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A mes parents, qui m'ont appris à écrire et compter.

Et à Granny, qui refuse de croire que ces résultats ne servent à rien.

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ABSTRACT

We study growth of Betti numbers in towers of cocompact arithmetic lattices in unitary groups $U(a, b)$. In the middle degree of cohomology, the Betti numbers grow proportionally to the volume of the manifold, but away from the middle degree, the growth is known to be sub-linear in the volume. After rephrasing the problem into representation-theoretic terms, we give upper bounds on the growth of cohomology in small degrees coming from certain families of representations. These upper bounds are achieved in the framework of the endoscopic classification of representations: we use Arthur's stable trace formula to bound the growth in terms of multiplicities of discrete series representations on endoscopic groups.

CHAPTER 1

INTRODUCTION.

Let G be a semisimple Lie group and $\Gamma(p^n) \subset G$ a congruence tower of cocompact arithmetic lattices. The problem motivating this thesis is the computation of the Betti numbers $h^i(p^n) = \dim(H^i(\Gamma(p^n), \mathbf{C}))$, and more precisely of the growth rate of $h^i(p^n)$ as $n \to \infty$. Cohomology of arithmetic groups is computed representation-theoretically via Matsushima's formula [32]:

$$h^i(p^n) = \sum_\pi m(\pi, p^n) h^i(\mathfrak{g}, K; \pi).$$

Here π is a unitary irreducible representation of G, with $m(\pi, p^n)$ its multiplicity in the regular representation of G on $L^2(\Gamma(p^n) \backslash G)$ and $h^i(\mathfrak{g}, K; \pi)$ the dimension of its ith so-called $(\mathfrak{g}, K)$-cohomology group. The finitely many cohomological representations for each group have been classified by Vogan-Zuckerman in [46]. This reduces the question of cohomology growth to that of finding the limit multiplicity of cohomological representations, i.e. the rate of growth of $m(\pi, p^n)$ as $n \to \infty$.

Multiplicity growth rates are best understood for discrete series representations, who contribute to cohomology only in the middle degree. DeGeorge-Wallach [11] have shown that if π is discrete series, then $m(\pi, p^n)$ grows as fast as possible: proportionally to the index $[\Gamma(p^n) : \Gamma(1)]$ or, equivalently, to the volumes of the associated locally symmetric space. If the group G does not have discrete series, no exact rates of growth are known, see for example [10]. Even for groups which admit discrete series representations (which will be our focus here), this leaves open the question of multiplicity growth for cohomological representations in lower degrees. In general such representations are non-tempered: their matrix coefficients fail to decay fast enough. DeGeorge-Wallach show a weaker result for

non-tempered representations π. Their multiplicities $m(\pi, p^n)$ satisfy

$$m(\pi, p^n)/[\Gamma(p^n) : \Gamma(1)] \xrightarrow[n \to \infty]{} 0.$$

Thus the more specific question motivating this thesis is:

Question 1.0.1. Is it possible to compute the exact rate of growth of $m(\pi, p^n)$ for non-tempered cohomological representations π?

In [39], Sarnak-Xue made a prediction for upper bounds interpolating between the growth of discrete series representation and the (constant) multiplicity of the trivial representation:

Conjecture 1.0.2. (Sarnak-Xue) Let π be a unitary representation of G and let

$$p(\pi) = \inf\{p \geq 2 \,|\, \text{ the } K\text{-finite matrix coefficients of } \pi \text{ are in } L^p(G)\}.$$

Then

$$m(\pi, p^n) \ll_\epsilon [\Gamma(p^n) : \Gamma(1)]^{\frac{2}{p(\pi)} + \epsilon}.$$

By definition, the representation π is tempered if $p(\pi) = 2$. Thus Sarnak-Xue expect that the extent of the failure of π to be tempered dictates the slowness of the growth of $m(\pi, p^n)$.

1.1 Main Theorem

In this thesis, we give upper bounds on the multiplicity growth of certain cohomological representations. Let E/F be a CM extension of number fields and $\mathfrak{p}$ a large enough prime of F. Let $U(N - a, a)$ be a unitary group defined in terms of E/F from a Hermitian form, and let $\Gamma(\mathfrak{p}^n)$ be a sequence of cocompact lattices in $U(N - a, a)$. Our main theorem concerns a

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family π_k of representations of $U(a, N - a)$ contributing to cohomology in degrees

$$i = \begin{cases} a(N - 2k) & a \leq k \\ a(N - a) - k^2 & k \leq a. \end{cases}$$

As will be discussed in the body of the text, these representations live in packets corresponding to Arthur parameters whose restriction to Arthur's SL_2 is $\nu(2k) \oplus \nu(1)^{N-2k}$.

Theorem 1.1.1. *Let $\Gamma(\mathfrak{p}^n)$ be a tower of full-level cocompact lattices in $U(a, N - a)$ and let $i < a(N - a)$. Let $h_k^i(\mathfrak{p}^n)$ denote the dimension of the subspace of $H^i(\Gamma(\mathfrak{p}^n), \mathbf{C})$ coming from the contribution to $(\mathfrak{g}, K)$-cohomology of the representation π_k from global parameters with two irreducible summands. Then*

$$h_k^i(\mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}.$$

These are, as far as the author can tell, the first results on growth of cohomology in low degrees for unitary groups of arbitrary rank, and which hold for any prime $\mathfrak{p}$ large enough. The theorem is a consequence of the endoscopic classification of representations for unitary groups. The classification is a result of Mok [35] if the group is quasisplit, and of Kaletha–Minguez–Shin–White [23] for inner forms, building on the seminal work of Arthur [4] who gave such a classification for quasisplit classical groups.

Note that we are not the first to consider this specific family of cohomological representations. It encompasses all representations contributing to the so-called special cohomology studied by Bergeron–Millson–Moeglin [7] in their proof of the Hodge conjecture for arithmetic quotients of the complex ball.

1.1.1 Outline of the Proof

The result is proved in the framework of endoscopy, Arthur parameters, and the stable trace formula, on which we start by saying a few words. The endoscopic classification of representations for a group G/F gives a decomposition of the regular representation of the adèle group $G(\mathbf{A}_F)$ on the discrete spectrum:

$$L_{\mathrm{disc}}^2(G(F)\backslash G(\mathbf{A}_F)) \simeq \bigoplus_\psi \bigoplus_{\pi \in \Pi_\psi} m(\pi)\pi$$

where the irreducible summands $\pi = \otimes'_v \pi_v$ are automorphic representations appearing in the discrete spectrum with multiplicity $m(\pi)$. This decomposition is given in terms of sets of representations called Arthur packets Π_ψ indexed by Arthur parameters ψ. These parameters stand in for representations

$$\psi : L_F \times SL_2(\mathbf{C}) \to {}^L G$$

where ${}^L G$ is the L-group of G and L_F is the Langlands group of F, an object whose existence is at the present moment only hypothetical.

Endoscopy is a specific instance of the principle of functoriality in the Langlands program. It concerns certain groups H, the so-called endoscopic groups of G, and states that if ψ factors through an embedding ${}^L H \hookrightarrow {}^L G$, then there must be trace identities between the characters of the representations $\pi \in \Pi_\psi$ and those of representations π^H of H in a corresponding packet Π_ψ^H. The character identities are witnessed through the trace formula $I_{\mathrm{disc},\psi}(f)$, a distribution computing the trace of convolution by a smooth, compactly supported function f on the subspace of L_{disc}^2 spanned by the representations $\pi \in \Pi_\psi$. More specifically, the character identities appear in a decomposition of $I_{\mathrm{disc},\psi}(f)$ referred to as the stabilization of the trace formula:

$$I_{\mathrm{disc},\psi}(f) = \sum_H S_{\mathrm{disc},\psi}^H(f^H). \tag{1.1}$$

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Here the sum runs over all endoscopic groups H such that ψ factors through $^L H$. The distributions $S^H_{\text{disc},\psi}(f^H)$ are stable, meaning that they satisfy a strengthening of the conjugacy-invariance property of characters of representations.

The summands $S^H_{\text{disc},\psi}(f)$, initially defined inductively, can be expanded explicitly as linear combinations of traces $\text{tr}\,\pi(f)$ of the representations $\pi \in \Pi_\psi$: this is the so-called stable multiplicity formula. We write here a simplified version of the stable multiplicity formula in which we have omitted constants which can be ignored in the asymptotic questions we are concerned with:

$$S^H_{\text{disc},\psi}(f^H) = \sum_{\pi \in \Pi_\psi} \xi(\pi, H)\,\text{tr}\,\pi(f). \tag{1.2}$$

The coefficients $\xi(\pi, H)$ arise from characters of a 2-group $\mathcal{S}_\psi$, the group of connected components of the centralizer of the image of ψ. More precisely, there are two mappings

$$\{\text{representations } \pi \in \Pi_\psi\} \to \{\text{characters of } \mathcal{S}_\psi\}$$

$$\{H \text{ such that } \psi \text{ factors through } {}^L H\} \to \{\text{elements of } \mathcal{S}_\psi\},$$

the second of which is a bijection. In this way, the coefficient $\xi(\pi, H)$ in the decomposition of the stable term $S^H_{\text{disc},\psi}(f^H)$ is the value of the character associated to π on the group element corresponding to H.

In this context, the steps of the proof of Theorem 1.1.1 can be outlined as:

(i) (§3.3.3) Determine the parameters ψ associated to the packets containing the cohomological representations π_k. Specifically, compute the restriction ψ_∞ of the Arthur parameters of packet containing these representations. This relies on work of Arthur [3] and Adams-Johnson [1].

(ii) (§3.2.2) Write the dimension of cohomology as $\sum_\psi I_{\text{disc},\psi}(f(\mathfrak{p}^n))$ for a specific test

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function $f(\mathfrak{p}^n)$, summing over the parameters ψ computed in the first step.

(iii) (§2.4.2) Fix a cohomological parameter ψ. Use the stabilization of the trace formula to decompose

$$I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) = \sum_H S_{\mathrm{disc},\psi}^H(f(\mathfrak{p}^n)^H).$$

(iv) (§3.1.2) By interpreting the coefficients $\xi(\pi, H)$ appearing in the stable multiplicity formula (1.2) as values of characters of $\mathcal{S}_\psi$, conclude that there is an endoscopic group H_ψ whose contribution bounds that of all the others in (1.1), i.e. such that

$$I_{\mathrm{disc},\psi}(f(\mathfrak{p}^n)) \leq K(\psi) S_{\mathrm{disc},\psi}^{H_\psi}(f(\mathfrak{p}^n)^{H_\psi})$$

for a uniformly bounded $K(\psi)$. This group H_ψ corresponds to the identity element of the group $\mathcal{S}_\psi$ and depends only on $\psi(SL_2(\mathbf{C}))$. As such it is determined by the parameter ψ_∞ and ultimately by the choice of cohomological representations.

(v) (§3.2.4) Interpret the stable trace $S_{\mathrm{disc},\psi}^{H_\psi}(f(\mathfrak{p}^n)^{H_\psi})$ as the contribution of ψ to the multiplicity $m(\pi^{H_\psi}, \mathfrak{p}^n)$ for a family π^{H_ψ} of representations of H_ψ. This relies on the fundamental lemma, proved by Laumon-Ngô for unitary groups [29], but also on a variant for congruence subgroups due to Ferrari [14]. Then sum over all ψ with the right ψ_∞. This sum is now proportional to the multiplicity $m(\pi^{H_\psi}, \mathfrak{p}^n)$.

(vi) (§3.2.4 and §3.3.6) The representations π^{H_ψ} obtained via steps (i)-(v) from the family π_k are the product of a discrete series representation and a character. Their limit multiplicity is thus know by results of Savin [40], which gives the desired bounds.

Throughout the paper, many results are imported from the works [4, 23, 35] cited above. It is the author's hope that this thesis can serve as an introduction, however black box-filled, for someone hoping to use the stable trace formula for "concrete" applications.

This method is in the lineage of a body of recent work applying the framework of endoscopy to the question of growth of cohomology. Most notably, recent progress on multiplicity growth of non-tempered cohomological representations has been made by Marshall [30] when $G = U(2,1)$, and Marshall-Shin [31] for $G = U(N,1)$ and $\mathfrak{p}$ a prime splitting in the CM extension defining the unitary group.

1.2 Conditionality

Our results are conditional on the endoscopic classification of representations for inner forms of unitary groups, a result which remains to be fully proved in two distinct ways. As explained in the introduction of [23], the classification depends on upcoming work of Chaudouard-Laumon on the weighted fundamental lemma. Moreover, the proof of the classification in [23] is not itself complete: in particular, the results appearing in this thesis as Theorem 2.3.6 and Theorem 2.4.1 are only proved for generic parameters. A full proof is expected in [22].

1.3 Further Work

The main result of this thesis is far from answering Question 1.0.1. It fails to even give upper bounds for any particular degree of cohomology. It is nevertheless our belief that the representations for which we do compute the rate of growth yield asymptotically all the cohomology in the prescribed degrees. We lay out below some avenues for doing this, as well as possible generalizations of the work of this thesis.

The most immediate obstacle to proving more general bounds is the absence of control on the stable terms corresponding to groups $H \neq H_\psi$. This issue prevents us from considering anything beyond the simplest families of Arthur parameters, as is most clearly laid out in the discussion around Proposition 3.2.10. We expect to address this in the near future: our hope is to show that there are enough representations in the local Arthur packets for all

characters of the group $\mathcal{S}_\psi$ to appear. This would force the traces corresponding H to be bounded, much in the way that for a nontrivial element g of a finite abelian group G, the sum $\sum_{\xi \in \hat{G}} \xi(g)$ vanishes.

A second obstacle arises when the packet Π_ψ is stable, meaning that $H_\psi = G$. This is a more serious limitation of our technique since in that case we lose access to the entire inductive scaffolding of the stable trace formula. We hope to solve this by considering twisted transfer to GL_N, but this is a more long-term goal.

The endoscopic classification of representations holds for symplectic and orthogonal groups, following the work of Arthur [4]. We have written the first half of this thesis, up to and including Section 3.1, with the idea that the group G could fairly painlessly be taken to be orthogonal or symplectic.

Finally, we note that we have yet to state whether our main theorem corroborates Sarnak-Xue's conjecture. We expect that this is the case. This question boils down to computing $p(\pi)$ for cohomological representations, something that will be done in upcoming joint work with Simon Marshall [17].

CHAPTER 2

BACKGROUND ON ENDOSCOPY AND THE TRACE FORMULA

2.1 Unitary Groups and Their L-Groups

In this section we introduce unitary groups and their endoscopic groups, L-groups, automorphic representations, as well as Arthur parameters and the various objects attached to them.

We start with some notation. Let E/F be a CM extension of number fields with Galois group $\Gamma_{E/F}$, algebraic closure $\bar{F}$ and absolute Galois groups Γ_F and Γ_E. We denote the places of F and E by v and w respectively. If v is a place of F let $E_v = E \otimes_F F_v$. Let $F_\infty = F \otimes_\mathbf{Q} \mathbf{R}$ denote the product of all the archimedean completions of F. Let $\mathcal{O}_F$ and $\mathcal{O}_E$ be the respective rings of integers, and $\mathbf{A}_F$ and $\mathbf{A}_E$ be adèle rings. Let $\mathbf{A}_F^f$ be the finite adèles, so that we have $\mathbf{A}_F = F_\infty \times \mathbf{A}_F^f$. Let $\mathrm{Nm} : \mathbf{A}_E \to \mathbf{A}_F$ denote the norm map.

Fix χ_κ for $\kappa \in \{\pm 1\}$, a pair of Hecke characters of E. We fix χ_{+1} to be trivial and the character χ_{-1} is chosen so that its restriction to $\mathbf{A}_F/F^\times$ is the character associated to E by class field theory.

If F is a field and G/F is a reductive group, we will denote the center of G by Z_G, or by Z when the ambient group is clear from context. If F is global then for any place v of F, we denote $G(F_v)$ by G_v and $G(F_\infty)$ by G_∞. For $H \subset G(\mathbf{A}_F)$ a subgroup of the adelic points of G, we use the notation $H_f = H \cap G(\mathbf{A}_F^f)$. The complexified Lie algebra of G_∞ will be denoted $\mathfrak{g}_\infty$.

2.1.1 Quasisplit Unitary Groups

We now introduce unitary groups and their L-groups, following the exposition of Kaletha-Minguez-Shin-White in [23, §0]. Let E/F be a quadratic algebra: either the CM extension

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introduced above, or one of its localizations E_v/F_v, in which case we have $E \simeq F \times F$ when v is split. If this is the case, fix an isomorphism and identify $E = F \times F$. Let $\sigma \in \mathrm{Aut}_F(E)$ be the nontrivial element of $\Gamma_{E/F}$ if E is a field, and the map given by $\sigma(x, y) = (y, x)$ if $E = F \times F$.

Let Φ_N be the antidiagonal $N \times N$ matrix

$$\Phi_N = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & (-1)^{N-2} & & & \\ (-1)^{N-1} & & & & \end{pmatrix}. \tag{2.1}$$

In the case that E is a split quadratic algebra, set $\Gamma_E := \Gamma_F$. Let $U_{E/F}(N)$ be the reductive group over F whose group of $\bar{F}$-points is $GL_N(\bar{F})$, with Galois action

$$\tau_N(g) = \begin{cases} \tau(g) & \tau \in \Gamma_E \\ \mathrm{Ad}(\Phi_N)\tau(g)^{-t} & \tau \in \Gamma_F \setminus \Gamma_E \end{cases}.$$

When the context is clear, the group $U_{E/F}(N)$ will be denoted $U(N)$. Its F-points can be identified with

$$U_{E/F}(N, F) = \{g \in GL_N(E) \mid \mathrm{Ad}(\Phi_N)\sigma(g)^{-t} = g\}. \tag{2.2}$$

It is a quasisplit unitary group, with maximal (non-split) torus given by the group of diagonal matrices, and a Borel subgroup consisting of upper-triangular matrices. Note that in the case that $E = F \times F$, we have $U(N) \simeq GL_N$ and we fix an isomorphism to identify them. Additionally we have an identification $U(N, E) = GL_N(E)$.

If the field F is global, we can consider the various localizations of $U(N, F)$. If v splits in E, we have $U(N, F_v) \simeq GL_N(F_v)$. Otherwise $U(N, F_v)$ a quasisplit unitary group over F_v,

a condition that determines it uniquely up to isomorphism, as we shall see below.

2.1.2   Inner Forms

An inner form of $U(N)$ is a pair consisting of an algebraic group G/F together with an isomorphism

$$\xi : G(\bar{F}) \to U(N, \bar{F})$$

with the property that for all $\sigma \in \Gamma_F$, the automorphism $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1}$ is inner. In this thesis, we will always require that the inner forms be groups defined with respect to a Hermitian space over E. When we speak of G an inner form of $U(N)$, we always make a choice a twist ξ although it is most often implicit. Furthermore, we will denote $U(N)$ by G^* when we want to highlight that it is the quasisplit form of G. We now discuss which possible groups G can arise as inner forms of $U_{E/F}(N)$ in the cases where F is local or global.

Local Inner Forms and the Kottwitz Sign

If v is archimedean the classification of inner forms is well-known: a unitary group over $F_v = \mathbf{R}$ is determined by its signature $p + q = N$, with $U(p,q) \simeq U(q,p)$. The group $U(p,q)$ is quasisplit if and only if $|p - q| \in \{0, 1\}$. Note that since the notation $U(N)$ is reserved for quasisplit groups, we will denote the compact inner form of $U(N, \mathbf{R})$ by $U_N(\mathbf{R})$.

For v nonarchimedean, the classification of unitary groups coming from Hermitian forms over F_v is due to Landherr [27]: if N is odd, there is one class of Hermitian forms up to isomorphism, so the group $U(N, F_v)$ is the unique unitary group of rank N. If N is even, there are two isomorphism classes of unitary groups, only one of which (the one containing $U(N, \bar{F}_v)$) is quasisplit.

One can associate to an inner twist G_v of $U(N)_{E_v/F_v}$ a *Kottwitz sign* $e(G_v)$. We record the formulas for $e(G_v)$ depending on the base field, as computed in [25].

- For $F_v = \mathbf{R}$, let $q(G_v)$ be half the dimension of the symmetric space associated to the group G_v. Then $e(G_v) = (-1)^{q(G_v)-q(G_v^*)}$.

- For F_v non-archimedean, let $r(G_v)$ be the rank of G_v. Then $e(G_v) = (-1)^{r(G_v)-r(G_v^*)}$.

Lastly, Kottwitz proves in [25] that for any group G defined over a global field, the local signs cancel out and $\prod_v e(G_v) = 1$.

Global Inner Forms

We describe the classification of global forms of unitary groups, following the discussion in Section 0.3.3 of Kaletha-Minguez-Shin-White [23]. When N is odd, there is no global obstruction and any collection of local inner twists can be realized as the localization of a global inner twist.

When N is even, the behavior of the place v in the extension E determines the cohomological invariants attached to G_v. In any case, we have that $H^1(\Gamma_{F_v}, G_v^{*,ad}) \simeq \mathbf{Z}/2\mathbf{Z}$. If v is split in E, the invariant of G_v depends on the division algebra D_v such that $G_v = \mathrm{Res}_{F_v}^{D_v} GL_{M_v}$. Since we only work with unitary groups coming from Hermitian forms, this invariant will always be 0 for us. At finite nonsplit places, the quasisplit group $U(N)_v$ and its unique inner form correspond respectively to 0 and 1 in $\mathbf{Z}/2\mathbf{Z}$. At the infinite places, the invariant associated to the group G_v with signature (p,q) is $\frac{N}{2} + q \in \mathbf{Z}/2\mathbf{Z}$. The condition for a collection of local G_v to come from a global unitary group is that almost all of the invariants associated to G_v are zero, and that their sum is also zero. We record a consequence of this in a lemma.

Lemma 2.1.1. *Let F be a totally real field and E/F a CM extension. Then there exists a unitary group G over F with any prescribed choice of signature at the infinite places. Moreover, this group G can be chosen to be quasisplit outside of a set of places of size at most 1.*

Remark 2.1.2. The authors of [23] work with a refinement of the notion of inner form. Recall that isomorphism classes of inner forms of G are in bijection with $H^1(\Gamma_F, G_{\mathrm{ad}})$. In addition to this, they introduce the notion of pure inner form, a triple consisting of the group G, the map ξ and a cocycle $z \in Z^1(\Gamma, G)$ compatible with the inner twist in the sense that $\sigma \circ \xi \circ \sigma^{-1} = \mathrm{Ad}(z(\sigma))$. The map sending a pure inner form to z induces a bijection between the isomorphism classes of pure inner form and $H^1(\Gamma_F, G)$. Inner forms of unitary groups which can be realized as pure inner forms are those which come from a Hermitian space over F and not over a division algebra, and these are precisely the groups we work with. We will point out dependency on z in our results whenever it is relevant. Mainly, the definition of the pairing in local Arthur packets is given in terms of the localization z_v of the cocycle z, but this dependency on z_v cancels out globally.

2.1.3 L-groups

Throughout, we will work with the Weil group version of the L-group, primarily because it is well-suited to our description of local parameters. In terms of the actual definition of the L-group, this choice is purely cosmetic as the Galois actions involved will always factor through a quotient of order at most 2.

For G/F with F either local or global, fix a root datum. The L-group of G is a semidirect product

$$^L G = \hat{G} \rtimes W_F.$$

The group $\hat{G}$ is the complex dual group of G, i.e. the complex-valued points of the group whose root datum is dual to that of G. The action of W_F on $\hat{G}$ is then induced by the Galois action on the root datum of G. As a consequence, if G is split then $^L G = \hat{G} \times W_F$, and in particular,

$$^L GL_N(F) = GL_N(\mathbf{C}) \times W_F.$$

If G'/F is an inner form of G then by definition $G'(\bar{F}) \simeq G(\bar{F})$ and the corresponding Galois actions differ by an inner automorphism. These induce isomorphisms of root data and Galois actions, which in turn induce isomorphisms $^L G \simeq {}^L G'$.

When F is global, we will sometimes abuse notation and write, $^L G_v$ for the L-group of the base change of G to a completion F_v. In this situation, the embedding $W_{F_v} \to W_F$ induces a map $^L G_v \to {}^L G$ which is the identity on $\hat{G}$.

The L-group of $U(N)$ is defined as

$$^L U(N) = GL_N(\mathbf{C}) \rtimes W_F$$

where W_F acts through the order two quotient $\Gamma_{E/F}$. The non-trivial element σ of this quotient acts by the outer automorphism of GL_N preserving the standard diagonal splitting:

$$\sigma(g) = \Phi_N^{-1} g^{-t} \Phi_N,$$

where Φ_N was the matrix defined in (2.1). This L-group is shared by all inner forms of $U(N)$.

Morphisms of L-groups

If $^L H$ and $^L G$ are two L-groups, then a morphism of L-groups is a continuous morphism

$$\eta : {}^L H \to {}^L G$$

which commutes with the projections onto W_F. We will typically be concerned with L-embeddings, where $\hat{H} \hookrightarrow \hat{G}$.

In particular, many objects associated to a unitary group $U_{E/F}(N)$ depend on a choice of embedding of L-groups from $^L U_{E/F}(N)$ to $^L \mathrm{Res}_F^E GL_N$. The connected component of the

L-group of $\mathrm{Res}^E_F GL_N$ is the product of two copies of $GL(N, \mathbf{C})$, and W_F acts through $\Gamma_{E/F}$ via the automorphism that interchanges the two factors.

To define the L-embedding (often referred to as the base-change morphism) recall the characters χ_κ from the beginning of this section. If F is global, we will use this character, and if $F = F_v$ is local, we will momentarily also denote by χ_κ the restriction of χ_κ to $E_v^\times$. For each $\kappa \in \{\pm 1\}$ we choose an embedding

$$\eta_\kappa : {}^L U(N) \to {}^L \mathrm{Res}^E_F GL_N \tag{2.3}$$

as follows. Choose an element w_c of $W_F \setminus W_E$, and denote the identity $N \times N$ matrix by I_N. Then η_κ is defined as

$$\eta_\kappa(g \rtimes 1) = (g, {}^t g^{-1}) \rtimes 1, \quad g \in \hat{G}$$

$$\eta_\kappa(I_N \rtimes \sigma) = (\chi_\kappa(\sigma) I_N, \chi_\kappa^{-1}(\sigma) I_N) \rtimes \sigma, \quad \sigma \in W_E$$

$$\eta_\kappa(I_N \rtimes w_c) = (\kappa \Phi_N, \Phi_N^{-1}) \rtimes w_c.$$

The second class of embeddings we will consider is from the L-group of a product $U(N_1) \times ... \times U(N_r)$ of unitary groups with $\sum N_i = N$ into ${}^L U(N)$. These products of smaller unitary groups include the elliptic endoscopic groups of inner forms G of $U(N)$. In order te define the L-embeddings, put $\kappa_i = (-1)^{N-N_i}$ for each index i, and let $\underline{\kappa} = (\kappa_1, ..., \kappa_r)$. Given $\underline{\chi}$ with signature $\underline{\kappa}$, and for a choice of w_c as above, the embedding

$$\eta_{\underline{\kappa}} : {}^L(U(N_1) \times ... \times U(N_r)) \to {}^L U(N) \tag{2.4}$$

is defined by

$$\eta_{\underline{\kappa}}(g_1, ..., g_r \rtimes 1) = \mathrm{diag}(g_1, ..., g_r) \rtimes 1, \quad g_i \in GL(N_i, \mathbf{C})$$

$$\eta_{\underline{\kappa}}(I_{N_1}, ..., I_{N_r} \rtimes \sigma) = \mathrm{diag}(\chi_{\kappa_1}(\sigma)I_{N_1}, ..., \chi_{\kappa_r}(\sigma)I_{N_r}) \rtimes \sigma, \quad \sigma \in W_E$$

$$\eta_{\underline{\kappa}}(I_{N_1}, ..., I_{N_r} \rtimes w_c) = \mathrm{diag}(\kappa_1 \Phi_{N_1}, ..., \kappa_r \Phi_{N_r}) \cdot \Phi_N^{-1} \rtimes w_c.$$

Note that the composite embedding $\eta_\kappa \circ \eta_{\underline{\kappa}}$ gives an embedding $^L U(N_1) \times ... \times {}^L U(N_r) \to$ $^L \mathrm{Res}_F^E GL_N$ with signature $(\kappa\kappa_1, ..., \kappa\kappa_r)$.

The necessity to consider several embeddings depending on $\underline{\kappa}$ stems from the possibility that parameters for the pair $(U(N), \eta_+)$ may factor through different embeddings of the products of groups $U(N_i)$ associated to different signs. This will become apparent when we introduce endoscopic groups in 2.2.10.

2.2 Parameters

Here we introduce the discrete automorphic spectrum of a unitary group G, and the local and global parameters which will classify the (constituents of) these automorphic representations.

2.2.1 Automorphic Representations

Let (G, ξ) be an inner form of $U(N)$. Fix a character ω of $Z_G(\mathbf{A}_F)$ and a maximal compact subgroup of K of $G(\mathbf{A}_F^f)$, which in turn determines maximal compact subgroups K_v, hyperspecial at all unramified places. We consider the right-regular representation of $G(\mathbf{A}_F)$ on

$$L^2_{\mathrm{disc}}(G(F) \backslash G(\mathbf{A}_F), \omega),$$

the discrete part of the space of square-integrable functions which transform by ω under the action of the $Z_G(\mathbf{A}_F)$. We will sometimes drop the ω when we allow for any choice of

central character. In the cases of initial interest to us, G/F will be anisotropic, the central character ω will be trivial, and the space L^2_{disc} will be the entire automorphic spectrum of G. However for induction purposes we will have to allow for arbitrary central characters and for $L^2(G(F)\backslash G(\mathbf{A}_F), \omega)$ to have a continuous part. The discrete spectrum decomposes as

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbf{A}_F)) = \bigoplus m(\pi)\pi$$

where $m(\pi)$ denotes the multiplicity of π, and the irreducible constituents are automorphic representations. Each of these automorphic representations is a restricted tensor product $\pi = \otimes'_v \pi_v$ with each π_v an irreducible admissible unitary representation of each of the G_v. All but finitely many of the π_v are spherical with respect to K_v. The representation π_v is said to be tempered if its K_v-finite matrix coefficients belong to the $L^{2+\epsilon}(G_v)$ for all $\epsilon > 0$.

After fixing a maximal compact subgroup K_∞ of G_∞, we replace π_∞ by the dense subspace of K_∞-finite smooth vectors, which we view as an admissible $(\mathfrak{g}_\infty, K_\infty)$-module. This is no loss of information since unitary admissible representations are determined by their underlying $(\mathfrak{g}_\infty, K_\infty)$-modules, see [24, 9.2]. Thus in practice our automorphic representations will carry an action of $G(\mathbf{A}_F^f) \times \mathfrak{g}_\infty$.

2.2.2   Local Langlands Parameters

We now introduce the objects classifying automorphic representations and their constituents, beginning locally. Let F be a local field with Weil group W_F. The Langlands group L_F of F is defined as

$$L_F := \begin{cases} W_F & F \text{ is archimedean} \\ W_F \times SU(2, \mathbf{C}) & F \text{ is non-archimedean.} \end{cases}$$

A (local) Langlands parameter for the reductive group G/F is a continuous homomor-

phism

$$\varphi : L_F \to {}^L G$$

satisfying certain conditions (see [9] for a discussion):

(i) The map φ must commute with the natural projections $L_F \to W_F$ and ${}^L G \to W_F$.

(ii) In the non-archimedean case, the restriction $\varphi \mid_{SU(2,\mathbf{C})}$ must be algebraic.

(iii) The image of of W_F under φ must consist of semisimple elements of ${}^L G$.

(iv) If the image of φ in $\hat{G}$ factors through a parabolic subgroup of $\hat{G}$, then this parabolic subgroup must be the dual $\hat{P}$ of a parabolic subgroup P of G.

Continuous homomorphisms that satisfy condition (i) are known as L-homomorphisms. If they additionally satisfy conditions (ii)-(iv) they are called *admissible.* If they satisfy condition (iv), they are called *relevant,* or *G-relevant.* Note that this fourth condition is the only one depending on the choice of inner form G. Finally, we say that φ is *bounded* if W_F has bounded image in $\hat{G}$. We will consider two parameters equivalent if they are conjugate by an element of $\hat{G}$ and will denote the collection of equivalence class of Langlands parameters for G by $\Phi(G)$.

2.2.3   Local Arthur Parameters

In order to classify the non-tempered spectrum of G, we consider enhancements of local Langlands parameters known as local Arthur parameters. These are admissible L-homomorphisms

$$\psi : L_F \times SL_2(\mathbf{C}) \to {}^L G$$

such that $\psi \mid_{L_F}$ is bounded. Again, two Arthur parameters are equivalent if they differ by conjugation by an element of $\hat{G}$, and we denote the set of equivalence classes of Arthur

parameters by $\Psi(G)$. We will sometimes refer to the $SL_2(\mathbf{C})$ factor appearing in the above product as the Arthur SL_2. We will say that ψ is bounded if its restriction to the Arthur SL_2 is trivial.

To each Arthur parameter ψ we associate a Langlands parameter φ_ψ as follows. Recall (eg. [43]) that the Weil group W_F is naturally equipped with a norm homomorphism $|\cdot|$ to $\mathbf{C}^\times$. Then φ_ψ is defined as the composition

$$\varphi_\psi : W_F \to {}^L G, \quad \varphi_\psi(\sigma) = \psi\left(\sigma, \begin{pmatrix} |\sigma|^{1/2} & 0 \\ 0 & |\sigma|^{-1/2} \end{pmatrix}\right).$$

In the case where ψ is bounded, we have $\varphi_\psi = \psi\mid_{L_F}$.

We now describe Arthur parameters of unitary groups, following Section 2.2 of Mok [35]. The set $\Psi(U(N))$ is best understood in terms of $\Psi(\mathrm{Res}_F^E GL_N)$. To produce an element of $\Psi(\mathrm{Res}_F^E GL_N)$, one starts with an admissible N-dimensional representation ψ of $L_E \times SL_2(\mathbf{C})$ and promotes it to a homomorphism $L_F \to {}^L \mathrm{Res}_F^E GL_N$. This is done by first choosing an element $w_c \in W_F \setminus W_E$. The parameter $\psi' : L_F \times SL_2(\mathbf{C}) \to {}^L \mathrm{Res}_F^E GL_N$ is then defined by Mok. It satisfies

$$\psi'(\sigma, g) = (\psi(\sigma, g), \psi^c(\sigma, g)) \rtimes \sigma, \quad (\sigma, g) \in L_E \times SL_2,$$

where $\psi^c(\sigma, g) = \psi(w_c^{-1} \sigma w_c, g)$. If $\psi^c \simeq \psi^\vee$ where $\psi^\vee$ is the contragredient of ψ, then the parameter is called *conjugate self-dual*. There is a further notion of being conjugate self-dual of parity ± 1, which depends on the parity of the resulting bilinear form.

The map η_κ introduced in (2.3) then induces a mapping

$$\eta_{\kappa*} : \Psi(U(N)) \to \Psi(\mathrm{Res}_F^E GL_N) \tag{2.5}$$

$$\psi \mapsto \eta_\kappa \circ \psi. \tag{2.6}$$

This map $\eta_{\kappa*}$ is shown by Mok, following work of Gan-Gross-Prasad [16], to be an injection whose image is independent of the choice of w_c and consists precisely the set of self-dual representations of parity $(-1)^{N+1}\kappa$.

2.2.4 Global Arthur Parameters

When trying to extend the notion of Arthur parameter given above to a global field F, one is confronted with the current absence of a well-defined global Langlands group L_F. As a substitute for global parameters, Arthur [4, §1.4] introduces formal objects realized by combining cuspidal automorphic representations of GL_N with representations of the Arthur SL_2. In the case of unitary groups, the general linear group of reference is GL_N/E. Echoing the local discussion, global Arthur parameters are first defined in terms of $\mathrm{Res}_F^E GL_N$, and Arthur parameters for $U(N)$ are then the ones factoring through a fixed embedding of L-groups.

A global Arthur parameter for GL_N is a formal object consisting of a unordered sum

$$\psi^N = \boxplus_i \psi_i^{N_i}, \quad \psi_i^{N_i} = \mu_i \boxtimes \nu(m_i).$$

Here μ_i is a cuspidal automorphic representation of GL_{n_i} and $\nu(m_i)$ is a representation of $SL_2(\mathbf{C})$ as above, with $m_i n_i = N_i$ and $\sum_i N_i = N$. Departing from our references, we will immediately restrict our attention to the set of Arthur parameters such that the $\psi_i^{N_i}$ are pairwise distinct: we denote this set $\Psi(N)$ instead of $\Psi_{\mathrm{ell}}(N)$. The collection $\Psi(N)$ contains a distinguished subset $\Psi_{\mathrm{sim}}(N)$ consisting of simple parameters for which there is a unique summand ψ^N. Following the theorem of Mœglin-Waldspurger [33], this subset $\Psi_{\mathrm{sim}}(N)$

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parameterizes the discrete spectrum of GL_N.

We now give the construction of global Arthur parameters for a quasi-split unitary group $G = U(N)$, following the exposition of Section 1.3.4 of Kaletha-Minguez-Shin-White [23]. We start by restricting our attention to the set $\tilde{\Psi}(N) \subset \Psi(N)$ consisting of parameters for which each of the μ_i is conjugate self-dual. This means that the cuspidal automorphic representations μ_i of $GL_N(\mathbf{A}_E)$ satisfy $\mu_i = \bar{\mu}_i^{\vee}$ where $\bar{\mu} = \mu \circ \sigma$ and $\sigma \in \Gamma_{E/F}$.

Now to record not only the parameter, but also its relation to the embedding η_κ, we introduce the group $\mathcal{L}_\psi$. If ψ^N decomposes as a sum of $\mu_i \boxtimes \nu(m_i)$, we associate to each index a pair $(U_{E/F}(n_i), \eta_{\kappa_i})$ consisting of a quasisplit unitary group and an associated embedding as in 2.1.3. Here the choice of sign κ_i is determined by μ_i. Then $\mathcal{L}_\psi$ is the fiber product

$$\mathcal{L}_\psi = \prod_i ({}^L U_{E/F}(n_i) \to W_F).$$

There is a natural map $\tilde{\psi}^N : \mathcal{L}_\psi \times SL_2(\mathbf{C}) \to {}^L \mathrm{Res}_F^E GL_N$ given by the direct sum

$$\tilde{\psi}^N = \oplus(\eta_{\kappa_i} \otimes \nu(m_i)).$$

A global Arthur parameter for $(U_{E/F}(N), \eta_\kappa)$ is then defined as a pair $\psi = (\psi^N, \tilde{\psi})$ where $\psi^N \in \tilde{\Psi}(N)$, and
$$\tilde{\psi} : \mathcal{L}_\psi \times SL_2(\mathbf{C}) \to {}^L U_{E/F}(N)$$
is an L-homomorphism such that $\eta_\kappa \circ \tilde{\psi} = \tilde{\psi}^N$. This definition Arthur parameters as consisting of two pieces of data is rather cumbersome, but it is useful to remember that ψ^N encodes the arithmetic information of the cuspidal automorphic representations of GL_{n_i}, and that $\tilde{\psi}$ has the advantage of being an actual homomorphism. As such, we can (and will) discuss, for example, the centralizer of the image of $\tilde{\psi}$. We set two Arthur parameters

to be equivalent if they are conjugate, and denote the set of Arthur parameters ψ as above by $\Psi(U(N), \eta_\kappa)$. Note that once again we have broken off from our references in the choice of notation: our set $\Psi(U(N), \eta_\kappa)$ is the one that the authors of [23] denote $\Psi_2(U_{E/F}(N), \eta_\kappa)$. Finally, note that the map from $\psi \mapsto \psi^N$ is an injection: this allows us to regard $\Psi(U(N), \eta_\kappa)$ as a subset of $\Psi(N)$.

Remark 2.2.1. We have made two constraints on the set of parameters under consideration here which bear highlighting. We require:

(i) that the irreducible summands ψ_i be pairwise distinct. In Mok's description of the parameters in [35, §2.4] this amounts to requiring that all the $l_i = 1$.

(ii) that each of the irreducible summands be itself conjugate self-dual. This is more strict than requiring the whole parameter to be conjugate self-dual since we could have had $\mu_i^\vee \simeq \mu_j$.

Parameters satisfying these conditions are called *elliptic*. These restrictions will give us control on the group $\mathcal{S}_\psi$ to be introduced below, whose characters determine which products of local representations occur in the discrete spectrum. It is also the case that only the parameters in the set which we denote by $\Psi(U(N), \eta_\kappa)$ correspond to packets whose members actually appear in the decomposition of L^2_{disc}, although this fact is far from obvious and is one of the main theorems in [35] and [23]. Following this result, global elliptic parameters are also called *square-integrable*.

2.2.5   Localization

We now explain how global Arthur parameter $\psi \in \Psi(U(N), \eta_\kappa)$ gives rise to local Arthur parameters ψ_v at each place v. Each cuspidal representation μ of GL_N factors as a restricted tensor product $\mu = \otimes' \mu_v$ over all places v of F. These representations μ_v are admissible representations of $GL_N(F_v)$. The local Langlands correspondence for GL_N [19, 20, 41]

associates to each μ_v an bounded parameter $\varphi_{\mu_v} \in \Phi(GL_N)$. Following [4], we then define the localization of ψ at v as the direct sum

$$\psi_v = \oplus_i \psi_{v,i}, \quad \psi_{v,i} = \varphi_{\mu_{v,i}} \otimes \nu(m_i).$$

These localizations a priori only belong to $\Psi(\mathrm{Res}_F^E GL_N)$. The fact that they are indeed in the image of the map (2.5) is one of the central theorems of the endoscopic classification of representations proved by Mok in [35].

2.2.6 Parameters of Inner Forms

Let (G, ξ) be an inner form of $G^* = U(N)$. A local Arthur parameter for G is simply a parameter for $U(N)$ which is G-relevant, a notion that was introduced in 2.2.2. Globally, a parameter $\psi \in \Psi(G^*, \eta_\kappa)$ is G-relevant if it is so everywhere locally, see [23, §1.3.7]. We denote by $\Psi(G, \xi)$ the collection of parameters in $\Psi(G^*, \eta_\kappa)$ which are G-relevant. In summary, we have the following global chain of inclusions:

$$\Psi(G, \xi) \subset \Psi(G^*, \eta_\kappa) \subset \tilde{\Psi}(N) \subset \Psi(N),$$

where the parameters in $\tilde{\Psi}(N)$ are conjugate self-dual, those in $\Psi(G^*, \eta_\kappa)$ factor though the embedding η_κ, and those in $\Psi(G, \xi)$ are additionally G-relevant.

2.2.7 Parameters and Conjugacy Classes

We now explain how to attach families of conjugacy classes to objects introduced in the previous sections. For F global, G reductive, and any finite set S of places of F containing the archimedean ones, let $\mathcal{C}^S(G)$ denote the set of collections $c = \{c_v\}_{v \notin S}$, where each c_v is a semisimple conjugacy class in $\hat{G}$. For two sets S and S', let $c \sim c'$ if $c_v = c'_v$ for almost all v. Denote the set of such equivalence classes by $\mathcal{C}(G)$. In keeping with the notation for

parameters, denote the special case where $G = GL_N$ by $\mathcal{C}(N)$. We can associate elements of $\mathcal{C}(G)$ to automorphic representation π of G. Factoring $\pi = \otimes'_v \pi_v$, where π_v is unramified at all but finitely many (non-archimedean) places, let $c(\pi) = \{c(\pi_v)\}$ be the collection of the Satake parameters of all the unramified representations.

When $G = GL_N$ we do more and associate an element of $\mathcal{C}(N)$ to each parameter $\psi \in \Psi(N)$. Starting with simple parameters $\psi \in \Psi_{\mathrm{sim}}(N)$, we use the recipe for the representation π_ψ prescribed by Moeglin-Waldspurger's theorem [33] and let $c(\psi) := c(\pi_\psi)$. If the parameter ψ is not simple, we apply the above process to its simple constituents and associate to ψ the conjugacy class coming from the diagonally embedded product of the GL_{N_i} inside of GL_N. In this way we obtain a mapping

$$\Psi(N) \to \mathcal{C}(N), \quad \psi \mapsto c(\psi).$$

Following the work of Jacquet-Shalika [21], this mapping is injective. We denote its image by $\mathcal{C}_{aut}(N)$.

2.2.8   Stabilizers and Quotients

We recall the definition of some centralizer groups attached to a parameter ψ. Their characters will determine both the identities between representations of G and of its endoscopic groups, as well as the multiplicity $m(\pi)$ of automorphic representations inside the discrete spectrum.

For ψ either local or global, we have

$$S_\psi := \mathrm{Cent}(\mathrm{Im}(\psi), \hat{G}), \tag{2.7}$$

$$\bar{S}_\psi := S_\psi / Z(\hat{G})^{W_F}, \tag{2.8}$$

$$\mathcal{S}_\psi := \pi_0(\bar{S}_\psi). \tag{2.9}$$

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As mentionned above, when ψ is global, $\mathrm{Im}(\psi)$ really means $\mathrm{Im}(\tilde{\psi})$. Localization of parameters $\psi \mapsto \psi_v$ induces a mapping $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$.

When G is a unitary group, the centralizer quotients $\mathcal{S}_\psi$ can be readily computed, as the four authors do in [23, p.63]. In particular, in the case of F global and $\psi \in \Psi(G^*, \eta_\kappa)$ decomposing as $\psi = \boxplus_{i=1}^r \psi_i$, we have

$$\mathcal{S}_\psi = (\mathbf{Z}/2\mathbf{Z})^{r-1}. \tag{2.10}$$

The reader who actually does open [23] to look at the computations will notice that here is where we use the two assumptions from Remark 2.2.1. They allow us to only consider centralizer groups that are purely orthogonal. The possibility of symplectic factors is eliminated by the assumption that $l_i = 1$ for all i, and that of general linear factors by the assumption that each summand is self-dual.

Finally, we introduce the distinguished element

$$s_\psi := \psi \left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \in S_\psi. \tag{2.11}$$

We will sometimes conflate s_ψ and its image in the quotient $\mathcal{S}_\psi$, and use the same notation for both.

Remark 2.2.2. In Kaletha-Minguez-Shin-White's classification of representations for inner forms of unitary groups [23], the authors introduce a new centralizer quotient $S_\psi^\natural$. In the case that G is a local unitary group, the two groups $S_\psi^\natural$ and $\mathcal{S}_\psi$ agree [23, §1.3.4]. When the local group G is isomorphic to GL_N (the only possibility for us when the corresponding place is split, since we only work with unitary groups that arise from global hermitian forms) then the group $S_\psi^\natural$ is isomorphic to $\mathbf{C}^\times$. However, in that situation, the only representation of $S_\psi^\natural$ which arises in character identities will be the trivial one, as will be discussed in

Section 2.3.5. There is thus no loss to instead work with the group $\mathcal{S}_\psi = \{1\}$. As for the global situation, the characters of $S_\psi^\natural$ that arise all factor through $\mathcal{S}_\psi$ [23, p. 89]. Finally, we point out that we follow Mok and Arthur's convention by denoting by $\mathcal{S}_\psi$ the group that the four authors of [23] denote $\bar{\mathcal{S}}_\psi$.

2.2.9   Epsilon Factors

The last invariant attached to a global parameter ψ is ϵ_ψ. It is a character of the group $\mathcal{S}_\psi$ and is defined by Arthur in [4, §1.5]. The definition involves the symplectic root number $\epsilon(1/2, \mu_\alpha)$ of an automorphic L-function $L(s, \mu_\alpha)$ for a product of general groups. The representation μ_α is associated to the parameter into $GL(\mathfrak{g}_{\hat{G}})$ obtained by composing ψ with the adjoint representation. As such, the arithmetic properties of the decomposition of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$ are encoded through ϵ_ψ. Finally, note that ϵ_ψ only depends on the parameter ψ and in particular is independent of the inner form of G^* under consideration, as is discussed in [23, p.89].

2.2.10   Endoscopic Data

An endoscopic datum for G is a triple (ξ, H, s) where

- s is a semisimple element of $\hat{G}$,

- H/F is a connected, quasisplit group whose dual group $\hat{H}$ is the connected component of the centralizer of s in $\hat{G}$,

- $\xi : {}^L H \to {}^L G$ is an L-embedding.

We will work only with *elliptic* endoscopic data, characterized by the finiteness of $Z(\hat{H})^{W_F}$. As such, we will denote the set of elliptic endoscopic data for G up to conjugation by $\mathcal{E}(G)$, dropping the "ell" subscript that appears in our references. We will also frequently abuse

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notation and refer to the group H as a stand-in for the full datum, sometimes denoting the other two elements of the triple by ξ_H and s_H. Lastly, we will also use the formalism of endoscopic data for our unitary groups and denote by $\tilde{\mathcal{E}}(N)$ the set of pairs consisting of a quasisplit unitary group $U(N)$ together with the L-embedding η_κ from (2.3), extending this pair to an endoscopic triple via the element $s = I_N$.

We now describe endoscopic groups of unitary groups. For any inner form G of $U_{E/F}(N)$, the set $\mathcal{E}(G)$ consists of pairs

$$(H, \xi) = (U(N_1) \times U(N_2), \eta_{\underline{\kappa}}), \quad N_1, N_2 \geq 0, \ N_1 + N_2 = N,$$

where the embedding was defined in (2.4). The signature $\underline{\kappa} = ((-1)^{N-N_1}, (-1)^{N-N_2})$ depends on the respective ranks of the groups. The equivalence class of endoscopic data is then uniquely determined by N_1 and N_2, see [35, §2.4].

Endoscopic Data and Parameters

We now import the first result concerning the objects introduced so far: the group $\mathcal{S}_\psi$ parametrizes endoscopic groups such that ψ factors through ξ_H. Let $(H, \xi_H, s_H) \in \mathcal{E}(G)$ be an endoscopic datum and $\psi^H \in \Psi(H, \eta_\kappa \circ \xi_H)$ be an Arthur parameter. Let $\psi = \xi_H \circ \psi^H$. Since the element s_H commutes with H, it also commutes with the image of ψ. In this way, we get an mapping

$$(H, \psi^H) \mapsto (\xi_H \circ \psi^H, s_H) \tag{2.12}$$

from the pairs (H, ψ^H) onto the set of pairs consisting of a parameter ψ for G, together with an element s of the centralizer S_ψ. The importance of the quotient $\mathcal{S}_\psi$ comes from the fact that for each parameter ψ the map from (2.12) descends to a bijection between $\mathcal{S}_\psi$ and the set of endoscopic data such that ψ factors through ξ_H. We state this result below and refer to the proof in [23], which is an adaptation of Arthur's proof in [4, §1.4]. To simplify

statements, we will restrict ourselves to the result we will use: a global unitary group G and a square-integrable parameter ψ.

Lemma 2.2.3. *Let F be global and $G^* = U_{E/F}(N)$. Let $\psi \in \Psi(G^*, \eta_\kappa)$. The map (2.12) induces a bijection*

$$(H, \psi^H) \leftrightarrow (\psi, s)$$

where the left-hand side runs over pairs where H is stands in for an endoscopic datum (H, ξ, s) and ψ^H is a parameter of H such that $\psi = \xi \circ \psi^H$, and the right-hand side runs over elements of $\mathcal{S}_\psi$.

Proof. The proof is the content of section 1.4 of [23], and the above statement is a reformulation of Lemma 1.4.3. Our simplifying assumption that ψ is square-integrable implies that S_ψ and a fortiori $\bar{S}_\psi$ are finite. Thus the groups $\bar{S}_\psi$ and $\mathcal{S}_\psi$ are one and the same and we use the latter in our bijection. $\qquad\square$

Example 2.2.4. To fix ideas, we give an example of this bijection. Let $G = U(5)$ be the quasisplit unitary group in 5 variables. Fix a parameter

$$\psi = \psi_1 \boxplus \psi_2 \boxplus \psi_2 = (\mu_1 \boxtimes \nu(1)) \boxplus (\mu_2 \boxtimes \nu(2)) \boxplus (\mu_3 \boxtimes \nu(2))$$

where each μ_i is cuspidal automorphic representation of $U_{E/F}(1)$, i.e. a Grossencharacter of $\mathbf{A}_E^{\mathrm{Nm}=1}$. From section 2.10 we compute that $\mathcal{S}_\psi = (\mathbf{Z}/2\mathbf{Z})^2$. A choice of representatives in S_ψ for the elements of $\mathcal{S}_\psi$ are diagonal matrices with entries

$$s_1 = (1, 1, 1, 1, 1), \quad s_2 = (-1, 1, 1, 1, 1), \quad s_3 = (-1, -1, -1, 1, 1), \quad s_4 = (1, -1, -1, 1, 1).$$

The associated endoscopic pairs are (up to a twist of ψ^H induced by the embedding ξ_H):

$$(H_1, \psi^{H_1}) = (U(5), \psi_1 \boxtimes \psi_2 \boxtimes \psi_3)$$

$$(H_2, \psi^{H_2}) = (U(1) \times U(4), \psi_1 \times (\psi_2 \boxtimes \psi_3)),$$

$$(H_3, \psi^{H_3}) = (U(3) \times U(2), (\psi_1 \boxtimes \psi_2) \times \psi_3),$$

$$(H_4, \psi^{H_4}) = (U(3) \times U(2), (\psi_1 \boxtimes \psi_3) \times \psi_2).$$

Note that despite our notation (H, ψ^H), the same endoscopic group H appears twice, associated to two different parameters. Observe also that we have $s_\psi = s_4$ in the quotient $\mathcal{S}_\psi$, and that (as is always the case), the identity element of $\mathcal{S}_\psi$ corresponds to G.

2.3   Packets

The parameters introduced above serve to classify the admissible (in the local case) or automorphic (in the global case) representations of the group G. In this section, we introduce the set of representations associated to an Arthur parameter, known as an A-packet or Arthur packet. We also give the character identities relating the traces of the representations in a packet to corresponding representations for endoscopic groups.

2.3.1   Local Arthur Packets

Let G/F be a unitary group over a local field. The main local results of Mok [35, Theorem 2.5.1] and Kaletha-Minguez-Shin-White [23, Theorem 1.6.1] associate to each Arthur parameter ψ for the group G a finite set Π_ψ of irreducible unitary representations of $G(F)$ called a local Arthur packet. This packet Π_ψ is empty if ψ is not relevant, and it contains only tempered representations when ψ is bounded. If Π_ψ is nonempty, it is equipped with a pairing

$$\langle \, , \, \rangle : \mathcal{S}_\psi \times \Pi_\psi \to \{\pm 1\}. \tag{2.13}$$

Through this pairing, every representation $\pi \in \Pi_\psi$ gives rise to a character of the group $\mathcal{S}_\psi$. In particular, unramified representations correspond to the trivial character. The pairing depends on the triple (G, ξ, z) realizing G as a pure inner twist as discussed in Remark 2.1.2.

Remark 2.3.1. In the case where F is archimedean, the representations contained in the packet Π_ψ all have the same infinitesimal character. In the case of cohomological representations, these are explicitly described by Adams-Johnson in [1]. In particular, one can speak of "the infinitesimal character" of the parameter ψ.

We recall a result of Mok that the central character is the same for all representations in Π_ψ, and is determined by both the parameter ψ and the choice of embedding η_κ.

Proposition 2.3.2 (Proposition 1.5.2, 2. [23]). *For each* $\pi \in \Pi_\psi$ *the central character* $\omega_\pi :$ $Z(G^*)(F) \to \mathbf{C}^\times$ *has a Langlands parameter given by the composition*

$$L_F \xrightarrow{\varphi_\psi} {}^L G^* \xrightarrow{(\det \rtimes id) \circ \eta_\kappa} \mathbf{C}^\times \rtimes W_F.$$

2.3.2   Global Arthur Packets

Global parameters also have sets of representations attached to them. Let $\psi \in \Psi(G, \xi)$ be a global parameter with localizations ψ_v. The global Arthur packet Π_ψ is then defined both in terms of the local packets and of the pairings as

$$\Pi_\psi = \left\{ \pi = \otimes_v \pi_v \mid \pi_v \in \Pi_{\psi_v}, \ \langle \cdot, \pi_v \rangle_{\psi_v} = 1 \text{ for almost all } v \right\}.$$

The packet Π_ψ is equipped with a pairing

$$\langle \ , \ \rangle_\psi : \mathcal{S}_\psi \times \Pi_\psi \to \{\pm 1\}, \quad \langle \ , \pi \rangle_\psi = \prod_v \langle \ , \pi_v \rangle_{\psi_v} \tag{2.14}$$

determined by the maps $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ induced by localization. We note once again that the

pairing depends on the full inner twist (G, ξ). On the other hand, the local dependence on the pure inner twist, i.e. the dependency on the cocycle z, which appeared in the local definition of the pairing, cancels out globally. This is explained in [23, §1.7].

2.3.3   Test Functions

We introduce test functions in order to state the character identities relating packets for various groups, following the exposition of Section 1.5 of Arthur's book [4]. Continuing with F global, we fix compatible Haar measures μ on $G(\mathbf{A}_F)$ and μ_v on G_v for all v. We also fix a maximal compact subgroup K of $G(\mathbf{A}_F)$. The group K determines a maximal compact subgroup $K_v \subset G_v$ at each place v, and we choose K so that at all the unramified finite v the subgroup K_v is hyperspecial. Finally, we fix a character ω of $Z(\mathbf{A}_F)$, which in turn determines local characters ω_v.

The local Hecke algebra $\mathcal{H}(G_v, \omega_v)$ is the algebra of smooth compactly supported functions on G_v, which transform under the center of G_v by the character ω_v. At the archimedean places we further require that they be K_v-finite. We will call elements of $\mathcal{H}(G_v, \omega_v)$ local test functions.

The global Hecke algebra is defined as the restricted tensor product $\mathcal{H}(G, \omega) = \otimes_v' \mathcal{H}(G_v, \omega_v)$. It is the algebra of smooth, compactly supported, K-finite functions which transform under the action $Z(\mathbf{A}_F)$ by the character ω. Each such test function is a finite sum of factorizable test functions of the form $f = \prod_v f_v$, where each $f_v \in \mathcal{H}(G_v, \omega_v)$ and all but finitely many f_v are the characteristic function of the maximal compact subgroup K_v.

A smooth, admissible representation π_v of G_v on a Hilbert space can be promoted to an $\mathcal{H}(G_v, \omega_v)$-module, and the two categories are equivalent. The operator on π_v given by convolution by f_v is then of trace class, and we denote its trace by $\operatorname{tr} \pi_v(f_v)$. Likewise globally, the algebra $\mathcal{H}(G, \omega)$ acts on $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F), \omega)$ and on its irreducible constituents π. We denote this by $R(f)$ (when considering the right-regular representation

on $L^2_{\text{disc}}(G(F), G(\mathbf{A}_F), \omega))$ or by $\operatorname{tr} \pi(f)$ (when f is acting on the irreducible representation π.) Finally note that both locally and globally, the trace of f on a representation π vanishes if $f \in \mathcal{H}(G, \omega)$ for ω different from the central character of π. We will sometimes suppress the notation of the character and simply denote the Hecke algebra by $\mathcal{H}(G)$.

2.3.4   Stable Distributions and Transfer

We introduce the notions of stable distributions on the local and global Hecke algebras, following Sections 3.1 and 4.2 of [35], respectively. Let γ be a conjugacy class in $G(F_v)$. The stable conjugacy class of γ is the union of all the (finitely many) conjugacy classes of $G(F_v)$ that are conjugate to γ over $G(\bar{F}_v)$. A local test function $f_v \in \mathcal{H}(G_v)$ is determined by all its so-call orbital integrals $f_{v,G}(\gamma)$ over regular conjugacy classes in G.

We start with a sketch of the definition of local transfer. Let G_v be a quasisplit unitary group. Each stable conjugacy class δ gives rise to a linear functional

$$f_v^G(\delta) = \sum_{\gamma} \Delta_v(\delta, \gamma) f_{v,G}(\gamma), \tag{2.15}$$

where the sum is taken over all the conjugacy classes γ of $G(F_v)$. The factor $\Delta(\delta, \gamma)$ is equal to 1 if γ belongs to the stable conjugacy class δ and to 0 otherwise. In other words, the right-hand side is the sum of orbital integrals over the conjugacy classes belonging to the stable conjugacy class δ. This construction gives a map from $\mathcal{H}(G_v)$ to the ring of functions on stable conjugacy classes. Denote the image of this map by $\mathcal{S}(G_v)$. We will say that a linear functional on $\mathcal{H}(G_v)$ is *stable* if it factors through the quotient $\mathcal{S}(G_v)$.

Now let G_v be an arbitrary unitary group. For each endoscopic group H_v of G_v (including G_v^*), there is a map $\mathcal{H}(G_v) \to \mathcal{S}(H_v)$ whose definition is formally identical to (2.15) with δ a stable conjugacy class on H_v, but in which the transfer factors are much more delicate and are were defined by Langlands-Shelstad [28] and Kottwitz-Shelstad [26].

This gives us a system of maps from the Hecke algebras $\mathcal{H}(G_v)$ to their stable counterparts $\mathcal{S}(H_v)$. If H_v is an endoscopic group of G_v, we will say that two functions $f_v \in \mathcal{H}(G_v)$ and $f_v^{H_v} \in \mathcal{H}(H_v)$ form a *transfer pair* if they have the same image under their respective maps to $\mathcal{S}(H_v)$. Although the function $f_v^{H_v}$ is not uniquely determined by H_v (we could for example take any conjugate), we will sometimes abuse terminology and call one choice of $f_v^{H_v}$ the transfer of f_v.

In order to extend the notion of transfer to global test functions, it is first necessary to know that the transfer of characteristic functions of maximal compact subgroups of G_v are the corresponding functions on H_v. This is the fundamental lemma, now a theorem due to Laumon-Ngô [29] in the case of unitary groups, and to Ngô [36] in general, after reductions by Waldspurger [47, 48].

Theorem 2.3.3 (Fundamental Lemma). *Let G_v and H_v be unramified reductive groups over a non-archimedean local field F_v. Let $K(G_v)$ and $K(H_v)$ be respective choices of hyperspecial maximal compact subgroups. Then their characteristic functions $f_v = 1_{K(G_v)}$ and $f_v^{H_v} = 1_{K(H_v)}$ form a transfer pair.*

With this in mind, the transfer of a factorizable global test function $f = \prod_v f_v \in \mathcal{H}(G_v)$ is the product $f^H = \prod_v f_v^{H_v}$ of its transfers, and we extend this definition to all of $\mathcal{H}(G)$ linearly. We will likewise define the global stable Hecke algebra $\mathcal{S}(G^*, \omega) := \otimes_v' \mathcal{S}(G_v^*, \omega_v)$, where the restriction is that all but finitely many tensors must come from the characteristic function of a hyperspecial maximal compact subgroup. Finally, we will say that a global distribution on $\mathcal{H}(G^*)$ is stable if it factors through the quotient $\mathcal{S}(G^*)$.

2.3.5   Local Character Identities

The transfer of representations between G and its endoscopic groups H is encoded via identities between linear combinations of characters; the coefficients are determined by the pairings (2.13). We collect the relevant results below; here F a local field.

We start with the existence of a distribution $f^G(\psi)$ on $\mathcal{H}(G)$. Let G^*/F be a quasisplit unitary group or a product thereof, and ψ an Arthur parameter of G^*. Then Mok shows the existence of a stable linear form associated to the packet Π_ψ.

Theorem 2.3.4 (Theorem 3.2.1 (a), [35])**.** *Let $\psi \in \Psi(G^*)$. Then there exists a unique stable linear form*

$$f \mapsto f^{G^*}(\psi)$$

on $\mathcal{H}(G^*)$ determined by transfer properties to GL_N. If $G^* = G_1^* \times G_2^*$ and $\psi = \psi_1 \times \psi_2$, then $f^{G^*}(\psi) = f^{G_1^*} \times f^{G_2^*}$.

We will not discuss in detail the character identities relating $f^{G^*}(\psi)$ to traces on GL_N as they do not come into play for us, although they are critical to establishing the endoscopic classification of representations. It suffices to say that this distribution is related to the trace $\operatorname{tr} \pi_{\psi,N}(f)$ for a representation $\pi_{\psi,N}$ associated to the parameter ψ. Our focus will be on the relation between the $f^H(\psi^H)$ for the groups $H \in \mathcal{E}(G)$, and the characters of representations in the packet Π_ψ. If $G = G^*$ is a quasisplit unitary group, these identities were established by Mok. Recall that s_ψ is the distinguished element of $\mathcal{S}_\psi$ defined in (2.11).

Theorem 2.3.5 (Theorem 3.2.1 (b), [35])**.** *Let G^* be a quasisplit unitary group, let $\psi \in \Psi(G^*)$, and let Π_ψ be the associated Arthur packet equipped with the pairing $\langle \cdot, \cdot \rangle$. Let s be a semisimple element of $\mathcal{S}_\psi$ and let (H, ψ^H) correspond to (ψ, s_H) as in Lemma 2.2.3. Then for a transfer pair (f, f^H) we have*

$$f^H(\psi^H) = \sum_{\pi \in \Pi_\psi} \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

When the group is not quasisplit, the corresponding result is due to Kaletha-Minguez-Shin-White [23].

Theorem 2.3.6 (Theorem 1.6.1, [23])**.** *Let (G, ξ) be an inner form of $U(N)$ and let ψ, Π_ψ, H, s_H, and (f, f^H) be as above. Then*

$$f^H(\psi^H) = e(G) \sum_{\pi \in \Pi_\psi} \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f)$$

where $e(G)$ is the Kottwitz sign introduced in 2.1.2.

Remark 2.3.7. Here we recall something that we discussed in the introduction: the theorems stated in [23] are not in fact all fully proved. For example, Theorem 2.3.6 is only proved in the case that the parameter ψ^N is bounded, i.e. trivial on the SL_2 factor. The authors of [23] anticipate that they will provide the full proof in a pair of upcoming papers, the first of which [22] concerns unitary groups defined with respect to hermitian forms and will contain the results used in this thesis.

Local Packets for General Linear Groups

As was discussed in Section 2.1.1, if F is a local field associated to a place which splits in the CM field defining our unitary group, then $G \simeq GL_N$. In this situation, the local Arthur packet and the pairing are especially simple.

Theorem 2.3.8 (Section 2, [35])**.** *If $G = GL_N$ and ψ is an Arthur parameter for G, then the packet Π_ψ contains one element: the irreducible representation associated to φ_ψ by the local Langlands correspondence. The character $\langle \, , \pi_\psi \rangle$ is trivial.*

We now consider the character of identities between representations of G and that of its endoscopic groups. They are alluded to in [35] and [23], but we give a more explicit description based on Shin's exposition in [42, §3.3]. For $G = GL_N$, stable and regular conjugacy classes coincide and the stable quotients $\mathcal{S}(G)$ are equal to $\mathcal{H}(G)$. Since the global extension giving rise to our unitary group is CM, we need only consider the case of GL_N/F

when F is non-archimedean. If $H = GL_{N_1} \times GL_{N_2}$ with $N_1 + N_2 = N$, then the embedding $\eta_{\underline{\kappa}}$ that is part of the endoscopic datum can be used to realize H as a Levi subgroup of G. Let $P = HN$ be a parabolic subgroup of G containing H. Given a function $f \in \mathcal{H}(G)$, define the constant term along P as

$$f^P(h) := \delta_P^{1/2}(h) \int_N \int_{K^H} f(khnk^{-1})dxdn, \quad h \in H(F).$$

Here the integrals are taken with respect to suitably normalized Haar measures and δ_P is the modulus character for the parabolic P. The function f_v^P is smooth and compactly supported, and following results of van Dijk [44], it satisfies the requisite identity of orbital integrals to be a transfer of f. We thus let $f^H := f^P$. Furthermore, if f is unramified, then f^H is the image of f under the map $\mathcal{H}(G)^{ur} \to \mathcal{H}(H)^{ur}$ induced by the Satake isomorphism. Thus this notion of transfer satisfies the fundamental lemma.

For a parameter ψ of G, we let $f^G(\psi) = \operatorname{tr} \pi_\psi(f)$ [23, §1.5] for the unique $\pi_\psi \in \Pi_\psi$ and extend this definition as a product to pairs of general linear groups. We let π_ψ^H be the unique representation of H in the packet associated to ψ^H. Then it follows from the local Langlands correspondence (see for example [19, p.6] and note that the twist therein is accounted here by the one coming from the embedding $\eta_{\underline{\kappa}}$) that $\pi_\psi = \mathcal{I}_P(\pi_\psi^H)$, where $\mathcal{I}_P$ denotes normalized parabolic induction with respect to P. In view of this and of Theorem 2.3.8, the local character identities for GL_N amount to an equality of traces between $\operatorname{tr} \pi(f^H)$ and the trace of f on the corresponding induced representation. Again this is a result of van Dijk, which we record below.

Theorem 2.3.9 (Section 5, [44]). *Let G, H, P, and f^H be as above. Let π ba a unitary irreducible representation of H and let $\mathcal{I}_P(\pi)$ be its normalized parabolic induction with respect to P. Then $\operatorname{tr} \pi(f^H) = \operatorname{tr} \mathcal{I}_P(\pi)(f)$.*

2.4 The Trace Formula and its Stabilization

We now discuss global identities between the automorphic representations of a unitary group G and its endoscopic groups. These are encoded in Arthur's trace formula I_{disc}. This distribution, and in particular its restriction to the contributions of certain parameters, is described in [23, §3.1] and in greater detail in [4, §3.1] and we do not go in more detail than is needed for our applications. In particular, our arguments only make use of representations with regular infinitesimal character. As such, the contribution of proper Levi subgroups to the definition of I_{disc} will never come into play.

It suffices to say that I_{disc} is a distribution on the Hecke algebra $\mathcal{H}(G, \omega)$, defined in terms of the traces of intertwining operators on variants of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$. These traces are indexed by a system of Levi subgroups of G, and we will follow Arthur [4, §3.4] in denoting the contribution of the group G itself by R_{disc}. This latter distribution computes the trace convolution by a function $f \in \mathcal{H}(G, \omega)$ induced by the right-regular representation of $G(\mathbf{A}_F)$ on $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F), \omega)$, and as such

$$R_{\mathrm{disc}}(f) = \sum_\pi m(\pi) \operatorname{tr}(\pi)(f).$$

Here, the sum is taken over all representations π appearing in the decomposition of the space $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F), \omega)$ as a $G(\mathbf{A}_F)$-representation.

2.4.1 Contribution of a Parameter

The interplay between the endoscopic and spectral decompositions of the trace formula drives our arguments, and we start by describing the latter, that is, how I_{disc} splits into a sum of contributions indexed by parameters $\psi \in \Psi(G)$.

The first level of decomposition is concerned with the archimedean places and is embedded in the definition of the distribution of I_{disc}. Indeed, the initial definition of the trace formula

is initially given (see for example [4, §3.1]) as

$$I_{\mathrm{disc}}(f) := \sum_t I_{\mathrm{disc},t}(f)$$

where $t \geq 0$. Representations contributing to $I_{\mathrm{disc},t}$ are the ones whose infinitesimal character μ_π satisfies $|\mathrm{Im}\mu_\pi| = t$, under an appropriate choice of metric on the dual of a Cartan subalgebra of $\mathfrak{g}_\infty$.

The next level or refinement relies on the identification of parameters ψ with collections of conjugacy classes $c(\psi)$. Let $c \in \mathcal{C}(G)$, the set of families of conjugacy classes introduced in 2.2.7. The distribution $I_{\mathrm{disc},t,c}$ described in [23, §3.1] is given by the restriction of the traces defining $I_{\mathrm{disc},t}$ to the subset of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$ consisting of representations whose Satake parameters at almost all unramified places correspond to the components of c. This results in a decomposition

$$I_{\mathrm{disc},t}(f) = \sum_{c \in \mathcal{C}(G)} I_{\mathrm{disc},t,c}(f).$$

Note that for a given function $f \in \mathcal{H}(G)$, all but finitely many summands on the right-hand side vanish. To go from conjugacy classes to parameters, recall that in 2.2.7 we have described an identification of the set $\Psi(N)$ with the subset $\mathcal{C}_{aut}(N) \subset \mathcal{C}(N)$. To each $\psi^N \in \Psi(N)$ is thus associated an element $c(\psi^N) \in \mathcal{C}_{aut}(N)$. Likewise, as stated in Remark 2.3.1, we can talk about the infinitesimal character associated to ψ^N and associate to ψ^N a real number $t(\psi^N)$. Finally, recall that the map η_κ induces an injection $\Psi(G^*, \eta_\kappa) \to \Psi(N)$ given by $\psi = (\tilde{\psi}, \psi^N) \mapsto \psi^N$. Thus for each parameter $\psi \in \Psi(G^*, \eta_\kappa)$, we follow [23, p. 149] and define

$$I_{\mathrm{disc},\psi}(f) = \sum_{c \mapsto c(\psi^N), t \mapsto t(\psi^N)} I_{\mathrm{disc},t,c}(f).$$

The sum runs over the elements $c \in \mathcal{C}(G)$ which map to $c(\psi^N)$ under the map $\mathcal{C}(G) \to \mathcal{C}(N)$ induced by η_κ.

We can similarly restrict the trace of the right-regular representation to Hecke eigen-components of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F), \omega)$ indexed by $\psi \in \Psi(G)$ in order to obtain distributions $R_{\mathrm{disc},t,c}$ and $R_{\mathrm{disc},\psi}$. An essential step along the proof of the main global theorem of the endoscopic classification of representations is showing that $R_{\mathrm{disc},\psi}$ does indeed compute the trace of the representations in the packet Π_ψ, provided that ψ is square-integrable, i.e. satisfies the two conditions of Remark 2.2.1.

Theorem 2.4.1 (From [35], (5.7.27), and [23], Theorem 5.0.5.). *Let ψ be a global square-integrable parameter with associated Arthur packet Π_ψ, and let f be a global test function. Then*

$$R_{\mathrm{disc},\psi}(f) = \sum_{\pi \in \Pi_\psi} m(\pi) \operatorname{tr} \pi(f).$$

More is in fact true, although we won't strictly make use of it: the multiplicity $m(\pi)$ is either 0 or 1, the latter occurring exactly when the global character $\langle \cdot, \pi \rangle$ from (2.14) is equal to the character ϵ_ψ introduced in 2.2.9. Finally, note that Kaletha-Minguez-Shin-White's Theorem 5.0.5 of is one of the results that is stated, but not fully proved, in the case of non-generic parameters, as mentioned in the introduction and in Remark 2.3.7.

The definition of R_{disc} is more straightforward than that of I_{disc} as it only involves traces on the group G. However, following a result of Bergeron-Clozel [6], the ψ-summands of both distributions are in fact equal, provided that the infinitesimal character of ψ is regular. Note that the parameters associated to our main objects of interest, namely cohomological representations, have regular infinitesimal characters.

Theorem 2.4.2 (Theorem 6.2, [6]). *Let ψ be a global Arthur parameter such that the infinitesimal character associated to ψ_∞ is regular. Then the contributions of the Levi subgroups $M \neq G$ to the distribution $I_{\mathrm{disc},\psi}$ vanish. In particular for all $f \in \mathcal{H}(G)$ we have*

$$I_{\mathrm{disc},\psi}(f) = R_{\mathrm{disc},\psi}(f).$$

We now consider the decomposition that will drive our theorems: the stabilization of the distribution $I_{\text{disc},\psi}$. This stabilization expresses $I_{\text{disc},\psi}(f)$ as a sum of stable traces of the transfers f^H for the endoscopic groups $H \in \mathcal{E}(G)$. We refer to Arthur for the statement of the stabilization, but the versions for unitary groups are formally identical, see for example (3.3.2) of Kaletha-Minguez-Shin-White [23] and (4.2.1) of Mok [35]. Recall that $\tilde{\Psi}(N)$ is the set of conjugate self-dual parameters, and $\Psi(G, \xi) \subset \tilde{\Psi}(N)$.

Theorem 2.4.3 ([4], Corollary 3.3.2(b))**.** *Suppose that $\psi \in \tilde{\Psi}(N)$ and let $f \in \mathcal{H}(G)$. Then for each endoscopic group $H \in \mathcal{E}(G)$ there is a constant $\iota(G, H)$ and stable distributions $S_{disc,\psi}^H$ on $\mathcal{H}(H)$ such that*

$$I_{\text{disc},\psi}(f) = \sum_{H \in \mathcal{E}(G)} \iota(G, H) S_{disc,\psi}^H(f^H). \tag{2.16}$$

In the case of quasisplit groups, the stable distributions $S_{\text{disc},\psi}^H$ are defined inductively.

Remark 2.4.4. For unitary groups, the global factor $\iota(G, H)$ appearing in the stabilization of the trace formula is introduced in [35, §4.2] and [23, §3.1]. These factors are independent of the inner form G. If $G = U(N)$ and $H = U(N_1) \times U(N_2)$ is the group appearing in an endoscopic datum for G, then following [35, 4.2] we have

$$\iota(G, H) = \begin{cases} 1 & N_1 N_2 = 0 \\ \frac{1}{2} & N_1, N_2 \neq 0, N_1 \neq N_2 \\ \frac{1}{4} & N_1 = N_2 \neq 0. \end{cases} \tag{2.17}$$

2.5 Notation Changes.

In the upcoming chapter we will move towards applications to multiplicity growth, but we make a small stop to discuss notation. Having had to grapple with this herself, the author aware that keeping track of it is one of the challenges of engaging with the material introduced above. We take a few lines to highlight the notational choices that we have made which differ from Kaletha-Minguez-Shin-White [23], of from Arthur [4] or Mok [35].

- Reductive groups: we have chosen to keep the long form $\mathrm{Res}_F^E GL_N$ instead of the shorter $G(N)$ appearing in [23] and [35].

- Collections of Arthur parameters: recall that a parameter is elliptic if the quotient of the centralizer S_ψ by $Z(G)^{\Gamma_F}$ is finite.

    - For $U(N)$ local, the symbol $\Psi(U(N))$ denotes collection of conjugacy classes of parameters in $\Psi(\mathrm{Res}_F^E GL_N)$ which factor through the embedding $\xi_\kappa : {}^L U(N) \to {}^L \mathrm{Res}_F^E GL_N$. The parameters in this set are self-dual but not necessarily elliptic. This set is also denoted $\Psi(U(N))$ in the four-author paper [23, p. 61].

    - For $U(N)$ over a number field, the set $\Psi(U(N), \eta_\kappa)$ is the set of equivalence classes of self-dual, elliptic parameters of $\mathrm{Res}_F^E GL_N$ that factor through the embedding η_κ. This set is denoted $\Psi_2(UN), \eta_\kappa)$ on [23, p. 69].

    - For (G, ξ) an inner twist of $U(N)$, we denote the subset of $\Psi(G^*, \eta_\kappa)$ consisting of parameters that are G-relevant by $\Psi(G, \xi)$. In [23, §1.3.7], it is denoted by $\Psi_2(U(N), \eta_\kappa)_{(G,\xi)-\mathrm{rel}}$.

- We denote the set of elliptic endoscopic data of G by $\mathcal{E}(G)$, dropping the "ell" subscript used by Kaletha-Minguez-Shin-White in [23, §1.1.1]

- If S_ψ is the centralizer of the image of ψ, we denote the component group $\pi_0(S_\psi/Z(G)^{\Gamma_F})$ by $\mathcal{S}_\psi$ as in [35, §2.2], rather than by $\bar{\mathcal{S}}_\psi$ as in [23, §1.3.4].

CHAPTER 3

BOUNDS ON LIMIT MULTIPLICITY AND COHOMOLOGY

OF ARITHMETIC GROUPS

In this last chapter we produce bounds on limit multiplicity for certain classes of representations. The results rely on the framework introduced in Chapter 2, specifically on the local character identities and the stabilization of the stable trace formula. In 3.1 we obtain upper bounds on the traces of certain classes of test functions on $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F))$ using the stable trace formula. In 3.2 we specialize the test functions and deduce results on limit multiplicity for representations in certain Arthur packets. Finally in 3.3 we give applications to cohomological representations.

3.1 Upper Bounds from the Stabilization

In this section, we examine in further detail the various summands of the stabilization of the trace formula. We extract bounds on the trace of test functions from the character identities involved. We remind the reader that throughout, we will be working with elliptic parameters ψ whose stabilizer group S_ψ is finite and whose infinitesimal character is regular.

3.1.1 The Stable Multiplicity Formula

Recall from (2.16) that the distribution $I_{\mathrm{disc},\psi}(f)$ giving the trace of f on representations in the Arthur packet Π_ψ decomposes as

$$I_{\mathrm{disc},\psi}(f) = \sum_{H\in\mathcal{E}(G)} \iota(G,H)S^H_{\mathrm{disc},\psi}(f^H). \tag{3.1}$$

A concrete expression for each $S^H_{\mathrm{disc},\psi}$ is given by the stable multiplicity formula. Since all the endoscopic groups appearing in the stabilization of the trace formula are quasisplit, the

relevant results are those of [35].

We introduce some notation. If f_v is a local test function and ψ_v a local parameter, the formula for $f^{H_v}(\psi_v)$ was given in Section 2.3.5. If $f = \prod_v f_v$ and ψ are global, we write $f^H(\psi) := \prod_v f^{H_v}(\psi_v)$. Recall the group $\mathcal{S}_\psi$ and the element s_ψ from 2.2.8, as well as the character ϵ_ψ from 2.2.9. The stable multiplicity formula is the following expression:

Theorem 3.1.1 ([35], 5.1.2). *For $\psi \in \tilde{\Psi}(N)$, there is a constant $\sigma(\bar{S}_\psi^0)$ such that*

$$S_{disc,\psi}^G(f) = |\mathcal{S}_\psi|^{-1}\epsilon_\psi^G(s_\psi)\sigma(\bar{S}_\psi^0)f^G(\psi).$$

The term $\sigma(\bar{S}_\psi^0)$ is a special instance of a constant $\sigma(S)$ defined by Arthur in [4, §4.1] for any complex reductive group S. We will not define it beyond pointing out that the centralizers S_ψ of our square-integrable parameters ψ are always finite. In that case $\bar{S}_\psi^0$ is trivial and $\sigma(\bar{S}_\psi^0) = 1$, see [35, Remark 5.1.4].

The stable multiplicity formula is initially only stated for G a unitary group, but extends to products $H = U(N_1) \times U(N_2)$ as discussed in [35, §5.6]. It then takes the form

$$S_{disc,\psi^H}^H(f^H) = \sum_{\psi^H \in \Psi(H,\psi^N)} \frac{1}{|\mathcal{S}_{\psi^H}|}\epsilon_\psi^H(s_\psi^H)\sigma(\bar{S}_{\psi^H}^0)f^H(\psi^H). \qquad (3.2)$$

Here $\Psi(H, \psi^N)$ is the set consisting parameters ψ of H such that $\xi_H \circ \psi = \psi^N$. We have given an example at the end 2.2.10 of how this set can contain more than one element. We combine equations (3.1) and (3.2) to get the expression

$$I_{\text{disc},\psi}(f) = \sum_{H \in \mathcal{E}(G)} \iota(G,H) \sum_{\psi^H \in \Psi(H,\psi^N)} \frac{1}{|\mathcal{S}_{\psi^H}|}\epsilon_\psi^H(s_\psi^H)\sigma(\bar{S}_{\psi^H}^0)f^H(\psi^H).$$

This can in turn be written as a sum over pairs (H, ψ^H) where H stands in for the endoscopic

datum (H, ξ, s) and ψ^H factors through ξ. We get

$$I_{\text{disc},\psi}(f) = \sum_{(H,\psi^H)} \iota(G, H) \frac{1}{|\mathcal{S}_{\psi^H}|} \epsilon_\psi^H(s_\psi^H) \sigma(\bar{S}_{\psi^H}^0) f^H(\psi^H). \tag{3.3}$$

Lemma 3.1.2. *Let $\psi \in \Psi(G^*, \eta_\kappa)$ and let $(H, \xi, s) \in \mathcal{E}(G)$ be an endoscopic datum such that ψ factors through ξ. Then there is a positive constant $C(\psi, H)$ such that the contribution of (H, ψ^H) to the sum (3.3) is equal to*

$$C(\psi, H) \epsilon_\psi^H(s_\psi^H) f^H(\psi^H).$$

Moreover, $C(\psi, H)$ is bounded above and away from zero uniformly in H and ψ.

Proof. This is immediate from (3.3) and it suffices to show that the product

$$C(\psi, H) := \frac{\iota(G, H)}{|\mathcal{S}_{\psi^H}|} \sigma(\bar{S}_{\psi^H}^0)$$

is bounded. From above we have that $\sigma(\bar{S}_\psi^0) = 1$ since ψ is elliptic. Additionally, we have uniform bounds on $\iota(G, H)$ in (2.17) and on $|\mathcal{S}_\psi|$ in (2.10). $\qquad \square$

Recall from Lemma 2.2.3 that the indexing set of pairs (H, ψ^H) is in bijection with the centralizer quotient $\mathcal{S}_\psi$. We can thus re-index the sum (3.3) and obtain the expression

$$I_{\text{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \epsilon_\psi^H(s_\psi^H) f^H(\psi^H). \tag{3.4}$$

At this point, the invariants appearing in the stable trace formula depend on parameters and representations of H. We would like to reformulate the entire expression in terms of characters of representations of G and of elements of $\mathcal{S}_\psi$. The relevant result in the case of the epsilon factors is reproduced below: Mok refers to it as the endoscopic sign lemma.

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Lemma 3.1.3 (Theorem 5.6.1, [35]). *Let $(H, \xi, s_H) \in \mathcal{E}(G)$ and $\psi \in \Psi(G^*, \eta_\kappa)$ be such that (H, ψ^H) corresponds to (ψ, s_H). Let $\epsilon_\psi^{G^*}$ and ϵ_ψ^H be the respective characters of ψ and ψ^H. Let s_ψ^H be the image of $\psi^H(-I)$ in the quotient $\mathcal{S}_\psi^H$ associated to H. Then we have*

$$\epsilon_\psi^H(s_\psi^H) = \epsilon^{G^*}(s_\psi s_H).$$

We will now combine the results of this section together with the character identities of Section 2.3.5 relating $f^H(\psi^H)$ to the traces of the representations in the packet Π_ψ. This reformulation of the stabilization of the trace formula will be conducive to producing bounds on limit multiplicity.

Proposition 3.1.4. *Let $\psi \in \Psi(G, \xi)$ be an Arthur parameter such that the representations in the associated Arthur packet Π_ψ have regular infinitesimal character. Let $f \in \mathcal{H}(G)$ be a factorizable test function. Then we have*

$$I_{\text{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \epsilon_\psi^{G^*}(s_\psi s_H) \prod_v \left(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \right) \qquad (3.5)$$

$$= \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f). \qquad (3.6)$$

Proof. We start from the equality (3.4):

$$I_{\text{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \epsilon_\psi^H(s_\psi^H) f^H(\psi^H).$$

The distribution $f^H(\psi^H)$ was defined in the beginning of this section as $f^H(\psi^H) = \prod_v f_v^{H_v}(\psi_v^H)$. Each local factor can be written in terms of the trace of representations in Π_{ψ_v} by the local character identities of Theorems 2.3.5 and 2.3.6 if G_v is a unitary group, and of Theorem 2.3.9

if G_v is a general linear group. In all cases, the identity is:

$$f_v^{H_v}(\psi_v^{H_v}) = e(G_v) \sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v).$$

The local Kottwitz signs cancel out globally, and using the Endoscopic Sign Lemma 3.1.3, we have rewritten the expression as

$$I_{\operatorname{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \epsilon_\psi^{G^*}(s_\psi s_H) \prod_v \left(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \right).$$

At all but finitely many places v, the function $f_v = 1_{K_v}$ is the characteristic function of a hyperspecial maximal compact subgroup. At these places, $\operatorname{tr} \pi_v(f_v)$ is only nonzero on unramified representations π_v which we remind the reader are associated to the trivial character of $\mathcal{S}_\psi$. Unramified local packets contain exactly one unramified representation following [23, Proposition 1.5.2 (5)] so we can interchange the sum and product to get

$$\prod_v \left(\sum_{\pi_v \in \Pi_{\psi_v}} \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \operatorname{tr} \pi_v(f_v) \right) = \sum_{\pi \in \Pi_\psi} \left(\prod_v \langle s_{\psi_v} s_{H_v}, \pi_v \rangle \right) \operatorname{tr} \pi(f)$$

where $\pi = \otimes_v \pi_v$. The global characters were defined as $\langle \cdot, \pi \rangle = \prod_v \langle \cdot, \pi_v \rangle$. This allows us to rewrite

$$I_{\operatorname{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

$\square$

We consider the product $\epsilon_\psi^{G^*}(\cdot)\langle \cdot, \pi \rangle$ as a single character of $\mathcal{S}_\psi$ depending on π. The global statement of the endoscopic classification of representations is that π appears in the discrete spectrum precisely when this character is trivial. For our purposes, we will fix an element $s \in \mathcal{S}_\psi$ and vary the characters coming from the representations $\pi \in \Pi_\psi$.

3.1.2 Upper Bounds and the Dominant Group

We once again recall the bijection from 2.2.10

$$(H, \psi^H) \leftrightarrow (\psi, s_H)$$

between endoscopic groups and elements of $\mathcal{S}_\psi$ associated to a parameter ψ. We now single out one object on either side of this bijection, and show that its contribution to the distribution $I_{\mathrm{disc}, \psi}(f)$ is larger than the others, subject to conditions on the choice of the function f. Recall that the element $s_\psi \in \mathcal{S}_\psi$ was the image of the matrix $-I \in SL_2$ under ψ.

Definition 3.1.5. Let (H_ψ, ψ^{H_ψ}) be the pair corresponding to the pair (ψ, s_ψ) containing the distinguished element s_ψ under the bijection $(H, \psi^H) \leftrightarrow (\psi, s_H)$.

Note that it is possible that $H_\psi = G$, for example when the character ψ is bounded, i.e. when $\psi(SL_2)$ is trivial.

We now introduce a new piece of notation for the contribution of a pair (H, ψ^H) to the stabilization of the trace formula.

Definition 3.1.6. Let $\psi \in \Psi(G^*, \eta_\kappa)$ and let (H, ξ, s_H) be such that ψ factors through ξ. Let f be a global test function. Then define

$$S(\psi, s_H, f) = C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H) \langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f).$$

We have now collected all the information leading up to our main technical result.

Theorem 3.1.7. *Let $\psi \in \Psi(G^*, \eta_\kappa)$ have regular infinitesimal character, and let $f \in \mathcal{H}(G)$ be a factorizable test function such that $\operatorname{tr} \pi(f)$ is nonnegative for all $\pi \in \Pi_\psi$. Then there exist a constant $C(\psi)$, which can be bounded above and below independently of ψ, such that*

$$I_{\mathrm{disc}, \psi}(f) \leq C(\psi) S(\psi, s_H, f).$$

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Proof. We will compare the various terms appearing in the decomposition (3.6):

$$I_{\text{disc},\psi}(f) = \sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f)$$

$$= \sum_{s_H \in \mathcal{S}_\psi} S(\psi, s_H, f).$$

Ignoring for a moment the constants $C(\psi, s_H)$, the terms $S(\psi, s_H, f)$ only differ from one another via the signs $\epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi \rangle \in \{\pm 1\}$ appearing as coefficients of the traces $\operatorname{tr} \pi(f)$. Looking specifically at the term coming from s_ψ, we see that

$$\epsilon_\psi^{G^*}(s_\psi^2)\langle s_\psi^2, \pi \rangle = \epsilon_\psi^{G^*}(0)\langle 0, \pi \rangle = 1$$

since the group $\mathcal{S}_\psi$ is a product of copies of $\mathbf{Z}/2\mathbf{Z}$ as seen in Section 2.2.8. Thus we have

$$S(\psi, s_\psi, f) = C(\psi, s_\psi) \sum_{\pi \in \Pi_\psi} \operatorname{tr}(\pi)(f).$$

For any other term indexed by $s_H \in \mathcal{S}_\psi$, the coefficients $\epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi \rangle$ have the potential to be equal to -1. Thus if the trace of f is nonnegative on all representations in the packet Π_ψ, we have

$$S(\psi, s_H, f) = C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \epsilon_\psi^{G^*}(s_\psi s_H)\langle s_\psi s_H, \pi \rangle \operatorname{tr} \pi(f)$$

$$\leq C(\psi, s_H) \sum_{\pi \in \Pi_\psi} \operatorname{tr}(\pi)(f)$$

$$= \frac{C(\psi, s_H)}{C(\psi, s_\psi)} \cdot S(\psi, s_\psi, f).$$

Summing over the S_H we get

$$I_{\text{disc},\psi}(f) \leq \left(\frac{\sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)}{C(\psi, s_\psi)} \right) S(\psi, s_\psi, f).$$

We of course let $C(\psi) := \frac{\sum_{s_H \in \mathcal{S}_\psi} C(\psi, s_H)}{C(\psi, s_\psi)}$. We saw in Lemma 3.1.2 that the $C(\psi, s_H)$ are bounded above and away from zero uniformly in ψ and H. The cardinality of $\mathcal{S}_\psi$ is also bounded between 1 and 2^N as we saw in Section 2.2.8, which allows us to conclude. $\qquad\square$

In practice, the group H_ψ is easily computed from $\psi \mid_{SL_2}$.

Lemma 3.1.8. *Let* $\psi = \boxplus_i(\mu_i \boxtimes \nu(m_i))$ *be a global square-integrable Arthur parameter. Then the group* H_ψ *is*

$$H_\psi = U(N_1) \times U(N_2)$$

where $N_1 = \sum_{m_i \equiv 1 \bmod 2} m_i$ *and* $N_2 = N - N_1$.

Proof. The element $s_\psi \in GL_N$ is defined as $\psi(1, -I)$ there I is the identity matrix of SL_2. The image of $-I$ under the m-dimensional representation of SL_2 is $-I_m$ if m is even, and I_m if m is odd. Thus $s_\psi = \text{diag}(-I_{N_1}, I_{N_2})$, where $N_1 = \sum_{m_i \equiv 1 \bmod 2} m_i$ and $N_2 = N - N_1$, with centralizer $GL_{N_1} \times GL_{N_2}$. $\qquad\square$

Remark 3.1.9. The image $\psi(SL_2)$, and by extension the dominant endoscopic group H_ψ, are determined by any localization $\psi_v(SL_2)$. In Section 3.3, we will use this, together with the well-understood (archimedean) parameters of cohomological representations, to give bounds on growth of cohomology.

3.1.3 Towards Lower Bounds

The description of the stable terms of the trace formula in terms of representations of $\mathcal{S}_\psi$ points the way to an approach to produce lower bounds. Such a proof would rest on showing

that asymptotically, the stable trace $S(\psi, s_\psi, f)$ not only bounds $I_{\text{disc},\psi}(f)$ but that the two are in fact proportional. The argument should boil down to showing that for groups $s_H \neq s_\psi$, the size in absolute value of the stable summand $S(\psi, s_H, f)$ is asymptotically negligible. A possible path to proving such a statement would be to achieve control of the coefficients $\epsilon_\psi(s_\psi s_H)\langle s_\psi s_H, \pi \rangle$ and to show that they are positive on average half the time. In order for such a statement to hold for any s_H in H, we would need to show that asymptotically, the characters $\epsilon_\psi(s_\psi s_H)\langle s_\psi s_H, \pi \rangle$ are evenly distributed. Since the characters are defined globally as a product, it would in fact suffice to show that at one finite place v, the Arthur packet contains enough representations π for the assignment $\langle \pi, \cdot \rangle$ to surject onto $\hat{\mathcal{S}}_\psi$. In summary, this would require a more explicit understanding of the contents of local Arthur packets, as well as of the definition of the pairing $\langle \pi, \cdot \rangle$. This type of explicit description is given in Rogawski's endoscopic classification of representations of $U(2,1)$ [38]. It is used by Marshall in [30] to give sharp rates of growth for cohomological representations.

Showing that $I_{\text{disc}}(f)$ is proportional to $S(\psi, s_H, f)$ is desirable beyond the goal of achieving lower bounds. Such asymptotics will also prove necessary to get general results on upper bounds in the case where the step of passing to an endoscopic group requires to be iterated. We will see in the following section that our current inability to iterate our induction restricts the types of representations for which we can bound limit multiplicity.

3.2 Limit Multiplicity

Here we apply the results of the previous section to the limit multiplicity problem.

3.2.1 Level Structures

To formulate the question of limit multiplicity, we define level structures and tower of congruence subgroups of our unitary groups. Let $\mathcal{O}_E$ and $\mathcal{O}_F$ be the rings of integers of the global fields E and F. We introduce some collections of places of F:

- S_f is a finite set of finite places of F, containing the places which ramify in E as well as the places below those where the character χ_- introduced in Section 2.1 is ramified.

- S_∞ is the set of all infinite places of F.

- $S_0 \subsetneq S_\infty$ is a nonempty subset of the infinite places.

- $S = S_f \cup S_\infty$.

Note that implicit in the third requirement is the assumption that $F \neq \mathbf{Q}$.

Let $\mathfrak{p}$ be an ideal of F such that the associated place $v_\mathfrak{p}$ is not in the set S and such that the residue characteristic of $\mathfrak{p}$ is strictly greater than $9[F : \mathbf{Q}] + 1$. We define the subgroups $U(N, \mathfrak{p}^n) \subset U(N, \mathbf{A}_F^f)$ to be

$$U(N, \mathfrak{p}^n) := \{g \in U(N, \hat{\mathcal{O}}_F) \subset GL(N, \hat{\mathcal{O}}_E) \mid g \equiv I_N \ (\mathfrak{p}^n \mathcal{O}_E)\}.$$

For any finite place v of F, let $U(N, \mathfrak{p}^n)_v = U(N, \mathfrak{p}^n) \cap U(N)_v$. At the expense of possibly including additional primes in the set S, note that for all $v \notin S \cup \{v_\mathfrak{p}\}$, the subgroup $U(N, \mathfrak{p}^n)_v$ is a hyperspecial maximal compact subgroup of $U(N)_v$. This gives level structures on the quasisplit group $U(N)$. If $H = U(N_1) \times U(N_2)$ is a product of quasisplit unitary groups, we define level subgroups $H(\mathfrak{p}^n) = U(N_1, \mathfrak{p}^n) \times U(N_2, \mathfrak{p}^n)$.

We now discuss level structures on inner forms of $U(N)$. Let (G, ξ) be an inner form of $U(N, F)$ defined with respect to a Hermitian inner product and with prescribed signatures $U(a_v, b_v)$ at the archimedean places. We will require that G_v be compact at the archimedean places contained in S_0: this ensures that the group G is anisotropic. Following the classification of global inner forms stated in Proposition 2.1.1, we have that if N is odd, the group G can be chosen so that G_v is quasisplit at all finite places. If N is even, then G is determined by choosing at most one place $v \in S_f$, up to again enlarging S_f. Once that choice is made, the group G can be chosen to be quasisplit away from $\{v\} \cup S_\infty$. In both cases, this group G is realized as an inner form (G, ξ) as in Section 2.1.2.

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For each finite $v \notin S_f$, there are isomorphisms $\xi_v : G_v \to U(N)_v$, induced by the inner twist. For each natural number n, we fix a compact subgroup $K(\mathfrak{p}^n) = \prod_v K_v(\mathfrak{p}^n)$ of $G(\mathbf{A}_F)$ as follows: at all finite $v \notin S$, we let $K_v(\mathfrak{p}^n) = \xi_v^{-1}(U(N, \mathfrak{p}^n)_v)$; at $v \in S_f$, the subgroup $K_v(\mathfrak{p}^n)$ is an arbitrary open compact subgroup fixed once and for all independently of n; at the archimedean places we let $K_v(\mathfrak{p}^n) \simeq U_{a_v}(\mathbf{R}) \times U_{b_v}(\mathbf{R})$ be a maximal compact subgroup. Note that once again at the finite places $v \notin S_f$, the subgroup $K_v(\mathfrak{p}^n)$ is a hyperspecial maximal compact subgroup. Let $K_f(\mathfrak{p}^n) = \prod_{v < \infty} K_v(\mathfrak{p}^n)$ and $K_\infty(\mathfrak{p}^n) = \prod_{v | \infty} K_v(\mathfrak{p}^n)$. For simplicity, we will sometimes use the notation K_v instead of $K_v(\mathfrak{p}^n)$ for $v \neq v_{\mathfrak{p}}$. We extend these definitions to products of unitary groups.

We now define the (cocompact since G is anisotropic) lattices

$$\Gamma(\mathfrak{p}^n) = G(F) \cap K_f(\mathfrak{p}^n).$$

Recall that $G_\infty = \prod_{v|\infty} G_v$ and let $X_G = G_\infty/K_\infty Z_{G_\infty}$. Assume that G_∞ has at least one noncompact factor. The diagonal embedding $\Gamma(\mathfrak{p}^n) \hookrightarrow \prod_{v|\infty} G_v$ induces an action $\Gamma(\mathfrak{p}^n) \curvearrowright X_G$, and we will consider the quotients $X(\mathfrak{p}^n) := \Gamma(\mathfrak{p}^n) \backslash X_G$. We start by relating these quotients to their disconnected counterparts realized as adelic double quotients. Let

$$Y(\mathfrak{p}^n) = G(F) \backslash G(\mathbf{A}_F) / K(\mathfrak{p}^n) Z_G(\mathbf{A}_F).$$

The quotient $Y(\mathfrak{p}^n)$ is a disjoint union of finitely many copies of $X(\mathfrak{p}^n)$.

Proposition 3.2.1. *Let G be an inner form of $U(N)$ and $Y(\mathfrak{p}^n)$ be defined as above. The cardinality of the set of components $\pi_0(Y(\mathfrak{p}^n))$ is bounded independently of n.*

Proof. We adapt an argument from [13, §2]. Considering G as a subgroup of GL_N/E, let $\det : G \to U(1, E/F)$ be the determinant map and let $G^1 = \ker(\det)$. This map induces a fibering of $Y(\mathfrak{p}^n)$ over

$$U(1, F) \backslash U(1, \mathbf{A}_F) / \det(Z(\mathbf{A}_F) K(\mathfrak{p}^n)).$$

The fibers are adelic double quotients for the group G^1, which is simply connected and has at least one noncompact factor at infinity. So by [37, 7.12], the group G^1 satisfies strong approximation with respect to the set S_∞ and $G^1(F)$ is dense in $G^1(\mathbf{A}_F^f)$, making the fibers connected. Thus we find that

$$\pi_0(Y(\mathfrak{p}^n)) \simeq U(1, F)\backslash U(1, \mathbf{A}_F)/\det(Z(\mathbf{A}_F)K(\mathfrak{p}^n)) = E^1\backslash \mathbf{A}_E^1/\det(Z(\mathbf{A}_F)K(\mathfrak{p}^n)).$$

Now the image $\det(Z(\mathbf{A}_F))$ is the subgroup $(\mathbf{A}_E^1)^N$ of $\mathbf{A}_E^1$. For each finite place w, the factor corresponding to E_w in the quotient $\mathbf{A}_E^1/(\mathbf{A}_E^1)^N$ is a finite set. It follows that by increasing the level in powers of a single prime $\mathfrak{p}$, one can only produce a bounded number of components. $\square$

We now fix a unitary irreducible admissible representation π_∞ of G_∞ with trivial central character, and let $m(\pi_\infty, \mathfrak{p}^n)$ denote the multiplicity of π_∞ in right-regular representation of G_∞ on the space $L^2(\Gamma(\mathfrak{p}^n)\backslash G_\infty)$. We will be interested in the asymptotics of the multiplicities $m(\pi_\infty, \mathfrak{p}^n)$ as $n \to \infty$.

Corollary 3.2.2. *Let G, $\Gamma(\mathfrak{p}^n)$, and π_∞ be as above. Then*

$$m(\pi_\infty, \mathfrak{p}^n) \asymp \sum_{\pi = \pi_\infty \otimes \pi_f} m(\pi) \dim \pi_f^{K_f(\mathfrak{p}^n)}$$

where the sum is taken over automorphic representations π with the prescribed π_∞, and $m(\pi)$ is the multiplicity of the representation π in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}_F), 1)$.

Proof. The $K_f(\mathfrak{p}^n)$-fixed vectors of the representation π count precisely the occurrences of the archimedean part π_∞ in the quotient $Y(\mathfrak{p}^n)$. By the above, the number of components of $Y(\mathfrak{p}^n)$ is bounded independently of n. $\square$

3.2.2 Choice of Test Functions.

Here we define test functions whose traces will compute the multiplicity of archimedean representations at level $\mathfrak{p}^n$. At all non-archimedean places, they are the characteristic functions of compact subgroups. Recall that μ_v denotes the Haar measure on G_v.

Definition 3.2.3. At each finite place v, let $f_v(\mathfrak{p}^n)$ be the scaled characteristic function

$$f_v(\mathfrak{p}^n) = \frac{1_{K(\mathfrak{p}^n)_v}}{\mu_v(K(\mathfrak{p}^n)_v)}.$$

Of course this definition is only dependent on n for $v = v_{\mathfrak{p}}$.

Definition 3.2.4. Let $v \in S_0$ be (by definition of S_0) an archimedean place such that the group G_v is compact. Then define $f_v(\mathfrak{p}^n)$ to be the constant function

$$f_v = \frac{1}{\mu_v(G_v)}.$$

The traces of these test functions count the dimension of $K(\mathfrak{p}^n)$-fixed vectors. At the archimedean places $v \in S_0$, they only detect the trivial representation and have vanishing trace on all other representations of G_v. We want functions that play the same role at the non-compact archimedean places: they should detect representations π_v contained in a specific subset $\Pi^0_{\psi_v} \subset \Pi_{\psi_v}$ and vanish at all other representations. The key is that we will only be working with Arthur packets attached to parameters ψ having a specific ψ_v for v archimedean. As such, the test function at an infinite place v only needs to isolate $\pi_v \in \Pi^0_v$ from the other finitely many representations in the same packet.

Lemma 3.2.5. *Let ψ_v be a local Arthur parameter with associated Arthur packet Π_{ψ_v}. Fix*

a subset $\Pi^0_{\psi_v} \subset \Pi_{\psi_v}$. *Then there exists a function* $f^0_v \in \mathcal{H}(G_v)$ *such that*

$$
\operatorname{tr} \pi_v(f^0_v) = \begin{cases} 1 & \pi_v = \Pi^0_{\psi_v} \\ 0 & otherwise, \end{cases} \qquad \pi_v \in \Pi_v.
$$

Proof. This follows directly from linear independence of characters for admissible representations. If v is archimedean this was proved by Harish-Chandra in [18]. $\qquad\square$

Definition 3.2.6. Let v be a non-compact archimedean place, let ψ_v be an Arthur parameter and fix a subset $\Pi^0_{\psi_v} \subset \Pi_{\psi_v}$. Let $f_v(\mathfrak{p}^n) = f_v(\mathfrak{p}^n, \Pi^0_{\psi_v})$ be the function f^0_v described above.

Definition 3.2.7. Let the function $f(\mathfrak{p}^n)$ be defined as $f(\mathfrak{p}^n) = \prod_v f_v(\mathfrak{p}^n_v)$. We will also denote $f_f(\mathfrak{p}^n) = \prod_{v \nmid \infty} f_v(\mathfrak{p}^n_v)$.

Note that the function $f(\mathfrak{p}^n)$ satisfies the assumption of Theorem 3.1.7: it is a factorizable test function whose traces on all representations contained in the packet Π_{ψ_v} are nonnegative.

Proposition 3.2.8. *Let* $\psi \in \Psi(G^*, \eta_\kappa)$ *be an Arthur parameter. For each* $v \in S_\infty \setminus S_0$, *fix a subset* $\Pi^0_{\psi_v}$ *and a corresponding function* $f(\mathfrak{p}^n) = f(\mathfrak{p}^n, \Pi^0_{\psi_v})$. *The we have*

$$
R_{\mathrm{disc}, \psi}(f(\mathfrak{p}^n)) = \sum_\pi m(\pi) \dim \pi_f^{K_f(\mathfrak{p}^n)}
$$

where the sum is taken over representations $\pi \in \Pi_\psi$ *of the form* $\pi = (\otimes_{v \mid \infty} \pi_v) \otimes \pi_f$ *such that for archimedean* v, *the representation* π_v *is trivial if* $v \in S_0$ *and* $\pi_v \in \Pi^0_v$ *otherwise.*

Proof. As stated in Theorem 2.4.1, the distribution $R_{\mathrm{disc}, \psi}(f)$ computes the sum of $\operatorname{tr} \pi(f) = \prod_v \operatorname{tr} \pi_v(f_v)$ over all representations in the packet Π_ψ. At the finite places, the trace of convolution by the characteristic function of a compact open subgroup K_v is equal to the product $\mu_v(K_v) \cdot \dim \pi_v^{K_v}$. In the case of the archimedean places $v \in S_0$, the representations π_v are finite-dimensional so the only representation with a K_v-fixed vector is the

trivial representation. At $v \in S_\infty \setminus S_0$, the function $f_v(\mathfrak{p}^n)$ was chosen precisely to detect the representations π_v belonging to the subset $\Pi^0_{\psi_v}$. $\qquad\qquad\qquad\qquad\qquad\square$

The key input allowing us to compare multiplicity growth on G and on its endoscopic group H is a version of the fundamental lemma for congruence subgroups, proved by Ferrari [14]. It states that provided that $\mathfrak{p}$ is large enough, congruence subgroups of the same level form a transfer pair.

Theorem 3.2.9 (Theorem 3.2.3, [14]). *Let $\mathfrak{p}$ be a prime of F with localization $F_{v_{\mathfrak{p}}}$ and residue field $k_{\mathfrak{p}}$. Let $\mathrm{Nm}(\mathfrak{p})$ be the cardinality of $k_{\mathfrak{p}}$ and let p be its characteristic. Assume that $p > 9[F : \mathbf{Q}] + 1$. Let H be an endoscopic group of G. Let $d(G, H) = \frac{\dim G - \dim H}{2}$. Then the functions*

$$f(\mathfrak{p}^n)_{v_{\mathfrak{p}}} = \frac{1_{K_{v_{\mathfrak{p}}}(\mathfrak{p}^n)}}{\mu_{v_{\mathfrak{p}}}(K_{v_{\mathfrak{p}}}(\mathfrak{p}^n))}$$

and

$$f(\mathfrak{p}^n)^H_{v_{\mathfrak{p}}} = \mathrm{Nm}(\mathfrak{p})^{-n \cdot d(G,H)} \frac{1_{K_{v_{\mathfrak{p}}}(\mathfrak{p}^n)^H}}{\mu_{v_{\mathfrak{p}}}(K_{v_{\mathfrak{p}}}(\mathfrak{p}^n))}$$

form a transfer pair.

3.2.3   The Stable Term and Characters of Representations

Here we give assumptions under which the dominant stable term of Theorem 3.1.7 also computes limit multiplicity on a locally symmetric space associated to H_ψ. This will allow us to compare multiplicity growth on different groups. We start by expanding the stable distribution on H as an actual trace of representations of H for any endoscopic group H. The results are stated in terms of arbitrary test functions with nonnegative trace, but will be applied to $f = f(\mathfrak{p}^n)$ in the next section.

Lemma 3.2.10. *Let $\psi \in \Psi(G, \xi)$ be an Arthur parameter and let $H \in \mathcal{E}(G)$ be such that (H, ψ^H) corresponds to (ψ, s_H). Let $f \in \mathcal{H}(G)$ be a factorizable test function. Assume that the function f^H_v has nonnegative trace on all elements of the packet $\Pi_{\psi^H_v}$ for*

non-archimedean v, and that $f_v^H(\psi_v^H)$ is positive for all archimedean v. Then we have the inequality

$$S(\psi, s_H, f) \le C(\psi, s_H) \prod_{v|\infty} f_v^H(\psi_v^H) \prod_{v<\infty} \left(\sum_{\pi_v \in \Pi_{\psi_v}^H} \operatorname{tr} \pi_v(f_v^H) \right). \qquad (3.7)$$

Furthermore, equality holds $H = H_\psi$.

Proof. This follows from the definition of $S(\psi, s_H, f)$ and the local character identities of Section 2.3.5, applied this time to the group H instead of to G. From the proof of Proposition 3.1.4, we have

$$S(\psi, s_H, f) = C(\psi, s_H)\epsilon_\psi^H(s_\psi^H)f^H(\psi^H).$$

We then factor $f^H(\psi^H) = \prod_v f_v^H(\psi_v^H)$. At the finite places we use the identities of Theorems 2.3.6 and 2.3.9 to get

$$S(\psi, s_H, f) = C(\psi, s_H)\epsilon_\psi^H(s_\psi^H) \prod_{v|\infty} f_v^H(\psi_v^H) \cdot \prod_{v<\infty} \left(\sum_{\pi_v \in \Pi_{\psi_v}^H} \langle s_{\psi_v}^H s_H^H, \pi_v \rangle \operatorname{tr} \pi_v(f_v^H) \right).$$

The element s_H^H is by definition the image in $\mathcal{S}_{\psi^H}$ of an element s of the centralizer of ψ^H, the defining property of s being that it is centralized by the whole group $\hat{H}$. Thus s_H^H is the image in $\mathcal{S}_{\psi^H}$ of a central element, and as such it is the trivial element of $\mathcal{S}_{\psi^H}$. We can now rewrite $S(\psi, s_H, f)$ as the expression

$$C(\psi, s_H)\epsilon_\psi^H(s_\psi^H) \prod_{v|\infty} f_v^H(\psi_v^H) \cdot \prod_{v<\infty} \sum_{\pi_v \in \Pi_{\psi_v}^H} \langle s_{\psi_v}^H, \pi_v \rangle \operatorname{tr} \pi_v(f_v^H).$$

Since all the traces are nonnegative by assumption, the expression above only differs from the right-hand side of (3.7) by the presence of possible -1's coming from the characters $\epsilon_\psi^H(s_\psi^H)$ and $\langle s_{\psi_v}^H, \pi_v \rangle$ and the inequality of (3.7) holds. Moreover, it is an equality precisely

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if $\epsilon_\psi^H(s_\psi^H) = 1$ and $\langle s_{\psi_v}^H, \pi_v \rangle = 1$ for all π_v. This happens when the element s_ψ^H is trivial in the quotient $\mathcal{S}_\psi$, i.e. where $H = H_\psi$, the centralizer of $\psi(-I)$.

$\square$

The right-hand side of Lemma 3.2.10 computes the sum of the traces of all the representations in the packet $\Pi_{\psi H}$, i.e. all possible products coming from all the local packets. Yet the relevant representations for applications to multiplicity are the automorphic ones which appear in $L^2_{\mathrm{disc}}(H(F)\backslash H(\mathbf{A}_F))$ with nonzero multiplicity. However, if the parameter ψ is the sum of two irreducible pieces, all products of local representations of H are automorphic.

Corollary 3.2.11. *Under the assumptions of Lemma 3.2.10, if the parameter $\psi = \psi_1 \boxplus \psi_2$ has only two simple components and $H = H_\psi \neq G^*$, then we have*

$$S(\psi, s_H, f) = C(\psi, s_H) \prod_{v \in S} f_v^H(\psi_v^H) \sum_{\pi \in \Pi_\psi^H} m(\pi) \prod_v \operatorname{tr} \pi_v(f_v^H).$$

Proof. If $H_\psi \neq G$, which implies that $H = U(N_1) \times U(N_2)$ with ψ_i an irreducible parameter of $U(N_i)$, then the centralizer quotient $\mathcal{S}_{\psi H}$ is the product $\mathcal{S}_{\psi H} = \mathcal{S}_{\psi_1^H} \times \mathcal{S}_{\psi_2^H}$. From the computations of centralizers in (2.10), this is the product of two copies of the trivial group and characters ϵ_ψ^H and $\langle \cdot, \pi \rangle$ have no choice but to be identically 1 for all $\pi \in \Pi_{\psi H}$. The representations $\pi \in \Pi_\psi$ which appear in the discrete spectrum are precisely the ones satisfying $\epsilon_{\psi H} = \langle \cdot, \pi \rangle$ as characters of $\mathcal{S}_{\psi H}$, a condition which is here trivially satisfied. So $m(\pi) = 1$ for all π and the formula is equivalent to that of Lemma 3.2.10. $\square$

Remark 3.2.12. We hope to prove an asymptotic analogue of Corollary 3.2.10 for more general parameters. This would consist of a statement that the stable piece of the trace formula computes the traces of automorphic representations as the level grows, as opposed to the traces of all representations in $\Pi_{\psi H}$. This would require showing that in the limit, all stable terms for $H' \neq H_\psi$ in the trace formula for H_ψ are negligible. This could follow

from the methods by which we hope to get lower bounds discussed in Section 3.1.3, namely showing that there are enough representations to exhaust the characters of the group $\mathcal{S}_{\psi H}$.

3.2.4   Limit Multiplicity Results

We now show that if our parameter is a sum of two irreducible pieces, we can bound the limit multiplicity of certain representations of G by the multiplicities of a corresponding set of representations on one of its endoscopic groups H. Note that the fundamental lemma together with Theorem 3.2.9 ensure that the non-archimedean contributions of the function $f(\mathfrak{p}^n)$ satisfy the assumptions of Lemma 3.2.10.

Theorem 3.2.13. *Let $\psi \in \Psi(G, \xi)$ be a global parameter with prescribed localization ψ_∞, and such that $\psi = \psi_1 \boxplus \psi_2$ is a sum of two simple components. Assume that the infinitesimal character of ψ_∞ is regular. Let $\Pi^0_{\psi_\infty} \subset \Pi_{\psi_\infty}$ be a fixed subset and let Π^0_ψ be the subset of Π_ψ consisting of representations such that $\pi_\infty \in \Pi^0_{\psi_\infty}$. Then there exists a compact open subgroup $K^{H_\psi}(\mathfrak{p}^n)$ depending only on our choice of $f(\mathfrak{p}^n)$, and a positive constant C depending only on ψ_∞, H_ψ, and $K(\mathfrak{p}^n)$ such that*

$$\sum_{\pi \in \Pi^0_\psi} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \le C \cdot \mathrm{Nm}(\mathfrak{p}^n)^{d(G, H_\psi)} \sum_{\pi \in \Pi^{H_\psi}_\psi} m(\pi) \dim \pi_f^{K^{H_\psi}(\mathfrak{p}^n)}.$$

Moreover, at all places $v \notin S$, the subgroup $K_v^{H_\psi}(\mathfrak{p}^n)$ agrees with the one introduced at the beginning of 3.2.1.

Proof. If $\psi = \psi_1 \oplus \psi_2$, then $|\mathcal{S}_\psi| = 2$ as stated in (2.10), so there are two possible endoscopic groups for this parameter: G^* and a group $H = U(N_1) \times U(N_2)$ with $N_1 N_2 > 0$. If $H_\psi = G^*$ then the result is trivial so assume that $H_\psi \ne G^*$. Let $f(\mathfrak{p}^n)$ be the function introduced

above section so that $f_\infty(\mathfrak{p}^n)$ detects the representations in the subset $\Pi^0_{\psi_\infty}$. Then we have

$$\sum_{\pi \in \Pi^0_\psi} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} = R_{\text{disc},\psi}(f(\mathfrak{p}^n))$$

by Proposition 3.2.8. Then by Theorem 3.1.7, and using Theorem 2.4.2 and the assumption that the infinitesimal character is regular to switch from $I_{\text{disc},\psi}$ to $R_{\text{disc},\psi}$, we get the inequality

$$\sum_{\pi \in \Pi^0_\psi} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \leq C(\psi) S(\psi, s_H, f).$$

The parameter ψ satisfies the assumption of Corollary 3.2.10 so we can, ignoring the constant $C(\psi)$ for a moment, rewrite the right-hand side as

$$S(\psi, s_H, f) = f_\infty^H(\psi_\infty^H) \prod_{v \in S_f} f_v^H(\psi_v) \prod_{v \notin S} \sum_{\pi_v \in \Pi_v^H} \text{tr}\, \pi_v(f_v^H).$$

We will now relate the contribution of each place v in the product to a sum of traces of representations on v.

At the infinite places, we have chosen $f_\infty = f_\infty(\mathfrak{p}^n)$ so that $f_\infty^H(\psi_\infty) = k$, where $k = |\Pi^0_{\psi_\infty}|$. Up to a constant depending on ψ_∞, a dependency which we allow in the theorem, this is proportional to $|\Pi^H_{\psi_\infty}|$.

We now consider the places $v \in S_f$. Recall that these are finite places at which the group G_v is potentially ramified, so that the fundamental lemma does not apply. We need to show (i) that $f_v^H(\psi_v)$ is bounded above independently of ψ_v and (ii) that there is a compact open subgroup $K_v^H \subset H_v$ such that if $f_v^H(\psi_v) \neq 0$ then $\sum_{\pi \in \Pi_v^H} \pi_v^{K_v^H} \neq 0$. For (i), recall that $f_v^H(\psi_v) = \sum_{\pi_v \in \Pi_{\psi_v}} \pi_v^{K_v}$. The possible dimensions of K_v-fixed vectors in π_v is bounded only in terms of K_v (and in particular independently of π_v) by Bernstein's uniform admissibility theorem, see [8]. Furthermore, Mœglin's arguments in [34] can be adapted to show that there is a uniform upper bounds on the size of packets Π_v, proving (i). The result (ii) is more

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delicate, and is a consequence of Lemmas 5.2 and 5.4 of [31]. The combination of the two statements (i) and (ii), together with another application of the arguments of (i), this time to the packet $\Pi^H_{\psi_v}$, imply that $f^H_v(\psi_v)$ is a uniformly bounded multiple of $\sum_{\pi \in \Pi^H_{\psi_v}} \pi^{K^H_v}$.

Finally, at the places $v \notin S$, the function f^H is the characteristic function of $K_v(\mathfrak{p}^n)$, scaled by its volume, by the fundamental lemma 2.3.3 if $v \neq v_\mathfrak{p}$ and by the result of Ferrari [14] cited in Theorem 3.2.9 for $v = v_p$. At $v_\mathfrak{p}$ we also pick up the factor of $\mathrm{Nm}(\mathfrak{p}^n)^{d(G,H_\psi)}$.

Collecting these results and combining all the constants in the constant C (recall that the $C(\psi)$ we imported from Theorem 3.1.7 was bounded independently of ψ) we get that

$$S(\psi, s_H, f) \leq C \cdot \mathrm{Nm}(\mathfrak{p}^n)^{d(G,H)} \left(\sum_{\pi_\infty \in \Pi^H_{\psi_\infty}} 1 \right) \prod_{v < \infty} \left(\sum_{\pi_v \in \Pi^H_{\psi_v}} \pi_v^{K^H_v} \right).$$

Expanding the product, we get a sum over all possible representations in the packet Π^H_ψ, recalling that the fundamental lemma ensures that the representations which contribute are unramified at all but finitely many places. But following our assumptions on ψ which leads to $|\mathcal{S}^H_\psi| = 1$ as in Corollary 3.2.11, all representations in the packet Π^H_ψ are automorphic, i.e. they satisfy $m(\pi) = 1$. This allows is to conclude. $\qquad \square$

We are now ready to give upper bounds for limit multiplicities of a representation π_v of a unitary group $U(a, b)$. We first restrict the sets of parameters with which we work. We also fix the choice $\kappa = 1$ of sign determining our embedding of L-group ^LG as in Section 2.1.3. We now define some restricted subsets of parameters.

1. Let v be any place of F, and let ψ_v be a local parameter. Define $\Psi(\psi_v) \subset \Psi(G)$ to be the collection of parameters of G whose localization at v is ψ_v.

2. Let ψ_∞ denote a choice of parameters at all places $v \mid \infty$. Define $\Psi(\psi_\infty) \subset \Psi(G)$ to be the collection of parameters of G whose localization at the infinite places is ψ_∞.

3. Let $\Psi^2(G)$ be the collection of parameters satisfying the additional condition that $\psi = \psi_1 \oplus \psi_2$ is a sum of two irreducible parameters.

4. Finally, let $\Psi^2(\psi_v) = \Psi(\psi_v) \cap \Psi^2(G)$, and similarly for $\Psi^2(\psi_\infty)$.

While the notation $\Psi^2(G)$ seemed to us the most natural, it bears pointing out that there can be a risk of confusing such a set with $\Psi_2(G)$, a notation which we have not been using, but which denotes the set of square-integrable parameters in both [23] and [35].

We will work with Arthur parameters ψ with regular infinitesimal characters at infinity, and such that

$$\psi \mid_{SL_2(\mathbf{C})} = \nu(2k) \oplus \nu(1)^{N-2k}.$$

We now bound limit multiplicity of representations of G_∞ appearing in certain packets Π_{ψ_∞} associated to these composite parameters.

Theorem 3.2.14. *Let ψ_∞ be an Arthur parameter with regular infinitesimal character at all infinite places, and such that $\psi_\infty \mid_{SL_2(\mathbf{C})} = \nu(2k) \oplus \nu(1)^{N-2k}$. Fix a subset $\Pi^0_{\psi_\infty}$ in Π_{ψ_∞}. For each $\psi \in \Psi^2(\psi_\infty)$, let*

$$\Pi^0_\psi = \{\pi = \otimes'_v \pi_v \in \Pi_\psi \mid \pi_\infty \in \Pi^0_{\psi_\infty}.\}$$

Then

$$\sum_{\psi \in \Psi^2(\psi_\infty)} \sum_{\pi \in \Pi^0_\psi} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}. \tag{3.8}$$

Proof. We first give bounds for each parameter $\psi \in \Psi^2(\psi_\infty)$. Note that following the restrictions on the SL_2 and on the number of summands, any such parameter ψ must satisfy

$$\psi^N = (\nu(2k) \boxtimes \mu_1) \boxplus (\nu(1) \boxtimes \mu_{N-2k}),$$

where μ_1 is a Grossencharacter, and μ_{N-2k} is a cuspidal automorphic representation of GL_{N-2k}. From Section 2.2.8 we have that $\mathcal{S}_\psi \simeq \mathbf{Z}/2\mathbf{Z}$, and since $\nu(2k)$ is even-dimensional, the image of $-I \in SL_2$ under ψ is not central. It follows that s_ψ is the nontrivial element of $\mathbf{Z}/2\mathbf{Z}$. Thus we have $H_\psi = U_{E/F}(2k) \times U_{E/F}(N-2k)$ independently of ψ. Following Theorem 3.2.13, the contribution of ψ to (3.8) satisfies

$$\sum_{\pi \in \Pi_\psi^0} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \leq C \cdot \mathrm{Nm}(\mathfrak{p}^n)^{d(G,H)} \sum_{\pi \in \Pi_\psi^{H_\psi}} m(\pi) \dim \pi_f^{K^{H_\psi}(\mathfrak{p}^n)}.$$

Summing over $\psi \in \Psi^2(\psi_\infty)$, and recalling that the constants C are bounded independently of ψ, we get

$$\sum_{\psi \in \Psi^2(\psi_\infty)} \sum_{\pi \in \Pi_\psi^0} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \leq C \cdot \mathrm{Nm}(\mathfrak{p}^n)^{d(G,H)} \sum_{\psi \in \Psi^2(\psi_\infty)} \sum_{\pi \in \Pi_\psi^{H_\psi}} m(\pi) \dim \pi_f^{K^{H_\psi}(\mathfrak{p}^n)}.$$

$$(3.9)$$

Our goal is now to understand the right-hand side of (3.9). We momentarily ignore the scaling factor coming from transfer, and note that the rest is bounded by the sum where we consider all parameters in $\Psi(\psi_\infty)$, not only the ones in the subset $\Psi^2(\psi_\infty)$. In other words, we have

$$\sum_{\psi \in \Psi^2(\psi_\infty)} \sum_{\pi \in \Pi^0(\psi)} m(\pi) \dim \pi_f^{K(\mathfrak{p}^n)} \leq \sum_{\psi \in \Psi(\psi_\infty)} \sum_{\pi \in \Pi_\psi^{H_\psi}} m(\pi) \dim \pi_f^{K^{H_\psi}(\mathfrak{p}^n)}. \qquad (3.10)$$

To get a count on fixed vectors on the right-hand side of (3.10), we will consider more carefully the representations π which contribute to the sum. Let $N_1 = 2k$ and $N_2 = N - 2k$. The representations π appearing in the sum are of the form $\pi = \pi_1 \otimes \pi_2$, with π_i a representation of $U(N_i)$. Moreover, their central character is determined by Proposition 2.3.2 together with the central character of the representations of G corresponding to the param-

eters ψ. Let χ_i be the central character of π_i for $i = 1, 2$. Since the representations of G were assumed to have trivial central character, we find that the product $\chi_1\chi_2$ is determined by the characters χ_κ we used in our embeddings in 2.4, as well as by the parity of $N - N_i$. Going back to the formula of Proposition 2.3.2 and remembering that N_1 is even, we see that

$$\chi_1\chi_2 = \chi_\kappa^{N_1} = \begin{cases} \mathbf{1} & N \text{ even} \\ \chi_-^{N_1} & N \text{ odd,} \end{cases}$$

where $\mathbf{1}$ denotes the trivial character. In either case, the product $\chi_1\chi_2$ is by assumption unramified outside of the set S_f of finite places introduced in the beginning of this section.

The multiplicity of π is a product $m(\pi) = m(\pi_1)m(\pi_2)$ with $m(\pi_i)$ the multiplicity of π_i in

$$L^2_{\text{disc}}(U(N_i, F)\backslash U(N_i, \mathbf{A}_F), \chi_i).$$

Since $\psi_1(SL_2) = \nu(N_1)$ is maximally large, the representation π_1 is a character and by the computations of (2.10), the group $\mathcal{S}_{\psi_1}$ is always trivial so that $m(\pi_1) = 1$. Moreover, the character π_1 must be of the form $\theta \circ \det$ where θ is a Hecke character of $U(1, \mathbf{A}_F) = (\mathbf{A}_E)^{\text{Nm}=1}$ which we now describe. The character θ_∞ is determined by ψ_∞ and as such is known. The characters θ_f with fixed vectors of level $K(\mathfrak{p}^n)$ must have conductor dividing $\mathfrak{a}\mathfrak{p}^n$, where the ideal $\mathfrak{a}$ is determined by the level subgroup at the ramified places $v \in S_f$. In particular, the ideal $\mathfrak{a}$ can be fixed once and for all independently of n. The number of characters with these ramification restrictions is asymptotically proportional to $\text{Nm}(\mathfrak{p}^n)$. Finally we note that the central character of π_1 must be equal to $\theta \circ \chi^n$ where χ^n denotes the nth power map on $(\mathbf{A}_E)^{\text{Nm}=1}$.

We now count the dimension of fixed vectors coming from the parameter ψ_2 mapping into $^LU(N_2)$. This parameter ψ_2 is bounded by assumption. Mok shows that this implies that the local constituents of $\pi_2 \in \Pi_{\psi_2}$ are tempered [35, Thm. 2.5.1(b)], which implies by results of Wallach [49] that they occur in the cuspidal part of the discrete spectrum. We now

fix the character θ giving rise to π_1. This determines, among other things, a central character χ_1 and a central character $\chi_2 = \chi_\kappa^{N_1}\chi_1^{-1}$ for $U(N_2)$. As it is the case for χ_1, the archimedean part of χ_2 is fixed and determined by ψ_∞. Let $K_2(\mathfrak{p}^n) = U(N_2, \mathbf{A}_F) \cap K^{H_\psi}(\mathfrak{p}^n)$. We now want to count, for each $\pi_\infty \in \Pi_{\psi_{2,\infty}}$ the quantity

$$
m(\pi_\infty, \mathfrak{p}^n, \chi_2) := \sum_{\psi_2 \in \Psi(\psi_{2,\infty})} \sum_{\substack{\pi = \pi_\infty \times \pi_f \\ \omega(\pi) = \chi_2}} m(\pi) \dim \pi_f^{K_2(\mathfrak{p}^n)}
$$

$$
\leq \dim \mathrm{Hom}(\pi_\infty, L^2_{cusp}(Y_2^*(\mathfrak{p}^n), \chi_2)),
$$

where $Y_2^*(\mathfrak{p}^n) = U(N_2, F) \backslash U(N_2, \mathbf{A}_F) / K_{2,f}(\mathfrak{p}^n)$ and the inequality in the second line holds for n large enough, depending on χ_2. We explain this inequality: the space $Y_2^*(\mathfrak{p}^n)$ carries a commuting action of the groups G_∞ and $\bar{K}_2(\mathfrak{p}^n) := K_2(1)/K_2(\mathfrak{p}^n)$, the latter acting by deck transformations. This induces a representation of $G_\infty \times \bar{K}_2(\mathfrak{p}^n)$ on $L^2_{cusp}(Y_2^*(\mathfrak{p}^n))$, where the representation of latter group is isomorphic to the induction of the trivial representation. We then abuse notation and denote $\chi_2 \mid_{K_2(1)}$ by χ_2 again. This character is trivial on $K_{2,f}(\mathfrak{p}^n)$ for n large enough. In that case we define the space $L^2_{cusp}(Y_2(\mathfrak{p}^n), \chi_2)$ to be the χ_2-eigenspace for the center of $\bar{K}_2(\mathfrak{p}^n)$.

As in the Lemma 3.2.1, the space $Y_2(\mathfrak{p}^n)$ is a finite disjoint union of copies of the locally symmetric space $X_2(\mathfrak{p}^n)$ defined with respect to the same level for $SU(N_2)$. The representation π_∞ restricts to a finite sum of representations of $SU(N_2, F_\infty)$, which we will call ρ_∞. By the result of Savin [40], for each of these components, the multiplicity $m(\rho_\infty, \mathfrak{p}^n)$ grows at most like the volume of $X_2(\mathfrak{p}^n)$, and proportionally to the volume of $X_2(\mathfrak{p}^n)$ if ρ_∞ is discrete series. Summing over all components, whose number is bounded independently of n by Proposition 3.2.1, we find that the multiplicity $m(\pi_\infty, \mathfrak{p}^n)$ grows at most like the volume of $Y_2(\mathfrak{p}^n)$, i.e. the index $[K_2(\mathfrak{p}^n) : K_2(1)]$. In order to restrict ourselves to central character χ_2, we will think of this index as the dimension of the regular representation of $\bar{K}_2(\mathfrak{p}^n)$ on

$\mathbf{C}[\bar{K}_2(\mathfrak{p}^n)]$. The character χ_2 of $Z_{\bar{K}_2(\mathfrak{p}^n)}$ appears inside $\mathbf{C}[\bar{K}_2(\mathfrak{p}^n)]$ with multiplicity

$$[\bar{K}_2(\mathfrak{p}^n) : Z_{\bar{K}_2(\mathfrak{p}^n)}] = \frac{[K_2(\mathfrak{p}^n) : K_2(1)]}{[Z_{K_2(\mathfrak{p}^n)} : Z_{K_2(1)}]} \sim \mathrm{Nm}(\mathfrak{p}^n)^{N_2-1}.$$

We then take the sum over characters χ_1. The product $\chi_1\chi_2$ is unramified at $\mathfrak{p}$, so in order for π_1 to have $K(\mathfrak{p}^n)$-fixed vectors, the character θ must have conductor dividing $\mathfrak{p}^n$. The number of such characters grows asymptotically like $\mathrm{Nm}(\mathfrak{p}^n)$, giving us a growth for fixed vectors of at most $\mathrm{Nm}(\mathfrak{p}^n)^{N_2^2}$. Thus we have

$$\sum_{\psi\in\Psi^2(\psi_\infty)} \sum_{\pi\in\Pi_\psi^0} m(\pi)\dim\pi_f^{K(\mathfrak{p}^n)} \ll \mathrm{Nm}(\mathfrak{p}^n)^{N_2^2+d(G,H)}.$$

Finally, we compute $d(G,H) = 2Nk - 4k^2$ and recall that $N_2^2 = (N-2k)^2 = N^2 - 4Nk + 4k^2$ which gives us the desired bounds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A note on the proof: the quantitative results we import from Savin's work [40] are upper bounds in general but exact asymptotics if the representations of $U(N - 2k)$ are discrete series. In the following section where we consider specifically cohomological representations, one can combine the recipe for the embeddings of L-groups in (2.4) and the construction of cohomological parameters 3.3.3 to see that the representations of $U(N - 2k)$ are indeed discrete series.

3.3   Applications to Growth of Cohomology.

We now give an application of the results of Section 3.2 to growth of cohomology of arithmetic groups in congruence towers. We start with a discussion of cohomological representations and their Arthur parameters. The description of these representations is longer than strictly necessary for the proof of the final theorem, but we hope it can serve as an entry point for a reader interested in computing cohomological representations of unitary groups.

3.3.1   Cohomological Representations

The cohomology of lattices Γ in Lie groups is computed from the multiplicity of automorphic representations via Matsushima's formula, which we now state.

Theorem 3.3.1 (Matsushima's formula, [32]). *Let G be a connected semisimple Lie group with maximal compact subgroup K and Lie algebra $\mathfrak{g}$. Let $\Gamma \subset G$ be a cocompact lattice with associated compact locally symmetric space X_Γ. Denote the multiplicity of a unitary representation π of G in the right-regular representation $L^2(\Gamma \backslash G)$ by $m(\pi, \Gamma)$. Then the dimension of the ith cohomology of X_Γ is:*

$$\dim(H^i(X_\Gamma, \mathbf{C})) = \sum_\pi m(\pi, \Gamma) \dim(H^i(\mathfrak{g}, K; \pi)).$$

The $H^i(\mathfrak{g}, K; \pi)$ which appear above are the so-called $(\mathfrak{g}, K)$ cohomology groups of the representation π. We say that an irreducible representation π of G is *cohomological* if $H^*(\mathfrak{g}, K; \pi) \neq 0$. Cohomological representations of all semisimple Lie groups have been classified by Vogan and Zuckerman.

Theorem 3.3.2 ([46]). *Let G be a semisimple Lie group with complexified Lie algebra $\mathfrak{g}$. Let K a maximal compact subgroup of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. There are finitely many cohomological representations π of G, and an irreducible representation π satisfies $H^i(\mathfrak{g}, K; \pi) \neq 0$ if and only if the following two conditions hold:*

(i) π has the same infinitesimal character as the trivial representation of G;

(ii) $Hom_K(\pi, \wedge^i \mathfrak{p}) \neq 0$,

where the action of K on $\wedge^i \mathfrak{p}$ is induced by the adjoint representation.

Vogan and Zuckerman parameterize cohomological representations in terms of so-called θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$. Their initial results apply only to semisimple groups: it

is extended to groups $U(a, b)$ in [45] and condition (ii) above implies that the central character of the cohomological representation must be trivial. We now describe the parametrization of cohomological representations of unitary groups as concretely as possible. Aside from the initial results of [46], the rest of this subsection is based on computations which can be found in Chapter 5 of Bergeron-Clozel's book [5]. In short, cohomological representations of $U(a, b)$ are parametrized by refinements of partitions of N which are compatible with the signature (a, b).

Cohomological Representations and Bipartitions

Let $\mathfrak{t}$ be the Lie algebra of a compact torus of G contained in K. A θ-stable parabolic subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ determined by an element $\alpha \in i\mathfrak{t}$ in the following manner:

(i) $\mathfrak{l}$ is the zero eigenspace of the adjoint action of α on $\mathfrak{g}$;

(ii) $\mathfrak{u}$ is the sum of the positive eigenspaces of this same action.

To each θ-stable parabolic subalgebra $\mathfrak{q}$, Vogan and Zuckerman attach a representation $A_\mathfrak{q}$, and show that this construction yields all cohomological representations up to isomorphism. We now specialize their results to $\mathfrak{g}$, the Lie algebra of a unitary group $G = U(a, b)$ with $a + b = N$. (For this section only, we return to the more classical notation in which a unitary group over $\mathbf{R}$ is identified by its signature.) We embed $K_\infty = U(a) \times U(b)$ block-diagonally and take the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ to be diagonal. This $\mathfrak{t}$ is the Lie algebra of a compact torus T and the element $\alpha \in i\mathfrak{t}$ giving rise to $A_\mathfrak{q}$ is then of the form

$$\alpha = \mathrm{diag}(H_1, ..., H_a, H'_1, ..., H'_b)$$

where the coordinates are real numbers, the first a entries belong to the Lie algebra of $U(a)$ and rest to that of $U(b)$. Up to the action of the Weyl group of K_∞, we can assume that α

is such that $H_i \geq H_{i+1}$ and similarly for H'_i. Then the Lie algebra $\mathfrak{q}$ is determined by the relative sizes of the H_i and the H'_i. More specifically, we have:

Proposition 3.3.3 ([6], Section 5). *Let α be as above and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the associated θ-stable subalgebra. Let $Z = \{z_1 > ... > z_r\}$ be the set of distinct values taken on by the H_i and H'_i and let*

$$a_i = \#\{H_j = z_i\}, \quad b_i = \#\{H'_j = z_i\}.$$

Then $\mathfrak{l}$ is the Lie algebra of the Levi subgroup

$$L = \prod_{z_i \in Z} U(a_i, b_i)$$

defined over $\mathbf{R}$, and $\mathfrak{q}$ is determined completely by the ordered tuple

$$B = ((a_1, b_1), ..., (a_r, b_r))$$

of pairs of nonnegative integers such that $\sum_{i=1}^r a_i = a$ and $\sum_{i=1}^r b_i = b$.

We will call these tuples B of pairs *bipartitions* of (a, b) and will denote the associated Levi subgroup L_B.

Remark 3.3.4. The bipartitions of (a, b) *almost* parametrize the cohomological representation of $U(a, b)$, but there is redundancy. Specifically, two bipartitions B and B' give rise to the same representations if B' has adjacent pairs of the form $(a_1, 0), (a_2, 0)$ (resp. $(0, b_1)(0, b_2)$) which are collapsed into $(a_1 + a_2, 0)$ (resp. $(0, b_1 + b_2)$) in B. For example, the two bipartitions

$$B = ((1, 2)(1, 0)(2, 0)), \quad \text{and} \quad B' = ((1, 2)(3, 0))$$

give rise to the same cohomological representation of $U(4, 2)$. These pairs where either a or b is zero correspond to compact factors in the Levi L_B. Thus bipartitions of (a, b) parameterize

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representation with cohomology up to collapsing the compact factors. We will soon see that this redundancy dictates the possible Arthur packets to which a representation belongs.

Keeping the above redundancy in mind, we will denote the cohomological representation associated to the partition B by π_B.

3.3.2 Computation of Cohomology

The dimensions of cohomology of a representation can be computed from the bipartition B. We start with the result giving the dimensions of cohomology for a general group.

Proposition 3.3.5 ([46], Proposition 3.2). *Let the Lie algebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be as above and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Let $R = \dim \mathfrak{u} \cap \mathfrak{p}$. Then*

$$H^i(\mathfrak{g}, K, A_\mathfrak{q}) \simeq Hom_{\mathfrak{l} \cap \mathfrak{k}}(\wedge^{i-R}\mathfrak{p}, \mathbf{C}).$$

In particular, the smallest nonvanishing degree of cohomology of $A_\mathfrak{q}$ is R, for which we now give an explicit recipe in terms of the bipartition B, still following [5]. The summand $\mathfrak{p}$ of the Lie algebra decomposes as

$$\mathfrak{p} = (\mathfrak{l} \cap \mathfrak{p}) \oplus (\mathfrak{u} \cap \mathfrak{p}) \oplus (\mathfrak{u}^- \cap \mathfrak{p}),$$

where $\mathfrak{u}^-$ is the negative eigenspace for the element α and $\dim(\mathfrak{u} \cap \mathfrak{p}) = \dim(\mathfrak{u}^- \cap \mathfrak{p})$ since they are exchanged by the involution θ. Thus if $\mathfrak{q}$ corresponds to the bipartition

$$B = ((a_1, b_1), ..., (a_r, b_r))$$

and $\mathfrak{l}$ is the Lie algebra of $L = U(a_1, b_1) \times ... \times U(a_r, b_r)$, we have

$$R = \frac{\dim(\mathfrak{p}) - \dim(\mathfrak{p} \cap \mathfrak{l})}{2} = ab - \sum_{i=1}^{r} a_i b_i.$$

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In particular, if $a_i b_i = 0$ for all pairs, i.e. if L is compact, then the discrete series representation $A_{\mathfrak{q}}$ only has cohomology in the middle degree ab.

Remark 3.3.6. In fact, we can say more. The locally symmetric spaces associated to unitary groups are complex varieties and Bergeron-Clozel [5] give a recipe not only for the degree of cohomology but also for the Hodge bidegrees to which a representation contributes. Let $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ be the decomposition $\mathfrak{p}$ into a holomorphic and antihomolorphic part. Then $R = R^+ + R^-$ with

$$R^+ = \dim \mathfrak{u} \cap \mathfrak{p}^+ = \sum_{i<j} a_i b_j, \quad R^- = \dim \mathfrak{u} \cap \mathfrak{p}^- = \sum_{i>j} a_i b_j,$$

and the contribution of $A_{\mathfrak{q}}$ that appears in lowest degree is to H^{R^+, R^-}. Note that the degrees of cohomologyof $A_{\mathfrak{q}}$ depend on the unordered bipartition, but that the Hodge bidegrees are determined by the ordering.

3.3.3 Arthur Parameters of Cohomological Representations

We turn our attention to the parameters ψ whose Arthur packets at the archimedean places contain cohomological representations. These will be obtained via a choice of embedding of L-groups from parameters associated to the trivial representation of Levi subgroups of G. We will also give a description of the packets associated to these parameters. Their construction was given by Adams-Johnson, in [1] in conversation with work of Arthur [3], and in a language that predates the current formulation of the endoscopic classification of representations. Recently, Arancibia-Moeglin-Renard [2] have shown that Adams-Johnson's construction yields the same packets as the endoscopic classification by Mok [35] and Kaletha-Minguez-Shin-White [23].

To begin, note that there is a natural way to associate to an (ordered) bipartition B

of (a, b) an (ordered) partition P_B of N, namely by letting

$$P_B = (N_1, ..., N_r), \quad N_i = a_i + b_i.$$

Let L be the Levi subgroup associated to the bipartition B of (a, b). Then $\hat{L} \simeq \prod_i GL(N_i, \mathbf{C})$, is determined by the partition P_B, together with an embedding $\hat{L} \hookrightarrow \hat{G}$. The description of ${}^L L$, i.e. of the Galois action on $\hat{L}$, is given in 2.1.3.

Cohomological Arthur parameters are realized as the composition of parameters associated to the trivial representation of L with embeddings ${}^L L \hookrightarrow {}^L G$. To promote the embedding of dual groups to an L-homomorphism $\xi_{\hat{L}, \hat{G}}$, it suffices to give the image of $W_\mathbf{R}$ inside of ${}^L G$. We give Arthur's construction from Section 5 of [3]. Let T be the compact maximal torus with Lie algebra $\mathfrak{t}$ and let

$$\psi_{\hat{L}, \hat{G}} : W_\mathbf{R} \to {}^L G$$

be the map sending $W_\mathbf{C}$ into $\hat{T}$ so that for any $\lambda^\vee \in X_*(T)$, we have

$$\lambda^\vee(\psi_{\hat{L}, \hat{G}}(z)) = z^{\langle \rho_Q, \lambda^\vee \rangle} \bar{z}^{-\langle \rho_Q, \lambda^\vee \rangle}$$

where $\rho_Q = \rho_{\hat{G}} - \rho_{\hat{L}}$. Let the element $(1 \rtimes \sigma)$ map to $n_Q \rtimes \sigma$, where for any group G, n_G is an element in the derived group of $\hat{G}$ such that $ad\, n_G$ interchanges the positive and negative roots of $(\hat{G}, \hat{T})$, and with $n_Q = n_L^{-1} n_G$. Putting this together and denoting the embedding of $\hat{L}$ into $\hat{G}$ by ι, define

$$\xi_{\hat{L}, \hat{G}}(g, w) = \iota(g) \psi_{\hat{L}, \hat{G}}(w).$$

Now let $\psi_{0, \hat{L}} : SL_2(\mathbf{C}) \times W_\mathbf{R} \to {}^L L$ be the Arthur parameter of the packet containing the trivial representation of L. It is trivial on $W_\mathbf{R}$ and sends SL_2 to the principal SL_2 of $\hat{L}$.

Then the Arthur parameter of G corresponding to the Levi subgroup $\hat{L}$ is the composition

$$\psi_{\hat{L}} := \xi_{\hat{L},\hat{G}} \circ \psi_{0,\hat{L}} : SL_2 \times W_{\mathbf{R}} \to {}^L G.$$

Example 3.3.7. We now work out a few examples in the case of $G = U(2,2)$ to fix ideas. For additional examples, Bergeron-Clozel's concrete computations of the parameters φ_ψ for $G = U(2,1)$ can be found in [5, §4.6].

The description of the L-group of $U(2,2)$ is the specialization to $N = 4$ and $F = \mathbf{R}$ of the definitions of Section 2.1.3. We have

$$^L U(2,2) = GL_4(\mathbf{C}) \rtimes W_{\mathbf{R}},$$

where $W_{\mathbf{R}}$ acts through its $\mathbf{Z}/2\mathbf{Z}$ quotient. The nontrivial element σ acts by

$$\sigma(g) = \Phi_4^{-1} \bar{g}^{-t} \Phi_4, \quad \Phi_4 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

Each Levi subgroup is associated to an ordered partition. In the case where $P = (4)$, the associated Levi is $L = G = U(2,2)$, the embedding of L-groups is the identity and we have $\psi_{\hat{G}} = \psi_{0,\hat{G}}$.

If $P = (3,1)$, then $\hat{L} = GL_3 \times GL_1$ is embedded block-diagonally in $\hat{G}$ and we compute

$$\psi_{\hat{L},\hat{G}}(z \rtimes 1) = \begin{pmatrix} \left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} & & & \\ & \left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} & & \\ & & \left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} & \\ & & & \left(\frac{z}{\bar{z}}\right)^{\frac{-3}{2}} \end{pmatrix}, \quad n_Q = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{pmatrix}.$$

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From this we obtain the embedding $\xi_{\hat{L},\hat{G}}$ as described above, and in turn get $\psi_{\hat{L}} = \xi_{\hat{L},\hat{G}} \circ \psi_{0,\hat{L}}$.
The restriction to $W_{\mathbf{C}}$ of the corresponding Langlands parameter $\varphi_{\psi_{\hat{L}}}$ is

$$\varphi_{\psi_{\hat{L}}}(z \rtimes 1) = \begin{pmatrix} z^{\frac{3}{2}} \bar{z}^{\frac{1}{2}} & & & \\ & z^{\frac{1}{2}} \bar{z}^{\frac{-1}{2}} & & \\ & & z^{\frac{-1}{2}} \bar{z}^{\frac{-3}{2}} & \\ & & & z^{\frac{-3}{2}} \bar{z}^{\frac{3}{2}} \end{pmatrix}.$$

Lastly we look at the partition $(1, 1, 1, 1)$. In this case we have $L = T$ and the embedding of L-groups is determined by

$$\psi_{\hat{L},\hat{G}}(z \rtimes 1) = \begin{pmatrix} \left(\frac{z}{\bar{z}}\right)^{\frac{3}{2}} & & & \\ & \left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}} & & \\ & & \left(\frac{z}{\bar{z}}\right)^{\frac{-1}{2}} & \\ & & & \left(\frac{z}{\bar{z}}\right)^{\frac{-3}{2}} \end{pmatrix}, \quad n_Q = \Phi_4.$$

Since in this case $\psi_{0,\hat{T}}$ is the trivial L-morphism, we conclude that $\varphi_{\psi_T}(z \rtimes 1) = \psi_{\hat{L},\hat{G}}(z \rtimes 1)$.
This parameter is bounded: it corresponds to a packet of discrete series representations.

3.3.4   Structure of Cohomological Arthur Packets

Having now given the construction of cohomological Arthur parameters, we described the packets of representations to which they are attached.

Proposition 3.3.8 ([1], §3.3). *Let $\hat{L}$ be the Levi subgroup of $\hat{G}$ determined by a partition P. The parameter $\psi_{\hat{L}} = \xi_{\hat{G},\hat{L}} \circ \psi_{0,\hat{L}}$ is associated to a packet Π_{ψ} of representations of G such that Π_{ψ} only contains cohomological representations.*

We describe the specific representations contained in the packet $\Pi_{\psi_{\hat{L}}}$. Unsurprisingly, they correspond precisely to Levi subgroups L of G whose dual is $\hat{L}$. This is explained in

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both [1] and [3], but we spell out the consequences for our parametrizations of cohomological representations and Levi subgroups in terms of ordered (bi)-partitions.

Proposition 3.3.9. *Let $P = (N_1, ..., N_r)$ be an ordered partition of N and $\psi_P := \psi_{\hat{L}_P}$ be the corresponding parameter. Then the packet $\Pi_P := \Pi_{\psi_P}$ consists precisely of the cohomological representations π_B corresponding to bipartitions $B = ((a_1, b_1), ..., (a_r, b_r))$ such that $P_B = P$.*

We will call such bipartitions B *refinements* of P.

Proof. We have explained above how a Levi subgroup L_B gives rise to a morphism $\psi_{\hat{L}_B}$. The parameters attached to B and B' will be equivalent if they are conjugate by an element of $\hat{G}$. The isomorphism classes of representations π_B correspond to Levi subgroups L_B containing the fixed torus T, so we need only consider conjugation by $N_{\hat{G}}(\hat{T})$. This action induces an action of the Weyl group $W(\hat{T}, \hat{G})$ on $\hat{T}$ and on the root datum $(X_*(T), \Delta(T), X_*(\hat{T}), \Delta(\hat{T}))$. Note that the action of conjugation by $\hat{T}$ on cohomological Arthur parameters will only modify $\psi_{\hat{L}}$ by scaling the entries of n_Q. This has no impact on the parameter since n_Q was only specified up to scalars in the construction of $\psi_{\hat{L}}$.

Thus to determine which Levi subgroups L (i.e. bipartitions) give rise to the conjugacy class of $\hat{L}$, we consider the action of $W(\hat{G}, \hat{T})$ (denoted $W(\mathfrak{g}, \mathfrak{t})$ in [1] since it is the Weyl group of the complexified Lie algebra of G) on the bipartitions. Recall that bipartitions are determined ultimately by an element $\alpha \in \mathfrak{t}$. The entries of conjugate elements $w \cdot \alpha$ will have the same values, but these values will be distributed differently among the two pieces of $\mathfrak{t}$ belonging to $U(a)$ and $U(b)$. Recall from Proposition 3.3.3 that we denote the values appearing in the entries of α by z_i. The data being preserved by conjugation is the number of entries $a_i + b_i$ which are associated to the same value z_i, as well as the ordering of the z_i. Transitivity of the Weyl group action then ensures that all the possible bipartitions obtained as a refinement of P give rise to the parameter ψ_P. $\square$

The next natural question is to get a description of the elements in the packet. Denote

by $W(G,T)$ the Weyl group of the maximal compact subgroup of G. As was alluded to when we introduced the parametrization by bipartitions, we can act on the element α by $W(G,T)$ until α is in the form

$$\alpha = \text{diag}(H_1, ..., H_a, H_1', ..., H_b')$$

where $H_i \geq H_{i+1}$ and similarly for H_i'. This action will preserve the Levi subgroup L since this Levi is determined by which values z_i appear in the first a entries. Thus once we have fixed α, the action of $W(\hat{G}, \hat{T})$ on the conjugate Levis by permuting coordinates amounts to the left-action on $W(\hat{G}, \hat{T})/W(G,T)$. For this action, the stabilizer of an element $g \cdot \alpha$ is the subgroup permuting all the entries with identical values. The multiplicities of these values are precisely encoded in the partition P, i.e. on the Levi $\hat{L}_P$. Thus $hg \cdot \alpha = g \cdot \alpha$ exactly when $h \in W(\hat{L}, \hat{T})$. This discussion recovers Adams-Johnson's parametrization of representations inside of a packet.

Lemma 3.3.10 ([1], Section 2.). *Representations in the packet $\Pi_{\psi_{\hat{L}}}$ are in bijection with*

$$W(\hat{L}, \hat{T}) \backslash W(\hat{G}, \hat{T}) / W(G,T).$$

A particular case will be of interest to us: If the group G is compact, we have $W(G,T) = W(\hat{G}, \hat{T})$ and each of the cohomological Arthur packets contains a unique representation. In each of the packets, this representation is the only finite-dimensional cohomological representation, namely the trivial one.

Example 3.3.11. As we alluded to, the overlap between cohomological Arthur packets can be understood via the redundancy in the parametrization of representations. For example if $G = U(3,1)$, we can consider $P = (3,1)$ with refinements $B_1 = ((3,0),(0,1))$ and $B_2 = ((2,1)(1,0))$. The associated packet has two elements: π_{B_2} is non-tempered but π_{B_1} is a discrete series also associated to the bipartitions $((2,0),(1,0),(0,1))$, $((1,0),(2,0),(0,1))$, and $((1,0),(1,0),(1,0),(0,1))$ and as such appears in three other packets.

3.3.5 A Recipe for the Dimensions of Cohomology Inside a Packet.

Here we give a method to easily compute the exact dimensions of cohomology coming from the representations inside a packet Π_ψ. This follows Section 9 of [3], and was first explained to the author by Simon Marshall. It does not strictly have a bearing on the proof of the main theorem of this section, but as far as we can tell this recipe is not explicitly written down anywhere, so we record it here.

In theorem 9.1 of [3], Arthur proves the existence of an isomorphism between two representations ρ_ψ and σ_ψ of $W_{\mathbf{C}} \times SL_2(\mathbf{C}) \times S_\psi$ on the space

$$V_\psi = \oplus_{\pi \in \Pi_\psi} H^*(\mathfrak{g}, K; \pi).$$

The representation ρ_ψ is realized by constructing a representation of each of the three groups in the product and showing they commute. The representation of S_ψ on each cohomology group $H^i(\mathfrak{g}, K; \pi)$ is a character with values in $\{\pm 1\}$ coming from the pairing between Π_ψ and the quotient $\mathcal{S}_\psi$. The representation of $W_{\mathbf{C}}$ was initially defined by Langlands and is a sum of characters determined by the Hodge bidegree. Finally, the representation of $SL_2(\mathbf{C})$ is the traditional "Lefschetz SL_2" acting on the cohomology of complex varieties: at the level of the Lie algebra, the degree-raising operator X is given by the wedge product with a certain element of $\mathrm{Hom}_K(\mathfrak{p}^+ \times \mathfrak{p}^-, \mathbf{C})$. There is a corresponding lowering operator Y, and $H = XY - YX$ has eigenvalue $k - ab$ on $H^k(\mathfrak{g}, K, \pi)$.

The second representation σ_ψ is obtained from a combination of the parameter ψ and the Shimura X datum associated to $G' = \mathrm{Res}^F_{\mathbf{Q}} G$. First, fix a basepoint x of X associated to the choice of maximal compact subgroup K. This amounts to the choice of a mapping $h : \mathbf{S} \to G'(\mathbf{R})$ where $\mathbf{S} = \mathrm{Res}^{\mathbf{C}}_{\mathbf{R}} \mathbf{G}_m$ is the Deligne torus. The image of this mapping should be contained in the diagonal torus T. We have $\mathbf{S}(\mathbf{C}) = \mathbf{C}^\times \times \mathbf{C}^\times$ and the restriction of h to the first factor gives an element of $X_*(T)$, the cocharacter group of T, and by

duality a character $\mu \in X^*(\hat{T})$. This character is the highest weight of a finite-dimensional representation (ρ_μ, V_μ) of $\hat{G}$. The map

$$\psi : SL_2 \times W_{\mathbf{R}} \to \hat{G}',$$

whose image commutes with S_ψ by construction, makes V_μ into a $SL_2(\mathbf{C}) \times W_{\mathbf{C}} \times S_\psi$-representation denoted σ_ψ. The content of [3, Theorem 9.1] is that the two representations (σ_ψ, V_μ) and (ρ_ψ, V_ψ) are isomorphic. We now compute ρ_μ for certain unitary groups.

Lemma 3.3.12. *If $G'(\mathbf{R}) = U(a,b) \times U_N(\mathbf{R})^{[F:\mathbf{Q}]-1}$, then the representation (ρ_μ, V_μ) described above is the representation $\wedge^a W \oplus 1^{[F:\mathbf{Q}]-1}$, where W is the standard N-dimensional representation of $GL_N(\mathbf{C})$.*

Proof. Following the axioms for a Shimura variety [12] we find that in the case of $U(a,b)$ a choice of h corresponding to K is given by

$$h(z) = \begin{pmatrix} \left(\frac{z}{\bar{z}}\right) I_a & \\ & I_b \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. For the compact factors h can be taken to be trivial. Then the weight μ is $(1, ..., 1, 0, ..., 0)$ with a entries labeled 1 for $U(a,b)$, and trivial on the other factors. This first weight is the highest weight of $\wedge^a W$, see for example [15, §15]. $\quad\square$

In the examples below we will restrict our attention to the factor $U(a,b)$ for which the representations are not trivial.

Example 3.3.13. The above theorem tells us that we can compute the degrees of cohomology inside the Arthur packet Π_{ψ_P} associated to a partition P by computing the weights in the representation $\sigma_{\psi_P} \mid SL_2(\mathbf{C})$. We work with $G = U(2,2)$ and compute the degrees of cohomology associated to the parameters from Example 3.3.7.

In the case of $\hat{L} = \hat{G}$, i.e. of $P = (4)$, the representation $\psi \mid_{SL_2(\mathbf{C})}$ is $\nu(4)$. The restriction of σ_ψ to SL_2 is

$$\wedge^2 \nu(4) = \nu(5) \oplus \nu(1)$$

and the nonvanishing dimensions of cohomology in the packet are

$$h^0 = h^2 = h^6 = h^8 = 1, \quad h^4 = 2.$$

Note that all the cohomology in this case comes from the unique representation contained in Π_ψ, namely the trivial representation.

In the case of the parameter corresponding to $P = (3, 1)$, we compute

$$\wedge^2(\nu(3) \oplus \nu(1)) = \nu(3) \oplus \nu(3)$$

and find that the nonvanishing dimensions in the packet are

$$h^2 = h^4 = h^6 = 2.$$

In this case, there are two representations in the packet, corresponding to the Levi subgroups $U(2, 1) \times U(0, 1)$ and $U(1, 2) \times U(1, 0)$ and they each contribute in all three degrees, but in different Hodge bidegrees. For example, we can see from Remark 3.3.6 that the representation $\pi_{((2,1),(0,1))}$ has cohomology in $H^{2,0}$ and that $\pi_{((1,2)(1,0))}$ contributes to $H^{0,2}$.

Finally we consider the partition $P = (1, 1, 1, 1)$. In this case the restriction of σ_ψ to SL_2 is the sum of six copies of the trivial representation $\nu(0)$. As such, each of the six discrete series in Π_ψ contribute one dimension to the middle degree h^4.

The combinatorics giving the weights of tensor powers of representations of SL_2 rapidly get out of hand, rendering difficult the task of giving a general recipe for degrees of cohomology in terms of a partition. Yet any specific example is readily computed, and it is also

straightforward to obtain results in specific families.

Example 3.3.14. Let $G = U(a, b)$ and let $P_N = (2, 1, ..., 1)$ be a partition of N where 1 appears with multiplicity $N - 2$. Then one can check that the representation σ_{ψ_P} of SL_2 acting on the cohomology groups $\oplus_{\pi \in \Pi_{\psi_{P_N}}} H^*(\mathfrak{g}, K; \pi)$. decomposes into

$$\sigma_{\psi_{P_N}} = \nu(2)^{\binom{N-2}{a-1}} \oplus \nu(1)^{\binom{N-2}{a-2} + \binom{N-2}{a}}$$

as soon as N is large enough relative to a for the binomial coefficients to be defined. Thus the nonzero degrees of cohomology are h^{ab} and $h^{ab\pm 1}$, with multiplicities prescribed from the dimensions above.

3.3.6 Limit Multiplicity for Packets of Cohomological Representations

We now give results on growth of cohomology. Note that we return to the notation of most of this document, in which F is global and for which the subscript "∞" denotes the collection of all the archimedean places. Fix the set S_0 so that it contains all but one archimedean place v_0. Let G be the inner form of $U_{E/F}(N)$ such that $G_{v_0} \simeq U(a, b)$ and all the other factors at infinity are compact. Define the group $K(\mathfrak{p}^n)$ and the cocompact lattices as in Section 3.2.1. By Matsushima's formula and Lemma 3.2.2, we have

$$h^i(\mathfrak{p}^n) := \dim(H^i(\Gamma(\mathfrak{p}^n), \mathbf{C}) \asymp \sum_{\pi \in \Pi_\psi} m(\pi) h^i(\mathfrak{g}_{v_0}, K_{v_0}; \pi_{v_0}) \dim \pi_f^{K_f(\mathfrak{p}^n)}$$

where the sum is taken over representations π such that π_v is trivial (the only cohomological representation of a compact Lie group) at all places $v \in S_0$. We can now give our theorem for growth of cohomology. We compute the contribution of parameters ψ_∞ associated to a certain partition, and will discuss below why these parameters are of a particular interest. Additionally our current results on growth constrain us to counting only the contribution of

global parameters of the form $\psi = \psi_1 \boxplus \psi_2$. We explained in Remark 3.2.12 how we hope to lift this restriction.

Theorem 3.3.15. *Let ψ_∞ be the cohomological parameter of G_∞ associated to an ordered partition with one entry equal to $2k$ and all the other $N - 2k$ entries equal to 1. Let*

$$h^{i,2}_{\psi_\infty}(\mathfrak{p}^n) = \sum_{\psi \in \Psi^2(\psi_\infty)} \sum_{\pi \in \Pi_\psi} m(\pi) h^i(\mathfrak{g}_{v_0}, K_{v_0}; \pi_{v_0}) \dim \pi_f^{K_f(\mathfrak{p}^n)}.$$

Then

$$h^{i,2}_{\psi_\infty}(\mathfrak{p}^n) \ll \mathrm{Nm}(\mathfrak{p}^n)^{N(N-2k)}.$$

Proof. Since the possible contribution to cohomology of a given representation π_{v_0} is bounded, the expression $h^{i,2}_{\psi_\infty}(\mathfrak{p}^n)$ grows like the contribution of parameters $\psi = \psi_1 \boxplus \psi_2$ to the multiplicity growth of representations $\pi_\infty \simeq \pi_{v_0} \otimes \mathbf{1}^{[F:\mathbf{Q}]-1}$ of G_∞. Thus the result is a direct consequence of Theorem 3.2.13, provided that cohomological parameters satisfy its assumptions. From Theorem 3.3.2, cohomological representations have the same infinitesimal character as the trivial representations. In particular, it is regular. Following the discussion in Section 3.3.3, partitions obtained as reorderings of $(2k, 1, ..., 1)$ correspond to parameters for which $\psi(SL_2) = \nu(2k) \oplus \nu(1)^{N-2k}$. Thus the assumptions are satisfied and the result follows. $\square$

Remark 3.3.16. One can wonder about the extent of the restriction posed by considering parameters in $\Psi^2(\psi_\infty)$ rather than $\Psi(\psi_\infty)$, that is, only counting parameters whose global shape is $\psi = \psi_1 \boxplus \psi_2$. The global parameters $\psi \in \Psi(\psi_\infty) \setminus \Psi^2(\psi_\infty)$ which contribute to $h^i_{\psi_\infty}(\mathfrak{p}^n)$ will be of the form

$$\psi = \psi_1 \boxplus \psi_2 \boxplus ... \boxplus \psi_r \tag{3.11}$$

where $\psi_1 = \mu_1 \boxtimes \nu(2k)$ for μ_1 a Grossencharacter. For $i > 1$ we have $\psi_i = \mu_i \boxtimes \nu(1)$ with μ_i

a cuspidal automorphic representation of GL_{N_i}. In that case, we can estimate the growth as if $GL_{N_1} \times ... \times GL_{N_r}$ was the dual group of an endoscopic group of $U(N)$:

- As in the proof of Theorem 3.2.13, the factor of ψ_1 should contribute $\mathrm{Nm}(\mathfrak{p}^n)$, namely an exponent of 1 to the growth.

- Each other factor should contribute the growth of the discrete series representation on $U(N_i)$, namely an exponent of N_i^2 following the result of Savin [40].

- We then subtract 1 from the exponent to account for the fixed central character.

- Ferrari's result (Theorem 3.2.9) adds a $\frac{N^2 - \sum_i N_i^2}{2}$ to the exponent, coming from the transfer of test functions.

Of course one cannot simply reproduce the argument of Theorem 3.2.13, chiefly on account that if there are more than two summands, the group $\prod_i U(N_i)$ is not an endoscopic group of $U(N)$. However, we hope to obtain this growth via an iterated application of the ideas of Sections 3.1 and 3.2. For now, we view this as a credible heuristic. It predicts a growth exponent of

$$\frac{N^2 - (2k^2) + \sum_{i=2}^r N_i^2}{2}$$

coming from parameters with shape as in (3.11). Recalling the restriction that $2k + \sum_{i=2}^r N_i = N$, one immediately sees that this exponent is maximized if there is a unique $N_2 = N - 2k$ and that any other contribution will be asymptotically negligible in comparison. We thus believe that the bounds of Theorem 3.3.15 should hold even when the outer sum is taken over $\psi \in \Psi(\psi_\infty)$.

The smallest degree of cohomology for $U(a, b)$ associated to a partition given by a reordering of $(2k, 1, ..., 1)$ is the one whose actual growth is most likely to be $N(\mathfrak{p}^n)^{N(N-2k)}$. It be computed from the formulas of Section 3.3.2. It is associated to the most split bipartition

realized as a reordering of

$$P = ((a_1, b_1), (1, 0), ..., (1, 0), (0, 1), ..., (0, 1)). \qquad (3.12)$$

We see that for (a_1, b_1) to be maximally split, if $a < b$, we will have $a_1 = \min\{a, k\}$. Thus the lowest degree i of cohomology associated to this partition is

$$i = \begin{cases} a(N - 2k) & a \leq k \\ ab - k^2 & k \leq a. \end{cases}$$

Remark 3.3.17. In the special case that the partition P from (3.12) has either $a_1 = a$ or $b_1 = b$, the representation π_P contributes to the so-called special cohomology considered by Bergeron–Millson–Mœglin in [7]. This portion of the cohomology can be realized as the image of a theta lift from smaller unitary groups. In the case where $a = 1$, this allows the authors to deduce the Hodge conjecture for the corresponding Shimura varieties.

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