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ABSTRACT

We show that the triangulated category of bounded constructible complexes on an algebraic variety X over an algebraically closed field is equivalent to the bounded derived category of the abelian category of constructible sheaves on X, extending a theorem of Nori to the case of positive characteristic. The coefficients are allowed to be finite of torsion, a finite extension of $\mathbf{Q}_\ell$, or the ring of integers of such. A series of reductions due to Nori, independent of the characteristic of the ground field, show that the aforementioned equivalence of categories is implied by the property that the (étale or ℓ-adic) cohomology in degrees strictly greater than zero of a constructible sheaf on an affine space $\mathbf{A}^n$ is effaceable by a monomorphism into another constructible sheaf, which is a theorem of Nori in characteristic zero.

To extend Nori's theorem on the effaceability of the cohohomology of a constructible sheaf on affine space to positive characteristic, we use a variation of a theorem of Achinger to show that the hypotheses needed to proceed with Nori's construction can be secured after twisting our given constructible sheaf $\mathscr{F}$ by an automorphism of $\mathbf{A}^n$. We show that we can find an automorphism g of $\mathbf{A}^n$ so that if we extend our twisted sheaf $g^*\mathscr{F}$ by zero to $\mathbf{A}^{n-1} \times \mathbf{P}^1$, the restriction of its singular support to a neighborhood of $\mathbf{A}^{n-1} \times \{\infty\}$ is either empty or is the union of the zero section and the conormal to this divisor. This condition implies that the formation of the direct image of $g^*\mathscr{F}$ along the open immersion $\mathbf{A}^n \hookrightarrow \mathbf{A}^{n-1} \times \mathbf{P}^1$ commutes with restriction to the fibers of the projection to $\mathbf{A}^{n-1}$. Following Nori, we show that given any constructible sheaf $\mathscr{G}$ on $\mathbf{A}^n$ that has this property and which is the extension by zero of a lisse sheaf on the complement of a closed subset of $\mathbf{A}^n$ finite and surjective over $\mathbf{A}^{n-1}$, $\mathscr{G}$ can be embedded in a constructible sheaf $\mathscr{H}$ so that $R\pi_*\mathscr{H} = 0$, where $\pi : \mathbf{A}^n \to \mathbf{A}^{n-1}$ is the projection. This is enough to make the rest of Nori's argument work in positive characteristic.

ACKNOWLEDGMENTS

1. Introduction

Let X/k be an algebraic variety (i.e. a separated scheme of finite type) over a base field k. Let p be the characteristic of k and ℓ a prime number, $\ell \neq p$. Denote by $D(X) = D(X, \Lambda)$ the triangulated category of bounded constructible complexes of Λ-sheaves on X; here the coefficient ring Λ is either finite with ℓ nilpotent in it, or, in the ℓ-adic setting, Λ is either a finite extension E of $\mathbf{Q}_\ell$ or its ring of integers R_E.[1] This is a t-category whose heart is the abelian category $C(X) = C(X, \Lambda)$ of constructible Λ-sheaves on X. Consider the bounded derived category $D^b(C(X))$ and the evident t-exact functor $F : D^b(C(X)) \to D(X)$ which identifies the hearts.

Theorem **(1.1).** — *If k is algebraically closed then* F *is an equivalence of categories.*

For Λ finite, the theorem is valid for any k and follows from [SGA4, VI 5.8, IX 2.9 (iii)]. In the ℓ-adic setting it was proved in Nori's remarkable article [N] under the assumption that $p = 0$. The aim of this note is to remove that assumption.

Remarks **(1.2).** — (i) In his paper, Nori embeds k into **C** and considers the corresponding Betti version of constructible sheaves with arbitrary coefficient rings R. If $R = R_E$ then these are the same as étale R_E-sheaves.

(ii) Nori's theorem 3 (b) asserts that the Betti version of (1.1) above is true when R is a field. But his theorem 3 (a) implies that (1.1) is true for $R = R_E$ as well. Indeed, torsion free constructible sheaves generate $D^b(C(X, R_E))$ as a triangulated category,[2] and so they generate $D(X, R_E)$ too (since its t-structure is bounded). By Nori's theorem 3 (a), for each $n \in \mathbf{Z}$, F yields isomorphisms between $\mathrm{Hom}(\mathscr{A}[n], \mathscr{B})$ when $\mathscr{A}$ and $\mathscr{B}$ are among the above generators, hence is an equivalence.

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1. If Λ is finite, $D(X, \Lambda)$ is the full subcategory of the bounded derived category of étale sheaves of Λ-modules on X consisting of those complexes with constructible cohomology sheaves. In the ℓ-adic setting, $D(X, \Lambda)$ was defined by Ekedahl [E] over an arbitrary k.

2. To see this, note that every constructible sheaf is an extension of a torsion free sheaf by a torsion sheaf, and every torsion sheaf admits a finite filtration with successive quotients of the form $j_!\mathscr{F}$, where $\mathscr{F}$ is a locally constant torsion sheaf on a locally closed subvariety $j : Y \hookrightarrow X$. It remains to find a surjection $\alpha : \mathscr{G} \twoheadrightarrow j_!\mathscr{F}$ where $\mathscr{G}$ is torsion free (its kernel $\mathscr{K}$ is torsion free then as well, and $j_!\mathscr{F} = \mathrm{Cone}(\mathscr{K} \to \mathscr{G})$). Let $p : T \to Y$ be a finite etale covering such that $p^*\mathscr{F}$ is constant. Pick a surjection $(R_E)_T^m \twoheadrightarrow p^*\mathscr{F}$. The promised α is the composition $j_!p_*(R_E)_T^m \twoheadrightarrow j_!p_*p^*\mathscr{F} \twoheadrightarrow j_!\mathscr{F}$.

Nori deduces theorem 1.1 from the following theorem 1.3 by an argument that works for arbitrary p.

Theorem **(1.3)**. — *Suppose k is algebraically closed. Then every constructible Λ-sheaf on $\mathbf{A}^n$ is a subsheaf of a constructible sheaf $\mathcal{G}$ with $H^q(\mathbf{A}^n, \mathcal{G}) = 0$ for $q > 0$.*

It is in the proof of theorem 1.3 that Nori uses the assumption $p = 0$. Precisely, the proof goes by induction on n. Given a constructible sheaf $\mathcal{F}$ on $\mathbf{A}^n$, we can assume that the projection $\mathbf{A}^n \to \mathbf{A}^{n-1}, (x_1, .., x_n) \mapsto (x_1, .., x_{n-1})$, becomes finite on the singular locus Y of $\mathcal{F}$. The key fact used by Nori is that then $\mathcal{F}$ is equisingular along $x_n = \infty$: the reason for this is that if $k = \mathbf{C}$, so that we can use the classical topology, then for any ball $U \subset \mathbf{A}^{n-1}$ and punctured disc D° in $\mathbf{A}^1$ around ∞, a local system on $U \times D^\circ$ is the same as a local system on D°. This assertion is in no sense valid in case $p > 0$ due to the effect of wild ramification. We show that, by a slight elucidation of Achinger's result [A, 3.6], the above equisingularity is achieved after turning $\mathcal{F}$ by an appropriate quadratic automorphism of $\mathbf{A}^n$. This is enough to make the rest of Nori's argument work for $p > 0$.

2. On a theorem of Achinger

We explain a version of Achinger's theorem [A, 3.6]. The constructions of the proof are Achinger's original ones; we streamline his argument (avoiding the use of a theorem of Deligne-Laumon) and obtain a precise description of the singular support.[3] In this section, the coefficient ring Λ is finite.

Let $\pi : \mathbf{A}^{n+1} \to \mathbf{A}^n, \bar{\pi} : \mathbf{A}^n \times \mathbf{P}^1 \to \mathbf{A}^n, j : \mathbf{A}^{n+1} \hookrightarrow \mathbf{A}^n \times \mathbf{P}^1$ be the evident projections and open embedding; set $D := \mathbf{A}^n \times \{\infty\} = \mathbf{A}^n \times \mathbf{P}^1 \setminus \mathbf{A}^{n+1}$. Let $\mathcal{F} \in D(\mathbf{A}^{n+1})$ be a constructible complex and Y a closed subset of $\mathbf{A}^{n+1}$, $Y \neq \mathbf{A}^{n+1}$, such that $\mathcal{F}$ is locally constant[4] on $\mathbf{A}^{n+1} \setminus Y$.

Theorem **(2.1)**. — *If k is infinite, then one can find an automorphism g of $\mathbf{A}^{n+1}$ such that*

(i) the restriction of π to $g^{-1}(Y)$ is finite, and

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3. For the notion of singular support, see Beilinson [B] and Saito [S].

4. i.e. all cohomology sheaves of $\mathcal{F}$ are locally constant.

(ii) the restriction of $\mathrm{SS}(j_! g^* \mathcal{F})$ to the complement of $g^{-1}(Y)$ is either empty (if $\mathcal{F}$ is supported on Y) or is the union of the zero section and the conormal to D.

If k is finite, then one can find g as above after a finite extension of k.

Proof. — As the theorem is automatically true when $n = 0$ and $g = \mathrm{id}$, assume $n > 0$ and let $x_1, \ldots, x_n$ and s be the linear coordinates on $\mathbf{A}^n$ and $\mathbf{A}^1$, and $s : t$ be the homogeneous coordinates on $\mathbf{P}^1$, so that $\bar{\pi}(x_1, \ldots, x_n, (s : t)) = (x_1, \ldots, x_n)$ and $j(x_1, \ldots, x_n, s) = (x_1, \ldots, x_n, (s : 1))$. Let $(y_0 : y_1 : \ldots : y_n : y_{n+1})$ be the homogeneous coordinates on $\mathbf{P}^{n+1}$ and $j' : \mathbf{A}^{n+1} \hookrightarrow \mathbf{P}^{n+1}$ be the embedding $(x_1, \ldots, x_n, s) \mapsto (1 : x_1 : \ldots : x_n : s)$; let $\overline{Y}$ be the closure of Y in $\mathbf{P}^{n+1}$. If $\mathcal{F}$ is supported on Y, we can take for g any linear transformation such that $g(0 : \ldots : 0 : 1) \notin \overline{Y}$. So we assume that $\mathcal{F}$ is generically nonzero. Our g will be of the form $g = ah$, where a is a linear automorphism of $\mathbf{A}^{n+1}$, and h is the automorphism $(x_1, \ldots, x_n, s) \mapsto (x_1 + s^2, x_2, \ldots, x_n, s)$. Notice that h extends to a map

$$\bar{h} : \mathbf{A}^n \times \mathbf{P}^1 \to \mathbf{P}^{n+1} \qquad (x_1, \ldots, x_n, (s : t)) \mapsto (t^2 : t^2 x_1 + s^2 : t^2 x_2 : \ldots t^2 x_n : st)$$

sending D to $c := (0 : 1 : 0 : \ldots : 0) \in \mathbf{P}^{n+1}$. Let W be a vector space with coordinates $z, x_2, \ldots, x_n, s$ and σ be the map

$$\sigma : W \to \mathbf{P}^{n+1} \setminus (y_1 = 0) \qquad (z, x_2, \ldots, x_n, s) \mapsto (z^2 : 1 : x_2 : \ldots : x_n : s);$$

note that $\sigma(0) = c$.

Step 1. We formulate conditions (a)–(c) on $\mathcal{F}$ and prove that they assure that our theorem holds with $g = h$. The cases $p \neq 2$ and $p = 2$ are considered separately.

Case $p \neq 2$: Our conditions are (a) $c \notin \overline{Y}$, (b) the fiber $\mathrm{SS}(\sigma^* j'_! \mathcal{F})_0$ has dimension 1, i.e. $\mathrm{SS}(\sigma^* j'_! \mathcal{F})_0$ is the union of a finite nonempty set of lines, and (c) $\tau := (1, 0, \ldots, 0, 1) \in T_0 W = W$ does not lie in the union of hyperplanes orthogonal to the lines from (b).

Condition (a) implies that $h^{-1}(Y) = \bar{h}^{-1}(\overline{Y})$ is closed in $\mathbf{A}^n \times \mathbf{P}^1 \to \mathbf{A}^n$, so that (i) of

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theorem 2.1 holds. Let us check (ii). We need to show that the restriction of $SS(j_!h^*\mathscr{F})$ to D is the conormal to D. It is enough to check our claim replacing $\mathbf{A}^n \times \mathbf{P}^1$ by an étale neighborhood V of D; we choose V to be an étale covering of $\mathbf{A}^n \times \mathbf{P}^1 \smallsetminus ((s = 0) \cup (t^2 x_1 + s^2 = 0))$ obtained by adding v, $v^2 = 1 + (t/s)^2 x_1$, to the ring of functions; the embedding $D \hookrightarrow V$ is $t \mapsto 0, v \mapsto 1$. The restriction of $\overline{h}$ to V lifts to a map

$$\kappa : V \to W \qquad (x_1, \ldots, x_n, (1 : t), v) \mapsto (t/v, (t/v)^2 x_2, \ldots, (t/v)^2 x_n, t/v^2)$$

sending D to $0 \in W$ and the normal vector ∂_t at any point of D to τ. Now (b) and (c) mean that κ is properly $SS(\sigma^* j_!' \mathscr{F})$-transversal on a neighborhood of D, and so, by Saito [S, 8.15],[5] $SS(\kappa^* \sigma^* j_!' \mathscr{F}) = \kappa^\circ SS(\sigma^* j_!' \mathscr{F})$, which is the union of the zero section and the conormal to D. We are done since $\kappa^* \sigma^* j_!' \mathscr{F}$ is the pullback of $j_! h^* \mathscr{F}$ to V.

Case $p = 2$: Consider the purely inseparable map $\mathrm{Fr}_1 : \mathbf{A}^{n+1} \to \mathbf{A}^{n+1}, (x_1, \ldots, x_n, s) \mapsto (x_1^2, x_2, \ldots, x_n, s)$. Set $F := h \, \mathrm{Fr}_1 \, h^{-1}$ and $\mathscr{F}' := F_* \mathscr{F}$, so $F^* \mathscr{F}' = \mathscr{F}$. Our conditions are (a) $c \notin \overline{Y}$, (b) $\dim SS(\sigma^* j_!' \mathscr{F}')_0 = 1$, and (c) τ does not lie in the union of hyperplanes orthogonal to the lines in $SS(\sigma^* j_!' \mathscr{F}')_0$. Let us check that they imply that theorem 2.1 holds with $g = h$. As above, (a) implies (i) of the theorem. The map Fr_1 extends to a map $\mathbf{A}^n \times \mathbf{P}^1 \to \mathbf{A}^n \times \mathbf{P}^1$ also called Fr_1; notice that $\overline{h} \, \mathrm{Fr}_1$ lifts to a map $\chi : \mathbf{A}^n \times \mathbf{P}^1 \smallsetminus ((s = 0) \cup (tx_1 + s = 0)) \to W$ given in coordinates by

$$(x_1, \ldots, x_n, (1 : t)) \mapsto \left(\frac{t}{tx_1 + 1}, \frac{t^2 x_2}{t^2 x_1^2 + 1}, \ldots, \frac{t^2 x_n}{t^2 x_1^2 + 1}, \frac{t}{t^2 x_1^2 + 1} \right).$$

As before, $\chi(D) = 0$ and χ is properly $SS(\sigma^* j_!' \mathscr{F}')$-transversal, so $SS(\chi^* \sigma^* j_!' \mathscr{F}') = \chi^\circ SS(\sigma^* j_!' \mathscr{F}')$ is again the union of the zero section and the conormal to D. As $\mathrm{Fr}_1^* h^* \mathscr{F}' = h_*^{-1} \mathscr{F} = h^* \mathscr{F}$, we are done.

STEP 2. It remains to find a linear transformation a of $\mathbf{A}^{n+1}$ such that $a^* \mathscr{F}$ satisfies the above conditions (a)–(c). Let $G \subset GL(n+1)$ be the group of linear transformations a such that $a^*(x_1) = x_1$.

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5. In the formulation of [S, 8.15], Saito assumes that k is perfect. This assumption is redundant since singular support is compatible with change of base field by [B, 1.4 (iii)], and so Saito's assertion amounts to the one for the base change of the data to the perfect closure of k.

The action of G on $\mathbf{P}^{n+1}$ lifts to an action of G on W fixing the coordinate z. Let $D' \subset W$ denote the hyperplane $(z = 0)$. The vector bundle $TW|_{D'}$ over D' decomposes as the direct sum of the tangent bundle to D' and the normal line bundle $\ker(d\sigma)$; the action of G preserves this decomposition. Let $\mathbf{P}(TW)|^{\circ}_{D'}$ be the complement in $\mathbf{P}(TW)|_{D'}$ of the projectivization of these subbundles. One checks that the action of G on this open subset is transitive. So, since the point in $\mathbf{P}(TW)$ that corresponds to τ lies in $\mathbf{P}(TW)|^{\circ}_{D'}$, its G-orbit is Zariski open in $\mathbf{P}(TW)|_{D'}$.

Let $C \subset T^*W$ be $\mathrm{SS}(\sigma^* j'_! \mathscr{F})$ if $p \neq 2$ or $\mathrm{SS}(\sigma^* j'_! \mathscr{F}')$ if $p = 2$. There is a nonempty open subset Q of D' such that every fiber of C over Q is the union of a finite nonempty set of lines. Indeed, since $\dim C = n + 1$ by [B], one can find Q such that the fibers of C over Q have dimension ≤ 1; their dimension equals 1 since $\sigma^* j'_! \mathscr{F}$, or $\sigma^* j'_! \mathscr{F}'$, is not locally constant at the generic point of D' (recall that $\mathscr{F}$ does not vanish at the generic point of $\mathbf{A}^{n+1}$). Shrinking Q, we may assume it does not intersect $\overline{Y}$. By the above, we can find $a \in G(k)$ such that $a(c) \in Q$ and $a(\tau)$ lies in the complement of the hyperplanes orthogonal to the lines in the fiber of C at $a(c)$, at least when k is infinite. When k is finite, we may need to pass to a finite extension of k to find the promised a. $\quad\square$

Remarks (**2.2**). — (i) The assertion of theorem 2.1 and its proof remain valid in the setting of holonomic $\mathscr{D}$-modules. (The reference to [S, 8.15] should be replaced by theorem 4.7 (3) in [K, §4.4].) This answers positively Kontsevich's question from [A, §1.3].

(ii) Achinger's theorem [A, 3.6] asserts that in the case of locally constant $\mathscr{F}$, $\overline{\pi}$ is locally acyclic rel. $j_! \mathscr{F}$. This follows immediately from theorem 2.1.

3. Adapting Nori's argument to characteristic p

We explain how to modify Nori's proof of the theorems from §1 to make it work in any characteristic. In fact, only one step of the proof of theorem 1.3, which goes by induction on n, requires modification. We present it as proposition 3.2 below; it replaces Nori's proposition 2.2.

As in §1, the coefficient ring Λ is either finite or ℓ-adic. For a map ? between algebraic varieties, we write $?_*$ instead of R$?_*$, etc.

Suppose we are in the situation of §2; we follow the notation there. For a geometric point s of $\mathbf{A}^n$ we denote by $i_s : \mathbf{A}^1 = \pi^{-1}(s) \hookrightarrow \mathbf{A}^{n+1}, \bar{i}_s : \mathbf{P}^1 = \bar{\pi}^{-1}(s) \hookrightarrow \mathbf{A}^n \times \mathbf{P}^1$ the embeddings of the s-fibers; let $j_s : \mathbf{A}^1 \hookrightarrow \mathbf{P}^1$ be the restriction of j to the s-fibers.

Lemma (**3.1**). — *If k is infinite, then one can find an automorphism g of $\mathbf{A}^{n+1}$ such that $g^{-1}(\mathrm{Y})$ is finite over $\mathbf{A}^n$, and for every s as above the base change morphism $\bar{i}_s^* j_* g^* \mathscr{F} \to j_{s*} i_s^* g^* \mathscr{F}$ is an isomorphism. If k is finite, then such g exists after a finite extension of k.*

Proof. — Assume for the moment that Λ is finite. Our g is the automorphism from theorem 2.1 applied to the Verdier dual $\mathrm{D}\mathscr{F}$ of $\mathscr{F}$. We check our claim on the complement of $g^{-1}(\mathrm{Y})$, which is a neighborhood of the divisor D. By (2.1), $\bar{i}_s$ is $\mathrm{SS}(j_! g^* \mathrm{D}\mathscr{F})$-transversal there, and so one has

$$
\begin{aligned}
\bar{i}_s^!(j_! g^* \mathrm{D}\mathscr{F}) &= \bar{i}_s^*(j_! g^* \mathrm{D}\mathscr{F})(-n)[-2n] \\
&= j_{s!} i_s^*(g^* \mathrm{D}\mathscr{F})(-n)[-2n] \\
&= j_{s!} \mathrm{D}(i_s^* g^* \mathscr{F}) \\
&= \mathrm{D}(j_{s*} i_s^* g^* \mathscr{F});
\end{aligned}
$$

here the first equality comes from [S, 8.13]. Applying D to both sides, we are done.

The case of ℓ-adic coefficients follows from that of finite coefficients. Indeed, for $\mathscr{F} \in \mathrm{D}(\mathbf{A}^{n+1}, \mathrm{R}_{\mathrm{E}})$ the assertion of the lemma for $\mathscr{F} \otimes^{\mathbf{L}} \mathbf{Z}/\ell$ amounts to the one for $\mathscr{F}$ (with the same g and Y),[6] and implies the one for $\mathscr{F} \otimes \mathbf{Q}_\ell$. $\qquad\square$

Henceforth $\mathscr{F}$ is a constructible sheaf (not a complex). Replacing $\mathscr{F}$ by $g^* \mathscr{F}$, we assume lemma 3.1 holds for $g = \mathrm{id}$. Enlarging Y (so that it is still finite over $\mathbf{A}^n$) we assume that $\pi(\mathrm{Y}) = \mathbf{A}^n$. Replacing $\mathscr{F}$ by its extension by zero from $\mathbf{A}^{n+1} \setminus \mathrm{Y}$, we assume $\mathscr{F}|_{\mathrm{Y}} = 0$.

Proposition (**3.2**). — *Under the above conditions, one can find an embedding of constructible sheaves $\delta : \mathscr{F} \hookrightarrow \mathscr{C}$ such that $\pi_* \mathscr{C} = 0$.*

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6. To see this, notice that $\mathscr{G} \in \mathrm{D}(\mathrm{X}, \mathrm{R}_{\mathrm{E}})$ vanishes iff $\mathscr{G} \otimes^{\mathbf{L}} \mathbf{Z}/\ell = 0$. Thus, a morphism $\mathscr{K} \to \mathscr{L}$ in $\mathrm{D}(\mathrm{X}, \mathrm{R}_{\mathrm{E}})$ is an isomorphism iff $\mathscr{K} \otimes^{\mathbf{L}} \mathbf{Z}/\ell \to \mathscr{L} \otimes^{\mathbf{L}} \mathbf{Z}/\ell$ is an isomorphism (since being an isomorphism amounts to the vanishing of the cone). Now apply this remark to the morphism from the lemma.

Proof. — Nori provides a natural construction of the sheaf $\mathcal{C}$ and the arrow δ. Let $p_1, p_2 : \mathbf{A}^{n+2} \to \mathbf{A}^{n+1}$ denote the canonical projections. We define a sheaf $\mathcal{B}$ on $\mathbf{A}^{n+2}$ via the exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow p_1^* \mathcal{F} \longrightarrow \Delta_* \mathcal{F} \longrightarrow 0. \tag{†}$$

Applying p_{2*} to this sequence, the long exact sequence of cohomology gives the arrow

$$\delta : \mathcal{F} = p_{2*} \Delta_* \mathcal{F} \to H^1 p_{2*} \mathcal{B} =: \mathcal{C}.$$

Let us check the promised properties. One has $p_{2*} p_1^* \mathcal{F} \xleftarrow{\sim} \pi^* \pi_* \mathcal{F}$ by smooth base change. Therefore for every geometric point t of $\mathbf{A}^{n+1}$, putting $s := \pi(t)$ one has

$$(p_{2*} p_1^* \mathcal{F})_t = (\pi_* \mathcal{F})_s = (\overline{\pi}_* j_* \mathcal{F})_s = R\Gamma(\mathbf{A}^1, i_s^* \mathcal{F}),$$

the last equality following from proper base change and lemma 3.1. By Artin's theorem, the latter complex is acyclic off degrees 0 and 1; as $i_s^* \mathcal{F}$ is the extension by zero of a locally constant sheaf from the complement of a nonempty finite set Y_s, its degree 0 cohomology vanishes, and the complex $p_{2*} p_1^* \mathcal{F}$ is in fact concentrated in degree 1. This implies that p_{2*} transforms (†) into a distinguished triangle whose long exact sequence of cohomology reduces to the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\delta} \mathcal{C} \longrightarrow H^1 p_{2*} p_1^* \mathcal{F} \longrightarrow 0,$$

showing δ injective and $p_{2*} \mathcal{B}$ acyclic off degree 1, i.e. $\mathcal{C} = p_{2*} \mathcal{B}[1]$. If $\Pi := \pi p_1 = \pi p_2$,

$$\pi_* \mathcal{C} = \Pi_* \mathcal{B}[1] = \text{Cone}(\Pi_* p_1^* \mathcal{F} \longrightarrow \Pi_* \Delta_* \mathcal{F}) = 0,$$

and we have the proposition. □

Remark (**3.3**). — If k is no longer supposed algebraically closed, but of cohomological

dimension ≤ 1, the proof of theorem 1.3 gives that if $\mathcal{F}$ is a constructible sheaf on $\mathbf{A}^n$, there exists a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ into a constructible $\mathcal{G}$ inducing the null morphism $H^q(\mathbf{A}^n, \mathcal{F}) \to H^q(\mathbf{A}^n, \mathcal{G})$ for $q > 0$, and theorem 1.1 continues to hold over such a k.

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