

THE UNIVERSITY OF CHICAGO

HECKE OPERATORS AND GALOIS SYMMETRY IN RATIONAL CONFORMAL FIELD
THEORY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

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CHICAGO, ILLINOIS

JUNE 2020

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This thesis is dedicated to the people who inspired and supported me throughout my education.

It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power.

— Paul Dirac

TABLE OF CONTENTS

LIST OF FIGURES	vii
LIST OF TABLES	viii
ACKNOWLEDGMENTS	ix
ABSTRACT	x
1 INTRODUCTION	1
2 CONFORMAL FIELD THEORY IN A NUTSHELL	4
2.1 Conformal symmetry	4
2.2 Extended algebras	12
3 HECKE OPERATORS AND GALOIS SYMMETRY	16
3.1 Simple current	18
3.2 Hecke operators for $\Gamma(N)$	20
3.3 Galois permutations	26
3.4 Applications and examples	32
3.5 Bilinear relations	38
4 GALOIS SYMMETRY ON MTCS	41
4.1 Simple-current reduction of affine algebra	42
4.1.1 Rank three	43
4.1.2 Rank four	46
4.2 MTCs of higher rank	50
5 PICTURE OF EFFECTIVE CENTRAL CHARGE	55
5.1 Unified method	55
5.2 Fusion rules of Hecke image	60
5.3 $M(3, 5)$ as an example	62
6 DUALITY TRANSFORMATION OF CONFORMAL BLOCKS	69
6.1 Chiral vertex operators and conformal blocks	69
6.2 Fusion and braiding symmetries	70
7 DUALITY MATRICES AND GALOIS SYMMETRY	77
7.1 Fusion Matrices	77
7.1.1 Analytical results in two-channel fusion	77
7.1.2 Galois symmetry in fusion matrices	81
7.2 Braiding Matrices	86
8 CONCLUSIONS AND SUMMARY	95

A	MODULAR LINEAR DIFFERENTIAL EQUATIONS	97
B	GALOIS SYMMETRY INTERPOLATED MODULAR REPRESENTATIONS	100
C	DERIVATION OF MTC DATA	104
D	HECKE OPERATION AND GALOIS ACTION ON MINIMAL MODELS	107
	D.1 Overview of minimal models	107
	D.2 Some examples	108
	D.2.1 Minimal model $M(2, 9)$	108
	D.2.2 Minimal model $M(2, 11)$	111
	D.2.3 Minimal model $M(2, 13)$	114
	D.2.4 Minimal model $M(2, 15)$	118
	D.2.5 Minimal model $M(2, 17)$	123
	D.2.6 Minimal model $M(2, 19)$	131
	D.2.7 Minimal model $M(3, 7)$	136
	D.2.8 Minimal model $M(3, 8)$	139
	REFERENCES	143

LIST OF FIGURES

3.1	Fundamental domain of $SL(2, \mathbb{Z})$	17
6.1	A geometric illustration of fusion, as the composition of two 4-point functions.	70
6.2	Fusion matrix between blocks. The labels of the matrix entries, i.e. p and q , take the positions of the “propagator”.	72
6.3	Braiding matrix between blocks. The labeling of the matrix entries, i.e. p and q , take the positions of the “propagator”.	73
6.4	A simple loop transformation of conformal block.	73
7.1	Braiding eigenvalues.	87

LIST OF TABLES

3.1	This table summarizes some data for a few minimal models. N is the conductor of the theory, which is also equal to the order of $\rho(T)$. $n = \mathcal{I} $ denotes the number of primary fields. \mathcal{G} stands for the Galois group of fusion rules. We put an asterisk on $M(2, 15)$, where \mathcal{G} does not agree with C_2 because $\rho(T)$ has degenerate eigenvalues for fields that are not complex conjugates.	31
7.1	This table summarizes some data for RCFTs derived from the $M(3, 5)$ MTC. As always, the subscripts of primary fields denote the conformal weights. The field content is also evaluated in the effective picture of $M(3, 5)$, so are the fusion rules. The approach of effective central charge is essentially the Galois conjugation plus an additional simple current permutation. The data in MTC, including F - and B -matrices, embody the Galois symmetry as well.	94

ACKNOWLEDGMENTS

I appreciate my advisor Prof. Jeffrey Harvey for kindly guiding and supporting me along the PhD journey. Without him, I would not have been exposed to the realm of string theory and conformal field theory, and not have made a contribution to the knowledge of mankind. I would like to thank Prof. Edward Blucher, Prof. Michael Levin and Prof. Emil Martinec for serving the thesis committee and offering valuable feedbacks. I am grateful to my parents Fumin Wu and Caifang Wu, who cultivate me to be a person of integrity and give me endless love while apart. I also thank my graduate fellows, especially Nelson Leung and Xining Zhang, who endorse me during this precious period. Finally I appreciate Sydney Oswald for the very best of times.

ABSTRACT

We define Hecke operators on vector-valued modular forms of the type that appear as characters of rational conformal field theories (RCFTs). We apply our results to derive a number of relations between characters of known RCFTs with different central charges, and extend the previously studied Galois symmetry of modular representations and fusion algebras. We show that the conductor N of a RCFT and the quadratic residues modulo N play an important role in the computation and classification of Galois permutations. We establish a field correspondence in different theories through the picture of effective central charge, which combines Galois inner automorphisms and the structure of simple currents. We then make a first attempt to extend Hecke operators to the full data of modular tensor categories. The Galois symmetry encountered in the modular data transforms the fusion and the braiding matrices as well, and yields isomorphic structures in theories related by Hecke operators.

CHAPTER 1

INTRODUCTION

Two-dimensional rational conformal field theories (RCFTs) have found applications in the world-sheet description of classical string theory backgrounds, as well as in many areas in condensed matter physics such as quantum Hall systems and the study of boundary modes in topological insulators. The characters of RCFT are partition functions on the torus, and record the number of physical states. Because of the modular properties under the action of $SL(2, \mathbb{Z})$, the characters are also modular functions and thus also encode fascinating number theoretic features.

We discover that Hecke operators relate characters of certain RCFTs with different central charges [1]. Hecke operators and their eigenfunctions play a crucial role in the mathematical theory of modular forms, thus it is natural to expect that they should play a role in RCFT as well. In math literature, standard Hecke operators map modular forms of $SL(2, \mathbb{Z})$ to modular forms of the same weights, and thus preserve modularity. While Hecke operators for congruence subgroups are probably less familiar to physicists. References we found useful include [56, 19, 57]. These Hecke operators extend the known Galois symmetry connecting modular representations. They act on vector-valued modular functions which may be characters of one RCFT and often produce characters of another RCFT which is not obviously related to the original RCFT. Two RCFTs whose characters are related by Hecke operators are clearly not the same RCFT since they have different central charges, but it could be that some algebraic structure related to the two RCFTs is the same. In particular we will provide evidence that the modular tensor categories (MTCs) related to the two RCFTs are either the same or closely related. We explore this possibility in a number of simple cases in this thesis.

MTCs arise as representation categories and encode the topological structures of vertex operator algebras (VOAs) and CFTs [3, 43, 44, 54]. Extensive applications of MTC are found in condensed matter physics, where they offer tools for studying anyonic systems and topological quantum computation [29, 30, 31, 33].

Our first main result consists of a purely mathematical statement about a particular class of

Hecke operators that act on vector-valued weakly holomorphic modular functions of a particular type. This definition can obviously be extended to non-zero weight, but for simplicity we restrict to weight zero since this is the case arising in RCFT. We define a level N vector-valued modular function f with respect to a representation ρ to be a set $f_i(\tau)$, $i = 1, \dots, n$ of weakly holomorphic modular functions for $\Gamma(N)$ which transform under $SL(2, \mathbb{Z})$ as

$$f_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij} f_j(\tau) \quad (1.1)$$

where ρ is an n -dimensional unitary representation

$$\rho : SL(2, \mathbb{Z}) \rightarrow GL(V(\rho)) \quad (1.2)$$

with V a complex vector space of finite dimension n and where N is the finite order of $\rho(T)$ and $\Gamma(N)$ is in the kernel of ρ . Here

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (1.3)$$

are the generators of $SL(2, \mathbb{Z})$. All characters of RCFTs are level N modular functions for some finite-dimensional representation ρ by a theorem of Bantay [2], but the converse is not true. If f is a level N modular function with respect to the representation ρ , then for each p such that $\gcd(p, N) = 1$ there are Hecke operators T_p such that $(T_p f)_i$ are the components of a level N modular function with respect to the representation $\rho^{(p)}$. The modular representation $\rho^{(p)}$ is derived, and is related to ρ via Galois symmetry.

Our second main result concerns the structure of the Galois permutations induced by Hecke operators. All the Galois information is traced back to the conductor N , and the unit group $(\mathbb{Z}/N\mathbb{Z})^\times$ can be represented by the Frobenius maps. Upon the Hecke operation, a Frobenius map acts on the modular representation. However the permutations are characterized by the quadratic residues

in $(\mathbb{Z}/N\mathbb{Z})^\times$, in other words the quadratic subgroup determines the Galois group of fusion rules. With effective central charges less than 1, the Virasoro minimal models are nice candidates to probe the Hecke images of the characters, as well as Galois conjugates of modular representations. A number of examples are presented in Chapter 3. We note that the modular representation of the minimal model $M(2, k + 2)$ coincides with the $(-k)$ -th Galois conjugate of $SU(2)_k$. Consequently, $M(2, k + 2)$ has identical fusion rules as the integral spins in $SU(2)_k$, which explains this observation in condensed matter physics.

The final main result is that the RCFTs related by Hecke operators embody Galois symmetry in their fusion and braiding matrices. Given a RCFT and its MTC, the Hecke image of the characters gives rise to a Galois conjugate of the initial MTC. As the structure of MTC contains the duality transformation of conformal blocks, the Galois symmetry applies to the duality property naturally. To interpret this phenomenon, we exploit the picture of effective central charge as an intermediate step, and show that the initial RCFT and the Hecke image theory share identical fusion rules. The fusion and braiding matrices in each image theory obey the same pentagon and hexagon system of equations, whose different solutions are related by Galois symmetry.

The thesis is organized as follows. We first review the general structure of RCFT briefly in Chapter 2. In Chapter 3 we give the definition of Hecke operators for $\Gamma(N)$, and discuss Galois permutations. We then describe in Chapter 4 the Hecke images and the Galois symmetry with several examples of RCFT and MTC. Chapter 5 introduces the picture of effective central charge, which facilitates the derivation of the fusion rules of the Hecke image theory. In Chapter 6 we review duality transformations in RCFT, which include both fusing and braiding. We then turn in Chapter 7 to the Galois symmetry on fusion and braiding matrices of the Hecke image theory. Finally in Chapter 8, we conclude and suggest relevant problems for future study.

CHAPTER 2

CONFORMAL FIELD THEORY IN A NUTSHELL

Conformal field theory (CFT) has been an active field in theoretical physics over the last decades. Historically the first impetus came from statistical mechanics, where it described and classified critical phenomena at phase transitions. Later on CFT develops rapidly due to its application in string theory. There also has been important input from mathematics, in particular VOA and Lie algebras.

2.1 Conformal symmetry

Assume the general coordinate invariance in d dimensions. We can show that the energy-momentum tensor is conserved. Vary the action S under changes of the spacetime metric

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \quad (2.1)$$

we get the energy-momentum tensor (stress tensor) $T^{\mu\nu}$.

$$\delta S = \frac{1}{2} \int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu}. \quad (2.2)$$

If the theory is invariant under general coordinate transformations, Noether's theorem tells us that

$$\partial_\nu T^{\mu\nu} = 0 \quad (2.3)$$

in the flat coordinates. In CFT we are interested in a different symmetry called the Weyl invariance, whose transformation reads

$$g_{\mu\nu}(x) \rightarrow (1 + \delta\omega(x))g_{\mu\nu}(x). \quad (2.4)$$

The invariance of S yields the condition

$$T^\mu{}_\mu = 0. \quad (2.5)$$

A conformal transformation is defined as a coordinate transformation which acts on the metric as a Weyl transformation. To find what a conformal transformation satisfies in d dimensions, we consider infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$, and get the condition

$$\partial_\mu \epsilon^\nu + \partial_\nu \epsilon^\mu = g_{\mu\nu} \frac{2}{d} \partial_\alpha \epsilon^\alpha. \quad (2.6)$$

For $d > 2$, $\epsilon^\mu(x)$ is shown to be quadratic in x at most. All the possibilities are:

- $\epsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$ constitute the Poincaré group, where a^μ is constant and ω is antisymmetric.
- $\epsilon^\mu = \lambda x^\mu$ are the scale transformations.
- $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x$ are the so-called special conformal transformations (SCTs).

These transformations form the conformal group in d dimensions and can be realized by differential operators. For the spacetime with signature (p, q) , the generators of conformal transformation are

$$P_\mu = -i\partial_\mu, \quad (2.7a)$$

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad (2.7b)$$

$$D = -ix^\mu \partial_\mu, \quad (2.7c)$$

$$K_\mu = i(x^2 \partial_\mu - 2x_\mu x_\nu \partial^\nu). \quad (2.7d)$$

P_μ stands for translations. The Lorentz generators $M_{\mu\nu}$ form the algebra $so(p, q)$. The complete set of generators form a closed algebra which is isomorphic to $so(p+1, q+1)$. D and K_μ generate scalings and SCTs.

Things change greatly for $d = 2$. In two dimensions we use the complex coordinates $z, \bar{z} =$

$x^1 \pm ix^2$. The above condition amounts to

$$\partial_z \bar{\epsilon}(z, \bar{z}) = 0, \quad \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0. \quad (2.8)$$

The corresponding global transformation is

$$z \rightarrow f(z), \quad \bar{z} \rightarrow \bar{f}(\bar{z}), \quad (2.9)$$

where $f(z)$ is a holomorphic function of z . These transformations are generated by the differential operators

$$l_n = -z^{n+1} \partial_z, \quad n \in \mathbb{Z}. \quad (2.10)$$

Similarly for the anti-holomorphic sector. As the Weyl symmetry is generated by the stress tensor $T(z)$, the operators L_n 's, the quantum version of l_n 's, are in fact the modes of $T(z)$, i.e.

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n. \quad (2.11)$$

Equivalently L_n is recovered by the contour integral

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z). \quad (2.12)$$

The quantum stress tensor $T(z)$ has the operator product expansion (OPE) with itself as

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \mathcal{O}(1). \quad (2.13)$$

We can compute the commutator of the modes L_n 's

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}, \quad (2.14)$$

where the number c is called the central charge and depends on the theory.

To probe the quantization in 2D CFT, we begin with the Euclidean space and time coordinates (σ^1, σ^0) . In order to eliminate any infrared divergences, we compactify the spatial direction:

$$\sigma^1 \equiv \sigma^1 + 2\pi. \quad (2.15)$$

We may use $u, \bar{u} = \sigma^1 \mp i\sigma^0$ as coordinates on a cylinder, in which the time σ^0 goes from $-\infty$ to $+\infty$. It is convenient to perform a conformal map

$$z = \exp(iu) = \exp(\sigma^0 + i\sigma^1), \quad (2.16)$$

which maps the cylinder to the complex plane coordinatized by z . Then the time coordinate propagates from the origin to the infinity on the complex plane. A time-ordered product is equivalently radially ordered. The dilatation generator on the conformal plane can be regarded as the Hamiltonian for the system, and the Hilbert space is built up on surfaces of constant radius. Under the conformal map eq(2.16), the stress tensor transforms as

$$T_{\text{cyl}}(u) = \left(\frac{\partial z}{\partial u}\right)^2 T(z(u)) + \frac{c}{12} S(z, u) = -\left[z^2 T(z) - \frac{c}{24}\right], \quad (2.17)$$

where

$$S(z, u) = \frac{z'''(u)}{z'(u)} - \frac{3}{2} \left(\frac{z''(u)}{z'(u)}\right)^2 \quad (2.18)$$

is the Schwarzian derivative. And analogously for the anti-holomorphic component. We can further show that the Hamiltonian on the cylinder is

$$H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{\bar{c}}{24}. \quad (2.19)$$

The shift by $-\frac{c}{24}$ will be seen in the definition of the characters.

Generalize the conformal transformation on the coordinates to the form

$$\Phi(z, \bar{z}) \rightarrow \Phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \quad (2.20)$$

This transformation property defines what is known as a primary field Φ of conformal weight (h, \bar{h}) . The infinitesimal version is

$$\delta_{\epsilon, \bar{\epsilon}} \Phi(z, \bar{z}) = ((h\partial\epsilon + \epsilon\partial) + (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})) \Phi(z, \bar{z}) \quad (2.21)$$

under $z \rightarrow z + \epsilon(z)$ and $\bar{z} \rightarrow \bar{z} + \bar{\epsilon}(\bar{z})$. We expect that the conserved charge

$$Q_{\epsilon, \bar{\epsilon}} = \frac{1}{2\pi i} \oint dz \epsilon(z) T(z) + \frac{1}{2\pi i} \oint d\bar{z} \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}) \quad (2.22)$$

generates the conformal transformation

$$\begin{aligned} \delta_{\epsilon, \bar{\epsilon}} \Phi(w, \bar{w}) &= [Q_{\epsilon, \bar{\epsilon}}, \Phi(w, \bar{w})] \\ &= \frac{1}{2\pi i} \oint_w dz \epsilon(z) R(T(z)\Phi(w, \bar{w})) + \frac{1}{2\pi i} \oint_{\bar{w}} d\bar{z} \bar{\epsilon}(\bar{z}) R(\bar{T}(\bar{z})\Phi(w, \bar{w})), \end{aligned} \quad (2.23)$$

where $R(\dots)$ are the radially ordered product. In the holomorphic sector the OPE of $T(z)$ with $\Phi(w)$ takes the form

$$T(z)\Phi(w) = \frac{h}{(z-w)^2} \Phi(w) + \frac{1}{z-w} \partial_w \Phi(w) + \mathcal{O}(1). \quad (2.24)$$

where h is the conformal weight of Φ . The representations of the Virasoro algebra are of interest as physical realizations, and are labelled by two real numbers, h and c . By definition, a highest weight representation is a representation containing a state with a smallest value of L_0 . Such a state is called the highest weight state, denoted by $|h\rangle$.

$$L_0|h\rangle = h|h\rangle. \quad (2.25)$$

$|h\rangle$ is annihilated by all generators L_n with positive n . The negative modes L_n ($n < 0$) generate other states in the representation. The state $L_{-n}|h\rangle$ ($n > 0$) has the L_0 -value $h + n$. Usually such states are referred to as descendants. The vacuum $|0\rangle$ satisfies

$$L_n|0\rangle = 0, \quad \text{for } n \geq -1, \quad (2.26)$$

especially it is $SL(2, \mathbb{R})$ invariant. (We only spell out the holomorphic sector of the primary field. Similarly one can write the OPE of the primary field with anti-holomorphic stress-energy tensor $\bar{T}(\bar{z})$, as well as the states acted on by the Virasoro generators \bar{L}_n 's.)

Ground states of Virasoro representations are generated from the vacuum by conformal fields, which are also known as (Virasoro) primary fields. For a primary field $\phi(z, \bar{z})$ with weight (h, h') , we define

$$|h, \bar{h}\rangle := \lim_{z, \bar{z} \rightarrow 0} \phi(z, \bar{z})|0\rangle = \phi(0, 0)|0\rangle. \quad (2.27)$$

We compute $L_n|h, \bar{h}\rangle$ and find

$$\begin{aligned} [L_n, \phi(0, 0)] &= 0, \quad n > 0, \\ [L_0, \phi(0, 0)] &= h \phi(0, 0). \end{aligned} \quad (2.28)$$

It follows that $|h, \bar{h}\rangle$ is a highest weight state. The state $|h, \bar{h}\rangle$ indeed has the L_0 -eigenvalue h , as the notation suggests. This is known as the operator-state correspondence in 2D CFT. The descendant fields generate descendant states from the vacuum. They can be constructed by the OPE with the stress tensor

$$T(z)\phi(w, \bar{w}) = \sum_{k \geq 0} (z - w)^{k-2} \phi^{(-k)}(w, \bar{w}). \quad (2.29)$$

Each term in the expansion is extracted by

$$\phi^{(-k)}(w, \bar{w}) = \frac{1}{2\pi i} \oint dz \frac{1}{(z-w)^{k-1}} T(z) \phi(w, \bar{w}), \quad (2.30)$$

and generates the L_{-k} descendant of $|h, \bar{h}\rangle$, namely

$$\phi^{(-k)}(0, 0)|0\rangle = L_{-k}\phi(0, 0)|0\rangle. \quad (2.31)$$

In 2D CFT we compute the correlation function on the complex plane, with infinity added as a single point (a.k.a. the Riemann sphere). First consider correlation functions of primary fields

$$\langle 0|\phi_1(z_1) \cdots \phi_n(z_n)|0\rangle, \quad (2.32)$$

where the operators are radially ordered implicitly. Conformal invariance puts strong constraints on such correlators. The two-point, three-point and four-point functions take specific forms that respect the conformal symmetry. Apply the infinitesimal conformal transformation δ_ϵ .

$$\langle 0|\oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_1(z_1) \cdots \phi_n(z_n)|0\rangle, \quad (2.33)$$

where the contour encircles all the point z_i . We may deform the contour to encircle each z_i separately and obtain

$$\begin{aligned} & \sum_i \langle 0|\phi_1(z_1) \cdots \oint_{z_i} \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_i(z_i) \cdots \phi_n(z_n)|0\rangle \\ &= \sum_i \langle 0|\phi_1(z_1) \cdots \delta_\epsilon \phi_i(z_i) \cdots \phi_n(z_n)|0\rangle. \end{aligned} \quad (2.34)$$

We insert the form of $\delta_\epsilon \phi_i(z_i)$ and get the unintegrated property

$$\langle 0|T(z)\phi_1(z_1) \cdots \phi_n(z_n)|0\rangle = \sum_i \left[\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right] \langle 0|\phi_1(z_1) \cdots \phi_n(z_n)|0\rangle. \quad (2.35)$$

which is called the conformal Ward identity. The equation is used to show that 4-point functions satisfy hypergeometric differential equations. The above computation can be generalized to the correlators of descendants.

We now illustrate the formalism developed so far in the cases of a single massless free boson and a free fermion, respectively. The Lagrangian of a free boson is

$$\mathcal{L}_b = \frac{1}{2\pi} \partial X \bar{\partial} X. \quad (2.36)$$

$X(z, \bar{z})$ splits into two the holomorphic and anti-holomorphic parts

$$X(z, \bar{z}) = \frac{1}{2} (x(z) + \bar{x}(\bar{z})). \quad (2.37)$$

The two-point function is

$$\langle x(z)x(w) \rangle = -\log(z-w). \quad (2.38)$$

$x(z)$ is not itself a conformal field, however $\partial x(z)$ is a primary field of weight 1. We define the stress tensor for the system via normal ordering

$$T_b(z) = \frac{1}{2} : \partial x(z) \partial x(z) : . \quad (2.39)$$

One can verify that the central charge is 1 here. We move on to the massless free fermion, which has central charge 1/2.

$$\mathcal{L}_f = \frac{1}{8\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}). \quad (2.40)$$

The equation of motion determines that $\psi(z)$ and $\bar{\psi}(\bar{z})$ are left- and right-moving respectively. The

holomorphic stress tensor is

$$T_f(z) = \frac{1}{2} : \psi(z) \partial \psi(z) : . \quad (2.41)$$

We shall choose to consider periodic (P) and anti-periodic (A) boundary conditions on the fermion $\psi(z)$ as z rotates by 2π about the origin. With the mode expansion $\psi(z) = \sum_n \psi_n z^{-n-1/2}$, the two boundary conditions select half-integer and integer modes respectively.

$$\psi(e^{2\pi i} z) = \begin{cases} +\psi(z), & n \in \mathbb{Z} + \frac{1}{2}, \quad (\text{P}); \\ -\psi(z), & n \in \mathbb{Z}, \quad (\text{A}). \end{cases} \quad (2.42)$$

The two-point function in period case is

$$\langle \psi(z) \psi(w) \rangle_{\text{P}} = \langle 0 | \psi(z) \psi(w) | 0 \rangle = -\frac{1}{z-w}. \quad (2.43)$$

To compute that in the anti-period case, we may insert the twist operator $\sigma(w)$.

$$\langle \psi(z) \psi(w) \rangle_{\text{A}} = \langle 0 | \sigma(\infty) \psi(z) \psi(w) \sigma(0) | 0 \rangle = -\frac{1}{2(z-w)} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right). \quad (2.44)$$

The twist operator will be seen the Ising model and is also known as the spin field of conformal weight $1/16$.

2.2 Extended algebras

Extended algebras can group an infinite number of Virasoro characters into extended algebra representations. Rational conformal field theory (RCFT) is any CFT with a finite number of primary operators with respect to the action of its chiral algebra. Chiral algebras can be much larger than the Virasoro algebra. Well-known examples include affine Lie algebras and superconformal algebras. In RCFT all conformal weights and the central charge are rational numbers. We refer the

reader to [3, 4, 5] for an overview of RCFT.

The affine Lie algebras (Kac-Moody algebras) are extensions of the chiral algebra by a set of holomorphic spin-1 currents. In an affine Lie algebra (G, k) , the integer k denotes the central extension called the level. The central charge is

$$c = \frac{k \cdot \dim(G)}{k + h^\vee}, \quad (2.45)$$

where h^\vee is the dual Coxeter number of G . We have the following OPEs

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if_c^{ab}J^c(w)}{z-w} + \mathcal{O}(1), \quad (2.46)$$

$$T(z)J^b(w) = \frac{1}{(z-w)^2}J^b(w) + \frac{1}{z-w}\partial_w J^b(w) + \mathcal{O}(1). \quad (2.47)$$

The Virasoro algebra eq(2.14) is supplemented by the commutation relations

$$[J_m^a, J_n^b] = i \sum_c f_c^{ab} J_{m+n}^c + k m \delta^{ab} \delta_{m+n,0}, \quad (2.48a)$$

$$[L_n, J_m^a] = -m J_{n+m}^a, \quad (2.48b)$$

where f_{abc} are the structure constants of the Lie algebra G . Affine Lie algebras are the symmetry algebras of WZW models. In WZW models the Virasoro generators can be written in terms of J_n^a by Sugawara construction. Given a simple representation of the Lie algebra of G , the generators are represented by $r(t^a)$. An affine primary field $\Phi(z)$ is a field that takes values in the representation space of r , such that

$$J^a(z)\Phi(w) = \frac{r(t^a)\Phi(w)}{z-w} + \mathcal{O}(1). \quad (2.49)$$

An affine primary field is also a primary field for the Virasoro algebra that results from the Sug-

awara construction. The conformal weight of the ground state is

$$h(r) = \frac{C_2(r)}{2(k + h^\vee)}, \quad (2.50)$$

where $C_2(r)$ is the quadratic Casimir operator. In the WZW models, the infinite number of Virasoro primaries are reorganized into a finite number of affine Lie algebra representations. Hence, we set up a relation between the highest weight states and the affine primaries, which will be identified as the conformal blocks in modular transformations.

Another important class of extensions of the chiral algebra is by currents of spin 3/2. It leads to the superconformal field theory (SCFT). There are two sectors, Neveu-Schwarz and Ramond, and there may be square root branch cuts in operator products. The spin-3/2 currents of this algebra generate the superconformal transformations and are called the supercurrents. In $\mathcal{N} = 1$ superconformal algebra, there is a spin-3/2 current T_F in addition to the Virasoro generators. The set of OPEs is

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \mathcal{O}(1), \quad (2.51a)$$

$$T(z)T_F(w) = \frac{3}{2(z-w)^2}T_F(w) + \frac{1}{z-w}\partial_w T_F(w) + \mathcal{O}(1), \quad (2.51b)$$

$$T_F(z)T_F(w) = \frac{c}{6(z-w)^3} + \frac{1}{2(z-w)}\partial_w T(w) + \mathcal{O}(1). \quad (2.51c)$$

$T(z)$ is the stress tensor as usual. Denote by G_r the modes of the supercurrent $T_F(z)$. The full algebra in terms of modes is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \quad (2.52a)$$

$$[L_m, G_r] = \left(\frac{1}{2}m - r\right)G_{m+r}, \quad (2.52b)$$

$$\{G_s, G_r\} = 2L_{s+r} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}. \quad (2.52c)$$

The fermionic currents $T_F(w)$ can be half-integer-moded (Neveu-Schwarz) or integer-moded (Ra-

mond). The ground states are annihilated by G_r with $r > 0$. In the Neveu-Schwarz sector the ground state $|0\rangle$ is annihilated by $G_{-1/2}$, while a Ramond ground state is annihilated by G_0 .

CHAPTER 3

HECKE OPERATORS AND GALOIS SYMMETRY

We first give a brief introduction of RCFT characters and modular symmetry before defining Hecke operators.

In a two-dimensional RCFT, the Hilbert space decomposes into a finite sum of irreducible representations $V_i, \bar{V}_{\bar{i}}$ of the chiral algebras \mathcal{A} and $\bar{\mathcal{A}}$, namely

$$\mathcal{H} = \bigoplus_{i \in \mathcal{I}, \bar{i} \in \bar{\mathcal{I}}} \mathcal{N}_{i, \bar{i}} V_i \otimes \bar{V}_{\bar{i}}, \quad (3.1)$$

where $\mathcal{N}_{i, \bar{i}} \in \mathbb{Z}_{\geq 0}$ and $\mathcal{I}, \bar{\mathcal{I}}$ are finite index sets labelling irreducible representations of \mathcal{A} and $\bar{\mathcal{A}}$. In each representation V_i , one has the character

$$\chi_i(\tau) = \text{Tr}_{V_i} q^{L_0 - c/24}, \quad q = e(\tau), \quad \tau \in \mathbb{H}, \quad (3.2)$$

where c is the central charge and \mathbb{H} is the upper half complex plane. Here we are using the standard convention in number theory that $e(x) \equiv e^{2\pi i x}$. Following the decomposition of \mathcal{H} , the partition function is a sesquilinear form of the characters $\chi_i(\tau)$:

$$Z = \sum_{i \in \mathcal{I}, \bar{i} \in \bar{\mathcal{I}}} \mathcal{N}_{i, \bar{i}} \chi_i(\tau) \overline{\chi_{\bar{i}}(\tau)}. \quad (3.3)$$

The full modular group $SL(2, \mathbb{Z})$ acts on τ in the upper half plane \mathbb{H} by

$$\tau \rightarrow \gamma\tau := \frac{a\tau + b}{c\tau + d} \quad (3.4)$$

with

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (3.5)$$

\mathbb{H} is divided into multiple regions by the action of $SL(2, \mathbb{Z})$. The fundamental domain of $SL(2, \mathbb{Z})$ is shown in Figure 3.1.

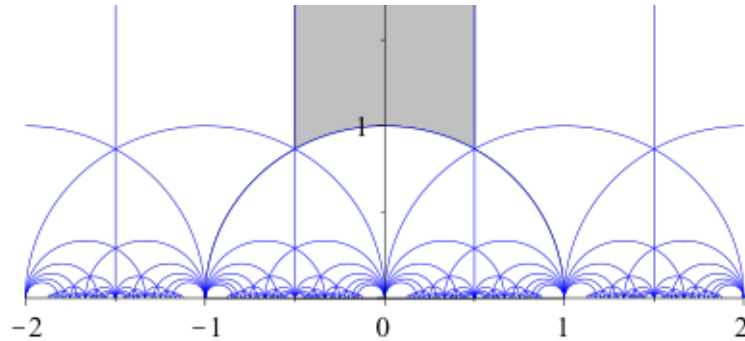


Figure 3.1: Fundamental domain of $SL(2, \mathbb{Z})$.

We call $f(\tau)$ a (weakly holomorphic) modular form of weight k for the modular group $\Gamma = SL(2, \mathbb{Z})$ if $f: \mathbb{H} \rightarrow \mathbb{C}$ is holomorphic (except for a possible pole as $\tau \rightarrow i\infty$) in \mathbb{H} and obeys the transformation law

$$f(\gamma\tau) = (c\tau + d)^k f(\tau). \quad (3.6)$$

It suffices to know the action by the $SL(2, \mathbb{Z})$ generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.7)$$

In RCFT, the individual characters $\chi_i(\tau)$ are weakly holomorphic modular functions for the principal congruence subgroup $\Gamma(N)$ for a finite N defined below. Under the $SL(2, \mathbb{Z})$ transformation γ , the characters transform as

$$\chi_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij} \chi_j(\tau). \quad (3.8)$$

Here, ρ is a finite-dimensional representation of $SL(2, \mathbb{Z})$:

$$\rho: SL(2, \mathbb{Z}) \rightarrow GL(V), \quad (3.9)$$

which is completely determined by its values on the $SL(2, \mathbb{Z})$ generators

$$S : \tau \rightarrow -1/\tau, \quad T : \tau \rightarrow \tau + 1. \quad (3.10)$$

The partition function Z must be modular invariant. As an $SL(2, \mathbb{Z})$ representation, ρ obeys the consistency condition

$$\rho(S)^2 = (\rho(T)\rho(S))^3 = \mathcal{C}, \quad (3.11)$$

where \mathcal{C} is the charge conjugation matrix. In [1] the first and third authors studied the Hecke operators for $\Gamma(N)$ modular forms. These Hecke operators act nicely on the Fourier expansion of the characters

$$\chi_i(\tau) = q^{h_i - \frac{c}{24}} \sum_{n=0}^{\infty} a_i(n) q^n, \quad q = e(\tau), \quad (3.12)$$

where c is the central charge and h_i is the conformal weight. The partition function eq(3.3) is modular invariant *iff*

$$[\mathcal{N}, \rho(T)] = [\mathcal{N}, \rho(S)] = 0. \quad (3.13)$$

3.1 Simple current

The fusion coefficients ${}_0N_{ij}^k$ which govern the fusion of primary operators ϕ_i as

$$\phi_i \times \phi_j = \sum_k {}_0N_{ij}^k \phi_k \quad (3.14)$$

are determined by the Verlinde formula

$${}_0N_{ij}{}^k = \sum_m \frac{\rho(S)_{im}\rho(S)_{jm}\rho(S^{-1})_{km}}{\rho(S)_{0m}} \quad (3.15)$$

[7], where the label 0 emphasizes the special role played by the vacuum entry [12]. The fusion coefficients in a unitary RCFT must be non-negative. The fusion rules for the field i are gathered into the matrix N_i with the element

$$(N_i)_{j,k} = {}_0N_{ij}{}^k. \quad (3.16)$$

A simple current J permutes the fields by the fusion rule $J \times a = Ja$, where there is only one term on the right hand side. In Chapter 4.1 we shall see another usage of the simple current, where it reduces the structure of affine algebras. In this chapter we investigate its role in the effective description of RCFT. Invoke the property of simple currents

$$\rho(S)_{Ja,b} = e[Q_J(b)]\rho(S)_{a,b}, \quad (3.17)$$

where $Q_J(b)$ is the monodromy charge of the field b under the current J [27]. If J is of order n , the monodromy charge $Q_J(b) \in \frac{1}{n}\mathbb{Z}$. In this paper, $Q_J(b)$ is a half-integer and thus $e[Q_J(b)] = \pm 1$. The positive column of the minimal primary requires $Q_J(o) \in \mathbb{Z}$ for all simple currents J , which boils down to $Q_J(0) \in \mathbb{Z}$ in unitary theories. The monodromy charges are determined by the conformal weights of the fields on the simple current orbit

$$h_a + h_J - h_{Ja} \equiv Q_J(a) - Q_J(0) \pmod{1} \quad (3.18)$$

[26]. Unlike in the unitary theories, $Q_J(0)$ can be a half-integer when the theory is non-unitary.

This yields modifications of the selection rule applying to the unitary theories

$$Q_J(a) + Q_J(b) \equiv Q_J(c) + Q_J(0) \pmod{1} \quad (3.19)$$

if ${}_0N_{ab}^c \neq 0$. Apart from the above constraint on fusion rules, the property eq(3.17) of simple currents demands ${}_0N_{Ji, J-1j}^k = {}_0N_{ij}^k$. The existing fusion rule eq(3.14) then implies another:

$$\phi_{Ji} \times \phi_{J-1j} = \sum_k {}_0N_{ij}^k \phi_k. \quad (3.20)$$

3.2 Hecke operators for $\Gamma(N)$

Since $\chi_i(\tau)$ is a q -series with leading term $q^{h_i - c/24}$, the matrix $\rho(T)$ is diagonal with entries $e(h_i - c/24)$. In RCFT the conformal weights h_i and the central charge c are rational [6]. Hence $\rho(T)$ has a finite order N , which is the least common denominator of $h_i - c/24$. Theorem 1 of Bantay [2] states that the kernel of ρ contains the principal congruence subgroup $\Gamma(N)$ defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}. \quad (3.21)$$

In other words, $\chi_i(\tau)$ are invariant under $\tau \rightarrow \gamma\tau$ for $\gamma \in \Gamma(N)$. There is a natural homomorphism

$$\mu_N : SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/N\mathbb{Z}) \quad (3.22)$$

done by reduction mod N of each element $\gamma \in SL(2, \mathbb{Z})$. Because the kernel of μ_N is precisely $\Gamma(N)$, the map μ_N does not affect the modular representation. Hence, ρ can be also regarded as a representation of $SL(2, \mathbb{Z}/N\mathbb{Z})$.

The Hecke operator T_p for $SL(2, \mathbb{Z})$ modular forms has been discussed in textbooks on number theory [15, 18]. However, characters in RCFT are modular functions for $\Gamma(N)$ and transform according to the representation ρ under $SL(2, \mathbb{Z})$, that is they are vector-valued forms rather than

strictly modular forms. The Hecke operators on them should be compatible with their vector structure. To define the Hecke operators for $\Gamma(N)$, we introduce the set of orbit representatives

$$\begin{aligned}\Delta_N^{(p)} &= \left\{ \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & bN \\ 0 & p \end{pmatrix} \mid 0 \leq b \leq p-1 \right\} \\ &= \sigma_p \circ \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \sigma_{\bar{p}} \begin{pmatrix} 1 & bN \\ 0 & p \end{pmatrix} \mid 0 \leq b \leq p-1 \right\},\end{aligned}\tag{3.23}$$

where p is a prime number with $\gcd(p, N) = 1$ [19]. Here σ_p denotes the preimage of

$$\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}\tag{3.24}$$

under μ_N , and \bar{p} is the multiplicative inverse of p modulo N . The occurrence of σ_p reflects the nature of $\Gamma(N)$. Properties of $\rho(\sigma_p)$ will be addressed shortly.

Define the Hecke operator T_p acting on weight zero vector-valued modular form f relative to a representation ρ for p prime

$$(T_p f)_i(\tau) := \sum_{\delta \in \Delta_N^{(p)}} f_i(\delta\tau) = \sum_j \rho_{ij}(\sigma_p) f_j(p\tau) + \sum_{b=0}^{p-1} f_i\left(\frac{\tau + bN}{p}\right)\tag{3.25}$$

[1]. It can be loose expressed as

$$(T_p \chi)(\tau) = \chi(\sigma_p \circ p\tau) + \sum_{b=0}^{p-1} \chi\left(\frac{\tau + bN}{p}\right),\tag{3.26}$$

where $\chi(\tau)$ is understood as a vector-valued modular form for $\Gamma(N)$. The normalization of T_p differs from the traditional T_p for scalar modular forms in order to preserve the integrality of coefficients. One essential feature of these Hecke operators is that they do lead to a nice action on

the Fourier coefficients. If

$$f_i(\tau) = \sum_n b_i(n)q^{n/N} \quad (3.27)$$

then using

$$\sum_{b=0}^{p-1} e(bn/p) = \begin{cases} p & p|n \\ 0 & \text{otherwise} \end{cases} \quad (3.28)$$

in eq(3.25) gives

$$(\mathbb{T}_p f)_i(\tau) = \sum_n b_i^{(p)}(n)q^{n/N} \quad (3.29)$$

with

$$b_i^{(p)}(n) = \begin{cases} p b_i(pn) & p \nmid n \\ p b_i(pn) + \sum_j \rho_{ij}(\sigma_p) b_j(n/p) & p|n \end{cases}. \quad (3.30)$$

An essential ingredient in defining Hecke operators for $\Gamma(N)$ is the representation matrix of the $SL(2, \mathbb{Z})$ element σ_p , which also constitutes the modular representation of the Hecke image. Under the Hecke operation \mathbb{T}_p , the induced modular representation $\rho^{(p)}$ is related to the original representation ρ via

$$\rho^{(p)}(T) = \rho(T^{\bar{p}}), \quad \rho^{(p)}(S) = \rho(\sigma_p S). \quad (3.31)$$

Though σ_p is not unique, two choices differ by the action of $\Gamma(N)$ which is in the kernel of ρ . For this reason, the representation $\rho^{(p)}$ is uniquely determined by any choice of σ_p . Since $\gcd(\bar{p}, N) = 1$, $\rho^{(p)}(T)$ has the same order as $\rho(T)$. Therefore the action of \mathbb{T}_p preserves the value of N and the dimension of the representation.

We now prove that if f is a level N vector-valued modular function with representation ρ then $(\mathbb{T}_p f)$ is a level N modular function with representation $\rho^{(p)}$ of the form claimed earlier. First consider the action of $\gamma \in \Gamma(N)$. The fact that each component of $(\mathbb{T}_p f)$ is a modular form for $\Gamma(N)$ follows from our discussion of Hecke operators for $\Gamma(N)$ and the fact that $\Gamma(N) \subset \ker(\rho)$. To determine the representation $\rho^{(p)}$ it suffices to determine it on the generators T, S of $SL(2, \mathbb{Z})$.

Write $\mathcal{M}_n(\mathbb{Z})$ for the set of two by two matrices with entries in \mathbb{Z} and determinant n . In [19] it is shown that

$$\mathcal{M}_p(\mathbb{Z}) = \Gamma(1) \cdot \mathcal{T}_p^+ = \mathcal{T}_p^+ \cdot \Gamma(1) \quad (3.32)$$

where $\Gamma(1) = SL(2, \mathbb{Z})$ and \mathcal{T}_p^+ is a right transversal given by

$$\mathcal{T}_p^+ = \{ \sigma_p \beta_p; U_b, 0 \leq b \leq p-1 \}, \quad (3.33)$$

where

$$\beta_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad U_\nu = \begin{pmatrix} 1 & \nu N \\ 0 & p \end{pmatrix}, \quad (3.34)$$

In light of this, for each $\gamma \in \Gamma$ and $\delta \in \Delta_N^{(p)}$ we have $\delta\gamma = \gamma'\delta'$ for some $\gamma' \in \Gamma$ and $\delta' \in \Delta_N^{(p)}$.

We want to determine γ' for $\gamma = T$ and S modulo the action of $\Gamma(N)$.

In particular we show that

$$\begin{aligned} \Delta_N^{(p)} \circ T &= T^{\bar{p}} \circ \Delta_N^{(p)}, \\ \Delta_N^{(p)} \circ S &= \sigma_p S \circ \Delta_N^{(p)}. \end{aligned} \quad (3.35)$$

which shows that the Hecke image $T_p f$ transforms under $SL(2, \mathbb{Z})$ according to the representation $\rho^{(p)}$ with $\rho^{(p)}(S) = \rho(\sigma_p S)$ and $\rho^{(p)}(T) = \rho(T^{\bar{p}})$.

We first consider the modular transformation $T : \tau \rightarrow \tau + 1$ and its right action on the second set of representatives in eq(3.33)

$$U_\nu T = \begin{pmatrix} 1 & \nu N \\ 0 & p \end{pmatrix}, \quad (3.36)$$

where ν runs over a complete set of residues mod p . We have

$$U_\nu T = T^{\bar{p}} U_{\nu'(\nu)}, \quad (3.37)$$

with

$$\nu'(\nu) = \nu - \frac{p\bar{p} - 1}{N}. \quad (3.38)$$

Since $p\bar{p} \equiv 1 \pmod{N}$, when ν runs over a complete set of residues mod p so does ν' .

To deal with the first representative in eq(3.33) we note that

$$\sigma_p \beta_p T = \gamma' \sigma_p \beta_p \quad (3.39)$$

with

$$\gamma' = \sigma_p \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \sigma_p^{-1} \quad (3.40)$$

and that $\gamma' \in SL(2, \mathbb{Z})$ since $\sigma_p \in SL(2, \mathbb{Z})$. Furthermore working mod N we have

$$\gamma' = \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix} = \begin{pmatrix} 1 & \bar{p} \\ 0 & 1 \end{pmatrix} \quad (3.41)$$

so $\gamma' = T^{\bar{p}}$ up to the action of $\Gamma(N)$. This establishes that

$$\Delta_N^{(p)} \circ T = T^{\bar{p}} \circ \Delta_N^{(p)}, \quad (3.42)$$

and verifies the modular representation of T for the Hecke image $\mathbb{T}_p f$,

$$\rho^{(p)}(T) = \rho(T^{\bar{p}}). \quad (3.43)$$

Next we consider the transformation $S : \tau \rightarrow -1/\tau$ and first deal with the action on the representatives U_ν for $\nu \not\equiv 0 \pmod{p}$. For each $\nu \not\equiv 0 \pmod{p}$, we have

$$U_\nu S = \gamma'_\nu U_{\nu'} \quad (3.44)$$

with

$$\gamma'_\nu \equiv U_\nu S U_{\nu'}^{-1} = \begin{pmatrix} N\nu & -(1 + N^2\nu\nu')/p \\ p & -N\nu' \end{pmatrix}. \quad (3.45)$$

To find a ν' such that $\gamma'_\nu \in SL(2, \mathbb{Z})$ we note we must have

$$1 + N^2\nu\nu' = 0 \pmod{p} \quad (3.46)$$

which has the solution $\nu' = -(N^2\nu)^{p-2}$ by Fermat's little theorem. With this choice of ν' , γ'_ν has integral entries and determinant one and so is in $SL(2, \mathbb{Z})$. One then checks that $\gamma'_\nu S^{-1}$ is congruent to $\sigma_p \pmod{N}$ so we can take $\gamma'_\nu = \sigma_p S$. Since p is a prime number, we have that $\{\nu'(\nu) | 1 \leq \nu \leq p-1\}$ is a permutation of the elements of $(\mathbb{Z}/p\mathbb{Z})^\times$.

There are two representatives in $\Delta_N^{(p)}$ left to deal with, i.e.

$$U_0 \quad \text{and} \quad \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.47)$$

One easily checks that

$$\begin{aligned} U_0 S &= \gamma'_0 \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} S &= \gamma'_\sigma U_0, \end{aligned} \quad (3.48)$$

with γ'_0 and γ'_σ in $SL(2, \mathbb{Z})$ and equal to $\sigma_p S$ up to the action of $\Gamma(N)$.

Having verified

$$\Delta_N^{(p)} \circ S = \sigma_p S \circ \Delta_N^{(p)}, \quad (3.49)$$

we find the modular representation of S for the Hecke image $T_p f$,

$$\rho^{(p)}(S) = \rho(\sigma_p S). \quad (3.50)$$

and thus verify eq(3.31).

Given the form of T_p for p prime, one can construct Hecke operators T_n for n coprime to N but not necessarily prime. We restate the results in the appendix of [1]. Hecke operators T_m, T_n for m, n relatively prime obey

$$T_m T_n f = T_{mn} f \quad (3.51)$$

and for $n = p^m$ with p prime and $m \geq 1$

$$T_{p^{m+1}} f = T_p T_{p^m} f - p \sigma_p \circ T_{p^{m-1}} f. \quad (3.52)$$

We keep in mind that these Hecke operators act on RCFT characters, which are weight-zero modular forms. The form of the equation above slightly differs from math textbooks due to the normalization of the Hecke operator T_p for RCFT characters.

3.3 Galois permutations

A crucial consequence of the Hecke operation is the induced Galois symmetry, which relates $SL(2, \mathbb{Z})$ representations and thereby fusion rules in RCFTs. In a nondegenerate RCFT, we show that the Galois group of fusion rules is fully determined by the conductor.

Denote by \mathbb{K} the number field obtained by adjoining all matrix elements of the modular representation to \mathbb{Q} . De Boer and Goeree show that \mathbb{K} is a finite Abelian extension of \mathbb{Q} [8]. Write $\xi_m = e(1/m)$. Denoting by $\mathbb{Q}[\xi_m]$ the cyclotomic field that is an extension of \mathbb{Q} by a primitive m th root of unity, the smallest integer m such that $\mathbb{K} \subset \mathbb{Q}[\xi_m]$ is called the conductor of the RCFT. Bantay shows that the conductor equals precisely N , the order of $\rho(T)$ [2]. Moreover, since \mathbb{K}

contains the N th roots of unity as the diagonal entries of $\rho(T)$, \mathbb{K} is exactly $\mathbb{Q}[\xi_N]$. The automorphisms of \mathbb{K} over \mathbb{Q} furnish the Galois group $\mathcal{G}_N = \text{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$, which is isomorphic to the unit group $(\mathbb{Z}/N\mathbb{Z})^\times$, the group of multiplicative units in $\mathbb{Z}/N\mathbb{Z}$. Each element ℓ in $(\mathbb{Z}/N\mathbb{Z})^\times$ gives rise to a Frobenius map $f_{N,\ell}$ which takes ξ_N to ξ_N^ℓ while leaving \mathbb{Q} fixed.

We write \bar{p} for the multiplicative inverse of p in $(\mathbb{Z}/N\mathbb{Z})^\times$. As discussed in [11], the Frobenius element $f_{N,\bar{p}}$ acts on the representation matrices $\rho(T), \rho(S)$ as

$$f_{N,\bar{p}}(\rho(T)) = \rho(T)^{\bar{p}}, \quad (3.53a)$$

$$f_{N,\bar{p}}(\rho(S)) = \rho(S) G_p = G_p^{-1} \rho(S). \quad (3.53b)$$

The matrix G_p coincides with $\rho(\sigma_{\bar{p}})$ [1], proving that the modular representation $\rho^{(p)}$ is equivalent to the Galois action $f_{N,\bar{p}}$ on ρ :

$$f_{N,\bar{p}}(\rho(T)) = \rho^{(p)}(T), \quad f_{N,\bar{p}}(\rho(S)) = \rho^{(p)}(S). \quad (3.54)$$

The Hecke operator T_p extends $f_{N,\bar{p}}$ to an action on the characters of the RCFT rather than just on the modular representation.

Let $N = \prod_{i=1}^r p_i^{k_i}$ be the prime factorization of the conductor. The matrices $G_p = \rho(\sigma_{\bar{p}})$ reveal intriguing features as the representation of $SL(2, \mathbb{Z}/N\mathbb{Z})$. The finite-dimensional representation ρ is completely reducible, and each irreducible component ω of ρ has the unique product decomposition

$$\omega = \otimes_{i=1}^n \pi(p_i^{k_i}) \quad (3.55)$$

[24]. Here $\pi(p_i^{k_i})$ is an irreducible representation of $SL(2, \mathbb{Z}/p_i^{k_i}\mathbb{Z})$. The homomorphism $p \rightarrow \rho(\sigma_p)$ defines an n -dimensional representation of $(\mathbb{Z}/N\mathbb{Z})^\times$, where $n = |\mathcal{I}|$.

An explicit computation gives

$$T^p S^{-1} T^{\bar{p}} S T^p S = \begin{pmatrix} (1 - p\bar{p})p + p & p\bar{p} - 1 \\ 1 - p\bar{p} & \bar{p} \end{pmatrix} \equiv \begin{pmatrix} p & 0 \\ 0 & \bar{p} \end{pmatrix} \pmod{N}, \quad (3.56)$$

which establishes that $T^p S^{-1} T^{\bar{p}} S T^p S$ is the preimage of $\sigma_{\bar{p}}$ under the mod N map from $SL(2, \mathbb{Z})$ to $SL(2, \mathbb{Z}/N\mathbb{Z})$. In practice, G_p can be evaluated from the expression

$$G_p = \rho(T^p S^{-1} T^{\bar{p}} S T^p S) = \rho(T)^p \rho(S)^{-1} \rho(T)^{\bar{p}} \rho(S) \rho(T)^p \rho(S). \quad (3.57)$$

As it turns out, G_p is a monomial matrix with the elements

$$(G_p)_{i,j} = \varepsilon_p(i) \delta_{\pi_p(i),j}, \quad (3.58)$$

where π_p is some permutation of \mathcal{I} and ε_p is a map from \mathcal{I} to $\{+1, -1\}$ [10, 11]. When $p \equiv l^2 \pmod{N}$ for some $l \in (\mathbb{Z}/N\mathbb{Z})^\times$, $f_{N,p}$ induces the inner automorphism of $\rho(\gamma)$:

$$f_{N,p}(\rho(T)) = G_l \rho(T) G_l^{-1}, \quad f_{N,p}(\rho(S)) = G_l \rho(S) G_l^{-1}. \quad (3.59)$$

Hence, f_{N,l^2} shuffles the diagonal entries of $\rho(T)$ by

$$\rho(T)_{aa}^{l^2} = \rho(T)_{\pi_l(a); \pi_l(a)}, \quad a \in \mathcal{I}. \quad (3.60)$$

G_p 's form the Galois group of fusion rules, denoted by \mathcal{G} , under matrix multiplication. The homomorphism $p \rightarrow G_p$ shows that the matrices G_p are an n -dimensional representation of $(\mathbb{Z}/N\mathbb{Z})^\times$. For the abelian group $(\mathbb{Z}/N\mathbb{Z})^\times$, all the irreducible representations over \mathbb{C} are one-dimensional, which are known as the Dirichlet characters mod N . Therefore the n -dimensional representation G_p can be diagonalized over \mathbb{C} to give a finite sum of Dirichlet characters mod N .

Next we demonstrate that the Galois permutations are determined by the quadratic residues

modulo N . For $l_1^2 \equiv l_2^2 \pmod{N}$, one verifies that

$$G_{l_1} \rho(T) G_{l_1}^{-1} = G_{l_2} \rho(T) G_{l_2}^{-1}, \quad (3.61)$$

and finds that $\rho(T)$ is invariant under conjugation by $G_{\bar{l}_1 l_2}$, i.e.

$$\rho(T) = G_{\bar{l}_1 l_2} \rho(T) G_{l_1 \bar{l}_2}. \quad (3.62)$$

In terms of the permutation, this equation states that

$$\rho(T)_{aa} = \rho(T)_{\pi_{\bar{l}_1 l_2}(a); \pi_{l_1 \bar{l}_2}(a)}. \quad (3.63)$$

The permutation $\pi_{\bar{l}_1 l_2}$ only shuffles the fields with same twist. In most cases we encounter nondegenerate modular fusion algebras where independent characters are associated with different twists θ_i , and thus all the eigenvalues of $\rho(T)$ are distinct [24]. As a result, $\pi_{\bar{l}_1 l_2}$ is an identity permutation, and $G_{\bar{l}_1 l_2}$ must be diagonal with entries ± 1 . Lemma 5 in [2] states that if G_p is diagonal then it must be $\pm \mathbb{I}$, hence we deduce

$$G_{\bar{l}_1 l_2} = \pm \mathbb{I}, \quad \text{for } (\bar{l}_1 l_2)^2 \equiv 1 \pmod{N}. \quad (3.64)$$

This reasoning leads to the following result:

In a nondegenerate modular fusion algebra we have the relation

$$G_{l_1} = \pm G_{l_2}, \quad (3.65)$$

if $l_1^2 \equiv l_2^2 \pmod{N}$. In particular, $G_l = \pm \mathbb{I}$ for every l such that $l^2 \equiv 1 \pmod{N}$.

This shows that G_l is specified by the congruence class of l^2 modulo N , up to a parity sign $\varepsilon_l(0)$ which does not affect the Galois permutation π_l . In some RCFTs there can be complex-conjugate primary fields which have the same character. We may regard them as a single neutral primary

and reduce the dimensionality of the modular representation, prior to imposing the nondegeneracy condition. Then the Hecke operators will act on the reduced vector-valued modular form. See examples in Chapter 4.2.

In applying the above result it is useful to discuss the structure of $(\mathbb{Z}/N\mathbb{Z})^\times$ and the group of quadratic residues modulo N . With the prime factorization $N = \prod_{i=1}^r p_i^{k_i}$, the unit group $(\mathbb{Z}/N\mathbb{Z})^\times$ is the direct product of the unit groups associated with each prime power factor

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_{i=1}^r (\mathbb{Z}/p_i^{k_i}\mathbb{Z})^\times \quad (3.66)$$

by the Chinese Remainder Theorem. Each prime sector can be expressed by the cyclic group C_m .

$$(\mathbb{Z}/p^k\mathbb{Z})^\times \cong \begin{cases} 1 & \text{if } p = 2 \text{ and } k = 1, \\ C_2 \times C_{2^{k-2}} & \text{if } p = 2 \text{ and } k \geq 2, \\ C_{p^{k-1}(p-1)} & \text{if } p > 2. \end{cases} \quad (3.67)$$

Define the group of quadratic residues modulo N

$$[(\mathbb{Z}/N\mathbb{Z})^\times]^2 := \{m^2 \mid m \in (\mathbb{Z}/N\mathbb{Z})^\times\}, \quad (3.68)$$

which is evidently a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. The group $[(\mathbb{Z}/N\mathbb{Z})^\times]^2$ can be calculated by folding the components in eq(3.66).

A wide class of RCFTs are the Virasoro minimal models $M(p_1, p_2)$, which are labeled by a pair of coprime integers (p_1, p_2) with $p_1, p_2 > 1$. The model $M(p_1, p_2)$ is unitary *iff* $|p_1 - p_2| = 1$. Both unitary and non-unitary minimal models will be considered in this work. Their conductors are computed in [2]. We list in Table 3.1 the unit group $(\mathbb{Z}/N\mathbb{Z})^\times$ and the group of quadratic residues $[(\mathbb{Z}/N\mathbb{Z})^\times]^2$ for a number of minimal models, as well as their Galois groups.

Affine Lie algebras (Kac-Moody algebras) are also important examples of RCFT. They are infinite dimensional algebras that extend simple Lie algebras, and appear as current algebras in

(p_1, p_2)	N	n	$(\mathbb{Z}/N\mathbb{Z})^\times$	$[(\mathbb{Z}/N\mathbb{Z})^\times]^2$	\mathcal{G}
(2, 5)	60	2	$C_2 \times C_2 \times C_4$	C_2	C_2
(2, 7)	42	3	$C_2 \times C_6$	C_3	C_3
(2, 9)	36	4	$C_2 \times C_6$	C_3	C_3
(2, 11)	33	5	$C_2 \times C_{10}$	C_5	C_5
(2, 13)	156	6	$C_2 \times C_2 \times C_{12}$	C_6	C_6
(2, 15)*	30	7	$C_2 \times C_4$	C_2	C_4
(2, 17)	204	8	$C_2 \times C_2 \times C_{16}$	C_8	C_8
(2, 19)	57	9	$C_2 \times C_{18}$	C_9	C_9
(3, 4)	48	3	$C_2 \times C_4 \times C_2$	C_2	C_2
(3, 5)	40	4	$C_2 \times C_2 \times C_4$	C_2	C_2
(3, 7)	168	6	$C_2 \times C_2 \times C_2 \times C_6$	C_3	C_3
(3, 8)	32	7	$C_2 \times C_8$	C_4	C_4

Table 3.1: This table summarizes some data for a few minimal models. N is the conductor of the theory, which is also equal to the order of $\rho(T)$. $n = |\mathcal{I}|$ denotes the number of primary fields. \mathcal{G} stands for the Galois group of fusion rules. We put an asterisk on $M(2, 15)$, where \mathcal{G} does not agree with C_2 because $\rho(T)$ has degenerate eigenvalues for fields that are not complex conjugates.

the WZW models. The characters of an affine Lie algebra transform among themselves under the modular group. In affine Lie algebras there is a Galois symmetry acting on highest weight representations [12, 13], and the resulting fusion rule automorphism is discussed in [11].

The Galois symmetry appears in the induced modular representations of Hecke images, as well as in the definition of Hecke operators for $\Gamma(N)$. Since the second line of eq(3.23) is merely an alternative form of $\Delta_N^{(p)}$, one can rewrite the Hecke operation as

$$\begin{aligned}
(\mathbb{T}_p f)_i(\tau) &= \sum_j \rho(\sigma_p)_{ij} \sum_{\delta \in \sigma_{\bar{p}} \circ \Delta_N^{(p)}} f_j(\delta\tau) \\
&= \sum_j \rho(\sigma_p)_{ij} \left(f_j(p\tau) + \sum_k \rho(\sigma_{\bar{p}})_{jk} \sum_{b=0}^{p-1} f_k\left(\frac{\tau + bN}{p}\right) \right). \tag{3.69}
\end{aligned}$$

From the physical point of view, the Hecke operation by \mathbb{T}_p changes the Fourier coefficients, followed by a signed permutation by $\rho(\sigma_p) = G_p^{-1}$. Along the way, the conformal weights in RCFT

are multiplied by p modulo \mathbb{Z} . As shown in [1], $\mathbb{T}_p f$ transforms in the modular representation

$$\rho^{(p)}(\gamma) = f_{N,\bar{p}}(\rho(\gamma)), \quad \gamma \in SL(2, \mathbb{Z}). \quad (3.70)$$

The Frobenius map $f_{N,\bar{p}}$ is a composition of $f_{N,p}$ and f_{N,\bar{p}^2} , both of which have interpretations.

The action of f_{N,\bar{p}^2} amounts to conjugation under G_p , i.e.

$$f_{N,\bar{p}}(\rho(\gamma)) = f_{N,\bar{p}^2} \circ f_{N,p}(\rho(\gamma)) = G_p^{-1} f_{N,p}(\rho(\gamma)) G_p. \quad (3.71)$$

While the remaining $f_{N,p}$ causes the observed relations between the conformal weights:

$$\tilde{\theta}_i \mapsto \theta_i^{(p)} = f_{N,p}(\tilde{\theta}_i), \quad (3.72)$$

or equivalently $h_i^{(p)} \equiv p \tilde{h}_i \pmod{1}$, where $\tilde{\theta}_i$ and \tilde{h}_i are respectively the twists and the conformal weights in the effective description to be introduced in Chapter 5. The twists $\theta_i^{(p)}$ and $\tilde{\theta}_i$ are roots of unity of same order, and their associated primary fields share similar statistical (braiding) properties. We will see this essential fact in the full structure of RCFT under Hecke operations.

3.4 Applications and examples

We now apply the Hecke operators of the previous section to derive relations between the characters of RCFTs with more than one character. These examples in general involve mixing between the different components of the characters. We will start with $M(2, 5)$, a.k.a. the Yang-Lee model.

The characters are given by

$$\chi_0^{\text{YL}}(q) = q^{-1/60} G(q), \quad (3.73a)$$

$$\chi_{1/5}^{\text{YL}}(q) = q^{11/60} H(q), \quad (3.73b)$$

where the functions $G(q)$, $H(q)$ obey the Rogers-Ramanujan identities

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \quad (3.74a)$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \quad (3.74b)$$

The representations of S and T are

$$\rho^{\text{YL}}(S) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & -\sin(\frac{2\pi}{5}) \end{pmatrix}, \quad \rho^{\text{YL}}(T) = \text{diag}(\xi_{60}^{-1}, \xi_{60}^{11}). \quad (3.75)$$

We thus see that the conductor is $N = 60$.

For $p = 7, 13$ we have

$$\rho^{\text{YL}}(\sigma_7) = \rho^{\text{YL}}(\sigma_{13}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.76)$$

We can then use eqn.(3.30) to derive the coefficients $b_i^{(p)}(n)$ of the Hecke image. Although the Hecke relations between coefficients are most easily seen when characters are written as a series in $q^{1/N}$, it is more conventional in conformal field theory to write characters in terms of a leading fractional power of q and a series with integer powers of q . Thus for the Yang-Lee model we define coefficients of characters as

$$\chi_0^{\text{YL}} = q^{-1/60} \sum_{n=0}^{\infty} c_0^{\text{YL}}(n) q^n, \quad (3.77a)$$

$$\chi_{1/5}^{\text{YL}} = q^{11/60} \sum_{n=0}^{\infty} c_{1/5}^{\text{YL}}(n) q^n, \quad (3.77b)$$

and we similarly define coefficients $c_0^{G_2}(n)$ and $c_{2/5}^{G_2}(n)$ of the vacuum character and character corresponding to the primary of conformal weight $2/5$ for level one affine G_2 and coefficients

$c_0^{F_4}(n)$ and $c_{3/5}^{F_4}(n)$ for level one affine F_4 characters.

We find that the Hecke relations $\chi^{G_2} = T_7 \chi^{\text{YL}}$ and $\chi^{F_4} = T_{13} \chi^{\text{YL}}$ with the relations

$$\begin{aligned} c_0^{G_2}(n) &= \begin{cases} 7c_{1/5}^{\text{YL}}(7n-1) & \text{if } 7 \nmid n, \\ 7c_{1/5}^{\text{YL}}(7n-1) + c_0^{\text{YL}}(\frac{n}{7}) & \text{if } 7|n; \end{cases} \\ c_{2/5}^{G_2}(n) &= \begin{cases} 7c_0^{\text{YL}}(7n+2) & \text{if } 7 \nmid (n-1), \\ 7c_0^{\text{YL}}(7n+2) - c_{1/5}^{\text{YL}}(\frac{n-1}{7}) & \text{if } 7|(n-1). \end{cases} \end{aligned} \quad (3.78)$$

$$\begin{aligned} c_0^{F_4}(n) &= \begin{cases} 13c_{1/5}^{\text{YL}}(13n-3) & \text{if } 13 \nmid n, \\ 13c_{1/5}^{\text{YL}}(13n-3) + c_0^{\text{YL}}(\frac{n}{13}) & \text{if } 13|n; \end{cases} \\ c_{3/5}^{F_4}(n) &= \begin{cases} 13c_0^{\text{YL}}(13n+5) & \text{if } 13 \nmid (n-2), \\ 13c_0^{\text{YL}}(13n+5) - c_{1/5}^{\text{YL}}(\frac{n-2}{13}) & \text{if } 13|(n-2). \end{cases} \end{aligned} \quad (3.79)$$

We now show that the modular representation and characters of affine level one G_2 and F_4 are the Hecke images of the Yang-Lee characters under the Hecke operators T_7 and T_{13} respectively.

The relation between modular representations is well known. We have

$$\begin{aligned} \rho^{G_2}(S) &= \rho^{F_4}(S) = \rho^{\text{YL}}(\sigma_p S) = \rho^{\text{YL}}(\sigma_p) \rho^{\text{YL}}(S) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\left(\frac{2\pi}{5}\right) & \sin\left(\frac{\pi}{5}\right) \\ \sin\left(\frac{\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \end{pmatrix} = \frac{2}{\sqrt{5}} \begin{pmatrix} -\sin\left(\frac{\pi}{5}\right) & \sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) & \sin\left(\frac{\pi}{5}\right) \end{pmatrix}. \end{aligned} \quad (3.80)$$

and

$$\begin{aligned} \rho^{G_2}(T) &= \rho^{\text{YL}}(T^{\bar{7}}) = \rho^{\text{YL}}(T^{43}) = \text{diag}\left(\xi_{60}^{17}, \xi_{60}^{-7}\right), \\ \rho^{F_4}(T) &= \rho^{\text{YL}}(T^{\bar{13}}) = \rho^{\text{YL}}(T^{37}) = \text{diag}\left(\xi_{60}^{23}, \xi_{60}^{-13}\right). \end{aligned} \quad (3.81)$$

Note that the ordering of the G_2 and F_4 characters comes out with the vacuum character in the second entry due to our choice of ordering of the Yang-Lee characters.

We explore a few examples of relations between RCFTs with three independent characters. The most well-known RCFT with three independent characters is $M(3, 4)$, a.k.a. the Ising model with $c = 1/2$. It is equivalent to a free Majorana fermion. The characters of the Ising model are given by

$$\begin{aligned}\chi_0^{\text{Ising}} &= \frac{1}{2} \left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \right) = q^{-1/48} (1 + q^2 + q^3 + 2q^4 + 2q^5 + \dots), \\ \chi_{1/2}^{\text{Ising}} &= \frac{1}{2} \left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}} \right) = q^{23/48} (1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \dots), \\ \chi_{1/16}^{\text{Ising}} &= \frac{1}{\sqrt{2}} \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} = q^{1/24} (1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \dots).\end{aligned}\tag{3.82}$$

In the ordering (I, ψ, σ) , the modular representation is

$$\rho^{\text{Ising}}(T) = \text{diag}(\xi_{48}^{-1}, \xi_{48}^{23}, \xi_{48}^2),\tag{3.83a}$$

$$\rho^{\text{Ising}}(S) = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}.\tag{3.83b}$$

The characters and representation matrices of the Ising model show that the conductor is $N = 48$. Denoting the Ising model modular representation matrices by ρ^{Ising} , there are only two different values of $\rho^{\text{Ising}}(\sigma_p)$ that occur depending on whether $p^2 = 1 \pmod{48}$ or $p^2 = 25 \pmod{48}$:

$$\rho^{\text{Ising}}(\sigma_p) = \mathbb{I}_3, \quad p^2 = 1 \pmod{48}.\tag{3.84}$$

$$\rho^{\text{Ising}}(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad p^2 = 25 \pmod{48}. \quad (3.85)$$

The Hecke images $\mathbb{T}_p \chi^{\text{Ising}}$ for all prime p with $(p, 48) = 1$ lead to characters obeying all of the consistency conditions for RCFT characters. We mention here some examples where we have been able to match the Hecke images to the characters of known RCFT. For all $p < 24$ the Hecke images are the characters of affine $\text{spin}(p)$ algebras at level 1. While $\mathbb{T}_p \chi^{\text{Ising}}$ is a shift from the $\text{spin}(p)$ characters for $24 < p < 48$:

$$\begin{pmatrix} \chi_0 \\ \chi_v \\ \chi_s \end{pmatrix}^{\text{spin}(p)} - \begin{pmatrix} \chi_0 \\ \chi_v \\ \chi_s \end{pmatrix}^{Y=(p)} = p \mathcal{P} \cdot \begin{pmatrix} \chi_0 \\ \chi_v \\ \chi_s \end{pmatrix}^{\text{spin}(p-24)}, \quad (3.86)$$

where

$$\mathcal{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.87)$$

The super-index $Y = (p)$ represents the Hecke image under \mathbb{T}_p . The sub-indices $0, v, s$ of χ stand for the vacuum, vector and spinor representation respectively.

For $p = 31$ the Hecke image of the Ising model characters are the characters of affine E_8 at level 2:

$$\begin{aligned} \chi_0^{E_8^{(2)}} &= q^{-31/48} (1 + 248q + 31124q^2 + 871627q^3 + 13496501q^4 + \dots), \\ \chi_{3/2}^{E_8^{(2)}} &= q^{41/48} (3875 + 181753q + 3623869q^2 + 46070247q^3 + \dots), \\ \chi_{15/16}^{E_8^{(2)}} &= q^{7/24} (248 + 34504q + 1022752q^2 + 16275496q^3 + \dots). \end{aligned} \quad (3.88)$$

For $p = 47$ case the Hecke images are the characters of the Baby Monster vertex algebra constructed in [59]. The Baby Monster characters are

$$\begin{aligned}\chi_0^{\text{BM}} &= q^{-47/48}(1 + 96256q^2 + 9646891q^3 + 366845011q^4 + \dots), \\ \chi_{3/2}^{\text{BM}} &= q^{25/48}(4371 + 1143745q + 64680601q^2 + 1829005611q^3 + \dots), \\ \chi_{31/16}^{\text{BM}} &= q^{23/24}(96256 + 10602496q + 420831232q^2 + 9685952512q^3 + \dots).\end{aligned}\tag{3.89}$$

We now give a few example of Hecke images of the minimal model $M(2, 7)$ characters. The representations of S and T are

$$\rho^{\text{M}(2,7)}(T) = e\left(\frac{17}{42}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e\left(-\frac{3}{7}\right) & 0 \\ 0 & 0 & e\left(-\frac{2}{7}\right) \end{pmatrix},\tag{3.90a}$$

$$\rho^{\text{M}(2,7)}(S) = \frac{2 \sin\left(\frac{\pi}{7}\right)}{\sqrt{7}} \begin{pmatrix} d & 1 & 1 - d^2 \\ 1 & d^2 - 1 & d \\ 1 - d^2 & d & -1 \end{pmatrix}.\tag{3.90b}$$

For integers p with $\gcd(p, 42) = 1$ we find

$$\rho^{\text{M}(2,7)}(\sigma_p) = \mathbb{I}_3, \quad p^2 = 1 \pmod{42}.\tag{3.91}$$

$$\rho^{\text{M}(2,7)}(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad p^2 = 25 \pmod{42}.\tag{3.92}$$

$$\rho^{M(2,7)}(\sigma_p) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad p^2 = 37 \pmod{42}. \quad (3.93)$$

The Hecke image under $p = 5$ gives rise to a vector-valued modular form that is relevant to the WRT invariant of the Brieskorn homology sphere $\Sigma(2, 3, 7)$ [58]. The Hecke image under $p = 59$ recovers the three character model without Kac-Moody symmetry

$$\begin{aligned} \chi_0^{(59)} &= q^{-59/42}(1 + 63366q^2 + 46421200q^3 + 5765081101q^4 + \dots), \\ \chi_{16/7}^{(59)} &= q^{37/42}(715139 + 257698784q + 24078730130q^2 + \dots), \\ \chi_{17/7}^{(59)} &= q^{43/42}(848656 + 232637826q + 19201964416q^2 + \dots), \end{aligned} \quad (3.94)$$

obtained in [63] by analyzing the third-order MLDE.

3.5 Bilinear relations

In [63] some bilinear relations between RCFT characters and the modular J function were exhibited. These turn out to follow simply from properties of characters and their Hecke images. Let $f_1(\tau)$ and $f_2(\tau)$ be the Hecke images of some RCFT character $\chi(\tau)$:

$$f_i(\tau) = (\mathbb{T}_{p_i}\chi)(\tau), \quad i = 1, 2. \quad (3.95)$$

The induced modular representations are

$$\rho^{(p_i)}(T) = \rho(T^{\bar{p}_i}), \quad \rho^{(p_i)}(S) = G_{p_i}^{-1}\rho(S) = \rho(S)G_{p_i}. \quad (3.96)$$

Under the action of $SL(2, \mathbb{Z})$ generators T and S , the bilinear form $f_2(\tau)^T G_\ell f_1(\tau)$ transforms

as

$$\begin{aligned}
f_2(\tau + 1)^T G_\ell f_1(\tau + 1) &= f_2(\tau)^T [\rho(T^{\bar{p}_2}) G_\ell \rho(T^{\bar{p}_1})] f_1(\tau), \\
f_2(-1/\tau)^T G_\ell f_1(-1/\tau) &= f_2(\tau)^T [G_{p_2}^{-1} \rho(S) G_\ell \rho(S) G_{p_1}] f_1(\tau).
\end{aligned} \tag{3.97}$$

and will be modular invariant provided that

$$\begin{aligned}
\rho(T^{\bar{p}_2}) G_\ell \rho(T^{\bar{p}_1}) &= G_\ell, \\
G_{p_2}^{-1} \rho(S) G_\ell \rho(S) G_{p_1} &= G_\ell,
\end{aligned} \tag{3.98}$$

or equivalently

$$\rho(T^{\bar{p}_2 + \bar{p}_1 \ell^2}) = \mathbb{I}_n, \tag{3.99}$$

$$G_{p_2}^{-1} G_{-\bar{\ell}} G_{p_1} G_{\bar{\ell}} = \mathbb{I}_n. \tag{3.100}$$

Since the set of G_ℓ is homomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$ and hence abelian, eq(3.100) amounts to

$$G_{-p_1 \bar{p}_2 \bar{\ell}^2} = \mathbb{I}_n. \tag{3.101}$$

While eqn.(3.99) states the fact

$$\bar{p}_2 + \bar{p}_1 \ell^2 \equiv 0 \pmod{N}, \tag{3.102}$$

which necessarily implies $-p_1 \bar{p}_2 \bar{\ell}^2 \equiv 1 \pmod{N}$ therefore solves eq(3.100).

For the Ising model with conductor N equals 48 one can also construct modular invariant combinations. For example, when ℓ is taken to be 1, eqn.(3.102) boils down to

$$p_1 + p_2 = 0 \pmod{48}. \tag{3.103}$$

As a result, $(\mathbb{T}_{p\chi})^T (\mathbb{T}_{48-p\chi})$ is modular-invariant for $p \in (\mathbb{Z}/48\mathbb{Z})^\times$. Since it has the most

singular term q^{-1} , this bilinear form also gives $J(q)$ up to an additive constant, namely

$$(\mathbb{T}_p\chi)^T(\mathbb{T}_{48-p}\chi) = J(q) + c_0^{(p)}(1) + c_0^{(48-p)}(1). \quad (3.104)$$

CHAPTER 4

GALOIS SYMMETRY ON MTCS

MTCs of low rank are classified in [53, 34]. See [35] for a catalog of known MTCs. The basic data in a MTC include the twists (topological spins) which are exponentials of the conformal weights

$$\theta_i = e(h_i), \quad (4.1)$$

and the quantum dimensions d_i which are ratios of elements of the modular S matrix

$$d_i = \rho(S)_{0i} / \rho(S)_{00}. \quad (4.2)$$

A MTC is called unitary if each quantum dimension d_i is the Frobenius-Perron eigenvalue of the corresponding fusion matrix N_i . When the MTC is unitary, $\rho(S)_{00}$ is positive and the d_i are positive numbers greater than or equal to 1. The positive number

$$D = \sqrt{\sum_i d_i^2} = 1/|\rho(S)_{00}| \quad (4.3)$$

is called the total quantum order. The (topological) central charge c is related to the twists and the quantum dimensions by

$$e\left(\frac{c}{8}\right) = \rho(S)_{00} \sum_i \theta_i d_i^2. \quad (4.4)$$

A MTC may arise from more than one RCFT, since the MTC only fixes $h_i \pmod{1}$ and $c \pmod{8}$. A MTC is also equipped with duality matrices obeying the consistency conditions known as pentagon and the hexagon identities [43, 44].

In RCFT, real and pseudo-real primary fields have a difference in sign when their conformal blocks of two-point functions get braided. The reality property can be characterized by introducing

the Frobenius-Schur indicator (FSI). The FSI of the primary field a reads

$$\begin{aligned}\kappa_a &= \sum_{r,s} {}_0N_{rs}^a \rho(S)_{0r} \rho(S)_{0s} \theta_s^2 \theta_r^{-2} \\ &= \frac{1}{D^2} \sum_{r,s} {}_0N_{rs}^a d_r d_s \theta_s^2 \theta_r^{-2},\end{aligned}\tag{4.5}$$

which solely depends on the modular representation. Here, the FSI κ_a is ± 1 if the field a is self-conjugate, and is 0 if it is complex.

4.1 Simple-current reduction of affine algebra

We turn to a number of less familiar RCFT characters and explore the structure of their corresponding MTCs upon Hecke operations. In condensed matter physics, the (2+1)-dimensional (2+1D) bosonic topological orders are classified by unitary MTCs [32, 33, 34], and simple-current reduction is an important tool in this construction [30, 31]. A simple current J by definition has a single primary field $J\phi$ appearing in the fusion of J with any primary field ϕ , thus J permutes the fields by $J \times \phi = J\phi$, and divides the field content into orbits under the action of J . Because there are a finite number of primary fields, there exists a smallest positive integer n such that $J^n = I$ in the sense of fusion. This n is called the order of the simple current J . See [27] for a general discussion of simple currents.

In (A_1, k) with odd k , the spin- $\frac{k}{2}$ primary field $\varphi_{\frac{k}{2}}$ is a simple current with the fusion rules

$$\varphi_j \times \varphi_{\frac{k}{2}} = \varphi_{\frac{k}{2}-j}, \quad j = 0, \frac{1}{2}, 1, \dots, \frac{k}{2}.\tag{4.6}$$

It maps the half-integer representations onto the integer representations, and vice versa. The MTC $(A_1, k)_{\frac{1}{2}}$ for k odd consists of the primary fields of integer spin in (A_1, k) , and is called the even half of (A_1, k) [31, 34]. We use the notation $\overline{(A_1, k)}$ for the MTC whose twists and modular representation are the complex conjugate of those of (A_1, k) . With $c = c(A_1, k) \mp 1$, the MTC is

understood as the tensor product

$$(A_1, k) = \begin{cases} (A_1, k)_{\frac{1}{2}} \otimes (A_1, 1), & \text{if } k - 1 \equiv 0 \pmod{4}; \\ (A_1, k)_{\frac{1}{2}} \otimes \overline{(A_1, 1)}, & \text{if } k + 1 \equiv 0 \pmod{4}. \end{cases} \quad (4.7)$$

The first nontrivial example is $(A_1, 3)_{\frac{1}{2}}$, the integer subset of $(A_1, 3)$. It contains the primary fields φ_0 and φ_1 with the fusion rules

$$\varphi_0 \times \varphi_1 = \varphi_1, \quad \varphi_1 \times \varphi_1 = \varphi_0 + \varphi_1, \quad (4.8)$$

which are isomorphic to those of the Fibonacci theory. Moreover, the modular data confirm that $(A_1, 3)_{\frac{1}{2}}$ sits in the Fibonacci MTC like $(G_2, 1)$.

Next we study two specific examples, $(A_1, 5)_{\frac{1}{2}}$ and $(A_1, 7)_{\frac{1}{2}}$. Their modular representations are closely related to the minimal models $M(2, 7)$ and $M(2, 9)$ respectively. We study the Hecke images of their characters and modular representations, as well as the realizations of these Hecke images in VOAs and MTCs.

4.1.1 Rank three

The $(A_1, 5)_{\frac{1}{2}}$ MTC is realized at central charge

$$c \left[(A_1, 5)_{\frac{1}{2}} \right] = c[(A_1, 5)] - c[(A_1, 1)] = 15/7 - 1 = 8/7, \quad (4.9)$$

and its conformal weights are computed from $(A_1, 5)$ as

$$h_{j=0} = 0, \quad h_{j=1} = \frac{2}{7}, \quad h_{j=2} = \frac{6}{7}. \quad (4.10)$$

With this basis ordering, the modular representation $\rho^{(A_1,5)\frac{1}{2}}$ is determined by

$$\rho^{(A_1,5)\frac{1}{2}}(T) = e\left(-\frac{1}{21}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e(\frac{2}{7}) & 0 \\ 0 & 0 & e(\frac{6}{7}) \end{pmatrix}, \quad (4.11a)$$

$$\rho^{(A_1,5)\frac{1}{2}}(S) = \frac{2 \sin(\frac{\pi}{7})}{\sqrt{7}} \begin{pmatrix} 1 & d^2 - 1 & d \\ d^2 - 1 & -d & 1 \\ d & 1 & 1 - d^2 \end{pmatrix}, \quad (4.11b)$$

where $d = 2 \cos(\frac{\pi}{7})$. From $\rho^{(A_1,5)\frac{1}{2}}(T)$ we see that the conductor is $N = 21$.

The minimal model $M(2, 7)$ has central charge $c[M(2, 7)] = -68/7$ and conformal weights

$$(h_{1,1}, h_{3,1}, h_{5,1}) = \left(0, -\frac{3}{7}, -\frac{2}{7}\right). \quad (4.12)$$

The labeling of conformal weights will be discussed in the next chapter. $M(2, 7)$ has conductor is $N' = 42$ and the modular representation is given before. Hecke images of the $M(2, 7)$ characters in some cases are vector-valued modular forms that have appeared in other context, but they fail to be the characters of a unitary RCFT, because of the existence of negative Fourier coefficients and fusion coefficients [1].

Three-dimensional modular representations whose kernels contain congruence subgroups have been classified by Theorem 2 in [24]. According to the classification, $\rho^{M(2,7)}$ and $f_{42,-5}(\rho^{(A_1,5)\frac{1}{2}})$ differ only by a one-dimensional representation

$$\rho^{1d}(T) = e\left(\frac{1}{6}\right), \quad \rho^{1d}(S) = -1. \quad (4.13)$$

This explains why $(A_1, 5)\frac{1}{2}$ has fusion rules that are isomorphic to those of $M(2, 7)$.

The complete set of $\rho^{M(2,7)}(\sigma_p)$ are provided in [1]. Explicit computation leads to the following $\rho^{(A_1,5)\frac{1}{2}}(\sigma_p)$ for all p , with $1 \leq p < 21$ and $\gcd(p, 21) = 1$.

When $p = 1, 8, 13, 20$,

$$\rho^{(A_1,5)_{\frac{1}{2}}}(\sigma_p) = \mathbb{I}_3, \quad p^2 = 1 \pmod{21}. \quad (4.14)$$

When $p = 2, 5, 16, 19$,

$$\rho^{(A_1,5)_{\frac{1}{2}}}(\sigma_p) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad p^2 = 4 \pmod{21}. \quad (4.15)$$

When $p = 4, 10, 11, 17$,

$$\rho^{(A_1,5)_{\frac{1}{2}}}(\sigma_p) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad p^2 = 16 \pmod{21}. \quad (4.16)$$

Among all the $p \in (\mathbb{Z}/N\mathbb{Z})^\times$, the series $p = 1, 8, 13, 20$ give rise to two inequivalent unitary MTCs that are complex conjugates. These $\rho^{(p)}$ have VOA realizations [36], whose characters are not Hecke images of any primitive characters under \mathbb{T}_p though. For $p = 8, 13, 20$, the central charge inferred from the characters is $8p/7$; while for $p = 1$ there is no Kac-Moody sub-VOA for $(A_1, 5)_{\frac{1}{2}}$ at central charge $8/7$ [40]. Nevertheless there exists a three-character corresponding to $c = 22 * 8/7 \equiv 8/7 \pmod{24}$ and thereby the modular representation $\rho^{(A_1,5)_{\frac{1}{2}}}$. The case with $c = 8 * 8/7$ is realized as a simple-current reduction of $(A_1, 5) \otimes (E_7, 1)$. However for $p = 13, 20, 22$, the characters associated to these unitary MTCs still lack RCFT interpretations. They are not linked by Hecke operations neither. When $p^2 = 4$ or $16 \pmod{21}$, the induced MTCs by \mathbb{T}_p are non-unitary, and there are no Hecke image interpretations.

4.1.2 Rank four

We present a similar relation between $M(2, 9)$ and $(A_1, 7)_{\frac{1}{2}}$, which have the common conductor $N = 36$. The $(A_1, 7)_{\frac{1}{2}}$ MTC has central charge $c[(A_1, 7)_{\frac{1}{2}}] = 10/3$ and twists

$$\{\theta_j\} = \{e(h_j)\} = \left\{1, e\left(\frac{2}{9}\right), e\left(\frac{2}{3}\right), e\left(\frac{1}{3}\right)\right\}. \quad (4.17)$$

With this ordering of the twists, the modular representation is determined by

$$\rho^{(A_1, 7)_{\frac{1}{2}}}(T) = e\left(-\frac{5}{36}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e\left(\frac{2}{9}\right) & 0 & 0 \\ 0 & 0 & e\left(\frac{2}{3}\right) & 0 \\ 0 & 0 & 0 & e\left(\frac{1}{3}\right) \end{pmatrix}, \quad (4.18a)$$

$$\rho^{(A_1, 7)_{\frac{1}{2}}}(S) = \frac{2 \sin\left(\frac{\pi}{9}\right)}{3} \begin{pmatrix} 1 & r^2 - 1 & r + 1 & r \\ r^2 - 1 & 0 & 1 - r^2 & r^2 - 1 \\ r + 1 & 1 - r^2 & r & -1 \\ r & r^2 - 1 & -1 & -r - 1 \end{pmatrix}, \quad (4.18b)$$

where $r = 2 \cos\left(\frac{\pi}{9}\right)$. The minimal model $M(2, 9)$ has central charge $c[M(2, 9)] = -46/3$ and conformal weights

$$(h_{1,1}, h_{3,1}, h_{5,1}, h_{7,1}) = \left(0, -\frac{5}{9}, -\frac{2}{3}, -\frac{1}{3}\right). \quad (4.19)$$

The modular representation of $M(2, 9)$ is

$$\rho^{M(2,9)}(T) = e\left(\frac{23}{36}\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e\left(-\frac{5}{9}\right) & 0 & 0 \\ 0 & 0 & e\left(-\frac{2}{3}\right) & 0 \\ 0 & 0 & 0 & e\left(-\frac{1}{3}\right) \end{pmatrix}, \quad (4.20a)$$

$$\rho^{M(2,9)}(S) = \frac{2 \sin\left(\frac{\pi}{9}\right)}{3} \begin{pmatrix} -r & 1-r^2 & 1 & r+1 \\ 1-r^2 & 0 & r^2-1 & 1-r^2 \\ 1 & r^2-1 & r+1 & r \\ r+1 & 1-r^2 & r & -1 \end{pmatrix}. \quad (4.20b)$$

Note that $\rho^{M(2,9)}$ differs from $f_{36,-7} \left(\rho^{(A_1,7)_{\frac{1}{2}}} \right)$ by a one-dimensional representation

$$\rho^{1d}(T) = e\left(\frac{-1}{3}\right), \quad \rho^{1d}(S) = 1, \quad (4.21)$$

yielding the same fusion rules.

Explicit computations lead to the following $\rho(\sigma_p)$ for all p , with $1 \leq p < 36$ and $\gcd(p, 36) =$

1. The bases are ordered following eq(4.19) and (4.17) respectively.

When $p = 1, 35$, ($p^2 = 1 \pmod{36}$)

$$\rho^{M(2,9)}(\sigma_p) = \rho^{(A_1,7)_{\frac{1}{2}}}(\sigma_p) = \mathbb{I}_4. \quad (4.22)$$

When $p = 17, 19$, ($p^2 = 1 \pmod{36}$)

$$\rho^{M(2,9)}(\sigma_p) = \rho^{(A_1,7)_{\frac{1}{2}}}(\sigma_p) = -\mathbb{I}_4. \quad (4.23)$$

When $p = 5, 31, (p^2 = 25 \bmod 36)$

$$\rho^{\mathbf{M}(2,9)}(\sigma_p) = \rho^{(A_1,7)\frac{1}{2}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.24)$$

When $p = 13, 23, (p^2 = 25 \bmod 36)$

$$\rho^{\mathbf{M}(2,9)}(\sigma_p) = \rho^{(A_1,7)\frac{1}{2}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4.25)$$

When $p = 7, 29, (p^2 = 49 \bmod 36)$

$$\rho^{\mathbf{M}(2,9)}(\sigma_p) = \rho^{(A_1,7)\frac{1}{2}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (4.26)$$

When $p = 11, 25, (p^2 = 49 \bmod 36)$

$$\rho^{\mathbf{M}(2,9)}(\sigma_p) = \rho^{(A_1,7)\frac{1}{2}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (4.27)$$

Again, the observation that $\rho^{\mathbf{M}(2,9)}(\sigma_p) = \rho^{(A_1,7)\frac{1}{2}}(\sigma_p)$ is explained by the underlying Galois symmetry between the two MTCs.

Some Hecke images of $\chi^{M(2,9)}$ give the characters of affine Lie algebras:

$$\mathbb{T}_7\chi^{M(2,9)} = \chi^{(G_2,2)}, \quad (4.28)$$

$$\mathbb{T}_{29}\chi^{M(2,9)} = \chi^{(C_{5,3}) \otimes (A_{1,1})}. \quad (4.29)$$

These four-character theories are listed in Table 3 of [39]. Their bilinear form

$$\mathbb{T}_7\chi^{M(2,9)} \cdot \mathbb{T}_{29}\chi^{M(2,9)} = J(\tau) + 72 \quad (4.30)$$

reproduces the partition function of No. 21 in Schelleken's list of $c = 24$ meromorphic CFTs [42], where

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots \quad (4.31)$$

is the modular J function.

$(A_1, 7)_{\frac{1}{2}}$ has a VOA realization as the simple-current reduction of $(A_1, 7) \otimes (A_1, 1)$. Its characters are constructed as

$$\chi_j^{(A_1,7)_{\frac{1}{2}}}(\tau) = \chi_j^{(A_1,7)}(\tau)\chi_0^{(A_1,1)}(\tau) + \chi_{\frac{7}{2}-j}^{(A_1,7)}(\tau)\chi_{\frac{1}{2}}^{(A_1,1)}(\tau), \quad (4.32)$$

where the subscript $j = 0, 1, 2, 3$ stands for the spin- j representation of $SU(2)$. The characters afford the q -series expansions

$$\chi_0^{(A_1,7)_{\frac{1}{2}}}(\tau) = q^{-\frac{5}{36}}(1 + 6q + 38q^2 + 112q^3 + 347q^4 + \dots), \quad (4.33a)$$

$$\chi_1^{(A_1,7)_{\frac{1}{2}}}(\tau) = q^{\frac{1}{12}}(3 + 30q + 114q^2 + 384q^3 + 1065q^4 + \dots), \quad (4.33b)$$

$$\chi_2^{(A_1,7)_{\frac{1}{2}}}(\tau) = q^{\frac{19}{36}}(13 + 62q + 230q^2 + 692q^3 + 1874q^4 + \dots), \quad (4.33c)$$

$$\chi_3^{(A_1,7)_{\frac{1}{2}}}(\tau) = q^{\frac{7}{36}}(4 + 23q + 102q^2 + 319q^3 + 886q^4 + \dots). \quad (4.33d)$$

In principle, the Hecke images of the $(A_1, 7)_{\frac{1}{2}}$ characters can be calculated by the standard algorithm. Various Galois-conjugate representations of $(A_1, 7)_{\frac{1}{2}}$ are listed in Table 11 of [24], which summarizes four-dimensional simple strongly-modular fusion algebras up to one-dimensional modular representations.

Moreover, the MTCs of $(A_1, 7)_{\frac{1}{2}}$ and $(G_2, 2)$ are complex conjugate [31]. Their characters make up the bilinear form

$$\chi^{(A_1, 7)_{\frac{1}{2}}}(\tau) \cdot \chi^{(G_2, 2)}(\tau) = j(\tau)^{1/3}, \quad (4.34)$$

where $j(\tau) = J(\tau) + 744$ is the j -invariant.

4.2 MTCs of higher rank

The previous examples suggest a connection between $M(2, k+2)$ and $(A_1, k)_{\frac{1}{2}}$, since their fusion rules are isomorphic. This connection offers a series of examples that Galois conjugations convert non-unitary RCFTs to unitary ones, and vice versa. Moreover, both theories are related to critical behaviors of chains of antiferromagnetically coupled anyons as pointed out in [41].

The primary fields in $M(2, k+2)$ are denoted by $\phi_{(u,1)}$ with u an odd integer satisfying $1 \leq u < k+2$. They respect the fusion rules

$$\phi_{(u_1,1)} \times \phi_{(u_2,1)} = \sum_{\substack{u=1+|u_1-u_2| \\ u \text{ odd}}}^{u_{\max}} \phi_{(u,1)}, \quad (4.35)$$

where $u_{\max} = \min(u_1 + u_2 - 1, 2k + 3 - u_1 - u_2)$ [28]. The fusion rules of (A_1, k) resemble the compositions of $SU(2)$ angular momenta, namely

$$\varphi_{l_1} \times \varphi_{l_2} = \sum_{l=1+|l_1-l_2|}^{l_{\max}} \varphi_l, \quad (4.36)$$

where $l_{\max} = \min(l_1 + l_2 - 1, 2k + 3 - l_1 - l_2)$ [41]. The label $l = 2j + 1$ is the number of states

for integral spin- j , and is odd in $(A_1, k)_{\frac{1}{2}}$. Evidently, the fusion rules of $M(2, k+2)$ and $(A_1, k)_{\frac{1}{2}}$ are isomorphic with the identification $\phi_{(l,1)} \sim \varphi_l$.

More fundamentally, the isomorphism of fusion rules stems from the Galois symmetry between $M(2, k+2)$ and $(A_1, k)_{\frac{1}{2}}$. To conduct a general analysis, we set $N = 24(k+2)$, which is an integral multiple of both conductors. All the modular data are in the number field $\mathbb{Q}[\xi_N]$. We claim that $\rho^{M(2,k+2)}$ differs from the $(-k)$ -th Galois conjugate of $\rho^{(A_1,k)_{\frac{1}{2}}}$ by the one-dimensional representation ρ^{1d} , where

$$\rho^{1d}(T) = e\left(\frac{t}{6}\right), \quad \rho^{1d}(S) = (-1)^t, \quad (4.37)$$

if $k = 4t + 1$, or

$$\rho^{1d}(T) = e\left(-\frac{t}{6}\right), \quad \rho^{1d}(S) = (-1)^t, \quad (4.38)$$

if $k = 4t - 1$ with $t \in \mathbb{Z}_+$. The proof of this assertion is left to Appendix B. For the moment, the physical meaning of ρ^{1d} is unclear here. One-dimensional representations of modular fusion algebras are possible for central charge c a multiple of 4 [24], but RCFT/VOA realizations are only known for c a multiple of 8. When $c \equiv 0 \pmod{8}$, a RCFT such as affine E_8 at level one corresponds to the trivial MTC.

In summary, the $M(2, k+2)$ characters are known for general odd k and their Hecke images are computable. Though not unitary, $M(2, k+2)$ has a unitarization realized as $(A_1, k)_{\frac{1}{2}}$ by Galois symmetry. When $k \equiv -1 \pmod{4}$, the $(A_1, k)_{\frac{1}{2}}$ characters are constructed in an analogous way to those for $(A_1, 7)_{\frac{1}{2}}$, and can be acted on by Hecke operators. Although there is no standard way to realize $(A_1, k)_{\frac{1}{2}}$ via a VOA when $k \equiv 1 \pmod{4}$, Hecke operators can still be implemented on the level of character.

As mentioned earlier, some MTCs have ranks greater than four and involve complex-conjugate pairs of primaries. In such cases, we may identify each pair of complex primaries and reduce the modular representation to a smaller dimensionality, before acting with the Hecke operators. For

example, affine $SU(3)$ at level 1 has two primaries that create fields in the 3 and $\bar{3}$ of $SU(3)$, but since they have the same character one can construct a two-dimensional representation of the modular group given by the modular transformation of the vacuum character and one of these characters. As a more complicated example of this technique, we start with the six-character ψ associated to a special RCFT, which has $c = 8/5$ and $N = 15$. Let M_I be the MTC of this RCFT.

$$\psi_0(\tau) = q^{-\frac{1}{15}}(1 + 4q + 8q^2 + 20q^3 + 37q^4 + \dots), \quad (4.39a)$$

$$\psi_{\frac{2}{15}}(\tau) = q^{\frac{1}{15}}(1 + 2q + 7q^2 + 12q^3 + 26q^4 + \dots), \quad (4.39b)$$

$$\psi_{\frac{2}{15}}^*(\tau) = \psi_{\frac{2}{15}}(\tau), \quad (4.39c)$$

$$\psi_{\frac{4}{5}}(\tau) = q^{\frac{11}{15}}(3 + 4q + 10q^2 + 20q^3 + 38q^4 + \dots), \quad (4.39d)$$

$$\psi_{\frac{1}{3}}(\tau) = q^{\frac{4}{15}}(2 + 5q + 12q^2 + 23q^3 + 46q^4 + \dots), \quad (4.39e)$$

$$\psi_{\frac{1}{3}}^*(\tau) = \psi_{\frac{1}{3}}(\tau), \quad (4.39f)$$

where the sub-index of ψ refers to the conformal weight. This RCFT has two pairs of complex primaries, which are of conformal weights $2/15$ and $1/3$ respectively. It is constructed as an intermediate vertex sub-algebra like those in [16, 17]. Moreover, its modular T matrix is of odd order, though the orders of $\rho(T)$ tend to be even in generic RCFTs. The components of

$$\psi = \left(\psi_0, \psi_{\frac{2}{15}}, \psi_{\frac{4}{5}}, \psi_{\frac{1}{3}} \right) \quad (4.40)$$

are solutions to a fourth-order modular linear differential equation (MLDE), and are closed under the $SL(2, \mathbb{Z})$ transformations since the MLDE is modular invariant. The differential equation involves three free parameters μ_1, μ_2, μ_3 , and takes the form

$$(\mathcal{D}^4 + \mu_1 E_4 \mathcal{D}^2 + \mu_2 E_6 \mathcal{D} + \mu_3 E_8) f = 0, \quad (4.41)$$

where $\mathcal{D} = d/d\tau - \frac{1}{6}i\pi k E_2$ is the Serre derivative acting on weight- k modular forms, and E_j is

the Eisenstein series of weight j [14]. We denote by $\ell(W)/6$ the number of zeros in the Wronskian $W = W_n$. The form of eq(4.41) implies $\ell(W) = 0$ here. The $(A_2, 1)$ MTC is the tensor product of M_I and the Yang-Lee model, plus a simple-current reduction. Neither M_I nor ψ is listed in the VOA encyclopedia [35] because M_I is non-unitary. ($c = 8/5$ should be understood as the effective central charge to be introduced later.) We anticipate a connection to the three-state Potts model due to resemblance of the field contents as well as the modular representations. Inspired by Consequence 4 in [26], we predict that the VOA of ψ contains the \mathcal{W}_3 algebra.

As a vector-valued modular form, ψ has Hecke images that are characters of affine Lie algebras:

$$\mathbb{T}_2\psi = \chi^{(A_2,2)}, \quad (4.42)$$

$$\mathbb{T}_{13}\psi = \chi^{(F_4,6)}. \quad (4.43)$$

They are also obtained by solving eq(4.41) [39]. When treated as six-character theories, $(A_2, 2)$ and $(F_4, 6)$ each have two complex-conjugate fields, in accord with the preimage ψ . Their MTCs are complex conjugates, and the bilinear form of their characters is modular invariant.

$$\begin{aligned} & \chi_0^{(A_2,2)} \chi_0^{(F_4,6)} + 2\chi_{\frac{4}{15}}^{(A_2,2)} \chi_{\frac{26}{15}}^{(F_4,6)} + \chi_{\frac{3}{5}}^{(A_2,2)} \chi_{\frac{7}{5}}^{(F_4,6)} + 2\chi_{\frac{2}{3}}^{(A_2,2)} \chi_{\frac{4}{3}}^{(F_4,6)} \\ & = q^{-1} + 60 + 196884q + 21493760q^2 + \dots \equiv J(\tau) + 60. \end{aligned} \quad (4.44)$$

The multiplicity 2 accounts for the complex primaries and is crucial to attain the modular invariance. This bilinear form produces Schellekens No. 14 [42]. The Hecke image of ψ under \mathbb{T}_{14} yields positive q -series, which can also be constructed by acting with \mathbb{T}_7 on $\chi^{(A_2,2)} = \mathbb{T}_2\psi$ since

$T_{mn} = T_m T_n$ for $\gcd(m, n) = 1$.

$$(T_{14}\psi)_0 = q^{-\frac{14}{15}}(1 + 56q + 87836q^2 + 7358176q^3 + \dots), \quad (4.45a)$$

$$(T_{14}\psi)_{\frac{28}{15}} = q^{\frac{14}{15}}(26730 + 2694384q + 99032220q^2 + \dots), \quad (4.45b)$$

$$(T_{14}\psi)_{\frac{28}{15}}^* = (T_{14}\psi)_{\frac{28}{15}}, \quad (4.45c)$$

$$(T_{14}\psi)_{\frac{6}{5}} = q^{\frac{4}{15}}(308 + 147280q + 9692893q^2 + \dots), \quad (4.45d)$$

$$(T_{14}\psi)_{\frac{5}{3}} = q^{\frac{11}{15}}(13608 + 1927233q + 82069848q^2 + \dots), \quad (4.45e)$$

$$(T_{14}\psi)_{\frac{5}{3}}^* = (T_{14}\psi)_{\frac{5}{3}}. \quad (4.45f)$$

ψ and $T_{14}\psi$ correspond to complex-conjugate MTCs, which are non-unitary. Their bilinear form gives the identical modular invariant $J(\tau) + 60$ as eq(4.44).

Remarkably for any odd integer $m > 1$, the cyclotomic field $\mathbb{Q}[\xi_m]$ is identical to $\mathbb{Q}[\xi_{2m}]$ because a $2m$ -th root of unity can be expressed as a power of an m -th

$$\xi_{2m} = -\xi_m^{\frac{m+1}{2}}. \quad (4.46)$$

It can also be seen from the fact that $(\mathbb{Z}/m\mathbb{Z})^\times$ and $(\mathbb{Z}/2m\mathbb{Z})^\times$ are isomorphic due to eq(3.67).

The above $T_2\psi$ and $T_{14}\psi$ seem to be examples of T_p with $p|N$. It is still open how to construct Hecke operators T_p with $p|N$.

CHAPTER 5

PICTURE OF EFFECTIVE CENTRAL CHARGE

We start with the characters χ of a RCFT (unitary or non-unitary) with effective central charge c_{eff} . If $T_p\chi$ are also the characters of a RCFT, then it follows from the formula for the Hecke transform that the central charge is

$$c^{(p)} = p c_{\text{eff}}, \quad (5.1)$$

as long as the criteria of unitarity are met [1]. Moreover, the conformal weights also change upon Hecke operations, as seen in eq(3.72). However the Hecke operation does not necessarily map the vacuum character of the original to the vacuum character of the new theory. To clarify the Hecke operation, we seek a systematic approach to the picture of effective central charge. This method aligns the fields in the Hecke image with the initial theory by similar statistics, and helps to locate the vacuum entry.

We use the notation that if X refers to a quantity in the initial theory, then \tilde{X} and $X^{(p)}$ stand for the counterparts in the effective description and the Hecke image under T_p respectively.

5.1 Unified method

The Virasoro minimal models were briefly mentioned in Chapter 3.3. They have well-known characters, and these characters provide an interesting class of vector-valued modular forms that can be acted on by Hecke operators. The minimal model $M(p_1, p_2)$ has central charge

$$c[M(p_1, p_2)] = 1 - 6 \frac{(p_1 - p_2)^2}{p_1 p_2} \quad (5.2)$$

and conformal weights

$$h_{r,s} = \frac{(p_1 r - p_2 s)^2 - (p_1 - p_2)^2}{4p_1 p_2} \quad (5.3)$$

for the primary fields labeled by (r, s) with $0 < r < p_2, 0 < s < p_1$. In a non-unitary minimal model, the central charge and the conformal weights can be negative. To provide a general analysis Gannon defines the minimal primary $o = (r_o, s_o)$ to be the primary field of lowest conformal weight, which corresponds to the unique $(r, s) \in \mathcal{I}$ obeying $p_1 r - p_2 s = 1$ [26]. He also shows that o has a positive $\rho(S)$ column. The effective central charge c_{eff} and the shifted conformal weights \mathfrak{h} are defined so that the character of o has leading singularity $q^{-c_{\text{eff}}/24}$ while other primaries have $q^{\mathfrak{h}-c_{\text{eff}}/24}$ as $q \rightarrow 0$. In what follows we denote by $\tilde{M}(p_1, p_2)$ the effective description of $M(p_1, p_2)$. For the minimal model $M(p_1, p_2)$, one has

$$c_{\text{eff}}[M(p_1, p_2)] \equiv c[\tilde{M}(p_1, p_2)] = 1 - \frac{6}{p_1 p_2} \quad (5.4)$$

and the shifted conformal weights

$$\mathfrak{h}_{r,s} = \frac{(p_1 r - p_2 s)^2 - 1}{4p_1 p_2}, \quad (r, s) \in \mathcal{I}. \quad (5.5)$$

$\mathfrak{h}_{r,s}$ is a mere constant shift from $h_{r,s}$, and should not be confused with the effective conformal weights to be presented later. The conductor remains the same in this description.

We can extend the analysis for minimal models to generic RCFTs. There are two generic ways to find symmetric S matrices which diagonalize the fusion coefficients N_i : simple currents and the Galois symmetry. Under either of them the symmetry condition of $\rho(S)$ is preserved. There exists a unique chiral primary o called the minimal primary, which has the lowest conformal weight in the RCFT [26]. Let c_{eff} still denote the effective central charge. As always, the character of the o primary has the leading term $q^{-c_{\text{eff}}/24}$. By Gannon's definition [26], a RCFT is said to have the Galois shuffle (GS) property if there is a simple current J_o (possibly the identity) and a Galois automorphism σ_o (possibly the identity), such that the precise relationship

$$o = J_o \times \sigma_o 0 \quad (5.6)$$

holds, where 0 is the vacuum primary. The right hand side is understood as the fusion of the fields J_o and $\sigma_o 0$. Moreover, J_o is of order 1 or 2 (so $4h_{J_o} \in \mathbb{Z}$). The GS property obviously holds for unitary theories. Gannon proves that the GS property is possessed by all \mathcal{W}_N minimal models, in particular \mathcal{W}_2 also known as the Virasoro minimal models.¹ There are modular representations that do not obey the GS property, for instance the one-dimensional representation $\rho(T) = -1$, $\rho(S) = -1$.

The concept of simple current is introduced in Chapter 3.1. The other element of the GS property is the Galois automorphism σ_o , which is chosen to be the permutation π_ℓ of fields labeled by some $\ell \in (\mathbb{Z}/N\mathbb{Z})^\times$. Recall that $(G_\ell)_{i,j} = \varepsilon_\ell(i) \delta_{\pi_\ell(i),j}$. For instance in the Virasoro minimal models, it permutes the primary fields according to

$$\pi_\ell : (r, s) \rightarrow (\ell r, \ell s). \quad (5.7)$$

It is convenient to require $\ell \in (\mathbb{Z}/8p_1p_2\mathbb{Z})^\times$ as well, hence ℓ is odd. On the level of modular representations, the permutation π_ℓ gives rise to the inner automorphism f_{N,ℓ^2} on $\mathbb{Q}[\xi_N]$, namely

$$f_{N,\ell^2}(\rho(\gamma)) = G_\ell \rho(\gamma) G_\ell^{-1}. \quad (5.8)$$

The GS property implies the relation

$$-\frac{c_{\text{eff}}}{24} \equiv -\frac{c}{24} \ell^2 + h_{J_o} \equiv \left(-\frac{c}{24} + h_{J_o}\right) \ell^2 \pmod{1} \quad (5.9)$$

in the $\rho(T)$ entries, where the second congruence comes from the fact that $4h_{J_o} \in \mathbb{Z}$. We call this relation the GS equation. Both J_o and σ_o yield signed permutations of the characters. There are

1. Akin to the Virasoro minimal models, \mathcal{W}_N minimal models are generated by fields of conformal weight N [51, 52].

only a finite number of quadratic residues

$$\ell^2 \in [(\mathbb{Z}/N\mathbb{Z})^\times]^2 \quad (5.10)$$

that validate the GS property. The choice of ℓ is not unique for each quadratic residue. Once the inner automorphism by suitable ℓ^2 has been chosen, the simple current J_o is uniquely determined. In what follows we omit the subscript of J_o and write the Galois automorphism σ_o as π_ℓ with explicit dependence on ℓ .

In the effective description, the modular representation reads

$$\tilde{\rho}(\gamma) = f_{N,\ell^2}(P \rho(\gamma) P^{-1}) = P f_{N,\ell^2}(\rho(\gamma)) P^{-1} = P G_\ell \rho(\gamma) (P G_\ell)^{-1}, \quad (5.11)$$

where P is the permutation matrix $P_{ab} = \delta_{J_a,b}$. The orthogonal matrices P and G_ℓ commute, implying that they are different types of permutation. Hence, one can write $\tilde{\rho}(\gamma)$ in the form of inner automorphism

$$f_{N,\ell^2}(P \rho(\gamma) P^{-1}) = G_\ell (P \rho(\gamma) P^{-1}) G_\ell^{-1}, \quad (5.12)$$

and regard $P \rho(\gamma) P^{-1}$ as a new modular matrix. In terms of the $SL(2, \mathbb{Z})$ generators, the matrix elements are

$$\tilde{\rho}(T)_{ab} = f_{N,\ell^2}(\rho(T)_{J_a,J_b}) = \delta_{ab} (\rho(T)_{J_a,J_a})^{\ell^2}, \quad (5.13a)$$

$$\tilde{\rho}(S)_{ab} = f_{N,\ell^2}(\rho(S)_{J_a,J_b}) = e[\ell^2 Q_J(Jb)] \epsilon_{\ell^2}(Jb) \rho(S)_{a,\pi_{\ell^2} Jb}. \quad (5.13b)$$

As such, $\tilde{\rho}(T)$ is obtained by shuffling the diagonal elements of $\rho(T)$. The effective conformal

weights \tilde{h}_a are inferred from $\tilde{\rho}(T)$:

$$\begin{aligned}\tilde{h}_a - \frac{c_{\text{eff}}}{24} &\equiv \ell^2 \left(h_{Ja} - \frac{c}{24} \right) \pmod{1} \\ \Rightarrow \tilde{h}_a &\equiv \ell^2 (h_{Ja} - h_J) \pmod{1}.\end{aligned}\tag{5.14}$$

\tilde{h}_a is not the shifted conformal weight \mathfrak{h}_a in eq(5.5) for any $a \in \mathcal{I}$. Instead, one deduces from the shuffling rule $\tilde{\rho}(T)_{aa} = \rho(T)_{\pi_\ell Ja, \pi_\ell Ja}$ that

$$\begin{aligned}e \left(\tilde{h}_a - \frac{c_{\text{eff}}}{24} \right) &= e \left(h_{\pi_\ell Ja} - \frac{c}{24} \right) = e \left(\mathfrak{h}_{\pi_\ell Ja} - \frac{c_{\text{eff}}}{24} \right) \\ \Rightarrow \tilde{h}_a &\equiv \mathfrak{h}_{\pi_\ell Ja} \pmod{1}.\end{aligned}\tag{5.15}$$

We anticipate that $\tilde{\theta}_a$ and θ_a are roots of unity of the same order. The first row/column of $\tilde{\rho}(S)$ need not be positive, because $\tilde{\rho}(S)$ does not necessarily transform positive characters. As will be seen shortly, the form of $\tilde{\rho}(S)$ enables non-negative fusion rules.

Based on the GS property, Gannon proposed a method called ‘‘unitarization’’, which converts RCFTs to unitary ones with identical fusion rules [26]. Given a RCFT with central charge c , its unitarization usually has central charge that is an integral multiple of c . In this case, the unitarization is essentially equivalent to the method of Hecke operation, where the unitarization focuses on the MTC aspect while the Hecke operation deals with the characters. Without the integral relation of central charges, there is no interpretation for the unitarization in terms of the Hecke image.

Our approach differs from Gannon’s in that the effective description exploits the GS property but does not unitarize the initial theory. There could be several effective descriptions (ℓ^2, J) with distinct representations $\tilde{\rho}(\gamma)$, but they correspond to the unique c_{eff} . By the form of $\tilde{\rho}(\gamma)$, the effective description does not alter the conductor. Moreover, c_{eff} are known for the Virasoro minimal models, which allows us to solve (ℓ^2, J) in the GS equation.

A crucial subset of Hecke operators are \mathbb{T}_p with $\bar{p}^2 \equiv \ell^2 \pmod{N}$, where (ℓ^2, J) is an effective description. In this case, the Hecke operation \mathbb{T}_p comprises two steps as in eq(3.69). The signed permutation $\rho(\sigma_p) = G_p^{-1}$ implements the same transformation as the inner automorphism

f_{N,ℓ^2} does in the effective description. The effective picture imposes a criteria for unitary Hecke images. In practice, once the (effective) central charges are known, we only need to check the following equation in order to know whether T_p produces characters of unitary RCFT.

$$\frac{1}{24}(p^2 c_{\text{eff}} - c) \equiv -h_J \pmod{1}, \quad (5.16)$$

where J is a simple current in RCFT. Like in the effective description, the conductor stays invariant under the Hecke operation. As we will see, the fusion rules do not change under the joint action by J and π_ℓ , with the same ordering in the representation matrix. In the rest of this chapter, we impose the constraint $\bar{p}^2 \equiv \ell^2 \pmod{N}$ so as to get unitary theories under T_p .

5.2 Fusion rules of Hecke image

In this subsection we discuss the fusion rules for the Hecke image which are related to the couplings of the primary fields in the Hecke image. The derived fusion rules are one physical implication of Hecke relations and pave the way for computing the duality properties algebraically.

The Hecke operation T_p takes the characters of a RCFT with effective central charge c_{eff} to their image characters, which may be characters of a RCFT which has central charge $c^{(p)} = p c_{\text{eff}}$. The modular representations of the two theories are related by the Frobenius map $f_{N,\bar{p}}$. It might seem that the initial theory and its Hecke image have same fusion rules, since the Frobenius map acts trivially on the integral fusion coefficients. However this reasoning is not accurate, because the fusion coefficients depend on the vacuum row as shown in the Verlinde formula. Though the Hecke image has the modular S matrix $\rho^{(p)}(S) = f_{N,\bar{p}}(\rho(S))$,

$$\{f_{N,\bar{p}}(\rho(S)_{0i}) \mid i \in \mathcal{I}\}$$

is no longer the vacuum row as the primaries have been shuffled.

Thus to study fusion rules of the Hecke image we should ensure that the field content is prop-

erly aligned so that we can identify the new vacuum character. To do so, we first translate the initial modular data to the picture of effective central charge. With the representation $\tilde{\rho}$ defined in eq(5.13), we compute the fusion rules ${}_0\tilde{N}_{ab}{}^c$ directly.

$$\begin{aligned}
{}_0\tilde{N}_{ab}{}^c &= \sum_m \frac{\tilde{\rho}(S)_{a,m} \tilde{\rho}(S)_{b,m} \tilde{\rho}(S)_{c,m}^{-1}}{\tilde{\rho}(S)_{0,m}} \\
&= f_{N,\ell^2} \left(\sum_m \frac{\rho(S)_{a,m} \rho(S)_{b,m} \rho(S)_{c,m}^{-1}}{\rho(S)_{0,m}} e^{[Q_J(a) + Q_J(b) - Q_J(c) - Q_J(0)]} \right) \quad (5.17) \\
&= f_{N,\ell^2}({}_0N_{ab}{}^c) = {}_0N_{ab}{}^c,
\end{aligned}$$

where the selection rule eq(3.19) is used. Hence, $\tilde{\rho}$ leads to the identical fusion rules as in the initial theory, which are of course non-negative. There are not necessarily physical fields that give rise to the representation $\tilde{\rho}$. In fact, the q -series that transform under $\tilde{\rho}$ are the initial characters with a signed permutation, and could have negative Fourier coefficients.

Being brought to the effective picture, it remains to multiply the effective conformal weights by p along with changing the Fourier coefficients. It causes an action of $f_{N,p}$ on the modular representation, rendering the fusion rules invariant. The fusion rule in the image theory is

$$\begin{aligned}
{}_0N_{ab}{}^{c(p)} &= \sum_m \frac{f_{N,p}(\tilde{\rho}(S))_{am} f_{N,p}(\tilde{\rho}(S))_{bm} f_{N,p}(\tilde{\rho}(S)^{-1})_{cm}}{f_{N,p}(\tilde{\rho}(S))_{0m}} \\
&= f_{N,p} \left(\sum_m \frac{\tilde{\rho}(S)_{a,m} \tilde{\rho}(S)_{b,m} \tilde{\rho}(S)_{c,m}^{-1}}{\tilde{\rho}(S)_{0,m}} \right) \quad (5.18) \\
&= f_{N,p}({}_0\tilde{N}_{ab}{}^c) = {}_0\tilde{N}_{ab}{}^c.
\end{aligned}$$

The fields in the Hecke image are aligned in the same way as before. The first row of $f_{N,p}(\tilde{\rho}(\gamma))$ still corresponds to the vacuum, in the sense of $\mathbb{T}_p\chi$. We therefore confirm that the fusion rules are preserved under suitable Hecke operators, i.e. \mathbb{T}_p with $\tilde{p}^2 \equiv \ell^2 \pmod{N}$. In summary,

$${}_0N_{ab}{}^{c(p)} = {}_0\tilde{N}_{ab}{}^c = {}_0N_{ab}{}^c. \quad (5.19)$$

This result will prove essential in establishing the polynomial equations for various RCFTs.

5.3 $M(3, 5)$ as an example

We now illustrate the technique of effective picture using the minimal model $M(3, 5)$ as an example. The non-unitary minimal model $M(3, 5)$ has (effective) central charge

$$c[M(3, 5)] = -3/5, \quad c_{\text{eff}}[M(3, 5)] = 3/5. \quad (5.20)$$

The primary fields are

$$\phi_0, \phi_{-\frac{1}{20}}, \phi_{\frac{1}{5}}, \phi_{\frac{3}{4}}, \quad (5.21)$$

where the subscripts denote the conformal weights. $\phi_{-\frac{1}{20}}$ has the smallest conformal weight and is recognized as the minimal primary o . The vacuum field ϕ_0 is the trivial simple current, while $\phi_{\frac{3}{4}}$ has order 2 and permutes the primaries by

$$\left(\phi_0, \phi_{-\frac{1}{20}}, \phi_{\frac{1}{5}}, \phi_{\frac{3}{4}}\right) \times \phi_{\frac{3}{4}} = \left(\phi_{\frac{3}{4}}, \phi_{\frac{1}{5}}, \phi_{-\frac{1}{20}}, \phi_0\right). \quad (5.22)$$

The modular representation is

$$\rho^{M(3,5)}(T) = \text{diag}(\xi_{40}, \xi_{40}^{-1}, \xi_{40}^9, \xi_{40}^{-9}), \quad (5.23a)$$

$$\rho^{M(3,5)}(S) = \begin{pmatrix} x & y & -y & -x \\ y & x & x & y \\ -y & x & -x & y \\ -x & y & y & -x \end{pmatrix}, \quad (5.23b)$$

where

$$x = \sqrt{\frac{2}{5}} \sin\left(\frac{2\pi}{5}\right), \quad y = \sqrt{\frac{2}{5}} \sin\left(\frac{\pi}{5}\right). \quad (5.24)$$

With $N = 40$, the quadratic subgroup is $[(\mathbb{Z}/N\mathbb{Z})^\times]^2 = \{1, 9\}$.

With the trivial simple current, the GS equation does not hold for any $\ell^2 \in [(\mathbb{Z}/N\mathbb{Z})^\times]^2$. The GS property requires the simple current $\phi_{\frac{3}{4}}$ and the quadratic element $\ell^2 \equiv 9 \pmod{N}$. Following eq(5.13), one converts $\rho^{\tilde{M}(3,5)}(\gamma)$ to the representation in the effective picture.

$$\rho^{\tilde{M}(3,5)}(T) = \text{diag}(\xi_{40}^{-1}, \xi_{40}, \xi_{40}^{-9}, \xi_{40}^9), \quad (5.25a)$$

$$\rho^{\tilde{M}(3,5)}(S) = \begin{pmatrix} x & -y & -y & x \\ -y & x & -x & y \\ -y & -x & -x & -y \\ x & y & -y & -x \end{pmatrix}. \quad (5.25b)$$

The effective twists are computed from $\rho^{\tilde{M}(3,5)}(T)$ as

$$\tilde{\theta} = 1, e\left(\frac{1}{20}\right), e\left(\frac{4}{5}\right), e\left(\frac{1}{4}\right). \quad (5.26)$$

In terms of field content, $\tilde{M}(3, 5)$ is viewed as the tensor product of affine algebra $(A_1, 1)$ and $\overline{\tilde{M}(2, 5)}$. The latter is the complex conjugate of $\tilde{M}(2, 5)$ (Yang-Lee model), which has

$$c[\tilde{M}(2, 5)] = c_{\text{eff}}[M(2, 5)] = 2/5. \quad (5.27)$$

As a result, the modular representation of $\tilde{M}(3, 5)$ is simply the Kronecker product of $(A_1, 1)$ and $\overline{\tilde{M}(2, 5)}$. The $(A_1, 1)$ MTC has the vacuum and the semion as primary fields. The semion has conformal weight $1/4$ and serves as the simple current in the tensor product structure. Note that $M(3, 5)$ has conductor $N = 40$.

The characters of $M(3, 5)$ are given by

$$\chi_0^{M(3,5)}(\tau) = q^{\frac{1}{40}}(1 + q^2 + q^3 + 2q^4 + 2q^5 + \dots), \quad (5.28a)$$

$$\chi_{-\frac{1}{20}}^{M(3,5)}(\tau) = q^{-\frac{1}{40}}(1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + \dots), \quad (5.28b)$$

$$\chi_{\frac{1}{5}}^{M(3,5)}(\tau) = q^{\frac{9}{40}}(1 + q + 2q^2 + 2q^3 + 3q^4 + 4q^5 + \dots), \quad (5.28c)$$

$$\chi_{\frac{3}{4}}^{M(3,5)}(\tau) = q^{\frac{31}{40}}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \dots). \quad (5.28d)$$

Among the components, $\chi_{-\frac{1}{20}}^{M(3,5)}$ contains the most singular term and corresponds to the minimal primary o , while $\chi_0^{M(3,5)}$ is the vacuum character due to its leading term $q^{-c[M(3,5)]/24}$. Moreover, the true vacuum is invariant under the Poincaré group and in particular under translations. Hence, the Virasoro generator L_{-1} annihilates the vacuum, i.e. $L_{-1}|0\rangle = 0$ and there is thus no q^1 term in the vacuum character.

The characters of $(A_1, 1)$ and $M(2, 5)$ have Hecke images which were computed in [1]. It is interesting to explore Hecke images of the $M(3, 5)$ characters as well. Explicit computation by eq(3.57) provides the list of $G_{\bar{p}}^{M(3,5)} = \rho^{M(3,5)}(\sigma_p)$ for all $p \in (\mathbb{Z}/N\mathbb{Z})^\times$.

When $p = 1, 11, 29, 39$,

$$\rho^{M(3,5)}(\sigma_p) = \mathbb{I}_4, \quad p^2 = 1 \pmod{40}. \quad (5.29)$$

When $p = 9, 19, 21, 31$,

$$\rho^{M(3,5)}(\sigma_p) = -\mathbb{I}_4, \quad p^2 = 1 \pmod{40}. \quad (5.30)$$

When $p = 3, 7, 33, 37$,

$$\rho^{M(3,5)}(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad p^2 = 9 \pmod{40}. \quad (5.31)$$

When $p = 13, 17, 23, 27$,

$$\rho^{M(3,5)}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad p^2 = 9 \pmod{40}. \quad (5.32)$$

The four distinct values of $\rho^{M(3,5)}(\sigma_p)$ form the cyclic group C_4 under multiplication. As we shall see later, $(A_1, 3)$ is the Hecke image theory of $M(3, 5)$ under T_3 . The matrices $\rho^{(A_1,3)}(\sigma_p)$ are the same as $\rho^{M(3,5)}(\sigma_p)$ with proper ordering of the basis. The authors in [11] computed the four distinct values of $\rho^{(A_1,3)}(\sigma_p)$ and regarded C_4 as the Galois group on primary fields. But the Galois group of fusion rules we refer to is basically the permutation within the matrices $\{N_i | i \in \mathcal{I}\}$, where the permutation is given by eq(19) in [11] and N_i is defined by eq(3.16). Since $\rho(\sigma_p)$ only tells how the primary fields are shuffled given its non-zero entries, the overall sign of $\rho(\sigma_p)$ does not affect the field permutation. Hence the fusion rule automorphism is characterized by $\pm \rho^{M(3,5)}(\sigma_p)$ in $M(3, 5)$ or $(A_1, 3)$, and the Galois group of fusion rules is exactly $\mathcal{G} = C_2$, in agreement with the group of quadratic residues $[(\mathbb{Z}/N\mathbb{Z})^\times]^2$.

Not every Hecke image corresponds to a unitary RCFT. A unitary RCFT or MTC requires non-negative integral fusion coefficients that are determined by the Verlinde formula eq(3.15). If the constraint of unitarity is relaxed, there could be negative fusion coefficients though with positive q -series. For simplicity, here we focus on the Hecke images which have interpretations as the characters of unitary RCFTs. They correspond to the series $p = 3, 7, 33, 37 \pmod{40}$. The Hecke

images $\mathbb{T}_p\chi^{\mathbb{M}(3,5)}$ with $p = 3, 7$ provide the characters of two affine Lie algebras.

$$\mathbb{T}_3\chi^{\mathbb{M}(3,5)} = \chi^{(A_1,3)}, \quad (5.33)$$

$$\chi_0^{(A_1,3)} = q^{-\frac{3}{40}}(1 + 3q + 9q^2 + 22q^3 + 42q^4 + 81q^5 + \dots), \quad (5.33a)$$

$$\chi_{\frac{3}{20}}^{(A_1,3)} = q^{\frac{3}{40}}(2 + 6q + 18q^2 + 36q^3 + 78q^4 + 144q^5 + \dots), \quad (5.33b)$$

$$\chi_{\frac{2}{5}}^{(A_1,3)} = q^{\frac{13}{40}}(3 + 9q + 20q^2 + 45q^3 + 90q^4 + 170q^5 + \dots), \quad (5.33c)$$

$$\chi_{\frac{3}{4}}^{(A_1,3)} = q^{\frac{27}{40}}(4 + 6q + 18q^2 + 34q^3 + 72q^4 + 126q^5 + \dots); \quad (5.33d)$$

$$\mathbb{T}_7\chi^{\mathbb{M}(3,5)} = \chi^{(C_3,1)}, \quad (5.34)$$

$$\chi_0^{(C_3,1)} = q^{-\frac{7}{40}}(1 + 21q + 126q^2 + 511q^3 + 1743q^4 + \dots), \quad (5.34a)$$

$$\chi_{\frac{7}{20}}^{(C_3,1)} = q^{\frac{7}{40}}(6 + 70q + 336q^2 + 1302q^3 + 4186q^4 + \dots), \quad (5.34b)$$

$$\chi_{\frac{3}{5}}^{(C_3,1)} = q^{\frac{17}{40}}(14 + 105q + 483q^2 + 1764q^3 + 5523q^4 + \dots), \quad (5.34c)$$

$$\chi_{\frac{3}{4}}^{(C_3,1)} = q^{\frac{23}{40}}(14 + 78q + 378q^2 + 1288q^3 + 4032q^4 + \dots). \quad (5.34d)$$

The affine Lie algebras $(A_1, 3)$ and $(C_3, 1)$ have central charges $p c_{\text{eff}}$ with $p = 3$ and 7 , respectively. However there is no obvious way to realize the RCFTs for $p = 33, 37$, though the derived MTCs by Galois conjugation are unitary. They are perhaps intermediate vertex subalgebras similar to the $E_{7\frac{1}{2}}$ theory [16, 17]. As expected, all four MTCs in this series enter into the classification of topological orders [30, 31]. Notably Hecke images of $\chi^{\mathbb{M}(3,5)}$ can be solved from the MLDE eq(4.41). The modular representation for $\mathbb{T}_p\chi^{\mathbb{M}(3,5)}$ is

$$\rho^{(p)}(\gamma) = f_{N,p} \left(\rho^{\tilde{\mathbb{M}}(3,5)}(\gamma) \right). \quad (5.35)$$

In particular for $p = 3, 7, 33, 37$,

$$\rho^{(p)}(S) = \begin{pmatrix} y & x & x & y \\ x & y & -y & -x \\ x & -y & -y & x \\ y & -x & x & -y \end{pmatrix} \quad (5.36)$$

meets all the requirements of unitary MTC, i.e. the non-negative fusion coefficients and the quantum dimensions $d_i^{(p)} \geq 1$.

We infer the fusion rules in the Hecke images of $M(3, 5)$ by the analysis earlier. They are expressed in terms of the matrices N_i defined in eq(3.16), where a super-index labels the RCFT and a sub-index indicates the conformal weight as usual. The primary fields are arrayed in the same order as before.

$$N_0^{M(3,5)} = N_0^{\tilde{M}(3,5)} = N_0^{(A_1,3)} = N_0^{(C_3,1)} = \mathbb{I}_4. \quad (5.37)$$

$$N_{-\frac{1}{20}}^{M(3,5)} = N_{\frac{1}{20}}^{\tilde{M}(3,5)} = N_{\frac{3}{20}}^{(A_1,3)} = N_{\frac{7}{20}}^{(C_3,1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (5.38)$$

$$N_{\frac{1}{5}}^{M(3,5)} = N_{\frac{4}{5}}^{\tilde{M}(3,5)} = N_{\frac{2}{5}}^{(A_1,3)} = N_{\frac{3}{5}}^{(C_3,1)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (5.39)$$

$$N_{\frac{3}{4}}^{M(3,5)} = N_{\frac{1}{4}}^{\tilde{M}(3,5)} = N_{\frac{3}{4}}^{(A_1,3)} = N_{\frac{3}{4}}^{(C_3,1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (5.40)$$

They agree perfectly with the fusion rules calculated from the modular data in these RCFTs.

Let us not forget that $M(3, 5)$, as a member of non-unitary minimal models, has long been known to describe critical phases of 2D classical statistical mechanics models, such as the “restricted solid-on-solid” (RSOS) models [25]. In condensed matter physics, $M(3, 5)$ is of particular interest as it describes the critical behavior of a chain of antiferromagnetically coupled Yang-Lee anyons [41]. The earlier discovered Galois conjugation relations between Yang-Lee and Fibonacci anyons serve as a first example of the broader Galois symmetries induced by Hecke relations between different RCFTs we present in this thesis.

CHAPTER 6

DUALITY TRANSFORMATION OF CONFORMAL BLOCKS

Besides modular invariance, duality is another distinctive property of RCFT. In this chapter, we describe the duality transformations in RCFT and build the formalism for probing Galois symmetry. This chapter is largely a review of the literature, and set up the notation for the following chapter.

6.1 Chiral vertex operators and conformal blocks

In preparation for our discussion of duality, we first define the chiral vertex operators (CVOs). Their correlation functions are conformal blocks for physical correlation functions. The exchange symmetries of conformal blocks are described by duality transformations. See [43, 45, 44] for mathematical details.

The physical Hilbert space $\mathcal{H}_{\text{phys}}$ is a direct sum over irreducible representations of $\mathcal{A} \times \overline{\mathcal{A}}$, as is reviewed in eq(3.1). Every state in the decomposition transforms as the representation (V_i, \overline{V}_i) . The CVO is the intertwining operator for chiral representations, with dependence on the coordinate z on the complex plane. Given three representations labeled by $i, j, k \in \mathcal{I}$, we define the operator

$$\Phi_t(z) : (V_i)^\vee \otimes V_j \otimes V_k \rightarrow \mathbb{C}, \quad (6.1)$$

where $(V_i)^\vee$ is the dual of V_i . The representations are ordered such that j, k refer to the incoming states and i labels the outgoing one. Such operators are called of type $(i; j, k)$, and the subscript t distinguishes between different operators of the same type. In general the CVOs of type $(i; j, k)$ span a vector space V_{jk}^i , which has dimensionality

$$\dim V_{jk}^i = N_{jki^\vee} = N_{jk}^i. \quad (6.2)$$

The numbers N_{jk}^i are the fusion rules determined by the Verlinde formula eq(3.15), and their dependence on the vacuum 0 is omitted occasionally. The case $N_{jk}^i \leq 1$ contains most essential

features of RCFT and affords a simpler description. In this situation, there is only one operator of type $(i; j, k)$, which can be written as Φ_{jk}^i for brevity.

In RCFT conformal blocks form a basis for physical 4-point functions. Each conformal block is computed by gluing two CVOs at points which we label as z_2, z_3 , with the initial and the final state at 0 and ∞ respectively [3].

$$\mathcal{F}_p^{ijkl}(z_2, z_3) := \langle i | \Phi_{ip}^j(z_2) \Phi_{pl}^k(z_3) | l \rangle \quad (6.3)$$

Figure 6.1 gives a graphical description, where the indices i, j, k, l stand for the external legs while p labels the field in the mediated channel. In the diagonal theory, the physical correlation function is

$$\langle \phi_i(\infty, \infty) \phi_j(z_2, \bar{z}_2) \phi_k(z_3, \bar{z}_3) \phi_l(0, 0) \rangle = \sum_{p \in \mathcal{I}} D_p |\mathcal{F}_p^{ijkl}(z_2, z_3)|^2, \quad (6.4)$$

where D_p are constants independent of z and \bar{z} .

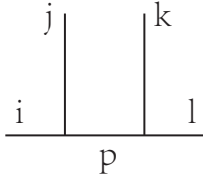


Figure 6.1: A geometric illustration of fusion, as the composition of two 4-point functions.

6.2 Fusion and braiding symmetries

The axiom of duality states that physical correlation functions do not depend on the choice of the basis of conformal blocks. The conformal block for any diagram is a linear combination of conformal blocks for any other [44]. In particular, duality of the 4-point functions implies the existence of fusion and braiding matrices, which are induced by F- and B-moves respectively.

When acting on $\mathcal{F}_p^{ijkl}(z_2, z_3)$, the F- and B-moves cause the change

$$F \begin{bmatrix} j & k \\ i & l \end{bmatrix} : \oplus_p V_{jp}^i \otimes V_{kl}^p \rightarrow \oplus_q V_{ql}^i \otimes V_{jk}^q, \quad (6.5)$$

$$B \begin{bmatrix} j & k \\ i & l \end{bmatrix} : \oplus_p V_{jp}^i \otimes V_{kl}^p \rightarrow \oplus_q V_{kq}^i \otimes V_{jl}^q, \quad (6.6)$$

where the matrix elements F_{pq}, B_{pq} specify the initial and the final terms in the direct sum [44]. Any duality transformation are expressible by these two basic moves. We will elucidate the fusion and the braiding matrix explicitly in terms of OPE.

Let z_{ij} be shorthand for $z_i - z_j$. The fusion matrix F is defined by

$$\Phi_{ip}^j(z_2)\Phi_{pl}^k(z_3) = \sum_{q \in \mathcal{I}} F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \sum_{Q \in V_q} \Phi_{il}^{q,Q}(z_3) \langle Q | \Phi_{qk}^j(z_{23}) | k \rangle, \quad (6.7)$$

where $Q \in V_q$ denotes the descendant states in the module V_q [3]. To obtain the OPE on the right hand side, we use the translation and scaling invariance. Figure 6.2 characterizes the s - t duality schematically. Two successive F-moves are equivalent to the identity transformation, leading to the quadratic relation

$$\sum_q F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} F_{qp'} \begin{bmatrix} l & k \\ i & j \end{bmatrix} = \delta_{pp'}. \quad (6.8)$$

The braiding matrix B is defined by

$$\Phi_{ip}^j(z_2)\Phi_{pl}^k(z_3) = \sum_{q \in \mathcal{I}} B_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \Phi_{iq}^k(z_3)\Phi_{ql}^j(z_2) \quad (6.9)$$

[3]. Figure 6.3 provides the graphical illustration for the s - u duality. In fact B_{pq} is the monodromy matrix for the vector of blocks $\mathcal{F}_p^{ijkl}(z_2, z_3)$ when z_2 circles around z_3 . The braiding matrix is

$$\begin{array}{c} j \\ | \\ i \text{---} \text{---} \text{---} l \\ p \end{array} = \sum_q F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \begin{array}{c} j \\ | \\ q \text{---} \text{---} l \\ p \end{array}$$

Figure 6.2: Fusion matrix between blocks. The labels of the matrix entries, i.e. p and q , take the positions of the “propagator”.

independent of z in each connected region of the common domain. Given two regions separated by a branch cut, there are two transformations

$$B(\epsilon), \quad \epsilon = \text{sgn}(\Im(z_{23})) \quad (6.10)$$

with the consistency condition

$$\sum_q B_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} (\epsilon) B_{qp'} \begin{bmatrix} k & j \\ i & l \end{bmatrix} (-\epsilon) = \delta_{pp'}. \quad (6.11)$$

It should be stressed that B^2 is not the identity matrix because of the cuts. If the sign ϵ is omitted, we are referring to $B(+)$. For a coupling t of type $(i; j, k)$, we define the operators Ω and Θ

$$\Omega(\pm) : V_{jk}^i \cong V_{kj}^i, \quad \Omega(\pm)(t) = e^{\pm i\pi\Delta_t} \varsigma_{23}(t), \quad (6.12a)$$

$$\Theta(\pm) : V_{jk}^i \cong V_{ji}^{k\vee}, \quad \Theta(\pm)(t) = \varsigma_{13}(e^{\pm i\pi\Delta_t} t), \quad (6.12b)$$

Here $\Delta_t = \Delta_j + \Delta_k - \Delta_i$, with Δ_i the smallest L_0 eigenvalue of the states in V_i . ς_{ij} is a transposition of i and j with $\varsigma_{ij}^2 = 1$. The extra phase $e^{\pm i\pi\Delta_t}$ compensates the phase arising from swapping the external legs, done by $z \rightarrow e^{\pm i\pi} z$ depending on the cut. Ω and Θ are special cases of the B-move. The operation Ω is also referred to as the R-move, and its eigenvalues are called the braiding eigenvalues or the R -matrices.

We start with a 4-point function and perform the duality transformations of the CVOs in two

$$\begin{array}{c} j \quad k \\ | \quad | \\ i \quad l \\ \hline p \end{array} = \sum_q B_{pq} \begin{array}{c} j \quad k \\ | \quad | \\ i \quad l \\ \hline q \end{array}$$

Figure 6.3: Braiding matrix between blocks. The labeling of the matrix entries, i.e. p and q , take the positions of the “propagator”.

ways as depicted in Figure 6.4. Ending in the same configuration, we build

$$B_{pp'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} (\epsilon) = \sum_{q \in \mathcal{I}} F_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} e^{-i\pi\epsilon(\Delta_k + \Delta_j - \Delta_q)} F_{qp'} \begin{bmatrix} l & j \\ i & k \end{bmatrix}, \quad (6.13)$$

or symbolically

$$B(\epsilon) = F^{-1} [1 \otimes \Omega(-\epsilon)] F. \quad (6.14)$$

The B -move is simply a combined operation of F - and R -moves. As a consequence, eigenvalues of the B -matrices are square roots of mutual locality factors and are deduced as half-monodromies.

$$\begin{array}{ccc} \begin{array}{c} j \quad k \\ | \quad | \\ i \quad l \\ \hline \end{array} & \xrightarrow{B} & \begin{array}{c} k \quad j \\ | \quad | \\ i \quad l \\ \hline \end{array} \\ F \downarrow & & \uparrow F^{-1} \\ \begin{array}{c} j \quad k \\ | \quad | \\ i \quad l \\ \hline \end{array} & \xrightarrow{1 \otimes \Omega} & \begin{array}{c} k \quad j \\ | \quad | \\ i \quad l \\ \hline \end{array} \end{array}$$

Figure 6.4: A simple loop transformation of conformal block.

The duality matrices are usually computed as follows. We first determine the fusion rules by the Verlinde formula and find all the fusion channels. Given any five-point function, we can

formulate different sequences of F-moves from the same starting fusion basis decomposition to the same ending decomposition. These consistency conditions build the polynomial equations called the *pentagon equations*. The solution to the pentagon equations is organized into the F -matrices, whose entries are known as the $6j$ symbols [34]. Likewise consistency relations arise if the R-moves act on the fusion space of three particles in different ways, ending in the commutative hexagon diagrams. The hexagon diagrams contain both F- and R-moves, making the braidings compatible with the fusions. They give rise to the *hexagon equations*. In practice, we first solve the pentagon equations to gain all the fusion matrices. We then insert the solved fusion matrices into the hexagon equations and determine all the braiding eigenvalues. Despite the several sets of solutions, we pick the desired one by inspecting typical braiding eigenvalues in that MTC. (A complete set of fusion matrices does not determine the MTC, and could incorporate many several sets of consistent braiding eigenvalues.)

Using global conformal symmetry, we rewrite the correlator in terms of the cross-ratio $z = z_{12}z_{34}/z_{13}z_{24}$. If the coordinates of the external legs are chosen to be

$$z_1 = \infty, \quad z_2 = 1, \quad z_3 = z, \quad z_4 = 0, \quad (6.15)$$

the cross-ratio reduces to z . There are two other cross-ratios

$$1 - z = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \frac{z}{1 - z} = \frac{z_{12}z_{34}}{z_{14}z_{23}}. \quad (6.16)$$

Duality transformations are done by permuting the positions of CVOs. The F-move results in the permutation ς_{1234} on the external legs, which amounts to $z \rightarrow 1 - z$ on the coordinates. The R-move is simply done by the transposition ς_{23} , which takes z to $1/z$. Thus the B-move causes the transformation $z \rightarrow z/(z - 1)$, courtesy of eq(6.14).

Without loss of generality, we consider the 4-point function

$$\mathcal{G}(z_i, \bar{z}_i) = \langle \phi_A(z_1, \bar{z}_1) \phi_A(z_2, \bar{z}_2) \phi_A(z_3, \bar{z}_3) \phi_A(z_4, \bar{z}_4) \rangle \quad (6.17)$$

of a real primary field ϕ_A . By conformal symmetry, $\mathcal{G}(z_i, \bar{z}_i)$ factors into

$$\mathcal{G}(z_i, \bar{z}_i) = (z_{14}z_{32}\bar{z}_{14}\bar{z}_{32})^{-2h_A} G(z, \bar{z}), \quad (6.18)$$

where h_A is the conformal weight of ϕ_A . For convenience we adopt the shorthand notation

$$f_\alpha(z) = \mathcal{F}_\alpha^{AAAA}(z), \quad (6.19)$$

where $\mathcal{F}_\alpha^{AAAA}(z)$ is perceived as the conformal block with the z_{ij} powers factored out. The conformally invariant part $G(z, \bar{z})$ is a sum over conformal blocks f_α :

$$G(z, \bar{z}) = \sum_{\alpha \in \mathcal{I}} d_{AA\alpha}^2 f_\alpha(z) \bar{f}_\alpha(\bar{z}) = \sum_{\alpha \in \mathcal{I}} d_{AA\alpha}^2 |f_\alpha(z)|^2, \quad (6.20)$$

where $d_{AA\alpha}$ are the OPE coefficients. Unitary RCFTs require $d_{AA\alpha}^2$ to be positive, while $d_{AA\alpha}^2$ could be negative in a non-unitary RCFT. The normalization of conformal blocks depends on the OPE coefficients, and only the product $|d_{AA\alpha} f_\alpha(z)|$ is definite. For this reason, we have freedom in choosing the off-diagonal entries of the F - and the B -matrices. Such freedom is referred to as a change of gauge [3]. The gauge transformation is parameterized by the relative fugacity matrix $\Lambda = \text{diag}(\lambda_\alpha^2)$, and takes the form

$$f_\alpha(z) \rightarrow \lambda_\alpha f_\alpha(z), \quad F \rightarrow \Lambda^{-1} F \Lambda, \quad (6.21)$$

where F is any fusion matrix [9]. In the literature, the conventional gauge is chosen such that the F -matrices are symmetric. Furthermore, whether or not an entry of the F -matrix vanishes is a gauge-invariant property [33].

To describe the gauge dependence, we take the Fibonacci-type fusion rule $\phi \times \phi = I + \phi$ as an

example. The nontrivial fusion matrix reads

$$F \begin{bmatrix} \phi & \phi \\ \phi & \phi \end{bmatrix} = \begin{pmatrix} a_{\pm} & 1 \\ a_{\pm} & -a_{\pm} \end{pmatrix}, \quad (6.22)$$

where $a_{\pm} = (-1 \pm \sqrt{5})/2$ [44]. The choice of a_+ corresponds to the G_2 or the F_4 theory. While the choice of a_- yields an imaginary OPE coefficient, thus any RCFT with this monodromy is non-unitary. This verifies the non-unitarity of the Yang-Lee theory and the $E_{7\frac{1}{2}}$ theory. If we choose the symmetric normalization, the F -matrix takes the familiar form as in [41].

$$F_{\text{sym}} \begin{bmatrix} \phi & \phi \\ \phi & \phi \end{bmatrix} = \begin{pmatrix} a_{\pm} & \sqrt{a_{\pm}} \\ \sqrt{a_{\pm}} & -a_{\pm} \end{pmatrix}. \quad (6.23)$$

For the Fibonacci-type theory, it is impossible to achieve unitary F -matrix and abelian Galois extension of \mathbb{Q} simultaneously. But with different F -matrices, each can be obtained separately [65].

The conformal fields and the correlation functions are manifestly gauge invariant [43]. It is gauge invariant as well for the pentagon and hexagon system of equations, i.e. the polynomial equations originating from various closed loop diagrams. For any solution to these equations, there exists a continuous family of solutions that are gauge equivalent to it.

CHAPTER 7

DUALITY MATRICES AND GALOIS SYMMETRY

Fusion and braiding are two basic duality transformations, as introduced in the last chapter. In this chapter we demonstrate how the duality matrices are related in different Hecke image theories, whose MTCs are Galois conjugates.

7.1 Fusion Matrices

The fusion matrices inherit the Galois symmetry from the pentagon equations reviewed in the last chapter. We begin the analysis by visiting the two-channel fusion, which affords explicit calculation of the conformal blocks. The fusion matrices computed thereof obey the Galois symmetry consistently. We then study the MTCs of some familiar RCFTs as evidence for general cases. For simplicity we will confine ourselves to the fusion rules for which each fusion coefficient ${}_0N_{ij}^k$ equals 0 or 1.

7.1.1 Analytical results in two-channel fusion

The physical correlation function eq(6.4) remains invariant under the crossing $z \rightarrow 1 - z$. Meanwhile, the holomorphic conformal blocks transform into themselves as

$$\mathcal{F}_p^{ijkl}(1-z) = \sum_{q \in \mathcal{I}} \mathcal{M}_{pq} \mathcal{F}_q^{iljk}(z). \quad (7.1)$$

The fusion matrix is computable, provided $\mathcal{F}_q^{ijkl}(z)$ is known. It can be taken to a unitary matrix by gauge transformation for unitary RCFT, which amounts to choosing an orthonormal basis for the conformal blocks. Then the matrix elements \mathcal{M}_{pq} appear as the probability amplitudes.

We explain the idea with the 4-point function of a real primary ϕ_A . Assume that there are at most two conformal blocks as is true for a number of RCFTs. The OPE of ϕ_A with itself must

contain the identity operator, since ϕ_A is real and Hermitian. The assumed fusion rule would be

$$\phi_A \times \phi_A = I + \phi_B, \quad (7.2)$$

where the identity I and one other field ϕ_B flow in the intermediate channels. Denote by h_A and h_B the conformal weights of ϕ_A and ϕ_B respectively. We shall calculate the conformal blocks of $\langle \phi_A \phi_A \phi_A \phi_A \rangle$ and extract the fusion matrices following the analytical approach in [38].

In order for $\langle \phi_A \phi_A \phi_A \phi_A \rangle$ to be non-vanishing, there are restrictions on the fusion channels. In particular,

$$N = 8h_A + 1 - 3h_B \quad (7.3)$$

must be a non-negative integer. For a RCFT with finitely many chiral primaries, each primary field reorganizes an infinite number of Virasoro primaries. Referring to the definition eq(6.7), we thus need to consider the descendant states of the chiral primary in the intermediate channel. For $0 \leq n \leq N$, the integer n labels the lowest secondary that flows in the ϕ_B channel, while $N - n$ measures that in the vacuum channel. The part $G(z, \bar{z})$ in the correlation function eq(6.18) is expanded into the irreducible components $G^{(n)}(z, \bar{z})$ labeled by n . With the given fusion rules, each $G^{(n)}$ is the sum of two conformal blocks

$$G^{(n)}(z, \bar{z}) = \sum_{\alpha=0,1} \left(d_{AA\alpha}^{(n)} \right)^2 f_{\alpha}^{(n)}(z) \bar{f}_{\alpha}^{(n)}(\bar{z}). \quad (7.4)$$

Here, the index $\alpha = 0, 1$ labels the vacuum component and the ϕ_B channel respectively. For each

n , the conformal block $f_\alpha^{(n)}(z)$ solves the differential equation

$$\begin{aligned} \frac{d^2}{dz^2}f + \frac{2}{3}\left(2h_A + 1 + \frac{\mathbf{N}}{2}\right)\left(\frac{1}{z} + \frac{1}{z-1}\right)\frac{d}{dz}f + \left\{-\frac{2}{3}h_A(2h_A + 1 - \mathbf{N})\left(\frac{1}{z^2} + \frac{1}{(z-1)^2}\right) \right. \\ \left. + \left(\frac{4h_A}{3}(2h_A + 1 - 2n + \mathbf{N}) + \frac{1}{3}(\mathbf{N} - n)(1 + 3n - \mathbf{N})\right)\frac{1}{z(z-1)}\right\}f = 0. \end{aligned} \quad (7.5)$$

This differential equation arises from studying the singular behavior of Wronskians, without knowledge of null vectors [38]. It is a variant of the hypergeometric equation and admits two fundamental solutions around the point $z = 0$, i.e.

$$f_0^{(n)}(z) = [z(1-z)]^{-2h_A} {}_2F_1(\mathbf{a}, \mathbf{b}; \mathbf{c}; z), \quad (7.6a)$$

$$f_1^{(n)}(z) = \mathcal{N}^{(n)} [z(1-z)]^{-2h_A} z^{1-\mathbf{c}} {}_2F_1(\mathbf{a} - \mathbf{c} + 1, \mathbf{b} - \mathbf{c} + 1; 2 - \mathbf{c}; z), \quad (7.6b)$$

where

$$\mathbf{a} = \frac{1 - 4h_A - \mathbf{N}}{3} + n, \quad \mathbf{b} = -4h_A + \mathbf{N} - n, \quad \mathbf{c} = \frac{2(1 - 4h_A) + \mathbf{N}}{3}. \quad (7.7)$$

${}_2F_1(\mathbf{a}, \mathbf{b}; \mathbf{c}; z)$ is the hypergeometric function and $\mathcal{N}^{(n)}$ is a normalization constant. The singularities in various coincident limits confirm that $f_0^{(n)}(z)$ and $f_1^{(n)}(z)$ correspond to the intermediate channels I and ϕ_B respectively.

The differential equation (7.5) is invariant under the crossing $z \rightarrow 1 - z$, thus the conformal blocks transform linearly. That yields the fusion matrix $\mathcal{M}^{(n)}$, which tells the transformation law of the hypergeometric function.

$$\begin{pmatrix} f_0^{(n)}(1-z) \\ f_1^{(n)}(1-z) \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{00}^{(n)} & \mathcal{M}_{01}^{(n)} \\ \mathcal{M}_{10}^{(n)} & \mathcal{M}_{11}^{(n)} \end{pmatrix} \begin{pmatrix} f_0^{(n)}(z) \\ f_1^{(n)}(z) \end{pmatrix} \quad (7.8)$$

Shifting n by one unit flips the sign of $\mathcal{M}^{(n)}$. The case $n = \mathbf{N}$ is most relevant, where it is exactly

the Virasoro vacuum that flows in the conformal primary of the identity.¹ The fusion matrix has the diagonal entries

$$\mathcal{M}_{00}^{(n=N)} = -\mathcal{M}_{11}^{(n=N)} = \frac{\sin [(h_B - 4h_A)\pi]}{\sin [h_B\pi]}, \quad (7.9)$$

which are gauge invariant. Though the off-diagonal elements depend on the relative normalization $\mathcal{N}^{(n)}$, one has the fixed product

$$\mathcal{M}_{10}^{(n=N)} \mathcal{M}_{01}^{(n=N)} = \frac{\sin [(2h_B - 4h_A)\pi] \sin [4h_A\pi]}{\sin^2 [h_B\pi]}, \quad (7.10)$$

which is obviously in $\mathbb{Q}[\xi_N]$. Appropriate choice of $\mathcal{N}^{(n)}$ makes the fusion matrix $\mathcal{M}^{(n)}$ unitary, yielding $\mathcal{M}_{01}^{(n)} = \mathcal{M}_{10}^{(n)}$. It corresponds to the symmetric normalization. Alternatively, it is possible to choose $\mathcal{M}_{10}^{(n)}$ and $\mathcal{M}_{01}^{(n)}$ such that they both sit in $\mathbb{Q}[\xi_N]$.

The analysis of the fusion matrices applies to the effective picture and the Hecke images as well. It is noteworthy that the conformal weights enter into the parameters of the conformal blocks. Upon the Hecke operation T_p , the effective conformal weights \tilde{h} get multiplied by p modulo \mathbb{Z} , namely

$$h_A^{(p)} \equiv p\tilde{h}_A, \quad h_B^{(p)} \equiv p\tilde{h}_B \quad (\text{mod } 1). \quad (7.11)$$

Equivalently the twists are acted with the Frobenius map $f_{N,p}$. In the Hecke image theory, the fusion matrix has the diagonal entries

$$\frac{\sin [(h_B^{(p)} - 4h_A^{(p)})\pi]}{\sin [h_B^{(p)}\pi]} = \frac{\sin [p(\tilde{h}_B - 4\tilde{h}_A)\pi]}{\sin [p\tilde{h}_B\pi]} = f_{N,p} \left(\frac{\sin [(\tilde{h}_B - 4\tilde{h}_A)\pi]}{\sin [\tilde{h}_B\pi]} \right). \quad (7.12)$$

The effective conformal weights \tilde{h} are evaluated by eq(5.14). While the selection rule demands

1. We thank S. Mukhi for confirming this fact.

that $2Q_J(A) \equiv Q_J(B) + Q_J(0) \pmod{1}$. These relations help to establish that

$$\frac{\sin [(\tilde{h}_B - 4\tilde{h}_A)\pi]}{\sin [\tilde{h}_B\pi]} = f_{N,\ell^2} \left(\frac{\sin[(h_B - 4h_A)\pi]}{\sin[h_B\pi]} \right). \quad (7.13)$$

The Frobenius maps in the two steps combines to

$$f_{N,\ell^2} \circ f_{N,p} = f_{N,\bar{p}}, \quad (7.14)$$

which is precisely the map between the modular representations upon the Hecke operation \mathbb{T}_p . It confirms that $f_{N,\bar{p}}$ transforms the diagonal entries of the fusion matrix. The off-diagonal entries are not uniquely fixed. We could let them undergo the same Frobenius map if they are in $\mathbb{Q}[\xi_N]$. The determinant as well as various polynomial equations of the F -matrix are maintained under the Frobenius map. By doing so, the normalizations of conformal blocks are naturally fixed in both the effective picture and the image theory.

Let us consider the general case of the fusion with m channels ($m \geq 2$). For RCFTs with multi-component primaries, their conformal blocks satisfy the BPZ equation [37]. The solutions of this equation are known to be hypergeometric functions. Therefore, we expect to extend what we have worked out to these cases as well. However, RCFTs with m fusion channels, as appeared in theories such as WZW theories with high levels or latter members of minimal model series, come with at least m types of anyons. This makes their physical realizations hard to achieve. The cases of complex primaries can be worked out similarly [38]. We will leave them for future work.

7.1.2 Galois symmetry in fusion matrices

We mentioned the philosophy of solving the F - and the B -matrices in Chapter 6. For a given set of fusion rules, the solutions are discrete with the fixed gauge, including both unitary and non-unitary RCFTs. For instance, there are eight distinct solutions with the Ising-type fusion rules [32, 31]. All of them can be realized by affine $\text{spin}(p)$ at level 1 with $p \in (\mathbb{Z}/16\mathbb{Z})^\times$, thereby being unitary. However, the Ising model has conductor $N = 48$, and there exist Hecke images $\mathbb{T}_p\chi^{\text{Ising}}$ for all

prime p with $\gcd(p, 48) = 1$. The VOA of $\mathbb{T}_p\chi^{\text{Ising}}$ sits in the $\text{spin}(p)$ MTC and therefore obeys the same duality transformations as the $\text{spin}(p)$ theory.

In the RCFT with character $\mathbb{T}_p\chi^{\text{Ising}}$, we consider the correlation function $\langle \sigma^{(p)}\sigma^{(p)}\sigma^{(p)}\sigma^{(p)} \rangle$, where $\sigma^{(p)}$ is the spin field. Denote the vacuum by $I^{(p)}$ and the fermion field by $\psi^{(p)}$. The fusion rules are isomorphic to those of the Ising model, in particular

$$\sigma^{(p)} \times \sigma^{(p)} = I^{(p)} + \psi^{(p)}. \quad (7.15)$$

Hence, there are two conformal blocks with $I^{(p)}$ and $\psi^{(p)}$ as the intermediate channels. The associated fusion matrix is evaluated from the analytic method. A distinguished entry is

$$F_{00} = \frac{\sin [(h_B^{(p)} - 4h_A^{(p)})\pi]}{\sin [h_B^{(p)}\pi]} = \cos \left(\frac{p}{4}\pi \right) = \left(\frac{2}{p} \right) \frac{1}{\sqrt{2}}, \quad (7.16)$$

where $\left(\frac{a}{n} \right)$ stands for the Jacobi symbol. The entire fusion matrix reads

$$F \begin{bmatrix} \sigma^{(p)} & \sigma^{(p)} \\ \sigma^{(p)} & \sigma^{(p)} \end{bmatrix} = \left(\frac{2}{p} \right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (7.17)$$

Similar results are listed in [32]. With the property

$$f_{48, \bar{p}}(\sqrt{2}) = \left(\frac{2}{\bar{p}} \right) \sqrt{2} = \left(\frac{2}{p} \right) \sqrt{2}, \quad (7.18)$$

eq(7.17) is translated to

$$F \begin{bmatrix} \sigma^{(p)} & \sigma^{(p)} \\ \sigma^{(p)} & \sigma^{(p)} \end{bmatrix} = f_{48, \bar{p}} \left(F \begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix} \right) = \left(\frac{2}{p} \right) F \begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix}. \quad (7.19)$$

The parity $\left(\frac{2}{p} \right) = \pm 1$ is critical and cannot be gauged away. In the MTC perspective, this sign

corresponds to the Frobenius-Schur indicator (FSI). In general,

$$F_A \equiv F_{00} \begin{bmatrix} A & A \\ A & A \end{bmatrix} = 1/\kappa_A d_A, \quad (7.20)$$

where d_A and κ_A are the quantum dimension and the FSI of the primary field ϕ_A respectively [45, 43]. We study the FSI in more detail later.

There is a mathematical explanation for the above example. As shown in Chapter 5.2, the Hecke image theories have identical fusion rules, therefore the duality matrices obey the same set of polynomial equations. The Galois symmetry of fusion matrices originates from the algebraic structure in pentagon equations. By Ocneanu rigidity [47, 48], for any set of fusion rules there are only finitely many gauge equivalence classes of solutions to the polynomial equations. We have a finite number of solutions with the fixed gauge [33]. For the pentagon equations, each solution corresponds to an individual MTC and is characterized by $\{\sqrt{d_i}\}$, where d_i are the quantum dimensions in that MTC. According to the theory of algebraic equations, these solutions are Galois conjugates. In retrospect, Galois conjugations do not alter the algebraic structure of MTC. Therefore, the same fusion rules hold, and the polynomial equations of the F -matrices are preserved.

We first examine the derived MTC from the Yang-Lee theory. For the Fibonacci-type fusion rule $\phi \times \phi = I + \phi$, there are a total of four MTC solutions. They correspond to the Yang-Lee, G_2 , F_4 and $E_{7+\frac{1}{2}}$ theory respectively, with the common conductor $N = 60$. The less-known $E_{7+\frac{1}{2}}$ is an intermediate vertex subalgebra [16, 17]. In each of the four MTCs, the entry $F_{00} = F_\phi$ is calculated by eq(7.9). These entries are indeed related via the Frobenius maps $f_{N,p}$, explicitly

	YL	G_2	F_4	$E_{7+\frac{1}{2}}$	
$1/F_\phi$	$-1/g$	g	g	$-1/g$,
$f_{N,p}$	$f_{60,1}$	$f_{60,7}$	$f_{60,13}$	$f_{60,19}$	

(7.21)

where g is the golden ratio.

$$g = e^{2\pi i/5} + e^{-2\pi i/5} + 1 = e^{\pi i/5} + e^{-\pi i/5} = (1 + \sqrt{5})/2. \quad (7.22)$$

Remarkably, the property $d_i d_j = \sum_k {}_0N_{ij}^k d_k$ implies the quadratic equation $x^2 = 1 + x$, which admits g and $-1/g$ as Galois-conjugate solutions.

Another example is the derived MTC from $M(3, 5)$. The Hecke images of $M(3, 5)$ include the affine algebras $(A_1, 3)$ and $(C_3, 1)$. The $M(3, 5)$ MTC has a tensor product structure, which should be maintained under Hecke operations. Furthermore, the anti-semion in $M(3, 5)$ is a simple current of order 2 and has counterparts in the Hecke images. Among the fusion rules we focus on two types of fusion, which are referred to as type I and type II. We then compute the F -matrices of the correlators. The field contents and fusion rules are listed in Table 7.1. In each theory, the fields in the two types of fusion sit on the (anti-)semion orbit, and the conformal blocks have the same intermediate channels by eq(3.20). In type I fusion, the F -matrices follow from eq(7.9):

$$F^{M(3,5)} \begin{bmatrix} \phi_{\frac{-1}{20}} & \phi_{\frac{-1}{20}} \\ \phi_{\frac{-1}{20}} & \phi_{\frac{-1}{20}} \end{bmatrix} = g \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.23a)$$

$$F^{\tilde{M}(3,5)} \begin{bmatrix} \tilde{\phi}_{\frac{1}{20}} & \tilde{\phi}_{\frac{1}{20}} \\ \tilde{\phi}_{\frac{1}{20}} & \tilde{\phi}_{\frac{1}{20}} \end{bmatrix} = g \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.23b)$$

$$F^{(A_1,3)} \begin{bmatrix} \varphi_{\frac{3}{20}} & \varphi_{\frac{3}{20}} \\ \varphi_{\frac{3}{20}} & \varphi_{\frac{3}{20}} \end{bmatrix} = -\frac{1}{g} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.23c)$$

$$F^{(C_3,1)} \begin{bmatrix} \psi_{\frac{7}{20}} & \psi_{\frac{7}{20}} \\ \psi_{\frac{7}{20}} & \psi_{\frac{7}{20}} \end{bmatrix} = -\frac{1}{g} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}. \quad (7.23d)$$

In type II fusion, the F -matrices are evaluated as

$$F^{\mathbb{M}(3,5)} \begin{bmatrix} \phi_{\frac{1}{5}} & \phi_{\frac{1}{5}} \\ \phi_{\frac{1}{5}} & \phi_{\frac{1}{5}} \end{bmatrix} = -g \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.24a)$$

$$F^{\tilde{\mathbb{M}}(3,5)} \begin{bmatrix} \tilde{\phi}_{\frac{4}{5}} & \tilde{\phi}_{\frac{4}{5}} \\ \tilde{\phi}_{\frac{4}{5}} & \tilde{\phi}_{\frac{4}{5}} \end{bmatrix} = -g \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.24b)$$

$$F^{(A_{1,3})} \begin{bmatrix} \varphi_{\frac{2}{5}} & \varphi_{\frac{2}{5}} \\ \varphi_{\frac{2}{5}} & \varphi_{\frac{2}{5}} \end{bmatrix} = \frac{1}{g} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}, \quad (7.24c)$$

$$F^{(C_{3,1})} \begin{bmatrix} \psi_{\frac{3}{5}} & \psi_{\frac{3}{5}} \\ \psi_{\frac{3}{5}} & \psi_{\frac{3}{5}} \end{bmatrix} = \frac{1}{g} \begin{pmatrix} 1 & * \\ * & -1 \end{pmatrix}. \quad (7.24d)$$

Because of the aforementioned gauge dependence, we do not spell out the off-diagonal entries but denote them by asterisks instead. Notice the Frobenius maps between the algebraic numbers g and $-1/g$.

$$f_{40,3}(g) = f_{40,7}(g) = -1/g. \quad (7.25)$$

For the $\mathbb{M}(3, 5)$ theory, we justify that $f_{N,p}$ interpolates the F -matrices in the effective picture and the Hecke image under \mathbb{T}_p , as claimed.

We are curious how the FSIs are related in the image theories. Later on, the general treatment is based on the picture of effective central charge. In eq(7.20) the product $\kappa_i d_i$ seems an instructive combination, and there are the Galois relations

$$\tilde{\kappa}_i \tilde{d}_i = f_{N,\ell^2}(\kappa_i d_i) = \kappa_i f_{N,\ell^2}(d_i), \quad (7.26)$$

$$\kappa_i^{(p)} d_i^{(p)} = f_{N,\bar{p}}(\kappa_i d_i) = \kappa_i f_{N,\bar{p}}(d_i), \quad (7.27)$$

according to Appendix C. Given the F -matrices in the original theory, we acquire their Galois

conjugates by making the replacement

$$\{d_i\} \rightarrow \{f_{N,\bar{p}}(d_i)\} \quad (7.28)$$

in all the occurrences of d_i [9]. The new values obtained are the counterparts in the Hecke image theory under T_p .

There remains a subtlety about the number field of data in MTC. The solution to the polynomial equations involves $\{\sqrt{d_i}\}$ under the symmetric normalization. However, $\mathbb{Q}[\sqrt{d}]$ is a non-abelian extension, which cannot be acted on by the Frobenius map. In this case it is not straightforward to find the Galois conjugates of duality matrices. As pointed out in [34], all the data of MTC can be presented over certain finite-degree Galois extension of \mathbb{Q} , probably over an abelian Galois extension of \mathbb{Q} if normalized appropriately. That being said, every modular category defined over \mathbb{C} is conjectured to have a cyclotomic defining number field [49]. The conjecture is restated in [50]. If the conjecture holds, one can avoid the non-abelian extension $\mathbb{Q}[\sqrt{d}]$ and restrict the F -matrices in $\mathbb{Q}[\xi_N]$. The Frobenius maps are then applied unambiguously.

7.2 Braiding Matrices

The braiding matrices (B -matrices) describe unitary transformations of degenerate ground states when the positions of the anyons are fixed. Akin to the fusion, they exhibit Galois relations upon the Hecke operations. The Frobenius-Schur indicators play a ubiquitous role in such relations.

The braiding matrix is linked to the eigenvalues of R-move by similarity transformation ,as shown in eq(6.14). These eigenvalues arise from interchanging two particles, as illustrated by Figure 7.1. They serve as one-dimensional representations of the braid group, and indicate the statistics of anyons. (The B -matrices do not commute and imply non-abelian statistics.) The braiding eigenvalues are more accessible than the braiding matrices, because they are just square roots of mutual locality factors and are gauge invariant. Again we restrict ourselves to the fusion rules $|N_{ab}^c| \leq 1$. In terms of CVO, the vector space V_{ab}^c is at most one-dimensional for any

$a, b, c \in \mathcal{I}$.

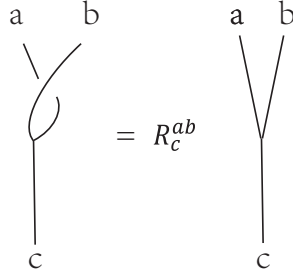


Figure 7.1: Braiding eigenvalues.

The FSIs occur in the study of braiding matrices, like fusion matrices. When solving the pentagon equations, one cannot fully specify the signs of $6j$ symbols. The signs depend on the FSIs and are chosen correctly by solving the hexagon equations. For any field a , the braiding eigenvalue R_0^{aa} is the phase obtained when two identical particles a are exchanged:

$$R_0^{aa} = \kappa_a \theta_a^{-1} = \kappa_a e(-h_a). \quad (7.29)$$

The FSI can be interpreted in terms of angular momentum. For a composite object of zero topological charge formed by two identical anyons, the FSI tells whether its total angular momentum is even or odd, as is evident from eq(7.29) [32]. Assuming rotational invariance, a rotation of the composite object by π is the same as exchanging the two anyons with physical spin $s_a \equiv h_a \pmod{1}$. The rotation then results in a phase factor $e^{i\pi s_a} e^{i\pi s_a} R_0^{aa} = \kappa_a$ for the whole system. Therefore, $\kappa_a = \pm 1$ determines the parity of the total angular momentum. Bantay derives the expression for FSI from the trace of the braiding operator, and finds that $\kappa_J = \theta_J^2$ for the simple current J in unitary RCFT [46]. Hence, $\kappa_J = 1$ if h_J is an integer or half-integer, while $\kappa_J = -1$ if $h_J \equiv \pm \frac{1}{4} \pmod{1}$. To incorporate non-unitary theories, the expression for κ_J needs slight modification. By Appendix C, the FSI reads

$$\kappa_J = e[Q_J(0)] \theta_J^2. \quad (7.30)$$

It generalizes Bantay's formula with a phase factor from the monodromy charge of the vacuum. Denote κ_a the FSI of the generic primary field a . Using the modular data, we get the FSI

$$\tilde{\kappa}_a(\ell^2, J) = e[Q_J(a) + Q_J(0)] \kappa_a \quad (7.31)$$

in the effective picture (ℓ^2, J) . As demonstrated before, the map $f_{N,p}$ connects precisely the modular data in the effective picture (ℓ^2, J) and the Hecke image theory under \mathbb{T}_p . Moreover, it acts on the integer $\tilde{\kappa}_a$ trivially.

$$\kappa_a^{(p)} = f_{N,p}(\tilde{\kappa}_a(\ell^2, J)) = \tilde{\kappa}_a(\ell^2, J). \quad (7.32)$$

That being said, $\kappa_a^{(p)}$ does not depend on specific choice of p , as long as $\bar{p}^2 \equiv \ell^2 \pmod{N}$.

We now explore the braiding eigenvalues like R_b^{aa} . In the last chapter, we computed the four point function $\langle aaaa \rangle$ and studied its fusion matrix, in the case with no more than two channels. The braiding symmetry of $\langle aaaa \rangle$ is characterized by the eigenvalues R_b^{aa} . When $N_{aa}^b = 1$, R_b^{aa} has a compact expression

$$R_b^{aa} = \theta_a^{-1} \sum_{r,s} {}_0N_{rs}^a \rho(S)_{br} \rho(S)_{0s} \theta_s^2 \theta_r^{-2}. \quad (7.33)$$

This formula holds in general, no matter how many fusion channels there are. We translate the expression to the effective picture and the Hecke image respectively. It is not hard to verify that

$$\tilde{R}_b^{aa} = f_{N,\ell^2}(R_b^{aa}), \quad (7.34)$$

$$R_b^{aa(p)} = f_{N,p}(\tilde{R}_b^{aa}). \quad (7.35)$$

An instructive example is the Hecke image of the Ising model under \mathbb{T}_p , which sits in the

$\text{spin}(p)$ MTC. With the same notation in Chapter 7.1.2, we have the twists

$$\theta_{\psi}^{(p)} = -1, \quad \theta_{\sigma}^{(p)} = e\left(\frac{p}{16}\right). \quad (7.36)$$

The symbol (p) labels the $\text{spin}(p)$ MTC and is omitted for the Ising model itself. The spin field $\sigma^{(p)}$ has the nontrivial FSI

$$\kappa_{\sigma}^{(p)} = \left(\frac{2}{p}\right) = \begin{cases} 1, & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1, & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases} \quad (7.37)$$

As we know, the FSI for a primary field is 1, 0 or -1 if the field is real, complex or quaternionic (a.k.a. pseudo-real) respectively [34]. The values of $\kappa_{\sigma}^{(p)}$ demonstrate the mathematical fact that the spinor representations of $\text{spin}(p)$ are quaternionic when $p \equiv \pm 3 \pmod{8}$. For $p \in (\mathbb{Z}/48\mathbb{Z})^{\times}$, a little arithmetic verifies that

$$p^2 \equiv 1 \pmod{48}, \quad \text{if } p \equiv \pm 1 \pmod{8}, \quad (7.38a)$$

$$p^2 \equiv 25 \pmod{48}, \quad \text{if } p \equiv \pm 3 \pmod{8}. \quad (7.38b)$$

With the Ising fusion rule, there are a total of 16 theories divided into two groups according to the FSI of the spinor. They correspond to two effective pictures for the Ising model, labeled by $(\ell^2, J) = (1, I)$ and $(\ell^2, J) = (25, \psi)$ respectively. The monodromy charges under the current I are trivial; while under $J = \psi$ the charges are

$$e[Q_{\psi}(I)] = e[Q_{\psi}(\psi)] = 1, \quad e[Q_{\psi}(\sigma)] = -1. \quad (7.39)$$

The FSIs $\kappa_{\sigma}^{(p)}$ are reproduced with these monodromy charges and fall into the two effective pic-

tures. Following eq(7.34) and (7.35), the nontrivial braiding eigenvalues are

$$R_0^{\psi\psi(p)} = -1, \quad (7.40a)$$

$$R_\sigma^{\sigma\psi(p)} = R_\sigma^{\psi\sigma(p)} = e\left(-\frac{p}{4}\right) = -i^p, \quad (7.40b)$$

$$R_0^{\sigma\sigma(p)} = \left(\frac{2}{p}\right) \cdot e\left(-\frac{p}{16}\right), \quad (7.40c)$$

$$R_\psi^{\sigma\sigma(p)} = \left(\frac{2}{p}\right) \cdot e\left(\frac{3p}{16}\right). \quad (7.40d)$$

A similar result is due to Kitaev [32].

Lastly we turn to $M(3, 5)$, whose Hecke images are computed in Chapter 5.1 as

$$\mathbb{T}_3\chi^{M(3,5)} = \chi^{(A_1,3)}, \quad (7.41)$$

$$\mathbb{T}_7\chi^{M(3,5)} = \chi^{(C_3,1)}. \quad (7.42)$$

Table 7.1 lists the field content and the fusion rules in these theories. Besides the fusion symmetries, $M(3, 5)$ has the following braiding properties. The primary fields are labeled by $a = (r, s)$ as usual.

M(3, 5)				
a	(1, 1)	(2, 1)	(3, 1)	(4, 1)
θ_a	1	$e\left(\frac{-1}{20}\right)$	$e\left(\frac{1}{5}\right)$	$e\left(\frac{3}{4}\right)$
κ_a	1	1	1	1
R_0^{aa}	1	$e\left(\frac{1}{20}\right)$	$e\left(\frac{-1}{5}\right)$	$e\left(\frac{-3}{4}\right)$

(7.43)

We provide the quantities needed to compute the braiding eigenvalues for $(A_1, 3)$ and $(C_3, 1)$, as well as the effective picture of $M(3, 5)$. In general let a be any primary field in the original RCFT. The effective picture (ℓ^2, J) amounts to the combined action $a \mapsto \pi_\ell J a$, or equivalently $J \pi_\ell a$

since J and π_ℓ commute. In $(A_1, 3)$ the symbol j means the spin- j representation as usual; while in $(C_3, 1)$ the primary fields are labeled by the null root α_0 and the simple roots $\alpha_1, \alpha_2, \alpha_3$.

$\tilde{M}(3, 5)$				
a	(1, 1)	(2, 1)	(3, 1)	(4, 1)
$\pi_\ell J a$	(2, 1)	(1, 1)	(4, 1)	(3, 1)
$\tilde{\theta}_a$	1	$e(\frac{1}{20})$	$e(\frac{4}{5})$	$e(\frac{1}{4})$
$\tilde{\kappa}_a$	1	-1	1	-1
\tilde{R}_0^{aa}	1	$-e(\frac{-1}{20})$	$e(\frac{-4}{5})$	$-e(\frac{-1}{4})$

(7.44)

As a consistency check, the twists $\tilde{\theta}_a$ are also evaluated by shuffling the field content, namely $\tilde{h}_a \equiv \mathfrak{h}_{\pi_\ell J a} \pmod{1}$. After bringing $M(3, 5)$ to its effective picture, we then perform the $f_{N,p}$ map to obtain quantities in the Hecke image, such as the twists, FSIs and braiding eigenvalues etc.

$(A_1, 3)$				
j	0	$\frac{1}{2}$	1	$\frac{3}{2}$
θ_j	1	$e(\frac{3}{20})$	$e(\frac{2}{5})$	$e(\frac{3}{4})$
κ_j	1	-1	1	-1
R_0^{jj}	1	$-e(\frac{-3}{20})$	$e(\frac{-2}{5})$	$-e(\frac{-3}{4})$

(7.45)

($C_3, 1$)				
α	α_0	α_1	α_2	α_3
θ_α	1	$e\left(\frac{7}{20}\right)$	$e\left(\frac{3}{5}\right)$	$e\left(\frac{3}{4}\right)$
κ_a	1	-1	1	-1
$R_0^{\alpha\alpha}$	1	$-e\left(\frac{-7}{20}\right)$	$e\left(\frac{-3}{5}\right)$	$-e\left(\frac{-3}{4}\right)$

(7.46)

So far we have seen the Galois relations in R_b^{aa} . Without the expression for R_c^{ab} , it seems difficult to find the Galois conjugates of general braiding eigenvalues, though the same Galois symmetry is expected to hold. Nevertheless, R_c^{ab} squares to the mutual locality factor.

$$(R_c^{ab})^2 = e(h_c - h_a - h_b), \quad a, b, c \in \mathcal{I}. \quad (7.47)$$

For $R_c^{ab(p)}$ in the image theory under \mathbb{T}_p , the above relation implies

$$\left(R_c^{ab(p)}\right)^2 = f_{N,p} \left((\tilde{R}_c^{ab})^2 \right), \quad (7.48)$$

due to the Galois relation between the twists. In a similar vein, we establish

$$(\tilde{R}_c^{ab})^2 = f_{N,\ell^2} \left((R_c^{ab})^2 \right) \quad (7.49)$$

for \tilde{R}_c^{ab} in the effective picture, where the selection rule eq(3.19) is used. They support the conjecture that the braiding eigenvalues are related by the same Frobenius map for the modular represen-

tations. We have the neat relations

$$\tilde{R}_c^{ab} = f_{N,\ell^2} \left(R_c^{ab} \right), \quad (7.50)$$

$$R_c^{ab(p)} = f_{N,p} \left(\tilde{R}_c^{ab} \right). \quad (7.51)$$

They are argued as follows. Because of the identical fusion rules, the braiding eigenvalues in the effective picture saturate the same hexagon equations, but with Galois-conjugate F -matrices inserted. The solved braiding eigenvalues are then related by the same Galois symmetry for the F -matrices. To be precise,

$$\left\{ f_{N,\ell^2} \left(R_c^{ab} \right) \mid a, b, c \in \mathcal{I} \right\}$$

constitute the solution in the effective picture. Similarly,

$$\left\{ f_{N,\bar{p}} \left(R_c^{ab} \right) \mid a, b, c \in \mathcal{I} \right\}$$

are the braiding eigenvalues for the Hecke image under T_p . As compositions of the F -matrices and the braiding eigenvalues, the B -matrices obey the same Frobenius map.

Based on the study of fusion and braiding, we finally reach the conclusion that the Galois symmetry in the Hecke relations also connects the duality quantities of RCFTs.

RCFT	$M(3, 5)$	$\tilde{M}(3, 5)$	$(A_1, 3)$	$(C_3, 1)$
unitarity	non-unitary	—	unitary	unitary
c	$-3/5$	$3/5$	$9/5$	$21/5$
field content	$\phi_0, \phi_{\frac{-1}{20}}, \phi_{\frac{3}{4}}, \phi_{\frac{1}{5}}$	$\tilde{\phi}_0, \tilde{\phi}_{\frac{1}{20}}, \tilde{\phi}_{\frac{1}{4}}, \tilde{\phi}_{\frac{4}{5}}$	$\varphi_0, \varphi_{\frac{3}{20}}, \varphi_{\frac{3}{4}}, \varphi_{\frac{2}{5}}$	$\psi_0, \psi_{\frac{7}{20}}, \psi_{\frac{3}{4}}, \psi_{\frac{3}{5}}$
(anti-)semion orbit	$\phi_{\frac{-1}{20}} \times \phi_{\frac{3}{4}} = \phi_{\frac{1}{5}}$	$\tilde{\phi}_{\frac{1}{20}} \times \tilde{\phi}_{\frac{1}{4}} = \tilde{\phi}_{\frac{4}{5}}$	$\varphi_{\frac{3}{20}} \times \varphi_{\frac{3}{4}} = \varphi_{\frac{2}{5}}$	$\psi_{\frac{7}{20}} \times \psi_{\frac{3}{4}} = \psi_{\frac{3}{5}}$
type I fusion	$\phi_{\frac{-1}{20}} \times \phi_{\frac{-1}{20}} = \phi_0 + \phi_{\frac{1}{5}}$	$\tilde{\phi}_{\frac{1}{20}} \times \tilde{\phi}_{\frac{1}{20}} = \tilde{\phi}_0 + \tilde{\phi}_{\frac{4}{5}}$	$\varphi_{\frac{3}{20}} \times \varphi_{\frac{3}{20}} = \varphi_0 + \varphi_{\frac{2}{5}}$	$\psi_{\frac{7}{20}} \times \psi_{\frac{7}{20}} = \psi_0 + \psi_{\frac{3}{5}}$
type II fusion	$\phi_{\frac{1}{5}} \times \phi_{\frac{1}{5}} = \phi_0 + \phi_{\frac{1}{5}}$	$\tilde{\phi}_{\frac{4}{5}} \times \tilde{\phi}_{\frac{4}{5}} = \tilde{\phi}_0 + \tilde{\phi}_{\frac{4}{5}}$	$\varphi_{\frac{2}{5}} \times \varphi_{\frac{2}{5}} = \varphi_0 + \varphi_{\frac{2}{5}}$	$\psi_{\frac{3}{5}} \times \psi_{\frac{3}{5}} = \psi_0 + \psi_{\frac{3}{5}}$

Table 7.1: This table summarizes some data for RCFTs derived from the $M(3, 5)$ MTC. As always, the subscripts of primary fields denote the conformal weights. The field content is also evaluated in the effective picture of $M(3, 5)$, so are the fusion rules. The approach of effective central charge is essentially the Galois conjugation plus an additional simple current permutation. The data in MTC, including F - and B -matrices, embody the Galois symmetry as well.

CHAPTER 8

CONCLUSIONS AND SUMMARY

In this work we defined the Hecke operators that acts on vector-valued modular forms of $\Gamma(N)$. Our Hecke operators take representations n -dimensional of $SL(2, \mathbb{Z})$ to representations of the same dimension and are defined without much of the mathematical machinery employed in [20, 21]. The Hecke operators reveal novel relations between the characters of a number of interesting RCFTs. We explored these connections for a few theories with small numbers of independent characters. In general it appears that given the characters χ of a known RCFT with n independent characters then there will be an infinite sequence of characters $T_p\chi$ that provide characters that obey all the conditions required for them to be characters of a RCFT, that is vacuum degeneracy one, positive integer coefficients in the q expansion and non-negative integer fusion coefficients. We have derived the set of p such that $T_p\chi$ are valid RCFT characters.

In addition to relating characters, the Hecke operators also induce Galois symmetries between modular representations, thus connecting analytic and algebraic number theory in the context of RCFT. It is natural to wonder whether these Hecke relations are a sign of a deeper number theoretic relation between certain RCFTs. A preliminary step towards answering this question is to ask whether the MTCs of two RCFTs whose characters are related by Hecke relations are isomorphic. In the thesis we have shown that this is the case for special classes of MTCs related to minimal models or with only two fusion channels by utilizing the duality properties in the Hecke image theory where the Galois symmetry relating modular representations is extended to the duality matrices. In our framework, the picture of effective central charge occurs as a significant intermediate step, and is useful for identifying unitary Hecke images. Specifically, physical quantities in the effective picture and the initial theory are related through Galois inner automorphism and simple current permutation. For the Hecke image under T_p , modularity and the duality properties are then deduced by acting with the Frobenius map $f_{N,p}$ on the data in the effective picture. As part of this procedure we also provided a unified study of the Frobenius-Schur indicator of the MTC and its Hecke images.

We also formulated a relation between the modular representations of the minimal models $M(2, k+2)$ and the simple-current reduced affine algebras $(A_1, k)_{\frac{1}{2}}$, connecting non-unitary RCFTs to unitary ones by Galois symmetry. This relation could prove useful in condensed matter theory where the unitarity of the theory is determined by tuning the couplings in the Hamiltonian.

There are further aspects of Hecke operators in RCFT that deserve exploring. Firstly, one naturally asks about the physical origin of the Hecke operator T_p and what it means by changing the complex structure τ in Hecke operations. Secondly, since SCFTs describe special points in Calabi-Yau moduli space, Hecke operations could provide new geometrical relations. We may study the Hecke image of the Gepner model, and see whether the Hecke images relate the SCFTs underlying different Calabi-Yau manifolds at special points. Finally we are pleased to see applications of Hecke operations in condensed matter theory and topological quantum computation.

APPENDIX A

MODULAR LINEAR DIFFERENTIAL EQUATIONS

The characters of RCFT satisfy modular linear differential equations (MLDEs). Solutions of an n th order MLDE with positive integers coefficients of their q expansion will necessarily form an n -dimensional representation of the modular group and are thus candidate characters of a RCFT. This point of view was taken in [60, 61, 62] where it was used to classify RCFT with two independent characters and in [63] where it was used to classify possible RCFT characters without dimension one operators.

Let E_j be the Eisenstein series of weight j . Acting on a modular form of weight k the Serre derivative $\mathcal{D}_k = d/d\tau - \frac{1}{6}i\pi k E_2$ produces a modular form of weight $k + 2$. Therefore the iterated derivative

$$\mathcal{D}^n = \mathcal{D}_{2n-2}\mathcal{D}_{2n-4}\cdots\mathcal{D}_2\mathcal{D}_0 \tag{A.1}$$

increases the weight of modular forms by $2n$. A general n th order MLDE then takes the form

$$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau) \mathcal{D}^k f = 0 \tag{A.2}$$

with the $\phi_k(\tau)$ modular forms of weight $2(n-k)$. The coefficients $\phi_k(\tau)$ can be expressed in terms of the Wronskian determinants constructed out of the n linearly independent solutions f_1, \dots, f_n of the MLDE as

$$\phi_k = (-1)^{n-k} W_k / W, \tag{A.3}$$

where W_k are the Wronskians and $W = W_n$ is called the modular Wronskian. Given an n th-order

MLDE, the Wronskians are constructed out of the n linearly independent solutions as

$$W_k = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \mathcal{D}f_1 & \mathcal{D}f_2 & \cdots & \mathcal{D}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1}f_1 & \mathcal{D}^{k-1}f_2 & \cdots & \mathcal{D}^{k-1}f_n \\ \mathcal{D}^{k+1}f_1 & \mathcal{D}^{k+1}f_2 & \cdots & \mathcal{D}^{k+1}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^n f_1 & \mathcal{D}^n f_2 & \cdots & \mathcal{D}^n f_n \end{vmatrix}. \quad (\text{A.4})$$

In RCFT the characters are holomorphic in the upper half plane except possibly at the cusp at infinity. Thus the Wronskians W_k cannot have poles except at infinity. However the coefficient functions ϕ_k may have poles coming from the zeros of W . It is useful to classify solutions by the “number of zeros” of W as has been done in earlier literature [60, 62, 64]. Define $\text{ord}_\infty(W)$ be the leading coefficient α in the q expansion $W = q^\alpha \sum_{n=0}^\infty a(n)q^n$. W transforms under the modular group with a multiplier system given by $\det \rho(\gamma)$ for $\gamma \in SL(2, \mathbb{Z})$. If the leading behaviors of the solutions f_i are q^{α_i} , then $\text{ord}_\infty(W) = \sum_i \alpha_i$. We integrate dW/W around the boundary of the fundamental domain of the modular group, and arrive at

$$\sum_i \alpha_i + \frac{\ell(W)}{6} = \frac{n(n-1)}{12}. \quad (\text{A.5})$$

For $n = 2$ the general equation with $\ell = 0$ involves one free parameter μ and takes the form

$$D^2 f - \frac{1}{6} E_2 Df - \frac{\mu}{4} E_4 f = 0 \quad (\text{A.6})$$

in terms of

$$D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}. \quad (\text{A.7})$$

While the 2nd-order equation with $\ell = 2$ reads

$$\left(D^2 - \frac{1}{6}E_2D + \mu_1 \frac{E_6}{E_4}D + \mu_2 E_4\right) f = 0. \quad (\text{A.8})$$

where μ_1 must be set to $1/3$ due to modular invariance.

For $n = 3$ and $\ell = 0$ the MLDE involves two free parameters μ_1 and μ_2 , and takes the form

$$\left(D^3 - \frac{1}{2}E_2D^2 + \left(\frac{1}{2}(DE_2)D + \left(\frac{1}{18} - \frac{\mu_1}{4}\right)E_4D - \frac{\mu_2}{8}E_6\right)\right) f = 0. \quad (\text{A.9})$$

For $n = 4$ and $\ell = 0$ the differential equation involves three free parameters μ_1, μ_2, μ_3 , and takes the form

$$\begin{aligned} D^4 f - E_2 D^3 f + \left[3DE_2 + \left(\frac{11}{36} + \mu_1\right)E_4\right] D^2 f \\ + \left[\left(\mu_2 - \frac{1}{6}\mu_1 - \frac{1}{36}\right)E_6 - \left(\frac{1}{2}\mu_1 + \frac{11}{72}\right)DE_4 - D^2 E_2\right] Df + \mu_3 E_8 f = 0. \end{aligned} \quad (\text{A.10})$$

For $n = 5$ and $\ell = 0$ the differential equation involves four free parameters $\mu_1, \mu_2, \mu_3, \mu_4$, and takes the general form

$$(\mathcal{D}^5 + \mu_1 E_4 \mathcal{D}^3 + \mu_2 E_6 \mathcal{D}^2 + \mu_3 E_8 \mathcal{D} + \mu_4 E_{10}) f = 0. \quad (\text{A.11})$$

It can be rewritten in terms of the differential operator D .

MLDEs of order greater than 6 are will not be of practical use, since they are cumbersome to write down in terms of D .

APPENDIX B

GALOIS SYMMETRY INTERPOLATED MODULAR REPRESENTATIONS

In this section we explore the Galois connection between the $M(2, k+2)$ and the $(A_1, k)_{\frac{1}{2}}$ modular representations with odd k . In addition to the conductor N , we set N_0 to be the least common denominator of the conformal weights. Proposition 5 in [2] states that

$$N = \epsilon N_0, \tag{B.1}$$

where the integer ϵ divides 12. In addition, $\gcd(\epsilon, N_0) = 1$ or 2.

For the basis $\phi_{(u,1)}$ in $M(2, k+2)$, the representation is determined by

$$\rho^M(T)_{u,v} = \delta_{u,v} e^{\left(\frac{(k+2-2u)^2}{8(k+2)} - \frac{1}{24} \right)}, \tag{B.2a}$$

$$\rho^M(S)_{u,v} = \frac{2}{\sqrt{k+2}} (-1)^{1+u+v} \sin\left(\frac{uv}{k+2} 2\pi\right) \sin\left(\frac{k+2}{2} \pi\right). \tag{B.2b}$$

While the modular representation of $(A_1, k)_{\frac{1}{2}}$ is

$$\rho^A(T)_{l,l'} = \delta_{l,l'} e^{\left(\frac{l^2}{4(k+2)} - \frac{1}{8} \pm \frac{1}{24} \right)}, \tag{B.3a}$$

$$\rho^A(S)_{l,l'} = \frac{2}{\sqrt{k+2}} \sin\left(\frac{ll'}{k+2} \pi\right), \tag{B.3b}$$

where the signs \pm correspond to the cases $k \equiv \pm 1 \pmod{4}$ respectively and l, l' are odd integers that satisfy $1 \leq l, l' \leq k$.

We will not directly apply the Frobenius map $f_{N,-k}$ on $\rho(\gamma)$, for fear that k may not be coprime to the conductor. Instead, we exploit the techniques in MTC [34]. Rather than working with the CFT-normalized T matrix, we use an appropriate surjective restriction: $(\mathbb{Z}/N\mathbb{Z})^\times \rightarrow (\mathbb{Z}/N_0\mathbb{Z})^\times$.

Define the diagonal matrix

$$\varrho(T) = e\left(\frac{c}{24}\right)\rho(T) \quad (\text{B.4})$$

for the T transformation in MTC. The diagonal entries of $\varrho(T)$ consist of all the twists and take values in $\mathbb{Q}[\xi_{N_0}]$. We then have

$$\varrho^{\text{M}}(T)_{u,v} = \delta_{u,v} e\left(\frac{(k+2-2u)^2 - k^2}{8(k+2)}\right), \quad (\text{B.5})$$

$$\varrho^{\text{A}}(T)_{l,l'} = \delta_{l,l'} e\left(\frac{l^2 - 1}{4(k+2)}\right). \quad (\text{B.6})$$

Since the indices are odd, we find $N_0 = k + 2$ in both MTCs. Similarly, we write $\varrho(S) = \rho(S)$. In some sense, the pair $(\varrho(T), \varrho(S))$ also characterizes $SL(2, \mathbb{Z})$ as $(\rho(T), \rho(S))$ does. The constraints for $(\varrho(T), \varrho(S))$ are

$$\varrho(S)^2 = \mathcal{C}, \quad (\text{B.7a})$$

$$(\varrho(T)\varrho(S))^3 = \mathcal{C} e\left(\frac{c}{8}\right), \quad (\text{B.7b})$$

where \mathcal{C} is the charge conjugation matrix. We also learn that $\varrho(S)$ is in $\mathbb{Q}[\xi_{4(k+2)}]$ for both MTCs [2, 28]. Therefore the extension of \mathbb{Q} by either $\varrho^{\text{M}}(\gamma)$ or $\varrho^{\text{A}}(\gamma)$ leads to $\mathbb{Q}[\xi_{4(k+2)}]$, on which there exists the Frobenius map $f_{4(k+2), -k}$.

We proceed to act the Frobenius map $f_{4(k+2), -k}$ on $\varrho^{\text{A}}(T)$ and $\varrho^{\text{A}}(S)$ respectively.

$$\begin{aligned} f_{4(k+2), -k}(\varrho^{\text{A}}(T)_{l,l'}) &= \delta_{l,l'} e\left(-k \frac{l^2 - 1}{4(k+2)}\right) \\ &= \delta_{l,l'} e\left(\frac{(k+2-2l)^2 - k^2}{8(k+2)} - \left(\frac{l-1}{2}\right)^2\right) \\ &= \delta_{l,l'} e\left(\frac{(k+2-2l)^2 - k^2}{8(k+2)}\right) \equiv \varrho^{\text{M}}(T)_{l,l'}. \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned}
f_{4(k+2),-k}(\varrho^A(S)_{l,l'}) &= \left(\frac{k+2}{k}\right) \frac{2}{\sqrt{k+2}} i^{k+1} \sin\left(-\frac{kl'l'}{k+2}\pi\right) \\
&= \left(\frac{2}{k}\right) \frac{2}{\sqrt{k+2}} i^{k+1} (-1)^{ll'} \sin\left(ll'\pi - \frac{kl'l'}{k+2}\pi\right) \\
&= \left(\frac{2}{k}\right) \frac{2}{\sqrt{k+2}} i^{k-1} \sin\left(\frac{ll'}{k+2}2\pi\right) \equiv \left(\frac{2}{k}\right) \varrho^M(S)_{l,l'}.
\end{aligned} \tag{B.9}$$

In the derivations we bear in mind that l, l' are odd. The fusion rule isomorphism is validated by the shown Galois symmetry between the modular S matrices.

From the MTC point of view, $M(2, k+2)$ is same as the Galois conjugate of $(A_1, k)_{\frac{1}{2}}$, up to a one-dimensional modular representation of central charge

$$\begin{aligned}
c_{\text{tot}} &= c[M(2, k+2)] + k \cdot c\left[(A_1, k)_{\frac{1}{2}}\right] \\
&= 1 - \frac{6k^2}{2(k+2)} + k\left(\frac{3k}{k+2} \mp 1\right) = 1 \mp k.
\end{aligned} \tag{B.10}$$

This central charge leads to

$$\rho^{\text{1d}}(T) = e\left(-\frac{c_{\text{tot}}}{24}\right) = e\left(\frac{\pm k - 1}{24}\right) = \begin{cases} e\left(\frac{t}{6}\right), & \text{if } k = 4t + 1, \\ e\left(-\frac{t}{6}\right), & \text{if } k = 4t - 1, \end{cases} \tag{B.11a}$$

$$\rho^{\text{1d}}(S) = \left(\frac{2}{k}\right) = (-1)^{\frac{k^2-1}{8}} = (-1)^t, \quad \text{for } k = 4t \pm 1, \tag{B.11b}$$

where t is a positive integer. ρ^{1d} agrees with the one-dimensional representations classified by Lemma 5 of [24].

There is another approach to understand this Galois symmetry. With the effective description, the $(-k)$ -th Galois conjugate of $\tilde{M}(2, k+2)$ differs from $\rho^{(A_1, k)_{\frac{1}{2}}}$ by a one-dimensional representation, which has central charge

$$\begin{aligned}
c'_{\text{tot}} &= k \cdot c[\tilde{M}(2, k+2)] + c\left[(A_1, k)_{\frac{1}{2}}\right] \\
&= k\left(1 - \frac{3}{k+2}\right) + \frac{3k}{k+2} \mp 1 = k \mp 1.
\end{aligned} \tag{B.12}$$

Gannon provides the unitarization of $M(2, k + 2)$, which is slightly different from the above [26]. Our analysis demonstrates that the unitarization of $M(2, k + 2)$ leads to the $(A_1, k)_{\frac{1}{2}}$ MTC.

APPENDIX C

DERIVATION OF MTC DATA

In this appendix we provide the derivation for some data in the Hecke image theory. We restrict ourselves to RCFTs without degenerate twists.

Denote by d_b , \tilde{d}_b and $d_b^{(p)}$ respectively the quantum dimensions in the original theory, the effective description (ℓ^2, J) and the Hecke image under \mathbb{T}_p . As always, ℓ is odd and $\bar{p}^2 \equiv \ell^2 \pmod{N}$. For each individual p , \mathbb{T}_p produces positive quantum dimensions in the image theory, which offer evidence for unitary RCFTs. By definition, $d_b^{(p)}$ takes the form

$$d_b^{(p)} = \frac{\rho^{(p)}(S)_{0b}}{\rho^{(p)}(S)_{00}}. \quad (\text{C.1})$$

It is simplified as follows.

$$\begin{aligned} d_b^{(p)} &= f_{N,p} \left(\frac{\tilde{\rho}(S)_{0b}}{\tilde{\rho}(S)_{00}} \right) \\ &= f_{N,p} \circ f_{N,\ell^2} \left(\frac{\rho(S)_{J0,Jb}}{\rho(S)_{J0,J0}} \right) = f_{N,\ell} \left(\frac{\rho(S)_{J0,Jb}}{\rho(S)_{J0,J0}} \right) \\ &= \frac{\rho(S)_{\pi_\ell J0,Jb}}{\rho(S)_{\pi_\ell J0,J0}} = \frac{\rho(S)_{o,Jb}}{\rho(S)_{o,J0}} = \frac{\rho(S)_{o,b}}{\rho(S)_{o,0}}. \end{aligned} \quad (\text{C.2})$$

In the second line above, we use the fact that $f_{p\ell}$ yields the identity permutation of the fields when $(p\ell)^2 \equiv 1 \pmod{N}$, c.f. Chapter 3.3. (We do not know how to analyze in RCFTs with degenerate twists.) In the last line, the minimal primary is reached from the vacuum by $o = \pi_\ell J0$. Gannon defines $\rho(S)_{o,b}/\rho(S)_{o,0}$ as the quantum dimension for non-unitary cases, and shows that

$$\rho(S)_{o,b}/\rho(S)_{o,0} \geq 1 \quad (\text{C.3})$$

[26]. Nevertheless, we stick to the definition eq(4.2) for quantum dimensions. Independent of specific choices of p , the quantum dimensions $d_b^{(p)}$ are inherently encoded in the initial modular S matrix. The values $d_b^{(p)} \geq 1$ suggest that the image theory is unitary. While in the effective picture

the quantum dimensions are

$$\tilde{d}_b = \frac{\tilde{\rho}(S)_{0b}}{\tilde{\rho}(S)_{00}} = f_{N,\ell^2} \left(\frac{\rho(S)_{J0,Jb}}{\rho(S)_{J0,J0}} \right) = e[Q_J(b) - Q_J(0)] f_{N,\ell^2} \left(\frac{\rho(S)_{0,b}}{\rho(S)_{0,0}} \right), \quad (\text{C.4})$$

which need not be positive.

In the effective description (ℓ^2, J) , the FSI reads

$$\tilde{\kappa}_a(\ell^2, J) = \sum_{r,s} {}_0\tilde{N}_{rs}^a \tilde{\rho}(S)_{0r} \tilde{\rho}(S)_{0s} \tilde{\theta}_s^2 \tilde{\theta}_r^{-2}, \quad (\text{C.5})$$

where ${}_0\tilde{N}_{rs}^a$ equals ${}_0N_{rs}^a$ by eq(5.17). We act on this formula by $f_{N,\bar{\ell}^2}$ and notice that ${}_0N_{rs}^a = {}_0N_{Jr,J_s}^a$ for $J^2 = I$.

$$\begin{aligned} f_{N,\bar{\ell}^2}(\tilde{\kappa}_a(\ell^2, J)) &= \sum_{r,s} {}_0N_{rs}^a \rho(S)_{J0,Jr} \rho(S)_{J0,Js} (\theta_{Js} \theta_J^{-1})^2 (\theta_{Jr} \theta_J^{-1})^{-2} \\ &= \sum_{r,s} {}_0N_{Jr,J_s}^a e[Q_J(Jr) + Q_J(Js)] \rho(S)_{0,Jr} \rho(S)_{0,Js} \theta_{Js}^2 \theta_{Jr}^{-2} \\ &= e[Q_J(a) + Q_J(0)] \sum_{r,s} {}_0N_{Jr,J_s}^a \rho(S)_{0,Jr} \rho(S)_{0,Js} \theta_{Js}^2 \theta_{Jr}^{-2} \\ &\equiv e[Q_J(a) + Q_J(0)] \kappa_a \end{aligned} \quad (\text{C.6})$$

It confirms that $f_{N,\bar{\ell}^2}(\tilde{\kappa}_a(\ell^2, J))$ and therefore $\tilde{\kappa}_a(\ell^2, J)$ are integers. The FSIs in the effective description and the original theory are related by

$$\tilde{\kappa}_a(\ell^2, J) = e[Q_J(a) + Q_J(0)] \kappa_a. \quad (\text{C.7})$$

Since the monodromy charges vanish under the trivial simple current, the FSIs are preserved in this particular scenario. Moreover, the FSI in the image theory is translated invariantly from the effective description.

$$\kappa_a^{(p)} = f_{N,p}(\tilde{\kappa}_a(\ell^2, J)) = \tilde{\kappa}_a(\ell^2, J). \quad (\text{C.8})$$

It remains to study the orbit of the simple current J . We assume $J^2 = I$, in which case J could play a role in the effective description. For the quantum dimensions, we have

$$d_{Jb} = \frac{\rho(S)_{0,Jb}}{\rho(S)_{0,0}} = e[Q_J(0)] \frac{\rho(S)_{0,b}}{\rho(S)_{0,0}} = e[Q_J(0)] d_b. \quad (\text{C.9})$$

On a simple current orbit, the fields have quantum dimensions of the same magnitude. In particular they are equal for unitary RCFTs, in which the vacuum has trivial monodromy charge. Given the fusion rule $\phi_r \times \phi_s = \sum_b {}_0N_{rs}^b \phi_b$, we find

$$\phi_{Jr} \times \phi_s = \sum_b {}_0N_{rs}^b \phi_{Jb}, \quad (\text{C.10})$$

where ${}_0N_{Jr,s}^{Jb} = {}_0N_{rs}^b$ by straightforward computation. For the FSI, there is

$$\begin{aligned} \kappa_{Jb} &= \sum_{r,s} {}_0N_{rs}^{Jb} \rho(S)_{0r} \rho(S)_{0s} \theta_s^2 \theta_r^{-2} \\ &= \sum_{r,s} {}_0N_{Jr,s}^{Jb} \rho(S)_{0,Jr} \rho(S)_{0s} \theta_s^2 \theta_{Jr}^{-2} \\ &= e[Q_J(0)] \sum_{r,s} {}_0N_{rs}^b \rho(S)_{0r} \rho(S)_{0s} \theta_s^2 \theta_r^{-2} \theta_J^2 \\ &= e[Q_J(0)] \theta_J^2 \kappa_b. \end{aligned} \quad (\text{C.11})$$

To see the application of this formula, we return to any individual theory in the $M(3, 5)$ series. In type I and type II fusions, the external fields sit on the (anti-)semion orbit, implying

$$\theta_J^2 = e(2h_J) = -1. \quad (\text{C.12})$$

With the quantum dimensions and the FSIs inserted, eq(7.20) explains why the overall signs of the F -matrices are different for the two types of fusion.

APPENDIX D

HECKE OPERATION AND GALOIS ACTION ON MINIMAL MODELS

D.1 Overview of minimal models

The Virasoro minimal models are specified by a pair of coprime integers (p_1, p_2) . The minimal model $\mathbb{M}(p_1, p_2)$ is unitary *iff* $|p_1 - p_2| = 1$. It has the central charge

$$c[\mathbb{M}(p_1, p_2)] = 1 - 6 \frac{(p_1 - p_2)^2}{p_1 p_2} \quad (\text{D.1})$$

and conformal weights

$$h_{r,s} = \frac{(p_1 r - p_2 s)^2 - (p_1 - p_2)^2}{4p_1 p_2} \geq \frac{1 - (p_1 - p_2)^2}{4p_1 p_2} \quad (\text{D.2})$$

for the primary fields labeled by (r, s) with $0 < r < p_2, 0 < s < p_1$. There are a total of $n = (p_1 - 1)(p_2 - 1)$ of primary fields due to the degeneracy $(r, s) \cong (p_2 - r)(p_1 - s)/2$.

The character $\chi_{r,s}(\tau)$ associated with the field (r, s) has the leading term

$$\chi_{r,s}(\tau) \sim q^{h_{r,s} - \frac{1}{24}c[\mathbb{M}(p_1, p_2)]}. \quad (\text{D.3})$$

The exponent can also be written as

$$h_{r,s} - \frac{1}{24}c[\mathbb{M}(p_1, p_2)] = \frac{(p_1 r - p_2 s)^2}{4p_1 p_2} - \frac{1}{24} \geq \frac{1}{4p_1 p_2} - \frac{1}{24}. \quad (\text{D.4})$$

There exists a pair (r_0, s_0) such that $|p_1 r_0 - p_2 s_0| = 1$. If we choose a new vacuum labeled by this particular pair, the effective central charge and conformal weights become

$$c_{\text{eff}}[\mathbb{M}(p_1, p_2)] = 1 - \frac{6}{p_1 p_2}, \quad \mathfrak{h}_{r,s} = \frac{(p_1 r - p_2 s)^2 - 1}{4p_1 p_2}. \quad (\text{D.5})$$

For a non-unitary minimal model, the primary $(1, 1)$ is no longer the vacuum in the effective

picture. The effective vacuum is (r_0, s_0) , a.k.a. the minimal primary. The shifted conformal weight $\mathfrak{h}_{r,s}$ should not be confused with $\tilde{h}_{r,s}$ in the effective picture.

In general, the character $\chi_{r,s}$ takes the form

$$\chi_{r,s}(\tau) = \left(\vartheta_{p_1 p_2, p_1 r - p_2 s}(\tau) - \vartheta_{p_1 p_2, p_1 r + p_2 s}(\tau) \right) / \eta(\tau), \quad (\text{D.6})$$

where the theta function is defined by

$$\vartheta_{a,b}(\tau) := \sum_{n \in \mathbb{Z}} q^{a(n+b/2a)^2}, \quad a, b \in \frac{1}{2}\mathbb{Z}^+. \quad (\text{D.7})$$

The character $\chi_{r,s}$ does exhibit the form as eq(D.2).

D.2 Some examples

Since $\rho^M(\sigma_p)$ is a monomial matrix, it is invariant under the Frobenius map $f_{N,\ell}$. We notice that some Hecke images of the minimal models can be realized by WZW models based on the affine Lie algebras \hat{g} . Perhaps $\rho^M(\sigma_p)$ also contains information about $\mathcal{O}(\hat{g})$, i.e. the outer automorphism group of \hat{g} .

D.2.1 Minimal model $M(2, 9)$

For the minimal model $M(2, 9)$, the four primary fields are labeled by

$$(r, s) = (4, 1), (3, 1), (2, 1), (1, 1). \quad (\text{D.8})$$

The conductor is $N = 36$, and the effective central charge is $c_{\text{eff}} = 2/3$.

$$\rho^{\text{M}}(T) = \begin{pmatrix} \xi_N^{-1} & & & \\ & \xi_N^3 & & \\ & & \xi_N^{11} & \\ & & & \xi_N^{23} \end{pmatrix} \quad (\text{D.9})$$

Explicit computations lead to following $\rho^{\text{M}}(\sigma_p)$ for all p , with $1 \leq p < N$ and $\text{gcd}(p, N) = 1$.

When $p = 1, 35$,

$$\rho^{\text{M}}(\sigma_p) = \mathbb{I}_4. \quad p^2 = 1 \text{ Mod } N \quad (\text{D.10})$$

When $p = 17, 19$,

$$\rho^{\text{M}}(\sigma_p) = -\mathbb{I}_4. \quad p^2 = 1 \text{ Mod } N \quad (\text{D.11})$$

When $p = 5, 31$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad p^2 = 25 \text{ Mod } N \quad (\text{D.12})$$

When $p = 13, 23$,

$$\rho^{\text{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad p^2 = 25 \text{ Mod } N \quad (\text{D.13})$$

When $p = 7, 29$,

$$\rho^{\mathbb{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 49 \text{ Mod } N \quad (\text{D.14})$$

When $p = 11, 25$,

$$\rho^{\mathbb{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 49 \text{ Mod } N \quad (\text{D.15})$$

Some Hecke images coincide with characters of affine Lie algebras:

$$\begin{aligned} \mathbb{T}_7 \chi[\mathbb{M}(2, 9)] &= \chi[(G_2, 2)], \\ \mathbb{T}_{29} \chi[\mathbb{M}(2, 9)] &= \chi[(C_5, 3) \otimes (A_1, 1)]. \end{aligned} \quad (\text{D.16})$$

The vacuum character of $\mathbb{T}_{29} \chi[\mathbb{M}(2, 9)]$ is

$$q^{-\frac{29}{36}}(1 + 58q + \dots).$$

The bilinear of $\chi[(G_2, 2)]$ and $\mathbb{T}_{29} \chi[\mathbb{M}(2, 9)]$ yields $J(\tau) + 72$, the partition function of No. 21 in Schelleken's list of $c = 24$ meromorphic CFTs [42]. We also notice that $\chi[(C_5, 3) \otimes (A_1, 1)]$ has the vacuum character of the same form, and hence confirm that it is $\mathbb{T}_{29} \chi[\mathbb{M}(2, 9)]$. These two four-character theories with $\ell = 0$ are listed in Table 3 of [39]. Various representations $\rho^{(p)}$ are listed in Table 11 of [24], as four-dimensional simple strongly-modular fusion algebras. In the work by Schoutens and Wen [31], the unitary MTCs induced by \mathbb{T}_7 and \mathbb{T}_{29} also enter the classification of bosonic topological orders at rank 4.

D.2.2 Minimal model $M(2, 11)$

For the minimal model $M(2, 11)$, the five primary fields are labeled by

$$(r, s) = (5, 1), (4, 1), (3, 1), (2, 1), (1, 1). \quad (\text{D.17})$$

The conductor is $N = 33$, and the effective central charge is $c_{\text{eff}} = 8/11$.

$$\rho^M(T) = \begin{pmatrix} \xi_N^{-1} & & & & \\ & \xi_N^2 & & & \\ & & \xi_N^8 & & \\ & & & \xi_N^{17} & \\ & & & & \xi_N^{29} \end{pmatrix} \quad (\text{D.18})$$

The solutions to a fifth-order MLDE requires

$$\frac{1}{N}(-1 + 2 + 8 + 17 + 29) = \frac{1}{12}5(5 - 1), \quad (\text{D.19})$$

which is true.

Explicit computations lead to following $\rho^M(\sigma_p)$ for all p , with $1 \leq p < N$ and $\text{gcd}(p, N) = 1$.

When $p = 1, 10, 23, 32$,

$$\rho^M(\sigma_p) = \mathbb{I}_5. \quad p^2 = 1 \text{ Mod } N \quad (\text{D.20})$$

When $p = 2, 13, 20, 31$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 4 \text{ Mod } N \quad (\text{D.21})$$

When $p = 4, 7, 26, 29$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad p^2 = 16 \text{ Mod } N \quad (\text{D.22})$$

When $p = 5, 16, 17, 28$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad p^2 = 25 \text{ Mod } N \quad (\text{D.23})$$

When $p = 8, 14, 19, 25$,

$$\rho^{\mathbf{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 64 \text{ Mod } N \quad (\text{D.24})$$

Some Hecke images coincide with characters of known RCFTs:

$$\begin{aligned} \mathbb{T}_2 \chi[\mathbf{M}(2, 11)] &= \chi[D_{3C}], \\ \mathbb{T}_{13} \chi[\mathbf{M}(2, 11)] &= \chi[(F_4, 2)], \\ \mathbb{T}_{20} \chi[\mathbf{M}(2, 11)] &= \chi \left[(A_8, 2)_{\frac{1}{9}} \right], \\ \mathbb{T}_{31} \chi[\mathbf{M}(2, 11)] &= \chi[VT^{\natural}]. \end{aligned} \quad (\text{D.25})$$

$\mathbb{T}_2 \chi[\mathbf{M}(2, 11)]$ describes the 3C conjugacy class of the Monster. Moreover, it is a character realization of $(A_1, 9)_{\frac{1}{2}}$. VT^{\natural} is the commutant of $\mathcal{W}_{D_{3C}}$ and encodes an action by the Thompson sporadic group [55]. The **unitary** MTCs induced by \mathbb{T}_{13} and \mathbb{T}_{31} enter the classification of bosonic topological orders at rank 5 [31]. However $\mathbb{T}_{31} \chi[\mathbf{M}(2, 11)]$ is not $\chi[(E_8, 3)]$, though they are in the same MTC. Because the dimensions of many irreps of the E_8 Lie algebra are multiples of 31, the Hecke image should have a connection to $\chi[(E_8, 3)]$. We solve the $\chi[(E_8, 3)]$ characters from a fifth-order MLDE and find

$$\chi[(E_8, 3)] = \chi[VT^{\natural}] - 248 G_k \chi[\mathbf{M}(2, 11)], \quad (\text{D.26})$$

where $\bar{k}^2 \equiv 64 \pmod{N}$. The bilinear

$$\chi[D_{3C}]^T \cdot \chi[(E_8, 3)] = J(\tau) + 744 \quad (\text{D.27})$$

is not listed in [42]. The bilinear of $\chi[(F_4, 2)]$ and $\chi[(A_8, 2)_{\frac{1}{9}}]$ yields $J(\tau) + 132$, the partition function of No. 32 in Schelleken's list of $c = 24$ meromorphic CFTs [42]. Because the MLDE have solutions with zero poles, it is not satisfied by neither $T_2 \chi[M(2, 11)]$ nor $T_{20} \chi[M(2, 11)]$. In addition, T_p and T_{p+11} induce the same MTC since $11 \times \frac{8}{11} \equiv 0 \pmod{8}$. Hence we only have to compute the induced MTCs under T_{13} and T_{31} respectively.

$T_2 \chi[M(2, 11)]$ have same central charge as the \mathbb{Z}_9 parafermion. $\mathcal{W}_{D_{3C}}$ is an extension of the \mathbb{Z}_9 parafermion theory by its two irreducible modules with integral highest weight. In general, $M(2, k+2)$ have half the central charge as the \mathbb{Z}_k parafermion. Do they have a connection?

D.2.3 Minimal model $M(2, 13)$

For the minimal model $M(2, 13)$, the six primary fields are labeled by

$$(r, s) = (6, 1), (5, 1), (4, 1), (3, 1), (2, 1), (1, 1). \quad (\text{D.28})$$

The conductor is $N = 156$, and the effective central charge is $c_{\text{eff}} = 10/13$.

$$\rho^M(T) = \begin{pmatrix} \xi_N^{-5} & & & & & \\ & \xi_N^7 & & & & \\ & & \xi_N^{31} & & & \\ & & & \xi_N^{67} & & \\ & & & & \xi_N^{115} & \\ & & & & & \xi_N^{175} \end{pmatrix} \quad (\text{D.29})$$

The solutions to a sixth-order MLDE requires

$$\frac{1}{N}(-5 + 7 + 31 + 67 + 115 + 175) = \frac{1}{12}6(6 - 1), \quad (\text{D.30})$$

which is true.

Explicit computations lead to following $\rho^M(\sigma_p)$ for all p , with $1 \leq p < N$ and $\gcd(p, N) = 1$.

When $p = 1, 53, 103, 155$,

$$\rho^M(\sigma_p) = \mathbb{I}_6 . \quad p^2 = 1 \text{ Mod } N \quad (\text{D.31})$$

When $p = 25, 77, 79, 131$,

$$\rho^M(\sigma_p) = -\mathbb{I}_6 . \quad p^2 = 1 \text{ Mod } N \quad (\text{D.32})$$

When $p = 5, 47, 109, 151$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 25 \text{ Mod } N \quad (\text{D.33})$$

When $p = 31, 73, 83, 125$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 25 \text{ Mod } N \quad (\text{D.34})$$

When $p = 7, 59, 97, 149$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad p^2 = 49 \text{ Mod } N \quad (\text{D.35})$$

When $p = 19, 71, 85, 137$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \quad p^2 = 49 \text{ Mod } N \quad (\text{D.36})$$

When $p = 11, 41, 115, 145$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \text{ Mod } N \quad (\text{D.37})$$

When $p = 37, 67, 89, 119$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \text{ Mod } N \quad (\text{D.38})$$

When $p = 17, 35, 121, 139$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 133 \text{ Mod } N \quad (\text{D.39})$$

When $p = 43, 61, 95, 113$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 133 \text{ Mod } N \quad (\text{D.40})$$

When $p = 23, 29, 127, 133$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 61 \text{ Mod } N \quad (\text{D.41})$$

When $p = 49, 55, 101, 107$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 61 \text{ Mod } N \quad (\text{D.42})$$

In [31], the **unitary** MTCs induced by $\mathbb{T}_{37}, \mathbb{T}_{67}, \mathbb{T}_{89}$ and \mathbb{T}_{119} yields two primitive bosonic topological orders at rank 6. In addition, \mathbb{T}_p and \mathbb{T}_{p+52} induce the same MTC, since $52 \times \frac{10}{13} \equiv 0 \pmod{8}$.

D.2.4 Minimal model $M(2, 15)$

For the minimal model $M(2, 15)$, the seven primary fields are labeled by

$$(r, s) = (7, 1), (6, 1), (5, 1), (4, 1), (3, 1), (2, 1), (1, 1). \quad (\text{D.43})$$

When $p = 11, 19$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 1 \text{ Mod } N \quad (\text{D.48})$$

When $p = 7, 23$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} . \quad p^2 = 49 \text{ Mod } N \quad (\text{D.49})$$

When $p = 13, 17$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 49 \text{ Mod } N \quad (\text{D.50})$$

Though $11^2 \equiv 1^2 \pmod{30}$, the relation $\rho^M(\sigma_{11}) = \pm \mathbb{I}_7$ does not hold. Actually the group of Galois permutations ¹

$$\mathcal{G} = \left\{ \rho^M(\sigma_p) \mid p \in (\mathbb{Z}/N\mathbb{Z})^\times \right\} / \mathbb{Z}_2 \quad (\text{D.51})$$

is here isomorphic to C_4 , rather than

$$[(\mathbb{Z}/N\mathbb{Z})^\times]^2 \cong C_2.$$

In this circumstance the non-degeneracy assumption breaks down, which yields the contradiction above. Evaluating the roots $\gamma_i^{(j)}$ explicitly, we find their Galois group to be exactly C_4 .

For the sake of illustration, we give an example how to compute the Galois group of fusion rules. Though this theory is not unitary, the method applies as in the cases when the fusion rules are non-negative integers. The primary field $\phi_{\frac{1}{15}}$ has fusion rules

$$N_{\frac{1}{15}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad (\text{D.52})$$

whose characteristic polynomial is

$$\lambda \left(\lambda^2 - \lambda - 1 \right) \left(\lambda^4 - 5\lambda^3 + 5\lambda^2 + 5\lambda - 5 \right). \quad (\text{D.53})$$

1. It is not the Galois group $\mathcal{G}_N = (\mathbb{Z}/N\mathbb{Z})^\times$, because some elements of \mathcal{G}_N yield the same permutation.

Solving this polynomial, we find roots

$$\begin{aligned}
\lambda_1 &= \frac{1}{4} \left(5 + \sqrt{5} + \sqrt{6(5 - \sqrt{5})} \right), & \lambda_2 &= \frac{1}{4} \left(5 + \sqrt{5} - \sqrt{6(5 - \sqrt{5})} \right), \\
\lambda_3 &= \frac{1}{4} \left(5 - \sqrt{5} + \sqrt{6(5 + \sqrt{5})} \right), & \lambda_4 &= \frac{1}{4} \left(5 - \sqrt{5} - \sqrt{6(5 + \sqrt{5})} \right), \\
\lambda_5 &= g, & \lambda_6 &= -1/g, & \lambda_7 &= 0,
\end{aligned} \tag{D.54}$$

where g is the golden ratio. We look for the Galois group \mathcal{G} of these roots by acting various Frobenius maps $f_{N,p}$ (denoted by f_p for short), since f_p form the group \mathcal{G} under function composition. Note that

$$p \in (\mathbb{Z}/N\mathbb{Z})^\times = \{1, 7, 11, 13, 17, 19, 23, 29\} \tag{D.55}$$

and $f_{N-p} = f_p$. The Galois group \mathcal{G} has four elements at most. It suffices to find the way f_1, f_{11}, f_7, f_{13} permute $\{\lambda_i\}$:

$$f_1 = \text{identity map}, \tag{D.56a}$$

$$f_{11}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (\lambda_2, \lambda_1, \lambda_4, \lambda_3, \lambda_5, \lambda_6, \lambda_7), \tag{D.56b}$$

$$f_7(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (\lambda_4, \lambda_3, \lambda_1, \lambda_2, \lambda_6, \lambda_5, \lambda_7), \tag{D.56c}$$

$$f_{13}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (\lambda_3, \lambda_4, \lambda_2, \lambda_1, \lambda_6, \lambda_5, \lambda_7). \tag{D.56d}$$

f_p permutes the roots in the same way as G_p permutes the fusion rules. Therefore, we verify $\mathcal{G} = C_4$ with element identifications

$$(\{f_1, f_{11}, f_7, f_{13}\}, *) \cong (\{0, 2, 1, 3\}, +). \tag{D.57}$$

$M(2, 15)$ has no Hecke images that are characters of unitary RCFTs, as there is no unitary Galois-conjugate MTC listed in [31]. Moreover with level-rank duality, it explains why $(A_1, 13)_{\frac{1}{2}}$ and its complex conjugation $(A_{12}, 2)_{\frac{1}{13}}$ occur pairwise in the classification of bosonic topological

When $p = 35, 101, 103, 269$,

$$\rho^{\text{M}}(\sigma_p) = -\mathbb{I}_8. \quad p^2 = 1 \pmod{N} \quad (\text{D.62})$$

When $p = 5, 73, 131, 199$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 25 \pmod{N} \quad (\text{D.63})$$

When $p = 29, 97, 107, 175$,

$$\rho^{\text{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 25 \pmod{N} \quad (\text{D.64})$$

When $p = 7, 61, 143, 197$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot p^2 = 49 \pmod{N} \quad (\text{D.65})$$

When $p = 41, 95, 109, 163$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot p^2 = 49 \pmod{N} \quad (\text{D.66})$$

When $p = 11, 79, 135, 193,$

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \pmod{N} \quad (\text{D.67})$$

When $p = 23, 91, 113, 181,$

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \pmod{N} \quad (\text{D.68})$$

When $p = 13, 55, 149, 191$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 169 \pmod{N} \quad (\text{D.69})$$

When $p = 47, 89, 115, 157$,

$$\rho^{\text{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 169 \pmod{N} \quad (\text{D.70})$$

When $p = 19, 49, 155, 185$,

$$\rho^{\mathbf{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 157 \pmod{N} \quad (\text{D.71})$$

When $p = 53, 83, 121, 151$,

$$\rho^{\mathbf{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 157 \pmod{N} \quad (\text{D.72})$$

When $p = 25, 43, 161, 179$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \quad p^2 = 13 \pmod{N} \quad (\text{D.73})$$

When $p = 59, 77, 127, 145$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \quad p^2 = 13 \pmod{N} \quad (\text{D.74})$$

When $p = 31, 37, 167, 173$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 145 \pmod{N} \quad (\text{D.75})$$

When $p = 59, 77, 127, 145$,

$$\rho^{\text{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 145 \pmod{N} \quad (\text{D.76})$$

By the picture of effective central charge, we learn that there are unitary MTCs induced by T_p with $p = 53, 83, 121, 151$ respectively.

D.2.6 Minimal model $M(2, 19)$

For the minimal model $M(2, 19)$, the nine primary fields are labeled by

$$(r, s) = (9, 1), (8, 1), (7, 1), (6, 1), (5, 1), (4, 1), (3, 1), (2, 1), (1, 1). \quad (\text{D.77})$$

The conductor is $N = 57$, and the effective central charge is $c_{\text{eff}} = 16/19$.

$$\rho^M(T) = \begin{pmatrix} \xi_N^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_N^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_N^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_N^{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_N^{28} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_N^{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_N^{61} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_N^{82} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_N^{106} \end{pmatrix} \quad (\text{D.78})$$

The solutions to a ninth-order MLDE requires

$$\frac{1}{N}(-2 + 1 + 7 + 16 + 28 + 43 + 61 + 82 + 106) = \frac{1}{12}9(9 - 1), \quad (\text{D.79})$$

which is true.

Explicit computations lead to following $\rho^M(\sigma_p)$ for all p , with $1 \leq p < N$ and $\text{gcd}(p, N) = 1$.

When $p = 1, 20, 37, 56$,

$$\rho^M(\sigma_p) = \mathbb{I}_9. \quad p^2 = 1 \pmod{N} \quad (\text{D.80})$$

When $p = 2, 17, 40, 55$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 4 \text{ Mod } N \quad (\text{D.81})$$

When $p = 4, 23, 34, 53$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 16 \text{ Mod } N \quad (\text{D.82})$$

When $p = 5, 14, 43, 52$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 25 \text{ Mod } N \quad (\text{D.83})$$

When $p = 7, 26, 31, 50$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 49 \text{ Mod } N \quad (\text{D.84})$$

When $p = 8, 11, 46, 49$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad p^2 = 7 \text{ Mod } N \quad (\text{D.85})$$

When $p = 10, 28, 29, 47$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \quad p^2 = 43 \text{ Mod } N \quad (\text{D.86})$$

When $p = 13, 25, 32, 44$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 55 \text{ Mod } N \quad (\text{D.87})$$

When $p = 16, 22, 35, 41$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 28 \text{ Mod } N \quad (\text{D.88})$$

By the picture of effective central charge, we learn that there are unitary MTCs induced by $\mathbb{T}_2, \mathbb{T}_{17}, \mathbb{T}_{40}$ and \mathbb{T}_{55} respectively. That means the situation $p^2 \equiv 4 \pmod{N}$, when N is odd. We may wonder if the derivations of FSI and quantum dimensions still hold, since they involve f_{N,ℓ^2} acting on $e[Q_J(a)]$. Note that the denominator of h_J must divide N_0 and therefore N . For odd N , h_J can not be congruent to $\frac{1}{2}$ nor $\pm\frac{1}{4} \pmod{\mathbb{Z}}$. Hence there cannot be any non-trivial

simple current of order 2 in the initial RCFT. In that case, $e[Q_J(a)]$ are trivial and are not affected by possible even ℓ^2 .

D.2.7 Minimal model $M(3, 7)$

For the minimal model $M(3, 7)$, the six primary fields are labeled by

$$(r, s) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1). \quad (\text{D.89})$$

The conductor is $N = 168$, and the effective central charge is $c_{\text{eff}} = 5/7$.

$$\rho^M(T) = \begin{pmatrix} \xi_N^{25} & & & & & \\ & \xi_N^{-5} & & & & \\ & & \xi_N^1 & & & \\ & & & \xi_N^{43} & & \\ & & & & \xi_N^{121} & \\ & & & & & \xi_N^{235} \end{pmatrix} \quad (\text{D.90})$$

The solutions to a sixth-order MLDE requires

$$\frac{1}{N}(25 - 5 + 1 + 43 + 121 + 235) = \frac{1}{12}6(6 - 1), \quad (\text{D.91})$$

which is true.

Explicit computations lead to following $\rho^M(\sigma_p)$ for all p , with $1 \leq p < N$ and $\text{gcd}(p, N) = 1$.

When $p = 1, 41, 55, 71, 97, 113, 127, 167$,

$$\rho^M(\sigma_p) = \mathbb{I}_6. \quad p^2 = 1 \pmod{N} \quad (\text{D.92})$$

When $p = 13, 29, 43, 83, 85, 125, 139, 155$,

$$\rho^M(\sigma_p) = -\mathbb{I}_6 . \quad p^2 = 1 \text{ Mod } N \quad (\text{D.93})$$

When $p = 5, 19, 37, 61, 107, 131, 149, 163$,

$$\rho^M(\sigma_p) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \quad p^2 = 25 \text{ Mod } N \quad (\text{D.94})$$

When $p = 23, 47, 65, 79, 89, 103, 121, 145$,

$$\rho^M(\sigma_p) = - \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \quad p^2 = 25 \text{ Mod } N \quad (\text{D.95})$$

When $p = 11, 53, 59, 67, 101, 109, 115, 157$,

$$\rho^{\mathbf{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \text{ Mod } N \quad (\text{D.96})$$

When $p = 17, 25, 31, 73, 95, 137, 143, 151$,

$$\rho^{\mathbf{M}}(\sigma_p) = - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 121 \text{ Mod } N \quad (\text{D.97})$$

Clearly $(\ell^2, J) = (25, \phi_{(6,1)})$ saturates the GS property

$$\frac{1}{24}(\bar{\ell}^2 c_{\text{eff}} - c) + h_J \equiv 0 \pmod{\mathbb{Z}}. \quad (\text{D.98})$$

We can construct an effective picture based on that and narrow down the allowed T_p . Moreover with the sign of Fourier coefficients, $T_p \chi$ are associated with unitary RCFTs only when $p = 11, 53, 59, 67, 101, 109, 115, 157$. Some Hecke images coincide with characters of known RCFTs:

$$T_{11} \chi[\mathbf{M}(3, 7)] = \chi[(C_5, 1)]. \quad (\text{D.99})$$

In [31], the unitary MTCs induced by T_{11}, T_{53}, T_{59} and T_{101} enter the classification of bosonic

topological orders at rank 6. In addition, \mathbb{T}_p and \mathbb{T}_{p+56} induce the same MTC, since $56 \times \frac{5}{7} \equiv 0 \pmod{8}$. $\mathbb{T}_{59} \chi[\mathbb{M}(3, 7)]$ produces the same MTC as $(A_1, 5)$.

D.2.8 Minimal model $\mathbb{M}(3, 8)$

For the minimal model $\mathbb{M}(3, 8)$, the seven primary fields are labeled by

$$(r, s) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1). \quad (\text{D.100})$$

The conductor is $N = 32$, and the effective central charge is $c_{\text{eff}} = 3/4$.

$$\rho^{\mathbb{M}(T)} = \begin{pmatrix} \xi_N^7 & & & & & & \\ & \xi_N^0 & & & & & \\ & & \xi_N^{-1} & & & & \\ & & & \xi_N^4 & & & \\ & & & & \xi_N^{15} & & \\ & & & & & \xi_N^{32} & \\ & & & & & & \xi_N^{55} \end{pmatrix} \quad (\text{D.101})$$

has degenerate eigenvalues. The solutions to a seventh order MLDE requires

$$\frac{1}{N}(7 + 0 - 1 + 4 + 15 + 32 + 55) = \frac{1}{12}7(7 - 1), \quad (\text{D.102})$$

which is true.

Explicit computations lead to following $\rho^{\mathbb{M}(\sigma_p)}$ for all p , with $1 \leq p < N$ and $\text{gcd}(p, N) = 1$.

When $p = 1, 15, 17, 31$,

$$\rho^{\mathbb{M}(\sigma_p)} = \mathbb{I}_7. \quad p^2 = 1 \pmod{N} \quad (\text{D.103})$$

When $p = 3, 13, 19, 29$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 9 \text{ Mod } N \quad (\text{D.104})$$

When $p = 5, 11, 21, 27$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad p^2 = 25 \text{ Mod } N \quad (\text{D.105})$$

When $p = 7, 9, 23, 25$,

$$\rho^{\text{M}}(\sigma_p) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad p^2 = 49 \text{ Mod } N \quad (\text{D.106})$$

There are 16 unitary MTCs relevant to the initial theory [31]. Simple currents are crucial in the GS property, which take a non-unitary RCFT to the pictures of effective central charge [26]. Due to the associativity of fusion product, the product of two simple currents is again a simple current. Simple currents thus form an abelian group under multiplication which is called the center of a RCFT [27]. There are two simple currents

$$\{\Phi_0, \Phi_{3/2}\} \quad (\text{D.107})$$

by checking the vacuum row of the modular S matrix. $\Phi_{1/4}$ and $\Phi_{-1/4}$ are not simple currents though with $4h \in \mathbb{Z}$. The GS equation admits two solutions:

$$(\ell^2, J) = (9, \Phi_0), (25, \Phi_{3/2}). \quad (\text{D.108})$$

The allowed p^2 are 25 and 9 (mod N) respectively. However $(9, \Phi_0)$ does not lead to unitary Hecke images since $\epsilon_p(\nu) = -1$ with $p = 5, 11, 21, 27$, where ν is the minimal primary. With Gannon's approach, the unitarized theory has central charge

$$\hat{c} = \ell(c - 24h_J) + 12\epsilon \pmod{24}. \quad (\text{D.109})$$

where $(-1)^\epsilon := \epsilon_\ell(0)$. In the MTC approach we may shift the central charge by 12 after implementing the effective picture $(9, \Phi_0)$ and the Frobenius map $f_{N,p}$.

$$\hat{c} = \ell c + 4 \pmod{8}, \quad \ell = 3, 13, 19, 29. \quad (\text{D.110})$$

Adding the central charge by 4 amounts to tensoring a 1d modular representation [24]. It produces four unitary MTCs of $c = \pm 1/4, \pm 15/4$. Then the sign of $\epsilon_p(\nu)$ flips, and we get unitary MTCs with consistent twists as in [31]. For the effective picture $(25, \Phi_{3/2})$, \mathbb{T}_p with $p = 3, 13, 19, 29$ are

allowed, and give rise to unitary Hecke images with $c = \pm 7/4, \pm 9/4$.

$$\begin{aligned} T_3 \chi[M(3, 8)] &= \chi[(A_1, 6)], \\ T_{13} \chi[M(3, 8)] &= \chi[(C_6, 1)]. \end{aligned} \tag{D.111}$$

The T_3 case is verified by the author. The Hecke images under T_{19} and T_{29} correspond to unitary RCFTs as well.

The other eight MTCs of rank 7 cannot be obtained using the effective picture of $M(3, 8)$ and are not listed in [35] either. Perhaps they correspond to some other primitive MTC and have conductor other than 32. By [30], the MTC of $c = 5/4$ arise from the \mathbb{Z}_6 parafermion, which has conductor $N = 96$. Again, the field of twist $e(\frac{1}{2})$ is a simple current, denoted by J . For the \mathbb{Z}_6 parafermion, there are two effective picture and they are $(1, I)$ and $(49, J)$ respectively. T_p can be applied in either effective picture and induce unitary MTCs. The analysis exhausts the sixteen unitary MTCs of same non-abelian type.

Is there a connection between $M(3, k + 2)$ and the $(k - 1)$ -th Galois conjugation of the \mathbb{Z}_k parafermion, when $k - 1$ is odd? They have total central charge

$$c_{\text{tot}} = 1 - \frac{6(k-1)^2}{3(k+2)} + (k-1)\frac{2(k-1)}{k+2} = 1. \tag{D.112}$$

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