

THE UNIVERSITY OF CHICAGO

P-ADIC SHTUKAS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

ANTHONY YUMING WANG

CHICAGO, ILLINOIS

AUGUST 2020

Copyright © 2020 by Anthony Yuming Wang
All Rights Reserved

Table of Contents

ACKNOWLEDGMENTS	iv
ABSTRACT	v
1 INTRODUCTION	1
1.1 Outline	3
2 ADIC SPACES	5
2.1 Huber Rings	5
2.2 Banach Rings	9
2.3 Open Mapping Theorem	15
2.4 Adic Spectrum	18
2.5 Structure Sheaf	23
3 DIAMONDS	28
3.1 Integral Tilting	28
3.2 Perfectoid Spaces	29
3.3 Topologies	38
3.4 Diamonds	40
4 FARGUES-FONTAINE CURVES	42
4.1 Period Rings	42
4.2 The Analytic and Schematic Fargues-Fontaine Curves	50
4.3 Slope Theory	57
5 SHTUKAS FOR ALGEBRAIC GROUPS G	67
5.1 B_{dR}^+ -affine Grassmannian	67
5.2 Shtukas and (Pro-)Étaleness of the Period Morphism	75
5.3 An Example	83
REFERENCES	91

ACKNOWLEDGMENTS

I would like to thank my advisor Kazuya Kato for being a kind and generous mentor.

I would like to thank Matt Emerton for being my second reader. I would also like to thank Madhav Nori for being my secondary advisor and for helping me on my mathematical journey.

Finally, I would like to thank my loving and caring family.

ABSTRACT

We give an extension of Scholze's theory of shtukas to non-reductive groups. We compute a simple example of such a moduli of shtukas, and show that in this case, the period morphism is related to the logarithm. We also give an exposition of the objects appearing in this theory, and mention two joint results of the author: the open mapping theorem for complete analytic Huber rings, and a characterization of extensions of semistable vector bundles on the Fargues-Fontaine curve.

CHAPTER 1

INTRODUCTION

In recent years, p -adic Hodge theory has experienced many breakthroughs. One of these is the introduction of the Fargues-Fontaine curve, \mathcal{X}_{FF} , a geometric object which has brought some clarity to the many period rings of Fontaine, first introduced in order to properly state the p -adic analogues of classical results in Hodge theory over \mathbb{C} . Another is the introduction of perfectoid spaces by Scholze, who realized how the almost mathematics of Faltings can be geometrized, which allows him to pass between characteristic 0 and characteristic p : associated to any perfectoid space in (possibly in characteristic 0) is a perfectoid space in characteristic p called its tilt. A perfectoid space tilting to a given perfectoid space S in characteristic p is called an untilt of S . He then generalized the notion of a perfectoid space to the notion of a diamond, which can be thought of as an analogue of algebraic spaces, but for characteristic p perfectoid spaces instead of for schemes. Scholze defines the proétale topology for this purpose, and this topology allows him to define the tilt of any analytic adic space over $\text{Spa } \mathbb{Q}_p$ (not just perfectoid spaces) as a diamond.

The two stories are closely related For a given algebraically closed perfectoid field C^b in characteristic p , using Scholze's theory of diamonds, we can define a space

$$\mathcal{Y}_{[0,\infty)} = \text{Spd}(C^b) \times \text{Spd } \mathbb{Z}_p.$$

This space can be thought of as parametrizing all the perfectoid spaces which tilt to $\text{Spa}(C^b)$.

There is a subset

$$\mathcal{Y}_{(0,\infty)} = \text{Spd}(C^b) \times \text{Spd } \mathbb{Q}_p \subset \mathcal{Y}_{[0,\infty)}$$

which can be thought of as parametrizing the untilts of $\text{Spa}(C^b)$ which are of characteristic 0. The Frobenius φ on C^b then induces Frobenius automorphisms φ on $\mathcal{Y}_{[0,\infty)}$ and $\mathcal{Y}_{(0,\infty)}$. There is a continuous map $\kappa : \mathcal{Y}_{[0,\infty)} \rightarrow [0, \infty)$ with $\kappa(\varphi(x)) = p\kappa(x)$. We can write

$\mathcal{Y}_I = \kappa^{-1}(I)$ for intervals $I \subset [0, \infty)$, and $\mathcal{Y}_{(0, \infty)}$ is of this form. The Fargues-Fontaine curve \mathcal{X}_{FF} is then the quotient of $\mathcal{Y}_{(0, \infty)}$ under the action of φ , and so can be thought of as parametrizing characteristic 0 untilts of $\text{Spa}(C^b)$ up to a Frobenius.

Using his theory of diamonds, Scholze then introduced the notion of local p -adic shtukas, in analogy of the usual characteristic p shtukas introduced as part of the geometric Langlands program. Shtukas with one leg at a specified point, also known as Breuil-Kisin-Fargues modules, can be thought of as integral p -adic Hodge structures. The relation of the Fargues-Fontaine curve with p -adic Hodge theory can be viewed through this lens. Explicitly, a p -adic shtuka with one leg is a vector bundle \mathcal{F} on $\mathcal{Y}_{[0, \infty)}$ along with a meromorphic isomorphism

$$\varphi : \mathcal{F}|_{\mathcal{Y}_{[0, \infty)} \setminus \varphi^{-1}(x)} \rightarrow \varphi^* \mathcal{F}|_{\mathcal{Y}_{[0, \infty)} \setminus \varphi^{-1}(x)}$$

for some given point x on $\mathcal{Y}_{(0, \infty)}$. Given a shtuka we can descend the φ -action “near ∞ ” to get a vector bundle on \mathcal{X}_{FF} and descend the φ -action “near 0” to get a different vector bundle on \mathcal{X}_{FF} . There is an essentially surjective functor from isocrystals to vector bundles on \mathcal{X}_{FF} . Moreover, there is a good theory of slopes for vector bundles on \mathcal{X}_{FF} with the category of slope zero vector bundles equivalent to the category of \mathbb{Q}_p -vector spaces. If \mathcal{B} is a vector bundle on \mathcal{X}_{FF} , then the pullback of \mathcal{B} to $\mathcal{Y}_{(0, \infty)}$ can be extended to a φ -equivariant vector bundle on $\mathcal{Y}_{[0, \infty)}$ if and only if \mathcal{B} is of slope 0, with choices of extensions corresponding to \mathbb{Z}_p -lattices in the \mathbb{Q}_p -vector space associated to the slope 0 bundle \mathcal{B} . One sees from this that the vector bundle on \mathcal{X}_{FF} coming from the descent “near 0” of a shtuka with one leg is of slope 0 and is associated to a \mathbb{Q}_p -vector space.

All of this is related to p -adic Hodge theory. Given a smooth projective scheme over K a finite extension of \mathbb{Q}_p , we can associate a shtuka with one leg to the n th cohomology. Étale cohomology with \mathbb{Z}_p -coefficients gives us \mathbb{Z}_p -modules, which is associated to what happens for this shtuka “near 0”. Tensoring with \mathbb{Q}_p gives us a crystalline $\text{Gal}(K)$ -representation. Taking D_{crys} gives us crystalline cohomology, which is a φ -module over K_0 and defines an

isocrystal. This isocrystal then gives us a vector bundle on \mathcal{X}_{FF} , which is the descent of the shtuka “near ∞ ”. Then D_{dR} gives the deRham cohomology. Essentially, it tells us how to modify the vector bundle coming from descent “near ∞ ” at the image of the pleg of the shtuka to get the vector bundle coming from descent “near 0”.

Scholze then defines moduli spaces of shtukas for reductive groups G as a diamond, with the vector bundle of rank n case corresponding to GL_n . In this paper, we define such moduli spaces for shtukas of a general algebraic group with one leg at a specified point. To do this we define the B_{dR}^+ affine grassmannian for such G , which can be done similarly to the usual affine grassmannian for non-reductive groups, once some results about B_{dR}^+ -loop and B_{dR}^+ -positive loop groups are known.

We then compute the example of

$$G = \begin{bmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & 1 \end{bmatrix}$$

Inside the space of shtukas for this G , we will specify a subspace which can be interpreted parametrizing certain $\mathbb{Q}_p(1)$ extensions of \mathbb{Q}_p . If $C = \mathbb{C}_p$, we shall show that the C points of this space are given by $1 + \mathfrak{m}_{C^b}$. The natural period morphism to the affine grassmannian then corresponds to

$$1 + \mathfrak{m}_{C^b} \rightarrow (1 + \mathfrak{m}_{C^b})/\mathbb{Z}_p(1) = 1 + \mathfrak{m}_C \xrightarrow{\log} C.$$

1.1 Outline

In chapter 2, we give an exposition of Huber’s theory of adic spaces. §2.1 is about Huber rings, the topological commutative algebra underlying the theory. §2.2 is about Banach rings, which gives another way of looking at Huber rings, and allows us to easily construct objects such as the uniform completion. In §2.3, we give a proof of a joint result with Atticus

Christensen and Zonglin Jiang showing the open mapping theorem for complete analytic Huber rings. §2.4, we describe the topological space associated to a Huber pair. §2.5, we give a definition of a structure presheaf on these topological spaces, restrict attention to those Huber pairs for which this presheaf is a sheaf, and then use this to define the category of adic spaces.

In chapter 3, we give an exposition of Scholze’s theory of perfectoid spaces and diamonds. In §3.1, we describe the tilt operation integrally. In §3.2, we define perfectoid spaces and the tilt operation, and then talk about how the tilt operation preserves much geometry. §3.3, we define topologies on the category of perfectoid spaces, which we then use in §3.4 to define diamonds.

In chapter 4, we give an exposition of Fargues-Fontaine curves, including the relative situation of Kedlaya-Liu. In 4.1, we give the analytic definitions of many period rings. In 4.2, we define the Fargues-Fontaine curve and vector bundles on the curve. In 4.3, we describe slope theories, and apply it to vector bundles on the Fargues-Fontaine curve. We also briefly mention a joint result with Christopher Birkbeck, Tony Feng, David Hansen, Serin Hong and Lynelle Ye about which extensions can occur for vector bundles on the Fargues-Fontaine curve (essentially all those not forbidden by slope theory can occur).

In chapter 5, we give the theory for shtukas of algebraic groups. §5.1, we define the B_{dR}^+ -affine grassmannian for such groups. §5.2 defines the moduli space of shtukas and shows that the period morphism is pro-étale. In §5.3, we compute for the example

$$G = \begin{bmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & 1 \end{bmatrix}.$$

CHAPTER 2

ADIC SPACES

In this section we briefly go over the theory of adic spaces. Adic spaces were introduced by Huber to give a good theory of rigid analytic spaces, and has been the theory which Scholze uses to build his theory of diamonds.

2.1 Huber Rings

Definition 2.1.1. A *Huber ring* is a topological (commutative) ring A such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal I of A_0 such that the ideals I, I^2, \dots form an open neighborhood basis of 0. Any such $A_0 \subset A$ is called a *ring of definition*, and any such ideal $I \subset A_0$ is called an *ideal of definition*.

We will mostly be considering complete, uniform, analytic Huber rings, so we describe what these are.

Definition 2.1.2. A topological ring A is a topological group under $+$. This topological group structure gives A the structure of a uniform space. We say that A is *complete* if it is Hausdorff and complete with respect to this uniformity.

Proposition 2.1.3. [20, 1.1.1] *Suppose A is a complete Huber ring. Then*

1. *the set of units is open.*
2. *every maximal ideal is closed.*

Proposition 2.1.4. [14, 1.6] *The Cauchy completion $A \rightarrow \widehat{A}$ is the universal uniformly continuous function from A to a complete, Hausdorff uniform space. It is also the universal continuous ring homomorphism from A to a complete Huber ring.*

Definition 2.1.5. A subset $S \subset A$ of a Huber ring is *bounded* if for every open neighborhood U of 0, there exists an open neighborhood V of 0 such that $VS \subset U$. Since A has an open neighborhood basis consisting of additive subgroups, we may assume that U is open and additively closed, and check that $\sum VS \subset U$.

Lemma 2.1.6. [28, 3.1.5] *If $T \subset A$ is a finite subset of a Huber ring such that $\sum TM$ is open for some open additive subgroup $M \subset A$, then $\sum TM'$ is open for every open additive subgroup $M' \subset A$.*

Proposition 2.1.7. [14, Proposition 1(ii)] *A subring $A_0 \subset A$ of a Huber ring is a ring of definition if and only if it is open and bounded.*

Definition 2.1.8. Let $a \in A$.

1. a is *topologically nilpotent* if the sequence $1, a, a^2, a^3, \dots$ converges to 0.
2. a is *powerbounded* if the set $\{1, a, a^2, \dots\}$ is bounded.

Definition 2.1.9. [14, 1.3(iii)] We define A° to be the set of all powerbounded elements of A . This ring is the union of all rings of definition $A_0 \subset A$. We say A is *uniform* if A° is bounded, i.e. it is itself a ring of definition.

Definition 2.1.10. We let $A^{\circ\circ}$ denote the set of all topologically nilpotent elements of A . For any open, integrally closed subring $A^+ \subset A^\circ$, $A^{\circ\circ} \subset A^+$ is an ideal.

Proposition 2.1.11. *A finite set T is a set of generators for an ideal of definition if and only if the following two conditions are satisfied:*

1. *all elements of T are topologically nilpotent.*
2. *$\sum TM$ is open for some (equiv. every) open additive subgroup $M \subset A$.*

If this is the case, then for any ring of definition A_0 containing T , $\sum TA_0 \subset A_0$ is an ideal of definition.

Definition 2.1.12. Let A be Huber ring.

1. A is *Tate* if there exists a topologically nilpotent unit ϖ .
2. A is *analytic* if there is a finite set T of topologically nilpotent elements such that $\sum TA = A$, i.e. the topologically nilpotent elements generate the unit ideal.

It is clear that Tate implies analytic as the Tate case is the case where the finite set T has exactly one element.

Proposition 2.1.13. *If A is a Huber ring and $J \subset A$ is an ideal, then A/J with the quotient topology is a Huber ring. Moreover, if A is complete, then the completion of A/J is A/\bar{J} with the quotient topology.*

Construction 2.1.14. If A is a Huber ring, then we can make $A[T_1, \dots, T_n]$ into a topological abelian group by letting $U[T_1, \dots, T_n] \subset A[T_1, \dots, T_n]$ be open neighborhoods of 0 as U ranges over all open neighborhoods of 0 in A .

We let $A\langle T_1, \dots, T_n \rangle$ denote the completion of $A[T_1, \dots, T_n]$.

Proposition 2.1.15. *[15, 3.3] $A[T_1, \dots, T_n]$ and $A\langle T_1, \dots, T_n \rangle$ are Huber rings. Moreover,*

1. *If B is a Huber A -algebra, then continuous A -algebra homomorphisms $A[T_1, \dots, T_n] \rightarrow B$ are in bijection with functions $\{1, \dots, n\} \rightarrow B^\circ$.*
2. *If B is a complete Huber A -algebra, then continuous A -algebra homomorphisms from $A\langle T_1, \dots, T_n \rangle$ to B are in bijection with functions $\{1, \dots, n\} \rightarrow B^\circ$.*
3. *If A is complete, then we can write*

$$A\langle T_1, \dots, T_n \rangle = \left\{ \sum_{I \in \mathbb{Z}_{\geq 0}^n} a_I T^I \mid a_I \in A, a_I \rightarrow 0 \text{ as } I \rightarrow \infty \right\}.$$

Proposition 2.1.16. *Let A be a complete Huber ring and*

$$A\langle T^{\pm 1} \rangle := A\langle T, S \rangle / \overline{(TS - 1)}.$$

1. *If B is a complete Huber A -algebra, then continuous A -algebra homomorphisms from $A\langle T^{\pm 1} \rangle$ to B correspond to powerbounded elements in B which have a powerbounded inverse.*

2. *We have*

$$A\langle T^{\pm 1} \rangle = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \mid a_n \in A, a_n \rightarrow 0 \text{ as } n \rightarrow \pm\infty \right\}.$$

Remark 2.1.17. Huber [15, Section 3] introduces some other versions of polynomial and completed polynomial rings to capture some different convergence behavior of the coefficients. When A is analytic, which is the case most of interest to us, these other versions are not necessary.

In general, pushouts do not exist in the category of Huber rings. However, they do exist when the morphisms are adic, which will always be the case when working with analytic Huber rings.

Definition 2.1.18. A continuous ring homomorphism $f : A \rightarrow B$ between Huber rings is called *adic* if for some (equiv. for every) finite set $T \subset A$ which is a set of generators for an ideal of definition, the set $f(T) \subset B$ is a set of generators for an ideal of definition.

Example 2.1.19. We have the following examples:

1. Completion maps $A \rightarrow \widehat{A}$ are adic.
2. Quotient maps $A \rightarrow A/J$ are adic.
3. Polynomial algebra maps $A \rightarrow A[T_1, \dots, T_n]$ and their completions $A \rightarrow A\langle T_1, \dots, T_n \rangle$ are adic.

Proposition 2.1.20. [14, 1.5] [20, 1.1.7] *If $f : A \rightarrow B$ is a continuous ring homomorphism and A is analytic (resp. Tate), then B is analytic (resp. Tate) and f is adic.*

Proposition 2.1.21. [14, 1.8(i)] *Adic morphisms $f : A \rightarrow B$ map bounded sets to bounded sets.*

Construction 2.1.22. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be adic morphisms. Pick a ring of definition $A_0 \subset A$. We give $B \otimes_A C$ the additive group topology where the images of $U \otimes_{A_0} V \rightarrow B \otimes_A C$ form an open neighborhood basis of 0 as U ranges over all open A_0 -submodules of B and V ranges over all open A_0 -submodules of C . (Note: these maps $U \otimes_{A_0} V \rightarrow B \otimes_A C$ are not necessarily injective).

We define $B \widehat{\otimes}_A C$ to be the completion of $B \otimes_A C$.

Theorem 2.1.23. [28, 5.1.5(2)] *Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be adic morphisms. Then*

1. *$B \otimes_A C$ is their pushout in the category of Huber rings with continuous (resp. with adic) maps.*
2. *If A, B, C are complete, then $B \widehat{\otimes}_A C$ is their pushout in the category of complete Huber rings with continuous (resp. with adic) maps.*

2.2 Banach Rings

Complete, uniform, analytic Huber rings can also be considered from the point of view of Banach rings, where certain constructions and proofs become easier due to the existence of a norm. Approaching adic spaces from this point of view is given by Kedlaya [19] and Kedlaya-Liu [22] and can be thought of as being related to Berkovich's theory [2] except we don't need to work over a non-archimedean field.

Definition 2.2.1. Let M be an abelian group. A (*nonarchimedean*) *seminorm* on M is a function $|\cdot| : M \rightarrow \mathbb{R}_{\geq 0}$ such that

1. $|0| = 0$.
2. $|x - y| \leq \max(|x|, |y|)$.

A seminorm is called a *norm* if $|x| = 0$ implies $x = 0$.

A seminormed ring can be naturally given the structure of a topological abelian group, which is Hausdorff iff the seminorm is a norm.

Definition 2.2.2. A *seminormed ring* is a (commutative) ring B equipped with a seminorm $|\cdot| : B \rightarrow \mathbb{R}_{\geq 0}$ such that

1. $|1| = 1$
2. (submultiplicativity) $|xy| \leq |x||y|$.

The seminormed ring is *power multiplicative* if it satisfies

3. $|x^n| = |x|^n$

and it is *multiplicative* if

- 2'. $|xy| = |x||y|$.

A seminormed ring is a *normed ring* if the seminorm is a norm, and a normed ring is a *Banach ring* if B is complete with respect to the topology induced by the norm.

Definition 2.2.3. A group homomorphism $f : M \rightarrow N$ between seminormed abelian groups is called *bounded* if there exists a $c > 0$ such that for all $m \in M$,

$$|f(m)| \leq c|m|.$$

Proposition 2.2.4. *Bounded morphisms are continuous.*

Remark 2.2.5. Since we are not working with modules for algebras over nonarchimedean fields, continuity does not imply boundedness in general.

Proposition 2.2.6. *Let M be a seminormed abelian group, and $N \subset M$ be a subgroup. Then*

1. *The restriction of $|\cdot|$ to N gives it the structure of a seminormed abelian group.*

Moreover,

- *If $|\cdot|$ is a norm on M , then the restriction norm is a norm.*
- *If M is complete and N is closed, then N is complete.*

2. *The map $M/N \rightarrow \mathbb{R}_{\geq 0}$ defined by*

$$|m + N| = \inf_{n \in N} |m + n|$$

makes M/N into a seminormed abelian group. Moreover,

- *If $|\cdot|$ is a norm and N is closed, then this quotient seminorm is a norm.*
- *If M is complete, then M/N is complete.*

3. *If $M = B$ is a Banach ring and $N = J \subset B$ is a proper ideal, then the quotient seminorm on B/J makes this into a seminormed ring.*

Proof Sketch. 1. and 2. are clear. For 3., use the fact that B is complete to show that $|1| = 1$ in the quotient. □

Definition 2.2.7. If B is a seminormed ring, we can define the notions of a *bounded subset*, of *powerbounded elements*, of *nilpotent elements* using the same topological definitions we used for Huber rings. We can then also define the subring B° and say B is *uniform* if B° is bounded.

We say that a subset $S \subset B$ is *bounded in norm* if $|S| \subset \mathbb{R}_{\geq 0}$ is bounded. It is clear that any subset which is bounded in norm is bounded.

Definition 2.2.8. If B is a seminormed ring, then we say that B is *analytic* if the topologically nilpotent units generate the unit ideal in A .

Proposition 2.2.9. [20, 1.5.5] *Suppose B is an analytic seminormed ring. Then the underlying topological ring of B is an analytic Huber ring, and the ball of radius 1 is a ring of definition.*

By the following construction, every Huber ring can be seen as the underlying topological ring of a seminormed ring.

Construction 2.2.10. [20, 1.5.4] Let A be a Huber ring which is not indiscrete and choose a ring of definition A_0 and an ideal of definition I . We can define a function $|\cdot| : A \rightarrow e^{\mathbb{Z}} \cup \{0\}$ by describing the balls around 0:

- the ball $\{x : |x| \leq e^{-n}\}$ is equal to I^n for $n \geq 0$.
- the ball $\{x : |x| \leq e^{-n}\}$ is equal to $\{x : xI^{-n} \subset A_0\}$ for $n < 0$.

Proposition 2.2.11. *The construction gives A the structure of a seminormed ring.*

Remark 2.2.12. Construction 2.2.10 depends on a choice of A_0 and I . The resulting Banach ring structure on A depends on these choices: different choices can result in non-isomorphic Banach rings in the category of Banach rings with bounded maps[20, 1.5.9].

We see that we can promote a Huber ring to a seminormed ring. This is useful because we can then use the seminorm to prove statements about the underlying topological ring. However, notions which depend on the seminorm structure can depend on our choice of promotion.

One useful application of the theory of Banach rings is that uniformity is linked to the spectral norm.

Definition 2.2.13. If B is a seminormed ring, then we define the *spectral seminorm* on B by

$$|x|_{\text{sp}} = \liminf_{n \rightarrow \infty} |x^n|^{1/n}.$$

Proposition 2.2.14. *We have the following:*

1. $|x|_{\text{sp}} = \lim_{n \rightarrow \infty} |x^n|^{1/n}$.
2. (Fekete's lemma) the spectral seminorm is a power multiplicative seminorm.
3. x is topologically nilpotent if and only if $|x|_{\text{sp}} < 1$.

Definition 2.2.15. If B is a seminormed ring, the *Gelfand spectrum* of B , denoted $\mathcal{M}(B)$ is the set of all multiplicative seminorms $|\cdot|_m$ bounded by the norm of B , i.e.

$$|x|_m \leq |x| \text{ for all } x \in B.$$

There is a natural evaluation map $\mathcal{M}(B) \rightarrow \mathbb{R}_{\geq 0}^B$ which is an inclusion, and we give $\mathcal{M}(B)$ the subspace topology under this map.

Theorem 2.2.16. [2, 1.3.1] *We have the following:*

1. $\mathcal{M}(B)$ is compact.
2. $\mathcal{M}(B) \neq \emptyset$.
3. $|x|_{\text{sp}} = \sup_{\beta \in \mathcal{M}(B)} |\beta(x)| = \max_{\beta \in \mathcal{M}(B)} |\beta(x)|$.

Theorem 2.2.17. [20, 1.5.13] *Let B be a seminormed ring.*

1. *If the topology of B agrees with the topology induced by the spectral seminorm, then B is uniform.*
2. *If B is an analytic, uniform seminormed ring, then the topology of B is induced by the spectral seminorm.*

We can now use this characterization to prove facts about Huber rings. For example, the next proposition shows that we can take the uniform completion.

Proposition 2.2.18. [22, 2.8.13] *Let A be an analytic Huber ring. Then there is a map $A \rightarrow A_{\text{sp}}$ which is initial among continuous ring homomorphisms from A to uniform Huber rings. The map to the completion $A \rightarrow \widehat{A}_{\text{sp}}$ is initial among continuous ring homomorphisms from A to complete, uniform Huber rings.*

Proof. A_{sp} is the ring A but with topology defined by the spectral seminorm. If $A \rightarrow B$ is a continuous ring homomorphism, then since A is analytic, the map is adic. This implies that there is an ideal of definition $I \subset A_0$ of A and a ring of definition $f(A_0) \subset B_0 \subset B$ such that IB_0 is an ideal of definition. Promote A and B to Banach rings via Construction 2.2.10 using these ideals and rings of definitions. Then $|f(a)| \leq |a|$ which implies that $|f(a)|_{\text{sp}} \leq |a|_{\text{sp}}$. Since B is analytic and uniform, the spectral seminorm on B defines the topology, so we see that $A_{\text{sp}} \rightarrow B$ is continuous. \square

Proposition 2.2.19. *Suppose A_i is a filtered diagram of uniform, analytic Huber rings. Then $\underline{\text{colim}} A_i$ their colimit in the category of rings can be made into a Huber ring, such that it becomes a colimit of the diagram in the category of uniform, analytic Huber rings.*

Let $\widehat{\underline{\text{colim}}} A_i$ denote the completion of $\underline{\text{colim}} A_i$. When the A_i are complete, uniform, analytic Huber rings, this is the colimit of the diagram in the category of complete, uniform, analytic Huber rings.

Proof. WLOG, we may assume the diagram has an initial object A_0 . Let $I \subset A_0^\circ$ be an ideal of definition, so that $IA_i^\circ \subset A_i^\circ$ is an ideal of definition for all i . Promote the A_i to Banach rings via Construction 2.2.10 with $IA_i^\circ \subset A_i^\circ$, and equip them with their spectral seminorms, which define the same topology since the A_i are uniform and analytic. For $a \in \underline{\text{colim}} A_i$, define $|a|$ to be the infimum of $|a_i|$ over all $a_i \in A_i$ mapping to a . The argument of the previous proposition shows that the norm is decreasing along maps $A_i \rightarrow A_j$, so $|a|$ is the limit of a convergent net in $\mathbb{R}_{\geq 0}$. We can use this limit characterization to show that $|\cdot|$ defines a power multiplicative seminorm on $\underline{\text{colim}} A_i$. Then $\underline{\text{colim}} A_i$ is an analytic, seminormed ring with topology given by its spectral seminorm (by power multiplicativity),

so its underlying topological ring is a uniform, analytic Huber ring. To see it is a colimit, argue as in the previous proposition. \square

Example 2.2.20. Let A be a complete, uniform, analytic Huber ring. For X a topological space, we can endow the ring of continuous functions $C^0(X, A)$ with the compact-open topology.

Let $S = \varprojlim S_i$ be a profinite set (where the S_i are finite). Then the rings $C^0(S, A)$ and $C^0(S_i, A) \cong A^{S_i}$ are complete, uniform, analytic Huber rings, and

$$C^0(S, A) \cong \widehat{\operatorname{colim}} C^0(S_i, A).$$

2.3 Open Mapping Theorem

Modules for a complete, analytic Huber ring satisfy an open mapping theorem. As this was joint work of the author with Atticus Christensen and Zonglin Jiang at the 2017 Arizona Winter School, and the proof is short, we give it here.

Theorem 2.3.1 (Open Mapping Theorem). *[17, 2.2.5] Let A be a complete analytic Huber ring. Suppose M and N are complete, first countable topological A -modules. If $f : M \rightarrow N$ is a continuous surjective morphism of A -modules, then it is open.*

We need some standard results, which we repeat here:

Proposition 2.3.2 (Birkhoff-Kakutani Theorem). *A topological group G is metrizable if and only if it is Hausdorff and first countable. Moreover, we can take the metric to be right invariant.*

Definition 2.3.3. Let X be a topological space. A *meager subset* $Y \subset X$ is a subset which can be written as a countable union of nowhere dense sets. X is a *Baire space* if every meager subset has empty interior.

Theorem 2.3.4 (Baire Category Theorem). *If X is completely metrizable, then X is a Baire space.*

The proof of the open mapping theorem relies on the following lemma:

Lemma 2.3.5. *Let A be an analytic Huber ring, and let $f : M \rightarrow N$ be a continuous A -module homomorphism of topological A -modules. If the image $f(M) \subset N$ is not meager, then for every open neighborhood U of 0 in M , the closure $\overline{f(U)}$ of the image of U contains an open neighborhood of 0 in N .*

Proof. First, find an open neighborhood $0 \in U' \subset U$ such that $U' + U' \subset U$ and $U' = -U'$. This is possible since we can find an open neighborhood U'' satisfying the first condition by continuity of addition, and then take $U' = U'' \cap -U''$. We claim that it suffices to show that $\overline{f(U')}$ contains an interior point. Indeed, if W is an open neighborhood of 0 and $y \in \overline{f(V)}$ is a point such that $y + W \subset \overline{f(V)}$ then $W = (-y) + (y + W) \subset \overline{f(U')} + \overline{f(U')} \subset \overline{f(U') + f(U')} = \overline{f(U' + U')} \subset \overline{f(U)}$. This shows the claim.

Let x_1, \dots, x_k be topologically nilpotent units generating the unit ideal in A . By continuity of addition, find an open neighborhood $V \subset U'$ of 0 such that

$$\underbrace{V + \dots + V}_{k \text{ times}} \subset U'.$$

Let

$$W_n := \{\nu \in N \mid x_1^n \nu, \dots, x_k^n \nu \in \overline{f(V)}\}.$$

Note that W_n is closed since multiplication by elements of A is continuous on N . Since these elements are topologically nilpotent, for every $\mu \in M$, there exists a large enough n such that

$$x_1^n \mu, \dots, x_k^n \mu \in V.$$

This shows that $f(M) \subset \bigcup_{n=1}^{\infty} W_n$. Since $f(M)$ is not meager and each W_n is closed, we

conclude that some W_n contains an interior point z . Let B be an open neighborhood of $0 \in N$ such that $z + B \subset W_n$.

Since x_1, \dots, x_k generate the unit ideal in A , (x_1^n, \dots, x_k^n) is not contained in a maximal ideal, so by a standard axiom of choice argument, it must also be the unit ideal. Thus, we can find $a_1, \dots, a_k \in A$ such that

$$a_1 x_1^n + \dots + a_k x_k^n = 1.$$

The $a_i^{-1}(B)$ are all open neighborhoods of 0, so

$$B' := a_1^{-1}(B) \cap \dots \cap a_k^{-1}(B)$$

is an open neighborhood of 0. Let

$$y = (x_1^n + \dots + x_k^n)z.$$

We claim that $y + B' \subset \overline{f(U')}$, thus showing this set has an interior point. Indeed, if $r \in B'$, then

$$\begin{aligned} y + r &= (x_1^n + \dots + x_k^n)z + (a_1 x_1^n + \dots + a_k x_k^n)r \\ &= x_1^n(z + a_1 r) + \dots + x_k^n(z + a_k r) \\ &\subset x_1^n(z + B) + \dots + x_k^n(z + B) \\ &\subset x_1^n W_n + \dots + x_k^n W_n \\ &\subset \overline{f(V)} + \dots + \overline{f(V)} \subset \overline{f(U')} \end{aligned}$$

by construction of V . This shows the lemma. □

Proof of Theorem 2.3.1. By Proposition 2.3.2 and Theorem 2.3.4, we see that M and N are

metrizable Baire spaces, and there exists a right invariant metric on both. By Lemma 2.3.5, we see that for any open neighborhood U of 0 in M , the set $\overline{f(U)} \subset N$ contains an open neighborhood of 0.

For $r > 0$ let $B_r(x)$ be the open ball of radius r around x in M . We then see that there exists a $\rho(r) > 0$ such that $B_{\rho(r)}(0) \subset \overline{f(B_r(0))}$. By right invariance of the metric, this implies that for any $x \in M$, the ball $B_{\rho(r)}(f(x))$ centered around $f(x)$ is a subset of $\overline{f(B_r(x))}$.

We now claim that for all $r > 0$, we have $B_{\rho(r/2)}(f(x)) \subset \overline{f(B_r(x))}$. Let y be any element of $B_{\rho(r/2)}(f(x))$. We find a Cauchy sequence $z_1, z_2, \dots \in B_r(x)$ converging to $z \in \overline{B_r(x)}$ such that $f(z) = y$.

Let $z_1 = x$. Since $y \in B_{\rho(r/2)}(f(z_1))$, there exists a $z_2 \in B_{r/2}(z_1)$ such that $d(y, f(z_2)) < \min(1/2^2, \rho(r/2^2))$.

For $n \geq 2$, suppose we have found z_n such that $z_n \in B_{r/2^{n-1}}(z_{n-1})$ and $d(y, f(z_n)) < \min(1/2^n, \rho(r/2^n))$. Then since $y \in B_{\rho(r/2^n)}(f(z_n))$, we see that there exists a $z_{n+1} \in B_{r/2^n}(z_n)$ such that $d(y, f(z_{n+1})) < \min(1/2^{n+1}, \rho(r/2^{n+1}))$.

By construction, we see that $f(z_n)$ converges to y . Moreover, since $d(z_n, z_{n+1}) < r/2^n$, we see that the z_n form a Cauchy sequence, and each z_n is in $B_r(x)$. This shows $B_{\rho(r/2)}(f(x)) \subset \overline{f(B_r(x))}$ for all r .

To finish the proof of the theorem, since f is a continuous morphism of groups, it suffices to show that for any open neighborhood U of 0 in M , the image $f(U)$ contains an open neighborhood of 0. But U contains $B_s(0)$ for some s , and so it contains $\overline{B_r(0)}$ for every $0 < r < s$. In particular, $f(U)$ contains $B_{\rho(r/2)}(0)$, thus showing the theorem. \square

2.4 Adic Spectrum

Definition 2.4.1. A *Huber pair* (A, A^+) is a pair where A is a Huber ring, and $A^+ \subset A^\circ$ is an open, integrally closed subring. Such a ring A^+ is called a *ring of integral elements*

We say a Huber pair is *Hausdorff* (*resp. complete, analytic, etc.*) if the ring A has the corresponding property.

Morphisms of Huber pairs $(A, A^+) \rightarrow (B, B^+)$ are continuous ring homomorphisms $f : A \rightarrow B$ with $f(A^+) \subset B^+$, and we say the morphism is *adic* if f is.

Definition 2.4.2. A valuation x on a ring A is a totally ordered abelian group Γ (written multiplicatively) called the *value group* along with a map

$$x : A \rightarrow \{0\} \cup \Gamma,$$

which we often denote by $a \mapsto |a(x)|$, such that

1. $|(ab)(x)| = |a(x)||b(x)|$, $|0(x)| = 0$, $|1(x)| = 1$
2. $|(a + b)(x)| \leq \max(|a(x)|, |b(x)|)$
3. Γ is generated by $\text{Image}(x) \cap \Gamma$.

There is a natural notion of equivalence of valuations.

When A is a topological ring, we say a valuation $x : A \rightarrow \{0\} \cup \Gamma$ is *continuous* if, for every $\gamma \in \Gamma$ the set of all $a \in A$ with $|a(x)| < \gamma$ is open.

Definition 2.4.3. The *rank* of a valuation (Γ, x) is the length of a maximal chain

$$\{1\} = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_\ell = \Gamma$$

of totally ordered abelian groups.

Definition 2.4.4. If A is a ring, we define

$$\text{Spv}(A) = \{\text{equivalence classes of valuations on } A\}.$$

For elements f_1, \dots, f_n, g of A

$$U\left(\frac{f_1, \dots, f_n}{g}\right) = \{x \in \text{Spv}(A) : |g(x)| \neq 0 \text{ and } |f_i(x)| \leq |g(x)| \text{ for all } i\}.$$

We will also denote this by $U(\frac{T}{g})$ where $T = \{f_1, \dots, f_n\}$. These subsets are stable under finite intersections, and we topologize $\text{Spv}(A)$ so that they form a basis of opens.

Definition 2.4.5. If A is a Huber ring, we define

$$\text{Cont}(A) = \{x \in \text{Spv}(A) : x \text{ is continuous}\}.$$

If (A, A^+) is a Huber pair, we define

$$\text{Spa}(A) = \{x \in \text{Cont}(A) : |x(a)| \leq 1 \text{ for all } a \in A^+\}.$$

These inherit the subspace topology from $\text{Spv}(A)$. We abuse notation and will also use the symbol $U(\frac{T}{g})$ for the opens $U(\frac{T}{g}) \cap \text{Cont}(A) \subset \text{Cont}(A)$ and $U(\frac{T}{g}) \cap \text{Spa}(A, A^+) \subset \text{Spa}(A, A^+)$.

Definition 2.4.6. For A a Huber ring (resp. (A, A^+) a Huber pair), a rational open is a subset of $\text{Cont}(A)$ (resp. $\text{Spa}(A, A^+)$) of the form $U(\frac{T}{g})$ where $\sum TA \subset A$ is open. Since we may add g to T without changing the set $U(\frac{T}{g})$, we can alternatively define rational opens as subsets of the form $U(\frac{T}{g})$ where $\sum(T \cup \{g\})A \subset A$ is open.

Theorem 2.4.7. [14, 2.6, 3.2, 3.5] *We have the following:*

1. Spv (resp. Cont , resp. Spa) defines a contravariant functor from the category of rings (resp. Huber rings, resp. Huber pairs) to the category of spectral spaces. (Here, the category of Huber rings and Huber pairs have continuous maps as their morphisms, and the category of spectral spaces have spectral maps as their morphisms.)

2. The $U(\frac{T}{g})$ are quasicompact open in $\text{Spv}(A)$ (resp. $\text{Cont}(A)$, resp. $\text{Spa}(A, A^+)$).
3. The rational opens are stable under finite intersections and form a basis of quasicompact opens in $\text{Cont}(A)$ (resp. $\text{Spa}(A, A^+)$).

Definition 2.4.8. Let $x \in \text{Spa}(A, A^+)$. We define the *support* of x to be

$$\text{supp}(x) = \{a \in A : |a(x)| = 0\}.$$

By multiplicativity, this is a prime ideal.

We say that x is *non-analytic* if $\text{supp}(x)$ is open. Otherwise, we say that x is *analytic*. We define $\text{Spa}(A, A^+)^{\text{an}} \subset \text{Spa}(A, A^+)$ to be the subset of all analytic points. We say $\text{Spa}(A, A^+)$ is analytic if every point is analytic.

Proposition 2.4.9. [14, discussion after 3.5] Let $x \in \text{Spa}(A, A^+)$ be an analytic point with value group Γ . Then

1. Γ has rank at least 1.
2. The generalizations x' of x are in bijective correspondence with proper convex subgroups of Γ .
3. x has a unique maximal generalization, denoted by \tilde{x} , which is of rank 1.

Theorem 2.4.10. [14, 3.8] We have the following:

1. $\text{Spa}(A, A^+)^{\text{an}} \subset \text{Spa}(A, A^+)$ is quasicompact open.
2. (A, A^+) is analytic if and only if $\text{Spa}(A, A^+)$ is analytic.
3. If $f : (A, A^+) \rightarrow (B, B^+)$ is continuous, then $f^* : \text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ sends non-analytic points to non-analytic points.

4. If $f : (A, A^+) \rightarrow (B, B^+)$ is adic, then $f^* : \text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ sends analytic points to analytic points and maps rational subsets to rational subsets.
5. Suppose $f : (A, A^+) \rightarrow (B, B^+)$ is a continuous morphism inducing the map $f^* : \text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ and suppose (B, B^+) is complete. If f^* maps analytic points to analytic points, then f is adic.

Theorem 2.4.11. [14, 3.9] Let (A, A^+) be a Huber pair with completion $(\widehat{A}, \widehat{A}^+)$. Then the induced map

$$\text{Spa}(\widehat{A}, \widehat{A}^+) \rightarrow \text{Spa}(A, A^+)$$

is a homeomorphism. Moreover, the notions of analytic points, non-analytic points and rational subsets are identified via this homeomorphism.

Theorem 2.4.12. [14, 3.6] Let (A, A^+) be a complete Huber pair. Then

1. $\text{Spa}(A, A^+)^{\text{an}} = \emptyset$ if and only if A has the discrete topology.
2. $\text{Spa}(A, A^+) = \emptyset$ if and only if A is the zero ring.
3. If $a \in A$ is an element such that $|a(x)| \neq 0$ for all $x \in \text{Spa}(A, A^+)$, then a is invertible.

Theorem 2.4.13. Let A be a Huber ring. There is a contravariant Galois connection between subsets $B \subset A$ and subsets $X \subset \text{Cont}(A)$ defined by

$$X(B) := \{x \in \text{Cont}(A) : |x(b)| \leq 1 \text{ for all } b \in B\}$$

$$B(X) := \{b \in A : |x(b)| \leq 1 \text{ for all } x \in X\}.$$

The sets of the form $B(X) \subset A$ are precisely the open and integrally closed subrings of A . In particular,

$$A^+ = \{a \in A : |a(x)| \leq 1 \text{ for all } x \in \text{Spa}(A, A^+)\}.$$

Moreover, $X(A^\circ) \subset \text{Cont}(A)$ is dense, thus implying that $\text{Spa}(A, A^+) \subset \text{Cont}(A)$ is dense for any Huber pair (A, A^+) .

Definition 2.4.14. We define

1. A *nonarchimedean field* is a complete, non-discrete topological field K whose topology is induced by a valuation.
2. An *affinoid field* is a Huber pair (K, K^+) where K is either a nonarchimedean field or a discrete field, and $K^+ \subset K$ is an open and bounded valuation subring.

Given an affinoid field, the valuation subring $K^+ \subset K$ defines a distinguished closed point of $\text{Spa}(K, K^+)$ which is a specialization of every other point of $\text{Spa}(K, K^+)$.

Proposition 2.4.15. [16, 1.1.8] *Let (A, A^+) be a Huber pair. There is a natural bijection between points $x \in \text{Spa}(A, A^+)$ and isomorphism classes of continuous morphisms from (A, A^+) to an affinoid field (K, K^+) such that the subfield of K generated by the image of A is dense in K .*

Under this bijection, analytic points of $\text{Spa}(A, A^+)$ correspond to maps to (K, K^+) for K a nonarchimedean field.

2.5 Structure Sheaf

In this section, we give the topological space $\text{Spa}(A, A^+)$ a structure sheaf of rings.

Theorem 2.5.1. [15, 1.1, 1.3, 1.5] *We have the following:*

1. *Let A be a Huber ring, and $U = U(\frac{T}{g}) \subset X = \text{Cont}(A)$ be a rational subset. Then there exists a map $A \rightarrow A\langle\frac{T}{g}\rangle$ which is initial among continuous ring homomorphisms from A to a complete Huber ring B such that the induced morphism $\text{Cont}(B) \rightarrow \text{Cont}(A)$ factors through $U(\frac{T}{g})$. Moreover, $\text{Cont}(A\langle\frac{T}{g}\rangle) = U(\frac{T}{g}) \subset \text{Cont}(A)$.*

2. Let (A, A^+) be a Huber pair, and $U = U(\frac{T}{g}) \subset X = \text{Spa}(A, A^+)$ be a rational subset. Then there exists $A^+ \langle \frac{T}{g} \rangle \subset A \langle \frac{T}{g} \rangle$ and a map $(A, A^+) \rightarrow (A \langle \frac{T}{g} \rangle, A^+ \langle \frac{T}{g} \rangle)$ which is initial among maps of Huber pairs from (A, A^+) to complete Huber pairs (B, B^+) such that the induced morphism $\text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ factors through $U(\frac{T}{g})$. Moreover, $\text{Spa}(A \langle \frac{T}{g} \rangle, A^+ \langle \frac{T}{g} \rangle) = U(\frac{T}{g}) \subset \text{Spa}(A, A^+)$.

The maps $A \rightarrow A \langle \frac{T}{g} \rangle$ and $(A, A^+) \rightarrow (A \langle \frac{T}{g} \rangle, A^+ \langle \frac{T}{g} \rangle)$ are called *rational localizations*.

Example 2.5.2. When A is an analytic Huber ring and $T = \{f_1, \dots, f_n\}$, we can show that

$$A \left\langle \frac{T}{g} \right\rangle = A \langle T_1, \dots, T_n \rangle / \overline{(gT_1 - f_1, \dots, gT_n - f_n)}.$$

Moreover, if (A, A^+) is an analytic Huber pair, then $A^+ \langle \frac{T}{g} \rangle$ is the integral closure of the image of $A^+ \langle T_1, \dots, T_n \rangle$ in $A \langle \frac{T}{g} \rangle$.

Definition 2.5.3. Let A be a Huber ring (resp. (A, A^+) a Huber pair), and $X = \text{Cont}(A)$ (resp. $X = \text{Spa}(A, A^+)$). We define a presheaf \mathcal{O}_X of topological rings on X by

$$\mathcal{O}_X(V) := \varprojlim_{U(\frac{T}{g}) \subset V \text{ rational}} A \left\langle \frac{T}{g} \right\rangle.$$

Here, the limit is taken in the category of topological rings (and $\mathcal{O}_X(V)$ is not necessarily a Huber ring).

When $X = \text{Spa}(A, A^+)$, we also have a presheaf

$$\begin{aligned} \mathcal{O}_X^+(V) &= \varprojlim_{U(\frac{T}{g}) \subset V \text{ rational}} A^+ \left\langle \frac{T}{g} \right\rangle \\ &= \{f \in \mathcal{O}_X(V) : |f(x)| \leq 1 \text{ for all } x \in V\}. \end{aligned}$$

Definition 2.5.4. We say A is *sheafy* if \mathcal{O}_X is a sheaf on $X = \text{Cont}(A)$, or equivalently, if \mathcal{O}_X is a sheaf on $X = \text{Spa}(A, A^+)$ for some (equivalently any) Huber pair (A, A^+) .

If A is sheafy, the second description of \mathcal{O}_X^+ above shows that \mathcal{O}_X^+ is also a sheaf on $\mathrm{Spa}(A, A^+)$ for any Huber pair (A, A^+) .

Definition 2.5.5. We say an analytic Huber ring A is *strongly Noetherian* if $A\langle T_1, \dots, T_n \rangle$ is Noetherian for all nonnegative integers n .

Definition 2.5.6. We say that an analytic Huber ring A is *stably uniform* if $A\langle \frac{T}{g} \rangle$ is uniform for every rational open $U(\frac{T}{g})$.

Theorem 2.5.7. [15, 2.2] [6, 7] [23] [22, 2.8.10] [20, 1.2.13] *Suppose A is a Huber ring. Then A is sheafy in each of the following cases:*

- A is discrete.
- A has a Noetherian ring of definition.
- A is analytic and strongly Noetherian.
- A is analytic and stably uniform.

Proposition 2.5.8. *If A is sheafy, then $\mathcal{O}_X(X) = \widehat{A}$ for $X = \mathrm{Cont} A$ or $X = \mathrm{Spa}(A, A^+)$.*

Definition 2.5.9. We define the category of locally v -ringed spaces as follows:

1. The objects are triples $(X, \mathcal{O}_X, \{|\cdot(x)|\}_{x \in X})$ where X is a topological space, \mathcal{O}_X is a sheaf of complete topological rings on X , and for each $x \in X$, the stalk

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U)$$

(with the colimit taken in the category of topological rings) is a local ring, and $|\cdot(x)|$ is a continuous valuation on $\mathcal{O}_{X,x}$.

2. The morphisms $(X, \mathcal{O}_X, \{|\cdot(x)|\}_{x \in X}) \rightarrow (Y, \mathcal{O}_Y, \{|\cdot(y)|\}_{y \in Y})$ are pairs (f, μ) where $f : X \rightarrow Y$ is a continuous morphism, and $\mu : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a map of sheaves such

that for every $x \in X$, the induced morphism $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a map of local rings and there exists a morphism of the value groups $\Gamma_{f(x)} \rightarrow \Gamma_x$ making the diagram

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \{0\} \cup \Gamma_{f(x)} & \longrightarrow & \{0\} \cup \Gamma_x \end{array}$$

commute.

It is clear that if $(X, \mathcal{O}_X, \{|\cdot(x)|\}_{x \in X})$ is a locally v -ringed space and $U \subset X$ is open, then U inherits the structure of a locally v -ringed space. By abuse of notation, we shall often write X for $(X, \mathcal{O}_X, \{|\cdot(x)|\}_{x \in X})$, and will sometimes write $|X|$ for the underlying topological space of a locally v -ringed space X written in this way.

Definition 2.5.10. For a locally v -ringed space X , we can define a sheaf \mathcal{O}_X^+ of complete topological rings by

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1 \text{ for all } x \in U\}.$$

Construction 2.5.11. Let (A, A^+) be a sheafy Huber pair. We give $X = \text{Spa}(A, A^+)$ the structure of a locally v -ringed space by taking \mathcal{O}_X to be the structure sheaf. For any $x \in X$ and any rational open $U(\frac{T}{g})$ containing x , we have a natural valuation $\mathcal{O}_X(U(\frac{T}{g})) = A(\frac{T}{g}) \rightarrow \{0\} \cup \Gamma_x$. Putting these together give us the desired valuations $\mathcal{O}_{X,x} \rightarrow \{0\} \cup \Gamma_x$.

We will let $\text{Spa}(A)$ denote $\text{Spa}(A, A^\circ)$.

Definition 2.5.12. An *adic space* is a locally v -ringed space X for which there exists an open cover $X = \bigcup_{i \in I} U_i$ such that each U_i is isomorphic to $\text{Spa}(A, A^+)$ as locally v -ringed spaces for some sheafy Huber pair (A, A^+) . An adic space is *affinoid* if it is isomorphic to $\text{Spa}(A, A^+)$ for some sheafy (A, A^+) .

Morphisms of adic spaces are morphisms of the underlying locally v -ringed spaces.

Theorem 2.5.13. *[15, 2.1(ii)] If (A, A^+) is a sheafy Huber pair, and X is an adic space, then*

$$\mathrm{Hom}(X, \mathrm{Spa}(A, A^+)) \cong \mathrm{Hom}((A, A^+), (\mathcal{O}_X(X), \mathcal{O}_X^+(X))).$$

Here, homomorphisms on the left are taken in the category of adic spaces, and homomorphisms on the right are taken in the category of pairs of complete topological rings.

In particular, if (A, A^+) and (B, B^+) are sheafy, then homomorphisms $\mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ are in bijection with maps $(\widehat{A}, \widehat{A}^+) \rightarrow (\widehat{B}, \widehat{B}^+)$.

Remark 2.5.14. In the above theorem, we do not get an adjunction because $\mathcal{O}_X(X)$ may not be a Huber ring.

CHAPTER 3

DIAMONDS

3.1 Integral Tilting

Definition 3.1.1. By a *p-adically complete topological ring*, we mean a topological ring A^+ equipped with a continuous homomorphism $\mathbb{Z}_p \rightarrow A^+$ such that the natural morphism

$$A^+ \rightarrow \varprojlim A^+/p^n A^+$$

is an isomorphism of rings (equivalently, of topological rings).

Definition 3.1.2. By a *perfect topological \mathbb{F}_p -algebra*, we mean a topological \mathbb{F}_p -algebra R^+ whose Frobenius φ is a homeomorphism.

Definition 3.1.3. Given a *p-adically complete topological ring* A^+ , we define the *tilt* of A^+ to be the perfect topological \mathbb{F}_p -algebra

$$(A^+)^{\flat} = \varprojlim_{\varphi} A^+/pA^+ = \{(a_0, a_1, a_2, \dots) \in (A^+/pA^+)^{\mathbb{N}} \mid a_n^p = a_{n-1} \text{ for all } n \geq 1\}.$$

The construction of tilting was first introduced by Fontaine and Wintenberger [12], and eventually gets its name from Scholze [25].

Lemma 3.1.4. *Suppose A^+ is a topological ring and $I \subset A^+$ is an ideal containing p such that $A^+ \xrightarrow{\cong} \varprojlim A^+/I^n$ as topological rings. Then the natural reduction maps*

$$\varprojlim_{x \mapsto x^p} A^+ \rightarrow \varprojlim_{\varphi} A^+/pA^+ = (A^+)^{\flat} \rightarrow \varprojlim_{\varphi} A^+/I$$

are homeomorphisms.

Definition 3.1.5. For a topological ring R , its ring of Witt vectors $W(R)$ can be made into a topological ring by taking the product topology on the Witt components $W(R) \cong R^{\mathbb{N}}$.

If R^+ is a perfect topological \mathbb{F}_p -algebra, then since $W(\mathbb{F}_p) = \mathbb{Z}_p$, we see that $W(R^+)$ is a p -adically complete topological ring.

Lemma 3.1.6. *We have the following:*

1. *If R is a complete topological ring, then so is $W(R)$.*
2. *If R^+ is a perfect topological \mathbb{F}_p -algebra equipped with the I -adic topology for some finitely generated ideal $I = (r_1, \dots, r_n)$, then $W(R^+)$ is equipped with the J -adic topology for $J = (p, [r_1], \dots, [r_n])$.*

Theorem 3.1.7. *Taking Witt vectors and tilting gives an adjunction between the categories of perfect topological \mathbb{F}_p -algebras R^+ and p -adically complete topological rings A^+ :*

$$\mathrm{Hom}(W(R^+), A^+) \cong \mathrm{Hom}(R^+, (A^+)^{\flat}).$$

Definition 3.1.8. For A^+ a p -adically complete topological ring, we define the theta map $\theta : W((A^+)^{\flat}) \rightarrow A^+$ to be the counit of the above adjunction.

We define the sharp map $x \mapsto x^{\sharp} : (A^+)^{\flat} \rightarrow A^+$ to be the composition

$$(A^+)^{\flat} \xrightarrow{x \mapsto [x]} W((A^+)^{\flat}) \xrightarrow{\theta} A^+.$$

Under the identification $(A^+)^{\flat} \cong \varprojlim_{x \mapsto x^p} A^+$, the sharp map sends (x_0, x_1, \dots) to x_0 .

3.2 Perfectoid Spaces

Definition 3.2.1. A topological ring A^+ is *integral perfectoid* if it is I -adically complete (and has the I -adic topology) for some ideal $I = (\varpi)$ satisfying

1. $p \in I^p$
2. the Frobenius $\varphi : A^+/I^p \rightarrow A^+/I^p$ is surjective.

Definition 3.2.2. A uniform, Tate Huber ring A is *perfectoid* if A° is integral perfectoid (equivalently, if A^+ is integral perfectoid for some/any ring of integral elements A^+). We call a Huber pair (A, A^+) with A perfectoid a *perfectoid pair*.

Remark 3.2.3. There are more general definitions of perfectoid and integral perfectoid rings. One such definition is that A^+ is integral perfectoid if it is I -adically complete for some finitely generated ideal I satisfying the conditions of definition 3.2.1. Then a perfectoid ring is a uniform analytic Huber ring such that A° is integral perfectoid. Many of the following statements hold for this more general notion of perfectoid ring, however the proofs are sometimes more complicated and not everything is known. See [20] for statements and proofs along these lines.

Since every analytic adic space is locally Tate, the more general definition of perfectoid ring above does not change the resulting category of perfectoid spaces. One benefit of the more general definition is that we have a larger class of “affinoid perfectoid spaces”. For example, the class of adic spaces which are both affinoid and perfectoid must at least include the Spa of these more general perfectoid rings. It is a conjecture that every uniform Tate (or analytic) ring whose Spa is a perfectoid space is perfectoid (See [18, Problem 2]). Buzzard-Verberkmoes [6, Proposition 13] give a counterexample without the uniform assumption.

Theorem 3.2.4. *Suppose A^+ and A are of characteristic p .*

1. *If A^+ is I -adically complete (with the I -adic topology) for $I = (\varpi)$, then A^+ is integral perfectoid if and only if A^+ is perfect.*
2. *If A is a complete, Tate Huber ring, then A is perfectoid if and only if A is perfect.*

Definition 3.2.5. Let A be a perfectoid ring. We define the *tilt* of A to be the topological \mathbb{F}_p -algebra whose underlying multiplicative monoid is given by

$$\varprojlim_{x \mapsto x^p} A = \{(a_0, a_1, \dots) \in A^{\mathbb{N}} \mid a_n^p = a_{n-1} \text{ for all } n \geq 1\}.$$

Addition is defined by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (c_0, c_1, \dots)$$

where $c_n = \lim_{i \rightarrow \infty} a_{n+i}^{p^i} + b_{n+i}^{p^i}$. Under the identification $(A^+)^{\flat} \cong \varprojlim_{x \mapsto x^p} A^+$, we have an embedding of topological rings $(A^+)^{\flat} \subset A^{\flat}$.

Definition 3.2.6. Let R be a perfectoid ring of characteristic p . We define the *bounded Witt vectors* $W^b(R) \subset W(R)$ to be the set of all elements

$$[a_0] + [a_1]p + [a_2]p^2 + \dots \in W(R)$$

such that $\{a_0, a_1, a_2, \dots\} \subset R$ is bounded.

Definition 3.2.7. [11] Let R^+ be an integral perfectoid ring of characteristic p . An element

$$\xi = [\xi_0] + [\xi_1]p + [\xi_2]p^2 + \dots \in W(R^+)$$

is called *primitive of degree n* if ξ_0, \dots, ξ_{n-1} are topologically nilpotent, and ξ_n is a unit. This is equivalent to saying the image of ξ under the natural map $W(R^+) \rightarrow W(R^+/R^{\circ\circ})$ generates the ideal (p^n) .

Example 3.2.8. The primitive elements of degree 0 are precisely the units of $W(R^+)$.

Proposition 3.2.9. [10, Section 2.2.1] Let R^+ be integral perfectoid of characteristic p , and

$a, b, c \in W(R^+)$ be elements with $ab = c$. If two of a, b, c are primitive, then so is the third, and $\deg a + \deg b = \deg c$.

Definition 3.2.10. An ideal $I \subset W(R^+)$ is *primitive* if it is generated by an element which is primitive of degree 1.

Theorem 3.2.11. [25][20, 2.3.9][22][21][Algebraic Tilting Equivalence] There is an equivalence of categories between

$$\{\text{perfectoid pairs } (A, A^+)\} \Leftrightarrow \left\{ (R, R^+, I) \left| \begin{array}{l} (R, R^+) \text{ perfectoid pair over } \mathbb{F}_p \\ I \subset W(R^+) \text{ primitive} \end{array} \right. \right\}$$

given by

$$(A, A^+) \mapsto (A^{\flat}, (A^+)^{\flat}, \ker \theta : W(A^{+\flat}) \rightarrow A^+)$$

and

$$(W^{\flat}(R)/IW^{\flat}(R), W(R^+)/I) \leftarrow (R, R^+, I).$$

Definition 3.2.12. If (R, R^+) is a perfectoid pair in characteristic p , an *untilt* of (R, R^+) is a perfectoid pair (A, A^+) along with an isomorphism $(A^{\flat}, (A^+)^{\flat}) \cong (R, R^+)$.

Corollary 3.2.13. If (A, A^+) is an untilt of (R, R^+) , there is an equivalence of categories

$$\{\text{perfectoid pairs over } (A, A^+)\} \Leftrightarrow \{\text{perfectoid pairs over } (R, R^+)\}.$$

The tilting equivalence also preserves much geometry on both sides.

Theorem 3.2.14 (Tilting equivalence for rational subsets). [25] [20, 2.5.2] [22] [21] Let (A, A^+) be a perfectoid pair.

1. Every rational localization of (A, A^+) is a perfectoid pair.

2. *The map of sets*

$$\begin{aligned} \mathrm{Spa}(A, A^+) &\rightarrow \mathrm{Spa}(A^{\flat}, (A^{\flat})^+) \\ v &\mapsto v \circ \sharp \end{aligned}$$

is a homeomorphism.

3. *The tilting equivalence induces an equivalence between the rational localizations of (A, A^+) and $(A^{\flat}, (A^{\flat})^+)$ compatibly with the homeomorphism above.*

Corollary 3.2.15. *A perfectoid pair is stably uniform, thus sheafy.*

Definition 3.2.16. An *affinoid perfectoid space* is an adic space of the form $\mathrm{Spa}(A, A^+)$ for a perfectoid pair (A, A^+) . A *perfectoid space* X is an adic space which is covered by affinoid perfectoids. Since tilting preserves rational subsets, we can define the tilt of a perfectoid space X to get a perfectoid space X^{\flat} of characteristic p .

Remark 3.2.17. As mentioned in Remark 3.2.3, the Spa of a non-Tate “analytic perfectoid ring” is perfectoid and affinoid but not affinoid perfectoid.

Lemma 3.2.18. *[20, 2.4.1] [25] [21] If $A \rightarrow B$ and $A \rightarrow C$ are morphisms of perfectoid rings, then $B \widehat{\otimes}_A C$ is a perfectoid ring with tilt $B^{\flat} \widehat{\otimes}_{A^{\flat}} C^{\flat}$.*

Definition 3.2.19. [27] Let (A, A^+) be a complete, uniform, analytic Huber pair. Let B be the uniform completion of A/I , and B^+ be the integral closure of the image of A^+ in B . Then $(A, A^+) \rightarrow (B, B^+)$ is initial among maps f from (A, A^+) to complete, analytic, uniform Huber pairs (C, C^+) with the property that $f(I) = 0$. We write

$$V(I) := \mathrm{Spa}(B, B^+) \subset \mathrm{Spa}(A, A^+)$$

and call it a *Zariski closed embedding*.

Theorem 3.2.20 (Tilting equivalence for Zariski closed). [3, 7.4,7.5][20, 2.4.4,2.9.12] Suppose (A, A^+) is perfectoid, and let $f : \mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ be a map induced by $f^* : (A, A^+) \rightarrow (B, B^+)$.

1. If f is a Zariski closed embedding, then (B, B^+) is perfectoid and $f^* : A \rightarrow B$ is surjective.
2. If B is perfectoid, then f is a Zariski closed embedding if and only if $f^* : A \rightarrow B$ is surjective and B^+ is the integral closure of the image of A^+ .
3. f is a Zariski closed embedding if and only if f^\flat is a Zariski closed embedding.

Remark 3.2.21. For general (not necessarily perfectoid) complete, uniform, analytic Huber rings (A, A^+) with $V(I) = \mathrm{Spa}(B, B^+)$, we only know that the image of $A \rightarrow B$ is dense.

Definition 3.2.22. Let A be a topological ring and M be a finitely generated A -module. The *natural topology* on M is the quotient topology under any surjection $A^n \rightarrow M$. This is the topology for which continuous A -module homomorphisms from M to a topological A -module N are in bijection with maps of the underlying A -modules.

Proposition 3.2.23 (Huber). *Suppose A is a Huber ring, and let B be a finite type A -algebra. Then B with its natural topology is a Huber ring, and the map $A \rightarrow B$ is adic.*

Proof. Let $A_0 \subset A$ be a ring of definition. Let b_1, \dots, b_n generate B as an A -module and $q : A^n \rightarrow B$ be the corresponding A -module homomorphism. Write $b_i b_j = \sum_k a_{ijk} b_k$ for $a_{ijk} \in A$. There is then a small enough ideal of definition I such that $I a_{ijk} \subset A_0$ for all i, j, k . Then

$$B_0 := A_0 + I b_1 + \dots + I b_n$$

will be an open subring of B .

By definition, if $V \subset B$ were open, then V contains $I^N b_1 + \dots + I^N b_n$ for some large N . Write $1 = a_1 b_1 + \dots + a_n b_n$. Find a large enough $M \geq N$ such that $I^M a_i \subset I^N$ for

$i = 1, \dots, n$. Then

$$I^M B_0 = I^M(a_1 b_1 + \dots + a_n b_n) + I^{M+1} b_1 + \dots + I^{M+1} b_n \subset I^N b_1 + \dots + I^N b_n \subset V.$$

Thus, we see that B_0 is a ring of definition, and IB_0 is an ideal of definition. \square

Definition 3.2.24. A map of Huber rings $A \rightarrow B$ is *finite* (resp. *finite étale*) if the underlying map of rings is finite (resp. finite étale) and B is equipped with its natural topology as an A -module.

A map of Huber pairs $(A, A^+) \rightarrow (B, B^+)$ is *finite* (resp. *finite étale*) if $A \rightarrow B$ is a finite (resp. finite étale) map of Huber rings and B^+ is the integral closure of the image of A^+ in B .

Lemma 3.2.25. *If $(A, A^+) \rightarrow (B, B^+)$ is a finite étale map of perfectoid pairs, then $\mathrm{Spa}(B, B^+) \rightarrow \mathrm{Spa}(A, A^+)$ has open image.*

Theorem 3.2.26 (Almost Purity). [20, 2.5.9] [25][9] [22] *We have the following:*

1. *A finite étale algebra over a perfectoid pair is a perfectoid pair.*
2. *For perfectoid (A, A^+) , the tilting equivalence induces an equivalence between the categories of finite étale algebras over (A, A^+) and finite étale algebras over its tilt.*

Definition 3.2.27. [24] A morphism $f : X \rightarrow Y$ of perfectoid spaces is *finite étale* if for every affinoid perfectoid $U = \mathrm{Spa}(A, A^+) \subset Y$, then inverse image $V = f^{-1}(U) \subset X$ is affinoid perfectoid and the morphism of perfectoid pairs inducing $V \rightarrow U$ is a finite étale morphism of Huber pairs.

Definition 3.2.28. [24] A morphism $f : X \rightarrow Y$ of perfectoid spaces is *étale* if for every $x \in X$, there exist open neighborhoods $x \in V \subset X$ and $f(x) \in U \subset Y$ with $f(V) \subset U$ such

that $f|_V : V \rightarrow U$ factors as

$$\begin{array}{ccc} V & \longrightarrow & W \\ & \searrow f|_V & \downarrow \\ & & U \end{array}$$

where $V \rightarrow W$ is finite étale and $W \rightarrow U$ is an open embedding.

Remark 3.2.29. Such a definition of étaleness is not good for schemes, but it works in this case because of the analytic nature of perfectoid spaces.

Kedlaya expects that there is a more algebraic definition of étaleness whenever we are working with what he calls stably sheafy Huber rings, which is a class of Huber rings including the perfectoid rings [20, A.5] The subcase where the rings are strongly Noetherian was considered by Huber [16, 1.1.1, 1.7, 1.8], who showed that in that case, the more algebraic definition is equivalent to the above definition.

Lemma 3.2.30. *Étale maps $X \rightarrow Y$ have open image.*

Theorem 3.2.31. *For perfectoid spaces X , tilting induces an equivalence between perfectoid spaces étale over X and perfectoid spaces étale over X^\flat .*

Definition 3.2.32. [24, 7.8] A morphism $(A, A^+) \rightarrow (B, B^+)$ of perfectoid pairs is *affinoid pro-étale* if there exists a filtered system $(A, A^+) \rightarrow (B_i, B_i^+)$ of perfectoid pairs over (A, A^+) such that $\mathrm{Spa}(B_i, B_i^+) \rightarrow \mathrm{Spa}(A, A^+)$ is étale and

$$(B, B^+) \cong \widehat{\mathrm{colim}} (B_i, B_i^+)$$

as (A, A^+) -algebras. This means that $B \cong \widehat{\mathrm{colim}} B_i$ and B^+ is the smallest open integrally closed subring of B containing the images of the B_i^+ (which exists by Theorem 2.4.13).

If $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$ and $Y_i = \mathrm{Spa}(B_i, B_i^+)$, we say that $Y = \varprojlim Y_i \rightarrow X$ is a *pro-étale presentation* of the map $Y \rightarrow X$.

Theorem 3.2.33. [24, 7.10] *If $X = \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid space, then the*

pro-category of the category of étale morphisms from affinoid perfectoid spaces to X and the category of affinoid pro-étale morphisms over X are naturally equivalent.

Definition 3.2.34. A morphism $X \rightarrow Y$ of perfectoid spaces is *pro-étale* if for every $x \in X$, there exist affinoid perfectoids $x \in V$ and $f(x) \in U$ such that $f(V) \subset U$ and the morphism of perfectoid pairs inducing $f|_V : V \rightarrow U$ is an affinoid pro-étale morphism.

Theorem 3.2.35. *For perfectoid spaces X , tilting induces an equivalence between perfectoid spaces pro-étale over X and perfectoid spaces pro-étale over X^\flat .*

Example 3.2.36. [24, Discussion after 7.8] For S a topological space, let \underline{S} denote the presheaf sending a perfectoid space X to the continuous morphisms from X to S . If S is profinite, and $X = \text{Spa}(A, A^+)$ is an affinoid perfectoid space, then

$$\underline{S} \times X \rightarrow X$$

is affinoid pro-étale, with a pro-étale presentation given by

$$(A, A^+) \rightarrow \widehat{\text{colim}} (C^0(S_i, A), C^0(S_i, A^+)) \cong (C^0(S, A), C^0(S, A^+))$$

where the diagram is over all finite quotients $S \rightarrow S_i$.

Example 3.2.37. [24, 7.9] Zariski closed embeddings are affinoid pro-étale. Indeed, if $Z \subset X = \text{Spa}(A, A^+)$ is defined by the ideal $I \subset A$, then

$$Z \cong \varprojlim U \left(\frac{T}{1} \right) \rightarrow X$$

where T ranges over all finite subsets of I .

3.3 Topologies

Definition 3.3.1. We define symbols for categories:

1. Let Perf be the category of perfectoid spaces of characteristic p .
2. Let Perfd be the category of perfectoid spaces.
3. For X a perfectoid space of characteristic p , let Perf_X denote the category of perfectoid spaces over X .
4. For X a perfectoid space, let Perfd_X denote the category of perfectoid spaces over X . This is equivalent to Perf_{X^\flat} by tilting.
5. For X a perfectoid space, let X_{an} denote the category of open immersions $U \rightarrow X$.
6. For X a perfectoid space, let $X_{\text{ét}}$ denote the category of étale morphisms $Y \rightarrow X$.
7. For X a perfectoid space, let $X_{\text{proét}}$ denote the category of pro-étale morphisms $Y \rightarrow X$.

Lemma 3.3.2. [24] *For each class of morphism*

$$\mathcal{C} \in \{\text{open immersions, étale morphisms, pro-étale morphisms, morphisms}\}$$

in Perfd , we have

1. *isomorphisms are in \mathcal{C} ,*
2. *compositions of morphisms in \mathcal{C} are in \mathcal{C} ,*
3. *pullbacks of a morphism in \mathcal{C} along an arbitrary morphism in Perfd are in \mathcal{C} .*

Definition 3.3.3. In the category Perfd , we say that $\{f_i : X_i \rightarrow X\}$ is an

1. *analytic cover* if the f_i are open immersions and $\bigcup f_i(|X_i|) = |X|$.
2. *étale cover* if the f_i are étale and $\bigcup f_i(|X_i|) = |X|$.
3. *proétale cover* if the f_i are pro-étale and for every quasicompact $U \subset X$, there exists finitely many indices i and quasicompact opens $U_i \subset X_i$ such that $|U| \subset \bigcup f_i(|U_i|)$.
4. *v-cover* if for every quasicompact $U \subset X$, there exists finitely many indices i and quasicompact opens $U_i \subset X_i$ such that $|U| \subset \bigcup f_i(|U_i|)$.

Using these notions of covers, we can define:

- analytic sites on $\text{Perf}, \text{Perfd}, \text{Perf}_X, \text{Perfd}_X$ and X_{an} .
- étale sites on $\text{Perf}, \text{Perfd}, \text{Perf}_X, \text{Perfd}_X$ and $X_{\text{ét}}$.
- proétale sites on $\text{Perf}, \text{Perfd}, \text{Perf}_X, \text{Perfd}_X$ and $X_{\text{proét}}$.
- v -sites on $\text{Perf}, \text{Perfd}, \text{Perf}_X$ and Perfd_X .

Remark 3.3.4. One should technically cut off the size of these categories. We will ignore all such issues. See Scholze's paper [24] for more careful definitions dealing with size issues.

Theorem 3.3.5. [24, 8.6] *The v -topology on Perfd is subcanonical. In particular, this implies that every topology defined above is subcanonical.*

In the following, we will abuse notation and use Y to denote the presheaf $h_Y(-) = \text{Hom}(-, Y)$ for Y in any of the above categories.

Theorem 3.3.6. [24, 8.6] *The functors*

$$X \mapsto \mathcal{O}_X(X) \quad \text{and} \quad X \mapsto \mathcal{O}_X^+(X)$$

are sheaves in the v -topology.

By definition, the notion of being a pro-étale morphism $Y \rightarrow X$ is analytic local on the target, i.e. if there is an analytic cover $U_i \rightarrow X$ such that $Y \times_X U_i \rightarrow U_i$ are pro-étale, then $Y \rightarrow X$ is pro-étale. However, being a pro-étale morphism is not pro-étale local on the target. This is remedied by the notion of quasi-pro-étale morphism.

Definition 3.3.7. [24, 7.15,7.16] A perfectoid space is *strictly totally disconnected* if it is affinoid and every connected component is isomorphic to $\mathrm{Spa}(K, K^+)$ for a perfectoid affinoid field (K, K^+) for which K is algebraically closed.

Definition 3.3.8. A morphism $Y \rightarrow X$ of perfectoid spaces is *quasi-pro-étale* if for every strictly totally disconnected space T and every morphism $T \rightarrow X$, the space $Y \times_X T$ is affinoid perfectoid and $Y \times_X T \rightarrow T$ is pro-étale.

Theorem 3.3.9. [28, 9.2] *A morphism of affinoid perfectoid spaces $Y \rightarrow X$ is quasi-pro-étale if and only if there exists a pro-étale cover $\{X_i \rightarrow X\}$ by affinoids X_i such that the morphisms $Y \times_X X_i \rightarrow X_i$ are affinoid pro-étale.*

3.4 Diamonds

Definition 3.4.1. A pro-étale sheaf D on Perf is called a *diamond* if there exists a pro-étale equivalence relation $R \rightrightarrows X$ of perfectoid spaces such that

$$D \simeq \mathrm{coeq}(R \rightrightarrows X)$$

in the category of pro-étale sheaves. If this is the case, then the natural morphism $X \rightarrow D$ is quasi-pro-étale.

For a diamond $D \cong \mathrm{coeq}(R \rightrightarrows X)$, we define the underlying topological space of D to be $|D| = \mathrm{coeq}(|R| \rightrightarrows |X|)$ in the category of topological spaces. This is independent of choice of presentation.

Theorem 3.4.2. [24, 11.5] *A pro-étale sheaf D is a diamond if and only if there exists a quasi-pro-étale surjection $X \rightarrow D$ from a perfectoid space X .*

Theorem 3.4.3. [24, 11.7] *If $E \rightarrow D$ is a quasi-pro-étale map of pro-étale sheaves and D is a diamond, then E is a diamond.*

Theorem 3.4.4. [24, 11.9] *If D is a diamond, then D is a v -sheaf.*

Definition 3.4.5. Let X be an adic space over $\mathrm{Spa}(\mathbb{Z}_p)$. We define a presheaf on Perf

$$X^\diamond(Y) = \left\{ (f, \iota) \left| \begin{array}{l} f : Y^\sharp \rightarrow X \text{ a morphism of adic spaces and} \\ \iota : Y \rightarrow (Y^\sharp)^\flat \text{ an isomorphism} \end{array} \right. \right\} / \cong.$$

If (A, A^+) is an affinoid Huber pair over $(\mathbb{Z}_p, \mathbb{Z}_p)$ (resp. A is a Huber \mathbb{Z}_p -algebra), we write

$$\mathrm{Spd}(A, A^+) = \mathrm{Spa}(A, A^+)^\diamond$$

$$\mathrm{Spd} A = \mathrm{Spa}(A)^\diamond.$$

Theorem 3.4.6. [24, 15.1, 15.6] *For X an adic space over $\mathrm{Spa}(\mathbb{Z}_p)$, the presheaf X^\diamond is a v -sheaf. If X is an analytic adic space over $\mathrm{Spa}(\mathbb{Z}_p)$, then X^\diamond is a diamond.*

CHAPTER 4

FARGUES-FONTAINE CURVES

4.1 Period Rings

In this section, let $(R, |\cdot|)$ be a characteristic p , perfect Banach ring with a power-multiplicative norm. We define $v : R \rightarrow (-\infty, \infty]$ by $v(x) = -\log_p |x|$.

Notation 4.1.1. Let us define order-reversing inverse functions $r : [0, 1] \rightarrow [0, \infty]$ and $\rho : [0, \infty] \rightarrow [0, 1]$ by the formula

$$r(\rho) = -\log_p \rho \quad \text{and} \quad \rho(r) = p^{-r}.$$

In the following, we will use these to pass between different ways of indexing concepts. Many objects are indexed either by $r, 1/r$ or ρ . We will try to consistently write things so that r corresponds to ρ under the above isomorphisms. This will mean that sometimes, what other people write as r , we will write as $1/r$.

Definition 4.1.2. Let $x = \sum_{n=0}^{\infty} [x_n]p^n \in W(R)$. For $\rho \in (0, 1]$ and $r \in [0, \infty)$, we define functions $|\cdot|_{\rho} : W(R) \rightarrow [0, \infty]$ and $v_r : W(R) \rightarrow [-\infty, \infty]$ by

$$|x|_{\rho} = \sup_{n \geq 0} |x_n| \rho^n \quad \text{and} \quad v_r(x) = \inf_{n \geq 0} (v(x_n) + nr).$$

These are defined so that $v_r(x) = -\log_p |x|_{\rho(r)}$ and $|x|_{\rho} = p^{-v_r(x)}$. We say x *attains its ρ -norm* if the supremum in the definition of $|x|_{\rho}$ is a maximum, and we say that it *attains its r -valuation* if the infimum in the definition of $v_r(x)$ is a minimum.

Lemma 4.1.3. [10][22] *We have*

1. $|x + y|_{\rho} \leq \max(|x|_{\rho}, |y|_{\rho})$ and $v_r(x + y) \geq \max(v_r(x), v_r(y))$.

2. $|xy|_\rho \leq |x|_\rho |y|_\rho$ and $v_r(xy) \geq v_r(x) + v_r(y)$.

3. If x attains its ρ -norm (resp. attains its r -valuation), then $|x^n|_\rho = |x|_\rho^n$ (resp. $v_r(x^n) = nv_r(x)$).

4. If $|\cdot| : R \rightarrow [0, \infty)$ is multiplicative, and x, y attain their ρ -norms (resp. attain their r -valuations), then $|xy|_\rho = |x|_\rho |y|_\rho$ (resp. $v_r(xy) = v_r(x) + v_r(y)$).

Definition 4.1.4. For $\rho \in (0, 1]$ and $r \in [0, \infty)$, we define rings of bounded Witt vectors

$$W_\rho^b(R) = \{x \in W(R) : |x|_\rho < \infty\}$$

$$W^{b,1/r}(R) = \{x \in W(R) : v_r(x) > -\infty\}$$

so that $W^{b,1/r}(R) = W_{\rho(r)}^b(R)$ and $W_\rho^b(R) = W^{b,1/r(\rho)}(R)$.

Theorem 4.1.5. For $\rho \in (0, 1]$, $W_\rho^b(R)$ is a complete Banach ring with respect to the power-multiplicative norm $|\cdot|_\rho$. If the norm $|\cdot| : R \rightarrow [0, \infty)$ were multiplicative, then $|\cdot|_\rho$ is also multiplicative.

Definition 4.1.6. For $\rho \in (0, 1)$ and $r \in (0, \infty)$, we define

$$\tilde{\mathcal{R}}_\rho^{\text{int}}(R) = \left\{ \sum_{n \geq 0} [x_n] p^n \in W(R) \mid \lim_{n \rightarrow \infty} |x_n|_\rho^n = 0 \right\}$$

$$\tilde{\mathcal{R}}^{\text{int},1/r}(R) = \left\{ \sum_{n \geq 0} [x_n] p^n \in W(R) \mid \lim_{n \rightarrow \infty} (v(x_n) + nr) = \infty \right\}$$

so that $\tilde{\mathcal{R}}^{\text{int},1/r}(R) = \tilde{\mathcal{R}}_{\rho(r)}^{\text{int}}(R)$ and $\tilde{\mathcal{R}}_\rho^{\text{int}}(R) = \tilde{\mathcal{R}}^{\text{int},1/r(\rho)}(R)$. These rings are called *integral Robba rings*.

It is clear that for $0 < \rho < \rho' \leq 1$, we have $W_{\rho'}^b(R) \subset \tilde{\mathcal{R}}_\rho^{\text{int}}(R) \subset W_\rho^b(R)$.

Theorem 4.1.7. For $\rho \in (0, 1)$, the subset $\tilde{\mathcal{R}}_\rho^{\text{int}}(R) \subset W_\rho^b(R)$ is the closure of the subring $W_1^b(R)$. In particular, $\tilde{\mathcal{R}}_\rho^{\text{int}}(R)$ is a Banach ring with power-multiplicative norm $|\cdot|_\rho$.

Definition 4.1.8. We define the *integral Robba ring* to be

$$\begin{aligned}\tilde{\mathcal{R}}^{\text{int}}(R) &= \bigcup_{1 \geq \rho > 0} \tilde{\mathcal{R}}_{\rho}^{\text{int}}(R) = \bigcup_{1 \geq \rho > 0} W_{\rho}^b(R) \\ &= \{x \in W(R) : |x|_{\rho} < \infty \text{ for some } \rho \in (0, 1]\}.\end{aligned}$$

We give it the colimit topology.

Proposition 4.1.9. *The p -adic completion of each of the topological rings $W_{\rho}^b(R)$, $\tilde{\mathcal{R}}_{\rho}^{\text{int}}(R)$ and $\tilde{\mathcal{R}}^{\text{int}}(R)$ is the topological ring $W(R)$.*

Definition 4.1.10. We define the following rings gotten by inverting p :

$$\begin{aligned}\tilde{\mathcal{E}}(R) &= W(R)[\frac{1}{p}] = \left\{ \sum_{n \gg -\infty} [x_n]p^n \mid x_n \in R \right\} \\ \tilde{\mathcal{E}}_{\rho}^b(R) &= W_{\rho}^b(R)[\frac{1}{p}] = \left\{ \sum_{n \gg -\infty} [x_n]p^n \in \tilde{\mathcal{E}}(R) \mid \sup_{n \in \mathbb{Z}} |x_n|_{\rho^n} < \infty \right\} \\ \tilde{\mathcal{R}}_{\rho}^{\text{bd}}(R) &= \tilde{\mathcal{R}}_{\rho}^{\text{int}}(R)[\frac{1}{p}] = \left\{ \sum_{n \gg -\infty} [x_n]p^n \in \tilde{\mathcal{E}}(R) \mid \lim_{n \rightarrow \infty} |x_n|_{\rho^n} = 0 \right\} \\ \tilde{\mathcal{R}}^{\text{bd}}(R) &= \tilde{\mathcal{R}}^{\text{int}}(R)[\frac{1}{p}] = \bigcup_{\rho \in (0, 1]} \tilde{\mathcal{R}}_{\rho}^{\text{bd}}(R) = \bigcup_{\rho \in (0, 1]} \tilde{\mathcal{E}}_{\rho}^b(R).\end{aligned}$$

We also define $\tilde{\mathcal{E}}^{b, 1/r}(R) = \tilde{\mathcal{E}}_{\rho(r)}^b(R)$ and $\tilde{\mathcal{R}}^{\text{bd}, 1/r}(R) = \tilde{\mathcal{R}}_{\rho(r)}^{\text{bd}}(R)$. The ring $\tilde{\mathcal{E}}_1^b(R) = \tilde{\mathcal{E}}^{b, \infty}(R)$ is also denoted by $B^b(R)$.

Definition 4.1.11. Let $x = \sum_{n \gg -\infty} [x_n]p^n \in \tilde{\mathcal{E}}(R)$. We for $\rho \in (0, 1]$ and $r \in [0, \infty)$, we can extend the definitions of $|\cdot|_{\rho}$ and v_r by setting

$$|x|_{\rho} = \sup_{n \in \mathbb{Z}} |x_n|_{\rho^n} \in [0, \infty] \quad \text{and} \quad v_r(x) = \inf_{n \in \mathbb{Z}} (v(x_n) + nr) \in [-\infty, \infty]$$

so that again we have $v_r(x) = -\log_p |x|_{\rho(r)}$ and $|x|_{\rho} = p^{-v_r(x)}$.

Proposition 4.1.12. [22][10] For each $\rho \in (0, 1]$, the function $|\cdot|_\rho$ is a power-multiplicative norm on $\tilde{\mathcal{E}}_\rho^b(R)$, (and thus, all rings contained in this ring), which is multiplicative if $|\cdot| : R \rightarrow [0, \infty)$ were multiplicative.

Remark 4.1.13. In contrast with the situation where we have not inverted p , these rings are not complete with respect to these norms.

Definition 4.1.14. For $x = \sum_{n \gg 0} [x_n]p^n \in \tilde{\mathcal{R}}^{\text{bd}}(R)$, we define the *Newton polygon* of x , denoted $\text{Newt}(x)$, to be the decreasing convex hull of the set

$$\{(n, v(x_n)) \mid n \in \mathbb{Z}\}.$$

Definition 4.1.15. We define a Legendre transform

$$\mathcal{L} : \{\text{convex functions } f : \mathbb{R} \rightarrow (-\infty, \infty)\} \leftrightarrow \{\text{concave functions } g : \mathbb{R} \rightarrow [-\infty, \infty)\} : \mathcal{L}^{-1}$$

so that if f and g correspond, then for all (x, α) lying above the graph of f and all (y, β) lying below the graph of g , we have

$$-\alpha + \beta - xy \leq 0.$$

Explicitly, this means that

$$\begin{aligned} \mathcal{L}f(y) &= \inf_{x \in \mathbb{R}} (f(x) + xy) \\ \mathcal{L}^{-1}g(x) &= \sup_{y \in \mathbb{R}} (g(y) - xy). \end{aligned}$$

Proposition 4.1.16. [10, Definition 1.5.2] For $x \neq 0$ in $\tilde{\mathcal{R}}^{\text{bd}}(R)$, the Legendre transform

of the Newton polygon of x is given by

$$\mathcal{L} \text{Newt}(x)(r) = \begin{cases} -\infty & \text{if } r < 0 \\ v_r(x) & \text{if } r \geq 0. \end{cases}$$

Corollary 4.1.17. *Let $x \in \tilde{\mathcal{R}}^{\text{bd}}(R)$. Then for any $\rho \in [\rho_1, \rho_2] \subset (0, 1]$, and any $r \in [r_1, r_2] \subset [0, \infty)$, we have*

$$|x|_\rho \leq \max(|x|_{\rho_1}, |x|_{\rho_2}) \quad \text{and} \quad v_r(x) \geq \min(v_{r_1}(x), v_{r_2}(x)).$$

Corollary 4.1.18. *The function $r \mapsto v_r(x)$ is continuous on the set $\{r : v_r(x) > -\infty\}$.*

Definition 4.1.19. For $I \subset (0, 1]$ a nonempty interval, we define $B_I(R)$ to be the Frechet completion of the ring $B^b(R)$ with respect to the norms $|\cdot|_\rho$ for $\rho \in I$. Then,

$$B_I(R) = \varprojlim_{[\rho_1, \rho_2] \subset I} B_{[\rho_1, \rho_2]}(R)$$

is a limit of Banach algebras.

We define

$$B(R) = B_{(0,1)}(R)$$

$$B_\rho(R) = B_{[\rho, \rho]}(R).$$

Definition 4.1.20. For $\rho \in (0, 1)$, we define

$$\tilde{\mathcal{R}}_\rho(R) = B_{(0, \rho]}(R) = \left\{ \text{Frechet completion of } \tilde{\mathcal{R}}_\rho^{\text{bd}}(R) \text{ with respect } |\cdot|_{\rho'} \text{ for } \rho' \in (0, \rho] \right\}.$$

For $r \in (0, \infty)$, we define $\tilde{\mathcal{R}}^{1/r}(R) = \tilde{\mathcal{R}}_{\rho(r)}(R)$. We define the *Robba ring*

$$\tilde{\mathcal{R}}(R) = \bigcup_{r \in (0, \infty)} \tilde{\mathcal{R}}^r(R).$$

Remark 4.1.21. The “projection maps” $\pi_\ell : \sum_{n \gg 0} [x_n]p^n \mapsto x_\ell$ are often not uniformly continuous in the $|\cdot|_\rho$ -topologies, so we cannot extend these maps from $B^b(R)$ to $B(R)$. (Note that since these maps do not preserve addition, continuity and uniform continuity are not equivalent). This means we do not necessarily have a “power series expansions” for every of $B(R)$, and that even when we do, the expansions may not be unique.

Definition 4.1.22. Let $x \in B^b(R)$. Let $\partial^- v_r(x)$ and $\partial^+ v_r(x)$ denote the left and right derivative of the concave function $r \mapsto v_r(x)$, defined for $r \in [0, \infty)$.

Proposition 4.1.23. [10, Lemma 1.6.10] Let $r \in [0, \infty)$ and suppose x_1, x_2, \dots is a Cauchy sequence in $B^b(R)$ for the $|\cdot|_{\rho(r)}$ -topology which does not converge to 0. Then the sequences $v_r(x_n), \partial^- v_r(x_n)$ and $\partial^+ v_r(x_n)$ are eventually constant.

Proposition 4.1.24. [10, Proposition 1.6.15] For $\emptyset \neq I \subset J$, the map $B_J(R) \rightarrow B_I(R)$ is injective. In particular, for $x \in B_I(R)$, there exists a largest interval $J \subset (0, 1]$ such that $x \in B_J(R)$.

Theorem 4.1.25. [10, Proposition 1.6.16] Suppose $I \subset (0, 1]$ is an interval. Let x be a nonzero element of $B_I(R)$ and $x_1, x_2, \dots \in B^b(R)$ be a Cauchy sequence converging to x . Then for every compact subinterval $K \subset I$ there exists a large enough N such that $v_{r(\rho)}(x_n) = v_{r(\rho)}(x_N)$ for all $\rho \in K$ and $n \geq N$.

Definition 4.1.26. [10, Definition 1.6.21, Example 1.6.22] For $r \in (0, \infty)$ and $\rho \in (0, 1)$, the valuation v_r and norm $|\cdot|_\rho$ extends by continuity from $B^b(R)$ to the completion $B(R)$. For $x \neq 0$ in $B(R)$, the theorem implies that $r \mapsto v_r(x)$ is concave down and finite on

$(0, \infty)$. We can extend this function by right continuity to define $v_0(x) \in [-\infty, \infty)$, and we let $|x|_1 = p^{-v_0(x)}$.

We define the *Newton polygon* $\text{Newt}(x)$ of x so that its Legendre transform is given by

$$\mathcal{L} \text{Newt}(x)(r) = \begin{cases} -\infty & \text{if } r < 0 \\ v_r(x) & \text{if } r \geq 0. \end{cases}$$

In the following, we let $R^\circ \subset R$ denote the closed unit ball. When R is analytic (and using the fact that $|\cdot|$ is power-multiplicative), one can show that the closed unit ball R° is precisely the ring of powerbounded elements.

Definition 4.1.27. We define $B^{b,+}(R)$, $B_I^+(R)$, $B^+(R)$ and $B_\rho^+(R)$ to be $B^b(R^\circ)$, $B_I(R^\circ)$, $B(R^\circ)$ and $B_\rho(R^\circ)$, respectively.

Definition 4.1.28. [10, Section 1.10.3] For $\rho \in (0, 1)$, let $\mathfrak{a}_\rho \subset R^\circ$ denote the closed ball of radius ρ , which is an ideal in R° . Let \mathfrak{b}_ρ denote the kernel of the composite

$$W(R^+) \rightarrow R^+ \rightarrow R^+/\mathfrak{a}_\rho,$$

i.e. $x = \sum_{n \geq 0} [x_n]p^n \in \mathfrak{b}_\rho$ if and only if $|x_0| \leq \rho$. We define $A_{\text{crys},\rho}^0(R)$ to be the PD thickening of $W(R^\circ)$ with respect to the ideal \mathfrak{b}_ρ , and $A_{\text{crys},\rho}(R)$ to be the p -adic completion of $A_{\text{crys},\rho}^0(R)$. We define $B_{\text{crys},\rho}^+(R)$ to be $A_{\text{crys},\rho}(R)[\frac{1}{p}]$.

Example 4.1.29. [10, Example 1.10.3] Suppose $\varpi \in R^\circ$ is an element such that $\mathfrak{a}_\rho = \varpi R^\circ$. Then the closed unit $|\cdot|_\rho$ -ball in $B^{b,+}(R)$ is equal to $W(R^\circ) \left[\frac{[\varpi]}{p} \right] \subset W(R)$, and the restriction of the $|\cdot|_\rho$ -topology to this ball is just the p -adic topology. We can then show that

$$B_\rho^+(R) = W(R^\circ) \left[\frac{[\varpi]}{p} \right]^\wedge \left[\frac{1}{p} \right]$$

where the $^\wedge$ denotes the p -adic completion.

Example 4.1.30. [10, Section 1.10.3] Suppose $\varpi \in R^\circ$ is an element such that $\mathfrak{a}_\rho = \varpi R^\circ$. Let $\xi = \sum_{n \geq 0} [\xi_n] p^n \in W(R^\circ)$ be any element with $[\xi_0] = [\varpi]$. Then $\mathfrak{b}_\rho = (\xi, p)$, so we see that $A_{\text{crys}, \rho}^0(R) = W(R^\circ) \left[\frac{\xi^n}{n!} \right]_{n \geq 1}$, and that

$$B_{\text{crys}, \rho}^+(R) = W(R^\circ) \left[\frac{\xi^n}{n!} \right]_{n \geq 1}^\wedge \left[\frac{1}{p} \right]$$

with $^\wedge$ denoting the p -adic completion. In particular, by taking $\xi = [\varpi]$, we see that

$$B_{\rho^p}^+(R) \subset B_{\text{crys}, \rho}^+(R) \subset B_\rho^+(R).$$

Proposition 4.1.31. [10, Lemma 1.10.4, Proposition 1.10.5, Proposition 1.10.7] *We have:*

1. Let $x \in B^+(R)$. For $0 < \rho \leq \rho' < 1$ and $0 < r' \leq r < \infty$, we have

$$|x|_{\rho'} \leq |x|_\rho^{\log_\rho \rho'} \quad \text{and} \quad v_{r'}(x) \geq \frac{r'}{r} v_r(x).$$

2. For $0 < \rho < 1$, we have $B_\rho^+(R) = B_{[\rho, 1]}^+(R)$.

3. If $0 < \rho \leq \rho' < 1$, then $B_\rho^+(R) \subset B_{\rho'}^+(R)$.

4. $B^+(R) = \bigcap_{0 < \rho < 1} B_\rho^+(R)$.

5. An element x of $B(R)$ is in $B^+(R)$ if and only if $|x|_1 \leq 1$ if and only if $\text{Newt}(x) \geq 0$.

Definition 4.1.32. The Frobenius $\varphi : R \rightarrow R$ is a homeomorphism and sends the norm $|\cdot|$ to the norm $|\cdot|^p$. Let $\tilde{\varphi} : \tilde{\mathcal{E}}(R) \rightarrow \tilde{\mathcal{E}}(R)$ be the induced morphism. We then see that $|\varphi(x)|_{\rho^p} = |x|_\rho^p$, so φ induces homeomorphisms $B(R) \rightarrow B(R)$, $B^+(R) \rightarrow B^+(R)$, $B_{[\rho_1, \rho_2]}^+(R) \rightarrow B_{[\rho_1^p, \rho_2^p]}^+(R)$, etc.

Proposition 4.1.33. [10] Let $\rho \in (0, 1)$ be such that there exists ϖ with $\mathfrak{a}_\rho = \varpi R^\circ$. Then

1. We have

$$B^+(R) = \bigcap_{n \geq 0} \varphi^n(B_\rho^+(R)) = \bigcap_{n \geq 0} \varphi^n(B_{\text{crys}, \rho}^+(R)).$$

2. For all $x \in B(R)$, we have

$$\text{Newt}(\varphi^n(x))(y) = p^n \text{Newt}(x)(y) \quad \text{and} \quad \text{Newt}(p^m x)(y) = \text{Newt}(x)(y - m).$$

3. For all integers $d \geq 0$ and $h > 0$, we see that

$$B^{\varphi^h = p^d}(R) = (B^+)^{\varphi^h = p^d}(R) = (B_\rho^+)^{\varphi^h = p^d}(R) = (B_{\text{crys}, \rho}^+)^{\varphi^h = p^d}(R).$$

4.2 The Analytic and Schematic Fargues-Fontaine Curves

Definition 4.2.1. Let S be a perfectoid space of characteristic p . We be considering the following presheaves on Perf :

$$\mathcal{Y}_{[0, \infty)}(S)^\diamond = S \times \text{Spd } \mathbb{Z}_p$$

$$\mathcal{Y}_{(0, \infty)}(S)^\diamond = S \times \text{Spd } \mathbb{Q}_p$$

$$\mathcal{X}_{\text{FF}, S}^\diamond = (S/\varphi^{\mathbb{Z}}) \times \text{Spd } \mathbb{Q}_p.$$

For now, the left sides are symbols, but we shall soon see that they are the diamonds of analytic adic spaces over \mathbb{Q}_p .

Definition 4.2.2. Let $S = \text{Spa}(R, R^+)$ be affinoid perfectoid of characteristic p . Consider

$$\mathcal{Y}_{[0, \infty]}(S) := \text{Spa}(W(R^+))^{\text{an}}$$

If $\varpi \in R^+$ be a pseudouniformizer, then $W(R^+)$ has the $(p, [\varpi])$ -adic topology, so the

analytic subspace is the subspace of valuations x such that at least one of $|x(p)|$ or $|x([\varpi])|$ is nonzero. Thus the analytic subspace is covered by $U(\frac{[\varpi]}{p})$ and $U(\frac{p}{[\varpi]})$. It can be shown that $W(R^+)(\frac{[\varpi]}{p})$ and $W(R^+)(\frac{p}{[\varpi]})$ are sheafy, so we see that $\mathcal{Y}_{[0,\infty]}$ is an analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p)$.

Theorem 4.2.3. [28, 11.2.1, Discussion after 15.2.5] *Let $S = \mathrm{Spa}(R, R^+)$ be affinoid perfectoid of characteristic p .*

1. *Let $S \dot{\times} \mathrm{Spa} \mathbb{Z}_p$ denote the open subspace $\mathcal{Y}_{[0,\infty]}(S) \setminus \{[\varpi] = 0\} \subset \mathcal{Y}_{[0,\infty]}(S)$. Then*

$$(S \dot{\times} \mathrm{Spa} \mathbb{Z}_p)^\diamond = \mathcal{Y}_{[0,\infty]}(S)^\diamond = S \times \mathrm{Spd} \mathbb{Z}_p.$$

We also use the symbol $\mathcal{Y}_{(0,\infty)}(S) \subset \mathcal{Y}_{[0,\infty]}(S)$ to denote this analytic adic space over $\mathrm{Spa}(\mathbb{Q}_p)$.

2. *Let $S \dot{\times} \mathrm{Spa} \mathbb{Q}_p$ denote the open subspace $\mathcal{Y}_{[0,\infty]}(S) \setminus \{[\varpi]p = 0\} \subset \mathcal{Y}_{[0,\infty]}(S)$. Then*

$$(S \dot{\times} \mathrm{Spa} \mathbb{Q}_p)^\diamond = \mathcal{Y}_{(0,\infty)}(S)^\diamond = S \times \mathrm{Spd} \mathbb{Q}_p.$$

We shall also use the symbol $\mathcal{Y}_{(0,\infty)}(S) \subset \mathcal{Y}_{[0,\infty]}(S)$ to denote this analytic subspace.

Moreover, these constructions behave well under localization, so they can be glued together to give corresponding analytic spaces for an arbitrary perfectoid space S of characteristic p .

Definition 4.2.4. Let $S = \mathrm{Spa}(R, R^+)$ be affinoid perfectoid of characteristic p and $\varpi \in R$ be a pseudouniformizer. Let $(\varpi) \subset R^\circ$ be the ideal generated by ϖ in the ring of power-bounded elements. We define a function $\kappa_{(\varpi)} : |\mathcal{Y}_{[0,\infty]}(S)| \rightarrow [0, \infty]$ by sending a point $x \in |\mathcal{Y}_{[0,\infty]}(S)|$ to

$$\kappa_{(\varpi)}(x) = \frac{\log |[\varpi](\tilde{x})|}{\log |p(\tilde{x})|} \in [0, \infty]$$

where \tilde{x} is the maximal rank one generalization of the analytic point x . This map is then

continuous. Moreover, it only depends on the ideal $(\varpi) \subset R^\circ$ as $u \in (R^\circ)^\times$ have norm 1 at all rank 1 points.

For $I \subset [0, \infty]$ a subinterval, we define $\mathcal{Y}_{I,(\varpi)}(S)$ to be adic space whose underlying space is the inverse image of I under $\kappa_{(\varpi)}$. We remark that this indexing corresponds to $1/r$ in the $r, 1/r, \rho$ indexing from the previous section.

We see that $\mathcal{Y}_{[0,\infty)}(S) = \mathcal{Y}_{[0,\infty),(\varpi)}(S)$ and $\mathcal{Y}_{(0,\infty)}(S) = \mathcal{Y}_{(0,\infty),(\varpi)}(S)$ for any choice of pseudouniformizer ϖ .

Theorem 4.2.5. *[28, 15.2.6, 15.2.7] We have*

1. *For $S = \text{Spa}(R, R^+)$ affinoid perfectoid of characteristic p , $\varphi : S \rightarrow S$ induces an automorphism $\varphi : \mathcal{Y}_{[0,\infty]}(S) \rightarrow \mathcal{Y}_{[0,\infty]}(S)$. Then $\kappa_{(\varpi)}(\varphi(x)) = p\kappa_{(\varpi)}(x)$. In particular, the action of φ is properly discontinuous on $\mathcal{Y}_{(0,\infty)}(S)$.*
2. *Let $\mathcal{X}_{\text{FF},S}$ denote the quotient $\mathcal{Y}_{(0,\infty)}(S)/\varphi^{\mathbb{Z}}$. Then the diamond of $\mathcal{X}_{\text{FF},S}$ is $\mathcal{X}_{\text{FF},S}^\diamond$.*

This $\mathcal{X}_{\text{FF},S}$ is called the Fargues-Fontaine curve over S .

Theorem 4.2.6. *[10][22] Let $S = \text{Spa}(R, R^+)$ be affinoid perfectoid of characteristic p and $\varpi \in S$ be a pseudouniformizer. Promote R to a Banach ring using the ideal $(\varpi) \subset R^+$ as in Construction 2.2.10, except use p instead of e in the construction. Then replace the resulting norm by the power-multiplicative spectral norm.*

1. *Let $I \subset (0, \infty)$ be an interval all of whose closed endpoints (if they exist) are in $\mathbb{Z}[\frac{1}{p}]$. Let $I^{-1} \subset (0, \infty)$ denote the interval of multiplicative inverses of elements of I . Let $J = \rho(I^{-1}) \subset (0, 1)$. Then*

$$H^0(\mathcal{Y}_{I,(\varpi)}(S), \mathcal{O}_{\mathcal{Y}_{I,(\varpi)}(S)}) = B_J(R)$$

In particular, we see that $H^0(\mathcal{Y}_{(0,\infty)}(S), \mathcal{O}_{\mathcal{Y}_{(0,\infty)}(S)}) = B(R)$.

2. Let $1/r \in \mathbb{Z}[\frac{1}{p}]$. Then

$$H^0(\mathcal{Y}_{[0,1/r],(\varpi)}(S), \mathcal{O}_{\mathcal{Y}_{[0,1/r],(\varpi)}(S)}) = \tilde{\mathcal{R}}^{\text{int},1/r}(R).$$

3. Let $1/r \in \mathbb{Z}[\frac{1}{p}]$. Then

$$H^0(\mathcal{Y}_{[1/r,\infty],(\varpi)}(S), \mathcal{O}_{\mathcal{Y}_{[1/r,\infty],(\varpi)}(S)}) = B_{[\rho(r),1]}^+(R).$$

4. Let $1/r \in [0, \infty)$. Then

$$H^0(\mathcal{Y}_{(1/r,\infty],(\varpi)}(S), \mathcal{O}_{\mathcal{Y}_{(1/r,\infty],(\varpi)}(S)}) = B_{(\rho(r),1]}^+(R).$$

In particular, we see that $H^0(\mathcal{Y}_{(0,\infty]}(S), \mathcal{O}_{\mathcal{Y}_{(0,\infty]}(S)}) = B^+(R)$.

Definition 4.2.7. For $\psi : B \rightarrow B$ an endomorphism of a ring (resp. $\psi : X \rightarrow X$ and endomorphism of an adic space), a ψ -module over B (resp. a ψ -module on X) is a finite projective R -module M (resp. vector bundle \mathcal{E} on X) along with an isomorphism $\psi^*M \xrightarrow{\cong} M$ (resp. $\psi^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$).

Theorem 4.2.8. [22] Let $S = \text{Spa}(R, R^+)$, and promote S into a Banach ring as in Theorem 4.2.6. The following tensor categories are equivalent:

- | | |
|--|--|
| 1. vector bundles on $\mathcal{X}_{\text{FF},S}$, | $(\rho, 1)$ or $[\rho, 1)$ with $\rho \in (0, 1)$, |
| 2. φ -modules on $\mathcal{Y}_{(0,\infty)}(S)$, | 6. φ^{-1} -modules on $\mathcal{Y}_{I,(\varpi)}(S)$ for I of the |
| 3. φ -modules over $B(R)$, | form $(0, \frac{1}{r})$ or $(0, \frac{1}{r}]$ with $r \in (0, \infty)$, |
| 4. φ -modules on $\mathcal{Y}_{I,(\varpi)}(S)$ for I of the | 7. φ^{-1} -modules over $B_J(R)$ for J of the |
| form $(\frac{1}{r}, \infty), [\frac{1}{r}, \infty)$ with $r \in (0, \infty)$ | form $(0, \rho)$ or $(0, \rho]$ with $\rho \in (0, 1)$, |
| 5. φ -modules over $B_J(R)$ for J of the form | 8. φ -modules over $\tilde{\mathcal{R}}(R)$. |

Here, the endomorphism $\varphi : B_{(\rho,\infty)}(R) \rightarrow B_{(\rho,\infty)}(R)$ is defined to be the composite

$$B_{(\rho,\infty)}(R) \xrightarrow[\varphi]{\cong} B_{(\rho^p,\infty)}(R) \subset B_{(\rho,\infty)}(R).$$

We can similarly define the other endomorphisms.

Example 4.2.9. Let $S = \text{Spa}(R, R^+)$ with R as in Theorem 4.2.6. For integers $d \geq 0$, we can define a rank one φ -module M_d over $B(R)$ to be the rank one module basis v with $\varphi_{M_d}(xv) = p^{-d}\varphi(x)v$. Let $\mathcal{O}(d)$ denote the corresponding line bundle on $\mathcal{X}_{\text{FF},S}$.

Proposition 4.2.10. [10][22] Let $S = \text{Spa}(R, R^+)$ with R as in Theorem 4.2.6. For integers $d \geq 0$, we have

$$H^0(\mathcal{X}_{\text{FF},S}, \mathcal{O}(d)) = B^{\varphi=p^d}(R).$$

Definition 4.2.11. Let $S = \text{Spa}(R, R^+)$ with R as in Theorem 4.2.6. We define the *schematic Fargues-Fontaine curve* to be

$$X_{\text{FF},S} = \text{Proj} \left(\bigoplus_{d \geq 0} B^{\varphi=p^d}(R) \right).$$

Note that the schematic Fargues-Fontaine curve only depends on R .

Theorem 4.2.12. [22, 8.7.7] Let $S = \text{Spa}(R, R^+)$ as before. There exists a map of locally ringed spaces

$$\mathcal{X}_{\text{FF},S} \rightarrow X_{\text{FF},S}.$$

Pullback along this map identifies the categories of vector bundles and preserves their cohomology.

Definition 4.2.13. Let $S^\sharp = \text{Spa}(A, A^+)$ be an affinoid perfectoid of characteristic 0, and let $S = \text{Spa}(R, R^+)$ be its tilt in characteristic p . By the algebraic perfectoid equivalence, S^\sharp gives us a primitive ideal $(\xi) \subset W(R^+)$ generated by a primitive element $\xi = [\xi_0] + p[\xi_1] + \dots$

of degree 1. Then ξ_0 is a topologically nilpotent unit in A^+ and changing our choice of ξ only changes ξ_0 by a unit. Given the data of such an untilt S^\sharp of S , we shall promote R to a Banach ring as in Theorem 4.2.6 using the ideal (ξ_0) .

The map $(W(R^+), W(R^+)) \rightarrow (A, A^+)$ is adic, so it induces a map

$$S^\sharp \rightarrow \mathcal{Y}_{[0, \infty]}(S).$$

Moreover, $\kappa(S^\sharp) = 1$, so we also get maps $S^\sharp \rightarrow \mathcal{Y}_{I, (\xi_0)}(S)$ for all intervals I containing 1, and moreover, we get a map

$$S^\sharp \rightarrow \mathcal{X}_{\text{FF}, S}$$

by descending such an above map for a small enough interval I mapping $\mathcal{Y}_{I, (\xi_0)}(S)$ homeomorphically to an open subset of $\mathcal{X}_{\text{FF}, S}$. We shall define $\mathcal{Y}_I(S^\sharp)$, and $\mathcal{X}_{\text{FF}, S^\sharp}$ to mean the adic spaces $\mathcal{Y}_{I, (\xi_0)}(S)$ and $\mathcal{X}_{\text{FF}, S}$, respectively. These spaces come equipped with a morphism $S^\sharp \rightarrow \mathcal{Y}_I(S^\sharp)$ when $1 \in I$ and a morphism $S^\sharp \rightarrow \mathcal{X}_{\text{FF}, S^\sharp}$. Similarly, we can define the schematic Fargues-Fontaine curve X_{FF, S^\sharp} to be $X_{\text{FF}, S}$. These constructions globalize for arbitrary perfectoid spaces S^\sharp of characteristic 0.

Definition 4.2.14. Let $S^\sharp = \text{Spa}(A, A^+)$ be affinoid perfectoid of characteristic 0. Promote $R = A^b$ to a Banach ring as in the previous definition. For each of the period rings $\mathcal{PR} \in \{W_\rho^b, W^{b, 1/r}, \tilde{\mathcal{R}}_\rho^{\text{int}}, \tilde{\mathcal{R}}^{\text{int}, 1/r}, \tilde{\mathcal{R}}^{\text{int}}, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}^b, \mathcal{E}^{b, 1/r}, \tilde{\mathcal{R}}_\rho^{\text{bd}}, \tilde{\mathcal{R}}^{\text{bd}, 1/r}, \tilde{\mathcal{R}}^{\text{bd}}, \tilde{\mathcal{R}}_\rho, \tilde{\mathcal{R}}^{1/r}, \tilde{\mathcal{R}}, B^b, B^{b, +}, B_I, B_I^+, B, B^+, B_\rho, B_\rho^+, B_{\text{crys}, \rho}^+\}$ of the previous section, we define $\mathcal{PR}(A, A^+)$ to be $\mathcal{PR}(R)$. Moreover, we define $B_{\text{crys}}^+(A, A^+)$ to mean $B_{\text{crys}, p^{-1}}^+(R)$.

Let $I \subset W(R^+)$ be the primitive ideal associated to (A, A^+) . We define

$$\begin{aligned} B_{\text{dR}}^+(A, A^+) &= \varprojlim W^b(R)/I^n W^b(R) \\ &= \varprojlim W(R^+)[\frac{1}{p}]/(\xi^n) \end{aligned}$$

where ξ is a primitive element of degree 1 generating I . We define

$$B_{\mathrm{dR}}(A, A^+) = B_{\mathrm{dR}}^+(A, A^+)[\frac{1}{\xi}].$$

Theorem 4.2.15. [22, 8.8.18, 8.8.19] *Let S^\sharp be a perfectoid space of characteristic 0. There exists a global section*

$$t_{S^\sharp} \in H^0(\mathcal{X}_{\mathrm{FF}, S^\sharp}, \mathcal{O}(1)) = B^{\varphi=p}(R, R^+)$$

which cuts out the closed embedding.

$$S^\sharp \rightarrow \mathcal{X}_{\mathrm{FF}, S^\sharp}.$$

Now we can also view t_{S^\sharp} as a global section of the corresponding line bundle $\mathcal{O}(1)$ on the schematic Fargues-Fontaine curve. It cuts out an effective Cartier divisor $D \subset X_{\mathrm{FF}, S^\sharp}$. If $S^\sharp = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid, then D is contained in an affine open, and the coordinate ring of the formal scheme corresponding to $D \subset X_{\mathrm{FF}, S^\sharp}$ is $B_{\mathrm{dR}}^+(A, A^+)$.

Definition 4.2.16. For $S^\sharp = \mathrm{Spa}(A, A^+)$ an affinoid perfectoid of characteristic 0, we define

$$B_{\mathrm{crys}}(A, A^+) = B_{\mathrm{crys}}^+(A, A^+)[\frac{1}{t_{S^\sharp}}].$$

We have natural inclusions $B_{\mathrm{crys}}^+(A, A^+) \subset B_{\mathrm{dR}}^+(A, A^+)$ and $B_{\mathrm{crys}}(A, A^+) \subset B_{\mathrm{dR}}(R, R^+)$.

Theorem 4.2.17. [10] *For $S^\sharp = \mathrm{Spa}(A, A^+)$ an affinoid perfectoid of characteristic 0, the open scheme $X_{\mathrm{FF}, S^\sharp} \setminus D$ is affine with coordinate ring*

$$B_e(A, A^+) := B_{\mathrm{crys}}^{\varphi=1}(A, A^+).$$

Proposition 4.2.18. *We have:*

1. The sequence

$$0 \rightarrow \underline{\mathbb{Q}_p}(A, A^+) \rightarrow B_e(A, A^+) \rightarrow (B_{\mathrm{dR}}/B_{\mathrm{dR}}^+)(A, A^+) \rightarrow 0$$

is an exact sequence of pro-étale sheaves (here the quotient on the right is the quotient in the category of pro-étale sheaves).

2. For $d \geq 0$ a nonnegative integer,

$$B_{\mathrm{crys}}^{+, \varphi=p^d}(A, A^+) = B_{\mathrm{crys}}^{\varphi=p^d}(A, A^+) \cap B_{\mathrm{dR}}^+(A, A^+) \subset B_{\mathrm{dR}}(A, A^+).$$

Theorem 4.2.19. [10][22, 8.9.6] Let $S^\sharp = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid of characteristic 0. Then the category of vector bundles on $X_{\mathrm{FF}, S^\sharp}$ is equivalent to the category of triples $(M_e, M_{\mathrm{dR}}^+, \iota)$ where M_e is a finite projective $B_e(A, A^+)$ -module, M_{dR}^+ is a finite projective $B_{\mathrm{dR}}^+(A, A^+)$ -module, and $\iota : M_e \otimes_{B_e(A, A^+)} B_{\mathrm{dR}}(A, A^+) \xrightarrow{\cong} M_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+(A, A^+)} B_{\mathrm{dR}}(A, A^+)$ is an isomorphism of $B_{\mathrm{dR}}(A, A^+)$ -modules.

4.3 Slope Theory

We will follow the formalism of Harder-Narasimhan filtrations in [7].

Definition 4.3.1. Let \mathcal{C} be a category with a 0 object, kernels and cokernels. Then

1. an *admissible mono* is a morphism $K \rightarrow M$ which is the kernel of some morphism $M \rightarrow N$.
2. an *admissible epimorphism* is a morphism $M \rightarrow C$ which is the cokernel of some morphism $L \rightarrow M$.

3. a *short exact sequence* is a sequence

$$0 \rightarrow K \xrightarrow{\alpha} M \xrightarrow{\beta} C \rightarrow 0$$

where β is the cokernel of α and α is the kernel of β .

4. For $M \in \text{Obj}(\mathcal{C})$, let $\text{Sub}(M)$ denote the skeleton of the category of admissible monos $K \rightarrow M$. We call the objects of $\text{Sub}(M)$ *subobjects* of M . Note that if $K \xrightarrow{\alpha} M$ and $K' \xrightarrow{\alpha'} M$ are admissible monos, there can be at most one morphism $f : K \rightarrow K'$ such that $\alpha = \alpha' \circ f$, and f must be an admissible mono if it exists. Thus, we see that $\text{Sub}(M)$ is a poset.

5. For $M \in \text{Obj}(\mathcal{C})$, we define the *length of M* , denoted $\text{length}(M)$ to be the largest ℓ such that there exists a chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M.$$

Here $M_i \subsetneq M_{i+1}$ means an admissible mono $M_i \rightarrow M_{i+1}$ which is not an isomorphism.

We say M has infinite length if there exist such chains of arbitrarily large length.

6. Let $M \in \text{Obj}(\mathcal{C})$ have finite length. A \mathbb{R} -*filtration* on M is an order reversing map from \mathbb{R} to $\text{Sub}(M)$ denoted $r \mapsto F^r(M)$ such that

- $F^r(M) = 0$ for $r \gg 0$ and $F^r(M) = M$ for $r \ll 0$.
- for all $r \in \mathbb{R}$, $F^r(M) = \lim_{s \rightarrow r^-} F^s(M)$.

Note that since M has finite length, $F^s(M)$ is constant for s on a small enough interval $(r - \varepsilon, r)$, so the second point makes sense.

A morphism $(M, F_M^\bullet) \rightarrow (N, F_N^\bullet)$ of \mathbb{R} -filtered objects is a morphism $M \rightarrow N$ such that for all $r \in \mathbb{R}$, the morphism $F_M^r M \rightarrow N$ factors through $F_N^r N \rightarrow N$.

7. Suppose M is of finite length and F^\bullet is an \mathbb{R} -filtration on M . For each r , we define

$$\mathrm{gr}_F^r M = F^r M / \lim_{s \rightarrow r^+} F^s M.$$

Note that since M has finite length, F^s is constant for s in some small interval $(r, r + \varepsilon)$.

Moreover, finiteness implies that $\mathrm{gr}_F^r M = 0$ for all but finitely many r . We define the

associated graded of (M, F^\bullet) to be $\bigoplus_{r \in \mathbb{R}} \mathrm{gr}_F^r M$.

Definition 4.3.2. Let \mathcal{C} be a category with a 0-object, kernels and cokernels, and suppose the composition of two admissible monos is an admissible mono. A *rank function* $\mathrm{rk} : \mathrm{Obj}(\mathcal{C}) \rightarrow \mathbb{Z}_{\geq 0}$ is a function satisfying

1. rk is additive on short exact sequences, i.e. if

$$0 \rightarrow K \rightarrow M \rightarrow C \rightarrow 0$$

is short exact, then $\mathrm{rk}(M) = \mathrm{rk}(K) + \mathrm{rk}(C)$.

2. If $f : M \rightarrow N$ has zero kernel and cokernel, then $\mathrm{rk}(M) = \mathrm{rk}(N)$.
3. $\mathrm{rk}(M) = 0$ if and only if M is a zero object.

Proposition 4.3.3. [7, Lemma 2.1,3.1.7] *If \mathcal{C} admits a rank function, then every object has finite length, and the length function is a rank function.*

Definition 4.3.4. Let \mathcal{C} be a category with 0-objects, kernels and cokernels, and suppose the composition of two admissible monos is an admissible mono. A *degree function* $\mathrm{deg} : \mathrm{Obj}(\mathcal{C}) \rightarrow \mathbb{R}$ is a function satisfying

1. deg is additive on short exact sequences.
2. If $f : M \rightarrow N$ has zero kernel and cokernel, then $\mathrm{deg}(M) \leq \mathrm{deg}(N)$. Moreover, in this situation, f is an isomorphism if and only if $\mathrm{deg}(M) = \mathrm{deg}(N)$.

Definition 4.3.5. Suppose \mathcal{C} is a category with 0-objects, kernels and cokernels, and suppose the composition of two admissible monos is an admissible mono. Suppose rk and deg are rank and degree functions on \mathcal{C} . For a nonzero object M , we define the *slope* of M to be

$$\mu(M) = \frac{\text{deg } M}{\text{rk } M}.$$

For any object M and $\mu \in \mathbb{R}$, we say that M is *semistable* (resp. *stable*) if $\mu(M') \leq \mu(M)$ (resp. $\mu(M') < \mu(M)$) for every nonzero proper subobject $M' \in \text{Sub}(M)$.

Example 4.3.6. A 0-object is semistable of every slope.

Proposition 4.3.7. *If M is semistable of slope μ and N is semistable of slope ν for $\mu > \nu$, then $\text{Hom}(M, N) = 0$.*

Theorem 4.3.8. [7, 3.2.3] *Suppose \mathcal{C} is as above and is equipped with rank and degree functions. For every $M \in \text{Obj}(\mathcal{C})$, there exists a \mathbb{R} -filtration F_M on M such that $\text{gr}_F^r M$ is semistable of slope r for all $r \in \mathbb{R}$. Moreover, the map $M \mapsto (M, F_M)$ is functorial.*

Definition 4.3.9. The filtration (M, F_M) is called the *Harder-Narasimhan (HN) filtration* on M . The *Harder-Narasimhan (HN) polygon* of M is the convex hull of the points

$$(\text{rk}(F^r M), \text{deg}(F^r M)) \in \mathbb{R}^2$$

as r ranges over \mathbb{R} .

Definition 4.3.10. Let k be a discrete perfect ring of characteristic p . Let $K_0 = W(k)[\frac{1}{p}]$. We define the category Isoc_k of *isocrystals over k* to be the category of finite projective K_0 -modules M equipped with a φ -semilinear bijection $\varphi_M : M \rightarrow M$.

If k is a field, then K_0 is also a field. For an isocrystal (M, φ_M) over k , we define the rank to be the K_0 -dimension of M . If we pick a basis for M and let A be the matrix for φ_M in this basis, then

$$\text{deg}(M, \varphi_M) = v(\det(A))$$

is independent of our choice of basis. Here $v : W(k)[\frac{1}{p}] \rightarrow \mathbb{Z} \cup \{\infty\}$ is the p -adic valuation.

Example 4.3.11. For $\mu = \frac{m}{n} \in \mathbb{Q}$ written in lowest terms with $n > 0$, we define M_μ to be the isocrystal with basis e_1, \dots, e_n with $\varphi(e_i) = e_{i+1}$ for $1 \leq i \leq n-1$ and $\varphi(e_n) = p^m e_1$.

Theorem 4.3.12 (Dieudonné-Manin). [8] *Let k be a perfect field of characteristic p .*

1. *rk is a rank function on Isoc_k .*
2. *Both deg and $-\text{deg}$ are degree functions on Isoc_k . We will agree to use deg for our degree function.*
3. *For $\mu \in \mathbb{Q}$, M_μ is stable of degree μ .*
4. *For $\mu \neq \nu$, if M is semistable of slope μ and N is semistable of slope ν , then $\text{Hom}(M, N) = 0$.*
5. *We have a unique functorial decomposition*

$$M \cong \bigoplus_{r \in \mathbb{Q}} M^r$$

where M^r is semistable of slope r .

6. *If k is in addition algebraically closed, then every semistable isocrystal of slope μ is a direct sum of copies of M_μ .*

Semistable isocrystals of slope μ are also often also called *pure of slope μ* .

Definition 4.3.13. Let $S = \text{Spa}(R, R^+)$ be an affinoid perfectoid of characteristic p , and promote R to a Banach ring as in Theorem 4.2.6. Let $k = R^\circ/R^{\circ\circ}$. Then we can construct a functor from isocrystals over k to φ -modules over $B(R)$ by sending M to $M \otimes_{W(k)} B(R)$. In other words, we get a \otimes -functor from isocrystals over k to vector bundles on the Fargues-Fontaine curve.

For $\mu \in \mathbb{Q}$, we write $\mathcal{O}(\mu)$ for the image of M_μ . This notation agrees with our definition of $\mathcal{O}(d)$ for nonnegative integers d that we defined in Example 4.2.9.

Definition 4.3.14. Let $S^\sharp = \mathrm{Spa}(A, A^\circ)$ where A is a perfectoid field of characteristic 0, with tilt $S = \mathrm{Spa}(R, R^\circ)$. Then $B_{\mathrm{dR}} := B_{\mathrm{dR}}(A, A^\circ)$ is a discrete valuation field with ring of integral elements $B_{\mathrm{dR}}^+ := B_{\mathrm{dR}}^+(A, A^\circ)$ with maximal ideal (ξ) . Let $B_e = B_e(A, A^\circ)$. Then B_e is a Bézout domain, so every finite projective module is free.

Vector bundles on $X_{\mathrm{FF}, S^\sharp} = X_{\mathrm{FF}, S}$ are equivalent to (B_e, B_{dR}^+) -pairs (M_e, M_{dR}^+) . We define the rank of such a pair to be the B_{dR} -dimension of $M_e \otimes_{B_e} B_{\mathrm{dR}}$. Now pick a basis for M_e and a basis for M_{dR}^+ . If $\iota : M_e \otimes_{B_e} B_{\mathrm{dR}} \rightarrow M_{\mathrm{dR}}^+ \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$ is the isomorphism, let D be the matrix of ι under the two induced bases. We define

$$\mathrm{deg}(M_e, M_{\mathrm{dR}}^+) = v_\xi(\det D),$$

where v_ξ is the valuation on B_{dR} . Note that since $B_e \cap B_{\mathrm{dR}} = \mathbb{Q}_p$, this degree function is well defined.

Then these are rank and degree functions on the category of vector bundles on $X_{\mathrm{FF}, S}$, and these functions do not depend on our choice of untilt (A, A°) of (R, R°) . We thus get Harder-Narasimhan filtrations and polygons for vector bundles on $X_{\mathrm{FF}, S}$.

Theorem 4.3.15 (Fargues-Fontaine). *[10] Suppose $S = \mathrm{Spa}(L, L^\circ)$ where L is a perfectoid field and let $k = L^\circ/L^{\circ\circ}$. Then the functor*

$$\mathrm{Isoc}_k \rightarrow \{\text{vector bundles on } X_{\mathrm{FF}, S}\}$$

is essentially surjective. Moreover, it preserves rank and negates degrees. The restriction of this functor to the isocrystals pure of slope 0 gives an equivalence with the semistable vector bundles of slope 0 on $X_{\mathrm{FF}, S}$.

Remark 4.3.16. The functor is *not* fully faithful, so it is not an equivalence. As an example, for any untilt S^\sharp of S , there is a nonzero section $\mathcal{O}_{X_{\text{FF},S}} = \mathcal{O}(0) \rightarrow \mathcal{O}(1)$ cutting out $S^\sharp \subset X_{\text{FF},S}$, but $\text{Hom}(M_0, M_1) = 0$.

Proposition 4.3.17. [20, 3.6.15,3.6.16] *We have*

1. *If $R \rightarrow R'$ is a morphism of perfectoid fields, then pullback along $X_{\text{FF},\text{Spa}(R')} \rightarrow X_{\text{FF},\text{Spa}(R)}$ preserves semistability of vector bundles.*
2. *If R is a perfectoid field, then a tensor product of semistable vector bundles on the scheme $X_{\text{FF},\text{Spa}(R)}$ of slopes μ and μ' is semistable of slope $\mu + \mu'$.*

Definition 4.3.18. Let S be a perfectoid space of characteristic p . Let \mathcal{F} be a vector bundle on $X_{\text{FF},S}$. If $x : \text{Spa}(R, R^+) \rightarrow S$ is the morphism corresponding to a point of S where (R, R^+) is an affinoid perfectoid field, then we can pullback \mathcal{F} along $X_{\text{FF},\text{Spa}(R,R^+)} \rightarrow X_{\text{FF},S}$. We define $\text{HN}(\mathcal{F}, x)$ to be the HN polygon of this pullback. This then defines a function from S to HN polygons which is constant under specializations.

Theorem 4.3.19. [22][20, 3.7.2] *Let S be a perfectoid space of characteristic p , and \mathcal{F} a vector bundle on $X_{\text{FF},S}$.*

1. *For every polygon P , the set*

$$\{x \in S \mid \text{HN}(\mathcal{F}, x) \leq P\}$$

is open and stable under generalizations. Here $P_1 \leq P_2$ means that the endpoints of P_1 and P_2 are the same and that P_1 lies under P_2 .

2. *If HN is constant on S , then there exists a \mathbb{R} -filtration on \mathcal{F} which pulls back to the HN filtration at any point.*

By the first part of the theorem, the locus

$$\{x \in S \mid \text{HN}(\mathcal{F}, x) = 0\}$$

is an open sublocus stable under generalizations. This set is called the *étale locus* of \mathcal{F} . (The name comes from the fact that isocrystals over a perfect characteristic p field k which are pure of slope 0 are also known as étale φ -modules over $K_0 = W(k)[\frac{1}{p}]$).

Definition 4.3.20. If S is a perfectoid space, a *pro-étale \mathbb{Z}_p -local system* (resp. *pro-étale \mathbb{Q}_p -local system*) is a sheaf \mathbb{L} of \mathbb{Z}_p -modules (resp. \mathbb{Q}_p -modules) on $X_{\text{pro-ét}}$ for which there exists a pro-étale cover $\{X_i \rightarrow X\}$ such that the pullback of \mathbb{L} to X_i is a finite free \mathbb{Z}_p -modules (resp. \mathbb{Q}_p -module) for every i .

Theorem 4.3.21. [22] *Let S be perfectoid of characteristic p . The following categories are equivalent:*

1. φ -modules over $\mathcal{Y}_{[0, \infty)}(S)$
2. φ -modules over $\mathcal{Y}_{I, (\varpi)}(S)$ for I of the form $[0, \frac{1}{r})$ or $[0, \frac{1}{r}]$, $r \in (0, \infty)$
3. pro-étale \mathbb{Z}_p -local systems on S

If, in addition, $S = \text{Spa}(R, R^+)$ is affinoid perfectoid, then this category is also equivalent to the categories

4. φ -modules over $W(R)$
5. φ -modules over $\tilde{\mathcal{R}}^{\text{int}}(R)$
6. φ^{-1} -modules over $\tilde{\mathcal{R}}^{\text{int}, 1/r}(R)$ for $r \in (0, \infty)$.

Theorem 4.3.22. [22][20, 3.7.5] *Let S be perfectoid of characteristic p . The following categories are equivalent:*

1. semistable vector bundles of slope 0 on $X_{\text{FF},S}$

2. pro-étale \mathbb{Q}_p -local systems on S

Moreover, we have a functor from pro-étale \mathbb{Z}_p -local systems \mathbb{L} on S to pro-étale \mathbb{Q}_p -local systems on S gotten by $\mathbb{L} \mapsto \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Under the equivalences of this and the previous theorem, this functor is identified with the functor from φ -modules \mathcal{F} over $\mathcal{Y}_{[0,\infty)}(S)$ to semistable vector bundles of slope 0 on $X_{\text{FF},S}$ gotten by descending the φ -module $\mathcal{F}|_{\mathcal{Y}_{(0,\infty)}(S)}$ to the quotient $\mathcal{X}_{\text{FF},S}$ (and then applying GAGA).

Example 4.3.23. Let $S = \text{Spa}(L, L^+)$ for R a perfectoid field of characteristic p , and let G denote the Galois group of L . We can then recover the classical facts from p -adic Hodge theory that

1. continuous \mathbb{Z}_p -representations of G are equivalent to φ -modules over $W(L)$.
2. continuous \mathbb{Q}_p -representations of G are equivalent to étale φ -modules over $W(L)[\frac{1}{p}]$.

Finally, we state a joint result with Christopher Birkbeck, David Hansen, Tony Feng, Serin Hong, Qirui Li, Lynelle Ye which was the result of a project proposed by David Hansen to the other coauthors during the 2017 Arizona Winter School.

Theorem 4.3.24. [4] *Let $S = \text{Spa}(L, L^+)$ be an algebraically closed perfectoid field of characteristic p . Let \mathcal{E} be a vector bundle on $X_{\text{FF},S}$. For $k \geq 2$, if $\mathcal{F}_1, \dots, \mathcal{F}_k$ are semistable vector bundles on $X_{\text{FF},S}$ of slopes $\mu_1 < \dots < \mu_k$ with $\text{HN}(\mathcal{E}) \leq \text{HN}(\mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_k)$, then there exists a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_k = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{F}_i$ for $i = 1, \dots, k$.

Note here the slopes of the quotients are increasing along the filtration so such a filtration is not a Harder-Narasimhan filtration. The above result was proposed because David Hansen realized that it can be used to prove the following theorem:

Theorem 4.3.25 (Hansen). [13] *Let $S = \mathrm{Spa}(L, L^+)$ be an algebraically closed perfectoid field of characteristic p . Let P be a HN-polygon with width n . Then*

$$\overline{\mathrm{Bun}_n^P} = \mathrm{Bun}_n^{\geq P}$$

as substacks of Bun_n . Here Bun_n is a v -stack Perf_S which parametrizes rank n vector bundles on $X_{\mathrm{FF}, S}$, and the substacks Bun_n^P and $\mathrm{Bun}_n^{\geq P}$ are the substacks parametrizing vector bundles on $X_{\mathrm{FF}, S}$ whose Harder-Narasimhan polygon $= P$ and $\geq P$, respectively.

CHAPTER 5

SHTUKAS FOR ALGEBRAIC GROUPS G

5.1 B_{dR}^+ -affine Grassmannian

In this section we define the B_{dR} -affine Grassmannian Gr_G for a linear algebraic group G . We then show that if $H \subset G$ is an algebraic subgroup such that G/H is quasiaffine (i.e. an open of an affine space), then

$$\mathrm{Gr}_H \subset \mathrm{Gr}_G$$

is locally closed. The analogue of this statement for the usual affine grassmannian is given in [1, §4.5]. In the Scholze-Weinstein Berkeley lectures [28], it is shown that if $H \subset G$ is an inclusion of reductive groups, then $\mathrm{Gr}_H \subset \mathrm{Gr}_G$ is closed. Our argument is an adaptation of the argument in [1] to this situation. Once we have this statement, we can embed an algebraic group H in a reductive group $H \subset G$ so that the quotient is quasiaffine. Using the results in [28], we can then show that Gr_H has nice properties. For example, it is a v -sheaf, and after cutting down the size, it becomes a diamond.

Let C be a completed algebraic closure of \mathbb{Q}_p , with ring of bounded elements \mathcal{O}_C . In the following, we shall write Perf_C for the category of perfectoid (C, \mathcal{O}_C) -algebras which is isomorphic to the category of $(C^\flat, \mathcal{O}_C^\flat)$ -algebras. We will write down functors $\mathrm{Perf}_C \rightarrow \mathrm{Set}$ which can be viewed as presheaves on the category of affine perfectoid spaces over $\mathrm{Spa}(C)$. All these presheaves behave well under rational localization and can be naturally extended to presheaves on the category of all perfectoid spaces over $\mathrm{Spa}(C)$, but we shall just write their values on the affinoid perfectoid spaces for simplicity.

Theorem 5.1.1. [26, Theorem 6.5] *The functors $B_{\mathrm{dR}}^+, B_{\mathrm{dR}} : \mathrm{Perf}_C \rightarrow \mathrm{Set}$ are pro-étale sheaves.*

Definition 5.1.2. Let X be a \mathbb{Q}_p -scheme. We define the (B_{dR}^+) -loop group $LX : \mathrm{Perf}_C \rightarrow X$

to be the étale sheafification of

$$(R, R^+) \mapsto X(B_{\mathrm{dR}}(R, R^+)).$$

We define the (B_{dR}^+) -positive loop group L^+X to be the étale sheafification of

$$(R, R^+) \mapsto X(B_{\mathrm{dR}}^+(R, R^+)).$$

Proposition 5.1.3. *If X is affine, then*

$$(R, R^+) \mapsto X(B_{\mathrm{dR}}(R, R^+)) \quad \text{and} \quad (R, R^+) \mapsto X(B_{\mathrm{dR}}^+(R, R^+))$$

are already étale sheaves.

Proof. Let $B = B_{\mathrm{dR}}$ or B_{dR}^+ . Since these are pro-étale sheaves, for any étale cover $(R, R^+) \rightarrow (R_i, R_i^+)$, we have an equalizer sequence of sets and thus of rings

$$B(R, R^+) \rightarrow \prod_i B(R_i, R_i^+) \rightrightarrows \prod_{i,j} B((R_i, R_i^+) \widehat{\otimes}_{(R, R^+)} (R_j, R_j^+)).$$

Then applying the affine $X(-) = \mathrm{Hom}_{\mathrm{Ring}}(A, -)$ will preserve this limit. \square

Lemma 5.1.4. *If $U \subset X$ is an open subscheme of a \mathbb{Q}_p -scheme, then $L^+U \subset L^+X$ is an open subfunctor.*

Proof. We may assume that $X = \mathrm{Spec} A$ is affine and $U = D(a)$ for some $a \in A$. Fix a morphism $g : A \rightarrow B_{\mathrm{dR}}^+(R, R^+)$ corresponding to $\gamma : \mathrm{Spa}(R, R^+) \rightarrow L^+X$. Let $h : (R, R^+) \rightarrow (S, S^+)$ be a morphism so we have a diagram

$$\begin{array}{ccc} \mathrm{Spa}(S, S^+) & \xrightarrow{h^*} & \mathrm{Spa}(R, R^+) \\ \downarrow & & \downarrow \gamma \\ L^+U & \longrightarrow & L^+X. \end{array}$$

Then $\gamma \circ h^*$ corresponds to the morphism $B_{\text{dR}}^+(h) \circ g : A \rightarrow B_{\text{dR}}^+(S, S^+)$. This factors through $L^+U \subset L^+X$ if and only if $(B_{\text{dR}}^+(h) \circ g)(a)$ is invertible in $B_{\text{dR}}^+(S, S^+)$, which happens if and only if $(\theta \circ B_{\text{dR}}^+(h) \circ g)(a) = (h \circ \theta \circ g)(a)$ becomes invertible in S . We thus see that $\text{Spa}(R, R^+) \times_{L^+U} L^+X \subset \text{Spa}(R, R^+)$ is the locus where $(\theta \circ g)(a)$ becomes invertible, which is open in the analytic topology. \square

Lemma 5.1.5. *Suppose $X = \text{Spec } A$. Then $L^+X \subset LX$ is a closed subfunctor.*

Proof. Fix a morphism $g : A \rightarrow B_{\text{dR}}(R, R^+)$ corresponding to a morphism $\gamma : \text{Spa}(R, R^+) \rightarrow LX$. Let $h : (R, R^+) \rightarrow (S, S^+)$ be a morphism of perfectoid C -algebras. Then the morphism $\gamma \circ h^*$ factors through L^+X

$$\begin{array}{ccc} \text{Spa}(S, S^+) & \xrightarrow{h^*} & \text{Spa}(R, R^+) \\ \vdots \downarrow & & \downarrow \gamma \\ L^+X & \longrightarrow & LX \end{array}$$

if and only if $(B_{\text{dR}}(h) \circ g)(A) \subset B_{\text{dR}}^+(S, S^+)$.

We build a sequence of subspaces

$$\text{Spa}(R, R^+) \supset \text{Spa}(R_1, R_1^+) \supset \text{Spa}(R_2, R_2^+) \supset \dots$$

each of which is Zariski closed in the previous perfectoid space. Their intersection is then (Zariski) closed.

Let us set $(R_0, R_0^+) = (R, R^+)$. Suppose we have defined (R_n, R_n^+) , and let g_n be the composite $A \rightarrow (R, R^+) \rightarrow (R_n, R_n^+)$. For $a \in A$, let

$$i_n(a) = \begin{cases} 0 & \text{if } g_n(a) \in B_{\text{dR}}^+(R_n, R_n^+) \\ m & \text{if } g_n(a) \in \text{Fil}^m B_{\text{dR}}(R_n, R_n) \text{ but not in } \text{Fil}^m B_{\text{dR}}(R_n, R_n^+) \text{ for } m < 0. \end{cases}$$

Then, when $i_n(a) < 0$, we see that $\xi^{-i_n(a)} g_n(a) \in B_{\text{dR}}^+(R_n, R_n^+)$. Let $I_n \subset R_n$ be the ideal

generated by the set

$$\{\theta(\xi^{-i_n(a)} g_n(a)) \in R_n \mid a \in A, i_n(a) < 0\}.$$

We define (R_{n+1}, R_{n+1}^+) so that $\mathrm{Spa}(R_{n+1}, R_{n+1}^+) = V(I_n) \subset \mathrm{Spa}(R_n, R_n^+)$. One sees that a morphism $h^* : \mathrm{Spa}(S, S^+) \rightarrow \mathrm{Spa}(R_n, R_n^+)$ factors through $V(I_n)$ if and only if for all $a \in A$, either $g_n(a) \in B_{\mathrm{dR}}^+(R_n, R_n^+)$ or $(B_{\mathrm{dR}}(h) \circ g_n)(a) \in \mathrm{Fil}^{i_n(a)+1} B_{\mathrm{dR}}(S, S^+)$.

It is then immediate that $\mathrm{Spa}(R, R^+) \times_{LX} L^+X \subset \mathrm{Spa}(R, R^+)$ is the intersection of the $\mathrm{Spa}(R_i, R_i^+)$, thus giving the lemma. \square

Definition 5.1.6. Let G be a linear algebraic group over \mathbb{Q}_p . We define the (B_{dR}^+) -affine Grassmannian Gr_G to the étale sheafification of the functor $\mathrm{Perf}_C \rightarrow \mathrm{Set}$ given by

$$(R, R^+) \mapsto LG(R, R^+)/L^+G(R, R^+).$$

The reductive case was considered in [28]. We shall use that case to show that the affine grassmannian is a nice space in general.

We can use the following lemma to describe the affine Grassmannian in terms of torsors.

Lemma 5.1.7. *Let G be a linear algebraic group over \mathbb{Q}_p and (R, R^+) be a perfectoid ring over C . Let \mathcal{E} be a G -torsor for the étale topology on $\mathrm{Spec} B_{\mathrm{dR}}^+(R, R^+)$. If \mathcal{E} is trivial on $\mathrm{Spec} R \subset \mathrm{Spec} B_{\mathrm{dR}}^+(R, R^+)$, then \mathcal{E} is trivial.*

Proof. Since $(B_{\mathrm{dR}}^+(R, R^+), (\xi))$ is a Henselian pair, and G is a linear algebraic group over \mathbb{Q}_p , this follows from Theorem 2.1.4 in [5]. \square

Corollary 5.1.8. *A G -torsor \mathcal{E} for the étale topology on $\mathrm{Spec} B_{\mathrm{dR}}^+(R, R^+)$ is étale locally on $\mathrm{Spa}(R, R^+)$ trivial.*

Proof. Lemma 5.1.7 shows that \mathcal{E} is (algebraically) étale-locally trivial on $\mathrm{Spec}(R)$, so in particular, it is (analytically) étale locally trivial on $\mathrm{Spa}(R, R^+)$. \square

Proposition 5.1.9. *The (B_{dR}^+) -affine Grassmannian Gr_G takes (R, R^+) to the set of isomorphism classes of étale G -torsors \mathcal{E} on $\text{Spec } B_{\text{dR}}^+(R, R^+)$ equipped with a trivialization on the open subset $\text{Spec } B_{\text{dR}}(R, R^+)$.*

Proof. Given an étale G -torsor \mathcal{E} , by Corollary 5.1.8, we see that there is an étale cover of $\text{Spa}(R, R^+)$ on which \mathcal{E} is trivial. Then we get compatible elements of LG/L^+G over this cover, which glue together to an element of the sheaf. \square

Proposition 5.1.10. *Let $H \subset G$ be an algebraic subgroup of an algebraic group over \mathbb{Q}_p . Then $L^+(G/H)$ is the étale sheafification of the functor*

$$(R, R^+) \mapsto L^+G(R, R^+)/L^+H(R, R^+).$$

Proof. The map

$$G \rightarrow G/H$$

is a H -torsor in the fppf topology. Since we are working in characteristic 0, H is smooth, and it is also a H -torsor in the étale topology. Now let

$$f : \text{Spec } B_{\text{dR}}^+(R, R^+) \rightarrow G/H$$

be a morphism of schemes. The pullback of $G \rightarrow G/H$ along f gives us a étale H -torsor on $\text{Spec } B_{\text{dR}}^+(R, R^+)$ which is étale locally on $\text{Spa}(R, R^+)$ trivial by Corollary 5.1.8. This shows the proposition. \square

Theorem 5.1.11. *Suppose $H \subset G$ is an inclusion of linear algebraic groups over \mathbb{Q}_p such that G/H is quasiaffine. Then the corresponding map of (B_{dR}^+) -affine grassmannians*

$$\text{Gr}_H \rightarrow \text{Gr}_G$$

is a locally closed embedding.

Once we have the above facts for the B_{dR}^+ -affine grassmannian, the proof in Beilinson-Drinfeld [1, 4.5.1] in the case of the usual affine grassmannian works without change.

Proof. Let $G/H = U \subset Z$ where U is open in the affine scheme Z . We have a commutative diagram

$$\begin{array}{ccccc}
 L^+H & \longrightarrow & LH & & \\
 \downarrow & & \downarrow & & \\
 L^+G & \longrightarrow & L^+U & \xrightarrow{\text{open emb.}} & L^+Z \\
 \downarrow & & \downarrow & & \downarrow \text{closed emb.} \\
 LG & \longrightarrow & LU & \longrightarrow & LZ.
 \end{array}$$

Here, we note that the map $LH \rightarrow L^+U$ exists since $H \rightarrow G/H = U$ factors through the final object, so we have the commuting diagram

$$\begin{array}{ccccc}
 & & L^+\mathrm{pt} & \longrightarrow & L^+U \\
 & \nearrow & \downarrow \cong & & \downarrow \\
 LH & \longrightarrow & L\mathrm{pt} & \longrightarrow & LU.
 \end{array}$$

Let $Y \subset LG$ be the pullback of $L^+U \rightarrow LZ$ under $LG \rightarrow LZ$. This is a locally closed subfunctor which contains the closed subfunctor L^+G , so it is the inverse image of some locally closed $Y' \subset \mathrm{Gr}_G = LG/L^+G$

We claim that $Y' = LH/L^+H$. Indeed, $Y \rightarrow L^+U = LG^+/LH^+$ is LH^+ -equivariant. Moreover, $LH \cap L^+G = L^+H$, so we see that $LH \rightarrow Y' = Y/L^+G$ is an L^+H -torsor, as desired. \square

To finish, we use the following lemma:

Lemma 5.1.12. [1, Lemma after 4.5.1] *Let H be a linear algebraic group. Then there exists a reductive group G and an embedding $H \subset G$ such that G/H is quasiaffine.*

Proof. Pick a faithful representation $H \rightarrow \mathrm{GL}_V$ such that H is the stabilizer of some one dimensional subspace $\ell \subset V$. Then H acts by a character $\chi : H \rightarrow \mathbb{G}_m$. Let $G = \mathrm{GL}_V \times \mathbb{G}_m$ and consider the map $H \rightarrow G$ sending h to $(h, \chi(h)^{-1})$. Consider the action of $G = \mathrm{GL}_V \times \mathbb{G}_m$ on V where GL_V has its natural action on V and \mathbb{G}_m acts by scalars. Then for any nonzero $v \in \ell$, the stabilizer of v is the image of H in G . Thus G/H is the orbit of v under this action, and in particular it is quasiaffine. \square

Definition 5.1.13. Let G be an algebraic group over \mathbb{Q}_p , and suppose we have chosen closed embedding $G \subset G'$ as an algebraic subgroup of a reductive group G' where the quotient G'/G is quasiaffine. Let $T' \subset G'$ be a maximal torus, and $\mu \in X_\bullet(T')^+$. Then by the results of [28], we can define $\mathrm{Gr}_{G',\mu} \subset \mathrm{Gr}_{G'}$ and $\mathrm{Gr}_{G',\leq\mu} \subset \mathrm{Gr}_{G'}$ which are locally closed and closed subsheaves, respectively. These $\mathrm{Gr}_{G',\mu}$ and $\mathrm{Gr}_{G',\leq\mu}$ are then diamonds, and have nice geometric structure.

We now define

$$\mathrm{Gr}_{G \subset G',\mu} \quad \text{and} \quad \mathrm{Gr}_{G \subset G',\leq\mu}$$

to be the inverse image of $\mathrm{Gr}_{G',\mu}$ and $\mathrm{Gr}_{G',\leq\mu}$ under the embedding $\mathrm{Gr}_G \subset \mathrm{Gr}_{G'}$. It is immediate that $\mathrm{Gr}_{G \subset G',\mu}$ and $\mathrm{Gr}_{G \subset G',\leq\mu}$ are diamonds, as they are locally closed in a diamond.

Remark 5.1.14. Note that in the above, our data μ depends on a choice of nice embedding of G into a reductive group G' . It is an interesting question whether there is an intrinsic way of defining data μ allowing us to define subfunctors of Gr_G which are diamonds, without referring to such an embedding.

Let us give some remarks about what this entails. Let T', G' be as in the definition. Let $B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+(C, \mathcal{O}_C)$, and $B_{\mathrm{dR}} = B_{\mathrm{dR}}(C, \mathcal{O}_C)$. For $(C', \mathcal{O}_{C'})$ an algebraically closed affinoid perfectoid field over (C, \mathcal{O}_C) , let us write $B_{\mathrm{dR},C'}^+ = B_{\mathrm{dR}}^+(C', \mathcal{O}_{C'})$ and $B_{\mathrm{dR},C'} =$

$B_{\text{dR}}(C', \mathcal{O}_{C'})$. Then we have a bijections

$$X_{\bullet}(T')^+ = G'(B_{\text{dR}}^+) \setminus G'(B_{\text{dR}})/G'(B_{\text{dR}}^+) = G'(B_{\text{dR},C'}^+) \setminus G'(B_{\text{dR},C'})/G'(B_{\text{dR},C'})$$

where a cocharacter $\mu : \mathbb{G}_m \rightarrow T' \subset G'$ corresponds to the double coset of the element $\mu(t)$.

Then $\text{Gr}_{G',\mu}(R, R^+) \subset \text{Gr}_{G'}(R, R^+)$ is the subfunctor of points such that the pullback along every geometric point $\text{Spa}(C', \mathcal{O}_{C'})$ of $\text{Spa}(R, R^+)$ lies in the coset of $\mu(t)$ in $\text{Gr}_{G'}(C', \mathcal{O}_{C'}) = G'(B_{\text{dR},C'})/G'(B_{\text{dR},C'}^+)$. Thus, $\text{Gr}_{G \subset G',\mu}(R, R^+) \subset \text{Gr}_G(R, R^+)$ is the subfunctor such that the pullback along every geometric point $\text{Spa}(C', \mathcal{O}_{C'})$ gives an element of

$$G(B_{\text{dR},C'})/G(B_{\text{dR},C'}^+) = \text{Gr}_G(C', \mathcal{O}_{C'})$$

which maps to the coset of $\mu(t)$ under the map

$$G(B_{\text{dR},C'})/G(B_{\text{dR},C'}^+) \rightarrow G'(B_{\text{dR},C'}^+) \setminus G'(B_{\text{dR},C'})/G'(B_{\text{dR},C'}^+).$$

We are thus interested in the subsets of $G(B_{\text{dR},C'})/G(B_{\text{dR},C'}^+)$ which can be gotten as inverse images of double cosets from embeddings to reductive groups as above.

It is immediately clear that such an inverse image is the union of double cosets

$$G(B_{\text{dR},C'}^+) \setminus G(B_{\text{dR},C'})/G(B_{\text{dR},C'}^+).$$

In very special circumstances, for example when

$$G = \begin{bmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad G = \begin{bmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & \mathbb{G}_m \end{bmatrix}$$

there exists an embedding $G \subset G'$ into a reductive group with quasiaffine quotient such that

$$\begin{aligned} G(B_{\mathrm{dR}}^+) \backslash G(B_{\mathrm{dR}}) / G(B_{\mathrm{dR}}^+) &= G(B_{\mathrm{dR},C'}^+) \backslash G(B_{\mathrm{dR},C'}) / G(B_{\mathrm{dR},C'}^+) \\ &\hookrightarrow G'(B_{\mathrm{dR},C'}^+) \backslash G'(B_{\mathrm{dR},C'}) / G'(B_{\mathrm{dR},C'}^+), \end{aligned}$$

i.e. the collection of double cosets does not depend on the algebraically closed perfectoid field C' over C , and the map to the double cosets of the reductive group is injective. We see that in such special situations, these double cosets all define subfunctors of Gr_G which are diamonds.

However, such a strategy fails dramatically outside these very special cases. As an example, suppose $G \subset \mathrm{GL}_3$ is the subgroup of upper triangular matrices. For $x, y \in (B_{\mathrm{dR},C'}^+)^\times$, if the two matrices

$$\begin{bmatrix} 1 & t^{-1} & xt^{-2} \\ 0 & 1 & t^{-1} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & t^{-1} & yt^{-2} \\ 0 & 1 & t^{-1} \\ 0 & 0 & 1 \end{bmatrix}$$

are in the same double coset, then we must have $x \equiv y \pmod{t}$. We thus see that there are uncountably many cosets, so no map to $X_\bullet(T')^+$ can be injective. Moreover, we do not expect the collection of double cosets to be independent of the choice of algebraically closed perfectoid field C' over C .

5.2 Shtukas and (Pro-)Étaleness of the Period Morphism

In this section, we give two definitions of a moduli space, show they are equivalent, and show that a certain period morphism from this moduli space to an affine grassmannian is (pro-)étale. This will give this moduli space the structure of a diamond. The first definition of the moduli space is via exact tensor functors, and is related to Fontaine's theory of admissible

representations, while the second definition is in terms of p -adic shtukas (with one leg). Using the second definition, the proof of (pro-)étaleness is essentially the same as that given for reductive groups in [28].

In the following, K is a finite field extension of \mathbb{Q}_p , k its residue field, and $K_0 = W(k)[1/p] \subset K$. Let G be a connected affine algebraic group over \mathbb{Q}_p and b be an element of $G(K_0)$.

Definition 5.2.1. We define an exact \otimes -functor $D : \text{Rep}_{\mathbb{Q}_p}(G) \rightarrow \text{Isoc}_k$ by the map

$$D(E) = K_0 \otimes_{\mathbb{Q}_p} E$$

equipped with the Frobenius defined by

$$\varphi(x \otimes e) = F(b(x \otimes e))$$

where $F : K_0 \otimes_{\mathbb{Q}_p} E \rightarrow K_0 \otimes_{\mathbb{Q}_p} E$ is given by $F(x \otimes e) = \varphi(x) \otimes e$.

Proposition 5.2.2. [29] *The functors $A_{\text{crys}}^0, A_{\text{crys}}, B_{\text{crys}}^+$ and B_{crys} are pro-étale sheaves. Moreover, $H^i(\text{Spa}(R, R^+), B_{\text{crys}}) = 0$ for $i > 0$.*

Definition 5.2.3. We define a functor $\mathcal{M}_{G,b}$ which sends an affinoid perfectoid space $S = \text{Spa}(R, R^+)$ over $\text{Spa} C$ to the set of isomorphisms of exact \otimes -functors from $\text{Rep}_{\mathbb{Q}_p}(G)$ to locally free $B_e(R, R^+)$ -modules of finite rank:

$$\iota : (E \mapsto B_e(R, R^+) \otimes_{\mathbb{Q}_p} E) \xrightarrow{\cong} (E \mapsto (B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E))^{\varphi=1}).$$

Here, φ is the endomorphism of $B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E)$ given by

$$x \otimes (y \otimes e) \mapsto \varphi(x) \otimes \varphi(y \otimes e)$$

for $x \in B_{\text{crys}}(R, R^+)$, $y \in K_0$ and $e \in E$.

Definition 5.2.4. We define an endomorphism φ of $G(B_{\text{crys}}(R, R^+))$ which is compatible with the φ -action on $B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E)$. There is a natural action of $G(B_{\text{crys}}(R, R^+))$ on

$$B_{\text{crys}}(R, R^+) \otimes_{\mathbb{Q}_p} E \cong B_{\text{crys}}(R, R^+) \otimes_{K_0} (K_0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_p) = B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E).$$

Let F be the endomorphism of $G(B_{\text{crys}}(R, R^+))$ induced by the Frobenius φ on $B_{\text{crys}}(R, R^+)$. Define

$$\varphi(g) = F(bgb^{-1})$$

for $g \in G(B_{\text{crys}}(R, R^+))$.

Proposition 5.2.5. *We have*

$$\varphi(g(x \otimes (y \otimes e))) = \varphi(g)\varphi(x \otimes (y \otimes e))$$

for $g \in G(B_{\text{crys}}(R, R^+))$, $x \in B_{\text{crys}}(R, R^+)$, $y \in K_0$ and $e \in E$.

Proof. Let F be the endomorphism of $B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E) \cong B_{\text{crys}}(R, R^+) \otimes_{\mathbb{Q}_p} E$ given by

$$x \otimes (y \otimes e) \mapsto \varphi(x) \otimes (\varphi(y) \otimes e) = \varphi(x) \otimes F(y \otimes e).$$

By definition of F on $G(B_{\text{crys}}(R, R^+))$,

$$F(g(x \otimes (y \otimes e))) = F(g)F(x \otimes (y \otimes e)).$$

Moreover,

$$\begin{aligned}
\varphi(x \otimes (y \otimes e)) &= \varphi(x) \otimes F(b(y \otimes e)) \\
&= F(x \otimes b(y \otimes e)) \\
&= F(b(x \otimes (y \otimes e))),
\end{aligned}$$

where the last equality comes from functoriality of G . We now compute

$$\begin{aligned}
\varphi(g)\varphi(x \otimes (y \otimes e)) &= F(bgb^{-1})F(b(x \otimes (y \otimes e))) \\
&= F(bg(x \otimes (y \otimes e))) \\
&= \varphi(g(x \otimes (y \otimes e))). \quad \square
\end{aligned}$$

We now give an alternate definition of the functor we called \mathcal{M} above, and show that the two definitions are equivalent.

Definition 5.2.6. Let \mathcal{G} be a smooth group over \mathbb{Z}_p whose generic fiber is G and with connected special fiber. Recall that under our definition 4.2.13

$$\mathcal{Y}_{[0,\infty)}(S) = S^b \dot{\times} \mathrm{Spa} \mathbb{Z}_p$$

is an analytic adic space over $\mathrm{Spa}(\mathbb{Z}_p)$ which is equipped with

- a continuous map $\kappa = \kappa_{(\xi_0)} : |\mathcal{Y}_{[0,\infty)}(S)| \rightarrow [0, \infty]$.
- an automorphism $\varphi : \mathcal{Y}_{[0,\infty)} \rightarrow \mathcal{Y}_{[0,\infty)}$ such that $\kappa(\varphi(x)) = p\kappa(x)$ for $x \in |\mathcal{Y}_{[0,\infty)}(S)|$.
- a map $S \mapsto \mathcal{Y}_{[0,\infty)}(S)$ defining a closed Cartier divisor with $\kappa(S) = 1$.

We define $\mathrm{Sht}_{\mathcal{G},b}$ to be the functor sending $S = \mathrm{Spa}(R, R^+)$ to the set of isomorphism classes of triples $(\mathcal{P}, \varphi_{\mathcal{P}}, \iota)$ where

1. \mathcal{P} is a pro-étale \mathcal{G} -torsor on $\mathcal{Y}_{[0,\infty)}(S)$.
2. $\varphi_{\mathcal{P}}$ is an isomorphism

$$\varphi_{\mathcal{P}} : (\varphi^* \mathcal{P})|_{\mathcal{Y}_{[0,\infty)}(S) \setminus \varphi^{-1}(S)} \xrightarrow{\cong} \mathcal{P}|_{\mathcal{Y}_{[0,\infty)}(S) \setminus \varphi^{-1}(S)}$$

which is meromorphic along $\varphi^{-1}(S) \subset \mathcal{Y}_{[0,\infty)}(S)$.

3. ι is an isomorphism

$$\iota : \mathcal{P}|_{\mathcal{Y}_{[1/p,\infty)}(S)} \xrightarrow{\cong} G \times \mathcal{Y}_{[1/p,\infty)}(S)$$

under which $\varphi_{\mathcal{P}}$ gets identified with $b \times \varphi$.

Remark 5.2.7. Our definition of moduli of shtukas when restricted to reductive groups slightly differs from the one given in Scholze's Berkley lectures [28]. The difference is that in our situation, we are only concerned with shtukas with one leg, so we fix a characteristic 0 untilt C of C^b and we use this untilt to fix the position of our leg. In [28], they start out with C^b in characteristic p and allow n legs for n a nonnegative integer. The resulting moduli space will then have a natural map to $(\mathrm{Spa} \mathbb{Q}_p)^n$.

Definition 5.2.8. For a given $b \in G(K_0)$, we can define a G -torsor \mathcal{E}^b on X_{FF, S^b} by descending the trivial G -torsor

$$G \times \mathcal{Y}_{(0,\infty)}(S)$$

on $\mathcal{Y}_{(0,\infty)}(S)$ using the φ -equivariant map $b \times \varphi$.

Let us first mention the following theorem, which follows from Theorems 4.3.21 and 4.3.22 along with the Tannakian formalism [28, Appendix to Lecture XIX]

Theorem 5.2.9. [28, 22.5.3, 22.6.1] *The categories*

- *pro-étale $\underline{G}(\mathbb{Q}_p)$ -torsors on S^b*

- G -torsors on $\mathcal{X}_{\text{FF},S}$ trivial at every geometric point of S^b

are equivalent.

The categories

- pro-étale $\underline{\mathcal{G}(\mathbb{Z}_p)}$ -torsors \mathbb{P} on S^b
- φ -equivariant \mathcal{G} -torsors \mathcal{T} on $\mathcal{Y}_{[0,\infty)}(S)$
- φ^{-1} -equivariant \mathcal{G} -torsors on $\mathcal{Y}_{[0,r)}(S)$ for any $r \in (0, \infty)$

are equivalent.

Moreover, under the above equivalences, the map $\mathbb{P} \mapsto \mathbb{P} \otimes_{\underline{\mathcal{G}(\mathbb{Z}_p)}} \underline{G(\mathbb{Q}_p)}$ corresponds to the map sending \mathcal{T} to the descent of the φ -module $\mathcal{T}|_{\mathcal{Y}_{(0,\infty)}(S)}$.

Proposition 5.2.10. *The $S = \text{Spa}(R, R^+)$ points of $\text{Sht}_{\mathcal{G},b}$ can also be described as the triples $(\mathcal{E}, \alpha, \mathbb{P})$ where*

1. \mathcal{E} is a G -torsor on $\mathcal{X}_{\text{FF},S}$ which is trivial at every geometric point of S^b ,
2. α is an isomorphism of G -torsors

$$\mathcal{E}|_{\mathcal{X}_{\text{FF},S} \setminus S} \xrightarrow{\cong} \mathcal{E}^b|_{\mathcal{X}_{\text{FF},S} \setminus S}$$

which is meromorphic along the Cartier divisor $S \subset \mathcal{X}_{\text{FF},S}$.

3. \mathbb{P} is a $\underline{\mathcal{G}(\mathbb{Z}_p)}$ -lattice in the pro-étale $\underline{G(\mathbb{Q}_p)}$ -torsor on S^b corresponding to \mathcal{E} .

The proof in Scholze's Berkley lectures [28, Proposition 23.3.1] for reductive groups works without change here.

Proof. Given the data $(\mathcal{E}, \alpha, \mathbb{P})$, the G -torsor \mathcal{E} defines a φ -equivariant G -torsor on $\mathcal{Y}_{(0,\infty)}(S)$ and the $\underline{\mathcal{G}(\mathbb{Z}_p)}$ gives us an extension of this φ -equivariant torsor to a φ -equivariant \mathcal{G} -torsor \mathcal{P}' on $\mathcal{Y}_{[0,\infty)}(S)$ by the two equivalences above. To get \mathcal{P} , we glue \mathcal{P}' on $\mathcal{Y}_{[0,1)}(S)$ with

the pullback of \mathcal{E}^b on $\mathcal{Y}_{(1/p,\infty)}(S)$ using α to identify the two torsors on $\mathcal{Y}_{(1/p,1)}(S)$. The φ -structure on \mathcal{P}' and the φ -structure of the pullback of \mathcal{E}^b then give us the desired isomorphism $\varphi_{\mathcal{P}}$ defined away from $\varphi^{-1}(S)$, and ι exists by construction.

Conversely, given the data $(\mathcal{P}, \varphi_{\mathcal{P}}, \iota)$, we can take the restriction of \mathcal{P} to $\mathcal{Y}_{(0,1/p)}(S)$ along with $\varphi_{\mathcal{P}}$ restricted to this space. This allows us to descend to a G -torsor \mathcal{E} on $\mathcal{X}_{\text{FF},S}$. The restriction of \mathcal{P} to $\mathcal{Y}_{[0,1/p)}(S)$ then gives us \mathbb{P}^1 by . Finally, ι along with $\varphi_{\mathcal{P}}$ identifies $\mathcal{P}|_{\mathcal{Y}_{(0,\infty)}(S)}$ with $G \times \mathcal{Y}_{(0,\infty)}(S)$ away from $\bigcup_{n \in \mathbb{Z}} \varphi^n(S)$, which allows us to descend to the desired isomorphism α on $X_{\text{FF},S} \setminus S$. \square

We see that $\text{Sht}_{\mathcal{G}}$ only depends on $\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. Because of this, we can make the following definition:

Definition 5.2.11. For $K \subset G(\mathbb{Q}_p)$ a compact open subgroup, we define

$$\text{Sht}_{G,K,b}$$

to be the moduli space $\text{Sht}_{\mathcal{G},b}$ for any \mathcal{G} with $\mathcal{G}(\mathbb{Z}_p) = K \subset G(\mathbb{Q}_p)$. As stated above, this only depends on K and not the choice of \mathcal{G} . We define

$$\text{Sht}_{G,b} = \varprojlim_K \text{Sht}_{G,K,b}.$$

The following description of Sht_G can be easily seen from the previous proposition.

Proposition 5.2.12. *The $S = \text{Spa}(R, R^+)$ points of Sht_G can be described as isomorphisms*

$$\alpha : \mathcal{E}^1|_{\mathcal{X}_{\text{FF},S^b} \setminus S} \xrightarrow{\cong} \mathcal{E}^b|_{\mathcal{X}_{\text{FF},S^b} \setminus S}$$

which is meromorphic along S .

Proposition 5.2.13. *The Tannakian formalism gives us an identification of $\mathcal{M}_{G,b}$ with $\text{Sht}_{G,b}$.*

Proof. By the description of vector bundles on $\mathcal{X}_{\text{FF},S}$ as $(B_{\text{crys}}^{\varphi=1}, B_{\text{dR}})$ -pairs one sees that the restriction of \mathcal{E}^1 to $\text{Spec } B_e(R, R^+)$ corresponds under the tannakian formalism to the exact \otimes -functor $E \mapsto B_e(R, R^+) \otimes_{\mathbb{Q}_p} \mathcal{E}$. Similarly, by definition of D , the vector bundle $\mathcal{E}^b|_{\mathcal{X}_{\text{FF},S} \setminus S}$ corresponds to the exact \otimes -functor $E \mapsto (B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E))^{\varphi=1}$. The Tannakian formalism then gives the proposition. \square

Definition 5.2.14. We can define a morphism $\text{Sht}_{G,K,b} \rightarrow \text{Gr}_G$ by sending $(\mathcal{E}, \alpha, \mathbb{P})$ to the element of Gr_G induced by the meromorphic morphism $\alpha : \mathcal{E}^1|_{\mathcal{X}_{\text{FF},S} \setminus S} \xrightarrow{\cong} \mathcal{E}^b|_{\mathcal{X}_{\text{FF},S} \setminus S}$ telling us the relative position of the torsor \mathcal{E}^1 in \mathcal{E}^b at $S \subset \mathcal{X}_{\text{FF},S}$. These morphisms fit together to give a morphism $\text{Sht}_{G,b} \rightarrow \text{Gr}_G$.

Theorem 5.2.15. *The morphism $\text{Sht}_{G,K,b} \rightarrow \text{Gr}_G$ is étale. The morphism $\text{Sht}_{G,b} \rightarrow \text{Gr}_G$ is pro-étale.*

Proof. Given a point $S = \text{Spa}(R, R^+)$ of Gr_G , we can modify \mathcal{E}^b using this data to get a G -torsor \mathcal{E} on $\mathcal{X}_{\text{FF},S}$. The locus $T \subset S$ where the tensor functor $\text{Rep}_{\mathbb{Q}_p}(G) \rightarrow \mathcal{X}_{\text{FF},T}$ corresponding to the pullback of \mathcal{E} along $\mathcal{X}_{\text{FF},T} \rightarrow \mathcal{X}_{\text{FF},S}$ factors through the semistable vector bundles of slope 0 on $\mathcal{X}_{\text{FF},T}$ is an open subset $T = S^a \subset S$. Over $S^a \times \text{Spa } \mathbb{Q}_p$, \mathcal{E} gives us a pro-étale $\underline{G(\mathbb{Q}_p)}$ -torsor \mathbb{P}_η on S^a . If $U \rightarrow S$ is a pro-étale map under which the torsor \mathbb{P}_η is trivialized after pullback along $U^a = U \times_S S^a \rightarrow S^a$, we then see that over U , the above map is given by

$$U^a \times G(\mathbb{Q}_p)/K \rightarrow U.$$

It is thus clear that pro-étale locally, the map $\text{Sht}_{G,K,b} \rightarrow \text{Gr}_G$ is étale. We then also conclude that $\text{Sht}_G \rightarrow \text{Gr}_G$ is pro-étale. \square

Corollary 5.2.16. *For any subfunctor $X \subset \text{Gr}_G$ which is a diamond, the inverse image of X under $\text{Sht}_{G,b} \rightarrow \text{Gr}_G$ and $\text{Sht}_{G,K,b} \rightarrow \text{Gr}_G$ are diamonds.*

5.3 An Example

Let us consider the case where

$$G = \begin{bmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} p^{-m} & 0 \\ 0 & 1 \end{bmatrix}$$

for some integer m .

First, let us compute the shape of Gr_G . We have a natural embedding $G \subset \text{GL}_2$.

Proposition 5.3.1. *The quotient*

$$\text{GL}_2/G \simeq \mathbb{A}^2 \setminus \{(0, 0)\}$$

which is a quasiffine scheme.

Proof. GL_2 acts on \mathbb{A}^2 by

$$\rho(M)v = (M^{-1})^T v.$$

Then G is the stabilizer of $(0, 1)$ under this action, so the quotient is isomorphic to the orbit of $(0, 1)$, which is $\mathbb{A}^2 \setminus \{(0, 0)\}$. \square

Let us take $B \subset \text{GL}_2$ to be the upper triangular matrices, with $T \subset B$ the diagonal matrices. Then

$$\text{GL}_2(B_{\text{dR}}^+) \setminus \text{GL}_2(B_{\text{dR}}) / \text{GL}_2(B_{\text{dR}}^+)$$

correspond to positive cocharacters

$$X_{\bullet}(T)^+ = \{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha \geq \beta\},$$

with (α, β) corresponding to the cocharacter

$$x \mapsto \mu_{(\alpha, \beta)}(t) = \begin{bmatrix} x^\alpha & 0 \\ 0 & x^\beta \end{bmatrix}.$$

For such a cocharacter, $\mu_{(\alpha, \beta)}(t)$ gives us a double coset of $\mathrm{GL}_2(B_{\mathrm{dR}}^+) \backslash \mathrm{GL}_2(B_{\mathrm{dR}}) / \mathrm{GL}_2(B_{\mathrm{dR}}^+)$.

Proposition 5.3.2. *Suppose $(\alpha, \beta) \in X_\bullet(T)^+$ defined above. Then*

$$G(B_{\mathrm{dR}}) \cap \left(\mathrm{GL}_2(B_{\mathrm{dR}}^+) \mu_{(\alpha, \beta)}(t) \mathrm{GL}_2(B_{\mathrm{dR}}^+) \right)$$

is given by the following cases

1. if $\beta = 0$, then it is

$$\begin{bmatrix} t^\alpha (B_{\mathrm{dR}}^+)^\times & B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}$$

2. if $\alpha = 0$, then it is

$$\begin{bmatrix} t^\beta (B_{\mathrm{dR}}^+)^\times & t^\beta B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}$$

3. if $\alpha > 0$ and $\alpha + \beta < 0$, then it is

$$\begin{bmatrix} t^{\alpha+\beta} (B_{\mathrm{dR}}^+)^\times & t^\beta (B_{\mathrm{dR}}^+)^\times + t^{\alpha+\beta} B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}$$

4. if $\beta < 0$ and $\alpha + \beta \geq 0$, then it is

$$\begin{bmatrix} t^{\alpha+\beta} (B_{\mathrm{dR}}^+)^\times & t^\beta (B_{\mathrm{dR}}^+)^\times + B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}$$

5. if $\beta \leq \alpha < 0$ or $0 < \beta \leq \alpha$, then it is \emptyset .

Proof. Suppose $\beta \leq 0$ and $\alpha \geq 0$. Note that

$$\begin{bmatrix} 0 & 1 \\ 1 & t^{-\beta} \end{bmatrix} \begin{bmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{bmatrix} \begin{bmatrix} -1 & 0 \\ t^\alpha & 1 \end{bmatrix} = \begin{bmatrix} t^\alpha & t^\beta \\ t^\alpha & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ t^\alpha & 1 \end{bmatrix} = \begin{bmatrix} t^{\alpha+\beta} & t^\beta \\ 0 & 1 \end{bmatrix}.$$

Now for $u, v \in (B_{\text{dR}}^+)^\times$ and $x, y \in B_{\text{dR}}^+$, we have

$$\begin{aligned} \begin{bmatrix} v & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^{\alpha+\beta} & t^\beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} uv^{-1} & yv^{-1} \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} t^{\alpha+\beta}v & t^\beta v + x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} uv^{-1} & yv^{-1} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} t^{\alpha+\beta}u & t^\beta v + t^{\alpha+\beta}y + x \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This shows that in each of the cases, the claimed subset is contained in the corresponding intersection. Since we have accounted for every element of $G(B_{\text{dR}})$ in the cases, and we get a partition of $G(B_{\text{dR}})$ as (α, β) ranges over all elements of $X_\bullet(T)^+$, we conclude that the intersections are precisely the claimed subsets. \square

We note that Proposition 5.3.2 holds more generally for any discrete valuation field F in place of B_{dR} , with ring of integers \mathcal{O}_F replacing B_{dR}^+ and a uniformizer π replacing t . In particular, it holds for $B_{\text{dR}}(C', \mathcal{O}_{C'})$ for any algebraically closed perfectoid field C' over C . Moreover, the computation used in the proof of the proposition shows that

$$G(B_{\text{dR}}^+) \setminus G(B_{\text{dR}})/G(B_{\text{dR}}^+) \rightarrow \text{GL}_2(B_{\text{dR}}^+) \setminus \text{GL}_2(B_{\text{dR}})/\text{GL}_2(B_{\text{dR}}^+)$$

is injective.

Corollary 5.3.3. *For any double coset $c \in G(B_{\text{dR}}^+) \setminus G(B_{\text{dR}})/G(B_{\text{dR}}^+)$ (for the particular G of this section), let $\text{Sht}_{G,b,c} \subset \text{Sht}_{G,b}$ be the subsheaf of all points $\text{Spa}(R, R^+) \rightarrow \text{Sht}_{G,b}$ such that the pullback along any geometric point $\text{Spa}(C', \mathcal{O}_{C'}) \rightarrow \text{Spa}(R, R^+)$ is an element*

in the class of c . Then $\text{Sht}_{G,b,c}$ is a diamond.

Proof. Let $\mu \in X_\bullet(T)^+$ be the cocharacter corresponding to the image of c in the double cosets $\text{GL}_2(B_{\text{dR}}^+) \backslash \text{GL}_2(B_{\text{dR}}) / \text{GL}_2(B_{\text{dR}}^+)$. By injectivity of the map of double cosets, we see that $\text{Sht}_{G,b,c}$ is the inverse image of $\text{Gr}_{G \subset \text{GL}_2, \mu} \subset \text{Gr}_G$ under the map $\text{Sht}_{G,b} \rightarrow \text{Gr}_G$, so this follows from corollary 5.2.16. \square

We now compute the Spa C points of $\mathcal{M}_{G,b}$.

Proposition 5.3.4. *We have*

$$\mathcal{M}_{G,b}(C, \mathcal{O}_C) \cong \begin{bmatrix} t^m (B_e)^\times & t^m B_e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t^m & 0 \\ 0 & 1 \end{bmatrix} G(B_e).$$

Proof. For $a \in B_e^\times$ and $b \in B_e$, for $E \in \text{Rep}_{\mathbb{Q}_p}(G)$, we can define

$$\iota_E : B_e(R, R^+) \otimes_{\mathbb{Q}_p} E \rightarrow (B_{\text{crys}}(R, R^+) \otimes_{K_0} D(E))^{\varphi=1}$$

given by

$$x \otimes e \mapsto \begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} \cdot (x \otimes 1 \otimes e)$$

for $x \in B_e(R, R^+) = B_{\text{crys}}^{\varphi=1}(R, R^+)$ and $e \in E$. To see that the right side is fixed by φ , just

note that

$$\begin{aligned}
\varphi \left(\begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} \cdot (x \otimes 1 \otimes e) \right) &= F(b)F \left(\begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} \right) F(b^{-1})F(b)(\varphi(x) \otimes \varphi(1) \otimes e) \\
&= \begin{bmatrix} p^{-m} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} ap^mt^m & bp^mt^m \\ 0 & 1 \end{bmatrix} \cdot (x \otimes 1 \otimes e) \\
&= \begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} \cdot (x \otimes 1 \otimes e).
\end{aligned}$$

We thus see that ι defines an element of $\mathcal{M}_{G,b}(C, \mathcal{O}_C)$. That this gives all elements of $\mathcal{M}_{G,b}(C, \mathcal{O}_C)$ follows from the Tannakian formalism, as the collection of isomorphisms between two exact \otimes -functors $\text{Rep}_{\mathbb{Q}_p}(G) \rightarrow \text{Mod}_{B_e}$ is a $G(B_e)$ -torsor if it is nonempty. \square

One sees from the proof of the above proposition and the definition of the period map, that the map $\mathcal{M}_{G,b}(C, \mathcal{O}_C) \rightarrow \text{Gr}_G(C, \mathcal{O}_C) = G(B_{\text{dR}})/G(B_{\text{dR}}^+)$ corresponds to

$$\begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} at^m & bt^m \\ 0 & 1 \end{bmatrix} G(B_{\text{dR}}^+).$$

Example 5.3.5. Let us now specialize to the case where $m = 1$, i.e.

$$b = \begin{bmatrix} p^{-1} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{M}_{G,b}(C, \mathcal{O}_C) = \begin{bmatrix} at & bt \\ 0 & 1 \end{bmatrix}.$$

Let $\mu : \mathbb{G}_m \rightarrow G$ be defined by

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}.$$

By abuse of notation, we shall also use μ to denote the composition $\mathbb{G}_m \rightarrow G \subset \text{GL}_2$. Let $c \in G(B_{\text{dR}}^+) \setminus G(B_{\text{dR}})/G(B_{\text{dR}}^+)$ be the double coset of $\mu(t)$. By Proposition 5.3.2, we see

that

$$\mathrm{Sht}_{G,b,c}(C, \mathcal{O}_C) = \begin{bmatrix} t\mathbb{Q}_p^\times & B_{\mathrm{crys}}^{\varphi=p} \cap B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}.$$

Proposition 5.3.6. *For μ as in 5.3.5, we have*

$$\mathrm{Gr}_{G \subset \mathrm{GL}_{2,\mu}}(C, \mathcal{O}_C) \cong C.$$

Proof. For each $x \in C$ pick a lift \tilde{x} to B_{dR}^+ . Then

$$\begin{bmatrix} t & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} tc & \tilde{x} + td \\ 0 & 1 \end{bmatrix}$$

so we see that

$$\begin{bmatrix} t & \tilde{x} \\ 0 & 1 \end{bmatrix} G(B_{\mathrm{dR}}^+) = \begin{bmatrix} t(B_{\mathrm{dR}}^+)^\times & \tilde{x} + B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}$$

By Proposition 5.3.2, we then conclude the proposition. \square

Example 5.3.7. Let $\mathbb{Z}_p(1)$ be the Tate module of \mathbb{G}_m and $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $\mathcal{G}_m = \mathbb{G}_{m,\mathbb{Z}_p}$ so that its generic fiber is $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{Q}_p}$. By p -adic Hodge theory, we can associate to $\mathbb{Z}_p(1)$ and $\mathbb{Q}_p(1)$ elements of $\mathrm{Sht}_{\mathcal{G}_m,[p-1]}$ and $\mathrm{Sht}_{\mathbb{G}_m,[p-1]}$, respectively. Let us explicitly say what happens for $\mathbb{Q}_p(1)$ here.

Let v be a basis element of $\mathbb{Q}_p(1)$. We have a φ -module over K_0

$$D_{\mathrm{crys}}(\mathbb{Q}_p(1)) := (\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{\mathrm{Gal}(K)} = K_0(v \otimes t^{-1})$$

with φ -action given by $\varphi(x(v \otimes t^{-1})) = \varphi(x)p^{-1}(v \otimes t^{-1})$. This comes naturally with an isomorphism of B_e -modules

$$\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_e \xrightarrow{\cong} (D_{\mathrm{crys}}(\mathbb{Q}_p(1)) \otimes B_{\mathrm{crys}})^{\varphi=1}$$

sending $v \otimes 1$ to $t(v \otimes t^{-1} \otimes 1)$.

Using an argument similar to that of Proposition 5.3.4, one can show that the space $\mathcal{M}_{\mathbb{G}_m, [p^{-1}]}(C, \mathcal{O}_C)$ is $[t]\mathbb{G}_m(B_e) = [tB_e^\times]$. Under this isomorphism, the above computation shows that the element corresponding to $\mathbb{Q}_p(1)$ is $[t]$. Now $X_\bullet(\mathbb{G}_m)^+ = \mathbb{Z}$, with $\alpha \in \mathbb{Z}$ corresponding to the cocharacter $\mu_\alpha : \mathbb{G}_m \rightarrow \mathbb{G}_m$ sending x to x^α , and $\mathrm{Gr}_{\mathbb{G}_m, \mu_\alpha} = t^\alpha B_{\mathrm{dR}}^\times$. Letting 1 be the cocharacter $x \mapsto x$, we thus see that we have an element of $\mathrm{Sht}_{\mathbb{G}_m, [p^{-1}], =1}(C, \mathcal{O}_C)$ associated to $\mathbb{Q}_p(1)$.

Example 5.3.8. Let X denote the pullback

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Sht}_{G, b, c} \\ \downarrow & & \downarrow \\ \mathrm{Spa}(C) & \longrightarrow & \mathrm{Sht}_{\mathbb{G}_m, [p^{-1}], 1} \end{array}$$

where $\mathrm{Spa}(C) \rightarrow \mathrm{Sht}_{\mathbb{G}_m, [p^{-1}], 1}$ is the point corresponding to $\mathbb{Q}_p(1)$ and the vertical map $\mathrm{Sht}_{G, b, c} \rightarrow \mathrm{Sht}_{\mathbb{G}_m, [p^{-1}], 1}$ is gotten by tensoring our G -torsors with the quotient $G/G_u = G_{\mathrm{red}} = \mathbb{G}_m$. At the level of tensor functors, this is precomposing our functor with the natural map $\mathrm{Rep}_{\mathbb{G}_m}(\mathbb{Q}_p) \rightarrow \mathrm{Rep}_G(\mathbb{Q}_p)$ induced from the morphism $G \rightarrow \mathbb{G}_m$. The corresponding map of (C, \mathcal{O}_C) -points is then

$$\begin{bmatrix} ta & b \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} ta \\ \end{bmatrix} \quad \text{where } a \in \mathbb{Q}_p^\times, b \in B_{\mathrm{crys}}^{\varphi=p} \cap B_{\mathrm{dR}}^+.$$

We thus see that

$$X(C, \mathcal{O}_C) = \begin{bmatrix} t & B_{\mathrm{crys}}^{\varphi=p} \cap B_{\mathrm{dR}}^+ \\ 0 & 1 \end{bmatrix}.$$

The period morphism $X(C, \mathcal{O}_C) \rightarrow \mathrm{Gr}_{G, \mu}(C, \mathcal{O}_C)$ can then be identified with the natural quotient map

$$B_{\mathrm{crys}}^{\varphi=p} \cap B_{\mathrm{dR}}^+ \rightarrow C.$$

It is known that $1 + \mathfrak{m}_{C^b} \cong B_{\text{crys}}^{\varphi=p} \cap B_{\text{dR}}^+$ via the map sending

$$x \mapsto \frac{1}{p^\ell} \log([x]^{p^\ell})$$

for ℓ large enough that $[x]^{p^\ell} \equiv 1 \pmod{(p, \xi)}$ where ξ is a primitive element corresponding to the untilt C of C^b . Such an ℓ exists because $x \in 1 + \mathfrak{m}_{C^b}$ and the logarithm of $[x]^{p^\ell}$ is in B_{crys}^+ by definition of this ring as the p -adic completion of the PD hull of $W(\mathcal{O}_{C^b})$ with respect to the ideal (p, ξ) .

Finally, we note that the quotient

$$X(C, \mathcal{O}_C) \Big/ \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}$$

corresponds to $(1 + \mathfrak{m}_{C^b})/\mathbb{Z}_p(1) \cong 1 + \mathfrak{m}_C$, where this isomorphism comes from the characterization of $1 + \mathfrak{m}_C^b$ as $\varprojlim_{x \mapsto x^p} 1 + \mathfrak{m}_C$. Under these isomorphisms, the period morphism from the quotient to $\text{Gr}_{G, \mu}(C, \mathcal{O}_C)$ then becomes identified with the logarithm $\log : 1 + \mathfrak{m}_C \rightarrow C$.

References

- [1] Alexander Beilinson and Vladimir Drinfeld. Quantization of Hitchin’s integrable system and Hecke eigensheaves, 2005.
- [2] Vladimir Berkovich. *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*. Number 33 in Mathematical Surveys and Monographs. American Mathematical Society, 1990.
- [3] Barghav Bhatt and Peter Scholze. Prisms and prismatic cohomology (preprint). Available at <https://arxiv.org/abs/1905.08229> (retrieved July 2020).
- [4] Christopher Birkbeck, Tony Feng, David Hansen, Serin Hong, Anthony Wang, and Lynelle Ye. Extensions of Vector Bundles on the Fargues-Fontaine Curve. *Journal of the Institute of Mathematics of Jussieu*, pages 1–46, 2020.
- [5] Alexis Bouthier and Kestutis Česnavičius. Torsors on loop groups and the hitchin fibration (preprint). Available at <https://arxiv.org/abs/1908.07480> (retrieved July 2020).
- [6] Kevin Buzzard and Alain Verberkmoes. Stably uniform affinoids are sheafy. *Journal für die Reine und Angewandte Mathematik*, 740:25–39, 2018.
- [7] Christophe Cornut. On Harder-Narasimhan filtrations and their compatibility with tensor products. *Confluentes Mathematici*, 10 no 2:3–49, 2018.
- [8] Yi Wen Ding and Yi Ouyang. A simple proof of Dieudonné-Manin classification theorem. *Acta Mathematica Sinica, English Sires*, 28:1553–1558, 2012.
- [9] Gerd Faltings. Almost étale extensions. *Astérisque*, 279:185–270, 2002.
- [10] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de hodge p-adique.
- [11] Jean-Marc Fontaine. Perfectoïdes, presque pureté et monodromie-poids (d’après Peter Scholze). Séminaire Bourbaki 64ème année, 2011-2012, no 1057.
- [12] Jean-Marc Fontaine and Jean-Pierre Wintenberger. Le “corps des normes” de certaines extensions algébriques de corps locaux. *Comptes rendus*, 288, série A:367–370, 1979.
- [13] David Hansen. Degenerating vector bundles in p-adic Hodge theory. *Journal of the Institute of Mathematics of Jussieu*, 2020 (to appear).
- [14] Roland Huber. Continuous valuations. *Mathematische Zeitschrift*, 212:455–477, 1993.
- [15] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Mathematische Zeitschrift*, 217:513–551, 1994.

- [16] Roland Huber. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, volume E 30 of *Aspects of Mathematics*. Springer Fachmedien Wiesbaden, 1996.
- [17] Zonglin Jiang. *Non-archimedean Analysis and GAGA*. PhD thesis, University of California San Diego, 2019.
- [18] Kiran S. Kedlaya. The nonarchimedean Scottish Book. https://scripts.mit.edu/~kedlaya/wiki/index.php?title=The_Nonarchimedean_Scottish_Book (retrieved July 2020).
- [19] Kiran S. Kedlaya. Reified valuations and adic spaces. *Research in Number Theory*, 1:1–42, 2015.
- [20] Kiran S. Kedlaya. Sheaves, stacks and shtukas. In *Perfectoid Spaces: Lectures from the 2017 Arizona Winter School*, volume 242 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2019.
- [21] Kiran S. Kedlaya and Ruochuan Liu. Relative p -adic Hodge Theory, II: Imperfect period rings (preprint). Available at <https://arxiv.org/abs/1601.06899> (retrieved July 2020).
- [22] Kiran S. Kedlaya and Ruochuan Liu. *Relative p -adic Hodge theory: Foundations*, volume 371. Astérisque, 2015.
- [23] Tomoki Mihara. On Tate acyclicity and uniformity of Berkovich spectra and adic spectra. *Israel Journal of Mathematics*, 216:61–105, 2016.
- [24] Peter Scholze. Étale cohomology of Diamonds (preprint). Available at <https://www.math.uni-bonn.de/people/scholze/Publikationen.html> (retrieved July 2020).
- [25] Peter Scholze. Perfectoid spaces. *Publications mathématiques de l’IHÉS*, 116:245–313, 2012.
- [26] Peter Scholze. p -adic Hodge theory for rigid analytic varieties. *Forum of Mathematics, Pi*, 1, 2013, e1.
- [27] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Annals of Mathematics*, 182 Issue 3:945–1066, 2015.
- [28] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p -adic Geometry*, volume 207 of *Annals of Mathematics Studies*. Princeton University Press, 2020.
- [29] Fuchen Tan and Jilong Tong. Crystalline comparison isomorphisms in p -adic hodge theory: the absolutely unramified case. *Algebra Number Theory*, 13(7):1509–1581, 2019.