THE UNIVERSITY OF CHICAGO

THE KERNEL OF THE MONODROMY OF THE UNIVERSAL FAMILY OF DEGREE D SMOOTH PLANE CURVES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
REID HARRIS

CHICAGO, ILLINOIS
JUNE 2020

To Buddy and Patsy

CONTENTS

ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor Benson Farb for his continued support, patience, and encouragement throughout my graduate career as well as his seeming inability to contain his excitement for mathematics. I cannot think of anyone else who has such a contagious enthusiasm for learning and I could not have hoped for a better advisor for the past several years.

I would additionally like to thank Howard Masur for our conversations and for specifically pointing me towards Weil-Petersson Metric and geometric length functions. I also thank Shmuel Weinberger for his mentorship and guidance. I would further like to acknowledge the contributions of Madhav Nori, Scott Wolpert, Tara Brendle, Stephen Bigelow, Nick Salter, and Neil Fullarton for their input and advice during my graduate career.

I would also like to acknowledge and thank Ralph Howard, Jesse Kass, and Anton Schep, for my years as an undergraduate student at the University of South Carolina, and Kana Sriskandarajah, Clyde Smith, Mark Godwin, and Gary Salazar, for my years as a student at South Carolina Governor's School for Science and Mathematics.

ABSTRACT

We consider the parameter space $\mathcal{U}_d$ of smooth plane curves of degree d. The universal smooth plane curve of degree d is a fiber bundle $\mathcal{E}_d \to \mathcal{U}_d$ with fiber diffeomorphic to a surface Σ_g. This bundle gives rise to a monodromy homomorphism $\rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Mod}(\Sigma_g)$, where $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the mapping class group of Σ_g. The main result of this paper is that the kernel of $\rho_4 : \pi_1(\mathcal{U}_4) \to \mathrm{Mod}(\Sigma_3)$ is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is a free group of countably infinite rank. In the process of proving this theorem, we show that the complement $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ of the hyperelliptic locus $\mathcal{H}_g$ in Teichmüller space $\mathrm{Teich}(\Sigma_g)$ has the homotopy type of an infinite wedge of spheres. As a corollary, we obtain that the moduli space of plane quartic curves is aspherical. The proofs use results from the Weil-Petersson geometry of Teichmüller space together with results from algebraic geometry.

CHAPTER 1

INTRODUCTION

Let $\mathbb{P}\left(\mathrm{Sym}^d\left(\mathbb{C}^3\right)\right) = \mathbb{P}^N$, where $N = \binom{d+2}{2} - 1$, be the parameter space of plane curves of degree $d > 0$. Elements of $\mathbb{P}^N$ are homogeneous degree d polynomials in variables x, y, z. Let $\mathcal{U}_d$ denote the *parameter space of smooth plane curves of degree d*. More precisely, $\mathcal{U}_d = \mathbb{P}^N \setminus \Delta_d$ is the complement of the *discriminant locus* $\Delta_d \subset \mathbb{P}^N$ which is the set of polynomials f such that the curve $V(f) = \{p \in \mathbb{P}^2 : f(p) = 0\}$ is singular.

The *universal smooth plane curve of degree d* is the fiber bundle $\mathcal{E}_d \to \mathcal{U}_d$ defined by

$$\mathcal{E}_d := \{(f, p) \in \mathcal{U}_d \times \mathbb{P}^2 : f(p) = 0\} \to \mathcal{U}_d$$
$$(f, p) \mapsto f$$

There exists a monodromy homomorphism

$$\rho_d : \pi_1\left(\mathcal{U}_d\right) \to \mathrm{Mod}(\Sigma_g),$$

where $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the mapping class group. We omit reference to the basepoint in $\pi_1\left(\mathcal{U}_d\right)$, however, it can be taken to be the Fermat curve $f_F(x, y, z) = x^d + y^d + z^d = 0$. The homomorphism ρ_d is called the *geometric monodromy of the universal smooth plane curve of degree d*. A finite presentation for $\pi_1(\mathcal{U}_d)$ has been given by Lönne [Lö9, Main Theorem].

Two natural questions are to determine the image $\mathrm{Im}(\rho_d)$ and kernel $K_d := \ker(\rho_d)$. Dolgachev and Libgober have given a description of $\pi_1(\mathcal{U}_3)$ as an extension

$$0 \to \mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}) \to \pi_1(\mathcal{U}_3) \xrightarrow{\rho_3} \mathrm{Mod}(\Sigma_1) \to 0$$

[DL81, Exact Squence 4.8] of $\mathrm{Mod}(\Sigma_1)$ by the $\mathbb{Z}/3\mathbb{Z}$-points of the 3-dimensional Heisenberg group [DL81, Page 12]

$$\mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{Z}/3\mathbb{Z} \right\}$$

The action $\mathrm{Mod}(\Sigma_1) \circlearrowleft H_1(\mathrm{Heis}_3(\mathbb{Z}/3\mathbb{Z}); \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2$ is the action on the Weierstraß points of the elliptic curve. This action is exactly the composition $\mathrm{Mod}(\Sigma_1) \xrightarrow{\Psi_1} \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, where $\Psi_1 : \mathrm{Mod}(\Sigma_1) \cong \mathrm{SL}_2(\mathbb{Z})$ is the action on $H_1(\Sigma_1; \mathbb{Z})$, see [FM12, Theorem 2.5], and $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is the natural projection.

For higher degrees $d \geq 4$, there is an exact sequence

$$0 \to K_d \to \pi_1(\mathcal{U}_d) \xrightarrow{\rho_d} \mathrm{Mod}(\Sigma_g).$$

The map ρ_d is, in general, not surjective. However, Salter [Sal19, Theorem A] has shown that $\mathrm{Im}(\rho_d)$ always has finite index in $\mathrm{Mod}(\Sigma_g)$. For $d = 4$, Kuno has shown that $\mathrm{Im}(\rho_4) = \mathrm{Mod}(\Sigma_3)$ and that K_4 is infinite [Kun08, Proposition 6.3]. For $d = 5$, Salter [Sal16, Theorem A] shows that $\mathrm{Im}(\rho_5)$ is the stabilizer $\mathrm{Mod}(\Sigma_6)[\phi]$ of a certain spin structure ϕ on Σ_6, the spin structure $\phi = e^*\mathcal{O}(1)$ induced on Σ_6 by its embedding $e : \Sigma_6 \to \mathbb{P}^2$ as a plane curve. For odd $d \geq 5$, Salter shows that the monodromy group $\mathrm{Im}(\rho_d)$ is the stabilizer of a $(d-3)$-spin structure on Σ_g, for $g = \binom{d-1}{2}$. For even $d \geq 6$, $\mathrm{Im}(\rho_d)$ is only known to be finite index in this stabilizer, hence in $\mathrm{Mod}(\Sigma_g)$ [Sal19, Theorem A].

Another result in this vein $\pi_1(\mathcal{U}_d)$ can be found in [CT99]. Recall that $\mathrm{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g; \mathbb{Z})$ preserving the intersection form. This gives rise to the *symplectic representation* $\Psi_g : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. Consider the composition

$$\Psi_g \circ \rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Sp}_{2g}(\mathbb{Z}).$$

This representation is called the *algebraic monodromy of the universal smooth plane curve of degree d*. Carlson and Toledo show that $\tilde{K}_d := \ker(\Psi_g \circ \rho_d)$ is *large* [CT99, Theorem 1.2], i.e. there is a homomorphism $\tilde{K}_d \to G$ to a noncompact semisimple real algebraic Lie group G with Zariski-dense image.

We prove the following theorem, which is a refinement of Kuno's theorem [Kun08, Proposition 6.3] that K_4 is infinite. In the statement, $\mathrm{SMod}(\Sigma_g) < \mathrm{Mod}(\Sigma_g)$ denotes the centralizer of a fixed hyperelliptic involution, the homotopy class of an order 2 homeomorphism $\tau : \Sigma_g \to \Sigma_g$ which acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by -1.

Theorem 1.0.1. *The group K_4 is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is an infinite rank free group. Moreover, F_∞ has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group* $\mathrm{SMod}(\Sigma_3)$*, and*

$$H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$$

as $\mathrm{Mod}(\Sigma_3)$*-modules.*

The idea for the proof of Theorem 1.0.1 is to exhibit the cover $\mathcal{U}_4^{mark} \to \mathcal{U}_4$ corresponding to K_4 as a principal fiber bundle over the complement $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ of the hyperelliptic locus $\mathcal{H}_3$ in Teichmüller space $\mathrm{Teich}(\Sigma_3)$. The following theorem determines the homotopy type of $\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$.

Theorem 1.0.2. *Let $g \geq 3$. The hyperelliptic complement* $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ *has the homotopy type of a wedge $\bigvee_{i=1}^{\infty} S^n$ of infinitely many n-spheres, where $n = 2g - 5$.*

From Theorem 1.0.2, we can conclude that $\mathcal{U}_4^{mark} \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$ is trivial and Theorem 1.0.1 follows.

We will also show that the structure of the group K_d is closely related to that of the hyperelliptic mapping class group. The failure of our proof method in Theorem 1.0.1 for degrees $d > 4$ is due to the lack of knowledge of the topology of the locus of planar curves in the moduli space of Riemann surfaces; there are many more obstructions to being planar than being hyperelliptic.

This thesis is organized as follows. Chapter 2 recalls basic facts about the Weil-Petersson metric on Teichmüller space and the hyperelliptic locus. Chapter 3 introduces the geodesic length functions. These will then be used to prove Theorem 1.0.2. The proof of Theorem 1.0.1 is carried out in chapter 4.

CHAPTER 2

THE HYPERELLIPTIC LOCUS AND THE WEIL-PETERSSON METRIC

For the rest of the paper, let $g \geq 2$ unless otherwise stated. In this chapter we give the necessary background on Teichmüller space and its geometry. We review the Weil-Petersson metric on Teichmüller space and describe the geometric properties of the hyperelliptic locus in terms of this metric, see Proposition 2.3.1.

2.1 Teichmüller Space

We recall the basic theory of Teichmüller space and of the moduli space of Riemann surfaces of genus g. For additional background, see e.g. [FM12]. Let $\mathrm{Teich}(\Sigma_g)$ denote the Teichmüller space of genus $g \geq 2$ curves. That is, $\mathrm{Teich}(\Sigma_g)$ is the set of equivalence classes $[X, h]$ of pairs (X, h), where X is a complex curve of genus g and h is a *marking*, i.e. a homeomorphism $\Sigma_g \to X$. Two pairs (X, h) and (Y, g) are equivalent if $h \circ g^{-1} : Y \to X$ is isotopic to a biholomorphism. We will also denote such an equivalence class $[X, h]$ by $\mathcal{X}$. The (complex) dimension of $\mathrm{Teich}(\Sigma_g)$ is $3g - 3$.

The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on $\mathrm{Teich}(\Sigma_g)$ by

$$[f] \cdot [X, h] = [X, h \circ f^{-1}]$$

where $[f] \in \mathrm{Mod}(\Sigma_g)$. This action is properly discontinuous [FM12, Theorem 12.2] so that the quotient space $\mathcal{M}_g := \mathrm{Mod}(\Sigma_g) \backslash \mathrm{Teich}(\Sigma_g)$, the *moduli space of genus g Riemann surfaces*, is an orbifold. Let $\pi : \mathrm{Teich}(\Sigma_g) \to \mathcal{M}_g$ denote the quotient map. The space $\mathcal{M}_g$ can also be defined as the space of all complex curves of genus g, up to biholomorphism. Note that the orbifold fundamental group $\pi_1^{orb}(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $\mathrm{Mod}(\Sigma_g)$.

2.2 Weil-Petersson Metric

In this section we recall the Weil-Petersson (WP) metric and some of its properties. The WP metric is a certain Kähler metric on $\mathrm{Teich}(\Sigma_g)$ which gives rise to a Riemannian structure on $\mathrm{Teich}(\Sigma_g)$. For more on the Weil-Petersson metric, see the survey [Wol09].

The cotangent space $T^*_{\mathcal{X}}\mathrm{Teich}(\Sigma_g)$ at a point $\mathcal{X} = [X, h] \in \mathrm{Teich}(\Sigma_g)$ can be identified with the space $Q(X)$ of holomorphic quadratic differentials on X. Define a (co)metric on $T^*_{\mathcal{X}}\mathrm{Teich}(\Sigma_g)$ by

$$\langle\langle \varphi, \psi \rangle\rangle := \int_X \varphi\overline{\psi}(ds^2)^{-1},$$

where ds^2 is the hyperbolic metric on X and $(ds^2)^{-1}$ is its dual. The *Weil-Petersson (WP) metric* is defined to be the dual of $\langle\langle \cdot, \cdot \rangle\rangle$.

The WP metric is a $\mathrm{Mod}(\Sigma_g)$-invariant, incomplete [Wol75, Section 2], smooth Riemannian metric of negative sectional curvature [Tro86, Theorem 2]. Teichmüller space $\mathrm{Teich}(\Sigma_g)$ equipped with the WP metric is *geodesically convex* [Wol87, Subsection 5.4], meaning that any two points $\mathcal{X}, \mathcal{Y} \in \mathrm{Teich}(\Sigma_g)$ are connected by a unique geodesic. When referring to any metric properties of Teichmüller space, we will assume they are with respect to the WP metric unless otherwise stated.

2.3 Hyperelliptic Locus

A *hyperelliptic curve* X is a complex curve equipped with a biholomorphic involution $\tau : X \to X$ such that X/τ is isomorphic to $\mathbb{P}^1$. Such a map τ, if it exists, is called a *hyperelliptic involution*. An element $[\tau] \in \mathrm{Mod}(\Sigma_g)$ is called a *hyperelliptic mapping class* if $[\tau]^2 = 1$ and Σ_g/τ is homeomorphic to $\mathbb{P}^1$, or equivalently, if $[\tau]$ acts on $H_1(\Sigma_g; \mathbb{Z})$ by multiplication by -1.

Let $\overline{\mathcal{H}_g} \subset \mathcal{M}_g$ denote the locus of hyperelliptic curves and let $\mathcal{H}_g := \pi^{-1}(\overline{\mathcal{H}_g})$, where $\pi : \mathrm{Teich}(\Sigma_g) \to \mathcal{M}_g$ is the quotient map. The set $\mathcal{H}_g$ is called the *hyperelliptic locus*. It has (complex) dimension $2g-1$. Note that when $g=3$, the hyperelliptic locus $\mathcal{H}_3$ has complex codimension 1 in $\mathrm{Teich}(\Sigma_g)$.

The following proposition collects some facts that will be useful in later sections.

Proposition 2.3.1. *The locus $\mathcal{H}_g$ is a complex-analytic submanifold of $\mathrm{Teich}(\Sigma_g)$. Moreover, $\mathcal{H}_g$ has infinitely many connected components (see Figure 1). If H is any component of $\mathcal{H}_g$ then H is totally geodesic in $\mathrm{Teich}(\Sigma_g)$ and H is biholomorphic to $\mathrm{Teich}(\Sigma_{0,2g+2})$, the Teichmüller space of a sphere with $2g+2$ punctures. In particular, each component of $\mathcal{H}_g$ is contractible.*

Proof. Let $[\tau] \in \mathrm{Mod}(\Sigma_g)$ be a hyperelliptic mapping class. Then $[\tau]$ acts on $\mathrm{Teich}(\Sigma_g)$ with fixed set

$$\mathrm{Fix}([\tau]) := \{[Y,g] \in \mathrm{Teich}(\Sigma_g) : [Y,g] = [Y, g\circ\tau]\}.$$

First, we show that

$$\mathcal{H}_g = \bigcup_{[\tau] \text{ hyperelliptic}} \mathrm{Fix}([\tau]),$$

where the union is taken over all hyperelliptic mapping classes $[\tau] \in \mathrm{Mod}(\Sigma_g)$. If $[X,h] \in \mathrm{Fix}([\tau])$ then $\tau : X \to X$ is isotopic to a biholomorphism τ_b. The map τ_b must be a hyperelliptic involution, and so $[X,h] \in \mathcal{H}_g$. Conversely, if $[X,h] \in \mathcal{H}_g$ then there is a hyperelliptic involution $\tau : X \to X$ which is a biholomorphism and so $[X,h] \in \mathrm{Fix}([\tau])$.

If $[\tau]$ and $[\eta]$ are two distinct hyperelliptic mapping classes, then $\mathrm{Fix}([\tau]) \cap \mathrm{Fix}([\eta]) = \varnothing$. More explicitly, if $[X,h] \in \mathrm{Fix}([\tau]) \cap \mathrm{Fix}([\eta])$ then, $[\tau]$ and $[\eta]$ contain biholomorphic representatives $\tau_b, \eta_b : X \to X$. By [FK80, Section III.7.9, Corollary 2], we must have $\tau_b = \eta_b$.

Each set $\mathrm{Fix}([\tau])$ is totally geodesic in $\mathrm{Teich}(\Sigma_g)$. This follows from the uniqueness of geodesics in the WP metric: if γ is any geodesic with endpoints lying in $\mathrm{Fix}([\tau])$, then $[\tau]\cdot\gamma$ must be another geodesic with the same endpoints as γ, hence γ must be fixed by τ.

For a proof that $\mathcal{H}_g$ is a complex-analytic submanifold of $\mathrm{Teich}(\Sigma_g)$ and that each component is biholomorphic to $\mathrm{Teich}(\Sigma_{0,2g+2})$, we refer the reader to [Nag88, Section

4.1.5]. □

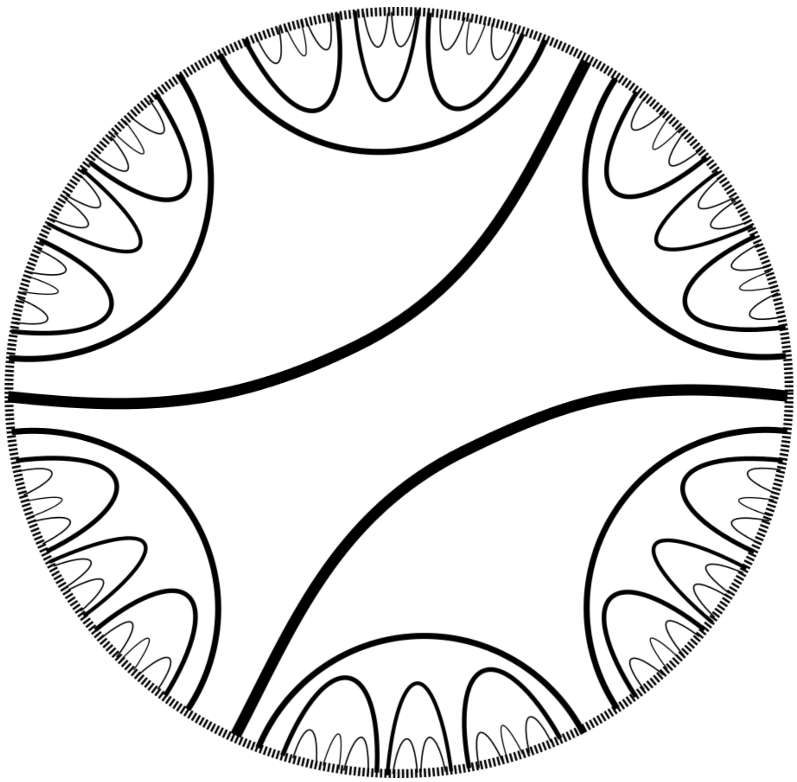

Figure 2.1: A schematic of the hyperelliptic locus $\mathcal{H}_g$ in $\text{Teich}(\Sigma_g)$. The submanifold $\mathcal{H}_g \subset \text{Teich}(\Sigma_g)$ has infinitely many connected components, each of which is totally geodesic with respect to the Weil-Petersson metric.

CHAPTER 3

HOMOTOPY TYPE OF THE HYPERELLIPTIC COMPLEMENT

In Section 3.1, we prove, Lemma 3.1.1, the existence of certain Morse functions on $\mathrm{Teich}(\Sigma_g)$. These functions will be used to prove Theorem 1.0.2 in Section 3.2.

3.1 Geodesic Length Functions

This section is devoted to proving the existence of sufficiently well-behaved functions on $\mathrm{Teich}(\Sigma_g)$.

Lemma 3.1.1. *Let $g \geq 3$. There exists a function $f : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+$ which satisfies the following properties.*

1. *The function f is proper, strictly convex and has positive-definite Hessian everywhere.*

2. *The function f has a unique critical point in $\mathrm{Teich}(\Sigma_g)$, denoted x_0.*

3. *For any component H of $\mathcal{H}_g$, the restriction $f|_H$ has a unique critical point, denoted x_H.*

4. *Any two critical values are distinct. That is, for any component H of $\mathcal{H}_g$, $f(x_H) \neq f(x_0)$. Also, if H' is any other component of $\mathcal{H}_g$, then $f(x_H) = f(x_{H'})$ if and only if $H = H'$.*

5. *The set of critical values*

$$\{f(x_H) : H \text{ is a component of } \mathcal{H}_g\} \cup \{f(x_0)\}$$

is a discrete subset of $\mathbb{R}_+$.

In particular, such a function f is Morse on $\mathrm{Teich}(\Sigma_g)$ *and for each component H of $\mathcal{H}_g$, the restriction $f|_H$ is Morse.*

Proof. The function f is built using *geodesic length functions*. These functions are defined as follows. Let α be a free homotopy class of simple closed curves on Σ_g and let $[X, h]$ be a point in $\mathrm{Teich}(\Sigma_g)$. Then $h(\alpha)$ is a free homotopy class of simple closed curves in X. Recall that $h(\alpha)$ contains a unique geodesic γ. The *geodesic length function* $\ell_\alpha : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+$ associated to α is defined by

$$\ell_\alpha(\mathcal{X}) := \text{length of the unique geodesic in the free homotopy class } h(\alpha) \text{ on } X,$$

where $\mathcal{X} = [X, h]$. Any other choice (X', h') of representative of $[X, h]$ would differ from (X, h) by an isometry, hence ℓ_α is well-defined. Fix a finite collection $\mathcal{A}$ of (homotopy classes of) simple closed curves which fills Σ_g, and let $\mathbf{c} = (c_\alpha) \in \mathbb{R}_+^{\mathcal{A}}$ be a collection of positive real numbers for each $\alpha \in \mathcal{A}$. For each choice of $\mathbf{c} \in \mathbb{R}_+^{\mathcal{A}}$, there is a function

$$\mathcal{L}_{\mathcal{A},\mathbf{c}} := \sum_{\alpha \in \mathcal{A}} c_\alpha \ell_\alpha : \mathrm{Teich}(\Sigma_g) \to \mathbb{R}_+.$$

The function f in the statement of the theorem will be defined to be $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ for a specific value of $\mathbf{c}$.

Wolpert [Wol87, Theorem 4.6] states that for any free homotopy class of simple closed curves α on Σ_g, the geodesic length function ℓ_α has positive-definite Hessian everywhere. In particular, ℓ_α is strictly convex along WP geodesics.

Recall that the Hessian operator Hess is given in local coordinates by

$$f \mapsto \left(\frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j,$$

where Γ_{ij}^k are the Christoffel symbols given by g. Thus, Hess is $\mathbb{R}$-linear. It follows that

$$\text{Hess } \mathcal{L}_{\mathcal{A},\mathbf{c}} = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\text{Hess } \ell_\alpha).$$

For any $v \in T_{\mathcal{X}}\text{Teich}(\Sigma_g)$,

$$\text{Hess } \mathcal{L}_{\mathcal{A},\mathbf{c}}(v,v) = \sum_{\alpha \in \mathcal{A}} c_\alpha \cdot (\text{Hess } \ell_\alpha)(v,v) > 0$$

and so Hess $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is positive-definite. This also shows that $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is strictly convex.

Let $\mathbf{1}$ denote the element of $\mathbb{R}_+^{\mathcal{A}}$ such that $c_\alpha = 1$ for all $\alpha \in \mathcal{A}$. For $\mathbf{c} = (c_\alpha) \in \mathbb{R}_+^{\mathcal{A}}$, let $c_{min} := \min_{\alpha \in \mathcal{A}} c_\alpha$. Then,

$$c_{min}\mathcal{L}_{\mathcal{A},\mathbf{1}} \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}.$$

Kerckhoff [Ker83, Lemma 3.1] states that the functions $\mathcal{L}_{\mathcal{A},\mathbf{1}}$ are proper. If $K = [a,b] \subset \mathbb{R}_+$ is compact, then

$$(\mathcal{L}_{\mathcal{A},\mathbf{c}})^{-1}(K) \subset (\mathcal{L}_{\mathcal{A},\mathbf{1}})^{-1}[0, b/c_{min}],$$

so $(\mathcal{L}_{\mathcal{A},\mathbf{c}})^{-1}(K)$ is a closed subset of a compact set, hence is compact. Thus, $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is proper. This proves (1) in the statement of the theorem.

If $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ has distinct critical points x_0 and x_0' in Teich(Σ_g), then these are local minima of $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ since Hess $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ is positive definite at both x_0 and x_0'. Without loss of generality, assume $\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0') \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0)$. However, by strict convexity, this is impossible. Let γ be the unique geodesic with $\gamma(0) = x_0$ and $\gamma(1) = x_0'$. Then

$$\mathcal{L}_{\mathcal{A},\mathbf{c}}(\gamma(t)) < (1-t)\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0) + t\mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0') \leq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_0)$$

for all $t \in (0,1]$, contradicting the fact that x_0 must be a local minimum. Hence $x_0 = x_0'$ and $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ has a unique critical point in Teich(Σ_g), denoted x_0. This proves property (2).

Since the components of $\mathcal{H}_g$ are totally geodesic in the WP metric, the same argument shows that the restriction $\mathcal{L}_{\mathcal{A},\mathbf{c}}|_H$ will have a unique critical point, denoted x_H, for each component H of $\mathcal{H}_g$. This proves property (3) of the theorem. Thus, properties (1) through (3) of the theorem above are satisfied by the function $\mathcal{L}_{\mathcal{A},\mathbf{c}}$ for any value of $\mathbf{c}$.

Let $S = \{H : H \text{ is a component of } \mathcal{H}_g\} \cup \{0\}$. For each pair $i, j \in S$ of distinct elements, there is an open dense subset $U_{i,j}$ of $\mathbb{R}_+^{\mathcal{A}}$ given by

$$U_{i,j} = \left\{\mathbf{c} \in \mathbb{R}_+^{\mathcal{A}} : \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_i) \neq \mathcal{L}_{\mathcal{A},\mathbf{c}}(x_j)\right\}.$$

By the Baire Category Theorem, $\bigcap_{i\neq j} U_{i,j}$ is open and dense in $\mathbb{R}_+^{\mathcal{A}}$. Let $\mathbf{c}' \in \bigcap_{i\neq j} U_{i,j}$. We now define $f := \mathcal{L}_{\mathcal{A},\mathbf{c}'}$. Then, f satisfies property (4).

Lastly, we wish to show that $f(S)$ is discrete. Choose a neighborhood U_0 of x_0 and U_H of x_H, for each component H of $\mathcal{H}_g$ which are mutually disjoint. Properness of f then implies that $f(S)$ is discrete. This shows that f satisfies property (5). □

3.2 Relative Morse theory of the pair $(\mathrm{Teich}(\Sigma_g), \mathcal{H}_g)$

The goal of this section is to prove Theorem 1.0.2. The idea is that the Morse function f found in Lemma 3.1.1 may be used to determine a handle decomposition of both $\mathcal{H}_g$ and $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. For a reference on relative Morse theory, see e.g. [Sha88, Section 3].

Theorem 1.0.2. *Let $g \geq 3$. The hyperelliptic complement $\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ has the homotopy type of a wedge $\bigvee_{i=1}^{\infty} S^n$ of infinitely many n-spheres, where $n = 2g - 5$.*

Note that since every curve of genus $g = 2$ is hyperelliptic, $\mathrm{Teich}(\Sigma_2) \setminus \mathcal{H}_2 = \varnothing$. The proof of Theorem 1.0.2 is similar to Mess's proof that the image of the period mapping on $\mathrm{Teich}(\Sigma_2)$ has the homotopy type of an infinite wedge of circles [Mes92, Proposition 4]. We now prove Theorem 1.0.2.

Proof. The idea behind relative Morse theory is that such a function as given by Lemma 3.1.1 can be used to determine a handle decomposition not only of $\mathcal{H}_g$, but of its complement $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$. Let f be the function that satisfies the conclusion of Lemma 3.1.1. We let x_0 denote the unique minimum point of f in $\text{Teich}(\Sigma_g)$. For each component H of $\mathcal{H}_g$, let x_H denote the unique critical point of $f|_H$. We refer to x_0 as a *critical point of f of type I* and each x_H are referred to as *critical points of f of type II*. The values $c_0 = f(x_0)$ and $c_H = f(x_H)$ are called *critical values of type I* and *II*, respectively.

For r a real number, let $X_r := \{\mathcal{X} \in \text{Teich}(\Sigma_g) : f(\mathcal{X}) \leq r\}$. If $(c_0, c_0 + \epsilon]$ contains no type II critical values, then $X_{c_0+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to a 0-handle, i.e. a closed ball. Consider an arbitrary interval $[a, b] \subset \mathbb{R}$. If $[a, b]$ contains no critical value of type I or II of f, then $X_a \setminus \mathcal{H}_g$ is diffeomorphic to $X_b \setminus \mathcal{H}_g$. To see this, we can construct a vector field V which is equal to $\text{grad}(f)$ outside a neighborhood of $\mathcal{H}_g$ and such that $V|_{\mathcal{H}_g}$ is equal to $\text{grad}(f|_{\mathcal{H}_g})$. The flow along this vector field gives the required diffeomorphism.

Let x be a critical point of type II, and let $c = f(x)$. By Lemma 3.1.1, the set of critical values of f is discrete, so there exists some $\epsilon > 0$ such that $[c - \epsilon, c + \epsilon]$ contains no other critical values of f. We wish to show that $X_{c+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to $X_{c-\epsilon} \setminus \mathcal{H}_g$ with an n-handle attached, where $n = 2g - 5$ (see Figure 2).

Let H be the component of $\mathcal{H}_g$ containing x. There exists a coordinate system $(u, y) \in \mathbb{R}^{2g-4} \times \mathbb{R}^{4g-2}$ in a neighborhood U of x such that [Sha88, 3.3]

1. $U \cap H$ is given by $u = 0$,
2. $f = c + \|y\|^2$ on $U \cap H$.

The coordinates y are "tangent" coordinates to H and the coordinates u are "normal" coordinates to H. Note that since H has complex dimension $2g - 1$, it has real dimension $4g - 2$.

Then, $X_{c+\epsilon} \setminus \mathcal{H}_g$ is diffeomorphic to the union of $X_{c-\epsilon} \setminus \mathcal{H}_g$ and a tubular neighborhood

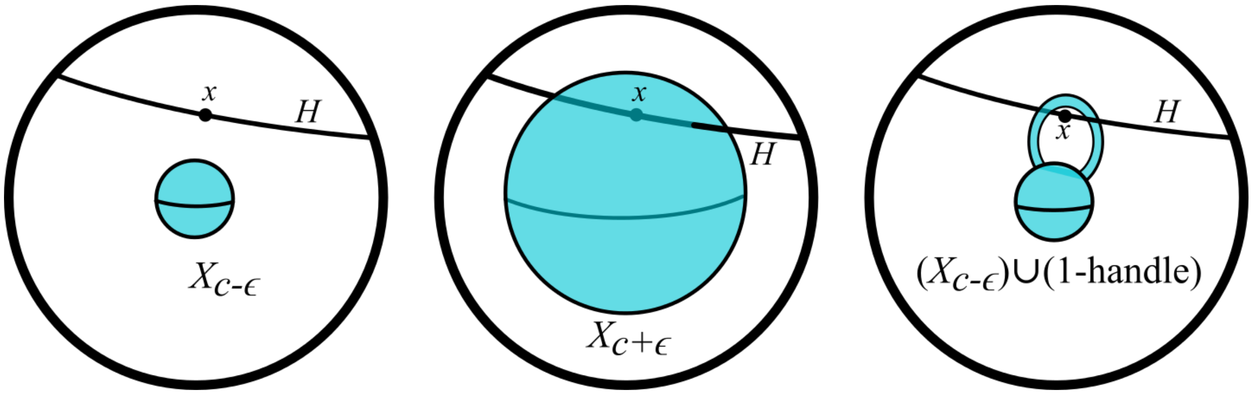

Figure 3.1: Start with $X_{c-\epsilon}$. As $c-\epsilon$ increases to $c+\epsilon$, the level set $X_{c+\epsilon}$ intersects exactly one more component H of $\mathcal{H}_g$, the component containing the critical point x. Here, the $g=3$ case is depicted.

of

$$\{(u,0) : \|u\|^2 = \delta\},$$

for some small $\delta > 0$. This tubular neighborhood deformation retracts to the $(2g-5)$-sphere $\{(u,0) : \|u\|^2 = \delta\}$. Hence, $\text{Teich}(\Sigma_g) \setminus \mathcal{H}_g$ has a handle decomposition consisting of a 0-handle with infinitely many (one for each component of $\mathcal{H}_g$) n-handles attached, where $n = 2g-5$. □

Let $\mathcal{M}_g^{nhyp}$ denote the moduli space of hyperelliptic curves of genus g. Since $\text{Teich}(\Sigma_3)\setminus\mathcal{H}_3$ is a covering space for $\mathcal{M}_3^{nhyp}$, the moduli space $\mathcal{M}_3^{nhyp}$ has contractible universal cover and $\mathcal{M}_3^{nhyp}$ is aspherical. If $g \geq 4$ then $\pi_n(\mathcal{M}_g^{nhyp})$, where $n = 2g-5 > 1$, is an infinite rank abelian group. In particular, $\mathcal{M}_g^{nhyp}$ is not aspherical for $g \geq 4$.

We can be even more precise. The components of the hyperelliptic locus $\mathcal{H}_g$ are enumerated by the set of cosets of the group $\text{SMod}(\Sigma_g)$ in $\text{Mod}(\Sigma_g)$. Recall that $\text{SMod}(\Sigma_g)$ is the centralizer in $\text{Mod}(\Sigma_g)$ of a fixed hyperelliptic involution $\tau \in \text{Mod}(\Sigma_g)$. The group $\text{SMod}(\Sigma_g)$ is called the *hyperelliptic mapping class group of genus g*. If η is another hyperelliptic involution, then the centralizers of τ and η are conjugate in $\text{Mod}(\Sigma_g)$.

Corollary 3.2.1. *Let $g \geq 3$. There is a homotopy equivalence*

$$\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g \cong \bigvee_{[h] \in \mathrm{Mod}(\Sigma_g)/\mathrm{SMod}(\Sigma_g)} S^{2g-5}.$$

In particular,

$$H_{2g-5}(\mathrm{Teich}(\Sigma_g) \setminus \mathcal{H}_g; \mathbb{Z}) \cong \mathbb{Z}[\mathrm{Mod}(\Sigma_g)/\mathrm{SMod}(\Sigma_g)]$$

as $\mathrm{Mod}(\Sigma_g)$*-modules.*

Proof. The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on the set of components of $\mathcal{H}_g$ by permutations. Let H_0 be a fixed component of $\mathcal{H}_g$. Then, there is a map

$$\begin{aligned} \mathrm{Orb}(H_0) &\to \mathrm{Mod}(\Sigma_g)/\mathrm{Stab}(H_0) \\ h \cdot H_0 &\mapsto h\mathrm{Stab}(H_0) \end{aligned}$$

from the orbit $\mathrm{Orb}(H_0)$ of H_0 to the left coset space of the stabilizer $\mathrm{Stab}(H_0)$. It suffices to show that $\mathrm{Stab}(H_0) = \mathrm{SMod}(\Sigma_g)$ and $\mathrm{Mod}(\Sigma_g)$ acts transitively on the set of components of $\mathcal{H}_g$.

First, since $H_0 = \mathrm{Fix}(\tau)$, the mapping class $h \in \mathrm{Stab}(H_0)$ if and only if

$$h \cdot \mathrm{Fix}(\tau) = \mathrm{Fix}(h\tau h^{-1}) = \mathrm{Fix}(\tau).$$

Since no hyperelliptic curve can have two distinct hyperelliptic involutions, it must follow that $h\tau h^{-1} = \tau$ so $h \in \mathrm{SMod}(\Sigma_g)$. Therefore, $\mathrm{Stab}(H_0) = \mathrm{SMod}(\Sigma_g)$.

Secondly, if H is any other component of $\mathcal{H}_g$, then $H = \mathrm{Fix}(\eta)$ for some hyperelliptic involution $\eta \in \mathrm{Mod}(\Sigma_g)$. Since hyperelliptic involutions in $\mathrm{Mod}(\Sigma_g)$ are conjugate, there exists some $h \in \mathrm{SMod}(\Sigma_g)$ such that

$$H = \mathrm{Fix}(\eta) = \mathrm{Fix}(h\tau h^{-1}) = h \cdot \mathrm{Fix}(\tau) = h \cdot H_0.$$

Therefore, $\mathrm{Mod}(\Sigma_g)$ acts transitively on the set of components of $\mathcal{H}_g$. $\square$

CHAPTER 4

THE PARAMETER SPACE OF SMOOTH PLANE CURVES

In this chapter, we prove Proposition 4.1.2, showing that the cover of $\mathcal{U}_d$ determined by the subgroup K_d of $\pi_1(\mathcal{U}_d)$ carries the structure of a principal fiber bundle. This will be critical in the proof of Theorem 1.0.1 in Section 4.2.

4.1 Covers of $\mathcal{U}_d$ and principal fiber bundles

The main result of this section is to prove Proposition 4.1.2, exhibiting a cover of $\mathcal{U}_d$ as a principal fiber bundle over a certain subspace of $\mathrm{Teich}(\Sigma_g)$.

Associating each point of $\mathcal{U}_d$ to the curve it determines gives rise to a map $\varphi_d : \mathcal{U}_d \to \mathcal{M}_g$ into the moduli space of Riemann surfaces of genus $g(d)$, where $g = g(d) := \binom{d-1}{2}$ by the degree-genus formula. Let $\mathcal{M}_g^{pl}$ denote the image of this map. For $d \geq 4$, the locus $\mathcal{M}_g^{pl} \subsetneq \mathcal{M}_g$ and for $d = 3$, $\mathcal{M}_1^{pl} = \mathcal{M}_1$.

There is a (disconnected) covering $\mathcal{U}_d^{mark}$ of $\mathcal{U}_d$ defined as follows. A point $(f, [h]) \in \mathcal{U}_d^{mark}$ is an ordered pair consiting of $f \in \mathcal{U}_d$ and a homotopy class $[h]$ of orientation-preserving homeomorphisms $h : \Sigma_g \to V(f)$ of some fixed Σ_g with the complex curve $V(f)$ given by $f(x, y, z) = 0$.

Let $\pi_1(\mathcal{U}_d^{mark})$ be the fundamental group of a chosen component of $\mathcal{U}_d^{mark}$. Note that $\pi_1(\mathcal{U}_d^{mark}) \cong K_d$.

Remark 4.1.1. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_d^{mark} & \xrightarrow{\tilde{\varphi}_d} & \mathrm{Teich}(\Sigma_g) \\ \downarrow & & \downarrow \pi \\ \mathcal{U}_d & \xrightarrow[\varphi_d]{} & \mathcal{M}_g \end{array}$$

The map $\varphi_d : \mathcal{U}_d \to \mathcal{M}_g$ lifts to a map $\tilde{\varphi}_d : \mathcal{U}_d^{mark} \to \mathrm{Teich}(\Sigma_g)$ into Teichmüller space defined by

$$\varphi_d : (f, [h]) \mapsto [V(f), h].$$

Let $\mathrm{Teich}(\Sigma_g)^{pl}$ denote the image of φ_d.

Recall that a *principal G-bundle* is a fiber bundle $P \to X$ with a G-action that acts freely and transitively on the fibers.

Proposition 4.1.2. *For $d \geq 4$, the map $\tilde{\varphi}_d : \mathcal{U}_d^{mark} \to \mathrm{Teich}(\Sigma_g)^{pl}$ is a principal $\mathrm{PGL}_3(\mathbb{C})$-bundle.*

Proof. First, $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathcal{U}_d^{mark}$ by $g \cdot (f, [h]) = (g \cdot f, [g \circ h])$ where $g \cdot f$ denotes the action of g on polynomials $f(x, y, z)$, by acting on the triple of variables (x, y, z). This induces a map $g : V(f) \to V(g \cdot f)$ and $g \circ h$ is the composition of this map with the marking $h : \Sigma_g \to V(f)$.

This action is free. Indeed, if $g \cdot (f, [h]) = (f, [h])$ then $g \cdot f = f$ and $[g \circ h] = [h]$. Thus g induces an automorphism on the curve $V(f)$. Moreover, this automorphism acts trivially on the marking, hence trivially on $H_1(V(f); \mathbb{Z})$. An automorphism of $V(f)$ acting trivially on homology must be the identity [FM12, Theorem 6.8]. The fixed set of any automorphism of $\mathbb{P}^2$ is a linear subspace, so any $g \in \mathrm{PGL}_3(\mathbb{C})$ point-wise fixing a smooth quartic curve must be the identity automorphism.

Next, we show that this action is transitive on fibers. It suffices to show that if $\tilde{\varphi}_d(f_1, [h_1]) = \tilde{\varphi}_d(f_2, [h_2])$, then the $(f_i, [h_i])$ lie in the same $\mathrm{PGL}_3(\mathbb{C})$-orbit. By assumption, $[V(f_1), h_1] = [V(f_2), h_2]$ and there is some biholomorphism $\psi : V(f_1) \to V(f_2)$ such that $[\psi \circ h_1] = [h_2]$. Then the pullback of the hyperplane bundle H along the embeddings $e_i : V(f_i) \to \mathbb{P}^2$ gives line bundles $L_i := e_i^*(H)$ on $V(f_i)$ of degree d with $h^0(L_i) = 3$.

A g_d^r *line bundle* is a line bundle $L \to C$ such that $\deg(L) = d$ and $h^0(L) \geq r + 1$. Smooth plane curves have a unique g_d^2 given by the pullback of the hyperplane bundle [Ser87, Theorem

3.13]. Therefore, L_1 and $\psi^* L_2$ are isomorphic line bundles on $V(f_1)$.

For any smooth curve C, there is a correspondence between maps $C \to \mathbb{P}^r$ up to the action of $\mathrm{PGL}_{r+1}(\mathbb{C})$ and pairs (L, V) where L is a line bundle over C and $V \subset H^0(C; L)$ is an $(r+1)$-dimensional subspace. The fact that there is a unique line bundle L on $V(f_1)$ with $h^0(L) \geq 3$ implies that there is only one such map $V(f_1) \to \mathbb{P}^2$ up to the action of $\mathrm{PGL}_3(\mathbb{C})$. Therefore, the two embeddings e_1 and $e_2 \circ \psi$ are equivalent up to the action of $\mathrm{PGL}_2(\mathbb{C})$, i.e. there is some $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $g \circ e_1 = e_2 \circ \psi$. This implies that $g \cdot f_1 = f_2$ and $g : V(f_1) \to V(f_2)$ coincides with ψ. Thus, $(f_1, [h_1])$ and $(f_2, [h_2])$ lie in the same $\mathrm{PGL}_3(\mathbb{C})$-orbit.

Finally, it remains to prove local triviality. This is a consequence of a much more general fact that if G acts on a manifold P freely such that P/G is a manifold, then $q : P \to P/G$ is locally trivial. Indeed, a local trivialization of $q : P \to P/G$ can be built over any contractible subset U by first taking a section $\sigma : U \to P$ and defining $\varphi : q^{-1}(U) \to U \times G$ by $\varphi(x) = (q(x), g(x))$, where $g(x) \in G$ is the unique element such that $x = g(x) \cdot \sigma(q(x))$. □

Proposition 4.1.3. *Let $d \geq 3$ and $g = \binom{d-1}{2}$. The space $\mathcal{U}_d^{mark}$ has finitely many components. Consequently,* $\mathrm{Teich}(\Sigma_g)^{pl}$ *has finitely many components.*

Proof. A single component of $\mathcal{U}_d^{mark}$ is the connected covering space of $\mathcal{U}_d$ corresponding to K_d. Hence, its deck transformation group is the image of the homomorphism $\rho_d : \pi_1(\mathcal{U}_d) \to \mathrm{Mod}(\Sigma_g)$. The components of $\mathcal{U}_d^{mark}$ are enumerated by the cosets of $\mathrm{Im}(\rho_d)$ in $\mathrm{Mod}(\Sigma_g)$. It was shown in and [Sal19, Theorem A] that the index $[\mathrm{Mod}(\Sigma_g) : \mathrm{Im}(\rho_d)] < \infty$. □

4.2 The kernel of the geometric monodromy of the universal quartic

In this section, we prove Theorem 1.0.1.

Theorem 1.0.1. *The group K_4 is isomorphic to $F_\infty \times \mathbb{Z}/3\mathbb{Z}$, where F_∞ is an infinite rank* *free group. Moreover,* F_∞ *has a free generating set in bijection with the set of cosets of the hyperelliptic mapping class group* $\mathrm{SMod}(\Sigma_3)$*, and*

$$H_1(K_4; \mathbb{Q}) \cong \mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$$

as $\mathrm{Mod}(\Sigma_3)$*-modules.*

Proof of Theorem 1.0.1. Classically, $\mathrm{Teich}(\Sigma_3)^{pl}$ is exactly the complement of the hyperelliptic locus $\mathcal{H}_3$ in $\mathrm{Teich}(\Sigma_3)$: the canonical map $C \to \mathbb{P}^2$ is an embedding precisely when C is nonhyperelliptic [GH94, Pages 246-7]. Consider the following principal fiber bundle.

$$\begin{array}{ccc} \mathrm{PGL}_3(\mathbb{C}) & \longrightarrow & \mathcal{U}_4^{mark} \\ & & \Big\downarrow \varphi_4 \\ & & \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \end{array}$$

Because $\rho_4 : \pi_1(\mathcal{U}_4) \to \mathrm{Mod}(\Sigma_3)$ is surjective [Kun08, Proposition 6.3], $\mathcal{U}_4^{mark}$ is connected.

By Theorem 1.0.2, $\mathrm{Teich}(\Sigma_3)\setminus\mathcal{H}_3$ is homotopy equivalent to an infinite wedge of circles and, since $\mathrm{PGL}_3(\mathbb{C})$ is connected, there must exist some continuous section $\sigma : \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3 \to \mathcal{U}_4^{mark}$. Because φ_4 is a principal $\mathrm{PGL}_3(\mathbb{C})$-bundle, the existence of such a section implies that $\mathcal{U}_4^{mark}$ is homeomorphic to $\mathrm{PGL}_3(\mathbb{C}) \times (\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3)$, and so

$$\pi_i(\mathcal{U}_4^{mark}) = \begin{cases} \mathbb{Z}/3\mathbb{Z} \times F_\infty, & \text{for } i = 1 \\ \pi_i(\mathrm{PGL}_3(\mathbb{C})), & \text{for } i > 1. \end{cases} \tag{4.2.1}$$

This also shows that $\pi_i(\mathcal{U}_4) \cong \pi_i(\mathrm{PGL}_3(\mathbb{C}))$ for $i \geq 2$.

We now wish to show that $H_1(K_4; \mathbb{Q})$ is isomorphic to $\mathbb{Q}\,[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$ as $\mathrm{Mod}(\Sigma_3)$-modules. The calculation of $K_4 \cong \pi_1(\mathcal{U}_4^{mark})$ in equation 4.2.1 shows that the projection

$$\mathcal{U}_4^{mark} \xrightarrow{\cong} \mathrm{PGL}_3(\mathbb{C}) \times (\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3) \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3$$

induces an isomorphism

$$H_1(K_4; \mathbb{Q}) \cong H_1(\mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3; \mathbb{Q}).$$

The action of $\mathrm{Mod}(\Sigma_3)$ on $\mathcal{U}_4^{mark}$ commutes with the projection map

$$\mathcal{U}_4^{mark} \to \mathrm{Teich}(\Sigma_3) \setminus \mathcal{H}_3,$$

so that the above isomorphism of $\mathbb{Q}$-vector spaces is an isomorphism of $\mathrm{Mod}(\Sigma_3)$-modules.

The group $H_1(\mathrm{Teich}(\Sigma_3)\setminus\mathcal{H}_3; \mathbb{Z})$ is the free abelian group on the set of cycles in $\mathrm{Teich}(\Sigma_3)\setminus \mathcal{H}_3$ represented by meridians around the components of the hyperelliptic locus $\mathcal{H}_3$; that is, the boundaries of disks transversely intersecting $\mathcal{H}_3$ in a single point. Such cycles are in bijection with the cosets of $\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)$ (see proof of Corollary 3.2.1). This bijection commutes with the action of $\mathrm{Mod}(\Sigma_3)$ and therefore this $\mathrm{Mod}(\Sigma_3)$-module is isomorphic to the permutation representation $\mathbb{Q}[\mathrm{Mod}(\Sigma_3)/\mathrm{SMod}(\Sigma_3)]$. □

The following table shows $\pi_i(\mathcal{U}_4) \cong \pi_i(\mathrm{PGL}_3(\mathbb{C}))$ for small values of $i \geq 2$ (c.f. [MT64, Introduction], where we have used the fact that $\mathrm{SL}_3(\mathbb{C})$ covers $\mathrm{PGL}_3(\mathbb{C})$ and is homotopy equivalent to $\mathrm{SU}(3)$).

i	2	3	4	5	6	7	8	9	10	11	12
$\pi_i(\mathcal{U}_4)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	0	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/30\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/60\mathbb{Z}$

BIBLIOGRAPHY

[CT99] James A. Carlson and Domingo Toledo. Discriminant complements and kernels of monodromy representations. *Duke Math. J.*, 97(3):621–648, 1999.

[DL81] Igor Dolgachev and Anatoly Libgober. On the fundamental group of the complement to a discriminant variety. In *Algebraic geometry (Chicago, Ill., 1980)*, volume 862 of *Lecture Notes in Math.*, pages 1–25. Springer, Berlin-New York, 1981.

[FK80] Hershel M. Farkas and Irwin Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1980.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Ker83] Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[Kun08] Yusuke Kuno. The mapping class group and the Meyer function for plane curves. *Math. Ann.*, 342(4):923–949, 2008.

[LÖ9] Michael Lönne. Fundamental groups of projective discriminant complements. *Duke Math. J.*, 150(2):357–405, 2009.

[Mes92] Geoffrey Mess. The Torelli groups for genus 2 and 3 surfaces. *Topology*, 31(4):775–790, 1992.

[MT64] Mamoru Mimura and Hirosi Toda. Homotopy groups of SU(3), SU(4) and Sp(2). *J. Math. Kyoto Univ.*, 3:217–250, 1963/64.

[Nag88] Subhashis Nag. *The complex analytic theory of Teichmüller spaces*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1988. A Wiley-Interscience Publication.

[Sal16] Nick Salter. On the monodromy group of the family of smooth plane curves. *arXiv e-prints*, page arXiv:1610.04920, Oct 2016.

[Sal19] Nick Salter. Monodromy and vanishing cycles in toric surfaces. *Invent. Math.*, 216(1):153–213, 2019.

[Ser87] Fernando Serrano. Extension of morphisms defined on a divisor. *Math. Ann.*, 277(3):395–413, 1987.

[Sha88] R. W. Sharpe. Total absolute curvature and embedded Morse numbers. *J. Differential Geom.*, 28(1):59–92, 1988.

[Tro86] A. J. Tromba. On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric. *Manuscripta Math.*, 56(4):475–497, 1986.

[Wol75] Scott Wolpert. Noncompleteness of the Weil-Petersson metric for Teichmüller space. *Pacific J. Math.*, 61(2):573–577, 1975.

[Wol87] Scott A. Wolpert. Geodesic length functions and the Nielsen problem. *J. Differential Geom.*, 25(2):275–296, 1987.

[Wol09] Scott A. Wolpert. The Weil-Petersson metric geometry. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 47–64. Eur. Math. Soc., Zürich, 2009.