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SURFACES IN GRAPHS OF GROUPS AND THE STABLE COMMUTATOR LENGTH

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Dedicated to my family

Grant me the courage to prove the things I can figure out,

serenity to ignore the things I cannot,

and wisdom to know the difference.

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ABSTRACT

In this thesis, we study stable commutator length (scl) in graphs of groups by analyzing surfaces in the corresponding graphs of spaces. We give a linear programming algorithm computing scl in a large class of graphs of groups including those with cyclic vertex and edge groups. The algorithm implies that the unit norm ball of scl is a rational polyhedron. We also establish a linear programming duality method to show sharp lower bounds of scl in graphs of groups, subject to a local n-relatively torsion-free condition on the inclusion of edge groups into vertex groups. As an application we prove a uniform sharp lower bound of scl in graph products.

CHAPTER 1

INTRODUCTION

Given a second homology class $\alpha \in H_2(X; \mathbb{Q})$ in a space X, the Gromov–Thurston norm is the minimal complexity of surfaces representing α, measured in terms of Euler characteristic. The unit norm ball is a rational polyhedron when X is a 3-manifold [41].

For a given loop $\gamma \subset X$, its stable commutator length (scl) is a relative version of the Gromov–Thurston norm, which is the minimal complexity of surfaces bounding γ. It only depends on the conjugacy class g representing γ in the fundamental group $G = \pi_1 X$ and can be defined algebraically (Section 2.1), so we denote it as $\mathrm{scl}_G(g)$.

Calegari [12] discovered an algorithm to compute scl in free groups, which is a notoriously hard task in general. The nature of his algorithm shows that in this case the unit norm ball of scl is a rational polyhedron, and has led to positive partial answers to Gromov's question about surface subgroups in hyperbolic groups [10, 17, 44]. Calegari asked if similar statements hold true more generally for hyperbolic groups or groups acting on hyperbolic spaces.

We give a positive answer to this question for a large class of groups acting on simplicial trees, the simplest δ-hyperbolic spaces, Such groups have the structure of *graphs of groups* [38], where vertex and edge groups are vertex and edge stabilizers. Our result for instance applies to those with cyclic vertex and edge stabilizers, such as the Baumslag–Solitar groups

$$\mathrm{BS}(M, L) := \left\langle a, t \mid a^M = t a^L t^{-1} \right\rangle.$$

The method is geometric and relies on a (simple) normal form that we introduce to understand all (relative) maps of surfaces into graphs of groups. For free groups, this is analogous to the normal forms obtained by Wicks [36] and Culler [23].

The main difficulty we overcome is the fact that the space of admissible surfaces has a lot of information but very little structure. More structure arises when we put surfaces in normal

form, which makes use of two operations on surfaces: cut-and-paste and compression. There is another operation, taking finite covers, which is analogous to working with $\mathbb{Q}$ instead of $\mathbb{Z}$. This is crucial for us to deal with graphs of groups with *infinite* edge groups.

Understanding surfaces in graphs of groups also allows us to prove sharp lower bounds of scl and *spectral gap* properties. A group G has a spectral gap $C > 0$ if scl_G does not take any value in the interval $(0, C)$. Many groups are known to have such a gap together with a characterization of elements with zero scl, such as word-hyperbolic groups [14] and mapping class groups [4]; see Example 2.24 for a more detailed list. Such gaps can be thought of the homological analog of Margulis constants or quantitative obstructions for the existence of certain group homomorphisms; see Sect. 2.3. They are also related to the spectrum of simplicial volume [31] and bi-orderability of groups [33].

We obtain such sharp estimates using a linear programming duality method that proves uniform lower bounds of the complexity of all surfaces bounding the given loop. This is very different from the more common approach by constructing "effective" *quasimorphisms* and applying Bavard's duality [3]. In this way we avoid the difficult problem of finding quasimorphisms that give the sharp estimate.

1.1 Statement of Results

1.1.1 Rationality and Computation

We develop a method to compute scl in many graphs of groups using linear programming. The nature of the algorithm shows that scl is piecewise rational linear on chains (See Sect. 2.1 for definitions).

One main result is:

Theorem 5.25 (rationality). Let G be a graph of groups with vertex groups $\{G_v\}$ and edge groups $\{G_e\}$ where

(1) $\mathrm{scl}_{G_v} \equiv 0$; and

(2) the images of the edge groups in each vertex group are *central* and *mutually commensurable*.

Then scl_G is piecewise rational linear, and $\mathrm{scl}_G(c)$ can be computed via linear programming for each rational chain $c \in B_1^H(G)$.

The theorem above applies to many interesting groups. Bullet (1) holds true for amenable groups [11, Theorem 2.47], irreducible lattices in higher rank Lie groups [6, 7] (see also [11, Theorem 5.26]), and certain transformation groups like $\mathrm{Homeo}^+(\mathrm{S}^1)$ [28, Proposition 5.11] and subgroups of $\mathrm{PL}^+(\mathrm{I})$ [8, Theorem A]. In particular, Theorem 5.25 applies to all graphs of groups with vertex and edge groups isomorphic to $\mathbb{Z}$ (also known as generalized Baumslag–Solitar groups). See Example 5.1 for more groups covered by this theorem.

In particular, it generalizes most known rationality results, listed in Example 2.23.

A map of a surface group to a graph of groups is represented geometrically by a map of a surface to a graph of spaces. The surface can be cut into pieces along curves mapping to the edge spaces. Simplifying these pieces we can put the surface into a normal form.

If we understand the edge groups we can give conditions under which such pieces may be reassembled. When edge groups are infinite, such gluing conditions depend a priori on an infinite amount of data which we refer to as "winding numbers".

The key to our method is to keep track of a *sufficient* but *finite* amount of information about winding numbers. What makes this approach possible is a method to solve gluing conditions *asymptotically*. This is a geometric covering space technique and depends on residual properties of the fundamental group of the underlying graph. It also relies on an elementary but crucial observation about *stability* of *virtual isomorphisms* of abelian groups (see Subsection 5.1.1).

We also obtain two isometric embedding Theorems 3.6 and 3.10 from the norm form, which can be used to reduce the computation of scl in complicated graphs of groups to

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simpler ones.

As an example, there is a striking relation between scl in $\mathrm{BS}(M,L)$ and in $\mathbb{Z}/M\mathbb{Z}*\mathbb{Z}/L\mathbb{Z}$. A word $w \in \mathrm{BS}(M,L) = \left\langle a, t \mid a^M = ta^L t^{-1} \right\rangle$ is t-*alternating* if it can be written as $w = a^{u_1}ta^{v_1}t^{-1}a^{u_2}ta^{v_2}t^{-1}\cdots a^{u_n}ta^{v_n}t^{-1}$, where the generator t alternates between t and t^{-1}.

Corollary 3.14. Let $\mathbb{Z}/M\mathbb{Z}*\mathbb{Z}/L\mathbb{Z} = \left\langle x, y \mid x^M = y^L = 1 \right\rangle$. For any t-alternating word in $\mathrm{BS}(M,L)$, we have

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(a^{u_1}ta^{v_1}t^{-1}a^{u_2}ta^{v_2}t^{-1}\cdots a^{u_n}ta^{v_n}t^{-1}) = \mathrm{scl}_{\mathbb{Z}/M\mathbb{Z}*\mathbb{Z}/L\mathbb{Z}}(x^{u_1}y^{v_1}\cdots x^{u_n}y^{v_n}).$$

1.1.2 Spectral Gaps

We study spectral gaps of groups acting on trees. Our estimates depend on the types of elements. An element of the group is called *elliptic* if it stabilizes some vertex and *hyperbolic* otherwise.

We say that a pair of a group G and a subgroup $H \leq G$ is n-*relatively torsion-free* (n-RTF) if there is no $1 \leq k < n$, $g \in G \setminus H$ and $\{h_i\}_{1 \leq i \leq k} \subset H$ such that

$$gh_1 \cdots gh_k = 1_G,$$

and simply *relatively torsion-free* if we can take $n = \infty$; see Definition 4.3.

Theorem 4.8 (estimates for hyperbolic elements). Let G be a graph of groups such that each edge group is n-RTF in the adjacent vertex groups. If $g \in G$ is hyperbolic, then

$$\begin{aligned}
\mathrm{scl}_G(g) &\geq \frac{1}{2} - \frac{1}{n}, \text{ if } n \in \mathbb{N} \text{ and} \\
\mathrm{scl}_G(g) &\geq \frac{1}{2}, \text{ if } n = \infty.
\end{aligned}$$

Our estimates are sharp, strengthening the estimates in [22] and generalizing all other

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spectral gap results for graph of groups known to the author [19, 24, 30].

We actually prove a stronger version that gives the estimates for individual elements under weaker assumptions; see Theorem 4.7.

The proof is based on a *linear programming duality method* that we develop to uniformly estimate the Euler characteristic of all admissible surfaces in X in *simple normal form* (Section 3.3). The normal form is obtained by cutting the surface along edge spaces and simplifying the resulting surfaces, similar to the one in [20]. The linear programming duality method is a generalization of the argument for free products in [19].

As an application, we obtain sharp spectral gaps in *graph product*.

Let Γ be a simple and not necessarily connected graph with vertex set V and let $\{G_v\}_{v \in V}$ be a collection of groups. The graph product G_Γ is the quotient of the free product $\star_{v \in V} G_v$ subject to the relations $[g_u, g_v]$ for any $g_u \in G_u$ and $g_v \in G_v$ such that u, v are adjacent vertices. This is a rich class of groups includes right-angled Artin groups; see Example 4.20.

Theorem 4.22 (spectral gap of graph products). Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ ($m \geq 1$) is in cyclically reduced form and there is some $3 \leq n \leq \infty$ such that $g_i \in G_{v_i}$ has order at least n for all $1 \leq i \leq k$. Then either

$$\mathrm{scl}_{G_\Gamma}(g) \geq \frac{1}{2} - \frac{1}{n},$$

or Γ contains a complete subgraph Λ with vertex set $\{v_1, \ldots, v_m\}$. In the latter case, we have

$$\mathrm{scl}_{G_\Gamma}(g) = \mathrm{scl}_{G_\Lambda}(g) = \max \mathrm{scl}_{G_i}(g_i).$$

The estimate is sharp; see Remark 4.23.

In particular, for a collection of groups with a uniform spectral gap and without 2-torsion, their graph product has a spectral gap. In the special case of right-angled Artin groups, this provides a new proof that is topological in nature of the sharp 1/2 gap [30].

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We also characterize and estimate scl of elliptic elements (Section 4.3). In the special case of an amalgamated free product $G = A \star_C B$, the unit norm ball of scl_G on C has a geometric description in terms of unit norm balls of scl_A and scl_B on C; see Theorem 4.35.

1.2   Organization

We review basic definitions and properties of scl and introduce relative scl in Chapter 2. We analyze surfaces in graphs of groups in Chapter 3 to obtain (simple) normal forms. Along the way we prove two isometric embedding theorems. In Chapter 4, we introduce a duality method to give sharp lower bounds of scl in graphs of groups for hyperbolic elements, and apply the result to obtain sharp spectral gap results for graph products. We also characterize scl of elliptic elements in Section 4.3. Finally in Chapter 5, we give an algorithm to compute scl and prove rationality for graphs of groups as in Theorem 5.25. We give explicit examples and formulas in the case of Baumslag–Solitar groups in Section 5.3.

Most results presented here are originally obtained in my published work [20] or in my joint work with Nicolaus Heuer [21]. Many figures are reused from these two papers.

CHAPTER 2

BACKGROUND

In this chapter we first recall some basic definitions and properties of stable commutator length. Then we introduce the notion of relative stable commutator length, which is helpful for our later use.

2.1 Stable Commutator Length

We give three equivalent definitions of stable commutator length and discuss some useful basic properties. A nice general reference is [11, Chap. 2].

2.1.1 Group-theoretic Definition

For any group G, any element g in the commutator subgroup $[G, G]$ can be written as $g = [a_1, b_1] \cdots [a_k, b_k]$. The smallest possible k is called the *commutator length* of g, denoted $\mathrm{cl}_G(g)$. The sequence $\{\mathrm{cl}_G(g^n)\}_n$ is subadditive and thus the limit in the following definition exists and equals the infimum.

Definition 2.1. The *stable commutator length* of $g \in [G, G]$ is defined to be

$$\mathrm{scl}_G(g) := \lim_{n \to \infty} \frac{\mathrm{cl}_G(g^n)}{n} = \inf_n \frac{\mathrm{cl}_G(g^n)}{n}.$$

As functions on $[G, G]$, both cl_G and scl_G are *characteristic* in the sense that, for any isomorphism $\phi : G \to H$ we have $\mathrm{scl}_G(g) = \mathrm{scl}_H(\phi(g))$ for all $g \in [G, G]$. In particular, $\mathrm{scl}_G(g)$ only depends on the conjugacy class of g.

2.1.2   Topological Definition

Let X be a topological space with fundamental group G. An expression $g = [a_1, b_1] \cdots [a_k, b_k]$ in G can be represented by a continuous map $f : S \to X$ from a compact connected oriented surface S of genus k with one boundary component to X such that ∂S is sent to a loop γ representing the conjugacy class of g.

Thus one should consider surfaces in X with boundary representing powers of g when computing $\mathrm{scl}_G(g)$. This leads to the definition of admissible surfaces, and we define them more generally for *chains*.

Recall that a real (rational, integral resp.) chain is a formal finite sum $\sum t_j g_j$ of elements $g_j \in G$ with real (rational, integral resp.) coefficients t_j. Represent each g_j by some loop $\gamma_j : S_j^1 \to X$.

Definition 2.2. An *admissible surface* (S, f) for a chain $c = \sum t_i g_i$ of degree $n = n(S, f) \in \mathbb{Z}_+$ is a map $f : S \to X$ from a compact connected oriented surface S to X such that the following diagram commutes and the homology class $\partial f_*[\partial S] = n \sum t_j [S_j^1]$,

$$
\begin{array}{ccc}
\partial S & \xrightarrow{\;i\;} & S \\
{\scriptstyle \partial f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\sqcup S_j^1 & \xrightarrow{\;\sqcup \gamma_j\;} & X
\end{array}
$$

where i is the inclusion map and ∂f is the map on the boundary.

Note that admissible surfaces are allowed to be disconnected and each component may have several boundary components. They exist if and only if the chain c is rational and null-homologous.

We measure the "complexity" of an admissible surface (S, f) of degree n by $-\chi^-(S)/2n$, where $\chi^-(S)$ is the Euler characteristic of S after removing all sphere and disk components. One obvious reason to use the negative part of Euler characteristic of S is to prevent dummy

admissible surfaces by adding lots of spheres and disks with arbitrarily negative complexity. The deeper reason is the fact that $-2\chi^-(S)$ is the simplicial volume of S; see [11, Theorem 1.14].

This complexity has the nice property that any finite cover $\tilde{S}$ of S with the induced map to X, which is also admissible, has the same complexity. This makes it convenient passing to a finite cover; see Example 2.4 below. Taking finite covers also plays a central role in the rationality problem; see Chapter 5.

Definition 2.3. For a rational chain $c = \sum t_i g_i$, its *stable commutator length* (scl) is

$$\mathrm{scl}_G(c) := \inf_{(S,f)} \frac{-\chi^-(S)}{2n(S,f)},$$

where the infimum is taken over all admissible surfaces (S, f). We make the convention that $\mathrm{scl}_G(c) = +\infty$ if c has nontrivial rational homology, i.e. it has no admissible surface.

In the case where $c = g$ is a single element in $[G, G]$, then this agrees with the algebraic Definition 2.1. The key reason is that one can take a finite cover $\tilde{S}$ of arbitrarily large degree of any admissible surface without increasing the number of connected components and boundary components. This allows one to modify $\tilde{S}$ into a connected surface $\tilde{S}'$ with one boundary component at the cost of changing $-\chi^-(\tilde{S})$ by a *bounded* amount. Then this cost is negligible once we divide $-\chi^-(\tilde{S}')$ by the its (huge) degree, and similarly for the difference $1/2$ between $-\chi^-(\tilde{S}')/2$ and the genus of $\tilde{S}'$. See [11, Proposition 2.10] for a detailed proof.

In the sequel, we will only consider admissible surfaces where each boundary component is an orientation-preserving covering map of some loop in the chain. This does not affect the computation of scl [11, Proposition 2.13].

Example 2.4. Consider a commutator $g = [a, b] \in G$. There is an obvious admissible surface S of degree one: the once-punctured torus sending the standard generators of its π_1 to a and b respectively. From the algebraic definition, this only justifies the trivial fact that

$\mathrm{cl}_G(g) \leq 1$. However, by the topological definition, we have $\mathrm{scl}_G(g) \leq 1/2$.

To get a glimpse of the equivalence of the two definitions in this case, note that for each odd positive integer n, there is a degree n cover $\tilde{S}_n$ of S that has only one boundary component. Then by counting the Euler characteristic, we see $\tilde{S}_n$ has genus $(n+1)/2$ while its boundary is sent to g^n, expressing g^n as a product of $(n+1)/2$ commutators. Concretely, when $n = 3$, this yields Culler's identity [23]: $[a,b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2]$. Hence in a limit, we observe algebraically that $\mathrm{scl}_G(g) \leq 1/2$ as well.

One advantage of using the topological definition is to avoid applying this kind of tricks over and over again.

The topological definition (or the trick above) also implies $\mathrm{scl}_G(g) \leq \mathrm{cl}_G(g^n)/n - 1/2n$. This shows that the infimum in the algebraic Definition 2.1 is never achieved. In contrast, the bound $1/2$ above given by the once-punctured torus is sharp in many cases, for instance when a, b are non-commuting elements in a non-abelian free group.

Definition 2.5. An admissible surface is *extremal* if it achieves the infimum in Definition 2.3.

Proposition 2.6. *Extremal surfaces are π_1-injective (on each component).*

Proof. We give a sketch; see [11, Proposition 2.104] for details. If the induced map $\pi_1 S \to \pi_1 X$ is not injective, then some finite cover $\tilde{S}$ of S has a simple closed loop mapping to a null-homotopic loop in X since free groups are LERF. Thus one can compress $\tilde{S}$ along such a loop to obtain a simper admissible surface, contradicting the fact that any finite cover of an extremal surface is also extremal. $\square$

This relates the study of scl to the problem of finding surface subgroups.

2.1.3 Quasimorphisms and Bavard's Duality

Denote by $C_1(G)$ the space of real chains in G. Let $H(G)$ be the subspace spanned by chains of the forms $g - hgh^{-1}$ and $g^n - n \cdot g$ with $g, h \in G$ and $n \in \mathbb{Z}$, and let $C_1^H(G) := C_1(G)/H(G)$. Then there is a well-defined linear map $h_G : C_1^H(G) \to H_1(G)$ taking chains to the homology classes they represent. Denote the kernel of h_G as $B_1^H(G) \leq C_1^H(G)$.

Note that finite-order elements can be removed from a chain without changing scl. Thus we will often assume elements in chains to have infinite order.

Using the topological definition, one can show that scl is subadditive, i.e. $\mathrm{scl}_G(c_1 + c_2) \leq \mathrm{scl}_G(c_1) + \mathrm{scl}_G(c_2)$ for any two (rational) chains, and it vanishes on $H(G)$. Thus scl_G extends uniquely in a continuous way to a well-defined semi-norm on $B_1^H(G)$; See [11, Sect. 2.7].

Christophe Bavard found a concrete description of the dual space as homogeneous quasi-morphisms [3], which turns out to be closely related to the second bounded cohomology.

Definition 2.7. A map $\varphi : G \to \mathbb{R}$ is a *quasimorphism* if

$$D(\varphi) := \sup_{a,b} |\varphi(a) + \varphi(b) - \varphi(ab)| < +\infty.$$

The quantity $D(\varphi)$ is called the *defect* of φ. If in addition $\varphi(g^n) = n\varphi(g)$ for all $g \in G$ and $n \in \mathbb{Z}$, we say φ is *homogeneous*.

Denote the space of homogeneous quasimorphisms by $\mathcal{Q}(G)$. The defect is a semi-norm on it and vanishes exactly on the space of homomorphisms to $\mathbb{R}$, i.e. the first cohomology $H^1(G)$ with $\mathbb{R}$ coefficient.

Theorem 2.8 (Bavard's duality)**.** *The space $\mathcal{Q}(G)/H^1(G)$ equipped with $2D$ is the dual space of $(C_1^H(G), \mathrm{scl}_G)$, where D the defect norm. Concretely, for any chain c, we have*

$$\mathrm{scl}_G(c) = \sup_{\varphi \in \mathcal{Q}(G)/H^1(G)} \frac{\varphi(c)}{2D(\varphi)}.$$

See [11, Sections 2.4 and 2.5] for detailed discussions.

Definition 2.9. We say a group G has trivial scl if scl_G vanishes on $C_1^H(G)$. By Bavard's duality, this holds if and only if $\mathcal{Q}(G) = H^1(G)$, i.e. G has no "nontrivial" quasimorphisms.

Proposition 2.10. *For a group G, the following sequence is exact*

$$0 \longrightarrow H^1(G) \longrightarrow \mathcal{Q}(G) \longrightarrow H_b^2(G) \overset{c}{\longrightarrow} H^2(G), \tag{2.1}$$

where $H_b^2(G)$ is the second bounded cohomology and all cohomology groups are taken with $\mathbb{R}$ coefficients.

This shows that G has trivial scl if and only if the comparison map $c : H_b^2(G) \to H^2(G)$ is injective. See [11, Theorem 2.50] for a proof.

Theorem 2.11. *The following groups have trivial scl:*

(1) *Amenable groups [11, Theorem 2.47];*

(2) *Irreducible lattices in higher rank semisimple Lie groups [6, 7] [11, Theorem 5.26]; and*

(3) *Some transformation groups like* $\mathrm{Homeo}^+(S^1)$ *[28, Proposition 5.11] and subgroups of* $\mathrm{PL}^+([0, 1])$ *[8].*

With the help of Bavard's duality, quasimorphisms can be used to show lower bounds of scl. There are quasimorphisms seemingly of very different nature, including Brook's quasimorphisms and its generalizations to groups acting on trees, de Rham quasimorphisms, and the rotation quasimorphism. See [11, Sect. 2.3] for details.

2.1.4 Basic Properties and Isometric Embeddings

Now we define isometric embeddings and collect some basic properties of scl.

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Lemma 2.12. *Let G be a group.*

(1) *(Monotonicity) We have $\mathrm{scl}_G(c) \geq \mathrm{scl}_H(\phi(c))$ for any homomorphism $\phi : G \to H$ and any chain c;*

(2) *If $G = \oplus G_\lambda$, then $\mathrm{scl}_G(g) = \max_\lambda \mathrm{scl}_{G_\lambda}(g_\lambda)$ for any $g \in G$ with factors $g_\lambda \in G_\lambda$.*

Proof. The monotonicity is obvious for rational chains by pushing forward admissible surfaces. By continuity, it also holds for real chains. When G is a direct product, expressing different factors g_λ as products of commutators do not interfere each other, thus the number of commutators needed is given by the element with the largest (stable) commutator length. $\qquad\square$

The monotonicity makes scl a nice obstruction for certain homomorphisms; See Section 2.3 for a detailed discussion. It also forces equality of scl in many situations.

Definition 2.13. A homomorphism $\phi : G \to H$ is an *isometric embedding* if ϕ is *injective* and $\mathrm{scl}_G(c) = \mathrm{scl}_H(\phi(c))$ for all chains $c \in B_1^H(G)$.

Example 2.14. In the following cases, the injection $\phi : G \to H$ is an isometric embedding:

(1) The map ϕ admits a retract $r : H \to G$, i.e. $r \circ \phi = id_G$. This follows from monotonicity.

(2) The group G is abelian. This is trivially true since $B_1^H(G)$ is trivial.

(3) The group G has trivial scl. This also follows from monotonicity.

Isometric embeddings allow one to pull back admissible surfaces at arbitrarily small cost.

Lemma 2.15. *Let $\phi : G \to H$ be an isometric embedding, realized by a map $\phi : X_G \to X_H$ between $K(G, 1)$ and $K(H, 1)$ spaces. Suppose $f' : S' \to X_H$ is a surface without sphere components in X_H such that $\partial f' \partial S' = \phi(\gamma)$ for a collection of loops γ in X_G whose sum is a null-homologous chain c. Then for any $\epsilon > 0$, there is a surface $f : S \to X_G$ satisfying the following properties:*

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(1) S has no sphere components;

(2) There is a (disconnected) covering map $\pi : \partial S \to \partial S'$ of a certain degree $n > 0$ such that the following diagram commutes;

$$
\begin{array}{ccc}
\partial S & \xrightarrow{\ \pi\ } & \partial S' \\
\partial f \downarrow & & \partial f' \downarrow \\
X_G & \xrightarrow{\ \phi\ } & X_H
\end{array}
$$

(3) We have

$$\frac{-\chi(S)}{n} \leq -\chi(S') + \epsilon.$$

Proof. Let D' be the union of disk components in S'. Since ϕ is an isometric embedding and thus π_1-injective, the loops in $\underline{\gamma}$ corresponding to $\partial D'$ bound a collection of disks D in X_G accordingly. Let S_0' be the remaining components of S'. Then $\partial S_0'$ represents a null-homologous chain c_0 equivalent to c in $B_1^H(G)$ and $\chi(S_0') = \chi^-(S_0')$. Since ϕ preserves scl, there is an admissible surface S_0 for c_0 of a certain degree $n > 0$ without disk or sphere components such that

$$\frac{-\chi(S_0)}{n} \leq 2\mathrm{scl}_G(c_0) + \epsilon = 2\mathrm{scl}_H(\phi(c_0)) + \epsilon \leq -\chi(S_0') + \epsilon.$$

Then the surface $S = S_0 \sqcup nD$ has the desired properties. $\qquad\square$

2.2 Relative Stable Commutator Length

Now we introduce the notion of *relative* stable commutator length. This is what comes up naturally and is convenient to use in the situation of graphs of groups. The materials of this section is mostly based on [20, Sect. 2.2] and [21, Sect. 2.2].

Definition 2.16 (Relative scl). Let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of subgroups of G. Denote by $C_1(\mathcal{G})$ the subspace of $C_1(G)$ consisting of chains of the form $\sum_\lambda c_\lambda$ with $c_\lambda \in C_1(G_\lambda)$, where all but finitely many c_λ vanish in each summation.

For any chain $c \in C_1(G)$, define its *relative stable commutator length* to be

$$\mathrm{scl}_{(G,\mathcal{G})}(c) := \inf\{\mathrm{scl}_G(c + c') : c' \in C_1(\mathcal{G})\}.$$

Let $H_1(\mathcal{G}) \le H_1(G; \mathbb{R})$ be the subspace of homology classes represented by chains in $C_1(\mathcal{G})$. Recall that we have a linear map $h_G : C_1^H(G) \to H_1(G)$ taking chains to their homology classes. Denote $B_1^H(G, \mathcal{G}) := h_G^{-1} H_1(\mathcal{G})$, which contains $B_1^H(G)$ as a subspace. Then $\mathrm{scl}_{(G,\mathcal{G})}$ is finite on $B_1^H(G, \mathcal{G})$ and is a pseudo-norm.

The following properties of relative scl are basic.

Lemma 2.17. *Let G be a group and $\{G_\lambda\}$ be a collection of subgroups.*

(1) $\mathrm{scl}_G(c) \ge \mathrm{scl}_{(G,\{G_\lambda\})}(c)$.

(2) *If g^n conjugates into some G_λ for some integer $n \ne 0$, then $\mathrm{scl}_{(G,\{G_\lambda\})}(g) = 0$.*

(3) *(Stability) For any $g \in G$, we have $\mathrm{scl}_{(G,\{G_\lambda\})}(g^n) = n \cdot \mathrm{scl}_{(G,\{G_\lambda\})}(g)$.*

(4) *(Monotonicity) Let $\phi : G \to H$ be a homomorphism such that $\phi(G_\lambda) \subset H_\lambda$ for a collection of subgroups H_λ of H, then for any $c \in C_1(G)$ we have*

$$\mathrm{scl}_{(G,\{G_\lambda\})}(c) \ge \mathrm{scl}_{(H,\{H_\lambda\})}(\phi(c)).$$

Proof. Bullets (3) and (4) follow immediately from the corresponding properties of scl in the absolute sense. Bullet (1) is by definition. Bullet (2) is immediate from the definition when $n = 1$ and the general case follows from bullet (3). $\qquad\square$

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For a rational chain, its relative scl can be described using *relative admissible surfaces*. This will be a main tool that we use later to compute or estimate scl in graphs of groups.

Definition 2.18. Let $c \in B_1^H(G, \{G_\lambda\})$ be a rational chain. A surface S together with a specified collection of boundary components $\partial_0 \subset \partial S$ is called *relative admissible* for c of degree $n > 0$ if ∂_0 represents $[nc] \in C_1^H(G)$ and every other boundary component of S represents an element conjugate into some G_λ.

Lemma 2.19. *For any rational chain $c \in B_1^H(G, \{G_\lambda\})$, we have*

$$\mathrm{scl}_{(G, \{G_\lambda\})}(c) = \inf \frac{-\chi^-(S)}{2n}, \tag{2.2}$$

where the infimum is taken over all relative admissible surfaces for c.

Proof. On the one hand, any relative admissible surface S for c of degree n is admissible for a chain $c + \sum c_\lambda$ of degree n, where $n \sum c_\lambda$ is the chain represented by the boundary components of S outside of the specified components ∂_0. This proves the "$\leq$" direction.

On the other hand, first consider a chain $c + \sum c_\lambda$ with all c_λ rational. Any admissible surface S for $c + \sum c_\lambda$ of degree n is a relative admissible surface for c of degree n by taking the boundary components representing nc to be the specified components ∂_0. Thus for such a rational chain, $\mathrm{scl}(c + \sum c_\lambda)$ is no less than the right-hand side of the desired equality. Then the "$\geq$" direction follows by continuity. □

A relative admissible surface is called *extremal* if it obtains the infimum in Lemma 2.19. Such surfaces are also π_1-injective, analogous to Proposition 2.6 and proved in the same way.

Proposition 2.20. *Any extremal relative admissible surface $f : S \to X$ induces an injective map on the fundamental groups for each component of S.*

There is also an analog of Bavard's duality for relative scl.

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Lemma 2.21 (Relative Bavard's Duality). *For any chain $c \in B_1^H(G, \{G_\lambda\})$, we have*

$$\text{scl}_{(G,\{G_\lambda\})}(c) = \sup \frac{f(c)}{2D(f)},$$

where the supremum is taken over all homogeneous quasimorphisms f on G that vanish on $C_1(\{G_\lambda\})$.

Proof. Denote the space of homogeneous quasimorphisms on G by $\mathcal{Q}(G)$. Let $N(G)$ be the subspace of $B_1(G)$ where scl vanishes. Then the quotient $B_1(G)/N(G)$ with induced scl becomes a normed vector space. Denote the quotient map by $\pi : B_1(G) \to B_1(G)/N(G)$. Bavard's duality shows that the dual space of $B_1(G)/N(G)$ is exactly $\mathcal{Q}(G)/H^1(G)$ equipped with the norm $2D(\cdot)$ (Theorem 2.8). Then scl further induces a norm $\|\cdot\|$ on the quotient space V of $B_1(G)/N(G)$ by the closure of $\pi(C_1(\{G_\lambda\}) \cap B_1(G))$. By definition we have $\|\bar{c}\| = \text{scl}_{(G,\{G_\lambda\})}(c)$ for any $c \in B_1(G)$, where $\bar{c}$ is the image in V. It is well known that the dual space of V is naturally isomorphic to the subspace of $\mathcal{Q}(G)/H^1(G)$ consisting of linear functionals that vanish on $C_1(\{G_\lambda\}) \cap B_1(G)$. Any $\bar{f} \in \mathcal{Q}(G)/H^1(G)$ with this vanishing property can be represented by some $f \in \mathcal{Q}(G)$ that vanishes on $C_1(\{G_\lambda\})$. This proves the assertion assuming $c \in B_1^H(G)$. The general case easily follows since any $c \in B_1^H(G, \{G_\lambda\})$ can be replaced by $c + c' \in B_1^H(G)$ for some $c' \in C_1(\{G_\lambda\})$ without changing both sides of the equation. $\qquad\square$

2.3   The scl Spectrum and Spectral Gaps

Lots of studies have been attracted to understand the *scl spectrum*.

Definition 2.22. For a group G, its scl *spectrum* is the image of scl_G (as a function on G). We say G has a *spectral gap* $C > 0$ if for any $g \in G$ either $\text{scl}_G(g) \geq C$ or $\text{scl}_G(g) = 0$. If in addition, the case $\text{scl}_G(g) = 0$ only occurs when g is torsion, we say G has a *strong spectral gap* C.

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On the one hand, among the known examples, the spectrum often appears to be a subset of $\mathbb{Q}_{\geq 0}$ for finitely presented groups except a few cases related to dynamics (see [45] and [11, Chapter 5]. Such rationality of scl is necessary for the existence of extremal surfaces, which are π_1-injective (see Proposition 2.6). This has led to good positive partial answers to Gromov's question about surface subgroups in hyperbolic groups [10, 17, 44].

Here is a list of groups where scl is previously shown to be piecewise rational linear.

Example 2.23.

(1) Free groups, by Calegari [12].

(2) Free products of cyclic groups, by Walker [43].

(3) Free products of free abelian groups, by Calegari [13].

(4) Free products $*_\lambda G_\lambda$ with $\mathrm{scl}_{G_\lambda} \equiv 0$ for all λ, by the author [18].

(5) Amalgams of free abelian groups, by Susse [39].

(6) t-alternating words in Baumslag–Solitar groups, by Clay–Forester–Louwsma [22].

Theorem 5.25 is a generalization of all the rationality results above. Corollary 3.14 provides an easier way to understand and compute scl of t-alternating words in Baumslag–Solitar groups.

On the other hand, many classes of groups are known to have a spectral gap.

Example 2.24.

(1) G trivially has a spectral gap C for any $C > 0$ if G has trivial scl (see Theorem 2.11 for a list of examples); the gap is strong if G is abelian;

(2) G has a strong spectral gap $1/2$ if G is residually free [24];

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(3) δ-hyperbolic groups have gaps depending on the number of generators and δ [14]; the gap is strong if the group is also torsion-free;

(4) Any finite index subgroup of the mapping class group of a (possibly punctured) closed surface has a spectral gap [4];

(5) All right-angled Artin groups have a strong gap $1/2$ [30] (see [26, 27] for earlier weaker estimates);

(6) All Baumslag–Solitar groups have a gap $1/12$ [22].

(7) The fundamental group of any closed oriented closed 3-manifold has a gap depending on the manifold [21, Theorem C].

Theorem 4.24 generalizes the gap for right-angled Artin groups to graph products and provides a topological proof.

The spectral gap phenomenon together with monotonicity provides obstructions for the existence of certain homomorphisms. For example the following rigidity theorem has a conceptually simple proof using (more precise versions of) spectral gap results.

Theorem 2.25 (Farb–Kaimanovich–Masur [25, 35]). *For any irreducible lattice G of higher rank and any closed surface S, any homomorphism $\phi : G \to \mathrm{MCG}(S)$ has finite image.*

Proof. By [4, Theorem B], there is a subgroup $H \leq \mathrm{MCG}(S)$ of finite index such that any $h \neq id \in H$ has $\mathrm{scl}_H(h) > 0$. For any homomorphism $\phi : G \to \mathrm{MCG}(S)$, up to replacing G by a finite index subgroup $\phi^{-1}(G)$, we may assume $\mathrm{Im}\phi \subset H$. Since $\mathrm{scl}_G(g) = 0$ for any g by Theorem 2.11 and the fact that $H_1(G) = 0$, monotonicity of scl forces $\phi(g) \equiv id_H$ for all g. $\qquad\square$

The gap property is essentially preserved under taking free products.

Lemma 2.26 (Clay–Forester–Louwsma)**.** *Let $G = \star_\lambda G_\lambda$ be a free product. Then for any $g \in G$ not conjugate into any factor, we have either $\mathrm{scl}_G(g) = 0$ or $\mathrm{scl}_G(g) \geq 1/12$. Moreover, $\mathrm{scl}_G(g) = 0$ if and only if g is conjugate to g^{-1}. Thus if the groups G_λ have a uniform spectral gap $C > 0$, then G has a gap $\min\{C, 1/12\}$.*

Proof. If g does not conjugate into any factor, then it either satisfies the so-called well-aligned condition in [22] or is conjugate to its inverse. In this case, it follows from [22, Theorem 6.9] that either $\mathrm{scl}_G(g) \geq 1/12$ or $\mathrm{scl}_G(g) = 0$, corresponding to the two situations. Assuming the factors have a uniform gap C, if an element $g \in G$ conjugates into some factor G_λ, then $\mathrm{scl}_G(g) = \mathrm{scl}_{G_\lambda}(g) \geq C$ since G_λ is a retract of G. $\square$

The constant $1/12$ is optimal in general, but can be improved if there is no torsion of small order. See [19] or [34].

Many other groups have a uniform positive lower bound on *most* elements. They often satisfy a spectral gap in a relative sense.

Definition 2.27. For a collection of subgroups $\{G_\lambda\}$ of G and a positive number C, we say $(G, \{G_\lambda\})$ has a *strong relative spectral gap C* if either $\mathrm{scl}_{(G,\{G_\lambda\})}(g) \geq C$ or $\mathrm{scl}_{(G,\{G_\lambda\})}(g) = 0$ for all $g \in G$, where the latter case occurs if and only if g^n conjugates into some G_λ for some $n \neq 0$.

Some previous work on spectral gap properties of scl can be stated in terms of or strengthened to strong relative spectral gap.

Theorem 2.28.

(1) *[19, Theorem 3.1] Let $n \geq 3$ and let $G = \star_\lambda G_\lambda$ be a free product where G_λ has no k-torsion for all $k < n$. Then $(G, \{G_\lambda\})$ has a strong relative spectral gap $\frac{1}{2} - \frac{1}{n}$.*

(2) *[30, Theorem 6.3] Suppose we have inclusions of groups $C \hookrightarrow A$ and $C \hookrightarrow B$ such that both images are left relatively convex subgroups (see Definition 4.12). Let $G = A \star_C B$ be the associated amalgam. Then $(G, \{A, B\})$ has a strong relative spectral gap $\frac{1}{2}$.*

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(3) [14, Theorem A′] Let G be δ-hyperbolic with symmetric generating set S. Let a be an element with $a^n \neq ba^{-n}b^{-1}$ for all $n \neq 0$ and all $b \in G$. Let $\{a_i\}$ be a collection of elements with translation lengths bounded by T. Suppose a^n does not conjugate into any $G_i := \langle a_i \rangle$ for any $n \neq 0$, then there is $C = C(\delta, |S|, T) > 0$ such that $\mathrm{scl}_{(G, \{G_i\})}(a) \geq C$.

(4) [9, Theorem C] Let M be a compact 3-manifold with tori boundary (possibly empty). Suppose the interior of M is hyperbolic with finite volume. Then $(\pi_1 M, \pi_1 \partial M)$ has a strong relative spectral gap $C(M) > 0$, where $\pi_1 \partial M$ is the collection of peripheral subgroups.

Proof. Our Theorem 4.8 immediately implies (1) and (2). Part (3) is an equivalent statement of the original theorem [14, Theorem A′] in view of Lemma 2.21.

Part (4) is stated stronger than the original form [9, Theorem C]. See [21, Theorem 8.10] for a proof of this stronger version. ☐

We will prove spectral gaps for certain graphs of groups relative to the vertex groups; see Theorem 4.7.

CHAPTER 3

SURFACES IN GRAPHS OF GROUPS

The goal of this chapter is to analyze surfaces in graphs of groups and develop their normal forms. This lays the foundation for the next two chapters, where we compute and give good estimates of scl in graphs of groups.

We will first recall the basic definitions of graphs of groups and setup notation in Section 3.1. Then in Section 3.2, we develop the normal form of (admissible) surfaces in graphs of groups. Finally in Section 3.3, we make simplifications to obtain the simple normal form of (admissible) surfaces.

Along the way, we will use the (simple) normal form to prove two isometric embedding theorems (Theorems 3.6 and 3.10), which reduce the computations of scl in complicated groups to those in simper ones.

The main bulk of this section is largely based on [20, Sect. 3].

3.1   Graphs of Groups

Throughout this chapter, we consider graphs $\Gamma = (V, E)$ with vertex and edge sets V and E. All edges are oriented and we always include in E both orientations of each edge. So we have an involution $e \mapsto \bar{e}$ on E without fixed point by reversing edge orientations, i.e. e and $\bar{e}$ represent the same edge but with opposite orientations. We will use $\{e, \bar{e}\}$ to denote an edge without preferred orientation. We also have maps $o, t : E \to V$ taking origin and terminus vertices of edges respectively, such that $t(e) = o(\bar{e})$.

3.1.1   Standard Realization

Suppose we have a graph $\Gamma = (V, E)$ and two collections of groups $\{G_v\}_{v \in V}$ and $\{G_e\}_{e \in E}$ indexed by vertices and edges such that $G_e = G_{\bar{e}}$. Suppose we also have injections $o_e : G_e \to$

$G_{o(e)}$ and $t_e : G_e \to G_{t(e)}$ for each oriented edge e satisfying $o_{\bar{e}} = t_e$. Let X_v and $X_e = X_{\bar{e}}$ be $K(G_v, 1)$ and $K(G_e, 1)$ spaces with base points b_v and $b_e = b_{\bar{e}}$ respectively. For each edge e, the injections o_e and t_e determine (up to homotopy) maps $o_e : (X_e, b_e) \to (X_{o(e)}, b_{o(e)})$ and $t_e : (X_e, b_e) \to (X_{t(e)}, b_{t(e)})$ respectively.

Each map t_e determines a mapping cylinder with X_e on top and $X_{t(e)}$ on bottom. For each vertex v, take the disjoint union of all such mapping cylinders with $t(e) = v$ and identify the bottom spaces $X_{t(e)}$. Refer to the resulting space $N(X_v)$ as the *thickened vertex space*. Now take the disjoint union of all $N(X_v)$ and identify the top spaces via $X_e = X_{\bar{e}}$. Denote the resulting space by X.

Explicitly, X is the space obtained from the disjoint union of $\sqcup_{e \in E} X_e \times [-1, 1]$ and $\sqcup_{v \in V} X_v$ by gluing $X_e \times \{1\}$ to $X_{t(e)}$ via t_e and identifying $X_e \times \{s\}$ with $X_{\bar{e}} \times \{-s\}$ for all $s \in [-1, 1]$ and $e \in E$.

We call X the *graph of spaces* associated to the given data. Identify X_v with its image in X, referred to as the *vertex space*. Similarly identify X_e with the image of $X_e \times \{0\}$, called the *edge space*.

When Γ is connected, we call $G = \pi_1(X)$ the (fundamental group of) *graph of groups* and X the *standard realization* of G. We use the notation $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ to specify the dependence on the underlying data. It is a fact that G_v and G_e sit inside G as subgroups via the inclusions [38], referred to as *vertex groups* and *edge groups*.

Example 3.1.

(1) Let Γ be the graph with a single vertex v and an edge $\{e, \bar{e}\}$ connecting v to itself. Let $G_e \cong G_v \cong \mathbb{Z}$. Fix nonzero integers m, ℓ, and let the edge inclusions $o_e, t_e : G_e \hookrightarrow G_v$ be given by $o_e(1) = m$ and $t_e(1) = \ell$. Let the edge space X_e and vertex space X_v be circles S_e^1 and S_v^1 respectively. Then the standard realization X is obtained by gluing the two boundary components of a cylinder $S_e^1 \times [-1, 1]$ to the circle S_v^1 wrapping around m and ℓ times respectively. See the left of Figure 3.1. The fundamental group
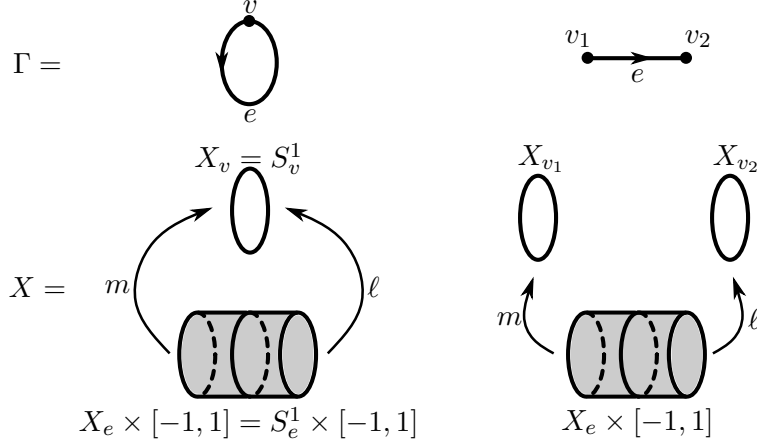
Figure 3.1: On the left we have the underlying graph Γ and the standard realization X of $\mathrm{BS}(m, \ell)$; on the right we have the graph Γ and the realization X for the amalgam $\mathbb{Z} \star_{\mathbb{Z}} \mathbb{Z}$ associated to $\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$. This originally appears in [21, Figure 1].

is the *Baumslag–Solitar group* $\mathrm{BS}(m, \ell)$, which has presentation

$$\mathrm{BS}(m, \ell) = \langle a, t \mid a^m = t a^\ell t^{-1} \rangle.$$

In general, with the same graph Γ, for any groups $G_e = C$ and $G_v = A$ together with two inclusions $t_e, o_e : C \hookrightarrow A$, the corresponding graph of groups is the *HNN extension* $G = A_{\star C}$.

(2) Similarly, if we let Γ be the graph with a single edge $\{e, \bar{e}\}$ connecting two vertices $v_1 = o(e)$ and $v_2 = t(e)$, the graph of groups associated to two inclusions $t_e : G_e \to G_{v_1}$ and $o_e : G_e \to G_{v_2}$ is the *amalgam* $G_{v_1} \star_{G_e} G_{v_2}$. See the right of Figure 3.1 for an example where all edge and vertex groups are $\mathbb{Z}$.

In general, each connected subgraph of Γ gives a graph of groups, whose fundamental group *injects* into G, from which we see that each separating edge of Γ splits G as an amalgam and each non-separating edge splits G as an HNN extension. Hence G arises as a sequence of amalgamations and HNN extensions.

It is a fundamental result of the Bass–Serre theory that there is a correspondence between groups acting on trees (without inversions) and graphs of groups, where vertex and edge

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stabilizers correspond to vertex and edge groups respectively. See [38] for more details about graphs of groups and their relation to groups acting on trees.

3.1.2   Elliptic and Hyperbolic Elements

Let X be the standard realization of a graph of groups G as in the previous subsection.

We say an element $g \in G$ and its conjugacy class are *elliptic* if g conjugates into some vertex group, otherwise they are *hyperbolic*. Geometrically, g is elliptic if and only if it is represented by a loop supported in some vertex space.

Let γ be a loop in X representing the conjugacy class of an element $g \in G$. We can homotope γ so that it is either disjoint from $X_e \times \{t\}$ or intersects it only at $b_e \times \{t\}$ transversely, for all $t \in (-1, 1)$ and any edge e. Then the edge spaces cut γ into finitely many arcs, unless γ is supported in a vertex space.

Each arc a is supported in some thickened vertex space $N(X_v)$ and decomposes into three parts (see Figure 3.2): an arc parameterizing $b_e \times [0, 1]$, a based loop in X_v, and an arc parameterizing $b_{e'} \times [-1, 0]$, where $t(e) = v = o(e')$. We refer to the element $w(a) \in G_v$ represented by the based loop as *the winding number* of a, and denote e, e' by $e_{in}(a), e_{out}(a)$ respectively. We say γ *trivially backtracks* if for some arc a as above $\overline{e_{out}(a)} = e_{in}(a)$ and $w(\alpha)$ lies in $t_e(G_e)$. In this case, γ can be simplified by a homotopy reducing the number of arcs. See Figure 3.2. After finitely many simplifications, the loop γ does not trivially backtrack, which we call a *tight* loop.

In the case where g is elliptic, γ is tight if and only if it is supported in some vertex space. Moreover, instead of a collection of arcs, we have a single loop γ, whose winding number $w(\gamma)$ is only well-defined up to conjugacy.
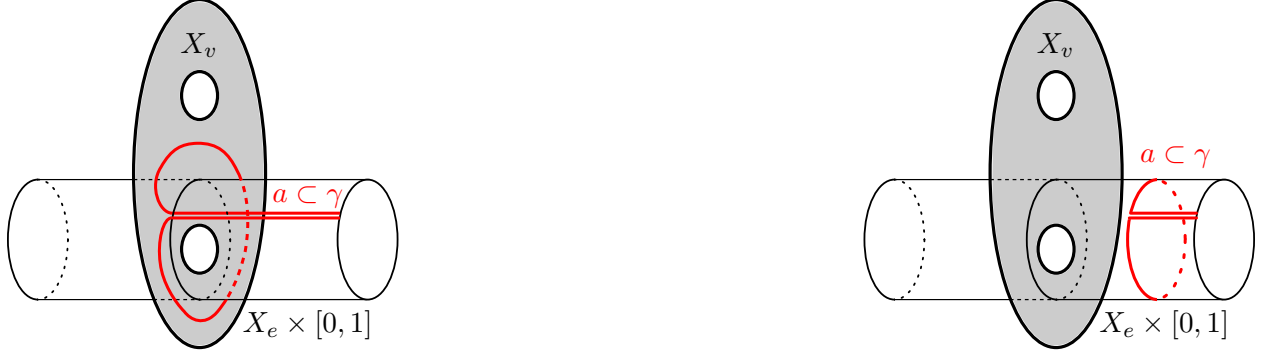
Figure 3.2: A loop γ trivially backtracks at an arc a supported in the thickened vertex space $N(X_v)$ shown on the left. It can be pushed off the vertex space X_v by a homotopy shown on the right. This appears originally in [20, Fig. 1].

3.1.3   Scl Relative to Vertex Groups

In a graph of groups G with vertex groups $\{G_v\}$, the naive lower bound $\mathrm{scl}_G(c) \geq \mathrm{scl}_{(G,\{G_v\})}(c)$ allows us to estimate scl using relative scl. If scl vanishes on the vertex groups, this estimate is actually accurate.

Proposition 3.2. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a a graph of groups with underlying graph $\Gamma = (V, E)$. Suppose $\mathrm{scl}_G(c) = 0$ for any $c \in B_1^H(G_v)$ and vertex v. Then for any null-homologous chain $c \in B_1^H(G)$, we have*

$$\mathrm{scl}_{(G,\{G_v\})}(c) = \mathrm{scl}_G(c).$$

Proof. The standard realization X of G can be written as the union of thickened vertex spaces and regular neighborhoods of all edge spaces. Then it follows from the Mayer–Vietoris exact sequence that

$$
\begin{aligned}
H_1(G;\mathbb{R}) \;\cong\;& H_1(\Gamma;\mathbb{R}) \\
&\oplus\; \left(\bigoplus_v H_1(G_v;\mathbb{R}) \right) \Big/ \langle o_{e*}(c_e) - t_{e*}(c_e) : c_e \in H_1(G_e;\mathbb{R}) \rangle . \qquad (3.1)
\end{aligned}
$$

Consider any null-homologous chain $c + \sum c_i$ where $c_i \in C_1(G_{v_i})$ for some vertex v_i. It

follows that $\sum c_i$ is null-homologous since c is. Thus by the homology calculation (3.1), there exist chains $c_e = -c_{\bar{e}} \in C_1(G_e)$ such that, for each vertex v, the chain

$$c_v := \sum_{i:v_i=v} c_i + \sum_{e:t(e)=v} t_e(c_e) \quad \in C_1(G_v)$$

is null-homologous in $H_1(G_v; \mathbb{R})$. Clearly, for each e the chains $t_{\bar{e}}(c_{\bar{e}}) = o_e(-c_e)$ and $-t_e(c_e)$ are equivalent to each other as they are geometrically homotopic. Hence $c + \sum c_i = c + \sum_{v \in V} c_v \in C_1^H(G)$, and

$$\mathrm{scl}_G\left(c + \sum c_i\right) = \mathrm{scl}_G\left(c + \sum c_v\right) = \mathrm{scl}_G(c),$$

where the last equality holds since $\mathrm{scl}_G(c_v) = 0$ for all v by assumption. Then the result follows from the definition of relative scl. $\quad\square$

In the sequel, all relative admissible surfaces will be understood to be relative to vertex groups unless stated otherwise.

3.2   Normal Form

Throughout this section, let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups and let X be its standard realization as in Section 3.1.1. Let $\underline{g} = \{g_1, \ldots, g_m\}$ be a set of infinite order elements in G represented by tight loops $\underline{\gamma} = \{\gamma_1, \ldots, \gamma_m\}$ in X. The goal is to analyze surfaces admissible for rational chains supported on $\underline{\gamma}$.

Recall that edge spaces cut hyperbolic tight loops into arcs, while each elliptic tight loop is already supported in some vertex space. For each vertex v, denote by A_v the collection of arcs supported in $N(X_v)$ obtained by cutting hyperbolic loops in $\underline{\gamma}$. Arcs from different loops or different parts of the same loop are considered as distinct elements. Let $L_v \subset \underline{\gamma}$ be the set of elliptic loops supported in X_v.

For each $\alpha \in A_v \cup L_v$, let $i(\alpha)$ be the index such that α sits on $\gamma_{i(\alpha)}$. Denote by $[w(\alpha)]$ the homology class of the winding number $w(\alpha)$ in $H_1(G_v; \mathbb{R})$.

Lemma 3.3. *Let $c = \sum_i r_i g_i$ be a null-homologous chain. Then there exists $\beta_e \in H_1(G_e; \mathbb{R})$ for each edge e such that $\beta_e = -\beta_{\bar{e}}$ and*

$$\sum_{\alpha \in A_v \cup L_v} r_{i(\alpha)}[w(\alpha)] = \sum_{t(e)=v} t_{e*}\beta_e$$

holds for each vertex v.

Proof. This directly follows from equation (3.1). $\qquad\qquad\square$

3.2.1   Cut into Pieces

With the above setup, fix a rational homologically trivial chain $c = \sum_i r_i g_i$ with $r_i \in \mathbb{Q}$. Let $f : S \to X$ be an arbitrary admissible surface for c without sphere components. Put S in general position so that it is transverse to all edge spaces. Then $F = f^{-1}(\cup_e X_e)$ is a proper submanifold of codimension 1, that is, a union of embedded loops and proper arcs. Eliminate all trivial loops in F (innermost first) by homotopy and then compress S along a loop in F whose image is trivial in X if any. Since this process decreases $-\chi(S)$, all loops in F are non-trivial in X after finitely many repetitions. All proper arcs in F are essential since loops in $\underline{\gamma}$ are tight.

Now cut S along F into surfaces with corners, each component mapped into $N(X_v)$ for some vertex v. Let S_v be the union of components mapped into $N(X_v)$. The boundary components of S_v fall into two types (See Figure 3.3).

(1) Polygonal boundary: these are the boundary components divided by corners of S_v into segments alternating between proper arcs in F and arcs in A_v.
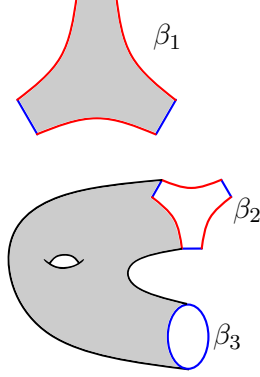
Figure 3.3: An example of S_v consisting of two components and three boundary components. On the boundary, the blue parts are supported in edge spaces and the red in the interior of $N(X_v)$. The boundaries β_1 and β_2 are polygonal, with arcs curly and turns straight. The boundary β_3 is a loop boundary. This appears originally in [20, Fig. 2].

(2) Loop boundary: these are the components disjoint from corners of S_v, and thus each is a loop in either F or L_v.

Note that any disk component in S_v must bound a polygonal boundary since loop boundaries are non-trivial in X by the simplification process above and the assumption that $\underline{g}$ consists of infinite order elements.

Let α be any proper arc in F that appears on a polygonal boundary of S_v. If α with the orientation induced from S_v starts from an arc $a_v \in A_v$ and ends in $a'_v \in A_v$, we say α is a *turn* from a_v to a'_v. With the induced orientation, α represents an element $w \in G_e$, called the *winding number* of this turn, where $e = e_{out}(a_v) = \overline{e_{in}(a'_v)}$. The triple (a_v, w, a'_v) determines the *type* of the turn α. There are possibly many turns in S_v of the same type.

Suppose $u = t(e)$ is the other end of the edge e above. Then α, viewed from S_u, gives rise to a turn in S_u from a_u to a'_u, where $a_u, a'_u \in A_u$ are arcs such that a_v is followed by a'_u and a_u is followed by a'_v on $\underline{\gamma}$. See Figure 3.4. Since the two sides induce opposite orientations on α, the winding numbers of the two turns are inverses. We refer to such two types of turns (a_v, w, a'_v) and (a_u, w^{-1}, a'_u) as *paired turns*.

Similar analysis can be done for loops in F except that they represent conjugacy classes in G_e instead of elements.
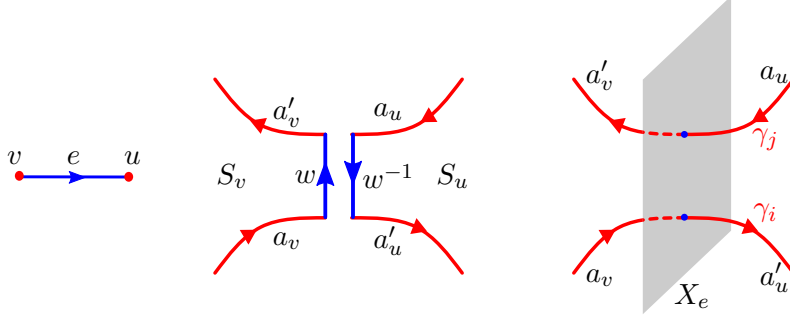
Figure 3.4: Two paired turns on an edge space X_e, with arcs $a_v, a'_u \subset \gamma_i$ and $a'_v, a_u \subset \gamma_j$, $e = e_{out}(a_v)$. This appears originally in [20, Fig. 3].

Then the collection of boundaries $\{\partial S_v\}$ satisfies the following

Gluing condition: Turns of paired types have equal numbers of instances. (3.2)

Moreover, the loop boundaries together represent a trivial chain in $B_1^H(G)$.

Definition 3.4. Such a decomposition into subsurfaces S_v, one for each vertex v, is called the *normal form* of S.

Recall the orbifold Euler characteristic of a surface Σ with corners is

$$\chi_o(\Sigma) := \chi(\Sigma) - \frac{1}{4}\#\text{corners}.$$

Then we obtain a surface S' admissible for a chain equivalent to c in $B_1^H(G)$ of the same degree as S by gluing up turns of paired types arbitrarily on polygonal boundaries in the normal form of S, and

$$-\chi(S') = -\sum \chi_o(S_v) \leq -\chi(S).$$

We summarize the discussion above as the following lemma.

Lemma 3.5. *Every admissible surface S can be decomposed into the normal form after throwing away sphere components, compressions, homotopy and cutting along edge spaces.*

By gluing up paired turns arbitrarily on polygonal boundaries in the normal form of S, we obtain a surface S' relative admissible for c of the same degree as S and

$$-\chi(S') = -\sum_v \chi_o(S_v) \leq -\chi(S).$$

3.2.2 The First Isometric Embedding Theorem

The following theorem pieces isometric embeddings of vertex groups into a global one.

Theorem 3.6 (the first isometric embedding). *Let $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ and $\mathcal{G}(\Gamma', \{G'_v\}, \{G'_e\})$ be graphs of groups. Suppose there is a graph isomorphism $h : \Gamma \to \Gamma'$, homomorphisms $h_v : G_v \to G'_{h(v)}$ and isomorphisms $h_e : G_e \to G'_{h(e)}$ such that*

(1) each h_v is an isometric embedding (Definition 2.13);

(2) for each v, the map induced by h_v on homology is injective on H_v, where H_v is the sum of $\mathrm{Im} t_{e*}$ over all edges e with $t(e) = v$; and

(3) the following diagram commutes for all edge pairs (e, e') with $e' = h(e)$.

$$
\begin{array}{ccccc}
G_{o(e)} & \xleftarrow{\;o_e\;} & G_e & \xrightarrow{\;t_e\;} & G_{t(e)} \\
\Big\downarrow{\scriptstyle h_{o(e)}} & & \Big\downarrow{\scriptstyle h_e}\;{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle h_{t(e)}} \\
G'_{o(e')} & \xleftarrow{\;o_{e'}\;} & G'_{e'} & \xrightarrow{\;t_{e'}\;} & G'_{t(e')}
\end{array}
$$

Then the induced homomorphism $h : \mathcal{G}(\Gamma, \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma', \{G'_v\}, \{G'_e\})$ is an isometric embedding.

Proof. It easily follows from assumptions (1) and (3) that h is well defined and injective. Now we show h preserves scl.

Consider an arbitrary rational homologically trivial chain $c = \sum_i r_i g_i$ with $r_i \in \mathbb{Q}$ and $g_i \in G$ of infinite order. Represent each g_i by a tight loop γ_i in the standard realization X

of G. Define γ, A_v, L_v as in the discussion above.

Let X' be the standard realization of G' and let $\eta : X \to X'$ be the map representing h. Then the arcs obtained by cutting $\sqcup_i \eta(\gamma_i)$ along the edge spaces of X' are exactly $\sqcup_v \eta(A_v)$.

Let S' be an admissible surface for $c' = h(c)$ of degree n'. It suffices to show, for any $\epsilon > 0$, there is an admissible surface S for c of degree n such that

$$\frac{-\chi^-(S)}{n} \leq \frac{-\chi^-(S')}{n'} + C(S')\epsilon$$

for some constant $C(S')$.

By Lemma 3.5, we may assume S' to be in its normal form and obtained by gluing surfaces with corner $S'_{v'}$, where $S'_{v'}$ is supported in the thickened vertex space $N(X'_{v'})$. Let $v = h^{-1}(v')$. Now we pull back $\partial S'_{v'}$ in a natural way and show that the pull back is homologically trivial in G_v. First consider each polygonal boundary of $S'_{v'}$. Each arc is in $\eta(A_v)$ and pulls back via h_v to a unique arc in A_v. Each turn is in $\mathrm{Im} t_{e'}$ for some e' with $t(e') = v'$, and thus, by assumption (3), pulls back via h_v to a unique turn in $\mathrm{Im} t_e$, where $e = h^{-1}(e')$. Similarly consider each loop boundary of $S'_{v'}$. If such a loop lies in $\eta(L_v)$, then it uniquely pulls back to a loop in L_v. Otherwise, it is obtained by cutting along the preimage of edge spaces, thus uniquely pulls back to a loop represented by an element in $\mathrm{Im} t_e$. By construction, the homology of the pull back of $\partial S'_{v'}$ lies in

$$n' \sum_{\alpha \in A_v \cup L_v} r_{i(\alpha)}[w(\alpha)] + H_v,$$

which is exactly H_v by Lemma 3.3 since c is homologically trivial. Thus assumption (2) implies the pull back of $\partial S'_{v'}$ is homologically trivial in G_v. By assumption (1) and Lemma 2.15, there is a surface S_v (with corners induced by those on $S'_{v'}$) mapped into X_v such that

$h_v(\partial S_v) = n_v \partial S'_{v'}$ for some integer $n_v > 0$ and

$$\frac{-\chi(S_v)}{n_v} \leq -\chi(S'_v) + \epsilon.$$

The analogous inequality holds with χ replaced by χ_o.

Now let $N = \prod_v n_v$ and take N/n_v copies of S_v for every vertex v. These pieces satisfy the gluing condition (3.2) and can be glued to form an admissible surface for c of degree $n = N \cdot n'$ since $h_v(\partial S_v) = n_v \partial S'_{v'}$ and $\{S'_{v'}\}_{v'}$ glues up to S', which is admissible of degree n'. Note that S_v has no disk components with loop boundary, and thus S has no sphere components. Each g_i is of infinite order, so S has no disk components either. Hence $\chi^-(S) = \chi(S)$ and

$$
\begin{aligned}
\frac{-\chi(S)}{n} &= \frac{\sum_v -\chi_o(S_v) \cdot \frac{N}{n_v}}{N \cdot n'} \\
&\leq \frac{\sum_{v'}[-N\chi_o(S'_{v'}) + N\epsilon]}{N \cdot n'} \\
&\leq \frac{-\chi^-(S')}{n'} + \frac{\#\{v'\}}{n'} \cdot \epsilon,
\end{aligned}
$$

where the summations are taken over vertices v' where S' has nonempty intersection with $X_{v'}$ and $\#\{v'\}$ is the number of such vertices, which is finite by compactness of S'. $\qquad\square$

In the special case of free products, this is [18, Theorem B], whose applications can be found in [18, Section 5].

Corollary 3.7. Let w be an element in a group G representing a non-trivial rational homology class. Let $G(w, M, L)$ be the HNN extension $G*_{\mathbb{Z}}$ given by the inclusions $o_e, t_e : \mathbb{Z} \to G$ sending the generator 1 to w^M and w^L respectively, where $M, L \neq 0$. Then the homomorphism $h : \mathrm{BS}(M, L) \to G(w, M, L)$ is an isometric embedding.

Proof. Consider both groups as graphs of groups where the underlying graph has one vertex and one edge; see Example 3.1. The inclusion of the vertex group $\mathbb{Z} = \langle a \rangle \to G$ with $a \mapsto w$

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is injective since $[w] \neq 0 \in H_1(G; \mathbb{Q})$, and is an isometric embedding since $\mathbb{Z}$ is abelian (See Example 2.14). The non-triviality of $[w]$ also implies that condition (2) of Theorem 3.6 holds. Then it is easy to see that Theorem 3.6 applies. $\qquad\square$

3.3 Simple Normal Form

In this section, we further simplify the normal form for relative admissible surfaces into the so-called simple normal form. Such surfaces can be used to compute scl in graphs of groups relative to the vertex groups. In general this only ends up with a lower bound of (absolute) scl. However, Proposition 3.2 shows that equality holds when $\text{scl}_G(c) = 0$ for all $c \in B_1^H(G_v)$ and all vertices v, which is the case for example by monotonicity when all vertex groups have trivial scl.

Definition 3.8. A normal form of an admissible surface S relative to vertex groups is called the *simple normal form* if each component in S_v has exactly one polygonal boundary and is either a disk or an annulus with the former case occurring if and only if the polygonal boundary is null-homotopic in S_v. For simplicity, we refer to such a surface as a *simple relative admissible surface*.

We first explain how to obtain simple normal form from a (relative) admissible surface in normal form. Assume each $g_i \in \underline{g}$ to be hyperbolic. Let $c = \sum r_i g_i$ be a rational chain whose homology class $[c]$ is in the kernel of the projection $H_1(G; \mathbb{R}) \to H_1(\Gamma; \mathbb{R})$.

Lemma 3.9. *For any relative admissible surface S for c, there is another S' of the same degree in simple normal form, such that*

$$-\chi(S') \leq -\chi(S).$$

Thus one can compute $\text{scl}_{(G,\{G_v\})}(c)$ by taking the infimum in equation (2.2) over simple relative admissible surfaces.
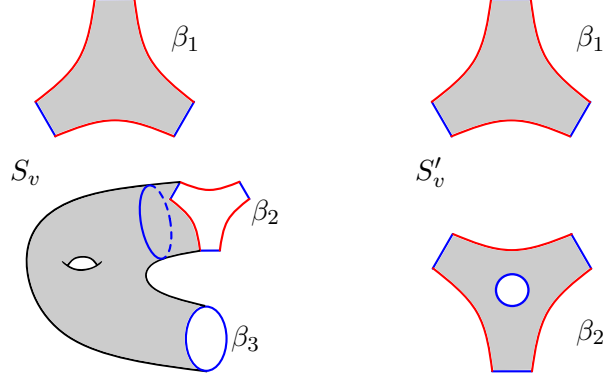
Figure 3.5: With the S_v in Figure 3.3, we cut out a neighborhood of β_2, shown on the left, and throw away the component without polygonal boundary to obtain S_v' on the right. This appears originally in [20, Fig. 4].

Proof. By Lemma 3.5, we may assume S to be in its normal form. For each subsurface S_v, in any of its components other than disks, take out a small collar neighborhood of each polygonal boundary. Let S_v' be the disjoint union of these collar neighborhoods and disk components of S_v. See Figure 3.5. Then each component of the subsurfaces in S_v that we throw away to obtain S_v' has at least one boundary component, has no corners, and cannot be a disk. Thus

$$-\chi_o(S_v') \leq -\chi_o(S_v).$$

Since S_v' has all its polygonal boundaries taken from S_v, it satisfies the gluing condition (3.2). Recall that disk components of S_v must have polygonal boundaries. It follows that each component of S_v' has exactly one polygonal boundary. Moreover, each loop boundary in S_v' is homotopic to a polygonal boundary by construction, and thus to a loop in X_v. If the loop boundary is null-homotopic, replace the interior by a disk realizing the null-homotopy. Since each $g_i \in \underline{g}$ is hyperbolic, gluing up paired turns in S_v' produces a relative admissible surface S' for c of the same degree as S with the desired properties. $\square$

3.3.1   The Second Isometric Embedding Theorem

Theorem 3.10 (the second isometric embedding). *Let $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ and $\mathcal{G}(\Gamma', \{G'_v\}, \{G'_e\})$ be graphs of groups where scl vanishes on vertex groups. Suppose there is a graph homomorphism $h : \Gamma \to \Gamma'$, injective homomorphisms $h_v : G_v \to G'_{h(v)}$ and isomorphisms $h_e : G_e \to G'_{h(e)}$ such that*

(1) *h is injective on the set of edges (Definition 2.13);*

(2) *the map induced by h_v on homology is injective on H_v, where H_v is the sum of $\mathrm{Im} t_{e*}$ over all edges e with $t(e) = v$; and*

(3) *the following diagram commutes for all edge pairs (e, e') with $e' = h(e)$.*

$$
\begin{array}{ccccc}
G_{o(e)} & \xleftarrow{\;o_e\;} & G_e & \xrightarrow{\;t_e\;} & G_{t(e)} \\
{\scriptstyle h_{o(e)}}\big\downarrow & & {\scriptstyle h_e}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle h_{t(e)}}\big\downarrow \\
G'_{o(e')} & \xleftarrow{\;o_{e'}\;} & G'_{e'} & \xrightarrow{\;t_{e'}\;} & G'_{t(e')}
\end{array}
$$

Then the induced homomorphism $h : \mathcal{G}(\Gamma, \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma', \{G'_v\}, \{G'_e\})$ is an isometric embedding.

Proof. The proof is similar to that of Theorem 3.6 by considering an arbitrary relative admissible surface S' for $c' = h(c)$ in simple normal form instead of normal form, where $c = \sum r_i g_i$ is a rational chain in G with all g_i hyperbolic. Elliptic elements are ignored since it suffices to show that h preserves scl relative to vertex groups according to Proposition 3.2. The difference is that, the graph homomorphism h is now allowed to collapse vertices, so there might be several vertices of Γ mapped to the same vertex v' in Γ'. However, each component of $S'_{v'}$ has a polygonal boundary on which each turn is supported in some $X'_{e'}$ connecting two arcs in $A'_{v'}$, where $t(e') = v'$. Then the two arcs must be the image of two arcs in A_v under h_v for the vertex $v = t(e) \in h^{-1}(v)$, where $e = h^{-1}(e')$. It follows that all

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the arcs on each polygonal boundary come from the same vertex v and thus each polygonal boundary of $S'_{v'}$ can be pulled back to a polygonal boundary in a unique X_v with $h(v) = v'$. Then the rest of the proof is the same as that of Theorem 3.6. $\qquad \square$

An immediate corollary is the following proposition, which is useful to simplify the computation of scl.

Proposition 3.11 (restriction of domain)**.** *Let $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups where scl vanishes on vertex groups. Then the inclusion $i : \mathcal{G}(\Gamma', \{G_v\}, \{G_e\}) \to \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ associated to any connected subgraph Γ' of Γ is an isometric embedding.*

We can use Theorem 3.10 to compute scl of t-alternating words in HNN extensions of abelian groups.

Definition 3.12. Given injective homomorphisms $i, j : E \to V$, let G be the associated HNN extension, i.e. $G = V *_E = \langle G, t \mid i(e) = t\, j(e)t^{-1} \rangle$. An element g is t-*alternating* if it is conjugate to a cyclically reduced word of the form $a_1 t b_1 t^{-1} \ldots a_n t b_n t^{-1}$ for some $n \geq 0$.

Let $H = V_1 *_E V_2$ be the amalgam with $V_1 = V_2 = V$ and injections i, j above, which has a natural inclusion into G, whose image is the set of t-alternating words.

Proposition 3.13 (t-alternating words)**.** *Let V and E be abelian groups with inclusions $i, j : E \to V$, then we have an isometric embedding $h : V_1 *_E V_2 \to V *_E$ where $V_1 = V_2 = V$. In particular, with $T = t^{-1}$, we have*

$$
\begin{aligned}
\mathrm{scl}_{(V*_E, V)}(a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T) &= \mathrm{scl}_{(V_1 *_E V_2, \{V_1, V_2\})}(a_1 b_1 \cdots a_n b_n) \\
&= \mathrm{scl}_{(V_1/i(E) * V_2/j(E), \{V_1/i(E), V_2/j(E)\})}(\bar{a}_1 \bar{b}_1 \cdots \bar{a}_n \bar{b}_n).
\end{aligned}
$$

Proof. Consider $V_1 *_E V_2$ and $V *_E$ as graphs of groups where the underlying graphs are a segment and a loop respectively. Let h be the graph homomorphism taking the two end points of the segment to the vertex on the loop. Let h_v and h_e be identities; see Example

3.1. It is easy to check that the assumptions of Theorem 3.10 are satisfied since V and E are abelian. Thus we obtain an induced isometric embedding $h : V_1 *_E V_2 \to V *_E$ with $h(a_1 b_1 \cdots a_n b_n) = a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T$.

Let $a_0 \in V_1$ and $b_0 \in V_2$ be elements such that the chain $c = a_0 + b_0 + a_1 b_1 \cdots a_n b_n$ is null-homologous. Then by Proposition 3.2, the isometric embedding h provides

$$
\begin{aligned}
\mathrm{scl}_{(V_1 *_E V_2, \{V_1, V_2\})}(a_1 b_1 \cdots a_n b_n) &= \mathrm{scl}_{V_1 *_E V_2}(a_0 + b_0 + a_1 b_1 \cdots a_n b_n) \\
&= \mathrm{scl}_{V *_E}(a_0 + b_0 + a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T) \\
&= \mathrm{scl}_{(V *_E, V)}(a_1 t b_1 T a_2 t b_2 T \cdots a_n t b_n T)
\end{aligned}
$$

The other equality can be proved similarly using the scl-preserving projection π [39, Proposition 4.3] in the central extension

$$
1 \to E \to V_1 *_E V_2 \xrightarrow{\pi} V_1/i(E) * V_2/j(E) \to 1.
$$

$\square$

In particular, scl of t-alternating words in Baumslag–Solitar groups can be computed as scl in free products of cyclic groups, which is better understood and easier to compute.

Corollary 3.14. Let $\mathbb{Z}/M\mathbb{Z} * \mathbb{Z}/L\mathbb{Z} = \left\langle x, y \mid x^M = y^L = 1 \right\rangle$. For any t-alternating word in $\mathrm{BS}(M, L)$, we have

$$
\mathrm{scl}_{\mathrm{BS}(M,L)}(a^{u_1} t a^{v_1} t^{-1} a^{u_2} t a^{v_2} t^{-1} \cdots a^{u_n} t a^{v_n} t^{-1}) = \mathrm{scl}_{\mathbb{Z}/M\mathbb{Z}*\mathbb{Z}/L\mathbb{Z}}(x^{u_1} y^{v_1} \cdots x^{u_n} y^{v_n}).
$$

Proof. It directly follows from Proposition 3.2 and Proposition 3.13. $\square$

Corollary 3.14 implies that the scl spectrum (Definition 2.22) of $\mathbb{Z}/M\mathbb{Z}*\mathbb{Z}/L\mathbb{Z}$ is a subset of the spectrum of $\mathrm{BS}(M, L)$. Due to our limited understanding of the scl spectrum, it is not

clear if this is a proper subset and how big the difference is. We do know that the smallest positive elements in the two spectra exist and agree if M and L are odd; See Corollary 4.9 and [19, Remark 3.6].

Here are a few examples, where we write $T = t^{-1}$ and $A = a^{-1}$ for simplicity.

Example 3.15. With the notation in Corollary 3.14, the product formula [11, Theorem 2.93] implies

$$\operatorname{scl}_{\mathrm{BS}(M,L)}(atAT) = \operatorname{scl}_{\mathbb{Z}/M * \mathbb{Z}/L}(xy^{-1}) = \frac{1}{2}\left(1 - \frac{1}{M} - \frac{1}{L}\right).$$

This explains why [22, Proposition 5.5] resembles the product formula.

Example 3.16. Similarly, using [18, Proposition 5.6] instead of the product formula, we have

$$\operatorname{scl}_{\mathrm{BS}(M,L)}(ataTAtAT) = \operatorname{scl}_{\mathbb{Z}/M * \mathbb{Z}/L}([x,y]) = \frac{1}{2} - \frac{1}{\min(M,L)}.$$

Example 3.17. Finally, for explicit M and L, one can use Corollary 3.14 and the computer program `scallop` [15] to quickly compute scl of rather long t-alternating words. For example,

$$\operatorname{scl}_{\mathrm{BS}(7,5)}(ataTataTatATatATataTatATatATatATataT) = \frac{123}{70}$$
$$\operatorname{scl}_{\mathrm{BS}(5,11)}(ataTataTatATatATataTatATatATatATataTataTatAT) = \frac{102}{55}.$$

Proposition 3.13 also provides a new perspective and a shorter proof for [16, Proposition 4.4] as follows.

Proposition 3.18 (Calegari–Walker [16]). *The homomorphism*

$$h : F_2 = \langle x, y \rangle \rightarrow F_2 = \langle a, t \rangle$$

given by $h(x) = a$ and $h(y) = tat^{-1}$ is an isometric embedding.

Proof. Consider the domain F_2 as a free product and the co-domain F_2 as a free HNN extension of $\mathbb{Z}$. Then the conclusion immediately follows from Proposition 3.13 and Proposition 3.2. □

CHAPTER 4

SPECTRAL GAPS OF STABLE COMMUTATOR LENGTH

In this chapter, we show lower bounds of scl in graphs of groups, possibly relative to vertex groups. We will focus mostly on hyperbolic elements, where we develop a duality method to give a uniform lower bound on the complexity of (relative) admissible surfaces in simple normal form (see Section 4.1). The lower bounds on (relative) scl obtained this way are often sharp. For example, we prove sharp bounds for graph products by splitting them as amalgams in Section 4.2. Finally in Section 4.3 we discuss estimates for scl of elliptic elements.

The contents are largely based on my joint work with Nicolaus Heuer [21, Sect. 5, 6, 7].

4.1 Scl of Hyperbolic Elements: A Duality Method

Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups, and let X be its standard realization as in Sect. 3.1.1. Let $\underline{g} = \{g_i\}_{i \in I}$ be a set of *hyperbolic* elements in G represented by tight loops $\underline{\gamma} = \{\gamma_i\}_{i \in I}$ in X, indexed by a *finite* set I.

Recall from Lemma 3.9 that one can compute $\mathrm{scl}_{(G, \{G_v\})}(c)$ for any rational chain $c = \sum r_i g_i$ with $r_i > 0$ by taking the infimum in equation 2.2 over *simple* relative admissible surfaces. This provides a lower bound of $\mathrm{scl}_G(c)$ by Lemma 2.17, and equality holds when scl vanishes on vertex groups.

Recall from Section 3.2 that a simple relative admissible surface S decomposes as $S = \sqcup_v S_v$, where S_v consists of components (each of which we refer to as a *piece*) supported in a thickened vertex space $N(X_v)$. Each piece C has exactly one polygonal boundary, and topologically speaking C is either a disk or an annuli, depending on whether the polygonal boundary represents the identity element. Thus $\chi(C) = 0, 1$.

For each vertex v, recall from Sect. 3.1.2 that A_v denotes the collection of arcs supported

in $N(X_v)$ obtained by cutting hyperbolic loops in $\underline{\gamma}$. As is discussed in Sect. 3.2.1, for a piece C in S_v, each turn travels from an arc $a_1 \in A_v$ to another $a_2 \in A_v$ as a based loop supported in an edge space X_e representing some $w \in G_e$, where e is an edge adjacent to v. The type of such a turn is determined by the triple (a_1, w, a_2). Two turn types (a_1, w, a_2) and (a'_1, w', a'_2) are paired if they can be glued together, depicted in Figure 3.4.

On the boundary of each piece, the number of turns is half of the number of corners.

Let $t_{(a_1,w,a_2)}$ be the number of turns of type (a_1, w, a_2) that appear in $\sqcup S_v$ *divided by* k if S is a degree k admissible surface. With this notation, the *gluing condition* (3.2) becomes

Gluing condition:

$$t_{(a_1,w,a_2)} = t_{(a'_1,w',a'_2)} \text{ for any paired turn types } (a_1, w, a_2) \text{ and } (a'_1, w', a'_2). \qquad (4.1)$$

Let $i(\alpha) \in I$ be the index such that the arc α lies on $\gamma_{i(\alpha)}$. Recall that r_i is the coefficient of γ_i in the chain c. Then we also have the following

Normalizing condition:

$$\sum_{a_2,w} t_{(a_1,w,a_2)} = r_{i(a_1)} \text{ for any } a_1 \text{ and } \sum_{a_1,w} t_{(a_1,w,a_2)} = r_{i(a_2)} \text{ for any } a_2. \qquad (4.2)$$

Lemma 4.1. *For any rational chain* $c = \sum r_i g_i$ *as above, let* S *be a simple relative admissible surface of degree* k. *Then we have*

$$\frac{-\chi^-(S)}{k} = \frac{1}{2} \sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} + \sum \frac{-\chi(C)}{k} = \frac{1}{2} \sum_i r_i |\gamma_i| + \sum \frac{-\chi(C)}{k}, \qquad (4.3)$$

where $|\gamma_i|$ *denotes the number of arcs on the hyperbolic tight loop* γ_i, *and* C *ranges over all pieces in the decomposition of* S.

Proof. For a simple relative admissible surface S for the chain c, since it cannot have sphere

or disk components, we have

$$-\chi^-(S) = -\chi(S) = \sum -\chi_o(C).$$

Recall that $\chi_o(C) = \chi(C) - \frac{1}{4}\#\{\text{corners in } C\} = \chi(C) - \frac{1}{2}\#\{\text{turns in } C\}$, and the normalizing condition (4.2) implies that the total number of turns is

$$\sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} = \sum_i r_i|\gamma_i|.$$

Then equation (4.3) easily follows. $\qquad\qquad\square$

Note that in equation (4.3), the quantity $\frac{1}{2}\sum_i r_i|\gamma_i|$ is purely determined by the chain c, thus the key to estimate $\mathrm{scl}_G(c)$ (or $\mathrm{scl}_{(G,\{G_v\})}(c)$) is to control $\sum \frac{-\chi(C)}{k}$. Since each C is either a disk or an annulus, the problem comes down to maximizing the number of disk pieces.

4.1.1   Lower Bounds from Linear Programming Duality

We introduce a general strategy to obtain lower bounds of scl in graphs of groups relative to vertex groups using the idea of linear programming duality. This was first developed by the author in the special case of free products to obtain sharp uniform lower bounds of scl [19].

Recall that we are considering a rational chain $c = \sum r_i g_i$ with $r_i \in \mathbb{Q}_{>0}$ and $\underline{g} = \{g_i, i \in I\}$ consisting of finitely many *hyperbolic* elements represented by tight loops $\underline{\gamma} = \{\gamma_i, i \in I\}$. With notation as above, for each turn type (a_1, w, a_2), we assign a *non-negative cost* $c_{(a_1,w,a_2)} \geq 0$.

Suppose S is a simple admissible surface of degree k for c relative to the vertex groups. Then the assignment above induces a *non-negative* cost for each piece C in the decomposition of S by linearity: its cost is the sum of costs of its turns on the polygonal boundary.

Lemma 4.2. *Let S be any simple relative admissible surface for c of degree k. With the $t_{(a_1,w,a_2)}$ notation (normalized number of turns) in the estimate (4.3), if every disk piece C in the normal form of S has cost at least 1, then the normalized total cost*

$$\sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} c_{(a_1,w,a_2)} \geq \frac{1}{2} \sum r_i |\gamma_i| - \frac{-\chi^-(S)}{k}.$$

Proof. By equation (4.3), it suffices to prove

$$\sum \frac{\chi(C)}{k} \leq \sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} c_{(a_1,w,a_2)}.$$

Note that, for each piece C in the normal form of S, either $\chi(C) \leq 0$ which does not exceed its cost, or C is a disk piece and $\chi(C) = 1$ which is also no more than its cost by our assumption. The desired estimate follows by summing up these inequalities and dividing by k. $\qquad\square$

In light of Lemmas 3.9, 4.1 and 4.2, to get lower bounds of scl relative to vertex groups, one strategy is to come up with suitable cost assignments $c_{(a_1,w,a_2)} \geq 0$ such that

(1) every possible disk piece has cost at least 1; and

(2) one can use the gluing condition (4.1) and normalizing condition (4.2) to bound the quantity $\sum_{(a_1,w,a_2)} t_{(a_1,w,a_2)} c_{(a_1,w,a_2)}$ from above by a constant.

4.1.2   Uniform Lower Bounds

Now we use the duality method above to prove sharp uniform lower bounds of scl in graphs of groups relative to vertex groups. The results are subject to some local conditions introduced as follows.

Definition 4.3. Let H be a subgroup of G. For $2 \leq k < \infty$, an element $g \in G \setminus H$ is a *relative k-torsion* for the pair (G, H) if

$$gh_1 \ldots gh_k = id \tag{4.4}$$

for some $h_i \in H$. For $3 \leq n \leq \infty$, we say the subgroup H is *n-relatively torsion-free* (*n-RTF*) in G if there is no relative k-torsion for all $2 \leq k < n$. Similarly, we say g is *n-RTF* in (G, H) if $g \in G \setminus H$ is not a relative k-torsion for any $2 \leq k < n$.

By definition, m-RTF implies n-RTF whenever $m \geq n$.

Example 4.4. If H is normal in G, then g is a relative k-torsion if and only if its image in G/H is a k-torsion. In this case, H is n-RTF if and only if G/H contains no k-torsion for all $k < n$, and in particular, H is ∞-RTF if and only if G/H is torsion-free. Concretely, the subgroup $H = 6\mathbb{Z}$ in $G = \mathbb{Z}$ is not n-RTF for all $n \geq 3$ since z^3 is a relative 2-torsion, where z is a generator of $\mathbb{Z}$; more generally, $H = m\mathbb{Z}$ is p_m-RTF if m is odd and p_m is the smallest prime factor of $m \in \mathbb{Z}_+$.

For a less trivial example, let S be an orientable closed surface of positive genus and let $g \in \pi_1(S)$ be an element represented by a simple closed curve. Then the cyclic subgroup $\langle g \rangle$ is ∞-RTF in $\pi_1(S)$. One can see this from either Lemma 4.14 or Lemma 4.17 below.

The equation (4.4) can be rewritten as

$$g \cdot \tilde{h}_1 g \tilde{h}_1^{-1} \cdot \ldots \cdot \tilde{h}_{k-1} g \tilde{h}_{k-1}^{-1} = \tilde{h}_k^{-1}, \tag{4.5}$$

where $\tilde{h}_i := h_1 \ldots h_i$. This is closely related to the notion of generalized k-torsion.

Definition 4.5. For $k \geq 2$, an element $g \neq id \in G$ is a *generalized k-torsion* if

$$g_1 g g_1^{-1} \cdots g_k g g_k^{-1} = id$$

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for some $g_i \in G$.

If equation (4.4) holds with $\tilde{h}_k = h_1 h_2 \ldots h_k = id$, then g is a generalized k-torsion.

It is observed in [33, Theorem 2.4] that a generalized k-torsion cannot have scl exceeding $1/2 - 1/k$ for a reason similar to Proposition 4.6 below. On the other hand, it is well known and easy to note that the existence of any generalized torsion is an obstruction for a group G to be bi-orderable, i.e. to admit a total order on G that is invariant under left and right multiplications.

The n-RTF condition is closely related to lower bounds of relative scl.

Proposition 4.6. *Let H be a subgroup of G. If*

$$\mathrm{scl}_{(G,H)}(g) \geq \frac{1}{2} - \frac{1}{2n},$$

then $g \in G$ is n-RTF in (G, H).

Proof. Suppose equation (4.5) holds for some $k \geq 2$. Then this gives rise to an admissible surface S in G for g of degree k relative to H, where S is a sphere with $k + 1$ punctures: k of them each wraps around g once, and the other maps to $\tilde{h}_k$. This implies

$$\frac{1}{2} - \frac{1}{2n} \leq \mathrm{scl}_{(G,H)}(g) \leq \frac{-\chi(S)}{2k} = \frac{1}{2} - \frac{1}{2k},$$

and thus $k \geq n$. $\square$

Conversely, the n-RTF condition implies a lower bound for relative scl in the case of graphs of groups, which is the technical version of the main theorem of this section.

Theorem 4.7. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. Let γ be a tight loop cut into arcs $a_1, \ldots, a_L$ by the edge spaces, where a_i is supported in a thickened vertex space $N(X_{v_i})$ and $v_1, e_1, \ldots, v_L, e_L$ form a loop in Γ with $o(e_i) = v_i$ and $t(e_i) = v_{i+1}$, indices taken mod*

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L. Suppose for some $n \geq 3$, whenever $e_{i-1} = \bar{e}_i$, the winding number $w(a_i)$ of a_i is n-RTF in $(G_{v_i}, o_{e_i}(G_{e_i}))$. Then

$$\mathrm{scl}_{(G, \{G_v\})}(g) \geq \frac{1}{2} - \frac{1}{n}.$$

Proof. Let S be any relative admissible surface for γ of degree k in it simple normal form with pieces C. We follow the strategy and notation in Sect. 4.1.1 and assign costs in a way that does not depend on winding numbers. That is, for any $1 \leq i, j \leq L$, the cost $c_{(a_i, w, a_j)} = c_{ij}$ where

$$c_{ij} := \begin{cases} 1 - \frac{1}{n}, & \text{if } i < j \\ \frac{1}{n}, & \text{if } i \geq j. \end{cases}$$

Let t_{ij} be normalized the total number of turns of the form (a_i, w, a_j), i.e. $t_{ij} = \sum_w t_{(a_i, w, a_j)}$.

For any disk piece C, let $a_{i_1}, \ldots a_{i_s}$ be the arcs of γ on the polygonal boundary of C in cyclic order. There are two cases:

(1) All $i_j = i$ for some i. Then we necessarily have $e_{i-1} = \bar{e}_i$, and $a_i \in G_{v_i} \backslash o_{e_i}(G_{e_i})$ since γ is tight. For each $1 \leq j \leq s$, let $w_j \in o_{e_i}(G_{e_i})$ be the winding number of the turn from a_{i_j} to $a_{i_{j+1}}$, where the j-index is taken mod s. Since C is a disk, we have $w(a_i)w_1 \cdots w(a_i)w_s = id \in G_{v_i}$. Then we must have $s \geq n$ since $w(a_i)$ is n-RTF in $(G_{v_i}, o_{e_i}(G_{e_i}))$ by assumption, and thus the cost

$$c(C) = s \cdot c_{ii} = \frac{s}{n} \geq 1.$$

(2) Some $i_j \neq i_{j'}$. Then there is some $j_m \neq j_M$ such that $i_{j_m} < i_{j_m+1}$ and $i_{j_M} > i_{j_M+1}$, where $j_m + 1$ and $j_M + 1$ are interpreted mod s. Hence the cost

$$c(C) \geq c_{j_m, j_m+1} + c_{j_M, j_M+1} = \left(1 - \frac{1}{n}\right) + \left(\frac{1}{n}\right) = 1.$$

In summary, we have $c(C) \geq 1$ for any disk piece C. Hence by Lemma 4.2, we have

$$\frac{-\chi^-(S)}{2k} \geq \frac{L}{4} - \frac{1}{2}\sum_{ij} c_{ij}t_{ij}$$

since $|\gamma| = L$.

On the other hand, for any $1 \leq i, j \leq L$, we have $t_{ij} = t_{j-1,i+1}$ by the gluing condition (4.1) and $\sum_i t_{ij} = \sum_j t_{ji} = 1$ by the normalizing condition (4.2), indices taken mod L. We also have $t_{i,i+1} = 0$ since γ is tight and intersects edge spaces transversely. Thus

$$\sum_{i,j} c_{ij}t_{ij} = \sum_{i,j} \frac{1}{n}t_{ij} + \left(1 - \frac{2}{n}\right)\sum_{i<j} t_{ij} = \frac{L}{n} + \left(1 - \frac{2}{n}\right)\sum_{i<j} t_{ij}.$$

and

$$
\begin{aligned}
2\sum_{i<j} t_{ij} &= \sum_{1 \leq i < j \leq L} t_{ij} + \sum_{1 \leq i < j \leq L} t_{j-1,i+1} \\
&= \sum_{\substack{1 \leq i \leq L-1 \\ 2 \leq j \leq L}} t_{ij} + \sum_{1 \leq i \leq L-1} t_{i,i+1} \\
&= \left[\sum_{1 \leq i,j \leq L} t_{ij} - \sum_i t_{i1} - \sum_j t_{Lj}\right] + [0] \\
&= L - 2,
\end{aligned}
$$

where the first two equalities can be visualized in Figure 4.1.

Putting the equations above together, we have

$$\sum_{ij} c_{ij}t_{ij} = \frac{L}{n} + \left(1 - \frac{2}{n}\right)\frac{L-2}{2} = \frac{L}{2} - \left(1 - \frac{2}{n}\right),$$

and

$$\frac{-\chi^-(S)}{2k} \geq \frac{L}{4} - \frac{1}{2}\sum_{ij} c_{ij}t_{ij} = \frac{1}{2} - \frac{1}{n}$$

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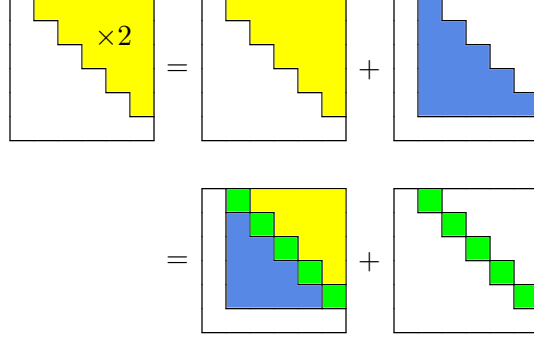
Figure 4.1: Visualization of the summation in the case $L = 6$, where the first equality uses the gluing condition $t_{ij} = t_{j-1,i+1}$. This appears originally in [21, Figure 9].

for any simple relative admissible surface S. Thus the conclusion follows from Lemma 3.9. □

Theorem 4.8. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. Let $\{\Gamma_\lambda\}$ be a collection of mutually disjoint connected subgraphs of Γ and let G_λ be the graph of groups associated to Γ_λ. If for some $3 \leq n \leq \infty$ the inclusion of each edge group into vertex group is n-RTF, then*

$$\mathrm{scl}_G(g) \geq \mathrm{scl}_{(G, \{G_\lambda\})}(g) \geq \frac{1}{2} - \frac{1}{n}$$

unless $g \in G$ conjugates into some G_λ. Hence $(G, \{G_\lambda\})$ has a strong relative spectral gap $\frac{1}{2} - \frac{1}{n}$.

In particular,

$$\mathrm{scl}_G(g) \geq \mathrm{scl}_{(G, \{G_v\})}(g) \geq \frac{1}{2} - \frac{1}{n}$$

for any hyperbolic element $g \in G$.

Proof. Each hyperbolic element g is represented by a tight loop γ satisfying the assumptions of Theorem 4.7 since the inclusions of edge groups are n-RTF. This implies the special case where $\{\Gamma_\lambda\}$ is the set of vertices of Γ.

To see the general case, by possibly adding some subgraphs each consisting of a single vertex, we assume each vertex is contained in some Γ_λ. By collapsing each Γ_λ to a single

vertex, we obtain a new splitting of G as a graph of groups where $\{G_\lambda\}$ is the new collection of vertex groups. Each new edge group is some G_e for some edge e of Γ connecting two Γ_λ's. Let Γ_λ be the subgraph containing the vertex $v = t(e)$, then (G_λ, G_v) is n-RTF by Lemma 4.17 in the next subsection and (G_v, G_e) is n-RTF by assumption. Thus (G_λ, G_e) is also n-RTF by Lemma 4.11. Therefore the edge group inclusions in this new splitting also satisfy the n-RTF condition, so the general case follows from the special case proved above. $\qquad\square$

Under the stronger assumption that the inclusion of each edge group into vertex group is left relatively convex (Definition 4.12), there are quasimorphisms detecting the lower bounds above in Theorem 4.8; see [21, Sect. 5.5] for details.

4.1.3 Examples

In the case of an amalgam $G = A \star_C B$, since left relatively convex implies ∞-RTF (Lemma 4.14), Theorem 4.8 implies [30, Theorem 6.3], which is the main input to obtain gap $1/2$ in all right-angled Artin groups in [30]. We will prove a generalization in Section 4.2.

For the moment, let us consider the case of Baumslag–Solitar groups.

Corollary 4.9. Let $\mathrm{BS}(m, \ell) := \langle a, t \mid a^m = t a^\ell t^{-1} \rangle$ be the Baumslag–Solitar group, where $|m|, |\ell| \geq 2$. Let p_m and p_ℓ be the smallest prime factors of $|m|$ and $|\ell|$ respectively. Then $\mathrm{BS}(m, \ell)$ has strong spectral gap $1/2 - 1/\min(p_m, p_\ell)$ relative to the subgroup $\langle a \rangle$ if m, l are both odd. This estimate is sharp since for $g = a^{m/p_m} t a^{\ell/p_\ell} t^{-1} a^{-m/p_m} t a^{-\ell/p_\ell} t^{-1}$ we have

$$\mathrm{scl}_{\mathrm{BS}(m,\ell)}(g) = \mathrm{scl}_{(\mathrm{BS}(m,\ell), \langle a \rangle)}(g) = 1/2 - 1/\min(p_m, p_\ell).$$

Proof. Let $n = \min(p_m, p_\ell)$. The Baumslag–Solitar group $\mathrm{BS}(m, \ell)$ is the HNN extension associated to the inclusions $\mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z}$ (Example 3.1), which are both n-RTF. Thus the strong relative spectral gap follows from Theorem 4.8. The example achieving the lower bound follows from Proposition 3.2, Corollary 3.14, and [18, Proposition 5.6]. $\qquad\square$

If at least one of m and ℓ is even, the word g above has scl value 0, and thus one cannot have a strong relative spectral gap. However, we do have a (relative) spectral gap $1/12$ by [22, Theorem 7.8], which is sharp for example when $p_m = 2$ and $p_\ell = 3$. When $p_m = 2$ and $p_\ell \geq 3$, the smallest known positive scl in $\mathrm{BS}(m, \ell)$ is $1/4 - 1/2p_\ell$ achieved by $a^{m/p_m} t a^{\ell/p_\ell} t^{-1}$.

Question 4.10. If $p_m = 2$ and $p_\ell \geq 3$, does $\mathrm{BS}(m, \ell)$ have (relative) spectral gap $1/4 - 1/2p_\ell$?

4.1.4 The n-RTF Condition

The goal of this subsection is to investigate the n-RTF condition that plays an important role in Theorem 4.7 and Theorem 4.8.

Let us start with some basic properties.

Lemma 4.11. *Suppose we have groups $K \leq H \leq G$.*

(1) If (G, K) is n-RTF, then so is (H, K);

(2) If $g \in G \setminus H$ is n-RTF in (G, H), then g is also n-RTF in (G, K);

(3) If both (G, H) and (H, K) are n-RTF, then so is (G, K).

Proof. (1) and (2) are clear from the definition. As for (3), suppose $gk_1 \ldots gk_i = id$ for some $1 \leq i \leq n$ where each $k_j \in K$. Then $g \in H$ since (G, H) is n-RTF, from which we get $g \in K$ since (H, K) is n-RTF. $\qquad\square$

The n-RTF condition is closely related to orders on groups.

Definition 4.12. A subgroup H is *left relatively convex* in G if there is a total order on the left cosets G/H that is G-invariant, i.e. $gg_1 H \prec gg_2 H$ for all g if $g_1 H \prec g_2 H$.

The definition does not require G to be left-orderable, i.e. G may not have a total order invariant under the left G-action. Actually, if H is left-orderable, then H is left relatively convex in G if and only if G has a left G-invariant order $\prec$ such that H is convex, i.e.

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$h \prec g \prec h'$ for some $h, h' \in H$ implies $g \in H$. Many examples and properties of left relatively convex subgroups are discussed in [1].

Example 4.13 ([1])**.** Let G be a surface group, a pure braid group or a subgroup of some right-angled Artin group. Let H be any maximal cyclic subgroup of G, that is, there is no cyclic subgroup of G strictly containing H. Then H left relatively convex in G.

The n-RTF conditions share similar properties with the left relatively convex condition, and they are weaker.

Lemma 4.14. *If H is left relatively convex in G, then (G, H) is ∞-RTF.*

Proof. Suppose for some $g \in G$ we have $gh_1 \ldots gh_n = id$ for some $n \geq 2$ and $h_i \in H$ for all $1 \leq i \leq n$. Suppose $gH \succ H$. Then $gh_{n-1}gH \succ gh_{n-1}H = gH \succ H$ by left-invariance. By induction, we have $gh_1 \ldots gh_{n-1}gH \succ H$, but $gh_1 \ldots gh_{n-1}gH = gh_1 \ldots gh_nH = H$, contradicting our assumption. A similar argument shows that we cannot have $gH \prec H$. Thus we must have $g \in H$. $\qquad\square$

The n-RTF condition has nice inheritance in graphs of groups (Lemma 4.17). To prove it together with a more precise statement (Lemma 4.15), we first briefly introduce the reduced words of elements in graphs of groups. See [38] for more details. For a graph of groups $G(\Gamma) = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$, let $F(\Gamma)$ be the quotient group of $(\star G_v) \star F_E$ by relations $\bar{e} = e^{-1}$ and $et_e(g)e^{-1} = o_e(g)$ for any edge $e \in E$ and $g \in G_e$, where F_E is the free group generated by the edge set E. Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be any oriented path (so $o(e_i) = v_{i-1}, t(e_i) = v_i$), and let $\mu = (g_0, \ldots, g_k)$ be a sequence of elements with $g_i \in G_{v_i}$. We say any word of the form $g_0 e_1 g_1 \cdots e_k g_k$ is *of type* (P, μ), and it is *reduced* if

(1) $k \geq 1$ and $g_i \notin \mathrm{Im} t_{e_i}$ whenever $\bar{e}_i = e_{i+1}$; or

(2) $k = 0$ and $g_0 \neq id$.

It is known that every reduced word represents a nontrivial element in $F(\Gamma)$. Fix any base vertex v_0, then $G(\Gamma)$ is isomorphic to the subgroup of $F(\Gamma)$ consisting of words of type (L, μ) for any oriented loop L based at v_0 and any μ. Moreover, any nontrivial element is represented by some reduced word of type (L, μ) as above.

Lemma 4.15. *With notation as above, let $g \in G = G(\Gamma)$ be an element represented by a reduced word of type (L, μ) with an oriented loop $L = (v_0, e_1, v_1, \ldots, e_j, v_j = v_0)$ and $\mu = (g_0, \ldots, g_j)$, $j \geq 1$. If j is odd, then g is ∞-RTF in (G, G_{v_0}); if j is even and $g_{j/2}$ is n-RTF in $(G_{v_{j/2}}, \mathrm{Im} t_{e_{j/2}})$ for some $n \geq 3$, then g is n-RTF in (G, G_{v_0}).*

Proof. If j is odd, then the projection $\bar{g}$ of g in the free group F_E is represented by a word of odd length, and thus must be of infinite order. It follows that the projection of $gh_1 \cdots gh_k$ is $\bar{g}^k$ for any $h_i \in G_{v_0}$ and $k > 0$, which must be nontrivial.

Now suppose $j = 2m$ is even and consider $w := gh_1 \cdots gh_k$ for some $1 \leq k < n$ and $h_i \in G_{v_0}$. We claim that there cannot be too much cancellation between the suffix and prefix of two nearby copies of g, more precisely, $g_m e_{m+1} \cdots e_j g_j h g_0 e_1 \cdots e_m g_m$ for any $h \in G_{v_0}$ can be represented by

(1) $g_m g'_m g_m$ for some $g'_m \in \mathrm{Im} t_{e_m}$; or

(2) a reduced word $g_m e_{m+1} \cdots g_{j-s-1} e_{j-s} g'_{j-s} e_{s+1} g_{s+1} \cdots e_m g_m$ with $0 \leq s < m$.

In fact, if either $e_j \neq \bar{e}_1$, or $e_j = \bar{e}_1$ and $g_j h g_0 \notin \mathrm{Im} t_{e_j}$, then we have case (2) with $s = 0$ and $g'_j = g_j h g_0$. If $e_j = \bar{e}_1$ and $g_j h g_0 \in \mathrm{Im} t_{e_j}$, then $v_{j-1} = v_1 = o(e_j)$ and we can replace $g_{e_{j-1}} e_j g_j h g_0 e_1 g_1$ by $g_{e_{j-1}} o_{e_j} t_{e_j}^{-1}(g_j h g_0) g_1 \in G_{v_{j-1}} = G_{v_1}$ to simplify w to a word of shorter length. This simplification procedure either stops in s steps with $s < m$ and we end up with case (2) or it continues until we arrive at $g_m o_{e_{m+1}}(g_m^*) g_m$ for some $g_m^* \in G_{e_{m+1}}$. Note that in the latter case, we must have $\bar{e}_m = e_{m+1}$ since the simplification continues all the way. Thus $g'_m := o_{e_{m+1}}(g_m^*) = t_{e_m}(g_m^*) \in \mathrm{Im} t_{e_m}$.

For each $1 \le i \le k$, write $w_i := e_{m+1} \cdots e_j g_j h_i g_0 e_1 \cdots e_m$ in a reduced form so that $g_m w_i g_m$ is of the form as in the claim above. Then a conjugate of w in $F(\Gamma)$ is represented by $g_m w_1 \cdots g_m w_k$. If $g_m w_i g_m$ is of the form (1) above for all i, then $w_i \in \operatorname{Im} t_{e_m}$ and $g_m w_1 \cdots g_m w_k \ne id$ since g_m is n-RTF in $(G_{v_m}, \operatorname{Im} t_{e_m})$ by assumption. Now suppose $i_1 < i_2 < \cdots < i_{k'}$ are the indices i such that $g_m w_i g_m$ is of the form (2) above, where $k' \ge 1$. Up to a cyclic conjugation, assume $i_{k'} = k$ and let $i_0 = 0$. We write

$$g_m w_1 \cdots g_m w_k = \tilde{g}_1 w_{i_1} \cdots \tilde{g}_k w_{i_{k'}},$$

where $\tilde{g}_s := g_m w_{i_{s-1}+1} \cdots g_m w_{i_s-1} g_m$. Note by the definition of the i_j, each w_i that appears in $\tilde{g}_s$ (i.e. $i_{s-1} + 1 \le i \le i_s - 1$) lies in $\operatorname{Im} t_{e_m}$. It follows that each $\tilde{g}_s \in G_{v_m} \setminus \operatorname{Im} t_{e_m}$ since g_m is n-RTF in $(G_{v_m}, \operatorname{Im} t_{e_m})$ by assumption and $i_s - i_{s-1} - 1 < n$. Thus the expression above puts a conjugate of w in reduced form, and hence $w \ne id$. $\qquad\square$

Corollary 4.16. With notation as above, let $g \in G = G(\Gamma)$ be an element represented by a reduced word of type (L, μ) with an oriented loop $L = (v_0, e_1, v_1, \dots, e_j, v_j = v_0)$ and $\mu = (g_0, \dots, g_j)$, $j \ge 1$. Suppose for some $n \ge 3$ each g_i is n-RTF in (G_{v_i}, G_e) for any edge e adjacent to v_i. Then g is n-RTF in (G, G_{v_0}).

Proof. This immediately follows from Lemma 4.15. $\qquad\square$

Lemma 4.17. *Let* $G(\Gamma) = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ *be a graph of groups. If the inclusion of each edge group into an adjacent vertex group is n-RTF, then for any connected subgraph $\Lambda \subset \Gamma$, the inclusion of* $G(\Lambda) := \mathcal{G}(\Lambda, \{G_v\}, \{G_e\}) \hookrightarrow G(\Gamma)$ *is also n-RTF.*

Proof. The case where Λ is a single vertex v immediately follows from Corollary 4.16 by choosing v to be the base point in the definition of $G(\Gamma)$ as a subgroup of $F(\Gamma)$.

Now we prove the general case with the additional assumption that $\Gamma \setminus \Lambda$ contains only finitely many edges. We proceed by induction on the number of such edges. The assertion is trivially true for the base case $\Lambda = \Gamma$. For the inductive step, let e be some edge outside

of Λ. If e is non-separating, then $G(\Gamma)$ splits as an HNN extension with vertex group $G(\Gamma - \{e\})$. In this case, the inclusion of the edge group G_e is n-RTF in $G_{o(e)}$, which is in turn n-RTF in $G(\Gamma - \{e\})$ by the single vertex case above. Thus by Lemma 4.11, the inclusion $G_e \hookrightarrow G(\Gamma - \{e\})$ is also n-RTF. The same holds for the inclusion of G_e into $G(\Gamma - \{e\})$ through $G_{t(e)}$. Therefore, using the single vertex case again for the HNN extension, we see that $(G(\Gamma), G(\Gamma - \{e\}))$ is n-RTF. Together with the induction hypothesis that $(G(\Gamma - \{e\}), G(\Lambda))$ is n-RTF, this implies that $(G(\Gamma), G(\Lambda))$ is n-RTF by Lemma 4.11. If e is separating, then $G(\Gamma)$ splits as an amalgam with vertex groups $G(\Gamma_1)$ and $G(\Gamma_2)$ such that $\Gamma = \Gamma_1 \sqcup \{e\} \sqcup \Gamma_2$ and $\Lambda \subset \Gamma_1$. The rest of the argument is similar to the previous case.

Finally the general case easily follows from what we have shown, as any $g \in G(\Gamma) \setminus G(\Lambda)$ can be viewed as an element in $G(\Gamma') \setminus G(\Lambda)$ for some connected subgraph Γ' of Γ with only finitely many edges in $\Gamma' \setminus \Lambda$. $\qquad\square$

See Lemma 4.21 for a discussion on the n-RTF conditions in graph products.

One can also use geometry to show that the peripheral subgroups of the fundamental group of certain compact 3-manifolds are 3-RTF.

Lemma 4.18. *Let M be a compact 3-manifold with tori boundary and let T be a boundary component. Suppose the interior of M is hyperbolic with finite volume. Then $\pi_1(T)$ is 3-RTF in $\pi_1(M)$.*

Proof. The hyperbolic structure gives $\pi_1(M)$ a discrete and faithful representation into $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, such that up to a conjugation $H := \pi_1(T)$ is a subgroup of

$$
P := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{C} \right\} \cong (\mathbb{C}, +)
$$

As a result, each $h \in H$ has a unique square root $\sqrt{h} \in P$, i.e. $(\sqrt{h})^2 = h$. Also note that $P \cap \pi_1(M) = H$.

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Suppose $g \in \pi_1(M)$ satisfies $gh_1gh_2 = id$ for some $h_1, h_2 \in H$. We need to show $g \in H$. Let $h^* := \sqrt{h_1h_2} \in P$ and $g^* := gh^* \in \mathrm{Isom}(\mathbb{H}^3)$. Then we have

$$(g^*)^{-1} = (h^*)^{-1}g^{-1} = (h^*)^{-1}h_1gh_2 = ((h^*)^{-1}h_1)g^*((h^*)^{-1}h_1)^{-1},$$

where the last equality uses the fact that P is abelian. We have three cases:

(1) g^* is elliptic or identity. Then g^* fixes a geodesic subspace X in $\mathbb{H}^3$ preserved by $(h^*)^{-1}h_1$. This is impossible unless

(1a) $(h^*)^{-1}h_1 = id$; or

(1b) $X = \mathbb{H}^3$.

In the first subcase, we get $h^* = h_1 = h_2$ and $g^* = (g^*)^{-1}$, but now $g^* = gh^* = gh$ is an element of $\pi_1(M)$, a torsion-free group. So $g^* = id$ and $g = h^{-1} \in P$. In the second subcase, we have $g^* = id$, i.e. $g = (h^*)^{-1}$ which lies in $P \cap \pi_1(M) = \pi_1(T)$.

(2) g^* is parabolic. Then g^* fixes a unique point on $\partial \mathbb{H}^3$ fixed by $(h^*)^{-1}h_1$. So either

(2a) $(h^*)^{-1}h_1 = id$; or

(2b) g^* fixes the unique common fixed point of P.

The first subcase (2a) is similar to (1a). In the second subcase, we have $g^* \in P$, and thus $g = g^*(h^*)^{-1} \in P \cap \pi_1(M) = \pi_1(T)$.

(3) g^* is hyperbolic. Then $(h^*)^{-1}h_1$ must switch the two unique points on $\partial \mathbb{H}^3$ fixed by g^*, which is impossible since $(h^*)^{-1}h_1$ is parabolic.

$\square$

4.2 Spectral Gaps in Right-angled Artin Groups and Graph Products

In this section we apply Theorem 4.8 to obtain sharp gaps of scl in graph products, which are groups obtained from given collections of groups generalizing both free products and direct products.

Definition 4.19. Let Γ be a *simple* graph (not necessarily connected or finite) and let $\{G_v\}$ be a collection of groups, one for each vertex of Γ. The *graph products* G_Γ is the quotient of the free product $\star G_v$ by the set of relations $\{[g_u, g_v] = 1 \mid g_u \in G_u, g_v \in G_v, u, v \text{ are adjacent}\}$.

Example 4.20. Here are some well known examples.

(1) If Γ has no edges at all, then G_Γ is the free product $\star_v G_v$;

(2) If Γ is a complete graph, then G_Γ is the direct product $\oplus_v G_v$;

(3) If each $G_v \cong \mathbb{Z}$, then G_Γ is called the *right-angled Artin group* (RAAG for short) associated to Γ;

(4) If each $G_v \cong \mathbb{Z}/2\mathbb{Z}$, then G_Γ is called the *right-angled Coxeter group* associated to Γ.

We first introduce some terminologies necessary for the statements and proofs. Denote the vertex set of Γ by $V(\Gamma)$. For any $V' \subset V(\Gamma)$, the *full subgraph* on V' is the subgraph of Γ whose vertex set is V' and edge set consists of all edges of Γ connecting vertices in V'. Any full subgraph Λ gives a graph product denoted G_Λ which is naturally a subgroup of G_Γ. It is actually a retract of G_Γ, by trivializing G_v for all $v \notin \Lambda$. Denote by $lk(v)$ the *link* of a vertex v, which is the full subgraph of $\{w \mid w \text{ is adjacent to } v\}$. The *star* $st(v)$ is the full subgraph of $\{v\} \cup \{w \mid w \text{ is adjacent to } v\}$.

Finally, each element $g \in G_\Gamma$ can be written as a product $g_1 \cdots g_m$ with $g_i \in G_{v_i}$. Such a product is *reduced* if

(1) $g_i \neq id$ for all i, and

(2) $v_i \neq v_j$ whenever we have $i \leq k < j$ such that $[g_i, g_t] = id$ for all $i \leq t \leq k$ and $[g_t, g_j] = id$ for all $k + 1 \leq t \leq j$.

It is known that every nontrivial element of G_Γ can be written in a reduced form, which is unique up to certain operations (syllable shuffling) [29, Theorem 3.9]. In particular, any g expressed in the reduced form above is nontrivial in G_Γ.

Lemma 4.21. *Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ ($m \geq 1$) is in reduced form such that for some $n \geq 3$ each $g_i \in G_{v_i}$ has order at least n, $1 \leq i \leq m$. Then g is n-RTF in (G_Γ, G_Λ) for a full subgraph $\Lambda \subset \Gamma$ unless $v_i \in \Lambda$ for all $1 \leq i \leq m$.*

Proof. We proceed by induction on m. The base case $m = 1$ is obvious using the retract from G_Γ to G_{v_1}. For the inductive step, we show g is n-RTF in (G_Γ, G_Λ) if $v_1 \notin \Lambda$, and the other cases are similar. It suffices to prove that g is n-RTF in $(G_\Gamma, G_{\Lambda_1})$ where Λ_1 is the full subgraph of the complement of v_1 in Γ since $G_\Lambda \leq G_{\Lambda_1}$ and using Lemma 4.11 (2).

Consider G_Γ as an amalgam $A \star_C B$ with $A = G_{st(v_1)}$, $C = G_{lk(v_1)}$ and $B = G_{\Lambda_1}$. Then there is a unique decomposition of g into $g = a_1 b_1 \cdots a_\ell b_\ell$ with $a_i \in A$ and $b_i \in B$, where each a_i is a maximal subword of $g_1 \cdots g_m$ that stays in $A - C$. To be precise, there is some $\ell \geq 1$ and indices $0 = \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_\ell \leq \beta_\ell \leq m$, such that $g = a_1 b_1 \cdots a_\ell b_\ell$, where $a_i = g_{\beta_{i-1}+1} \cdots g_{\alpha_i} \in A$ and $b_i = g_{\alpha_i+1} \cdots g_{\beta_i} \in B$ for $1 \leq i \leq \ell$, and such that

(1) $b_\ell = id$ if $\alpha_\ell = m$;

(2) For each $1 \leq i \leq \ell$, we have $v_j \in st(v_1)$ for all $\beta_{i-1}+1 \leq j \leq \alpha_i$, and $v_j = v_1$ for some $\beta_{i-1}+1 \leq j \leq \alpha_i$; and

(3) For each $1 \leq i \leq \ell$ (or $i < \ell$ if $\alpha_\ell = m$), we have $v_j \neq v_1$ for all $\alpha_i + 1 \leq j \leq \beta_i$, and $v_{\alpha_i+1}, v_{\beta_i} \notin st(v_1)$.

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Since g is reduced, so are each a_i and b_i. Thus each a_i (resp. b_i, except the case $b_\ell = id$) is n-RTF in (A, C) (resp. (B, C)) by the induction hypothesis, and thus g is n-RTF in (G_Γ, B) by Lemma 4.15, unless $\ell = 1$ and $b_\ell = id$. In the exceptional case we have $v_i \in st(v_1)$ for all $1 \le i \le m$, and the assertion is obvious using the direct product structure of $G_{st(v_1)}$ and the fact that $g = g_1 \cdots g_m$ is reduced. $\square$

Theorem 4.22. *Let G_Γ be a graph product. Suppose $g = g_1 \cdots g_m \in G_\Gamma$ ($m \ge 1$) is in cyclically reduced form such that for some $n \ge 3$ each $g_i \in G_{v_i}$ has order at least n, $1 \le i \le m$. Then either*

$$\mathrm{scl}_{G_\Gamma}(g) \ge \frac{1}{2} - \frac{1}{n},$$

or the full subgraph Λ on $\{v_1, \ldots, v_m\}$ in Γ is a complete graph. In the latter case, we have

$$\mathrm{scl}_{G_\Gamma}(g) = \mathrm{scl}_{G_\Lambda}(g) = \max \mathrm{scl}_{G_i}(g_i).$$

Proof. Fix any v_k, similar to the proof of Lemma 4.21, we express G_Γ as an amalgam $A \star_C B$, where A, C and B are the graph products associated to $st(v_k)$, $lk(v_k)$ and the full subgraph on $V(\Gamma) - \{v_k\}$ respectively. If there is some $v_i \notin st(v_k)$, then up to a cyclic conjugation, $g = a_1 b_1 \cdots a_\ell b_\ell$ where $\ell \ge 1$, each a_i and b_i is a product of consecutive g_j's such that $b_i \in B - C$ and each $a_i \in A - C$ is of maximal length. Since g is cyclically reduced, each a_i (resp. b_i) is n-RTF in (A, C) (resp. (B, C)) by Lemma 4.21. It follows from Theorem 4.7 that $\mathrm{scl}_{G_\Gamma}(g) \ge 1/2 - 1/n$.

Therefore, the argument above implies $\mathrm{scl}_{G_\Gamma}(g) \ge 1/2 - 1/n$ unless $v_i \in st(v_k)$ for all i, which holds for all k only when the full subgraph Λ on $\{v_1, \ldots, v_m\}$ in Γ is complete. In this case, G_Γ retracts to $G_\Lambda = \oplus G_{v_i}$. Then $v_i \ne v_j$ whenever $i \ne j$ since g is reduced, thus the conclusion follows from Lemma 2.12 (2).

$\square$

Remark 4.23. The estimate is sharp in the following sense. For any $g_v \in G_v$ of order $n \ge 2$

and any $g_u \in G_u$ of order $m \geq 2$ with u not equal or adjacent to v, then the retract from G_Γ to $G_u \star G_v$ gives

$$\mathrm{scl}_{G_\Gamma}([g_u, g_v]) = \mathrm{scl}_{G_u \star G_v}([g_u, g_v]) = \frac{1}{2} - \frac{1}{\min(m, n)}$$

by [18, Proposition 5.6].

Theorem 4.24. *Let G_Γ be the graph product of $\{G_v\}$. Suppose for some $n \geq 3$ and $C > 0$, each G_v has no k-torsion for all $2 \leq k \leq n$ and has strong gap C. Then G_Γ has strong gap $\min\{C, 1/2 - 1/n\}$.*

Proof. For any nontrivial $g \in G_\Gamma$ written in reduced form, by Theorem 4.22, we either have $\mathrm{scl}_{G_\Gamma}(g) \geq 1/2 - 1/n$ or $\mathrm{scl}_{G_\Gamma}(g) = \max \mathrm{scl}_{G_i}(g_i) \geq C$. $\square$

Corollary 4.25. For $n \geq 3$, any graph product of abelian groups without k-torsion for all $2 \leq k \leq n$ have strong gap $1/2 - 1/n$. In particular, all right-angled Artin groups have strong gap $1/2$.

Unfortunately, our result does not say much about the interesting case of right-angled Coxeter groups due to the presence of 2-torsion.

Question 4.26. Is there a spectral gap for every right-angled Coxeter group? If so, is there a uniform gap?

4.3   Scl of Elliptic Elements

Our estimate of scl of hyperbolic elements is based on scl relative to vertex groups. Thus it does not give any meaningful information for elliptic elements. The goal of this section is to characterize and give some simple estimates of scl of elliptic elements, and more generally, chains supported in vertex groups.
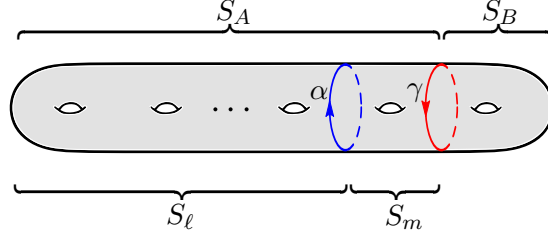
Figure 4.2: Illustration of Example 4.27. This appears originally in [21, Figure 10].

Let us start with simple examples. For a free product $G = A \star B$, we know that G has a retract to each factor, and thus $\mathrm{scl}_G(a) = \mathrm{scl}_A(a)$ for any $a \in A$. This is no longer true in general for amalgams.

Example 4.27. Let S be a closed surface of genus $g > 4$. Let γ be a separating simple closed curve that cuts S into S_A and S_B, where S_A has genus $g-1$ and S_B has genus 1. Let a be an element represented by a simple closed loop α in S_A that bounds a twice-punctured torus S_m with γ; see Figure 4.2. Then $G = \pi_1(S)$ splits as $G = A \star_C B$, where $A = \pi_1(S_A)$, $B = \pi_1(S_B)$ and C is the cyclic group generated by an element represented by γ. In this case, the element a is supported in A and the corresponding loop α does bound a surface S_ℓ in S_A of genus $g-2$, which is actually a retract of S_A and we have $\mathrm{scl}_A(a) = g - 5/2$. However, α also bounds a genus two surface from the B side, which is the union of S_B and S_m, showing that $\mathrm{scl}_G(a) \leq 3/2$, which is smaller than $\mathrm{scl}_A(a)$ since $g > 4$.

The example above shows that one can use chains in edge groups to adjust the given chain in vertex groups to a better one before evaluating it in individual vertex groups. Actually, scl is obtained by making the best adjustment of this kind.

Theorem 4.28. *Let $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups with $\Gamma = (V, E)$. For any finite collection of chains $c_v \in C_1^H(G_v)$, we have*

$$\mathrm{scl}_G(\sum_v c_v) = \inf \sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e), \tag{4.6}$$

where the infimum is taken over all finite collections of chains $c_e \in C_1^H(G_e)$ satisfying

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$c_e + c_{\bar{e}} = 0$ *for each* $e \in E$.

Proof. By the homology calculation in equation (3.1), we observe that $\mathrm{scl}_G(\sum_v c_v)$ is finite if and only if the right-hand side of (4.6) is. Thus we will assume both to be finite in the sequel. We will prove the equality for an arbitrary collection of rational chains c_v. Then the general case will follow by continuity.

Let X be the standard realization of G as in Subsection 3.1. Represent each chain c_v by a rational formal sum of elliptic tight loops in the corresponding vertex space X_v. Let $f : S \to X$ be any admissible surface in normal form (see Definition 3.4) of degree n for the rational chain $\sum c_v$ in $C_1^H(G)$. For each $v \in V$, let S_v be the union of components in the decomposition of S that are supported in the thickened vertex space $N(X_v)$. Note that S_v only has loop boundary since our chain is represented by elliptic loops. Moreover, any loop boundary supported in some edge space is obtained from cutting S along edge spaces. For each edge e with $t(e) = v$, let $c_e \in C_1^H(G_e)$ be the integral chain that represents the union of loop boundary components of S_v supported in X_e. Then S_v is admissible of degree n for the chain $c_v + \frac{1}{n}\sum_{t(e)=v} c_e$ in $C_1^H(G_v)$ for all $v \in V$. See Figure 4.3. Note that we must have $c_e + c_{\bar{e}} = 0$ since loops in c_e and $c_{\bar{e}}$ are paired and have opposite orientations. Since S is in normal form and $\sqcup S_v$ has no polygonal boundary, there are no disk components and hence we have

$$\frac{-\chi(S_v)}{2n} = \frac{-\chi^-(S_v)}{2n} \geq \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e), \quad \text{and}$$

$$\frac{-\chi(S)}{2n} = \sum_v \frac{-\chi(S_v)}{2n} \geq \sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e).$$

Since S is arbitrary, this proves the "$\geq$" direction in (4.6).

Conversely, consider any collection of chains $c_e \in C_1^H(G_e)$ satisfying $c_e + c_{\bar{e}} = 0$. With an arbitrarily small change of $\sum_v \mathrm{scl}_{G_v}(c_v + \sum_{t(e)=v} c_e)$, we replace this collection by another with the additional property that each c_e is a rational chain. This can be done since each
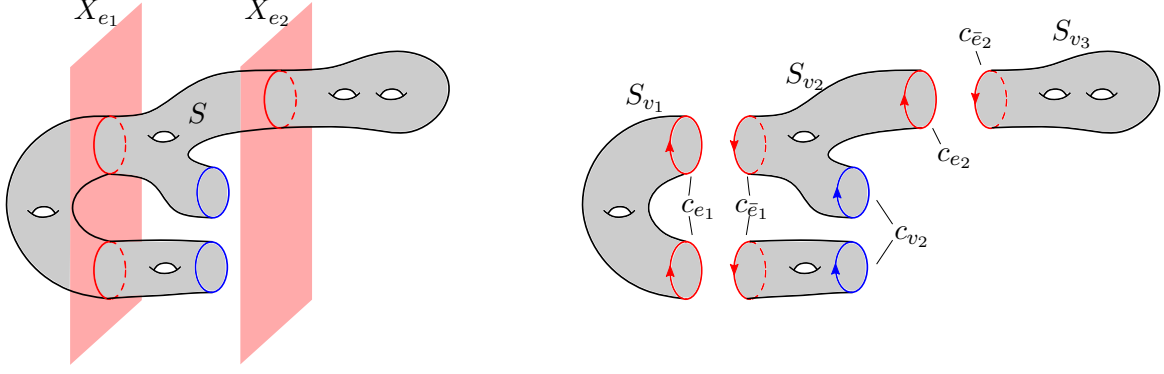
Figure 4.3: This is an example where $\sum c_v = c_{v_2}$ is supported in a single vertex group G_{v_2}, represented as the formal sum of the blue loops in X_{v_2}. On the left we have an admissible surface S for c_{v_2} of degree 1. The edge spaces X_{e_1}, X_{e_2} cut S in to $S_{v_1}, S_{v_2}, S_{v_3}$ shown on the right. The edges are oriented so that $t(e_1) = v_1$ and $t(e_2) = v_2$. Then S_{v_1} is admissible for its boundary, which is the chain c_{e_1}, and similarly S_{v_3} is admissible for $c_{\bar{e}_2}$,. The surface S_{v_2} is admissible for $c_{v_2} + c_{\bar{e}_1} + c_{e_2}$, where $c_{\bar{e}_1} = -c_{e_1}$ due to opposite orientations induced from S_{v_2} and S_{v_1}, and similarly $c_{e_2} = -c_{\bar{e}_2}$. Thus the sum of all boundary components of $\sqcup_i S_{v_i}$ is equal to c_{v_2} in $C_1^H(G)$. This appears originally in [21, Figure 11].

c_v is rational. For each $v \in V$, let S_v be any admissible surface for the rational chain $c_v + \sum_{e:\ t(e)=v} c_e$. By taking suitable finite covers, we may assume all S_v to be of the same degree n. Since $c_e + c_{\bar{e}} = 0$ for all $e \in E$, the union $\sqcup_v S_v$ is an admissible surface of degree n for a chain equivalent to $\sum_v c_v$ in $C_1^H(G)$. Since the S_v and the collection c_e are arbitrary, this proves the other direction of (4.6). $\qquad\square$

Remark 4.29. If some $e \in E$ has $\mathrm{scl}_{G_e} \equiv 0$ (e.g. when G_e is amenable), then $\mathrm{scl}_{G_{t(e)}}$ and scl_G both vanish on $B_1^H(G_e)$ by monotonicity. Given this, the typically infinite-dimensional space $C_1^H(G_e)$ in Theorem 4.28 can be replaced by the quotient $C_1^H(G_e)/B_1^H(G_e) \cong H_1(G_e, \mathbb{R})$, which is often (e.g. when G_e is finitely generated) finite-dimensional, for which Theorem 4.28 is still valid. Thus if all edge groups have vanishing scl and scl_{G_v} is understood in each finite-dimensional subspace of $C_1^H(G_v)$ for all v, then scl_G in vertex groups can be practically understood by equation (4.6), which is a convex programming problem.

Corollary 4.30. Let $G = A \star_C B$ be an amalgam. If C has trivial scl and $H_1(C; \mathbb{R}) = 0$, then $\mathrm{scl}_G(a) = \mathrm{scl}_A(a)$ for any $a \in C_1^H(A)$.

Proof. For any chain $c \in C_1^H(C)$, we have $\mathrm{scl}_C(c) = 0$ by our assumption, and thus $\mathrm{scl}_A(c) = \mathrm{scl}_B(c) = 0$ by monotonicity of scl. The conclusion follows readily from Theorem 4.28. $\square$

It is clear from Example 4.27 that the assumption $H_1(C; \mathbb{R}) = 0$ is essential in Corollary 4.30.

When the chain is supported in a single vertex group, we obtain the following simple estimate:

Lemma 4.31. *For any vertex v and any chain $c \in C_1(G_v)$, we have*

$$\mathrm{scl}_G(c) \geq \mathrm{scl}_{(G_v, \{G_e\}_{t(e)=v})}(c).$$

Proof. The right-hand side is a term in the infimum of equation (4.6). $\square$

4.3.1 Scl in Edge Groups

By inclusion of edge groups into vertex groups, Theorem 4.28 also applies to chains supported in edge groups. We carry this out carefully to reveal some interesting behaviors of scl in edge groups. For this purpose, it is convenient to view scl as a *degenerate norm*.

Definition 4.32. A *degenerate norm* $\|\cdot\|$ on a vector space V is a pseudo-norm on a linear subspace V^f, called the *domain* of $\|\cdot\|$, and is $+\infty$ outside V^f. The unit norm ball B of $\|\cdot\|$ is the (convex) set of vectors v with $\|v\| \leq 1$. The *vanishing locus* V^z is the subspace consisting of vectors v with $\|v\| = 0$. Note that $V^z \subset B \subset V^f$.

In the sequel, norms refer to degenerate norms unless emphasized as genuine norms. A norm in a finite-dimensional space with rational vanishing locus automatically has a "spectral gap".

Lemma 4.33. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. If the vanishing locus V^z is a rational subspace, then $\|\cdot\|$ satisfies a gap property on $\mathbb{Z}^n$: there exists $C > 0$ such that either $\|P\| = 0$ or $\|P\| \geq C$ for all $P \in \mathbb{Z}^n$.*

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Proof. Extend a rational basis $e_1, \ldots, e_m$ of V^z to a rational basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, where $m \le n$. Then there is some integer $N > 0$ such that any $P = \sum_i P_i e_i \in \mathbb{Z}^n$ has $NP_i \in \mathbb{Z}$ for all i. The restriction of $\| \cdot \|$ to the subspace spanned by $e_{m+1}, \ldots, e_n$ has trivial vanishing locus and thus there is a constant $C > 0$ such that $\| \sum_{j=m+1}^n Q_j e_j \| \ge NC$ for any integers $Q_{m+1}, \ldots, Q_n$ unless they all vanish. Therefore, if $P = \sum_{i=1}^n P_i e_i \in \mathbb{Z}^n \setminus V^z$, then $\|P\| = \| \sum_{j=m+1}^n NP_j e_j \| / N \ge C$. $\qquad\square$

Recall from Sect. 2.1.3 that scl_G is a (degenerate) norm on $C_1^H(G)$. If G is a subgroup of $\tilde{G}$, then $\mathrm{scl}_{\tilde{G}}$ also restricts to a norm on $C_1^H(G)$.

Let us set up some notation. For a graph of groups $G = \mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ with $\Gamma = (V, E)$. For each vertex $v \in V$, let $C_v := \bigoplus_{t(e)=v} C_1^H(G_e)$ be the space parameterizing chains in edge groups adjacent to v. Let $C_E := \bigoplus_{\{e, \bar{e}\}} C_1^H(G_e)$ parameterize chains in all edge groups.

Define $\|(c_{(e,\bar{e})})\|_E := \mathrm{scl}_G(\sum c_{e,\bar{e}})$ for any $(c_{(e,\bar{e})}) \in C_E$. Equivalently, $\| \cdot \|_E$ is the pullback of scl_G via

$$\bigoplus_{\{e,\bar{e}\}} C_1^H(G_e) \longrightarrow \bigoplus_{\{e,\bar{e}\}} C_1^H(G) \xrightarrow{+} C_1^H(G),$$

where the former map is inclusion on each summand and the latter map takes the summation.

Similarly, for each vertex $v \in V$, we pull back scl_{G_v} to get a norm $\| \cdot \|_v$ on $C_v = \bigoplus_{t(e)=v} C_1^H(G_e)$ via the composition

$$\bigoplus_{e: \, t(e)=v} C_1^H(G_e) \xrightarrow{\oplus t_{e*}} \bigoplus_{e: \, t(e)=v} C_1^H(G_v) \xrightarrow{+} C_1^H(G_v),$$

Note that $\bigoplus_{v \in V} C_v$ is naturally isomorphic to $\bigoplus_{\{e,\bar{e}\}} (C_1^H(G_e) \oplus C_1^H(G_{\bar{e}}))$ which has a surjective map π to C_E whose restriction on any summand is

$$C_1^H(G_e) \oplus C_1^H(G_{\bar{e}}) = C_1^H(G_e) \oplus C_1^H(G_e) \xrightarrow{+} C_1^H(G_e).$$

The kernel of π is exactly the collections of chains c_e over which we take infimum in equation (4.6).

Equipping $\bigoplus_{v \in V} C_v$ with the ℓ^1 product norm given by $\|(x_v)\|_1 := \sum_{v \in V} \|x_v\|_v$ induces a norm $\|\cdot\|$ on its quotient C_E via $\|x\| := \inf_{x = \pi(y)} \|y\|_1$.

Corollary 4.34. With the notation above, the two norms $\|\cdot\|$ and $\|\cdot\|_E$ on C_E agree.

Proof. This is simply an equivalent statement of Theorem 4.28 for chains in edge groups. $\square$

This is particularly simple in the special case of amalgams $G = A \star_C B$ so that the unit norm ball has a precise description. To do so, we need the following notion from convex analysis. The *algebraic closure* of a set A consists of points x such that there is some $v \in V$ so that for any $\epsilon > 0$, there is some $t \in [0, \epsilon]$ with $x + tv \in A$. If A is convex, the algebraic closure of A coincides with $\mathrm{lin}(A)$ defined in [32, p. 9], which is also convex. If A is in addition finite-dimensional, then its algebraic closure agrees with the topological closure of A [32, p. 59].

Theorem 4.35. *Let $G = A \star_C B$ be the amalgamated free product associated to inclusions $\iota_A : C \to A$ and $\iota_B : C \to B$. Then for any chain $c \in C_1^H(C)$, we have*

$$\mathrm{scl}_G(c) = \inf_{\substack{c_1, c_2 \in C_1^H(C) \\ c_1 + c_2 = c}} \{\mathrm{scl}_A(c_1) + \mathrm{scl}_B(c_2)\}. \tag{4.7}$$

The unit ball of scl *on C equals the algebraic closure of* $\mathrm{conv}(B_A \cup B_B)$*, where* $\mathrm{conv}(\cdot)$ *denotes the convex hull, B_A and B_B are the unit norm balls of the pullbacks of* scl_A *and* scl_B *via ι_A and ι_B on $C_1^H(C)$ respectively. If $C \cong \mathbb{Z}$ with generator t, then*

$$\mathrm{scl}_G(t) = \min\{\mathrm{scl}_A(t), \mathrm{scl}_B(t)\}.$$

Proof. Equation (4.7) is an explicit equivalent statement of Corollary 4.34 in our case. The assertion on the unit norm ball is an immediate consequence of (4.7) and Lemma 4.38 that

we will prove below. When $C \cong \mathbb{Z} = \langle t \rangle$ and $c = t$, we have $c_1 = \lambda t$ and $c_2 = (1 - \lambda)t$ for some $\lambda \in \mathbb{R}$ and $\mathrm{scl}_G(t) = \inf_\lambda \{|\lambda|\mathrm{scl}_A(t) + |1 - \lambda|\mathrm{scl}_B(t)\}$, where the optimization is achieved at either $\lambda = 0$ or $\lambda = 1$. $\square$

In the special case where the edge group is $\mathbb{Z}$, we can construct extremal quasimorphisms for edge group elements; see [21, Proposition 6.9] for details.

Now we use the simple formula (4.7) to describe the unit norm ball of $(\mathrm{scl}_G)|_{C_1^H(C)}$ in the case of amalgams. To accomplish this, we introduce the following definition.

Definition 4.36. For two degenerate norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space V. The ℓ^1-*mixture norm* $\| \cdot \|_m$ is defined as

$$\|v\|_m = \inf_{v_1 + v_2 = v} (\|v_1\|_1 + \|v_2\|_2).$$

Then the norm $(\mathrm{scl}_G)|_{C_1^H(C)}$ is the ℓ^1-mixture of scl_A and scl_B by (4.7) in Theorem 4.35.

Let V_i^f, V_i^z and B_i be the domain, vanishing locus and unit ball of the norm $\| \cdot \|_i$ for $i = 1, 2$ respectively. Note that the domain V_m^f of $\| \cdot \|_m$ is $V_1^f + V_2^f$, and the vanishing locus V_m^z of $\| \cdot \|_m$ contains $V_1^z + V_2^z$ as a subspace.

Lemma 4.37. *The vanishing locus V_m^z of $\| \cdot \|_m$ is $V_1^z + V_2^z$ if V is finite-dimensional.*

Proof. Fix an arbitrary genuine norm $\| \cdot \|$ on V. Let E_1 be a subspace of V_1^f such that V_1^f is the direct sum of E_1 and V_1^z. Then $\| \cdot \|_1$ is a genuine norm on E_1, a finite-dimensional space, and thus there exists $r_1 > 0$ such that any $u \in E_1$ with $\|u\|_1 \leq 1$ has $\|u\| \leq r_1$. It follows that every vector v in B_1 can be written as $v_0 + u$ where $v_0 \in V_1^z$ and $\|u\| \leq r_1$. A similar result holds for $\| \cdot \|_2$ with some constant $r_2 > 0$. Then for any $v \in V_m^z$, for any $\epsilon > 0$, we have $v = v_1 + v_2 + u_1 + u_2$ for some $v_i \in V_i^z$ and u_i satisfying $\|u_i\| \leq \epsilon r_i$, $i = 1, 2$. Hence V_m^z is contained in the closure of $V_1^z + V_2^z$. But $V_1^z + V_2^z$ is already closed since V is finite-dimensional. $\square$

The unit norm ball B_m of an ℓ^1-mixture norm $\|\cdot\|_m$ has a simple description.

Lemma 4.38. *The unit norm ball B_m is the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$, where $\mathrm{conv}(\cdot)$ takes the convex hull. If the underlying space is finite-dimensional, then we can take the topological closure of $\mathrm{conv}(B_1 \cup B_2)$ instead.*

Proof. Fix any $v \in B_m$. For any $\epsilon > 0$, there exist v_1, v_2 with $v = v_1 + v_2$ and $\|v_1\|_1 + \|v_2\|_2 < 1 + \epsilon$. Let $u_i \in B_i$ be $v_i/\|v_i\|_i$ if $\|v_i\|_i \neq 0$, and 0 otherwise. With $d = \max(1, \|v_1\|_1 + \|v_2\|_2)$, we have

$$\frac{v}{d} = \frac{\|v_1\|_1}{d} \cdot u_1 + \frac{\|v_2\|_2}{d} \cdot u_2 + (1 - \frac{\|v_1\|_1 + \|v_2\|_2}{d}) \cdot 0 \in \mathrm{conv}(B_1 \cup B_2).$$

It follows that for any $\epsilon > 0$, there is some $0 \leq t < \epsilon$ such that $(1 - t)v \in \mathrm{conv}(B_1 \cup B_2)$. Thus v is in the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$.

Conversely, fix any v in the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$. Note that $\mathrm{conv}(B_1 \cup B_2)$ is a subset of $V_1^f + V_2^f$ and that any linear subspace is algebraically closed, so the algebraic closure of $\mathrm{conv}(B_1 \cup B_2)$ is a subset of $V_1^f + V_2^f$. Then by definition, there is some $u = u_1 + u_2$ with $u_i \in V_i^f$ such that for any $\epsilon > 0$, we have $v + tu \in \mathrm{conv}(B_1 \cup B_2)$ for some $0 \leq t \leq \epsilon$. Thus $v = \lambda v_1 + (1 - \lambda)v_2 - tu = [\lambda v_1 - tu_1] + [(1 - \lambda)v_2 - tu_2]$ for some $\lambda \in [0, 1]$ and $v_i \in B_i$. We see

$$\begin{aligned}
\|v\|_m &\leq \|\lambda v_1 - tu_1\|_1 + \|(1 - \lambda)v_2 - tu_2\|_2 \\
&\leq \lambda\|v_1\|_1 + (1 - \lambda)\|v_2\|_2 + t(\|u_1\|_1 + \|u_2\|_2) \\
&\leq 1 + \epsilon(\|u_1\|_1 + \|u_2\|_2).
\end{aligned}$$

Since ϵ is arbitrary and $\|u_1\|_1 + \|u_2\|_2$ is finite, we get $v \in B_m$. $\square$

Remark 4.39. It is necessary to take the algebraic closure. On $\mathbb{R}^2 = \{(x, y)\}$, let $\|(x, y)\|_1 = \infty$ if $y \neq 0$ and $\|(x, 0)\|_1 = |x|$, and let $\|(x, y)\|_2 = \infty$ if $x \neq 0$ and $\|(0, y)\|_2 = 0$. Then

their ℓ^1-mixture has formula $\|(x, y)\|_m = |x|$. Thus the unit balls are $B_1 = [-1, 1] \times \{0\}$, $B_2 = \{0\} \times \mathbb{R}$ and $B_m = [-1, 1] \times \mathbb{R}$. Hence $\text{conv}(B_1 \cup B_2) = (-1, 1) \times \mathbb{R} \cup \{(\pm 1, 0)\}$ does not agree with B_m but its algebraic closure does.

Lemma 4.38 confirms the assertion on the unit norm ball in Theorem 4.35 and finishes its proof. This allows us to look at explicit examples showing how scl behaves under surgeries.

Example 4.40. Let Σ be a once-punctured torus with boundary loop γ. Then $X_A := S^1 \times \Sigma$ is a compact 3-manifolds with torus boundary T_A. Let γ_A be a chosen section of γ in T_A and let τ_A be a simple closed curve on T_A representing the S_1 factor. Let $C := \pi_1(T_A) = \langle \gamma_A, \tau_A \rangle \cong \mathbb{Z}^2$ be the peripheral subgroup of $A := \pi_1(X_A)$, where we abuse the notation and use γ_A, τ_A to denote their corresponding elements in $\pi_1(T_A)$.

Let X_B be another copy of X_A, where T_B, γ_B and τ_B correspond to T_A, γ_A and τ_A respectively. For any coprime integers p, q, there is an orientable closed 3-manifold $M_{p,q}$ (not unique) obtained by gluing T_A and T_B via a map $\phi : \pi_1(T_B) \to \pi_1(T_A) = C$ taking γ_B to $p\gamma_A + q\tau_A$. Then $\pi_1(M_{p,q})$ is an amalgam $A \star_C B$ where $B := \pi_1(X_B)$ and the inclusion $C \to B$ is given by $C \xrightarrow{\phi^{-1}} \pi_1(T_B) \hookrightarrow B$.

Identify $H_1(C; \mathbb{R})$ with $\mathbb{R}^2$ with $(1, 0)$ representing $[\gamma_A]$ and $(0, 1)$ representing $[\tau_A]$. According to Remark 4.29, scl_A and scl_B induce norms on $H_1(C; \mathbb{R}) \cong C_1^H(C)/B_1^H(C)$. Then the norm scl_A on $H_1(C; \mathbb{R})$ has an one-dimensional unit norm ball, which is the segment connecting $(-2, 0)$ and $(2, 0)$ since $\text{scl}_A(\gamma_A) = \text{scl}_\Sigma(\gamma) = 1/2$. Similarly the unit norm ball of scl_B on $H_1(C; \mathbb{R})$ is the segment connecting $(2p, 2q)$ and $(-2p, -2q)$. By Theorem 4.35, the unit norm ball of $\text{scl}_{M_{p,q}}$ on $H_1(C; \mathbb{R})$ is the convex hull of $\{(\pm 2, 0), \pm(2p, 2q)\}$ (which is already closed), which intersects the positive y-axis at $(0, \frac{q}{p+1})$ when $p, q > 0$. In this case, we have $\text{scl}_{M_{p,q}}(\tau_A) = \frac{p+1}{q}$.

In particular, with $p = 1$ and $q \in \mathbb{Z}_+$, the image of the loop τ_A has positive scl $2/q$ converging to 0 as q goes to infinity. In this example, the image of the torus T_A is a JSJ torus, cutting the manifold into X_A and X_B, both trivially Seifert fibered. Thus we exhibit a family

of graph manifolds $M_{1,q}$ which contain elements with positive scl converging to 0 by changing the gluing of the Seifert pieces. This is analogous in spirit to the known construction of closed hyperbolic 3-manifolds containing elements with positive scl converging to 0, obtained from different Dehn fillings on a fixed knot complement [14, Example 2.4].

CHAPTER 5

RATIONALITY AND COMPUTATION OF STABLE

COMMUTATOR LENGTH

Throughout this chapter, we consider graphs of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where

(1) each vertex group G_v has trivial scl, and

(2) the images of edge groups in each vertex group are *central* and *mutually commensurable*.

We develop an algorithm to compute scl in such groups, and the nature of the algorithm implies that scl takes rational values.

The first assumption can be weakened to $\mathrm{scl}_G(c) = 0$ for any $c \in B_1^H(G_v)$ and any v, but it is usually hard to check without having $\mathrm{scl}_{G_v} \equiv 0$. See Theorem 2.11 for a list of such groups.

Example 5.1. The following families of groups satisfy both assumptions above.

(1) Any graphs of groups with vertex and edge groups isomorphic to $\mathbb{Z}$, since all non-trivial subgroups of $\mathbb{Z}$ are commensurable. These groups are also known as generalized Baumslag–Solitar groups.

(2) Amalgams of abelian groups, since the commensurability assumption is vacuous for vertices of valence one. This includes the groups studied in [39].

(3) Graphs of groups where vertex groups are isomorphic to the Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ or fundamental groups of irreducible 3-manifolds with Nil geometry, and edge groups are isomorphic to $\mathbb{Z}$ and maps into the central $\mathbb{Z}$ subgroup of the vertex groups generated by a regular Seifert fiber (see [37]). Note that these vertex groups are amenable (actually virtually solvable [42, Theorem 4.7.8]), and thus have trivial scl.

(4) Free products of groups with trivial scl. These are the groups studied in [18].

In Section 5.1, we define disk-like pieces. These hold a finite amount of information about winding numbers, which turns out to be sufficient to solve the gluing conditions asymptotically. Based on this, we describe a linear programming algorithm computing scl and prove rationality in Section 5.2. Finally in Section 5.3, we study examples in Baumslag–Solitar groups and establish several explicit formulas.

5.1 Asymptotic Promotion

In our graph of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$, let $\underline{g}$ be a finite collection of hyperbolic elements and $\underline{\gamma}$ be their tight loop representatives (see Sect. 3.1.2). We would like to compute scl of a rational chain $c = \sum r_i g_i$ relative to vertex groups. By Lemma 3.9, it comes down to understanding components that might appear in simple relative admissible surfaces. The essential difficulty is that there are too many possible components since the winding number of each turn could have infinitely many choices when some edge group is infinite. One can simply ignore the winding numbers of turns to estimate the Euler characteristic to get lower bounds of scl. Clay–Forester–Louwsma [22] and Susse [39] show such a lower bound turns out to be equality in certain cases. In general, we cannot completely ignore the winding number.

Surprisingly, it turns out that recording winding numbers of turns mod certain *finite index* subgroup of edge groups is sufficient (Lemma 5.15) to asymptotically recover the surface by adjusting the winding numbers. This is the goal of this section and is the heart of the rationality theorem (Theorem 5.25). To this end, we investigate adjustment of winding numbers in terms of *transition maps* and *adjustment maps* which we will define.

For the moment, we will also assume Γ to be *locally finite* for convenience. We can always reduce the general situation to this case using restriction of domain (Proposition 3.11).

For each vertex v, let W_v be the subgroup generated by the images of adjacent edge groups. By our assumption and local finiteness, W_v is *central* in G_v and each adjacent edge

group is *finite index* in W_v.

Definition 5.2. A *virtual isomorphism* $\phi : H \to H'$ is an isomorphism $\phi : H_0 \to H'_0$ of finite index subgroups $H_0 \leq H$ and $H'_0 \leq H'$. The domain $\mathrm{Dom}\phi = H_0$ and the image $\mathrm{Im}\phi = H'_0$ are part of the data of ϕ. Typically ϕ is not defined for elements outside $\mathrm{Dom}\phi$. In the case $H = H'$, we say ϕ is a *virtual automorphism.*

Two virtual isomorphisms $\phi : H_1 \to H_2$ and $\psi : H_2 \to H_3$ form a composition $\psi\phi$ with $\mathrm{Dom}\psi\phi = \phi^{-1}(\mathrm{Im}\phi \cap \mathrm{Dom}\psi)$ and $\mathrm{Im}\psi\phi = \psi(\mathrm{Im}\phi \cap \mathrm{Dom}\psi)$, which is again a virtual isomorphism. Each virtual isomorphism has an inverse in the obvious way.

A similar notion appears in [40], which agrees with ours in the context of finitely generated abelian groups.

It follows from the commensurability assumption and local finiteness of Γ that the inclusion $t_e = o_{\bar{e}} : G_e \to W_v$ is a virtual isomorphism for all edges e with $t(e) = v$. For each edge e connecting vertices $u = o(e)$ and $v = t(e)$ (possibly $u = v$), we form a *transition map* $\tau_e : W_u \to W_v$ via $\tau_e := -t_e \circ o_e^{-1}$, which is a virtual isomorphism. The negative sign makes sense since W_u and W_v are abelian, and the reason we add it will be clear when we define adjustment maps in Sect. 5.1.2.

Moreover, for any oriented path $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ in Γ with $o(e_i) = v_{i-1}$ and $t(e_i) = v_i$, we have a virtually isomorphic *transition map* $\tau_P : W_{v_0} \to W_{v_n}$ defined as

$$\tau_P := \tau_{e_n} \circ \cdots \circ \tau_{e_1}.$$

For each vertex v, recall from Sect. 3.1.2 that A_v denotes the collection of arcs supported in $N(X_v)$ obtained by cutting hyperbolic loops in $\underline{\gamma}$. For any arc $a_v \in A_v$ on $\gamma_i \in \underline{\gamma}$, the loop γ_i projects to an oriented cycle $P(a_v) = (v, e_1, v_1, \ldots, e_n, v)$ in Γ, and thus gives rise to a transition map $\tau_{P(a_v)} : W_v \to W_v$ as above.

There is a stability result of virtual automorphisms that is important for our argument.

5.1.1 Stability of Virtual Automorphisms

Let $\phi : H \to H$ be a virtual automorphism. Typically both $\mathrm{Dom}\phi^p$ and $\mathrm{Im}\phi^p$ will keep getting smaller as $p \to \infty$. However, when H is an abelian group, the subgroup $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q$ generated by $\mathrm{Dom}\phi^p$ and $\mathrm{Im}\phi^q$ will stabilize to a finite index subgroup.

Lemma 5.3 (Stability). *Let H be an abelian group with a virtual automorphism ϕ. There is a finite index subgroup H_0 of H such that $H_0 \subset \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q$ for any $p, q \geq 0$.*

We first look at the simplest example with $H = \mathbb{Z}$, where everything is quite explicit.

Example 5.4. Let X, Y be non-zero integers. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be the virtual automorphism given by $\phi(X) = Y$ with $\mathrm{Dom}\phi = X\mathbb{Z}$ and $\mathrm{Im}\phi = Y\mathbb{Z}$. Let $d = \gcd(|X|, |Y|)$, $x = |X|/d$ and $y = |Y|/d$. Then $\mathrm{Dom}\phi^2 = \phi^{-1}(\mathrm{Im}\phi \cap \mathrm{Dom}\phi) = \phi^{-1}(dxy\mathbb{Z}) = dx^2\mathbb{Z}$ and $\mathrm{Im}\phi^2 = \phi(\mathrm{Im}\phi \cap \mathrm{Dom}\phi) = \phi(dxy\mathbb{Z}) = dy^2\mathbb{Z}$.

More generally, we have $\mathrm{Dom}\phi^p = dx^p\mathbb{Z}$ and $\mathrm{Im}\phi^q = dy^q\mathbb{Z}$, both keep getting smaller as $p, q \to \infty$. However, $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q = \gcd(dx^q, dy^q)\mathbb{Z} = d\mathbb{Z}$ for all $p, q \geq 1$.

We prove Lemma 5.3 by computing the index. Denote the index of $B \leq A$ by $|A : B|$. Recall the following basic identities, which we will use.

Lemma 5.5. *Let A be an abelian group with finite index subgroups B and C.*

(1) If C is a subgroup of B, then $|B : C| = |A : C|/|A : B|$.

(2)
$$|A : B + C| = \frac{|A : B|}{|B + C : B|} = \frac{|A : B|}{|C : B \cap C|}.$$

(3) If ϕ is an injective homomorphism defined on A, then $|A : B| = |\phi(A) : \phi(B)|$.

Lemma 5.6. *Let H be an abelian group with a virtual automorphism ϕ. Let $I_p := |H : \mathrm{Im}\phi^p|$.*

(1) We have
$$|H : \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q| = \frac{I_p I_q}{I_{p+q}}.$$

(2) There exist integers $r \geq 1$ and $N_\phi \geq 1$ such that $I_{q+1}/I_q = r$ for all $q \geq N_\phi$.

Proof. Note that ϕ^p isomorphically maps $\mathrm{Dom}\phi^p \cap \mathrm{Im}\phi^q$ to $\mathrm{Im}\phi^{p+q}$ and $\mathrm{Dom}\phi^p$ to $\mathrm{Im}\phi^p$. By the formulas in Lemma 5.5, we have

$$|H : \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q| = \frac{|H : \mathrm{Im}\phi^q|}{|\mathrm{Dom}\phi^p : \mathrm{Dom}\phi^p \cap \mathrm{Im}\phi^q|} = \frac{I_p I_q}{I_{p+q}}.$$

To prove the second assertion, let $p = 1$ in the equation above. We have

$$\frac{I_{q+1}}{I_q} = \frac{I_1}{|H : \mathrm{Dom}\phi + \mathrm{Im}\phi^q|}.$$

Since the sequence of *integers* $|H : \mathrm{Dom}\phi + \mathrm{Im}\phi^q|$ is increasing in q with upper bound $|H : \mathrm{Dom}\phi|$, it must stabilize when $q \geq N_\phi$ for some $N_\phi \geq 1$ and thus I_{q+1}/I_q stabilizes to some r when $q \geq N_\phi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proof of Lemma 5.3. With notation as in Lemma 5.6, we have $I_q = r^{q-N_\phi} I_{N_\phi}$ for all $q \geq N_\phi$. Thus

$$|H : \mathrm{Dom}\phi^p + \mathrm{Im}\phi^q| = \frac{I_p I_q}{I_{p+q}} = \frac{r^{p+q-2N_\phi} I_{N_\phi}^2}{r^{p+q-N_\phi} I_{N_\phi}} = \frac{I_{N_\phi}}{r^{N_\phi}},$$

for any $p, q \geq N_\phi$.

Since $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q \subset \mathrm{Dom}\phi^{p'} + \mathrm{Im}\phi^{q'}$ whenever $p \geq p'$ and $q' \geq q$, the index computation above shows that there is an index I_{N_ϕ}/r^{N_ϕ} subgroup H_0 of H such that $\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q = H_0$ when $p, q \geq N_\phi$. Therefore, for arbitrary $p, q \geq 0$,

$$\mathrm{Dom}\phi^p + \mathrm{Im}\phi^q \supseteq \mathrm{Dom}\phi^{\max\{p,N_\phi\}} + \mathrm{Im}\phi^{\max\{q,N_\phi\}} = H_0.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.1.2   Disk-like Pieces

Recall that any arc $a_v \in A_v$ on $\gamma_i \in \underline{\gamma}$ determines an oriented cycle $P(a_v) = (v, e_1, v_1, \ldots, e_n, v)$ in Γ, and consequently a transition map $\tau_{P(a_v)}$ which is a virtual automorphism of W_v.

Consider a simple relative admissible surface S for the rational chain $c = \sum r_i g_i$. Refer to a component of a subsurface S_v as *a piece*. Recall that S is obtained by gluing pieces together along paired turns on the polygonal boundaries in a certain way that can be encoded by a graph Γ_S, where each vertex corresponds to a piece and each edge corresponds to two paired turns glued together in S. There is a graph homomorphism $\pi : \Gamma_S \to \Gamma$ taking a vertex $\hat{v}$ to v if $\hat{v}$ corresponds to a piece C in S_v. In this way, S admits an induced structure of a graph of spaces with underlying graph Γ_S.

Recall that each piece C in S_v has a unique polygonal boundary, which as a loop in X_v represents a conjugacy class $w(C)$ in G_v called *the winding number* of C. Recall that W_v is the subgroup generated by all $t_e(G_e)$ with $t(e) = v$, which is central, so it makes sense to say whether $w(C)$ lies in W_v.

Definition 5.7. We say a piece C in S_v is a *potential disk* if its winding number $w(C)$ lies in W_v.

Since S is in simple normal form, a piece C is a disk if and only if the winding number $w(C)$ is trivial. Then potential disks are exactly the pieces that can be made into disks by adjusting winding numbers of turns on their polygonal boundaries. However, this typically cannot be done for all potential disks *simultaneously*. Our remedy is to find a class of potential disks, called *disk-like pieces*, that can be made into disks simultaneously in an asymptotic sense. Moreover, we will make sure that keeping track of a finite amount of information suffices to tell whether a piece is disk-like.

Recall that each oriented edge $\hat{e}$ of Γ_S going from $\hat{v}$ to $\hat{u}$ represents a turn shared by the pieces represented by $\hat{v}$ and $\hat{u}$. Changing the winding number of the turn by $n \in G_e$ would adjust $w(\hat{v})$ and $w(\hat{u})$ by $o_e(n)$ and $-t_e(n)$ respectively, where e is the projection of $\hat{e}$ in Γ.

Here the negative sign is due to the opposite orientations on the turn induced by the two pieces, which is why we have a negative sign in the definition of transition maps.

For an oriented path $\hat{P} = (\hat{v}_0, \hat{e}_1, \hat{v}_1, \ldots, \hat{e}_n, \hat{v}_n)$ in Γ_S projecting to a path P in Γ, associate to $\hat{P}$ the adjustment map $\alpha(\hat{P}) := \tau_P$. Then for any $x \in \mathrm{Dom}\,\alpha(\hat{P})$, we can adjust the winding numbers of the turns represented by $\hat{e}_i$ $(1 \leq i \leq n)$ such that, for any $1 \leq i \leq n-1$, the changes to the winding number of $\hat{v}_i$ contributed by the adjustment on $\hat{e}_i$ and $\hat{e}_{i+1}$ cancel each other, and the net result of the adjustment is

(1) the winding numbers of $\hat{v}_0$ and $\hat{v}_n$ increase by x and $\alpha(\hat{P})(x)$ respectively;

(2) the winding number of $\hat{v}_i$ stays unchanged for all $1 \leq i \leq n-1$.

We say such an adjustment is supported on $\hat{v}_0$ and $\hat{v}_n$.

Definition 5.8. For each $a_v \in A_v$, fix a finite index subgroup $W(a_v)$ of W_v such that

$$W(a_v) \subset \mathrm{Dom}\,\tau^p_{P(a_v)} + \mathrm{Im}\,\tau^q_{P(a_v)} \quad \text{for any } p, q \geq 0. \tag{5.1}$$

Such a $W(a_v)$ exists by Lemma 5.3.

Consider any piece C with a copy of a_v on its polygonal boundary. If $\gamma_i \in \underline{\gamma}$ is the loop contain a_v, then the component B of ∂S which this copy of a_v sits on is a finite cover of γ_i, say of degree n. Then B successively passes through n copies of a_v contained in pieces C_j, $j = 0, \ldots, n-1$, where $C_0 = C$. See Figure 5.1. It might happen that $C_j = C_k$ for $j \neq k$, in which case its polygonal boundary contain the j-,k-th copies of a_v on B as distinct arcs.

The boundary component B visits a sequence of pieces and thus determines an oriented cycle ω in Γ_S, on which we have vertices $\hat{v}_j$ corresponding to C_j for $j = 0, \ldots, n-1$. Let ω_j and ω'_j be the subpaths on ω going from $\hat{v}_0$ to $\hat{v}_j$ in the positive and negative orientation respectively, such that under the projection $\pi : \Gamma_S \to \Gamma$, we have $\pi(\omega_j) = P(a_v)^j$ and $\pi(\omega'_j) = \overline{P(a_v)}^{n-j}$. Thus the adjustment maps are $\alpha(\omega_j) = \tau^j_{P(a_v)}$ and $\alpha(\omega'_j) = \tau^{j-n}_{P(a_v)}$.
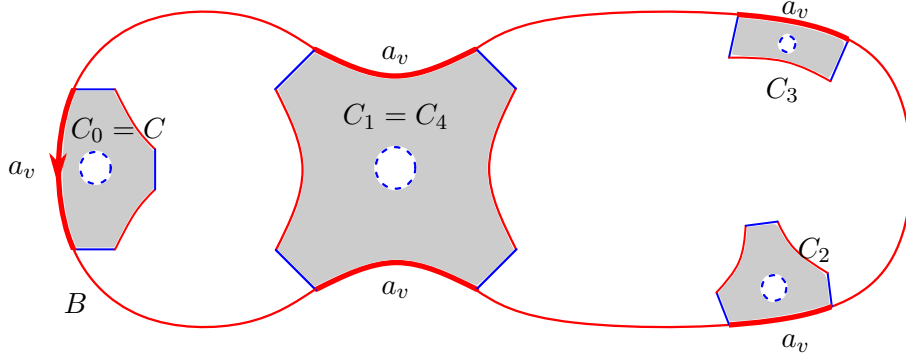
Figure 5.1: An example of a boundary component B of S with $n = 5$ copies of a_v, where C_1 coincides with C_4, and where pieces sitting along B not witnessing a_v are omitted. This appears originally in [20, Fig. 5].

Lemma 5.9. *With the above notation, let $C_j \neq C_0$ be a piece with $w(C_j) \in W(a_v)$. Then there is an adjustment of S supported on C_0 and C_j after which C_j has trivial winding number.*

Proof. By definition, we have $w(C_j) \in \mathrm{Dom}\tau_{P(a_v)}^{n-j} + \mathrm{Im}\tau_{P(a_v)}^{j} = \mathrm{Im}\alpha(\omega_j') + \mathrm{Im}\alpha(\omega_j)$. As a consequence, there exist $x, y \in W_v$ with $w(C_j) = \alpha(\omega_j')(x) + \alpha(\omega_j)(y)$. This gives rise to an adjustment that eliminates $w(C_j)$ and adjusts $w(C_0)$ to $w(C_0) - x - y$ without changing all other winding numbers. $\square$

Let $\widetilde{\Gamma}_S$ be a finite cover of Γ_S. This determines a cover $\widetilde{S}$ of S in simple normal form. Let $\widetilde{B}$ be a lift of B of degree m, which corresponds to a degree m lift $\widetilde{\omega}$ of ω in $\widetilde{\Gamma}_S$. Denote the m (distinct) lifts of C by $\widetilde{C}_0, \ldots, \widetilde{C}_{m-1}$ as vertices on $\widetilde{\omega}$.

Lemma 5.10. *With the above notation, suppose $w(C) \in W(a_v)$, then there is an adjustment of $\widetilde{S}$ supported on $\widetilde{C}_0, \ldots, \widetilde{C}_{m-1}$ after which $\widetilde{C}_k$ has trivial winding number for all $1 \leq k \leq m - 1$.*

Proof. For each $1 \leq k \leq m - 1$, apply Lemma 5.9 to the loop $\widetilde{\omega}$ to eliminate the winding number $w(\widetilde{C}_k)$ at the cost of changing $w(\widetilde{C}_0)$. $\square$

Once we normalize by the degree of admissible surfaces, the adjustment in Lemma 5.10 has the effect of making $1 - 1/m$ portion of an annulus piece C into disk pieces. This implies
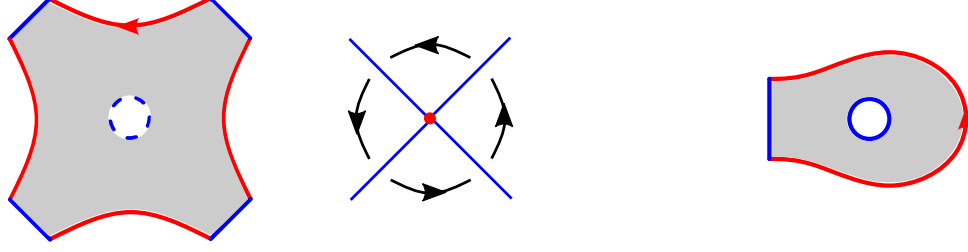
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Figure 5.2: The cyclic order on the edges around a vertex in Γ_S (left); A piece representing a vertex where the cycle ω backtracks (right). This appears originally in [20, Fig. 6].

that C can be asymptotically promoted to a disk as $m \to \infty$ without affecting other pieces if $w(C) \in W(a_v)$. In the exceptional case where ω is null-homotopic in Γ_S, we cannot find finite covers with $m \to \infty$ and need a different strategy to promote C.

Lemma 5.11. *The cycle ω representing the boundary component B of S in Γ_S backtracks at a vertex $\hat{u}$ if and only if $\hat{u}$ has valence one. In this case, the piece $\hat{u}$ has only one turn on the polygonal boundary and is not a potential disk.*

Proof. Let C' be a piece represented by a vertex $\hat{u}$ in Γ_S that ω passes through. Note that the edges adjacent to $\hat{u}$ correspond to the turns on the polygonal boundary of C' and thus have an induced cyclic order. See Figure 5.2. Since ω represents the boundary component, the two edges that ω enters and leaves $\hat{u}$ are adjacent in the cyclic order. Thus ω backtracks at $\hat{u}$ if and only if $\hat{u}$ has valence one, in which case the polygonal boundary of C' consists of one arc $a'_u \in A_u$ and one turn. Let $\hat{e}$ be the edge representing the gluing of this turn and let $e = \pi(\hat{e})$ so that $t(e) = \pi(\hat{u})$. Then $w(a'_u) \notin t_e(G_e)$ since $\underline{\gamma}$ consists of tight loops. It follows that C' cannot be a potential disk. $\qquad\square$

Lemma 5.12. *With the notation above, let B be the boundary component of S passing through a copy of a_v on a potential disk C. If the loop ω representing B in Γ_S is null-homotopic, then $P(a_v)$ is a null-homotopic loop in Γ and $\tau_{P(a_v)}$ is the identity on its domain. In particular, we have $W(a_v) \subset \mathrm{Dom}\,\tau_{P(a_v)}$.*

Proof. The conclusions follow immediately from the fact that $P(a_v)$ is the image of ω under the graph homomorphism $\pi : \Gamma_S \to \Gamma$. We deduce $W(a_v) \subset \mathrm{Dom}\tau_{P(a_v)}$ from equation (5.1). $\square$

Lemma 5.13. *With the notation above, suppose ω is null-homotopic and $w(C) \in W(a_v)$. Then there is an adjustment of S supported on C and a piece C' that is not a potential disk, such that $w(C)$ becomes 0 after the adjustment.*

Proof. Let C' be a piece as in Lemma 5.11 representing a vertex $\hat{u}$ where ω backtracks. Let ω_0 be the positively oriented subpath of ω going from C to C'. Considering ω as a cycle based at C, we have $\mathrm{Dom}\alpha(\omega) \subset \mathrm{Dom}\alpha(\omega_0)$ for the adjustment maps. Thus $w(C) \in W(a_v) \subset \mathrm{Dom}\alpha(\omega_0)$ by Lemma 5.12 and the claimed adjustment exists. $\square$

Definition 5.14. Given the choices of $W(a_v)$ for all arcs $a_v \in A_v$, a piece C of S_v is *a disk-like piece* if there is an arc $a_v \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v)$. Let

$$\widehat{\chi}(C) := \begin{cases} 1 & \text{if C is disk-like,} \\ 0 & \text{otherwise} \end{cases}$$

be the *over-counting Euler characteristic* that counts disk-like pieces as disks. Let $\widehat{\chi}_o(C) := -\frac{1}{4}\#\text{corners} + \widehat{\chi}(C)$ be the *over-counting orbifold Euler characteristic*. For a simple relative admissible surface S, define its *over-counting Euler characteristic* as

$$\widehat{\chi}(S) := \sum \widehat{\chi}_o(C),$$

where the sum is taken over all pieces C. Equivalently,

$$\widehat{\chi}(S) = \chi(S) + \#\{C : \text{disk-like but not a disk}\}.$$

Now we show that the over-counting is accurate via asymptotic promotion.

Lemma 5.15 (Asymptotic Promotion)**.** *Let G be a graph of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where*

(1) Γ is locally finite,

(2) each vertex group G_v has trivial scl, and

(3) the images of edge groups in each vertex group are central and mutually commensurable.

Let $\underline{g}$ be a finite collection of hyperbolic elements of G. With the notion of disk-like pieces and $\widehat{\chi}$ above (which depends on the choices of $W(a_v)$), we have the following: For any simple relative admissible surface S for a rational chain $c = \sum r_i g_i$ of degree n, and for any $\epsilon > 0$, there is a simple relative admissible surface S' of a certain degree n' such that

$$\frac{-\chi(S')}{2n'} \le \frac{-\widehat{\chi}(S)}{2n} + \epsilon.$$

Proof. For each disk-like piece C, say a component of S_v, fix an arc $a_v(C) \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v(C))$. Let $\omega(C)$ be the oriented cycle in Γ_S determined by the boundary component of S containing the copy of $a_v(C)$ on ∂C.

For those pieces C with null-homotopic $\omega(C)$, by applying the adjustment as in Lemma 5.13, we assume $w(C) = 0$ and C is a genuine disk.

For any given $\epsilon > 0$, choose a large integer $N > \frac{1}{2n\epsilon} \#\{C : \text{disk-like and } w(C) \neq 0\}$. For the finite collection of non-trivial loops $\Omega := \{\omega(C) : C \text{ disk-like and } w(C) \neq 0\}$, since free groups are residually finite, there is a finite cover $\widetilde{\Gamma}_S$ of Γ_S such that any lift of each $\omega \in \Omega$ has degree at least N. Let M be the covering degree.

Let $\widetilde{S}$ be the finite cover of S corresponding to $\widetilde{\Gamma}_S$. Applying the adjustment in Lemma 5.10 to each lift of each $\omega \in \Omega$, we observe that, for every disk-like piece C with $w(C) \neq 0$, at least $M(1 - 1/N)$ of its M preimages in $\widetilde{S}$ after adjustment bounds a disk.

Denote by S' the surface obtained by the adjustment above from $\widetilde{S}$. Then S' is simple

relative admissible of degree $n' = Mn$. We have

$$
\begin{aligned}
\frac{-\chi(S')}{2n'} &\leq \frac{-\chi(\widetilde{S}) - M(1 - 1/N)\#\{C : \text{disk-like and } w(C) \neq 0\}}{2nM} \\
&= \frac{-\chi(S) - (1 - 1/N)\#\{C : \text{disk-like and } w(C) \neq 0\}}{2n} \\
&= \frac{-\widehat{\chi}(S)}{2n} + \frac{\#\{C : \text{disk-like and } w(C) \neq 0\}}{2nN} \\
&< \frac{-\widehat{\chi}(S)}{2n} + \epsilon.
\end{aligned}
$$

$\square$

With Lemma 5.15 above, we can work with disk-like pieces and $\widehat{\chi}$ instead of genuine disks and χ. The advantage is that deciding whether a piece is disk-like or not is equivalent to checking whether an element in a finite abelian group vanishes or not, which only requires keeping track of a finite amount of information.

5.2 Determining scl by Linear Programming

With the help of Lemma 5.15, there are several known methods of encoding to compute scl via linear programming [12, 5, 22, 13, 43]. In this section, we use an encoding similar to those in [13, 18] to optimize our rationality result. We will use the notation from Section 5.1 and the setup in Lemma 5.15. We further assume the graph Γ to be *finite* for our discussion until Theorem 5.25, where we use restriction of domain (Proposition 3.11).

Given the choices of the finite index subgroups $W(a_v)$ of W_v satisfying (5.1) for all arcs $a_v \in A_v$ for any vertex v, let $D_v := \cap_{a_v \in A_v} W(a_v)$ be the intersection, which is also finite index in W_v. For each edge e of Γ, let $D_e := o_e^{-1} D_{o(e)} \cap t_e^{-1} D_{t(e)}$ and $W_e := G_e/D_e$. Then W_e is a finite abelian group with induced homomorphisms $\overline{o_e} : W_e \to W_{o(e)}/D_{o(e)}$ and $\overline{t_e} : W_e \to W_{t(e)}/D_{t(e)}$.

We are going to consider all abstract pieces that potentially appear as a component of S_v

for some simple relative admissible surface S for some rational chain supported on finitely many hyperbolic elements $\underline{g} = \{g_i : i \in I\}$ represented by tight loops $\underline{\gamma} = \{\gamma_i : i \in I\}$. Thus we extend some previous definitions as follows.

Definition 5.16. For each vertex v, *a piece C at v* is a surface with corners in the thickened vertex space $N(X_v)$ satisfying the following properties:

(1) C has a unique polygonal boundary where edges alternate between arcs in A_v and turns connecting them, where a turn (with the induced orientation) going from a_v to a'_v is an arc supported on X_e connecting the terminus point of a_v and initial point of a'_v.

(2) C is either a disk or an annulus depending on its winding number $w(C)$ as follows, where the winding number is the conjugacy class in G_v represented by the polygonal boundary of C.

  (a) C is a disk if $w(C)$ is trivial in G_v. The map in the interior of C is a null-homotopy of its polygonal boundary in $N(X_v)$.

  (b) C is an annulus if its winding number $w(C)$ is non-trivial, where the other boundary has no corners and represents a loop in X_v homotopic to the polygonal boundary in $N(X_v)$, and the homotopy gives the interior of C.

As before, we define a piece C at v to be *disk-like* if there is an arc $a_v \in A_v$ on the polygonal boundary of C such that $w(C) \in W(a_v)$.

For each vertex v of Γ, let T_v be the set of triples $(a_v, \bar{w}, a'_v)$ where $a_v, a'_v \in A_v$ are arcs with terminus point of a_v and initial point of a'_v on a common edge space X_e, and $\bar{w} \in W_e$. Note that T_v is a *finite* set since there are finitely many pairs of (a_v, a'_v) as above and $\bar{w}$ lies in a finite group given each such a pair. We use such a triple to record a turn from a_v to a'_v with winding number in the coset $\bar{w}$.

Consider a piece C at vertex v with m turns. For each turn, say supported on X_e and going from a_v to a'_v with winding number $w \in G_e$, the triple $(a_v, \bar{w}, a'_v)$ is in T_v, where $\bar{w}$ is the image of w in the quotient W_e. Regard this triple as a basis vector in $\mathbb{R}^{T_v}$ and let $x(C) \in \mathbb{R}^{T_v}$ be the sum of such triples over the m turns of C.

Let $\partial : \mathbb{R}^{T_v} \to \mathbb{R}^{A_v}$ be the rational linear map given by $\partial(a_v, \bar{w}, a'_v) := a'_v - a_v$ for all $(a_v, \bar{w}, a'_v) \in T_v$. Then $\partial(x(C)) = 0$ as the polygonal boundary of C closes up.

There is a pairing on the set $\cup_v T_v$ similar to the pairing of turns (Figure 3.4). For each triple $(a_v, \bar{w}, a'_v) \in T_v$ with $\bar{w} \in W_e$, there is a unique triple $(a_u, -\bar{w}, a'_u) \in T_u$ such that a_v and a_u are followed by a'_u and a'_v in γ respectively, where u is necessarily the vertex adjacent to v via e. We say such two triples are *paired*.

Denote the convex rational polyhedral cone $\mathbb{R}^{T_v}_{\geq 0} \cap \ker \partial$ by $\mathcal{C}_v$. For any simple relative admissible surface S for any rational chain supported on $\underline{\gamma}$, let $x(S_v) = \sum x(C)$, where the sum is taken over all pieces C of S_v. Then $x(S_v)$ is an integer point in $\mathcal{C}_v$ and paired turns are encoded by paired triples. Let $x(S) \in \prod_v \mathcal{C}_v$ be the element with v-coordinate $x(S_v)$.

Let $\mathcal{C}(\underline{\gamma})$ be the subspace of $\prod_v \mathcal{C}_v$ consisting of points satisfying the *gluing condition* which we now describe. This is the analog of (3.2) on $\prod_v \mathcal{C}_v$. For each triple $(a_v, \bar{w}, a'_v) \in T_v$, let $\#_{(a_v, \bar{w}, a'_v)}$ be the linear function taking the $(a_v, \bar{w}, a'_v)$ coordinate. We say $x \in \mathcal{C}(\underline{\gamma})$ satisfies the gluing condition if $\#_{(a_v, \bar{w}, a'_v)}(x) = \#_{(a_u, -\bar{w}, a'_u)}(x)$ for all paired triples $(a_v, \bar{w}, a'_v)$ and $(a_u, -\bar{w}, a'_u)$. It follows that $\mathcal{C}(\underline{\gamma})$ is a rational polyhedral cone.

For each $\gamma_i \in \underline{\gamma}$, fix an arc a_i in some A_v supported on γ_i and let $\#_{\gamma_i} = \sum \#_{(a_i, w, a'_v)}$, where the sum is taken over all triples in T_v starting with a_i. For any $x \in \mathcal{C}(\underline{\gamma})$, the gluing condition implies $\#_{\gamma_i}(x)$ is independent of the choice of arc a_i on γ_i. Roughly speaking, the rational linear function $\#_{\gamma_i}(x)$ counts how many times x winds around γ_i.

For $\boldsymbol{r} = (r_i) \in \mathbb{Q}^I_{\geq 0}$ (I is the index set of $\underline{\gamma}$), let $c(\boldsymbol{r})$ be the rational chain $\sum r_i g_i$, and let $\mathcal{C}(\boldsymbol{r})$ be the set of $x \in \mathcal{C}(\underline{\gamma})$ satisfying the *normalizing condition* $\#_{\gamma_i}(x) = r_i$ for all $i \in I$. Let $h(\boldsymbol{r})$ be the homology class of $c(\boldsymbol{r})$ under the projection $H_1(G; \mathbb{R}) \to H_1(\Gamma; \mathbb{R})$. Then h

is a rational linear map.

Lemma 5.17. *The set $\mathcal{C}(\boldsymbol{r})$ is nonempty if and only if $\boldsymbol{r} \in \ker h$, in which case, it is a compact rational polyhedron depending piecewise linearly on $\boldsymbol{r}$.*

Proof. The chain $c(\boldsymbol{r})$ bounds an admissible surface relative to vertex groups if and only if $h(\boldsymbol{r}) = 0$, according to the computation (3.1). Whenever a relative admissible surface exists, it can be simplified to one in simple normal form, which is encoded as a vector in $\mathcal{C}(\boldsymbol{r})$. Then in this case, each $\mathcal{C}(\boldsymbol{r})$ is the intersection of a rational polyhedral cone $\mathcal{C}(\gamma)$ with a rational linear subspace $\cap_i\{x : \#_{\gamma_i}(x) = r_i\}$ that depends linearly on $\boldsymbol{r}$. Thus $\mathcal{C}(\boldsymbol{r})$ is a closed rational polyhedron depending piecewise linearly on $\boldsymbol{r}$. It is compact since the equations $\#_{\gamma_i}(x) = r_i$ impose upper bounds on all coordinates. $\qquad\square$

Definition 5.18. For each vertex v, an integer point $d \in \mathcal{C}_v$ is a *disk-like vector* if $d = x(C)$ for some disk-like piece C at v. Let $\mathcal{D}(v)$ be the set of disk-like vectors.

For any $x \in \mathcal{C}_v$, we say $x = x' + \sum t_j d_j$ is an *admissible expression* if $x' \in \mathcal{C}_v$, $d_j \in \mathcal{D}(v)$ and $t_j \geq 0$. Define

$$\kappa_v(x) := \sup\left\{\sum t_j \mid x = x' + \sum t_j d_j \text{ is an admissible expression}\right\}.$$

The key point of our encoding and the definitions of D_v, D_e, W_e is to have enough information on the winding numbers of turns to tell whether a piece is disk-like.

Lemma 5.19. *For each $\bar{w} \in W_e$, fix an arbitrary lift $w \in G_e$. Then any disk-like vector $d \in \mathcal{C}_v$ can be realized as a disk-like piece C at v such that every turn from a_v to a'_v on the polygonal boundary of C representing a triple $(a_v, \bar{w}, a'_v)$ has winding number w.*

Proof. By definition, there is some disk-like piece C_0 at v realizing the given disk-like vector d. Thus, for some $a_{v,0} \in A_v$, the winding number $w(C_0)$ lies in $W(a_{v,0}) \subset W_v$, which is central in G_v. Suppose C_0 contains a turn from a_v to a'_v supported on the edge space

X_e representing a triple $(a_v, \bar{w}, a'_v)$ with actual winding number $w_0 \in G_e$ and $e = e_{out}(a_v)$. Then $w - w_0 \in D_e$ and changing the winding number of the turn from w_0 to w would change the winding number $w(C_0)$ of the piece C_0 by $o_e(w - w_0) \in D_v$. Since $D_v \subset W(a_{v,0})$, such a change preserves the property of being disk-like. After finitely many such changes, we modify C_0 to a disk-like piece C with the desired winding numbers of turns. $\qquad \square$

Any admissible surface S naturally provides an admissible expression for $x(S_v)$ by sorting out disk-like pieces among components of S_v. Hence $\kappa_v(x(S_v))$ is no less than the number of disk-like pieces in S_v.

For each vector $x \in \mathcal{C}_v \subset \mathbb{R}^{T_v}_{\geq 0}$, denote its ℓ^1-norm by $|x|$. Then $|\cdot|$ on $\mathcal{C}_v$ coincides with the linear function taking value 1 on each basis vector. Thus $\sum_v |x(S_v)|$ is the total number of turns in S, which is twice the number of corners.

Recall from Definition 5.14 that $\widehat{\chi}(S)$ is the over-counting Euler characteristic that counts disk-like pieces as disks in a simple relative admissible surface S.

Lemma 5.20. *Fix any $\boldsymbol{r}$ as above.*

(1) For any S simple relative admissible for $c(\boldsymbol{r})$ of degree n, we have $x(S)/n \in \mathcal{C}(\boldsymbol{r})$ and

$$\frac{-\widehat{\chi}(S)}{2n(S)} \geq \sum_v \frac{1}{4}|x(S_v)/n| - \sum_v \frac{1}{2}\kappa_v(x(S_v)/n).$$

(2) For any rational point $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$ and any $\epsilon > 0$, there is a simple relative admissible surface S for $c(\boldsymbol{r})$ of a certain degree n such that $x(S_v)/n = x_v$ and

$$\frac{-\widehat{\chi}(S)}{2n} \leq \sum_v \frac{1}{4}|x_v| - \sum_v \frac{1}{2}\kappa_v(x_v) + \epsilon.$$

Proof. (1) It is easy to see $x(S)/n \in \mathcal{C}(\boldsymbol{r})$ from the definition. To obtain the inequality,

recall that

$$-\widehat{\chi}(S) = -\sum_C \widehat{\chi}_o(C) = \frac{1}{4}\#\text{corners} - \sum_C \widehat{\chi}(C) = \frac{1}{2}\#\text{turns} - \#\text{disk-like pieces}.$$

Since $\kappa_v(x(S_v))$ is no less than the number of disk-like pieces in S_v, the inequality follows.

(2) By finiteness of Γ, there are admissible expressions $x_v = x'_v + \sum_j t_{j,v} d_{j,v}$ with each $t_{j,v} \in \mathbb{Q}_{\geq 0}$ such that $\sum_{j,v} t_{j,v} + 2\epsilon > \sum_v \kappa_v(x_v)$. Note that each x'_v is rational as each $t_{j,v}$ and x_v are. Choose an integer n so that each $nt_{j,v}$ is an integer and nx'_v is an integer vector in $\mathcal{C}_v$. For each $\bar{w} \in W_e$, fix a lift $w \in G_e$. We can choose the lifts such that $-w$ is the lift of $-\bar{w}$. By Lemma 5.19, we can realize $nt_{j,v} d_{j,v}$ as the union of $nt_{j,v}$ pieces that are disk-like, such that each turn from a_v to a'_v representing a triple $(a_v, \bar{w}, a'_v)$ has winding number w. We can also realize nx'_v as the union of some other pieces at v with turns satisfying the same property. Let S_v be the disjoint union of these pieces at v. Then $x \in \mathcal{C}(\boldsymbol{r})$ and our choice of the lifts imply that the surface $\sqcup_v S_v$ satisfies the gluing condition (3.2) and glues to a simple relative admissible surface S for $c(\boldsymbol{r})$ of degree n with $x(S_v)/n = x_v$. Noticing that the number of disk-like pieces in S is no less than $n\Sigma_{j,v} t_{j,v} > n\Sigma_v \kappa_v(x_v) - 2n\epsilon$, the estimate of $-\widehat{\chi}(S)/2n$ easily follows from a computation similar to the one in the first part.

$\square$

Let $\text{conv}(E)$ denote the convex hull of a set E in some vector space. Denote the Minkowski sum of two sets E and F by $E + F := \{e + f \mid e \in E \text{ and } f \in F\}$. Note that $\text{conv}(E + F) = \text{conv}(E) + \text{conv}(F)$.

The following lemma is the analog of [13, Lemma 3.10] and has the same proof.

Lemma 5.21 (Calegari [13]). *The function κ_v on $\mathcal{C}_v$ is a non-negative concave homogeneous function which takes value 1 exactly on the boundary of $\text{conv}(\mathcal{D}(v)) + \mathcal{C}_v$ in $\mathcal{C}_v$.*

It is an important observation in [18] that both $\operatorname{conv}(\mathcal{D}(v)) + \mathcal{C}_v$ and κ_v are nice, no matter how complicated $\mathcal{D}(v)$ is.

Lemma 5.22 (Chen [18]). *There is a* finite *subset D' of $\mathcal{D}(v)$ such that $D' + \mathcal{C}_v = \mathcal{D}(v) + \mathcal{C}_v$. Consequently, the function κ_v is the minimum of* finitely *many rational linear functions.*

Proof. The first assertion follows from [18, Lemma 4.7]. Then we have

$$\operatorname{conv}(\mathcal{D}(v)) + \mathcal{C}_v = \operatorname{conv}(\mathcal{D}(v) + \mathcal{C}_v) = \operatorname{conv}(D' + \mathcal{C}_v) = \operatorname{conv}(D') + \mathcal{C}_v.$$

Note that $\operatorname{conv}(D')$ is a compact rational polyhedron since D' is a finite set of integer points. Hence $\operatorname{conv}(D') + \mathcal{C}_v$ is a rational polyhedron as it is the sum of two such polyhedra (see the proof of [2, Theorem 3.5]). Then $\operatorname{conv}(\mathcal{D}(v)) + \mathcal{C}_v = \mathcal{C}_v \cap (\cap_i \{f_i \geq 1\})$ for a finite collection of rational linear functions $\{f_i\}$. Combining with Lemma 5.21, we have $\kappa_v(x) = \min_i f_i(x)$ for all $x \in \mathcal{C}_v$. $\qquad\square$

Lemma 5.23. *The optimization $\min \sum_v \frac{1}{4}|x_v| - \sum_v \frac{1}{2}\kappa_v(x_v)$ among $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$ can be computed via linear programming. The minimum is $\operatorname{scl}_{(G, \{G_v\})}(c(\boldsymbol{r}))$, which depends piecewise rationally linearly on $\boldsymbol{r} \in \ker h$ and is achieved at some rational point in $\mathcal{C}(\boldsymbol{r})$.*

Proof. Recall that $|x_v|$ is a rational linear function for $x_v \in \mathcal{C}_v$. Combining with Lemma 5.22, for each vertex v, there are finitely many rational linear functions $f_{j,v}$ such that $|x_v|/4 - \kappa_v(x_v)/2 = \max_j f_{j,v}(x_v)$. By introducing slack variables $y = (y_v)$, the optimization is equivalent to minimizing $\sum_v y_v$ subject to $y_v \geq f_{j,v}(x_v)$ for all j, v and $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$, which is a rational linear programming problem in variables (x, y). The minimum depends piecewise rationally linearly on $\boldsymbol{r} \in \ker h$ by Lemma 5.17 and is achieved at a rational point.

The minimum is $\operatorname{scl}_{(G, \{G_v\})}(c(\boldsymbol{r}))$ by Lemma 5.20 and Lemma 5.15. $\qquad\square$

Remark 5.24. The function $\sum_v |x_v|$ is actually a constant $\sum_i r_i|\gamma_i|$ for $x = (x_v) \in \mathcal{C}(\boldsymbol{r})$, where $|\gamma_i|$ is the number of arcs that the edge spaces cut γ_i into. Thus for a fixed chain, the problem comes down to maximizing the number of disk-like pieces.

Now we return to full generality without assuming Γ to be finite or locally finite.

Theorem 5.25 (Rationality). *Let G be a graph of groups $\mathcal{G}(\Gamma, \{G_v\}, \{G_e\})$ where*

(1) $\mathrm{scl}_{G_v} \equiv 0$, *and*

(2) the images of edge groups in each vertex group are central and mutually commensurable.

Then scl_G is piecewise rational linear, and $\mathrm{scl}_G(c)$ can be computed via linear programming for each rational chain $c \in B_1^H(G)$.

Proof. We compute $\mathrm{scl}_G(\sum r_i g_i)$ for an arbitrary finite set of element $\underline{g} = \{g_1, \dots, g_m\} \subset G$ with $r_i \in \mathbb{Q}_{>0}$ so that the chain $\sum r_i g_i$ is null-homologous in G. By Proposition 3.2, we may assume each g_i to be hyperbolic and consider $\mathrm{scl}_{(G, \{G_v\})}(c(\boldsymbol{r}))$ with $\boldsymbol{r} \in \ker h$ instead. By restriction of domain (Proposition 3.11), we further assume Γ to be finite. Then the result follows from Lemma 5.23. $\square$

Remark 5.26. Theorem 5.25 holds with the weaker assumption $\mathrm{scl}_G(c_v) = 0$ for all $c_v \in B_1^H(G_v)$ in place of $\mathrm{scl}_{G_v} \equiv 0$ and all vertices v.

Remark 5.27. Let G be a graph of groups where each edge group is $\mathbb{Z}$ and each vertex group G_v is itself a graph of groups as in Theorem 5.25 with vanishing $H_2(G_v; \mathbb{R})$. Using the method in [10], it follows from Theorem 5.25 that the Gromov–Thurston norm on $H_2(G; \mathbb{R})$ has a rational polyhedral unit ball and can be computed via linear programming.

Remark 5.28. It involves more work to characterize the existence of extremal surfaces (Definition 2.5), which requires the asymptotic promotion realizing disk-like pieces as genuine disks to be achieved at a *finite* stage (see the proof of [20, Lemma 6.1]). In the case of Baumslag–Solitar groups, this is carefully worked out in [20, Sect. 6.2], which involves lots of covering tricks similar to those in Sect. 5.1.2.

A direct implementation of the method above to compute scl would probably result in an algorithm with run time doubly exponential on the word length (see [13, Subsection

4.5]). However, in the case of Baumslag–Solitar groups, there is an algorithm computing $\text{scl}_{\text{BS}(M,L)}(c)$ with run time polynomial in the length of c if the complexity $\rho(c)$ (defined below) is fixed [20, Sect. 6.5]. It uses the idea developed by Walker [43] to efficiently compute scl in free products of cyclic groups. Unfortunately, the algorithm is not fast enough in practice to carry out computer experiments.

5.3 Examples in Baumslag–Solitar Groups

Our method can be used to practically compute scl in various kind of graphs of groups and prove explicit formulas. In the case of free products, several formulas are obtained in [18, Sect. 5].

Here we will focus on the harder and more interesting example of Baumslag–Solitar groups $G = \text{BS}(M, L) = \left\langle a, t \mid a^M = t a^L t^{-1} \right\rangle$ with integers $M, L \neq 0$. We are not interested in the case where $|M| = 1$ or $|L| = 1$ since $\text{BS}(M, L)$ is solvable and $\text{scl}_{\text{BS}(M,L)} \equiv 0$ in such cases. Most of the results and tools are applicable for any graphs of groups with abelian vertex groups, but we will not pursue such generalizations.

5.3.1 Basic Setup

Let $d := \gcd(|M|, |L|)$, $m := M/d$ and $\ell := L/d$. Denote by $h : G \to \mathbb{Z}$ the homomorphism given by $h(a) = 0$ and $h(t) = 1$. An element g is *t-balanced* if $h(g) = 0$.

We denote the only vertex and edge by v and $\{e, \bar{e}\}$ respectively, where e is oriented to represent the generator t. See Figure 3.1. Notation from Section 5.2 will be used.

Let $g = a^{p_1} t^{\epsilon_1} \ldots a^{p_n} t^{\epsilon_n}$ be a cyclically reduced word, where $\epsilon_i = \pm 1$ for all i. Then g is represented by a tight loop γ in X_G cut into n arcs $A_v = \{a_i \mid 1 \leq i \leq n\}$ where a_i has winding number $w(a_i) = p_i$. Note that we have the transition map $\tau_e : |M|\mathbb{Z} \to |L|\mathbb{Z}$ with $\tau_e(x) = -Lx/M$. For each $1 \leq i \leq n$, let $\mu_i := \max_{0 \leq k \leq n} \sum_{j=1}^{k} \epsilon_{i+j}$ and $\lambda_i := -\min_{0 \leq k \leq n} \sum_{j=1}^{k} \epsilon_{i+j}$, where indices are taken mod n and the summation is 0 when $k = 0$.

It is straightforward to see that $\operatorname{Dom}\tau_{P(a_i)} = dm^{\mu_i}\ell^{\lambda_i}\mathbb{Z}$, $\operatorname{Im}\tau_{P(a_i)} = dm^{\mu_i - h(g)}\ell^{\lambda_i + h(g)}\mathbb{Z}$, and $\tau_{P(a_i)}(x) = (-1)^{h(g)}\ell^{h(g)}x/m^{h(g)}$. With

Setup: Let $W(a_i) := \begin{cases} dm^{\mu_i - |h(g)|}\ell^{\lambda_i}\mathbb{Z} & \text{if } h(g) \geq 0, \\ dm^{\mu_i}\ell^{\lambda_i - |h(g)|}\mathbb{Z} & \text{if } h(g) \leq 0, \end{cases}$ for all i, $\qquad$ (5.2)

Example 5.4 (with $X = dm^{\mu_i}\ell^{\lambda_i}$ and $Y = \pm dm^{\mu_i - h(g)}\ell^{\lambda_i + h(g)}$) shows that we have $W(a_i) \subset \operatorname{Dom}\tau_{P(a_i)}^p + \operatorname{Im}\tau_{P(a_i)}^q$ for all $p, q \geq 0$.

We will always use Setup (5.2) in the sequel. Recall from Section 5.2 that the group D_v is defined as $\cap_i W(a_i)$.

For an explicit formula of D_v, let $\lambda := \max_i \lambda_i$ and $\mu := \max_i \mu_i$. Since $h(g) = \sum \epsilon_i$, it is easy to observe that $\mu - |h(g)| = \lambda \geq 0$ when $h(g) \geq 0$ and $\lambda - |h(g)| = \mu \geq 0$ when $h(g) \leq 0$. In any case, we have

$$D_v = dm^{\rho(g)}\ell^{\rho(g)}\mathbb{Z},$$

where $\rho(g) := \min(\mu, \lambda)$, which we call the *complexity* of g.

When $h(g) = 0$, this can be easily seen geometrically. The infinite cyclic cover $\widetilde{X}_G$ of X_G corresponding to $\ker h$ has a $\mathbb{Z}$-action by translation with fundamental domains projecting homeomorphically to the thickened vertex space $N(X_v)$. Since $h(g) = 0$, the tight loop γ representing g lifts to a loop $\tilde{\gamma}$ on $\widetilde{X}_G$, and $\rho(g) + 1$ is the number of fundamental domains that $\tilde{\gamma}$ intersects. In particular, when $h(g) = 0$, the element g is t-alternating if and only if the complexity $\rho(g) = 1$.

More generally, for a chain $c = \sum r_i g_i$ with each $r_i \neq 0$, define its complexity $\rho(c) := \max_i \rho(g_i)$. Then

$$D_v = dm^{\rho(c)}\ell^{\rho(c)}\mathbb{Z}, \qquad (5.3)$$

and $\rho(c)$ controls the amount of information we need to encode. Denote $|D_v| := d|m|^{\rho(c)}|\ell|^{\rho(c)}$.

Using the notation from Section 5.2, we have $D_e = m^{\rho(c)}\ell^{\rho(c)}\mathbb{Z}$ and $W_e = \mathbb{Z}/D_e$ in Setup

(5.2).

To better understand integer points in $\mathcal{C}_v$ and disk-like vectors, consider a directed graph Y with vertex set A_v, where each oriented edge from a_i to a_j corresponds to a triple $(a_i, \bar{w}, a_j) \in T_v$. See Proposition 5.33 and Figure 5.6 for an example. Then each vector $x \in \mathcal{C}_v$ assigns non-negative weights to edges in Y.

Define the support $\text{supp}(x)$ to be the subgraph containing edges with positive weights. Then $\text{supp}(x)$ is a union of positively oriented cycles in Y. Note that an integer point $x \in \mathcal{C}_v$ can be written as $x(C)$ for some piece C if and only if $\text{supp}(x)$ is connected. If two pieces C and C' are encoded by the same vector x, then their winding numbers $w(C)$ and $w(C')$ are congruent mod D_v since the vertex group is abelian. Moreover, fixing a lift $\widetilde{w} \in \mathbb{Z}$ of each $\bar{w} \in W_e$, we can compute this winding number $w(x)$ as a linear function on $\mathbb{R}^{T_v}$ determined by

$$w(a_i, \bar{w}, a_j) := w(a_j) + \iota(\widetilde{w}) \tag{5.4}$$

where $\iota = t_e$ if $e_{out}(a_i) = \bar{e}$ (ie a_i leaves v by following $\bar{e}$, see Figure 3.4) and $\iota = o_e$ if $e_{out}(a_i) = e$. Then $w(x)$ depends on the choice of lifts but $w(x)$ mod D_v does not. In particular, it makes sense to discuss whether $w(x) \in W(a_i)$ since $D_v \subset W(a_i)$.

Then in our setup, an integer point $x \in \mathcal{C}_v$ is disk-like if and only if $\text{supp}(x)$ is connected and $w(x) \in W(a_i)$ for some a_i in $\text{supp}(x)$.

We use $\mathcal{C}(c)$ instead of $\mathcal{C}(\boldsymbol{r})$ to denote the polyhedron encoding normalized simple relative admissible surfaces for c, as we will not consider families of chains with varying $\boldsymbol{r}$.

5.3.2   Lower Bounds from Duality

To obtain explicit formulas for scl, especially when we consider chains in $\text{BS}(M, L)$ with parameters or with varying M and L, it is often too complicated to work out the linear programming problems. Proving a sharp lower bound is usually the main difficulty.

Here we slightly modify the duality method in Sect. 4.1.1 for our Setup (5.2).

To each turn $(a_i, \bar{w}, a_j)$ we assign a *non-negative cost* $q_{i,\bar{w},j}$. This defines a linear cost function q on $\mathcal{C}_v$. As before, the cost of a piece is the sum of the costs of the turns on its polygonal boundary.

Recall from Lemma 5.23 and Remark 5.24 that computing scl is equivalent to maximizing the function κ_v counting the (normalized) number of disk-like pieces on $\mathcal{C}(c)$, where $c = \sum r_i g_i$. Hence giving upper bounds of κ_v produces lower bounds of scl.

Lemma 5.29. *If $q(d) \geq 1$ for any disk-like vector d, then $\kappa_v(x) \leq q(x)$ for any $x \in \mathcal{C}_v$.*

Proof. For any admissible expression $x = x' + \sum t_i d_i$ with d_i disk-like, $t_i \geq 0$, and $x' \in \mathcal{C}_v$, we have $q(x) = q(x') + \sum t_i q(d_i) \geq \sum_i t_i$. Thus $q(x) \geq \kappa_v(x)$. □

For each vector $x \in \mathcal{C}(c)$, we write it as $\sum t_{i,\bar{w},j}(a_i, \bar{w}, a_j)$ where $t_{i,\bar{w},j}$ is the coordinate corresponding to the basis element $(a_i, \bar{w}, a_j) \in T_v$. We think of $t_{i,\bar{w},j}$ as the normalized number of turns of type $(a_i, \bar{w}, a_j)$. Recall from Section 5.2 that the gluing condition requires $t_{i,\bar{w},j} = t_{i',\bar{w}',j'}$ if $(a_i, \bar{w}, a_j)$ and $(a_{i'}, \bar{w}', a_{j'})$ are paired triples, and the normalizing condition implies $\sum_{\bar{w},j} t_{i,\bar{w},j} = r_k$ if $a_i \subset \gamma_k$ and $\sum_{i,\bar{w}} t_{i,\bar{w},j} = r_k$ if $a_j \subset \gamma_k$.

If we can choose the costs so that

(1) $q(d) \geq 1$ for any disk-like vector d, and

(2) for any $x \in \mathcal{C}(c)$ expressed in the form above, $q(x) = \sum q_{i,\bar{w},j} t_{i,\bar{w},j}$ is equal to or bounded above by a constant K on $\mathcal{C}(c)$ by the gluing and normalizing conditions,

then Lemma 5.29 implies $\kappa_v \leq K$ on $\mathcal{C}(c)$. Combining with Lemma 5.23 and Remark 5.24, this gives a lower bound of scl.

5.3.3 Examples with Explicit Formulas

In this subsection, we compute three examples of complexities $\rho = 0, 1, 2$ that are not t-alternating words. As we will see, the computations get more complicated as the complexity increases. We will use Setup (5.2) in the sequel.
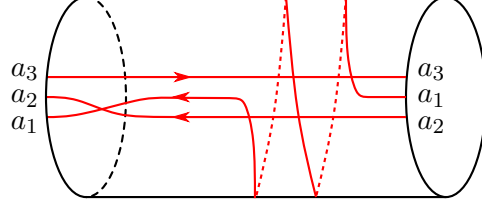
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Figure 5.3: The three arcs of $a^k t^2 + 2t^{-1}$ in the thickened vertex space $N(X_v)$ with $k = 2$. This appears originally in [20, Fig. 13].

Proposition 5.30. *For $d = \gcd(|M|, |L|)$, we have*

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(a^k t^2 + 2t^{-1}) = \frac{1}{2} - \frac{\gcd(|k|, d)}{2d}.$$

Proof. Let $n_k = \frac{d}{\gcd(|k|,d)}$, which is the order of $[k]$ in $\mathbb{Z}/d\mathbb{Z}$. We use the notation introduced in Sect. 5.3.1.

Let γ_1 be the tight loop representing $g_1 = a^k t^2$ consisting of two arcs a_1 and a_2 with winding numbers k and 0 respectively. Let γ_2 be the tight loop representing $g_2 = t^{-1}$ consisting of a single arc a_3 with winding number 0. The three arcs are depicted in Figure 5.3. In Setup (5.2), we have $W(a_1) = W(a_2) = W(a_3) = d\mathbb{Z}$, $\rho(g_1 + 2g_2) = \rho(g_1) = \rho(g_2) = 0$ and $D_v = d\mathbb{Z}$. As a consequence, we have $W_e = \{1\}$. That is, we ignore the winding numbers of turns and use a pair (a_i, a_j) instead of a triple $(a_i, \bar{w}, a_j)$ to represent a turn.

We have a turn (a_1, a_3) paired with (a_3, a_2), and a turn (a_2, a_3) paired with (a_3, a_1). The defining equation $\partial = 0$ implies that $\mathcal{C}_v$ consists of vectors of the form $\xi(x, y) = x(a_1, a_3) + x(a_3, a_1) + y(a_2, a_3) + y(a_3, a_2)$ with $(x, y) \in \mathbb{R}^2_{\geq 0}$, which has winding number $kx \bmod d$. Thus such a vector is disk-like if and only if $(x, y) \neq (0, 0) \in \mathbb{Z}^2_{\geq 0}$ and $kx \in d\mathbb{Z}$. This describes the set $\mathcal{D}(v)$ of disk-like vectors, from which we get $\mathcal{D}(v) + \mathcal{C}_v = \{\xi(n_k, 0), \xi(0, 1)\} + \mathcal{C}_v$ (See Figure 5.4). It follows that $\kappa_v(\xi(x, y)) = x/n_k + y$. For the chain $c = g_1 + 2g_2$, the normalizing condition requires $x = y = 1$, so $\xi(1, 1)$ is the only vector in $\mathcal{C}(c)$. Thus by
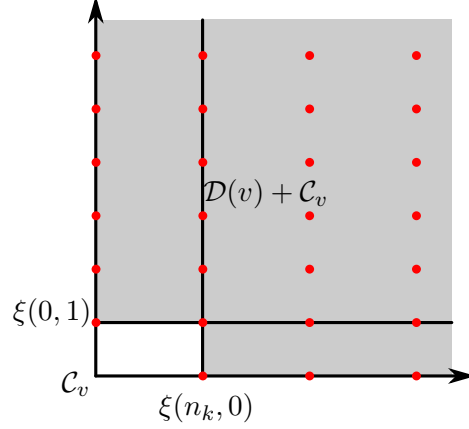
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Figure 5.4: The red dots are disk-like vectors in $\mathcal{D}(v)$. The gray region is the set $\mathcal{D}(v) + \mathcal{C}_v$. This appears originally in [20, Fig. 14].

Lemma 5.23 and Remark 5.24, we have

$$\mathrm{scl}(c) = 1 - \frac{1}{2}\kappa_v(\xi(1,1)) = \frac{1}{2} - \frac{1}{2n_k}.$$

$\square$

Proposition 5.31. *For all $|M|, |L| \geq 2$, we have*

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \leq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}.$$

The equality holds if $d = \gcd(|M|, |L|)$ satisfies $d \geq \frac{|M|+|L|}{2\min\{|M|,|L|\}}$.

Proof. We have 3 arcs a_1, a_2, a_3 on the loop γ_1 representing $atta^{-1}t^{-1}$ with winding numbers $1, 0, -1$ respectively, and have another arc a_4 on the other loop γ_2 with winding number 0. It easily follows from the definitions in Sect. 5.3.1 that $\rho(\gamma_1) = 1$, $\rho(\gamma_2) = 0$, so our chain c has complexity $\rho(c) = 1$.

With Setup (5.2), we have $W(a_1) = dm\mathbb{Z}$, $W(a_2) = W(a_4) = d\mathbb{Z}$, $W(a_3) = d\ell\mathbb{Z}$. Hence

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$D_v = dm\ell\mathbb{Z}$ and $W_e = \mathbb{Z}/m\ell\mathbb{Z}$. Turns are paired up as follows:

$$(a_1, \bar{w}, a_1) \quad \leftrightarrow \quad (a_3, -\bar{w}, a_2) \qquad (a_1, \bar{w}, a_4) \quad \leftrightarrow \quad (a_4, -\bar{w}, a_2)$$

$$(a_2, \bar{w}, a_4) \quad \leftrightarrow \quad (a_4, -\bar{w}, a_3) \qquad (a_2, \bar{w}, a_1) \quad \leftrightarrow \quad (a_3, -\bar{w}, a_3)$$

These are the only turns, so each piece C falls into exactly one of the following three types.

(1) The polygonal boundary contains both a_2 and a_4. It is disk-like if and only if $w(C)$ is divisible by d since $W(a_2) = d\mathbb{Z}$. Changing the winding numbers of turns will change $w(C)$ by a multiple of $|M|$ or $|L|$, both divisible by d.

(2) The polygonal boundary contains a_1 only. Then the winding number $w(C) \equiv k$ mod $|M|$ if there are k copies of a_1 on the boundary, which does not depend on the winding numbers of turns. Thus it is disk-like if and only if $w(C) \in |M|\mathbb{Z} = W(a_1)$, ie k is divisible by $|M|$.

(3) The polygonal boundary contains a_3 only. Similar to the previous case, it is disk-like if and only if the number of copies of a_3 on the boundary is divisible by $|L|$.

In summary, whether a piece is disk-like does not depend on the winding numbers of turns on its polygonal boundary. Thus we simply assume all turns to have winding numbers 0 in what follows.

We prove the upper bound by constructing a simple relative admissible surface S of degree $|2ML|$ consisting of the following disk-like pieces described by the turns on the polygonal boundaries (also see Figure 5.5).

(1) $(a_2, 0, a_4) + (a_4, 0, a_2)$, take $|ML|$ copies of this piece;

(2) $(a_2, 0, a_1) + (a_1, 0, a_4) + (a_4, 0, a_3) + (a_3, 0, a_2)$, take $|ML|$ copies of this piece;
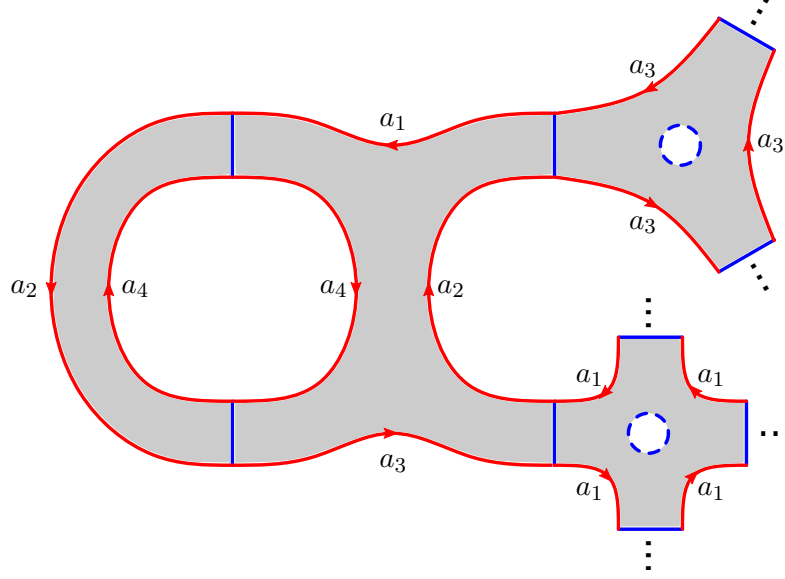
Figure 5.5: Part of a relative admissible surface involving the four disk-like pieces constructed to give the upper bound, illustrating the case $M = 4$ and $L = 3$. This appears originally in [20, Fig. 15].

(3) $|M|(a_1, 0, a_1)$, take $|L|$ copies of this piece;

(4) $|L|(a_3, 0, a_3)$, take $|M|$ copies of this piece.

It is easy to check that these are disk-like pieces and that the gluing conditions hold. With notation as in Lemma 5.15, we get $\widehat{\chi}(S) = -2|ML| + |M| + |L|$ and obtain the upper bound

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \leq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}.$$

To obtain the lower bound, we use the duality method in Sect 5.3.2. We assign the cost q_{ij} independent of the winding number $\bar{w}$ to each turn $(a_i, \bar{w}, a_j)$ as in the following matrix

$Q = (q_{ij})$, where $*$ appears if there are no such turns.

$$Q = \begin{pmatrix} \frac{1}{|M|} & * & * & 1 - \left(\frac{1}{|M|} + \frac{1}{|L|}\right) \\ \frac{1}{|M|} & * & * & 1 - \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right) \\ * & \frac{1}{|L|} & \frac{1}{|L|} & * \\ * & \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right) & 0 & * \end{pmatrix}$$

We first check that every disk-like piece costs at least 1 when $d \geq \frac{|M|+|L|}{2\min\{|M|,|L|\}}$. According to the classification of pieces above, only those of type (1) requires some attention. Let C be such a piece, which must contain a turn ending at a_4. Assume $|M| \leq |L|$.

(1) Suppose C contains a turn from a_1 to a_4.

  (a) If this turn is followed by another from a_4 to a_3, then we must also have a turn from a_3 to a_2 (to leave a_3) and another from a_2 to a_1 so that the boundary closes up. In this case, the cost is at least $q_{14} + q_{43} + q_{32} + q_{21} = 1$.

  (b) If this turn is followed by another from a_4 to a_2 instead, then the cost is at least $q_{14} + q_{42} + \min(q_{21}, q_{24}) \geq 1$ since $2 \leq |M| \leq |L|$.

(2) Suppose C does not contain a turn from a_1 to a_4. Then C does not visit a_1 and must contain a turn from a_2 to a_4. If C also contains a turn from a_4 to a_2, then the cost will be at least $q_{24} + q_{42} = 1$. Otherwise, C is encoded as $n_1(a_2, 0, a_4) + n_1(a_4, 0, a_3) + n_2(a_3, 0, a_3) + n_1(a_3, 0, a_2)$ with integers $n_1 \geq 1$, $n_2 \geq 0$, and costs $n_1(q_{24} + q_{43} + q_{32}) + n_2 q_{33}$. Then C has winding number $w(C) \equiv -(n_1 + n_2) \mod d$. For C to be disk-like, we have $n_1 + n_2 \geq d$. Note that $q_{24} \geq 1/2 \geq q_{33}$ since $|M|, |L| \geq 2$. Therefore, the cost

$$n_1(q_{24} + q_{43} + q_{32}) + n_2 q_{33} \geq q_{24} + q_{43} + q_{32} + (d-1)q_{33} = 1 + \frac{d - \frac{1}{2}}{|L|} - \frac{1}{2|M|} \geq 1$$

since $d \geq \frac{|M|+|L|}{2|M|}$.

The other case $|M| \geq |L|$ is similar.

Now let $t_{ij} = \sum_{\bar{w}} t_{i,\bar{w},j}$, where $t_{i,\bar{w},j}$ is the normalized number of the turn $(a_i, \bar{w}, a_j)$. Then we obtain $t_{14} = t_{42}$ from the gluing conditions, which implies the total cost

$$\sum_{i,j} q_{ij} t_{ij} = \left[1 - \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right)\right](t_{14} + t_{24}) + \frac{1}{|M|}(t_{11} + t_{21}) + \frac{1}{|L|}(t_{32} + t_{33}).$$

The normalizing conditions imply $t_{14} + t_{24} = t_{11} + t_{21} = t_{32} + t_{33} = 1$ and thus

$$\sum_{i,j} q_{ij} t_{ij} = 1 + \frac{1}{2}\left(\frac{1}{|M|} + \frac{1}{|L|}\right),$$

which is an upper bound of $\kappa_v(x)$ for all $x \in \mathcal{C}(c)$ by Lemma 5.29. Hence

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \geq \frac{1}{2} - \frac{1}{4|M|} - \frac{1}{4|L|}$$

by Lemma 5.23 and Remark 5.24. □

Remark 5.32. A slightly weaker lower bound

$$\mathrm{scl}_{\mathrm{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \geq \frac{1}{2} - \frac{1}{2\min\{|M|, |L|\}}.$$

holds for all $|M|, |L| \geq 2$, which can be proved in a similar way with much simpler computations using cost matrix

$$Q = \begin{pmatrix} \frac{1}{\min\{|M|,|L|\}} & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & \frac{1}{\min\{|M|,|L|\}} & * \\ * & 1 & 1 & * \end{pmatrix}.$$

This bound is sharp when $d = 1$ by constructing admissible surfaces in simple normal form.

In particular, we observe that

$$\text{scl}_{\text{BS}(M,L)}(atta^{-1}t^{-1} + t^{-1}) \to \text{scl}_{F_2}(atta^{-1}t^{-1} + t^{-1}) = 1/2$$

as $|M|, |L| \to \infty$. In contrast, Proposition 5.30 shows that for certain chains like $c = at^2 + 2t^{-1}$, the convergence depends on how $|M|, |L|$ go to infinity, governed by how $d = \gcd(|M|, |L|)$ behaves. Such convergence of $\text{scl}_{\text{BS}(M,L)}$ to scl_{F_2} holds true in general for any fixed chain as $d = \gcd(|M|, |L|)$; see [20, Theorem 6.29]. This is a homological analog of the phenomenon of geometric convergence in hyperbolic Dehn surgery.

In contrast to the two examples above, the winding numbers of turns cannot be ignored in the following example, which has higher complexity.

Proposition 5.33.
$$\text{scl}_{\text{BS}(2,3)}([a, t^2]) = \frac{5}{24}.$$

Proof. We have four arcs $a_1, \ldots, a_4$ with winding numbers $1, 0, -1, 0$ respectively. With Setup (5.2), we have $W(a_1) = 4\mathbb{Z}$, $W(a_2) = W(a_4) = 6\mathbb{Z}$, and $W(a_3) = 9\mathbb{Z}$. Then $D_v = D_e = 36\mathbb{Z}$, $W_e = \mathbb{Z}/36\mathbb{Z}$ and the complexity $\rho(c) = 2$. The turns are paired up as follows.

$$
\begin{aligned}
(a_1, \bar{w}, a_1) &\leftrightarrow (a_4, -\bar{w}, a_2) \\
(a_2, \bar{w}, a_4) &\leftrightarrow (a_3, -\bar{w}, a_3) \\
(a_2, \bar{w}, a_1) &\leftrightarrow (a_4, -\bar{w}, a_3) \\
(a_1, \bar{w}, a_4) &\leftrightarrow (a_3, -\bar{w}, a_2)
\end{aligned}
$$

To get $5/24$ as an upper bound, we present a simple relative admissible surface S of degree 36 consisting of the following disk-like pieces described by the turns on the polygonal boundaries. By the computation above and our orientation on e, the maps $\overline{o_e}, \overline{t_e} : \mathbb{Z}/36\mathbb{Z} \to$

$\mathbb{Z}/36\mathbb{Z}$ are given by $\overline{o_e}(\bar{w}) = 2\bar{w}$ and $\overline{t_e}(\bar{w}) = 3\bar{w}$.

(1) $(a_1, 0, a_1) + (a_1, 1, a_1)$, which is disk-like since its winding number $2w(a_1) + \overline{o_e}(0+1) \in 4 + D_v$ lies in $W(a_1) = 4\mathbb{Z}$. Take 18 copies of this piece;

(2) $(a_2, 0, a_4) + (a_4, 0, a_2)$, which is disk-like since its winding number $w(a_2) + w(a_4) + \overline{o_e}(0) + \overline{t_e}(0) \equiv 0 \mod 36$ lies in $W(a_2) = 6\mathbb{Z}$. Take 18 copies of this piece.

(3) $(a_2, 1, a_4) + (a_2, 2, a_4) + 2(a_4, -1, a_2)$, which is disk-like since its winding number $2w(a_2) + 2w(a_4) + \overline{o_e}(1 + 2) + 2\overline{t_e}(-1) \equiv 0 \mod 36$ lies in $W(a_2) = 6\mathbb{Z}$. Take 9 copies of this piece.

(4) $(a_3, 0, a_3) + 2(a_3, -1, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(2(-1)) \equiv -9 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. Take 4 copies of this piece.

(5) $(a_3, -2, a_3) + 2(a_3, 0, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(-2) \equiv -9 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. Take 7 copies of this piece.

(6) $(a_3, -1, a_3) + 2(a_3, -2, a_3)$, which is disk-like since its winding number $3w(a_3) + \overline{t_e}(-1 - 2 - 2) \equiv -18 \mod 36$ lies in $W(a_3) = 9\mathbb{Z}$. Take 1 copy of this piece.

It is easy to see that the gluing condition is satisfied. Note that S is relative admissible of degree 36 with $\widehat{\chi}(S)/36 = 2 - 57/36 = 5/12$, and thus

$$\mathrm{scl}_{\mathrm{BS}(2,3)}([a, t^2]) \leq \frac{5}{24}.$$

To establish the lower bound, assign cost $q_{i,\bar{w},j}$ to the turn $(a_i, \bar{w}, a_j)$ with

$$q_{1,\bar{w},1} = \begin{cases} 1/4 & \text{if } \bar{w} \text{ is even}, \\ 3/4 & \text{otherwise}, \end{cases} \qquad q_{4,\bar{w},2} = \begin{cases} 1 & \text{if } \bar{w} \text{ is even}, \\ 1/2 & \text{otherwise}, \end{cases}$$
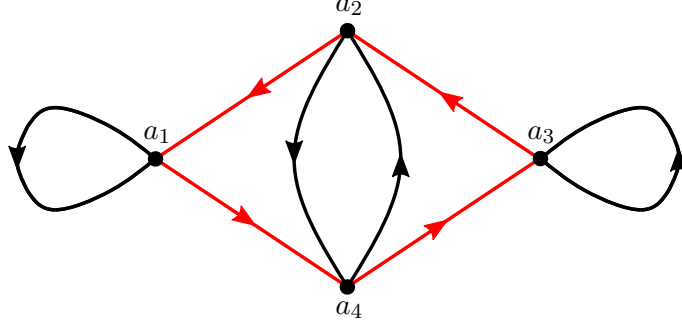
Figure 5.6: The graph Y indicating turns between arcs, except that every single edge represents $|W_e| = 36$ multiedges. This appears originally in [20, Fig. 16].

and for any $\bar{w}$

$$q_{2,\bar{w},4} = 0, \quad q_{3,\bar{w},3} = 1/3, \quad q_{2,\bar{w},1} = 1/3, \quad q_{4,\bar{w},3} = 0, \quad q_{1,\bar{w},4} = 1/4, \quad q_{3,\bar{w},2} = 1.$$

We check each disk-like piece C costs at least 1.

(1) Suppose C does not contain any turns of forms $(a_2, \bar{w}, a_1)$, $(a_4, \bar{w}, a_3)$, $(a_1, \bar{w}, a_4)$, or $(a_3, \bar{w}, a_2)$. That is, the red turns in Figure 5.6 are excluded. Then there are three cases:

   (a) The boundary only contains a_1. By formula (5.4), $w(a_1, \bar{w}, a_1) = w(a_1) + \overline{o_e}(\bar{w}) = 1 + 2\bar{w} \in \mathbb{Z}/36\mathbb{Z}$. Thus we have $w(C) \equiv n_0 + 3n_1 \mod 4$, where n_0 (resp. n_1) is the total number of turns $(a_1, \bar{w}, a_1)$ on C with $\bar{w}$ even (resp. odd). For C to be disk-like, we have $w(C) \in W(a_1) = 4\mathbb{Z}$. Thus the cost is $(n_0 + 3n_1)/4 = k$ for some integer $k \geq 1$.

   (b) The boundary only contains a_2 and a_4. Then $w(C) \equiv 2n_0 + n_1 \mod 2$, where n_0 (resp. n_1) is the total number of turns $(a_4, \bar{w}, a_2)$ on C with $\bar{w}$ even (resp. odd). For C to be disk-like, we need $w(C) \in W(a_2) = 6\mathbb{Z}$. Hence $2n_0 + n_1$ must be even and the cost $(2n_0 + n_1)/2 = k$ for some integer $k \geq 1$.

   (c) The boundary only contains a_3. Then $w(C) \equiv n \mod 3$, where n is the total number of turn on C. For C to be disk-like, we need $w(C) \in W(a_3) = 9\mathbb{Z}$. Hence

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n must be divisible by 3 and the cost $n/3 = k$ for some integer $k \geq 1$.

(2) Now suppose C is a disk-like piece containing at least one of the turns $(a_2, \bar{w}, a_1)$, $(a_4, \bar{w}, a_3)$, $(a_1, \bar{w}, a_4)$ or $(a_3, \bar{w}, a_2)$.

(a) If C contains a_3 on the boundary, then it must have a turn $(a_3, \bar{w}, a_2)$ which already has cost $q_{3,\bar{w},2} = 1$.

(b) If C does not contain a_3, then it includes 3 turns of the forms $(a_2, \bar{w}, a_1)$, $(a_1, \bar{w}', a_4)$ and $(a_4, \bar{w}'', a_2)$ respectively. In this case, the cost of C is at least

$$q_{2,\bar{w},1} + q_{1,\bar{w}',4} + q_{4,\bar{w}'',2} \geq \frac{1}{3} + \frac{1}{4} + \frac{1}{2} > 1.$$

In summary, any disk-like piece has cost at least 1. Let $t_{i,\bar{w},j}$ be the normalized number of turns $(a_i, \bar{w}, a_j)$ in a vector $x \in \mathcal{C}(c)$. Then the gluing conditions imply $t_{1,\bar{w},1} = t_{4,-\bar{w},2}$, $t_{1,\bar{w},4} = t_{3,-\bar{w},2}$ and $t_{2,\bar{w},1} = t_{4,-\bar{w},3}$. Therefore, the total cost

$$
\begin{aligned}
\sum_{i,\bar{w},j} q_{i,\bar{w},j} t_{i,\bar{w},j} &= \left(\frac{1}{4} + 1\right) \left(\sum_{\bar{w} \text{ even}} t_{1,\bar{w},1} + \sum_{\bar{w}} t_{1,\bar{w},4}\right) + \left(\frac{3}{4} + \frac{1}{2}\right) \sum_{\bar{w} \text{ odd}} t_{1,\bar{w},1} \\
&\quad + \frac{1}{3} \sum_{\bar{w}} (t_{3,\bar{w},3} + t_{4,\bar{w},3}) \\
&= \frac{5}{4} \sum_{\bar{w}} (t_{1,\bar{w},1} + t_{1,\bar{w},4}) + \frac{1}{3} \sum_{\bar{w}} (t_{3,\bar{w},3} + t_{4,\bar{w},3}).
\end{aligned}
$$

Combining with the normalizing condition, we have

$$\kappa_v(x) \leq \sum_{i,\bar{w},j} q_{i,\bar{w},j} t_{i,\bar{w},j} = \frac{5}{4} + \frac{1}{3},$$

by the duality method. Hence by Lemma 5.23, we have $\mathrm{scl}_{\mathrm{BS}(2,3)}([a, t^2]) \geq \frac{5}{24}$. $\qquad \square$

If M and L are coprime, one can see that any reduced word of the form

$$g = a^{u_1}ta^{u_2}t\cdots a^{u_n}ta^{v_1}Ta^{v_2}T\ldots a^{v_n}T$$

can be rewritten as a reduced word $g = a^u t^n a^v T^n$, where $T = t^{-1}$. A trick using the Chinese remainder theorem shows $\mathrm{scl}_{\mathrm{BS}(M,L)}(g) = \mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ for any such g. Thus it would be interesting to know how the sequence $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ behaves as $n \to \infty$, where the complexity increases unboundedly. For example, does $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$ converges to some limit? If so, how fast does it converge?

One can lift $[a, t^n]$ to the infinite cyclic cover corresponding to $\ker h$, which is an infinite amalgam with presentation $\widetilde{G}(M, L) = \langle a_k, k \in \mathbb{Z} \mid a_k^M = a_{k+1}^L \rangle$. Then $g_n = a_0 a_n^{-1}$ is a lift of $[a, t^n]$ for each n. One can use techniques similar to the computations above to show that $\mathrm{scl}_{(\widetilde{G}(M,L),\langle a_0\rangle)}(g_n)$ converges *exponentially* in n to $1/2$. It is not clear whether a similar convergence holds for $\mathrm{scl}_{\mathrm{BS}(M,L)}([a, t^n])$. An exponential convergence seems unusual for scl of a family of words with linear word length growth, eg in free groups.

REFERENCES

[1] Yago Antolín, Warren Dicks, and Zoran Šunić. Left relativity convex subgroups. In *Topological methods in group theory*, volume 451 of *London Math. Soc. Lecture Note Ser.*, pages 1–18. Cambridge Univ. Press, Cambridge, 2018.

[2] Alexander Barvinok. *Integer points in polyhedra*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[3] Christophe Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[4] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Stable commutator length on mapping class groups. *Ann. Inst. Fourier (Grenoble)*, 66(3):871–898, 2016.

[5] Noel Brady, Matt Clay, and Max Forester. Turn graphs and extremal surfaces in free groups. In *Topology and geometry in dimension three*, volume 560 of *Contemp. Math.*, pages 171–178. Amer. Math. Soc., Providence, RI, 2011.

[6] M. Burger and N. Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)*, 1(2):199–235, 1999.

[7] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[8] Danny Calegari. Stable commutator length in subgroups of $\mathrm{PL}^+(I)$. *Pacific J. Math.*, 232(2):257–262, 2007.

[9] Danny Calegari. Length and stable length. *Geom. Funct. Anal.*, 18(1):50–76, 2008.

[10] Danny Calegari. Surface subgroups from homology. *Geometry & Topology*, 12(4):1995–2007, 2008.

[11] Danny Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[12] Danny Calegari. Stable commutator length is rational in free groups. *J. Amer. Math. Soc.*, 22(4):941–961, 2009.

[13] Danny Calegari. scl, sails, and surgery. *J. Topol.*, 4(2):305–326, 2011.

[14] Danny Calegari and Koji Fujiwara. Stable commutator length in word-hyperbolic groups. *Groups Geom. Dyn.*, 4(1):59–90, 2010.

[15] Danny Calegari and Alden Walker. `scallop`, computer program available at authors' websites, and from computop.org.

[16] Danny Calegari and Alden Walker. Isometric endomorphisms of free groups. *New York J. Math.*, 17:713–743, 2011.

[17] Danny Calegari and Alden Walker. Random groups contain surface subgroups. *Journal of the American Mathematical Society*, 28(2):383–419, 2015.

[18] Lvzhou Chen. Scl in free products. *Algebr. Geom. Topol.*, 18(6):3279–3313, 2018.

[19] Lvzhou Chen. Spectral gap of scl in free products. *Proc. Amer. Math. Soc.*, 146(7):3143–3151, 2018.

[20] Lvzhou Chen. Scl in graphs of groups. *Inventiones mathematicae*, Jan 2020.

[21] Lvzhou Chen and Nicolaus Heuer. Spectral gap of scl in graphs of groups and 3-manifolds, 2019.

[22] Matt Clay, Max Forester, and Joel Louwsma. Stable commutator length in Baumslag-Solitar groups and quasimorphisms for tree actions. *Trans. Amer. Math. Soc.*, 368(7):4751–4785, 2016.

[23] Marc Culler. Using surfaces to solve equations in free groups. *Topology*, 20(2):133–145, 1981.

[24] Andrew J. Duncan and James Howie. The genus problem for one-relator products of locally indicable groups. *Math. Z.*, 208(2):225–237, 1991.

[25] Benson Farb and Howard Masur. Superrigidity and mapping class groups. *Topology*, 37(6):1169–1176, 1998.

[26] Talia Fernós, Max Forester, and Jing Tao. Effective quasimorphisms on right-angled Artin groups. *Ann. Inst. Fourier (Grenoble)*, 69(4):1575–1626, 2019.

[27] Max Forester, Ignat Soroko, and Jing Tao. Genus bounds in right-angled artin groups. *Publ. Mat.*, To appear.

[28] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.

[29] Elisabeth Ruth Green. *Graph products of groups*. PhD thesis, University of Leeds, 1990.

[30] Nicolaus Heuer. Gaps in SCL for amalgamated free products and RAAGs. *Geom. Funct. Anal.*, 29(1):198–237, 2019.

[31] Nicolaus Heuer and Clara Loeh. The spectrum of simplicial volume, 2019.

[32] Richard B. Holmes. *Geometric functional analysis and its applications*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 24.

[33] Tetsuya Ito, Kimihiko Motegi, and Masakazu Teragaito. Generalized torsion and decomposition of 3-manifolds. *Proc. Amer. Math. Soc.*, 147(11):4999–5008, 2019.

[34] Sergei V. Ivanov and Anton A. Klyachko. Quasiperiodic and mixed commutator factorizations in free products of groups. *Bull. Lond. Math. Soc.*, 50(5):832–844, 2018.

[35] Vadim A. Kaimanovich and Howard Masur. The Poisson boundary of the mapping class group. *Invent. Math.*, 125(2):221–264, 1996.

[36] R. C. Lyndon and M. J. Wicks. Commutators in free groups. *Canad. Math. Bull.*, 24(1):101–106, 1981.

[37] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[38] Jean-Pierre Serre. *Trees.* Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell.

[39] Tim Susse. Stable commutator length in amalgamated free products. *J. Topol. Anal.*, 7(4):693–717, 2015.

[40] Simon Thomas. On the complexity of the quasi-isometry and virtual isomorphism problems for finitely generated groups. *Groups Geom. Dyn.*, 2(2):281–307, 2008.

[41] William P. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.

[42] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[43] Alden Walker. Stable commutator length in free products of cyclic groups. *Exp. Math.*, 22(3):282–298, 2013.

[44] Henry Wilton. Essential surfaces in graph pairs. *Journal of the American Mathematical Society*, 31(4):893–919, 2018.

[45] Dongping Zhuang. Irrational stable commutator length in finitely presented groups. *J. Mod. Dyn.*, 2(3):499–507, 2008.