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CHAPTER 1

INTRODUCTION

One of the fascinating features of number theory is the deep connection between an object's arithmetic properties and the analytic properties of its L-function. Perhaps, one of the oldest examples of such a beautiful connection is the class number formula, which we now give an overview.

Let K be a finite extension of the rational number field $\mathbb{Q}$. Suppose $[K : \mathbb{Q}] = r_1 + 2r_2$ where r_1 (respectively r_2) is the number of real (respectively complex) places of K. We can attach to K its Dedekind zeta function which is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s},$$

where the sume is taken over all nonzero integral ideals of K. This zeta function has a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. For K, one of its most important and interesting invariances is its class number h_K which tells the failure of unique factorization in its subring of algebraic intergers $\mathcal{O}_K$. The class number formula says that

$$\lim_{s \to 1} \frac{\zeta_K(s)}{s-1} = \frac{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}_K h_K}{w_K\sqrt{|D_K|}},$$

where Reg_K, w_K, D_K are respectively the regulator, the number of roots of unity in K, the discriminant of K. This simple and elegant formula provides a bridge between the world of analytic functions and the world of algebraic numbers.

In the 1960s, Birch and Swinnerton-Dyer did some numerical computations with the number of points of an elliptic curve E over a finite field. These numerical computations led them to make a conjecture which relates the special values $L(E,1)$ in terms of certain invariants of the elliptic curves, including the rank of E and the size of the Tate-Shafarevich group. Their conjecture can be considered as another instance of the class number formula.

In the second half of the 20th century, many attempts were made to put these two examples into a unified picture: how can we understand $L(M,n)$ where M is a (mixed) motive, and n is an integer? In his seminal work, Beilinson (and independently Deligne) made a breakthrough by putting this question in the framework of mixed motives and motivic cohomology. Unfortunately, this work only predicts L-values up to an undetermined rational factor. Later on, with the advances in p-adic Hodge theory initiated by the pioneering work of Fontaine; Bloch and Kato [3] formulated a more precise conjecture about the L-values, which generalized the previous works of Beilinson and Deligne. They also verified their conjecture for the Riemann zeta function and provided partial results for $M = H^1(E)(2)$ where E is an elliptic curve with complex multiplication. Bloch and Kato's approach was inspired by the study of Tamagawa number of an algebraic group; hence their work is often referred to as the Tamagawa number conjecture. Shortly afterward, Fontaine and Perrin-Riou, and independently Kato reformulated and generalized Bloch-Kato's work in the framework of Galois cohomology and determinants of perfect complexes. More recently, in [13], K. Kato defined and studied heights of motives with attempts to look at the Tamagawa number conjecture from another perspective. More precisely, he expects that it is possible to consider the Tamagawa number conjecture as to the problem of counting the numbers of mixed motives of bounded heights with a fixed graded filtration. We will investigate Kato's approach in the first part of this thesis.

The analogies between number fields and function fields of one variable over finite fields often provide an effective way to test a conjecture. For example, in the case of the Tamagawa number conjecture mentioned in the previous paragraph, Bloch and Kato partially verified their conjecture over number fields by considering an analogous problem over function fields of characteristics p. More precisely, for the ℓ-adic realization of a motive M over a function field F with $\ell \neq p$, Bloch and Kato showed that over function fields the Tamagawa number conjecture is a consequence of the Grothendieck's formula. When $\ell = p$, Bloch and Kato explained in their paper that their method would not work, and new ideas were needed. Very recently, by using the work of Ogus on F-crystals, Kato essentially resolved the Tamagawa number conjecture when $\ell = p$ by constructing certain syntomic complexes whose cohomology groups are closely related to Selmer groups associated with F. On another direction, the success of classical Iwasawa theory in solving the Tamagawa number conjecture over number fields suggests that it might be interesting to generalize Kato's work for a family of F-crystals over function fields. This will be carried out in the second part of this thesis.

CHAPTER 2

HEIGHTS AND TAMAGAWA NUMBER CONJECTURE FOR MOTIVES

2.1 Introduction and main results

K. Kato has recently defined and studied heights of mixed motives and proposed some interesting questions (see [13]). In this chapter, we relate the study of heights to the Tamagawa number conjecture for motives. More precisely, we have the following theorem.

Theorem 1. *Let M be a pure motives with integer coefficients of weight $-d$ such that $d \geq 3$. We assume further that M has semistable reduction at all places. Then*

$$\lim_{B\to\infty} \frac{\#\{x \in B(\mathbb{Q}) | H_{\diamond,d}(x) \leq B\}}{\mu\left(x \in \prod'_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(x) \leq B\right)} = \frac{1}{Tam(M)}.$$

Here $B(\mathbb{Q})$ $B(\mathbb{Q}_p)$ are as defined in Bloch-Kato's paper [3].

In theorem 1, we restrict ourselves to the case of pure motives because the Tamagawa number conjecture was formulated this way in [3]. We expect that a similar statement happens if M is a mixed motive. To demonstrate this point, we will give a concrete computation (up to a power of 2) of the terms appeared in left hand side of theorem 1 when M is a mixed Tate motive with graded quotients $\mathbb{Z}(m), \mathbb{Z}(n)$. More precisely, we have the following.

Theorem 2. *Let m, n be two natural number such that $m - n \geq 2$, m is even and n is odd.*

Let D be a mixed Tate motive with graded quotients $\mathbb{Z}(m)$ and $\mathbb{Z}(n)$. Then

$$\#\{x \in B(\mathbb{Q}) | H_{\star,\diamond}(x) \leq B\} \sim 2^t \frac{Ш(D)}{(n-1)!\zeta(n)\zeta(1-m)} \log(B)^n.$$

This theorem answers one of Kato's questions [13] about the number of mixed motives of bounded heights.

Bloch-Kato's approach is to study $B(\mathbb{Q})$, which should be considered the extension group $\mathrm{Ext}^1(\mathbb{Z}; M)$ of a mixed motive, and how it is distributed in its corresponding extension groups of the realizations. It is natural to ask whether it is possible to generalize this study to allow more fixed graded quotients. To show that this question is interesting and important, we will provide a concrete computation with the set of motives with graded quotients $\mathbb{Z}(12), \mathbb{Z}(3), \mathbb{Z}$. More precisely, we have the following theorem.

Theorem 3. *Let X be the set of all mixed motives with graded quotients $\mathbb{Z}(12), \mathbb{Z}(3), \mathbb{Z}$. Then, we have*

$$\#\{x \in X | H_{\star,\diamond}(x) \leq B\} \sim \frac{1}{8!2!\binom{12}{3}} \frac{Ш(3)}{\zeta(3)} \frac{Ш(9)}{\zeta(9)} \frac{Ш(12)}{\zeta(-11)} \left(\frac{2}{691} - \frac{1}{691^2}\right) \log(B)^{12}.$$

In the above example, there are no contributions to heights from the non-archimedean places. In general, this will not be the case. As K. Kato suggested, in order to understand contributions from archimedean places (respectively non-archimedean places), we need to study period domains (respectively p-adic period domains) classifying Hodge structures (respectively p-adic Hodge structures). This study has been carried out in [14].

2.2 Tamagawa numbers of motives

In this section, we briefly review the definition of the Tamagawa measure associated to a motive with $\mathbb{Z}$ coefficients. We note that there is still no universally agreed definition of mixed motives even up to this day. For our purpose, we will follow Jannsen's definition: that is a mixed motive with $\mathbb{Q}$ coefficients over a number field F is a collection of realizations with weight and Hodge filtrations, and these realizations are related by comparison isomorphisms which are compatible with both these filtrations (for more details, see [12]). For simplicity, we will assume that $F = \mathbb{Q}$ as Bloch-Kato did in their paper. Also, to define heights, we will assume that each graded quotient $\mathrm{gr}^W_\omega(M)$ (which is a pure motive of weight ω) is equipped with a polarization, $\mathrm{gr}^W_\omega(M) = 0$ for $w \geq -2$, and M has semistable reduction everywhere. In particular, each graded quotient $\mathrm{gr}^W_\omega(M)$ is a polarized pure motive with semistable reduction. We note that the condition $\mathrm{gr}^W_\omega(M) = 0$ for all $\omega \geq -2$ guarantees that the Tamagawa measure is well-defined (the product of local Tamagawa measure converges). In principle, our argument should work even without this restriction.

To define Tamagawa measures and heights, we will restrict ourselves to motives with $\mathbb{Z}$-coefficients: that is a pair (M, Θ) where M is a mixed motive with $\mathbb{Q}$ coefficients, and Θ is a free $\mathbb{Z}$-module of finite rank equipped with a linear action of $G_\mathbb{Q}$ on $\widehat{\Theta} = \widehat{\mathbb{Z}} \otimes \Theta$ and an isomorphism $\mathbb{Q} \otimes \Theta \cong M_B$ where M_B is the Betti realization of M.

Remark 1. This definition of mixed motives with $\mathbb{Z}$ coefficients is almost the same as the one defined in section 11 of Fontaine's paper [10].

Following Bloch-Kato, we define

$$A(\mathbb{Q}_p) = \begin{cases} H^1_f(\mathbb{Q}_p, \widehat{\Theta}) & \text{if } p < \infty \\ (D_\infty \otimes_{\mathbb{R}} \mathbb{C})/(\mathrm{Fil}^0 D_\infty \otimes_{\mathbb{R}} \mathbb{C}) + \Theta)^+ & \text{if } p = \infty \end{cases}.$$

We define $B(\mathbb{R}) = A(\mathbb{R})$ and $B(\mathbb{Q}_p)$ to be the inverse image in $H^1_g(\mathbb{Q}_p, \widehat{\Theta})$ of

$$\mathrm{Im}\left(\Psi \to H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})\right).$$

Here $\mathbf{A}^f_{\mathbb{Q}}$ is the ring of finite adele of $\mathbb{Q}$. We equip $B(\mathbb{Q}_p)$ with the unique topology such that the topology of $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ is discrete. Roughly speaking, these are elements of $H^1_g(\mathbb{Q}_p, \widehat{\Theta})$ such that the monodromy operators have motivic origin (for more details on this see discussions in section 2.3.4 as well as section 2.1.6 of [13]). Finally, we define $A(\mathbb{Q}) \subset B(\mathbb{Q})$ using Ψ insted of Φ. The Tamagawa number of (M, Θ) is defined to be

$$\mathrm{Tam}(M) = \mu\left(\prod_p A(\mathbb{Q}_p)/A(\mathbb{Q})\right).$$

The Tamagawa number conjecture of Bloch-Kato says that

Conjecture 1.

$$\mathrm{Tam}(M) = \frac{\#H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}\,/\,\mathbb{Z}(1))}{\#\text{Ш}(M)}.$$

Note that the above sets all depend on the choice of Θ, so we should write $B_\Theta(\mathbb{Q})$ instead of $B(\mathbb{Q})$. However, for simplicity, we will drop Θ in all related notations.

2.3 Heights of mixed motives

Let (M, Θ) be a motive with $\mathbb{Z}$-coefficients whose graded quotients are polarized. In this section, we review the some key facts about the height functions $H_{\diamond,p,d}$ on local spaces $B(\mathbb{Q}_p)$, the height function $H_{\diamond,d}$ on its associated adelic space, and the height function on the global space $B(\mathbb{Q})$.

The definition of height functions depends on the validity of some conjectures in arithmetic geometry. We refer to section 1.7.2 of [13] for precise statements about these conjectures. Throughout this chapter, we will assume that these conjectures hold.

2.3.1 Height function on $B(\mathbb{Q}_p)$

Let $\widehat{M}$ be a mixed motive with $\mathbb{Q}$ coefficients who graded quotients are polarized. As mentioned earlier, we will also assume that $\widehat{M}$ has semistable reductions. Let v be a place of $\mathbb{Q}$ associated to a prime number p (we will often identify them) and $\widehat{M}_\ell$ be a semistable ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Define $A = \mathbb{Q}_\ell$, $\Upsilon = \widehat{M}_\ell$ if $\ell \neq p$, and $A = \mathbb{Q}_p^{\mathrm{ur}}$, $\Upsilon = D_{st}(\widehat{M}_\ell)$ if $\ell = p$. Since $\widehat{M}_\ell$ is a semistable representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, there is a nilpotent monodromy operator $N : \Upsilon \to \Upsilon(-1)$ preserving the induced weight filtration on Υ. In what follows, we will assume that N induces an increasing filtration $\mathcal{W}_m$ (the relative monodromy filtration) on Υ with the following properties:

1. $N\,\mathcal{W}_m \subset \mathcal{W}_{m-2}$ for all $m \in \mathbb{Z}$.

2. For every $\omega \in \mathbb{Z}$ and $m \geq 0$, N^m induces an isomorphism

$$N^m : \mathrm{gr}^{\mathcal{W}}_{\omega+m}\,\mathrm{gr}^{W}_{\omega}(\Upsilon) \cong \mathrm{gr}^{\mathcal{W}}_{\omega-m}\,\mathrm{gr}^{W}_{\omega}(\Upsilon).$$

3. The weight monodromy conjecture holds for Υ with for this filtration.

We refer to section 1.7.2 of [13] for a precise statement of the third condition. For simplicity, we call these properties WMC. By some linear algebra arguments, we can show that if $\widehat{M}$ is an extension of two motives which verify WMC and the class $[\widehat{M}]$ satisfies some local conditions (see example 1 for a more thorough discussion of this) then $\widehat{M}$ also has semistable reductions and verifies WMC. In general, because $\widehat{M}$ is a successive extension of its graded quotients, it is enough to check that its graded pieces' realizations satisfy WMC. For pure motives of the form $H^n(X)(r)$ where X is either a curve, surface, abelian variety, or a complete intersection in projective spaces, the WMC holds by works of many mathematicians, see [22] for a survey of known results on this conjecture.

As explained in [13], for each $d \geq 2$, we have a canonical element

$$N_{v,d} \in \left(\mathrm{gr}^{\mathcal{W}}_{-2}\,\mathrm{gr}^{W}_{-d}\,\mathrm{Hom}_A(\Upsilon,\Upsilon)\right)_{\mathrm{prim}}. \tag{2.3.1}$$

We also have a non-degenerate A-linear pairing

$$\langle,\rangle_{N_v} : \left(\mathrm{gr}^{\mathcal{W}}_{-2}\,\mathrm{gr}^{W}_{-d}\,\mathrm{Hom}_A(\Upsilon,\Upsilon)\right)_{\mathrm{prim}} \times \left(\mathrm{gr}^{\mathcal{W}}_{-2}\,\mathrm{gr}^{W}_{-d}\,\mathrm{Hom}_A(\Upsilon,\Upsilon)\right)_{\mathrm{prim}} \to A,$$

defined as

$$\langle a,b\rangle_{N_v} = \langle \mathrm{Ad}(N_v)^{d-2}(a), b\rangle_{N_v}.$$

Finally, define $\ell_v = |\langle N_{v,d}, N_{v,d}\rangle_{N_v}|^{1/d}$ and

$$H_{\diamond,d,v}(M_\ell) := N(v)^{l_v}. \tag{2.3.2}$$

We expect that the following is true.

Conjecture 2. If M_ℓ is the ℓ-adic realization of a mixed motive M then ℓ_v does not depends on ℓ. Moreover, $\langle N_{v,d}, N_{v,d}\rangle_{N_v} \in \mathbb{Q}_{\geq 0}$. It is zero if and only if $N_{v,d} = 0$.

Suppose $M = H^n(X)(r)$ where X is a smooth projective variety over $\mathbb{Q}$. We assume further that X has proper strictly semistable reduction at p. In this case, using the weight spectral sequence of Rapoport-Zink (for $\ell \neq p$) or the weight spectral sequence of Mokrane (for $\ell = p$), we have a rather concrete understanding of the above monodromy operator N. If $\dim(X) \leq 2$, then we can show that conjecture 2 holds.

Let us give a concrete example of the above discussion.

Example 1. Let $\widehat{M}$ be a mixed motive which is a successive extension of a pure motive M of weight $-d$ and the Tate motive $\mathbb{Q}$. By taking the ℓ-adic realization, we have a short exact sequence of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-modules

$$0 \to M_\ell \to \widehat{M}_\ell \to \mathbb{Q}_\ell \to 0. \tag{2.3.3}$$

Define $\widehat{\Upsilon}$ and Υ to be the A-module associated with $\widehat{M}$ and M as explained above. Then, $\widehat{\Upsilon}$ has the induced filtration with graded quotients $\mathrm{gr}^W_{-d}\widehat{\Upsilon} = \Upsilon$ and $\mathrm{gr}^W_0\widehat{\Upsilon} = \mathbb{Q}_\ell$. We have

$$\mathrm{gr}^W_{-d}\mathrm{Hom}(\widehat{\Upsilon}, \widehat{\Upsilon}) = \oplus_w \mathrm{Hom}(\mathrm{gr}^W_w \widehat{\Upsilon}, \mathrm{gr}^W_{w-d}\widehat{\Upsilon}).$$

In our case, only $\omega = 0$ gives a nontrivial summand. Therefore, we have

$$\mathrm{gr}^W_{-d}\mathrm{Hom}(\widehat{\Upsilon}, \widehat{\Upsilon}) = \mathrm{Hom}(\mathbb{Q}_l, \Upsilon) = \Upsilon.$$

This isomorphism is compatible with the relative filtration on both sides, so we have

$$\left(\mathrm{gr}_i^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}(\widehat{\Upsilon},\widehat{\Upsilon})\right) = \left(\mathrm{gr}_i^{\mathcal{W}}\,\Upsilon\right), \forall i \in \mathbb{Z}\,.$$

Moreover, under this isomorphism $\mathrm{Ad}(\widehat{N}_v)$ corresponds to N and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{gr}_{-2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}(\widehat{\Upsilon},\widehat{\Upsilon}) \arrow[r] \arrow[d, "\mathrm{Ad}(\widehat{N}_v)^{d-2}"] & gr_{-2}^{\mathcal{W}}\Upsilon \arrow[d, "N^{d-2}"] \\
\mathrm{gr}_{-2d+2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}(\widehat{\Upsilon},\widehat{\Upsilon}) \arrow[r] & gr_{-2d+2}^{\mathcal{W}}\Upsilon
\end{tikzcd}$$

With this identification, we can interpret the pairing

$$\left(\mathrm{gr}_{-2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}(\widehat{\Upsilon},\widehat{\Upsilon})\right)_{\mathrm{prim}} \times \left(\mathrm{gr}_{-2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}(\widehat{\Upsilon},\widehat{\Upsilon})\right)_{\mathrm{prim}} \to A,$$

as the pairing

$$\left(gr_{-2}^{\mathcal{W}}\Upsilon\right)_{\mathrm{prim}} \times \left(gr_{-2}^{\mathcal{W}}\Upsilon)\right)_{\mathrm{prim}} \to A.$$

Concretely, this is described by

$$\langle v, w\rangle = \langle N^{d-2}(v), w\rangle, \forall v, w \in gr_{-2}^{\mathcal{W}}\Upsilon.$$

More generally, if M has a weight filtration W_ω then the pairing

$$\langle,\rangle_{N_v} : \left(\mathrm{gr}_{-2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}_A(\Upsilon,\Upsilon)\right)_{\mathrm{prim}} \times \left(\mathrm{gr}_{-2}^{\mathcal{W}}\,\mathrm{gr}_{-d}^{W}\,\mathrm{Hom}_A(\Upsilon,\Upsilon)\right)_{\mathrm{prim}} \to A,$$

is the sum of the induced pairing on the graded quotients; i.e

$$\langle , \rangle_{N_v} = \sum_{\omega \in \mathbb{Z}} \langle , \rangle_{\omega, N_v}.$$

Here $\langle , \rangle_{\omega, N_v}$ is the pairing associated with $\mathrm{gr}_\omega^W M$ explained above.

For each ℓ adic relization, there is class $[\widehat{M_\ell}] \in H^1(\mathbb{Q}_p, M_\ell)$ associated with the short exact sequence of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ modules

$$0 \to M_\ell \to \widehat{M}_\ell \to \mathbb{Q}_\ell \to 0.$$

Suppose M has semistable reductions everywhere. Then, $\widehat{M}$ has semi-stable reduction everywhere if and only $[\widehat{M_\ell}] \in H^1_g(\mathbb{Q}_p, M_\ell)$ for all ℓ. Here we use the convention that

$$H^1_g(\mathbb{Q}_p, M_\ell) = \begin{cases} H^1(\mathbb{Q}_p, M_\ell) & \text{if } \ell \neq p \\ \ker(H^1(\mathbb{Q}_p, M_p) \to H^1(\mathbb{Q}_p, M_p \otimes B_{dr})) & \text{if } \ell = p. \end{cases}$$

Next, we explain how to define the height function

$$H_{v,\diamond,d,\ell} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 1}.$$

By definition, there is a canonical map

$$B(\mathbb{Q}_p) \to H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}}) := \prod_\ell H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell).$$

Under the assumption that M has semistable reductions, for $\ell \neq p$, there is an isomorphism

$$H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell) \cong (M_\ell(-1)/NM_\ell)^{\varphi=1}.$$

Here N is the monodromy operator associated with M_ℓ. Furthermore, if M_ℓ satisfies the weight monodromy conjecture, then we have a canonical isomorphism

$$(M_\ell(-1)/NM_\ell)^{\varphi=1} = \left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1}.$$

Let $a_\ell \in H^1_g(\mathbb{Q}_p, M_\ell)$. Then a_ℓ corresponds to an extension of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-modules:

$$0 \to M_\ell \to (E_a)_\ell \to \mathbb{Q}_\ell \to 0.$$

The image of a_ℓ under the isomorphism

$$H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell) \cong \left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1},$$

is nothing but the monodromy operator $N_{a_\ell,v,d}$ of $(E_a)_\ell$, see 2.3.1. Here we use the identification (see example 1)

$$\left(\mathrm{gr}^{\mathcal{W}}_{-2}\,\mathrm{gr}^{W}_{-d}\,\mathrm{Hom}((E_a)_\ell,(E_a)_\ell)\right)_{\mathrm{prim}} = \left(\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell\right)_{\mathrm{prim}}.$$

For $a \in B(\mathbb{Q}_p)$ let a_ℓ be the image of a in $\in H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell)$. As before, we define

$$l_{v,d}(a_\ell) = \langle N_{a_\ell,v,d}, N_{a_\ell,v,d}\rangle^{1/d}.$$

Similarly, when $\ell = p$ there is a canonical isomorphism

$$H^1_g(\mathbb{Q}_p, M_p)/H^1_f(\mathbb{Q}_p, M_p) \cong (\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1))^{\varphi=1}.$$

Let $a_p \in H^1_g(\mathbb{Q}_p, M_p)$ be the image of $a \in B(\mathbb{Q}_p)$ in $H^1_g(\mathbb{Q}_p, M_p)$. Then a_p corresponds to an extension of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ modules

$$0 \to M_p \to (E_a)_p \to \mathbb{Q}_p \to 0.$$

Moreover, under the isomorphism

$$H^1_g(\mathbb{Q}_p, M_p)/H^1_f(\mathbb{Q}_p, M_p) \cong (\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1))^{\varphi=1},$$

a_p is mapped to the monodromy operator $N_{a_p,v,d}$ of $(E_a)_p$. Here we again use the identification

$$\left(\mathrm{gr}^{\mathcal{W}}_{-2}\,\mathrm{gr}^{W}_{-d}\,\mathrm{Hom}(D_{st}((E_a)_p), D_{st}((E_a)_p))\right)_{\mathrm{prim}} = \left(\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p)\right)_{\mathrm{prim}}.$$

Define

$$l_v(a_p) = \langle N_{a_p,v,d}, N_{a_p,v,d}\rangle^{1/d}.$$

Conjecture 2 then implies that for $a \in B(\mathbb{Q}_p)$, $l_v(a_\ell)$ is a non-negative real number and is independent of ℓ. Moreover, it is 0 if and only if a_ℓ belongs to $H^1_f(\mathbb{Q}_p, M_\ell)$ for some ℓ (hence for all ℓ). From now on, we will write $l_v(a)$ for this value without referring to the choice of

ℓ.

Remark 2. With the above interpretation, it would be reasonable to think about $H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})$ as the "space of monodromy operators" at the place v. Similarly, we can think about $B(\mathbb{Q}_p)$ as the space of monodromy operators which have motivic origins.

We have the following lemma.

Lemma 1. *Under the above assumptions, $H^1_g(\mathbb{Q}_p, \widehat{\Theta})/H^1_f(\mathbb{Q}_p, \widehat{\Theta})$ is a free $\widehat{\mathbb{Z}}$-module of finite rank.*

Proof. Let $\widehat{\Theta}_\ell$ be the ℓ component of $\widehat{\Theta}$. To prove this lemma, it is enough to show that $H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)/H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$ is a free $\mathbb{Z}_\ell$ module of finite rank and moreover, this rank does not depend on ℓ. First, we prove that $H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)/H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$ is a torsion free $\mathbb{Z}_\ell$-module. Suppose $a \in H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)$ and $m \in \mathbb{Z}$, $m \neq 0$ such that $ma \in H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$. By definition, the image of ma in $H^1(\mathbb{Q}_p, M_\ell)$ belongs to $H^1_f(\mathbb{Q}_p, M_\ell)$. Because $H^1(\mathbb{Q}_p, M_\ell)$ is a vector space over $\mathbb{Q}_\ell$ and $m \neq 0$, the image of a must belong to $H^1_f(\mathbb{Q}_p, M_\ell)$ as well. Hence, $a \in H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$. This shows that $H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)/H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$ is torsion free.

We have

$$\frac{H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)}{H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \frac{H^1_g(\mathbb{Q}_p, M_\ell)}{H^1_f(\mathbb{Q}_p, M_\ell)}.$$

By previous discussions, we have the following isomorphisms

$$H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell) = \begin{cases} \left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1} & \text{if } \ell \neq p \\ \left((\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1)\right)^{\varphi=1} & \text{if } \ell = p. \end{cases}$$

As a corollary of conjecture 1.7.2 in [13], which we assume throughout this article, we have

$$\dim_{\mathbb{Q}_\ell}\left((\mathrm{gr}^{\mathcal{W}}_{-2} M_\ell(-1))_{\mathrm{prim}}\right)^{\varphi=1} = \dim_{\mathbb{Q}_p}\left((\mathrm{gr}^{\mathcal{W}}_{-2} D_{st}(M_p))_{\mathrm{prim}}(-1)\right)^{\varphi=1}.$$

Hence, $\dim_{\mathbb{Q}_\ell}\left(H^1_g(\mathbb{Q}_p, M_\ell)/H^1_f(\mathbb{Q}_p, M_\ell)\right)$ is independent of ℓ. We conclude that

$$H^1_g(\mathbb{Q}_p, \widehat{\Theta}_\ell)/H^1_f(\mathbb{Q}_p, \widehat{\Theta}_\ell)$$

is a free $\mathbb{Z}_\ell$ module of finite rank which is independent of ℓ. □

We have the following inclusions

$$A(\mathbb{Q}_p) := H^1_f(\mathbb{Q}_p, \widehat{\Theta}) \subset B(\mathbb{Q}_p) \subset H^1_g(\mathbb{Q}_p, \widehat{\Theta}).$$

By definition $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ is the inverse image in $H^1_g(\mathbb{Q}_p, \widehat{\Theta})/H^1_f(\mathbb{Q}_p, \widehat{\Theta})$ of

$$\mathrm{Im}(\Psi \to H^1_g(\mathbb{Q}_p, \widehat{\Theta} \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \widehat{\Theta} \otimes \mathbf{A}^f_{\mathbb{Q}})).$$

By lemma 1, we can see that $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ is a free $\mathbb{Z}$-module of finite rank. Hence, as a topological group $B(\mathbb{Q}_p)$ is isomorphic to the direct product of $A(\mathbb{Q}_p) \times \mathbb{Z}^{s_p}$ for some $s_p \in \mathbb{Z}_{\geq 0}$. The measure on $B(\mathbb{Q}_p)$ is the tensor product of the measure on $A(\mathbb{Q}_p)$ defined in [3] and the counting measure on $\mathbb{Z}^{s_p}$.

It is natural to ask the following question.

Question 1. Suppose $M = H^n(X)(r)$ where X smooth projective variety over $\mathbb{Q}$. Let $\mathfrak{X}$ be a proper regular model of X over $\mathbb{Z}$. Can we describe s_p from the geometry of the special fiber $\mathfrak{X}_p$ of $\mathfrak{X}$ at p?

Under the above assumption, we now can define a height function

$$H_{\diamond,v,d} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}. \tag{2.3.4}$$

For $a \in B(\mathbb{Q}_p)$

$$H_{\diamond,p,d} = N(v)^{l_v(a)} = p^{l_v(a)}.$$

Definition 1.

1. We call $H_{\diamond,p,d} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 1}$ the height function on $B(\mathbb{Q}_p)$.

2. Let $h_{\diamond,p,d} : B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}$ be defined by

$$h_{\diamond,p,d} := (\ln H_{\diamond,p,d}).$$

We call $h_{\diamond,p,d}$ the logarithmic height function on $B(\mathbb{Q}_p)$.

By definition, $H_{\diamond,p,d}$ and $h_{\diamond,p,d}$ are trivial when restricting to $A(\mathbb{Q}_p)$. Therefore, they are well-defined on the quotient $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ which is isomorphic to $\mathbb{Z}^{s_p}$ for some integer s_p. We use the same notations $H_{\diamond,p,d}$ and $h_{\diamond,p,d}$ to denote the induced height function and logarithmic height function on $B(\mathbb{Q}_p)/A(\mathbb{Q}_p)$. One we fix a basis for $\mathbb{Z}^{s_p}$, $h^d_{\diamond,p,d}$ the $d-$power of the logarithmic height function is given by a positive definite quadratic form $Q_{\diamond,p,d}$ on $\mathbb{Z}^{s_p}$. A direct consequence of this observation is the following.

Proposition 1. *For each positive real number B*

$$\mu(a \in B(\mathbb{Q}_p) | H_{\diamond,p,d}(a) \leq B) < \infty.$$

Proof. Let $a \in B(\mathbb{Q}_p)$, we denote by (a_0, n) the image of a under the identification $B(\mathbb{Q}_p) = A(\mathbb{Q}_p) \times \mathbb{Z}^{s_p}$. Then

$$\begin{aligned}\{a \in B(\mathbb{Q}_p) | H_{\diamond,p,d}(a) < B\} &= A(\mathbb{Q}_p) \times \{n \in \mathbb{Z}^r \, | H_{\diamond,p,d}(n) < B\} \\ &= A(\mathbb{Q}_p) \times \{n \in \mathbb{Z}^r \, | h_{\diamond,p,d}(n) < \ln(B)\} \\ &= A(\mathbb{Q}_p) \times \{n \in \mathbb{Z}^r \, | Q_{\diamond,p,d}(n) < \ln(B)^d\}.\end{aligned}$$

Because $Q_{\diamond,p,d}$ is a positive definite quadratic form, there are only finitely many $n \in \mathbb{Z}^r$ such that $Q_{\diamond,p,d}(n) < \log(B)^d$. Therefore

$$\mu(a \in B(\mathbb{Q}_p) | H_{\diamond,p,d}(a) < B) = |\{n \in \mathbb{Z}^r \, | Q_{\diamond,p,d}(n) < \log(B)^d\}| \mu(A(\mathbb{Q}_p)) < \infty.$$

□

We end this subsection with the following general remark.

Remark 3. If M is a motive of the form $H^n(X)(r)$ where X is a smooth projective variety over $\mathbb{Q}$, then by the proof of Weil's conjecture we can easily see that $H^1_g(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}})/H^1_f(\mathbb{Q}_p, \Theta \otimes \mathbf{A}^f_{\mathbb{Q}}) = 0$ if X has good reduction at p. Consequently, we have $B(\mathbb{Q}_p) = A(\mathbb{Q}_p)$ and the height function on $B(\mathbb{Q}_p)$ is trivial. Hence, the height functions $H_{\diamond,v,d}$ are only interesting at places v where X has bad reductions.

When X is a curve or a surface, by using the Rapoport-Zink spectral sequence for ℓ-adic cohomology (when $\ell \neq p$) and the Mokrane spectral sequence for p-adic cohomology, we have a concrete understanding of the height functions $H_{\diamond,v,d}$ from the geometry of the special fibers at p. It is expected that such an understanding also exists in higher dimensions.

2.3.2 Height function on $B(\mathbb{R})$

We define a height function $H_{\diamond,\infty,d}$ on $B(\mathbb{R})$ as an analogue of the height functions $H_{\diamond,v,d}$ on $B(\mathbb{Q}_p)$ discussed in the previous section. Let $v = \infty$ the the only archimedean place of $\mathbb{Q}$ corresponding to the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$.

We give some preparations. Let $\widehat{H}$ be a mixed $\mathbb{R}$-Hodge structure with a weight filtration W and a Hodge filtration F. As explained in section 1.7.6 of [13], there exists a unique pair (s, δ) such that

1. s is a splitting $s : \mathrm{gr}^W \widehat{H}_{\mathbb{R}} \cong \widehat{H}_{\mathbb{R}}$.
2. δ is a nilpotent linear map $\delta : \mathrm{gr}^W \widehat{H}_{\mathbb{R}} \to \mathrm{gr}^W \widehat{H}_{\mathbb{R}}$.
3. The filtration F of $\widehat{H}$ is given by $F = s(\exp(i\delta)\,\mathrm{gr}^W F)$. Moreover, the Hodge (p, q) components $\delta_{p,q}$ of δ is 0 unless $p < 0$ and $q < 0$.

For $d \geq 2$, Let

$$\widehat{H}' = \bigoplus_{\omega \in \mathbb{Z}} ((\mathrm{gr}^W_{\omega} \widehat{H})^* \otimes \mathrm{gr}^W_{\omega-d} \widehat{H}) = \mathrm{gr}^W_{-d} \mathrm{Hom}(\widehat{H}, \widehat{H}).$$

Note that $\widehat{H}'$ is polarized Hodge structure of weight $-d$. Let $N_{v,d} \in \widehat{H}'_{\mathbb{R}}$ be the weight

$(-d)$-component of the nilpotent linear map δ_v associated with $\widehat{H}_{\mathbb{R}}$. By definiton,

$$N_{v,d} \in \bigoplus_{\omega \in \mathbb{Z}} \operatorname{Hom}(\operatorname{gr}^W_\omega \widehat{H}_{\mathbb{R}}, \operatorname{gr}^W_{\omega-d} \widehat{H}_{\mathbb{R}}) = \widehat{H}'.$$

Because $\widehat{H}'$ is a polarized Hodge structure of weight $-d$, it is possible to define

$$l(\infty, d) = l_{v,d}(\widehat{H}) = \langle N_{v,d}, N_{v,d} \rangle^{1/d}.$$

Remark 4. In his paper [13], K. Kato uses the notation $\delta_{v,d}$ for $N_{v,d}$. We change his notation because we want to emphasize the analogies between height functions at archimedean and non-archimedean places.

We will give a concrete example to demonstrate the above discussion. This example is an analogue of example 1.

Example 2. Let H be a $\mathbb{R}$-Hodge structure of pure weight $-d < 0$. Let $\widehat{H}$ be a mixed $\mathbb{R}$-Hodge structure with graded quotients $\operatorname{gr}^W_{-d} \widehat{H} = H$ and $\operatorname{gr}^W_0 \widehat{H} = \mathbb{C}$. By definition, $N_{v,d'} = 0$ for all $d' \neq d$ and

$$N_{v,d} \in \bigoplus_{\omega \in \mathbb{Z}} \operatorname{Hom}(\operatorname{gr}^W_\omega \widehat{H}_{\mathbb{R}}, \operatorname{gr}^W_{\omega-d} \widehat{H}_{\mathbb{R}}) = \operatorname{Hom}(\mathbb{R}, H_{\mathbb{R}}) = H_{\mathbb{R}}.$$

There is an exact sequence in the category of mixed $\mathbb{R}$-Hodge structure which is a natural analogue of the exact sequence 2.3.3:

$$0 \to H \to \widehat{H} \to \mathbb{R} \to 0$$

in the category $\mathbb{R}\,\mathrm{MHS}$ of mixed $\mathbb{R}$-Hodge structure. By definition, $[\widehat{H}]$ defines a class $[\widehat{H}]$ in $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H)$. We have a canonical isomorphism

$$\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H) \cong H_{\mathbb{C}}/(H_{\mathbb{R}} + F^0 H_{\mathbb{C}}).$$

Moreover, if we define

$$H_{\mathbb{R}}^{\leq -1, \leq -1} := H_{\mathbb{R}} \cap \bigoplus_{p,q \leq -1, p+q=-d} H_{\mathbb{C}}^{p,q},$$

then we also have a canonical isomorphism

$$H_{\mathbb{C}}/(H_{\mathbb{R}} + F^0 H_{\mathbb{C}}) \cong H_{\mathbb{R}}^{\leq -1, \leq -1}.$$

Therefore, we have an isomorphism

$$\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H) \cong H_{\mathbb{R}}^{\leq -1, \leq -1}.$$

Under this isomorphism, the class $[\widehat{H}]$ is mapped to the $N_{v,d}$ of $\widehat{H}$.

Consequently, we can define a height function $H_{\diamond,\infty,d} : \mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H) \to \mathbb{R}_{\geq 1}$ as follow. For each $a \in \mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H)$, let E_a be the corresponding mixed $\mathbb{R}$-Hodge structure. We define

$$H_{\diamond,\infty,d}(a) = \exp(l_{\infty,d}(E_a)).$$

Definition 2.

1. We call $H_{\diamond,\infty,d} : \mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H) \to \mathbb{R}_{\geq 1}$ the height function on $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H)$.

2. Let $h_{\diamond,p,d} : \mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H) \to \mathbb{R}_{\geq 0}$ be defined by

$$h_{\diamond,p,d} := (\ln H_{\diamond,p,d}).$$

   We call $h_{\diamond,p,d}$ the logarithmic height function on $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H)$.

We remark that $h^d_{\diamond,\infty,d}$ the $d-$ power of the logarithmic height function is given by a positive definite quadratic form $Q_{\diamond,p,d}$ on $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, H)$.

Remark 5. We consider a special case of the above discussion in the case $H_{\mathbb{R}}$ is the mixed $\mathbb{R}$-Hodge structure arising from the motive M. In this case, $H_{\mathbb{R}}$ is $V_\infty = H_B(M) \otimes \mathbb{R}$ in Bloch-Kato's paper [3]. Here $H_B(M)$ is the Betti realization of M. In this case, the above isomorphism can be written as

$$\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty) \cong H_{\mathbb{R}}^{\leq -1, \leq -1} = V_\infty \cap \bigoplus_{p,q \leq -1, p+q=-d} H_{\mathbb{C}}^{p,q}.$$

In particular, we can define the height function $H_{\diamond,\infty,d}$ and the logarithmic height function $h_{\diamond,\infty,d}$ on $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$.

To define height function and logarithmic height function on $B(\mathbb{R})$, we will construct a map from $B(\mathbb{R})$ to $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$. First, following section 2.1.10 of [13] we introduce the following notation.

Definition 3. We define

$$E_\infty^{\mathbb{R}} = V_\infty^{\sigma=-1} \cap \bigoplus_{p,q\leq -1, p+q=-d} H_{\mathbb{C}}^{p,q}.$$

By definition $E_\infty^{\mathbb{R}}$ is a subset of $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$.

First, we observe that there is a canonical map

$$p : B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} = D_\infty/(\mathrm{Fil}^0 D_\infty + V_\infty^{\sigma=1}) \to E_\infty^{\mathbb{R}}.$$

We have

$$D_\infty = (V_\infty \otimes \mathbb{C})^{\sigma=1} = (V_\infty^{\sigma=1} \otimes \mathbb{R}) \oplus (V_\infty^{\sigma=-1} \otimes \mathbb{R}\, i).$$

In particular, there is a projection map

$$D_\infty \to V_\infty^{\sigma=-1} \otimes \mathbb{R}\, i.$$

By composing this map with the map $V_\infty^{\sigma=-1} \otimes \mathbb{R}\, i \to V_\infty^{\sigma=-1}$ sending $e \otimes i \to e$, we get a map $D_\infty \to V_\infty^{\sigma=-1}$. We then get a map

$$p : D_\infty \to E_\infty^{\mathbb{R}} := (V_\infty^{\sigma=-1} \cap \bigoplus_{p,q\leq -1, p+q=-d} H_{\mathbb{C}}^{p,q}).$$

It is easy to see that $(\mathrm{Fil}^0 D_\infty + V_\infty^{+})$ belongs to the kernel of this map. Hence, p induces a map

$$p : B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} \to E_\infty^{\mathbb{R}}.$$

Because $E_{\infty}^{\mathbb{R}}$ is a subset of $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$, we can consider p as a map from $B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$ to $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$. By composing p with the projection map $B(\mathbb{R}) \to B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$ we can define a canonical map

$$p : B(\mathbb{R}) \to \mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty).$$

Finally, by composing p with the height function $H_{\diamond,\infty,d}$ and the logarithmic height function $h_{\diamond,\infty,d}$ on $\mathrm{Ext}^1_{\mathbb{R}\,\mathrm{MHS}}(\mathbb{R}, V_\infty)$ we a height function and a logarithmic height function on $B(\mathbb{R})$. We will still denote them by $H_{\diamond,\infty,d}$ and $h_{\diamond,\infty,d}$.

Question 2. We will probably need to assume that $p : B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} \to E_{\infty}^{\mathbb{R}}$ an isomorphism?

By choosing a lift, we have an decomposition

$$B(\mathbb{R}) = \mathbb{R}^{s_\infty} \times B(\mathbb{R})_{\mathrm{cpt}}.$$

By definition, $h^d_{\diamond,\infty,d}$ the d- power of the logarithmic height function on $B(\mathbb{R})$ is given by a positive definite quadratic form $Q_{\diamond,\infty,d}$ on $\mathbb{R}^{s_\infty}$. Moreover, it is trivially 0 on $B(\mathbb{R})_{\mathrm{cpt}}$. With this observation, the following statement is obvious. We remark that it is the archimedean analogue of proposition 1.

Proposition 2. *For each positive real number B*

$$\mu(a \in B(\mathbb{R}) | H_{\diamond,p,d}(a) \leq B) < \infty.$$

One can ask a similar question to question 1.

Question 3. Suppose $M = H^n(X)(r)$ where X smooth projective variety over $\mathbb{Q}$. Can we describe s_∞ from the geometry of X?

2.3.3 Height function on adelic space $\prod_p B(\mathbb{Q}_p)$

Having defined height functions and logarithmic height functions at all local places $B(\mathbb{Q}_p)$ we can now define a height function as well as logarithmic height function on the adelic space $\prod_p B(\mathbb{Q}_p)$. More precisely, we define

$$H_{\diamond,d} : \prod_{p\leq\infty} B(\mathbb{Q}_p) \to \mathbb{R}_{\geq 1},$$

by sending $a = (a_p) \in \prod_{p\leq\infty} B(\mathbb{Q}_p)$ to

$$H_{\diamond,d}(a) = \prod_{p\leq\infty} H_{\diamond,p,d}(a_p).$$

By remark 3, all but finitely many terms in this product are 1 so the product makes sense. Similarly, we define the logarithmic height $h_{\diamond,d} = \ln H_{\diamond,d}$, for $a = (a_p)$

$$h_{\diamond,d}(a) = \sum_{p\leq\infty} h_{\diamond,p,d}(a_p).$$

We can describe $h_{\diamond,d}$ concretely as follows. For $a \in \prod_p B(\mathbb{Q}_p)$ let $([a]_p)$ be the image of a under the projection

$$\prod_p B(\mathbb{Q}_p) \to \big(B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}\big) \times \prod_{p<\infty} B(\mathbb{Q}_p)/A(\mathbb{Q}_p) = \mathbb{R}^{s_\infty} \times \prod_{p<\infty} \mathbb{Z}^{s_p}.$$

Then

$$h_{\diamond,d}(a) = \sum_{p\leq\infty} h_{\diamond,p,d}([a]_p) = Q_{\diamond,d}([a]_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}([a]_p)^{1/d}.$$

With this description, the following statement is relatively straightforward. We give proof because we will use it later.

Proposition 3. *For each positive real number B, the set*

$$\{a \in \prod_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B\},$$

is compact. In particular,

$$\mu(a \in \prod_{p\leq\infty} B(\mathbb{Q}_p) | H_{\diamond,d}(a) \leq B) < \infty.$$

Proof. Let S_B be the above set. Then

$$S = \{a \in \prod_{p\leq\infty} B(\mathbb{Q}_p) | h_{\diamond,d}(a) \leq \ln(B)\}$$

Under the identification $B(\mathbb{Q}_p) = A(\mathbb{Q}_p) \times \mathbb{Z}^{s_p}$ and $B(\mathbb{R}) = B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} \times B(\mathbb{R})_{\mathrm{cpt}}$, it is the same as the following set

$$\left\{ (a_\infty \times B(\mathbb{R})_{\mathrm{cpt}}) \times \prod_{p<\infty} ([a]_p \times A(\mathbb{Q}_p)) \right\},$$

where

$$Q_{\diamond,d}([a]_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}([a]_p)^{1/d} \leq \ln B.$$

For each p, $Q_{\diamond,p,d}$ is a positive definite quadratic form, there are only finitely many $[a]_p \in \mathbb{Z}^{s_p}$ verifying the above inequality. We also know that $A(\mathbb{Q}_p)$ is compact for all $p < \infty$. Therefore, S is compact. Moreover, if we equip $\mathbb{R}^{s_\infty} \times \prod_{p<\infty} \mathbb{Z}^{s_p}$ with the product measure, then it is easy to see that

$$\mu(S) = \mu(B(\mathbb{R})_{\mathrm{cpt}}) \times \prod_{p<\infty} \mu(A(\mathbb{Q}_p)) \times \mu(T_B),$$

where

$$T_B = \left\{ ([a]_p) \in \mathbb{R}^{s_\infty} \times \prod_{p<\infty} \mathbb{Z}^{s_p} \middle| Q_{\diamond,d}([a]_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}([a]_p)^{1/d} \leq \ln B \right\}.$$

$\square$

2.3.4 Height function on $B(\mathbb{Q})$

Having defined height functions on local spaces $B(\mathbb{Q}_p)$, we now define a height function on the global space $B(\mathbb{Q})$. First, we recall that $B(\mathbb{Q})$ is defined by the following fiber product diagram

$$\begin{array}{ccc} B(\mathbb{Q}) & \longrightarrow & H^1_g(\mathbb{Q}, \widehat{\Theta}) \\ \downarrow & & \downarrow \\ \Psi & \longrightarrow & H^1_g(\mathbb{Q}, \widehat{\Theta} \otimes \mathbf{A}^f_{\mathbb{Q}}) \end{array}$$

We also recall that $A(\mathbb{Q})$ is defined to be the fiber product diagram

$$\begin{array}{ccc} A(\mathbb{Q}) & \longrightarrow & H^1_f(\mathbb{Q}, \widehat{\Theta}) \\ \downarrow & & \downarrow \\ \Psi & \longrightarrow & H^1_g(\mathbb{Q}, \widehat{\Theta} \otimes \mathbf{A}^f_{\mathbb{Q}}) \end{array}$$

Remark 6. Philosophically, we think of Ψ as $\mathrm{Ext}^1_{\mathcal{MM}}(1, M)$ where $\mathcal{MM}$ is the category of mixed motives with $\mathbb{Q}$-coefficients. By definition, an element of $B(\mathbb{Q})$ is of the form (a, b) where $a \in \Psi$ and $b \in H^1_g(\mathbb{Q}, \widehat{\Theta})$ such that the images of a and b in $H^1_g(\mathbb{Q}, \widehat{\Theta} \otimes \mathbf{A}^f_{\mathbb{Q}})$ coincide. By definition, a corresponds to a mixed motives M_a that fits into a short exact sequence

$$0 \to M \to M_a \to \mathbb{Q} \to 0.$$

Similarly b corresponds to an extension of $\widehat{\mathbb{Z}}$-modules with a continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action

$$0 \to \widehat{\Theta} \to \widehat{\Theta}_b \to \widehat{\mathbb{Z}} \to 0.$$

The compatibility of a and b means that $(M_a, \widehat{\Theta}_b)$ is a motive with $\mathbb{Z}$-coefficients. Therefore, we should think about $B(\mathbb{Q})$ as the extension group $\mathrm{Ext}^1(\mathbb{Z}, (M, \Theta))$ in the category of motives with $\mathbb{Z}$-coefficients. Similarly, we should think about $A(\mathbb{Q})$ the extension group

$\mathrm{Ext}^1(\mathbb{Z}, (M, \Theta))$ in the category of motives with $\mathbb{Z}$-coefficients with local conditions at non-archimedean places.

In what follows, we will use the following convention. If M is finitely generated abelian group, then M_{tor} is the torsion subgroup of M and $M_{\mathrm{free}} = M/M_{\mathrm{tor}}$. By definition, M_{free} is a free $\mathbb{Z}$-module of finite rank.

By definition, $A(\mathbb{Q}) \subset B(\mathbb{Q})$ and both are a finitely generated abelian groups. Moreover we have

$$A(\mathbb{Q}) \otimes \widehat{\mathbb{Z}} = H^1_f(\mathbb{Q}, \widehat{\Theta}), A(\mathbb{Q}) \otimes \mathbb{Q} = \Phi, B(\mathbb{Q}) \otimes \widehat{\mathbb{Z}} = H^1_g(\mathbb{Q}, \widehat{\Theta}), B(\mathbb{Q}) \otimes \mathbb{Q} = \Psi.$$

We have a global analogue of lemma 1 that $B(\mathbb{Q})/A(\mathbb{Q})$ is a free ablian group of finite rank. Conjectures of Beilinson, Bloch, Kato, and Jannsen (see [2], [3], and [12]) predict that

1. The regular map induces an isomorphism

$$A(\mathbb{Q}) \otimes \mathbb{R} = \Phi \otimes \mathbb{R} \cong B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} = E^{\mathbb{R}}_{\infty}.$$

   In particular, there is a map $A(\mathbb{Q}) \to B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$. One can lift this map to be a map $h : A(\mathbb{Q}) \to B(\mathbb{R})$. In particular, the image of $A(\mathbb{Q})_{\mathrm{tor}}$ belongs to $B(\mathbb{R})_{\mathrm{cpt}}$. It is easy to see that we can choose a splitting $B(\mathbb{R}) = B(\mathbb{R})_{\mathrm{cpt}} \times B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$ and $A(\mathbb{Q}) = A(\mathbb{Q})_{\mathrm{free}} \times A(\mathbb{Q})_{\mathrm{tor}}$ in such a way that $h = (h_1, h_2)$ where $h_1 : A(\mathbb{Q})_{\mathrm{free}} \to B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$ and $h_2 : A(\mathbb{Q})_{\mathrm{tor}} \to B(\mathbb{R})_{\mathrm{cpt}}$. The above isomorphism implies that h_1 is an embedding and it identifies $A(\mathbb{Q})_{\mathrm{free}}$ with a lattice in $B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$.

2. The canonical projection map

$$B(\mathbb{Q})/A(\mathbb{Q}) \to \prod_{p<\infty} B(\mathbb{Q}_p)/A(\mathbb{Q}_p),$$

is an isomorphism.

As a corollary, we conclude that $\operatorname{rank}_{\mathbb{Z}} A(\mathbb{Q}) = \operatorname{rank}_{\mathbb{Z}} \Phi = B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}} = s_\infty$ and $s_{\mathrm{fin}} = \operatorname{rank}(B(\mathbb{Q})/A(\mathbb{Q})) = \sum_{p<\infty} s_p$. Moreover, it is easy to we can define a splitting (up to a choice of basis)

$$B(\mathbb{Q}) = A(\mathbb{Q}) \times \mathbb{Z}^{s_{\mathrm{fin}}},$$

in such a way that $\mathbb{Z}^{s_{\mathrm{fin}}}$ belongs to the kernel of the regulator map $R_\infty : \Psi \otimes \mathbb{R} \to B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}}$. We then have

$$\mathbb{Z}^{s_{\mathrm{fin}}} \cong \prod_{p<\infty} B(\mathbb{Q}_p)/A(\mathbb{Q}_p) = \prod_{p<\infty} \mathbb{Z}^{s_p}.$$

We define a height function and a logarithmic height function on $B(\mathbb{Q})$. As explained above, there is an evident map

$$B(\mathbb{Q}) \to \prod_{p\leq\infty} B(\mathbb{Q}_p).$$

By composing this map with the height function and the logarithmic height functions defined in section 2.3.4, we can define a height function and a logarithmic height function on $B(\mathbb{Q})$ as follows

Definition 4. $H_{\diamond,d} : B(\mathbb{Q}) \to \mathbb{R}_{\geq 1}$ is defined by

$$H(a) = \prod_{p \leq \infty} H_{\diamond,p,d}(a_p).$$

Similarly, $h_{\diamond,d} : B(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$ is defined to be

$$h_{\diamond,d} = \ln(H(a)) = \sum_{p \leq \infty} h_{\diamond,p,d}(a_p) = Q_{\diamond,\infty,d}(a_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}(a_p)^{1/d}.$$

As explained in section 2.3.3, we can describe logarithmic height function $h_{\diamond,p,d}$ concretely as follows. For $a \in B(\mathbb{Q})$ we write $a = \sum_p [a]_p$ under the splitting: $B(\mathbb{Q}) = A(\mathbb{Q}) \times \prod_p \mathbb{Z}^{s_p}$, then $([a]_\infty)_p \in A(\mathbb{Q}_p)$ for all $p < \infty$, hence $h_{\diamond,p,d}(b_\infty) = 0$. Similarly, if $p < \infty$, then $R_\infty([a]_p) = 0$ in $B(\mathbb{R})/B(\mathbb{R})_{\text{cpt}}$, hence $h_{\diamond,\infty,d}([a]_p) = 0$. From this observation, we conclude that

$$h_{\diamond,d}(a) = \sum_{p \leq \infty} h_{\diamond,p,d}([a]_p).$$

We also note that $B(\mathbb{Q})_{\text{tor}} = A(\mathbb{Q})_{\text{tor}}$ is a finite group, and if $b \in B(\mathbb{Q})_{\text{tor}}$ then $h_{\diamond,d}(a) = 0$. Note that there is a splitting $A(\mathbb{Q}) = A(\mathbb{Q})_{\text{tor}} \times A(\mathbb{Q})_{\text{free}}$. Hence, there is also a splitting

$$B(\mathbb{Q}) = B(\mathbb{Q})_{\text{free}} \times B(\mathbb{Q})_{\text{tor}} = A(\mathbb{Q})_{\text{tor}} \times A(\mathbb{Q})_{\text{free}} \times \prod_{p<\infty} \mathbb{Z}^{s_p} = A(\mathbb{Q})_{\text{tor}} \times \prod_{p \leq \infty} \mathbb{Z}^{s_p}.$$

With this observation, the following statement is a direct corollary of 3.

Proposition 4. *For each positive integer B, the set*

$$\{b \in B(\mathbb{Q}) | H_{\diamond,d}(b) < B\},$$

is finite.

Proof. We have

$$\{b \in B(\mathbb{Q}) | H_{\diamond,d}(b) < B\} = B(\mathbb{Q})_{\mathrm{tor}} \times \{b \in B(\mathbb{Q})_{\mathrm{free}} | H_{\diamond,d}(b) < B\}.$$

Under the above identification, the second component can be described as

$$\{([a]_p) \in \prod_{p \le \infty} \mathbb{Z}^{s_p} \,| Q_{\diamond,\infty,d}(a_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}(a_p)^{1/d} \le \ln B\}.$$

This is evidently a finite set. Hence $\{b \in B(\mathbb{Q}) | H_{\diamond,d}(b) < B\}$ is a finite set as well.

□

2.4 Proof of main theorem

We give a proof for theorem 1. First, we recall the definition of Tam(M) as defined in [3]

$$\mathrm{Tam}(M) = \mu\left(\prod_p A(\mathbb{Q}_p)/A(\mathbb{Q})\right).$$

Because $A(\mathbb{Q}_p)$ and $A(\mathbb{R})_{\mathrm{cpt}}$ is compact for all $p < \infty$, this can be rewrite as

$$\mathrm{Tam}(M) = |A(\mathbb{Q})_{\mathrm{tor}}|^{-1} \times \mu((B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}})/A(\mathbb{Q})_{\mathrm{free}}) \times \mu(A(\mathbb{R})_{\mathrm{cpt}}) \times \prod_p \mu(A(\mathbb{Q}_p)).$$

We know from previous discussions that thet set $\{x \in B(\mathbb{Q}) | H_{\diamond,d}(x) \le B\}$ is finite and

$\mu(a \in \prod_p B(\mathbb{Q}_p)|H_{\diamond,d}(a) \leq B) < \infty$. Hence, the quotient

$$\frac{\#\{a \in B(\mathbb{Q})|H_{\diamond,d}(b) \leq B\}}{\mu(a \in \prod_p B(\mathbb{Q}_p)|H_{\diamond,d}(a) \leq B)}$$

makes sense. By the discussions around proposition 3 and proposition 4 we have

$$\mathrm{LHS} = |A(\mathbb{Q})_{\mathrm{tor}}| \times N_B,$$

where

$$N_B = \#\{([a]_p) \in \prod_{p \leq \infty} \mathbb{Z}^{s_p} \,|Q_{\diamond,\infty,d}(a_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}(a_p)^{1/d} \leq \ln B\}.$$

Also

$$\mathrm{RHS} = \mu(B(\mathbb{R})_{\mathrm{cpt}}) \times \prod_{p<\infty} \mu(A(\mathbb{Q}_p)) \times \mu(T_B),$$

where

$$T_B = \left\{([a]_p) \in \mathbb{R}^{s_\infty} \times \prod_{p<\infty} \mathbb{Z}^{s_p} \Big| Q_{\diamond,d}([a]_\infty)^{1/d} + \sum_{p<\infty} (\ln p) Q_{\diamond,p,d}([a]_p)^{1/d} \leq \ln B \right\}.$$

We then have

$$\frac{\mathrm{LHS}}{\mathrm{RHS}} = \frac{\mu((B(\mathbb{R})/B(\mathbb{R})_{\mathrm{cpt}})/A(\mathbb{Q})_{\mathrm{free}}) \times N_B}{\mathrm{Tam}(M) \times \mu(T_B)}.$$

Therefore, theorem 1 is equivalent to

$$\lim_{B\to\infty} \frac{N_B}{\mu(T_B)} = (\mu((B(\mathbb{R})/B(\mathbb{R})_{\text{cpt}})/A(\mathbb{Q})_{\text{free}}))^{-1}.$$

Once we fix a basis, we can assume that $A(\mathbb{Q}_{\text{free}})$ is the standard basis for $B(\mathbb{R})/B(\mathbb{R})_{\text{cpt}}$. The above statement then becomes $\lim_{B\to\infty} \dfrac{N_B}{\mu(T_B)} = 1$ which is a consequence of the following general statement.

Proposition 5. *Let $f = (f_1, f_2) : \mathbb{R}^{r_1+r_2} = \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \to \mathbb{R}$. Suppose for each $i = 1, 2$ the following conditions are satisfied*

1. *$f_i(x) \geq 0$ for all x. Moreover, $f_i(x) = 0$ if and only if $x = 0$.*
2. *$f_i(x)$ is continuous.*
3. *There exists a positive real number c such that $f(\lambda x) = |\lambda|^c f(x)$ for all $x \in \mathbb{R}^{r_i}$ and $\lambda \in \mathbb{R}$.*

We equip $\mathbb{Z}^{r_1}$ with the counting measure $d\mu_0$ and $\mathbb{R}^{r_1}, \mathbb{R}^{r_2}$ the usual Lebesgue measures $d\mu_1, d\mu_2$. Let

$$I(B) = ((n, y) \in \mathbb{Z}^{r_1} \times \mathbb{R}^{r_2} \,|\, f(n, y) = f_1(n) + f_2(y) \leq B),$$

and

$$V(B) = ((x, y) \in \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \,|\, f_1(x) + f_2(y) \leq B).$$

Then

$$\lim_{B\to\infty} \frac{\mu(I(B)}{\mu(V(B))} = 1.$$

Proof. By propety (iii) of f_1, f_2, $f(x,y) \leq B$ is equivalent to

$$f_1\left(\frac{x}{B^{1/c}}\right) + f_2\left(\frac{y}{B^{1/c}}\right) \leq 1.$$

From this observation

$$\mu(V(B)) = (B^{1/c})^{r_1+r_2}\mu(S_1),$$

where

$$S_1 = ((x,y) \in \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \,|f_1(x) + f_2(y) \leq 1).$$

Let $D = \{x \in \mathbb{R}^{r_1} \,|f_1(x) \leq 1\}$. For each x in D, we define

$$g(x) = \int_{f_2(y)\leq 1-f(x)} 1 d\mu_2.$$

By our assumption, $g(x)$ is an integrable function on D. We have

$$\mu(S_1) = \int_{\mathbb{R}^{r_1+r_2}} 1_{S_1}.$$

By Fubini's theorem, we have

$$\mu(S_1) = \int_D g(x) d\mu_1.$$

We also have Similarly,

$$I(B) = \{(n,y) \in \mathbb{Z}^{r_1} \times \mathbb{R}^{r_2} \,|f_1\left(\frac{n}{B^{1/c}}\right) + f_2\left(\frac{y}{B^{1/c}}\right) \leq 1\}.$$

By a similar argument as above we have

$$\mu(I(B)) = (B^{1/c})^{r_2} \times \sum_{f_1(n)\le B} \left(\int_{f_2(y)\le 1-f_1(\frac{n}{B^{1/c}})} 1 d\mu_2 \right)$$
$$= (B^{1/c})^{r_2} \times \sum_{a\in \frac{1}{B^{1/c}}\mathbb{Z}^{r_1}\cap D} g(a)$$
$$= (B^{1/c})^{r_1+r_2} \times \sum_{a\in \frac{1}{B^{1/c}}\mathbb{Z}^{r_1}\cap D} g(a) \left(\frac{1}{B^{1/c}}\right)^{r_1}.$$

We observe that second term is nothing but the Riemann sum of g with respect to the partition $\frac{1}{B^{1/c}}\mathbb{Z}^{r_1}\cap D$. Since $g(x)$ is integrable on D, we have

$$\lim_{B\to\infty} \left[\sum_{a\in \frac{1}{B^{1/c}}\mathbb{Z}^{r_1}\cap D} g(a) \left(\frac{1}{B^{1/c}}\right)^{r_1} \right] = \mu(S_1).$$

The above statement follows immediately from this.

□

2.5 Some computations with mixed Tate motives

In what follows, let us fix a mixed motive $M = (V, D)$ with $\mathbb{Z}$ coefficients with graded quotients $\mathbb{Q}(12), \mathbb{Q}(3)$. We will compute the numbers of $x \in \mathrm{Ext}^1(\mathbb{Z}, (V, D))$ with height bounded by a positive number B. More generally, we will do computations with a mixed motives D with graded quotients $\mathbb{Z}(m), \mathbb{Z}(n)$ where m is an even positive integer, and n is an odd positive integer and $m \ge n + 2$. The reason we include this concrete example is that we want to compare it with the situation in the next section where (V, D) also moves among mixed motives with $\mathbb{Z}$-coefficients and with graded quotients $\mathbb{Q}(12), \mathbb{Q}(3)$.

For each prime p, let us denote D_p the p-adic component of of $D \otimes \widehat{Z}$ abd $V_p = D_p \otimes \mathbb{Q}_p$. By definition D_p is a Galois stable $\mathbb{Z}_p$ lattice inside V_p. Up to a power of 2, we have the following.

Proposition 6. *Let* $X = Ext^1(\mathbb{Z}, D)$. *Then*

$$\#\{x \in X | H_{\star,\diamond}(x) \leq B\} \sim 2^a \frac{\text{Ш}(D)}{\zeta(3)\zeta(-11)} \log(B)^3.$$

We recall the definition of Ш of a mixed motive. For a fixed prime p and for each place v of $\mathbb{Q}$, we have special subgroups of $H^1_f(\mathbb{Q}_v, D_p), H^1_f(\mathbb{Q}_v, V_p)$ and $H^1_f(\mathbb{Q}_v, V_p/D_p)$ of $H^1(\mathbb{Q}_v, D_p)$, $H^1(\mathbb{Q}_v, V_p)$ and $H^1_f(\mathbb{Q}_v, V_p/D_p)$ respectively. For example:

$$H^1_f(\mathbb{Q}_v, D_p) = \begin{cases} H^1_{ur}(\mathbb{Q}_v, D_p) & \text{if } v \nmid p \\ \\ \ker(H^1(\mathbb{Q}_v, D_p) \to H^1(\mathbb{Q}_v, D_p \otimes B_{\text{cris}})) & \text{if } v \mid p. \end{cases}$$

The Selmer group of D_p is defined to be

$$H^1_f(\mathbb{Q}, V_p/D_p) = \{x \in H^1(\mathbb{Q}, V_p/D_p) | x_v \in H^1_f(\mathbb{Q}_v, V_p/D_p), \forall v\}.$$

Similarly, we define

$$H^1_f(\mathbb{Q}, V_p) = \{x \in H^1(\mathbb{Q}, V_p) | x_v \in H^1_f(\mathbb{Q}_v, V_p), \forall v\}.$$

By definition, we have a canonical map $H^1_f(\mathbb{Q}, V_p) \to H^1_f(\mathbb{Q}, V_p/D_p)$.

Definition 5. Define $Ш(D_p)$ to be the cokernel of the map

$$H^1_f(\mathbb{Q}, V_p) \to H^1_f(\mathbb{Q}, V_p/D_p).$$

We define the global $Ш(D)$ to be:

$$Ш(D) = \prod_p Ш(D_p).$$

It is conjectured to be a finite group. Recall that V_p fits into an exact sequence

$$0 \to \mathbb{Q}_p(12) \to V_p \to \mathbb{Q}_p(3) \to 0.$$

We have the following simple lemma.

Lemma 2. *As a Galois represenation of $Gal(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$, V_p is unramifed if $v \nmid p$ and it is crystalline if $v \mid p$. Moreover,*

1. *If $v \nmid p$ then $H^i(\mathbb{Q}_v, D_p) = 0$ for $i = 0, 1, 2$.*
2. *If $v \mid p$ then $H^0(\mathbb{Q}_v, D_p) = H^2(\mathbb{Q}_v, D_p) = 0$. Also $H^1_f(\mathbb{Q}_v, V_p) = H^1(\mathbb{Q}_v, V_p)$.*

We also have the following proposition.

Proposition 7. *Let V_p, D_p be defined as above. Then*

$$Ш(D_p) = \ker\left(H^2(\mathbb{Z}[1/p], D_p) \to H^2(\mathbb{Q}_p, D_p)\right).$$

Proof. By lemma 1, we have

$$H^1_f(\mathbb{Q}, V_p) = H^1(\mathbb{Z}[1/p], V_p).$$

The long exact sequence associated with the short exact sequence

$$0 \to D_p \to V_p \to V_p/D_p \to 0$$

gives

$$\begin{aligned} H^1_f(\mathbb{Q}_v, V_p/D_p) &= \mathrm{Im}\left(H^1(\mathbb{Q}_v, V_p) \to H^1(\mathbb{Q}_v, V_p/D_p)\right) \\ &= \ker(H^1(\mathbb{Q}_v, V_p/D_p) \to H^2(\mathbb{Q}_v, D_p). \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^1_f(\mathbb{Q}, V_p/D_p) &= \ker(H^1(\mathbb{Q}, V_p/D_p) \to \bigoplus_v H^2(\mathbb{Q}_v, D_p)) \\ &= \ker\left(H^1(\mathbb{Z}[1/p], V_p/D_p) \to H^2(\mathbb{Q}_p, D_p)\right). \end{aligned}$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} H^1(\mathbb{Z}[1/p], V_p) \arrow[r] \arrow[d] & H^1(\mathbb{Z}[1/p], V_p/D_p) \arrow[r] \arrow[d] & H^2(\mathbb{Z}[1/p], D_p) \arrow[d] \\ H^1_f(\mathbb{Q}, V_p/D_p) \arrow[r] & H^1(\mathbb{Z}[1/p], V_p/D_p) \arrow[r] & H^2(\mathbb{Q}_p, D_p) \end{tikzcd}$$

By snake lemma, we see that

$$Ш(D_p) = \ker \left(H^2(\mathbb{Z}[1/p], D_p) \to H^2(\mathbb{Q}_p, D_p) \right).$$

$\square$

We want to relate $Ш(D_p)$ with $Ш(\mathbb{Z}_p(3))$ and $Ш(\mathbb{Z}_p(12))$. For simplicity, we will denote the later groups by $Ш_p(3)$ and $Ш_p(12)$ respectively. Taking long exact sequences associated to

$$0 \to \mathbb{Z}_p(12) \to D_p \to \mathbb{Z}_p(3) \to 0,$$

we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12))/\mathrm{im}(\delta) \arrow[r] \arrow[d] & H^2(\mathbb{Z}[1/p], D_p) \arrow[r] \arrow[d] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(3)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(\mathbb{Q}_p, \mathbb{Z}_p(12))/\mathrm{im}(\delta) \arrow[r] & H^2(\mathbb{Q}_p, D_p) \arrow[r] & H^2(\mathbb{Q}_p, \mathbb{Z}_p(3)) \arrow[r] & 0
\end{tikzcd}$$

here δ are the connecting homomorphisms

$$\delta : H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(3)) \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12))$$

and

$$\delta : H^1(\mathbb{Q}_p, \mathbb{Z}_p(3)) \to H^2(\mathbb{Q}_p, \mathbb{Z}_p(12)).$$

Note that if we think about D_p as an element $[a]$ in $\mathrm{Ext}^1(\mathbb{Z}_p(3), \mathbb{Z}_p(12))$ then δ is the cup product of the $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(3))$ with $[a]$. Also, by Tate duality

$$H^2(\mathbb{Q}_p, \mathbb{Z}_p(12)) = H^0(\mathbb{Q}_p, \mathbb{Q}_p \,/\, \mathbb{Z}_p(-11))^{\vee} = 0.$$

So by snake lemma, we have the short exact sequence

$$0 \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12))/\mathrm{im}(\delta) \to Ш(D_p) \to Ш(\mathbb{Z}_p(3)) \to 0. \tag{2.5.1}$$

To complete our proof, we will use the following zeta elements in K groups of $\mathbb{Z}$.

Lemma 3. *(see [6]) Let m be a positive odd integer. Then, there exists an element b_m such that*

1. *b_m is the generator of $K_{2m-1}(\mathbb{Z})$ modulo torsion.*
2. *$r_\infty(b_m) = 2^a \dfrac{(m-1)!\zeta(m)}{Ш(m)}$. where a is an unknown number.*

For simplicity we will write $b_{m,p}$ the image of b_m under the Chern class map

$$K_{2m-1}(\mathbb{Z}) \otimes \mathbb{Z}_p \to H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m)).$$

As a consequence of Quillen-Lichtenbaum theorem, the above Chern class map is an isomorphism for p odd. Consequently, $b_{m,p}$ is the generator of $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m))$ for all p odd.

Lemma 4.

1. We have the following

$$H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12)) = \begin{cases} 0 & \textit{if} \quad p \neq 691 \\ \\ \mathbb{Z}/691 & \textit{if} \quad p = 691. \end{cases}$$

2. The cup product of $b_{3,p} \cup b_{9,p} \in H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12))$ *is non-zero at* $p = 691$ *and is* 0 *at* $p \neq 691$*.*

With these preparations, let us prove proposition 6. First, we know that the class of D is an element in $\mathrm{Ext}^1(\mathbb{Z}(3), \mathbb{Z}(12)) = \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}(9)) = \mathbb{Z}\, b_9 \oplus \mathbb{Z}/2$. Under this identification let us assume that the class of D in $\mathrm{Ext}^1(\mathbb{Z}(3), \mathbb{Z}(12))$ is (ub_9, v) where $u \in \mathbb{Z}$ and $v \in \mathbb{Z}/2$. There are two cases that we need to deal with separately.

Case 1: $691|u$**.** By lemma 4, the connecting homorphism

$$\delta : H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(3)) \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12)),$$

is 0. Therefore, the short exact sequence 2.5.1 becomes

$$0 \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12)) \to Ш(D_p) \to Ш_p(3) \to 0.$$

Combing all p, we get the short exact sequence

$$0 \to Ш(12) \to Ш(D) \to Ш(3) \to 0.$$

In particular, $\#Ш(D) = \#Ш(12) \times \#Ш(3)$.

Note that because $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}(12)) = 0$, we can easily compute that $X = \mathrm{Ext}^1(\mathbb{Z}, D) = \mathbb{Z}\, b_3 \oplus T$ where T is the torsion part. If we ignore the prime $p = 2$ then

$$T = \bigoplus_p H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(12)).$$

By a theorem Mazur-Wiles, which is a consequence of their proof for the Iwasawa main conjecture, we know that

$$\zeta(-11)_p = \frac{\#H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(12))}{\#H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(12)} = \frac{\#\text{Ш}_p(12)}{\#H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(12))}.$$

Therefore, we have

$$\#H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(12)) = \frac{\#\text{Ш}_p(12)}{\zeta(-11)}.$$

Hence

$$\#\{x \in X | H_{\star,\diamond}(x) < B\} = |T| \#\{m \in \mathbb{Z} \,|\, \exp(|r_\infty(ub_9)|^{1/9} + |mr_\infty(b_3)|^{1/3}) \leq B\}.$$

This is asymptotically equal to

$$\frac{\#\text{Ш}(12)}{\zeta(-11)} \frac{\#\text{Ш}(3)}{2\zeta(3)} \log(B)^3 = 2^a \frac{\text{Ш}(D)}{\zeta(3)\zeta(-11)} \log(B)^3.$$

Note that we do not know the exact power of 2 in this formula.

Case 2: $691 \nmid u$**.** In this case, by lemma 4 the connecting homomorphism δ is a surjection.

Therefore, the short exact sequence 2.5.1 becomes

$$Ш(D_p) = Ш_p(3).$$

Combining all p we have $Ш(D) = Ш(3)$. As in case 1 we can easily compute that

$$X = \mathbb{Z}\,691 b_3 \oplus T.$$

Here as above $\#T = \frac{Ш(12)}{\zeta(-11)}$. By a similar computation as in case 1 we see that the number

$$\#\{x \in X | H_{\star,\diamond}(x) \leq B\}$$

is asymptotically equal to

$$\frac{1}{691} \frac{Ш(3)}{2\zeta(3)} \frac{Ш(12)}{\zeta(-11)} \log(B)^3 = 2^a \frac{Ш(D)}{\zeta(3)\zeta(-11)} \log(B)^3.$$

One can generalize proposition 6 to a more general situation.

Proposition 8. *Let m, n be two natural number such that $m-n \geq 2$, m is even and n is odd. Let D be a mixed Tate motive with graded quotients $\mathbb{Z}(m)$ and $\mathbb{Z}(n)$ and $X = Ext^1(\mathbb{Z}, D)$. Then*

$$\#\{x \in X | H(x) \leq B\} \sim 2^t \frac{Ш(D)}{(n-1)!\zeta(n)\zeta(1-m)} \log(B)^n.$$

Proof. By a similar argument, we have a short exact sequence

$$0 \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(m))/\mathrm{im}(\delta) \to Ш(D_p) \to Ш(\mathbb{Z}_p(n)) \to 0. \tag{2.5.2}$$

where δ is the connecting homomorphism

$$\delta : H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(n)) \to H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(m)).$$

Note that because $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(n))$ is a free $\mathbb{Z}_p$ module of rank 1, the $\mathrm{im}(\delta)$ is a cyclic subgroup of $H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(m))$. Suppose that the order of $\mathrm{im}(\delta)$ is p^a. Note that we have an exact sequence

$$H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m)) \hookrightarrow H^1(\mathbb{Z}[1/p], D_p) \to H^1(\mathbb{Z}[1/p]. \mathbb{Z}_p(n)) \xrightarrow{\delta} H^2(\mathbb{Z}[1/p], \mathbb{Z}_p(m))$$

So,

$$\ker(\delta) = p^a H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m)) = |\mathrm{im}(\delta)| H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m)).$$

Let $|\delta| = \prod_p |\mathrm{im}(\delta)_p|$. As before, we can see that

$$X = \mathbb{Z}\,|\delta| b_n \oplus T,$$

where b_n the the generator of $K_{2n-1}(\mathbb{Z})$ described in lemma 3 and

$$T = H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m))$$

au. Again, by Mazur-Wiles theorem we have

$$\#T = \prod_p \#H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(m)) = \frac{\#\text{Ш}(m)}{\zeta(1-m)}.$$

Also, by the exact sequence 2.5.2 we have

$$\#\text{Ш}(D) = \frac{\#\text{Ш}(m)\#\text{Ш}(n)}{|\delta|}.$$

Therefore, the order of the set $\{x \in X | H(x) < B\}$ is given by

$$|T| \times \#\{z \in \mathbb{Z} \,|\, \exp(|r_\infty(ub_{m-n})|^{1/m-n} + |zr_\infty(|\delta|b_n)|^{1/n}) \leq B\}.$$

This is asymptotically equal to

$$\begin{aligned} |T|\frac{\log(B)^n}{|\delta|r_\infty(b_n)} &= 2^t \frac{\#\text{Ш}(m)}{\zeta(1-m)} \frac{\#\text{Ш}(n)}{|\delta|(n-1)!\zeta(n)} \log(B)^n \\ &= 2^t \frac{\#\text{Ш}(D)}{(n-1)!\zeta(n)\zeta(1-m)} \log(B)^n. \end{aligned}$$

Note that we do not know the exact power of 2 in this formula. □

In the above proof, we do not have to know whether the cup product is 0 or not. In the end, the factor $|\delta|$ is canceled. An important point here is that $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(n))$ is a free $\mathbb{Z}_p$-module when n is odd.

2.6 Heights of motives with graded quotients $\mathbb{Z}(12), \mathbb{Z}(3), \mathbb{Z}$.

In this section, we answer Kato's questions about the number of mixed motives with fixed graded quotients $\mathbb{Q}(12), \mathbb{Q}(3), \mathbb{Q}$ and bounded heights. Unlike the situation in the previous section, the mixed motives with $\mathbb{Z}$-coefficients and with graded quotients $\mathbb{Q}(12), \mathbb{Q}(3)$ will also vary. More precisely, if we denote by X the set of mixed motives with $\mathbb{Z}$-coefficients and with graded quotients $\mathbb{Q}(12), \mathbb{Q}(3), \mathbb{Z}$ then

$$X = \bigcup_{M \in \mathrm{Ext}^1(\mathbb{Z}(3), \mathbb{Z}(12))} \mathrm{Ext}^1(\mathbb{Z}, M).$$

For the definition of heights in this case, see [13] for more details. We note that in this case, only archimedean places contribute to heights. Therefore, this problem is purely a problem of finding a "basis" for X. The rest is a basic counting problem.

2.6.1 Some lemmas on counting integer points

We discuss some simple lemmas about counting integer points in bounded domains in $\mathbb{R}^2$.

Lemma 5. *Let f be a continuously differential function on $[y, x]$. Then*

$$\sum_{y<n\leq x} [f(n)] = \int_y^x f(u)dt + \int_y^x (u - [u]) f'(u) du + f(x)\{x\} - f(y)\{y\}.$$

Proof. See [1]. □

We will use the following lemma repeatedly.

Lemma 6. *Let s, t be positive integer. For each positive real number X, define*

$$d(X) = \#\{(m, n) \in \mathbb{N} \times \mathbb{N} \,|\, am^{1/s} + bn^{1/t} \leq X\}.$$

Then

$$d(X) \sim \frac{1}{\binom{s+t}{t}} \frac{1}{a^s b^t} X^{s+t}.$$

Proof. We see that $am^{1/s} + bn^{1/t} \leq X$ is equivalent to

$$m \leq \left(\frac{X - bn^{1/t}}{a} \right)^s.$$

Therefore

$$d(X) = \sum_{0<n\leq(X/b)^t} \left[\left(\frac{X - bn^{1/t}}{a} \right)^s \right].$$

Let $f(X) = \left(\frac{X - bu^{1/t}}{a} \right)^s$, then by lemma 5 we have

$$d(X) = \int_0^{(X/b)^t} f(u)du - \int_0^{(X/b)^t} (u - [u])f'(u)du).$$

Because $0 \leq u - [u] < 1$ we have

$$\left| \int_0^{(X/b)^t} (u - [u])f'(u)du \right| \leq \int_0^{(X/b)^t} |f'(u)|du = \left(\frac{X}{a} \right)^s.$$

We will estimate the first term

$$\int_0^{(X/b)^t} \left(\frac{X - bu^{1/t}}{a}\right)^s du.$$

By changing variable $u = (X/b)^t v^t$, we see that this integral is equal to

$$\frac{X^{s+t}}{a^s b^t} \int_0^1 t(1-v)^s v^{t-1} dv.$$

The above integral is a Gamma function; and by checking the table of values we see that

$$d(X) \sim \frac{1}{\binom{s+t}{t}} \frac{1}{a^s b^t} X^{s+t}, \text{as} \quad X \to \infty.$$

$\square$

2.6.2 *Proof of theorem* 3

Let $Y = \mathrm{Ext}^1(\mathbb{Z}(3), \mathbb{Z}(12)) = \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}(9))$. By the previous section we know that as a group

$$Y = \mathbb{Z}\, b_9 \oplus \mathbb{Z}/2.$$

Here b_9 is the zeta element described in lemma 3. By definition of X, we have a canonical projection map

$$p : X \to Y.$$

Therefore, as a set

$$X = \bigsqcup_{a \in Y} X_a,$$

where X_a is the fiber of p over a point a. In the following, our convention is follow: for each notation appeared in the previous section we add a letter a to indicate that the corresponding object depends on a. With this convention we have

$$\#\{x \in X | H_{\star,\diamond}(x) \leq B\} = \sum_{a \in Y} \#\{x \in X_a | H_{\star,\diamond}(x) \leq B\}.$$

Suppose that a correspond to ub_9 then the order of $\{x \in X_a | H(x) < B\}$ is given by

$$|T_a| \#\{m \in \mathbb{Z} \,|\, \exp(|r_\infty(ub_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq B\}.$$

We know that for all a

$$\#T_a = \prod_p \#H^1(\mathbb{Z}[1/p], \mathbb{Z}_p(12)) = \frac{\text{Ш}(12)}{\zeta(-11)}.$$

Therefore up to some power of 2 we have

$$\#\{x \in X | H(x) \leq B\} = \frac{\text{Ш}(12)}{\zeta(-11)} \#\{(m, u) \in \mathbb{N} \times \mathbb{N}\},$$

where

$$\exp(|ur_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq B\}.$$

In this particular case, $\delta_{a,p}$ can only be nontrivial when $p = 691$ by lemma 4. For simplicity, let us define

$$S(B) = \{(m,u) \in \mathbb{N}\times\mathbb{N} \,|\, \exp(|ur_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq B\}$$
$$= \{(m,u) \in \mathbb{N}\times\mathbb{N} \,||ur_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq \log(B)\}$$

As δ_a depends on a we consider two cases.

Case 1: $691|u$**.** Then, $\delta_{a,p} = 0$ for all p. In this case

$$X_a = \mathbb{Z}\, b_3 \oplus T_a.$$

Case 2: $691 \nmid u$**.** Then $\delta_{a,p} \neq 0$ when $p = 691$ by lemma 4. In this case X_a is a little smaller then the previous case; i.e

$$X_a = 691\,\mathbb{Z}\, b_3 \oplus T_a.$$

Combining these two cases we have

$$|S(B)| = |S_1(B)| + |S_2(B)| - |S_1(B) \cap S_2(B)|,$$

where

$$S_1 = \{(m,u) \in \mathbb{N}\times\mathbb{N} \,||ur_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq \log(B); 691|u\}$$

and

$$S_1 = \{(m,u) \in \mathbb{N} \times \mathbb{N} \, || u r_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq \log(B); 691|m\}.$$

By letting $u = 691u'$ in the first case we have

$$S_1 = \{(m,u') \in \mathbb{N} \times \mathbb{N} \, ||691u' r_\infty(b_9)|^{1/9} + (|m|\delta_a|r_\infty(b_3)|)^{1/3}) \leq \log(B)\}.$$

Therefore, by the counting lemma 5 we have

$$|S_1(B)| \sim \frac{1}{691\binom{12}{3}} \frac{1}{r_\infty(b_9) r_\infty(b_3)} \log(B)^{12}.$$

Similarly, we have

$$|S_2(B)| \sim \frac{1}{691\binom{12}{3}} \frac{1}{r_\infty(b_9) r_\infty(b_3)} \log(B)^{12},$$

and

$$|S_1(B) \cap S_2(B)| = \frac{1}{691^2\binom{12}{3}} \frac{1}{r_\infty(b_9) r_\infty(b_3)} \log(B)^{12}.$$

Consequently, we have

$$\begin{aligned} |S(B)| &\sim \frac{1}{\binom{12}{3}} \frac{1}{r_\infty(b_9) r_\infty(b_3)} \left(\frac{2}{691} - \frac{1}{691^2} \right) \log(B)^{12} \\ &\sim \frac{1}{8!2!\binom{12}{3}} \frac{\text{Ш}(3)}{\zeta(3)} \frac{\text{Ш}(9)}{\zeta(9)} \left(\frac{2}{691} - \frac{1}{691^2} \right) \log(B)^{12}. \end{aligned}$$

In conclusion

$$\#\{x \in X | H_{\star,\diamond}(x) \leq B\} \sim \frac{1}{8!2!\binom{12}{3}} \frac{\text{Ш}(3)}{\zeta(3)} \frac{\text{Ш}(9)}{\zeta(9)} \frac{\text{Ш}(12)}{\zeta(-11)} \left(\frac{2}{691} - \frac{1}{691^2}\right) \log(B)^{12}.$$

Remark 7. In general, the same computation can be performed for mixed motives with graded quotients $\mathbb{Z}(m), \mathbb{Z}(n), \mathbb{Z}$ where m (respectively n) are even (respectively odd) natural number and $m \geq n+2$.

CHAPTER 3

TAMAGAWA NUMBER CONJECTURE FOR UNIFORM F-CRYSTALS OVER FUNCTION FIELDS

The study of special values of L-functions of uniform F-crystals has been initiated by Kato in [15] to treat the remaining cases of the Tamagawa number conjecture over function fields. More precisely, for each uniform F-crystal over a smooth projective variety X over a finite field of characteristics p, Kato constructed a syntomic complex associated to F and expressed the special values of the L-function attached to F in terms of the cohomology groups of this syntomic complex. In this chapter, we will give a quick overview of Kato's work and generalize it to a family of F-crystals. Specifically, we will first reformulate proposition 5.3 in [15] in using K groups. We then use the same approach to treat the case of a tower of F-crystals.

3.1 Uniform F-crystals

Let X a smooth projective variety over a finite field k of characteristics p. Let us denote by $W(k)$ the ring of Witt vectors associated with k.

Definition 6. An F-crystal on X is a pair (D, Φ) where D is a crystal on the crystalline site of $X/W(k)$ and

$$\Phi : F^*D_{\mathbb{Q}_p} \to D_{\mathbb{Q}_p},$$

is an isomorphism in the category which Hom is replaced by $\mathbb{Q}_p \otimes$Hom. Here F is the Frobenius map $F : X \to X$ and F^*D is the pullback of D by F.

For each $q \in \mathbb{Z}$, let $\omega_X^q := \Omega_X^q$ the sheaf of differential q-forms on X. Let D_X be the vector bundle together with an integrable connection ∇ on X associated with D:

$$\nabla : D_X \to D_X \otimes_{\mathcal{O}_X} \omega_X^1.$$

Let $X \subset Y$ be a closed immersion of X into a p-adic formal scheme Y over $W(k)$. Let $D_X(Y)$ the PD envelop of X in Y and $\mathcal{O}_{D_X(Y)}$ be the structure sheaf of the formal scheme $D_X(Y)$. Let $\omega_Y^q := \Omega_Y^q$ be the sheaf of differential forms on Y. Let D_Y be the locally free $\mathcal{O}_{D_X(Y)}$-module together with an integrable connection ∇ associated with D which is defined by

$$\nabla : D_Y \to D_Y \otimes_{\mathcal{O}_Y} \omega_Y^1.$$

For an F-crystal (D, Φ) we will use the same notation Φ for the induced isomorphism:

$$\Phi : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} F^* D_Y \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} D_Y.$$

We remark that Φ is compatible with the integrable connection ∇ endowed with D_Y.

To define syntomic complexes, we first need to introduce several filtration on D_Y and $F^* D_Y$.

Definition 7. For each $r \in \mathbb{Z}$, we define

$$N_r(D)_Y := D_Y \cap p^{-r}\Phi(F^* D_Y),$$

$$N^r(D)_Y := D^Y \cap p^r \Phi(F^* D_Y),$$

and

$$M^r(D)_Y := \{x \in F^*D_Y | p^{-r}\Phi(x) \in D_Y\} \subset F^*D_Y.$$

We have the following proposition.

Proposition 9. *([21], 1.10.1) For $r \in \mathbb{Z}$*

1. *The image of $N_r(D)_Y$ in D_X is independent of the choice of an embedding $X \subset Y$. We will denote this image by $N_r(D_X)$*

2. *The image of $M^r(D)_Y$ in F^*D_X is independent of the choice of an embedding $X \subset Y$. We will denote this image by $M^r(D_X)$*

Following Ogus, we introduce the following definition of uniform F-crystals.

Definition 8. We say that $D = (D, \Phi)$ is a uniform F-crystal if one the following equivalent conditions holds

1. For all $r \in \mathbb{Z}$, $N_r(D_X)$ is locally a direct summand of D_X.

2. For all $r \in \mathbb{Z}$, $M^r(D_X)$ is locally a direct summand of F^*D_X

Remark 8. One of the main reasons that we restrict our study to uniform F-crystals is that uniform F-crystals form a good category. They also have good integral properties which will allow us to define integral syntomic complexes.

In fact, we have the following proposition.

Proposition 10. *([15], proposition 2.14) The category of uniform F-crystal is stable under $\oplus, \otimes$, the dual, and Tate twists.*

Most "geometric" F-crystals are uniform. More precisely, suppose $\pi : X \to Y$ is a proper smooth morphism of smooth schemes then under mild assumptions, the higher direct image $R^m\pi_*(\mathcal{O}_X)_{\text{cris}}$ is a uniform F-crystal on X (for more details, see [16], [17], [21], [19]).

3.2 Syntomic complexes

Let X be a projective smooth scheme over a finite field k of characteristics p and F is a uniform F-crystals on X. As explained in the previous section, we fix a closed immersion $X \subset Y$ where Y is a p-adic formal scheme over $W(k)$.

First, we explain the construction of some de Rham complexes using the filtration on D_Y and F^*D_Y constructed in the previous section. Following Deligne and Kato, we make the following convention. Let C be a filtered complex. We define $\overline{C}$ to be the following complex

$$\overline{C}^q = \{x \in ({}^qC)^q | dx \in ({}^{q+1}C)^q\}.$$

A key property of this construction is that if if $f : C \to C'$ is a homomorphism of filtered complexes such that for all q $f :^q C \to^q C'$ is a quasi-isomorphism then the induced map $\overline{C} \to \overline{C'}$ is a quasi-isomorphism.

Recall from the previous section that we have the de Rham complex associated with D:

$$\mathrm{DR}(D)_Y = [D_Y \xrightarrow{\nabla} D_Y \otimes_{\mathcal{O}_Y} \omega_Y^1 \xrightarrow{\nabla} D_Y \otimes_{\mathcal{O}_Y} \omega_Y^2 \xrightarrow{\nabla} \ldots]$$

Using the filtration on D_Y and $F^* D_Y$ defined in 7, we define the following complexes.

$$N_r \operatorname{DR}(D)_Y = \overline{C}, \text{where} \quad {}^q C = \operatorname{DR}(N_{r-q}(D))_Y,$$

$$N^r \operatorname{DR}(D)_Y = N_{-r} \operatorname{DR}(D)_Y,$$

and

$$M^r \operatorname{DR}(D)_Y = \overline{C}, \text{where} \quad {}^q C = p^q \operatorname{DR}(M^{r-q}(D))_Y.$$

We remark that

$$N_{(-\infty)} \operatorname{DR}(D)_Y = N^{(\infty)} \operatorname{DR}(D)_Y = p^{-r} N^r \operatorname{DR}(D)_Y,$$

is independent of $r \gg 0$.

We define the syntomic complex $\mathcal{S}(D)_Y$ as the mapping fiber of the map

$$1 - \Phi\eta : N_0 \operatorname{DR}(D)_Y \to N_{(-\infty)} \operatorname{DR}(D)_Y.$$

Here 1 is the inclusion map $N_0 \operatorname{DR}(D)_Y \to N_{(-\infty)} \operatorname{DR}(D)_Y$ and $\Phi\eta$ is the map defined as follow. First, $\eta : D_Y \to F^* D_Y$ is the map sending $x \in D_Y$ to $\eta(x) = 1 \otimes x \in F^*_{D_X(Y)} D_Y = F^* D_Y$. We then define $\Phi\eta$ as the induced map from $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \operatorname{DR}(D)_Y \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \operatorname{DR}(D)_Y$ whose degree q-part is given by

$$x \otimes \omega \mapsto \Phi(\eta(x)) \otimes F_Y(w).$$

The constructions of the complexes $N_0 \operatorname{DR}(D)_Y, N_{(-\infty)} \operatorname{DR}(D)_Y, \mathcal{S}(D)_Y$, are local in nature. However, we can glue these local constructions to get global complexes $\mathcal{S}(D)$ in the derived category of sheaves of abelian groups on the etale site of X. In other words, we have the following distinguished triangle.

$$\mathcal{S}(D) \to N_0 \operatorname{DR}(D) \to N_{(-\infty)} \operatorname{DR}(D) \to \mathcal{S}(D)[1].$$

We also define $\mathcal{T}(D)$ as the complex obtained by gluing the following local complexes

$$\mathcal{T}(D)_Y = N_{(-\infty)} \operatorname{DR}(D)_Y / N_0 \operatorname{DR}(D)_Y.$$

As above, we have the following distinguished triangle.

$$N_0 \operatorname{DR}(D) \to N_{(-\infty)} \operatorname{DR}(D) \to \mathcal{T}(D).$$

3.3 Formulation of Tamagawa number conjecture using K-theory

Let R, R' be two Noetherian rings equipped with a ring homomorphism $R \to R'$. We then have the following exact sequence of abelian group (see section 1.1 of [4] for more details).

$$K_1(R) \to K_1(R') \xrightarrow{\partial_{R,R'}} K_0(R, R') \to K_0(R) \to K_0(R'). \tag{3.3.1}$$

Example 3. Let $R = \mathbb{Z}_p$ and $R' = \mathbb{Q}_p$ then $K_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ and $K_0(\mathbb{Z}_p, \mathbb{Q}_p) = \mathbb{Z}$ and the boundary map

$$\partial_{\mathbb{Z}_p,\mathbb{Q}_p} : \mathbb{Q}_p^\times \to \mathbb{Z},$$

is nothing but the p-adic evaluation map.

Let X, D be as in the previous sections. We are now ready to reformulate proposition 5.4 of [15] in terms of K-theory.

Proposition 11. *([15], proposition* 5.4*) Suppose that*

$$1 - \Phi : H^m(X, DR(D)_{\mathbb{Q}_p} \to H^m(X, DR(D)_{\mathbb{Q}_p})$$

is an isomorphism for all m. Then, the complexes $\mathcal{S}(D)$ and $\mathcal{T}(D)$ over the etale site of X have finite cohomology groups. Let $[R\Gamma(X, \mathcal{S}(D))]$ and $[R\Gamma(X, \mathcal{T}(D))]$ be the corresponding classes in $K_0(\mathbb{Z}_p, \mathbb{Q}_p)$. Then, we have

$$\partial L(D, 1) = [R\Gamma(X, \mathcal{S}(D))] + [R\Gamma(X, \mathcal{T}(D))].$$

Here ∂ is the boundary map defined in 3.3.1*.*

3.4 Tamagawa number conjecture for a p-adic family of F-crystals, Preparations

In this section, we will assume that X is a projective smooth variety over a finite field k and $D = (D, \Phi)$ is a uniform F-crystals on X.

We know that $\mathrm{Gal}(\bar{k}/k) = \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. By Galois theory, there exists a p-adic Galois extension k_∞/k such that $\mathrm{Gal}(k_\infty/k) = \Gamma = \mathbb{Z}_p$. Let us denote by Γ_n the abelian group $\mathbb{Z}/p^n\mathbb{Z}$ and by k_n the unique subfield of k_∞ such that $\mathrm{Gal}(k_n/k) = \Gamma_n$. We call k_n the n-th layer of k_∞/k. Let X_n be the variety $X \otimes k_n$. We define the following total complexes

$$I_n = R\Gamma(X_n, N_0 DR(D)),$$

$$P_n = R\Gamma(X_n, N_{-\infty} DR(D)),$$

$$N_n = R\Gamma(X_n, \mathcal{S}(D)),$$

$$L_n = R\Gamma(X_n, \mathcal{T}(D)).$$

We also define N_∞ be the inverse limits of (N_n)

$$N_\infty = R \varprojlim_n N_n.$$

For each $S \in \{L, I, P\}$, we define S_∞ similarly.

For $S \in \{N, L, I, P\}$, S_n is a natural module over $\mathbb{Z}_p[\Gamma_n]$. Consequently, S_∞ is a module over the Iwasawa algebra $\Lambda(\Gamma)$ where

$$\Lambda(\Gamma) = \varprojlim_n \mathbb{Z}_p[\Gamma_n].$$

We denote by $\underline{N} = (N_n)$ the object in the derived category $D^b(\underline{\Gamma}(\mathbb{Z}_p - mod))$ of normic systems along the profinite group Γ (see section 2 of [23] for the precise definition of normic systems).For each $S \in \{L, I, P\}$, we define $\underline{S}$ similarly.

By construction, we have the following proposition (see proposition 5.1 of [23] for a similar statement).

Proposition 12. *Let* $\underline{W} = (W(k_n)) \in_{\underline{\Gamma}} (\mathbb{Z}_p - mod)$ *be the natural normic system of* $\mathbb{Z}_p$*-modules along* Γ*. For* $X \in \{I, P, L\}$*, there is a canonical isomorphism in* $D^b({}_{\underline{\Gamma}}(\mathbb{Z}_p - mod))$*:*

$$\underline{W} \overset{L}{\bigotimes} X_0 \cong \underline{X}.$$

Remark 9. Note that this is note true for $\underline{N}$.

We have the following result, which is a natural generalization of proposition 11 to a family of F-crystals.

Proposition 13. N_∞ *and* L_∞ *are torsion over* $\Lambda(\Gamma)$*. Furthermore, there exists an element* $\mathcal{L}_D \in K_1(\Lambda(\Gamma))$ *such that*

$$\partial(\mathcal{L}_D) = [N_\infty] + [L_\infty],$$

where $[N_\infty]$ *and* $[L_\infty]$ *are the corresponding classes of* N_∞ *and* L_∞ *in* $K_0(\Lambda(\Gamma))$ *and* ∂ *is the boundary map defined in* 3.3.1*.*

We provide a proof for the first statement. The second statement will be proved in a more general setting discussed in the next section. Our proof is based on the ideas of the paper [23] and we are thankful to its authors of for the innovative ideas. We refer the readers to section 2 and section 3 of that paper for some of the notions that we use here.

Proof. By the above construction, we have the following distinguished triangles in $D(\Lambda(\Gamma))$:

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} N_\infty \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_\infty \xrightarrow{1-\phi} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_\infty \to N_\infty[1],$$

and

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_\infty \xrightarrow{\mathbf{1}} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_\infty \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_\infty \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_\infty[1].$$

By proposition 12, we can rewrite these triangles as

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} N_\infty \to W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_0) \to W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_0) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} N_\infty[1], \tag{3.4.1}$$

and

$$\begin{tikzcd}
W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_0) \arrow[r, "\mathbf{1}"] & W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_0) \arrow[d] & \\
& W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_0) \arrow[r] & W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_0)[1].
\end{tikzcd}$$

Because L_0 is $\mathbb{Z}_p$-torsion, we have $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L_0 = 0$. Consequently, $\mathbf{1}$ gives an isomorphism

$$W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_0) \cong W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_0).$$

Let us denote by ι the inverse of the the isomorphism $\mathbf{1}$. Because $\mathbb{Q}_p$ and W_∞ are flat over $\mathbb{Z}_p$, the long exact sequence associated with the triangle 3.4.1 can be written as

$$\ldots \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(N_\infty) \to W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(P_\infty)) \xrightarrow{1-\phi\iota} W_\infty \otimes^L_{\mathbb{Z}_p} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(P_\infty))$$

By lemma 5.3 of [23], the map $1 - \phi\iota$ is injective. Consequently, the above sequence gives rise to the following short exact sequence

$$0 \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{i+1}(P_\infty) \xrightarrow{1-\phi\iota} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{i+1}(P_\infty) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{i}(N_\infty) \to 0. \qquad (3.4.2)$$

Let us denote by $Q(\Gamma)$ the total quotient field of $\Lambda(G)$. By applying $Q(\Gamma) \otimes_{\Lambda(\Gamma)} (-)$ to the short exact sequence 3.4.2, we have

$$0 \to Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^i(P_\infty) \xrightarrow{1-\phi\iota} Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^i(P_\infty) \to Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^{i+1}(N_\infty) \to 0.$$

Because $H^i(P_\infty)$ is a finitely generated $\Lambda(\Gamma)$-module, $Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^i(P_\infty)$ is a finite dimensional vector space over $Q(\Gamma)$. Consequently, the map

$$Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^{i+1}(P_\infty) \xrightarrow{1-\phi\iota} Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^{i+1}(P_\infty)$$

must be an isomorphism. We conclude that $Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^{i+1}(N_\infty) = 0$ for all i, and hence N_∞ is torsion over $\Lambda(\Gamma)$.

$\square$

3.5 Tamagawa number conjecture for a p-adic family of F-crystals

In this section, we provide a generalization of proposition 13. We will follow the notation in [8].

Let K be the function field of X. Let k be the total constant field of K and let k_∞ be the unique $\mathbb{Z}_p$-extension of k introduced in section 2.4. Let K_∞ be a Galois extension of K satisfying the following conditions (i)–(iii).

(i) The extension K_∞/K is unramified at every point of X.

(ii) $K_\infty \supset Kk_\infty$.

(iii) Let $G := \mathrm{Gal}(K_\infty/K)$. Then G is a p-adic Lie group having no element of order p.

Let $H := \mathrm{Gal}(K_\infty/Kk_\infty)$, $\Gamma := G/H = \mathrm{Gal}(Kk_\infty/K)$. Let $\Lambda(G) = \mathbb{Z}_p[[G]]$ be the completed group ring of G and define $\Lambda(H)$ and $\Lambda(\Gamma)$ similarly.

Take finite Galois extensions K_n of K in K_∞ ($n \geq 1$) such that $K_n \subset K_{n+1}$ and $K_\infty = \cup_n K_n$. Let X_n be the integral closure of X in K_n, so X_n is a finite étale Galois covering of X. This X_n is a natural generalization of X_n of §2.4.

Let S be the set of all elements f of $\Lambda(G)$ such that $\Lambda(G)/\Lambda(G)f$ are finitely generated $\Lambda(H)$-modules. Let $S^* = \cup_{n\geq 0}\, p^n S$. Then by [8] Theorem 2.4, S^* is a multiplicatively closed left and right Ore set in $\Lambda(G)$ and all elements of S^* are non-zero-divisors of $\Lambda(G)$. Hence we have the ring of fractions $\Lambda(G)_{S^*}$ by inverting all elements of S^* and the canonical homomorphism $\Lambda(G) \to \Lambda(G)_{S^*}$ is injective.

As in [8], let $\mathfrak{M}_H(G)$ be the category of finitely generated S^*-torsion $\Lambda(G)$-modules. If M is a finitely generated $\Lambda(G)$-module and $M(p)$ denotes the $\Lambda(G)$-submodule of M consisting of all elements killed by some powers of p, M belongs to $\mathfrak{M}_H(G)$ if and only if $M/M(p)$ is finitely generated as a $\Lambda(H)$-module.

We have a long exact sequence of K-groups (see [8], (24))

$$\cdots \to K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_G} K_0(\mathfrak{M}_H(G)) \to K_0(\Lambda(G)) \to \dots .$$

We have the following lemma.

Lemma 7. *Let $D = (D, \Phi)$ be a uniform F-crystal on X. We have*

(i) $R\varprojlim_n R\Gamma(X_n, N_0 DR(D))$ *and* $R\varprojlim_n R\Gamma(X_n, N_{-\infty} DR(D))$ *are bounded complexes and their cohomology groups are finitely generated* $\Lambda(G)$*-modules.*

(ii) $R\varprojlim_n R\Gamma(X_n, \mathcal{T}(D))$ *is a bounded complex and its cohomology groups belong to* $\mathfrak{M}_H(G)$.

(iii) $R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D))$ *is a bounded complex.*

(iv) Assume that there is a finite extension K' *of* K *in* K_∞ *such that* $Gal(K_\infty/K')$ *is pro-*p *and such that if* X' *denotes the integral closure of* X *in* K' *and* Γ' *denotes* $Gal(K'k_\infty/K')$*, then the* μ*-invariants of the cohomology groups* $R\varprojlim_n R\Gamma(X' \otimes_k k_n, \mathcal{S}(D))$*, which are finitely generated torsion* $\Lambda(\Gamma')$*-modules by Proposition 13, is zero. Then the cohomology groups of* $R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D))$ *belong to* $\mathfrak{M}_H(G)$.

Proof. Let K' be as in the assumption of (iv). Let $G' = \mathrm{Gal}(K_\infty/K')$, $H' = \mathrm{Gal}(K_\infty/K'k_\infty)$. These are pro-$p$ groups. Let X' be the integral closures of X in K'.

By Theorem 2.11 of [23], we have

$$\mathbb{Z}_p \otimes^L_{\Lambda(G')} R\varprojlim_n R\Gamma(X_n, N_0 DR(D)) = R\Gamma(X', N_0 DR(D)).$$

The right hand side is a finitely generated $\mathbb{Z}_p$-module. Hence (i) follows from theorem 2.11 of [23] and Nakayama's lemma applied to the local ring $\Lambda(G')$ (this is a local ring because G' is a pro-p group). The proof for $N_{-\infty} DR(D)$ is similar. (ii) and (iii) follow from (i) by the distinguished triangles

(1) $R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D)) \to R\varprojlim_n R\Gamma(X_n, N_0 DR(D)) \to R\varprojlim_n R\Gamma(X_n, N_{-\infty} DR(D)) \to,$

(2) $R\varprojlim_n R\Gamma(X_n, N_0 DR(D)) \to R\varprojlim_n R\Gamma(X_n, N_{-\infty} DR(D)) \to R\varprojlim_n R\Gamma(X_n, \mathcal{T}(D)) \to$

By the assumption of (iv), cohomology groups of $R\varprojlim_n R\Gamma(X' \otimes_k k_n, \mathcal{S}(D))$ are finitely generated $\mathbb{Z}_p$-modules. By theorem 2.11 of [23], we have also

$$\mathbb{Z}_p \otimes_{\Lambda(H')} R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D)) = R\varprojlim_n R\Gamma(X' \otimes_k k_n, \mathcal{S}(D)).$$

By Nakayama's lemma applied to the local ring $\Lambda(H')$, we have that all cohomology groups of $R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D))$ are finitely generated $\Lambda(H')$-modules and hence finitely generated $\Lambda(H)$-modules. This proves (iv). $\square$

In the rest of this §2.5, we assume that the assumption of 7 (iv) is satisfied.

We define the p-adic L-function $\mathcal{L}_D \in K_1(\Lambda(G)_{S^*})$ as the minus of the class of the automorphism $1 - \phi\iota$ of $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} R\varprojlim_n R\Gamma_{\mathrm{crys}}(X_n, N_{-\infty} DR(D))$.

The special values of $\mathcal{L}_D$ in the sense of [8] §3 are L-values of D:

$$\rho(\mathcal{L}_D) = L(X, D, \rho^\vee, 0)$$

for every finite dimensional continuous p-adic representation ρ of G which factors through a finite quotient of G. This is reduced to proposition 5.4 of [14] by the proof of [23] Theorem 1.1 (ii).

By the above distinguished triangles (1) and (2), we have (a version of Iwasawa main conjecture)

Proposition 14.

$$\partial_G(\mathcal{L}_D) = [R\varprojlim_n R\Gamma(X_n, \mathcal{S}(D))] + [R\varprojlim_n R\Gamma(X_n, \mathcal{T}(D)))]$$

in $K_0(\mathfrak{M}_H(G))$.

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