of the words ω, ω', the set $T \subset W^s_{\omega,\text{loc}}(x)$ and the constant c_1 (for Step 4)

Lemma 2.3.15. [LX, Lem. 4.4.20] For any $x_0 \in M$ and positive $r < R_0$ (from subsection 2.3.2), let $x \in B(x_0, r) \cap B(\nu) \setminus \{x_0\}$ be a regular point. Then there exist words $\omega, \omega' \in \text{Diff}^2(M)^{\mathbb{N}}$, a subset $T \subset W^s_{\omega,\text{loc}}(x) \cap B(x_0, r)$ and a constant $0 < c_1 = c_1(\alpha) < 1$ with the following properties.

1. x is ω-regular,
2. $W^s_{\omega,\text{loc}}(x)$ is a nice curve, i.e. $\omega \in \Lambda_x$ as in Lemma 2.3.8,
3. the set of ω'-regular points has full volume in M,
4. the leaf-volume of $T \subset W^s_\omega(x)$ is at least $c_1 r$,
5. for any $y \in T$,
   (a) y is ω-regular and ω'-regular,
   (b) $W^s_{\omega',\text{loc}}(y)$ is a nice curve.
   (c) $d(y, \partial B(x_0, r)) > c_1 r$,
   (d) $\measuredangle(E^s_\omega(y), E^s_{\omega'}(y)) > \alpha$,

Proof. We have the following properties of x:

(i) for $\mu^{\mathbb{N}}$-a.e. ω, x is ω-regular and almost every $y \in W^s_\omega(x)$ is ω-regular since x is regular.

(ii) for at least 99% of the words ω (with respect to $\mu^{\mathbb{N}}$), $W^s_{\omega,\text{loc}}(x)$ is a nice curve by Lemma 2.3.8.

Note that $x \neq x_0$. Let v be the initial vector in T_xM of the geodesic from x to x_0, and $v^\perp \in \mathbb{P}(T_xM)$ be the orthogonal complement of v in T_xM.

(iii) for at least 99% of the words ω (with respect to $\mu^{\mathbb{N}}$), $\measuredangle(E^s_\omega(x), v^\perp) > \alpha$ by Lemma 2.3.7.

Choice of ω: Let ω be one of the 99% words that satisfy (i), (ii) and (iii). Since $W^s_{\omega,\mathrm{loc}}(x)$ is a nice curve, it contains an ℓ-neighborhood of x with $\ell > 10r$, and we have a uniform bound on angle change of $\exp_x^{-1}(W^s_{\omega,\ell}(x))$.

Choice of c_1: Using Euclidean geometry, (iii) implies that there exist $c_1 = c_1(\alpha) > 0$ and a C^2-segment $\gamma \subset W^s_{\omega,\mathrm{loc}}(x)$ such that

(iv) $\mathrm{vol}_{W^s}(\gamma) > 2c_1 r$,

(v) for all $y \in \gamma$, $d(y, \partial B(x_0, r)) > c_1 r$.

Now for the almost every $y \in \gamma$ that is ω-regular, we have the following properties of y:

(vi$_y$) for $\mu^{\mathbb{N}}$-a.e. ω', y is ω'-regular by Remark 2.3.11.

(vii$_y$) for at least 99% of the words ω' (with respect to $\mu^{\mathbb{N}}$), $W^s_{\omega',\mathrm{loc}}(y)$ is a nice curve by Lemma 2.3.8.

(viii$_y$) for at least 99% of the words ω' (with respect to $\mu^{\mathbb{N}}$), $\measuredangle(E^s_\omega(y), E^s_{\omega'}(y)) > \alpha$ by Lemma 2.3.7.

Now consider the set

$$G := \{(\omega', y) \in \mathrm{Diff}^2(M)^{\mathbb{N}} \times \gamma \mid y \text{ is } \omega\text{-regular, and } \omega' \text{ satisfies (vi}_y\text{), (vii}_y\text{), (viii}_y\text{)}\}.$$

Almost every $y \in \gamma$ is ω-regular by (i). For each $y \in \gamma$, at least 98% of the words ω' satisfy (vi$_y$), (vii$_y$), (viii$_y$). Thus by Fubini's theorem,

$$\mu^{\mathbb{N}} \times \mathrm{vol}_{W^s}(G) \geq 0.98\, \mathrm{vol}_{W^s}(\gamma).$$

By Fubini's theorem again,

(ix) for at least 96% of the words ω',

$$\mathrm{vol}_{W^s}(\{y \in \gamma \mid y \text{ is } \omega\text{-regular, and } \omega' \text{ satisfies (vi}_y\text{), (vii}_y\text{), (viii}_y\text{)}\}) > 0.5\, \mathrm{vol}_{W^s}(\gamma) > c_1 r.$$

Here in the last inequality, we have used (iv). Recall that

(x) for $\mu^{\mathbb{N}}$-almost every word ω', $\mathrm{vol}(\{z \in M \mid z \text{ is } \omega'\text{-regular}\}) = 1$ by Lemma 2.3.12.

Choice of ω' and T: Let ω' be one of at least 96% words that satisfy (ix) and (x). Let

$$T := \{y \in \gamma \mid y \text{ is } \omega\text{-regular, and } \omega' \text{ satisfies } (\mathrm{vi}_y), (\mathrm{vii}_y), (\mathrm{viii}_y)\}.$$

We can verify each property:

1. This follows from (i).

2. This follows from (ii).

3. This follows from (x).

4. This follows from (ix).

5. for $y \in T$,

   (a) This follows from the definition of T and (vi_y).

   (b) This follows from (vii_y).

   (c) This follows from (v).

   (d) This follows from (viii_y).

□

Choice of the direction $E \in \mathbb{P}(T_{x_0}M)$, the ball $B(y_0, c_2 r)$, and the set $T' \subset T \subset W^s_{\omega,\mathrm{loc}}(x)$

Let $r < R_0$. Now lift the ball $B(x_0, r)$ to the tangent space at x_0 via the inverse of the exponential map $\exp_{x_0}^{-1}$. Let $x \in B(x_0, r)$. Recall that

1. since $W^s_{\omega,\mathrm{loc}}(x)$ is a nice curve by Lemma 2.3.15 (2), the angle change of $\exp_x^{-1} W^s_{\omega,\ell}(x)$ is less than $\alpha/100$.

2. Also by Lemma 2.3.15 (5d), for any $y \in T$, $\measuredangle(E^s_\omega(y), E^s_{\omega'}(y)) > \alpha$.

By the choice of R_0, we have

1. the angle change of $\exp_{x_0}^{-1} W^s_{\omega,\ell}(x)$ is less than $\alpha/99$,

2. for all $y \in T$, $\measuredangle(\exp_{x_0}^{-1} W^s_\omega(y), \exp_{x_0}^{-1} W^s_{\omega'}(y)) > \alpha/2$.

Choice of E: By compactness of $\mathbb{P}(T_{x_0}M)$ and $\mathrm{vol}_{W^s}(T) > 0$, there exists a direction $E \in \mathbb{P}(T_{x_0}(M))$ such that

1. $\mathrm{vol}_{W^s}(\{y \in T \mid \measuredangle(E, \exp_{x_0}^{-1} W^s_{\omega'}(y)) < \alpha/100\}) > 0$, and

2. for each $E' \in \mathbb{P}(T_{x_0}(M))$ with $\measuredangle(E, E') < \alpha/100$, and each tangent vector v to the curve $\exp_{x_0}^{-1} W^s_{\omega,\mathrm{loc}}(x)$ on $T_{x_0}M$, we have $\measuredangle(E', v) > \alpha/4$.

Choice of c_2: Now take a constant $c_2 = c_2(\alpha, c_1) > 0$ small enough so that $c_2 < c_1/2$ and the following property holds: for any $y_0 \in M$ and $z_1, z_2 \in B(y_0, c_2 r)$, if two C^1-curves γ_1 and γ_2 on M satisfy the following properties:

1. $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$,

2. γ_i contains an $(c_1 r/2)$-neighborhood (within the curve) of z_i for $i = 1, 2$,

3. the angle changes of $\exp_{y_0}^{-1} \gamma_1$ and $\exp_{y_0}^{-1} \gamma_2$ are less than $\alpha/99$,

4. $\measuredangle(\exp_{y_0}^{-1} \gamma_1, \exp_{y_0}^{-1} \gamma_2) > \alpha/8$,

then γ_1 and γ_2 intersect at least once.

Choice of y_0: Take $y_0 \in T$ such that

$$\mathrm{vol}_{W^s}(\{y \in T \cap B(y_0, c_2 r) \mid \measuredangle(E, \exp_{x_0}^{-1} W^s_{\omega'}(y)) < \alpha/100\}) > 0.$$

Choice of T': Let $T' := \{y \in T \cap B(y_0, c_2 r) \mid \measuredangle(E, \exp_{x_0}^{-1} W^s_{\omega'}(y)) < \alpha/100\}$. Then $\mathrm{vol}_{W^s}(T') > 0$.

Choice of the good set $U' \subset B(x_0, r)$, the word ω'' and the constant c_3 (for Step 5)

Lemma 2.3.16. Define R_0 as in subsection 2.3.2, the words ω, ω' as in subsection 2.3.2, and E, y_0, c_2, T' as in subsection 2.3.2. Let $r < R_0$. Then there exists a uniform constant $c_3 = c_3(c_2) > 0$, a measurable set $U' \subset B(x_0, r)$ with $\mathrm{vol}(U') > c_3 \mathrm{vol}(B(x_0, r))$ and a word ω'' such that for all $p \in U'$,

(a) $W^s_{\omega'',\mathrm{loc}}(p)$ is a nice curve.

(b) almost every $z \in W^s_{\omega''}(p)$ is ω'-regular, where ω' is the chosen word in subsection 2.3.2.

(c) p is ω''-regular and almost every point in $W^s_{\omega''}(p)$ is ω''-regular.

(d) The angle

$$\measuredangle(D_p \exp_{x_0}^{-1} E^s_{\omega''}(p), E) > \alpha/2,$$

where $E \in \mathbb{P}(T_{x_0}M)$ is the direction chosen in subsection 2.3.2.

Proof. We first collect a few facts that hold for vol-almost every points $p \in M$ and a large set of words ω''.

(a) By Lemma 2.3.8, for any $p \in M$, for at least 99% of the words ω'', $W^s_{\omega'',\mathrm{loc}}(p)$ is a nice curve.

(b) By (AC2) and Lemma 2.3.15(3), for vol-almost every $p \in M$ and $\mu^{\mathbb{N}}$-a.e. ω'', almost every $z \in W^s_{\omega''}(p)$ is ω'-regular.

(c) By Lemma 2.3.13, vol-almost every point $p \in M$ is regular, i.e. for $\mu^{\mathbb{N}}$-a.e. ω'', p is ω''-regular and almost every point in $W^s_{\omega''}(p)$ is ω''-regular.

(d) By Lemma 2.3.7 and the choice of R_0, for any $p \in B(x_0, r)$, for at least 99% of the words ω'',

$$\measuredangle(D_p \exp_{x_0}^{-1} E^s_{\omega''}(p), E) > \alpha/2,$$

where $E \in \mathbb{P}(T_{x_0}M)$ is the direction in subsection 2.3.2.

Hence for vol-a.e. $p \in B(x_0, r)$, there are at least 98% of the words ω'' such that (a)-(d) hold. Now consider the small ball $B(y_0, c_2 r)$ chosen in subsection 2.3.2. Since $c_2 < c_1$ and $d(y_0, \partial B(x_0, r)) > c_1 r$ (since $y_0 \in T$), $B(y_0, c_2 r) \subset B(x_0, r)$.

Choice of U' and ω'': By Fubini's theorem, there exists a word ω'' such that the subset

$$U' := \{p \in B(y_0, c_2 r) \mid \text{ (a)-(d) hold for } p \text{ with respect to } \omega''\} \subset B(x_0, r)$$

has volume $\mathrm{vol}(U') > 0.5\,\mathrm{vol}(B(y_0, c_2 r))$, where $B(y_0, c_2 r)$ is the ball from subsection 2.3.2.

Choice of c_3: Now we can take a uniform constant $c_3 = c_3(c_2) > 0$ such that $\mathrm{vol}(U') > c_3 \mathrm{vol}(B(x_0, r))$. □

The set U' and the word ω'' are related to the ω'-local stable curves through T' in the following manner.

Lemma 2.3.17. Define $T' \subset W^s_{\omega,\mathrm{loc}}(x)$ as in subsection 2.3.2. Let $U := \bigcup_{y \in T'} W^s_{\omega',\mathrm{loc}}(y)$. Then for all $p \in U'$,

$$\mathrm{vol}_{W^s}(\{z \in W^s_{\omega'',\mathrm{loc}}(p) \cap U \mid z \text{ is } \omega'\text{-regular and } \omega''\text{-regular}\}) > 0.$$

Proof. Let $p \in U'$ and $y \in T'$. Note that $p, y \in B(y_0, c_2 r)$. Let $z_1 = p$ and $z_2 = y$. We verify properties 1-4 in the choice of c_2 in subsection 2.3.2 for the local stable curves

$$\gamma_1 := W^s_{\omega'',c_1 r/2}(p) \subset W^s_{\omega'',\mathrm{loc}}(p) \qquad \text{and} \qquad \gamma_2 := W^s_{\omega',c_1 r/2}(y) \subset W^s_{\omega',\mathrm{loc}}(y).$$

Note that since $c_2 < c_1/2$, $d(y_0, \partial B(x_0, r)) > c_1 r$ (since $y_0 \in T$) and $p, y \in B(y_0, c_2 r)$, we have $\gamma_1, \gamma_2 \subset B(x_0, r)$.

1. Clearly $p \in \gamma_1$ and $y \in \gamma_2$.

2. By definition of γ_1 and γ_2, γ_i is the $(c_1 r/2)$-neighborhood of z_i in the local stable curve.

3. Note that $W^s_{\omega'',\mathrm{loc}}(p)$ and $W^s_{\omega',\mathrm{loc}}(y)$ are nice curves by Lemma 2.3.16(a) and Lemma 2.3.15 (5b), so γ_1 and γ_2 have bounded angle change in their respective tangent spaces. Now using the choice of R_0 applied to the tangent space at y_0, we conclude the bound on angle changes in $T_{y_0}M$.

4. By the choice of R_0, E and T', one can readily verify that $\measuredangle(\exp_{x_0}^{-1}\gamma_1, \exp_{x_0}^{-1}\gamma_2) > \alpha/4$. Apply the choice of R_0 again, $\measuredangle(\exp_{y_0}^{-1}\gamma_1, \exp_{y_0}^{-1}\gamma_2) > \alpha/8$.

Therefore properties 1-4 in the choice of c_2 are satisfied, thus $W^s_{\omega'',\mathrm{loc}}(p)$ intersects $W^s_{\omega',\mathrm{loc}}(y)$ for all $y \in T'$, with angle at least $\alpha/4$ on $T_{x_0}M$.

Now $W^s_{\omega,\mathrm{loc}}(x)$ is uniformly transverse to $W^s_{\omega',\mathrm{loc}}(y)$ for $y \in T'$ by Lemma 2.3.15 (5d). Apply (AC1) to the holonomy h_{W^s} between the transversals $W^s_{\omega'',\mathrm{loc}}(p)$ and $W^s_{\omega,\mathrm{loc}}(x)$ along the family of local stable curves $\{W^s_{\omega',\mathrm{loc}}(y)\}_{y\in T'}$. By the previous paragraph, h_{W^s} is a bijection from $W^s_{\omega'',\mathrm{loc}}(p) \cap U$ to $T' \subset W^s_{\omega,\mathrm{loc}}(x)$. Since T' has positive leaf-volume in $W^s_{\omega,\mathrm{loc}}(x)$, by (AC1), $W^s_{\omega'',\mathrm{loc}}(p) \cap U$ has positive leaf-volume.

Now the conclusion holds since almost every point in $W^s_{\omega'',\mathrm{loc}}(p)$ is ω'-regular and ω''-regular by Lemma 2.3.16 (b, c). □

Conclude the proof of Proposition 2.3.4 and Proposition 2.3.1

Proof of Proposition 2.3.4. The proof goes by performing the Hopf argument in a local ball $B(x_0, r)$ with $r < R_0$, combining the pieces built in previous sections.

Step 1: It suffices to show that the basin $B(\nu)$ has full volume.

By Lemma 2.3.9, to show that $\nu = m$, it suffices to show that $\mathrm{vol}(B(\nu)) = 1$.

Step 2: It suffices to show that the basin has nontrivial density in each small local ball $\mathfrak{B} = B(x_0, r)$.

Note that $\mathrm{vol}(B(\nu)) \geq \mathrm{vol}(A) > 0$ since a full volume subset of A is in the basin $B(\nu)$ by the pointwise ergodic theorem and that $\nu = \frac{1}{m(A)} m|_A$. By Lemma 2.3.10, to show that $B(\nu)$ has full volume, it suffices to show that there exist $c > 0$ and $R_0 > 0$ such that for all $x_0 \in M$ and positive $r < R_0$ that satisfy $\mathrm{vol}(B(x_0, r) \cap B(\nu)) > 0$, we have

$$\mathrm{vol}(B(x_0, r) \cap B(\nu)) > c\, \mathrm{vol}(B(x_0, r)).$$

We choose R_0 as in subsection 2.3.2, and will choose $c = c_3$ from subsection 2.3.2 in **Step 6**. In particular $R_0 < \ell/10$.

In the rest of the proof we fix $x_0 \in M$ and $r \in (0, R_0)$. Let $\mathfrak{B} := B(x_0, r)$.

Step 3: Choose a regular point x in the local ball $\mathfrak{B}$.

By Lemma 2.3.13, the set of regular points in M has full volume. Thus for fixed $x_0 \in M$ and $r < R_0$ with $\mathrm{vol}(B(x_0, r) \cap B(\nu)) > 0$, one can choose a regular point $x \in B(x_0, r) \cap B(\nu) \smallsetminus \{x_0\}$.

Step 4: Choose two words ω, ω' with transverse local stable manifolds in $\mathfrak{B}$.

Choose words $\omega, \omega' \in \mathrm{Diff}^2(M)^{\mathbb{N}}$ as in subsection 2.3.2 and a subset $T' \subset W^s_{\omega,\mathrm{loc}}(x)$ as in subsection 2.3.2.

Let

$$U := \bigcup_{y \in T'} W^s_{\omega',\mathrm{loc}}(y).$$

Step 5: Choose a good set U' with positive density in $\mathfrak{B}$, a word ω'' and a subset $T' \subset T$ with positive leaf-volume.

We choose the good set $U' \subset \mathfrak{B}$, the word ω'' and the subset $T' \subset T$ as in subsection 2.3.2.

Step 6: The good set U' is contained in the basin $B(\nu)$.

Let $p \in U'$. Now we claim that $p \in B(\nu)$. In fact

(i) $x \in B(\nu)$ by the choice in **Step 3**.

(ii) For all $y \in T'$, note that $T' \subset W^s_\omega(x)$ and x, y are ω-regular by Lemma 2.3.15 (1, 5a). Therefore by Lemma 2.3.14, $y \in B(\nu)$.

(iii) Suppose $z \in W^s_{\omega'',\mathrm{loc}}(p) \cap U$ is ω'-regular. By the definition of U, there exists $y \in T'$ such that $z \in W^s_{\omega',\mathrm{loc}}(y)$. Recall that $y \in T'$ is ω'-regular from Lemma 2.3.15 (5a). Therefore by Lemma 2.3.14, $z \in B(\nu)$.

(iv) By Lemma 2.3.17, the set of points in $W^s_{\omega'',\mathrm{loc}}(p) \cap U$ that are ω'-regular and ω''-regular has positive leaf-volume. Let z be one such point. Note that $p \in W^s_{\omega''}(z)$, and p is ω''-regular by Lemma 2.3.16(c). Therefore by Lemma 2.3.14, $p \in B(\nu)$.

Therefore $U' \subset \mathfrak{B} \cap B(\nu)$, hence

$$\mathrm{vol}(\mathfrak{B} \cap B(\nu)) \geq \mathrm{vol}(U') > c_3\, \mathrm{vol}(\mathfrak{B})$$

by Lemma 2.3.16, as desired.

□

Proof of Proposition 2.3.1. Since μ is uniformly expanding, by Proposition 2.2.2, any ergodic μ-stationary measure ν has positive Lyapunov exponent. Hence in the case of volume-preserving diffeomorphisms on surfaces, it is hyperbolic. Now by [BRH17, Thm. 3.4], either ν is finitely supported, the stable distribution is non-random, or ν is the restriction of m to a positive volume subset. By Lemma 2.3.3, the second possibility is eliminated. In the third possibility, by Proposition 2.3.4, we have $\nu = m$. The result follows. □

2.3.3 Comparison with Brown-Rodriguez Hertz

The following proposition may be viewed as a motivation for the assumption of uniform expansion, in view of the theorem [BRH17, Thm. 3.4].

Proposition 2.3.18. Let M be a closed surface, μ be a Borel probability measure on $\mathrm{Diff}^2(M)$. If μ is not uniformly expanding, then there exists an ergodic μ-stationary measure ν on M and a μ-almost surely invariant ν-measurable subbundle $\hat{V} \subset TM$ in which the top Lyapunov exponent is nonpositive.

In particular, if μ is supported on $\mathrm{Diff}^2_m(M)$ for some smooth measure m on M, then μ is uniformly expanding if and only if every ergodic μ-stationary measure ν on M has a positive Lyapunov exponent and the stable distribution is not non-random with respect to ν.

To prove this proposition, we first note that each map $f \in \mathrm{Diff}^2(M)$ induces the projective action on the unit tangent bundle T^1M by

$$f \cdot (x, v) = \left(f(x), \frac{D_x f(v)}{\|D_x f(v)\|} \right).$$

From now on we shall abuse the notation and write $f(x, v) := f \cdot (x, v)$.

In the case that μ is uniformly expanding, we first construct an ergodic stationary measure on T^1M which does not exhibit exponential growth on average.

Lemma 2.3.19. If μ is not uniformly expanding, then there exists an ergodic μ-stationary measure $\overline{\nu}'$ on T^1M such that

$$\iint \log \|D_x f(v)\| d\mu(f) d\overline{\nu}'(x, v) \leq 0.$$

Proof. Fix $\varepsilon > 0$. Since μ is not uniformly expanding, for all positive integer N, there exists $(x_N, v_N) \in T^1M$ such that

$$\int \log \|D_{x_N} f(v_N)\| d\mu^{(N)}(f) < \varepsilon. \tag{2.3.1}$$

Let

$$\nu_N := \frac{1}{N} \sum_{n=0}^{N-1} \int \delta_{f(x_N, v_N)} d\mu^{(n)}(f),$$

and let $\overline{\nu}$ be any weak-* limit point of $\{\nu_N\}$. Note that $\overline{\nu}$ is a μ-stationary measure on T^1M since

$$\mu * \nu_N = \frac{1}{N} \sum_{n=0}^{N-1} \int \mu * \delta_{f(x_N, v_N)} d\mu^{(n)}(f) = \frac{1}{N} \sum_{n=0}^{N-1} \int \delta_{f(x_N, v_N)} d\mu^{(n+1)}(f)$$

and hence as $N \to \infty$,

$$\mu * \nu_N - \nu_N = \frac{1}{N}\left(\int \delta_{f(x_N,v_N)} d\mu^{(N)}(f) - \delta_{(x_N,v_N)}\right) \to 0.$$

For $f \in \mathrm{Diff}^2(M)$ and $(x,v) \in T^1M$, let

$$\Phi(f,(x,v)) := \log \|D_x f(v)\|.$$

Note that for each $N \in \mathbb{N}$ and $\omega = (\omega_0, \omega_1, \omega_2, \ldots) \in \mathrm{Diff}^2(M)^{\mathbb{N}}$,

$$\log \|D_x f_\omega^N(v)\| = \sum_{n=0}^{N-1} \Phi(\omega_n, f_\omega^n(x,v)). \tag{2.3.2}$$

Since the first argument of $\Phi(\omega_n, f_\omega^n(x,v))$ depends only on the $(n+1)$-th coordinate of ω, and the second argument depends only on the first n coordinates of ω, we have

$$\int \log \|D_x f_\omega^N(v)\| d\mu^{\mathbb{N}}(\omega) = \sum_{n=0}^{N-1} \int \Phi(\omega_n, f_\omega^n(x,v)) d\mu^{\mathbb{N}}(\omega) = \sum_{n=0}^{N-1} \int \Phi(g, f(x,v)) d\mu(g) d\mu^{(n)}(f).$$

On the other hand, the left hand side is $\displaystyle\int \log \|D_x f(v)\| d\mu^{(N)}(f)$. Therefore if we set $(x,v) = (x_N, v_N)$, by the definition of ν_N and (2.3.1), for all $N \in \mathbb{N}$,

$$\int\int \Phi(g,(x,v))\, d\mu(g)\, d\nu_N(x,v) < \frac{\varepsilon}{N}.$$

By continuity of Φ and weak-* convergence, we have upon taking limit

$$\int\int \Phi\, d\mu\, d\overline{\nu} \leq 0.$$

Let $\overline{\nu}'$ be an ergodic component of $\overline{\nu}$ such that

$$\int\int \Phi\, d\mu\, d\overline{\nu}' \leq 0,$$

which exists since $\overline{\nu}$ is a convex combination of its ergodic components. This measure $\overline{\nu}'$ satisfies the desired properties. □

Proof of Proposition 2.3.18. Assume that μ is not uniformly expanding. Consider the measure $\overline{\nu}'$ given by Lemma 2.3.19. Let $\nu := \pi_* \overline{\nu}'$, where $\pi : T^1M \to M$ is the natural projection. Then note that ν is an ergodic μ-stationary measure on M since π is equivariant with respect to the action by $\mathrm{Diff}^2(M)$. Let $\{\overline{\nu}'_x\}$ be a family of conditional measures of $\overline{\nu}'$ along the partition of T^1M into fibers over M.

Let F be the skew product map on $\mathrm{Diff}^2(M)^{\mathbb{N}} \times T^1M$ defined by $F(\omega, x) = (\sigma(\omega), \omega_0(x))$. Recall that $\overline{\nu}'$ is an ergodic μ-stationary measure on T^1M if and only if $\mu^{\mathbb{N}} \times \overline{\nu}'$ is an ergodic F-invariant measure on $\mathrm{Diff}^2(M)^{\mathbb{N}} \times T^1M$ ([Kif86, Lem. I.2.3, Thm. I.2.1]). Consider the following map on $\mathrm{Diff}^2(M)^{\mathbb{N}} \times T^1M$,

$$\Psi(\omega, (x, v)) := \log \|D_x\omega_0(v)\|.$$

By the pointwise ergodic theorem, for ν-a.e. $x \in M$ and $\overline{\nu}'_x$-a.e. $v \in T^1_xM$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Psi(\sigma^n(\omega), f^n_\omega(x, v)) = \int\int \Psi \; d\mu^{\mathbb{N}} \; d\overline{\nu}' \qquad \text{for } \mu^{\mathbb{N}}\text{-a.e. } \omega. \tag{2.3.3}$$

Note that since Ψ depends only on the first coordinate of ω, by Lemma 2.3.19,

$$\int\int \Psi \; d\mu^{\mathbb{N}} \; d\overline{\nu}' = \int\int \log \|D_xf(v)\| d\mu(f) \; d\overline{\nu}'(x, v) \leq 0. \tag{2.3.4}$$

Now the support of $\overline{\nu}'$ spans a μ-a.s. invariant ν-measurable subbundle $\hat{V} \subset TM$ (not necessarily proper). Apply (2.3.3) again, we have that the top Lyapunov exponent in $\hat{V}$ is nonpositive.

Finally, to show the second assertion, assume that μ is supported on $\mathrm{Diff}^2_m(M)$ for some smooth measure m on M and μ is not uniformly expanding.

In the volume preserving case, for each ergodic μ-stationary measure ν, either all exponents are zero for ν-a.e. x, or there is a positive and a negative exponent for ν-a.e. x. If all the Lyapunov exponents of ν are zero, we are done. Hence we may assume that ν has a positive exponent. By Oseledets theorem, for $\mu^{\mathbb{N}} \times \nu$-a.e. $(\omega, x) \in \mathrm{Diff}^2(M)^{\mathbb{N}} \times M$, the tangent vectors in T_xM outside of $E^s_\omega(x)$ have exponential growth. Since vectors in $\hat{V}(x)$ have nonpositive top exponent, $\hat{V}(x) \subset E^s_\omega(x)$ for ν-a.e. $x \in M$. Since $E^s_\omega(x)$ is one-dimensional, we have $\hat{V}(x) = E^s_\omega(x)$. Since $\hat{V}$ is μ-a.s. invariant, we have that the stable distribution $E^s_\omega(x)$ is non-random. This shows the "if" direction. The "only if" direction follows from Proposition 2.2.2 and Lemma 2.3.3.

□

2.4 Equidistribution and Orbit closures

We now prove an equidistribution statement from the measure rigidity result using the existence of a Margulis function, which follows from uniform expansion. We follow the strategy in [EMM15], the idea of which goes back to [EM04] and [EMM98]. The orbit closure classification then follows. The assumptions we make in this section are slightly weaker than Theorem D, though Theorem D suffices for the applications in the subsequent sections.

Proposition 2.4.1 (Equidistribution)**.** Let M be a closed surface, $\Gamma \subset \mathrm{Diff}^2(M)$ be a subsemigroup that preserves a smooth measure m on M. Let μ be a uniformly expanding probability measure on $\mathrm{Diff}^2_m(M)$ with $\mu(\Gamma) = 1$ satisfying

$$\int_{\mathrm{Diff}^2(M)} |f|^\delta_{C^2} + |f^{-1}|^\delta_{C^2}\, d\mu(f) < \infty \qquad \text{for all sufficiently small } \delta > 0. \tag{**}$$

Suppose $x \in M$ has infinite Γ-orbit. Then for any continuous function $\varphi \in C(M)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{\mathrm{Diff}^2(M)} \varphi(f(x))\, d\mu^{(k)}(f) = \int_M \varphi\, dm.$$

Clearly if μ is finitely supported, then (**) is satisfied. Also assumption (**) is stronger than (*).

Proposition 2.4.2 (Orbit Closures)**.** Let M be a closed surface, $\Gamma \subset \mathrm{Diff}^2(M)$ be a subsemigroup that preserves a smooth measure m on M. Let μ be a uniformly expanding probability measure on $\mathrm{Diff}^2_m(M)$ with $\mu(\Gamma) = 1$ satisfying (**). Then every orbit of Γ is either finite or dense.

The following lemma shows that if μ is uniformly expanding, then there exists a so-called Margulis function.

Lemma 2.4.3. Suppose μ is a uniformly expanding measure. Then there exists a proper continuous function $u : M \times M \setminus \Delta \to \mathbb{R}_+$, $c < 1$, $b > 0$ and a positive integer n_0 such that for all $(x, y) \in M \times M \setminus \Delta$,

$$\int u(f(x), f(y)) d\mu^{(n_0)}(f) \leq cu(x, y) + b.$$

Proof. The proof is similar to Lemma 10.8 of [Via14]. We can take

$$u(x, y) := d(x, y)^{-\delta},$$

where $\delta \in (0, 1)$ is a small number to be determined. Fix $x \in M$ and $v \in T_x M$. Consider the function

$$\phi_n(\delta) := \int_{\mathrm{Diff}^2(M)} \left(\frac{\|D_x f(v)\|}{\|v\|} \right)^{-\delta} d\mu^{(n)}(f).$$

This is a differentiable function in δ, with

$$\phi_n'(\delta) = -\int_{\mathrm{Diff}^2(M)} \left(\frac{\|D_x f(v)\|}{\|v\|} \right)^{-\delta} \log \left(\frac{\|D_x f(v)\|}{\|v\|} \right) d\mu^{(n)}(f).$$

By uniform expansion, there exists $C > 0$ and $N \in \mathbb{N}$ (independent of x and v) such that

$$\phi_N'(0) = -\int_{\mathrm{Diff}^2(M)} \log \left(\frac{\|D_x f(v)\|}{\|v\|} \right) d\mu^{(N)}(f) < -C.$$

Since $\phi_N(0) = 1$, for small enough $\delta > 0$ (can be chosen independent of x and v using the compactness of M and T^1M), we have

$$\phi_N(\delta) = \int_{\mathrm{Diff}^2(M)} \left(\frac{\|D_x f(v)\|}{\|v\|}\right)^{-\delta} d\mu^{(N)}(f) < 1 - \frac{C\delta}{2}.$$

Take such a δ in the definition of u, and let $n_0 = N$. Then we have

$$\int_{\mathrm{Diff}^2(M)} \left(\frac{\|D_x f(v)\|}{\|v\|}\right)^{-\delta} d\mu^{(n_0)}(f) < 1 - \frac{C\delta}{2}.$$

Let $c = 1 - C\delta/4$. Take $\varepsilon > 0$ small enough such that for all $x, y \in M \times M \setminus \Delta$ with $d(x,y) < \varepsilon$,

$$\int \frac{d(f(x), f(y))^{-\delta}}{d(x,y)^{-\delta}} d\mu^{(n_0)}(f) < 1 - \frac{C\delta}{4} = c.$$

For $0 < d(x,y) < \varepsilon$,

$$\int u(f(x), f(y)) d\mu^{(n_0)}(f) = \int d(f(x), f(y))^{-\delta} d\mu^{(n_0)}(f) < cd(x,y)^{-\delta} = cu(x,y).$$

Now using the moment condition (**) (take a smaller $\delta > 0$ if necessary), we can take some $b > 0$ so that for all $x, y \in M$ with $d(x,y) \geq \varepsilon$,

$$\int d(f(x), f(y))^{-\delta} d\mu^{(n_0)}(f) \leq b.$$

Hence for all $(x,y) \in M \times M \setminus \Delta$,

$$\int u(f(x), f(y)) d\mu^{(n_0)}(f) \leq cu(x,y) + b.$$

□

Corollary 2.4.4. Suppose μ is a uniformly expanding measure and $\mathcal{N} \subset M$ is a finite Γ-orbit. Then there exists a proper continuous function $f_{\mathcal{N}} : M \setminus \mathcal{N} \to \mathbb{R}_+$, $c < 1$, $b > 0$ and a positive integer n_0 such that for all $x \in M \smallsetminus \mathcal{N}$,

$$\int f_{\mathcal{N}}(f(x))d\mu^{(n_0)}(f) \leq c f_{\mathcal{N}}(x) + b.$$

Here c and b depend only on the size of $\mathcal{N}$. Moreover, for each $x \in M \smallsetminus \mathcal{N}$, there exists a positive integer $n(x)$ such that for all $n > n(x)$,

$$(\mu^{(n)} * \delta_x)(f_{\mathcal{N}}) = \int f_{\mathcal{N}}(f(x))d\mu^{(n)}(f) \leq b_1,$$

where $b_1 = b_1(b, c)$. For each compact subset $F \subset M \smallsetminus \mathcal{N}$, we can take $n(x)$ such that $\sup_{x \in F} n(x) < \infty$.

Proof. Let $u : M \times M \smallsetminus \Delta \to \mathbb{R}_+$ be the function as in Lemma 2.4.3 with the corresponding $c < 1$ and $b > 0$, and define the function $f_{\mathcal{N}} : M \smallsetminus \mathcal{N} \to \mathbb{R}$ by

$$f_{\mathcal{N}}(x) := \frac{1}{|\mathcal{N}|} \sum_{y \in \mathcal{N}} u(x, y).$$

Take the positive integer n_0 as in Lemma 2.4.3. Then for all $x \in M \smallsetminus \mathcal{N}$,

$$\begin{aligned}\int f_{\mathcal{N}}(f(x))d\mu^{(n_0)}(f) &= \frac{1}{|\mathcal{N}|} \int \sum_{y \in \mathcal{N}} u(f(x), y)d\mu^{(n_0)}(f) \\ &= \frac{1}{|\mathcal{N}|} \int \sum_{y \in \mathcal{N}} u(f(x), f(y))d\mu^{(n_0)}(f) \leq c f_{\mathcal{N}}(x) + b.\end{aligned}$$

Here we used that $\mathcal{N}$ is Γ-invariant in the second equality. This gives the first assertion.

For the second assertion, from the above, for all positive integer k and $x \in M \smallsetminus \mathcal{N}$,

$$(\mu^{(kn_0)} * \delta_x)(f_{\mathcal{N}}) = \int f_{\mathcal{N}}(f(x))d\mu^{(kn_0)}(f) \leq c^k f_{\mathcal{N}}(x) + \frac{b}{1-c}.$$

Therefore for all $n \geq 0$,

$$(\mu^{(n)} * \delta_x)(f_{\mathcal{N}}) = \int f_{\mathcal{N}}(f(x))d\mu^{(n)}(f) \leq c^{\lfloor n/n_0 \rfloor}\mu^{(i)} * \delta_x(f_{\mathcal{N}}) + \frac{b}{1-c},$$

where $i := n - n_0\lfloor n/n_0 \rfloor < n_0$. Now for any compact $F \subset M \smallsetminus \mathcal{N}$, there exists some positive integer m_F such that for all $n > m_F$,

$$c^{\lfloor n/n_0 \rfloor}\mu^{(i)} * \delta_x(f_{\mathcal{N}}) < \frac{b}{1-c} \qquad \text{for all} \quad 0 \leq i \leq n_0, \quad x \in F.$$

Then for any $n > m_F$ and $x \in F$,

$$(\mu^{(n)} * \delta_x)(f_{\mathcal{N}}) \leq \frac{2b}{1-c} =: b_1.$$

□

Corollary 2.4.5. Suppose μ is a uniformly expanding measure and $\mathcal{N} \subset M$ is a finite Γ-orbit. Take $f_{\mathcal{N}}, c, b$ as in Corollary 2.4.4. Suppose ν is an ergodic μ-stationary measure on M with $\nu(\{f_{\mathcal{N}} < \infty\}) > 0$. Then

$$\int f_{\mathcal{N}}(x)d\nu(x) \leq B,$$

where B depends only on b, c.

Proof. For each positive integer n, let $f_{\mathcal{N},n} := \min\{f_{\mathcal{N}}, n\}$. By the Birkhoff ergodic theorem, for $\mu^{\mathbb{N}} \times \nu$-a.e. $(\omega, x) \in \Gamma^{\mathbb{N}} \times M$,

$$\lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} f_{\mathcal{N},n}(f_\omega^k(x)) = \int f_{\mathcal{N},n}(x)d\nu(x),$$

where for $\omega = (\omega_0, \omega_1, \dots) \in \Gamma^{\mathbb{N}}$, $f_\omega^k := \omega_{k-1} \circ \omega_{k-2} \circ \cdots \circ \omega_0$. Pick a point $x_0 \in M \smallsetminus \mathcal{N}$ such that the convergence holds for $\mu^{\mathbb{N}}$-a.e. $\omega \in \Gamma^{\mathbb{N}}$ (note that we can pick such $x_0 \notin \mathcal{N}$ since $\nu(\{f_{\mathcal{N}} < \infty\}) = \nu(M \smallsetminus \mathcal{N}) > 0$). By Egorov's theorem, we can take a subset $\Gamma' \subset \Gamma^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\Gamma') \geq 1/2$ such that at $x = x_0$, the convergence is uniform on $\omega \in \Gamma'$. Then there exists a positive integer m_n such that for all $m > m_n$ and $\omega \in \Gamma'$,

$$\frac{1}{m}\sum_{k=1}^{m} f_{\mathcal{N},n}(f_\omega^k(x_0)) \geq \frac{1}{2}\int f_{\mathcal{N},n}(x)d\nu(x).$$

Integrating over $\omega \in \Gamma^{\mathbb{N}}$, we have for all $m > m_n$,

$$\frac{1}{m}\sum_{k=1}^{m}\int f_{\mathcal{N},n}(f(x_0))d\mu^{(k)}(f) \geq \frac{1}{4}\int f_{\mathcal{N},n}(x)d\nu(x).$$

By Corollary 2.4.4, for large enough m, the left hand side is at most some constant $B' = B'(b,c)$. Therefore for all n,

$$\int f_{\mathcal{N},n}(x)d\nu(x) \leq 4B'.$$

Taking the limit $n \to \infty$, we have the assertion. □

Proposition 2.4.6. The number of points with finite Γ-orbit is countable.

Proof. It suffices to show that for each positive integer n, there are finitely many Γ-orbits of size n. Suppose the contrary that there are infinitely many Γ-orbits of size n. Then by compactness of M, they have an accumulation point $x \in M$, hence there exists a sequence of points $x_i \in M$ with finite Γ-orbit of size n such that $d(x_i, x_{i+1}) \to 0$ as $i \to \infty$. Fix an $\varepsilon = \varepsilon(B, n, \delta) > 0$ (to be determined later), and a large enough j such that $d(x_j, x_{j+1}) < \varepsilon$. By deleting finitely many points from the sequence if necessary, we may assume x_j and x_{j+1} are in different Γ-orbits. For each $i \in \mathbb{N}$, let ν_i be the ergodic Γ-invariant (hence μ-stationary) measure on M supported on the Γ-orbit $\mathcal{N}_i$ of x_i with uniform distribution, i.e. $\nu_i(x) = 1/n$ for each $x \in \mathcal{N}_i$, and let $f_i := f_{\mathcal{N}_i}$ be the function defined in Corollary 2.4.4 with the corresponding upper bound $B = B(b,c)$ as in Corollary 2.4.5. As $x_{j+1} \notin \mathcal{N}_j$, $f_j(x_{j+1}) < \infty$. Hence $\nu_{j+1}(f_j < \infty) \geq 1/n > 0$. Therefore by Corollary 2.4.5,

$$\int f_j(x)d\nu_{j+1}(x) \leq B. \tag{***}$$

On the other hand, recall from definition that $f_j(x) = \frac{1}{|\mathcal{N}_j|}\sum_{y\in\mathcal{N}_j} u(x,y)$ where $u(x,y) =$

$d(x,y)^{-\delta}$ for some $\delta > 0$ chosen in the proof of Lemma 2.4.3. Thus

$$\int f_j(x) d\nu_{j+1}(x) = \frac{1}{n^2} \sum_{x \in \mathcal{N}_{j+1}} \sum_{y \in \mathcal{N}_j} u(x,y) \geq \frac{1}{n^2} u(x_{j+1}, x_j) > \frac{1}{n^2} \varepsilon^{-\delta}.$$

Taking ε small enough such that $\varepsilon^{-\delta} \geq 2Bn^2$, this leads to a contradiction to (***). □

Define

$$\overline{\mu}^{(n)} := \frac{1}{n} \sum_{k=1}^{n} \mu^{(k)}.$$

Lemma 2.4.7. Let $\mathcal{N}$ be a finite Γ-orbit in M. The for any $\varepsilon > 0$, there exists an open set $\Omega_{\mathcal{N},\varepsilon}$ containing $\mathcal{N}$ with $(\Omega_{\mathcal{N},\varepsilon})^c$ compact such that for any compact $F \subset M \smallsetminus \mathcal{N}$ there exists a positive integer n_F, such that for all $x \in F$ and $n > n_F$, we have

$$(\overline{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N},\varepsilon}) < \varepsilon.$$

Proof. The proof follows that of Proposition 3.3 in [EMM15]. Take the function $f_{\mathcal{N}} : M \smallsetminus \mathcal{N} \to \mathbb{R}_+$ as in Corollary 2.4.4 with the corresponding $c < 1$, $b > 0$ and positive integer n_0. Let

$$\Omega_{\mathcal{N},\varepsilon} := \left\{ x \in M : f_{\mathcal{N}}(x) > \frac{1}{\varepsilon} \left(\frac{2b}{1-c} + 1 \right) \right\}.$$

By Corollary 2.4.4, for each compact subset $F \subset M \smallsetminus \mathcal{N}$, there exists $b_1 = 2b/(1-c) > 0$ and positive integer m_F such that for all $n > m_F$ and $x \in F$,

$$(\mu^{(n)} * \delta_x)(f_{\mathcal{N}}) \leq b_1.$$

Therefore there exists a positive integer $n_F \geq m_F$ such that for all $n > n_F$ and $x \in F$,

$$(\overline{\mu}^{(n)} * \delta_x)(f_{\mathcal{N}}) \leq b_1 + 1.$$

Thus for all $n > n_F$, $x \in F$ and $L > 0$, we have

$$(\overline{\mu}^{(n)} * \delta_x)(\{p \in M : f_{\mathcal{N}}(p) > L\}) < \frac{b_1 + 1}{L}.$$

Therefore by the choice of $\Omega_{\mathcal{N},\varepsilon}$, we know that $(\overline{\mu}^{(n)} * \delta_x)(\Omega_{\mathcal{N},\varepsilon}) < \varepsilon$. Moreover, it is clear from the definition of $f_{\mathcal{N}}$ and the choice of u in Lemma 2.4.3 that

$$(\Omega_{\mathcal{N},\varepsilon})^c = \left\{ x \in M : f_{\mathcal{N}}(x) \le \frac{1}{\varepsilon}\left(\frac{2b}{1-c} + 1\right)\right\}$$

is compact. □

Proof of Proposition 2.4.1. Assume that the conclusion of the assertion does not hold. Then there exists $\varphi \in C(M)$, $\varepsilon > 0$, $x \in M$ with infinite Γ-orbit and a subsequence $n_k \to \infty$ such that

$$|(\overline{\mu}^{(n_k)} * \delta_x)(\varphi) - m(\varphi)| \ge \varepsilon.$$

By compactness of the space of probability measures on M with the weak-* topology, we may assume that $\overline{\mu}^{(n_k)} * \delta_x \to \nu$ for some probability measure ν.

First note that ν is a μ-stationary measure. By Proposition 2.4.6, there are at most countably many finite Γ-orbits. Therefore by Proposition 2.3.1, we have the ergodic decomposition of ν:

$$\nu = \sum_{\mathcal{N} \subset M} a_{\mathcal{N}} \nu_{\mathcal{N}} + am,$$

where the sum is over all finite Γ-orbit $\mathcal{N}$. Here $a, a_{\mathcal{N}} \in [0,1]$, and $\nu_{\mathcal{N}}$ is the probability measure supported on the finite Γ-orbit $\mathcal{N}$ with uniform distribution. It remains to show that $a_{\mathcal{N}} = 0$ for all finite Γ-orbit $\mathcal{N}$.

For each finite Γ-orbit $\mathcal{N}$, as $x \notin \mathcal{N}$ by assumption, we may apply Lemma 2.4.7 with $\mathcal{N}$ and compact $F = \{x\}$. Then for any $\varepsilon > 0$, there exists a positive integer n_x such that for all $n > n_x$, $(\overline{\mu}^{(n)} * \delta_x)((\Omega_{\mathcal{N},\varepsilon})^c) \ge 1 - \varepsilon$. Passing to the limit along the subsequence $n_k \to \infty$, we have

$$\nu((\Omega_{\mathcal{N},\varepsilon})^c) \geq 1 - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we have $\nu(\mathcal{N}) = 0$. Hence $a_{\mathcal{N}} \leq \nu(\mathcal{N}) = 0$. □

Proof of Proposition 2.4.2. This is an immediate consequence of Proposition 2.4.1, as every nonempty open subset of M has positive volume. □

2.5 Geometric interpretation of uniform expansion

In the rest of the paper, we study how to verify uniform expansion in concrete settings. In this section, we give a geometric perspective of uniform expansion by visualizing it on the hyperbolic disk.

2.5.1 Cartan decomposition and hyperbolic geometry

Let $F \in SL_2(\mathbb{R})$. Throughout we identify the real projective line $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{R})$ with $\mathbb{R}/\pi\mathbb{Z}$ as metric spaces, i.e. we identify each line in $\mathbb{R}^2$ through the origin with the angle it makes with the positive horizontal axis. Recall that the Cartan decomposition of F is given by

$$F = r_{-\varphi} a_\lambda r_\theta, \qquad \text{where } \; r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; \text{ and } \; a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for some $\lambda \geq 1$ and $\varphi, \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Moreover,

$$\lambda = \|F\| := \sup_{v \in \mathbb{R}^2 \setminus \{0\}} \frac{\|Fv\|}{\|v\|}$$

is the (operator) *norm* of the matrix F. We remark that if $\lambda = \|F\| > 1$, then φ and θ are uniquely defined modulo π, i.e. correspond to a unique element in $\mathbb{P}^1$. We call $\theta \in \mathbb{P}^1$ *the* *expanding direction* of F since

$$\|F(\theta)\| = \sup_{\theta' \in \mathbb{P}^1} \|F(\theta')\| = \lambda,$$

where $F(\theta)$ is the vector $F \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. It is easy to see that if we let $\theta_F := \theta + \pi/2 \in \mathbb{P}^1$, then

$$\|F(\theta_F)\| = \inf_{\theta' \in \mathbb{P}^1} \|F(\theta')\| = \lambda^{-1}.$$

Hence for $\|F\| > 1$, we call $\theta_F = \theta + \pi/2 \in \mathbb{P}^1$ the *contracting direction* of F. Notice also that $\varphi \in \mathbb{P}^1$ and $\varphi + \pi/2 \in \mathbb{P}^1$ are the contracting and expanding directions of F^{-1}.

In certain computation we find it helpful to have an explicit formula to compute the contraction direction and the norm given the matrix $F \in SL_2(\mathbb{R})$. This is given by the following simple lemma.

Lemma 2.5.1. Let $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ with $\|F\| > 1$. Then

(a) the contracting direction $\theta_F \in \mathbb{P}^1$ satisfies

$$\tan 2\theta_F = \frac{2(ab + cd)}{a^2 + c^2 - b^2 - d^2},$$

here we follow the convention that $1/0 = \infty$ and that $\tan\varphi = \infty$ implies $\varphi = \pi/2 \in \mathbb{P}^1$.

(b) The norm $\lambda := \|F\|$ satisfies

$$\lambda^2 + \lambda^{-2} = a^2 + b^2 + c^2 + d^2.$$

In particular, if $a^2 + b^2 + c^2 + d^2 \gg 1$, then

$$\lambda \sim \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Proof. Part (a) is a straightforward computation by considering the function

$$f(\theta) := \|F(\theta)\|^2 = \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\|^2 .$$

Notice that for $\|F\| > 1$, f is not a constant function, and the expanding and contracting directions are precisely the critical points of f, i.e. when $f'(\theta) = 0$.

For part (b), we observe that

$$\mathrm{tr}(F^T F) = a^2 + b^2 + c^2 + d^2.$$

On the other hand, if we write $F = r_{-\varphi} a_\lambda r_\theta$, then

$$F^T F = (r_{-\theta} a_\lambda r_\varphi)(r_{-\varphi} a_\lambda r_\theta) = r_{-\theta} a_\lambda^2 r_\theta.$$

Hence its trace equals $\lambda^2 + \lambda^{-2}$. $\square$

We also find it helpful to think of each $F \in SL_2(\mathbb{R})$ as a point of the unit tangent bundle of the hyperbolic plane in the disk model $T^1\mathbb{D}$, using the identification $T^1\mathbb{D} \leftrightarrow PSL_2(\mathbb{R}) := SL_2(\mathbb{R})/\{\pm I\}$ (Figure 2.2). Recall that the group $PSL_2(\mathbb{R})$ is the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, which can be identified isometrically with the hyperbolic disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ via the map $z \mapsto (z - i)/(z + i)$. $PSL_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $T^1\mathbb{D}$, hence one can identify $PSL_2(\mathbb{R})$ with $T^1\mathbb{D}$ so that the identity element e corresponds to the unit vector based at the origin pointing rightward. Moreover the identification is such that the isometry g on $T^1\mathbb{D}$

corresponds to the *right* multiplication by the inverse g^{-1} on $PSL_2(\mathbb{R})$. We visualize the base point on the disk model $\mathbb{D} \leftrightarrow SO(2)\backslash SL_2(\mathbb{R})$. For instance, the matrix $F = r_{-\varphi}a_\lambda r_\theta \in SL_2(\mathbb{R})$ corresponds to the point $P_F \in \mathbb{D}$ with polar coordinates $(2\log\lambda, 2\theta)$ (the first coordinate measured in hyperbolic distance) and the unit tangent vector with angle $2(\theta - \varphi)$ from the positive real axis.

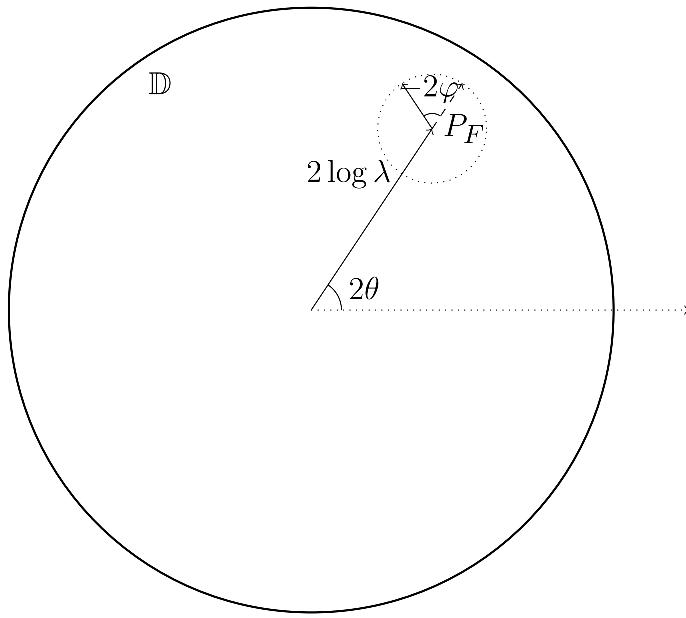


Figure 2.2: The matrix $F = r_{-\varphi}a_\lambda r_\theta \in SL_2(\mathbb{R})$ in the hyperbolic disk

Hence one can read off the norm of F from the distance between P_F and the origin, and read off the contracting direction from the angle from the positive axis.

Now we relate this picture with uniform expansion. From now on, we assume that μ is finitely supported, so that the uniform expansion condition reduces to a finite sum. For simplicity, for the moment we also assume that the maps in the support of μ have the same mass. Let $\Omega := \{f_1, f_2, \ldots, f_d\} \subset \mathrm{Diff}^2(M)$ be the support of μ. Then μ is uniformly expanding if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and $v \in T_xM$,

$$\sum_{\omega\in\Omega^N} \log \frac{\|D_x f_\omega^N(v)\|}{\|v\|} > C.$$

Here we recall that for $\omega = (\omega_1, \omega_2, \ldots, \omega_N) \in \Omega^N$ and $1 \le i \le N$, $f_\omega^i := \omega_i \circ \omega_{i-1} \circ \cdots \circ \omega_1$.

Note that by picking a measurably varying basis for the tangent bundle TM, we can identify $D_x f_\omega^N$ as an element in $SL_2(\mathbb{R})$. Note that if $\theta \in \mathbb{P}^1$ is the contracting direction of $D_x f_\omega^N$, then $\log \|D_x f_\omega^N(\theta)\| < 0$. In particular if for some $x \in M$, $\theta \in \mathbb{P}^1$ is close to the contracting direction of $D_x f_\omega^N$ for many words $\omega \in \Omega^N$, then uniform expansion cannot hold. Hence verifying uniform expansion amounts to checking that the contracting directions of $D_x f_\omega^N$ are "spread out" enough. On the hyperbolic disk, for each $x \in M$, we can draw the matrices $D_x f_\omega^N$ as endpoints of a tree from the origin, where each node with graph distance i from the origin corresponds to a matrix $D_x f_\omega^i$ (Figure 2.3, the dashed lines indicate the contracting directions of $D_x f_\omega^N$ for $N = 3$). Hence verifying uniform expansion reduces to studying the geometry of the contracting directions.

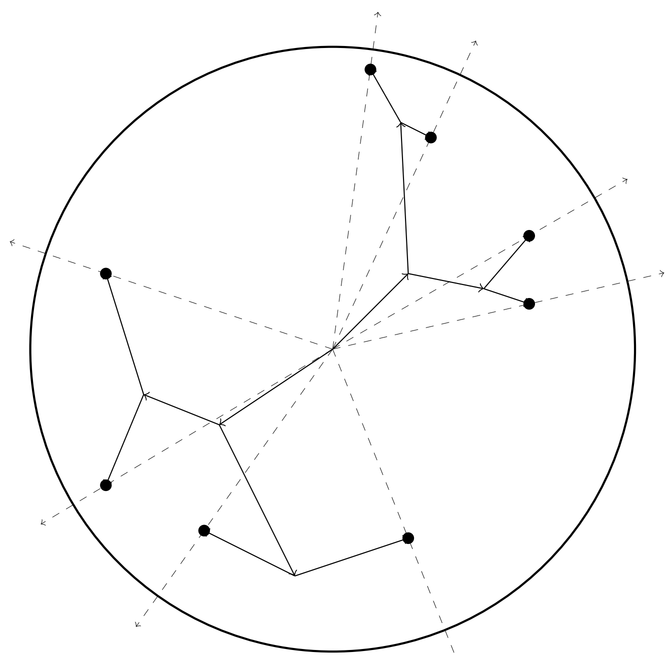


Figure 2.3: The tree representing the random walk after 3 steps

2.5.2 Estimates on changes of the contracting directions

The following lemma provides a lower bound on the expansion of a given matrix $F \in SL_2(\mathbb{R})$ in the direction θ, depending on the norm of F and how far θ is from the contracting direction of F.

Lemma 2.5.2. For all $F \in SL_2(\mathbb{R})$ with norm $\|F\| > 1$ and contracting direction $\theta_F \in \mathbb{P}^1$, we have

$$\|F(\theta)\| \geq \frac{2}{\pi}\|F\| \cdot d(\theta, \theta_F) \qquad \text{for all } \theta \in \mathbb{P}^1.$$

Here we recall that the metric d on $\mathbb{P}^1$ is given by the identification $\mathbb{P}^1 \leftrightarrow \mathbb{R}/\pi\mathbb{Z}$.

Proof. By the Cartan decomposition one may assume that F is a diagonal matrix with entries λ and λ^{-1}, with $\lambda = \|F\|$. The lemma now follows from a direct calculation. □

For matrices $M_1, M_2 \in SL_2(\mathbb{R})$, the following lemma shows that if M_2 has large norm λ_2, then as long as the contracting direction of M_1 is far away from the contracting direction of M_2^{-1}, as we vary the contracting direction of M_1, the contracting direction of the product M_1M_2 changes by $1/\lambda_2^2$ of that amount.

Lemma 2.5.3. Let $M_1, M_2 \in SL_2(\mathbb{R})$. Let $\lambda_i = \|M_i\| > 1$ for $i = 1, 2$ and $\varphi = \theta_{M_1} + \pi/2 - \theta_{M_2^{-1}}$, i.e. φ is the distance between the contracting direction of M_1 and the expanding direction of M_2^{-1}.

(a) If $\|M_1M_2\| > 1$, then

$$\frac{d\theta_{M_1M_2}}{d\theta_{M_2}} = 1,$$

where we treat $\theta_{M_1M_2}$ as a function of θ_{M_2} by fixing M_1, $\theta_{M_2^{-1}}$ and λ_2.

(b) If $\lambda_2 \gg 1$ and $d(\varphi, \pi/2) \gtrsim \lambda_2^{-1}$, then

$$\frac{d\theta_{M_1M_2}}{d\theta_{M_1}} \sim \frac{2(1 + k\cos 2\varphi)}{(k + \cos 2\varphi)^2}\frac{1}{\lambda_2^2}, \qquad \text{where} \qquad k = \frac{\lambda_1^2 + \lambda_1^{-2}}{\lambda_1^2 - \lambda_1^{-2}} = 1 + \frac{2}{\lambda_1^4 - 1}.$$

Here we treat $\theta_{M_1M_2}$ as a function of θ_{M_1} by fixing $\theta_{M_1^{-1}}$, λ_1 and M_2. Furthermore, if $\lambda_1 \gg 1$ and $d(\varphi, \pi/2) \gtrsim \lambda_1^{-1}$ as well, then

$$\frac{d\theta_{M_1M_2}}{d\theta_{M_1}} \sim \frac{2}{(1 + \cos 2\varphi)}\frac{1}{\lambda_2^2}.$$

Proof. For (a), write M_2 in its Cartan decomposition $M_2 = r_{-\varphi_2} a_{\lambda_2} r_{\theta_2}$, and write $M_1 r_{-\varphi_2} a_{\lambda_2}$ in its Cartan decomposition

$$M_1 r_{-\varphi_2} a_{\lambda_2} = r_{-\varphi'} a_{\lambda'} r_{\theta'}.$$

Then

$$M_1 M_2 = M_1 r_{-\varphi_2} a_{\lambda_2} r_{\theta_2} = r_{-\varphi'} a_{\lambda'} r_{\theta'+\theta_2}.$$

By the uniqueness of the Cartan decomposition (up to $\pm I$), we have $\theta_{M_1M_2} = \theta_{M_2} + \theta'$, where θ' depends only on M_1, $\varphi_2 = \theta_{M_2^{-1}}$ and λ_2, hence the result of (a). This statement can be visualized on the hyperbolic disk (Figure 2.4).

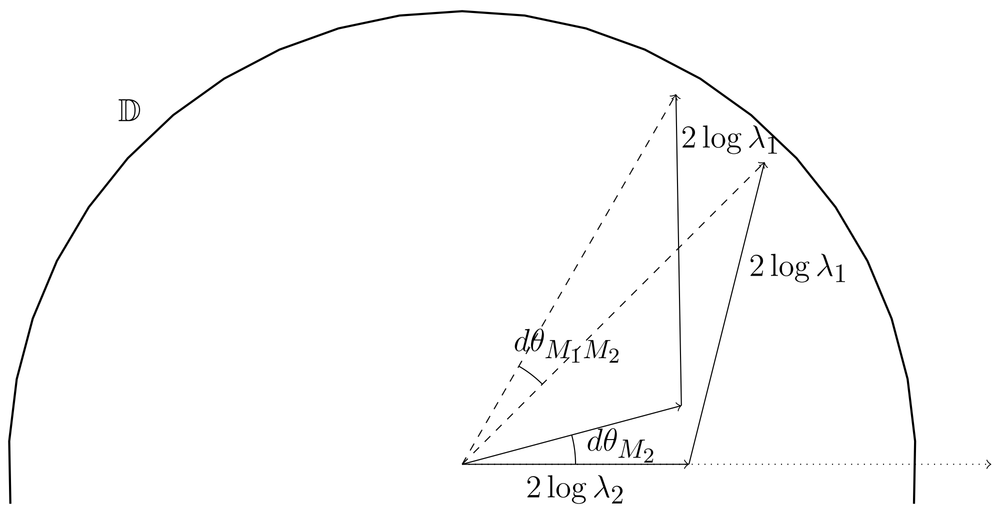


Figure 2.4: The change of $\theta_{M_1M_2}$ as θ_{M_2} varies.

For (b), the assumptions $d(\varphi, \pi/2) \gtrsim \lambda_2^{-1}$ and $\lambda_1, \lambda_2 > 1$ imply that $\|M_1M_2\| > 1$. Thus $\theta_{M_1M_2}$ is well-defined. By applying the Cartan decomposition, we may, without loss of generality, assume that $\theta := \theta_{M_1M_2}$ is the contracting direction of

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}.$$

Note that

$$\frac{d\theta_{M_1M_2}}{d\theta_{M_1}} = \frac{d\theta}{d\varphi}.$$

The statement can be illustrated on the hyperbolic disk (Figure 2.5).

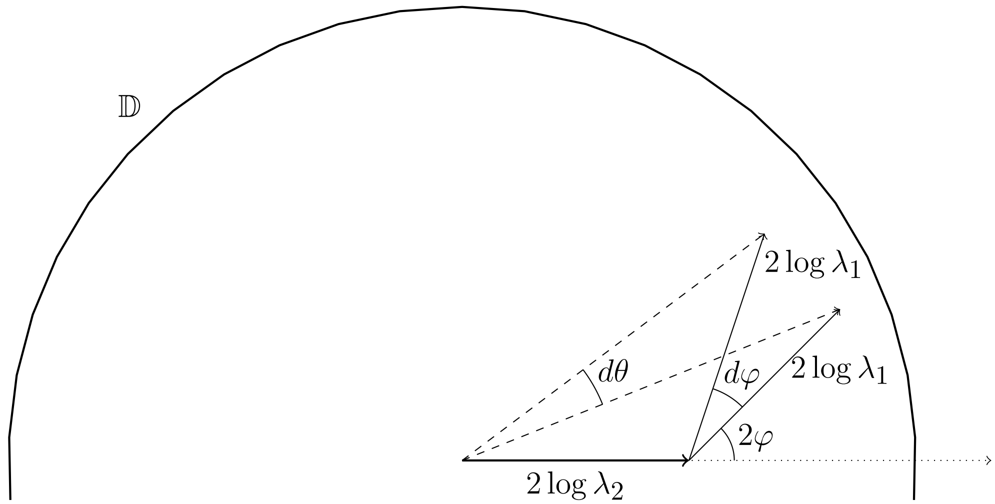


Figure 2.5: The change of θ as φ varies.

Using Lemma 2.5.1(a), one computes directly that

$$\cot 2\theta = \frac{1}{2}(\lambda_2^2 + \lambda_2^{-2})\cot 2\varphi + \frac{1}{2}\frac{(\lambda_1^2+\lambda_1^{-2})(\lambda_2^2-\lambda_2^{-2})}{\lambda_1^2-\lambda_1^{-2}}\csc 2\varphi.$$

Hence upon taking derivative, one gets

$$\frac{d\theta}{d\varphi} = \frac{\frac{(\lambda_1^2+\lambda_1^{-2})(\lambda_2^2-\lambda_2^{-2})}{\lambda_1^2-\lambda_1^{-2}}\cos 2\varphi + (\lambda_2^2+\lambda_2^{-2})}{2\sin^2 2\varphi + \frac{1}{2}\left(\frac{(\lambda_1^2+\lambda_1^{-2})(\lambda_2^2-\lambda_2^{-2})}{\lambda_1^2-\lambda_1^{-2}} + (\lambda_2^2+\lambda_2^{-2})\cos 2\varphi\right)^2}.$$

Thus for $\lambda_2 \gg 1$, let $k = (\lambda_1^2+\lambda_1^{-2})/(\lambda_1^2-\lambda_1^{-2})$, then

$$\frac{d\theta}{d\varphi} \sim \frac{2(1+k\cos 2\varphi)}{(k+\cos 2\varphi)^2}\frac{1}{\lambda_2^2}.$$

In addition, by taking $\lambda_1 \gg 1$, we have $k \sim 1$, so

$$\frac{d\theta}{d\varphi} \sim \frac{2}{(1+\cos 2\varphi)}\frac{1}{\lambda_2^2}.$$

It is clear from Figure 2.5 that when φ is close to $\pi/2$, the random walk “backtracks” towards the origin, so we do not expect a good estimate on $d\theta/d\varphi$. $\square$

2.5.3 A general criterion for uniform expansion

We finish this section with a sufficient condition for uniform expansion on one step of the random dynamics. As mentioned in the introduction, this criterion illustrates that overlap of contraction directions and maps close to rotations are essentially the two obstructions to uniform expansion. Even though we will not use this criterion in the rest of the paper, one may consider the verification in the next few sections as proving a more refined version of Proposition 2.5.4 (depending on the specific features of each application) and the verification of this more refined criterion.

Given $F \in SL_2(\mathbb{R})$, recall that we define $\lambda_F := \|F\|$ to be the norm of F with $\lambda_F > 1$, and $\theta_F \in \mathbb{P}^1$ to be the contracting direction.

Proposition 2.5.4. For all $\lambda_{\text{crit}} > 0$, $\lambda_{\max} > 0$ and small enough $\varepsilon > 0$ satisfying $\frac{1}{\sin\varepsilon}\sqrt{2+\frac{1}{\varepsilon}} < \lambda_{\text{crit}} \le \lambda_{\max}$, there exists $\eta = \eta(\lambda_{\text{crit}}, \lambda_{\max}, \varepsilon) \in (0, 1)$ such that if for all $(x, \theta) \in T^1M$,

$$\mu(\{f : d(\theta_{D_xf}, \theta) > \varepsilon \text{ and } \lambda_{D_xf} > \lambda_{\text{crit}}\}) > \eta, \quad \text{and} \quad \lambda_{D_xf} \le \lambda_{\max} \quad \text{for } \mu\text{-a.s. } f,$$

then μ is uniformly expanding. Furthermore, η can be made explicit.

We think of ε as measuring the separation of the contracting directions at each point $x \in M$, λ_{crit} as measuring how far D_xf is from a rotation, and $\lambda_{\max}$ as the maximum norm over all the points $x \in M$ and all the possible maps f in the support of μ.

The idea of the proposition is that if at every point, the contracting directions of the diffeomorphisms are spread out enough and most of the diffeomorphisms are far from being a rotation, then with high probability the random walk does not backtrack. Lemma 2.5.3(b) and the next two lemma then tell us that the contracting directions of the random walk will eventually be spread out as well. In this case, as long as none of the norms dominate the others (bounded by $\lambda_{\max}$), we can obtain uniform expansion. In particular, as we will see, η is an increasing function of $\lambda_{\max}$ and a decreasing function of λ_{crit} and ε.

Lemma 2.5.5. Fix $m > 1$. Let $M_1, M_2 \in SL_2(\mathbb{R})$. Let $\lambda := \|M_1\| > 1$ and $\tau := \|M_2\| > 1$ be the norm of M_1 and M_2, $\varphi = \theta_{M_1} + \pi/2 - \theta_{M_2^{-1}}$ be the difference between the contracting direction of M_1 and the expanding direction of M_2^{-1}. Then the norm of the product $M_1 M_2$ is at least $\lambda\tau/m$ if and only if

$$\cos 2\varphi \geq \frac{2((\lambda\tau/m)^2 + (\lambda\tau/m)^{-2})}{(\lambda^2 - \lambda^{-2})(\tau^2 - \tau^{-2})} - \frac{\lambda^2 + \lambda^{-2}}{\lambda^2 - \lambda^{-2}} \cdot \frac{\tau^2 + \tau^{-2}}{\tau^2 - \tau^{-2}}.$$

In particular, if $\lambda > \sqrt{m}$, $\tau > \sqrt{m}$ and $|\cos\varphi| \geq 1/m$, then the norm of $M_1 M_2$ is at least $\lambda\tau/m$.

Proof. The first equivalence is a calculation using the Cartan decomposition. Note that the norm λ of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ is the unique root of

$$\lambda^2 + \lambda^{-2} = a^2 + b^2 + c^2 + d^2$$

with $\lambda \geq 1$. In particular λ is an increasing function of $a^2 + b^2 + c^2 + d^2$. Now the norm of $M_1 M_2$ is the same as that of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} = \begin{pmatrix} \lambda\tau\cos\varphi & \lambda\tau^{-1}\sin\varphi \\ -\lambda^{-1}\tau\sin\varphi & \lambda^{-1}\tau^{-1}\cos\varphi \end{pmatrix}.$$

Thus $\|M_1 M_2\| \geq \lambda\tau/m$ if and only if

$$(\lambda\tau\cos\varphi)^2 + (\lambda\tau^{-1}\sin\varphi)^2 + (\lambda^{-1}\tau\sin\varphi)^2 + (\lambda^{-1}\tau^{-1}\cos\varphi)^2 \geq \left(\frac{\lambda\tau}{m}\right)^2 + \left(\frac{\lambda\tau}{m}\right)^{-2}. \quad (2.5.1)$$

Rearranging (2.5.1) gives the first assertion. Finally, the left hand side of (2.5.1) is an increasing function of $\cos^2\varphi$ for $\lambda > 1$ and $\tau > 1$. One can verify directly that (2.5.1) holds when $\lambda > \sqrt{m}$,

$\tau > \sqrt{m}$, $\cos^2\varphi = 1/m^2$, therefore it also holds for $\cos^2\varphi \geq 1/m^2$. □

The next lemma controls the contracting direction of M_1M_2 assuming no backtracking.

Lemma 2.5.6. Fix $m > 1$ large (an explicit lower bound will be obtained in the proof). Let $M_1, M_2 \in SL_2(\mathbb{R})$. Let $\lambda := \|M_1\| > 1$ and $\tau := \|M_2\| > 1$, $\varphi = \theta_{M_1} + \pi/2 - \theta_{M_2^{-1}} \in \mathbb{P}^1 = \mathbb{R}/\pi\mathbb{Z}$ as in the previous lemma. If $|\cos\varphi| \geq 1/m$ and $\tau \geq m$,

$$d(\theta_{M_2}, \theta_{M_1M_2}) \leq \frac{m^2}{\tau^2}.$$

If we further assume that $\tau \geq \sqrt{2}m$, the conclusion holds for all $m > 1$.

Proof. Note that if $\varphi = 0$, $d(\theta_{M_2}, \theta_{M_1M_2}) = 0$. Therefore we need to give an upper bound on the increment of $\theta_{M_1M_2}$ as we vary φ within the given range. Again by the Cartan decomposition, it suffices to consider the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} = \begin{pmatrix} \lambda\tau\cos\varphi & \lambda\tau^{-1}\sin\varphi \\ -\lambda^{-1}\tau\sin\varphi & \lambda^{-1}\tau^{-1}\cos\varphi \end{pmatrix},$$

and give an upper bound on the absolute value of its contracting direction θ. By Lemma 2.5.1 (a), one obtains,

$$\tan 2\theta = \frac{(\lambda^2 - \lambda^{-2})\sin 2\varphi}{\frac{1}{2}(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) + \frac{1}{2}(\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2})\cos 2\varphi}.$$

Since $|2\theta| \leq |\tan 2\theta|$, and also the right hand side is an odd function of φ, it remains to show that for $\varphi \in [0, \pi/2]$ with $|\cos\varphi| \geq 1/m$,

$$f(\varphi) := \frac{(\lambda^2 - \lambda^{-2})\sin 2\varphi}{(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) + (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2})\cos 2\varphi} \leq \frac{m^2}{\tau^2}. \tag{2.5.2}$$

Clearly $|\cos\varphi| \geq 1/m$ if and only if $\cos 2\varphi \geq -1 + 2/m^2$.

Case 1: $\lambda \leq \tau$. Then

$$(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) \geq (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2}).$$

Using the fact that $\cos 2\varphi \geq -1 + 2/m^2$, the denominator of $f(\varphi)$ has a lower bound

$$\begin{aligned}(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) + (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2}) \cos 2\varphi &\geq (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2})(1 + \cos 2\varphi) \\ &\geq \frac{2}{m^2}(\lambda^2 - \lambda^{-2})\tau^2,\end{aligned}$$

and (2.5.2) holds.

Case 2: $\lambda \geq \tau$. We let $k := (\lambda^2 + \lambda^{-2})/(\lambda^2 - \lambda^{-2}) > 1$ and write

$$f(\varphi) = \frac{\sin 2\varphi}{k(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2}) \cos 2\varphi}.$$

Since $\lambda \geq \tau$,

$$(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) \leq (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2}),$$

and therefore $k(\tau^2 - \tau^{-2}) \leq (\tau^2 + \tau^{-2})$. Now compute

$$f'(\varphi) = 2\frac{k(\tau^2 - \tau^{-2}) \cos 2\varphi + (\tau^2 + \tau^{-2})}{(k(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2}) \cos 2\varphi)^2} > 0.$$

On the other hand, note that the denominator of $f(\varphi)$ is positive for $\varphi \in [0, \pi/2]$ with $|\cos \varphi| \geq 1/m$:

$$\begin{aligned}&(\lambda^2 + \lambda^{-2})(\tau^2 - \tau^{-2}) + (\lambda^2 - \lambda^{-2})(\tau^2 + \tau^{-2}) \cos 2\varphi \\ &> (\lambda^2 - \lambda^{-2})[(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2}) \cos 2\varphi] \\ &\geq (\lambda^2 - \lambda^{-2})[(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2})(-1 + 2/m^2)] \\ &\geq (\lambda^2 - \lambda^{-2}) \left(\frac{2}{m^2}(\tau^2 + \tau^{-2}) - 2\tau^{-2} \right).\end{aligned}$$

Since $\tau \geq m$, $2\tau^2/m^2 \geq 2\tau^{-2}$, and hence the right hand side is positive. Therefore within the given range of φ, $f(\varphi)$ is a smooth increasing function of φ, hence its maximum occurs for $\varphi = \varphi_0$, where $\varphi_0 \in [0, \pi/2]$ is such that $\cos 2\varphi_0 = -1 + 2/m^2$, or equivalently $|\cos \varphi_0| = 1/m$. Now

$$\sin 2\varphi_0 = 2 \sin \varphi_0 \cos \varphi_0 < \frac{2}{m}.$$

Therefore recalling that $k > 1$,

$$\begin{aligned} f(\varphi_0) &= \frac{\sin 2\varphi_0}{k(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2}) \cos 2\varphi_0} < \frac{2/m}{(\tau^2 - \tau^{-2}) + (\tau^2 + \tau^{-2})(-1 + 2/m^2)} \\ &= \frac{m^2}{\tau^2} \left(\frac{1/m}{1 - (m^2 - 1)\tau^{-4}} \right). \end{aligned}$$

Finally, as $\tau \geq m$, we have

$$\frac{1/m}{1 - (m^2 - 1)\tau^{-4}} \leq \frac{1/m}{1 - (m^2 - 1)m^{-4}} = \frac{m^3}{m^4 + 1 - m^2}.$$

As $m \to \infty$, the right hand side goes to 0, therefore for large enough m, it is less than 1, hence for large enough m (can take, say, $m > 1.4$),

$$f(\varphi_0) \leq m^2/\tau^2,$$

and the result follows. If we assume that $\tau \geq \sqrt{2}m$, then we have instead

$$\frac{1/m}{1 + (1 - m^2)\tau^{-4}} \leq \frac{1/m}{1 + (1 - m^2)m^{-4}/4} = \frac{m^3}{m^4 + (1 - m^2)/4}.$$

The right hand side is a smooth decreasing function for all $m > 1$ and is exactly 1 at $m = 1$, hence it is at most 1 for all $m \geq 1$, and so $f(\varphi_0) \leq m^2/\tau^2$ for all $m \geq 1$. □

Proof of Proposition 2.5.4. Let $m_0 := 1/\sin \varepsilon$. Clearly $\lambda_{\text{crit}} > m_0$. Fix $x \in M$ and $\theta \in T^1_x M$.

Consider n maps $f_1, f_2, \ldots, f_n \in \mathrm{Diff}^2(M)$ satisfying

$$\lambda_{D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i} > \lambda_{\mathrm{crit}} \qquad \text{and} \qquad \lambda_{D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i} \leq \lambda_{\max} \qquad \text{for all } i, \tag{2.5.3}$$

and

$$d(\theta_{D_x f_1}, \theta) > \varepsilon, \qquad d(\theta_{D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i}, \theta_{(D_x f_{i-1}f_{i-2}\cdots f_1)^{-1}}) > \varepsilon \qquad \text{for all } i. \tag{2.5.4}$$

For each $i > 1$, we apply Lemma 2.5.5 with $M_1 = D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i$, $M_2 = D_x f_{i-1}f_{i-2}\cdots f_1$ and $m = m_0$. Then $M_1 M_2 = D_x f_i f_{i-1}\cdots f_1$. Note that the corresponding

$$\varphi = \theta_{D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i} + \pi/2 - \theta_{(D_x f_{i-1}f_{i-2}\cdots f_1)^{-1}}$$

satisfies $|\cos\varphi| = |\sin(\theta_{D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i} - \theta_{(D_x f_{i-1}f_{i-2}\cdots f_1)^{-1}})| \geq |\sin\varepsilon| = 1/m_0$. Also

$$\|D_{f_{i-1}f_{i-2}\cdots f_1(x)}f_i\| > \lambda_{\mathrm{crit}} > m_0 > \sqrt{m_0}$$

for all i, thus by induction using Lemma 2.5.5 we have

$$\lambda_{f_i f_{i-1}\cdots f_1} \geq \frac{\lambda_{\mathrm{crit}}^i}{m_0^{i-1}}$$

(note that the right hand side is greater than $\lambda_{\mathrm{crit}} > m_0 > \sqrt{m_0}$.) Since $\lambda_{\mathrm{crit}} > \sqrt{2}m_0$, by Lemma 2.5.6, we get that

$$d(\theta_{D_x f_{i-1}f_{i-2}\cdots f_1}, \theta_{D_x f_i f_{i-1}\cdots f_1}) \leq m_0^2 \left(\frac{\lambda_{\mathrm{crit}}^i}{m_0^{i-1}}\right)^{-2} = \left(\frac{m_0}{\lambda_{\mathrm{crit}}}\right)^{2i}.$$

Since $d(\theta_{D_x f_1}, \theta) > \varepsilon$, we have

$$d(\theta_{D_x f_n f_{n-1} \cdots f_1}, \theta) > \varepsilon - \left(\left(\frac{m_0}{\lambda_{\text{crit}}} \right)^2 + \left(\frac{m_0}{\lambda_{\text{crit}}} \right)^4 + \cdots + \left(\frac{m_0}{\lambda_{\text{crit}}} \right)^{2n} \right) > \varepsilon - \frac{(m_0/\lambda_{\text{crit}})^2}{1-(m_0/\lambda_{\text{crit}})^2}.$$

As $\frac{1}{\sin\varepsilon}\sqrt{2+\frac{1}{\varepsilon}} < \lambda_{\text{crit}}$, we have $\frac{(m_0/\lambda_{\text{crit}})^2}{1-(m_0/\lambda_{\text{crit}})^2} < \varepsilon/2$ (recall that $m_0 = 1/\sin\varepsilon$). Thus $d(\theta_{D_x f_n f_{n-1} \cdots f_1}, \theta) > \varepsilon/2$. By Lemma 2.5.2,

$$\log \|D_x f_n f_{n-1} \cdots f_1(\theta)\| \geq \log \left(\frac{2}{\pi} \lambda_{f_n f_{n-1} \cdots f_1} d(\theta_{D_x f_n f_{n-1} \cdots f_1}, \theta) \right) > \log \frac{\lambda_{\text{crit}}^n}{m_0^{n-1}} \frac{\varepsilon}{\pi}.$$

By assumption we know that the $\mu^{(n)}$-probability that the chosen $f_1, \ldots, f_n$ satisfy (2.5.3) and (2.5.4) is at least η^n. Moreover for $\mu^{(n)}$-almost every f, $\log \|D_x f(\theta)\| \geq -n \log \lambda_{\max}$. Hence

$$\int \log \|D_x f(\theta)\| d\mu^{(n)}(f) \geq \eta^n \left(\log \frac{\lambda_{\text{crit}}^n}{m_0^{n-1}} \frac{\varepsilon}{\pi} \right) + (1-\eta^n)(-n \log \lambda_{\max}). \tag{2.5.5}$$

Take n large enough so that

$$\log \frac{\lambda_{\text{crit}}^n}{m_0^{n-1}} \frac{\varepsilon}{\pi} > 0.$$

Now fix such n, as the right hand side of (2.5.5) increases to $\log \frac{\lambda_{\text{crit}}^n}{m_0^{n-1}} \frac{\varepsilon}{\pi}$ as $\eta \to 1$, there is some $\eta \in (0,1)$ such that the right hand side of (2.5.5) is positive. □

2.6 Discrete random perturbation of the standard map

In this section, we show an example of a random dynamical system satisfying uniform expansion.

Let $L \in \mathbb{R}$ be a parameter. The standard map Φ_L of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, given by

$$\Phi_L(I, \theta) = (I + L\sin\theta, \theta + I + L\sin\theta),$$

is a well-known example of a chaotic system for which it is hard to show positivity of Lyapunov exponents (with respect to the Lebesgue measure on $\mathbb{T}^2$). For $L \gg 1$, it has strong expansion and contraction on a large but non-invariant region. Nonetheless on two narrow strips near $\theta = \pm\pi/2$, vectors can be arbitrarily rotated. The area of these "bad regions" goes to zero as $L \to \infty$, so one expect the Lyapunov exponent to be roughly $\log L$, reflecting the expansion rate in the rest of the phase space. However, positivity of Lyapunov exponents has not been shown for any single L.

In [BXY17], the authors considered a kind of random perturbations of a family of maps including the standard map, and showed positivity of Lyapunov exponents for this perturbation for sufficiently large L. More precisely, under a linear change of coordinates $x = \theta$, $y = \theta - I$, the standard map is conjugate to the map

$$F(x, y) = (L \sin x + 2x - y, x) \tag{2.6.1}$$

on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. Note that F preserves the Lebesgue measure on $\mathbb{T}^2$. They considered the composition of random maps

$$F^n_{\underline{\omega}} = F_{\omega_n} \circ \cdots \circ F_{\omega_1} \qquad \text{for} \qquad n = 1, 2, 3, \ldots,$$

where

$$F_\omega = F \circ S_\omega, \qquad S_\omega(x, y) = (x + \omega, y),$$

and the sequence $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega^{\mathbb{N}}$ is chosen with the probability measure $\mu^{\mathbb{N}}$, where $\mu = \mathrm{Leb}_{[-\varepsilon,\varepsilon]}$ is the uniform distribution on the interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

For this Markov chain, any stationary measure is absolutely continuous with respect to Lebesgue measure. Hence they were able to use this in the subsequent estimates of the Lyapunov exponents, using the fact that the Lebesgue measure of the "bad regions" goes to zero as $L \to \infty$.

In this section, we consider a discrete version of the random perturbation, where at each step, one can choose from only finitely many maps with equal probability. In this case it is not *a priori* clear that every stationary measure is absolutely continuous with respect to Lebesgue. In particular it is possible that the stationary measure may have positive measure concentrated in the bad region. In fact, one of our results is a classification of the ergodic stationary measures of this perturbation.

We shall show that this random dynamical system satisfies uniform expansion. As a corollary we show that the maps have a Lyapunov exponent $\sim \log L$. Moreover, from the previous sections, it follows that the stationary measures are either finitely supported or Lebesgue, and the orbits are either finite or dense.

Let $r \in \mathbb{N}$ and $\Omega := \{k\varepsilon : k = 0, \pm 1, \pm 2, \ldots, \pm r\}$. We consider the composition of random maps

$$F^n_{\underline{\omega}} = F_{\omega_n} \circ \cdots \circ F_{\omega_1} \qquad \text{for} \qquad n = 1, 2, 3, \ldots,$$

where

$$F_\omega = F \circ S_\omega, \qquad S_\omega(x, y) = (x + \omega, y),$$

and the sequence $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega^{\mathbb{N}}$ is chosen with the probability measure $\mu^{\mathbb{N}} := \left(\frac{1}{|\Omega|} \sum_{\omega \in \Omega} \delta_\omega \right)^{\mathbb{N}}$. Here $\delta_{k\varepsilon}$ is the delta mass on $\mathrm{Diff}^2(\mathbb{T}^2)$ at the map $F_{k\varepsilon}$.

The main proposition in this section is the following.

Proposition 2.6.1. Let $\delta \in (0, 1)$. There exists an integer $r_0 = r_0(\delta) > 0$ such that if $r \geq r_0$ and $\varepsilon \in [L^{-1+\delta}, 1/(2r + 1))$, then the measure $\mu = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \delta_\omega$ is uniformly expanding on $\mathbb{T}^2$ for all large enough L.

Throughout this section, estimates containing $\gg, \gtrsim$ and $\sim$ are with respect to $L \to \infty$. More precisely, we write

- $f(L) \gg g(L)$ if $|f(L)/g(L)| \to \infty$ as $L \to \infty$.
- $f(L) \gtrsim g(L)$ if $\liminf_{L \to \infty} |f(L)/g(L)| > 0$ (possibly infinite).
- $f(L) \sim g(L)$ if $f(L)/g(L) \to 1$ as $L \to \infty$.

For $A \in \mathbb{R}$, let $G(A) := \begin{pmatrix} A & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{R})$. Note that $DF_{(x,y)} = \begin{pmatrix} L\cos x + 2 & -1 \\ 1 & 0 \end{pmatrix} =$

$G(L\cos x + 2)$. Let $n \in \mathbb{N}$ to be determined. By Lemma 2.5.1, we observe that if $A \gg 1$, then

$$\|G(A)\| \sim A, \qquad \theta_{G(A)} \sim \frac{\pi}{2}, \qquad \text{and} \qquad \theta_{G(A)^{-1}} \sim 0.$$

The next lemma estimates the change of the contracting direction of products of $G(A_i)$ as we vary one of A_i and fix the rest, assuming A_j is large for all $j \neq i$.

Lemma 2.6.2. Let θ_n be the contracting direction of $G(A_n)G(A_{n-1})\cdots G(A_2)G(A_1)$. If $A_i \gg 1$ for all $i = 1, 2, \ldots n$, then for each i with $1 \leq i \leq n$,

$$\frac{d\theta_n}{dA_i} \sim \frac{1}{A_1^2 A_2^2 \cdots A_i^2}.$$

More precisely, let θ_n' be the contracting direction of $G(A_n')G(A_{n-1}')\cdots G(A_2')G(A_1')$. For each $i = 1, 2, \ldots, n$, if $A_j = A_j' \gg 1$ for all $j \neq i$, and $A_i, A_i' \gg 1$, then

$$\theta_n' - \theta_n \sim \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2}\left(\frac{1}{A_i} - \frac{1}{A_i'}\right).$$

Proof. By Lemma 2.5.1(a), we know that

$$\tan 2\theta_{G(A)} = -\frac{2}{A}.$$

By differentiating in A,

$$\frac{d\theta_{G(A)}}{dA} = \frac{1}{A^2 + 4}.$$

By Lemma 2.5.3 (a), for all $1 \leq i \leq n$,

$$\frac{d\theta_{G(A_n)G(A_{n-1})\cdots G(A_i)}}{dA_i} = \frac{d\theta_{G(A_i)}}{dA_i} = \frac{1}{A_i^2 + 4}.$$

Moreover, using Lemma 2.5.1(b), one can show that for all $1 \leq i \leq n$,

$$\|G(A_i)G(A_{i-1})\cdots G(A_1)\| \sim A_1 A_2 \cdots A_i$$

since the top left corner of $G(A_i)G(A_{i-1})\cdots G(A_1)$ is $A_1A_2\cdots A_i$ and the other three entries are of strictly lower order if $A_k \gg 1$ for all $1 \le k \le i$. Also notice that

$$\theta_{G(A_n)G(A_{n-1})\cdots G(A_i)} \sim \pi/2 \qquad \text{and} \qquad \theta_{(G(A_{i-1})G(A_{i-2})\cdots G(A_1))^{-1}} \sim 0.$$

Apply Lemma 2.5.3(b) with $M_1 = G(A_n)G(A_{n-1})\cdots G(A_i)$ and
$M_2 = G(A_{i-1})G(A_{i-2})\cdots G(A_1)$, we have

$$\frac{d\theta_n}{d\theta_{G(A_n)G(A_{n-1})\cdots G(A_i)}} \sim \frac{1}{(A_1A_2\cdots A_{i-1})^2}.$$

Hence

$$\begin{aligned}\frac{d\theta_n}{dA_i} &= \frac{d\theta_n}{d\theta_{G(A_n)G(A_{n-1})\cdots G(A_i)}}\frac{d\theta_{G(A_n)G(A_{n-1})\cdots G(A_i)}}{dA_i}\\ &\sim \frac{1}{(A_1A_2\cdots A_{i-1})^2}\frac{1}{A_i^2+4} \sim \frac{1}{A_1^2A_2^2\cdots A_i^2}.\end{aligned}$$

□

The next lemma estimates the change of the contracting direction of $DF^n_{\underline{\omega}}$ if we fix the first $i-1$ letters in $\underline{\omega}$ and change ω_j for all $j \ge i$.

Lemma 2.6.3. Let $\underline{\omega}, \underline{\omega}' \in \Omega^{\mathbb{N}}$, $\varepsilon > L^{-1}$. Given $(x,y) \in \mathbb{T}^2$, for $i = 0,1,2,\ldots,n$, let

- $(x_i, y_i) := F^i_{\underline{\omega}}(x,y)$ and $(x_i', y_i') := F^i_{\underline{\omega}'}(x,y)$,
- $A_i := L\cos x_{i-1} + 2$ and $A_i' := L\cos x_{i-1}' + 2$ for $i = 1,2,3,\ldots$,
- θ, θ' be the contracting directions of $G(A_n)G(A_{n-1})\cdots G(A_2)G(A_1)$ and $G(A_n')G(A_{n-1}')\cdots G(A_2')G(A_1')$.

For each $i = 1,2,\ldots,n$, suppose $A_j = A_j' \gg 1$ for all $j < i$, $A_j, A_j' \gg 1$ for all $j \ge i$ and $A_i - A_i' \gtrsim \varepsilon L/2$. Then

$$\theta - \theta' \gtrsim \frac{1}{A_1^2A_2^2\cdots A_{i-1}^2}\frac{\varepsilon L/2}{A_iA_i'}. \tag{a}$$

As a result,

$$\|DF^n_{\underline{\omega}}(\theta')\| \gtrsim \frac{A_{i+1}A_{i+2}\cdots A_n}{A_1A_2\cdots A_{i-1}}\frac{\varepsilon L/2}{A'_i}. \tag{b}$$

Proof. Without loss of generality, assume that $A_i > A'_i$. For all $j \geq i$, let θ_j be the contracting direction of

$$G(A'_n)G(A'_{n-1})\cdots G(A'_j)G(A_{j-1})G(A_{j-2})\cdots G(A_1).$$

Then $\theta' = \theta_i$. We also use the notation $\theta_{n+1} := \theta$, the contracting direction of $G(A_n)G(A_{n-1})\cdots G(A_1)$. By Lemma 2.6.2, for all $i \leq j \leq n$,

$$\theta_j - \theta_{j+1} \sim \frac{1}{A_1^2A_2^2\cdots A_{j-1}^2}\left(\frac{1}{A_j} - \frac{1}{A'_j}\right).$$

For all $j > i$, since $A_j, A'_j \gg 1$, $A_i > A'_i$ and $\varepsilon > L^{-1}$, we have

$$\begin{aligned}\theta_j - \theta_{j+1} &\sim \frac{1}{A_1^2A_2^2\cdots A_{j-1}^2}\left(\frac{1}{A_j} - \frac{1}{A'_j}\right) \ll \frac{1}{A_1^2A_2^2\cdots A_{i-1}^2}\frac{\varepsilon L/2}{A_iA'_i}\\ &\lesssim \frac{1}{A_1^2A_2^2\cdots A_{i-1}^2}\left(\frac{1}{A_i} - \frac{1}{A'_i}\right) \sim \theta_i - \theta_{i+1}.\end{aligned}$$

Therefore $\theta_j - \theta_{j+1}$ is dominated by $\theta_i - \theta_{i+1}$ for all $i < j \leq n$. Hence

$$\theta' - \theta = \theta_i - \theta_{n+1} = (\theta_i - \theta_{i+1}) + (\theta_{i+1} - \theta_{i+2}) + \cdots (\theta_n - \theta_{n+1}) \sim \frac{1}{A_1^2A_2^2\cdots A_{i-1}^2}\left(\frac{1}{A_i} - \frac{1}{A'_i}\right).$$

The second statement follows from the first by Lemma 2.5.2 since $\|G(A_n)G(A_{n-1})\cdots G(A_1)\| \sim A_nA_{n-1}\cdots A_1$ by Lemma 2.5.1(b). $\square$

Proof of Proposition 2.6.1. We are now ready to prove the main proposition of the section. The idea is as follows: for each point $(x, y) \in \mathbb{T}^2$, since the elements in Ω are of distance at least $\varepsilon \geq L^{-1+\delta}$ apart, for each $k\varepsilon \in \Omega$, for all $k'\varepsilon \in \Omega \setminus \{k\varepsilon\}$, all except possibly one of them satisfy (let $A(x) := L\cos x + 2$ for $x \in \mathbb{R}/2\pi\mathbb{Z}$)

$$|A(x + k'\varepsilon) - A(x + k\varepsilon)| \gtrsim \varepsilon L/2 \qquad \text{and} \qquad |A(x + k'\varepsilon)| \gtrsim L^\delta. \tag{2.6.2}$$

Geometrically, this means that firstly, all except one of them has norm growing to infinity with L, and the contracting directions of the corresponding differential maps

$$DF_{(x+\omega,y)} = \begin{pmatrix} L\cos(x+\omega)+2 & -1 \\ 1 & 0 \end{pmatrix}$$

are all pointing in roughly the vertical direction. Moreover, each of the contracting direction is separated from all others (except one) by a significiant amount ($\sim \varepsilon/\|F_{(x+\omega,y)}\|$). Hence after n steps, for many of the words $\underline{\omega} \in \Omega^n$, the contracting directions are close to the vertical direction and yet well separated (Figure 2.6). Thus each $\theta \in \mathbb{P}^1$ has distance from all but one of these contracting direction bounded from below. From Lemma 2.6.3, we know that the distance between the contracting directions of two words are dominated by their distance at the first letter they differ, and yet the norm grows by at least L^δ after every step. Using Lemma 2.5.2, as long as the word does not enter a bad region (where the contracting direction is rotated drastically), the log expansion $\log \|DF^n_{\underline{\omega}}\|$ will eventually be large. Since most words do not enter a bad region, and those that do enter a bad region admit a trivial lower bound $\log \|DF^n_{\underline{\omega}}\| \geq -n \log L$, eventually we will obtain positive expansion on average.

We now make the above discussion precise using the previous lemmas. For $x \in \mathbb{R}/2\pi\mathbb{Z}$, let $A(x) = L\cos x + 2$. Recall that at each point $(x,y) \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, the differential map of $F(x,y) = (L\sin x + 2x - y, x)$ is

$$DF = \begin{pmatrix} L\cos x + 2 & -1 \\ 1 & 0 \end{pmatrix} = G(A(x)).$$

Let $\varepsilon \in [L^{-1+\delta}, 1/(2r+1))$. For each $\omega \in \Omega = \{k\varepsilon : k = 0, \pm 1, \pm 2, \dots, \pm r\}$,

$$F_\omega = F \circ S_\omega = (L\sin(x+\omega) + 2(x+\omega) - y, x+\omega).$$

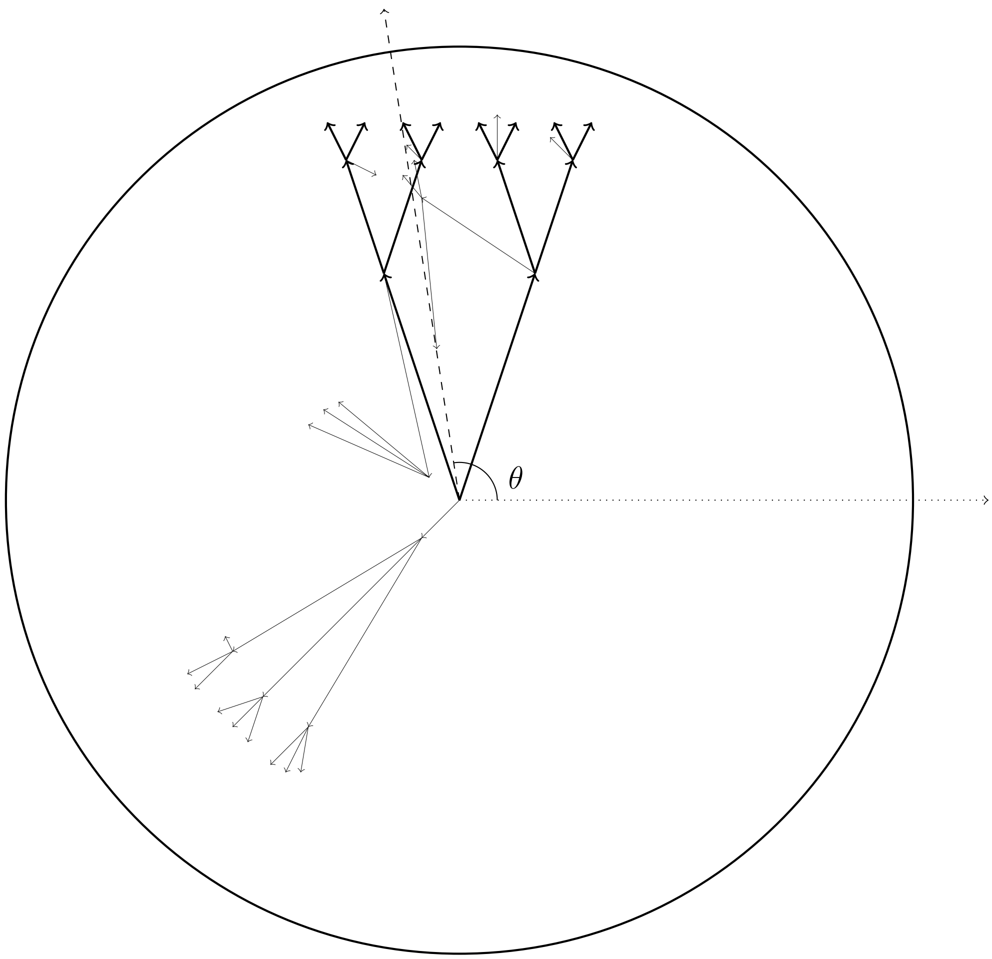


Figure 2.6: The random walk after 3 steps. The bold directions form a well-separated "tree".

Hence the differential DF_ω is

$$\begin{pmatrix} L\cos(x+\omega)+2 & -1 \\ 1 & 0 \end{pmatrix} = G(A(x+\omega)).$$

Fix a point $(x, y) \in \mathbb{T}^2$. For $\underline{\omega}, \underline{\omega}' \in \Omega^n$ and $0 \leq i \leq n$, let

$$(x_i, y_i) := F^i_{\underline{\omega}}(x, y) \qquad \text{and} \qquad (x'_i, y'_i) := F^i_{\underline{\omega}'}(x, y).$$

Let

$$A_i := L\cos x_{i-1} + 2, \qquad \text{and} \qquad A'_i := L\cos x'_{i-1} + 2.$$

We say that a word $\underline{\omega} \in \Omega^n$ is *long* (with respect to $(x, y) \in \mathbb{T}^2$) if

$$|A_i| \gtrsim L^\delta \qquad \text{for all } 1 \leq i \leq n.$$

For each word $\underline{\omega} \in \Omega^n$, let

$$\theta_{\underline{\omega}} := \theta_{DF^n_{\underline{\omega}}}$$

be the contracting direction of the matrix $DF^n_{\underline{\omega}}$.

Observe by (2.6.2) that for each long $\underline{\omega} \in \Omega^n$, there are at least $(|\Omega| - 2)(|\Omega| - 1)^{n-1}$ long words $\underline{\omega}' \in \Omega^n$ such that

$$|A_1 - A_1'| \gtrsim \frac{\varepsilon L}{2}.$$

By Lemma 2.6.3(a), since $A_1 \leq L + 2$,

$$|\theta_{\underline{\omega}} - \theta_{\underline{\omega}'}| \gtrsim \frac{\varepsilon L/2}{A_1' A_1} \gtrsim \frac{\varepsilon/2}{A_1'}.$$

Similarly, for all $1 \leq i \leq n$, there are at least $(|\Omega| - 2)(|\Omega| - 1)^{n-i}$ long words $\underline{\omega}' \in \Omega^n$ such that

$$\omega_j = \omega_j' \quad \text{for all } j < i, \qquad \text{and} \qquad |A_i - A_i'| \gtrsim \frac{\varepsilon L}{2}.$$

Thus again by Lemma 2.6.3(a),

$$|\theta_{\underline{\omega}} - \theta_{\underline{\omega}'}| \gtrsim \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2} \frac{\varepsilon L/2}{A_i A_i'} \gtrsim \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2} \frac{\varepsilon/2}{A_i'}.$$

For all $\theta \in \mathbb{P}^1$, take a long word $\underline{\omega} \in \Omega^n$ that minimizes $|\theta_{\underline{\omega}} - \theta|$ (among long words). Then from above, we know that for each $1 \leq i \leq n$, there are at least $(|\Omega| - 2)(|\Omega| - 1)^{n-i}$ long words $\underline{\omega}' \in \Omega^n$ such that

$$|\theta_{\underline{\omega}'} - \theta| \gtrsim \frac{1}{2} \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2} \frac{\varepsilon/2}{A_i'} = \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2} \frac{\varepsilon/4}{A_i'}.$$

Hence by Lemma 2.5.2 (note that $A_j = A'_j$ for $j < i$ since $\omega_j = \omega'_j$),

$$\begin{aligned}\|DF^n_{\underline{\omega}'}(\theta)\| &\gtrsim \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2}\frac{\varepsilon/4}{A'_i}\cdot\|DF^n_{\underline{\omega}'}\| \gtrsim \frac{1}{A_1^2 A_2^2 \cdots A_{i-1}^2}\frac{\varepsilon/4}{A'_i}(A'_1A'_2\cdots A'_n)\\ &\gtrsim \frac{A'_{i+1}A'_{i+2}\cdots A'_n}{A_1\cdots A_{i-1}}\frac{\varepsilon}{4} \gtrsim \frac{(L^\delta)^{n-i}}{L^{i-1}}\frac{L^{-1+\delta}}{4}\\ &\gtrsim L^{\delta(n-i+1)-i}.\end{aligned}$$

Thus for each direction $\theta \in \mathbb{P}^1$, for each $i = 1, 2, \ldots, n$, we have at least $(|\Omega|-2)(|\Omega|-1)^{n-i}$ words $\underline{\omega}$ in Ω^n such that

$$\log\|DF^n_{\underline{\omega}}(\theta)\| \gtrsim (\delta(n-i+1)-i)\log L.$$

For the remaining $|\Omega|^n - (|\Omega|-1)^n + 1$ words $\underline{\omega}$, we have

$$\log\|DF^n_{\underline{\omega}}(\theta)\| \geq -n\log L.$$

Hence

$$\int \log\|DF^n_{\underline{\omega}}(\theta)\| d\mu^{(n)}(\underline{\omega}) \gtrsim \frac{\Xi(|\Omega|, n, \delta)}{|\Omega|^n}\log L,$$

where $\Xi(|\Omega|, n, \delta) = \sum_{i=1}^{n}(|\Omega|-2)(|\Omega|-1)^{n-i}(\delta(n-i+1)-i) - (|\Omega|^n - (|\Omega|-1)^n + 1)n$.

The coefficient of $|\Omega|^n$ in $\Xi(|\Omega|, n, \delta)$ is $n\delta - 1$, hence is positive if $n > 1/\delta$. The coefficient of $|\Omega|^{n-1}$ is $-(\delta+1)(n^2+1)+n$. If $n > 1/\delta$, for large enough r (hence large enough $|\Omega| = 2r+1$), we have $\Xi(|\Omega|, n, \delta) > 1$. Hence μ is uniformly expanding for all large enough r (depending only on δ) and large enough L, with $N := \lceil 1/\delta \rceil$ and $C = |\Omega|^{-N}\log L$. Moreover, for $\delta \in (1/3, 1)$, we can take $n = 3$, and $|\Omega| \geq \dfrac{10\delta+7}{3\delta-1}$.

$\square$

2.7 Computer-assisted verification of uniform expansion

In this section we outline an algorithm to verify uniform expansion numerically, when μ is finitely supported on $\mathrm{Diff}^2(M)$. Uniform expansion is *a priori* an infinite condition in the sense that there are infinitely many points on the manifold and infinitely many directions on each fiber of the unit tangent bundle. Nonetheless since the maps in the support of μ are C^2 and the left hand side of the uniform expansion condition is Lipschitz in v, using the fact that the unit tangent bundle T^1M is compact, one can take a finite grid on T^1M, verify the uniform expansion at each grid point, and then prove uniform expansion on the whole T^1M by the Lipschitz condition.

This algorithm checks a sufficient condition of uniform expansion when $N = 1$. Nonetheless, by replacing $\mu^{(N)}$ with μ, one may in principle apply the same algorithm to verify uniform expansion for any N.

Let $f_1, \dots f_d$ be the maps in the support of μ and $\mu = c_1\delta_{f_1} + \dots + c_d\delta_{f_d}$ for $c_i \in (0, 1]$. For each $i = 1, 2, \dots, d$, $P \in M$ and $\theta \in \mathbb{P}^1$, we consider the function

$$F_i(P, \theta) := \log \|D_P f_i(\theta)\|.$$

Our goal is to verify that

$$F(P, \theta) := \sum_{i=1}^{d} c_i F_i(P, \theta) > C \tag{UE}$$

for some $C > 0$.

We now outline the algorithm.

Step 1: Choose local coordinates t_1, t_2 on M, and find $C_M, C_\theta > 0$ such that

$$\left|\frac{\partial F_i}{\partial t}\right| < C_M, \qquad \left|\frac{\partial F_i}{\partial \theta}\right| < C_\theta$$

for $t = t_1, t_2$. Such constants exist since F_i is C^1 and M is compact.

Step 2: Fix some $C > 0$.

Step 3: Pick $r, \rho > 0$ such that $rC_M < C/4$ and $\rho C_\theta < C/4$.

Step 4: Take a finite grid $\mathcal{G}$ on the unit tangent bundle T^1M that is r-dense on the manifold and ρ-dense on the unit tangent space T^1_PM for each grid point $P \in M$.

Step 5: Verify (UE) for each grid point $(P, \theta) \in \mathcal{G}$.

Step 6: From the derivative bounds in **Step 1** and the choices of r and ρ in **Step 3**, one can conclude that (UE) holds with C replaced by $C/4$.

2.8 Outer automorphism group action on character variety

2.8.1 Introduction

In this section, we consider an example of a random dynamical system where the uniform expansion property can be checked numerically using the algorithm outlined in Section 2.7.

Let F_n be a free group of rank $n > 1$, G be a compact Lie group. The natural volume form on $\mathrm{Hom}(F_n, G)$ is invariant under $\mathrm{Aut}(F_n)$. This form descends to a natural finite measure λ on the character variety $\mathrm{Hom}(F_n, G)//G$ that is invariant under $\mathrm{Out}(F_n)$. We refer the reader to [Gol07] for more details about ergodic properties of this system, and the celebrated work of Goldman [Gol97] for a detailed account in the case when F_n is replaced by the mapping class group of a surface.

Goldman [Gol07] proved that in the case when $G = \mathrm{SU}(2)$ and $n > 2$, the $\mathrm{Out}(F_n)$-action on $\mathrm{Hom}(F_n, G)//G$ is ergodic. On the other hand, the action is not ergodic when $n = 2$, since it preserves the surjective function

$$\kappa : \mathrm{Hom}(F_n, G)//G \to [-2, 2]$$
$$[\rho] \mapsto \mathrm{tr}(\rho([X, Y]))$$

where X, Y is a pair of free generators of F_2, and $[X, Y] := XYX^{-1}Y^{-1}$ is the commutator of X and Y. The ergodic components are the disintegration λ_s of λ on the fibers $\mathfrak{X}_s := \kappa^{-1}(s)$ of κ for $s \in [-2, 2]$.

In the case when $n = 2$, the topological dynamics of this action was studied by Previte and Xia [PX00], who proved, in particular, that on each shell $\mathfrak{X}_s$, the $\mathrm{Out}(F_2)$-invariant sets are either finite or dense. In fact, they classified all the finite $\mathrm{Out}(F_2)$-invariant sets, and gave a condition for when the invariant set is dense. On the other hand, Brown [Bro98] showed using standard KAM techniques that for any nontrivial cyclic subgroup $\Gamma \subset \mathrm{Out}(F_2)$ and s close enough to -2, there is a Γ-invariant set with positive measure on $\mathfrak{X}_s$ that is not dense. We refer our readers to [Gol97] and [PX02] for analogous analysis of the measurable and topological dynamics of the mapping class group $\mathrm{Out}(\pi_1(M))$-action on the character variety $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$.

The analysis in [PX00] relies crucially on the fact that $\mathrm{Out}(F_2)$ is generated by Dehn twists. In fact with minor modification their method also applies to the action of a subsemigroup $\Gamma \subset \mathrm{Out}(F_2)$ generated by at least two powers of distinct Dehn twists. In this section, we consider a set of generators $\mathcal{S}$ of a semigroup $\Gamma \subset \mathrm{Out}(F_2)$ that does not contain any Dehn twists or powers of Dehn twists, and attempt to show that the Γ-invariant sets are finite or dense by showing uniform expansion on $\mathcal{S}$ and applying Theorem D. The uniform expansion property is checked using a computer program. For s close to 2, the expansion is large enough that uniform expansion is observed after 1 iteration. However, for s close to -2, the expansion cannot be checked numerically due to the limitation of computational power. We will verify uniform expansion for a specific s as a proof of concept, though the same algorithm carries for other s close to 2 as well.

More precisely, consider the following two elements of $\mathrm{Out}(F_2)$:

$$\tau_X : X \mapsto X, \quad Y \mapsto XY, \qquad \tau_Y : X \mapsto YX, \quad Y \mapsto Y.$$

Note that τ_X and τ_Y generate a subgroup $\langle \tau_X, \tau_Y \rangle$ that has index 2 in $\mathrm{Out}(F_2)$. Let $\tau_{ABC} :=$

$\tau_A \circ \tau_B \circ \tau_C$ where $A, B, C \in \{X, Y\}$. Define the subsemigroup

$$\Gamma = \langle f_i : i = 1, 2, \ldots, 16 \rangle \subset \mathrm{Out}(F_2),$$

where

- $f_1 = \tau_{XXXXY}$
- $f_2 = \tau_{XXXYY}$,
- $f_3 = \tau_{XXYYY}$,
- $f_4 = \tau_{XYYYY}$,
- $f_5 = \tau_{YXXXX}$,
- $f_6 = \tau_{YYXXX}$,
- $f_7 = \tau_{YYYXX}$,
- $f_8 = \tau_{YYYYX}$,

and $f_i = f_{17-i}^{-1}$ for $i = 9, 10, \ldots, 16$. Now define the measure $\mu := \dfrac{1}{16}\left(\sum_{i=1}^{16} \delta_{f_i}\right)$ on $\mathrm{Out}(F_2)$.

The result of this section is the following.

Proposition 2.8.1. For $s = 1.99$, the measure μ is uniformly expanding as an action on the surface $\mathfrak{X}_s$.

Corollary 2.8.2. For $s = 1.99$, the Γ-invariants sets on $\mathfrak{X}_s$ are either finite or dense.

2.8.2 *Character variety as a subvariety of* $\mathbb{R}^3$

We now describe the character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2)$ in more explicit terms. The character variety $\mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2)$ injects into $\mathbb{R}^3$ under the trace coordinates

$$\begin{aligned} \mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2) &\to \mathbb{R}^3 \\ [\rho] &\mapsto \begin{pmatrix} \mathrm{tr}(\rho(X)) \\ \mathrm{tr}(\rho(Y)) \\ \mathrm{tr}(\rho(XY)) \end{pmatrix}. \end{aligned}$$

This is injective, with image

$$\mathfrak{X} := \{(x, y, z) \in \mathbb{R}^3 : -2 \leq x^2 + y^2 + z^2 - xyz - 2 \leq 2\}.$$

Hence we may identify $\mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2)$ with $\mathfrak{X}$. In these coordinates, the map $\kappa : \mathrm{Hom}(F_2, \mathrm{SU}(2))//\mathrm{SU}(2) \to [-2, 2]$ described in the introduction is then

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2.$$

For $s \in [-2, 2]$, the ergodic components are

$$\mathfrak{X}_s := \kappa^{-1}(s) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - xyz - 2 = s\}.$$

In trace coordinates, the maps τ_X and τ_Y are

$$\tau_X : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ z \\ xz - y \end{pmatrix}, \qquad \tau_Y : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z \\ y \\ yz - x \end{pmatrix}.$$

At each point $P = (x, y, z)$, a normal vector is given by $\mathbf{n}(P) = (2x - yz, 2y - zx, 2z - xy)$, with the unit normal $\mathbf{v_3}(P) = \dfrac{\mathbf{n}(P)}{\|\mathbf{n}(P)\|}$.

From [Gol07, Sect. 5.3], a cosymplectic structure on $\mathfrak{X}_t$ can be given explicitly by (up to a multiplicative constant)

$$(2x - yz)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + (2y - zx)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + (2z - xy)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

Since Γ preserves the symplectic structre, if we take the metric $\|\cdot\|_P := \|\mathbf{n}(P)\|^{-1/2}\|\cdot\|$ on $T_P\mathfrak{X}_s$, where $\|\cdot\|$ is the restriction of the Euclidean metric from $\mathbb{R}^3$ to the tangent space $T_P\mathfrak{X}_s$, then for each $f \in \mathrm{Out}(F_2)$, we have the area-preserving linear map

$$D_Pf : T_P\mathfrak{X}_s \to T_{f(P)}\mathfrak{X}_s.$$

Note that each element $f \in \mathrm{Out}(F_2)$ is the restriction of a map $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ to $\mathfrak{X}_s$ in terms of the trace coordinates. Therefore D_Pf can be expressed as the restriction of a volume-preserving linear map $D_Pf_0 : \mathbb{R}^3 \to \mathbb{R}^3$, i.e. an element of $SL_3(\mathbb{R})$, to $T_P\mathfrak{X}_s$. For instance, writing $P = (x, y, z)$,

$$D_P\tau_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ z & -1 & x \end{pmatrix}, \qquad D_P\tau_Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & z & y \end{pmatrix},$$

both restricted to the tangent space $T_P\mathfrak{X}_s$.

2.8.3 Choice of metric

We will choose a convenient metric to work with. To do so, it suffices to give an orthonormal basis at each point. For each $P = (x, y, z) \in \mathfrak{X}_s$, let $\mathbf{n}(P) = (n_1(P), n_2(P), n_3(P)) := (2x - yz, 2y - zx, 2z - xy)$ be the normal vector. Consider the following three tangent vectors in

$T_P(\mathfrak{X}_s)$

$$\mathbf{v}_1(P) = \begin{pmatrix} 0 \\ n_3(P) \\ -n_2(P) \end{pmatrix}, \qquad \mathbf{v}_2(P) = \begin{pmatrix} -n_3(P) \\ 0 \\ n_1(P) \end{pmatrix}, \qquad \mathbf{v}_3(P) = \begin{pmatrix} n_2(P) \\ -n_1(P) \\ 0 \end{pmatrix}.$$

Clearly these are tangent vectors at P. Moreover since the normal vector

$$\mathbf{n}(P) = (n_1(P), n_2(P), n_3(P))$$

is nonzero, at least one of $n_i(P)$, $i = 1, 2, 3$ is nonzero, thus at least two of $\mathbf{v}_1(P), \mathbf{v}_2(P), \mathbf{v}_3(P)$ are linearly independent. In fact, for $s < 2$, there is a positive lower bound $c = c(s)$ such that $\max_{i=1,2,3} |n_i(P)| \geq c(s)$ for all $P \in \mathfrak{X}_s$, so at least two of $\mathbf{v}_1(P), \mathbf{v}_2(P), \mathbf{v}_3(P)$ have Euclidean norm larger than $c(s)$.

Now at each $P \in \mathfrak{X}_s$, we define a positive definite inner product $\langle \cdot, \cdot \rangle_P$ on $T_P\mathfrak{X}_s$ such that

$$\left\{ \frac{\mathbf{v}_i(P)}{\sqrt{n_k(P)}}, \frac{\mathbf{v}_j(P)}{\sqrt{n_k(P)}} \right\}$$

form an orthonormal basis, where $k \in \{1, 2, 3\}$ is the index that maximizes $|n_k(P)|$, and $\{i, j, k\}$ form an even permutation of $\{1, 2, 3\}$ (we will comment on the normalizing factor $\sqrt{n_k(P)}$ in the next section). The map $P \mapsto \langle \cdot, \cdot \rangle_P$ is smooth except along the curves on $\mathfrak{X}_s$ where at least two of x, y, z are equal. Therefore strictly speaking they do not form a smooth metric. Nonetheless from the end of the previous paragraph, we know that there exists a constant $c'(s) > 0$ such that

$$c'(s)^{-1} \langle \cdot, \cdot \rangle \leq \langle \cdot, \cdot \rangle_P \leq c'(s) \langle \cdot, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product induced from $\mathbb{R}^3$. It is evident from the definition of uniform expansion that it is invariant under change of equivalent metrics, so it suffices to verify uniform expansion with respect to $\{\langle \cdot, \cdot \rangle_P\}_{P \in \mathfrak{X}_s}$.

The advantage of considering this metric is that, with respect to this metric and the specific orthonormal basis chosen above, $D_P\tau_X$ and $D_P\tau_Y$ (and hence the compositions) are 2×2 matrices such that up to the factor $n_k(P)$, the entries are polynomials in x, y, z. For instance,

$$\begin{aligned}
D_P\tau_X\mathbf{v}_1(P) &= \mathbf{v}_1(\tau_X(P)), \\
D_P\tau_X\mathbf{v}_2(P) &= \frac{n_1(P)}{n_3(\tau_X(P))}\mathbf{v}_1(\tau_X(P)) + \frac{n_3(P)}{n_3(\tau_X(P))}\mathbf{v}_2(\tau_X(P)), \\
D_P\tau_Y\mathbf{v}_1(P) &= \frac{n_3(P)}{n_3(\tau_Y(P))}\mathbf{v}_1(\tau_Y(P)) + \frac{n_2(P)}{n_3(\tau_Y(P))}\mathbf{v}_2(\tau_Y(P)), \\
D_P\tau_Y\mathbf{v}_2(P) &= \mathbf{v}_2(\tau_Y(P)).
\end{aligned}$$

The matrices with respect to other bases can be found using the identity

$$n_1(P)\mathbf{v}_1(P) + n_2(P)\mathbf{v}_2(P) + n_3(P)\mathbf{v}_3(P) = 0.$$

2.8.4 Derivative bounds

To choose the bounds C_M and C_θ in the algorithm, it is necessary to compute bounds on $|\partial F_i/\partial t|$ and $|\partial F_i/\partial \theta|$ for $F_i(P, \theta) = \log \|D_P f_i(\theta)\|$ and local coordinates $t = t_1, t_2$ near P. If we treat f_i as a function $\mathbb{R}^3 \to \mathbb{R}^3$, we can compute $D_P f_i$ as an element L_i of $SL_3(\mathbb{R})$.

With respect to the metric and the corresponding orthonormal basis chosen above, $D_P f_i$ can be written as a 2×2 matrix with entries being the square root of rational functions of x, y, z, say $D_P f_i = \begin{pmatrix} a_{i,P} & b_{i,P} \\ c_{i,P} & d_{i,P} \end{pmatrix}$. For instance, if the orthonormal basis for P is $\dfrac{\{\mathbf{v}_1(P), \mathbf{v}_2(P)\}}{\sqrt{n_3(P)}}$ and that of $f_i(P)$ is $\dfrac{\{\mathbf{v}_1(f_i(P)), \mathbf{v}_2(f_i(P))\}}{\sqrt{n_3(f_i(P))}}$, we can write explicitly that

$$D_P f_i = \frac{1}{\sqrt{n_3(P) n_3(f_i(P))}} \begin{pmatrix} (L_i \mathbf{v}_1)_2 & (L_i \mathbf{v}_2)_2 \\ -(L_i \mathbf{v}_1)_1 & -(L_i \mathbf{v}_2)_1 \end{pmatrix}.$$

In particular, $\sqrt{n_3(P) n_3(f_i(P))} D_P f_i$ has polynomial entries and

$$\det D_P f_i = 1$$

(the primary reason to have the normalizing factor $\sqrt{n_k(P)}$ is to ensure this matrix has determinant 1.) Similar expressions can be obtained for the other points where the other two orthonormal bases are chosen. Hence if we choose x and y to be the local coordinates near P (corresponding to the $\mathbf{v}_1$ and $\mathbf{v}_2$ directions), the derivatives with respect to x and y can be explicitly computed and bounded.

More explicitly, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, let $F_M(\theta) = \log \|M(\theta)\|$. Then

$$F_M(\theta) = \frac{1}{2} \log \left(\frac{1}{2}(a^2 + b^2 + c^2 + d^2) + \frac{1}{2}(a^2 - b^2 + c^2 - d^2) \cos 2\theta + (ab + cd) \sin 2\theta \right).$$

Thus $\partial F_M(\theta)/\partial t$ can be represented explicitly in terms of $a, b, c, d, a', b', c', d'$ and θ, where $a' = \partial a/\partial t$ etc. Since for all $P = (x, y, z) \in \mathcal{X}_s$, the coordinates x, y, z are in $[-2, 2]$, while a, b, c, d are polynomials in x, y, z divided by $\sqrt{n_3(P) n_3(f_i(P))}$, all these can be explicitly bounded. Furthermore by the choice of the orthonormal bases at P and $f_i(P)$ we know that $|n_3(P)| > |n_1(P)|, |n_2(P)|$ and similarly for $|n_3(f_i(P))|$, we have that $\sqrt{n_3(P) n_3(f_i(P))}$ is bounded below by an explicit positive number depending only on s. We shall omit the explicit expressions here as they are written in the program (see Program 1).

2.8.5 Choice of Parameters in the verification

In this section we choose the parameters in the algorithm to check that μ is uniformly expanding.

Proof of Proposition 2.8.1. We verify uniform expansion using the algorithm from the previous section. Let f_i be the maps in the support of μ with $i = 1, 2, \ldots, d$, where $d = 16$. We choose the grid $\mathcal{G}$ in the following process: recall that

$$\mathfrak{X}_s = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - xyz - 2 = s\}.$$

Let $\mathbf{n}(P) = (n_1(P), n_2(P), n_3(P)) = (2x - yz, 2y - zx, 2z - xy)$. Within the region $\{P \in M \mid |n_3(P)| = \max_{k=1,2,3} |n_k(P)|\}$, we use the x and y directions as local coordinates. This corresponds to using $\mathbf{v}_1$ and $\mathbf{v}_2$ as an orthonormal coordinate system. Similarly for the other two regions where $|n_1(P)|$ and $|n_2(P)|$ dominate. We verify uniform expansion for $s = 1.99$.

Step 1: We take $C_M = 600$ and $C_\theta = 600$.
(these are computed using the explicit expressions of $\partial F_M(\theta)/\partial t$ on a grid (**Program 1**) and then a naïve bound on second derivatives of $F_M(\theta)$.).

Step 2: Fix $C = 0.25$.

Step 3: Let $r = 0.0001 < C/(4C_M)$ and $\rho = 0.0001 < C/(4C_\theta)$.

Step 4: Take an r-dense grid on $\mathfrak{X}_s$ using the specified local coordinates. We fix a ρ-grid in the unit tangent space direction.

Step 5: We verify with **Program 2** that (UE) holds on the grid with $C = 0.25$ as in **Step 2**.

Step 6: From the derivative bounds in **Step 1** and the choices of r and ρ in **Step 3**, one can conclude that (UE) holds on the whole surface with C replaced by $C/4$.

The programs were run on the University of Chicago Midway compute cluster partition broadwl. Specification: 28 cores of Intel E5-2680v4 2.4 GHz. Memory: 64 GB. Runtime: 47714 seconds.

Program 1 (C^2 bounds in **Step 1**):

- Code: `http://math.uchicago.edu/~briancpn/derivative_single.cpp`
- Output: `http://math.uchicago.edu/~briancpn/secondderivative.txt`

Program 2 (C^1 bounds and (UE) in **Step 5**):

- Code: `http://math.uchicago.edu/~briancpn/actual.cpp`
- Output: `http://math.uchicago.edu/~briancpn/character_variety_test.txt`

□

CHAPTER 3

STATIONARY MEASURES ON VECTOR SPACES

3.1 Introduction

Let μ be a Borel probability measure on $G = GL(V)$, and let $\Gamma_\mu := \overline{\langle \mathrm{supp}\, \mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by the support of μ.

In this note, we are interested in studying the μ-stationary measures on the vector space V with respect to the Γ_μ-action on V by left multiplication.

Definition. We say that a Borel probability measure ν on V is *μ-stationary* if $\mu * \nu = \nu$, i.e.

$$\nu = \int_{GL(V)} g_* \nu \, d\mu(g).$$

Clearly if ν is Γ_μ-invariant then it is μ-stationary. Also note that since supp μ acts linearly on V, the origin of V is a fixed point, so the delta mass δ_0 at the origin of V is always a μ-stationary probability measure on V. We would like to understand when there are other μ-stationary probability measures on V, and if so whether we can classify all of them. In the rest of this note, we say that a μ-stationary measure ν on V is *nontrivial* if $\nu \neq \delta_0$.

In order to state our main classification result, we need the following two notions.

Definition. A Borel probability measure μ on $GL(V)$ has *finite first moment* if

$$\int_{GL(V)} \log \max(\|g\|, \|g^{-1}\|) d\mu(g) < \infty.$$

Here $\| \cdot \| := \| \cdot \|_{GL(V)}$ is the operator norm on $GL(V)$ with respect to a fixed norm on V.

Definition. We define the *top Lyapunov exponent* of μ on a Γ_μ-invariant subspace $W \subset V$ as

$$\lambda_{1,W} = \lambda_{1,W}(\mu) := \lim_{n \to \infty} \frac{1}{n} \int_{GL(V)} \log \|g\|_{GL(W)} d\mu^{(n)}(g),$$

where $\mu^{(n)} := \mu * \mu * \cdots * \mu$ is the n-th convolution power of μ, and for $g \in GL(V)$, $\|g\|_{GL(W)}$ denotes the operator norm of the restriction $g|_W$ in $GL(W)$.

The following result gives a necessary and sufficient condition for the existence of a nontrivial μ-stationary measure on V.

Theorem 3.1.1. Let μ be a Borel probability measure on $GL(V)$ with finite first moment. Then there exists a nontrivial μ-stationary measure ν on V if and only if there exist Γ_μ-invariant subspaces $W' \subsetneq W \subset V$ such that

(i) Γ_μ acts compactly on W/W', i.e. the image of $\rho_{W/W'} : \Gamma_\mu \to GL(W/W')$ is compact,

(ii) either $W' = 0$, or the top Lyapunov exponent of μ on W' is negative,

(iii) the support of every μ-stationary probability measure on V is in W.

The author only knew afterwards that the main proposition (Proposition 3.5.5) was already proved in the necessity direction of [Bou87, Thm. 5.1]. Theorem 3.1.1 follows directly from Proposition 3.5.5 (see Section 3.6) (can be shown that (i) in [Bou87, Thm. 5.1] can be improved to ensure $d_2 > 0$ if $d > 0$).

The following result classifies the stationary measures on V in terms of the compact Γ_μ-orbits on W/W'.

Theorem 3.1.2. Suppose there is a nontrivial μ-stationary measure on V and let $W' \subsetneq W \subset V$ be the Γ_μ-invariant subspaces from Theorem 3.1.1. Then the map $\nu \mapsto \operatorname{supp} \pi_*\nu$ gives a one-to-one correspondence between

$$\{\text{ergodic } \mu\text{-stationary measure on } V\} \quad \leftrightarrow \quad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

where $\pi : W \to W/W'$ is the quotient map.

We can describe the inverse map in a more explicit way in terms of the asymptotic behavior in law of the random walk on V induced by μ.

Theorem 3.1.3. For any compact Γ_μ-orbit $\mathcal{C}$ in W/W', let $m_{\mathcal{C}}$ be the Haar (probability) measure supported on $\mathcal{C}$. Let $s : W/W' \to W$ be a linear section, i.e. a linear map such that $\pi \circ s = \text{id}$. Then the weak-$*$ limit

$$\nu_{\mathcal{C}} := \lim_{n\to\infty} \mu^{(n)} * (s_* m_{\mathcal{C}})$$

exists and does not depend on the choice of the section s. Moreover, the map $\mathcal{C} \mapsto \nu_{\mathcal{C}}$ is the inverse map of the bijection in Theorem 3.1.2.

Using the classification of stationary measures, we can obtain the following equidistribution result.

Theorem 3.1.4. For all $x \in W$, let $\mathcal{C}$ is the compact Γ_μ-orbit of $x + W'$ in W/W'. Then

1. we have the weak-$*$ convergence

$$\frac{1}{n}\sum_{i=0}^{n-1} \mu^{(i)} * \delta_x \to \nu_{\mathcal{C}}.$$

2. For $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots) \in GL(V)^{\mathbb{N}}$, we have the convergence of the empirical measures

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{b_i b_{i-1}\ldots b_1 x} \to \nu_{\mathcal{C}} \qquad \text{as } n \to \infty.$$

The following definition is standard when considering stationary measures.

Proposition 3.1.5. [BL85, Lem. II.2.1] Let μ be a Borel probability measure on $G = GL(V)$ and ν be a μ-stationary measure on V. Then for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, there exists a probability measure ν_b on V such that for all $g \in \Gamma_\mu$,

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \ldots b_n g)_* \nu.$$

Moreover, we have

$$\nu = \int_{G^{\mathbb{N}}} \nu_b \, d\mu^{\mathbb{N}}(b).$$

The measure ν_b is sometimes called the *limit measure* of ν with respect to the word b.

We can describe the limit measures of any stationary measures on V.

Theorem 3.1.6. For each compact Γ_μ-orbit $\mathcal{C}$ in W/W', for $\mu^{\mathbb{N}}$-almost every word $b \in GL(V)^{\mathbb{N}}$, the limit measure

$$\nu_b = \lim_{n\to\infty}(b_1 b_2 \dots b_n)_* \nu_{\mathcal{C}}$$

is supported on the compact subset $p_b(\mathcal{C}) \subset W$ for some linear section $p_b : W/W' \to W$. In particular, ν_b is compactly supported on W.

If Γ_μ acts trivially on W/W', then ν_b is a delta mass $\delta_{\xi(b)}$ for $\mu^{\mathbb{N}}$-almost every word b, and thus ν is μ-proximal (cf. [BQ16, Sect. 2.7]).

The note is structured as follows.

1. In section 3.2, we recall a few preliminary facts about stationary measures and top exponents.

2. In section 3.3, we recall the situation when the action is irreducible, which will form the building blocks of the general case.

3. In section 3.4, we list a few properties of Γ_μ-actions that satisfy (i) and (ii) of Theorem 3.1.1. In particular most of Theorem 3.1.2, 3.1.3, 3.1.4 and 3.1.6 will be proved in this section.

4. In section 3.5, we study properties of the action on the span of the support of any given stationary measure on V. The main result in this section is Proposition 3.5.5, when we show that the action on this span satisfies (i) and (ii) of Theorem 3.1.1.

5. In section 3.6, we conclude by proving Theorem 3.1.1 using results from the previous sections.

3.2 Preliminary facts

We first recall that, in the case of a compact action, we have the standard fact that any stationary measure is invariant.

Proposition 3.2.1. [BQ11, Lem. 8.4] Let μ be a Borel probability measure on $G = GL(V)$ and ν be a μ-stationary measure on V. If Γ_μ acts compactly on V, then ν is Γ_μ-invariant. Moreover, if ν is ergodic, then the support of ν is a single compact Γ_μ-orbit in V, and ν is the unique μ-stationary measure supported on this orbit.

We recall the following general theorem by Furstenberg and Kesten, which follows from Kingman's subadditive ergodic theorem and the ergodicity of the Bernoulli shift.

Theorem 3.2.2. [FK60, Thm. 2], see also [BQ16, Lem. 4.27].
Let μ be a Borel probability measure on $GL(V)$ with finite first moment. For $\mu^{\mathbb{N}}$-a.e. $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|b_n \cdots b_1\| = \lim_{n\to\infty} \frac{1}{n} \log \|b_1 \cdots b_n\| = \lambda_{1,V}(\mu).$$

In particular, if $\lambda_{1,V} < 0$, then $\|b_1 \cdots b_n\| \to 0$ for $\mu^{\mathbb{N}}$-almost every word b.

To simplify notation, given a vector space V' with a homomorphism $\rho_{V'} : \Gamma_\mu \to GL(V')$, we say that μ has *negative top exponent on* V' if the top Lyapunov exponent $\lambda_{1,V'}$ of $\rho_{V'}$ with respect to μ is negative.

We need the following two lemmas that allow us to carry certain properties to invariant subspaces and quotients.

Lemma 3.2.3. Let μ be a Borel probability measure on $GL(V)$ with finite first moment. Let $W \subset V$ be a Γ_μ-invariant subspace of V. Then the following are equivalent:

(i) μ has negative top exponent on V.

(ii) μ has negative top exponent on W and V/W.

Proof. In fact the top exponent on V is the maximum of the top exponents on W and V/W. This is standard. See, for instance, [FK83, Lem. 3.6]. □

We also need the following elementary result about boundedness.

Lemma 3.2.4. Let μ be a Borel probability measure on $GL(V)$. Let $W \subset V$ be a Γ_μ-invariant subspace of V. Given a subset $B \subset \Gamma_\mu$, if B is bounded from above in $GL(V)$, then B is bounded from above in $GL(W)$ and $GL(V/W)$.

3.3 The irreducible case

We first recall the classification of stationary measures for *irreducible* Γ_μ-actions, i.e. the only Γ_μ-invariant subspaces of V are 0 and V.

Proposition 3.3.1. Let μ be a Borel probability measure on $GL(V)$. Suppose that Γ_μ acts irreducibly on V. Then there exists a nontrivial μ-stationary probability measure ν on V if and only if Γ_μ is compact in $GL(V)$.

Proof. If Γ_μ is compact in $GL(V)$ then clearly there is a nontrivial Γ_μ-invariant measure on V (by averaging via the finite Haar measure on Γ_μ), hence in particular μ-stationary.

The opposite direction was proved in [BL85, Prop. V.8.1].

□

We will also need another proposition that shows that for irreducible actions, assuming a boundedness condition, the only two options are negative top exponent and compact action.

Proposition 3.3.2. Let μ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that Γ_μ is irreducible. If for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, the sequence

$$\{b_n b_{n-1} \ldots b_1 \mid n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either μ has negative top exponent on V, or Γ_μ is compact in $GL(V)$.

Proof. The assumption implies that the top exponent is nonpositive by Theorem 3.2.2. Hence it suffices to consider the case when $\lambda_{1,V} = 0$.

Let $C : G^{\mathbb{N}} \to \mathbb{R}_+ \cup \{\infty\}$ be a measurable function such that

$$\|b_n b_{n-1} \dots b_1\| \leq C(b) \qquad \text{for all } n.$$

Then by assumption, we can take C to be finite $\mu^{\mathbb{N}}$-almost surely. If we take C' large enough, there is a subset $\mathcal{B} \subset G^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\mathcal{B}) > 1/2$ such that $C(b) < C'$ for all $b \in \mathcal{B}$. Now fix a μ-stationary measure $\nu_{\mathbb{P}}$ on $\mathbb{P}(V)$, and consider the dynamical system on $G^{\mathbb{N}} \times \mathbb{P}(V)$ with the map

$$T(b, v) := \left(\sigma(b), \log \frac{\|b_1 v\|}{\|v\|}\right),$$

where $\sigma : G^{\mathbb{N}} \to G^{\mathbb{N}}$ is the left shift map. Note that $\mu^{\mathbb{N}} \times \nu_{\mathbb{P}}$ is a T-invariant probability measure on $G^{\mathbb{N}} \times \mathbb{P}(V)$.

By the proof of the Atkinson's lemma ([Atk76], [Kes75], see e.g. [BQ16, Lem. 3.18]), for $\mu^{\mathbb{N}} \times \nu_{\mathbb{P}}$-almost every $(b, v) \in G^{\mathbb{N}} \times \mathbb{P}(V)$, there is an infinite sequence $\{n_k\}_k$ such that

$$\left|\log \frac{\|b_{n_k} \dots b_1 v\|}{\|v\|}\right| \leq 1. \tag{3.3.1}$$

Fix a nonzero $v \in V$ such that (3.3.1) holds for $\mu^{\mathbb{N}}$-almost every $b \in G^{\mathbb{N}}$. For each such word $b \in G^{\mathbb{N}}$, for each $n \geq 1$, take k large enough so that $n_k > n$. Then

$$\log \frac{\|b_n \dots b_1 v\|}{\|v\|} = \log \frac{\|b_{n_k} \dots b_1 v\|}{\|v\|} - \log \frac{\|b_{n_k} \dots b_1 v\|}{\|b_n \dots b_1 v\|}.$$

Now on the right hand side, the first term is at least -1 by (3.3.1), and the second term is at least $-\log C(\sigma^n(b))$ by definition of C. Therefore

$$\log \frac{\|b_n \dots b_1 v\|}{\|v\|} \geq -1 - \log C(\sigma^n(b)).$$

However note that $C(\sigma^n(b))$ does not depend on $b_1, b_2, \dots, b_n$. Therefore we can replace b by one of the words that starts with $b_1, b_2, \dots, b_n$ and satisfies $\sigma^n(b) \in \mathcal{B}$ so that $C(\sigma^n(b)) < C'$ for the uniform constant C' chosen above. Thus for $\mu^{\mathbb{N}}$-almost every word b, for all $n \geq 1$,

$$\log \frac{\|b_n \dots b_1 v\|}{\|v\|} \geq -1 - \log C'.$$

Now consider the sequence of measures on V

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{b_n b_{n-1} \dots b_1 v}.$$

Then any weak-$*$ limit ν is a μ-stationary measure on V by Breiman's Law of Large Numbers ([Bre60], also see e.g. [BQ16, Cor. 3.4]), and is a probability measure since there is a uniform bound from above on the sequence $\{b_n b_{n-1} \dots b_1 \mid n \geq 1\}$ by assumption. Since

$$\frac{\|b_n \dots b_1 v\|}{\|v\|} \geq C'' \qquad \text{for all } n \geq 1$$

for some uniform C'', ν is not δ_0, so it is a nontrivial μ-stationary probability measure on V. By Proposition 3.3.1, Γ_μ is compact in $GL(V)$. $\square$

The same is true if the order of the matrix product $b_1 b_2 \dots b_n$ is reversed.

Corollary 3.3.3. Let μ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that Γ_μ is irreducible. If for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, the sequence

$$\{b_1 b_2 \dots b_n \mid n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either μ has negative top exponent on V, or Γ_μ is compact in $GL(V)$.

Proof. Apply Proposition 3.3.2 to the pushforward μ^T of μ via the adjoint map $GL(V) \to GL(V^*)$ defined by $g \mapsto g^T$ (i.e. the matrix transpose). Note that $\|g\|_{GL(V)} = \|g^T\|_{GL(V^*)}$, so the first moments of μ and μ^T are the same. Similarly the top exponents of μ and μ^T are the same. Finally Γ_μ is irreducible if and only if Γ_{μ^T} is, and Γ_μ is compact if and only if Γ_{μ^T} is. □

3.4 Properties of a contracting-by-compact action

In this section, we list a few properties of subspaces with a contracting-by-compact action by μ, i.e. there is a proper subspace (possibly zero) with negative top exponent with respect to μ and Γ_μ acts compactly on the quotient.

The following proposition shows that for such action, almost every word is bounded from above with respect to the operator norm (though this bound may depend on the word).

Proposition 3.4.1. Let μ be a Borel probability measure on $GL(W)$ with finite first moment. Moreover there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Then there exists a measurable map $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots)$,

$$\|b_1 b_2 \ldots b_n\| < C(b) \qquad \text{for all } n \geq 1.$$

Proof. By choosing suitable basis, we can write each $b_i \in \operatorname{supp} \mu$ as

$$\begin{pmatrix} x_i & y_i \\ 0 & z_i \end{pmatrix},$$

where $x_i \in GL(W')$, $z_i \in GL(W/W')$ and $y_i \in \mathrm{Hom}(W/W', W')$.

Now we expand $b_1 b_2 \dots b_n$ in terms of x_i, y_i, z_i,

$$b_1 b_2 \dots b_n = \begin{pmatrix} X_n & Y_n \\ 0 & Z_n \end{pmatrix},$$

where

$$X_n = x_1 x_2 \dots x_n, \qquad Y_n = \sum_{k=1}^{n} x_1 \dots x_{k-1} y_k z_{k+1} \dots z_n, \qquad Z_n = z_1 z_2 \dots z_n.$$

Since μ has negative top exponent on W', $x_1 x_2 \dots x_n \to 0$ for $\mu^{\mathbb{N}}$-almost every word b by Theorem 3.2.2. Since Γ_μ acts compactly on W/W', Z_n is uniformly bounded by some constant C'. Hence it remains to find a bound on Y_n that is independent of n (but may depend on the word b).

If $W' = 0$, we are done. If $W' \neq 0$, let $\lambda_{1,W'} < 0$ be the top exponent of μ on W'. Then for $\mu^{\mathbb{N}}$-almost every word b,

$$\lim_{k\to\infty} \frac{1}{k} \log \|x_1 x_2 \dots x_k\| = \lambda_{1,W'} < 0.$$

Since μ has finite first moment in $GL(W)$, in particular, we have

$$\int_G \log^+(\|g\|)\, d\mu < \infty,$$

where $\log^+(x) := \max(\log(x), 0)$. This implies that (since $\|b_k\| \geq \|y_k\|$)

$$\sum_{k=1}^{\infty} \mu\left(\log^+(\|y_k\|) > -\frac{k\lambda_{1,W'}}{2}\right) \leq \sum_{k=1}^{\infty} \mu\left(\log^+(\|b_k\|) > -\frac{k\lambda_{1,W'}}{2}\right) < \infty.$$

By Borel-Cantelli Lemma, for $\mu^{\mathbb{N}}$-almost every word b,

$$\limsup_k \frac{1}{k} \log^+ \|y_k\| \leq -\frac{\lambda_{1,W'}}{2}.$$

This implies that

$$\limsup_k \frac{1}{k} \log \|x_1 \dots x_{k-1} y_k\| \leq \limsup_k \frac{1}{k} \log(\|x_1 \dots x_{k-1}\| \|y_k\|) < \frac{\lambda_{1,W'}}{2}.$$

Since $\lambda_{1,W'} < 0$, and z_i is in a compact subgroup of $GL(W/W')$ with a uniform upper bound C', there exist $n_0 = n_0(b)$ and $C'' = C''(b)$ such that for all large enough n,

$$\|Y_n\| \leq \sum_{k=1}^{n} \|x_1 \dots x_{k-1} y_k z_{k+1} \cdots z_n\| \leq C'' + C' \sum_{k=n_0}^{n} e^{k\lambda_{1,W'}/2} \leq C'' + \frac{C'}{1 - e^{\lambda_{1,W'}/2}} < \infty,$$

as desired. □

The following proposition shows that there is at least one nontrivial stationary measure in the subspace W.

Proposition 3.4.2. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Then for all $x \in W \smallsetminus W'$, any weak-$*$ limit point of the sequence of probability measures

$$\nu_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \mu^{(i)} * \delta_x$$

is a nontrivial μ-stationary probability measure on W.

Proof. Let $\widehat{W} := W \cup \{\infty\}$ be the one-point compactification of W. Then the space of probability measures $\mathcal{M}(\widehat{W})$ is compact, hence there exists a subsequence $\{n_k\}$ such that ν_{x,n_k}

converges to a probability measure $\nu \in \mathcal{M}(\widehat{W})$. Moreover,

$$\mu * \nu_{x,n_k} - \nu_{x,n_k} = \frac{1}{n_k}(\mu^{(n_k)} * \delta_x - \delta_x) \to 0.$$

Hence ν is μ-stationary. Since ∞ is a fixed point, we may consider ν as a μ-stationary measure on W (*a priori* may not be a probability measure). It remains to show that $\nu(W \smallsetminus \{0\}) = 1$. Let $\pi : W \to W/W'$ be the quotient map.

First of all since Γ_μ acts compactly on W/W' and $x \in W \smallsetminus W'$, $\Gamma_\mu \pi(x) \subset W/W'$ is compact and does not contain the origin in W/W'. Therefore there exists a compact subset $\mathcal{C}_x \subset W \smallsetminus W'$ such that $\Gamma_\mu x \subset \mathcal{C}_x + W'$. Note that $0 \notin \mathcal{C}_x + W'$. Now clearly the support of $\nu_{x,n}$ is contained in $\Gamma_\mu x \subset \mathcal{C}_x + W'$ for all n and hence the support of ν is also contained in the closed set $\mathcal{C}_x + W'$. In particular $\nu(\{0\}) = 0$.

It remains to show that for all $\varepsilon > 0$, there exists $C'' = C''(\varepsilon, x) > 0$ such that

$$\nu(\{w \in W \mid \|w\| < C''\}) > 1 - \varepsilon.$$

Since $\nu_{x,n_k} \to \nu$, applying this convergence to the indicator function $\mathbf{1}_{\{w \in W \mid \|w\| < C''\}}$, we have

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu^i(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_i x\| < C''\}) = \nu(\{w \in W \mid \|w\| < C''\}).$$

But the left hand side can be bounded from below using Fatou's lemma:

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu^i(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_i x\| < C''\}) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b) \\
&\geq \int \liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b)
\end{aligned}$$

Moreover, by Proposition 3.4.1, there exists a measurable function $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that, for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots)$,

$$\|b_1 b_2 \ldots b_n\| < C(b).$$

Now take a subset $\mathcal{B}_\varepsilon \subset G^{\mathbb{N}}$ and large enough $C'_\varepsilon > 0$ such that $\mu^{\mathbb{N}}(\mathcal{B}_\varepsilon) > 1 - \varepsilon$ and $C(b) < C'_\varepsilon$ for all $b \in \mathcal{B}_\varepsilon$. Let $C'' = C''(\varepsilon, x) := C'_\varepsilon \|x\|$. Then for all $b \in \mathcal{B}_\varepsilon$,

$$\liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b) = 1.$$

Thus

$$\nu(\{w \in W \mid \|w\| < C''\}) \geq \int \liminf_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbf{1}_{\|b_1 b_2 \ldots b_i x\| < C''}(b)\, d\mu^{\mathbb{N}}(b) \geq \mu^{\mathbb{N}}(\mathcal{B}_\varepsilon) > 1 - \varepsilon.$$

□

The following proposition shows that any stationary measure in such subspace W is uniquely determined by its pushforward on the quotient W/W'.

Proposition 3.4.3. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Let $W' \subsetneq W$ be a Γ_μ-invariant flag. Suppose

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Suppose that we have two μ-stationary measures ν and ν' on W that satisfy $\pi_*\nu = \pi_*\nu'$ for the quotient map $\pi : W \to W/W'$, then $\nu = \nu'$.

Proof. By Proposition 3.4.1, there exists a measurable map $C : G^{\mathbb{N}} \to \mathbb{R}_+$ such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, we have

$$\|b_1 b_2 \ldots b_n\|_{GL(W)} < C(b).$$

Also for almost every word b, we have the limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu.$$

Therefore we can take a limit point b_∞ of the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ in $\mathrm{End}(W)$, and

$$\nu_b = (b_\infty)_* \nu.$$

Similarly, we have, for almost every word b,

$$\nu'_b := \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu' = (b_\infty)_* \nu'.$$

Now since μ has negative top exponent on W', for almost every word b,

$$\lim_{n\to\infty} b_1 b_2 \dots b_n v = 0 \qquad \text{for every vector } v \in W'.$$

Therefore $W' \subset \ker b_\infty$, hence the map $b_\infty : W \to W$ factors through W/W', i.e. there exists a linear map $b'_\infty : W/W' \to W$ such that $b_\infty = b'_\infty \circ \pi$, where $\pi : W \to W/W'$ is the quotient map. Since $\pi_* \nu = \pi_* \nu'$, for $\mu^{\mathbb{N}}$-almost every word b, we have

$$\nu_b = (b_\infty)_* \nu = (b'_\infty)_* \pi_* \nu = (b'_\infty)_* \pi_* \nu' = (b_\infty)_* \nu' = \nu'_b.$$

Thus by Theorem 3.1.5,

$$\nu = \int_{G^{\mathbb{N}}} \nu_b d\mu^{\mathbb{N}}(b) = \int_{G^{\mathbb{N}}} \nu'_b d\mu^{\mathbb{N}}(b) = \nu'.$$

□

In particular the above proof shows that each limit measure ν_b is supported on a compact subset of W. We record this in the following proposition (which proves Theorem 3.1.6).

Proposition 3.4.4. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Let $W' \subsetneq W$ be a Γ_μ-invariant flag. Suppose

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Given an ergodic μ-stationary measure ν on W, for $\mu^{\mathbb{N}}$-almost every word b, the limit measure

$$\nu_b = \lim_{n\to\infty}(b_1 b_2 \dots b_n)_* \nu$$

is supported on the pushforward of a single compact Γ_μ-orbit on W/W' via a linear injection $p_b : W/W' \to W$. In particular, ν_b is compactly supported on W.

Proof. Take p_b to be the linear map b'_∞ defined in the proof of Proposition 3.4.3. Since $\pi_*\nu$ is an ergodic μ-stationary measure on W/W' and μ acts compactly on W/W', $\pi_*\nu$ is an ergodic Γ_μ-invariant measure and is supported on a single compact Γ_μ-orbit in W/W' by Proposition 3.2.1. Thus $\nu_b = (b'_\infty)_*\pi_*\nu$ is also compactly supported on W. □

Using Proposition 3.4.3, one can refine Proposition 3.4.2.

Proposition 3.4.5. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

For all $x \in W \setminus W'$, let

$$\nu_{x,n} := \frac{1}{n}\sum_{i=0}^{n-1} \mu^{(i)} * \delta_x.$$

Then the weak-$*$ limit

$$\nu_x := \lim_{n\to\infty} \nu_{x,n}$$

exists and is a nontrivial μ-stationary probability measure on W.

Proof. By Proposition 3.4.2, we know that any limit point of the sequence $\{\nu_{x,n}\}_n$ is a nontrivial μ-stationary measure on W. Moreover, since the projection of $\nu_{x,n}$ on W/W' lies in the compact Γ_μ-orbit of $x + W' \in W/W'$, any weak-$*$ limit point projects to a μ-stationary measure supported on the single compact orbit $\Gamma_\mu x + W' \subset W/W'$, hence is in fact the unique invariant measure supported on the compact set $\Gamma_\mu x + W'$. In particular, any limit point of $\{\nu_{x,n}\}_n$ is a μ-stationary probability measure that projects to the same measure on W/W'. By Proposition 3.4.3, all such limit points agree, so the sequence $\{\nu_{x,n}\}_n$ converges. □

In fact, if we start with any initial measure that projects to the Haar measure supported on a compact Γ_μ-orbit in W/W', then the convolution powers are not just Cesáro summable, but themselves converge.

Proposition 3.4.6. Let μ be a Borel probability measure on $G = GL(W)$ with finite first moment. Suppose there exists a proper Γ_μ-invariant subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

For all $x \in W \setminus W'$, let $\mathcal{C}_x$ be the Γ_μ-orbit of the image x in W/W', and m_x be the Haar (probability) measure on W/W' supported on $\mathcal{C}_x$. Then for any linear section $s : W/W' \to W$, i.e. a linear map such that $\pi \circ s = \mathrm{id}$, we have the following weak-$*$ convergence

$$\nu_x := \lim_{n\to\infty} \mu^{(n)} * (s_* m_x).$$

Moreover, ν_x is a nontrivial μ-stationary probability measure on W that does not depend on the choice of the linear section s. The map $x \mapsto \nu_x$ is constant along the orbit $\mathcal{C}_x$.

Proof. By Proposition 3.4.2, for all $x \in W \setminus W'$, there exists a nontrivial μ-stationary measure ν_x on W that projects to m_x on W/W'.

Similar to the proof of Proposition 3.4.3, there exists a measurable function $C : G^{\mathbb{N}} \to \mathbb{R}_+$

such that for $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots)$, we have

$$\|b_1 b_2 \ldots b_n\|_{GL(W)} < C(b), \qquad \text{and} \qquad \nu_b = \lim_{n\to\infty} (b_1 \ldots b_n)_* \nu_x$$

exists. Moreover, for any limit point b_∞ of $\{b_1 b_2 \ldots b_n \mid n \geq 1\}$ in $\mathrm{End}(W)$, there exists a linear map $b'_\infty : W/W' \to W$ such that $b_\infty = b'_\infty \circ \pi$. Let $\{n_k\}_k$ be the indices of the subsequence such that

$$\lim_{k\to\infty} b_1 b_2 \ldots b_{n_k} = b_\infty = b'_\infty \circ \pi.$$

Now for any linear section $s : W/W' \to W$, we have

$$\lim_{k\to\infty} (b_1 \ldots b_{n_k})_* (s_* m_x) = (b'_\infty)_* \pi_* s_* m_x = (b'_\infty)_* m_x$$

since $\pi \circ s = \mathrm{id}$. On the other hand since the stationary measure ν_x projects to m_x on W/W', we also have

$$\nu_b = \lim_{n\to\infty} (b_1 \ldots b_n)_* \nu_x = (b_\infty)_* \nu_x = (b'_\infty)_* \pi_* \nu_x = (b'_\infty)_* m_x.$$

Thus

$$\nu_b = (b'_\infty)_* m_x = \lim_{k\to\infty} (b_1 \ldots b_{n_k})_* (s_* m_x)$$

for any convergent subsequence $\{b_1 \ldots b_{n_k} \mid k \geq 1\}$. Since the left hand side does not depend on the subsequence, we have the convergence

$$\nu_b = \lim_{n\to\infty} (b_1 \ldots b_n)_* (s_* m_x).$$

Since this holds for $\mu^{\mathbb{N}}$-almost every b, we have

$$\nu_x = \int \nu_b \, d\mu^{\mathbb{N}}(b) = \int \lim_{n\to\infty} (b_1 \ldots b_n)_* (s_* m_x) \, d\mu^{\mathbb{N}}(b) = \lim_{n\to\infty} \mu^{(n)} * (s_* m_x).$$

□

3.5 Properties of the span of the support of a stationary measure

In this section, we prove a few properties of the action on the span of the support of a given stationary measure. The main statement is that the span of the support of a given stationary measure must have a contracting-by-compact action by μ (Proposition 3.5.5). An important auxiliary proposition leading towards this fact is Proposition 3.5.2.

Lemma 3.5.1. Let μ be a Borel probability measure on $GL(V)$, ν be a μ-stationary probability measure on V. Let W be the linear span of the support of ν. Then

(i) W is Γ_μ-invariant.

(ii) For $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$, the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is bounded from above in $GL(W)$.

Proof. (i) is clear since $\operatorname{supp} \nu$ is Γ_μ-invariant. The proof of (ii) is similar to the proof of [BP13, Lem. 3.3], using ideas of [Fur63, Thm. 1.2]. By considering the restriction of the action to W we may assume that $V = W$ and thus $G = GL(W)$ without loss of generality. For $b \in G^{\mathbb{N}}$ for which the limit measure ν_b exists, assume the contrary that the sequence $\{b_1 b_2 \dots b_n \mid n \geq 0\}$ is not bounded from above in $GL(W)$. Then we can find a subsequence $\{n_k \mid k \in \mathbb{N}\}$ and $b_\infty \in \operatorname{End}(W)$ with $\|b_\infty\|_{\operatorname{End}(W)} = 1$ such that

$$\lim_{n\to\infty} \|b_1 b_2 \dots b_{n_k}\|_{GL(W)} = \infty, \qquad \text{and} \qquad \lim_{k\to\infty} \frac{b_1 b_2 \dots b_{n_k}}{\|b_1 b_2 \dots b_{n_k}\|_{GL(W)}} = b_\infty.$$

Let $W_b := \ker b_\infty \subset W$. For $v \in W \smallsetminus W_b$, we have

$$\lim_{k\to\infty} \|b_1 b_2 \dots b_{n_k} v\|_W = \infty.$$

Thus for any continuous function $\phi : W \to \mathbb{R}$ with compact support, for all $v \in W \smallsetminus W_b$,

$$\phi(b_1 b_2 \dots b_{n_k} v) \to 0 \qquad \text{as} \qquad k \to \infty.$$

Therefore

$$\begin{aligned}
\int \phi(v)d\nu_b(v) &= \lim_{k\to\infty} \int \phi(v)d(b_1b_2\dots b_{n_k})_*\nu(v) \\
&= \lim_{k\to\infty} \int \phi(b_1b_2\dots b_{n_k}v)d\nu(v) \\
&= \lim_{k\to\infty} \int \mathbf{1}_{W_b}(v)\phi(b_1b_2\dots b_{n_k}v)d\nu(v) \\
&\le \nu(W_b) \sup_{v\in W} |\phi(v)|.
\end{aligned}$$

Since ϕ is an arbitrary continuous function on W with compact support, by taking a sequence of such ϕ supported on balls of radius $n \to \infty$ and takes value 1 within a slightly smaller open ball, we can conclude that $\nu(W_b) = 1$. Since W_b is closed, we have $\operatorname{supp} \nu \subset W_b$.

Since W_b is a subspace of W and $\operatorname{supp} \nu$ spans W, we have $\ker b_\infty = W_b = W$, i.e. b_∞ is the zero map. But this is a contradiction since $\|b_\infty\|_{\mathrm{End}(W)} = 1$. □

We shall show the following important auxiliary proposition.

Proposition 3.5.2. Let μ be a Borel probability measure μ on $G = GL(V)$ with finite first moment. Suppose there exists a μ-stationary measure ν on V such that V is the span of $\operatorname{supp} \nu$. Suppose there exist Γ_μ-invariant subspaces $0 \subset W' \subset W \subsetneq V$ such that

(i) Γ_μ acts compactly on W';

(ii) if $W' \neq W$, μ has negative top exponent on W/W';

(iii) Γ_μ acts compactly on V/W.

Then there is a Γ_μ-invariant splitting of V:

$$V = W' \oplus W''$$

for some Γ_μ-invariant subspace $W'' \subset V$.

We first prove a lemma which allows us to reduce the proposition to the case when the acting group Γ_μ is uniformly bounded from above in $GL(V)$.

Lemma 3.5.3. Under the assumptions of Proposition 3.5.2, if in addition, Γ_μ is unbounded from above with respect to the operator norm on $GL(V)$, i.e. there exists a sequence $\{g_k\} \subset \Gamma_\mu$ such that $\|g_k\|_{GL(V)} \to \infty$, then there is a nonzero Γ_μ-invariant subspace $W_0 \subset W$ such that

$$W' \cap W_0 = 0.$$

Proof. The proof is similar to that of Lemma 3.5.1(ii). By Lemma 3.5.1(ii), for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, the sequence

$$\{b_1 b_2 \dots b_n \mid n \geq 1\}$$

is bounded from above in $GL(V)$. Let b_∞ be a limit point of this sequence in $\mathrm{End}(V)$. Moreover, by Lemma 3.1.5, for all $g \in \Gamma_\mu$ and each positive integer k, we have

$$\nu_b = \lim_{n \to \infty} (b_1 b_2 \dots b_n g g_k)_* \nu = (b_\infty g g_k)_* \nu.$$

Let g_∞ be a limit point of the sequence $\{g_k / \|g_k\|\}_k$ in $\mathrm{End}(V)$. Then by the same argument as the proof of Lemma 3.5.1(ii), using the fact that $\|g_k\| \to \infty$, one can conclude that

$$b_\infty g g_\infty \equiv 0,$$

the zero map on V. Hence for all $g \in \Gamma_\mu$,

$$g g_\infty V \subset \ker b_\infty.$$

Let W_0 be the span of $\{g g_\infty V \mid g \in \Gamma_\mu\}$. Then $W_0 \subset \ker b_\infty$. Since $\|g_\infty\| = 1$, $g_\infty V$ is nonzero, so W_0 is a nonzero Γ_μ-invariant subspace of V. Moreover, since Γ_μ acts compactly on W' and b_∞ is in the closure of Γ_μ in $\mathrm{End}(V)$, $W' \cap \ker b_\infty = 0$.

On the other hand, we claim that $\ker b_\infty \subset W$. In fact, for $v \notin W$, since $b_\infty \in \Gamma_\mu$ acts compactly on V/W, we have $b_\infty v \notin W$, in particular $b_\infty v \neq 0$, so $v \notin \ker b_\infty$.

Now since $W_0 \subset \ker b_\infty$, we have that $W_0 \subset W$ and $W' \cap W_0 = 0$, as desired. □

We also need an algebraic fact about compact subsemigroups of $\mathrm{End}(V)$.

Lemma 3.5.4. [HM66, A.1.22] Let $\mathcal{S} \subset \mathrm{End}(V)$ be a nonempty compact subsemigroup. Then there exists $h \in \mathcal{S}$ such that

(a) h is idempotent, i.e. $h^2 = h$,

(b) $h\mathcal{S}h := \{hgh \mid g \in \mathcal{S}\}$ has the structure of a compact group with identity element h,

(c) there is a group action by $h\mathcal{S}h$ on hV.

For completeness we include a sketch of the proof here.

Sketch of Proof. Let r be the smallest rank among elements in $\mathcal{S}$, and let $\mathcal{S}_0 := \{g \in S \mid \mathrm{rank}(g) = r\}$. Then $\mathcal{S}_0$ is itself a compact subsemigroup of $\mathrm{End}(V)$ since the rank cannot increase when taking products and limits. By Ellis-Numakura lemma ([HM66, A.1.16]), any nonempty compact semigroup has an idempotent element, so there exists $h \in \mathcal{S}_0$ with $h^2 = h$. Then $h\mathcal{S}h$ is a compact semigroup with h acting as the identity element.

We claim that h is the only idempotent element in $h\mathcal{S}h$. In fact let h' be another idempotent element in $h\mathcal{S}h$. Then the image of h' is contained in the image of h. But h has minimal rank in $\mathcal{S}$ and $h\mathcal{S}h$ is contained in $\mathcal{S}$, so the images of h and h' are the same. Moreover, since h and h' are idempotents in $\mathrm{End}(V)$, we have the decompositions

$$V = \mathrm{im}\ h \oplus \ker h = \mathrm{im}\ h' \oplus \ker h'.$$

Since $h' \in h\mathcal{S}h$, $\ker h \subset \ker h'$. But since $\mathrm{im}\ h = \mathrm{im}\ h'$, the dimensions of $\ker h$ and $\ker h'$ agree, so $\ker h = \ker h'$. Any idempotent in $\mathrm{End}(V)$ is determined by its image and kernel, so $h = h'$.

On the other hand, one can check that if a compact semigroup K with identity has no other idempotent, then it is a compact group. In fact, for any $t \in K$, tK and Kt are nonempty compact subsemigroups of K, so they also have idempotent elements. But by assumption, this idempotent element must be the identity, so t has left and right inverses for all $t \in K$, as desired.

Thus we have shown that $K = h\mathcal{S}h$ is a compact group with identity h. $h\mathcal{S}h$ acts on hV since the identity element h acts trivially on hV. $\square$

Now we are ready to prove Proposition 3.5.2.

Proof of Proposition 3.5.2. We prove the statement by induction on $\dim V$.

Base case: $\dim V = 1$**.**

Since W is a proper subspace of V, we have $W' = W = 0$. Therefore we can take $W'' = V$.

Induction step.

If Γ_μ is unbounded from above in $GL(V)$, by Lemma 3.5.3, there exists a nonzero Γ_μ-invariant subspace $W_0 \subset W$ with $W' \cap W_0 = 0$. Now consider the Γ_μ-invariant flag

$$0 \subsetneqq W' \subset W/W_0 \subsetneqq V/W_0.$$

Since W_0 is nonzero, $\dim V/W_0 < \dim V$, so by the induction hypothesis, there exists a Γ_μ-invariant subspace $W_2 \subset V$ with $W_0 \subset W_2$ such that there is the Γ_μ-invariant splitting

$$V/W_0 = W' \oplus W_2/W_0.$$

Thus we can take $W'' = W_2$.

Hence in the remaining part of the proof we assume also that there exists $C > 0$ such that $\|g\| \leq C$ for all $g \in \Gamma_\mu$. Let $\overline{\Gamma_\mu}$ be the (topological) closure of Γ_μ in $\mathrm{End}(V)$, then $\overline{\Gamma_\mu}$ is a compact semigroup in $\mathrm{End}(V)$. By Lemma 3.5.4, there exists an idempotent $h \in \overline{\Gamma_\mu}$ (i.e. $h^2 = h$) such that

$$K := h\overline{\Gamma_\mu}h$$

is a compact group with identity h. Moreover K acts on hV, and preserves W' (note that $hW' = W'$ since Γ_μ acts compactly on W'). Since K is compact, there exists a K-invariant complementary subspace $W_1 \subset hV$ of W', i.e.

$$hV = W' \oplus W_1.$$

Note that $hW_1 = W_1$ since $h \in K$. Now let W'' be the span of $\{ghW_1 \mid g \in \overline{\Gamma_\mu}\}$. Then W'' is Γ_μ-invariant.

Let $v \in W'' \cap W'$. On one hand, $hv \in hW' = W'$, on the other hand,

$$hv \in \operatorname{span}(\{hghW_1 \mid g \in \overline{\Gamma_\mu}\}) = W_1$$

since $hgh \in K$ for $g \in \overline{\Gamma_\mu}$ and W_1 is K-invariant. Thus $hv \in W' \cap W_1 = 0$, i.e. $v \in \ker h$. Now since Γ_μ acts compactly on W', $\ker h \cap W' = 0$. But $v \in \ker h \cap W'$, so $v = 0$. Therefore $W'' \cap W' = 0$.

Hence we have found a Γ_μ-invariant subspace W'' with trivial intersection with W'. It remains to show that $W'' + W' = V$.

We first observe that $\ker h \subset W$. In fact, consider $v \notin W$. Since h acts compactly on V/W, $hv \neq 0$ in V/W, so $hv \neq 0$ in V, thus $v \notin \ker h$.

Since h is idempotent, we have that

$$V = \operatorname{im} h \oplus \ker h = W' \oplus W_1 \oplus \ker h.$$

Since $W_1 \subset W''$ and $W' \oplus \ker h \subset W$, we have

$$V = W'' + W.$$

If $W' = W$, we are done. If $W' \neq W$, by assumption, μ has negative top exponent on W/W'.

Now

$$V/W'' = (W'' + W)/W'' = W/(W'' \cap W).$$

Since W' is Γ_μ-invariant, $W' \subset W$ and $W' \cap W'' = 0$, we have the following Γ_μ-equivariant identification

$$V/(W'' \oplus W') = W/((W'' \cap W) \oplus W') = (W/W')/(W'' \cap (W/W')).$$

Since μ has negative top exponent on W/W', it also has negative top exponent on $(W/W')/(W'' \cap (W/W'))$, thus on $V/(W'' \oplus W')$. Therefore the only μ-stationary measure on $V/(W'' \oplus W')$ is δ_0. On the other hand, since ν is a μ-stationary measure on V with span(supp ν) $= V$, the pushforward of ν on $V/(W'' \oplus W')$ also spans. But this pushforward is μ-stationary on $V/(W'' \oplus W')$, so it equals δ_0. Therefore $V = W'' \oplus W'$, as desired. □

Now we are ready to prove that the μ-action on the span of the support of a stationary measure is contracting-by-compact.

Proposition 3.5.5. [Bou87, Thm. 5.1 necessity direction] Let μ be a Borel probability measure μ on $G = GL(V)$ with finite first moment, and ν be a nontrivial μ-stationary measure on V. Let W be the linear span of supp ν. Then there exists a Γ_μ-invariant proper subspace $W' \subsetneq W$ such that

(i) Γ_μ acts compactly on W/W', and

(ii) if $W' \neq 0$, μ has negative top exponent on W'.

Proof. We prove this by induction on $\dim W$.

Base case: $\dim W = 1$.

In this case, Γ_μ acts irreducibly on W. By Proposition 3.3.1, Γ_μ acts compactly on W and we can take $W' = 0$.

Induction step.

If Γ_μ acts irreducibly on W, then again by Proposition 3.3.1, Γ_μ acts compactly on W and we can take $W' = 0$.

If Γ_μ does not act irreducibly on W, take a minimal nonzero Γ_μ-invariant proper subspace $0 \subsetneq W_0 \subsetneq W$. The pushforward of ν under the map $W \to W/W_0$ is a stationary measure on W/W_0 whose support spans W/W_0. Since $\dim W/W_0 < \dim W$, by the induction hypothesis, we know that there exists a Γ_μ-invariant proper subspace $W_1 \subsetneq W$ such that

(i) $0 \subsetneq W_0 \subset W_1 \subsetneq W$,

(ii) Γ_μ acts compactly on W/W_1, and

(iii) either $W_1 = W_0$ or μ has negative top exponent on W_1/W_0.

By minimality of W_0, we know that Γ_μ acts irreducibly on W_0. Since W is the linear span of $\operatorname{supp} \nu$, by Lemma 3.5.1, for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, the sequence $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is bounded from above in $GL(W)$. By Lemma 3.2.4, $\{b_1 b_2 \dots b_n \mid n \geq 1\}$ is also bounded from above in $GL(W_0)$. Thus by Corollary 3.3.3, either μ has negative top exponent on W_0 or Γ_μ acts compactly on W_0.

Case 1: μ has negative top exponent on W_0.

We claim that in this case, μ has negative top exponent on W_1. The claim is clear if $W_1 = W_0$. If $W_0 \subsetneq W_1$, since μ has negative top exponent on W_1/W_0, by Lemma 3.2.3, μ also has negative top exponent on W_1. Thus we can take $W' = W_1$.

Case 2: μ acts compactly on W_0.

In this case, by Proposition 3.5.2, there exists a proper Γ_μ-invariant subspace $W_2 \subsetneq W$ such that

$$W = W_0 \oplus W_2.$$

Let $W_2' := W_1 \cap W_2$. Then we can Γ_μ-equivariantly identify W_2' and W_1/W_0. Thus either

$W_2' = 0$ or μ has negative top exponent on W_2', and Γ_μ acts compactly W_2/W_2'. Moreover, since

$$W/W_2' = W_0 \oplus W_2/W_2',$$

and Γ_μ acts compactly on W_0 and W_2/W_2', we have that Γ_μ acts compactly on W/W_2'. Therefore we can take $W' = W_2'$.

□

3.6 Proofs of the main theorems

Using properties proved in the previous two sections, we can now prove the main theorems.

Proof of Theorem 3.1.1. Let $W \subset V$ be the Γ_μ-invariant subspace of maximal dimension such that $W = \text{span}(\text{supp}\, \nu_0)$ for some μ-stationary measure ν_0 on V.

We now claim that every μ-stationary measure ν satisfies supp $\nu \subset W$. In fact, assume that there is some stationary measure ν' such that supp $\nu' \not\subset W$. Let $U = \text{span}(\text{supp}\, \nu')$. Now let $\nu'' = \frac{1}{2}\nu + \frac{1}{2}\nu'$. Then $W + U = \text{span}(\text{supp}\, \nu'')$. Since $W + U$ has strictly larger dimension than W, this contradicts the maximality of $\dim W$, hence condition (i) in the theorem holds.

By Proposition 3.5.5, there exists a Γ_μ-invariant proper subspace $W' \subsetneq W$ such that Γ_μ acts compactly on W/W', and if $W' \neq 0$, μ has negative top exponent on W'. Thus (ii) and (iii) in the theorem hold. □

Proof of Theorem 3.1.2. Let $\pi : W \to W/W'$ be the quotient map. By Theorem 3.1.1 and Proposition 3.2.1, the map

$$\Phi : \{\text{ergodic } \mu\text{-stationary measure on } V\} \quad \to \quad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

defined by $\Phi(\nu) := \text{supp}\, \pi_*\nu$ is well-defined.

- **Φ is injective.**

This follows from Proposition 3.4.3 and the uniqueness of the Γ_μ-invariant measure supported on a single compact Γ_μ-orbit.

- **Φ is surjective and determine Φ^{-1}**

  The origin 0 of W/W' is a compact invariant subset of W/W', and is the image of the invariant measure δ_0 on V. Now given a compact Γ_μ-invariant subset $\mathcal{C} \neq \{0\}$ in W/W', let $x \in \pi^{-1}(\mathcal{C}) \subset W \setminus W'$. By Proposition 3.4.6, $\nu_x = \lim_{n\to\infty} \mu^{(n)} * (s_* m_x)$ is a μ-stationary probability measure on V such that supp $\pi_*\nu_x$ is $\mathcal{C}$, where as we recall, $s : W/W' \to W$ is any linear section and m_x is the unique Γ_μ-invariant measure supported on $\mathcal{C}$. Thus $\mathcal{C} \mapsto \nu_x$ is the inverse of Φ.

□

Proof of Theorem 3.1.3. The first claim was proved in Proposition 3.4.6. The second claim was shown in the proof of Theorem 3.1.2. □

Proof of Theorem 3.1.4. The convergence of the limit in the first claim was shown in Proposition 3.4.5. That the limiting measure is $\nu_{\mathcal{C}}$ follows from the injectivity of Φ in Theorem 3.1.2. The second claim is true since by Breiman's law of large number [Bre60], for $\mu^{\mathbb{N}}$-almost every word $b \in G^{\mathbb{N}}$, every weak-* limit point of the empirical measures is a μ-stationary probability measure. Now the rest follows from the same argument as Proposition 3.4.5 and the injectivity of Φ in Theorem 3.1.2. □

Proof of Theorem 3.1.6. This follows from Proposition 3.4.4. □

CHAPTER 4

RANDOM WALKS ON HOMOGENEOUS SPACES WITH NONDISCRETE QUOTIENTS

4.1 Introduction

Let G be a (real) Lie group, $H \subset G$ is a closed unimodular subgroup. Let μ be a Borel probability measure on G, and let $\Gamma_\mu := \overline{\langle \operatorname{supp} \mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by the support of μ.

In this paper, we are interested in studying the μ-stationary measures on a homogeneous space G/H with respect to the Γ_μ-action by left multiplication. We first recall the definition of stationary measures.

Definition. Suppose G acts on a Borel space X. We say that a Borel probability measure ν on X is *μ-stationary* if $\mu * \nu = \nu$, i.e.

$$\nu = \int_G g_* \nu \; d\mu(g).$$

Clearly if ν is Γ_μ-invariant then it is μ-stationary. On the other hand, if Γ_μ is abelian (for instance if Γ_μ is isomorphic to $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ or $\mathbb{N}$), then every μ-stationary measure is Γ_μ-invariant by the classical Choquet-Deny theorem ([CD60], see [BQ16, Cor. 2.22] for a short proof). Therefore stationary measures can be considered a natural generalization of invariant measures, which is a major object of interest in dynamics.

It has long been observed in the literature that while the space of invariant measures of a typical dynamical system given by an $\mathbb{R}$ or $\mathbb{Z}$-action is rich and flexible (for instance there can be invariant measures whose support has arbitrary Hausdorff dimension up to the dimension of the space), the space of stationary measures of a random dynamical system given by a "large enough" semigroup Γ_μ is quite rigid. One of the first instances of such phenomena was observed by Furstenberg ([Fur67]), who famously conjectured that the only Borel probability measures on the circle S^1 invariant and ergodic under $\times 2$ and $\times 3$ are either finitely supported or the Lebesgue measure.

The first result that is close to our setting was given by Bourgain, Furman, Lindenstrauss and Mozes [BFLM11]. They considered the action of a semigroup $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ on the n-torus $\mathbb{T}^n$ such that Γ acts strongly irreducibly on $\mathbb{R}^d$ is proximal, and showed using techniques from Fourier analysis and additive combinatorics that for any probability measure μ that is supported on a set of generators of Γ and satisfies a suitable moment condition, the only ergodic μ-stationary measures on $\mathbb{T}^n$ are either finitely supported or the Lebesgue measure.

Later, Benoist and Quint [BQ11] gave another proof of this result without the proximality assumption using techniques from ergodic theory. Moreover, their proof also applies to the μ-action on homogeneous spaces of the form G/Λ, where G is a simple real Lie group, Λ is a lattice in G, and μ is a probability measure on G whose support is Zariski dense in G. They showed that in this case, the only ergodic μ-stationary measures are either finitely supported or the Haar measure. In a later paper [BQ13a], they generalized the result to the setting where G is a real Lie group (they also have analogous statements for S-arithmetic groups), Λ is a discrete subgroup of G, and μ is a compactly supported measure on G such that the Zariski closure of its support is semisimple, Zariski connected and has no compact factors. They showed that in this case, every ergodic μ-stationary measure on G/Λ is homogeneous.

This result was extended by Eskin and Lindenstrauss [ELa] using slightly different techniques in ergodic theory inspired by the ideas of Eskin and Mirzakhani [EM18] in the context of Teichmüller dynamics. They also considered the G-action on G/Λ, but they relaxed the assumption that the support of μ has semisimple Zariski closure to an assumption they called "uniform expansion". We state a special case of their main result as it will be relevant to our main statement.

Theorem (Eskin-Lindenstrauss)**.** [ELa, Thm. 1.7] Let G be a real Lie group and Γ be a discrete subgroup of G. Suppose that μ is a probability measure on G with finite first moment (to be defined in the next section), and let Γ_μ be the (topological) closure of the semigroup generated by the support of μ.

Let ν be an ergodic μ-stationary probability measure on G/Γ. Suppose that μ is uniformly expanding on $\mathfrak{g}$. Then one of the following holds.

(a) There exists a closed subgroup $H \subset G$ with $\dim(H) > 0$ and an H-homogeneous probability measure ν_0 on G/Γ such that the unipotent elements of H act ergodically on ν_0, and there exists a finite μ-stationary measure λ on G/H such that

$$\nu = \lambda * \nu_0 := \int_G g_* \nu_0 \, d\lambda(g).$$

(b) The measure ν is Γ_μ-invariant and finitely supported.

We record a few remarks about H in the statement. First, it is a nondiscrete closed subgroup of G since $\dim(H) > 0$. Second, H is unimodular by the existence of an H-homogeneous probability measure, i.e. a translate of an H-invariant probability measure on $H/H \cap g\Gamma g^{-1}$ for some $g \in G$, which implies that H admits a lattice subgroup $H \cap g\Gamma g^{-1}$. Third, H may have infinitely many connected components. Fourth, it follows easily from [BQ11, Prop. 6.7] that if G is connected and simple, and Γ_μ is Zariski dense in G, then the only possible H in (a) is G. Using this observation, the theorem of Eskin-Lindenstrauss easily implies the main statement of [BQ11] that every ergodic stationary measure is either finitely supported or Haar (this was already observed and used in the last step of [BQ11] - see [BQ11, Lem. 8.2]). A similar restriction on H was also observed in the case when the Zariski closure of Γ_μ is semisimple (see the proof of [BQ13a, Thm. 2.7], using [BQ13a, Prop. 5.19]), which allows one to conclude, for instance, that every ergodic stationary measure is homogeneous.

However, in the setting of Eskin-Lindenstrauss, without assuming that the Zariski closure of Γ_μ is semisimple, the possibility of H such that G/H admits a μ-stationary measure is much less restrictive. In particular there is not enough restriction on H to conclude that the stationary measure on G/Γ is homogeneous, unlike in the situation of [BQ13a]. In fact, it was already observed in [ELa] that there exists an example of μ and G/Γ that satisfies the assumptions of [ELa, Thm. 1.7] and admits a non-homogeneous stationary measure. This prompts the natural question:

Question. What are the possible stationary measures on G/H when H is a closed nondiscrete unimodular subgroup of G?

The purpose of this paper is threefold:

1. Generalize the result of Eskin-Lindenstrauss to study the possible μ-stationary measures on G/H, where H is a closed nondiscrete unimodular subgroup of G, under suitable easily verifiable assumptions on μ that is analogous to the "uniform expansion" assumption introduced in [ELa].

2. Combine such an understanding of stationary measures on G/H with [ELa, Thm. 1.7] to understand a clearer picture of stationary measures on G/Γ under suitable assumptions.

3. Demonstrate how to apply the technique of Eskin-Lindenstrauss, the main ideas of which first introduced in Eskin-Mirzakhani [EM18], to a fiber bundle where stationary measures on the base are classified and well-understood, to generate extra invariance in the fiber direction.

4.1.1 Main Statement

We will need the following definitions to state the main results.

Definition. A Borel probability measure μ on G has *finite first moment* if

$$\int_G \log \max(\|g\|, \|g^{-1}\|) d\mu(g) < \infty.$$

As in [ELa], we shall use the following definition of an H-envelope.

Definition. Given a Lie group G and $H \subset G$ a closed subgroup, let H° be the connected component of the identity in H. A subgroup $L \subset G$ is called an ***H*-envelope** if the following holds:

(i) $L \supset H$ and H° is normal in L.

(ii) The image of H in L/H° is discrete.

(iii) There exists a representation $\rho : G \to GL(W)$ and a vector $v_L \in W$ such that the stabilizer of v_L is L.

The point of this definition is that (i) and (ii) imply that $L/H = (L/H^\circ)/(H/H^\circ)$ is a discrete quotient of a real Lie group L/H°, while (iii) implies that there is a G-equivariant smooth injection from G/L to a vector space W (by sending $g \mapsto \rho(g)v_L$), therefore a stationary measure on G/L can be considered a stationary measure on W. For unimodular H, there are at least two common constructions of an H-envelope L.

1. Let ρ_H be a nonzero element in $\bigwedge^{\dim H} \mathfrak{h} \subset \bigwedge^{\dim H} \mathfrak{g}$. Define

$$L := \{g \in N_G(H^\circ) \mid g_* \rho_H = \rho_H\}.$$

   In words, L contains elements in the normalizer of the connected component H° that preserves the Haar measure on H. Since H is unimodular, $H \subset L$. The other conditions are satisfied, as can be readily checked.

2. If G is an algebraic group and Γ an arithmetic lattice, one can take

$$L = \text{ Zariski closure of } \Gamma \cap N_G(H^\circ) \text{ in } G.$$

   On one hand, by a theorem of Chevalley (e.g. [Hum75, Thm. 11.2]), there exists a representation $\rho : G \to GL(W)$ and a one dimensional subspace $\ell \subset W$ such that L is the stabilizer of ℓ. On the other hand, L has no nontrivial character , therefore L fixes ℓ pointwise, hence we can take v_L to be any nonzero vector in ℓ.

Definition. Let μ be a Borel probability measure on G. Suppose $H \subset G$ is a closed subgroup and L is an H-envelope. We say that μ is **uniformly expanding on L/H at x** if for all

$\mathbf{v} \in \mathfrak{l}_x := \mathrm{Lie}(xLx^{-1})$, for $\mu^{\mathbb{N}}$-a.e $\omega^+ \in G^{\mathbb{N}}$, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n)\mathbf{v}\|_{(\mathfrak{l}/\mathfrak{h})_{T_\omega^n x}} > 0,$$

where $T_\omega^n := \omega_n \cdots \omega_1 \in G$, and $\|\cdot\|_{\mathfrak{l}_x/\mathfrak{h}_x}$ is a norm on $(\mathfrak{l}/\mathfrak{h})_{\hat{x}}$ induced by a fixed norm on $\mathfrak{g}$. For instance, define $\|\mathbf{v}\|_{(\mathfrak{l}/\mathfrak{h})_{\hat{x}}} := \|\mathbf{v} \wedge \rho_{\mathfrak{h}_x}\|_{\mathfrak{g}}/\|\rho_{\mathfrak{h}_x}\|_{\mathfrak{g}}$, where $\rho_{\mathfrak{h}_x}$ is a nonzero element in the one-dimensional subspace $\bigwedge^{\dim H} \mathfrak{h}_x \subset \bigwedge^{\dim H} \mathfrak{g}$, and $\|\cdot\|_{\mathfrak{g}}$ is a fixed norm on $\bigwedge^{\dim H+1} \mathfrak{g}$.

Theorem 4.1.1. Let G be a real linear algebraic group, and μ be a Borel probability measure on G with finite first moment. Let Γ_μ be the (topological) closure of the subsemigroup generated by the support of μ in G, and $\overline{\Gamma}_\mu^Z$ be the Zariski closure of Γ_μ.

Let $H \subset G$ be a closed unimodular subgroup, and H° be the connected component of the identity in H. Suppose there exists an H-envelope L and $x_0 \in G/L$ such that μ is uniformly expanding on L/H at x_0.

Let $\nu_{G/H}$ be an ergodic μ-stationary probability measure on $\overline{\Gamma}_\mu^Z x_0 L/H$. We also assume that

(†) There exists a closed normal subgroup $U \subset \overline{\Gamma}_\mu^Z$ and some $z_0 \in G$ with $z_0 L = x_0 L$ such that $\overline{\Gamma}_\mu^Z x_0 L = U z_0 H^\circ$ and $z_0^{-1} U z_0 \cap H^\circ = \{\mathrm{id}\}$.

Then one of the following holds:

(I) there exist a Lie subgroup $H' \subset G$ with $H^\circ \subset H' \subset L \subset G$ and $\dim(H'/H^\circ) > 0$, an H'-homogeneous probability measure $\nu_{L/H}$ on L/H and finite μ-stationary measure $\nu_{G/H'}$ on $\overline{\Gamma}_\mu^Z xL/H'$ such that

$$\nu_{G/H} = \nu_{G/H'} * \nu_{L/H} := \int_{G/H'} g_* \nu_{L/H} \, d\nu_{G/H'}(g).$$

(II) the stationary measure $\nu_{G/H}$ can be written as

$$\nu_{G/H} = \int_{G/L} \nu_x \, d\overline{\nu}(x),$$

where

(a) $\overline{\nu}$ is a generalized μ-Bernoulli measure (see Definition 4.3.9) supported on $\overline{\Gamma}_\mu^Z x_0 L/L$,

(b) there exists a positive integer k such that for $\overline{\nu}$-almost every $x \in G/L$, ν_x is the uniform measure on k points in $\pi^{-1}(x) = xL/H$, where $\pi : G/H \to G/L$ is the natural quotient map,

(c) there exists a Γ_μ-invariant locally Zariski closed subset $\mathcal{F}$ such that supp $\nu_{G/H} \subset \mathcal{F}$, and $\mathcal{F}$ has finite intersection with xL/H for all $x \in \overline{\Gamma}_\mu^Z x_0 L/L$ (the set $\mathcal{F}$ is defined dynamically and can be made more explicit and computable - see Theorem 4.4.9).

We remark that if H is a discrete subgroup of G, this statement recovers [ELa, Thm. 1.7] for trivial Z (in this case (†) is satisfied with $U = \overline{\Gamma}_\mu^Z$).

Remark 4.1.2. We have the following remarks regarding the assumptions of the theorem.

1. The assumption of uniform expansion on L/H is the main assumption of the theorem, and is analogous to the uniform expansion assumption in [ELa]. Note that we only require uniform expansion in the fiber direction above a single point $x \in G/L$, in particular it is readily verifiable. However the tradeoff is that we can only consider stationary measures supported on $\overline{\Gamma}_\mu^Z x_0 L/H$.

2. The reason to assume that $\nu_{G/H}$ is a stationary measure on $\overline{\Gamma}_\mu^Z x_0 L/H$ rather than on G/H is twofold: firstly a simple ergodicity argument shows that any ergodic measure on G/L is supported on a single $\overline{\Gamma}_\mu^Z$-orbit. Now any ergodic stationary measure ν on G/H induces a ergodic stationary measure $\pi_*\nu$ on G/L, so $\pi_*\nu$ is supporte on $\overline{\Gamma}_\mu^Z x_0 L/L$ for some $x \in G/L$, and hence ν is supported on $\pi^{-1}(\overline{\Gamma}_\mu^Z x_0 L/L) = \overline{\Gamma}_\mu^Z x_0 L/H$. On the other hand, the assumption of uniform expansion on L/H at x_0 ensures uniform expansion on L/H at x' for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, thus focusing only on measures supported on $\overline{\Gamma}_\mu^Z x_0 L/H$ ensures that the assumption on that one fiber above x_0 is relevant to ν-almost every point.

3. The assumption (†) is only used in Case II (see Theorem 4.4.9). In particular it is only

used in Section 4.11. It is our intention to remove this assumption in the final version. Subsection 4.11.1 records all the conseqeuences we need from assumption (†).

Theorem 4.1.1 together with [ELa, Thm. 1.7] form one step of an induction scheme, which allows one to obtain more information about measure rigidity even in the special case considered in [ELa] (with extra assumptions in the form of uniformly expanding on L/H). In some cases this would be enough to completely classify the ergodic stationary measures, and we will demonstrate one such example (and it will be clear how to generalize the example to a family of such) in Section 4.2 where all the ergodic stationary measures can be classified.

4.1.2 Ideas of the proof

As mentioned in the introduction, the main idea is to apply a bundle version of the technique of Eskin-Lindenstrauss. More precisely, we consider the bundle $\pi : G/H \to G/L$ given by the natural quotient map, where L is an H-envelope. We have the following observations based on the definition of an H-envelope.

1. Let ν be an ergodic μ-stationary measure on G/H, then $\overline{\nu} := \pi_*\nu$ is an ergodic μ-stationary measure on G/L. By the remarks following the definition of an H-envelope, we know that there is an algebraic homomorphism $\rho : G \to GL(V)$ that induces a G-equivariant injection $\rho : G/L \to GL(V)$ given by $g \mapsto \rho(g)v$, where L is the stabilizer of v. Thus $\overline{\nu}$ induces an ergodic μ-stationary measure $\rho_*\overline{\nu}$ on $V \setminus \{0\}$. We will see that (Theorem 4.3.1) unless $\Gamma_\mu \subset L$, the existence of such stationary probability measure imposes severe restrictions on Γ_μ. Furthermore such ergodic stationary measures on V can be completely classified and explicitly described (see Section 4.3) as self-affine measures on V. The pullback of such measures on G/L will be called *generalized μ-Bernoulli measures.*

2. The definition of an H-envelope implies that $L/H \cong (L/H^\circ)/(H/H^\circ)$ is a homogeneous space of the Lie group L/H° (since H° is normal in L) with the discrete subgroup $H/H^\circ \subset L/H^\circ$. Since each fiber of the bundle $\pi : G/H \to G/L$ is a translate of L/H, each fiber falls in the setting of Eskin-Lindenstrauss. Furthermore, the assumption of uniform

expansion on L/H at some $x \in G/L$ is the same as the uniform expansion assumption of Eskin-Lindenstrauss applied to the fiber xL/H.

The second remark suggest a simple way to adapt the drift method of Eskin-Lindenstrauss - one performs the drift argument by taking two points in the same L/H-fiber, and run the drift to gain extra invariance in the fiber direction. In fact, this was the approach taken by [SS19] in adapting the method of Benoist-Quint to the bundle of interest in their case. However, this adaptation will not give the Case II conclusion (c) in our main statement, since it does not relate the conditional measures on nearby L/H-fibers at all.

Our method, instead consider two starting points that may be in different fibers (though still stably related as in the method of Eskin-Lindenstrauss). This, however, imposes extra difficulty since unlike in the case of G/Γ, where there is a natural identification of the tangent spaces at every point on G/Γ with the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ via right multiplication, in the case of G/H where H has nontrivial identity component (that is also not normal in G in general), the tangent spaces at different points of G/H cannot be canonically identified. Such identification was used in certain constructions in Eskin-Lindenstrauss (for instance the P^- map in [ELa, Sect. 2.2] and linear map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ in [ELa, Sect. 4]) in a crucial way. Therefore to adapt their method in this situation, we need to apply a factorization technique, pioneered in the work of Eskin-Mirzakhani, to our setting. This is the content of Section 4.6 and is the main new ingredient of this work. We will also need to use the natural holonomy maps H_i^- (subsection 4.5.6) in this context to construct the P^- maps (subsection 4.5.7) and other constructions.

4.1.3 Outline

The outline of the paper is as follows:

- In Section 4.2, we will discuss an example where we can use Theorem 4.1.1 to completely classify the stationary measures, and where not all ergodic stationary measures are homogeneous. This is a canonical example of cases where measure classification can be done using Theorem 4.1.1.

- In Section 4.3, we summarize the results about the classification of stationary measures on finite-dimensional real vector spaces from Chapter 3 of this thesis. Most of the results are consequences of [Bou87, Thm. 5.1]. The main result that will be used in the future sections is Corollary 4.3.8, which gives a description of the Γ_μ-action on G/L and the possible stationary measures on G/L.

- In Section 4.4, we discuss the basic setup of our setting, and recall a few basic facts from [ELa]. This includes the general construction of a two-sided skew product from a stationary measure, a choice of metric on G/H, the stable and unstable manifolds, and how we split the two cases in the main argument.

- In Section 4.5, we recall the decomposition of the tangent spaces using Oseledets theorem and Zimmer's amenable reduction theorem. In particular we will discuss the relationship between the Lyapunov spaces of G/H, G and L/H. We also discuss the construction of the suspension space, the holonomy maps, the equivariant measurable flat connections P^-, and the dynamically defined norm on the tangent spaces.

- In Section 4.6, we describe the factorization procedure in the case of G/H, which is the main new ingredient of our paper. In particular we define the linear map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ which plays an important role in the main argument. The main result of this section is Theorem 4.6.5.

- In Section 4.7, we discuss a key divergence estimate (Proposition 4.7.2) of the norm of $\mathcal{A}(\hat{q}_1, u, \ell, t)$ under the assumption of Case I and uniform expansion on L/H. We also record a general lemma on conditional measures (Lemma 4.7.6) already appeared in [ELa] which allows us to choose points appropriately in a good compact set in the main argument.

- In Section 4.8, we describe the inert subbundle **E** of the tangent bundle. The main result is that under the assumption of Case I and uniform expansion on L/H, the image of $\mathcal{A}(\hat{q}_1, u, \ell, t)$ converges to the inert bundle (Proposition 4.8.7) as $\ell \to \infty$.

- In Section 4.9, we cite the necessary theorems from [ELa] that describe the tie-breaking procedure. Since the entire procedure happens in a single L/H fiber, the corresponding theorems can be quoted direclty from [ELa].
- In Section 4.10, we describe the main argument of case I using the Eskin-Mirzakhani scheme. We first give a detailed outline of the procedure, including how the main results in the previous section fit into the procedure, and then prove the claims afterwards.
- In Section 4.11, we prove Theorem 4.4.9, which describes the conclusion under the assumption of Case II.

4.2 Example with non-homogeneous stationary measures

Before presenting the proofs of the main theorem, we present an explicit example of a random walk where the classification of stationary measures can be done using Theorem 4.1.1 and (as far as we can tell) does not follow from previous results in the literature. One feature of this example is that the ergodic stationary measures are non-homogeneous.

Let $G = \mathrm{SL}_4(\mathbb{R})$,

$$H = \begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & * \\ 0 & \mathrm{SL}_2(\mathbb{Z}) \end{pmatrix}, \quad H^\circ = \begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & * \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & * \\ 0 & \mathrm{SL}_2(\mathbb{R}) \end{pmatrix}.$$

Note that $L/H \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and L is an H-envelope. Define the acting elements (i.e. the support of μ)

$$g_+ = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 4 & 0 & 1 & 1 \end{pmatrix}, \qquad g_- = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -4 & 0 & 1 & 2 \end{pmatrix}.$$

Then

$$\overline{\Gamma}_\mu^Z = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix} \in G : xy = 1 \right\}.$$

If we let

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix},$$

then one can check that (†) is satisfied with $x_0 = L \in G/L$ and $z_0 = \mathrm{id} \in G$ (i.e. U is normal in $\overline{\Gamma}_\mu^Z$, $\overline{\Gamma}_\mu^Z L = UH^\circ$ and $U \cap H^\circ = \{\mathrm{id}\}$). Furthermore μ is uniformly expanding on L/H at $x_0 = L$.

Our goal is to classify the μ-stationary measures on $\overline{\Gamma}_\mu^Z L/H$. We first remark that the μ-action on $U/(U \cap H) \cong UH/H = \overline{\Gamma}_\mu^Z L/H$ is given by the following calculation: for $A, X \in \mathrm{SL}_2(\mathbb{R}), \mathbf{b}, \mathbf{v} \in \mathbb{R}^2, \lambda \in \mathbb{R}$,

$$\begin{pmatrix} \lambda & 0 & \mathbf{0} \\ 0 & \lambda^{-1} & \mathbf{0} \\ \mathbf{b} & \mathbf{0} & A \end{pmatrix} \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{v} & \mathbf{0} & X \end{pmatrix} H = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{v}' & \mathbf{0} & AX \end{pmatrix} \begin{pmatrix} \lambda & 0 & \mathbf{0} \\ 0 & \lambda^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{pmatrix} H = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{v}' & \mathbf{0} & AX \end{pmatrix} H$$

where

$$\mathbf{v}' := \lambda^{-1}(A\mathbf{v} + \mathbf{b}). \tag{4.2.1}$$

Thus for any ergodic μ-stationary measure ν on $\overline{\Gamma}_\mu^Z L/H$, via the natural map $\overline{\Gamma}_\mu^Z L/H =$

$UH/H \cong U/(U\cap H) \to U/(U\cap L)$, one obtains a μ-stationary measure $\overline{\nu}$ on $U/(U\cap L)$, where $g_\pm$ acts on $U/(U\cap L)$ by (4.2.1), explicitly written as:

$$g_+ : \mathbf{v} \mapsto \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{v} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad g_- : \mathbf{v} \mapsto \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{v} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(Here we are using the identifications $U/(U\cap L) \leftrightarrow \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{v} & \mathbf{0} & X \end{pmatrix} L \leftrightarrow \mathbf{v}.$) Note that 4 is greater than the top Lyapunov exponent of the random walk given by the matrices $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (for instance it is larger than the norm of both matrices), therefore $g_\pm$ are contractions on $U/(U\cap L)$, hence the ergodic μ-stationary measures $\overline{\nu}$ on $\overline{\Gamma}_\mu^Z L/L$ can be completely classified (in terms of generalized μ-Bernoulli measures) using the statements in Section 4.3.

Now we apply Theorem 4.1.1. If Case I holds with $H' = L$, then we are done - there is only on L-homogeneous probability measure $\nu_{L/H}$ on L/H, namely the Haar measure $\mathrm{Haar}_{\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})}$ on $L/H \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ in our case, and we have a complete classification of stationary measures $\nu_{G/L}$ on G/L, therefore any μ-stationary probability measure ν on $\overline{\Gamma}_\mu^Z L/H$ is of the form

$$\nu = \int_{G/L} g_* \mathrm{Haar}_{\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})}\, d\nu_{G/L}(g).$$

If Case II holds, let

$$\mathcal{F}_{\overline{U}}^{\leq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{pmatrix}.$$

It can be computed that $\mathcal{F}_{G/H}^{\leq 0}[z] \cap \overline{\Gamma}_{\mu}^{Z} L/H = \mathcal{F}_{\overline{U}}^{\leq 0} z$ for all $z \in \overline{\Gamma}_{\mu}^{Z} L/H$. Thus by the Case II conclusion of Theorem 4.1.1 (in the form of Theorem 4.4.9), ν is supported on finitely many cosets of $\mathcal{F}_{\overline{U}}^{\leq 0}$ permuted by Γ_μ, moreover it projects to a stationary measure on $\overline{\Gamma}_{\mu}^{Z} L/L$ that was classified in Section 4.3. (There is a third possibility where Case I holds with $H' \subset L$ such that H'/H° is a one-parameter unipotent subgroup of L/H°, in which case one applies the theorem again to conclude that there is no μ-stationary measure that is not already included in the previous two cases. We omit this straightforward but tedious analysis in this expository section.)

4.3 Stationary measures on vector spaces

In this section, we summarize the statements about classifying stationary measures of linear actions on a vector space from Chapter 3, and deduce Corollary 4.3.8 from them. The results in this section will be a crucial input to the general statement. The key statements for the purpose of the future sections are Theorem 4.3.1, 4.3.2 and Corollary 4.3.8.

Let μ be a Borel probability measure on $G = GL(V)$ for some finite dimensional (real) vector space V. Let $\Gamma_\mu := \overline{\langle \mathrm{supp}\ \mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by the support of μ.

Consider the action of $GL(V)$ on V by left multiplication. In this section we classify the μ-stationary probability measures on V with respect to this action. The main input is the result [Bou87, Thm. 5.1].

Since the origin of V is a fixed point of this linear action, the delta mass δ_0 at the origin is always an Γ_μ-invariant measure (hence in particular μ-stationary). We say that a μ-stationary measure ν on V is *nontrivial* if $\nu \neq \delta_0$.

Definition. We define the *top Lyapunov exponent* of μ on a Γ_μ-invariant subspace $W \subset V$ as

$$\lambda_{1,W} = \lambda_{1,W}(\mu) := \lim_{n\to\infty} \frac{1}{n} \int_{GL(V)} \log \|g\|_{GL(W)} d\mu^{(n)}(g),$$

where $\mu^{(n)} := \mu * \mu * \cdots * \mu$ is the n-th measure power of μ, and for $g \in GL(V)$, $\|g\|_{GL(W)}$ denotes the operator norm of the restriction $g|_W$ in $GL(W)$.

The following result, which follows immediately from [Bou87, Thm. 5.1], gives a necessary and sufficient condition for the existence of a nontrivial μ-stationary measure on V.

Theorem 4.3.1 (Theorem 3.1.1 of this thesis)**.** Let μ be a Borel probability measure on $GL(V)$ with finite first moment. Then there exists a nontrivial μ-stationary measure ν on V if and only if there exist Γ_μ-invariant subspaces $W' \subsetneq W \subset V$ such that

(i) Γ_μ acts compactly on W/W', i.e. the image of $\rho_{W/W'} : \Gamma_\mu \to GL(W/W')$ is compact,

(ii) either $W' = 0$, or the top Lyapunov exponent of μ on W' is negative,

(iii) the support of every μ-stationary probability measure on V is in W.

The following result classifies the stationary measures on V in terms of the compact Γ_μ-orbits on W/W'.

Theorem 4.3.2 (Theorem 3.1.2 of this thesis)**.** Suppose there is a nontrivial μ-stationary measure on V and let $W' \subsetneq W \subset V$ be the Γ_μ-invariant subspaces from Theorem 4.3.1. Then the map $\nu \mapsto \operatorname{supp} \pi_*\nu$ gives a one-to-one correspondence between

$$\{\text{ergodic } \mu\text{-stationary measure on } V\} \quad \leftrightarrow \quad \{\text{compact } \Gamma_\mu\text{-orbit in } W/W'\},$$

where $\pi : W \to W/W'$ is the quotient map.

We can describe the inverse map in a more explicit way in terms of the asymptotic behavior in law of the random walk on V induced by μ.

Theorem 4.3.3 (Theorem 3.1.3 of this thesis)**.** For any compact Γ_μ-orbit $\mathcal{C}$ in W/W', let $m_{\mathcal{C}}$ be the Haar (probability) measure supported on $\mathcal{C}$. Let $s : W/W' \to W$ be a linear section, i.e. a linear map such that $\pi \circ s = \mathrm{id}$. Then the weak-$*$ limit

$$\nu_{\mathcal{C}} := \lim_{n\to\infty} \mu^{(n)} * (s_* m_{\mathcal{C}})$$

exists and does not depend on the choice of the section s. Moreover, the map $\mathcal{C} \mapsto \nu_{\mathcal{C}}$ is the inverse map of the bijection in Theorem 4.3.2.

Using the classification of stationary measures, we can obtain the following equidistribution result.

Theorem 4.3.4 (Theorem 3.1.4 of this thesis)**.** For all $x \in W$, let $\mathcal{C}$ is the compact Γ_μ-orbit of $x + W'$ in W/W'. Then

1. we have the weak-$*$ convergence

$$\frac{1}{n}\sum_{i=0}^{n-1} \mu^{(i)} * \delta_x \to \nu_{\mathcal{C}}.$$

2. For $\mu^{\mathbb{N}}$-almost every word $b = (b_1, b_2, \ldots) \in GL(V)^{\mathbb{N}}$, we have the convergence of the empirical measures

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{b_i b_{i-1}\ldots b_1 x} \to \nu_{\mathcal{C}} \qquad \text{as } n \to \infty.$$

The following definition is standard when considering stationary measures.

Proposition 4.3.5 (Proposition 3.1.5 of this thesis)**.** [BL85, Lem. II.2.1] Let μ be a Borel probability measure on $G = GL(V)$ and ν be a μ-stationary measure on V. Then for $\mu^{\mathbb{N}}$-almost every $b = (b_1, b_2, \ldots) \in G^{\mathbb{N}}$, there exists a probability measure ν_b on V such that for all

$g \in \Gamma_\mu$,

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n g)_* \nu.$$

Moreover, we have

$$\nu = \int_{G^{\mathbb{N}}} \nu_b \, d\mu^{\mathbb{N}}(b).$$

The measure ν_b is sometimes called the *limit measure* of ν with respect to the word b.

We can describe the limit measures of any stationary measures on V.

Theorem 4.3.6 (Theorem 3.1.6 of this thesis)**.** For each compact Γ_μ-orbit $\mathcal{C}$ in W/W', for $\mu^{\mathbb{N}}$-almost every word $b \in GL(V)^{\mathbb{N}}$, the limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 b_2 \dots b_n)_* \nu_{\mathcal{C}}$$

is the pushforward of $m_{\mathcal{C}}$ via a linear injection $p_b : W/W' \to W$, where $m_{\mathcal{C}}$ is the Haar (probability) measure supported on $\mathcal{C}$. In particular, ν_b is compactly supported on W. Moreover, the family $\{p_b\}_{b\in G^{\mathbb{N}}}$ is equivariant, in the sense that

$$b_1 \circ p_b = p_{Tb} \circ b_1,$$

where $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ is the left shift map given by $(Tb)_n = b_{n+1}$.

If Γ_μ acts trivially on W/W', then ν_b is a delta mass $\delta_{\xi(b)}$ on V for $\mu^{\mathbb{N}}$-almost every word b, and thus any ergodic μ-stationary measure ν on V is μ-proximal (cf. [BQ16, Sect. 2.7]). In general, we have the following classification, which follows immediately from Proposition 3.1.5 and Theorem 3.1.6.

Theorem 4.3.7. There exists a single compact Γ_μ-orbit $\mathcal{C}$ on W/W' and an equivariant family of sections $\{p_b : W/W' \to W\}_{b\in G^{\mathbb{N}}}\}$, i.e. for $\mu^{\mathbb{N}}$-a.e. $b = (b_1, b_2, \dots) \in G^{\mathbb{N}}$,

$$b_1 \circ p_b = p_{Tb} \circ b_1,$$

where $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ is the left shift map defined by $(Tb)_n = b_{n+1}$, such that

$$\nu = \int_{G^{\mathbb{N}}} (p_b)_* m_{\mathcal{C}} \, d\mu^{\mathbb{N}}(b),$$

where $m_{\mathcal{C}}$ is the Haar (probability) measure on W/W' supported on $\mathcal{C}$.

Corollary 4.3.8. Let $\rho : G \to GL(V)$ be a finite dimensional representation. Let X be a G-homogeneous space with a G-equivariant injection $\iota_V : X \to V$, and inherit a metric d_X on X from the Euclidean metric on V.

Let $\overline{\nu}$ be a μ-stationary measure on X and $\hat{\overline{\nu}}$ be the corresponding invariant measure on $\mathcal{S}^{\mathbb{Z}} \times X$. There exist

(i) a partition $\mathcal{W}$ of X (use $\mathcal{W}[x]$ to denote the atom of $\mathcal{W}$ containing $x \in X$),

(ii) a compact Γ_μ-homogeneous space $\mathcal{C}$ with its (unique) Γ_μ-invariant measure $m_{\mathcal{C}}$,

(iii) a measurable map $\hat{p} : \mathcal{S}^{\mathbb{Z}} \times \mathcal{C} \to \mathcal{S}^{\mathbb{Z}} \times X$ (let $p : \mathcal{S}^{\mathbb{Z}} \times \mathcal{C} \to X$ be the composition of p with the projection onto X) that projects to the identity on the $\mathcal{S}^{\mathbb{Z}}$ factor, i.e. $\hat{p}(\omega, x) = (\omega, p(\omega, x))$,

such that

(a) $\hat{p}$ is $\hat{T}$-equivariant, i.e. $\hat{T}(\hat{p}(\omega, x)) = \hat{p}(\hat{T}(\omega, x))$,

(b) $\hat{p}_*(\mu^{\mathbb{Z}} \times m_{\mathcal{C}}) = \hat{\overline{\nu}}$, so $p_*(\mu^{\mathbb{Z}} \times m_{\mathcal{C}}) = \overline{\nu}$,

(c) for $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, $x \mapsto p(\omega, x)$ is a continuous injection, thus $p(\{\omega\} \times \mathcal{C})$ is compact in X,

(d) for all $x \in \mathcal{C}$, for $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, $p(\mathcal{S}^{\mathbb{Z}} \times \{x\}) \subset \mathcal{W}[p(\omega, x)]$.

(e) for $\mu^{\mathbb{Z}}$-a.e. ω, for all $x \in X$ and $x' \in \mathcal{W}[x]$,

$$\limsup_{n\to\infty} \frac{1}{n} \log d_X(T_\omega^n(x), T_\omega^n(x')) < 0.$$

Proof. Take $\hat{p}(\omega, x) := (\omega, p_{\omega^-}(x))$, where p_{ω^-} is the map defined in Theorem 4.3.7, and for each $z \in X$, take $\mathcal{W}[z] := \iota_V^{-1}(\iota_V(z) + W')$, where W' is the Γ_μ-invariant subspace in Theorem 4.3.1. We verify each property here.

(a) This follows from the equivariance of p_{ω^-} in Theorem 4.3.7.

(b) This follows from Theorem 4.3.7 that $\overline{\nu}_{\omega^-} = (p_{\omega^-})_* m_{\mathcal{C}}$ and $\overline{\nu} = \int \overline{\nu}_{\omega^-}\, d\mu^{\mathbb{Z}}(\omega)$.

(c) Since p_{ω^-} is a linear injection, so in particular continuous.

(d) Since p_{ω^-} is a section for the projection $W \to W/W'$, for fixed $x \in \mathcal{C}$, for $\mu^{\mathbb{Z}}$-a.e. ω, $p_{\omega^-}(x) \in x + W'$, therefore $p(\omega, x)$ is in the same atom of $\mathcal{W}$ for $\mu^{\mathbb{Z}}$-a.e. ω.

(e) Since $x' \in \mathcal{W}[x]$ if and only if $\iota_V(x') - \iota_V(x) \in W'$ by the definition of $\mathcal{W}$, this property follows from the fact that W' is exponentially contracted by $\mu^{\mathbb{Z}}$-a.e. word ω since the top Lyapunov exponent of μ on W' is negative.

□

We use Corollary 4.3.8 to give a precise definition of "generalized μ-Bernoulli measure" stated in the main Theorem 4.1.1.

Definition 4.3.9. We say that a Borel probability measure $\overline{\nu}$ on a locally compact Borel G-space X is a *generalized μ-Bernoulli measure* if there exists a compact Γ_μ-space $\mathcal{C}$ with its (unique) uniform probability measure $m_{\mathcal{C}}$ and a measurable map $p : \mathcal{S}^{\mathbb{Z}} \times \mathcal{C} \to X$ such that $\overline{\nu} = p_*(\mu^{\mathbb{Z}} \times m_{\mathcal{C}})$, and the map $\hat{p} : \mathcal{S}^{\mathbb{Z}} \times \mathcal{C} \to \mathcal{S}^{\mathbb{Z}} \times X$ defined by $\hat{p}(\omega, x) := (\omega, p(\omega, x))$ is $\hat{T}$-equivariant: $\hat{p}(\hat{T}(\omega, x)) = \hat{T}(\hat{p}(\omega, x))$.

We note that this definition includes the classical self-affine measures on $\mathbb{R}^n$ (for the appropriate measure μ) and uniform measures on compact Γ_μ-spaces. If μ is finitely supported, $\mathcal{C}$ is trivial, and the elements in $\mathcal{S}$ form a contracting similarity IFS, then this also include the Bernoulli measures defined in [SW19, Sect. 8] (hence for specific choices of μ this also include the classical Hausdorff measures on certain fractal sets).

4.4 Setup

In this section, we lay down the foundations of the proof. In particular, towards the end of this section, we will state precisely the Case I and II assumptions, and the precise conclusions we will prove in Theorem 4.4.8 and 4.4.9. These two theorems together imply Theorem 4.1.1.

We record the following setup from Eskin-Lindenstrauss [ELa], which form the basis of the modified exponential drift argument in the fiber direction L/H.

4.4.1 The acting group $\overline{\Gamma}_\mu^Z$

Let $\mathcal{S} := \operatorname{supp}\,\mu$, and $\Gamma_\mu := \overline{\langle \operatorname{supp}\,\mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by $\mathcal{S}$. Let $\overline{\Gamma}_\mu^Z \subset G$ be the Zariski closure of Γ_μ in G. In cases that this paper considers, $\overline{\Gamma}_\mu^Z$ will not be the whole group G, nor semisimple.

4.4.2 Skew Product $\mathcal{S}^{\mathbb{Z}} \times G/H$

Consider the two-sided shift $(\mathcal{S}^{\mathbb{Z}}, \mu^{\mathbb{Z}}, T)$ with the map

$$T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$$

defined by the left shift $(T\omega)_n = \omega_{n+1}$ for $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \mathcal{S}^{\mathbb{Z}}$.
Given a locally compact Borel G-space X (for instance $X = G/L, G/H^\circ, G/H$), define the skew product map

$$\hat{T} : \mathcal{S}^{\mathbb{Z}} \times X \to \mathcal{S}^{\mathbb{Z}} \times X \qquad \text{by} \qquad (\omega, x) \mapsto (T\omega, \omega_0 x).$$

For $\omega \in \mathcal{S}^{\mathbb{Z}}$ and nonnegative integer n, if we let

$$T^n_\omega := \omega_{n-1} \cdots \omega_0, \qquad \text{and} \qquad T^{-n}_\omega := (T^n_{T^{-n}\omega})^{-1}.$$

then for any integer n, $\hat{x} = (\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times X$,

$$\hat{T}^n(\omega, x) = (T^n\omega, T^n_\omega x).$$

Given a μ-stationary measure ν on X and $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \mathcal{S}^{\mathbb{Z}}$, let

$$\nu_{\omega^-} := \lim_{n\to\infty} (\omega_{-1} \ldots \omega_{-n})_* \nu.$$

The existence of the limit follows from the martingale convergence theorem (cf. [BL85, Lem. II.2.1]). Hence ν_{ω^-} is a probability measure on X.

Moreover, one can define a $\hat{T}$-invariant probability measure $\hat{\nu}$ on $\mathcal{S}^{\mathbb{Z}} \times X$ by

$$d\hat{\nu}(\omega^-, \omega^+, x) := d\mu^{\mathbb{Z}}(\omega^-, \omega^+)\, d\nu_{\omega^-}(x). \tag{4.4.1}$$

Proposition 4.4.1. If ν is an ergodic μ-stationary measure on X, then $\hat{\nu}$ is an ergodic $\hat{T}$-invariant measure on $\mathcal{S}^{\mathbb{Z}} \times X$.

Proof. This follows from [Kif86, Lem. I.2.4, Thm. I.2.1, P.19-20], as in [ELa, Prop. 1.12]. □

As in [ELa], we also introduce a group $\mathcal{U}_1^+$ acting on $\mathcal{S}^{\mathbb{Z}}$ so that the $\mathcal{U}_1^+$-orbit of $(\omega^-, \omega^+) \in \mathcal{S}^{\mathbb{Z}}$ is $\{\omega^-\} \times \mathcal{S}^{\mathbb{N}}$, and extend to an $\mathcal{U}_1^+$-action on $\mathcal{S}^{\mathbb{Z}} \times X$ by acting trivially on the second factor. In particular for any $\hat{x} \in \mathcal{S}^{\mathbb{Z}} \times X$, $\mathcal{U}_1^+\hat{x}$ is naturally identified with $\mathcal{S}^{\mathbb{N}}$, thus can be endowed with the probability measure $\mu^{\mathbb{N}}$. We similarly define the group $\mathcal{U}_1^-$ that changes ω^- in $\omega \in \mathcal{S}^{\mathbb{Z}}$. Then

Proposition 4.4.2. A measure $\hat{\nu}$ on $\mathcal{S}^{\mathbb{Z}} \times X$ is $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant if and only if $\hat{\nu}$ is constructed from a μ-stationary measure ν on X as in (4.4.1).

Now we apply the above constructions to $X = G/H$ and G/L. Let ν be an ergodic μ-stationary measure on G/H. Let $\pi_{G/L} : G/H \to G/L$ be the natural quotient map. Let $\overline{\nu} := (\pi_{G/L})_*\nu$. Then $\overline{\nu}$ is an ergodic μ-stationary measure on G/L. We construct the ergodic $\hat{T}$-invariant measures $\hat{\nu}$ and $\hat{\overline{\nu}}$ on $\mathcal{S}^{\mathbb{Z}} \times G/H$ and $\mathcal{S}^{\mathbb{Z}} \times G/L$ respectively.

It can be verified that

1. The pushforward of $\hat{\nu}$ via $\mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}}$ is $\mu^{\mathbb{Z}}$.

2. The pushforward of $\hat{\nu}$ via $\mathcal{S}^{\mathbb{Z}} \times G/H \to G/H$ is ν.

3. The pushforward of $\hat{\nu}$ via $\mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}} \times G/L$ is $\hat{\bar{\nu}}$.

4. The pushforward of $\hat{\nu}$ via $\mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{N}} \times G/H$ is $\mu^{\mathbb{N}} \times \nu$.

5. $\hat{\nu} = \mu^{\mathbb{Z}} \times \nu$ if and only if ν is $\mathcal{S}$-invariant.

4.4.3 Notational Remark 1

To make our notations more suggestive, we will adopt a number of notational rules.

Throughout this paper, we will consider a number of spaces, each with their natural Borel probability measure, and have a T^t-action and $\mathcal{U}_1^+$-action that are measure-preserving. The ones we have seen so far are:

$$(\mathcal{S}^{\mathbb{Z}}, \mu^{\mathbb{Z}}), \qquad (\mathcal{S}^{\mathbb{Z}} \times G/L, \hat{\bar{\nu}}), \qquad (\mathcal{S}^{\mathbb{Z}} \times G/H, \hat{\nu}).$$

We will also consider cocycle actions over these dynamical systems (and some others introduced in future sections). For instance T^n_ω is a cocycle action on the trivial bundle $\mathcal{S}^{\mathbb{Z}} \times X$ over $\mathcal{S}^{\mathbb{Z}}$ for $X = G/L$ and G/H.

Elements

For an element in a measure-preserving system with $\mathcal{S}^{\mathbb{Z}}$ as a factor,

- we use a letter with a hat to denote that element, for instance $\hat{x}, \hat{y}, \hat{z}, \hat{q}$ etc.

- we use the same letter without the hat to denote the component in a G-space, for instance x, y, z, q etc.

- we use ω with sub/superscripts to denote the component in the $\mathcal{S}^{\mathbb{Z}}$ factor, for instance $\omega, \omega', \omega''$ etc.

Dynamics

Rather than using a different notation for the dynamics on various spaces, we use the following rules.

- We use T^n for the action on $(\mathcal{S}^{\mathbb{Z}}, \mu^{\mathbb{Z}})$.
- We use $\hat{T}^n$ (for $\mathbb{Z}$-action) and $\hat{T}^t$ (for $\mathbb{R}$-action) for the dynamics of any system except $(\mathcal{S}^{\mathbb{Z}}, \mu^{\mathbb{Z}})$.
- We use $T^t_{\hat{x}}$ (for $\mathbb{Z}$-action) or $T^n_{\hat{x}}$ (for $\mathbb{R}$-action) for any cocycle action over any measure-preserving system, where $\hat{x}$ is an element in the underlying space of the system.

Manifolds and vector spaces

We will consider various dynamically defined submanifolds and vector bundles.

- We use curly letters for submanifolds of a G-space or subsets of $\mathcal{S}^{\mathbb{Z}}$, often (though not always) with a subscript to indicate the manifold it is embedded in, and a superscript related to the exponential growth rate. For instance $\mathcal{W}^+_{G/H}, \mathcal{W}^-_{G/L}, \mathcal{F}^-_{G/H}$ etc.
- We use curly letters with a hat for subsets of a system with a $\mathcal{S}^{\mathbb{Z}}$ component and a G-space component. For instance $\hat{\mathcal{W}}^+_{G/H}, \hat{\mathcal{W}}^-_{G/L}$ etc.
- We use straight letters for vector bundles, sometimes with a superscript related to the exponential growth rate, or in boldface. For instance $W^{<\lambda_i}$, $\mathbf{F}$, $\mathbf{E}$ etc.
- We use straight letters for subgroups of G, for instance G, H, L etc.

See **Notational Remark 2** in the end of Section 4.5 for other remarks on the notation.

4.4.4 Metric on G/H

Let H° be the connected component of identity in H.

Definition (Metric)**.** Fix a representation $\rho : G \to GL(V)$ s.t. $L \subset G$ is the stabilizer of a nonzero vector $v \in V$. Then there is an injection $G/L \to V$ given by $g \mapsto gv$. Let

1. $d_{L/H}$ be a right invariant Riemannian metric on L/H°,
2. $d_{G/L}$ be the metric on G/L induced by the injection $G/L \to V$ from the Euclidean metric on V,

We need to choose a convenient metric on G/H to control the drift of points in different fibers. We use a metric similar to that in Sargent-Shapira [SS19, Sect. 6.1].

Let $\mathfrak{g} := \text{Lie}(G)$ and $\mathfrak{h} := \text{Lie}(H^\circ)$. For $x \in G/L$, let $\mathfrak{h}_x := x\mathfrak{h}x^{-1}$ (note that this is well-defined since L normalizes H°). Consider the orthogonal projection

$$\Pi_x : \mathfrak{g} \to (\mathfrak{h}_x)^\perp =: \mathfrak{m}_x$$

with respect to a fixed K-invariant inner product on $\mathfrak{g}$. Let $\pi : G/H^\circ \to G/L$ be the quotient map. Now for $z \in G/H^\circ$, define $\mathfrak{h}_z := \mathfrak{h}_{\pi(z)} = z\mathfrak{h}z^{-1}$.

We then define a metric on G/H° as follows:

- for $z \in G/H^\circ$, define $r_z : G \to G/H^\circ$ by $g \mapsto gz$.
- The derivative at the identity $d_e r_z : \mathfrak{g} \to T_z(G/H^\circ)$ has kernel $\mathfrak{h}_z$, hence gives a well-defined linear isomorphism $d_e r_z : \mathfrak{m}_{\pi(z)} \to T_z(G/H^\circ) = \mathfrak{g}/(\mathfrak{h}_z)$.
- We pushforward the metric on $\mathfrak{m}_{\pi(z)}$ inherited from $\mathfrak{g}$ via this isomorphism to obtain a metric on $T_z(G/H^\circ) = \mathfrak{g}/(\mathfrak{h}_z)$.
- Doing this for all $z \in G/H^\circ$ we obtain a metric $d_{G/H}$ on G/H°.
- We use $\|\cdot\|_0$ for the norm on $\mathfrak{g}/\mathfrak{h}_z$ induced by this metric to distinguish this from the dynamically defined norm to be constructed in subection 4.5.9.

A convenient way to view the dynamics is to use the following diagrams: for all $\gamma \in G$, $z \in G/H^\circ$,

$$\begin{array}{ccc} G & \xrightarrow{c_\gamma} & G \\ {\scriptstyle r_z}\downarrow & & \downarrow{\scriptstyle r_{\gamma z}} \\ G/H^\circ & \xrightarrow[z \mapsto \gamma z]{} & G/H^\circ \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_\gamma} & \mathfrak{g} \\ {\scriptstyle d_e r_z}\downarrow & & \downarrow{\scriptstyle d_e r_{\gamma z}} \\ T_z G/H^\circ & \xrightarrow[d_z \gamma]{} & T_{\gamma z} G/H^\circ \end{array}$$

where $c_\gamma : G \to G$ is conjugation by γ, i.e. $c_\gamma(g) = \gamma g \gamma^{-1}$.

The vertical maps restrict to an isometry (by definition of the metric) on $\mathfrak{m}_{\pi(z)}$ and $\mathfrak{m}_{\pi(\gamma z)}$ respectively, and has norm at most 1 on all of $\mathfrak{g}$. The top horizontal map induces a map $\mathrm{Ad}_\gamma : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{\gamma z}$. Note that this map does not map $\mathfrak{m}_{\pi(z)}$ to $\mathfrak{m}_{\pi(\gamma z)}$ in general.

Proposition 4.4.3. The metric $d_{G/H}$ on G/H° defined above has the following properties.

(a) $d_{G/H}$ is invariant under right multiplication by L/H°, i.e. $d_{G/H}(zg, z'g) = d_{G/H}(z, z')$ for $z, z' \in G/H^\circ$ and $g \in L/H^\circ$.

(b) On each fiber xL/H° for $x \in G/L$, if $z, z' \in xL/H^\circ$, then right multiplication gives an isometry on tangent spaces $T_z G/H^\circ \to T_{z'} G/H^\circ$. Under the identification above, it is an isometry $\mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}$, hence we can identify the tangent spaces with $\mathfrak{g}/\mathfrak{h}_x$ for all $z \in xL/H^\circ$.

(c) $d_{G/H}(gz, gz') \leq \|g\|_{\mathrm{Ad}(G)} d_{G/H}(z, z') \qquad$ for all $g \in G$ and $z, z' \in G/H^\circ$.

(d) $d_{G/L}(\pi(z), \pi(z')) \leq c_\rho \, d_{G/H}(z, z')$, where c_ρ is the Lipschitz constant of the map $\rho : G \to GL(V)$.

Here $\| \cdot \|_{\mathrm{Ad}(G)}$ is the operator norm on G given by the adjoint action $\mathrm{Ad} : G \to \mathrm{Ad}(\mathfrak{g})$.

For $(z, z') \in G/H^\circ \times G/H^\circ$ close enough, define the *orthogonal displacement vector* $o_{z,z'} \in \mathfrak{m}_{\pi(z)}$ as the unique vector $v \in \mathfrak{m}_{\pi(z)}$ such that $z' = \exp(v) z$.

Proposition 4.4.4. For any compact set $E \subset G$, and all $0 < c < 1$, there exists a neighborhood of the diagonal $\mathcal{U} \subset G/H^\circ \times G/H^\circ$ such that for all $(z, z') \in \mathcal{U}$, and $g \in E \cup E^{-1} \cup \{e\}$, we have

(a) The vector $o_{gz,gz'} \in \mathfrak{m}_{\pi(gz)}$ is well-defined.

(b) $c\|o_{gz,gz'}\|_0 \le d_{G/H}(gz, gz') \le c^{-1}\|o_{gz,gz'}\|_0$.

(c) For all nonzero $\rho_H \in \bigwedge^{\dim H} \mathfrak{h}$, we have

$$c\|o_{gz,gz'}\|_0 \le \frac{\|g(o_{z,z'} \wedge \rho_H)\|_0}{\|g\rho_H\|_0} \le c^{-1}\|o_{gz,gz'}\|_0.$$

4.4.5 Stable and Unstable manifolds

We consider stable and unstable manifolds for the dynamics on both $\mathcal{S}^{\mathbb{Z}} \times G/L$ and $\mathcal{S}^{\mathbb{Z}} \times G/H^\circ$.

Stable and unstable manifolds on $\mathcal{S}^{\mathbb{Z}} \times G/L$

For $(\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$, define the stable manifolds

$$\begin{aligned} \mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] &:= \{\omega' \in \mathcal{S}^{\mathbb{Z}} \mid (\omega')^+ = \omega^+\}, \\ \hat{\mathcal{W}}^-_{G/L}[\hat{x}] &:= \left\{(\omega', x') \in \mathcal{S}^{\mathbb{Z}} \times G/L \mid (\omega')^+ = \omega^+ \text{ and } \limsup_{n\to\infty} \frac{1}{n} \log d_{G/L}(T^n_\omega x, T^n_\omega x') < 0\right\}, \end{aligned}$$

Similarly define the unstable manifolds $\mathcal{W}^+_{\mathcal{S}^{\mathbb{Z}}}[\omega], \hat{\mathcal{W}}^+_{G/L}[\hat{x}]$.

Stable and unstable manifolds on $\mathcal{S}^{\mathbb{Z}} \times G/H^\circ$

Let $\pi = \pi_{G/L} : G/H^\circ \to G/L$ be the quotient map. For $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H^\circ$, define the stable manifolds

$$\begin{aligned} \hat{\mathcal{W}}^-_{L/H}[\hat{z}] &:= \Big\{(\omega', z') \in \mathcal{S}^{\mathbb{Z}} \times G/H^\circ \mid (\omega')^+ = \omega^+, \pi_{G/L}(z') = \pi_{G/L}(z), \text{ and} \\ &\qquad \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T^n_\omega z, T^n_\omega z') < 0\Big\}, \\ \hat{\mathcal{W}}^-_{G/H}[\hat{z}] &:= \left\{(\omega', z') \in \mathcal{S}^{\mathbb{Z}} \times G/H^\circ \mid (\omega')^+ = \omega^+, \text{ and } \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T^n_\omega z, T^n_\omega z') < 0\right\}. \end{aligned}$$

Similarly define the unstable manifolds $\hat{\mathcal{W}}^+_{L/H}[\hat{z}]$ and $\hat{\mathcal{W}}^+_{G/H}[\hat{z}]$. We remark that the manifolds are local in the $\mathcal{S}^{\mathbb{Z}}$ component and global in the G-space component.

Then for almost every $(\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$, there exist unipotent subgroups $N^+_f(\omega, x) = N^+_f(\omega^-, x)$ and $N^-_f(\omega, x) = N^-_f(\omega^+, x)$ of $x(L/H^\circ)x^{-1}$ such that for almost every $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H^\circ$, the *fiberwise stable set*

$$\hat{\mathcal{W}}^\pm_{L/H}[\hat{z}] = \mathcal{W}^\pm_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \{\pi_{G/L}(z)\} \times N^\pm_f(\omega^\mp, \pi(z))z.$$

Also write the *total stable set*

$$\hat{\mathcal{W}}^-_{G/H}[\hat{z}] = \mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{W}^-_{G/H}[\hat{z}].$$

where $\mathcal{W}^-_{G/H}[\hat{z}] \subset G/H^\circ$ denotes the stable manifold of z with respect to the combinatorial future ω^+, i.e.

$$\mathcal{W}^-_{G/H}[\hat{z}] := \left\{ z' \in G/H^\circ \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T^n_\omega z, T^n_\omega z') < 0 \right\}.$$

Note that $\pi_{G/L}(\mathcal{W}^-_{G/H}[(\omega, z)]) \subset \mathcal{W}^-_{G/L}[(\omega, \pi_{G/L}(z))]$ by Proposition 4.4.3(d). We can also describe $\mathcal{W}^-_{G/H}[\hat{z}]$ in terms of the stable unipotent subgroup $N^-(\omega) \subset G$ of G with respect to the word ω, namely

$$\mathcal{W}^-_{G/H}[\hat{z}] = N^-(\omega)zH^\circ.$$

This will be proved in Lemma 4.5.3.

Inert center-stable set

Definition. Define the *inert center-stable set* by, for $z \in G/H^\circ$,

$$\mathcal{F}^{\leq 0}_{G/H}[z] := \left\{ z' \in G/H^\circ \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T^n_\omega(z), T^n_\omega(z')) \leq 0 \quad \text{for a.e. } \omega^+ \in \mathcal{S}^{\mathbb{N}} \right\}.$$

For $x \in G/L$, define

$$\mathcal{F}^{\leq 0}_{G/L}[x] := \left\{ x' \in G/L \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/L}(T^n_\omega(x), T^n_\omega(x')) \leq 0 \quad \text{for almost every } \omega^+ \in \mathcal{S}^{\mathbb{N}} \right\}.$$

In words, $z' \in \mathcal{F}^{\leq 0}_{G/H}[z]$ if for almost every future ω^+, z and z' do not diverge exponentially.

Proposition 4.4.5. We have the following equivariance properties: for all $z \in G/H^\circ$,

(a) for μ-a.e. $g \in G$, we have $g\mathcal{F}^{\leq 0}_{G/H}[z] \subset \mathcal{F}^{\leq 0}_{G/H}[gz]$; Similar for $\mathcal{F}^{\leq 0}_{G/L}[x]$.

(b) for all $h \in L/H^\circ$, we have $\mathcal{F}^{\leq 0}_{G/H}[zh] = \mathcal{F}^{\leq 0}_{G/H}[z]h$, where L/H° acts on G/H° by right multiplication.

(c) $\pi(\mathcal{F}^{\leq 0}_{G/H}[z]) \subset \mathcal{F}^{\leq 0}_{G/L}[\pi(z)]$ where $\pi : G/H^\circ \to G/L$ is the quotient map.

Proof. For μ-a.e. $g \in G$, we have $g\mathcal{F}^{\leq 0}_{G/H}[z] \subset \mathcal{F}^{\leq 0}_{G/H}[gz]$ basically by definition. For part (b), the equivariance follows from the right invariance of the metric $d_{G/H}$ by L/H°. Part (c) follows from Proposition 4.4.3(d). □

Proposition 4.4.6. If μ is uniformly expanding on L/H at $x \in G/L$, and there is a μ-stationary measure ν on $\overline{\Gamma}^Z_\mu xH/H$, then for all $z \in \overline{\Gamma}^Z_\mu xH^\circ/H^\circ$, the intersection of $\mathcal{F}^{\leq 0}_{G/H}[z]$ and $x'L/H^\circ$ contains at most one point for all $x' \in \overline{\Gamma}^Z_\mu x_0 L/L$.

Proof. We defer the proof of this proposition to Section 4.8.2, as it will be a corollary of Proposition 4.8.4 (see Corollary 4.8.5). □

Proposition 4.4.7. For almost every $\hat{x} = (\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$, we have $\mathcal{W}^-_{G/L}[\hat{x}] = \mathcal{F}^-_{G/L}[x] = \mathcal{W}[x]$, where $\mathcal{W}$ is the partition in Corollary 4.3.8.

Proof. This follows from Corollary 4.3.8 (e). □

4.4.6 Two cases

We distinguish two cases, depending on the shape of the conditional measure $\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}$ on $\hat{\mathcal{W}}^-_{G/H}[(\omega, z)] = W^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{W}^-_{G/H}[(\omega^+, z)]$: For $\hat{\nu}$-almost every $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$,

Case I: the conditional measure $\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}$ is not supported on $W^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{F}^{\leq 0}_{G/H}[z]$.

Case II: the conditional measure $\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}$ is supported on $W^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{F}^{\leq 0}_{G/H}[z]$.

By ergodicity of $\hat{\nu}$, Case I and II are complementary. In Case I, we apply the modified exponential drift argument to pairs of points $(\omega, z), (\omega', z')$ in $\hat{\mathcal{W}}^-_{G/H}[(\omega, z)]$ such that the basepoints $z, z' \in G/H$ have different inert stable sets $\mathcal{F}^{\leq 0}_{G/H}[z] \neq \mathcal{F}^{\leq 0}_{G/H}[z']$. In Case II, a separate argument will be needed. In particular, the main theorem follows from the following two statements:

Suppose G, H, μ satisfying the conditions of Theorem 4.1.1 and there exists an H-envelope L that satisfying the conditions of Theorem 4.1.1. Let ν be an ergodic μ-stationary probability measure on G/H, which projects to a μ-stationary probability measure $\bar{\nu} := \pi_*\nu$ on G/L.

Theorem 4.4.8. If Case I holds, then there exist a Lie subgroup $H' \subset G$ with $H^\circ \subset H' \subset L \subset G$ and $\dim(H'/H^\circ) > 0$, an H'-homogeneous probability measure $\nu_{L/H}$ on L/H and finite μ-stationary measure $\nu_{G/H'}$ on G/H' such that

$$\nu_{G/H} = \nu_{G/H'} * \nu_{L/H} = \int_{G/H'} g_*\nu_{L/H} \, d\nu_{G/H'}(g).$$

Theorem 4.4.9. Assume that

(†) There exists a closed normal subgroup $U \subset \overline{\Gamma}^Z_\mu$ and some $z_0 \in G$ with $z_0 L = x_0 L$ such that $\overline{\Gamma}^Z_\mu x_0 L = U z_0 H^\circ$ and $z_0^{-1} U z_0 \cap H^\circ = \{\mathrm{id}\}$.

If Case II holds, then the stationary measure $\nu_{G/H}$ can be written as

$$\nu_{G/H} = \int_{G/L} \nu_x \, d\overline{\nu}(x),$$

where

1. $\overline{\nu}$ is a generalized μ-Bernoulli measure supported on $\overline{\Gamma}^Z_\mu x_0 L/L$.

2. there exists a positive integer k such that for $\overline{\nu}$-almost every $x \in G/L$, ν_x is the uniform measure on k points in $\pi^{-1}(x) \subset G/H$,

3. there exist finitely many $z_1, \dots, z_m \in \overline{\Gamma}_\mu^Z x_0 L/H$ such that for $\mathcal{F} := \bigcup_{i=1}^m \mathcal{F}_{G/H}^{\leq 0}[z_i]$, we have (i) supp $\nu_{G/H} \subset \mathcal{F}$, (ii) $\mathcal{F}$ has finite intersection with $x' L/H$ for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, and (iii) $\mathcal{F}$ is invariant under Γ_μ.

4.5 Refined Lyapunov Subspaces $W_{ij}(\omega, z)$

In this section, we will apply Oseledets multiplicative ergodic theorem and Zimmer's amenable reduction theorem to write the cocycle in a specific form, as was done in [ELa].

4.5.1 Suspension flow $T^t : \Omega_b \to \Omega_b$ *on* $\Omega_b = \mathcal{S}^{\mathbb{Z}} \times [0,1]$

In the process of the exponential drift, it would be convenient consider the $\overline{G}_{\mathcal{S}}^Z$-action not just as a $\mathbb{Z}$-action, but an $\mathbb{R}$-action. This motivates the use of a suspension flow.

It will be evident by definition that the dynamics between two times $t < t'$ is the identity map unless there is an integer between t and t'. Nonetheless, in subection 4.5.9, we will define a dynamical norm that varies for different times in the suspension. In particular, the operator norm of a certain linear map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ with respect to these dynamical norms will vary continuously with respect to t (rather than a nondecreasing step function in t that changes only at integer times, as in the usual norm). This will help us determine a stopping time t by setting $\|\mathcal{A}(\hat{q}_1, u, \ell, t)\| = \varepsilon$ for some constant $\varepsilon > 0$.

Let $\Omega_b = \mathcal{S}^{\mathbb{Z}} \times [0,1]$, let T^t be the suspension flow on Ω_b, obtained by descending the flow $(\omega, s) \mapsto (\omega, s+t)$ on $\mathcal{S}^{\mathbb{Z}} \times \mathbb{R}$ onto Ω_b with respect to the identification $(\omega, s+1) \sim (T\omega, s)$. Let $\mathrm{Leb}_{[0,1]}$ be the Lebesgue (probability) measure on $[0,1]$. Then the probability measure $\mu^{\mathbb{Z}} \times \mathrm{Leb}_{[0,1]}$ on Ω_b is invariant under the flow.

With this in mind we can extend the definitions of iteration of maps defined in Section 4.4.2 to a flow: for $\omega \in \Omega_b$ and $z \in G/H$, define

- $T_\omega^t : G/H \to G/H$ for any real number t, not just for integers $t = n$, by setting $T_\omega^t := T_\omega^{\lfloor t \rfloor}$.
- $\hat{T}^t : \Omega_b \times G/H \to \Omega_b \times G/H$ by $(\omega, z) \mapsto (T^t \omega, T_\omega^t z)$.

- $\hat{T}^t : \Omega_b \times T(G/H) \to \Omega_b \times T(G/H)$ by $(\omega, z, \mathbf{v}) \mapsto (T^t\omega, T^t_\omega z, (T^t_\omega)_*\mathbf{v})$, where $T(G/H)$ is the tangent bundle on G/H.

Analogous notations are used to extend the base dynamics on $\mathcal{S}^{\mathbb{Z}} \times G/L$ to $\Omega_b \times G/L$.

Similarly, define $\Omega_0 := \mathcal{S}^{\mathbb{Z}} \times G/L \times [0,1]$ with the natural measure $\hat{\hat{\nu}} \times \mathrm{Leb}_{[0,1]}$ and let T^t be the suspension flow on Ω_0 by the same construction.

4.5.2 Lyapunov subspaces $W^{\lambda_i}(\omega, z)$

Apply Oseledets theorem to the cocycle $\hat{T}^t : \Omega_b \times T(G/H) \to \Omega_b \times T(G/H)$ over the base Ω_b:

Proposition 4.5.1. There exists real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ such that for almost every $\hat{z} = (\omega, z, s) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, there exists a T^t-invariant splitting

$$T_z(G/H) = \bigoplus_{i=1}^{n} W^{\lambda_i}(\omega, z)$$

with the property that $\mathbf{v} \in W^{\lambda_i}(\omega, z) \setminus \{0\}$ if and only if

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \frac{\|(T^t_\omega)_*\mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i.$$

Here we think of each $W^{\lambda_i}(\omega, z)$ as a subspace of $\mathfrak{g}/\mathfrak{h}_z$ using the identification $T_z(G/H) = \mathfrak{g}/\mathfrak{h}_z$ from subsection 4.4.4. The numbers λ_i are called the *Lyapunov exponents*, $W^{\lambda_i}(\omega, z)$ are called the *Lyapunov subspaces* of $\mathfrak{g}/\mathfrak{h}_z$ with respect to the cocycle $\hat{T}^t$.

Also for any real number λ, define the vector bundles

$$W^{\leq\lambda}(\omega, z) := \{0\} \cup \left\{ \mathbf{v} \in \mathfrak{g}/\mathfrak{h}_z \setminus \{0\} \mid \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|(T^t_\omega)_*\mathbf{v}\|_0}{\|\mathbf{v}\|_0} \leq \lambda \right\},$$

$$W^{\geq\lambda}(\omega, z) := \{0\} \cup \left\{ \mathbf{v} \in \mathfrak{g}/\mathfrak{h}_z \setminus \{0\} \mid \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|(T^{-t}_\omega)_*\mathbf{v}\|_0}{\|\mathbf{v}\|_0} \leq -\lambda \right\}.$$

And similarly define the bundles $W^{<\lambda}$ and $W^{>\lambda}$. Then by Oseledets theorem, for each i, for almost every $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, the lim sup in the definition of $W^{\leq\lambda}$ and $W^{\geq\lambda}$ can be replaced by lim, and furthermore, we have

$$W^{\geq\lambda_i}(\omega,z) := \bigoplus_{j\leq i} W^{\lambda_j}(\omega,z), \qquad W^{\leq\lambda_i}(\omega,z) := \bigoplus_{j\geq i} W^{\lambda_j}(\omega,z),$$

$$W^{>\lambda_i}(\omega,z) := \bigoplus_{j<i} W^{\lambda_j}(\omega,z), \qquad W^{<\lambda_i}(\omega,z) := \bigoplus_{j>i} W^{\lambda_j}(\omega,z).$$

The filtration

$$0 \subsetneq W^{\leq\lambda_n} \subsetneq W^{\leq\lambda_{n-1}} \subsetneq \cdots \subsetneq W^{\leq\lambda_1} = T(G/H)$$

is called the *forward Lyapunov flag.* The filtration

$$0 \subsetneq W^{\geq\lambda_1} \subsetneq W^{\geq\lambda_2} \subsetneq \cdots \subsetneq W^{\geq\lambda_n} = T(G/H)$$

is called the *backward Lyapunov flag.* Note that for almost every $(\omega,z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$,

$$W^{\lambda_i} = W^{\geq\lambda_i} \cap W^{\leq\lambda_i}.$$

We also define the unstable and stable bundles

$$W^{+}(\omega,z) := \bigoplus_{\lambda_j>0} W^{\lambda_j}(\omega,z), \qquad W^{-}(\omega,z) := \bigoplus_{\lambda_j<0} W^{\lambda_j}(\omega,z).$$

Note that $W^{\geq\lambda}(\omega,z) = W^{\geq\lambda}(\omega',z)$ for $\omega' \in \mathcal{W}^{+}_{\mathcal{S}^{\mathbb{Z}}}[\omega]$, and $W^{\leq\lambda}(\omega,z) = W^{\leq\lambda}(\omega',z)$ for $\omega' \in \mathcal{W}^{-}_{\mathcal{S}^{\mathbb{Z}}}[\omega]$.

Also $W^{\lambda_i}(\omega,z) = W^{\lambda_i}(\omega,z')$ if $\pi_{G/L}(z) = \pi_{G/L}(z')$. Thus for $x \in G/L$, sometimes we write $W^{\lambda_i}(\omega,x) := W^{\lambda_i}(\omega,z)$ for any $z \in G/H$ such that $\pi_{G/L}(z) = x$.

4.5.3 Fiberwise Lyapunov subspaces $W^{\lambda_i}_{\mathfrak{l}/\mathfrak{h}}(\omega, x)$

Now for each $x \in G/L$, define $\mathfrak{l}_x := x\mathfrak{l}x^{-1}$. Consider the Oseledets splitting on the fiberwise subspace

$$\mathfrak{l}_x/\mathfrak{h}_x = \bigoplus_{i=1}^{n} W^{\lambda_i}_{\mathfrak{l}/\mathfrak{h}}(\omega, x)$$

defined by $W^{\lambda_i}_{\mathfrak{l}/\mathfrak{h}}(\omega, x) := W^{\lambda_i}(\omega, x) \cap (\mathfrak{l}_x/\mathfrak{h}_x)$ (possibly trivial).

Now since H° is a normal subgroup of L, $\mathfrak{l}_x/\mathfrak{h}_x$ is a Lie algebra. In particular, we can now define the unipotent subgroups $N^{\pm}_f(\omega, x)$ claimed in Section 4.4.5: they are the unipotent subgroups of the Lie group $x(L/H^\circ)x^{-1}$ such that

$$\mathrm{Lie}(N^+)(\hat{x}) = W^+_{\mathfrak{l}/\mathfrak{h}}(\omega, x) := \bigoplus_{\lambda_j > 0} W^{\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(\omega, x), \quad \mathrm{Lie}(N^-)(\hat{x}) = W^-_{\mathfrak{l}/\mathfrak{h}}(\omega, x) := \bigoplus_{\lambda_j < 0} W^{\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(\omega, x).$$

Here we are using the fact that the right hand sides are nilpotent subalgebras of $(\mathfrak{l}/\mathfrak{h})_{\hat{x}}$.

4.5.4 Relationship with Lyapunov subspaces on $\mathfrak{g}$

Recall from subsection 4.4.4 that under the identification $T_z(G/H^\circ) = \mathfrak{g}/\mathfrak{h}_z$, the differential map of left multiplication by γ on G/H° is given by $\mathrm{Ad}_\gamma : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{\gamma z}$, where we recall $\mathfrak{h}_z := \mathrm{Ad}_z\mathfrak{h} = z\mathfrak{h}z^{-1}$. Thus we can consider the dynamics using the short exact sequence of cocycles over $\mathcal{S}^{\mathbb{Z}} \times G/H$

$$\mathfrak{h}_z \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}_z$$

where the action on all three are given by conjugation by elements in G. Recall that the Lyapunov subspaces and exponents are canonical with respect to short exact sequences.

Proposition 4.5.2. Let

$$W \hookrightarrow V \twoheadrightarrow V/W$$

be a short exact sequence of cocycles over an ergodic base $(\Omega, \mu_\Omega, T^t)$. More precisely, for each bundle $E = W, V, V/W$, for $\omega \in \Omega$, label the cocycle by

$$T^t_\omega : E_\omega \to E_{T^t\omega},$$

where E_ω is the fiber of E above $\omega \in \Omega$. Then these cocycles are compatible with the bundle maps in the short exact sequence.

For each real number λ, let

$$E^{\leq\lambda}_\omega := \{0\} \cup \left\{ \mathbf{v} \in E_\omega \setminus \{0\} \mid \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|T^t_\omega \mathbf{v}\|}{\|\mathbf{v}\|} \leq \lambda \right\}.$$

Define the subbundle $E^{\leq\lambda} \subset E$ over Ω with fibers $E^{\leq\lambda}_\omega$. Then $E^{\leq\lambda}$ is a T^t-invariant subbundle of E, and

$$W^{\leq\lambda} = V^{\leq\lambda} \cap W, \qquad (V/W)^{\leq\lambda} = V^{\leq\lambda}/W^{\leq\lambda}$$

over μ_Ω-a.e. $\omega \in \Omega$.

Here we interpret $V^{\leq\lambda}/W^{\leq\lambda}$ as a subspace of V/W via the natural isomorphism $V^{\leq\lambda}/W^{\leq\lambda} \cong (V^{\leq\lambda} + W)/W$ once we have established $W^{\leq\lambda} = V^{\leq\lambda} \cap W$. The analogous statements hold if we replace $\leq \lambda$ by $< \lambda$, $\geq \lambda$ or $> \lambda$ everywhere.

Proof. The first two claims are clear from definition. It remains to show the third claim that

$$(V/W)^{\leq\lambda} \subset (V^{\leq\lambda} + W)/W, \tag{4.5.1}$$

i.e. for each $v \in (V/W)^{\leq\lambda}_\omega$, there is a representative $v' \in V^{\leq\lambda}_\omega + W_\omega$ that is in the coset $v + W_\omega$. The key is to apply [Fil19, Lem. 2.3.3], which states that "unusually large growth in invariant subbundle implies splitting".

Let $q : V \to V/W$ be the natural quotient map of bundles over Ω, and consider the subbundle $q^{-1}(V/W)^{\leq\lambda} \subset V$. By T^t-equivariance of q and T^t-invariance of $(V/W)^{\leq\lambda}$, we know that $q^{-1}(V/W)^{\leq\lambda}$ is a T^t-invariant subbundle of V that contains W as an invariant subbundle. Thus for the purpose of showing (4.5.1), we may assume that $V = q^{-1}(V/W)^{\leq\lambda}$ by restricting to this subbundle and show that $V = V^{\leq\lambda} + W$. We may further assume that $\lambda = \lambda_1^{V/W}$, the top exponent of V/W.

Let $\lambda_1^W > \lambda_2^W > \cdots > \lambda_n^W$ be the Lyapunov exponents of W. If the top exponent of W is at most $\lambda = \lambda_1^{V/W}$, then we are done. Otherwise, let λ_k^W be the smallest exponent such that $\lambda_k^W > \lambda_1^{V/W}$. Now apply [Fil19, Lem. 2.3.3] (see also [Mn87, Lem. 11.6]) successively to the Oseledets filtration of V/W and of $W/W^{<\lambda_k^W}$, we have an invariant splitting of the short exact sequence

$$0 \to W/W^{<\lambda_k^W} \to V/W^{<\lambda_k^W} \to V/W \to 0,$$

i.e. a section $\sigma : V/W \to V/W^{<\lambda_k^W}$ such that there is a T^t-invariant decomposition

$$V/W^{<\lambda_k^W} = \sigma(V/W) \oplus W/W^{<\lambda_k^W} \tag{4.5.2}$$

and the exponents of $\sigma(V/W)$ coincide with the exponents of V/W (since σ is tempered). In particular the top exponent of $\sigma(V/W)$ is $\lambda = \lambda_1^{V/W}$.

Let $V' \subset V$ be the preimage of $\sigma(V/W)$ under the quotient map $V \to V/W^{<\lambda_k^W}$. Then the top exponent of V' is the maximum of the top exponents of $\sigma(V/W)$ and $W^{<\lambda_k^W}$. Note that either $k = n$, in which case $W^{<\lambda_k^W}$ is trivial, or the top exponent of $W^{<\lambda_k^W}$ is λ_{k+1}^W, which is at most λ by the choice of k. In both cases, we can conclude that the top exponent of V' is λ, so $V' \subset V^{\leq\lambda}$. On the other hand, from the decomposition (4.5.2), we have

$$V = V' + W.$$

Therefore $V = V^{\leq\lambda} + W$, as desired.

□

In particular, we can read off the Lyapunov flags on $\mathfrak{g}/\mathfrak{h}_z$ from the Lyapunov flags on $\mathfrak{g}$

using this exact sequence. Namely, for $\omega \in \mathcal{S}^{\mathbb{Z}}$ and real number λ, let

$$W_{\mathfrak{g}}^{\leq\lambda}(\omega) := \{0\} \cup \left\{ \mathbf{v} \in \mathfrak{g} \mid \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|(T_\omega^t)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \leq \lambda \right\}.$$

Then for $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, the forward flag on $T_z G/H = \mathfrak{g}/\mathfrak{h}_z$ is given by

$$W^{\leq\lambda_i}(\hat{z}) := W_{\mathfrak{g}}^{\leq\lambda_i}(\omega)/(W_{\mathfrak{g}}^{\leq\lambda_i}(\omega) \cap \mathfrak{h}_z) \cong (W_{\mathfrak{g}}^{\leq\lambda_i}(\omega) + \mathfrak{h}_z)/\mathfrak{h}_z. \tag{4.5.3}$$

In particular, the successive quotient is given by

$$W^{\leq\lambda_i}(\hat{z})/W^{<\lambda_i}(\hat{z}) = \frac{W_{\mathfrak{g}}^{\leq\lambda_i}(\omega) + \mathfrak{h}_z}{W_{\mathfrak{g}}^{<\lambda_i}(\omega) + \mathfrak{h}_z} \cong \frac{W_{\mathfrak{g}}^{\leq\lambda_i}(\omega)}{W_{\mathfrak{g}}^{\leq\lambda_i}(\omega) \cap (W_{\mathfrak{g}}^{<\lambda_i}(\omega) + \mathfrak{h}_z)}.$$

4.5.5 Description of stable manifolds on G/H

We now describe the (un)stable manifolds on G/H° with respect to a given word ω in terms of the (un)stable unipotent subgroup of G with respect to ω, as claimed in Subsection 4.4.5.

Lemma 4.5.3. For all $zH^\circ \in G/H^\circ$ and $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, the stable manifold on G/H° through $zH^\circ \in G/H^\circ$ along the word ω is $N^-(\omega)zH^\circ$, where $N^-(\omega)$ is the (unipotent) stable subgroup of G with respect to the word ω.

More precisely, let

$$\mathcal{W}_{G/H}^-[(\omega, z)] := \left\{ z' \in G/H^\circ \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T_\omega^n z, T_\omega^n z') < 0 \right\}.$$

Then for all $zH^\circ \in G/H^\circ$ and $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, we have $\mathcal{W}_{G/H}^-[(\omega, z)] = N^-(\omega)zH^\circ$ and in particular is algebraic. The analogous statement holds for the unstable manifold.

Remark. It will be clear from the proof that the analogous statement is true for the strong (un)stable manifolds

$$\mathcal{W}_{G/H}^{\leq\lambda}[(\omega, z)] := \left\{ z' \in G/H^\circ \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T_\omega^n z, T_\omega^n z') \leq \lambda \right\}$$

for all $\lambda < 0$, though we will not need this fact. However, it is *not* true in general for the center-stable set $\mathcal{W}^{\leq 0}_{G/H}[(\omega, z)]$.

We first prove the following (elementary) lemma.

Lemma 4.5.4. Let V be a finite-dimensional real inner product space, and $\Lambda \in GL(V)$ be a self-adjoint real operator on V with all eigenvalues (real and) positive. Let $V^{\geq 0} \subset V$ be the direct sum of the eigenspaces corresponding to the eigenvalues at least 1.

Let $\{\mathbf{v}_n\}_{n\in\mathbb{N}} \subset V^{\geq 0}$ be a sequence of nonzero vectors such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathbf{v}_n\| < 0. \tag{4.5.4}$$

Then

$$\lim_{n\to\infty} \Lambda^{-n}\mathbf{v}_n \to 0.$$

Proof. Let $e^{\lambda_1} > e^{\lambda_2} > \cdots > e^{\lambda_m}$ be the set of eigenvalues of Λ, and let $V^{\lambda_i} \subset V$ be the eigenspace of e^{λ_i}. Note that the assumption (4.5.4) does not depend on the choice of a (equivalent) norm $\|\cdot\|$ on V, thus we may use the equivalent norm $\|\cdot\|'$ on V defined as follows: for each $\mathbf{v} \in V$, decompose $\mathbf{v} := \sum_{i=1}^m \mathbf{v}^{\lambda_i}$ as a sum of eigenvectors $\mathbf{v}^{\lambda_i}$ with eigenvalue e^{λ_i}, and define $\|\mathbf{v}\|' := \max_{1\leq i\leq m} \|\mathbf{v}^{\lambda_i}\|$.

If we write the eigenvector decomposition of $\mathbf{v}_n = \sum_{i=1}^m \mathbf{v}_n^{\lambda_i}$, then (4.5.4) (with the norm $\|\cdot\|'$) implies that for all $1 \leq i \leq m$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathbf{v}_n^{\lambda_i}\| < 0.$$

Since $\Lambda^{-n}\mathbf{v}_n^{\lambda_i} = e^{-n\lambda_i}\mathbf{v}_n^{\lambda_i}$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\Lambda^{-n}\mathbf{v}_n^{\lambda_i}\| = \limsup_{n\to\infty} \frac{1}{n} \log e^{-n\lambda_i}\|\mathbf{v}_n^{\lambda_i}\| = -\lambda_i + \limsup_{n\to\infty} \frac{1}{n} \log \|\mathbf{v}_n^{\lambda_i}\| < -\lambda_i.$$

Since $\mathbf{v}_n \in V^{\geq 0}$, $\mathbf{v}_n^{\lambda_i} = 0$ for all $\lambda_i < 0$. Thus we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\Lambda^{-n}\mathbf{v}_n\|' = \limsup_{n\to\infty} \frac{1}{n} \log \max_{1\leq i\leq m, \lambda_i \geq 0} \|\Lambda^{-n}\mathbf{v}_n^{\lambda_i}\| < 0.$$

Hence $\Lambda^{-n}\mathbf{v}_n \to 0$ as $n \to \infty$, as desired. □

Proof of Lemma 4.5.3. First of all since the metric $d_{G/H}$ on each tangent space $\mathfrak{g}/\mathfrak{h}_z$ is defined by pullback from the restriction of a fixed inner product on $\mathfrak{g}$ to $\mathfrak{m}_z := (\mathfrak{h}_z)^{\perp} \subset \mathfrak{g}$, we have for all $z, z' \in G$,

$$d_{G/H}(zH^\circ, z'H^\circ) \leq d_G(z, z').$$

In particular, we have

$$N^-(\omega)zH^\circ \subset \mathcal{W}^-_{G/H}[(\omega, z)].$$

For the other direction, let $z' \in \mathcal{W}^-_{G/H}[(\omega, z)]$. Then by definition, there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}} \subset G$ and $\{h_n\}_{n\in\mathbb{N}} \subset H^\circ$ such that $T^n_\omega z' = \delta_n T^n_\omega z h_n$ and $\limsup_{n\to\infty} \frac{1}{n} \log d_G(\delta_n, e) < 0$.

Now consider the adjoint action of μ on $\mathfrak{g}$. By assumption μ has finite first moment. Therefore we can apply [GM89, Thm. 1.2] to this random action, and get for $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, an element $\Lambda(\omega) \in G$ such that

1. $\Lambda(\omega) = \lim_{n\to\infty} ((T^n_\omega)^T (T^n_\omega))^{1/2n}$, in particular the adjoint action of $\Lambda(\omega)$ on $\mathfrak{g}$ is self-adjoint with positive eigenvalues,
2. $\lim_{n\to\infty} \frac{1}{n} \log \|\Lambda(\omega)^n (T^n_\omega)^{-1}\| = \lim_{n\to\infty} \frac{1}{n} \log \|(T^n_\omega)\Lambda(\omega)^{-n}\| = 0,$
3. let $\mathfrak{g}^{<0}(\omega) \subset \mathfrak{g}$ be the direct sum of the eigenspaces of $\mathrm{Ad}(\Lambda(\omega))$ on $\mathfrak{g}$ with eigenvalue less than 1. Then

   $$\mathfrak{g}^{<0}(\omega) = \left\{ \mathbf{v} \in \mathfrak{g} \,\middle|\, \limsup_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T^n_\omega)\mathbf{v}\| < 0 \right\}.$$

   Thus $\exp(\mathfrak{g}^{<0}(\omega)) = N^-(\omega)$.

Let $\mathfrak{g}^{\geq 0}(\omega) \subset \mathfrak{g}$ be the direct sum of the eigenspaces of $\mathrm{Ad}(\Lambda(\omega))$ with eigenvalue at least 1. Then $\mathfrak{g} = \mathfrak{g}^{\geq 0}(\omega) \oplus \mathfrak{g}^{<0}(\omega)$. Moreover, there exists an open neighborhood $U \subset G$ of the identity such that $U \subset \exp(\mathfrak{g}^{\geq 0}(\omega)) \exp(\mathfrak{g}^{<0}(\omega))$, i.e. for all $g \in U$, there exist $\mathbf{w}_g \in \mathfrak{g}^{\geq 0}(\omega)$ and $\mathbf{v}_g \in \mathfrak{g}^{<0}(\omega)$ such that $g = \exp(\mathbf{w}_g)\exp(\mathbf{v}_g)$ (the existence of such a neighborhood U follows, for instance, from the inverse function theorem applied to the map $\mathfrak{g}^{\geq 0}(\omega) \oplus \mathfrak{g}^{<0}(\omega) \to G$ defined by $(\mathbf{w}, \mathbf{v}) \mapsto \exp(\mathbf{w})\exp(\mathbf{v})$).

Since $\limsup_{n\to\infty} \frac{1}{n} \log d_G(\delta_n, e) < 0$ and $\lim_{n\to\infty} \frac{1}{n} \log \|\Lambda(\omega)^n (T_\omega^n)^{-1}\| = 0$, we have in particular

$$\limsup_{n\to\infty} \frac{1}{n} \log d_G(\Lambda(\omega)^n (T_\omega^n)^{-1} \delta_n (T_\omega^n) \Lambda(\omega)^{-n}, e) < 0.$$

Take a large enough $N > 0$ such that for all $n > N$, $\Lambda(\omega)^n (T_\omega^n)^{-1} \delta_n (T_\omega^n) \Lambda(\omega)^{-n} \in U$, and write

$$\Lambda(\omega)^n (T_\omega^n)^{-1} \delta_n (T_\omega^n) \Lambda(\omega)^{-n} = \exp(\mathbf{w}_n)\exp(\mathbf{v}_n) \qquad \text{with } \mathbf{w}_n \in \mathfrak{g}^{\geq 0}(\omega) \text{ and } \mathbf{v}_n \in \mathfrak{g}^{<0}(\omega).$$

Then

$$\delta_n = \exp(\mathrm{Ad}((T_\omega^n)\Lambda(\omega)^{-n})\mathbf{w}_n) \exp(\mathrm{Ad}((T_\omega^n)\Lambda(\omega)^{-n})\mathbf{v}_n) \text{ with } \mathbf{w}_n \in \mathfrak{g}^{\geq 0}(\omega) \text{ and } \mathbf{v}_n \in \mathfrak{g}^{<0}(\omega).$$

Moreover, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathbf{w}_n\| < 0,$$

and analogously for $\mathbf{v}_n$. By definition,

$$z' = (T_\omega^n)^{-1} \delta_n T_\omega^n z h_n = \exp(\mathrm{Ad}(\Lambda(\omega))^{-n}\mathbf{w}_n) \exp(\mathrm{Ad}(\Lambda(\omega))^{-n}\mathbf{v}_n) z h_n. \tag{4.5.5}$$

Now apply Lemma 4.5.4 to the adjoint action of $\Lambda(\omega)$ on $\mathfrak{g}$ and the sequence of vectors $\{\mathbf{w}_n\} \subset \mathfrak{g}^{\geq 0}(\omega)$, we have $\mathrm{Ad}(\Lambda(\omega))^{-n}\mathbf{w}_n \to 0$ as $n \to \infty$. Therefore by (4.5.5),

$$z' = \lim_{n\to\infty} \exp(\mathrm{Ad}(\Lambda(\omega))^{-n}\mathbf{v}_n) z h_n.$$

Since $\mathbf{v}_n \in \mathfrak{g}^{<0}(\omega)$, we have $\mathrm{Ad}(\Lambda(\omega))^{-n}\mathbf{v}_n \in \mathfrak{g}^{<0}(\omega)$. Since $\exp(\mathfrak{g}^{<0}(\omega)) = N^-(\omega)$ and $h_n \in H^\circ$, we conclude that z' is in the closure of $N^-(\omega)zH^\circ$.

Finally, by Chevalley's theorem and that H° has no nontrivial character , there exists a representation $G \to GL(V)$ such that H° is the stabilizer of an element $v \in V$. Thus $N^-(\omega)zH^\circ$ can be identified with a single $N^-(\omega)$-orbit in V, namely $N^-(\omega)zv \subset V$, via the injection $G/H^\circ \to V$ by $g \mapsto gv$. Since $N^-(\omega)$ is a unipotent algebraic group, any orbit of $N^-(\omega)$ on V is Zariski closed by Kostant-Rosenlicht theorem (see e.g. [Ros61, Thm. 2]). Therefore $N^-(\omega)zH^\circ$ is algebraic (and in particular, closed), and thus $z' \in N^-(\omega)zH^\circ$. Since $z' \in \mathcal{W}^-_{G/H}[(\omega, z)]$ is arbitrary, we have $\mathcal{W}^-_{G/H}[(\omega, z)] = N^-(\omega)zH^\circ$. □

Remark 4.5.5. In several occasions, we would want an "identification" map between the tangent spaces of two points $z, z' \in G/H$, i.e. a linear map $I(z, z') : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}$. Since G/H is homogeneous, there exists $g \in G$ such that $z' = gz$, thus one can define

$$I(z, z', g) : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}$$

by $\mathbf{v} \mapsto \mathrm{Ad}_g\mathbf{v} := g\mathbf{v}g^{-1}$ for such an identification map. However, this map is not canonical - it depends on the choice of such $g \in G$. In particular there is no canonical way to define them that is compatible with the dynamics. We will used it several times to define more canonical constructions, like the holonomy map in the next subsection.

4.5.6 Holonomy maps $H_i^-(\hat{z}, \hat{z}')$

Proposition 4.5.6. For $\hat{\nu}$-almost every $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$ and almost every $\hat{z}' := (\omega', z') \in \hat{\mathcal{W}}^-_{G/H}[\hat{z}]$, for each i, there exist a linear map $H_i^-(\hat{z}, \hat{z}') : W^{\leq\lambda_i}(\hat{z})/W^{<\lambda_i}(\hat{z}) \to W^{\leq\lambda_i}(\hat{z}')/W^{<\lambda_i}(\hat{z}')$ such that

1. $H_i^-(\hat{z}, \hat{z}) = \mathrm{id}$ and $H_i^-(\hat{z}, \hat{z}'') = H_i^-(\hat{z}', \hat{z}'') \circ H_i^-(\hat{z}, \hat{z}')$.

2. $(T^t_{\omega'})_* \circ H_i^-(\hat{z}, \hat{z}') = H_i^-(T^t\hat{z}, T^t\hat{z}') \circ (T^t_\omega)_*$.

3. $(\hat{z}, \hat{z}') \mapsto H_i^-(\hat{z}, \hat{z}')$ varies continuously.

4. $H_i^-(\hat{z}, \hat{z}')$ is the identity map if $\pi_{G/L}(z) = \pi_{G/L}(z')$, where $\pi_{G/L} : G/H \to G/L$ is the quotient map.

Proof. Let $\hat{z}' := (\omega', z'H) \in \hat{\mathcal{W}}_{G/H}^-[\hat{z}]$. By Lemma 4.5.3 (and that the exponential map is a diffeomorphism on the unipotent group $N^-(\omega)$), there exists $\mathbf{v} \in W_{\mathfrak{g}}^{<0}(\omega)$ such that $\exp(\mathbf{v})zH = z'H$.

Recall from Remark 4.5.5 the map

$$I(z, z', \exp(\mathbf{v})) := \mathrm{Ad}_{\exp(\mathbf{v})} : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}.$$

Since $\mathbf{v}$ is exponentially contracted, $I(z, z', \exp(v))$ induces a map

$$H_i^-(\hat{z}, \hat{z}') : W^{\leq \lambda_i}(\hat{z})/W^{<\lambda_i}(\hat{z}) \to W^{\leq \lambda_i}(\hat{z}')/W^{<\lambda_i}(\hat{z}').$$

Now recall from subsection 4.5.4 that

$$W^{\leq \lambda_i}(\hat{z})/W^{<\lambda_i}(\hat{z}) = \frac{W_{\mathfrak{g}}^{\leq \lambda_i}(\omega)}{W_{\mathfrak{g}}^{\leq \lambda_i}(\omega) \cap (W_{\mathfrak{g}}^{<\lambda_i}(\omega) + \mathfrak{h}_z)}.$$

For each $\mathbf{w} \in W_{\mathfrak{g}}^{\leq \lambda_i}(\omega)$, since $\mathbf{v} \in W_{\mathfrak{g}}^{<0}(\omega)$, we have

$$\mathrm{Ad}_{\exp(\mathbf{v})}(\mathbf{w}) - \mathbf{w} \in W_{\mathfrak{g}}^{<\lambda_i}(\omega) \subset W_{\mathfrak{g}}^{\leq \lambda_i}(\omega) \cap (W_{\mathfrak{g}}^{<\lambda_i}(\omega) + \mathfrak{h}_{z'}). \tag{4.5.6}$$

In particular $\mathrm{Ad}_{\exp(\mathbf{v})} W_{\mathfrak{g}}^{\leq \lambda_i}(\omega) = W_{\mathfrak{g}}^{\leq \lambda_i}(\omega)$ and $\mathrm{Ad}_{\exp(\mathbf{v})} W_{\mathfrak{g}}^{<\lambda_i}(\omega) = W_{\mathfrak{g}}^{<\lambda_i}(\omega)$. Also clearly $\mathrm{Ad}_{\exp(\mathbf{v})}$ maps $\mathfrak{h}_z$ to $\mathfrak{h}_{z'}$ as $\exp(\mathbf{v})zH = z'H$ (recall that $\mathfrak{h}_z := \mathrm{Ad}_z \mathfrak{h}$). Therefore $H_i^-(\hat{z}, \hat{z}')$ is well-defined and does not depend on the choice of representative $\mathbf{v}$ in the coset $\mathbf{v} + \mathfrak{h}_z \in W^{<0}(\hat{z})$ as along as it satisfies $\exp(\mathbf{v})zH = z'H$.

In particular since $H_i^-(\hat{z}, \hat{z}')$ does not depend on the choice of $\mathbf{v}$, one can readily verify properties 1-3. Property 4 holds since if $\pi_{G/L}(z) = \pi_{G/L}(z')$, then $\mathfrak{h}_z = \mathfrak{h}_{z'}$ and furthermore $\hat{z}' \in \hat{\mathcal{W}}_{G/H}^-[\hat{z}]$ then $W^{\leq \lambda_i}(\omega, z) = W^{\leq \lambda_i}(\omega', z')$ (see the end of subsection 4.5.2). By (4.5.6), the restriction of the identity map $\mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}$ coincides with $H_i^-(\hat{z}, \hat{z}')$ on each successive quotient. $\square$

4.5.7 Equivariant measurable flat connections $P^-(\hat{z}, \hat{z}')$

For $\hat{\nu}$-almost every $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$ and almost every $\hat{z}' := (\omega', z') \in \hat{\mathcal{W}}^-_{G/H}[(\omega, z)]$, we can define $P_i^-(\hat{z}, \hat{z}') : W^{\lambda_i}(\omega, z) \to W^{\lambda_i}(\omega', z')$ by the composition

$$W^{\lambda_i}(\omega, z) \to W^{\leq \lambda_i}(\omega, z)/W^{<\lambda_i}(\omega, z) \xrightarrow{H_i^-(\hat{z}, \hat{z}')} W^{\leq \lambda_i}(\omega', z')/W^{<\lambda_i}(\omega', z') \to W^{\lambda_i}(\omega', z'),$$

where we use the natural isomorphism $W^{\lambda_i}(\omega, z) \to W^{\leq i}(\omega, z)/W^{<\lambda_i}(\omega, z)$ given by injection and then quotient.

Define $P^-(\hat{z}, \hat{z}') : \mathfrak{g}/\mathfrak{h}_z \to \mathfrak{g}/\mathfrak{h}_{z'}$ be the unique linear map that restricts to $P_i^-(\hat{z}, \hat{z}')$ on each subspaces $W^{\lambda_i}(\omega, z)$. The following properties of P^- are clear from definition and Proposition 4.5.6.

Lemma 4.5.7. cf. [ELa, Lem. 2.1] For $\hat{\nu}$-almost every $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$ and almost every $\hat{z}' = (\omega', z') \in \hat{\mathcal{W}}^-_{G/H}[(\omega, z)]$,

(a) $P^-(\hat{z}, \hat{z}') W^{\lambda_i}(\omega, z) = W^{\lambda_i}(\omega', z')$,

(b) $P^-(\hat{z}, \hat{z}') = (T^{-t}_{T^t\omega'})_* \circ P^-(T^t\hat{z}, T^t\hat{z}') \circ (T^t_\omega)_*$,

(c) $P^-(\hat{z}, \hat{z}'') = P^-(\hat{z}', \hat{z}'') \circ P^-(\hat{z}, \hat{z}')$.

(d) $P^-(\hat{z}, \hat{z}') = \mathrm{id}$ if $\pi_{G/L}(z) = \pi_{G/L}(z')$ and $\omega = \omega'$.

Similarly define $P^+(\hat{z}, \hat{z}')$ for $\hat{z}' \in \hat{\mathcal{W}}^+_{G/H}[\hat{z}]$.

Remark 4.5.8. Due to property 3 and 4, for $\hat{x} = (\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$ and $\hat{x}' = (\omega', x') \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}]$, we may sometimes write $P^-(\hat{x}, \hat{x}) := P^-(\hat{z}, \hat{z}')$ for $\hat{z} = (\omega, z)$ and $\hat{z}' = (\omega', z')$ with $\pi_{G/L}(z) = x$ and $\pi_{G/L}(z') = x'$. Conversely, we may extend the definition of $P^-(\hat{z}, \hat{z}')$ to all $\hat{z} = (\omega, z), \hat{z}' = (\omega', z') \in \mathcal{S}^{\mathbb{Z}} \times G/H$ such that $(\omega', \pi_{G/L}(z')) \in \hat{\mathcal{W}}^-_{G/L}[(\omega, \pi_{G/L}(z))]$ (note that it follows from Lemma 4.5.3 that in this case, $\mathcal{W}^-_{G/H}[\hat{z}]$ has nonempty intersection z'' with

$z'L/H$, and we have $\hat{z}'' := (\omega', z'') \in \hat{\mathcal{W}}^-_{G/H}[\hat{z}]$. Thus we can first define $P^-(\hat{z}, \hat{z}'')$, then post-compose with the identity map $P^-(\hat{z}'', \hat{z}')$ to define $P^-(\hat{z}, \hat{z}')$). We will need this extension in the proof of Theorem 4.10.1 (see **Step 15** in the proof outline in subsection 4.10.1).

The following lemma is an important property of the map P^- (and the corresponding map P^+). The proof in [ELa, Lem. 2.5] applies in our setting. The main input is a theorem of Ledrappier [Led86, Thm. 1].

Lemma 4.5.9. cf. [ELa, Lem. 2.5] Let $\{M(\hat{x}) \subset \mathfrak{g}/\mathfrak{h}_x\}_{\hat{x} \in \mathcal{S}^{\mathbb{Z}} \times G/L}$ be a $\hat{T}$-equivariant subbundle over $\mathcal{S}^{\mathbb{Z}} \times G/L$. Then, up to a null set, for $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}]$,

$$M(\hat{x}') = P^-(\hat{x}, \hat{x}')M(\hat{x}).$$

The analogous property holds for $\hat{x}' \in \hat{\mathcal{W}}^+_{G/L}[\hat{x}]$ and the map $P^+(\hat{x}, \hat{x}')$.

Lemma 4.5.10. [ELa, Lem. 2.2] There exists $\alpha > 0$ depending only on the Lyapunov exponents, and for all $\delta > 0$ there exists a subset $K \subset \Omega$ with measure at least $1 - \delta$ such that for all $\hat{x} = (\omega, x) \in K$, $\hat{x}' = (\omega', x') \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}] \cap K$, $t > 0$ and any $g \in \exp(W^{<0}_{\mathfrak{g}}(\omega)) \subset G$ such that $gxL = x'L$, we have

$$\|P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}') - I(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}', (T^t_\omega)g(T^t_\omega)^{-1})\|_{0\to 0} \leq \|g\|_{\mathrm{Ad}(G)}C(\delta)e^{-\alpha t}.$$

Here $\|\cdot\|_{0\to 0}$ is the operator norm on a linear operator $\mathfrak{g}/\mathfrak{h}_{\hat{T}^t\hat{x}'} \to \mathfrak{g}/\mathfrak{h}_{\hat{T}^t\hat{x}'}$ with respect to the norm $\|\cdot\|_0$ defined in subsection 4.4.4. The identification map $I(x, x', g) := \mathrm{Ad}_g$ was defined as in Remark 4.5.5. Also $\|g\|_{\mathrm{Ad}(G)}$ is the operator norm of the image of g via the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$, with the norm of $\mathfrak{g}$ chosen in subsection 4.4.4.

Proof. Let $\varepsilon > 0$ be smaller than one third the smallest gap between consecutive Lyapunov exponents. Then by Oseledets theorem, for any $\delta > 0$, there exists $K \subset \Omega$ of measure at least $1 - \delta$ and constants $\sigma = \sigma(\delta) > 0, \rho = \rho(\delta) > 0$ such that for all $\hat{x} \in K$ and $t \geq 0$, we have

(a) for any subset S of Lyapunov exponents,

$$d_0 \left(\bigoplus_{i \in S} W^{\lambda_i}(\hat{T}^t \hat{x}), \bigoplus_{j \notin S} W^{\lambda_j}(\hat{T}^t \hat{x}) \right) \geq \sigma e^{-\varepsilon t}.$$

(b) for any i and $\mathbf{w}_i \in W^{\lambda_i}(\hat{x})$,

$$\rho e^{(\lambda_i - \varepsilon)t} \|\mathbf{w}_i\|_0 \leq \|(T^t_\omega)_* \mathbf{w}_i\|_0 \leq \rho^{-1} e^{(\lambda_i + \varepsilon)t} \|\mathbf{w}_i\|_0.$$

By (a), it suffices to show that for $\mathbf{v} \in W^{\lambda_i}(\hat{T}^t \hat{x})$ for some i, we have

$$\frac{\|P^-(\hat{T}^t \hat{x}, \hat{T}^t \hat{x}')\mathbf{v} - \mathrm{Ad}_{(T^t_\omega) g (T^t_\omega)^{-1}} \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \leq C(\delta) e^{-\alpha t}$$

for any $g \in \exp(W^{<0}_{\mathfrak{g}}(\omega)) \subset G$ such that $gxL = x'L$.

Let $\mathbf{w} \in W^{\lambda_i}(\hat{x})$ be such that $(T^t_\omega)_* \mathbf{w} = \mathbf{v}$. Then by (b),

$$\|\mathbf{v}\|_0 \geq \rho e^{(\lambda_i - \varepsilon)t} \|\mathbf{w}\|_0. \tag{4.5.7}$$

Now we recall the definition of P^- and the construction of the holonomy map $H^-_i(\hat{x}, \hat{x}')$ that

$$P^-(\hat{x}, \hat{x}')\mathbf{w} = \mathrm{Ad}_g \mathbf{w} + \sum_{j > i} \mathbf{w}_j, \qquad \mathbf{w}_j \in W^{\lambda_j}(\hat{x}'), \tag{4.5.8}$$

where g is *any* element in $\exp(W^{<0}_{\mathfrak{g}}(\omega))$ with $gxL = x'L$ (it was for $gxH = x'H$ but since the holonomy map is the identity along xL/H, we can relax to requiring $gxL = x'L$. The vectors $\mathbf{w}_j$ will depend on the choice of g). By equivariance of the P^- map (Proposition 4.5.7(b)), we have

$$P^-(\hat{T}^t \hat{x}, \hat{T}^t \hat{x}')\mathbf{v} = (T^t_{\omega'})_* P^-(\hat{x}, \hat{x}')\mathbf{w}.$$

Therefore

$$P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}')\mathbf{v} = (T^t_{\omega'})_*\mathrm{Ad}_g\mathbf{w} + \sum_{j>i}(T^t_{\omega'})_*\mathbf{w}_j.$$

Note that $(T^t_{\omega'})_*\mathrm{Ad}_g\mathbf{w} = \mathrm{Ad}_{(T^t_\omega)g(T^t_\omega)^{-1}}\mathbf{v}$ since $(T^t_{\omega'})_*$ acts by conjugation on $\mathfrak{g}/\mathfrak{h}_{\hat{x}'}$ (note that $T^t_\omega = T^t_{\omega'}$ since $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}]$, i.e. ω and ω' have the same combinatorial future). Therefore we have

$$P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}')\mathbf{v} - \mathrm{Ad}_{(T^t_\omega)g(T^t_\omega)^{-1}}\mathbf{v} = \sum_{j>i}(T^t_{\omega'})_*\mathbf{w}_j. \tag{4.5.9}$$

By (a) and (4.5.8), for all $j > i$, we have (note that $P^-(\hat{x}, \hat{x}') \in W^{\lambda_i}(\hat{x}')$ and $\mathbf{w}_j \in W^{\lambda_j}(\hat{x}')$)

$$\|\mathbf{w}_j\|_0 \le C_1(\delta)\|\mathrm{Ad}_g\mathbf{w}\|_0 \le C_1(\delta)\|g\|_{\mathrm{Ad}(G)}\|\mathbf{w}\|.$$

Thus by (b),

$$\|(T^t_{\omega'})_*\mathbf{w}_j\|_0 \le \rho^{-1}e^{(\lambda_j+\varepsilon)t}\|\mathbf{w}_j\|_0 \le C_1(\delta)\rho^{-1}e^{(\lambda_j+\varepsilon)t}\|g\|_{\mathrm{Ad}(G)}\|\mathbf{w}\|.$$

By (4.5.7) and (4.5.9), we have

$$\|P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}')\mathbf{v} - \mathrm{Ad}_{(T^t_\omega)g(T^t_\omega)^{-1}}\mathbf{v}\|_0 \le \sum_{j>i} C_1(\delta)\rho^{-2}e^{(\lambda_j-\lambda_i+2\varepsilon)t}\|g\|_{\mathrm{Ad}(G)}\|\mathbf{v}\|_0.$$

This implies the statement since $\lambda_j < \lambda_i$ for all $j > i$. □

4.5.8 Jordan Canonical Form of a cocycle

Recall that $\Omega_0 := \mathcal{S}^{\mathbb{Z}} \times G/L \times [0,1]$. We recall Zimmer's amenable reduction theorem in the case of a cocycle in $GL(n, \mathbb{R})$.

Theorem 4.5.11. cf. [ELa, Lem. 2.3] Suppose T^t is a linear cocycle over an ergodic action of $\mathbb{R}$ on Ω_0. Then there exists a finite set Σ and an extension of the flow T^t to $\Omega = \Omega_0 \times \Sigma$ s.t.:

For each i, for $\hat{\bar{\nu}}$-a.e. $\hat{x} = (\omega, x)$, there exists an invariant flag

$$\{0\} = W_{i,0}(\omega, x) \subset W_{i,1}(\omega, x) \subset \cdots \subset W_{i,n_i}(\omega, x) = W^{\lambda_i}(\omega, x),$$

and on each $W_{ij}(\omega, x)/W_{i,j-1}(\omega, x)$, there exists a nondegenerate quadratic form $\langle \cdot, \cdot \rangle_{ij,\hat{x}}$ and a cocycle $\lambda_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for all $\mathbf{u}, \mathbf{v} \in W_{ij}(\omega, x)/W_{i,j-1}(\omega, x)$,

$$\langle ((T^t_\omega)_* \mathbf{u}, (T^t_\omega)_* \mathbf{v} \rangle_{ij, T^t \hat{x}} = e^{\lambda_{ij}(\hat{x}, t)} \langle \mathbf{u}, \mathbf{v} \rangle_{ij,\hat{x}}.$$

The space Ω has a natural probability measure, defined as the product of the measure $\hat{\bar{\nu}} \times \mathrm{Leb}_{[0,1]}$ on Ω_0 and the uniform measure Unif_Σ on the finite set Σ.

In summary, we have the following spaces, each with their natural measure, and has a $\hat{T}$-action and a $\mathcal{U}_1^+$-action that are measure-preserving.

Space	Probability measure
$\Omega_b = \mathcal{S}^{\mathbb{Z}} \times [0, 1]$	$\mu^{\mathbb{Z}} \times \mathrm{Leb}_{[0,1]}$
$\Omega_0 = \mathcal{S}^{\mathbb{Z}} \times G/L \times [0, 1]$	$\hat{\bar{\nu}} \times \mathrm{Leb}_{[0,1]}$
$\Omega = \mathcal{S}^{\mathbb{Z}} \times G/L \times [0, 1] \times \Sigma$	$\hat{\bar{\nu}} \times \mathrm{Leb}_{[0,1]} \times \mathrm{Unif}_\Sigma$
$\hat{\Omega}_0 = \mathcal{S}^{\mathbb{Z}} \times G/H \times [0, 1]$	$\hat{\nu} \times \mathrm{Leb}_{[0,1]}$
$\hat{\Omega} = \mathcal{S}^{\mathbb{Z}} \times G/H \times [0, 1] \times \Sigma$	$\hat{\nu} \times \mathrm{Leb}_{[0,1]} \times \mathrm{Unif}_\Sigma$

Recall that $\mathcal{U}_1^+$ acts on $\omega \in \mathcal{S}^{\mathbb{Z}}$ by changing the future of ω, i.e. if $\omega' := u\omega$, then $(\omega')^- = \omega^-$ and $(\omega')^+$ is an arbitrary. Extend this to an action on $\Omega_b, \Omega_0, \Omega, \hat{\Omega}_0, \hat{\Omega}$ by acting trivially on the extra factors and we again have a natural measure on $\mathcal{U}_1^+ \hat{x}$ for any $\hat{x}$ in any of these spaces using its natural identification with $\mathcal{S}^{\mathbb{N}}$.

Notational remark 2: there are obvious projection maps between various spaces defined above that are compatible with the dynamics and the measures. From now on, objects defined on a certain factor will be automatically lifted as objects defined on spaces with a projection onto this factor.

For instance there is a projection map $\pi : \hat{\Omega} \to \mathcal{S}^{\mathbb{Z}} \times G/L$ defined by projecting onto the first two factors, and then apply the quotient map $G/H \to G/L$ on the second factor. Then if we define an object $f(\hat{x})$ for $\hat{x} \in \mathcal{S}^{\mathbb{Z}} \times G/L$, we may sometimes write $f(\hat{z}) := f(\pi(\hat{z}))$ for $\hat{z} \in \hat{\Omega}$ without referencing the projection map.

4.5.9 Dynamically defined norms $\|\cdot\|_{\hat{x}}$

We would like to construct a norm on the tangent spaces of the bundle $\hat{\Omega} \to \Omega$ so that the exponentially growth rate given by Oseledets theorem does not just hold asymptotically, but hold for all time t. To do so, we first recall a choice of Markov partition on the base as in [ELa], and then the proof in [ELa] applies.

A Markov partition

Proposition 4.5.12. [ELa, Prop. 2.8] Suppose $\mathcal{C} \subset \Omega$ is a set with positive measure, and $T_0 : \mathcal{C} \to \mathbb{R}^+$ is a measurable function that is finite a.e. Then there exists $\hat{x}_0 \in \Omega$, a subset $\mathcal{C}_1 \subset \hat{\mathcal{W}}^-_{G/L}[\hat{x}_0] \cap \mathcal{C}$ and for each $\hat{c} \in \mathcal{C}_1$ a subset $B^+[\hat{c}] \subset \hat{\mathcal{W}}^+_{G/L}[\hat{c}]$ depending measurably on $\hat{c}$, and a number $t(\hat{c}) > 0$ such that if we let

$$J_{\hat{c}} := \bigcup_{0 \leq t < t(\hat{c})} \hat{T}^{-t} B^+[\hat{c}],$$

then the following holds:

(a) $B^+[\hat{c}]$ is relatively open in $\hat{\mathcal{W}}^+_{G/L}[\hat{c}]$, and $\tilde{\mu}|_{\hat{\mathcal{W}}^+_{G/L}[\hat{c}]}(B^+[\hat{c}]) > 0$.

(b) $J_{\hat{c}} \cap J_{\hat{c}'} = \emptyset$ if $\hat{c} \neq \hat{c}'$.

(c) $\bigcup_{\hat{c} \in \mathcal{C}_1} J_{\hat{c}}$ is conull in Ω.

(d) For every $\hat{c} \in \mathcal{C}_1$, there exists $\hat{c}' \in \mathcal{C}_1$ such that $\hat{T}^{-t(\hat{c})} B^+[\hat{c}] \subset B^+[\hat{c}']$.

(e) $t(\hat{c}) > T_0(\hat{c})$ for all $\hat{c} \in \mathcal{C}_1$.

For $\hat{x} \in \Omega$, let $J[\hat{x}]$ denote the set $J_{\hat{c}}$ containing $\hat{x}$. Let $\mathfrak{B}_0[\hat{x}] := J[\hat{x}] \cap \hat{\mathcal{W}}^+_{G/L}[\hat{x}]$.

Dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$

Using the partition given by $\mathfrak{B}_0[\hat{x}]$, the proof of [ELa, Prop. 2.14] holds in our setting.

Proposition 4.5.13. cf. [ELa, Prop. 2.14.] There exists a T^t-invariant full measure set $\Omega' \subset \Omega$ such that for all $\hat{x} \in \Omega'$, there exists an inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$ on $\mathfrak{g}/\mathfrak{h}_x$ and cocycles $\lambda_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ with the following properties:

(a) For all $\hat{x} \in \Omega'$, the distinct eigenspaces $W^{\lambda_i}(\hat{x})$ are orthogonal.

(b) Let $W'_{ij}(\hat{x}) := W^{\perp}_{i,j-1} \subset W_{ij}(\hat{x})$. Then for all $\hat{x} \in \Omega'$, $t \in \mathbb{R}$ and $\mathbf{v} \in W'_{ij}(\hat{x}) \subset \mathfrak{g}/\mathfrak{h}_x$,

$$(T^t_{\hat{x}})_* \mathbf{v} = e^{\lambda_{ij}(\hat{x},t)} \mathbf{v}' + \mathbf{v}'',$$

where $\mathbf{v}' \in W'_{ij}(\hat{T}^t \hat{x})$, $\quad \mathbf{v}'' \in W_{i,j-1}(\hat{T}^t \hat{x}) \quad$ and $\quad \|\mathbf{v}'\|_{\hat{T}^t \hat{x}} = \|\mathbf{v}\|_{\hat{x}}$. In particular,

$$\|(T^t_{\hat{x}})_* \mathbf{v}\|_{\hat{T}^t \hat{x}} \geq e^{\lambda_{ij}(\hat{x},t)} \|\mathbf{v}\|_{\hat{x}}.$$

(c) There exists $\kappa > 1$ such that for all $\hat{x} \in \Omega'$, $t > 0$ and i with $\lambda_i > 0$, $\kappa^{-1} t \leq \lambda_{ij}(\hat{x}, t) \leq \kappa t$.

(d) There exists $\kappa > 1$ such that for all $\hat{x} \in \Omega'$,

$$\begin{aligned} e^{\kappa^{-1} t} \|\mathbf{v}\|_{\hat{x}} &\leq \|(T^t_{\hat{x}})_* \mathbf{v}\|_{\hat{T}^t \hat{x}} \leq e^{\kappa t} \|\mathbf{v}\|_{\hat{x}} && \text{for all } t \geq 0, \quad \mathbf{v} \in W^+(\hat{x}), \\ e^{-\kappa t} \|\mathbf{v}\|_{\hat{x}} &\leq \|(T^t_{\hat{x}})_* \mathbf{v}\|_{\hat{T}^t \hat{x}} \leq e^{-\kappa^{-1} t} \|\mathbf{v}\|_{\hat{x}} && \text{for all } t \geq 0, \quad \mathbf{v} \in W^-(\hat{x}), \\ e^{-\kappa |t|} \|\mathbf{v}\|_{\hat{x}} &\leq \|(T^t_{\hat{x}})_* \mathbf{v}\|_{\hat{T}^t \hat{x}} \leq e^{\kappa |t|} \|\mathbf{v}\|_{\hat{x}} && \text{for all } t \in \mathbb{R}, \quad \mathbf{v} \in \mathfrak{g}/\mathfrak{h}_x. \end{aligned}$$

In particular, the map $t \mapsto \|(T^t_\omega)_* \mathbf{v}\|_{\hat{T}^t \hat{x}}$ is continuous.

(e) For all $\hat{x} \in \Omega'$, $\hat{x}' \in \mathfrak{B}_0[\hat{x}] \cap \Omega'$ and $t \geq 0$, $\lambda_{ij}(\hat{x}, -t) = \lambda_{ij}(\hat{x}', -t)$.

(f) For a.e. $\hat{x} \in \Omega'$, a.e. $\hat{x}' \in \mathfrak{B}_0[\hat{x}] \cap \Omega'$ and $\mathbf{v}, \mathbf{w} \in \mathfrak{g}/\mathfrak{h}_x$, $\langle P^+(\hat{x}, \hat{x}')\mathbf{v}, P^+(\hat{x}, \hat{x}')\mathbf{w}\rangle_{\hat{x}'} = \langle \mathbf{v}, \mathbf{w}\rangle_{\hat{x}}$.

Remark 4.5.14. We remark that the proof of [ELa, Prop. 2.14] applies more generally to any finite dimensional linear cocycle V over Ω for the corresponding Lyapunov subspaces $W_V^{\lambda_{V,i}}$, refined Lyapunov subspaces $W_{V,ij}$ and corresponding maps P_V^+ for some constant $\kappa_V > 1$ using the same partition $\mathfrak{B}_0[\hat{x}]$ on the base.

From now on, we may drop the subscript when we refer to the dynamical norm $\|\cdot\| := \|\cdot\|_{\hat{x}}$, and will always use $\|\cdot\|_0$ to denote the norm defined in subsection 4.4.4.

At times we may need to compare the dynamical norm $\|\cdot\|_{\hat{x}}$ with the fixed norm $\|\cdot\|_0$ on $\mathfrak{g}/\mathfrak{h}_x$ defined in Section 4.4.4.

Lemma 4.5.15. cf. [ELa, Lem. 2.16] For every $\delta > 0$ and $\varepsilon > 0$, there exists a compact set $K(\delta) \subset \Omega$ with measure at least $1 - \delta$ and $C_1(\delta, \varepsilon) < \infty$ such that for all $\hat{x} \in K(\delta)$, $\mathbf{v} \in \mathfrak{g}/\mathfrak{h}_x$ and $t \in \mathbb{R}$,

$$C_1(\delta)^{-1} e^{-\varepsilon|t|} \leq \frac{\|\mathbf{v}\|_{T^t \hat{x}}}{\|\mathbf{v}\|_0} \leq C_1(\delta) e^{\varepsilon|t|}.$$

Proof. The proof is identical to that of [ELa, Lem. 2.16]. □

4.6 Factorization

4.6.1 Normal forms

In this subection, we briefly discuss the theory of normal forms in our particular setting, where a much simpler construction is available. The main purpose for us is to build the necessary tool for the factorization theorem in the next subsection. The general theory of normal forms has been studied extensively in the smooth ergodic theory literature (see for instance [KS17]).

In our particular setting, a choice of normal forms can be more explicitly chosen with extra properties (in particular it is a subspace in $\mathfrak{g}$). The main result of this section is Proposition 4.6.3.

Lemma on nilpotent Lie groups

The key to the construction of normal forms in the setting of G/H is the following elementary lemma about nilpotent Lie groups.

Lemma 4.6.1. Let N be a nilpotent real Lie group, and $U \subset N$ be a Lie subgroup. Let $\mathfrak{u} = \mathrm{Lie}(U)$ and $\mathfrak{n} = \mathrm{Lie}(N)$. Let the lower central series of $\mathfrak{n}$ be $\mathfrak{n} = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \cdots \supset \mathfrak{n}_k = 0$ (i.e. $\mathfrak{n}_{i+1} = [\mathfrak{n}, \mathfrak{n}_i]$).

If a complementary subspace $\mathfrak{v}$ of $\mathfrak{u}$ in $\mathfrak{n}$ satisfies

$$(\mathfrak{u} \cap \mathfrak{n}_i) \oplus (\mathfrak{v} \cap \mathfrak{n}_i) = \mathfrak{n}_i \qquad \text{for all } 0 \leq i \leq k-1,$$

then for $V := \exp(\mathfrak{v})$, the map $V \times U \to N$ defined by $(v, u) \mapsto vu$ is bijective with polynomial inverse.

Proof. For each $0 \leq i \leq k-1$, define a complementary subspace $\mathfrak{u}_i$ of $\mathfrak{u} \cap \mathfrak{n}_{i+1}$ in $\mathfrak{u} \cap \mathfrak{n}_i$. Similarly define the subspaces $\mathfrak{v}_i$. Thus

$$\mathfrak{u}_i \oplus (\mathfrak{u} \cap \mathfrak{n}_{i+1}) = \mathfrak{u} \cap \mathfrak{n}_i, \qquad \mathfrak{v}_i \oplus (\mathfrak{v} \cap \mathfrak{n}_{i+1}) = \mathfrak{v} \cap \mathfrak{n}_i.$$

By assumption, we have $(\mathfrak{u} \cap \mathfrak{n}_i) \oplus (\mathfrak{v} \cap \mathfrak{n}_i) = \mathfrak{n}_i$. Therefore this implies $(\mathfrak{u}_i \oplus \mathfrak{v}_i) \oplus \mathfrak{n}_{i+1} = \mathfrak{n}_i$. Let $\mathfrak{n}_i' := \mathfrak{u}_i \oplus \mathfrak{v}_i$. Then we have

$$\mathfrak{n}_i = \mathfrak{n}_i' \oplus \cdots \oplus \mathfrak{n}_{k-1}'. \tag{4.6.1}$$

Now note that the exponential map exp is polynomial with polynomial inverse log on nilpotent Lie algebra, therefore it suffices to show that the map $\psi : \mathfrak{v} \times \mathfrak{u} \to \mathfrak{n}$ defined by

$\psi(v,u) = \log(\exp(v)\exp(u))$ has polynomial inverse.

By (a weak form of) the Baker-Campbell-Hausdorff formula, $\psi(v,u) = v + u +$ iterated Lie brackets of v and u. Now given $n \in \mathfrak{n}$, we need to find $v \in \mathfrak{v}$ and $u \in \mathfrak{u}$ such that $\psi(v,u) = n$. Write $n = n_0 + n_1 + \cdots n_{k-1}$ using the decomposition in (4.6.1) so that $n_i \in \mathfrak{n}'_i$. Similarly write $u = u_0 + u_1 + \cdots + u_{k-1}$ and $v = v_0 + v_1 + \cdots + v_{k-1}$.

Since $[v_i, u_j] \in \mathfrak{n}_{\max\{i+j\}+1}$ (setting $\mathfrak{n}_t = 0$ if $t \geq k$), by comparing the $\mathfrak{n}'_0$-component of both sides of $n = \psi(v,u)$, we have $n_0 = v_0 + u_0$, and thus we can obtain u_0, v_0 as the $\mathfrak{u}_0$ and $\mathfrak{v}_0$ components of n_0 respectively. Inductively, if we already know u_j, v_j for $j < i$, we can decompose n using the decomposition $\mathfrak{n} = \mathfrak{n}'_0 \oplus \mathfrak{n}'_1 \oplus \cdots \oplus \mathfrak{n}'_{i-1} \oplus \mathfrak{n}_i$. Let $\tilde{n}_i$ be the $\mathfrak{n}_i$ component of n. Then we have $\tilde{n}_i \in v_i + u_i + \varepsilon(u_j, v_{j'} : j, j' < i) + \mathfrak{n}_{i+1}$, where ε is a linear combination of repeated Lie brackets of the terms u_j, v'_j. Thus we can extract u_i and v_i inductively by computing the remainder term ε using the induction hypothesis, extract its $\mathfrak{n}'_i$ component ε_i so that $n_i = v_i + u_i + \varepsilon_i$ (recall that $\mathfrak{n}_{i+1} = \mathfrak{n}'_{i+1} \oplus \cdots \oplus \mathfrak{n}'_{k-1}$). Since ε_i can be computed using the induction hypothesis, and n_i is given, we can compute $v_i + u_i \in \mathfrak{n}'_i = \mathfrak{u}_i \oplus \mathfrak{v}_i$, and then set u_i, v_i to be its $\mathfrak{u}_i$ and $\mathfrak{v}_i$ components respectively. Since ε is a polynomial, u_i and v_i are computed as polynomials of components of $n_0, n_1, \ldots, n_i$. Thus ψ has polynomial inverse.

□

The following proposition shows that for Lie subgroup U of a nilpotent Lie group N, there exists $\mathfrak{v}$ that satisfies the assumption of the Lemma.

Proposition 4.6.2. Let N be a nilpotent real Lie group, and $U \subset N$ be a Lie subgroup. Let $\mathfrak{u} = \mathrm{Lie}(U)$ and $\mathfrak{n} = \mathrm{Lie}(N)$.

Then there exists a complementary subspace $\mathfrak{v}$ of $\mathfrak{u}$ in $\mathfrak{n}$ such that for $V := \exp(\mathfrak{v})$, the map $V \times U \to N$ defined by $(v,u) \mapsto vu$ is bijective with polynomial inverse.

Proof. Let $\mathfrak{u}_i := \mathfrak{u} \cap \mathfrak{n}_i$. One can construct a flag of subspaces

$$\mathfrak{v}_0 \supset \mathfrak{v}_1 \supset \cdots \supset \mathfrak{v}_k = 0$$

such that $\mathfrak{n}_i = \mathfrak{u}_i \oplus \mathfrak{v}_i$ for all $0 \le i \le k-1$. This can be done inductively by, for instance, pick a finite set $\mathcal{B}_i$ of independent vectors in $\mathfrak{n}_i \setminus (\mathfrak{n}_{i+1} + \mathfrak{u}_i)$ such that $\mathfrak{n}_i = (\mathfrak{n}_{i+1} + \mathfrak{u}_i) \oplus \langle \mathcal{B}_i \rangle$, and then set $\mathfrak{v}_i := \mathfrak{v}_{i+1} + \langle \mathcal{B}_i \rangle$. Note that it follows from the defining property of $\mathfrak{v}_i$ that $\mathfrak{v}_i = \mathfrak{v}_0 \cap \mathfrak{n}_i$ for all i: on one hand we have the containment $\mathfrak{v}_0 \cap \mathfrak{n}_i \supset \mathfrak{v}_i$. On the other hand $(\mathfrak{v}_0 \cap \mathfrak{n}_i) \cap \mathfrak{u}_i \subset \mathfrak{v}_0 \cap \mathfrak{u}_0 = 0$, and thus $\dim(\mathfrak{v}_0 \cap \mathfrak{n}_i) \le \dim \mathfrak{n}_i - \dim \mathfrak{u}_i = \dim \mathfrak{v}_i$. Therefore the containment cannot be strict.

Thus we have obtained a complementary subspace $\mathfrak{v} := \mathfrak{v}_0$ such that $(\mathfrak{u} \cap \mathfrak{n}_i) \oplus (\mathfrak{v} \cap \mathfrak{n}_i) = \mathfrak{u}_i \oplus \mathfrak{v}_i = \mathfrak{n}_i$ for all i. Now apply Lemma 4.6.1 to get the result.

□

Stable normal form

Given two (real) vectors spaces V_1 and V_2, a *polynomial map* $p : V_1 \to V_2$ is an element in $\mathrm{Sym}^\bullet(V_1^\vee) \otimes V_2$. In more explicity terms, it is a map so that if we fix a basis on V_1 and on V_2, the coordinates of the map are polynomial in the coordinates on V_1 with respect to this basis on V_1. Clearly this notion does not depend on the choice of basis (since pre-/post-composing a polynomial with a linear map is still polynomial).

We say that a (real) vector space V is *filtered* if there is a filtration

$$0 =: V^{\le \lambda_{n+1}} \subsetneq V^{\le \lambda_n} \subsetneq V^{\le \lambda_{n-1}} \subsetneq \cdots \subsetneq V^{\le \lambda_1} = V,$$

and we say that $\mathbf{v} \in V$ has *weight* (or Lyapunov exponent) λ_i if $\mathbf{v} \in V^{\le \lambda_i} \setminus V^{\le \lambda_{i-1}}$, and write $\lambda_V(\mathbf{v}) := \lambda_i$. For our purpose, all the weights will be negative (as we will only apply such notion to the stable subspaces).

Given two filtered vector spaces V_1 and V_2, we say that a map $p : V_1 \to V_2$ is *subresonant* if $\lambda_{V_2}(p(\mathbf{v})) \le \lambda_{V_1}(\mathbf{v})$ for all $\mathbf{v} \in V_1$. In other words, the map p does not increase the weights of the vectors in V_1.

With these definitions, we can define the following notion of a *stable normal form*. We remark that this is a much simpler case than the general theory of normal form, which we refer the readers to, say, [KS17].

Proposition 4.6.3. For almost every $\hat{q} = (\omega, q) \in \mathcal{S}^{\mathbb{Z}} \times G/H$. there exists a subspace $V_{\hat{q}} \subset \mathfrak{g}$ and a diffeomorphism $\mathcal{N}_{\hat{q}} : V_{\hat{q}} \to \mathcal{W}^{-}_{G/H}[\hat{q}]$ such that:

(i) $V_{\hat{q}}$ can be made a filtered vector space where for all nonzero $\mathbf{v} \in V_{\hat{q}}$, $\lambda_{V_{\hat{q}}}(\mathbf{v})$ is the exponential growth rate of $d_{G/H}(T^n_\omega q, T^n_\omega \exp(\mathbf{v})q)$ as $n \to \infty$.

(ii) For all $\hat{q}$ where $V_{\hat{q}}$ is defined, for all t, the map

$$\mathcal{N}^{-1}_{\hat{T}^t\hat{q}} \circ \hat{T}^t \circ \mathcal{N}_{\hat{q}} : V_{\hat{q}} \to V_{\hat{T}^t\hat{q}}$$

is a subresonant polynomial map (with respect to the filtration defined in (i)).

(iii) For all $\hat{q}' \in \hat{\mathcal{W}}^{-}_{G/H}[\hat{q}] = \mathcal{W}^{-}_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{W}^{-}_{G/H}[\hat{q}]$, such that $V_{\hat{q}}$ and $V_{\hat{q}'}$ are defined, we have

$$\mathcal{N}^{-1}_{\hat{q}'} \circ \mathcal{N}_{\hat{q}} : V_{\hat{q}} \to V_{\hat{q}'}$$

is a subresonant polynomial map (with respect to the filtration defined in (i)).

Proof. Recall that for $\hat{q} := (\omega, qH) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, $\mathcal{W}^{-}_{G/H}[\hat{q}] = N^{-}(\omega)qH$, where $N^{-}(\omega) \subset G$ is the stable unipotent subgroup with respect to the word $\omega \in \mathcal{S}^{\mathbb{Z}}$ by Lemma 4.5.3.

Let $\mathfrak{n}^{-}(\omega) \subset \mathfrak{g}$ be the nilpotent subalgebra such that $\exp(\mathfrak{n}^{-}(\omega)) = N^{-}(\omega)$. Then $\mathfrak{n}^{-}(\omega) \cap q\mathfrak{h}q^{-1} \subset \mathfrak{g}$ is also a nilpotent subalgebra. Now apply Proposition 4.6.2 to $\mathfrak{n}^{-}(\omega) \cap q\mathfrak{h}q^{-1}$ and $\mathfrak{n}^{-}(\omega)$, we have a complementary subspace $V_{\hat{q}} \subset \mathfrak{g}$ such that the multiplication map $\exp(V_{\hat{q}}) \times (N^{-}(\omega) \cap qHq^{-1}) \to N^{-}(\omega)$ is bijective with polynomial inverse.

Now define $\mathcal{N}_{\hat{q}} : V_{\hat{q}} \to \mathcal{W}^{-}_{G/H}[\hat{q}] = N^{-}(\omega)qH$ by $v \mapsto \exp(v)qH$. $\mathcal{N}_{\hat{q}}$ is a diffeomorphism since $\exp(V_{\hat{q}}) \times (N^{-}(\omega) \cap qHq^{-1}) \to N^{-}(\omega)$ is bijective with polynomial inverse . Recall that $\hat{T}^t$ acts by left multiplication by some element $g \in G$. Then $\mathcal{N}^{-1}_{\hat{T}^t\hat{q}} \circ \hat{T}^t \circ \mathcal{N}_{\hat{q}}$ maps $v \in V_{\hat{q}}$ to the unique element w in $V_{\hat{T}^t\hat{q}}$ such that $\exp(w)gqH = g\exp(v)qH$, i.e. $\exp(w) \in \exp(gvg^{-1})(gq)H(gq)^{-1}$. Thus it suffices to find the unique elements $w \in V_{\hat{T}^t\hat{q}}$

and $u' \in (gq)H(gq)^{-1}$ such that $\exp(w)u' = \exp(gvg^{-1})$. From the choice of $V_{\hat{T}^t\hat{q}}$ using Proposition 4.6.2, we know that the map $\exp(gvg^{-1}) \mapsto \exp(w)$ is polynomial. Since gvg^{-1} and w are nilpotent, the exponential map and its inverse are polynomial. Thus $v \mapsto w$ is also polynomial, as desired. Subresonance follows from the fact that the dynamics preserve the stable manifolds with exponent at most $\leq \lambda$ for all $\lambda < 0$. This shows the first property.

For the second property, recall that if $\hat{q}' \in \hat{\mathcal{W}}^{-}_{G/H}[\hat{q}]$, there exists $n' \in N^{-}(\omega)$ such that $n'q'H = qH$. Now note that $\mathcal{N}_{\hat{q}'}^{-1} \circ \mathcal{N}_{\hat{q}}$ maps $v \in V_{\hat{q}}$ to the unique vector $w \in V_{\hat{q}'}$ such that $\exp(w)q'H = \exp(v)qH = \exp(v)n'q'H$. Thus it suffices to find the unique element $w \in V_{\hat{q}'}$ and $u' \in q'H(q')^{-1}$ such that $\exp(w)u' = \exp(v)n'$. But $v \mapsto w$ is polynomial by Proposition 4.6.2. Subresonance is clear. This shows the second property.

□

Remark 4.6.4. From now on we shall fix a choice of stable normal form. In particular some constants may implicitly depend on such a choice (in additional to dependence on μ, G/H and other parameters explicitly stated). Any other choice differ by a measurable family of subresonant automorphisms.

4.6.2 Factorization

Theorem 4.6.5. Fix a constant $\beta > 0$. There exist a linear cocycle $V = V(\beta)$ over $\mathcal{S}^{\mathbb{Z}} \times G/H$, a measurable family of smooth maps $F_{\hat{q}} : \hat{\mathcal{W}}^{-}_{G/H}[\hat{q}] \to V_{\hat{q}}$ for each $\hat{q} \in \mathcal{S}^{\mathbb{Z}} \times G/H$ and linear maps $\mathcal{A}(\hat{q}_1, u, \ell, t) : V_{\hat{T}^{-\ell}\hat{q}_1} \to (\mathfrak{l}/\mathfrak{h})_{\hat{T}^t u \hat{q}_1}$ for $\hat{q}_1 \in \mathcal{S}^{\mathbb{Z}} \times G/H$ and $u \in \mathcal{U}_1^{+}$, $\ell \geq 0$ and $t \geq 0$ such that

(a) ($F_{\hat{q}}$ **is centered**) $F_{\hat{q}}(\hat{q}) = 0$.

(b) (**Equivariance** of $F_{\hat{q}}$) $F_{\hat{q}}$ is equivariant:

$$F_{\hat{T}^t\hat{q}} \circ \hat{T}^t = T^t_{\hat{q}} \circ F_{\hat{q}}.$$

Here $T^t_{\hat{q}} : V_{\hat{q}} \to V_{\hat{T}^t\hat{q}}$ is the cocycle action on V.

(c) (**Equivariance** of $\mathcal{A}(\hat{q}_1, u, \ell, t)$) $\mathcal{A}(\hat{q}_1, u, \ell, t)$ are linear maps and satisfy: for all $\ell' > 0$ and $t' > 0$,

$$\mathcal{A}(\hat{q}_1, u, \ell + \ell', t + t') = T^{t'}_{\hat{T}^t u \hat{q}_1} \circ \mathcal{A}(\hat{q}_1, u, \ell, t) \circ T^{\ell'}_{\hat{T}^{-(\ell+\ell')}\hat{q}_1}.$$

Here the cocycle action $T^t_{\hat{q}} : (\mathfrak{l}/\mathfrak{h})_{\hat{q}} \to (\mathfrak{l}/\mathfrak{h})_{\hat{T}^t \hat{q}}$ on $(\mathfrak{l}/\mathfrak{h})_{\hat{q}}$ is given by conjugation by $\omega_{\lfloor t \rfloor - 1} \dots \omega_0 \in G$ for $\hat{q} = (\omega, q) \in \mathcal{S}^{\mathbb{Z}} \times G/H$ (i.e. the restriction of the derivative cocycle on the tangent bundle $\mathfrak{g}/\mathfrak{h}_{\hat{q}}$).

(d) (**Factorization**) For all $\delta > 0$, there exists a compact set $K = K(\delta) \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ of measure at least $1 - \delta$ and constants $C = C(K, \beta) > 0$ and $\alpha > 0$ depending only on the Lyapunov spectrum such that if $\hat{q}, \hat{q}', \hat{q}_1 := \hat{T}^\ell \hat{q}, \hat{q}'_1 := \hat{T}^\ell \hat{q}'$ are all in K, $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}]$, and $0 < t \leq \beta\ell$, let $\hat{q}_2 := \hat{T}^t u \hat{q}_1, \hat{q}'_2 := \hat{T}^t u \hat{q}'_1$, then

$$|d_{G/H}(\hat{q}_2, \hat{q}'_2) - \|\mathcal{A}(\hat{q}_1, u, \ell, t) F_{\hat{q}}(\hat{q}')\|| \leq C e^{-\alpha \ell} \tag{4.6.2}$$

Proof. The steps of the proof are as follows:

Step 0: Reduction to polynomial cocycle V and polynomial maps $\mathcal{A}(\hat{q}_1, u, \ell, t)$

We first show that it suffices to construct a subresonant polynomial cocycle V and polynomial maps $\mathcal{A}(\hat{q}_1, u, \ell, t)$ that are polynomial maps up to a fixed degree (the degree bound depends only on G/H).

Subresonant polynomial cocycle V: Suppose that we have a measurable family of smooth maps $F_{\hat{q}} : \hat{\mathcal{W}}^-_{G/H}[\hat{q}] \to V_{\hat{q}}$ and a cocycle V where the cocycle action $T^t_{\hat{q}} : V_{\hat{q}} \to V_{\hat{T}^t \hat{q}}$ is **subresonant polynomial**. Then we claim that we can find a **linear** cocycle V' over $\mathcal{S}^{\mathbb{Z}} \times G/H$ with an equivariant smooth embedding $V \hookrightarrow V'$.

The idea is to consider $PV_{\hat{q}} := \mathrm{Sym}^\bullet(V^\vee_{\hat{q}})^{\leq \lambda_1}$, the space of polynomial functions on $V_{\hat{q}}$ with weight at most λ_1, the top exponent of the dynamics on G/H, and then take

$V'_{\hat{q}} := (PV_{\hat{q}})^{\vee}$. The embedding $V \hookrightarrow V'$ is given by the evaluation map:

$$V_{\hat{q}} \xrightarrow{ev} PV_{\hat{q}}^{\vee} =: V'_{\hat{q}}$$
$$v \mapsto ev(v)(p) = p(v)$$

The cocycle action on $V'_{\hat{q}}$ is given by the following: given a subresonant polynomial map $f : V \to W$, it induces a map $f^* : PW \to PV$ by pullback. Since f is subresonant polynomial, the pullback of a polynomial in PW indeed does give an element in PV (with the same weight upper bound λ_1). This induces the dual map $(f^*)^{\vee} : V' \to W'$. By construction this is linear. It is immediate that this is compatible with the evaluation map $V_{\hat{q}} \to V'_{\hat{q}}$ and therefore gives a smooth embedding of cocycles $V \to V'$. Finally we post-compose the original maps $F_{\hat{q}}$ with this embedding to get the new maps $F'_{\hat{q}}$.

Polynomial maps $\mathcal{A}(\hat{q}_1, u, \ell, t)$: Suppose we already have a measurable family of smooth maps $F_{\hat{q}} : \hat{\mathcal{W}}^-_{G/H}[\hat{q}] \to V_{\hat{q}}$, a linear cocycle V over $\mathcal{S}^{\mathbb{Z}} \times G/H$, and **polynomial** maps $\mathcal{A}(\hat{q}_1, u, \ell, t)$ of degree at most d (that depends only on G/H) that satisfy the given properties. Then we claim that we can find a new linear cocycle V' and **linear** maps $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ that satisfy the given properties with an embedding of cocycles $V \to V'$.

In fact, any polynomial map $f : V \to W$ of degree at most d factors through the symmetric power:

$$V \xrightarrow{\mathrm{Sym}^{\leq d}} \mathrm{Sym}^{\leq d} V \xrightarrow{f'} W,$$

where $\mathrm{Sym}^{\leq d}$ can be thought of as the Veronese embedding (a polynomial map but not linear in general), and f' is a linear map.

Thus we can take $V'_{\hat{q}} := \mathrm{Sym}^{\leq d} V_{\hat{q}}$, and then take $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ to be the map $V'_{\hat{T}^{-\ell}\hat{q}_1} = \mathrm{Sym}^{\leq d} V_{\hat{T}^{-\ell}\hat{q}_1} \to (\mathfrak{l}/\mathfrak{h})_{\hat{T}^t(u\hat{q}_1)}$ factored from $\mathcal{A}(\hat{q}_1, u, \ell, t)$. Now any linear map $f : V \to W$ induces naturally a linear map $\mathrm{Sym}^{\leq d} f : \mathrm{Sym}^{\leq d} V \to \mathrm{Sym}^{\leq d} W$, therefore we have an embedding of linear cocycles $V \to V'$. The new maps $F'_{\hat{q}}$ is then given by post-composing the original maps $F_{\hat{q}}$ with $\mathrm{Sym}^{\leq d}$.

Centering of $F_{\hat{q}}$: Note that after the previous two procedures, $F_{\hat{q}}$ may not be centered (even if they were originally). Therefore we need to take $F'_{\hat{q}}(\hat{q}') := F_{\hat{q}}(\hat{q}') - F_{\hat{q}}(\hat{q})$.

Step 1: Construction of $F_{\hat{q}}$.

For each $\hat{q} = (\omega, q) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, we consider the *normal form* $\mathcal{N}_{\hat{q}} : V_{\hat{q}} \to \mathcal{W}^{-}_{G/H}[\hat{q}] = N^{-}(\omega)qH$ of its stable manifold as described in Proposition 4.6.3, and define $F_{\hat{q}} := \mathcal{N}_{\hat{q}}^{-1} \circ \pi_2 : \hat{\mathcal{W}}^{-}_{G/H}[\hat{q}] \to V_{\hat{q}}$, where $\pi_2 : \hat{\mathcal{W}}^{-}_{G/H}[\hat{q}] = \mathcal{W}^{-}_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{W}^{-}_{G/H}[\hat{q}] \to \mathcal{W}^{-}_{G/H}[\hat{q}]$ is the projection map onto the second factor.

The normal forms give a measurable vector bundle V over $\mathcal{S}^{\mathbb{Z}} \times G/H$ where each fiber is $V_{\hat{q}}$. The cocycle on V is given by

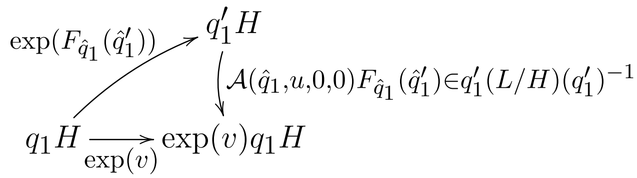
$$V_{\hat{q}} \xrightarrow{\mathcal{N}_{\hat{q}}} \mathcal{W}^{-}_{G/H}[\hat{q}] \xrightarrow{q'H \mapsto \omega_0 q'H} \mathcal{W}^{-}_{G/H}[\hat{T}\hat{q}] \xrightarrow{\mathcal{N}^{-1}_{\hat{T}\hat{q}}} V_{\hat{T}\hat{q}},$$

which is subresonant polynomial by a property of normal form coordinates. By the definition of the cocycle on V, $F_{\hat{q}} = \mathcal{N}_{\hat{q}}^{-1} \circ \pi_2$ is equivariant. Note that using this definition, $F_{\hat{q}}(\hat{q}) = 0$.

Step 2: Construction of the analytic map $\mathcal{A}'(\hat{q}_1, u, \ell, t)$

We first construct an analytic map $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ that satisfies all the properties of $\mathcal{A}(\hat{q}_1, u, \ell, t)$ except being only analytic (rather than polynomial). In future steps, we shall explain how to construct a polynomial map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ that approximates $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ with appropriate error and still satisfies the desired properties.

To construct the map $\mathcal{A}'(\hat{q}_1, u, \ell, t)$, the key is to first lift the stable manifold on G/L with respect to the new future in $u\hat{q}_1$ to an algebraic section of the bundle $G/H \to G/L$, and then use the projection along this stable section to the fiber $q_1'(L/H)(q_1')^{-1}$ through q_1'.



To describe the construction more precisely, we first describe the map $\mathcal{A}'(\hat{q}_1, u, 0, 0)$ (and then we extend this to other ℓ and t using the equivariance property).

Let $\mathcal{N}_{\hat{q},G/L} : V_{G/L,\hat{q}} \to \mathcal{W}^-_{G/L}[\hat{q}] = N^-(\omega)qL$ be a stable normal form coordinate on G/L as described in Proposition 4.6.3 (but for the space G/L rather than G/H). Then $V_{G/L,\hat{q}} \subset G$ gives a (algebraic) section $V_{G/L,\hat{q}}qH$ of the bundle $G/H \to G/L$ above $\mathcal{W}^-_{G/L}[\hat{q}]$. Now we consider the map

$$\pi_{\hat{q}_1,u\hat{q}_1} : V_{\hat{q}_1} \xrightarrow{\mathcal{N}_{\hat{q}_1}} N^-(\omega)q_1H \xrightarrow{q'H \mapsto q'L} N^-(\omega)q_1L = N^-(u\omega)q_1L \xrightarrow{\mathcal{N}^{-1}_{u\hat{q}_1,G/L}} V_{G/L,u\hat{q}_1}.$$

Here we use the key property about the base dynamics on G/L that two points that are stably related with respect to one future are also stably related with respect to almost every future by Proposition 4.4.7, so $N^-(\omega)q_1L = N^-(u\omega)q_1L$ holds almost surely (even though generically $N^-(\omega) \neq N^-(u\omega)$).

In other words, $\pi_{\hat{q}_1,u\hat{q}_1}$ maps $F_{\hat{q}_1}(\hat{q}'_1)$ to a vector v such that

(a) $v \in V_{G/L,u\hat{q}_1} \subset \log(N^-(u\omega)) \subset \mathfrak{g}$,

(b) $\exp(v)q_1H \in q'_1L/H$.

Note that this is a polynomial map . Now for $w \in V_{\hat{q}_1}$, we take

$$\mathcal{A}'(\hat{q}_1, u, 0, 0)w := \log(\exp(\pi_{\hat{q}_1,u\hat{q}_1}(w)) \exp(w)^{-1}) \in (\mathfrak{l}/\mathfrak{h})_{\hat{q}'_1}.$$

Note that the map before taking log is polynomial since w and $\pi_{\hat{q}_1,u\hat{q}_1}(w)$ are both nilpotent matrices in $\mathfrak{g}$. Finally, $\log : q'_1(L/H^\circ)(q'_1)^{-1} \to (\mathfrak{l}/\mathfrak{h})_{\hat{q}'_1} := q'_1(\mathfrak{l}/\mathfrak{h})(q'_1)^{-1}$ is well-defined and analytic since H° is normal in L (so L/H° is itself a Lie group with Lie algebra $\mathfrak{l}/\mathfrak{h}$).

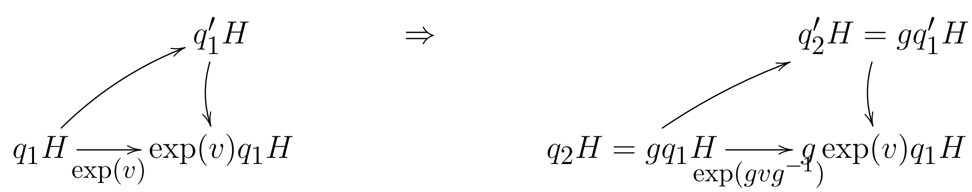
Both the domain and image of $\mathcal{A}'(\hat{q}_1, u, 0, 0) : V_{\hat{q}_1} \to (\mathfrak{l}/\mathfrak{h})_{u\hat{q}_1}$ have natural dynamics (V is a polynomial cocycle, $(\mathfrak{l}/\mathfrak{h})_{u\hat{q}_1}$ has dynamics given by conjugation), so we can extend the definition to other $\ell, t > 0$ by

$$\mathcal{A}'(\hat{q}_1, u, \ell, t) := T^t_{u\hat{q}_1} \circ \mathcal{A}'(\hat{q}_1, u, 0, 0) \circ T^{\ell}_{\hat{T}^{-\ell}\hat{q}_1}.$$

In particular the equivariance property of $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ is satisfied by construction.

Step 3: Factorization using $\mathcal{A}'(\hat{q}_1, u, \ell, t)$

Here we show that $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ satisfies the factorization property in the theorem. The point is that since the vector $v \in V_{G/L, u\hat{q}_1}$ is a nilpotent matrix in $\mathfrak{g}$ that is contracted by the future in $u\hat{q}_1$, the image of $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ gets exponentially close to the distance between $q_2 H = T^t_{u\hat{q}_1} q_1 H$ and $q_2' H = T^t_{u\hat{q}_1'} q_1' H$ (note that $u\hat{q}_1$ and $u\hat{q}_1'$ have the same future).



More precisely, let $v := \pi_{\hat{q}_1, u\hat{q}_1}(F_{\hat{q}_1}(\hat{q}_1')) \in V_{G/L, u\hat{q}_1}$. For all $\varepsilon > 0$, take a compact set $K' \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ of measure at least $1-\varepsilon$ and diameter at most some constant $C'(K') > 0$. Assume that $\hat{q} := \hat{T}^{-\ell}\hat{q}_1$ and $\hat{q}' := \hat{T}^{-\ell}\hat{q}_1'$ are both in K'. Then by Oseledets' theorem, there exists $\alpha' > 0$ depending only on the Lyapunov spectrum, and $C' = C'(K) > 0$ such that $d_{G/H}(q_1 H, q_1' H) \leq C' e^{-\alpha'\ell}$ (since $\hat{q}, \hat{q}'$ are stably related. Shrink K if necessary). Therefore $v = \pi_{\hat{q}_1, u\hat{q}_1}(F_{\hat{q}_1}(\hat{q}_1'))$ has norm at most $C' e^{-\alpha'\ell}$. Moreover, since v is contracting under the future of $u\hat{q}_1$, in particular we have $\|(T^t_{u\hat{q}_1})_* v\| \leq C''\|v\|$ for some $C''(K') > 0$ (shrink the compact set K' if necessary). Thus by the triangle inequality,

$$|d_{G/H}(q_2 H, q_2' H) - \|\mathcal{A}(\hat{q}_1, u, 0, t) F_{\hat{q}_1}(\hat{q}_1')\|| \leq \|(T^t_{u\hat{q}_1})_* v\| \leq C'' C' e^{-\alpha'\ell}.$$

Finally by the equivariance of $F_{\hat{q}}$ and the construction of $\mathcal{A}(\hat{q}_1, u, \ell, t)$, we know that for

$\hat{q} := \hat{T}^{-\ell}\hat{q}_1$ and $\hat{q}' := \hat{T}^{-\ell}\hat{q}_1'$, we have

$$\mathcal{A}(\hat{q}_1, u, \ell, t)F_{\hat{q}}(\hat{q}') = \mathcal{A}(\hat{q}_1, u, 0, t)F_{\hat{q}_1}(\hat{q}_1').$$

Combining this with the previous inequality yields the result. Note that so far we have not used the constant $\beta > 0$ at all (this will appear in later steps). In particular the factorization property holds for $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ for *all* $t \geq 0$ (without the upper bound $\beta\ell$).

Step 4: Construction of the polynomial map $\mathcal{A}(\hat{q}_1, u, \ell, t)$

Now we explain how to obtain a polynomial map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ from the analytic map $\mathcal{A}'(\hat{q}_1, u, \ell, t)$ constructed in the previous step. Note that as mentioned before the only intermediate map that is not polynomial in the construction of $\mathcal{A}'(\hat{q}_1, u, 0, 0)$ is the last map $\log : q_1'(L/H^\circ)(q_1')^{-1} \to (\mathfrak{l}/\mathfrak{h})_{\hat{q}_1'} := q_1'(\mathfrak{l}/\mathfrak{h})(q_1')^{-1}$. Nonetheless, we can approximate it using the Taylor expansion of log up to degree k by a polynomial map $\log|_k : q_1'(L/H^\circ)(q_1')^{-1} \to (\mathfrak{l}/\mathfrak{h})_{\hat{q}_1'} := q_1'(\mathfrak{l}/\mathfrak{h})(q_1')^{-1}$ such that

$$\|\log(v) - \log|_k(v)\| \leq O_k(\|v\|^{k+1}).$$

Define

$$\mathcal{A}(\hat{q}_1, u, 0, 0)w := \log|_k(\exp(w)\exp(\pi_{\hat{q}_1, u\hat{q}_1}(w))^{-1}) \in (\mathfrak{l}/\mathfrak{h})_{\hat{q}_1'}$$

for some $k = k(\beta)$ to be chosen in the last step.

Now again define $\mathcal{A}(\hat{q}_1, u, \ell, t) = T^t_{u\hat{q}_1} \circ \mathcal{A}(\hat{q}_1, u, 0, 0) \circ T^\ell_{\hat{T}^{-\ell}\hat{q}_1}$. We will show that $\mathcal{A}(\hat{q}_1, u, \ell, t)$ is still factorizable for large enough choices of $k = k(\beta)$.

Step 5: Factorization using $\mathcal{A}(\hat{q}_1, u, \ell, t)$

Here we show that $\mathcal{A}(\hat{q}_1, u, \ell, t)$ satisfies the factorization property in the theorem. Using **Step 3**, it suffices to show that

$$\|\mathcal{A}(\hat{q}_1, u, 0, t)F_{\hat{q}_1}(\hat{q}_1') - \mathcal{A}'(\hat{q}_1, u, 0, t)F_{\hat{q}_1}(\hat{q}_1')\| \leq O_{K,\beta}(e^{-\alpha''\ell})$$

for some constant $\alpha'' > 0$ depending only on the Lyapunov spectrum. In **Step 3** we have shown that we can take a large compact set $K \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ such that if $\hat{q} := \hat{T}^{-\ell}\hat{q}_1$ and $\hat{q}' := \hat{T}^{-\ell}\hat{q}_1'$ are in K and are stably related, then $\|F_{\hat{q}_1}(\hat{q}_1')\| \leq O_K(e^{-\alpha'\ell})$ for some $\alpha' > 0$ depending only on the Lyapunov spectrum. Thus we can use the error term in the approximation of log by $\log|_k$ to get

$$\|\mathcal{A}(\hat{q}_1, u, 0, 0)F_{\hat{q}_1}(\hat{q}_1') - \mathcal{A}'(\hat{q}_1, u, 0, 0)F_{\hat{q}_1}(\hat{q}_1')\| \leq O_{k,K}(e^{-\alpha'(k+1)\ell}).$$

Finally, let $\lambda_{\max} > 0$ be the top exponent of the dynamics on $(\mathfrak{l}/\mathfrak{h})_{\hat{q}_1}$. Then we have

$$\|\mathcal{A}(\hat{q}_1, u, 0, t)F_{\hat{q}_1}(\hat{q}_1') - \mathcal{A}'(\hat{q}_1, u, 0, t)F_{\hat{q}_1}(\hat{q}_1')\| \leq O_{k,K}(e^{\lambda_{\max}t-\alpha'(k+1)\ell}).$$

Finally, take $k = k(\beta)$ large enough such that $\lambda_{\max}\beta - \alpha'(k+1) < 0$. Since $t \leq \beta\ell$, the error term still decays exponentially with respect to ℓ (with rate at least $\alpha'' := -\lambda_{\max}\beta + \alpha'(k+1) > 0$) and we have the desired result.

□

We have the following contraction property of $F_{\hat{q}}$. We will use it later to prove Proposition 4.7.2 that certain equivariant subbundle in V is exponentially contracting.

Proposition 4.6.6 (Contraction of $F_{\hat{q}}$)**.** There exists a constant $\kappa' > 0$ depending only on the Lyapunov spectrum such that the following holds: for all $\delta > 0$, there exists a compact set $K = K(\delta) \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ of measure at least $1 - \delta$ and real number $C(\delta) > 0$ such that for all $\hat{q}, \hat{q}' \in K$ and $\ell > 0$ with $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}]$ and $\hat{T}^\ell\hat{q}, \hat{T}^\ell\hat{q}' \in K$, then

$$\|F_{\hat{T}^\ell\hat{q}}(\hat{T}^\ell\hat{q}') - F_{\hat{T}^\ell\hat{q}}(\hat{T}^\ell\hat{q})\| \leq C(\delta)e^{-\kappa'\ell}.$$

Proof. Given $\delta > 0$, take a compact set $K = K(\delta) \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ of measure at least $1 - \delta$ and a real number $C_1(\delta) > 0$ with the following properties:

1. K has diameter at most $C_1(\delta)$.

2. For all $\hat{q} \in K$, the Lipschitz constant $\|F_{\hat{q}}\| := \sup_{\hat{q}' \in B(\hat{q}, 1/100)} \frac{\|F_{\hat{q}}(\hat{q}') - F_{\hat{q}}(\hat{q})\|}{d(\hat{q}, \hat{q}')}$ is at most $C_1(\delta)$.

3. If $\hat{q}, \hat{q}', \hat{T}^\ell \hat{q}, \hat{T}^\ell \hat{q}' \in K$ and $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}]$ for some $\ell > 0$, we have $d(\hat{T}^\ell \hat{q}, \hat{T}^\ell \hat{q}') \leq C_1(\delta) e^{-\kappa' \ell} d(\hat{q}, \hat{q}')$, where κ' is 1/3 the largest negative Lyapunov exponent of the tangent bundle on $\mathcal{S}^{\mathbb{Z}} \times G/H$.

Here the first property holds since the entire space $\mathcal{S}^{\mathbb{Z}} \times G/H$ can be exhausted by compact sets of increasing finite diameter. The second property can be obtained since $F_{\hat{q}}$ is smooth. The third property can be obtained by Oseledets theorem.

With these three properties, the result follows:

$$\|F_{\hat{T}^\ell \hat{q}}(\hat{T}^\ell \hat{q}') - F_{\hat{T}^\ell \hat{q}}(\hat{T}^\ell \hat{q})\| \underset{\text{by } 2}{\leq} C_1(\delta) d(\hat{T}^\ell \hat{q}', \hat{T}^\ell \hat{q}) \underset{\text{by } 3}{\leq} C_1(\delta)^2 e^{-\kappa' \ell} d(\hat{q}, \hat{q}') \underset{\text{by } 1}{\leq} C_1(\delta)^3 e^{-\kappa' \ell}.$$

□

4.7 Preliminary divergence estimate

Throughout this section, we assume uniform expansion on L/H.

Define

- For $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H^\circ$, let $\hat{\mathcal{W}}^-_{\text{loc}}[\hat{z}] = \{(\omega', z') \in \hat{\mathcal{W}}^-_{G/H}[\hat{z}] \mid d_{G/H}(z, z') < 1\}$.

- Define $\mathcal{L}^-(\hat{z}) \subset V_{\hat{z}}$ as the smallest subspace of $V_{\hat{z}}$ such that the pushforward of $\hat{\nu}|_{\hat{\mathcal{W}}^-_{\text{loc}}[\hat{z}]}$ via the map $F_{\hat{z}} \circ \pi_2 : \hat{\mathcal{W}}^-_{G/H}[\hat{z}] \to V_{\hat{z}}$ is supported on $\mathcal{L}^-(\hat{z})$. Here $\pi_2 : \hat{\mathcal{W}}^-_{G/H}[\hat{z}] = \mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{W}^-_{G/H}[\hat{z}] \to \mathcal{W}^-_{G/H}[\hat{z}]$ is the projection map onto the second factor.

- In Case I, $\mathcal{L}^-(\hat{z})$ is not contained in $F_{\hat{z}}(\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times (\mathcal{F}^{\leq 0}_{G/H}[z] \cap \mathcal{W}^-_{G/H}[\hat{z}]))$ since $F_{\hat{z}}$ is injective. Here, we recall that $\mathcal{F}^{\leq 0}_{G/H}[z]$ is the set of elements $z' \in G/H$ that is in the center-stable manifold of z for almost every future word ω^+.

- In particular, since $z \in \mathcal{F}^{\leq 0}_{G/H}[z] \cap \mathcal{W}^-_{G/H}[\hat{z}]$, $\dim \mathcal{L}^-(\hat{z}) > 0$ almost surely in Case I.

From now on, we restrict the domain of the map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ defined in subection 4.6.2 to $\mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$, so it is a linear map $\mathcal{A}(\hat{q}_1, u, \ell, t) : \mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1) \to (\mathfrak{l}/\mathfrak{h})_{\hat{T}^t u \hat{q}_1}$.

We first show that the family of subspaces $\{\mathcal{L}^-(\hat{q})\}_{\hat{q}}$ form an equivariant subbundle of V.

Lemma 4.7.1. (cf. [ELa, Lem. 7.1]) For almost every $\hat{q} \in \mathcal{S}^{\mathbb{Z}} \times G/H$, and all $t \in \mathbb{R}$,

$$\mathcal{L}^-(\hat{T}^t\hat{q}) = T^t_{\hat{q}}\mathcal{L}^-(\hat{q}).$$

Proof. Note that by definition and equivariance of $F_{\hat{q}}$, for $t > 0$, we have $T^t_{\hat{q}}\mathcal{L}^-(\hat{q}) \subset \mathcal{L}^-(T^t\hat{q})$. Therefore $\hat{q} \mapsto \dim \mathcal{L}^-(\hat{q})$ is a bounded integer-valued function on $\mathcal{S}^{\mathbb{Z}} \times G/H$ that is non-decreasing under $\hat{T}^t$. By ergodicity, this function is constant almost surely. Therefore the statement holds. □

We then show that the cocycle restricted to $\mathcal{L}^-(\hat{q})$ is exponentially contracting. This follows from Proposition 4.6.6.

Proposition 4.7.2. There exists a constant $\kappa' > 0$ depending only on the Lyapunov spectrum of the dynamics on $\mathcal{S}^{\mathbb{Z}} \times G/H$ such that: the top Lyapunov exponent on the linear cocycle V restricted to $\mathcal{L}^-(\hat{q})$ is at most $-\kappa'$.

Proof. Recall that since $F_{\hat{q}}$ is centered, $F_{\hat{T}^\ell\hat{q}}(\hat{T}^\ell\hat{q}) = 0$. Since $F_{\hat{q}}$ is equivariant, $F_{\hat{T}^\ell\hat{q}}(\hat{T}^\ell\hat{q}') = \hat{T}^\ell_{\hat{q}}F_{\hat{q}}(\hat{q}')$. Therefore the inequality in Proposition 4.6.6 simplifies to $\|\hat{T}^\ell_{\hat{q}}F_{\hat{q}}(\hat{q}')\| \leq C(\delta)e^{-\kappa'\ell}$.

Now note that for large enough ℓ_0, at least $(1-2\delta)$ portion of ℓ in $[0, \ell_0]$ satisfies $\hat{T}^\ell\hat{q} \in K$ by the pointwise ergodic theorem. Therefore along a subsequence of ℓ in $\mathbb{R}^+$, $\frac{1}{\ell}\log\|\hat{T}^\ell_{\hat{q}}F_{\hat{q}}(\hat{q}')\|$ tends to a limit at most $-\kappa'$. Now by Oseledets theorem, the limit $\lim_{\ell\to\infty}\frac{1}{\ell}\log\|\hat{T}^\ell_{\hat{q}}F_{\hat{q}}(\hat{q}')\|$ exists for almost every $\hat{q}$, and thus this limit agrees with the limit along the previous subsequence, and hence is at most $-\kappa'$. Thus for all $\delta > 0$, there exists a compact set $K_\delta \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ with measure at least $1-\delta$ such that if $\hat{q}, \hat{q}' \in K_\delta$ and $\hat{q}' \in \mathcal{W}^s[\hat{q}]$, then

$$\lim_{\ell\to\infty}\frac{1}{\ell}\log\|\hat{T}^\ell_{\hat{q}}F_{\hat{q}}(\hat{q}')\| \leq -\kappa'.$$

Finally, if we take $\delta \to 0$, we obtain this inequality for almost every $\hat{q}$ and $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}]$. Since $\mathcal{L}^-(\hat{q})$ is the span of $F_{\hat{q}}(\text{supp } \hat{\nu}|_{\hat{\mathcal{W}}^-_{\text{loc}}[\hat{q}]})$, we conclude that the restriction of the cocycle to $\mathcal{L}^-(\hat{q})$ has top exponent at most $-\kappa'$. □

Proposition 4.7.3. cf. [ELa, Prop. 5.1] Recall the constants $\kappa, \kappa_V > 0$ from Proposition 4.5.13 and Remark 4.5.14 which depend only on the Lyapunov spectrum. For every $\delta > 0$, there exists a subset $K \subset \Omega$ with measure at least $1 - \delta$ such that for all $\hat{q}_1 \in K$, there exists $Q = Q(\hat{q}_1) \subset \mathcal{U}_1^+$ such that $Q\hat{q}_1$ has measure at least $1 - \delta$, and for $u \in Q$, $\ell > 0$ and $t > 0$, we have

$$\|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, t)\| \geq C(\delta) e^{-\kappa_V \ell + \kappa^{-1} t}.$$

Here the operator norm is with respect to the dynamical norms $\|\cdot\|_{\hat{T}^{-\ell}\hat{q}_1}$ on the domain and $\|\cdot\|_{\hat{T}^t u \hat{q}_1}$ on the target. $\pi_+ : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ is the orthogonal projection with respect to the dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$ defined in subection 4.5.9. In particular, since $(\mathfrak{l}/\mathfrak{h})_{\hat{x}} = W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x}) \oplus W^{\leq 0}_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ is an orthogonal decomposition, we also have

$$\|\mathcal{A}(\hat{q}_1, u, \ell, t)\| \geq C(\delta) e^{-\kappa_V \ell + \kappa^{-1} t}.$$

Remark. This statement is considerably different from the first part of [ELa, Prop. 5.1] since the analogous statement of [ELa, Prop. 5.1] does not hold if $\mathbf{v}$ is in the image of $\mathcal{F}^{\leq 0}_{G/H}[\hat{q}]$ via $F_{\hat{q}}$. In particular we only have a lower bound on the norm rather than the conorm of $\mathcal{A}(\hat{q}_1, u, \ell, t)$. We will see that this does not affect the main argument since we can apply Lemma 4.7.5 and Proposition 4.7.6, so that we can focus on points $\hat{q}'$ such that $F_{\hat{q}}(\hat{q}')$ grow roughly at the rate of $\|\mathcal{A}(\hat{q}_1, u, \ell, t)\|$ under $\mathcal{A}(\hat{q}_1, u, \ell, t)$.

Proof. Let $q_1' \in \text{supp } \hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[\hat{q}_1]}$ that is not in $\mathcal{F}^{\leq 0}_{G/H}[q_1]$ (which exists for almost every $\hat{q}_1$ in Case I), and let $\mathbf{v} := F_{\hat{q}_1}(\hat{q}_1')$. Since $q_1' \notin \mathcal{F}^{\leq 0}_{G/H}[q_1]$, for almost every u ,

$$\|\pi_+(\mathcal{A}(\hat{q}_1, u, 0, 0)\mathbf{v})\| > 0.$$

In particular, we have the operator norm $\|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, 0, 0)\| > 0$ for almost every $\hat{q}_1$ and almost every u. Thus for all $\delta > 0$, there exists a subset $K \subset \Omega$ with measure at least $1-\delta$ such that for all $\hat{q}_1 \in K$, there exists $Q = Q(\hat{q}_1) \subset \mathcal{U}_1^+$ with measure at least $1-\delta$ such that for $u \in Q$, $\|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, 0, 0)\| \geq C_1(\delta)$.

By Proposition 4.6.6 and the fact that the dynamics on V is bijective, this implies $\|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, 0)\| \geq C_2(\delta)e^{-\kappa'\ell}$. Finally Proposition 4.5.13(d) implies that for some $\mathbf{v} \in \mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$,

$$\|T^t_{u\hat{q}_1} \circ \pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, 0)\mathbf{v}\| \geq C_2(\delta)e^{-\kappa'\ell+\kappa t}.$$

Finally, since the dynamics $T^t_{u\hat{q}_1}$ restricts to $W^+_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1) \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{T}^t u\hat{q}_1)$ and $W^{\leq 0}_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1) \to W^{\leq 0}_{\mathfrak{l}/\mathfrak{h}}(\hat{T}^t u\hat{q}_1)$, $T^t_{u\hat{q}_1} \circ \pi_+ = \pi_+ \circ T^t_{u\hat{q}_1}$. Since $\mathcal{A}(\hat{q}_1, u, \ell, t) = T^t_{u\hat{q}_1} \circ \mathcal{A}(\hat{q}_1, u, \ell, 0)$, we have

$$\|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{v}\| \geq C_2(\delta)e^{-\kappa'\ell+\kappa t}.$$

□

Lemma 4.7.4. [ELa, Lem. 5.6] For all $\delta > 0$, there exists a compact set $K \subset \Omega$ with measure at least $1-\delta$ such that: for $t > 0$, let $\hat{x} \in K$, $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}] \cap K$, $\hat{T}^t\hat{x} \in K$ and $\hat{T}^t\hat{x}' \in \hat{T}^{[-a,a]}K$ for some $a \geq 0$. Then

$$|\lambda_{ij}(\hat{x}, t) - \lambda_{ij}(\hat{x}', t)| \leq C = C(a, \delta).$$

Proof. The proof is the same as that of [ELa, Lem. 5.6], using Lemma 4.5.10, Lemma 4.5.15, and Proposition 4.5.13 (b). □

We will need the following elementary linear algebra fact (see e.g. [ELa, Lem. 8.1]).

Lemma 4.7.5. For any $\rho > 0$, there exists a constant $c(\rho) > 0$ with the following property: let $A : W_1 \to W_2$ be a linear map between Euclidean spaces. Then there exists a proper subspace $W' \subset W_1$ such that for any $\mathbf{v}$ with $\|\mathbf{v}\| = 1$ and $d(\mathbf{v}, W') > \rho$, we have

$$\|A\| \geq \|A\mathbf{v}\| \geq c(\rho)\|A\|.$$

We also need a general lemma on conditional measures.

Proposition 4.7.6. cf. [ELa, Prop. 8.2] Let $\mathcal{B}$ be an arbitrary finite measure space.

For every $\delta > 0$, there exists constants $c_1(\delta) > 0$, $\varepsilon_1(\delta) > 0$ with $c_1(\delta) \to 0$ and $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$, and also constants $0 < \rho(\delta) < C(\delta) < \infty$ and $\rho'(\delta) > 0$ such that:

For any subset $K' \subset \hat{\Omega}$ with $\hat{\nu}(K') > 1 - \delta$, there exists a subset $K \subset K'$ with $\hat{\nu}(K) > 1 - c_1(\delta)$ such that:

suppose for each $\hat{q} \in \hat{\Omega}$, there is a measurable map from $\mathcal{B}$ to proper subspaces of $\mathcal{L}^-(\hat{q})$, written as $u \mapsto \mathcal{M}_u(\hat{q})$. Then for any $\hat{q} \in K$ there exists $\hat{q}' \in K'$ with $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}], F_{\hat{q}}(\hat{q}') \in \mathcal{L}^-(\hat{q})$ such that

$$\rho(\delta) \leq d_{G/H}(q, q') \leq 1/100, \qquad \rho(\delta) \leq \|F_{\hat{q}}(\hat{q}')\|_{\hat{q}} \leq C(\delta),$$

and

$$d_{\hat{q}}(F_{\hat{q}}(\hat{q}'), \mathcal{M}_u(\hat{q})) > \rho'(\delta) \qquad \text{for at least } (1 - \varepsilon_1(\delta))\text{-fraction of } u \in \mathcal{B}.$$

Proof. The proof can be adapted from the proof of [EM18, Prop. 5.3] (see also [ELa, Prop. 8.2]). In the proof, we define the measure $\tilde{\nu}_{\hat{x}} := (F_{\hat{x}})_*(\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[\hat{x}]})$. By definition of $\mathcal{L}^-(\hat{x})$, $\tilde{\nu}_{\hat{x}}$ restricted to a sufficiently small ball (say of diameter 1/10) is supported on $\mathcal{L}^-(\hat{x})$.

The rest of the proof follows from adapting the analogous statements of [EM18, Lem. 5.4, 5.5, 5.6], where we replace $\mathcal{L}_{ext}[x]^{(r)}$ by $\mathcal{L}^-(\hat{x})$ and $F(x)$ by $F_{\hat{x}}(\hat{x}) = 0$, and then follow the rest of the proof of [EM18, Prop. 5.3]. □

4.8 Inert Subbundle $\mathbf{E}(\hat{x})$

In this section, we define an $\hat{T}^t$ and $\mathcal{U}_1^+$-equivariant subbundle of $(\mathfrak{l}/\mathfrak{h})_{\hat{x}}$. In the end we will show that all the extra invariance is obtained within this bundle.

4.8.1 Inert subspaces $\mathbf{E}_j(\hat{x})$

For $\hat{x} = (\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$, for a real number λ, define

$$\mathbf{F}^{\leq\lambda}(\hat{x}) := \{\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_{\hat{x}} : \quad \text{for a.e. } u\hat{x} \in \mathcal{U}_1^+\hat{x},\ \mathbf{v} \in W^{\leq\lambda}(u\hat{x})\}.$$

Similarly define $\mathbf{F}^{<\lambda}$. In particular,

$$\mathbf{F}^{\leq 0}(\hat{x}) := \{\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_{\hat{x}} : \quad \text{for a.e. } u\hat{x} \in \mathcal{U}_1^+\hat{x},\ \mathbf{v} \in W^{\leq 0}(u\hat{x})\},$$

so for almost every future, vectors in $\mathbf{F}^{\leq 0}(u\hat{x})$ does not grow exponentially. Note that $\mathbf{F}^{\leq\lambda}(\hat{x})$ depends only on the point $x \in G/L$ and not the word ω, hence we will sometimes write $\mathbf{F}^{\leq\lambda}(x) := \mathbf{F}^{\leq\lambda}(\hat{x})$.

By definition, for $x \in G/L$, $\mathbf{F}^{\leq 0}(x) = 0$ if and only if μ is uniformly expanding on L/H at x_0. Then we have a flag

$$\{0\} \subset \mathbf{F}^{\leq\lambda_n}(\hat{x}) \subset \cdots \subset \mathbf{F}^{\leq\lambda_2}(\hat{x}) \subset \mathbf{F}^{\leq\lambda_1}(\hat{x}) = (\mathfrak{l}/\mathfrak{h})_{\hat{x}}.$$

Define

$$\mathbf{E}_j(\hat{x}) = \mathbf{F}^{\leq\lambda_j}(\hat{x}) \cap W^{\geq\lambda_j}(\hat{x}).$$

Define

$$\Lambda_{\mathbf{E}} = \{i : \mathbf{E}_i(\hat{x}) \neq \{0\} \text{ for a.e. } \hat{x}\}.$$

We quote the following basic properties from [ELa].

Lemma 4.8.1. [ELa, Lem. 3.1] For almost every $\hat{x} \in \Omega$, for all $\mathbf{v} \in \mathbf{E}_j(\hat{x}) \setminus \{0\}$ and almost every $u\hat{x} \in \mathcal{U}_1^+\hat{x}$, we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \frac{\|(T^t_{u\hat{x}})_*\mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_j.$$

In particular $\mathbf{E}_j(\hat{x}) \subset W^{\lambda_j}(\hat{x})$, hence $\mathbf{E}_j(\hat{x}) \cap \mathbf{E}_k(\hat{x}) = \{0\}$ for $j \neq k$.

Lemma 4.8.2. [ELa, Lem 3.2, 3.4] For almost every $\hat{x} \in \Omega$,

(a) For any $t \in \mathbb{R}$, $(T^t_{\hat{x}})_* \mathbf{E}_j(\hat{x}) = \mathbf{E}_j(T^t\hat{x})$, and $(T^t_{\hat{x}})_* \mathbf{F}^{\leq \lambda_j}(\hat{x}) = \mathbf{F}^{\leq \lambda_j}(T^t\hat{x})$.

(b) For almost every $u\hat{x} \in \mathcal{U}_1^+\hat{x}$, $\mathbf{E}_j(u\hat{x}) = \mathbf{E}_j(\hat{x})$, and $\mathbf{F}^{\leq \lambda_j}(u\hat{x}) = \mathbf{F}^{\leq \lambda_j}(\hat{x})$.

(c) For $\hat{x} = (\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$, $\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_{\hat{x}}$, let $Q_j(\mathbf{v}) = \{u\hat{x} \in \mathcal{U}_1^+\hat{x} : \mathbf{v} \in W^{\leq \lambda_j}(u\hat{x})\} \subset W^+_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \{x\}$. Then for each j, for a.e. $\hat{x}$, $Q_j(\mathbf{v})$ is either null or conull in $W^+_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \{x\}$ (in the latter case, $\mathbf{v} \in \mathbf{F}^{\leq \lambda_j}(\hat{x})$).

4.8.2 Consequence of uniform expansion on L/H

Proposition 4.8.3. Let $x_0 \in G/L$. Suppose that

(i) μ is uniformly expanding on L/H at x_0, and

(ii) for all $x \in \overline{\Gamma}^Z_\mu x_0 \subset G/L$, for $\mu^{\mathbb{N}}$-a.e $\omega^+ \in G^{\mathbb{N}}$, there exists a compact subset $K = K(\omega^+, x') \subset G/L$ such that for all $n \in \mathbb{N}$, $T^n_\omega x' \in K(\omega^+, x')$.

Then for all $x \in \overline{\Gamma}^Z_\mu x_0 \subset G/L$, μ is uniformly expanding on L/H at x, i.e. $\mathbf{F}^{\leq 0}(x) = 0$.

Proof. Let

$$\mathrm{NUE} := \{x \in G/L \mid \mathbf{F}^{\leq 0}(x) \neq 0\}.$$

Then it suffices to show that NUE is $\overline{\Gamma}^Z_\mu$-invariant. By Lemma 4.8.2(a), $\mathbf{F}^{\leq 0}$ is an equivariant bundle over G/H (and thus over G/L), in particular, $g_*\mathbf{F}^{\leq 0}(x) = \mathbf{F}^{\leq 0}(gx)$ for μ-a.e. $g \in G$. Therefore NUE is Γ_μ-invariant.

To show that NUE is $\overline{\Gamma}^Z_\mu$-invariant, it suffices to show that NUE is an algebraic subset of G/L. Here we use the norm $\|\mathbf{v}\|_{(\mathfrak{l}/\mathfrak{h})_{\hat{x}}} := \|\mathbf{v} \wedge \rho_{\mathfrak{h}_x}\|_{\mathfrak{g}}$, where $\rho_{\mathfrak{h}_x}$ is a nonzero element in the one-dimensional subspace $\bigwedge^{\dim H} \mathfrak{h}_x \subset \bigwedge^{\dim H} \mathfrak{g}$.

Using this norm, we have that $\mathbf{v} \in \mathbf{F}^{\leq 0}(x)$ if and only if for $\mu^{\mathbb{N}}$-a.e. $\omega^+ \in \mathcal{S}^{\mathbb{N}}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(\mathrm{Ad}(T^n_\omega)\mathbf{v}) \wedge \rho_{\mathfrak{h}_{T^n_\omega x}}\| \leq 0.$$

Since μ is bounded on G/L at x_0, it is also bounded on G/L at x for all $x \in \overline{\Gamma}_\mu^Z x_0 \subset G/L$. Since $\ell_* \rho_{\mathfrak{h}} = \rho_{\mathfrak{h}}$ for all $\ell \in L$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n x)\rho_{\mathfrak{h}}\| = 0.$$

Since $\mathrm{Ad}(x)\rho_{\mathfrak{h}}$ and $\rho_{\mathfrak{h}_x}$ are both nonzero vectors in the one-dimensional space $\bigwedge^{\dim H} \mathfrak{h}_x$, we also have

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n)\rho_{\mathfrak{h}_x}\| = 0.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log \|(\mathrm{Ad}(T_\omega^n)\mathbf{v}) \wedge \rho_{\mathfrak{h}_{T_\omega^n x}}\| = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n)(\mathbf{v} \wedge \rho_{\mathfrak{h}_x})\|.$$

Thus for $\mathbf{v} \in \mathfrak{l}_x$, $\mathbf{v} \in \mathbf{F}^{\leq 0}(x)$ if and only if for $\mu^{\mathbb{N}}$-a.e. $\omega^+ \in \mathcal{S}^{\mathbb{N}}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n)(\mathbf{v} \wedge \rho_{\mathfrak{h}_x})\| \leq 0.$$

Now consider the action of μ on the wedge power $\bigwedge^{\dim H+1} \mathfrak{g}$, and consider the corresponding subspace

$$\mathbf{F}^{\leq 0}_{\bigwedge \mathfrak{g}} := \left\{ \mathbf{w} \in \bigwedge^{\dim H+1} \mathfrak{g} \mid \limsup_{n\to\infty} \frac{1}{n} \log \|\mathrm{Ad}(T_\omega^n)\mathbf{w}\| \leq 0 \right\}.$$

Note that this space depends only on μ (and not on any word or basepoint).

Then $x \in \mathrm{NUE}$ if and only if $x(\mathfrak{l} \wedge \rho_{\mathfrak{h}})x^{-1} \cap \mathbf{F}^{\leq 0}_{\bigwedge \mathfrak{g}} \neq 0$. Now note that $\mathbf{F}^{\leq 0}_{\bigwedge \mathfrak{g}}$ is Γ_μ-invariant, therefore it is also $\overline{\Gamma}_\mu^Z$-invariant. Thus NUE is also $\overline{\Gamma}_\mu^Z$-invariant. Therefore so is its complement, as desired. □

Lemma 4.8.4. Suppose μ is uniformly expanding on L/H at $x \in G/L$ and there is a μ-stationary measure $\overline{\nu}$ on $\overline{\Gamma}_\mu^Z x_0 L/L$. Then for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, $\mathbf{F}^{\leq 0}(x') = \{0\}$. In particular, for a.e. $\hat{x} \in \Omega$, $\mathbf{F}^{\leq 0}(\hat{x}) = \{0\}$.

Proof. It suffices to verify the second assumption of Proposition 4.8.3, assuming the existence of a stationary measure on $\overline{\Gamma}_\mu^Z xH/H$.

Since there is a stationary measure $\overline{\nu}$ on $\overline{\Gamma}_\mu^Z x_0 L/L$, it projects to a stationary measure $\rho_*\overline{\nu}$ on V via the G-equivariant smooth injection $\rho : G/L \to V$ in the definition of L being an H-envelope. Note that this is isometric since the metric on G/L is induced from that of V.

Now by Theorem 4.3.1(iii), the support of $\rho_*\overline{\nu}$ is in the subspace W. Since W is Γ_μ-invariant, it is also $\overline{\Gamma}_\mu^Z$-invariant. In particular, if $\rho(x) \notin W$, then by G-equivariance of ρ, $\overline{\Gamma}_\mu^Z \rho(x) = \rho(\overline{\Gamma}_\mu^Z x)$ has empty intersection with W, and hence $\rho_*\overline{\nu}$ cannot be supported on W, a contradiction. Hence $\rho(x) \in W$ and $\overline{\Gamma}_\mu^Z \rho(x) = \rho(\overline{\Gamma}_\mu^Z x) \subset W$. Theorem 4.3.1 tells us that μ either acts compactly on W or there exists a proper subspace $W' \subsetneq W$ such that μ acts compactly on W/W' and has negative top exponent on W', in both cases we have that for $\mu^{\mathbb{N}}$-a.e. ω, the orbit $\{T_\omega^n \rho(x)\}_{n\in\mathbb{N}}$ is inside a compact subset $K_W = K_W(\omega, x) \subset W \smallsetminus \{0\}$, therefore the orbit $\{T_\omega^n x\}_{n\in\mathbb{N}}$ is in a compact subset $K = K(\omega, x) \subset G/L$, thus assumption (ii) in Proposition 4.8.3 is satisfied. Therefore by Proposition 4.8.3, for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, $\mathbf{F}^{\leq 0}(x') = \{0\}$. The last assertion is immediate from this. $\square$

Corollary 4.8.5. If μ is uniformly expanding on L/H at $x \in G/L$, and there is a μ-stationary measure ν on $\overline{\Gamma}_\mu^Z xH/H$, then for all $z \in \overline{\Gamma}_\mu^Z xH^\circ/H^\circ$, for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, the intersection of $\mathcal{F}_{G/H}^{\leq 0}[z]$ and $x'L/H^\circ$ contains at most one point; the intersection of $\mathcal{F}_{G/H}^{\leq 0}[z]H$ and $x'L/H$ contains at most one point.

Proof. If there exists distinct $z', z'' \in \mathcal{F}_{G/H}^{\leq 0}[z] \cap x'L/H^\circ$, then we also have $z'' \in \mathcal{F}_{G/H}^{\leq 0}[z']$. Moreover, since z', z'' are in the same coset of L/H°, there exists a nonzero vector $\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_{x'}$ such that $z'' = \exp(\mathbf{v})z'$. Since $z'' \in \mathcal{F}_{G/H}^{\leq 0}[z']$, $\mathbf{v} \in \mathbf{F}^{\leq 0}(x')$. Since ν projects to a μ-stationary measure $\overline{\nu}$ on $\overline{\Gamma}_\mu^Z x_0 L/L$, this contradicts Lemma 4.8.4. The same argument shows the last statement as well. $\square$

4.8.3 Inert subbundle $\mathbf{E}(\hat{x})$

Define the **inert subbundle E** by

$$\mathbf{E}(\hat{x}) := \bigoplus_{i \in \Lambda_{\mathbf{E}}^+} \mathbf{E}_i(\hat{x}) \subset W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x}), \quad \text{where } \Lambda_{\mathbf{E}}^+ := \{i : \mathbf{E}_i(\hat{x}) \neq \{0\} \text{ for a.e. } \hat{x} \quad \text{and} \quad \lambda_i > 0\}.$$

Notice $\mathbf{E}_1(\hat{x}) = W^{\lambda_1}(\hat{x}) = W^{\geq \lambda_1}(\hat{x}) \neq \{0\}$. We may have $\mathbf{E}_j(\hat{x}) = \{0\}$ if $j \neq 1$.

The following lemma follows immediately from Lemma 4.5.9.

Lemma 4.8.6. [ELa, Lem. 5.1] For almost every $\hat{x} \in \Omega$ and almost every $u\hat{x} \in \mathcal{U}_1^+ \hat{x}$, we have

$$P^+(\hat{x}, u\hat{x})\mathbf{E}(\hat{x}) = \mathbf{E}(u\hat{x}).$$

4.8.4 Convergence to the inert bundle $\mathbf{E}(\hat{x})$

The next proposition shows that for most $\hat{q}_2 \in \Omega$, and every $\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_{\hat{q}_2}$, we can take $\mathbf{v}$ exponentially close to the inert bundle by going backwards sufficiently far, change to one of most futures, and then go forward until the dynamical norm agrees with $\mathbf{v}$.

Proposition 4.8.7. For every $\delta > 0$, there exists a subset $K = K(\delta) \subset \Omega$ with measure at least $1 - \delta$ such that for all $\hat{q}_1 \in K$, $\ell > 0$ and $\mathbf{v} \in \mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$, there exists $Q = Q(\hat{q}_1, \mathbf{v}) \subset \mathcal{U}_1^+$ such that $Q\hat{q}_1$ has measure at least $1 - \delta$ and for $u \in Q$, $t > 0$,

$$d\left(\frac{\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{v}}{\|\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{v}\|}, \mathbf{E}(\hat{T}^t u\hat{q}_1)\right) \leq C(\delta)e^{-\alpha t},$$

where α depends only on the Lyapunov spectrum.

The proof closely follows the second part of [ELa, Prop. 4.1] and [EM18, Prop. 8.5a]. Since the argument is not too long and this result is important to the main argument, we include the proof here for completeness. In particular the proof clearly demonstrates how uniform expansion on L/H (in the stronger form of Lemma 4.8.4) is used in the argument.

We first quote a lemma from [ELa] which follows from the definitions of the equivariant bundles $\mathbf{E}$ and $\mathbf{F}$ and Lemma 4.8.2(c). For $j \in \Lambda_{\mathbf{E}}$, let $(\mathbf{F}^{<\lambda_j})^{\perp}(\hat{x})$ be the orthogonal complement of $\mathbf{F}^{<\lambda_j}(\hat{x})$ in $\mathfrak{g}/\mathfrak{h}_{\hat{x}}$ with respect to the dynamical inner product $\langle\cdot\rangle_{\hat{x}}$. Let $\mathbf{F}'_j(\hat{x}) := (\mathbf{F}^{<\lambda_j})^{\perp}(\hat{x}) \cap \mathbf{F}^{\leq\lambda_j}(\hat{x})$.

Lemma 4.8.8. [ELa, Lem. 3.5] Given $\delta > 0$, there exists a compact set $K_1 = K_1(\delta) \subset \Omega$ with measure at least $1-\delta$, $\beta(\delta) > 0$, $\beta'(\delta) > 0$ and for every $\hat{x} \in K_1$, $j \in \Lambda_{\mathbf{E}}$ and $\mathbf{v} \in (\mathbf{F}^{<\lambda_j})^{\perp}(\hat{x})$, a subset $Q_1 = Q_1(\hat{x}, \mathbf{v}/\|\mathbf{v}\|) \subset \mathcal{U}_1^+$ such that $Q_1\hat{x}$ has measure at least $1-\delta$, and for any $u \in Q_1$, we can write

$$\mathbf{v} = \mathbf{v}_u + \mathbf{w}_u, \qquad \mathbf{v}_u \in W^{\geq\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(u\hat{x}), \quad \mathbf{w}_u \in W^{<\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(u\hat{x}),$$

with $\|\mathbf{v}_u\| \geq \beta(\delta)\|\mathbf{v}\|$ and $\|\mathbf{v}_u\| > \beta'(\delta)\|\mathbf{w}_u\|$. Furthermore, if $j \in \Lambda_{\mathbf{E}}$ and $\mathbf{v} \in \mathbf{F}'_j(\hat{x})$, then $\mathbf{v}_u \in \mathbf{E}_j(u\hat{x})$.

Proof of Proposition 4.8.7. Let $\varepsilon > 0$ be less than 1/3 of the smallest gap between consecutive Lyapunov exponents on $(\mathfrak{l}/\mathfrak{h})_x$. By Oseledets theorem, there exists a compact subset $K_2 \subset \Omega$ with measure at least $1-\delta^2$ and $L > 0$ such that for $\hat{x} \in K_1$ and $\ell > L$,

$$\|(\hat{T}^t_{\hat{x}})_*\mathbf{v}\| \leq e^{(\lambda_j+\varepsilon)t} \quad \text{for } \mathbf{v} \in W^{\leq\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(\hat{x}), \quad \text{and} \quad \|(\hat{T}^t_{\hat{x}})_*\mathbf{v}\| \geq e^{(\lambda_j-\varepsilon)t} \quad \text{for } \mathbf{v} \in W^{\geq\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(\hat{x}).$$

By Fubini's theorem, there exists $K_3 \subset \Omega$ with measure at least $1-2\delta$ such that for $\hat{x} \in K_3$,

$$|\{u\hat{x} \in \mathcal{U}_1^+\hat{x} \mid u\hat{x} \in K_2\}| \geq (1-\delta/2)|\mathcal{U}_1^+\hat{x}|.$$

Let $K := K_1 \cap K_3$, where $K_1 = K_1(\delta/2)$ is the compact set in Lemma 4.8.8.

Let $\hat{q} := \hat{T}^{-\ell}\hat{q}_1$. It is clear from the definition of $\mathbf{F}'_j(\hat{x})$ that $(\mathfrak{l}/\mathfrak{h})_{\hat{x}} = \bigoplus_{j\in\Lambda_{\mathbf{E}}} \mathbf{F}'_j(\hat{x})$. For $\mathbf{v} \in \mathcal{L}^-(\hat{q})$, let $\mathbf{v}' := \mathcal{A}(\hat{q}_1, u, \ell, 0)\mathbf{v}$. We can write

$$\mathbf{v}' = \sum_{j\in\Lambda_{\mathbf{E}}} \mathbf{v}'_j, \qquad \mathbf{v}_j \in \mathbf{F}'_j(\hat{q}_1).$$

By Lemma 4.8.4, $\mathbf{F}^{\leq 0}(\hat{x}) = 0$ almost surely. Therefore in the decomposition above, $\mathbf{v}_j = 0$ if $j \notin \Lambda^+_{\mathbf{E}}$. Thus we can take the sum over only the indices in $\Lambda^+_{\mathbf{E}}$.

Suppose $\hat{q}_1 \in K$, $u \in Q_1(\hat{q}_1, \mathbf{v}')$ and $u\hat{q}_1 \in K_2$, where Q_1 is as in Lemma 4.8.8. By Lemma 4.8.8, we have

$$\mathbf{v}' = \sum_{j \in \Lambda^+_{\mathbf{E}}} (\mathbf{v}_j + \mathbf{w}_j), \tag{4.8.1}$$

where $\mathbf{v}_j \in \mathbf{E}_j(u\hat{q}_1), \mathbf{w}_j \in W^{<\lambda_j}_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1)$, and for all $j \in \Lambda^+_{\mathbf{E}}$,

$$\|\mathbf{v}_j\| \geq \beta'(\delta/2)\|\mathbf{w}_j\|.$$

Also we have

$$\|(T^t_{u\hat{q}_1})_*\mathbf{v}_j\| \geq e^{(\lambda_j-\varepsilon)t}\|\mathbf{v}_j\|, \qquad \text{and} \qquad \|(T^t_{u\hat{q}_1})_*\mathbf{w}_j\| \leq e^{(\lambda_{j+1}+\varepsilon)t}\|\mathbf{w}_j\|.$$

Thus for all $j \in \Lambda^+_{\mathbf{E}}$,

$$\|(T^t_{u\hat{q}_1})_*\mathbf{w}_j\| \leq e^{-(\lambda_j-\lambda_{j+1}-2\varepsilon)t}\beta'(\delta/2)^{-1}\|(T^t_{u\hat{q}_1})_*\mathbf{v}_j\|.$$

Since $(T^t_{u\hat{q}_1})_*\mathbf{v}_j \in \mathbf{E}_j(\hat{T}^t u\hat{q}_1)$ for each $j \in \Lambda^+_{\mathbf{E}}$, and $\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{v} = (T^t_{u\hat{q}_1})_*\mathcal{A}(\hat{q}_1, u, \ell, 0)\mathbf{v} = (T^t_{u\hat{q}_1})_*\mathbf{v}'$, this implies the proposition by (4.8.1). □

4.8.5 Bilipschitz estimates

For $\hat{q}_1 \in \hat{\Omega}$, $u \in \mathcal{U}^+_1$, $\ell > 0$ and $t > 0$, let $\hat{q} := \hat{T}^{-\ell}\hat{q}_1$ and $\hat{q}_2 := \hat{T}^t u\hat{q}_1$. Define

$$A(\hat{q}_1, u, \ell, t) := \|\pi_+\mathcal{A}(\hat{q}_1, u, \ell, t)|_{\mathcal{L}^-(\hat{q})}\|, \qquad \text{using the operator norm from } \|\cdot\|_{\hat{q}} \text{ to } \|\cdot\|_{\hat{q}_2}.$$

Here $\pi_+ : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ denote the orthogonal projection with respect to the dynamical inner product $\langle\cdot,\cdot\rangle_{\hat{x}}$ defined in subection 4.5.9.

For $\varepsilon > 0$, $\hat{q}_1 \in \hat{\Omega}$, $u \in \mathcal{U}_1^+$, and $\ell > 0$, let

$$\tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell) := \sup\{t \mid t > 0 \text{ and } A(\hat{q}_1, u, \ell, t) \leq \varepsilon\}.$$

By Proposition 4.7.3 and Proposition 4.8.7, $\tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)$ is finite almost surely.

There is a bilipschitz estimate on $\tilde{\tau}_\varepsilon$ in ℓ.

Proposition 4.8.9. cf. [ELa, Prop. 7.2] There exist constants $\kappa_\mathcal{T} > 1$ depending only on the Lyapunov spectrum on $\mathfrak{g}/\mathfrak{h}_x$ and on the cocycle V such that for almost all $\hat{q}_1 \in \hat{\Omega}$ and almost all $u\hat{q}_1 \in \mathcal{U}_1^+\hat{q}_1$, all $\varepsilon > 0$, $\ell > 0, s > 0$,

$$\tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell) + \kappa_\mathcal{T}^{-1} s < \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell + s) < \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell) + \kappa_\mathcal{T} s.$$

Proof. The proof is almost identical to the proof of [ELb, Prop. 4.2], [ELa, Prop. 7.2] and [EM18, Lem. 7.2, 7.3]. The main input is Proposition 4.5.13 and Remark 4.5.14. For this important result, we state precisely the modifications needed to be made to adapt the proof from [ELa, Prop. 7.2].

For $\hat{x} \in \hat{\Omega}$ and $t > 0$, let

- $\mathcal{A}_+(\hat{x}, t) : W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x}) \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{T}^t\hat{x})$ denote the restriction of $(T^t_{\hat{x}})_*$ on $(\mathfrak{l}/\mathfrak{h})_{\hat{x}}$ to $W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$,
- $\mathcal{A}_-(\hat{x}, t) : \mathcal{L}^-(\hat{x}) \to \mathcal{L}^-(\hat{T}^t\hat{x})$ denote the restriction of the cocycle on V to $\mathcal{L}^-$.

In the definitions of $\mathcal{A}_+$ and $\mathcal{A}_-$ we have used the fact that $W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x}) \subset (\mathfrak{l}/\mathfrak{h})_{\hat{x}}$ and $\mathcal{L}^- \subset V$ are equivariant in their corresponding cocycle (the latter follows from Lem 4.7.1).

It follows from Proposition 4.7.2 that $\mathcal{L}^-$ is in the stable bundle of V. Therefore by Proposition 4.5.13 and Remark 4.5.14, we have

$$e^{\kappa^{-1}t} \leq \|\mathcal{A}_+(\hat{x}, t)\| \leq e^{\kappa t}, \qquad e^{-\kappa t} \leq \|\mathcal{A}_+(\hat{x}, t)\| \leq e^{-\kappa^{-1}t},$$

$$e^{-\kappa_V^{-1}t} \geq \|\mathcal{A}_-(\hat{x}, t)\| \geq e^{-\kappa_V t}, \qquad e^{\kappa_V t} \geq \|\mathcal{A}_-(\hat{x}, t)\| \geq e^{\kappa_V^{-1}t}.$$

Note that since $(\mathfrak{l}/\mathfrak{h})_{\hat{x}} = W^{+}_{\mathfrak{l}/\mathfrak{h}}(\hat{x}) \oplus W^{\leq 0}_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ is an orthogonal decomposition (with respect to the dynamical inner product) that is equivariant under the dynamics, the equivariance property of $\mathcal{A}(\hat{q}_1, u, \ell, t)$ in Theorem 4.6.5(c) implies the following equivariance of $\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, t)$: for $\ell' > 0$ and $t' > 0$,

$$(\pi_+ \circ \mathcal{A})(\hat{q}_1, u, \ell + \ell', t + t') = T^{t'}_{\hat{T}^t u \hat{q}_1} \circ (\pi_+ \circ \mathcal{A})(\hat{q}_1, u, \ell, t) \circ T^{\ell'}_{\hat{T}^{-(\ell+\ell')}\hat{q}_1}.$$

The rest then follows from the argument in [ELa, Prop. 7.2], with $\kappa_\mathcal{T} = \kappa_V \kappa$. □

4.8.6 Jordan Canonical Form of cocycle on $\mathbf{E}(\hat{x})$: Invariant flag $\mathbf{E}_{ij}(\hat{x})$

Now we restrict the Jordan Canonical form of the cocycle from Theorem 4.5.11 to the inert subbundle $\mathbf{E}(\hat{x})$. For $i \in \Lambda^{+}_{\mathbf{E}}$, define

$$\mathbf{E}_{ij}(\hat{x}) := W_{ij}(\hat{x}) \cap \mathbf{E}_i(\hat{x})$$

to get an invariant flag

$$\{0\} = \mathbf{E}_{i0}(\hat{x}) \subset \mathbf{E}_{i1}(\hat{x}) \subset \cdots \subset \mathbf{E}_{i,n_i}(\hat{x}) = \mathbf{E}_i(\hat{x}).$$

Note that $\mathbf{E}_{ij}(\hat{x})$ might be the same as $\mathbf{E}_{i,j'}(\hat{x})$ for some $j < j'$. Remove redundant indices and relabel. Let $\Lambda''_{\mathbf{E}}$ be the set of new indices ij.

The following equivariance properties follow immediately from the corresponding equivariance properties of W_{ij} and $\mathbf{E}_i$.

Lemma 4.8.10. For $x \in \Omega_0$, all $t \in \mathbb{R}$ and a.e. $u\hat{x} \in \mathcal{U}_1^+ \hat{x}$,

$$(T^t_{\hat{x}})_* \mathbf{E}_{ij}(\hat{x}) = \mathbf{E}_{ij}(T^t \hat{x}), \qquad \mathbf{E}_{ij}(u\hat{x}) = \mathbf{E}_{ij}(\hat{x}).$$

In preparation for the tie-breaking procedure in the next section, we introduce the following notions for the cocycle on the inert bundle $\mathbf{E}$.

4.8.7 Flow $T^{ij,t}$ and time changes $\tilde{\tau}_{ij}(\hat{x},t)$

For each $ij \in \Lambda''_{\mathbf{E}}$, $\hat{x} \in \Omega$ and $t \in \mathbb{R}$, define $\tilde{\tau}_{ij}(\hat{x},t)$ be the unique number such that

$$\lambda_{ij}(\hat{x}, \tilde{\tau}_{ij}(\hat{x},t)) = \lambda_i t.$$

Define the **time changed flow** $T^{ij,t}\hat{x} := T^{\tilde{\tau}_{ij}(\hat{x},t)}\hat{x}$.

For each $\mathbf{v} \in \mathfrak{g}$, $\hat{x} \in \Omega$ and $t \in \mathbb{R}$, define $\tilde{\tau}_{\mathbf{v}}(\hat{x},t)$ be the unique number such that

$$\|(T_{\hat{x}}^{\tilde{\tau}_{\mathbf{v}}(\hat{x},t)})_* \mathbf{v}\|_{\hat{T}^{\tilde{\tau}_{\mathbf{v}}(\hat{x},t)}\hat{x}} = e^t \|\mathbf{v}\|_{\hat{x}}.$$

Define the **time changed flow** $T^{\mathbf{v},t}\hat{x} := T^{\tilde{\tau}_{\mathbf{v}}(\hat{x},t)}\hat{x}$.

4.8.8 Parallel transport $R(\hat{x},\hat{x}')$ and foliations $\mathcal{F}_{ij}$ and $\mathcal{F}_{\mathbf{v}}$

Let $\Omega' := \Omega_{ebp}^c \subset \Omega$, where Ω_{ebp} is the set of elements in Ω whose $\mathcal{S}^{\mathbb{Z}}$ component ω is "eventually backward periodic", i.e. there exist some $n > 0$ and $s > 0$ such that $\omega_{j+s} = \omega_j$ for all $j < -n$. Then Ω' is conull in Ω. For $\hat{x} \in \Omega'$, let

$$\mathcal{H}[\hat{x}] := \{\hat{T}^s u \hat{T}^{-t}\hat{x} : t, s \geq 0, u \in \mathcal{U}_1^+\} \subset \Omega.$$

For $\hat{x}' = \hat{T}^s u \hat{T}^{-t}\hat{x} \in \mathcal{H}[\hat{x}]$, define $R(\hat{x},\hat{x}') : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to (\mathfrak{l}/\mathfrak{h})_{\hat{x}'}$ by

$$R(\hat{x},\hat{x}')\mathbf{v} := (T^s_{u\hat{T}^{-t}\hat{x}})_* (T^{-t}_{\hat{x}})_* \mathbf{v}.$$

We remark that for $\hat{x} \in \Omega' = \Omega_{ebp}^c$, $R(\hat{x},\hat{x}')$ depends only on $\hat{x}, \hat{x}'$ and not on s, u and t.

Recall that $\mathfrak{B}_0[\hat{x}] := \mathfrak{B}_0[\hat{x}] := J[\hat{x}] \cap \hat{\mathcal{W}}^+_{G/H}[\hat{x}]$ is the local unstable set defined by the Markov partition J constructed in Proposition 4.5.12.

Define the local balls $\mathcal{F}_{ij}[\hat{x}, \ell]$ and the foliation $\mathcal{F}_{ij}[\hat{x}]$

$$\mathcal{F}_{ij}[\hat{x}, \ell] := \{\hat{x}' \in \mathcal{H}[\hat{x}] \mid T^{ij,-\ell}\hat{x}' \in \mathfrak{B}_0[T^{ij,-\ell}\hat{x}]\} \qquad \text{and} \qquad \mathcal{F}_{ij}[\hat{x}] := \bigcup_{\ell \geq 0} \mathcal{F}_{ij}[\hat{x}, \ell].$$

Similarly for $\mathbf{v} \in \mathbf{E}(\hat{x})$, define the balls $\mathcal{F}_{\mathbf{v}}[\hat{x}, \ell]$ and the foliation $\mathcal{F}_{\mathbf{v}}[\hat{x}]$

$$\mathcal{F}_{\mathbf{v}}[\hat{x}, \ell] := \{\hat{x}' \in \mathcal{H}[\hat{x}] \mid T^{\mathbf{v},-\ell}\hat{x}' \in \mathfrak{B}_0[T^{\mathbf{v},-\ell}\hat{x}]\} \qquad \text{and} \qquad \mathcal{F}_{\mathbf{v}}[\hat{x}] := \bigcup_{\ell \geq 0} \mathcal{F}_{\mathbf{v}}[\hat{x}, \ell].$$

Recall from Proposition 4.5.13(e) that $\lambda_{ij}(\hat{x}, -t) = \lambda_{ij}(\hat{x}', -t)$ for almost every $\hat{x} \in \Omega$ and $\hat{x}' \in \mathfrak{B}_0[\hat{x}]$, therefore (cf. [ELa, Lem. 5.2])

$$\mathcal{F}_{ij}[\hat{x}, \ell] \subset \mathcal{F}_{ij}[\hat{x}, \ell'] \qquad \text{and} \qquad \mathcal{F}_{\mathbf{v}}[\hat{x}, \ell] \subset \mathcal{F}_{\mathbf{v}}[\hat{x}, \ell'] \qquad \text{for all} \quad 0 \leq \ell \leq \ell'.$$

We also have the following properties which easily follow from the definitions (see [ELa, Sect. 5.3])).

Proposition 4.8.11. For $\hat{x} \in \Omega'$ and $\hat{x}' = \hat{T}^s u \hat{T}^{-t}\hat{x} \in \mathcal{H}[\hat{x}]$, let $\lambda_{ij}(\hat{x}, \hat{x}') := \lambda_{ij}(\hat{x}, -t) + \lambda_{ij}(uT^{-t}\hat{x}, s)$. Let $\mathbf{E}'_{ij}(\hat{x}) := \mathbf{E}^{\perp}_{i,j-1}(\hat{x}) \cap \mathbf{E}_{ij}(\hat{x})$, where we take orthogonal complement using the dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$.

(a) $R(\hat{x}, \hat{x}')\mathbf{v} = e^{\lambda_{ij}(\hat{x}, \hat{x}')}\mathbf{v}' + \mathbf{v}''$, where $\mathbf{v} \in \mathbf{E}'_{ij}(\hat{x})$, $\mathbf{v}' \in \mathbf{E}'_{ij}(\hat{x}')$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(\hat{x}')$ and $\|\mathbf{v}'\|_{\hat{x}'} = \|\mathbf{v}\|_{\hat{x}}$.

(b) $\lambda_{ij}(\hat{x}, \hat{x}') = 0$ if and only if $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$.

(c) [ELa, Lem. 5.3] Suppose $\hat{x} \in \Omega$ and $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$. Then for all ℓ large enough, $\mathcal{F}_{ij}[\hat{x}, \ell] = \mathcal{F}_{ij}[\hat{x}', \ell]$.

(d) [ELa, Lem. 5.4] For a.e. $\hat{x} \in \Omega$, any $\mathbf{v} \in \mathbf{E}(\hat{x})$ and a.e. $\hat{x}' \in \mathcal{F}_{\mathbf{v}}[\hat{x}]$, we have $\|R(\hat{x}, \hat{x}')\mathbf{v}\|_{\hat{x}'} = \|\mathbf{v}\|_{\hat{x}}$.

4.9 The tie-breaking procedure: Bounded subspaces $\mathbf{E}_{ij,\text{bdd}}(\hat{x})$ and Synchronized exponents $[ij]$

In this section, we collect and summarize the statements necessary to perform a tie-breaking procedure. The main statements are Proposition 4.9.1, Proposition 4.9.2 and Proposition 4.9.3. Since the entire argument for these statements happen within the Lie algebra $(\mathfrak{l}/\mathfrak{h})_{\hat{q}}$, the proofs are identical to that in the case of G/Γ considered in [ELa, Sect. 6]. Therefore this section contains only the necessary definitions and statements without proofs. We refer the reader to the corresponding statements in [ELa, Sect. 6].

Recall that Λ'' is the indices of the fine Lyapunov spectrum on $\mathbf{E}$. In this section, we define an equivalence relation called "synchronization" on Λ''. The equivalence class of $ij \in \Lambda''$ is denoted by $[ij]$ and the set of equivalence class is denoted by Λ_{sync}. For each $ij \in \Lambda''$ we define an $\hat{T}^t$-equivariant and $\mathcal{U}_1^+$-equivalent subbundle $\mathbf{E}_{ij,\text{bdd}}$ of the bundle $\mathbf{E}_i$ and we define

$$\mathbf{E}_{[ij],\text{bdd}}(\hat{x}) := \sum_{kr \in [ij]} \mathbf{E}_{kr,\text{bdd}}(\hat{x}).$$

In fact it can be shown that there exists a subset $[ij]' \subset [ij]$ such that

$$\mathbf{E}_{[ij],\text{bdd}}(\hat{x}) = \bigoplus_{kr \in [ij]'} \mathbf{E}_{kr,\text{bdd}}(\hat{x}).$$

The following are the main conclusions that will be used in future sections.

Proposition 4.9.1. [ELa, Prop. 6.1] There exists $\theta_1 \in (0,1)$ such that:
for all $\delta, \eta > 0$, there is $K = K(\delta, \eta) \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and $L_0 = L_0(\delta, \eta) > 0$ such that:
If $\hat{x} \in \Omega$, $\mathbf{v} \in \mathbf{E}(\hat{x})$, $L \geq L_0$ satisfy

$$|T^{[-1,1]}K \cap \mathcal{F}_{\mathbf{v}}[\hat{x}, L]| \geq (1 - \theta_1)|\mathcal{F}_{\mathbf{v}}[\hat{x}, L]|,$$

then for at least θ_1-fraction of $\hat{x}' \in \mathcal{F}_{\mathbf{v}}[\hat{x}, L]$,

$$d\left(\frac{R(\hat{x}, \hat{x}')\mathbf{v}}{\|R(\hat{x}, \hat{x}')\mathbf{v}\|}, \bigcup_{ij \in \Lambda_{\text{sync}}} \mathbf{E}_{[ij],\text{bdd}}(\hat{x}')\right) < \eta.$$

Proposition 4.9.2. [ELa, Prop. 6.2] There exists a measurable function $C : \Omega \to \mathbb{R}^+$ finite a.e. such that

for all $\hat{x} \in \Omega$, $\mathbf{v} \in \mathbf{E}_{[ij],\text{bdd}}(\hat{x})$ and $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$,

$$C(\hat{x})^{-1}C(\hat{x}')^{-1}\|\mathbf{v}\| \leq \|R(\hat{x}, \hat{x}')\mathbf{v}\| \leq C(\hat{x})C(\hat{x}')\|\mathbf{v}\|.$$

Proposition 4.9.3. [ELa, Prop. 6.3] There exists $\theta > 0$ (depending only on $\hat{\nu}$) and a co-null subset $\Psi \subset \Omega$ such that the following holds: Suppose $\hat{x} \in \Psi$, $\mathbf{v} \in W^{>0}_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$, and there exists $C > 0$ such that for all $\ell > 0$, and at least $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$,

$$\|R(\hat{x}, \hat{x}')\mathbf{v}\| \leq C\|\mathbf{v}\|.$$

Then $\mathbf{v} \in \mathbf{E}_{[ij],\text{bdd}}(\hat{x})$.

4.9.1 Synchronized exponents

Definition. Given $\theta > 0$ and $E \subset \Omega$ with $\tilde{\mu}(E) > 0$, we say $ij, kr \in \Lambda''$ are

(E, θ)−synchronized if: there exists $C < \infty$ such that for all $\hat{x} \in E$, $\ell > 0$, for at least $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$, we have

$$\rho(\hat{x}', \mathcal{F}_{kr}[\hat{x}]) < C.$$

We write $ij \sim kr$ if ij and kr are (E, θ)-synchronized for some $E \subset \Omega$ and some small $\theta > 0$.

Remark 4.9.4. Note that if ij and kr are (E, θ)-synchronized, then they are $(\bigcup_{|s|<t} T^s E, \theta)$-synchronized for all $t > 0$.

For $\mathbf{v} \in \mathbf{E}(\hat{x})$, we can decompose for some $I_{\mathbf{v}} \subset \Lambda''_{\mathbf{E}}$

$$\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}, \qquad \text{where } \mathbf{v}_{ij} \in \mathbf{E}_{ij}(\hat{x}) \smallsetminus \mathbf{E}_{i,j-1}(\hat{x}).$$

Lemma 4.9.5. [ELa, Lem. 6.11'] For a.e. $\hat{x} \in \Omega$, suppose there exists $C < \infty$ and $\mathbf{v} \in \mathbf{E}(\hat{x})$ such that

for all $\ell > 0$ and at least $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{\mathbf{v}}[\hat{x}, \ell]$,

$$\rho(\hat{x}', \mathcal{F}_{ij}[\hat{x}]) < C \qquad \text{for all} \quad ij \in I_{\mathbf{v}}.$$

Then all of $ij \in I_{\mathbf{v}}$ are synchronized.

Lemma 4.9.6. [ELa, Lem. 6.19'] Suppose ij and kr are synchronized, then there exists a function $C : \Omega \to \mathbb{R}^+$ finite $\tilde{\mu}$-a.e. such that for all $\hat{x} \in \Omega$, and all $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$,

$$\rho(\hat{x}', \mathcal{F}_{kr}[\hat{x}]) \leq C(\hat{x})C(\hat{x}').$$

4.9.2 Bounded subspaces $\mathbf{E}_{ij,\mathrm{bdd}}(\hat{x})$

Fix a sufficiently small $\theta > 0$.

Definition. Given $\hat{x} \in \Omega$, we say a vector $\mathbf{v} \in \mathbf{E}_{ij}(\hat{x})$ is **$(\boldsymbol{\theta}, \boldsymbol{ij})$-bounded** if:

there exists $C < \infty$ such that for all $\ell > 0$ and $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$,

$$\|R(\hat{x}, \hat{x}')\mathbf{v}\| \leq C\|\mathbf{v}\|.$$

Definition ($\mathbf{E}_{ij,bdd}(\hat{x})$)**.** Let $n = \dim \mathbf{E}_{ij}(\hat{x})$. Define

- $\mathbf{E}_{ij,\mathrm{bdd}}(\hat{x}) = \{0\}$ if there is no θ/n-bounded vector in $\mathbf{E}_{ij}(\hat{x}) \smallsetminus \mathbf{E}_{i,j-1}(\hat{x})$,
- Otherwise, $\mathbf{E}_{ij,\mathrm{bdd}}(\hat{x})$ is generated by θ/n-bounded vector in $\mathbf{E}_{ij}(\hat{x})$.

(The set of (θ, ij)-bounded vectors does not form a vector space in general.)

Lemma 4.9.7. [ELa, Lem. 6.17] Given $\theta > 0$.

Suppose for all $\delta > 0$, there exists $K = K(\delta) \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and $C_1 = C_1(\delta) < \infty$ such that

for all $\hat{x} \in K$, $\ell > 0$ and at least $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$, we have

$$\|R(\hat{x}, \hat{x}')\mathbf{v}\| \le C_1 \|\mathbf{v}\| \qquad \text{for all } \mathbf{v} \in \mathbf{E}_{ij,\mathrm{bdd}}(\hat{x}).$$

Then for all $\delta, \ell > 0$, there exists $K''(\ell) \subset \Omega$ with $\tilde{\mu}(K''(\ell)) > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and there exists $\theta'' = \theta''(\theta, \delta) > 0$ with $\theta'' \to 0$ as $\theta \to 0$ and $\delta \to 0$ such that

for all $\hat{x} \in K''(\ell)$, for at least $(1-\theta'')$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$,

$$C_1^{-1}\|\mathbf{v}\| \le \|R(\hat{x}, \hat{x}')\mathbf{v}\| \le C_1 \|\mathbf{v}\| \qquad \text{for all } \mathbf{v} \in \mathbf{E}_{ij,\mathrm{bdd}}(\hat{x}).$$

Lemma 4.9.8. [ELa, Lem. 6.18] There exists $C : \Omega \to \mathbb{R}^+$ finite a.e. such that for all $\hat{x} \in \Omega$, $\mathbf{v} \in \mathbf{E}_{ij,\mathrm{bdd}}(\hat{x})$ and $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$,

$$C(\hat{x})^{-1}C(\hat{x}')^{-1}\|\mathbf{v}\| \le \|R(\hat{x}, \hat{x}')\mathbf{v}\| \le C(\hat{x})C(\hat{x}')\|\mathbf{v}\|.$$

4.9.3 *Synchronized bounded subspaces* $\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x})$

Let $[ij] = \{kr : kr \text{ is synchronized with } ij\}$. Let

$$\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x}) := \sum_{kr \in [ij]} \mathbf{E}_{kr,\mathrm{bdd}}(\hat{x}).$$

Lemma 4.9.9. [ELa, Lem. 6.12] For $\tilde{\mu}$-a.e. $\hat{x} \in \Omega$, if $ij \sim ik$, $j < k$ and $\mathbf{E}_{ik,\mathrm{bdd}}(\hat{x}) \neq \{0\}$, then $\mathbf{E}_{ij,\mathrm{bdd}}(\hat{x}) \subset \mathbf{E}_{ik,\mathrm{bdd}}(\hat{x})$.

Thus there is a subset $[ij]' \subset [ij]$ with at most one ij for each i, such that

$$\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x}) = \bigoplus_{kr \in [ij]'} \mathbf{E}_{kr,\mathrm{bdd}}(\hat{x}).$$

Let Λ_{sync} be the equivalence classes in $\Lambda''_{\mathbf{E}}$.

Lemma 4.9.10. [ELa, Lem. 6.19] Suppose for all $\delta > 0$, there exists $K \subset \Omega$ with $\tilde{\mu}(K) > 1-\delta$ and $C < \infty$ s.t.

for all $\hat{x} \in K$, $\ell > 0$ and at least $(1-\theta)$-fraction of $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}, \ell]$, we have

$$\lambda_{kr}(\hat{x}, \hat{x}') \leq C.$$

Then, ij and kr are synchronized, and there exists a function $C : \Omega \to \mathbb{R}^+$ finite $\tilde{\mu}$-a.e. such that

for all $\hat{x} \in \Omega$, and all $\hat{x}' \in \mathcal{F}_{ij}[\hat{x}]$,

$$\rho(\hat{x}', \mathcal{F}_{kr}[\hat{x}]) \leq C(\hat{x})C(\hat{x}').$$

4.9.4 Conditional Measures $f_{ij}(\hat{z})$ on $\mathcal{E}_{ij}(\hat{x})$

As a consequence of Proposition 4.9.3, $\mathbf{E}_{[ij],\text{bdd}}(\hat{x})$ is in fact a nilpotent subalgebra of $(\mathfrak{l}/\mathfrak{h})_{\hat{x}}$ (see [ELa, Prop. 9.1]).

For $\tilde{\mu}$-almost every $\hat{x} \in \Omega$ with G/L component x, we define $\mathcal{E}_{ij}(\hat{x})$ to be the subgroup of $x(L/H^\circ)x^{-1}$ such that $\mathbf{E}_{[ij],\text{bdd}}(\hat{x}) = \text{Lie}(\mathcal{E}_{ij}(\hat{x}))$.

Lemma 4.9.11. [ELa, Lem. 9.3] For $\hat{x} \in \Omega$, $t \in \mathbb{R}$ and $u \in \mathcal{U}_1^+$,

$$\mathcal{E}_{ij}[\hat{T}^t \hat{x}] = \hat{T}^t \mathcal{E}_{ij}[\hat{x}], \qquad \text{and} \qquad \mathcal{E}_{ij}[u\hat{x}] = \mathcal{E}_{ij}[\hat{x}].$$

For $\hat{z} = (\omega, z) \in \Omega_b \times G/H^\circ$ with $\hat{x} = \hat{\pi}(\hat{z})$ for $\hat{\pi} : \Omega_b \times G/H^\circ \to \Omega_b \times G/L$, define $f_{ij}(\hat{z})$ to be a measure on $\mathcal{E}_{ij}(\hat{x})$ defined as the pullback of the conditional measure of $\hat{\nu}$ along $\mathcal{E}_{ij}(\hat{x})z$. More precisely, we consider the conditional measure $\hat{\nu}|_{zL/H}$ of $\hat{\nu}$ on the fiber zL/H, then apply the leafwise measure construction in [EL10, Sect. 6] to the unipotent subgroup $\mathcal{E}_{ij}(\hat{x}) \subset x(L/H^\circ)x^{-1}$ acting on zL/H to obtain a leafwise measure $f_{ij}(\hat{z})$ on $\mathcal{E}_{ij}(\hat{x})$. $f_{ij}(\hat{z})$ is well-defined for almost every $\hat{z} \in \Omega_b \times G/H$.

Lemma 4.9.12. [ELa, Lem. 9.4] We have for almost every $\hat{z} \in \hat{\Omega}$, $u \in \mathcal{U}_1^+$ and $s, t \in \mathbb{R}$,

$$f_{ij}(\hat{T}^t u \hat{T}^{-s} \hat{z}) \propto (T^t_{u\hat{z}})_* (T^{-s}_{\hat{z}})_* f_{ij}(\hat{z}).$$

Proof. This follows by the equivariance of conditional measures and leafwise measures. See e.g. [EL10, Thm. 6.3(iii)]. □

4.9.5 General lemmas

We use the following general elementary lemma a few times in the main argument in subection 4.10.1.

Lemma 4.9.13. Let (X, ν) be a Borel probability space with measurable partition $\mathcal{A}$ and the corresponding conditional measures $\{\nu_x\}_{x \in X}$, so in particular

$$\nu = \int \nu_x \, d\nu(x).$$

Let $a > 0$, and $K \subset X$ be a measurable subset with $\nu(K) > 1 - a$. Let

$$K_0 := \{x \in X \mid \nu_x(K) > 1 - b\}.$$

Then $\nu(K_0) > 1 - a/b$.

Proof. We have

$$\nu(K^c) = \int \nu_x(K^c) \, d\nu(x) \geq \int_{K_0^c} \nu_x(K^c) \, d\nu(x) \geq \int_{K_0^c} b \, d\nu(x) = b \, \nu(K_0^c).$$

By assumption, $\nu(K^c) \leq a$, therefore $b \, \nu(K_0^c) \leq a$ and the result follows. □

Lemma 4.9.14. Let $C > 1$. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing C-bilipschitz function, i.e. for all $\ell \geq 0$ and $s \geq 0$,

$$C^{-1} s \leq f(\ell + s) - f(\ell) \leq Cs.$$

Let $\delta > 0$ and $\ell_0 > 0$. Let $E \subset \mathbb{R}$ be a measurable subset with density at most δ for all $\ell > \ell_0$, i.e.

$$\mathrm{Leb}(E \cap [0,\ell]) < \delta\ell \qquad \text{for all } \ell > \ell_0.$$

Then the set $\{\ell \in \mathbb{R}_+ \mid f(\ell) \in E\}$ has density at most $C^2\delta$ for all $\ell > C\ell_0$.

Proof. Since f is C-bilipschitz, both f and f^{-1} are almost surely differentiable with derivative at most C. Also note that $f(0) \geq 0$, and f and f^{-1} are increasing. Thus

$$\begin{aligned}\mathrm{Leb}(\{\ell \in [0,t] \mid f(\ell) \in E\}) &= \int_0^t \mathbf{1}_E(f(\ell))\, d\ell = \int_{f(0)}^{f(t)} \mathbf{1}_E(s)\, \frac{df^{-1}(s)}{ds} ds \\ &\leq \int_0^{f(t)} \mathbf{1}_E(s)\, \frac{df^{-1}(s)}{ds} ds \leq C\mathrm{Leb}(E \cap [0, f(t)]).\end{aligned}$$

For all $t > C\ell_0$, we have $f(t) \geq f(t) - f(0) \geq C^{-1}t > \ell_0$. Since E has density at most δ for $\ell > \ell_0$, we have $\mathrm{Leb}(E \cap [0, f(t)]) < \delta f(t) \leq C\delta t$. Therefore

$$\mathrm{Leb}(\{\ell \in [0,t] \mid f(\ell) \in E\}) \leq C\mathrm{Leb}(E \cap [0, f(t)]) \leq C^2\delta t.$$

□

4.10 Main argument of Case I

4.10.1 Fit eight points into a compact set

For each $0 < \delta < 1$, let $K_* = K_*(\delta) \subset \hat{\Omega}$ be a compact subset with measure at least $1 - \delta$ such that f_{ij} is uniformly continuous on K_* for all $ij \in \Lambda_{\mathrm{sync}}$.

Proposition 4.10.1. [ELa, Prop. 10.2] There exist $0 < \delta < 0.1$ and $C = C(K_*(\delta)) > 1$ such that for every $0 < \varepsilon < C^{-1}/100$, there exists $E \subset K_*$ with measure at least δ such that for all $\hat{x} \in E$, there exists $ij \in \Lambda_{\mathrm{sync}}$ and $\hat{y} \in \mathcal{E}_{ij}[\hat{x}] \cap K_*$ with

$$C^{-1}\varepsilon \leq d_{G/H}(\hat{x}, \hat{y}) \leq C\varepsilon, \qquad \text{and} \qquad f_{ij}(\hat{y}) \propto f_{ij}(\hat{x}).$$

The proof mostly follows the scheme in [ELa, Prop. 10.2]. For expository purpose, we first write a detailed outline using claims without proof (and sometimes vaguely stated), and then prove the claims afterwards.

For the outline (and sometimes in proofs of the claims), we use the following shorthand notations:

let α denote a parameter (or a few parameters) and $\delta > 0$ is a distinguished parameter.

1. **large $\boldsymbol{A} \subset_{\boldsymbol{\alpha},\boldsymbol{\delta}} \boldsymbol{B}$** means $A = A(\alpha, \delta)$ is a measurable subset of B with measure (or density if $B = \mathbb{R}^+$) at least $1 - c(\delta)$ for some $c(\delta) \to 0$ as $\delta \to 0$ (sometimes $c(\delta)$ is more explicitly specified).

2. **for most $\boldsymbol{a} \in_{\boldsymbol{\alpha},\boldsymbol{\delta}} \boldsymbol{B}$** means there exists a large measurable subset $A \subset_{\alpha,\delta} B$ such that for all $a \in A$.

3. **for $\boldsymbol{t} \gg_{\boldsymbol{\alpha}} \mathbf{0}$** means there exists $T = T(\alpha) > 0$ such that for all $t > T(\alpha)$.

4. **$\boldsymbol{a} \leq_{\boldsymbol{\alpha}} \boldsymbol{b}$** means there exists $C = C(\alpha) > 0$ such that $a \leq Cb$.

5. **$\boldsymbol{a} \approx_{\boldsymbol{\alpha}} \boldsymbol{b}$** means there exists $C = C(\alpha) > 1$ such that $C^{-1}b \leq a \leq Cb$.

6. **$\boldsymbol{x} = \boldsymbol{O}_{\boldsymbol{\alpha}}(\boldsymbol{y})$** means $x \leq C(\alpha)y$ for some constant $C = C(\alpha) > 0$.

Proof outline of Proposition 4.10.1. Given $(\hat{q}_1, u, \ell) \in \hat{\Omega} \times \mathcal{U}_1^+ \times \mathbb{R}_{>0}$, a **$\boldsymbol{Y}$-configuration** $Y_{ij} = Y_{ij}(\hat{q}_1, u, \ell)$ is a quadruple $(\hat{q}, \hat{q}_1, \hat{q}_2, \hat{q}_{3,ij})$ s.t.

$$\hat{q}(Y_{ij}) = \hat{T}^{-\ell}\hat{q}_1, \qquad \hat{q}_1(Y_{ij}) = \hat{q}_1, \qquad \hat{q}_2(Y_{ij}) = \hat{T}^t u\hat{q}_1, \qquad \hat{q}_{3,ij}(Y_{ij}) = \hat{T}^{t_{ij}}\hat{q}_1,$$

where

$$t(Y_{ij}) = \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell) \qquad \text{satisfies} \qquad \|\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, t)\| = \varepsilon,$$
$$t_{ij}(Y_{ij}) = t_{ij}(\hat{q}_1, u, \ell) \qquad \text{satisfies} \qquad \lambda_{ij}(u\hat{q}_1, t(Y_{ij})) = \lambda_{ij}(\hat{q}_1, t_{ij}).$$

Note that for fixed $\hat{q}_1$ and u,

- $\ell \mapsto t(Y_{ij})$ is $\kappa_\mathcal{T}$-bilipschitz by **Proposition 4.8.9**,

- $t \mapsto t_{ij}(Y_{ij})$ is κ^2-bilipschitz by **Proposition 4.5.13** (c), so $\ell \mapsto t_{ij}(Y_{ij})$ is $\kappa_\mathcal{T}\kappa^2$-bilipschitz.

We define

- A Y-configuration Y_{ij} is **good** if $\hat{q}(Y_{ij}), \hat{q}_1(Y_{ij}), \hat{q}_2(Y_{ij}), \hat{q}_{3,ij}(Y_{ij}) \in K$ (the compact set $K \subset \hat{\Omega}$ will be defined in **Step 4**).

- $Y = Y_{ij}(\hat{q}_1, u, \ell)$ and $Y' = Y_{ij}(\hat{q}_1', u', \ell')$ are **coupled** if $\ell = \ell'$, $u = u'$, $\hat{q}(Y') \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}(Y)]$, and $\mathbf{w} := F_{\hat{q}}(\hat{q}') \in \mathcal{L}^-(\hat{q})$, $\|\mathbf{w}\| \approx_\delta 1$

$$\|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}\| \approx_\delta \|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, t)\|\|\mathbf{w}\| \approx_\delta \varepsilon \qquad \text{where } t = \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell).$$

  Recall that $\mathcal{A}(\hat{q}_1, u, \ell, t) : \mathcal{L}^-(\hat{q}) \to (\mathfrak{l}/\mathfrak{h})_{\hat{q}_2}$ is a linear map defined in Theorem 4.6.5 and then restricted to $\mathcal{L}^-(\hat{q})$ in the beginning of Section 4.7. Here $\pi_+ : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ denote the orthogonal projection with respect to the dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$ defined in subsection 4.5.9.

Fix $\theta_1 > 0$ as in **Proposition 4.9.1**, then fix $\delta > 0$ later and then fix sufficiently small $\varepsilon, \eta > 0$. The proof proceeds as follows:

1. Fix an arbitrary compact set $K_{00} \subset \hat{\Omega}$ with $\hat{\nu}(K_{00}) > 1 - 2\delta$ (See **Step 19** for why we start this way).

2. Recall that $K_* = K_*(\delta) \subset_\delta \hat{\Omega}$ is a compact subset with measure at least $1 - \delta$ such that f_{ij} is uniformly continuous on K_* for all $ij \in \Lambda_{\text{sync}}$.

3. Choose a large compact subset $K_0 \subset_\delta \hat{\Omega}$ such that

   (a) $K_0 \subset K_{00} \cap K_*$,

   (b) for $\hat{x} \in K_0$, $t \gg_{\delta,\varepsilon'} 0$, $\mathbf{v} \in W^{\lambda_i}(\hat{x})$, we have $e^{-(\lambda_i+\varepsilon')t}\|\mathbf{v}\| \le \|(\hat{T}^{-t}_{\hat{x}})_*\mathbf{v}\| \le e^{-(\lambda_i-\varepsilon')t}\|\mathbf{v}\|$ (Oseledets).

   Here ε' is a constant to be chosen in **Step 16** (see **Claim 4.10.7**), which depends only on the Lyapunov spectrum.

(c) for $\hat{x} \in K_0, \hat{x}' \in \mathcal{F}_{ij}[\hat{x}] \cap K_0$ and $\mathbf{v} \in \mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x})$, we have $\|R(\hat{x}, \hat{x}')\mathbf{v}\| \approx_\delta \|\mathbf{v}\|$. (**Prop. 4.9.2**)

(d) for $\hat{x} \in K_0$, $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}] \cap K_0$, $t > 0$ and $g \in \exp(W^{<0}_{\mathfrak{g}}(\hat{x}))$ such that $gxL = x'L$, we have $\|P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}') - \mathrm{Ad}_{(T^t_\omega)g(T^t_\omega)^{-1}}\|_{0\to 0} \leq_\delta \|g\|_{\mathrm{Ad}(G)}e^{-\alpha t}$ (**Lem. 4.5.10**).

4. Choose a large compact subset $K \subset_\delta \hat{\Omega}$ such that there exists $C = C(\delta, \varepsilon) > 1$ with

(a) For $\hat{x} \in K$ and $T \gg_\delta 0$, we have $\mathrm{Leb}(\{t \in [-T/2, T/2] \mid \hat{T}^t\hat{x} \in K_0\}) \geq 0.9T$ (Birkhoff),

(b) **Proposition 4.6.5(d)** holds ($d(\hat{q}_2, \hat{q}_2')$ is exponentially close to $\|\mathcal{A}(\hat{q}_1, u, \ell, t)F_{\hat{q}}(\hat{q}')\|$)

(c) **Proposition 4.7.3** holds (the norm of $\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, t)$ is lower bounded by $\geq_\delta e^{-\kappa'\ell+\kappa t}$)

(d) **Proposition 4.8.7** holds (any $\mathbf{v} \in \mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$ gets close to $\mathbf{E}$ via $\mathcal{A}(\hat{q}_1, u, \ell, t)$ as $t \to \infty$ for most u).

(e) **Proposition 4.9.1** holds (any $\mathbf{v} \in \mathbf{E}$ gets close to one of $\mathbf{E}_{[ij],\mathrm{bdd}}$ via $R(\hat{x}, \hat{x}')$ for many $\hat{x}' \in \mathcal{F}_{\mathbf{v}}[\hat{x}]$).

(f) For $\hat{x} \in K$, $\mathbf{v} \in \mathfrak{g}$ and $t \in \mathbb{R}$, we have $C^{-1}e^{-\varepsilon|t|} \leq \|\mathbf{v}\|_{\hat{T}^t\hat{x}}/\|\mathbf{v}\|_0 \leq Ce^{\varepsilon|t|}$ (**Lemma 4.5.15**).

(g) For $\hat{x} \in K$, $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}] \cap K$, $\hat{T}^t\hat{x} \in K$ and $\hat{T}^t\hat{x}' \in \hat{T}^{[-a,a]}K$, we have $|\lambda_{ij}(\hat{x}, t) - \lambda_{ij}(\hat{x}', t)| \leq C'(a, \delta)$. (**Lem. 4.7.4**)

(h) For $\hat{x} \in K, \hat{y} \in \mathcal{F}_{ij}[\hat{x}] \cap K$, $\mathbf{v} \in \mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x})$, we have $C^{-1}\|\mathbf{v}\| \leq \|R(\hat{x}, \hat{y})\mathbf{v}\| \leq C\|\mathbf{v}\|$ (**Prop. 4.9.2**).

(i) for $\hat{x} \in K$, $\hat{x}' \in \hat{\mathcal{W}}^-_{G/L}[\hat{x}] \cap K$, $t > 0$ and $g \in \exp(W^{<0}_{\mathfrak{g}}(\hat{x}))$ such that $gxL = x'L$, we have $\|P^-(\hat{T}^t\hat{x}, \hat{T}^t\hat{x}') - \mathrm{Ad}_{(T^t_\omega)g(T^t_\omega)^{-1}}\|_{0\to 0} \leq_\delta \|g\|_{\mathrm{Ad}(G)}e^{-\alpha t}$ (**Lem. 4.5.10**).

5. **Claim 4.10.1.** For most $\hat{q}_1 \in_\delta K \subset \hat{\Omega}$, most $u \in_{\delta,\hat{q}_1} \mathcal{U}_1^+$ and most $\ell \in_{\delta,\hat{q}_1,u} \mathbb{R}_{>0}$, we have $u\hat{q}_1 \in K$ and $\hat{q}_2(Y_{ij}), \hat{q}_3(Y_{ij}) \in K$ for $Y_{ij} := Y_{ij}(\hat{q}_1, u, \ell)$ for all $ij \in \Lambda_{\mathrm{sync}}$. (Proof: Pointwise ergodic theorem, and use bilipschitz estimates of $\ell \mapsto t, t_{ij}$.)

6. **Claim 4.10.2.** For most $\ell \in_\delta \mathbb{R}_{>0}$, most $\hat{q}_1 \in_{\delta,\ell} K \subset \hat{\Omega}$ and most $u \in_{\delta,\ell,\hat{q}_1} \mathcal{U}_1^+$, we have $\hat{q}_2(Y_{ij}), \hat{q}_3(Y_{ij}) \in K$ for $Y_{ij} := Y_{ij}(\hat{q}_1, u, \ell)$ for all $ij \in \Lambda_{\text{sync}}$ (Fubini).

7. **Claim 4.10.3.** For most $\ell \in_\delta \mathbb{R}_{>0}$, most $\hat{q}_1 \in_{\delta,\ell} K$, there exists $\hat{q}_1' = \hat{q}_1'(\ell, \hat{q}_1) \in K$ such that for most $u \in_{\delta,\ell,\hat{q}_1,\hat{q}_1'} \mathcal{U}_1^+$, we have $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are both good and coupled for all $ij \in \Lambda_{\text{sync}}$ (**Proposition 4.7.6**).

8. **Choice of parameters 1: $\boldsymbol{\ell, \hat{q}_1, \hat{q}_1', \hat{q}, \hat{q}'}$**: From now on, ℓ and $\hat{q}_1$ are chosen to satisfy **Claim 4.10.3**, i.e. there exist $\hat{q}_1' = \hat{q}_1'(\ell, \hat{q}_1) \in K$ such that for most $u \in_{\delta,\ell,\hat{q}_1,\hat{q}_1'} \mathcal{U}_1^+$, we have $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are both good and coupled for all $ij \in \Lambda_{\text{sync}}$. Let $\hat{q} := \hat{T}^{-\ell}\hat{q}_1, \hat{q}' := \hat{T}^{-\ell}\hat{q}_1'$.

With these choices, for all $ij \in \Lambda_{\text{sync}}$, let

$$\tau(u) := \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell), \qquad \tau'(u) := \tilde{\tau}_\varepsilon(\hat{q}_1', u, \ell), \qquad \mathbf{v}(u) := \mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)) F_{\hat{q}}(\hat{q}'),$$

$$t_{ij}(u) := t_{ij}(\hat{q}_1, u, \ell), \qquad t_{ij}'(u) := t_{ij}(\hat{q}_1', u, \ell).$$

9. **Claim 4.10.4.** For $\ell \gg_\delta 0$, most $\hat{q}_1 \in_{\delta,\ell} K$, most $u \in_{\delta,\ell,\hat{q}_1,\hat{q}_1'} \mathcal{U}_1^+$, we have (by **Proposition 4.7.3, 4.8.7** and **Theorem 4.6.5**)

$$\tau(u) \approx_\lambda \ell, \qquad d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(\hat{T}^{\tau(u)} u \hat{q}_1)\right) \leq_\delta e^{-\alpha' \ell},$$

$$d_{G/H}(\hat{T}^{\tau(u)} u \hat{q}_1, \hat{T}^{\tau(u)} u \hat{q}_1') \approx_\delta \|\mathbf{v}(u)\| \approx_\delta \varepsilon,$$

where $\alpha' > 0$ and the bilipschitz constant in $\approx_\lambda$ depend only on the Lyapunov spectrum.

10. **Claim 4.10.5.** For small enough $\delta > 0$ (depending on θ_1), for $\ell \gg_{\delta,\eta} 0$, $u \in_{\delta,\ell,\hat{q}_1,\hat{q}_1',\eta} \mathcal{U}_1^+$, there exists $ij \in \Lambda_{\text{sync}}$ such that (by **Proposition 4.9.1**)

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}_{[ij],\text{bdd}}(\hat{T}^{\tau(u)} u \hat{q}_1)\right) < 4\eta.$$

11. **Choice of parameters 2: $\boldsymbol{\delta, \ell, \hat{q}_1, \hat{q}'_1, \hat{q}, \hat{q}', ij}$:** Choose $\delta > 0$ to be a small enough number so that **Claim 4.10.1.5** holds. Choose $\ell, \hat{q}_1, u$ such that **Claim 4.10.3**, **Claim 4.10.4** and **Claim 4.10.5** hold. Then choose $\hat{q}'_1, \hat{q}, \hat{q}'$ as in **Choice of parameters 1**. Fix $ij \in \Lambda_{\text{sync}}$ such that **Claim 4.10.5** holds.

12. **Claim 4.10.6.** There exists some $C(\delta) > 0$ such that for $\ell \gg_\delta 0$, most $\hat{q}_1 \in_{\delta,\ell} K$, most $u \in_{\delta,\ell,\hat{q}_1,\hat{q}'_1} \mathcal{U}_1^+$, and all $ij \in \Lambda_{\text{sync}}$, we have

$$|\tau(u) - \tau'(u)| \leq C(\delta), \qquad \text{and} \qquad |t_{ij}(u) - t'_{ij}(u)| \leq C(\delta).$$

13. **Choice of parameters 3: $\boldsymbol{u, \hat{q}_2, \hat{q}'_2, \hat{q}_{3,ij}, \hat{q}'_{3,ij}, \tau, \tau_{ij}}$:** Recall that by **Choice of parameters 1**, we have

$$\hat{T}^{\tau(u)} u\hat{q}_1 \in K, \qquad \hat{T}^{\tau'(u)} u\hat{q}'_1 \in K, \qquad \hat{T}^{t_{ij}(u)} u\hat{q}_1 \in K, \qquad \hat{T}^{t'_{ij}(u)} u\hat{q}_1 \in K.$$

Thus by **Claim 4.10.6** and property (a) in the definition of K (**Step 4**), there exist $C(\delta) > 0$ and $s, s' \in [-C(\delta), C(\delta)]$ such that

$$\hat{q}_2 := \hat{T}^{\tau} u\hat{q}_1 \in K_0, \quad \hat{q}'_2 := \hat{T}^{\tau} u\hat{q}'_1 \in K_0, \quad \hat{q}_{3,ij} := \hat{T}^{\tau_{ij}} u\hat{q}_1 \in K_0, \quad \hat{q}'_{3,ij} := \hat{T}^{\tau_{ij}} u\hat{q}'_1 \in K_0,$$

where $\tau := s + \tau(u), \tau_{ij} := s' + t_{ij}(u)$.

14. Let $R := R(\hat{q}_{3,ij}, \hat{q}_2) = R(\hat{q}'_{3,ij}, \hat{q}'_2)$, since $\hat{q}, \hat{q}'$ have the same combinatorial future. Let

$$B : \mathbf{E}_{[ij],\text{bdd}}(\hat{q}_{3,ij}) \to \mathbf{E}_{[ij],\text{bdd}}(\hat{q}_2), \qquad B' : \mathbf{E}_{[ij],\text{bdd}}(\hat{q}'_{3,ij}) \to \mathbf{E}_{[ij],\text{bdd}}(\hat{q}'_2)$$

be the restrictions of R. By **Proposition 4.9.2**, there exists a constant $C'(\delta) > 0$ such that

$$\max(\|B\|, \|B^{-1}\|) \leq C'(\delta), \qquad \text{and} \qquad \max(\|B'\|, \|(B')^{-1}\|) \leq C'(\delta).$$

15. By **Lemma 4.5.10** and **Claim 4.10.4**,

$$\|P^-(\hat{q}_2, \hat{q}_2') - \mathrm{Ad}_{T^{\tau}_{\hat{q}_2} g}\| \leq_{\delta,g} e^{-\alpha'\ell}, \|P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}') - \mathrm{Ad}_{T^{\tau_{ij}}_{\hat{q}_3} g}\| \leq_{\delta,g} e^{-\alpha'\ell}.$$

Here $\alpha' > 0$ depends only on the Lyapunov spectrum. Note that $P^-(\hat{q}_2, \hat{q}_2')$ is well-defined since $\pi_{\mathcal{S}^{\mathbb{Z}}\times G/L}(\hat{q}_2') \in \hat{\mathcal{W}}^-_{G/L}[\pi_{\mathcal{S}^{\mathbb{Z}}\times G/L}(\hat{q}_2')]$ (by Proposition 4.4.7), even though $\hat{q}_2$ and $\hat{q}_2'$ may not be stably related (i.e. $\hat{q}_2'$ may not be in $\hat{\mathcal{W}}^-_{G/H}[\hat{q}_2]$). See **Remark 4.5.8**. $g \in \exp(W^{<0}_{\mathfrak{g}}(\omega)) \subset G$ is a choice of an element in $\exp(W^{<0}_{\mathfrak{g}}(\omega))$ such that $gq_1 L = q_1' L$.

16. **Claim 4.10.7**. For all $\mathbf{v} \in \mathbf{E}_{[ij],\mathrm{bdd}}(\hat{q}_{3,ij})$, $\|B'P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')\mathbf{v} - P^-(\hat{q}_2, \hat{q}_2')B\mathbf{v}\| \leq_\delta e^{-(\alpha/2)\ell}\|\mathbf{v}\|$.

17. **Claim 4.10.8**. There exists $c(\delta, \ell)$ with $c(\delta, \ell) \to 0$ as $\ell \to \infty$ such that

$$d(f_{ij}(\hat{q}_2), f_{ij}(\hat{q}_2')) \leq c(\delta, \ell).$$

18. Take $\eta \to 0$ in **Claim 4.10.5** (this necessarily implies $\ell \to 0$), let $\tilde{q}_2$, $\tilde{q}_2' \in K_0$ be the limit of $\hat{q}_2, \hat{q}_2'$, then by **Claim 4.10.5** and **Claim 4.10.8**,

$$\tilde{q}_2' \in \mathcal{E}_{ij}[\tilde{q}_2], \qquad d_{G/H}(\tilde{q}_2, \tilde{q}_2') \approx_\delta \varepsilon, \qquad f_{ij}(\tilde{q}_2) \propto f_{ij}(\tilde{q}_2').$$

19. Going back to **Step 1**: in summary, we have shown that (recall that $\hat{\nu}(K_*) > 1 - \delta$): for arbitrary $K_{00} \subset \hat{\Omega}$ with $\hat{\nu}(K_{00}) > 1 - 2\delta$, there exists $\hat{x} \in K_0 \subset K_{00} \cap K_*$ and $\hat{y} \in \mathcal{E}_{ij}[\hat{x}] \cap K_*$ such that

$$d_{G/H}(\hat{x}, \hat{y}) \approx_\delta \varepsilon, \qquad f_{ij}(\hat{x}) \propto f_{ij}(\hat{y}).$$

Thus there exists $E \subset K_*$ with $\hat{\nu}(E) > \delta$ such that for every $\hat{x} \in E$, there exists

$\hat{y} \in \mathcal{E}_{ij}[\hat{x}] \cap K_*$ such that

$$d_{G/H}(\hat{x}, \hat{y}) \approx_\delta \varepsilon, \qquad f_{ij}(\hat{x}) \propto f_{ij}(\hat{y}).$$

□

We now continue with the precise statements and proofs of the claims in the outline. For each claim, we assume all the choices of parameters (like $\delta, \varepsilon, \eta$) and sets (like K_*, K_0, K) in the steps preceding the claim in the outline. We will also reuse the notations K', Q, ℓ_0 in each claim with the understanding that unless otherwise stated, these letters mean different objects in different claims. We also reuse the constants $c(\delta), c'(\delta), c''(\delta)$ and they always satisfy $c(\delta), c'(\delta), c''(\delta) \to 0$ as $\delta \to 0$.

Claim 4.10.1. There exist

- a compact set $K' = K'(K_{00}, \delta, \varepsilon) \subset K$ of measure at least $1 - c(\delta)$,
- a subset $Q = Q(\hat{q}_1, K_{00}, \delta, \varepsilon, \eta) \subset \mathcal{U}_1^+$ with $Q\hat{q}_1$ of measure at least $1 - c'(\delta)$ for each $\hat{q}_1 \in K'$, and
- a real number $\ell_0 = \ell_0(K_{00}, \delta, \varepsilon, \eta) > 0$

such that if we let

$$E(\hat{q}_1, u) := \{\ell \in \mathbb{R}^+ : \hat{q}_2(Y_{ij}(\hat{q}_1, u, \ell)), \quad \hat{q}_3(Y_{ij}(\hat{q}_1, u, \ell)) \in K \quad \text{for all } ij \in \Lambda_{\text{sync}}\},$$

then for all $\hat{q}_1 \in K', u \in Q$ and $\ell > \ell_0$,

$$u\hat{q}_1 \in K, \qquad \text{and} \qquad |E(\hat{q}_1, u) \cap [0, \ell]| > (1 - c''(\delta))\ell.$$

Proof. The idea is that since K has almost full measure, by the pointwise ergodic theorem, for a large set of points $\hat{q}_1 \in \hat{\Omega}$, $\hat{T}^t \hat{q}_1$ enters K for almost full density of $t > 0$. Since $\{\mathcal{U}_1^+ \hat{x}\}_{\hat{x} \in \hat{\Omega}}$

form a partition of $\hat{\Omega}$, for a large set of $\hat{q}_1 \in \hat{\Omega}$ there is a large set of $u \in \mathcal{U}_1^+$, such that $\hat{T}^t u\hat{q}_1$ also enters K for almost full density of $t > 0$. Since $\ell \mapsto \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)$ and $\ell \mapsto t_{ij}(\hat{q}_1, u, \ell)$ are bilipschitz functions, using Lemma 4.9.14, we have that for almost full density of ℓ, $t_{ij}(\hat{q}_1, u, \ell)$ satisfies the first sentence, i.e. $\hat{q}_3 \in K$, and $\tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)$ satisfies the second sentence, i.e. $\hat{q}_2 \in K$, as desired.

Recall from **Step 4** of the main argument that $K \subset \hat{\Omega}$ has measure at least $1 - c_K(\delta)$ for some $c_K(\delta) \to 0$ as $\delta \to 0$. By the pointwise ergodic theorem, there exists compact $K_1(\delta) \subset \hat{\Omega}$ with measure at least $1 - \delta$, and $\ell_1 = \ell_1(\delta) > 0$ such that for all $\hat{q}_1 \in K_1$ and $L \geq \ell_1$,

$$\mathrm{Leb}(\{t \in [0, L] : \hat{T}^t \hat{q}_1 \in K\}) \geq (1 - 2c_K(\delta))L.$$

By Lemma 4.9.13, we know that the set

$$K_2 := \{\hat{x} \in \hat{\Omega} \mid \mu^{\mathbb{N}}(\mathcal{U}_1^+ \hat{x} \cap K_1 \cap K) > 1 - \sqrt{c_K(\delta) + \delta}\} \subset \hat{\Omega}$$

has measure at least $1 - \sqrt{c_K(\delta) + \delta}$. Let $K_3 := K_1 \cap K_2$, thus has measure at least $1 - 2c_K(\delta) - \sqrt{c_K(\delta) + \delta}$.

Suppose $\hat{q}_1 \in K_3$ and $u\hat{q}_1 \in K_1$.

Let $E_1 := \{t > 0 \mid \hat{T}^t u\hat{q}_1 \in K^c\}$. Since $u\hat{q}_1 \in K_1$, for $\ell > \ell_1$, the density of E_1 is at most $2c_K(\delta)$.

Let $E_2 := \{\ell > 0 \mid \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell) \in E_1\}$. Since $\ell \mapsto t(Y_{ij})$ is κ_τ-bilipschitz by Proposition 4.8.9, by Lemma 4.9.14, for $\ell > \kappa_\tau \ell_1$, the density of E_2 is at most $2\kappa_\tau^2 c_K(\delta)$.

Let $E_3 := \{t > 0 \mid \hat{T}^t \hat{q}_1 \in K^c\}$. Since $\hat{q}_1 \in K_3 \subset K_1$, for all $\ell > \ell_1$, the density of E_3 is at most $2c_K(\delta)$.

Let $E_{ij} := \{\ell > 0 \mid t_{ij}(\hat{q}_1, u, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)) \in E_3\}$ for each $ij \in \Lambda_{\mathrm{sync}}$. Since

$$\ell \mapsto t_{ij}(\hat{q}_1, u, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell))$$

is $\kappa_\tau \kappa^2$-bilipschitz by Proposition 4.8.9 and Proposition 4.5.13(c), by Lemma 4.9.14, for $\ell >$

$\kappa_\tau \kappa^2 \ell_1$, the density of E_{ij} is at most $2\kappa_\tau^2 \kappa^4 c_K(\delta)$.

Since $E(\hat{q}_1, u)$ is the complement of $E_2 \cup \bigcup_{ij\in\Lambda_{\text{sync}}} E_{ij}$, for $\ell > \kappa_\tau \kappa^2 \ell_1$, the density of $E(\hat{q}_1, u)$ is at least $1 - 2\kappa_\tau^2 c_K(\delta) - 2\kappa_\tau^2 \kappa^4 |\Lambda_{\text{sync}}| c_K(\delta)$.

Thus using the notations in the statement, we can take

- $K' := K_3$ with $c(\delta) := 2c_K(\delta) + \sqrt{c_K(\delta) + \delta}$,
- $Q(\hat{q}_1) := \{u \in \mathcal{U}_1^+ \mid \mathcal{U}_1^+ \hat{q}_1 \cap K_1 \cap K\}$ for each $\hat{q}_1 \in K'$, with $c'(\delta) := \sqrt{c_K(\delta) + \delta}$,
- $\ell_0 := \kappa_\tau \kappa^2 \ell_1$
- $c''(\delta) := 2\kappa_\tau^2 c_K(\delta) + 2\kappa_\tau^2 \kappa^4 |\Lambda_{\text{sync}}| c_K(\delta)$.

□

Claim 4.10.2. There exist

- a real number $\ell_0 = \ell_0(K_{00}, \delta, \varepsilon, \eta) > 0$,
- a set $\mathcal{D} = \mathcal{D}(K_{00}, \delta, \varepsilon, \eta) \subset \mathbb{R}^+$ such that $|\mathcal{D} \cap [0, \ell]| > (1 - c(\delta))\ell$ for $\ell > \ell_0$,
- a compact set $K' = K'(\ell, K_{00}, \delta, \varepsilon, \eta) \subset K$ of measure at least $1 - c'(\delta)$ for each $\ell \in \mathcal{D}$,
- a subset $Q = Q(\ell, \hat{q}_1) \subset \mathcal{U}_1^+$ with $Q\hat{q}_1$ of measure at least $1 - c''(\delta)$ for each $\ell \in \mathcal{D}$ and each $\hat{q}_1 \in K'$

such that for all $\hat{q}_1 \in K', u \in Q, \ell \in \mathcal{D}$, we have $u\hat{q}_1 \in K$ and $\ell \in E(\hat{q}_1, u)$, i.e.

$$\hat{q}_2(Y_{ij}(\hat{q}_1, u, \ell)), \quad \hat{q}_3(Y_{ij}(\hat{q}_1, u, \ell)) \in K \qquad \text{for all } ij \in \Lambda_{\text{sync}}.$$

Proof. This is a direct application of Fubini's theorem to the product $\{(\hat{x}, u\hat{x}) \mid \hat{x} \in \hat{\Omega}, u\hat{x} \in \mathcal{U}_1^+ \hat{x}\} \times \mathbb{R}^+$ using **Claim 4.10.1**. We just need to take care since we are using the density on $\mathbb{R}^+$ here.

Let $\ell_0 > 0$ be as in **Claim 4.10.1**. Let $L_1 < L_2 < L_3 < \cdots$ be an arithmetic progression with $L_1 = \ell_0 + 1$ and common difference 1. Now for each i, applying Fubini's theorem to the product $\{(\hat{x}, u\hat{x}) \mid \hat{x} \in \hat{\Omega}, u\hat{x} \in \mathcal{U}_1^+\hat{x}\} \times [0, L_i]$ using **Claim 4.10.1**, we get a set $\mathcal{D}_i \subset \mathbb{R}^+$ such that the set of $\ell \in \mathcal{D}_i$ satisfying the statement conditions has proportion at least $1 - c_1(\delta)$ for some $c_1(\delta) \to 0$ as $\delta \to 0$, depending only on the constants $c(\delta), c'(\delta), c''(\delta)$ in **Claim 4.10.1** (in particular does not depend on i). Now take $\mathcal{D} := \bigcup_i \mathcal{D}_i$, we have that $\mathcal{D}$ has density at least $1 - 2c_1(\delta)$ for $\ell > \ell_1(\delta)$, where $\ell_1(\delta)$ is some large enough constant depending only on $c_1(\delta)$. □

Claim 4.10.3. Let $\mathcal{D} \subset \mathbb{R}^+$ as in **Claim 4.10.2**. Then there exists

- a compact set $K' = K'(\ell, K_{00}, \delta, \varepsilon, \eta) \subset K$ of measure at least $1 - c(\delta)$,
- a point $\hat{q}_1' = \hat{q}_1'(\ell, \hat{q}_1) \in K$ for each $\ell \in \mathcal{D}$ and $\hat{q}_1 \in K'$,
- a subset $Q = Q(\ell, \hat{q}_1, \hat{q}_1', \delta) \subset \mathcal{U}_1^+$ with $Q\hat{q}_1$ of measure at least $1 - c'(\delta)$ for each $\ell \in \mathcal{D}$ and each $\hat{q}_1 \in K'$,

such that for all $\hat{q}_1 \in K'$, $u \in Q$, $\ell \in \mathcal{D}$ and $ij \in \Lambda_{\text{sync}}$, $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are both good and coupled. Also $u\hat{q}_1, u\hat{q}_1' \in K$.

Proof. The main idea is to apply Proposition 4.7.6. We just need to shrink K' so that $\hat{q}, \hat{q}', \hat{q}_1, \hat{q}_1' \in K$, and then use **Claim 4.10.2** (shrink K' again) to get $\hat{q}_2, \hat{q}_2', \hat{q}_3, \hat{q}_3', u\hat{q}_1, u\hat{q}_1' \in K$.

Let $K_1 \subset K$ and $Q_1 \subset \mathcal{U}_1^+$ be the corresponding sets K', Q in **Claim 4.10.2** respectively. In particular, for all $\hat{q}_1 \in K_1$, $u \in Q_1$ and $\ell \in \mathcal{D}$, we have

$$\hat{q}_2(Y_{ij}(\hat{q}_1, u, \ell)), \quad \hat{q}_3(Y_{ij}(\hat{q}_1, u, \ell)) \in K \qquad \text{for all } ij \in \Lambda_{\text{sync}}.$$

For any $\hat{q}_1 \in K_1$, $u \in Q_1$ and $\ell \in \mathcal{D}$, we apply Lemma 4.7.5 to the linear map $A := \pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)) : \mathcal{L}^-(\hat{q}) \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{q}_2)$ (here $\hat{q} := \hat{T}^{-\ell}\hat{q}_1$, $\hat{q}_2 := \hat{T}^t u\hat{q}_1$ and $\pi_+ : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ denote the orthogonal projection with respect to the dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$) and let $\mathcal{M}_u(\hat{q}) \subset \mathcal{L}^-(\hat{q})$ be the resulting proper subspace W'. Thus for any $\mathbf{v} \in \mathcal{L}^-(\hat{q})$ with $\|\mathbf{v}\|_{\hat{q}} = 1$ and $d(\mathbf{v}, \mathcal{M}_u) > \rho$, we have

$$\|\pi_+\mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell))\mathbf{v}\| \geq c_1(\rho)\|\pi_+\mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell))\| \tag{4.10.1}$$

for some $c_1(\rho) > 0$ that depends only on ρ.

Now for each $\ell \in \mathcal{D}$, we apply Proposition 4.7.6 with $K' := K \cap \hat{T}^{-\ell} K_1$ and the map $u \mapsto \mathcal{M}_u(\hat{q}) \subset \mathcal{L}^-(\hat{q})$. Let K_2 be the resulting subset $K \subset K'$, which has measure at least $1 - c_2(\delta)$ for some constant $c_2(\delta) \to 0$ as $\delta \to 0$. Then by Proposition 4.7.6, for any $\hat{q} \in K_2$, there exists $\hat{q}' \in K \cap \hat{T}^{-\ell} K_1$ with $\hat{q}' \in \hat{\mathcal{W}}^-_{G/H}[\hat{q}]$, $F_{\hat{q}}(\hat{q}') \in \mathcal{L}^-(\hat{q})$, such that

$$\rho(\delta) \leq d_{G/H}(q, q') \leq 1/100, \qquad \rho(\delta) \leq \|F_{\hat{q}}(\hat{q}')\|_{\hat{q}} \leq C(\delta),$$

and

$$d_{\hat{q}}(F_{\hat{q}}(\hat{q}'), \mathcal{M}_u(\hat{q})) > \rho(\delta) \qquad \text{for at least } (1 - \varepsilon_1(\delta))\text{-fraction of } u \in \mathcal{U}_1^+.$$

Here $\rho(\delta), C(\delta)$ are constants with $0 < \rho(\delta) < C(\delta) < \infty$, and $\varepsilon_1(\delta)$ is a constant with $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$. This together with (4.10.1) for $\rho = \rho(\delta)$ and the definition of $\tilde{\tau}_\varepsilon$ imply that

$$\|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)) F_{\hat{q}}(\hat{q}')\| \approx_\delta \|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell))\| = \varepsilon.$$

Therefore for all $\ell \in \mathcal{D}$, for all $\hat{q}_1 \in \hat{T}^\ell K_2 \subset \hat{T}^\ell K \cap K_1$, there exists $\hat{q}_1' := \hat{T}^\ell \hat{q}' \in \hat{T}^\ell K \cap K_1$ such that $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are coupled for most $u \in_{\ell, \hat{q}_1, \hat{q}_1'} \mathcal{U}_1^+$ for all $ij \in \Lambda_{\text{sync}}$. Since $\hat{q}_1, \hat{q}_1' \in \hat{T}^\ell K$, we have $\hat{q} := \hat{T}^{-\ell} \hat{q}_1, \hat{q}' := \hat{T}^{-\ell} \hat{q}_1' \in K$. Since $\hat{q}_1, \hat{q}_1' \in K_1 \subset K$, by **Claim 4.10.2**, we have $\hat{q}_1, u\hat{q}_1, \hat{q}_2, \hat{q}_3, \hat{q}_1', u\hat{q}_1', \hat{q}_2', \hat{q}_3' \in K$. Therefore both $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are good. Therefore we can take $K' := \hat{T}^\ell K_2$ in the statement, which has measure at least $1 - c_2(\delta)$ with $c_2(\delta) \to 0$ as $\delta \to 0$. □

Claim 4.10.4. Let $\mathcal{D} \subset \mathbb{R}^+$ as in **Claim 4.10.2**. Let $K' \subset K$ and Q as in **Claim 4.10.3**. There exist

- a real number $\ell_0 = \ell_0(\delta, \varepsilon) > 0$,
- a constant $C = C(\delta) > 1$,
- constants $\alpha > 1$ and $\alpha' > 0$ depending only on the Lyapunov spectrum,

such that for all $\hat{q}_1 \in K'$, $u \in Q$ and $\ell \in \mathcal{D}$ with $\ell > \ell_0$, let $\hat{q}_1' = \hat{q}_1'(\ell, \hat{q})$ as in **Claim 4.10.3**, then we have

(a)

$$\alpha^{-1}\ell \leq \tau(u) \leq \alpha\ell,$$

(b)

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(\hat{T}^{\tau(u)}u\hat{q}_1)\right) \leq C(\delta)e^{-\alpha'\ell},$$

(c)

$$C(\delta)^{-1}\varepsilon \leq \|\pi_+(\mathbf{v}(u))\| \leq C(\delta)\varepsilon \qquad \text{and} \qquad C(\delta)^{-1}\varepsilon \leq \|\mathbf{v}(u)\| \leq C(\delta)\varepsilon,$$

(d)

$$C(\delta)^{-1}\varepsilon \leq d_{G/H}(\hat{T}^{\tau(u)}u\hat{q}_1, \hat{T}^{\tau(u)}u\hat{q}_1') \leq C(\delta)\varepsilon.$$

Here we recall that $\mathbf{v}(u) := \mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell))F_{\hat{q}}(\hat{q}')$ and $\tau(u) := \tilde{\tau}_\varepsilon(\hat{q}_1, u, \ell)$.

Proof. Let $\ell \in \mathcal{D}$, $\hat{q}_1 \in K$ and $u \in Q$, where $\mathcal{D}, K'$ and Q are as in **Claim 4.10.3**.

(a) It suffices to show that $\alpha_1\ell \leq \tau(u) \leq \alpha_2\ell$ for some $\alpha_2 > \alpha_1 > 0$ depending only on the Lyapunov spectrum. In fact, for large enough $\ell_0 := \ell_0(\delta, \varepsilon) > 0$, the lower bound follows from Proposition 4.5.13(d) and Remark 4.5.14 with $\alpha_1 := (2\kappa_V\kappa)^{-1}$, the upper bound follows from Proposition 4.7.3 (recall from **Step 4c** that elements in $K' \subset K$ satisfy Proposition 4.7.3) with $\alpha_2 := \kappa_V\kappa$.

(b) Recall from **Step 4d** that elements in $K' \subset K$ satisfy Proposition 4.8.7. Therefore we have

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(\hat{T}^{\tau(u)}u\hat{q}_1)\right) \leq C(\delta)e^{-\alpha_3\tau(u)},$$

where α_3 depends only on the Lyapunov spectrum. Now apply the lower bound in part (a) to this inequality to get part (b).

(c) For $\ell \in \mathcal{D}$, $\hat{q}_1 \in K'$ and $u \in Q$, by **Claim 4.10.3**, $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are coupled, thus

$$\|\pi_+(\mathbf{v}(u))\| \approx_\delta \varepsilon.$$

Here we recall that for $\hat{x} := \hat{T}^{\tau(u)} u\hat{q}_1$, $\pi_+ : (\mathfrak{l}/\mathfrak{h})_{\hat{x}} \to W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{x})$ denote the orthogonal projection with respect to the dynamical inner product $\langle \cdot, \cdot \rangle_{\hat{x}}$ defined in subsection 4.5.9. By part (b), $\mathbf{v}(u)$ is exponentially close to $\mathbf{E}(\hat{T}^{\tau(u)} u\hat{q}_1) \subset W^+_{\mathfrak{l}/\mathfrak{h}}(\hat{T}^{\tau(u)} u\hat{q}_1)$, therefore there exists $\ell_0 = \ell_0(\delta)$ such that for all $\ell > \ell_0$, we have

$$\|\mathbf{v}(u)\| \approx_\delta \varepsilon.$$

(d) Recall from **Step 4a** that elements in $K' \subset K$ satisfy the Factorization theorem (4.6.5(d)). Take $\beta = \alpha_2$ from part (a) in Theorem 4.6.5(d). Then $d_{G/H}(\hat{T}^{\tau(u)} u\hat{q}_1, \hat{T}^{\tau(u)} u\hat{q}_1')$ is exponentially close to $\|\mathbf{v}(u)\|$. Thus using part (c), there exists $\ell_0 = \ell_0(\delta)$ such that for all $\ell > \ell_0$, we have

$$\|d_{G/H}(\hat{T}^{\tau(u)} u\hat{q}_1, \hat{T}^{\tau(u)} u\hat{q}_1')\| \approx_\delta \varepsilon.$$

□

Recall that $\mathfrak{B}_0[\hat{x}] := J[\hat{x}] \cap \hat{\mathcal{W}}^+_{G/H}[\hat{x}]$ is the local unstable set defined by the Markov partition J constructed in Proposition 4.5.12. Recall the parameter θ_1 from Proposition 4.9.1.

Claim 4.10.5. There exist

- a real number $\delta_0 = \delta_0(\theta_1) > 0$,
- a real number $\ell_0 = \ell_0(K_{00}, \delta, \varepsilon, \eta) > 0$ for each $0 < \delta < \delta_0$,
- a compact subset $K' = K'(\ell, K_{00}, \delta, \varepsilon, \eta) \subset K$ of measure at least $1 - c(\delta)$ for each $0 < \delta < \delta_0$ and $\ell > \ell_0$,
- a subset $Q = Q(\ell, \hat{q}_1, \hat{q}_1'(\ell, \hat{q}_1), K_{00}, \delta, \varepsilon, \eta) \subset \mathcal{U}_1^+$ such that $Q\hat{q}_1 \subset \mathfrak{B}_0[\hat{q}_1]$ with $Q\hat{q}_1$ of measure at least $(\theta_1/4)$-fraction of $\mathfrak{B}_0[\hat{q}_1]$ for each $0 < \delta < \delta_0$, $\ell > \ell_0$ and $\hat{q}_1 \in K'$,

such that for all $\delta < \delta_0$, $\hat{q}_1 \in K'$, $u \in Q$ and $\ell > \ell_0$, there exists $ij \in \Lambda_{\text{sync}}$ such that

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}_{[ij],\text{bdd}}(\hat{T}^{\tau(u)} u \hat{q}_1)\right) < \eta.$$

Proof sketch. The proof is identical to [ELa, Claim 10.11]. See **Choice of parameters #3** in [ELa, Sect. 10] for the choice of $\delta_0(\theta_1)$ (note that (10.20) in [ELa] is satisfied for all sufficiently small δ). The main idea is to apply Proposition 4.9.1. □

Claim 4.10.6. There exist

- a real number $\ell_0 = \ell_0(\delta, \varepsilon) > 0$,
- a compact subset $K' = K'(\ell, K_{00}, \delta, \varepsilon) \subset K$ of measure at least $1 - c(\delta)$ for each $\ell > \ell_0$,
- a subset $Q = Q(\ell, \hat{q}_1, \hat{q}_1'(\ell, \hat{q}_1), K_{00}, \delta, \varepsilon) \subset \mathcal{U}_1^+$ with $Q\hat{q}_1$ of measure at least $1 - c(\delta)$ for each $\ell > \ell_0$ and each $\hat{q}_1 \in K'$,
- a constant $C(\delta) > 0$,

such that for all $ij \in \Lambda_{\text{sync}}$, $\hat{q}_1 \in K'$, $u \in Q$, $\ell \in \mathcal{D}$ with $\ell > \ell_0$, let $\hat{q}_1' = \hat{q}_1'(\ell, \hat{q})$ as in **Claim 4.10.3**, then we have

(a)

$$|\tau(u) - \tau'(u)| \leq C(\delta),$$

(b)

$$|t_{ij}(u) - t_{ij}'(u)| \leq C(\delta).$$

Proof. In this proof, we write $\tau := \tau(u)$, $\tau' := \tau'(u)$, $\hat{q}_2 := \hat{T}^\tau u \hat{q}_1$ and $\hat{q}_2' := \hat{T}^\tau u \hat{q}_1$ (note that $\hat{q}_2'$ is not necessarily the same as $\hat{q}_2(Y_{ij}(\hat{q}_1', u, \ell))$ since we are using τ instead of τ').

(a) We first show that $|\tau(u) - \tau'(u)| \leq C(\delta)$. The idea is that by the choices of τ and τ', $d_{G/H}(\hat{q}_2, \hat{q}_2')$ and $d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2')$ are both $\approx_\delta \varepsilon$. Now the exponential rate of expansion (or contraction) by $\hat{T}^{\tau'-\tau}$ should be bounded by constants that depend only on the Lyapunov spectrum, therefore $|\tau' - \tau| = O_\delta(1)$. However to make this precise requires more work. For instance $d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2') \approx_\delta \varepsilon$ may not necessarily hold by the choices so far (we only have $F_{\hat{q}}(\hat{q}')$ avoids a proper subspace with strictly lower order growth using Proposition 4.7.6, but the same may not be true for $F_{\hat{q}'}(\hat{q})$). Also we need to first factorize $d_{G/H}(\hat{q}_2, \hat{q}_2')$ and $d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2')$ to get the precise bounds on exponential growth rates.

We first consider the case when $\tau' \geq \tau$. Note that since $\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, \tau'(u)) = \hat{T}^{\tau'-\tau} \circ \pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, \tau(u))$, by property (d) of the dynamical norm (Proposition 4.5.13(d)), we have

$$e^{\kappa^{-1}(\tau'-\tau)}\|\pi_+\mathbf{v}(u)\| \leq \|\pi_+\mathcal{A}(\hat{q}_1, u, \ell, \tau'(u))F_{\hat{q}}(\hat{q}')\| \leq e^{\kappa(\tau'-\tau)}\|\pi_+\mathbf{v}(u)\|.$$

By **Claim 4.10.4**(c), we know that $\|\pi_+\mathbf{v}(u)\| \approx_\delta \varepsilon$. On the other hand, by the Factorization theorem (Theorem 4.6.5),

$$\|\pi_+\mathcal{A}(\hat{q}_1, u, \ell, \tau'(u))F_{\hat{q}}(\hat{q}')\| = d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2') + O_\delta(e^{-\alpha\ell}).$$

Therefore for large enough $\ell \gg_{\delta,\varepsilon} 0$, we have

$$e^{\kappa^{-1}(\tau'-\tau)}\varepsilon \leq_\delta d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2').$$

By the choice of $\tau'(u)$, and the Factorization theorem (Theorem 4.6.5) for $\hat{q}'$, we also have

$$\begin{aligned}\varepsilon = \|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))\| &\geq \|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))F_{\hat{q}'}(\hat{q})\| \\ &= d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2') + O_\delta(e^{-\alpha\ell}).\end{aligned}$$

Thus for large enough $\ell \gg_{\delta,\varepsilon} 0$, we have $2\varepsilon \geq d_{G/H}(\hat{T}^{\tau'-\tau}\hat{q}_2, \hat{T}^{\tau'-\tau}\hat{q}_2') \geq_\delta e^{\kappa^{-1}(\tau'-\tau)}\varepsilon$, which implies $\tau' - \tau = O_\delta(1)$.

Now we consider the case when $\tau' < \tau$. Most of the above goes through by swapping the role of $\hat{q}$ and $\hat{q}'$. The main issue is that we don't necessarily have

$$\|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))F_{\hat{q}'}(\hat{q})\| \approx_\delta \|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))\|.$$

The remedy is to apply Proposition 4.7.6 to $\hat{q}'$ and obtain another point $\hat{q}'' \in \hat{\mathcal{W}}_{G/H}[\hat{q}']$. More precisely, we let $K_1 = K_1(\ell) \subset K$ be the compact set K' in **Claim 4.10.3** for K, $Q \subset \mathcal{U}_1^+$ be the subset in **Claim 4.10.3**, and let $\hat{q}_1'' := \hat{q}_1'(\ell, \hat{q}_1')$ be the corresponding point of $\hat{q}_1' \in K$. We then apply **Claim 4.10.3** with K_1 in place of K and let $K_2 \subset K_1$ be the resulting compact set K'. Then for $\hat{q}_1 \in K_2$, we have $\hat{q}_1' \in K_1$ and $\hat{q}_1'' \in K$. Moreover, for each $\ell \in \mathcal{D}$, and for a large set of $u \subset_\delta \mathcal{U}_1^+$, we have that $Y_{ij}(\hat{q}_1, u, \ell), Y_{ij}(\hat{q}_1', u, \ell)$ and $Y_{ij}(\hat{q}_1'', u, \ell)$ are good, $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are coupled, also $Y_{ij}(\hat{q}_1', u, \ell)$ and $Y_{ij}(\hat{q}_1'', u, \ell)$ are coupled. For the rest of the proof, we let $\ell \in \mathcal{D}$, $\hat{q}_1 \in K_2$ and $u \in Q$.

Then we have

$$\|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))F_{\hat{q}'}(\hat{q}'')\| \approx_\delta \|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))\| = \varepsilon.$$

Let $\mathbf{v}'(u) := \mathcal{A}(\hat{q}_1', u, \ell, \tau'(u))F_{\hat{q}'}(\hat{q}'')$. The above shows that $\|\pi_+\mathbf{v}'(u)\| \approx_\delta \varepsilon$.

Since $\pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, \tau(u)) = \hat{T}^{\tau - \tau'} \circ \pi_+ \circ \mathcal{A}(\hat{q}_1, u, \ell, \tau'(u))$, by property (d) of the dynamical norm (Proposition 4.5.13(d)), we have

$$e^{\kappa^{-1}(\tau - \tau')}\|\pi_+\mathbf{v}'(u)\| \le \|\pi_+\mathcal{A}(\hat{q}_1', u, \ell, \tau(u))F_{\hat{q}'}(\hat{q}'')\| \le e^{\kappa(\tau' - \tau)}\|\pi_+\mathbf{v}'(u)\|.$$

Let $\hat{q}_2'' := \hat{T}^{\tau(u)}u\hat{T}^\ell\hat{q}''$. By the Factorization theorem (Theorem 4.6.5),

$$d_{G/H}(\hat{q}_2', \hat{q}_2'') = \mathcal{A}(\hat{q}_1', u, \ell, \tau(u))F_{\hat{q}'}(\hat{q}'') + O_\delta(e^{-\alpha\ell}).$$

Since $\mathcal{A}(\hat{q}_1', u, \ell, \tau(u))F_{\hat{q}'}(\hat{q}'')$ gets exponentially close to $\mathbf{E} \subset W^+_{\mathfrak{l}/\mathfrak{h}}$ (Proposition 4.8.7), we have for $\ell \gg_{\delta,\varepsilon} 0$,

$$d(\hat{q}_2', \hat{q}_2'') \geq \frac{1}{2} e^{\kappa^{-1}(\tau-\tau')} \|\pi_+ \mathbf{v}'(u)\| \geq_\delta e^{\kappa^{-1}(\tau-\tau')} \varepsilon.$$

On the other hand, by the Factorization theorem, we have

$$d_{G/H}(\hat{q}_2, \hat{q}_2') = \|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, \tau(u)) F_{\hat{q}}(\hat{q}'')\| + O_\delta(e^{-\alpha\ell}).$$

Also by the choice of τ,

$$\|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, \tau(u)) F_{\hat{q}}(\hat{q}')\| \leq \|\pi_+ \mathcal{A}(\hat{q}_1, u, \ell, \tau(u))\| = \varepsilon.$$

Thus for $\ell \gg_{\delta,\varepsilon} 0$, $d_{G/H}(\hat{q}_2, \hat{q}_2') \leq 2\varepsilon$. By the same reasoning, we also have $d_{G/H}(\hat{q}_2, \hat{q}_2'') \leq 2\varepsilon$. Thus by the triangle inequality,

$$e^{\kappa^{-1}(\tau-\tau')} \varepsilon \leq_\delta d_{G/H}(\hat{q}_2, \hat{q}_2'') \leq d_{G/H}(\hat{q}_2, \hat{q}_2') + d_{G/H}(\hat{q}_2', \hat{q}_2'') \leq 4\varepsilon.$$

Therefore $\tau - \tau' = O_\delta(1)$, as desired.

(b) Recall that $t_{ij}(u)$ and $t_{ij}'(u)$ are defined (see the beginning of the proof outline of Theorem 4.10.1) to satisfy

$$\lambda_{ij}(u\hat{q}_1, \tau(u)) = \lambda_{ij}(\hat{q}_1, t_{ij}(u)), \qquad \text{and} \qquad \lambda_{ij}(u\hat{q}_1', \tau'(u)) = \lambda_{ij}(\hat{q}_1', t_{ij}'(u)).$$

The idea is to relate $|\lambda_{ij}(u\hat{q}_1, \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau'(u))| = |\lambda_{ij}(\hat{q}_1, t_{ij}(u)) - \lambda_{ij}(\hat{q}_1', t_{ij}'(u))|$ with $|\tau(u) - \tau'(u)|$ and $|t_{ij}(u) - t_{ij}'(u)|$, and then use the upper bound on the former from part (a) to give an upper bound on the latter.

Let K_1 be the compact set $K' \subset K$ in **Claim 4.10.3** and $Q \subset \mathcal{U}_1^+$ be the subset in **Claim 4.10.3**, . For $\ell \in \mathcal{D}, \hat{q}_1 \in K_1$ and $u \in Q$, by **Claim 4.10.3**, we have $\hat{q}_1, \hat{q}_1', u\hat{q}_1, u\hat{q}_1', \hat{T}^\tau u\hat{q}_1, \hat{T}^{\tau'} u\hat{q}_1' \in K$. By part (a), $|\tau - \tau'| \leq C(\delta)$, thus $\hat{T}^\tau u\hat{q}_1' \in$

$T^{[-C(\delta),C(\delta)]}K$. Note that since $\hat{q}_1' \in \hat{\mathcal{W}}^-_{G/H}(\hat{q}_1)$, by Proposition 4.4.7, we have

$$\pi_{\mathcal{S}^{\mathbb{Z}}\times G/L}(u\hat{q}_1') \in \hat{\mathcal{W}}^-_{G/L}[\pi_{\mathcal{S}^{\mathbb{Z}}\times G/L}(u\hat{q}_1)].$$

By Lemma 4.7.4, we have

$$|\lambda_{ij}(u\hat{q}_1, \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau(u))| = O_\delta(1).$$

On the other hand, by Proposition 4.5.13(c) (note that λ_{ij} is a cocycle), we have

$$|\lambda_{ij}(u\hat{q}_1', \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau'(u))| \le \kappa|\tau(u) - \tau'(u)| = O_\delta(1).$$

Therefore

$$|\lambda_{ij}(\hat{q}_1, t_{ij}(u)) - \lambda_{ij}(\hat{q}_1', t_{ij}'(u))| = |\lambda_{ij}(u\hat{q}_1, \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau'(u))| \le O_\delta(1).$$

Thus

$$|\lambda_{ij}(\hat{q}_1, t_{ij}(u)) - \lambda_{ij}(\hat{q}_1, t_{ij}'(u))| \le O_\delta(1).$$

Now apply Proposition 4.5.13(c) again, we have

$$|t_{ij}(u) - t_{ij}'(u)| \le \kappa|\lambda_{ij}(\hat{q}_1, t_{ij}(u)) - \lambda_{ij}(\hat{q}_1, t_{ij}'(u))| = O_\delta(1).$$

□

Claim 4.10.7. Assume the choices of $\delta, \ell, u, \hat{q}_1, \hat{q}_1', \hat{q}, \hat{q}', \hat{q}_2, \hat{q}_2', \hat{q}_{3,ij}, \hat{q}_{3,ij}', \tau, \tau_{ij}$ as in **Choice of parameters 2 and 3**. There exist

- a real number $\ell_0 = \ell_0(\delta, \varepsilon, \eta) > 0$,
- a constant $C = C(\delta) > 0$,
- a constant $\alpha > 0$ depending only on the Lyapunov exponents

such that for all $\mathbf{v} \in \mathbf{E}_{[ij],\mathrm{bdd}}(\hat{q}_{3,ij})$ and $\ell > \ell_0$,

$$\|R(\hat{q}'_{3,ij}, \hat{q}'_2)P^-(\hat{q}_{3,ij}, \hat{q}'_{3,ij})\mathbf{v} - P^-(\hat{q}_2, \hat{q}'_2)R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\|_{\hat{q}'_2} \leq C(\delta)e^{-\alpha\ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}}.$$

Proof. In this proof we will state the subscripts of the dynamical norm since it will play a role later in the proof.

Since $\mathbf{E}_{[ij],\mathrm{bdd}}$ is an equivariant bundle and $\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{x}) \subset (\mathfrak{l}/\mathfrak{h})_{\hat{x}}$, we have

$$\mathbf{E}_{[ij],\mathrm{bdd}} = \bigoplus_k (\mathbf{E}_{[ij],\mathrm{bdd}} \cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}})$$

where k runs through the indices of the Lyapunov exponents (note that if $\mathbf{E}_{[ij],\mathrm{bdd}} \cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}}$ is a nontrivial intersection, we must have $\lambda_k > 0$). Moreover, since the bundle $\mathbf{E}_{[ij],\mathrm{bdd}} \cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}}$ is preserved by $R(\hat{x}, \hat{x}')$ (since both are preserved by the dynamics) and $P^-(\hat{x}, \hat{x}')$ (by Lemma 4.5.9), it suffices to show the inequality for $\mathbf{v} \in (\mathbf{E}_{[ij],\mathrm{bdd}} \cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}})(\hat{q}_{3,ij})$. Furthermore, since the Lyapunov subspaces $W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}}$ are orthogonal to each other with respect to the dynamical norm, it suffices to show that for all $\mathbf{v} \in (\mathbf{E}_{[ij],\mathrm{bdd}} \cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}})(\hat{q}_{3,ij})$,

$$\|R(\hat{q}'_{3,ij}, \hat{q}'_2)P^-(\hat{q}_{3,ij}, \hat{q}'_{3,ij})\mathbf{v} - P^-(\hat{q}_2, \hat{q}'_2)R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v} + W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}'_2)\|_{\hat{q}'_2} \leq C(\delta)e^{-\alpha\ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}}, \tag{4.10.2}$$

where we understand the norm on the left as the norm induced on the quotient space $W^{\geq\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}'_2)/W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}'_2)$ by the dynamical norm on $\mathfrak{g}/\mathfrak{h}_{\hat{q}'_2}$.

Let $\mathbf{v}_0 \in V_{\hat{q}} \subset W^{<0}_{\mathfrak{g}}(\hat{q})$ be the unique vector in the normal form coordinate at $\hat{q}$ such that $q' = \exp(\mathbf{v}_0)q$. Note that by Proposition 4.7.6, we have in particular that $\|\mathbf{v}_0\| = O(\delta)$. Let $g := \exp(\mathbf{v}_0)$. Then one can show that $\|g\|_{\mathrm{Ad}(G)} = O(\delta)$ (using Lemma 4.5.15, see also **Step 4f**, to compare with the fixed norm $\|\cdot\|_0$, then use smoothness of the exponential map and Ad).

Recall that $R(\hat{q}_{3,ij}, \hat{q}_2) = R(\hat{q}'_{3,ij}, \hat{q}'_2) = (T^{\tau}_{u\hat{q}_1})_*(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*$. By equivariance of the map P^-, we know that (4.10.2) can be written as

$$\|(T^{\tau}_{u\hat{q}_1})_* P^-(\hat{q}_1,\hat{q}_1')(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v} - P^-(\hat{q}_2,\hat{q}_2')R(\hat{q}_{3,ij},\hat{q}_2)\mathbf{v} + W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}_2')\|_{\hat{q}_2'} \le C(\delta)e^{-\alpha\ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}}, \tag{4.10.3}$$

Thus it remains to show that for all $\mathbf{v}\in(\mathbf{E}_{[ij],\mathrm{bdd}}\cap W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}})(\hat{q}_{3,ij})$, (4.10.3) holds.

Fix $\varepsilon'>0$ that will be chosen in the end of the proof as a constant that depends only on the Lyapunov spectrum. We first treat the first term on the left hand side of (4.10.3). By **Step 3b**, since $\mathbf{v}\in W^{\lambda_k}(\hat{q}_{3,ij})$,

$$\|(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v}\|_{\hat{q}_1} \le e^{(-\lambda_k+\varepsilon')\tau_{ij}}\|\mathbf{v}\|_{\hat{q}_{3,ij}}. \tag{4.10.4}$$

By Lemma 4.5.10 and that $\|g\|_{\mathrm{Ad}(G)}=O(\delta)$, we have

$$\|P^-(\hat{q}_1,\hat{q}_1') - \mathrm{Ad}_{(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}}\|_{0\to 0} \le_\delta \|g\|_{\mathrm{Ad}(G)}e^{-\alpha\ell} \le_\delta e^{-\alpha_1\ell},$$

where $\alpha_1>0$ is the constant α in Lemma 4.5.10 that depends only on the Lyapunov spectrum.

By the norm comparison lemma (Lemma 4.5.15) and (4.10.4), we have

$$\|(P^-(\hat{q}_1,\hat{q}_1') - \mathrm{Ad}_{(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}})(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v}\|_{\hat{q}_1'} \le_\delta e^{-\alpha_1\ell}e^{(-\lambda_k+\varepsilon')\tau_{ij}}\|\mathbf{v}\|_{\hat{q}_{3,ij}}.$$

By the norm comparison lemma (Lemma 4.5.15) again, the norm on the left hand side can be taken with respect to $\|\cdot\|_{u\hat{q}_1'}$. Thus by taking a quotient by the subspace $W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1')$ on the left hand side, we have

$$\|(P^-(\hat{q}_1,\hat{q}_1') - \mathrm{Ad}_{(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}})(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v} + W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1')\|_{u\hat{q}_1'} \le_\delta e^{-\alpha_1\ell}e^{(-\lambda_k+\varepsilon')\tau_{ij}}\|\mathbf{v}\|_{\hat{q}_{3,ij}}.$$

Finally, we have the operator norm of $(T^{\tau}_{u\hat{q}_1})_* : (\mathfrak{l}/\mathfrak{h})_{u\hat{q}_1}/W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(u\hat{q}_1) \to (\mathfrak{l}/\mathfrak{h})_{\hat{q}_2}/W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}_2)$, with respect to the dynamical norm on both the range and target, is at most $e^{(\lambda_k+\varepsilon')\tau}$, therefore we have

$$\|(T^{\tau}_{u\hat{q}_1})_*(P^-(\hat{q}_1, \hat{q}'_1) - \mathrm{Ad}_{(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}})(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v} + W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}'_2)\|_{\hat{q}'_2} \tag{4.10.5}$$

$$\leq_\delta e^{-\alpha_1\ell}e^{(\lambda_k+\varepsilon')\tau}e^{(-\lambda_k+\varepsilon')\tau_{ij}}\|\mathbf{v}\|_{\hat{q}_{3,ij}}. \tag{4.10.6}$$

Now we treat the second term on the left hand side of (4.10.3). Since $R(\hat{q}_{3,ij}, \hat{q}_2)$ has operator norm $O(\delta)$ on $\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{q}_{3,ij})$ by Proposition 4.9.2 (see also **Step 3**), we have

$$\|R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\|_{\hat{q}_2} \leq_\delta \|\mathbf{v}\|_{\hat{q}_{3,ij}}. \tag{4.10.7}$$

Let $g' := (T^{\tau}_{u\hat{q}_1}T^{\ell}_{\hat{q}})g(T^{\tau}_{u\hat{q}_1}T^{\ell}_{\hat{q}})^{-1}$. By Lemma 4.5.10 (here we are using that $u\hat{q}_1, u\hat{q}'_1$ are stably related on G/L even though they are in general not stably related on G/H), we have

$$\|P^-(\hat{q}_2, \hat{q}'_2) - \mathrm{Ad}_{g'}\|_{0\to 0} \leq_\delta \|(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}\|_{\mathrm{Ad}(G)}e^{-\alpha_1\tau}. \tag{4.10.8}$$

Since $\|g\|_{\mathrm{Ad}(G)} = O(\delta)$ and $g \in \exp(W^{<0}_{\mathfrak{g}}(\hat{q}))$, we have $\|(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}\|_{\mathrm{Ad}(G)} = O(\delta)$ for large enough $\ell \gg_\delta 0$. By the norm comparison lemma (Lemma 4.5.15), (4.10.7) and (4.10.8), we have

$$\|(P^-(\hat{q}_2, \hat{q}'_2) - \mathrm{Ad}_{g'})R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\|_{\hat{q}'_2} \leq_\delta e^{-\alpha_1\tau}\|\mathbf{v}\|_{\hat{q}_{3,ij}}.$$

By **Claim 4.10.4a**, for large enough $\ell \gg_\delta 0$, we have $\tau > \frac{1}{2}\alpha_2^{-1}\ell$, where $\alpha_2 > 0$ is the constant in **Claim 4.10.4**. Therefore we have

$$\|(P^-(\hat{q}_2, \hat{q}'_2) - \mathrm{Ad}_{g'})R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\|_{\hat{q}'_2} \leq_\delta e^{-\alpha_3\ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}} \tag{4.10.9}$$

for the constant $\alpha_3 := \alpha_1\alpha_2^{-1}/2 > 0$ that depends only on the Lyapunov spectrum. Combine (4.10.5) and (4.10.9), using the fact that

$$\mathrm{Ad}_{g'}R(\hat{q}_{3,ij}, \hat{q}_2) = \mathrm{Ad}_{g'}(T^{\tau}_{u\hat{q}_1})_*(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_* = (T^{\tau}_{u\hat{q}_1})_*\mathrm{Ad}_{(T^{\ell}_{\hat{q}})g(T^{\ell}_{\hat{q}})^{-1}}(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*$$

(both sides are conjugation by the element $T^{\tau}_{u\hat{q}_1} T^{\ell}_{\hat{q}} g (T^{\ell}_{\hat{q}})^{-1} T^{-\tau_{ij}}_{\hat{q}_3}) \in G$), we have

$$\|(T^{\tau}_{u\hat{q}_1})_* P^-(\hat{q}_1, \hat{q}'_1)(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_* \mathbf{v} - P^-(\hat{q}_2, \hat{q}'_2) R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v} + W^{>\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}'_2)\|_{\hat{q}'_2} \tag{4.10.10}$$

$$\leq_\delta (e^{-\alpha_1 \ell} e^{(\lambda_k+\varepsilon')\tau} e^{(-\lambda_k+\varepsilon')\tau_{ij}} + e^{-\alpha_3 \ell})\|\mathbf{v}\|_{\hat{q}_{3,ij}}. \tag{4.10.11}$$

Thus to show (4.10.3), it remains to show that the right hand side of (4.10.10) is $\leq_\delta e^{-\alpha_4 \ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}}$ for some $\alpha_4 > 0$ that depends only on the Lyapunov spectrum.

To do so, note that on one hand, $R(\hat{q}_{3,ij}, \hat{q}_2) = (T^{\tau}_{u\hat{q}_1})_*(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*$ satisfies $\|R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\|_{\hat{q}_2} \leq C(\delta)\|\mathbf{v}\|_{\hat{q}_{3,ij}}$ by Proposition 4.9.2. On the other hand, apply **Step 3b** twice (since $\mathbf{v} \in W^{\lambda_k}_{\mathfrak{l}/\mathfrak{h}}(\hat{q}_{3,ij})$), we have

$$\|(T^{\tau}_{u\hat{q}_1})_*(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v}\|_{\hat{q}_2} \geq e^{(\lambda_k-\varepsilon')\tau}\|(T^{-\tau_{ij}}_{\hat{q}_{3,ij}})_*\mathbf{v}\|_{\hat{q}_1} \geq e^{(\lambda_k-\varepsilon')\tau} e^{(-\lambda_k-\varepsilon')\tau_{ij}}\|\mathbf{v}\|_{\hat{q}_{3,ij}}.$$

Therefore

$$e^{(\lambda_k-\varepsilon')\tau} e^{(-\lambda_k-\varepsilon')\tau_{ij}} \leq_\delta 1.$$

Thus by **Claim 4.10.4a**, for large enough $\ell \gg_\delta 0$, we have

$$e^{(\lambda_k+\varepsilon')\tau} e^{(-\lambda_k+\varepsilon')\tau_{ij}} \leq_\delta e^{2\varepsilon'(\tau+\tau_{ij})} \leq e^{8\varepsilon'\alpha_2 \ell},$$

where we recall $\alpha_2 > 0$ is the constant $\alpha > 0$ in **Claim 4.10.4** that depends only on the Lyapunov spectrum. Now if we take $\varepsilon' > 0$ small enough (depending only on the Lyapunov spectrum) so that $-\alpha_1 + 8\varepsilon'\alpha_2 < 0$ (say $\varepsilon' := \alpha_1/(16\alpha_2) > 0$), then the right hand side of (4.10.10) is $\leq_\delta e^{-\alpha_4 \ell}\|\mathbf{v}\|_{\hat{q}_{3,ij}}$ with $\alpha_4 := \min\{\alpha_1/2, \alpha_3\}$, as desired. □

Claim 4.10.8. There exists $c(\delta, \ell)$ with $c(\delta, \ell) \to 0$ as $\ell \to \infty$ such that

$$d(f_{ij}(\hat{q}_2), f_{ij}(\hat{q}'_2)) \leq c(\delta, \ell).$$

Proof. By Lemma 4.9.12, we have

$$f_{ij}(\hat{q}_2) \propto R(\hat{q}_{3,ij}, \hat{q}_2)_* f_{ij}(\hat{q}_{3,ij}), \qquad f_{ij}(\hat{q}_2') \propto R(\hat{q}_{3,ij}', \hat{q}_2')_* f_{ij}(\hat{q}_{3,ij}').$$

Since $\hat{q}_{3,ij}, \hat{q}_{3,ij}' \in K_0 \subset K_*$, f_{ij} is uniformly continuous, therefore

$$d(f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_{3,ij}')) \to 0 \qquad \text{as } \ell \to \infty.$$

Let $\mathbf{v}_1 \in V_{\hat{q}_1} \subset W_{\mathfrak{g}}^{<0}(\hat{q}_1)$ be the unique vector in the normal form coordinate at $\hat{q}_1$ such that $q_1' = \exp(\mathbf{v}_1)q_1$. Note that by Proposition 4.7.6, we have in particular that $\|\mathbf{v}_1\| = O(\delta)$ for large enough $\ell \gg_\delta 0$. Let $g := \exp(\mathbf{v}_1)$. Then one can show that $\|g\|_{\mathrm{Ad}(G)} = O(\delta)$ (using Lemma 4.5.15, see also **Step 4f**, to compare with the fixed norm $\|\cdot\|_0$, then use smoothness of the exponential map and Ad).

By continuity of $\mathrm{Ad}_{(T_{\hat{q}_1}^{\tau_{ij}})g(T_{\hat{q}_1}^{\tau_{ij}})^{-1}}$ and that $\tau_{ij} \to \infty$ as $\ell \to \infty$, we also have

$$d((\mathrm{Ad}_{(T_{\hat{q}_1}^{\tau_{ij}})g(T_{\hat{q}_1}^{\tau_{ij}})^{-1}})_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_{3,ij}')) \to 0 \qquad \text{as } \ell \to \infty.$$

By Lemma 4.5.10, we have

$$d(P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_{3,ij}')) \to 0 \qquad \text{as } \ell \to \infty.$$

Apply $R(\hat{q}_{3,ij}', \hat{q}_2')_*$ to both sides, noting that $R(\hat{q}_{3,ij}', \hat{q}_2')$ has operator norm $O(\delta)$ when restricted to $\mathbf{E}_{[ij],\mathrm{bdd}}(\hat{q}_{3,ij}')$, we get

$$d(R(\hat{q}_{3,ij}', \hat{q}_2')_* P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \qquad \text{as } \ell \to \infty.$$

By **Claim 4.10.7**, we have

$$d(P^-(\hat{q}_2, \hat{q}_2')_* R(\hat{q}_{3,ij}, \hat{q}_2)_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \qquad \text{as } \ell \to \infty.$$

By Lemma 4.5.10, we have

$$d((\mathrm{Ad}_{(T^{\tau}_{u\hat{q}_1})g(T^{\tau}_{u\hat{q}_1})^{-1}})_* R(\hat{q}_{3,ij},\hat{q}_2)_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \qquad \text{as } \ell \to \infty.$$

By continuity of $\mathrm{Ad}_{(T^{\tau}_{u\hat{q}_1})g(T^{\tau}_{u\hat{q}_1})^{-1})_*}$ and that $\tau_{ij} \to \infty$ as $\ell \to \infty$, we have

$$d(R(\hat{q}_{3,ij},\hat{q}_2)_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \qquad \text{as } \ell \to \infty.$$

Thus

$$d(f_{ij}(\hat{q}_2), f_{ij}(\hat{q}_2')) \to 0 \qquad \text{as } \ell \to \infty.$$

□

4.10.2 From the drift to extra invariance in a unipotent direction

Assuming **Proposition 4.10.1**, one prove the following (recall that $\mathcal{E}_{ij}(\hat{x})$ is a unipotent subgroup of $z(L/H^\circ)z^{-1}$).

Proposition 4.10.2. [ELa, Prop. 10.1] In Case I, there exists $ij \in \Lambda_{\text{sync}}$ such that for $\mathcal{E} := \mathcal{E}_{ij}$, for a.e. $\hat{x} = (\omega, x) \in \hat{\Omega}$, there exists a nontrivial unipotent subgroup $U^+_{new}(\hat{x}) \subset \mathcal{E}(\hat{x})$ such that:

(a) For a.e. $\hat{x} = (\omega, x) \in \hat{\Omega}$, a.e. $u \in \mathcal{U}^+_1$ and all $t \in \mathbb{R}$, $\hat{x} \mapsto U^+_{new}(\hat{x})$ is $\hat{T}^t$-equivariant and $\mathcal{U}^+_1$-invariant.

(b) For a.e. $\hat{x} = (\omega, x) \in \hat{\Omega}$, the conditional measure of $\hat{\nu}$ on $\{\omega\} \times \mathcal{E}(\hat{x})x$ is right invariant under $U^+_{new}(\hat{x}) \subset \mathcal{E}(\hat{x})$.

Proof outline. The proof goes by applying **Proposition 4.10.1**, following the proof of [ELa, Prop. 10.1]. Let $\delta > 0$ be the constant in **Proposition 4.10.1**.

1. Take $\varepsilon_n \to 0$. For each ε_n, take $E_n \subset K_*$ as in **Proposition 4.10.1**. We may assume that $ij \in \Lambda_{\text{sync}}$ is constant along this subsequence (by possibly replacing δ by $\delta/|\Lambda_{\text{sync}}|$).
2. Let $F := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \subset K_*$. Then $\hat{\nu}(F) > \delta$.

3. For $\hat{x} = (\omega, x) \in F$, there exists a subsequence $\hat{y}_n = (\omega, \gamma_n x) \in (\{\omega\} \times \mathcal{E}_{ij}(x)g) \cap K_*$ such that $\hat{y}_n \to \hat{x}$ but $\hat{y}_n \neq \hat{x}$ for all n, and $f_{ij}(\hat{x}) \propto f_{ij}(\hat{y}) = (r_{\gamma_n})_* f_{ij}(\hat{x})$ for all n, where r_g is the right multiplication by $g \in G$. This implies, $\gamma_n \to \mathrm{id}$ but $\gamma_n \neq \mathrm{id}$ for all n.

4. For $\hat{x} \in F$, let $U^+_{new}(\hat{x})$ be the maximal connected subgroup of $\mathcal{E}_{ij}(\hat{x})$ such that $f_{ij}(\hat{x}) \propto (r_g)_* f_{ij}(\hat{x})$ for all $g \in U^+_{new}(\hat{x})$.

5. We have the following properties of $U^+_{new}(\hat{x})$:

   (i) For $\hat{x} \in F$, $U^+_{new}(\hat{x})$ is non-trivial (Note that $U^+_{new}(\hat{x})$ is closed and not discrete by **Step 3**).

   (ii) $U^+_{new}(\hat{x})$ is constant on $\mathcal{E}_{ij}[\hat{x}] = \{\omega\} \times \mathcal{E}_{ij}(\hat{x})x$ by construction.

   (iii) $U^+_{new}(u\hat{x}) = U^+_{new}(\hat{x})$ for $\hat{x} \in F, u \in \mathcal{U}^+_1$ with $u\hat{x} \in F$ (by $\mathcal{U}^+_1$-invariance of $f(\hat{x})$)

   (iv) $\hat{T}^t_{T^{-t}\omega} U^+_{new}(\hat{T}^{-t}\hat{x}) = U^+_{new}(\hat{x})$ for $\hat{x} \in F, t > 0$ with $\hat{T}^{-t}\hat{x} \in F$ (by $\hat{T}$-equivariance of $f(\hat{x})$).

6. Since $\hat{\nu}(F) > \delta_0 > 0$ and $\hat{T}^t$ is ergodic, for a.e. $\hat{x} \in \hat{\Omega}$, there exists $t > 0$ with $\hat{T}^{-t}\hat{x} \in F$. Define $U^+_{new}(\hat{x}) := \hat{T}^t U^+_{new}(\hat{T}^{-t}\hat{x})$. Then $\hat{x} \mapsto U^+_{new}(\hat{x})$ is $\hat{T}$-equivariant and $\mathcal{U}^+_1$-invariant (this proves (a)).

7. By definition of U^+_{new}, there exists homomorphism $\beta_{\hat{x}} : U^+_{new}(\hat{x}) \to \mathbb{R}$ s.t. $g_* f_{ij}(\hat{x}) = e^{\beta_{\hat{x}}(g)} f_{ij}(\hat{x})$ for a.e. $\hat{x} \in \hat{\Omega}$.

8. By Step 5(iv), for a.e. $\hat{x} = (\omega, x) \in \hat{\Omega}$, $g \in U^+_{new}(\hat{x})$ and $t > 0$, $\beta_{\hat{T}^{-t}\hat{x}}(T^{-t}_\omega g T^t_{T^{-t}\omega}) = \beta_{\hat{x}}(g)$.

9. Since $T^{-t}_\omega g T^t_{T^{-t}\omega} \to e$ for all $g \in U^+_{new}(\hat{x})$, $t > 0$, by Poincaré recurrence, $\beta_{\hat{x}}(g) = 0$ for a.e. $\hat{x} \in \hat{\Omega}$ (this proves (b)).

□

4.10.3 From extra invariance to the measure classification

Proposition 4.10.3. In Case I, there exist a Lie subgroup $H' \subset G$ with $H^\circ \subset H' \subset L \subset G$ and $\dim(H'/H^\circ) > 0$, an H'-homogeneous probability measure ν_0 on L/H and finite μ-stationary measure $\nu_{G/H'}$ on G/H' such that

$$\nu = \nu_{G/H'} * \nu_0 = \int_{G/H'} g_* \nu_0 \, d\nu_{G/H'}(g).$$

Proof. The proof follows the proof of [ELa, Thm. 1.13(a)] closely. See also [BQ11, Sect. 8].

1. Let $\mathcal{P}(G/H)$ be the space of probability measures on G/H.

2. For $z \in G/H$, $\alpha \in \mathcal{P}(zL/H)$, let S_α be the connected component of the stabilizer of α w.r.t. $z(L/H^\circ)z^{-1}$ acting on zL/H. Recall that if $\pi_{G/L}(z) = \pi_{G/L}(z')$, then $zL/H = z'L/H$ and $z(L/H^\circ)z^{-1} = z'(L/H^\circ)(z')^{-1}$.

3. Let

$$\begin{aligned}\mathcal{F} =& \{\alpha \in \mathcal{P}(G/H) : \text{supp}\, \alpha \subset zL/H \text{ for some } z \in G/L, S_\alpha \neq \{1\} \\ & \text{and } \alpha \text{ supported on a } S_\alpha \text{ orbit}\}.\end{aligned}$$

   By the first condition, each $\alpha \in \mathcal{F}$ can be considered a probability measure on zL/H for some $z \in G/L$. By abuse of notation we let $\mathcal{F}_z := \mathcal{F} \cap \mathcal{P}(zL/H)$. Then $\mathcal{F} = \bigcup_{z \in G/L} \mathcal{F}_z$ as sets. Note that G acts on $\mathcal{F}$.

4. By Ratner's theorem applied to the $z(L/H^\circ)z^{-1}$-homogeneous space zL/H for each $z \in G/L$, $\mathcal{F}_z$ contains all measures invariant and ergodic under a connected non-trivial unipotent subgroup of $z(L/H^\circ)z^{-1}$ for some $z \in G/L$.

5. For a.e. $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, take $U^+_{new}(\hat{z}) \subset \mathcal{E}(\hat{z}) \subset z(L/H^\circ)z^{-1}$ as in Proposition 4.10.2. WLOG, assume $U^+_{new}(\hat{z})$ is the stabilizer in $\mathcal{E}(\hat{z})$ of $\hat{\nu}|_{\{\omega\}\times\mathcal{E}(\hat{z})z}$ (otherwise enlarge $U^+_{new}(\hat{z})$ to the stabilizer - the equivariance properties in **Proposition 4.10.2** still hold,

and it is nontrivial).

6. For $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, let

$$\Delta(\omega, z) := \{z' \in G/H : \pi_{G/L}(z) = \pi_{G/L}(z') \text{ and } U^+_{new}(\omega, z) = U^+_{new}(\omega, z')\}.$$

Here $\pi_{G/L} : G/H \to G/L$ is the quotient map.

7. For $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, let $\hat{\nu}_{\hat{z}}$ denote the conditional measure of $\hat{\nu}$ on $\{\omega\} \times zL/H$.

8. Disintegrate $\hat{\nu}$ under $(\omega, z) \mapsto (\omega, \pi_{G/L}(z), U^+_{new}(\omega, z))$. Then for a.e. $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, we get a probability measure $\tilde{\nu}_{\hat{z}}$ on zL/H supported on $\Delta(\omega, z)$ such that for a.e. $\hat{z} \in \mathcal{S}^{\mathbb{Z}} \times G/H$,

$$\hat{\nu}_{\hat{z}} = \int_{zL/H} \tilde{\nu}_{(\omega,z')} d\hat{\nu}_{\hat{z}}(z').$$

9. For a.e. $\hat{z} \in \mathcal{S}^{\mathbb{Z}} \times L/H$, $\tilde{\nu}_{\hat{z}}$ is $U^+_{new}(\hat{z})$-invariant.

10. Do simultaneous $U^+_{new}(\hat{z})$-ergodic decomposition of $\tilde{\nu}_{\hat{z}}$ for a.e. $\hat{z} = (\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, then

$$\tilde{\nu}_{\hat{z}} = \int_{zL/H} \zeta(\omega, z') d\tilde{\nu}_{\hat{z}}(z'),$$

where $\zeta : \mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{F}$ is constant along $\Delta(\omega, z)$ for a.e. $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$.

11. Integrate to get

$$\hat{\nu}_{\hat{z}} = \int_{zL/H} \zeta(\omega, z') d\hat{\nu}_{\hat{z}}(z').$$

12. The $\hat{T}$ and $\mathcal{U}^+_1$-equivariance of $\mathcal{E}(\hat{z})$ and $U^+_{new}(\hat{z})$ imply that for $t \in \mathbb{R}$ and $u \in \mathcal{U}^+_1$,

$$(T^t_{\hat{z}})_* \zeta(\omega, z) = \zeta(\hat{T}^t(\omega, z)), \qquad \text{and} \qquad \zeta(u\omega, z) = \zeta(\omega, z).$$

13. Define $\hat{\zeta} : \mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}} \times \mathcal{F}$ by $\hat{\zeta}(\omega, z) := (\omega, \zeta(\omega, z))$. Then $\hat{\eta} := \hat{\zeta}_* \hat{\nu}$ is an ergodic $\hat{T}^t$-invariant probability measure on $\mathcal{S}^{\mathbb{Z}} \times \mathcal{F}$.

14. By Ratner's theorem (for nondiscrete quotients - see e.g. [Wit94, Thm. 1.2] for the argument reducing this more general case to the version on discrete quotients in [Rat91, Thm. 1.1]), the set $\mathcal{G}$ of G-orbits on $\mathcal{F}$ is countable.

15. Since $\hat{T}_z^t$ acts trivially on $\mathcal{G}$ and $\hat{\eta}$ is ergodic, the pushforward of $\hat{\eta}$ on $\mathcal{S}^{\mathbb{Z}} \times \mathcal{G}$ via $\mathcal{S}^{\mathbb{Z}} \times \mathcal{F} \to \mathcal{S}^{\mathbb{Z}} \times \mathcal{G}$ is also ergodic and is supported on $\mathcal{S}^{\mathbb{Z}}$ times a single G-orbit, so $\hat{\eta}$ is supported on $\mathcal{S}^{\mathbb{Z}} \times G\nu_0$ for some $\nu_0 \in \mathcal{F}$.

16. Let $H' \subset G$ be the stabilizer of ν_0 in G. By definition of $\mathcal{F}$, ν_0 is supported on a single H'-orbit. Moreover, $\nu_0 \in \mathcal{P}(zL/H)$ for some $z \in G/L$ and $zH^\circ z^{-1} \subset H' \subset zLz^{-1}$.

17. Write $\zeta(\omega, z) = \theta(\omega, z)\nu_0$, where $\theta : \mathcal{S}^{\mathbb{Z}} \times G/H \to G/H'$. Then θ is $\hat{T}$-equivariant and $\mathcal{U}_1^+$-invariant.

18. Define $\hat{\theta} : \mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}} \times G/H'$ by $\hat{\theta}(\omega, z) := (\omega, \theta(\omega, z))$. Then $\hat{\lambda} := \hat{\theta}_*\hat{\nu}$ is a $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant measure on $\mathcal{S}^{\mathbb{Z}} \times G/H'$, therefore its projection $\nu_{G/H'}$ to G/H' is μ-stationary. Hence we have

$$\nu = \int_{G/H'} g_*\nu_0 \, d\nu_{G/H'}(g),$$

where ν_0 is an H'-homogeneous measure on zL/H for some $z \in G/L$.

□

4.11 Case II

In Case II, much of the proof is similar to [ELa, Sect. 11]. However there are two key distinctions. Firstly, in this setting, the stationary measure ν is not necessarily Γ_μ-invariant, therefore the analogue of [ELa, Prop. 11.1] cannot not hold. As a result, to adapt the Case II argument of [ELa, Sect. 11], we will work mostly with ν_{ω^-} rather than ν. A weaker analogue of [ELa, Prop. 11.1] will be proved in subsection 4.11.5 to finish the proof. Secondly, unlike in [ELa, Sect. 11], the measure ν is not necessarily compactly supported. The reason the argument does not carrry over directly is because μ acts compactly on Z in the case of [ELa, Sect. 11], but for us, the analogous partition $\mathcal{F}^{\leq 0}_{G/H}$ does not have a compact μ-action.

We first recall the assumption in Case II.

Case II Assumption: For $\hat{\nu}$-almost every $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, the conditional measure $\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}$ on the total stable subset of (ω, z) in the two-sided skew product is supported on $\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{F}^{\leq 0}_{G/H}[z]$.
Here we recall the definition of $\mathcal{F}^{\leq 0}_{G/H}[z]$:

$$\begin{aligned}\mathcal{F}^{\leq 0}_{G/H}[z] &:= \left\{ z' \in G/H^\circ \mid \limsup_{n\to\infty} \frac{1}{n} \log d_{G/H}(T^n_\omega(z), T^n_\omega(z')) \leq 0 \quad \text{for almost every } \omega^+ \in \mathcal{S}^{\mathbb{N}} \right\} \\ &= \{ z' \in G/H^\circ \mid z' \in \mathcal{W}^{\leq 0}[(\omega, z)] \quad \text{for almost every } \omega^+ \in \mathcal{S}^{\mathbb{N}} \}.\end{aligned}$$

Similarly define $\mathcal{F}^{\leq 0}_{G/L}[x] \subset G/L$ for $x \in G/L$.

The goal of this section is to prove the following theorem.

Theorem 4.11.1. Let G be a real linear algebraic group, and μ be a Borel probability measure on G with finite first moment. Let Γ_μ be the (topological) closure of the subsemigroup generated by the support of μ in G, and $\overline{\Gamma}^Z_\mu$ be the Zariski closure of Γ_μ.

Let $H \subset G$ be a closed unimodular subgroup. Suppose there exists an H-envelope L and $x_0 \in G/L$ such that μ is uniformly expanding on L/H at x_0.

Let ν be an ergodic μ-stationary probability measure on $\overline{\Gamma}^Z_\mu x_0 L/H$ and $\overline{\nu} := \pi_* \nu$ be its pushforward on G/L. Suppose that Case II holds. We also assume that

(†) There exists a closed normal subgroup $U \subset \overline{\Gamma}^Z_\mu$ and some $z_0 \in G$ with $z_0 L = x_0 L$ such that $\overline{\Gamma}^Z_\mu x_0 L = U z_0 H^\circ$ and $z_0^{-1} U z_0 \cap H^\circ = \{\mathrm{id}\}$.

Then the stationary measure $\nu_{G/H}$ can be written as

$$\nu_{G/H} = \int_{G/L} \nu_x \, d\overline{\nu}(x),$$

where

1. $\overline{\nu}$ is a generalized μ-Bernoulli measure supported on $\overline{\Gamma}_\mu^Z x_0 L/L$.

2. there exists a positive integer k such that for $\overline{\nu}$-almost every $x \in G/L$, ν_x is the uniform measure on k points in $\pi^{-1}(x) \subset G/H$,

3. there exist finitely many $z_1, \ldots, z_m \in \overline{\Gamma}_\mu^Z x_0 L/H$ such that for $\mathcal{F} := \bigcup_{i=1}^m \mathcal{F}_{G/H}^{\leq 0}[z_i]$, we have (i) supp $\nu_{G/H} \subset \mathcal{F}$, (ii) $\mathcal{F}$ has finite intersection with $x'L/H$ for all $x' \in \overline{\Gamma}_\mu^Z x_0 L/L$, and (iii) $\mathcal{F}$ is invariant under Γ_μ.

Theorem 4.11.1 follows from Proposition 4.11.2 and 4.11.15.

Proposition 4.11.2. Under the assumptions of Theorem 4.11.1, there exist finitely many points $z_1, \ldots, z_m \in G/H$ such that ν is supported on $\bigcup_{i=1}^{m} \mathcal{F}_{G/H}^{\leq 0}[z_i]$.

To show Proposition 4.11.2, we first establish the following proposition.

Proposition 4.11.3. Under the assumptions of Theorem 4.11.1, for $\mu^{\mathbb{Z}}$-almost every $\omega \in \mathcal{S}^{\mathbb{Z}}$, there exists finitely many points $z_1, \ldots, z_m \in G/H$ such that ν_{ω^-} is supported on $\bigcup_{i=1}^{m} \mathcal{F}_{G/H}^{\leq 0}[z_i]$.

The next few subsection will be dedicated to proving Proposition 4.11.3.

4.11.1 The inert center-stable set $\mathcal{F}_{G/H}^{\leq 0}$

In this subsection we record a few properties of the sets $\mathcal{F}_{\Gamma}^{\leq 0}[z] := \mathcal{F}_{G/H}^{\leq 0}[z] \cap \overline{\Gamma}_\mu^Z x_0 L/H^\circ$ under the assumption (†). Clearly it suffices to show Proposition 4.11.2 and 4.11.3 for $\mathcal{F}_{G/H}^{\leq 0}[z]$ replaced by $\mathcal{F}_{\Gamma}^{\leq 0}[z]$ (recall that ν is supported on $\overline{\Gamma}_\mu^Z x_0 L/H$). See Proposition 4.4.5 for general properties of $\mathcal{F}_{G/H}^{\leq 0}[z]$ even without the assumption (†).

Lemma 4.11.4. Under the assumption (†), there exists an algebraic subgroup $\mathcal{F}_U^{\leq 0} \subset U$ such that

(a) $\mathcal{F}_{\Gamma}^{\leq 0}[z] = \mathcal{F}_U^{\leq 0} z$ for all $z \in \overline{\Gamma}_\mu^Z x_0 L/H^\circ$.

(b) $\mathcal{F}_U^{\leq 0}$ is normalized by Γ_μ.

Proof. Using the assumption (†), we can identify

$$z_0^{-1}\overline{\Gamma}_\mu^Z x_0 L/H^\circ = z_0^{-1}Uz_0H^\circ/H^\circ \cong z_0^{-1}Uz_0/(z_0^{-1}Uz_0 \cap H) = z_0^{-1}Uz_0.$$

Using a right-invariant metric on $z_0^{-1}Uz_0$ and this identification, it is clear that $z_0^{-1}\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0]$ is a closed subgroup of $z_0^{-1}Uz_0$, and that $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z] = \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0]z_0^{-1}z$ for all $z \in \overline{\Gamma}_\mu^Z x_0 L/H^\circ$. Let $\mathcal{F}_{\overline{U}}^{\leq 0} := \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0]z_0^{-1}$. Then clearly (a) is satisfied.

Now by the definition of $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z]$ (see Proposition 4.4.5(a)), for μ-a.e. $g \in G$, $g\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0] \subset \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[gz_0]$. By (a), this implies that $g\mathcal{F}_{\overline{U}}^{\leq 0}g^{-1} \subset \mathcal{F}_{\overline{U}}^{\leq 0}$. On the other hand, both sides are Lie subgroups of U with same dimensions and same (finite) number of connected components, therefore the containment is in fact an equality. Therefore (b) holds. □

Corollary 4.11.5. Under the assumption (†), we have the following properties of the inert center-stable sets.

(a) $g\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z] = \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[gz]$ for all $z \in \overline{\Gamma}_\mu^Z x_0 L/H^\circ$.

(b) $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z]H$ is locally closed in $\overline{\Gamma}_\mu^Z x_0 L/H$, thus the action of $\mathcal{F}_{\overline{U}}^{\leq 0}$ on $\overline{\Gamma}_\mu^Z x_0 L/H$ is smooth (in the sense of [Zim84, Def. 2.1.9]), i.e. the quotient space $\mathcal{F}_{\overline{U}}^{\leq 0}\backslash\overline{\Gamma}_\mu^Z x_0 L/H$ is countably separated, so that $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z]$ is a measurable partition of $\overline{\Gamma}_\mu^Z x_0 L/H$.

Proof. (a) holds by Parts (a) and (b) of Lemma 4.11.4. To show part (b), by homogeneity it suffices to verify that at $z = z_0$, there exists a neighborhood $\mathcal{O}$ of z_0H in $\overline{\Gamma}_\mu^Z x_0 L/H$ such that $\mathcal{O} \cap \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0]H = \mathcal{O} \cap \mathcal{F}_{\overline{U}}^{\leq 0}z_0H$ is closed in $\mathcal{O}$. The key point here is to use the fact that this holds on G/L, and that $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z]H$ is homeomorphic to its image on G/L. More precisely, since $\mathcal{F}_{\overline{U}}^{\leq 0}$ is an algebraic subgroup, any orbit on G/L is locally (Zariski) closed by [Zim84, Thm. 3.1.1]. In particular $\mathcal{F}_{\overline{U}}^{\leq 0}z_0L$ is locally closed in G/L, i.e. there exists an open neighborhood $\mathcal{O}_L$ of z_0L in G/L such that $\mathcal{O}_L \cap \mathcal{F}_{\overline{U}}^{\leq 0}z_0L$ is closed in $\mathcal{O}_L$. Since $\mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z_0]H = \mathcal{F}_{\overline{U}}^{\leq 0}z_0H$ intersects z_0L/H at at most one point by Corollary 4.8.5, the quotient map $\pi : G/H \to G/L$ restricts to a bijective continuous open map on $\mathcal{F}_{\overline{U}}^{\leq 0}z_0H$, therefore it is a homeomorphism. Therefore

$\pi^{-1}\mathcal{O}_L$ is an open neighborhood of z_0 in $\overline{\Gamma}_\mu^Z x_0 L/H$ and $\pi^{-1}\mathcal{O}_L \cap \mathcal{F}_{\overline{U}}^{\leq 0} z_0 H$ is closed in $\pi^{-1}\mathcal{O}_L$, as desired. □

4.11.2 mod $\mathcal{F}^{\leq 0}$ local dimension

The goal of this subsection is to show Proposition 4.11.6, which shows that in Case II, under the assumption of uniform expansion on L/H, the local dimension of the measure ν in the L/H direction is 0.

For $z \in G/H$, $\varepsilon > 0$ and $r > 0$, define a local ball in the L/H fiber direction

$$B_{L/H}(z, r) := \{z' \in zL/H^\circ \mid d_{G/H}(z, z') < r\},$$

and

$$B_{/F}(z, r, \varepsilon) := \{z'' \in G/H^\circ \mid z'' \in B_{L/H}(z', r) \text{ for some } z' \in \mathcal{F}_{\overline{\Gamma}}^{\leq 0}[z] \text{ with } d_{G/H}(z, z') < \varepsilon\}.$$

For $\omega \in \mathcal{S}^{\mathbb{Z}}$ and $z \in G/H$, define the *mod $\mathcal{F}^{\leq 0}$ lower local dimension* as

$$\dim_{/F}(\nu, \omega, z) := \lim_{\varepsilon \to 0} \left(\liminf_{r \to 0} \frac{\log \nu_{\omega^-}(B_{/F}(z, r, \varepsilon))}{\log r} \right).$$

By ergodicity, for $\hat{\nu}$-a.e. $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, $\dim_{/F}(\nu, \omega, z)$ does not depend on (ω, z). Define this common value by $\dim_{/F}(\nu)$.

Proposition 4.11.6. In Case II with uniform expansion on L/H at $x \in G/L$, and ν a μ-stationary measure on $\overline{\Gamma}_\mu^Z zH/H$ for some $z \in G/H$ with $x = \pi_{G/L}(z)$, we have $\dim_{/F}(\nu) = 0$.

For positive integer n, $\omega \in \mathcal{S}^{\mathbb{Z}}$, $z \in G/H$ and $\varepsilon > 0$, we define the Bowen balls as

$$B^n(\omega, z, \varepsilon) := \{z' \in G/H^\circ \mid \text{ for } 0 \leq m \leq n, T_\omega^m z' \in B_{/F}(T_\omega^m z, \varepsilon, \varepsilon)\}.$$

We need the following consequence of uniform expansion on L/H, which has the same proof as [ELa, Lem. 11.9].

Lemma 4.11.7. Let $\varepsilon > 0$ and $x \in G/L$. Suppose that μ is uniformly expanding on L/H at x. Then for any unit vector $\mathbf{v} \in (\mathfrak{l}/\mathfrak{h})_x$, there exists a positive measure set $K(\mathbf{v}) \subset \mathcal{S}^{\mathbb{Z}}$ such that for all $\omega \in K(\mathbf{v})$, there exist $\eta(\mathbf{v}) > 0$ and $N(\mathbf{v}) > 0$ such that for all $n > N(\mathbf{v})$, any unit vector $\mathbf{w} \in (\mathfrak{l}/\mathfrak{h})_x$ with $\|\mathbf{v} - \mathbf{w}\|_0 < \eta(\mathbf{v})$ and any $z \in G/H$ that projects to x, we have

$$|\{t : \exp(t\mathbf{w})z \in B^n(\omega, z, \varepsilon)\}| \leq e^{-\alpha n},$$

where $\alpha > 0$ depends only on the Lyapunov spectrum.

We also recall the definition of fiber entropy for the bundle $\mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}}$.

Definition. Let ξ be a finite measurable partition of G/H. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} H_{\nu_{\omega^-}} \left(\bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\xi \right) := \lim_{n\to\infty} -\frac{1}{n} \sum_{A \in \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\xi} \nu_{\omega^-}(A) \log \nu_{\omega^-}(A)$$

exists and is constant for $\mu^{\mathbb{Z}}$-a.e. ω. We denote this value by $h^{G/H}_{\hat{\nu}}(\hat{T}, \xi)$, and define the *fiber entropy* $h^{G/H}_{\hat{\nu}}(\hat{T})$ to be the supremum over all finite measurable partition ξ of $h^{G/H}_{\hat{\nu}}(\hat{T}, \xi)$. Note that in Case II, we have $h^{G/H}_{\hat{\nu}}(\hat{T}) = 0$.

We will use the following relative version of the Brin-Katok local entropy formula, which computes the fiber entropy in terms of the Bowen balls.

Lemma 4.11.8. (cf. [Zhu09, Thm. 3.1]) For $\varepsilon > 0$, $\varepsilon' > 0$, $n \in \mathbb{N}$ and $\omega \in \mathcal{S}^{\mathbb{Z}}$, let $N(n, \omega, \varepsilon, \varepsilon')$ denote the smallest number of Bowen balls $B^n(\omega, z, \varepsilon) \subset G/H$ needed to cover a set of ν_{ω^-}-measure at least $1 - \varepsilon$. Then for $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$ and any $0 < \varepsilon' < 1$, we have

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(n, \omega, \varepsilon, \varepsilon') = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, \omega, \varepsilon, \varepsilon') = h^{G/H}_{\hat{\nu}}(\hat{T}).$$

Now using this formula and that the fiber entropy is zero in Case II, we obtain the following corollary.

Corollary 4.11.9. (cf. [ELa, Cor. 11.11]) Let $N(n,\omega,\varepsilon,\varepsilon')$ be as in Lemma 4.11.8. Then for any $\varepsilon > 0$, any $0 < \varepsilon' < 1$ and $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log N(n,\omega,\varepsilon,\varepsilon') = 0.$$

Proof of Proposition 4.11.6. Using Lemma 4.11.7 and Corollary 4.11.9, one can establish Proposition 4.11.6 as in the proof of [ELa, Prop. 11.8] (replacing ν with ν_{ω^-} everywhere in the proof, using critically the fact that ν_{ω^-} depends only on the past ω^- of the word ω and not the future). □

4.11.3 Margulis function

Just like in [ELa, Sect. 11], We need the construction of a Margulis function. We only present the proof (with the Margulis function) in the case when L/H is compact - the general case is a similar adaptation of [ELa, Lem. 11.14].

For $r > 0$ and $\delta > 0$, define the Margulis function $f_{r,\delta} : G/H \times G/H \to \mathbb{R}$ by

$$f_{r,\delta}(z,z') = \begin{cases} \min\{r, d_{G/H}(z', \mathcal{F}_\Gamma^{\leq 0}[z] \cap z'L/H)\}^{-\delta} & \text{if } \mathcal{F}_\Gamma^{\leq 0}[z] \cap z'L/H \neq \emptyset \\ r^{-\delta} & \text{otherwise.} \end{cases}$$

Suppose that for $\mu^{\mathbb{Z}}$-a.e. $\omega \in \mathcal{S}^{\mathbb{Z}}$, $\nu_{\omega^-}(\mathcal{F}_\Gamma^{\leq 0}[z]) = 0$ for ν_{ω^-}-a.e. $z \in G/H$. Then $f_{r,\delta}(z,z') < \infty$ for $\nu_{\omega^-} \times \nu_{\omega^-}$-a.e. (z,z').

We have the following Margulis inequality of this Margulis function $f_{r,\delta}$.

Proposition 4.11.10. Suppose H/H° is cocompact in L/H°. Assume that μ is uniformly expanding on L/H at some $x \in G/L$ and there exists a μ-stationary measure supported on $\overline{\Gamma}_\mu^Z xL/L$.

Then there exists $n = n(\mu) \in \mathbb{N}$, $\delta = \delta(\mu,n) > 0$, constants $c = c(\mu,n,\delta) < 1$ and

$b = b(\mu, n, \delta, r) > 0$ such that for all $z, z' \in G/H$ that project into $\overline{\Gamma}_\mu^Z x_0 L/L$, we have

$$\int_G f_{r,\delta}(gz, gz')\, d\mu^{(n)}(g) \leq c f_{r,\delta}(z, z') + b.$$

Proof. Firstly, notice that if $\mathcal{F}_\Gamma^{\leq 0}[z] \cap z'L/H$ consists of a single point $z'' \in G/H$, then $\mathcal{F}_\Gamma^{\leq 0}[gz] \cap gz'L/H$ also consists of the single point $gz'' \in G/H$ for all $g \in \Gamma_\mu$, and

$$d_{G/H}(z', \mathcal{F}_\Gamma^{\leq 0}[z] \cap z'L/H) = d_{G/H}(z', z'').$$

Also $\mathcal{F}_\Gamma^{\leq 0}[z]$ is an Γ_μ-equivariant partition by Corollary 4.11.5(a). Therefore it suffices to consider the case when $z' \in zL/H$. In this case, the proof is essentially [EM04, Lem. 4.2] by applying uniform expansion on L/H at $\pi_{G/H}(z)$ (which follows from the assumptions by Lemma 4.8.4).

□

The following is a standard consequence of the Margulis inequality (Proposition 4.11.10).

Proposition 4.11.11. Suppose H/H° is cocompact in L/H°. Assume that μ is uniformly expanding on L/H at some $x \in G/L$ and ν is a μ-stationary measure supported on $\overline{\Gamma}_\mu^Z zH/H$ for some $z \in G/H$ with $\pi_{G/L}(z) = x$.

Suppose that $\nu_{\omega^-}(\mathcal{F}_\Gamma^{\leq 0}[z]) = 0$ for $\hat{\nu}$-a.e. $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$. Then for any $\eta > 0$, there exists $K'' \subset \mathcal{S}^{\mathbb{Z}} \times G/H$ with $\hat{\nu}(K'') > 1 - \eta$ and a constant $C = C(\eta, r)$ such that for any $(\omega, z) \in K''$,

$$\int_{G/H} f_{r,\delta}(z, z')\, d\nu_{\omega^-}(z') < C.$$

Proof. By iterating Proposition 4.11.10, for any $z, z' \in G/H$ that project to $\overline{\Gamma}_\mu^Z x_0 L/L$, we have

$$\limsup_{k \to \infty} \int_G f_{r,\delta}(gz, gz') d\mu^{(kn)}(g) \leq \frac{b}{1 - c}. \tag{4.11.1}$$

Consider the probability measure $\tilde{\nu}$ on $\mathcal{S}^{\mathbb{Z}} \times G/H \times G/H$ defined by

$$d\tilde{\nu}(\omega, z, z') := d\nu_{\omega^-}(z) d\nu_{\omega^-}(z') d\mu^{\mathbb{Z}}(\omega).$$

This is invariant under the map $\tilde{T} : \mathcal{S}^{\mathbb{Z}} \times G/H \times G/H \to \mathcal{S}^{\mathbb{Z}} \times G/H \times G/H$ by $(\omega, z, z') \mapsto (\sigma(\omega), T_\omega z, T_\omega z')$. By the random ergodic theorem [Kif86, Cor. I.2.2], there exists a measurable function $\phi : \mathcal{S}^{\mathbb{Z}} \times G/H \times G/H \to \mathbb{R}$ such that

$$\int_{G/H \times G/H} \phi(\omega, z, z')\, d\tilde{\nu}(\omega, z, z') = \int_{G/H \times G/H} f_{r,\delta}(\omega, z, z')\, d\tilde{\nu}(\omega, z, z'),$$

and for $\tilde{\nu}$-a.e. $(\omega, z, z') \in \mathcal{S}^{\mathbb{Z}} \times G/H \times G/H$,

$$\phi(\omega, z, z') = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} f_{r,\delta}(\omega_{jn} \dots \omega_1 z, \omega_{jn} \dots \omega_1 z').$$

Integrating both sides with respect to $\tilde{\nu}$ over $\mathcal{S}^{\mathbb{Z}} \times G/H \times G/H$, using Fatou's lemma and (4.11.1), we have

$$\int_{\mathcal{S}^{\mathbb{Z}} \times G/H \times G/H} f_{r,\delta}\, d\tilde{\nu} \le \frac{b}{1-c}.$$

This implies the lemma. □

4.11.4 Proof of Proposition 4.11.3

Suppose that $\nu_{\omega^-}(\mathcal{F}_{\Gamma}^{\le 0}[z]) = 0$ for $\hat{\nu}$-a.e. $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$. By Proposition 4.11.11, for all $\varepsilon > 0$ and r' with $r > r' > 0$, and all $(\omega, z) \in K''$,

$$\nu_{\omega^-}(B_{/F}(z, r', \varepsilon)) \le C(\eta)(r')^{\delta}.$$

Thus

$$\frac{\log \nu_{\omega^-}(B_{/F}(z, r', \varepsilon))}{\log r'} \ge \delta - \frac{|\log C(\eta)|}{|\log r'|}.$$

Therefore $\dim_{/F}(\nu) \geq \delta > 0$, contradicting Proposition 4.11.6.

Therefore $\nu_{\omega^-}(\mathcal{F}_\Gamma^{\leq 0}[z]) > 0$ for a positive $\hat{\nu}$-measure set of $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$. Now consider the measurable function on $\mathcal{S}^{\mathbb{Z}} \times G/H$:

$$\phi(\omega, z) := \nu_{\omega^-}(\mathcal{F}_\Gamma^{\leq 0}[z]).$$

For $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, recall by definition of ν_{ω^-} that $(\omega_0)_* \nu_{\omega^-} = \nu_{\sigma(\omega)^-}$, and by Proposition 4.4.5(a), for μ-a.e. $g \in G$, $g_* \mathcal{F}_\Gamma^{\leq 0}[z] \subset \mathcal{F}_\Gamma^{\leq 0}[gz]$. Therefore $\phi(\hat{T}(\omega, z)) \geq \phi(\omega, z)$ for almost every (ω, z). Thus by ergodicity of $\hat{\nu}$ (applied to the level sets $\{(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H \mid \phi(\omega, z) \geq c\}$ for $0 \leq c \leq 1$), there exists $\varepsilon > 0$ such that $\nu_{\omega^-}(\mathcal{F}_\Gamma^{\leq 0}[z]) = \varepsilon$ for all (ω, z) in a $\hat{\nu}$-conull set $\Psi \subset \mathcal{S}^{\mathbb{Z}} \times G/H$.

Let $\Psi(\omega) := \{z \in G/H \mid (\omega, z) \in \Psi\}$ be the level sets of Ψ. Then $\nu_{\omega^-}(\Psi(\omega)) = 1$ for almost every $\omega \in \mathcal{S}^{\mathbb{Z}}$. Since $\varepsilon > 0$, one can pick finitely many $z_1, \ldots, z_m \in \Psi(\omega)$ such that $\nu_{\omega^-}\left(\bigcup_{i=1}^m \mathcal{F}_\Gamma^{\leq 0}[z_i]\right) = 1$. Hence ν_{ω^-} is supported on finitely many $\mathcal{F}_\Gamma^{\leq 0}[z]$, which proves Proposition 4.11.3.

We remark that throughout the proof of Proposition 4.11.3, we really only use an assumption weaker than the Case II assumption, namely that the fiber entropy is zero.

4.11.5 Proof of Proposition 4.11.2

To deduce Proposition 4.11.2 from Proposition 4.11.3, we need a key lemma (Lemma 4.11.13), whose proof is modelled on the proof of [ELa, Prop. 11.1] (see also [BRH17, Prop. 11.1]).

We first note that since $\hat{\mathcal{W}}_{G/H}^-[(\omega, z)] = \mathcal{W}_{\mathcal{S}^{\mathbb{Z}}}^-[\omega] \times \mathcal{W}_{G/H}^-[(\omega, z)]$ and by definition that $\hat{\nu} = \int \delta_\omega \times \nu_{\omega^-} d\mu^{\mathbb{Z}}(\omega)$, we can write

$$\hat{\nu}|_{\hat{\mathcal{W}}_{G/H}^-[(\omega,z)]} = \int_{\mathcal{W}_{\mathcal{S}^{\mathbb{Z}}}^-[\omega]} \delta_{\omega^-} \times \left(\nu_{\omega^-}|_{\mathcal{W}_{G/H}^-[(\omega,z)]}\right) d\tau_{(\omega^+,z)}(\omega^-) \qquad (4.11.2)$$

for some Borel probability measure $\tau_{(\omega^+,z)}$ on $\mathcal{W}_{\mathcal{S}^{\mathbb{Z}}}^-[\omega]$, which we consider as a measure on $\mathcal{S}^{-\mathbb{N}}$ via the identification $\mathcal{W}_{\mathcal{S}^{\mathbb{Z}}}^-[\omega] = \mathcal{S}^{-\mathbb{N}} \times \{\omega^+\} \leftrightarrow \mathcal{S}^{-\mathbb{N}}$.

Firstly, we have the following description of $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]}$ in Case II.

Lemma 4.11.12. Under the Case II assumption, for $\hat{\nu}$-a.e. (ω, z), $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]} = \delta_z$.

Proof. Let $\pi : G/H \to G/L$ be the quotient map. Recall that $\overline{\nu} := (\pi_{G/L})_*\nu$ is a μ-stationary measure on G/L. By Corollary 4.3.8, one can deduce that

$$\overline{\nu}_{\omega^-}|_{\mathcal{W}^-_{G/L}[(\omega,x)]} = \delta_x$$

for $\hat{\overline{\nu}}$-a.e. $(\omega, x) \in \mathcal{S}^{\mathbb{Z}} \times G/L$. Therefore $\nu_{\omega^-}|_{\pi^{-1}(\mathcal{W}^-_{G/L}[(\omega,x)])}$ is supported on $\pi^{-1}(x)$ for $\overline{\nu}$-a.e. x. Note that $\mathcal{W}^-_{G/H}[(\omega, z)] \subset \pi^{-1}(\mathcal{W}^-_{G/L}[(\omega, x)])$ if $x = \pi(z)$, therefore we also have that $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]}$ is supported on $\pi^{-1}(\pi(z)) = zL/H$ for ν-a.e. z. On the other hand, by the Case II assumption, we know that $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]}$ is supported on $\mathcal{F}^{\leq 0}_{\Gamma}[z]$, and therefore it is supported on $zL/H \cap \mathcal{F}^{\leq 0}_{\Gamma}[z]$. By Corollary 4.4.6, $zL/H \cap \mathcal{F}^{\leq 0}_{\Gamma}[z] = \{z\}$. Therefore we get that for $\hat{\nu}$-a.e. (ω, z),

$$\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]} = \delta_z.$$

□

The key lemma is the following.

Lemma 4.11.13. For $\hat{\nu}$-a.e. $(\omega, z) \in \mathcal{S}^{\mathbb{Z}} \times G/H$, we have $\tau_{(\omega^+,z)} = \mu^{-\mathbb{N}}$.

Proof. The idea is to use an argument similar to [Led84, Thm. 3.4] (see also [LY85, Sect. 6.1] and [BRH17, Prop. 11.1]). Define partitions $\hat{\eta}$ on $\mathcal{S}^{\mathbb{Z}} \times G/H$ and η on $\mathcal{S}^{\mathbb{Z}}$ such that the atoms are

$$\hat{\eta}[(\omega, z)] = \mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \mathcal{F}^{\leq 0}_{\Gamma}[z], \qquad \text{and} \qquad \eta[\omega] = \mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega].$$

It follows from Corollary 4.11.5(b) that $\hat{\eta}$ is a measurable partition. Note that since the partition $\mathcal{F}^{\leq 0}_{\Gamma}[z]$ is equivariant under μ-a.e. $g \in G$ by Corollary 4.11.5(a), we have $\hat{T}\hat{\eta}[(\omega, z)] = T(\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]) \times \mathcal{F}^{\leq 0}_{\Gamma}[z]$.

Under the Case II assumption, which in particular implies zero fiber entropy $h^{G/H}_{\hat{\nu}}(\hat{T}) = 0$, we have

$$h_{\mu^{\mathbb{Z}}}(T) = h_{\hat{\nu}}(\hat{T}) = H_{\hat{\nu}}(\hat{T}\hat{\eta} \mid \hat{\eta}). \tag{4.11.3}$$

Now using the Case II assumption, we compute that

$$H_{\hat{\nu}}(\hat{T}\hat{\eta} \mid \hat{\eta}) = -\int \log \frac{\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}(\hat{T}\hat{\eta}[(\omega,z)])}{\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]}(\hat{\eta}[(\omega,z)])}\, d\hat{\nu}(\omega,z) = -\int \log \tau_{(\omega^+,z)}(T\eta[\omega]) d\hat{\nu}(\omega,z),$$

and since $\hat{\nu}$ projects to $\mu^{\mathbb{Z}}$ under the map $\mathcal{S}^{\mathbb{Z}} \times G/H \to \mathcal{S}^{\mathbb{Z}}$, we have

$$h_{\mu^{\mathbb{Z}}}(T) = -\int \log \mu^{-\mathbb{N}}(T\eta[\omega])\, d\hat{\nu}(\omega,z).$$

Substituting both into (4.11.3), we have

$$\int \log \frac{\mu^{-\mathbb{N}}(T\eta[\omega])}{\tau_{(\omega^+,z)}(T\eta[\omega])}\, d\hat{\nu}(\omega,z) = 0. \tag{4.11.4}$$

For $s \in \mathcal{S}$, let $Y[s] := \{\omega \in \mathcal{S}^{-\mathbb{N}} \mid \omega_{-1} = s\}$. Now we note that for $\hat{\nu}$-a.e. (ω,z), we have

$$\begin{aligned}\int_{\hat{\eta}[(\omega,z)]} \log \frac{\mu^{-\mathbb{N}}(T\eta[\omega'])}{\tau_{(\omega,z)}(T\eta[\omega'])}\, d\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]} &= \int_{\eta[\omega]} \log \frac{\mu^{-\mathbb{N}}(T\eta[\omega'])}{\tau_{(\omega,z)}(T\eta[\omega'])}\, d\tau_{(\omega^+,z)}(\omega') \\ &= \sum_{s\in\mathcal{S}} \tau_{(\omega,z)}(Y[s]) \log \frac{\mu(\{s\})}{\tau_{(\omega,z)}(Y[s])} \le 0,\end{aligned}$$

where the last inequality follows from the convexity of log. But then the integral of the left hand side over (ω,z) with respect to $\hat{\nu}$ is 0 by (4.11.4), therefore the equality case holds for $\hat{\nu}$-a.e. (ω,z). Thus we have $\tau_{(\omega^+,z)}(Y[s]) = \mu^{-\mathbb{N}}(Y[s])$ for $\hat{\nu}$-a.e. (ω,z) and $s \in \mathcal{S}$. Now repeating the argument for $\hat{T}^k$ in lieu of $\hat{T}$ yields the claim. □

Using this description, and the Case II assumption, we have the following corollary.

Corollary 4.11.14. Under the Case II assumption, for $\hat{\nu}$-a.e. (ω,z), there exists $\phi_{\omega^+,z}(\omega^-) \in$

$\mathcal{W}^-_{G/H}[(\omega^+, z)] \cap \mathcal{F}^{\leq 0}_{\overline{\Gamma}}[z]$ such that

$$\hat{\nu}|_{\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]\times(\mathcal{W}^-_{G/H}[(\omega,z)]\cap\mathcal{F}^{\leq 0}_{\overline{\Gamma}}[z])} = \hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]} = \int_{\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]} \delta_{\omega^-} \times \delta_{\phi_{\omega^+,z}(\omega^-)} d\mu^{-\mathbb{N}}(\omega^-)$$
$$= (\mathrm{id} \times \phi_{\omega^+,z})_*(\mu^{-\mathbb{N}}).$$

Proof. By Lemma 4.11.13 and (4.11.2), we can write, for $\hat{\nu}$-a.e. (ω, z),

$$\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]} = \int_{\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]} \delta_{\omega^-} \times \left(\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]}\right) d\mu^{-\mathbb{N}}(\omega^-).$$

This in particular implies that for $\mu^{-\mathbb{N}}$-a.e. ω^-, $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]}$ is not the zero measure. By Lemma 4.11.12, we have $\nu_{\omega^-}|_{\mathcal{W}^-_{G/H}[(\omega,z)]} = \delta_z$. Hence for $\hat{\nu}$-a.e. (ω, z), there exists $\phi_{\omega^+,z}(\omega^-) \in \mathcal{W}^-_{G/H}[(\omega^+, z)] \cap \mathcal{F}^{\leq 0}_{\overline{\Gamma}}[z]$ such that

$$\hat{\nu}|_{\hat{\mathcal{W}}^-_{G/H}[(\omega,z)]} = \int_{\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]} \delta_{\omega^-} \times \delta_{\phi_{\omega^+,z}(\omega^-)} d\mu^{-\mathbb{N}}(\omega^-) = (\mathrm{id} \times \phi_{\omega^+,z})_*(\mu^{-\mathbb{N}}).$$

The first equality in the statement follows from the Case II assumption. □

Proof of Proposition 4.11.2. Let $\tilde{\nu}$ be the pushforward of ν via $\overline{\Gamma}^Z_\mu x_0 L/H \to \mathcal{F}^{\leq 0}_{\overline{U}} \backslash \overline{\Gamma}^Z_\mu x_0 L/H$, and $\tilde{\nu}_{\omega^-}, \hat{\tilde{\nu}}$ be the corresponding measures on $\mathcal{F}^{\leq 0}_{\overline{U}} \backslash \overline{\Gamma}^Z_\mu x_0 L/H$ and $\mathcal{S}^{\mathbb{Z}} \times \mathcal{F}^{\leq 0}_{\overline{U}} \backslash \overline{\Gamma}^Z_\mu x_0 L/H$ as in subsection 4.4.2. Corollary 4.11.14 implies that for $\hat{\tilde{\nu}}$-a.e. $(\omega, [z])$,

$$\hat{\tilde{\nu}}|_{\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega]\times\{[z]\}} = \mu^{-\mathbb{N}},$$

where we identify $\mathcal{W}^-_{\mathcal{S}^{\mathbb{Z}}}[\omega] \times \{[z]\} \leftrightarrow \mathcal{S}^{-\mathbb{N}}$. This implies that

$$d\hat{\tilde{\nu}}(\omega^-, \omega^+, [z]) = d\mu^{-\mathbb{N}}(\omega^-) d\theta(\omega^+, [z])$$

for some measure θ on $\mathcal{S}^{\mathbb{N}} \times \mathcal{F}^{\leq 0}_{\overline{U}} \backslash \overline{\Gamma}^Z_\mu x_0 L/H$. This implies that $\tilde{\nu} = \tilde{\nu}_{\omega^-}$ for almost every ω^-. Now this together with Proposition 4.11.3 implies Proposition 4.11.2. □

Proposition 4.11.15. Let μ be a probability measure on G with finite first moment, let $\Gamma_\mu := \overline{\langle \text{supp } \mu \rangle} \subset G$ be the (topological) closure of the semigroup generated by the support of μ and $\overline{\Gamma}_\mu^Z \subset G$ be the Zariski closure of Γ_μ.

Let L be an H-envelope such that μ is uniformly expanding on L/H at x_0 for some $x_0 \in G/L$. Let $\pi : G/H \to G/L$ be the quotient map. Suppose Case II and (†) holds.

Let ν be an ergodic μ-stationary probability measure on $\overline{\Gamma}_\mu^Z x_0 L/H$, and $\overline{\nu} := \pi_* \nu$ be its pushforward on G/L. We disintegrate ν with respect to the map π:

$$\nu = \int_{G/L} \nu_x \, d\overline{\nu}(x).$$

Then there exists a positive integer k such that for $\overline{\nu}$-almost every $x \in G/L$, ν_x is the uniform measure on k points in $\pi^{-1}(x) \subset G/H$.

Proof. We disintegrate the stationary measure ν into $\{\nu_x\}_{x \in G/L}$ with respect to the partition given by the fibers of the projection map $\pi : G/H \to G/L$. Since $\overline{\nu} = \pi_* \nu$, we have

$$\nu = \int_{G/L} \nu_x \, d\overline{\nu}(x).$$

Moreover, by Proposition 4.11.2, ν is supported on $\bigcup_{i=1}^{m} \mathcal{F}_\Gamma^{\leq 0}[z_i]$. Since $\mathcal{F}_\Gamma^{\leq 0}[z]$ intersects each fiber of π at at most one point by uniform expansion on L/H by Corollary 4.8.5, each ν_x is finitely supported at k_x points. Moreover, by ergodicity of $\overline{\nu}$, k_x is constant for $\overline{\nu}$-almost every $x \in G/L$, and that each ν_x is the uniform measure on k points in $\pi^{-1}(x)$ (see e.g. Sargent-Shapira [SS19] Section 4). □

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