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ZERO-CYCLES AND MEASURES OF IRRATIONALITY FOR ABELIAN VARIETIES

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ABSTRACT

In this thesis we make several advances in the study of the birational geometry of complex abelian varieties. We are mainly concerned with two birational invariants: the degree of irrationality and the covering gonality. The degree of irrationality of a projective variety X is the minimal degree of a dominant rational map $\varphi : X \dashrightarrow \mathbb{P}^{\dim X}$. The covering gonality of X is the minimal $g \in \mathbb{Z}_{\geq 0}$ such that X is birationally covered by a family of g-gonal curves, or equivalently such that a generic $x \in X$ is contained in a g-gonal curve. The degree of irrationality and the covering gonality measure respectively the failure of X to be rational and uniruled and are thus called measures of irrationality.

By studying rational equivalence of zero-cycles on abelian varieties, this work contributes new lower bounds on the covering gonality and the degree of irrationality of very general abelian varieties. In Chapter 2, we provide the reader with preliminary background on algebraic cycles and measures of irrationality. In Chapter 3, we present our first contribution: a proof of a conjecture of Voisin on the covering gonality of very general abelian varieties. In fact, in collaboration with E. Colombo, J. C. Naranjo, and G. P. Pirola, we prove an extension of this result which provides lower bounds on the degree of irrationality of subvarieties of very general abelian varieties. In Chapter 4, we derive a cohomological obstruction to the existence of low degree dominant rational maps from an abelian g-fold to a g-fold admitting a cohomological decomposition of the diagonal. In particular, using this obstruction we show that the degree of irrationality of a $(1, d)$-polarized abelian surface with Picard number one is 4 if d does not divide 6. This theorem and its generalization settle a conjecture of Chen, answer questions of Yoshihara, and provide the best known lower bound on the degree of irrationality of very general abelian varieties of large degree in all dimensions. Finally, in the Appendix C we present some new identities for zero-cycles on abelian varieties.

CHAPTER 1

INTRODUCTION

One of the first facts one learns about complex tori is that a holomorphic map from $\mathbb{P}^1$ to a compact complex torus is constant. Indeed, such a map must lift to the universal cover and Liouville's theorem along with compactness of $\mathbb{P}^1$ imply that this lift must be constant. Hence, an abelian variety does not contain any rational curves. One is naturally led to the following question: *How far from rational are curves on complex abelian g-folds?* Since many abelian varieties contain elliptic curves it is more interesting to study the complexity of curves on very general abelian varieties.

Question 1.0.1. *How far from rational are curves on very general complex abelian g-folds?*

There are at least two natural measures of complexity which can be used to measure how far a curve is from being rational: the genus and the gonality. The minimal genus of curves on very general abelian varieties has been studied in [BCV95]. However, the gonality seems like a more suitable measure of irrationality since it has a natural generalization to higher dimensions, namely the degree of irrationality. The degree of irrationality $\mathrm{irr}(X)$ of a projective variety X is the minimal degree of a dominant rational map $X \dashrightarrow \mathbb{P}^{\dim X}$.

The minimal gonality of curves on very general abelian varieties has been studied by Pirola [Pir89] who shows that a very general abelian variety of dimension at least 3 does not contain hyperelliptic curves. Using analogous methods, Alzati and Pirola [AP93] prove that a very general abelian variety of dimension at least 4 do not contain trigonal curves. These results suggest the following:

> For each $k \in \mathbb{Z}_{>0}$ there is an integer g_k such that a very general abelian variety of dimension at least g_k does not contain curves of gonality less than $k+1$.

This problem was posed in [BPE$^+$17] and answered positively by Voisin in [Voi18].

Theorem 1.0.2 (Voisin, Thm. 1.1 in [Voi18])**.** *A very general abelian variety of dimension at least*

$$2^{k-2}(2k-1) + (2^{k-2}-1)(k-2)$$

does not contain curves of gonality less than $k+1$*.*

Voisin provides some evidence suggesting that this bound can be improved significantly and conjectures the following linear bound on the gonality of curves on very general abelian varieties:

Conjecture 1.0.3 (Voisin, Conj. 1.2 in [Voi18])**.** *A very general abelian variety of dimension at least* $2k-1$ *does not contain curves of gonality less than* $k+1$*.*

The main result of this thesis is the proof of this conjecture.

Corollary 3.3.3 (Corollary 4.7 in [Mar20]). *A very general abelian variety of dimension at least* $2k-2$ *does not contain curves of gonality less than* $k+1$*.*

In fact, in collaboration with E. Colombo, J. C. Naranjo, and G. P. Pirola, we obtained a lower bound on the degree of irrationality of any subvariety of a very general abelian variety.

Corollary 3.3.4 (Corollary 1.2 in [CMNP19]). *If* A *is a very general abelian variety of dimension at least* 3 *and* $Z \subset A$ *is a subvariety, then*

$$\operatorname{irr}(Z) \geq \dim Z + \frac{\dim A + 1}{2}.$$

These results are obtained by studying rational equivalence of zero-cycles on abelian varieties. In particular, we are interested in fibers of the map:

$$\Sigma_k : A^k \to CH_0(A)$$
$$(a_1, \ldots, a_k) \mapsto \{a_1\} + \ldots + \{a_k\}.$$

A subvariety $Z \subset A$ which admits a dominant rational map of degree k to $\mathbb{P}^{\dim Z}$ gives rise to a $\dim Z$-dimensional fiber of Σ_k (see Section 2.2.4). Therefore, Theorems 3.3.3 and 3.3.4 follow at once from the following result.

Theorem 3.3.2 (Theorem 1.1 in [CMNP19]). *If A is a very general abelian variety of dimension at least 3, d is a positive integer, and $d \leq \dim A$ if $\dim A$ is odd, then all fibers of the map $\Sigma_k : A^k \to CH_0(A)$ have dimension less than d if*

$$k < d + \frac{\dim A + 1}{2}.$$

This theorem is obtained by an intricate induction argument which builds on ideas of [Pir89], [AP93], and [Voi18]. Let $\mathcal{A}/S$ be a locally complete family of abelian g-folds and $\mathcal{Z} \subset \mathcal{A}^k_S$ be a flat family of d-dimensional subvarieties such that the map

$$\mathcal{Z}_s \hookrightarrow \mathcal{A}^k_s \to CH_0(\mathcal{A}_s)$$

has only positive dimensional fibers for any $s \in S$. Given an abelian $(g-1)$-fold B, there are curves $S_\lambda(B) \subset S$ along which $\mathcal{A}_s \sim B \times E_s$, where E_s is an elliptic curve parametrized by $s \in S_\lambda(B)$. Given s on such a curve, we can project $\mathcal{Z}_s$ to B^k using the isogeny.

The key point is that, under suitable conditions (see Section 3.1.1):

(1) We can choose the curve $S_\lambda(B)$ appropriately so that the projection of $\mathcal{Z}_s \subset \mathcal{A}^k_s$ to B^k is generically finite on its image for generic $s \in S_\lambda(B)$.

(2) The image of this projection varies with $s \in S_\lambda(B)$.

If (1) and (2) hold, we obtain a subvariety $W \subset B^k$ of dimension $d+1$ such that

$$W \hookrightarrow B^k \to CH_0(B)$$

has only positive dimensional fibers. If g is large enough, we can proceed by induction to obtain an abelian surface B such that every point of B^k is contained in a positive dimensional fiber of $B^k \to CH_0(B)$. This contradicts Mumford's theorem 2.1.2 on rational equivalence of zero-cycles on surfaces with $p_g \neq 0$.

Before the author's work in [Mar20], this induction argument was implicitly suggested in [AP93] but ensuring that condition (1) holds seemed out of reach. The key insight of [Mar20] which led to the proof of Conjecture 1.0.3 is to exploit the structure of the inductive argument by keeping track of two projections (see Section 3.2).

The proof of Theorem 3.3.2 follows the same lines but the cases $d := \dim_S \mathcal{Z} > 2$ require the following lemma which is of independent interest and a consequence of the theory of generic vanishing:

Projection Lemma 3.2.6 (Projection Lemma 3.2 in [CMNP19]). *Let A be an abelian variety, l, r, k positive integers with $l \leq r$, $Z \subset [A^r]^k$ a subvariety which is not covered by subtori, and i_M the map $A^l \to A^r$ given by*

$$(a_1, \dots, a_l) \mapsto \Big(\sum_{j=1}^{l} m_{1j} a_j, \dots, \sum_{j=1}^{l} m_{rj} a_j \Big),$$

where $M = (m_{ij}) \in M_{r \times l}(\mathbb{Z})$ has maximal rank.

There is a non-empty Zariski open $U \subset M_{r \times l}(\mathbb{Z})$ such that if $M \in U$ either

- *the image of Z under the quotient map $p_M : [A^r]^k \to [A^r / i_M(A)]^k$ is covered by subtori,*
- *the restriction of p_M to Z is generically finite on its image.*

In the second half of this thesis, we turn to the following question:

Question 1.0.4. *What is the degree of irrationality of an abelian variety?*

Alzati and Pirola showed in [AP92] that the degree of irrationality of an abelian variety A is at least $\dim A + 1$. Tokunaga and Yoshihara later proved that this bound is sharp for abelian surfaces in [TY95]. They show that if an abelian surface contains a smooth curve of genus 3, then it admits a degree 3 dominant rational map to $\mathbb{P}^2$ (see Theorem 2.3.2). This prompted Yoshihara to ask the following questions:

Question 1.0.5 (Problem 10, [Yos96])**.** *Is there an abelian surface A satisfying $irr(A) \geq 4$?*

Question 1.0.6 (Problem 10, [Yos96])**.** *Do isogenous abelian surfaces have the same degree of irrationality?*

As far as upper bounds are concerned, Keum shows in [Keu90] that every Kummer surface is the K3 cover of an Enriques surface. It follows that every abelian surface admits a degree 8 dominant rational map to $\mathbb{P}^2$. This bound was recently improved by Chen and Chen-Stapleton who showed in [Che19] and [CS19] that the degree of irrationality of any abelian surface is at most 4 independently of the polarization type.

One of our main contributions to the field is an answer to Question 1.0.5.

Theorem 4.2.1 (Theorem 1.1 in [Mar19])**.** *If A is an abelian surface such that the image of the intersection pairing $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ does not contain 12, then*

$$\mathrm{irr}(A) = 4.$$

This cohomological obstruction to the existence of low degree dominant rational map to projective space can be generalized to higher dimensions. Given an abelian g-fold A, let

$$V_l \subset \mathrm{Hdg}^l(A^{k-1}) \tag{1.1}$$

be the sublattice of classes α supported on D_α^{k-1} for some closed proper subset $D_\alpha \subset A$. Consider the function

$$M_g(k) := (k-1)! \sum_{j=0}^{k} (-1)^j (j-1) \left(1-(1-j)^{g-1}\right)^2 \binom{k}{j}.$$

Theorem 4.2.4. *If the image of*

$$V_{(k-2)g} \longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z}) \cong \mathbb{Z}$$
$$\alpha \longmapsto \alpha \cdot [\ker(A^k \to A)]$$

does not contain $M_g(k)$, then A does not admit a dominant rational map of degree k to $\mathbb{P}^2$. Here the cohomology class $[\ker(A^k \to A)]$ is the cycle class of the kernel of the summation map.

One can show that this obstruction is non-trivial, namely $M_g(k) \neq 0$ for $g < k < 2g$ and the image of the map $V_{(k-2)g} \longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z})$ has high index for very general abelian varieties with a high degree polarization.

Theorem 4.2.7. *If A is a $(1, d_2, \ldots, d_g)$-polarized abelian g-fold with maximal special Mumford-Tate group the image of*

$$V_{(k-2)g} \longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z}) \cong \mathbb{Z}$$
$$\alpha \longmapsto \alpha \cdot [\ker(A^k \to A)]$$

is contained in $d_g\mathbb{Z}$.

Theorems 4.2.4 and 4.2.7 along with Corollary 3.3.4 give the following:

Theorem 4.2.11. *Let A be a very general $(1, d_2, \ldots, d_g)$-polarized abelian g-fold. If*

$$d_g \nmid \operatorname{lcm}\left\{M_g(k) : \frac{3g+1}{2} \leq k \leq 2g-1\right\},$$

then

$$\operatorname{irr}(A) \geq 2g.$$

One can make this theorem effective at the cost of a slightly stronger condition on the polarization.

Theorem 4.2.12. *Let A be a $(1, d_2, \ldots, d_g)$-polarized abelian g-fold with maximal special Mumford-Tate group. If*

$$d_g \nmid \operatorname{lcm}\left\{M_g(k) : g+1 \leq k \leq 2g-1\right\},$$

then

$$\operatorname{irr}(A) \geq 2g.$$

For the polarizations for which it applies, Theorem 4.2.11 give the best known lower bound on the degree of irrationality of very general abelian varieties. When Theorem 4.2.11 does not apply, Corollary 3.3.4 gives the best known lower bound of $(3 \dim A + 1)/2$. See Appendix B for tables of values of $M_g(k)$ and of the prime factorizations of the least common multiples appearing in Theorems 4.2.11 and 4.2.12 for low values of g and k.

CHAPTER 2

PRELIMINARIES

2.1 Algebraic cycles

In this section we review basic notions regarding algebraic cycles, focusing on Mumford's theorem and applications to families of suborbits for surfaces. Throughout this thesis we work with varieties over the complex numbers.

2.1.1 Basic definitions

Given a smooth projective variety $X/\mathbb{C}$, an algebraic cycle on X is a formal $\mathbb{Z}$-linear combination of subvarieties of X. We denote by $Z^d(X)$ (resp. $Z_d(X)$) the free abelian group on the set of codimension d (resp. dimension d) subvarieties of X. Elements of $Z_d(X)$ are called d-cycles. The group

$$Z^\bullet(X) := \sum_{d=0}^{\dim X} Z^d(X)$$

is equipped with a morphism

$$\begin{aligned} cl : Z^\bullet(X) &\to H^{2\bullet}(X, \mathbb{Z}) \\ \sum a_i W_i &\mapsto \sum a_i [W_i], \end{aligned}$$

called the cycle class map. For $W \subset X$, a subvariety of X, the cohomology class $[W]$ can be obtained by choosing a triangulation of W, considering the associated class in the Betti homology of X, and using Poincaré duality. Alternatively, one can follow the approach of [Voi02] p. 253 or [Voi14] 1.1.4. The support of a cycle $\sum a_i W_i \in Z^\bullet(X)$ is the set $\bigcup_{i:\, a_i \neq 0} W_i$.

To study intersection theory one endows a quotient of $Z^\bullet(X)$ with a ring structure making the cycle class map a ring homomorphism. Different choices of quotients correspond to different so-called adequate equivalence relations on cycles. We will mostly be concerned with rational equivalence. Two cycles A and B are rationally equivalent and one writes $A \sim B$ if there is a cycle $\Gamma = \sum a_i W_i \in Z^\bullet(X \times \mathbb{P}^1)$, with each W_i flat over $\mathbb{P}^1$, such that

$$\Gamma \cdot (X \times \{0\} - X \times \{\infty\}) = A - B,$$

where the product is the intersection product extended by linearity. If $A \sim B$, then

$$[A] - [B] = [Z] \cdot ([X \times \{0\}] - [X \times \{\infty\}]) = 0,$$

so that the cycle class map factors through $CH^\bullet(X) := Z^\bullet(X)/\sim$. One can define a ring structure on $CH^\bullet(X)$ making the cycle class map a homomorphism of graded rings. Given a subvariety $W \subset X$, we denote by $\{W\} \in CH^\bullet(X)$ the corresponding cycle.

Example 2.1.1. *The cycle class map*

$$cl : CH^\bullet(\mathbb{P}^n) \to H^\bullet(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[H]/(H^{n+1})$$

is an isomorphism. Given a subvariety $W \subset \mathbb{P}^n$, the rational equivalence class of the cycle $\{W\}$ is completely determined by the degree and dimension of W.

Example 2.1.2. *Let C be a smooth projective curve. Then*

$$CH^\bullet(C) = \mathrm{Pic}(C) \oplus \mathbb{Z} \cdot \{C\},$$

where $\mathrm{Pic}(C)$ is the Picard group of C, and the 1-cycle $\{C\}$ is the identity of the Chow ring. The cycle class map sends $\{C\}$ to the positive generator of $H^0(C, \mathbb{Z})$ and coincides with the first Chern class c_1 on $\mathrm{Pic}(C)$.

For any smooth projective variety X, $CH^1(X) \cong \mathrm{Pic}(X)$ and we have the exact sequence

$$1 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

where $\mathrm{Pic}^0(X)$ and $\mathrm{NS}(X)$ are respectively the kernel and the image of the cycle class map on $CH^1(X)$.

2.1.2 Mumford's theorem and decomposition of the diagonal

The two examples above are the exception rather than the rule. In general for $d > 1$ the group $CH^d(X)$ is gigantic and is a subtle and rich invariant of the variety X. Given a closed subset $Y \subset X$ and a desingularization $\widetilde{Y}$, the following sequence is exact and called the localization exact sequence:

$$CH_\bullet(\widetilde{Y}) \to CH_\bullet(X) \to CH_\bullet(X \setminus Y) \to 0.$$

We say $\alpha \in CH^\bullet(X)$ is supported on Y if it is in the image of the first map. Mumford shows in [Mum69] that if S is a surface with $p_g := h^0(S, \Omega^2) > 0$ then $CH_0(S)$ is not supported on a curve and, equivalently, is infinite-dimensional in a suitable sense. This work was later generalized by Roĭtman in [Roĭ71] and [Roĭ72]. Here we will present a more modern treatment of the theory as developed by Bloch and Srinivas (see [BS83] and [Voi02] Section 22.2). Recall that a correspondence $\Gamma \in CH^{\dim X}(X \times Y)$ gives rise to a morphism

$$\begin{aligned} \Gamma^* : CH_0(Y) &\longrightarrow CH_0(X) \\ z &\longmapsto \mathrm{pr}_{1*}(\Gamma \cdot \mathrm{pr}_2^*(z)). \end{aligned}$$

The main result we will use is the following:

Theorem 2.1.3 ([BS83], see [Voi02] Corollaire 22.20)**.** *Let X and Y be smooth projective varieties and consider $\Gamma \in CH^{\dim X}(X \times Y)$. Suppose that there exists a closed subset*

$X' \subset X$ such that for every $y \in Y$ the zero-cycle $\Gamma^*(\{y\})$ is supported on X'. Then, there is an integer $N \in \mathbb{N}$ and cycles Γ', Γ'' such that

$$N\Gamma = \Gamma' + \Gamma'' \in CH^{\dim X}(X \times Y),$$

Γ' is supported on $X' \times Y$, and the support of Γ'' does not dominate Y by the second projection.

In practice, we will mostly use Mumford's theorem 2.1.2 which is a well known corollary of this result. Consider the maps

$$\begin{aligned} \Sigma_k : X^k &\longrightarrow CH_0(X) \\ (x_1, \ldots, x_k) &\longmapsto \sum_{i=1}^{k} \{x_i\}, \\ \Sigma_{(k)} : \mathrm{Sym}^k(X) &\longrightarrow CH_0(X) \\ x_1 + \ldots + x_k &\longmapsto \sum_{i=1}^{k} \{x_i\}. \end{aligned} \tag{2.1}$$

Definition 2.1.4. *An orbit of degree k for rational equivalence is a fiber of Σ_k or $\Sigma_{(k)}$. We denote the orbit containing the point $z \in X^k$ (resp. $z \in \mathrm{Sym}^k(X)$) by $|z|$. A suborbit is a subvariety of an orbit and a subvariety Z of X^k or $\mathrm{Sym}^k(X)$ is foliated by codimension d suborbits if the fibers of $\Sigma_k|_Z$ (resp. $\Sigma_{(k)}|_Z$) have codimension at most d.*

The notion of suborbit is closely related to but not to be confused with that of constant cycle subvariety in the sense of Huybrechts [Huy14]. Indeed, a suborbit of degree one is exactly the analogue of a constant cycle subvariety as defined by Huybrechts for K3 surfaces. Nonetheless, a suborbit of degree k for X need not be a constant cycle subvariety of X^k; in the former case we consider rational equivalence of cycles in X, while in the latter rational equivalence of cycles in X^k.

Remark 2.1.5. In [Roĭ71] Roĭtman shows that the subset

$$R := \Big\{ \big((x_1, \ldots, x_k), (y_1, \ldots, y_k) \big) : \sum \{x_i\} \sim \sum \{y_i\} \Big\} \subset X^k \times X^k \tag{2.2}$$

is a countable union of Zariski closed subsets. In particular, fibers of Σ_k and $\Sigma_{(k)}$ are countable unions of Zariski closed subsets. Accordingly, since $\mathbb{C}$ is uncountable, one can define the dimension (resp. codimension) of such a fiber as the maximal (resp. minimal) dimension (resp. codimension) of irreducible components without ambiguity. Moreover, given a subvariety Z of X^k or $\mathrm{Sym}^k(X)$, there is a $d \in \mathbb{N}$ such that a very general point of Z is contained in a d-dimensional fiber of $\Sigma_k|_Z$ or $\Sigma_{(k)}|_Z$. All points of Z are then contained in fibers of dimension at least d.

We will call the following result Mumford's theorem, though [Mum69] only treats the case $d = 0$, $l = 2$, and $\dim X = 2$.

Mumford's theorem 2.1.6 (See [Voi02] Proposition 22.24)**.** *Let X be a smooth projective variety and $Z \subset X^k$ a subvariety foliated by codimension d suborbits. Given an integer $l > d$ and any $\omega \in H^0(X, \Omega^l_X)$, the holomorphic form*

$$\omega_k := \sum_{i=1}^{k} \mathrm{pr}_i^* \omega \in H^0(X^k, \Omega^l_X)$$

restricts to 0 *on Z.*

Proof. Consider the inclusion $\iota : Z \to X^k$, the transpose of its graph Γ^t_ι, along with the correspondence

$$\Gamma = \Gamma^t_\iota \circ \left(\sum_{i=2}^{k} \{\Delta_{1i}\} \right) = \sum_{i=1}^{k} \Big\{ \, (x, (x_1, \ldots, x_k)) : (x_1, \ldots, x_k) \in Z, x = x_i \Big\} \in CH^{\dim X}(X \times Z),$$

where

$$\Delta_{1i} = \{(x, x_1, \ldots, x_k) : x = x_i\} \subset X \times X^k.$$

Let $Z' \subset Z$ be a d-dimensional subvariety which is nef, e.g. a general complete intersection of ample divisors on Z. The closed subset $X' = \bigcup_{i=1}^{k} \mathrm{pr}_i(Z')$ had dimension at most d. We claim that for any $z \in Z$ the zero-cycle $\Gamma^*(\{z\}) \in CH_0(X)$ is supported on X'. Indeed, the fiber of Σ_k containing z has codimension at most d and so intersects Z' non-trivially at some point $z' \in Z'$. Since

$$\Gamma^*(\{z\}) = \Sigma_k(z) = \Sigma_k(z') = \Gamma^*(\{z'\}) \in CH_0(X)$$

and $\Gamma^*(\{z'\})$ is supported on X' the claim follows.

By Theorem 2.1.3, there is a positive integer N and cycles Γ' and Γ'' such that

$$N\Gamma = \Gamma' + \Gamma'' \in CH^{\dim X}(X \times Z), \tag{2.3}$$

Γ' is supported on $X' \times Z$, and Γ'' is supported on $X \times Z''$ for some closed proper subset $Z'' \subset X$. In particular, we have the equality of maps

$$(N\Gamma)_* = \Gamma'_* + \Gamma''_* : H^0(X, \Omega^l_X) \longrightarrow H^l(Z, \mathbb{C}).$$

The map Γ'_* factors through $H^0(\widetilde{X'}, \Omega^l)$, where $\widetilde{X'}$ is a desingularization of X'. In particular, it is zero for dimensional reasons when $l > d$. On the other hand, the map Γ''_* factors through a map

$$H^0(\widetilde{Z}'', \Omega^l) \to H^l(\widetilde{Z}, \mathbb{C}),$$

where $\widetilde{Z}''$ and $\widetilde{Z}$ are desingularizations of Z'' and Z respectively. This Gysin map has Hodge bidegree $(\mathrm{codim}_Z Z'', \mathrm{codim}_Z Z'')$ and so Γ''_* is zero. It follows from equation (2.3) that $\Gamma_* : H^0(X, \Omega^l_X) \to H^l(Z, \mathbb{C})$ is zero.

To finish the argument, it suffices to observe that

$$\{\Delta_{1i}\}_* = \mathrm{pr}_i^* : H^0(X, \Omega^l_X) \to H^0(X^k, \Omega^l_{X^k})$$

and that $\Gamma^t_{\iota *} = \Gamma^*_\iota = \iota^*$. □

Remark 2.1.7. Theorem 2.1.3 is already of interest in the case $Y = X$ and $\Gamma = \{\Delta\}$. Assume that $CH_0(X)$ is supported on $Z \subset X$. By Theorem 2.1.3 there is a positive integer N and cycles $\Gamma', \Gamma'' \in CH^{\dim X}(X \times X)$ such that

$$N\{\Delta\} = \Gamma' + \Gamma'' \in CH^{\dim X}(X \times X),$$

where the support of Γ' does not dominate X under the projection to the first factor and Γ'' is supported on $X \times Z$. Conversely, if $N\{\Delta\}$ admits such a decomposition, it is clear by considering the morphism

$$N\{\Delta\}^* : CH_0(X) \to CH_0(X)$$

that $CH_0(X)$ is supported on Z up to N-torsion.

We can argue essentially as in the proof of Mumford's theorem to show:

Theorem 2.1.8 (See [Voi02] Théorème 22.4)**.** *Let X be a smooth projective variety. If $CH_0(X)$ is supported on a d-dimensional closed subset $Y \subset X$, then $H^0(X, \Omega^l_X) = 0$ for all $l > d$.*

Bloch's conjecture conversely alleges that if $H^\bullet(X, \mathbb{C})$ has Hodge coniveau at least 1 in degrees $\bullet \neq 0, 2\dim X$, then $CH_0(X)_{\mathbb{Q}} \cong \mathbb{Q}$. Recall that the Hodge coniveau of a Hodge structure $(V_{\mathbb{Z}}, F^\bullet)$ is the largest c such that $F^c V_{\mathbb{C}} = V_{\mathbb{C}}$. This fits into a broad conjectural framework due to Bloch and Beilinson. The reader in invited to consult [Jan94] for an insightful presentation of the subject.

The discussion in Remark 2.1.7 motivates the following definition.

Definition 2.1.9. *A smooth projective variety X admits a decomposition of the diagonal if there is a positive integer N, a cycle $\Gamma' \in CH^{\dim X}(X \times X)$, and $\alpha \in CH_0(X)$, a zero-cycle of degree N, such that*

$$N\{\Delta\} = X \times \alpha + \Gamma' \in CH^{\dim X}(X \times X)$$

and the support of Γ' does not dominate X under the first projection. The smallest N for which a decomposition exists is called the torsion order of X. We say X has an integral decomposition of the diagonal if it has a decomposition of the diagonal and torsion order 1.

X admits a cohomological decomposition of the diagonal if there is a positive integer N and a cycle $\Gamma' \in CH^{\dim X}(X \times X)$ such that

$$N[\Delta] = N[X \times \{\text{pt}\}] + [\Gamma'] \in H^{2\dim X}(X \times X, \mathbb{Z}),$$

and the support of Γ' does not dominate X under the projection to the first factor. The smallest N for which a cohomological decomposition exists is called the cohomological torsion order of X. We say X has an integral cohomological decomposition of the diagonal if it has a cohomological decomposition of the diagonal and cohomological torsion order 1.

Note that by a theorem of Roĭtman [Roĭ80] the kernel of the degree map $CH_0(X) \to \mathbb{Z}$ injects into the albanese of X. If X admits a decomposition of the diagonal then $CH_0(X) \cong \mathbb{Z}$ up to torsion and by Theorem 2.1.8 the Albanese of X is trivial so that $CH_0(X) \cong \mathbb{Z}$. Hence, X has a decomposition of the diagonal if and only if the degree map $CH_0(X) \to \mathbb{Z}$ is an isomorphism.

Definition 2.1.10. *A smooth projective variety X is called CH_0-trivial if the degree map $CH_0(X) \to \mathbb{Z}$ is an isomorphism.*

Remark 2.1.11. Considering the action of the diagonal on cohomology and arguing as in the proof of Mumford's theorem 2.1.2, we can see that if X has a cohomological decomposition of the diagonal then its cohomology has coniveau at least one in all degrees except 0 and $2\dim X$. Bloch's conjecture then predicts that X is CH_0-trivial and so has a decomposition of the diagonal. Thus, conjecturally, X has a cohomological decomposition of the diagonal if and only if it has a decomposition of the diagonal.

2.1.3 Applications to surfaces

As described in the following two lemmas, given a smooth projective surface X with $p_g \neq 0$, Mumford's theorem 2.1.2 allows us to give upper bounds on the dimension of subvarieties of X^k foliated by positive dimensional suborbits.

Lemma 2.1.12. *Let X be a smooth projective surface and $\omega \in H^0(X, \Omega_X^2)$. If a subvariety $Z \subset X^k$ is foliated by codimension d suborbits, the form $\omega_k^{\lceil (d+1)/2 \rceil}$ restricts to zero on Z.*

Proof. Let R be as in (2.2) and consider the set-theoretic intersection

$$R \cap (Z \times Z) \subset Z \times Z.$$

This subset is a countable union of Zariski closed subsets and it has an irreducible component Γ which dominates the second factor with fibers Γ_z of codimension at most d. The set of points $z \in Z$ such that $z \in Z_{\mathrm{sm}} \cap \Gamma_{z,\mathrm{sm}}$ is clearly Zariski dense. Thus, it suffices to show that $\omega_k^{\lceil (d+1)/2 \rceil}$ restricts to zero on $T_{Z,z}$ for such a z.

Suppose that d is odd (the even case is treated in a similar way). The intersection of any $(d+1)$-dimensional subspace of $T_{Z,z}$ with the tangent space to $T_{\Gamma_z,z}$ is positive dimensional. Hence such a space admits a basis $v_1, \ldots, v_{d+1}$, with $v_1 \in T_{\Gamma_z, z}$. The number $\omega_k^{(d+1)/2}(v_1, \ldots, v_{d+1})$ consists of a sum of terms of the form

$$\pm \prod_{i \in I} \omega_k \left(v_i, v_{\sigma(i)} \right),$$

where $I \subset \{1, \ldots, d+1\}$ is a subset of cardinality $(d+1)/2$ and

$$\sigma : I \to \{1, \ldots, d+1\} \setminus I$$

is a bijection. But $\omega_k(v_1, v_j) = 0$ for any j by Mumford's theorem 2.1.2. Indeed, v_j is tangent to a curve $C \subset Z$ which is smooth at z and the variety $\mathrm{pr}_2|_\Gamma^{-1}(C) \subset \Gamma$ is foliated by suborbits of codimension at most 1. Finally, the subspace $\langle v_1, v_j \rangle$ is contained in the tangent space to $\mathrm{pr}_2^{-1}(C)$ at z. It follows from that $\omega_k^{(d+1)/2}(v_1, \ldots, v_{d+1}) = 0$ and so $\omega_k^{(d+1)/2}$ restricts to zero on Z. $\square$

Given a finite set S and a subset $I \subset S$, we let

$$\mathrm{pr}_I : X^S \to X^I$$

be the projection to the product of the factors with index in I. Moreover, given a positive integer k, we let

$$[k] := \{1, \ldots, k\}.$$

Lemma 2.1.13. *Let X be a smooth projective surface, $\omega \in H^0(X, \Omega^2)$ be non-zero, and $l \leq k$ be positive integers. If the form ω_k^l restricts to zero on the subvariety $Z \subset X^k$, then $\dim Z < k + l$.*

Proof. We proceed by induction on k and note that the case $k = 1$ is obvious. Write $D \subset X$

for the divisor $\operatorname{div}(\omega)$. We first show that we can assume that Z is not contained in

$$\bigcup_{i=1}^{k} \operatorname{pr}_i^{-1}(D).$$

Suppose without loss of generality that $Z \subset D \times X^{k-1}$. Since $\operatorname{pr}_1^* \omega|_D = 0$, the form ω_{k-1}^l restricts to zero on each irreducible component of

$$\operatorname{pr}_{[k]\setminus\{1\}} \left(Z \cap (\{x\} \times X^k) \right)$$

for any $x \in X$. By the induction hypothesis,

$$\dim Z \cap (\{x\} \times X^k) < k - 1 + l$$

so that $\dim Z < k + l$.

Now pick $z = (z_1, \ldots, z_k) \in Z_{\mathrm{sm}} \setminus \bigcup_{i=1}^k \operatorname{pr}_i^{-1}(D)$ and let $v_1, \ldots, v_m$ be a basis of $T_{Z,z}$. The section $\omega_k^k \in H^0(X^k, \Omega^{2k})$ does not vanish at z and we have

$$\omega_k^k(v_1, \ldots, v_m) \neq 0 \in \bigwedge^{2k-m} T^*_{X^k,z}.$$

If $m \geq k + l$, then $\omega_k^k(v_1, \ldots, v_m)$ is a sum of terms of the form

$$\pm \omega_k^l(v_{i_1}, \ldots, v_{i_{2l}}) \cdot \omega_k^{k-l}(v_{i_{2l+1}}, \ldots, v_{i_m}),$$

where $\{i_1, \ldots, i_m\} = [m]$. It follows from the non-vanishing of $\omega_k^k(v_1, \ldots, v_m)$ that ω_k^l does not restrict to zero on $T_{Z,z}$. $\square$

Corollary 2.1.14. *Let X be a smooth projective surface with $p_g \neq 0$ and suppose that the subvariety $Z \subset X^k$ is foliated by d-dimensional suborbits. Then,*

$$\dim Z \leq 2k - d.$$

Proof. By Lemma 2.1.12 $\omega^{\lceil (\dim Z - d + 1)/2 \rceil}$ restricts to zero on Z and so Lemma 2.1.13 implies

$$\dim Z < k + \lceil (\dim Z - d + 1)/2 \rceil - 1.$$

This inequality gives the stated bound for parity reasons. □

2.1.4 Suborbits for abelian varieties

Since the action of A on A^k by diagonal translation preserves rational equivalence, it is often convenient to consider normalized zero-cycle.

Definition 2.1.15. *Given an abelian variety A, the zero-cycle $(a_1, \ldots, a_k) \in A^k$ and the orbit $|\{a_1\} + \ldots + \{a_k\}|$ are called normalized if $a_1 + \ldots + a_k = 0_A$. We write $A^{k,0}$ for the kernel of the summation map, i.e. the set of normalized effective zero-cycles of degree k.*

Remark 2.1.16. If $|\{a_1\} + \ldots + \{a_k\}|$ is a normalized d-dimensional orbit of degree k and $a \in A$ is any point, then

$$|\{ka\} + \{a_1 - a\} + \ldots + \{a_k - a\}|$$

is a normalized orbit of dimension at least d. This was observed by Voisin in Example 6.3 of [Voi18].

We begin by stating analogues of Lemma 2.1.13 and Corollary 2.1.14 for normalized orbits. These are the main results on rational equivalence of zero-cycles that we will use in the proof of Theorem 3.3.2.

Lemma 2.1.17. *Let A be an abelian surface, $\omega \in H^0(A, \Omega^2_A)$ be a generator, and $Z \subset A^{k,0}$ be a subvariety such that ω_k^l restricts to zero on Z. Then,*

$$\dim Z < k + l - 1.$$

Proof. Given the following fact to be proved in Lemma 2.1.19, we can use the same argument as in the proof Lemma 2.1.13 : Let $\iota : A^{k,0} \to A^k$ be the inclusion. The form $\iota^*(\omega_k^{k-1})$ is a generator of $H^0(A^{k,0}, \Omega^{2k-2})$. □

Corollary 2.1.18. *Let A be an abelian surface and $Z \subset A^{k,0}$ be a subvariety foliated by d-dimensional suborbits. Then,*

$$\dim Z \leq 2(k-1) - d.$$

Given an abelian variety A, integers $1 \leq r \leq k$, and a matrix $M = (m_{ij}) \in M_{k\times r}(\mathbb{Z})$ of rank r, we denote by A^r_M (or A_M when $r = 1$) the image of A^r under the following embedding:

$$\begin{aligned} i_M : A^r &\longrightarrow A^k \\ (a_1, \dots, a_r) &\mapsto \Big(\sum_{j=1}^r m_{1j}a_j, \dots, \sum_{j=1}^r m_{kj}a_j\Big). \end{aligned}$$

Note that if A is simple all positive dimensional abelian subvarieties of A^k are of the form A^r_M. Given a $\mathbb{C}$-vector space V, we will use the same notation $V^r_M := i_M(V) \subset V^k$ for any $M \in M_{k\times r}(\mathbb{C})$.

Lemma 2.1.19. *Let A be an abelian variety and $\omega \in H^0(A, \Omega^2_A)$ be a generator. Then,*

$$(i_M^*\omega_k)^r \neq 0 \in H^0(A^r_M, \Omega^{2r}).$$

In particular, if $\dim A = 2$ the form ω_k restricts to a holomorphic symplectic form on A^r_M.

Proof. We treat the case $\omega = dz \wedge dw$, where z and w are two coordinates on A. The general case is easily obtained from this special case. Let z_i, w_i be the corresponding coordinates on the i^{th} factor of A^r. We have

$$i_M^* \omega_k = \sum_{i=1}^{k} \left(\sum_{j=1}^{r} m_{ij} dz_j \wedge \sum_{j'=1}^{r} m_{ij'} dw_{j'} \right).$$

A computation gives

$$(i_M^* \omega_k)^r = r! \det \mathbf{G} \; dz_1 \wedge dw_1 \wedge \ldots \wedge dz_r \wedge dw_r,$$

where

$$\mathbf{G} = (\langle M_i, M_j \rangle)_{1 \leq i,j \leq r}$$

is the Gram matrix of the columns M_i of the matrix M. Since M has maximal rank its Gram matrix has positive determinant. It follows that $(i_M^* \omega_k)^r \neq 0$. □

Lemmas 2.1.12 and 2.1.19 imply the following corollary which will play a crucial technical role in the proof of Theorem 3.3.2:

Corollary 2.1.20. *Given an abelian surface A, a subvariety of the form $A_M^r \subset A^k$ cannot be foliated by positive dimensional suborbits.*

2.1.5 Suborbits in families

We will not only be interested in suborbits of degree k for a smooth projective variety X, but in families of suborbits, as well as subvarieties of X^k foliated by suborbits. Moreover, it will be useful to study these notions in families.

An r-parameter family of effective d-dimensional cycles on a smooth projective variety X is an r-dimensional subvariety of a Chow variety $\mathrm{Chow}_d(X)$ which parametrizes effective pure d-dimensional cycles with a given cycle class. For simplicity, we will leave the cycle class unspecified in the notation for these Chow varieties. Similarly, an r-parameter family of effective d-dimensional cycles on $\mathcal{X}/S$ is a subvariety $\mathcal{D} \subset \mathrm{Chow}_d(\mathcal{X}/S)$ with relative dimension r over S. The subvariety $\mathcal{D}_s \subset \mathcal{D}$ parametrizes cycles with support lying over a point $s \in S$.

Definition 2.1.21.

1. *A subvariety $Z \subset X^k$ is foliated by d-dimensional suborbits if*

$$\dim |z| \cap Z \geq d, \qquad \forall z \in Z.$$

   Clearly, this agrees with Definition 2.1.4.

2. *Similarly, a locally closed subset $\mathcal{Z} \subset \mathcal{X}^k$ is foliated by d-dimensional suborbits if $\mathcal{Z}_s$ is foliated by d-dimensional suborbits for all $s \in S$.*

3. *A family of d-dimensional suborbits of degree k for X is a family of d-dimensional effective cycles on X^k whose supports are suborbits.*

4. *Similarly, a family of d-dimensional suborbits of degree k for $\mathcal{X}/S$ is a family $\mathcal{D} \subset \mathrm{Chow}_d(\mathcal{X}^k/S)$ of effective d-dimensional cycles on $\mathcal{X}^k$ such that $\mathcal{D}_s \subset \mathrm{Chow}_d(\mathcal{X}^k_s)$ is a family of d-dimensional suborbits of degree k for $\mathcal{X}_s$.*

Remark 2.1.22. Let $D \subset \mathrm{Chow}_d(X^k)$ be an r-parameter family of d-dimensional suborbits for a variety X and $\mathcal{Y} \to D$ be the corresponding family of cycles. The set

$$\bigcup_{t \in D} \mathcal{Y}_t \subset X^k$$

is foliated by d-dimensional suborbits but its dimension might be less than $(r+d)$. Conversely, any subvariety $Z \subset X^k$ foliated by d-dimensional suborbits arises in such a fashion from a family of d-dimensional suborbits after possibly passing to a Zariski open subset. Indeed, consider

$$\pi : R \cap (Z \times Z) \to Z,$$

the restriction of the projection of $Z \times Z$ onto the first factor to $R \cap (Z \times Z)$, where R is the graph of rational equivalence R in (2.2). Given a subvariety R' of $R \cap (Z \times Z)$ such that $\pi|_{R'} : R' \to Z$ has a generic fiber of dimension d, we can consider the map from an open set in Z to an appropriate Chow variety of $\mathrm{Chow}_d(X^k)$ taking $z \in Z$ to the cycle $\pi|_{R'}^{-1}(z)$. Letting D be the image of this morphism, we get a family of d-dimensional suborbits that covers an open in Z.

2.2 Measures of irrationality

This section presents the different measures of irrationality that we will study and surveys the literature on these birational invariants. The last subsection focuses on measures of irrationality for abelian varieties and reviews what was known about this topic prior to the contributions of this thesis.

2.2.1 Basic definitions

Definition 2.2.1. *A variety X is:*

- *rational if it birational to $\mathbb{P}^{\dim X}$,*
- *stably rational if there is a positive integer d such that $X \times \mathbb{P}^d$ is rational,*
- *unirational if there is a dominant rational map $Y \dashrightarrow X$ with Y rational,*
- *rationally connected if any two general points $x, y \in X$ can be connected by a (possibly singular) rational curve in X,*

- *uniruled if it admits a dominant and generically finite rational map from a family of rational curves, i.e., given a general point $x \in X$, there is a (possibly singular) rational curve in X going through x.*

We have the following implications between rationality properties of a variety X:

$$\text{rational} \implies \text{stably rational} \implies \text{unirational} \implies \text{rationally connected} \implies \text{uniruled} \tag{2.4}$$

For curves, all these notions agree. For surfaces, they agree aside from uniruledness which is clearly strictly weaker than rational connectedness. From dimension 3 onwards, it is known that all these notion differ aside possibly from rational connectedness and unirationality. It is an important open problem to exhibit rationally connected varieties which fail to be unirational.

To measure the failure of varieties to satisfy these different weakenings of rationality one introduces birational invariants called measures of irrationality.

Definition 2.2.2. *Let X be a projective variety.*

- *The degree of irrationality of X is*

$$\operatorname{irr}(X) := \min\{\deg \varphi : \varphi : X \dashrightarrow \mathbb{P}^{\dim X} \textit{ is dominant}\}.$$

  When X is a curve the degree of irrationality is also called the gonality of X and denoted gon(X).

- *The stable degree of irrationality of X is*

$$\operatorname{stab.irr}(X) := \min\{\operatorname{irr}(X \times \mathbb{P}^r) : r \in \mathbb{Z}_{\geq 0}\}.$$

- *The degree of unirationality of X is*

$$\text{uni.irr}(X) := \min\{\text{irr}(Y) : \exists\, \varphi : Y \dashrightarrow X \textit{ dominant and generically finite}\}.$$

- *The connecting gonality of X is*

$$\text{conn.gon}(X) := \min\left\{ g \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \textit{any two general points } x, y \in X \textit{ can be} \\ \textit{connected by an irreducible } g\textit{-gonal curve in } X \end{array} \right\}.$$

- *The covering gonality of X is*

$$\text{cov.gon}(X) := \min\left\{ g \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \textit{a general point } x \in X \textit{ is contained} \\ \textit{in an irreducible } g\textit{-gonal curve in } X \end{array} \right\}.$$

Clearly, these birational invariants are 1 if and only if X satisfies the corresponding weakening of rationality defined above, e.g., $\text{irr}(X) = 1 \iff X$ is rational. We have the following inequalities which in particular imply (2.4):

$$\text{cov.gon}(X) \leq \text{conn.gon}(X) \leq \text{uni.irr}(X) \leq \text{stab.irr}(X) \leq \text{irr}(X).$$

These inequalities are for the most part obvious but we single out the proof of some of them.

Lemma 2.2.3. *For any variety X,*

$$\text{conn.gon}(X) \leq \text{uni.irr}(X) \leq \text{stab.irr}(X).$$

Proof. Given a dominant generically finite rational map $Y \dashrightarrow X$, the connecting gonality of X is at most the connecting gonality of Y. Moreover, given a dominant degree k rational map $Y \dashrightarrow \mathbb{P}^{\dim Y}$ and a resolution of indeterminacy $\widetilde{Y} \to \mathbb{P}^{\dim Y}$, we can consider the pullback of lines on $\mathbb{P}^{\dim Y}$ to get a connecting family of curves of gonality at most k on $\widetilde{Y}$. It follows that:

$$\text{conn.gon}(X) \leq \text{uni.irr}(X).$$

To show the second inequality, consider a dominant rational map of degree k

$$X \times \mathbb{P}^r \dashrightarrow \mathbb{P}^{\dim X + r}.$$

We can resolve indeterminacy to get a morphism

$$\widetilde{X \times \mathbb{P}^r} \to \mathbb{P}^{\dim X + r}$$

and pullback a generic codimension r plane in $\mathbb{P}^{\dim X+r}$. The pushforward of this codimension r closed subset to $X \times \mathbb{P}^r$ dominates X by the first projection and every one of its components admits a dominant rational map to $\mathbb{P}^{\dim X}$ of degree at most k. □

Remark 2.2.4. Note that conn.gon(X), uni.irr(X), and stab.irr(X) are all stable birational invariants. This is trivial for stab.irr(X) so it suffices to check that for all $r \in \mathbb{Z}_{\geq 0}$

$$\text{conn.gon}(X \times \mathbb{P}^r) = \text{conn.gon}(X),$$
$$\text{uni.irr}(X \times \mathbb{P}^r) = \text{uni.irr}(X).$$

For the first equality, observe that a connecting family of g-gonal curves on $X \times \mathbb{P}^r$ can be projected to X to obtain a connecting family with gonality at most g so that

$$\text{conn.gon}(X \times \mathbb{P}^r) \geq \text{conn.gon}(X).$$

Now, consider general points $(x,t),(x',t') \in X \times \mathbb{P}^r$ and let $C \subset X$ be a curve of gonality cov.gon(X) containing x and x'. We can choose C such that its normalization $\widetilde{C}$ admits a dominant rational map $\varphi : \widetilde{C} \to \mathbb{P}^1$ of degree cov.gon(X) with $\varphi(\widetilde{x}) \neq \varphi(\widetilde{x}')$, where $\widetilde{x}$ and

$\widetilde{x}'$ lie over x and x' respectively. Let $L \subset \mathbb{P}^r$ be the line connecting t and t', $\iota : C \to X$ be the inclusion, and $\pi : \widetilde{C} \to C$ be the normalization map. One can choose an isomorphism $\mathbb{P}^1 \to L$ which takes $\varphi(\widetilde{x})$ to t and $\varphi(\widetilde{x}')$ to t' and we let $\varphi' : \widetilde{C} \to \mathbb{P}^r$ be the composition of φ with this isomorphism and the inclusion $L \to \mathbb{P}^r$. The curve

$$[(\iota \circ \pi) \times \varphi'](\Delta_{\widetilde{C}}) \subset X \times \mathbb{P}^r$$

is birational to C and connects (x, t) and (x', t'). It follows that

$$\text{conn.gon}(X \times \mathbb{P}^r) \leq \text{conn.gon}(X).$$

For the degree of unirationality, if $\varphi : Y \dashrightarrow X$ is a dominant rational map, then

$$\varphi \times \text{id}_{\mathbb{P}^r} : Y \times \mathbb{P}^r \dashrightarrow X \times \mathbb{P}^r$$

is a dominant rational map and $\text{irr}(Y \times \mathbb{P}^r) \leq \text{irr}(Y)$. It follows that

$$\text{uni.irr}(X \times \mathbb{P}^r) \leq \text{uni.irr}(X).$$

On the other hand, suppose that $\psi : W \dashrightarrow X \times \mathbb{P}^r$ is a dominant rational map and that W admits a degree d dominant rational map to $\mathbb{P}^{\dim X + r}$. The pull back of a generic codimension r linear subspace of $\mathbb{P}^{\dim X + r}$ is a variety with degree of irrationality at most d which dominates X. This shows that

$$\text{uni.irr}(X \times \mathbb{P}^r) = \text{uni.irr}(X).$$

For curves, all the measures of irrationality described above coincide with the gonality of X. For surfaces, the covering gonality does not always agree with the connecting gonality and we will see in Example 2.3.8 that the degree of unirationality can differ from the connecting gonality. Moreover, since there are examples of dominant rational maps $Y \dashrightarrow X$ between surfaces X and Y satisfying $\mathrm{irr}(Y) < \mathrm{irr}(X)$ (see [Yos98] and 4.2.3), the degree of irrationality cannot agree with the degree of unirationality for all surfaces. From dimension 3 onwards, it is clear from the discussion above that all these birational invariants are distinct.

Problem 2.2.5. *Find smooth projective surfaces X and Y with* $\mathrm{uni.irr}(X) = \mathrm{uni.irr}(Y)$ *and* $\mathrm{stab.irr}(X) \neq \mathrm{stab.irr}(Y)$. *Find smooth projective surfaces X and Y with* $\mathrm{stab.irr}(X) = \mathrm{stab.irr}(Y)$ *and* $\mathrm{irr}(X) \neq \mathrm{irr}(Y)$.

2.2.2 Examples

In this section we collect several example of computations of measures of irrationality. We refer to the literature for the proofs.

Example 2.2.6 (Max Noether's theorem)**.** *A theorem of Max Noether (see [Noe83] for the original source and [Cil84] and [Har86] for complete proofs) states that the gonality of a smooth curve C of degree $d \geq 4$ in $\mathbb{P}^2$ is $d-1$. Moreover, any dominant rational map of degree $d-1$ from C to $\mathbb{P}^1$ is given by projection from a point. Note that the gonality statement holds for $d \geq 2$ and the statement about uniqueness also holds for $d = 3$ if one rules out elliptic curves with j-invariant* 0 *and* 1728*, namely elliptic curves with extra automorphisms.*

Example 2.2.7 (High degree hypersurfaces)**.** *The previous example was generalized to hypersurfaces in [BCP14] and [BPE⁺17]. Given a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq n+3$, we have the inequalities*

$$d - n \leq \mathrm{irr}(X) \leq d - 1.$$

If $d \geq 2n+1$ and X is very general, then

$$\mathrm{irr}(X) = d - 1.$$

Moreover, if $d \geq 2n+2$ and X is very general, any dominant rational map $X \dashrightarrow \mathbb{P}^n$ of degree $d-1$ is given by projection from a point of X. It was further shown in [Yan19] that if X is very general of degree $d \geq 2n+2$, then

$$\text{uni.irr}(X) = d - 1.$$

As far as the connecting gonality is concerned, we have the following result for smooth surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 4$: For any $x \in X$, the intersection $C_x := T_x \cap X$ of X with the embedded tangent plane to X at x is an irreducible plane curve of degree d with a double point at x. Lopez and Pirola show in [LP95] that varying x yields a covering family of minimal gonality and that it is unique. Note that the projective dual of X is a surface $\check{X} \subset \check{\mathbb{P}}^3$. Given two very general points $y, y' \in X$, there is a line $l \subset \check{\mathbb{P}}^3$ parametrizing planes containing y and y'. Since this line intersects $\check{X}$ two very general points of X can be connected by a curve of the form C_x. It follows that

$$\text{conn.gon}(X) = \text{cov.gon}(X) = d - 2.$$

Finally, it was shown in [BCFS19] that if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n+2$, then

$$\text{cov.gon}(X) = d - \left\lfloor \frac{\sqrt{16n+1} - 1}{2} \right\rfloor$$

unless $n \in \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 | \alpha \in \mathbb{Z}_{>0}\}$, in which case the covering gonality may drop by one.

While measures of irrationality for hypersurfaces are well-understood, the results mentioned above rely heavily on geometric constructions particular to the case at hand. Notable extensions to hypersurfaces in some homogeneous spaces have been obtained by Stapleton in his thesis [Sta17].

Example 2.2.8 (Surfaces)**.** *Arguably the first step towards understanding measures of irrationality in dimension* > 1 *is to determine what values they can take for surfaces. In what follows,* S *is a smooth projective surface and we review what is known about measures of irrationality for the different classes of the Enriques-Kodaira classification.*

- $\kappa(S) = -\infty$
  - *(Rational): All measures of irrationality are equal to* 1*.*
  - *(Ruled):* S *is birational to* $C \times \mathbb{P}^1$ *for some curve* C *of positive genus. Clearly,* $\mathrm{irr}(S) \leq \mathrm{gon}(C)$ *and* $\mathrm{con.gon}(X) \geq \mathrm{gon}(C)$ *so that*

$$1 = \mathrm{cov.gon}(S) < \mathrm{con.gon}(S) = \mathrm{uni.irr}(S) = \mathrm{stab.irr}(S) = \mathrm{irr}(S) = \mathrm{gon}(C).$$

- $\kappa(S) = 0$
  - *(K3): A theorem of Bogomolov and Mumford ([MM83] p. 351) states that a K3 surface* S *is covered by (singular) elliptic curves. Since* S *is not uniruled it follows that*

$$\mathrm{cov.gon}(S) = 2.$$

If (X, L) *is a degree* $2g - 2$ *K3 surface, the Severi variety*

$$V^{L,g-2} \subset |L| \cong \mathbb{P}^g$$

is a locally closed surface which parametrizes irreducible curves in L *with* $g - 2$ *nodes as only singularities. Since such curves have geometric genus* 2 *and are thus hyperelliptic,*

$$\mathrm{conn.gon}(X) = 2.$$

The main open question regarding measures of irrationality for K*-trivial surfaces is the following:*

Conjecture 2.2.9 (Conjecture 4.2 in [BPE$^+$17]). *Let (S_d, L_d) denote a very general polarized $K3$ surface of degree d. Then,*

$$\limsup_{d\to\infty} \operatorname{irr}(S_d) = \infty.$$

In the opposite direction, Stapleton shows the following theorem in his thesis:

Theorem 2.2.10 ([Sta17] Theorem 5.1). *Let (S_d, L_d) denote a very general polarized $K3$ surface of degree d. There is a constant C such that*

$$\operatorname{irr}(S_d) \leq C\sqrt{d}.$$

We will see in Theorem 2.2.15 by Chen and Stapleton that the degree of irrationality of a K3 surface can only drop under specialization. There are several instance of $K3$ surfaces which have low degree of irrationality. For instance, Chen and Chen-Stapleton show that Kummer K3 surfaces have degree of irrationality 2 (see the sketch of proof for Theorem 2.3.6). Similarly, the degree of irrationality of a $K3$ surface admitting an elliptic fibration of index d is at most $2d$ (see the discussion of elliptic surface with $\kappa(S) = 1$ below).

- *(Abelian surfaces): For more details refer to Section 2.3. For every abelian surface A, we have the inequalities*

$$2 = \text{cov.gon}(A) < \text{conn.gon}(A) \leq \operatorname{irr}(A) \leq 4.$$

For simple abelian surfaces,

$$\text{conn.gon}(A) = \text{uni.irr}(A) = 3.$$

- *(Enriques): Enriques surfaces are branched double covers of the plane ([Enr06],* *see also [Dol16]). Since they are not ruled, all measures of irrationality considered in the previous section are equal to* 2*.*

– *(Hyperelliptic): Yoshihara has studied the degree of irrationality of hyperelliptic surfaces in [Yos00]. A hyperelliptic surface S is a quotient $E \times F/G$, where E and F are elliptic curves and G is a subgroup of F acting by translation on E. There are* 7 *families of hyperelliptic surfaces and their degrees of irrationality are listed in Table 2.1. Note that if $j(E) = 0$ or 1728, then E is respectively $\mathbb{C}/\mathbb{Z}[\omega]$ and $\mathbb{C}/\mathbb{Z}[i]$, where ω is a primitive third root of unity. In these cases,* $\mathrm{Aut}(E) = \langle -\omega \rangle = \mathbb{Z}/6\mathbb{Z}$ *and* $\mathrm{Aut}(E) = \langle i \rangle = \mathbb{Z}/4\mathbb{Z}$ *respectively. It does not seem to be known whether hyperelliptic surfaces can have degree of irrationality* 4*.*

Order of K_X	*j(E)*	G	*Action of G on E*	$\mathrm{irr}(S)$
2	*Any*	$\mathbb{Z}/2\mathbb{Z}$	$e \mapsto -e$	*2*
2	*Any*	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$e \mapsto -e,\ e \mapsto e + c,\ c \in E[2] \setminus \{0_E\}$	*2*
3	*0*	$\mathbb{Z}/3\mathbb{Z}$	$e \mapsto \omega e$	*3*
3	*0*	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$e \mapsto \omega e,\ e \mapsto e + c,\ \omega c = c$	*3 or 4*
4	*1728*	$\mathbb{Z}/4\mathbb{Z}$	$e \mapsto ie$	*3*
4	*1728*	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$e \mapsto ie,\ e \mapsto e + c,\ ic = c$	*3*
6	*0*	$\mathbb{Z}/6\mathbb{Z}$	$e \mapsto -\omega e$	*3*

Table 2.1: Degree of irrationality of hyperelliptic surfaces

- $\kappa(S) = 1$ *(Elliptic): We review what is known about measures of irrationality for elliptic surfaces regardless of Kodaira dimension. Clearly, if $S \to C$ is an elliptic surface* $\mathrm{cov.gon}(S) \leq 2$ *and*

$$\mathrm{conn.gon}(S) \geq \mathrm{gon}(C).$$

Yoshihara shows in Proposition 1 of [Yos96] that if $S \to C$ has a section then

$$\operatorname{irr}(S) \leq 2 \operatorname{gon}(C).$$

See Proposition 2.3.3 for the precise formulation. Of course, one can obtain more general results by considering the Jacobian fibration $J(S) \to C$ associated to an elliptic fibration $S \to C$. Recall that the index of an elliptic fibration $S \to C$ is the positive generator of the subgroup of $\mathbb{Z}$ generated by the degrees of intersections of curves on S with fibers of the fibration. If S admits an elliptic fibration of index d there is a dominant rational map of degree d^2 (see [BKL76] p. 138):

$$S \dashrightarrow J(S).$$

Since $J(S) \to C$ is an elliptic fibration with a section, $\operatorname{irr}(J(S)) \leq 2 \operatorname{irr}(C)$ *and*

$$\operatorname{irr}(S) \leq 2d^2 \operatorname{irr}(C).$$

Finally, one can use a similar method to get an upper bound on the degree of unirationality of S. Note that $S \to C$ is a torsor under $J(S) \to C$ so that we have a morphism

$$J(S) \times_C S \to S.$$

A multisection $D \subset S$ gives a covering

$$J(S) \times_C D \to S$$

and $J(S) \times_C D \to D$ is an elliptic fibration with a section. It follows that

$$\operatorname{irr}(J(S) \times_C D) \leq 2 \operatorname{gon}(D)$$

and

$$\text{uni.irr}(S) \leq 2\,\text{gon}(D).$$

In particular, if S is not rational and contains an infinite number of rational curves (say X is an elliptic K3 surface) then

$$\text{uni.irr}(S) = 2.$$

Problem 2.2.11. *Devise a method to give lower bounds on the degree of irrationality of elliptic surfaces. This is particularly interesting for elliptic K3 surfaces in light of Conjecture 2.2.9 and Theorem 2.2.15.*

- $\kappa(S) = 2$ *(General type): Aside from high degree hypersurfaces, there are only few scattered results, several of which we mention below.*

Example 2.2.12 (Symmetric square of a smooth curve)**.** *In [Bas12], Bastianelli studies measures of irrationality for the symmetric square of a smooth curve C of genus g. He shows that*

$$\text{irr}(\text{Sym}^2 C) \leq \min\left\{\text{gon}(C)^2, \frac{\delta_2(\delta_2-1)}{2}, \frac{(\delta_3-1)(\delta_3-2)}{2} - g\right\},$$

where δ_i is the minimal positive integer d such that C is birational to a non-degenerate curve of degree d in $\mathbb{P}^i$.

Moreover, if C is very general, then $\text{irr}(\text{Sym}^2 C) \geq g-1$*. Bastianelli also proves that if C is hyperelliptic then*

$$\begin{aligned} &\text{irr}(\text{Sym}^2 C) \in \{3,4\} && \textit{if } g \geq 2,\\ &\text{irr}(\text{Sym}^2 C) = 4 && \textit{if } g \geq 4. \end{aligned}$$

In particular, the degree of irrationality of the Jacobian of a genus 2 *curve is either* 3 *or* 4[1]. *Finally, Bastianelli proves that if* $g \geq 3$ *the covering gonality of* $\operatorname{irr}(\operatorname{Sym}^2 C)$ *coincides with the gonality of* C.

Example 2.2.13 (Fano surface of a cubic 3-fold)**.** *In [GK19], Gounelas and Kouvidakis study measures of irrationality for* S *the Fano surface of lines on a cubic* 3*-fold* X*. They show the following inequalities:*

$$3 \leq \text{cov.gon}(S) \leq \operatorname{irr}(S) \leq 6.$$

Moreover, if X *is very general, then*

$$4 = \text{cov.gon}(S) \leq 5 = \text{con.gon}(S) \leq 6 = \operatorname{irr}(S).$$

2.2.3 Measures of irrationality and specialization

It is has been shown recently in [NS19] and [KT19] that rationality and stable rationality specialize in smooth families. One might expect from these results and the behavior for curves that measures of irrationality can only drop under specialization in smooth families. This is indeed the case for the covering gonality (see Proposition 2.2 in [GK19]). However, if the degree of irrationality could only decrease under specialization, Theorem 2.3.2 below would imply that the degree of irrationality of the product of two very general elliptic curves is 3, instead of the expected value of 4.

Question 2.2.14. *Can the degree of irrationality or other measures of irrationality increase under specialization in smooth families?*

Despite these pessimistic prospects, it is known that the degree of irrationality can only drop on special fibers in some special instances:

1. In fact this holds for any abelian surface. See Section 2.3.

Theorem 2.2.15 ([CS19] Proposition C)**.** *Let $\mathscr{X} \to T$ be a smooth projective family over a smooth irreducible marked curve $(T, 0)$. If a very general fiber $\mathscr{X}_t$ is a regular surface (i.e. $q = H^1(\mathscr{X}_t, \mathcal{O}_{\mathscr{X}_t}) = 0$) or a strict Calabi-Yau 3-fold then*

$$\mathrm{irr}(\mathscr{X}_t) \leq d \implies \mathrm{irr}(\mathscr{X}_0) \leq d.$$

Sketch of proof. The proof proceeds by showing that the smallest degree of a dominant rational map to a ruled variety is lower semi-continuous. One then shows that regular surfaces and strict Calabi-Yau 3-folds do not admit generically finite dominant rational maps to non-rational ruled varieties.

If X is a regular surface, Y a smooth curve, and $\varphi : X \dashrightarrow Y \times \mathbb{P}^1$ a dominant rational map then regularity of X implies that $Y = \mathbb{P}^1$. If X is a strict Calabi-Yau 3-fold, Y a smooth surface, and $\varphi : X \dashrightarrow Y \times \mathbb{P}^1$ a dominant rational map, we can consider the MRC (maximally rationally connected) fibration $f : Y \times \mathbb{P}^1 \dashrightarrow B$. We can assume B is a smooth surface since $h^0(X, \Omega^1_X) = 0$. Moreover, the Kodaira dimension of B is 0 by the solution of the Iitaka conjecture in small dimension.

Indeed, if $\widetilde{X} \to B$ is a resolution of indeterminacy, the Iitaka conjecture gives the inequality

$$0 = \kappa(X) \geq \kappa(F) + \kappa(B),$$

where F is a general fiber. Since X and B are not uniruled, both the Kodaira dimension of F and B are non-negative. This forces $\kappa(F) = \kappa(B) = 0$.

Hence, there is proper étale cover $B' \to B$ with trivial canonical bundle. The fact that X is simply connected implies that $f \circ \varphi$ factors through $B' \to B$ and so that $h^0(X, \Omega^2_X) \neq 0$. This provides the desired contradiction. □

Remark 2.2.16. As observed in [GK19], the degree of irrationality and stable degree of irrationality are not lower semi-continuous as there are smooth 1-parameter families whose very general fiber is not stably rational but which have countably-many rational fibers. For instance, Hassett, Pirutka, and Tschinkel construct families of quadric surface bundles over $\mathbb{P}^2$ satisfying this property in [HPT18].

2.2.4 Zero-cycles and measures of irrationality

It will be useful to consider measures of irrationality that interpolate between the covering gonality and the degree of irrationality.

Definition 2.2.17. *Let X be a projective variety*

- *The covering degree of irrationality of X in dimension d is*

$$\mathrm{irr}_d(X) = \min\left\{ i \in \mathbb{Z}_{>0} \;\middle|\; \begin{array}{l} X \text{ is birationally covered by } d\text{-dimensional} \\ \text{subvarieties with degree of irrationality } i \end{array} \right\}.$$

- *Similarly, the covering degree of unirationality of X in dimension d is*

$$\mathrm{uni.irr}_d(X) = \min\left\{ i \in \mathbb{Z}_{>0} \;\middle|\; \begin{array}{l} X \text{ is birationally covered by } d\text{-dimensional} \\ \text{subvarieties with degree of unirationality } i \end{array} \right\}.$$

Clearly,

$$\mathrm{cov.gon}(X) = \mathrm{uni.irr}_1(X) = \mathrm{irr}_1(X),$$

and

$$\mathrm{irr}_{\dim X}(X) = \mathrm{irr}(X),$$

$$\mathrm{uni.irr}_{\dim X}(X) = \mathrm{uni.irr}(X).$$

Moreover, we have the obvious inequalities

$$\text{cov.gon}(X) \leq \text{uni.irr}_2(X) \leq \text{irr}_2(X) \leq \ldots \leq \text{uni.irr}(X) \leq \text{irr}(X).$$

One can define birational invariants $v_d(X)$ which are cycle-theoretic analogues of the invariants $\text{irr}_d(X)$ and $\text{uni.irr}_d(X)$. The invariant $v(X) := v_1(X)$ is known as the Voisin invariant in the literature (see [BPE$^+$17] Definition 2.1) and we will call $v_d(X)$ the d^{th} Voisin invariant of X.

Definition 2.2.18. *The d^{th} Voisin invariant is the smallest positive integer k for which there is a subvariety $Z \subset X^k$ such that $\Sigma_k|_Z$ has fibers of dimension at least d and $\text{pr}_1|Z : Z \to X$ is dominant.*

Remark 2.2.19. One can define analogously a cycle-theoretic analogue of the connecting gonality. We do not pursue this line of investigation here.

Lemma 2.2.20. *The following inequality holds for any $1 \leq d \leq \dim X$:*

$$v_d(X) \leq \text{uni.irr}_d(X).$$

Proof. Let $Y \subset X$ be a d-dimensional subvariety, W a d-dimensional variety, and $f : W \dashrightarrow Y$ a dominant rational map. Suppose that W admits a degree k dominant rational map $\varphi : W \dashrightarrow V$ to a CH_0-trivial d-fold V. There is a rational map

$$\text{Fib} : V \dashrightarrow \text{Sym}^k W$$
$$t \mapsto \varphi^{-1}(t),$$

which associates to a generic point $t \in V$ the fiber of φ over t. The image $\text{Fib}(V) \subset \text{Sym}^k W$ is contained in a fiber of $\Sigma_{(k)}$ since V is CH_0-trivial. The preimage $q_k^{-1}\text{Fib}(V)$ of $\text{Fib}(V)$ under the quotient map

$$q_k : W^k \to \text{Sym}^k(W)$$

is d-dimensional and contained in a fiber of Σ_k. The same holds for the image of this subset under the rational map $f^k : W^k \dashrightarrow Y^k$. Moreover, the image of $(f^k \circ q_k^{-1})(\mathrm{Fib}(V))$ under the projection to the first factor is Y.

By Remark 2.1.5, the subset

$$R_{\geq d} := \{(x_1, \ldots, x_k) \in X^k : \dim |\{x_1\} + \ldots + \{x_k\}| \geq d\} \subset X^k$$

is a countable union of closed subsets of X^k. If X is birationally covered by d-folds with degree of unirationality k, the above shows that $R_{\geq d}$ dominates X under the first projection. It follows from the uncountability of $\mathbb{C}$ that an irreducible component of $R_{\geq d}$ dominates X under the first projection, thereby showing that $v_d(X) \leq k$. In fact, we have shown that $v_d(X) \leq k$ as long as X is birationally covered by d-folds admitting a degree k dominant rational map to a CH_0-trivial variety. □

2.3 Measures of irrationality for abelian varieties

2.3.1 Degree of irrationality

In Section 4 of [BE78], Sommese shows by a topological argument that the minimal degree of a dominant morphism from an abelian g-fold to $\mathbb{P}^g$ is at least $g+1$. In [Yos90], Yoshihara proved that the degree of irrationality of abelian surfaces is at least 3. The first general lower bound on the degree of irrationality of an abelian variety is due to Alzati and Pirola in [AP92].

Theorem 2.3.1 (Alzati-Pirola bound, Corollary 3.7 in [AP92])**.** *Let A be an abelian variety. Then,*

$$\text{uni.irr}(A) \geq \dim A + 1,$$

and, in particular,

$$\mathrm{irr}(A) \geq \dim A + 1.$$

The original proof is an application of an inequality between the holomorphic lengths of varieties X and Y given the existence of a dominant rational map $Y \dashrightarrow X$. Instead, we give a cycle-theoretic proof in the spirit of Section 2.2.4.

Proof. Theorem 1.4 (i) of [Voi18] states that the dimension of an orbit of degree k on an abelian variety is at most $k-1$. Let $Y \dashrightarrow A$ be a dominant generically finite rational map. A degree k dominant rational map $Y \dashrightarrow X$ to a CH_0-trivial variety X gives a $\dim A$-dimensional suborbit of degree k for Y. The image of this orbit under $Y^k \to A^k$ is a $\dim A$-dimensional suborbit of degree k for A, so that $\dim A \leq k-1$. □

Tokunaga and Yoshihara showed that this bound is sharp in [TY95]:

Theorem 2.3.2 (Theorem 0.2 in [TY95])**.** *If an abelian surface A contains a smooth genus 3 curve, then* $\mathrm{irr}(A) = 3$.

Proof. By the adjunction formula $(C, C) = 4$ and by Riemann-Roch

$$\chi(\mathcal{O}(C)) = \chi(\mathcal{O}) + (C, C)/2 = 2.$$

Note that C must generate A and so is ample by the Nakai-Moishezon criterion. Kodaira vanishing then implies that $h^0(A, \mathcal{O}(C)) = \chi(\mathcal{O}_C) = 2$. Hence, C moves in a pencil and considering the blow up $\widetilde{A} \to A$ of the 4 basepoints, we get a morphism

$$\widetilde{\varphi} : \widetilde{A} \to \mathbb{P}^1.$$

The generic fiber of this morphism is a smooth trigonal genus 3 curve. Note that the generic fiber cannot be hyperelliptic since the only hyperelliptic deformations of a hyperelliptic curve on an abelian surface are translations. Indeed, a hyperelliptic curve on A with a Weierstraß point at 0_A give a rational curve on the Kummer surface of A (see [Pir89] Remark 1). Since any exceptional divisor in $\widetilde{A}$ is a section of $\widetilde{\varphi}$, we are in a position to use the following:

Proposition 2.3.3 (Proposition 1 in [Yos96])**.** *Let S be a smooth surface and C a smooth curve. Suppose that there is a surjective morphism $f : S \to C$ whose generic fiber F is a curve of genus g.*

(a) If $g = 0$ then $\mathrm{irr}(X) = \mathrm{gon}(C)$,

(b) If $g = 1$ and there is a divisor Γ *on S such that* $(\Gamma, F) = 2$*, then* $\mathrm{irr}(X) \leq 2\,\mathrm{gon}(C)$,

(c) If $g \geq 2$ and F is hyperelliptic, then $\mathrm{irr}(X) \leq 2\,\mathrm{gon}(C)$,

(d) If $g = 3$ and f has a section Γ'*, then* $\mathrm{irr}(X) \leq 3\,\mathrm{gon}(C)$.

For a discussion of how the section hypothesis in case 2 can be weakened at the expense of a worst upper bound, see the discussion of elliptic surfaces in Section 2.2.2.

Proof. For (a), S is birational to $C \times \mathbb{P}^1$ and so $\mathrm{irr}(S) = \mathrm{gon}(C)$. For the other cases the idea is to use the fact that the generic fiber has low gonality to obtain a degree $\mathrm{gon}(F)$ dominant rational map to a $\mathbb{P}^1$-bundle over C. In all cases, it is crucial that we can glue the low degree rational maps from the fibers to $\mathbb{P}^1$. This is achieved by means of a line bundle $\mathcal{L}$ on S such that $|\mathcal{L}_{|F}|$ is a $g^1_{\mathrm{gon}(F)}$ on the generic fiber. For case (b), any degree 2 line bundle on F is a g^1_2 by Riemann-Roch. For case (c), the canonical linear system $|K_F|$ is a g^1_2. For case (d), we can assume that F is not hyperelliptic, in which case given any $x \in F$ the linear system $|K_F - x|$ is a g^1_3.

Indeed, $|K_F|$ gives an embedding of F as a smooth quartic in $\mathbb{P}^3$ and projection from x gives a degree 3 dominant rational map to $\mathbb{P}^1$. By adjunction, we can let $\mathcal{L}$ be $\mathcal{O}_S(\Gamma), \mathcal{O}_S(K_S + F)$, and $\mathcal{O}_S(K_S + F - \Gamma')$ for cases $(b), (c)$, and (d) respectively[2].

2. Note that there appears to be a typographical mistake in [Yos96] where the divisor $K_S - \Gamma'$ is taken in the last case.

Finally, consider the coherent sheaf $f_*\mathcal{L}$ and the associated ruled surface $\mathbb{P}(f_*\mathcal{L})$ over C. By case (a),

$$\operatorname{irr}(\mathbb{P}(f_*\mathcal{L})) = \operatorname{gon}(C).$$

The result then follows from the fact that $\mathcal{L}$ gives a dominant rational map

$$\varphi : S \dashrightarrow \mathbb{P}(f_*\mathcal{O}_S(D))$$

which has degree 2 in cases (b) and (c) and degree 3 in case (d). □

□

Corollary 2.3.4. *The degree of irrationality of an abelian surface which admits a degree* 2 *isogeny to a Jacobian is* 3*. In particular, the degree of irrationality of a simple* $(1,2)$*-polarized abelian surface and the degree of unirationality of any simple abelian surface are both equal to* 3.

Proof. If $\varphi : A \to J(C)$ is a degree 2 isogeny to the Jacobian of a smooth genus 2 curve C the preimage $\varphi^{-1}(C)$ is a smooth curve of genus 3. For the second part, a simple $(1,2)$-polarized abelian surface has a degree 2 isogeny to a simple principally polarized abelian surface and such a surface is the Jacobian of a smooth genus 2 curve. Moreover, any simple abelian surface is isogenous to a simple $(1,2)$-polarized abelian surface. □

Example 2.3.5. *It is shown in [RS14] that the product of two non-CM elliptic curves* E *and* E' *which admit an isogeny of minimal degree* m *contains a smooth genus* 3 *curve if and only if* m *does not belong to the following set:*

$$\{1,3,5,9,11,15,21,29,35,39,51,65,95,105,165,231\} \cup S \subset \mathbb{N},$$

where S *is either empty of consists of a single odd number. Moreover, if the Generalized Riemann Hypothesis holds* S *is empty.*

The first significant upper bound on the degree of irrationality of abelian surfaces was provided by Keum in [Keu90] who proved that every Kummer surface is the K3 cover of some Enriques surface. Since Enriques surfaces have degree of irrationality 2 this shows that the degree of irrationality of every abelian surface is at most 8. Keum's result provides a counterexample to the analogue of Conjecture 2.2.9 for abelian surfaces which was formulated in [BPE$^+$17] some 27 years later. This upper bound was improved by N. Chen in [Che19] in the case of abelian surfaces with Picard number 1.

Theorem 2.3.6 (Chen's bound [Che19])**.** *If A is an abelian surface with Picard number* 1 *then*

$$\operatorname{irr}(A) \leq 4.$$

Sketch of proof. This bound is obtained by constructing explicitly a dominant rational map to a rational surface. Chen considers the space of even sections $H^0(A, 2L)^+$ and imposes base points at some but not all 2-torsion points to get a linear system $V \subset H^0(A, 2L)^+$. He shows that the image of A under the rational map φ associated to this linear system is a non-degenerate surface S and satisfies

$$\deg \varphi \cdot \deg S \leq 8. \tag{2.5}$$

By construction the rational map φ factors through the Kummer surface of A and thus $\deg \varphi$ is even. By the Alzati-Pirola bound $\deg \varphi$ cannot be 2 if S is rational. Accordingly, the only two possibilities are $\deg \varphi = 4$ and $\deg S = 2$ or $\deg \varphi = 2$ and $\deg S = 4$. In the first case, S is rational and we are done. In the second case, equality in (2.5) can be used to show that there is a morphism $\mathrm{Km}(A) \to S$ such that

$$\widetilde{A} \to \mathrm{Km}(A) \to S$$

resolves the indeterminacies of φ. Here $\widetilde{A}$ is the blow-up of A at 2-torsion points. Since V

does not have basepoints at some 2-torsion points, a rational curve in $\mathrm{Km}(A)$ gets contracted by the morphism to S. It follows that S has a double point and that projecting from a general codimension 3 plane containing this singular point gives a degree 2 dominant rational map to $\mathbb{P}^2$. □

Remark 2.3.7. Chen's bound was extended to all abelian surfaces in [CS19]. This follows from Theorem 2.2.15 along with the fact that the dominant rational map $\varphi : A \dashrightarrow \mathbb{P}^2$ constructed above factors through $\mathrm{Km}(A)$, so that a very general Kummer surface has degree of irrationality 2.

2.3.2 Covering and connecting gonality

The covering and connecting gonality take a particularly simple form for abelian varieties because of the translation action of an abelian variety on itself. Indeed, given any curve C on an abelian variety A, the translates of C form a covering family of A, so that

$$\mathrm{cov.gon}(A) = \min\{\mathrm{gon}(C) : C \subset A \text{ is an irreducible curve}\}.$$

Similarly, given a covering family of curves on A all of whose members go through 0_A, the translates of this family give a connecting family.

Consider the Jacobian $J(C)$ of a genus 2 curve C. There is a 1-parameter family of translates of the theta divisor which go through 0_A. Since any abelian surface A is covered by the Jacobian of a genus 2 curve, we see that

$$\mathrm{conn.gon}(A) = 2.$$

Example 2.3.8. *If A is a simple abelian surface, the discussion above along with Corollary 2.3.4 implies that*

$$\text{cov.gon}(\text{A}) = \text{conn.gon}(\text{A}) = 2 < \text{uni.irr}(A) = 3.$$

Note that this gives some heuristic evidence for the conjecture that rationally connected varieties need not be unirational.

For the covering gonality, we refer to the introduction for a discussion of the work of Pirola [Pir89], Alzati-Pirola [AP93], and Voisin [Voi18]. As discussed therein, one of the main contributions of this thesis is the proof of Conjecture 1.0.3 formulated by Voisin in [Voi18].

CHAPTER 3

DEGREE OF IRRATIONALITY OF SUBVARIETIES OF ABELIAN VARIETIES

In this section, we gather material from [Mar20] and [CMNP19] and provide proofs of Theorems 3.3.3 and 3.3.4.

In what follows, X is a smooth projective variety, $\mathcal{X}/S$ is a family of such varieties, A is an abelian g-fold with polarization type θ, and $\mathcal{A}/S$ is a family of such abelian varieties. In an effort to simplify notation, we will write $\mathcal{X}^k$ instead of $\mathcal{X}^k_S$ to denote the k-fold fiber product of $\mathcal{X}$ with itself over S.

Given $\mathcal{A}/S$, a family of abelian g-folds with polarization type θ, we will abuse notation to denote by $\varphi_{\mathcal{A}}$ both the morphism from S to the moduli stack of abelian g-folds with polarization type θ and the k-fold cartesian product of the natural map between $\mathcal{A}$ and the universal family over this stack, for any $k \in \mathbb{Z}_{>0}$. It will be clear from the context what is the meaning of $\varphi_{\mathcal{A}}$.

Definition 3.0.1. *A family $\mathcal{A}/S$ of abelian g-folds with polarization type θ is locally complete if the induced morphism $\varphi_{\mathcal{A}}$ from S to the moduli stack of abelian g-folds with polarization type θ is generically finite. It is almost locally complete if this morphism is dominant.*

In particular, if $\mathcal{A}/S$ is an almost locally complete family of abelian varieties, then $\varphi_{\mathcal{A}}(\mathcal{A})/\varphi_{\mathcal{A}}(S)$ is a locally complete family of abelian varieties. Note that the following remark will allow us to only ever deal with varieties and avoid stacks entirely.

Remark 3.0.2. In many of the arguments of this chapter we consider a family of varieties $\mathcal{X} \to S$ and a subvariety $\mathcal{Z} \subset \mathcal{X}$, such that $\mathcal{Z} \to S$ is flat with irreducible fibers. We will often need to base change by a generically finite morphism $S' \to S$. To avoid the growth of notation we denote the base changed family by $\mathcal{X} \to S$ again. Moreover, if $S_\lambda \subset S$ is a Zariski closed subset, the base change of S_λ under $S' \to S$ will also be denoted S_λ. Note that this applies also to the statement of theorems. For instance, if we say a statement holds for a family $\mathcal{X}/S$, we mean that it holds for some $\mathcal{X}_{S'}/S'$, where $S' \to S$ is generically finite.

Finally, observe that because of this convention we can think of isotrivial families of abelian varieties as trivial. Indeed, we can make a base change that corresponds to considering some large level structure. Since the corresponding moduli space is fine, triviality of the base changed family follows.

3.1 Specialization and projection

In this section we generalize Voisin's method from Section 2 of [Voi18] to powers of abelian varieties. The key difference is that our generalization requires technical assumptions which are automatically satisfied in Voisin's setting. Checking that these assumptions are also satisfied in the case at hand occupies the bulk of this chapter.

3.1.1 Setup and conditions $(*)$, $(**)$, *and* $(***)$

Just as in [Pir89],[AP93], and [Voi18], we will consider specializations to loci of non-simple abelian varieties. Given $\mathcal{A}/S$, a locally complete family of abelian varieties of dimension g, and a positive integer $l < g$, we denote by $S_\lambda \subset S$ the loci along which

$$\mathcal{A}_s \sim \mathcal{B}_s^\lambda \times \mathcal{E}_s^\lambda,$$

where $\mathcal{B}^\lambda/S_\lambda$ and $\mathcal{E}^\lambda/S_\lambda$ are locally complete families of abelian varieties of dimension l and $g - l$ respectively. The index λ encodes the structure of the isogeny and we let Λ_l (or $\Lambda_l(\mathcal{A})$ if there is a risk of confusion about the family $\mathcal{A}/S$) be the set of all such λ. Given a positive integer $l' < l$, we will also be concerned with loci $S_{\lambda,\mu} \subset S_\lambda$ along which

$$\mathcal{B}_s^\lambda \sim \mathcal{D}_s^{\lambda,\mu} \times \mathcal{F}_s^{\lambda,\mu},$$

where $\mathcal{D}^{\lambda,\mu}/S_{\lambda,\mu}$ and $\mathcal{F}^{\lambda,\mu}/S_{\lambda,\mu}$ are locally complete families of abelian varieties of dimension l' and $l - l'$ respectively. Again, the index μ encodes the structure of the isogeny and we let $\Lambda_{l'}^\lambda$ be the set of all such μ.

For our applications, we will mostly consider the case $(l, l') = (g-1, 2)$. Given a positive integer k and upon passing to a generically finite cover of S_λ and $S_{\lambda,\mu}$, we have projections

$$\begin{aligned} p_\lambda :& \mathcal{A}_{S_\lambda}^k \to (\mathcal{B}^\lambda)^k, \\ p_\mu :& (\mathcal{B}_{S_{\lambda,\mu}}^\lambda)^k \to (\mathcal{D}^{\lambda,\mu})^k, \end{aligned} \tag{3.1}$$

and we let

$$p_{\lambda,\mu} := p_\mu \circ p_\lambda : \mathcal{A}_{S_{\lambda,\mu}}^k \to (\mathcal{D}^{\lambda,\mu})^k, \qquad \text{for } \mu \in \Lambda_{l'}^\lambda. \tag{3.2}$$

For $\lambda \in \Lambda_l$, $\mu \in \Lambda_{l'}^\lambda$, and abelian varieties B, D, F in the families $\mathcal{B}^\lambda/S_\lambda$, $\mathcal{D}^{\lambda,\mu}/S_{\lambda,\mu}$, and $\mathcal{F}^{\lambda,\mu}/S_{\lambda,\mu}$, we can consider the loci

$$\begin{aligned} S_\lambda(B) &= \{s \in S_\lambda : \mathcal{B}_s^\lambda \cong B\} \subset S_\lambda, \\ S_{\lambda,\mu}(D, F) &= \{s \in S_{\lambda,\mu} : \mathcal{D}_s^{\lambda,\mu} \cong D, \mathcal{F}_s^{\lambda,\mu} \cong F\} \subset S_{\lambda,\mu}, \end{aligned} \tag{3.3}$$

which have the same dimension as the moduli stack of abelian $(g-l)$-folds.

The main result of this section is Proposition 3.1.1 which is a generalization of Proposition 2.4 from [Voi18]. It states the following:

Given a family of irreducible subvarieties $\mathcal{Z} \subset \mathcal{A}^k$ that satisfies certain technical *assumptions, there exists a* $\lambda \in \Lambda_l$ *such that the variety* $p_\lambda(\mathcal{Z}_s) \subset B^k$ *varies with* $s \in S_\lambda(B)$*, for* B *in the family* $\mathcal{B}^\lambda$*.*

The technical assumptions mentioned above are non-degeneracy conditions. Given a subvariety $\mathcal{Z} \subset \mathcal{A}^k/S$, we consider the following subsets of S:

$$\begin{aligned}
R_{gf} &= \bigcup_\lambda \{s \in S_\lambda : p_\lambda|_{\mathcal{Z}_s} : \mathcal{Z}_s \to (\mathcal{B}_s^\lambda)^k \text{ is generically finite on its image}\},\\
R_{ab} &= \bigcup_\lambda \{s \in S_\lambda : p_\lambda(\mathcal{Z}_s) \text{ is not an abelian subvariety of } (\mathcal{B}_s^\lambda)^k\},\\
R_{st} &= \bigcup_\lambda \{s \in S_\lambda : p_\lambda(\mathcal{Z}_s) \text{ is not stabilized by a pos. dim. ab. subvariety of } (\mathcal{B}_s^\lambda)^k\}.
\end{aligned} \tag{3.4}$$

We define three non-degeneracy conditions on $\mathcal{Z} \subset \mathcal{A}^k$:

$$\begin{aligned}
&(*) \qquad R_{gf} \subset S \text{ is dense},\\
&(**) \qquad R_{ab} \cap R_{gf} \subset S \text{ is dense},\\
&(***) \qquad R_{st} \cap R_{gf} \subset S \text{ is dense}.
\end{aligned}$$

These sets and conditions depend on $\mathcal{Z}$ and l and, while the subvariety $\mathcal{Z}$ should usually be clear from the context, we will say $(*)$ holds for a specified value of l. Note that we have the following obvious implications:

$$(***) \implies (**) \implies (*).$$

Assuming condition $(**)$ is satisfied, a simple argument will allow us to restrict ourselves to the case where condition $(***)$ is satisfied. Moreover, we will show in Lemma 3.1.4 that if $\mathcal{Z} \subset \mathcal{A}^k$ is foliated by positive dimensional suborbits then $(*) \iff (**)$. It follows that $(*)$ is the only crucial assumption on $\mathcal{Z}$ for our applications.

Recall that, given varieties X and S, an irreducible subvariety $\mathcal{Z} \subset X_S$ with generic fiber of dimension d over S gives rise to a morphism from (an open in) S to the Chow variety $\mathrm{Chow}_d(X)$. Here $\mathrm{Chow}_d(X)$ parametrizes effective cycles of class $[\mathcal{Z}_s]$ on X. We say that the variety $\mathcal{Z}_s$ varies with $s \in S$ if the corresponding morphism from S to $\mathrm{Chow}_d(X)$ is (generically) finite. We remind the reader of the notational convention of Remark 3.0.2 which allows us to remove the words in parenthesis from the previous sentences. The precise formulation of our main proposition is:

Proposition 3.1.1. *Let $\mathcal{Z} \subset \mathcal{A}^k$ be a variety that dominates S, has irreducible fibers of dimension d over S, and satisfies condition $(**)$. Then, there exists a $\lambda \in \Lambda_l$ such that the subvariety*

$$p_\lambda(\mathcal{A}_{S_\lambda(B)}) \subset \left(\mathcal{B}^\lambda_{S_\lambda(B)}\right)^k \cong B^k_{S_\lambda(B)}$$

gives rise to a finite morphism from $S_\lambda(B)$ to $\mathrm{Chow}_d(B^k)$ for all B in the family $\mathcal{B}^\lambda/S_\lambda$.

3.1.2 The induction argument

We propose to use Proposition 3.1.1 to prove Theorem 3.3.2. We will sketch the induction argument which is used. Consider a locally complete family of abelian g-folds $\mathcal{A}_1/S_1$ and suppose that a very general member has a d-dimensional orbit of degree k. Up to passing to a generically finite cover over S_1, we get a subvariety $\mathcal{Z}_1 \subset \mathcal{A}_1^k$ whose fibers over S_1 are irreducible d-folds, and such that $\mathcal{Z}_{1,s}$ is contained in a fiber of $\mathcal{A}^k_{1,s} \to CH_0(\mathcal{A}_{1,s})_{\mathbb{Q}}$ for any $s \in S_1$. Moreover, using translation, we can arrange for $\mathcal{Z}_1$ to be in the kernel of the summation map $\mathcal{A}_1^k \to \mathcal{A}_1$. Assuming that $(**)$ holds for $l = g-1$, we can use the Proposition 3.1.1 to get a $(d+1)$-fold foliated by codimension one suborbits:

$$\mathcal{Z}_2 := \varphi_{\mathcal{B}^\lambda}\left(p_\lambda(\mathcal{Z}_{1,S_\lambda})\right) \subset \varphi_{\mathcal{B}^\lambda}\left((\mathcal{B}^\lambda)^{k,0}\right) = \varphi_{\mathcal{B}^\lambda}(\mathcal{B}^\lambda)^{k,0}.$$

Indeed, observe that $p_\lambda(\mathcal{Z}_{1,s})$ varies with $s \in S_\lambda(B)$ if and only if the fiber of $\varphi_{\mathcal{B}^\lambda}(p_\lambda(\mathcal{Z}_{1,S_\lambda}))$ over the point $\varphi_{\mathcal{B}^\lambda}(S_\lambda(B)) \in \varphi_{\mathcal{B}^\lambda}(S_\lambda)$ has dimension greater than the relative dimension of

$p_\lambda(\mathcal{Z}_{1,S_\lambda})$ over S_λ. To see this, notice that the fiber of $\varphi_{\mathcal{B}^\lambda}(p_\lambda(\mathcal{Z}_{1,S_\lambda}))$ over this point is

$$\bigcup_{s\in S_\lambda(B)} p_\lambda(\mathcal{Z}_{1,s}) \subset B^k.$$

Rename the base $\varphi_{\mathcal{B}^\lambda}(S_\lambda)$ and the locally complete family $\varphi_{\mathcal{B}^\lambda}(\mathcal{B}^\lambda)/\varphi_{\mathcal{B}^\lambda}(S_\lambda)$ as S_2 and $\mathcal{A}_2/S_2$. We can hope to apply Proposition 3.1.1 to $\mathcal{Z}_2$ and proceed by induction. At the i^{th} step, one gets a variety $\mathcal{Z}_i \subset \mathcal{A}_i^{k,0}$, where $\mathcal{A}_i/S_i$ is a locally complete family of abelian $g-(i-1)$-folds. The variety $\mathcal{Z}_i$ has relative dimension $d+(i-1)$, irreducible fibers over S_i, and is foliated by d-dimensional suborbits. At the $(g-1)^{\text{st}}$ step, one obtains a variety $\mathcal{Z}_{(g-1)} \subset \mathcal{A}_{(g-1)}^{k,0}$ of relative dimension $d+g-2$ which is foliated by d-dimensional suborbits.

Since $\dim_{S_{(g-1)}} \mathcal{A}_{(g-1)} = 2$, Corollary 2.1.17 then provides the desired contradiction when g and d are large relative to k. The key issue with this induction argument is to ensure that condition $(**)$ is satisfied at each step. Since the relative dimension of the variety $\mathcal{Z}_i$ to which we apply Proposition 3.1.1 grows, it gets harder and harder to ensure generic finiteness of the projection $p_\lambda|_{\mathcal{Z}_{i,S_{i,\lambda}}}$. In Section 3.2, we will present a way around this using the fact that $\mathcal{Z}_i$ is obtained by successive specializations and projections. This methods requires the initial family $\mathcal{Z}_1$ to satisfy $(*)$ for $l=2$. We show in Section 3.2.2 that this condition is automatically satisfied.

3.1.3 $(*)$ is equivalent to $(**)$ for subvarieties foliated by suborbits

Before proceeding to give a summary of the proof of Proposition 3.1.1, we will establish some basic facts related to conditions $(*)$ and $(**)$. The main point is that if $\mathcal{A}/S$ is an almost locally complete family of abelian varieties and $\mathcal{Z} \subset \mathcal{A}^k$ is foliated by positive dimensional suborbits, then conditions $(*)$ and $(**)$ are equivalent for $l \geq 2$.

Given an abelian variety A, we denote by $T_A := T_{A,0_A}$ the tangent space to A at $0_A \in A$. We let

$$\begin{aligned} \mathscr{T} &:= T_{\mathcal{A}|S}, \\ G/S &:= \mathrm{Gr}(g-l, \mathscr{T})/S, \\ G'/S &:= \mathrm{Gr}(g-l', \mathscr{T})/S, \end{aligned} \tag{3.5}$$

and we consider the following sections

$$\begin{aligned} &\sigma_\lambda : S_\lambda \to G_{S_\lambda} = \mathrm{Gr}(g-l, \mathscr{T}_{S_\lambda}), && \sigma_\lambda(s) := T_{\ker(p_{\lambda,s})}, \\ &\sigma_{\lambda,\mu} : S_{\lambda,\mu} \to G'_{S_{\lambda,\mu}} = \mathrm{Gr}(g-l', \mathscr{T}_{S_{\lambda,\mu}}), && \sigma_{\lambda,\mu}(s) := T_{\ker(p_{\lambda,\mu,s})}. \end{aligned} \tag{3.6}$$

Finally, we will denote by $\mathcal{H}/G$ and $\mathcal{H}'/G'$ the universal families of hyperplanes and by $\mathcal{H}^\lambda/S_\lambda$ and ${\mathcal{H}'}^{\lambda,\mu}/S_{\lambda,\mu}$ the pullbacks of the universal families of hyperplanes by σ_λ and $\sigma_{\lambda,\mu}$.

Lemma 3.1.2. *For any $0 \le l' < l < g$, the following subsets are dense:*

$$\bigcup_{\lambda \in \Lambda_l} \sigma_\lambda(S_\lambda) \subset G$$

$$\bigcup_{\lambda \in \Lambda_l} \bigcup_{\mu \in \Lambda_{l'}^\lambda} \sigma_{\lambda,\mu}(S_{\lambda,\mu}) \subset G'.$$

Proof. We only show this for the first subset. The same argument can be used for the second. It suffices to consider the locus of abelian varieties isogenous to E^g for some elliptic curve E. This locus is dense in S and given s in this locus and any $M \in M_{k\times(g-l)}(\mathbb{Z})$ of rank $(g-l)$, the image of the tangent space $T_{E_M^{g-l}}$ under the differential of the isogeny is contained in $\sigma_\lambda(S_\lambda)$ for some $\lambda \in \Lambda_l$. To finish the proof, it suffices to observe that the following subset is dense:

$$\{T_{E_M^{g-l}} : M \in M_{k\times(g-l)}(\mathbb{Z}),\ \mathrm{rank}(M) = g-l\} \subset \mathrm{Gr}(g-l, T_{E^g}).$$

□

For the rest of this section, $\mathcal{A}/S$ is a locally complete family of abelian varieties of dimension at least 2 and $\mathcal{Z} \subset \mathcal{A}^k$ is a subvariety which dominates S and has irreducible fibers.

Lemma 3.1.3. *If $\mathcal{Z} \subset \mathcal{A}^k$ is foliated by positive dimensional suborbits, then for very general $s \in S$ the subset $\mathcal{Z}_s$ is not an abelian subvariety of the form A_M^r with $r > 0$.*

Proof. Consider the Zariski closed sets

$$\{s \in S : \mathcal{Z}_s = (\mathcal{A}_s)_M^r\} \subset S.$$

There are countably many such sets so it suffices to show that none of them is all of S. Suppose that $\mathcal{A}_M^r$ is foliated by positive dimensional suborbits. By the previous lemma, there is a $\lambda \in \Lambda_2$ such that $p_\lambda((\mathcal{A}_s)_M^r) = (\mathcal{B}_s^\lambda)_M^r$ is also foliated by positive dimensional suborbits for generic $s \in S_\lambda$. This contradicts Corollary 2.1.20. □

Lemma 3.1.4. *If $\mathcal{Z} \subset \mathcal{A}^k/S$ is foliated by positive dimensional orbits and $l \geq 2$ the conditions $(*)$ and $(**)$ are equivalent.*

Proof. Condition $(**)$ obviously implies condition $(*)$, so it suffices to show the other implication. First, observe that if $p_\lambda|_{\mathcal{Z}_s} : \mathcal{Z}_s \to (\mathcal{B}_s^\lambda)^k$ is generically finite on its image, then its image is foliated by positive dimensional suborbits. In this case, $R_{gf} \cap S_\lambda$ is open in S_λ and, for very general $s \in R_{gf} \cap S_\lambda$, the abelian variety $\mathcal{B}_s^\lambda$ is simple. By Lemma 2.1.20, for such an s the closed subset $p_\lambda(\mathcal{Z}_s)$ cannot be an abelian subvariety of $(\mathcal{B}_s^\lambda)^k$. Hence, $R_{gf} \cap R_{ab} \cap S_\lambda$ is dense in S_λ. This shows that $R_{ab} \cap R_{gf} \subset S$ is dense provided that R_{gf} is dense. □

3.1.4 Outline of the proof of Proposition 3.1.1

Let us now sketch the proof of Proposition 3.1.1 by highlighting the key steps.

- *Step (I)*: We reduce to the case where $(***)$ is satisfied. We will show that if $(**)$ is satisfied but $(***)$ is not, there is an $r \in \{1, \ldots, k-1\}$, an $M \in M_{k\times r}(\mathbb{Z})$, a family of irreducible subvarieties $\mathcal{Z}' \subset \mathcal{Z}$ of dimension $d - rl > 0$, and a non-empty open set $U \subset G$ such that the following holds:

  - $\mathcal{Z}'/\mathcal{A}^r_M \subset \mathcal{A}^k/\mathcal{A}^r_M$ satisfies conditions $(***)$ for l,
  - $p_\lambda(\mathcal{Z}'_s)/(\mathcal{B}^\lambda_s)^r_M = p_\lambda(\mathcal{Z}_s)/(\mathcal{B}^\lambda_s)^r_M$, for all $s \in \bigcup_{\lambda\in\Lambda_l} S_\lambda \cap \sigma_\lambda^{-1}(U)$.

  We can then proceed with the argument replacing $\mathcal{Z} \subset \mathcal{A}^k$ with $\mathcal{Z}'/\mathcal{A}^r_M \subset \mathcal{A}^k/\mathcal{A}^r_M$.

- *Step (II)*: As in the proof of Proposition 2.4 from [Voi18], we argue that, though the projections $p_\lambda|_{\mathcal{Z}_{S_\lambda}}$ are only defined along loci S_λ, there is a way to realize many of them birationally as specializations of a map defined over all of G. Namely, we show that there is a variety $\mathcal{Z}'_G/G$ and a generically finite morphism $p : \mathcal{Z}_G \to \mathcal{Z}'_G$ that identifies birationally to

  $$p_\lambda|_{\mathcal{Z}_{S_\lambda}} : (\mathcal{Z}_G)_{\sigma_\lambda(S_\lambda)} = \mathcal{Z}_{S_\lambda} \to p_\lambda(\mathcal{Z}_{S_\lambda})$$

  along many loci $\sigma_\lambda(S_\lambda) \subset G$. This uses crucially the density result from Lemma 3.1.2.

- *Step (III)*: Resolving the singularities of $\mathcal{Z}_G$ and $\mathcal{Z}'_G$, we obtain a morphism $\widetilde{p} : \widetilde{\mathcal{Z}}_G \to \widetilde{\mathcal{Z}'}_G$ which coincides with p up to birational equivalence. We examine the map

  $$\widetilde{p}_* \circ \widetilde{j}^* : \mathrm{Pic}^0(\mathcal{A}^k_t) \xrightarrow{\widetilde{j}^*} \mathrm{Pic}^0(\mathcal{Z}_t) \xrightarrow{\widetilde{p}_*} \mathrm{Pic}^0(\mathcal{Z}'_t),$$

  where $\widetilde{j}$ is the composition of the natural map $\rho : \widetilde{\mathcal{Z}_G} \to \mathcal{Z}_G$ and the inclusion $\mathcal{Z}_G \to \mathcal{A}^k_G$. This is a non-zero morphism of abelian varieties and, applying the convention of Remark 3.0.2, we see that there is a $r \in \{0, \dots, k-1\}$ and a matrix $M \in M_{k \times r}(\mathbb{Z})$ such that $\ker(\widetilde{p}_* \circ \widetilde{j}^*)$ contains $\mathrm{Pic}^0(\mathcal{A}_t)^r_M$ as a finite index subgroup for every $t \in G$.

But if $\widetilde{p}$ agrees birationally with the map $p_\lambda \circ \rho : (\widetilde{\mathcal{Z}_G})_{\sigma_\lambda(S_\lambda)} \to p_\lambda(\mathcal{Z}_{S_\lambda})$ along $\sigma_\lambda(S_\lambda)$ and $p_\lambda(\mathcal{Z}_s)$ does not vary with $s \in S_\lambda$, the abelian variety

$$\mathrm{Pic}^0\left(\widetilde{p_\lambda(\mathcal{Z}_s)}\right)$$

admits an isogeny from the abelian variety $\mathrm{Pic}^0\left((\mathcal{E}^\lambda_s)^k/(\mathcal{E}^\lambda_s)^r_M\right)$ for any $s \in S_\lambda(B)$. Since the set of abelian varieties in the locally complete family $\varphi_{\mathcal{E}^\lambda}(\mathcal{E}^\lambda)/\varphi_{\mathcal{E}^\lambda}(S_\lambda)$ admitting a non-zero morphism to a fixed abelian variety is discrete, we reach the desired contradiction.

3.1.5 Step (I): Reduction to the validity of condition $(***)$

Lemma 3.1.5. *Suppose $\mathcal{Z}$ satisfies condition $(**)$ for some l. There is an $r \in \{1, \dots, k-1\}$, an $M \in M_{k \times r}(\mathbb{Z})$, a family of irreducible subvarieties $\mathcal{Z}' \subset \mathcal{Z}$ of dimension $d - rl > 0$, and a non-empty open set $U \subset G$, such that the following holds:*

- *$\mathcal{Z}'/\mathcal{A}^r_M \subset \mathcal{A}^k/\mathcal{A}^r_M$ satisfies conditions $(***)$ for l,*
- *$p_\lambda(\mathcal{Z}'_s)/(\mathcal{B}^\lambda_s)^r_M = p_\lambda(\mathcal{Z}_s)/(\mathcal{B}^\lambda_s)^r_M$, for all $s \in \bigcup_{\lambda \in \Lambda_l} S_\lambda \cap \sigma_\lambda^{-1}(U)$.*

Proof. Since $(*)$ is satisfied, we can find a $\lambda_0 \in \Lambda_l$ and an $s_0 \in S_{\lambda_0}$ such that $p_{\lambda_0}|_{\mathcal{Z}_{s_0}}$ is generically finite on its image, $\mathcal{B}^{\lambda_0}_{s_0}$ is simple, and $p_{\lambda_0}(\mathcal{Z}_{s_0}) \subset (\mathcal{B}^{\lambda_0}_{s_0})^k$ is not an abelian subvariety. Suppose that $p_{\lambda_0}(\mathcal{Z}_{s_0})$ is stabilized by $(\mathcal{B}^{\lambda_0}_{s_0})^r_M$ but not by a larger abelian subvariety of $(\mathcal{B}_{s_0}^{\lambda_0})^k$. Then, the subvariety

$$p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r \subset (\mathcal{B}_{s_0}^{\lambda_0})^k/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$$

is not stabilized by any abelian subvariety of $(\mathcal{B}_{s_0}^{\lambda_0})^k/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$.

Let $\mathcal{Z}' \subset \mathcal{Z}$ be the intersection of $\mathcal{Z}$ with rl generic very ample divisors. Base changing by a generically finite morphism if needed, we can suppose that $\mathcal{Z}' \to S$ is flat with irreducible fibers. We claim that $\mathcal{Z}'/\mathcal{A}_M^r \subset \mathcal{A}^k/\mathcal{A}_M^r$ satisfies $(***)$ for l. Since $\mathcal{Z}'$ is an intersection with generic very ample divisors we have the following equality:

$$p_{\lambda_0}(\mathcal{Z}'_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r = p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r.$$

In particular, $\dim p_{\lambda_0}(\mathcal{Z}'_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r > 0$.

The idea is now to use the Gauss map to leverage the fact that the subvariety

$$p_{\lambda_0}(\mathcal{Z}'_{s_0})/(\mathcal{B}_{s_0}^{\lambda_0})_M^r \subset (\mathcal{B}_{s_0}^{\lambda_0})^k/(\mathcal{B}_{s_0}^{\lambda_0})_M^r$$

is not stabilized by an abelian subvariety in order to deduce similar information about $p_\lambda(\mathcal{Z}'_{S_\lambda}/(\mathcal{A}_{S_\lambda})_M^r)$ for $\lambda \neq \lambda_0$.

For each $\lambda \in \Lambda_l$ such that $p_\lambda|_{\mathcal{Z}_{S_\lambda}}$ is generically finite on its image, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}_{S_\lambda} & \overset{\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}(d, \mathscr{T}_{S_\lambda}^k) \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{H}^\lambda}} \\ p_\lambda(\mathcal{Z}_{S_\lambda}) & \overset{\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}\big(d, (\mathscr{T}_{S_\lambda}/\mathcal{H}^\lambda)^k\big), \end{array} \tag{3.7}$$

where $\mathscr{G}$ denotes the Gauss map and $\pi_{\mathcal{H}^\lambda}$ is the rational map induced by the quotient map

$\mathscr{T}^k_{S_\lambda} \to (\mathscr{T}_{S_\lambda}/\mathcal{H}^\lambda)^k$. Note that in this diagram and in what follows we take the liberty to write p_λ for the map $p_\lambda|_{\mathcal{Z}_{S_\lambda}} : \mathcal{Z}_{S_\lambda} \to p_\lambda(\mathcal{Z}_{S_\lambda})$. We can consider the analogous diagram obtained by restricting to $\mathcal{Z}'_{S_\lambda} \subset \mathcal{Z}_{S_\lambda}$ and quotienting by $(\mathcal{A}_{S_\lambda})^r_M$:

$$\begin{array}{ccc} \mathcal{Z}'_{S_\lambda}/(\mathcal{A}_{S_\lambda})^r_M & \overset{\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}\Big(d-rl, \mathscr{T}^k_{S_\lambda}/(\mathscr{T}_{S_\lambda})^r_M\Big) \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{H}^\lambda}} \\ p_\lambda(\mathcal{Z}'_{S_\lambda}/(\mathcal{A}_{S_\lambda})^r_M) & \overset{\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}\Big(d-rl, \mathscr{T}^k_{S_\lambda}/[(\mathcal{H}^\lambda)^k + (\mathscr{T}_{S_\lambda})^r_M]\Big). \end{array} \tag{3.8}$$

Here we abuse notation by writing p_λ and $\pi_{\mathcal{H}^\lambda}$ for the maps induced by p_λ and $\pi_{\mathcal{H}^\lambda}$ on the quotients.

Note that it is not a priori clear that both Gauss maps take value in the Grassmanians of $(d-rl)$-planes. We claim that there is a non-empty open set $U' \subset G$ such that these Gauss maps take value in the Grassmanians of $(d-rl)$-planes whenever $\sigma_\lambda(S_\lambda) \cap U' \neq \emptyset$. For the top Gauss map, one can observe that the dimension of $\mathcal{Z}'_s/(\mathcal{A}_s)^r_M$ is upper semi-continuous and that

$$\dim p_{\lambda_0}(\mathcal{Z}'_{s_0})/(\mathcal{B}^{\lambda_0}_{s_0})^r_M = \dim p_{\lambda_0}(\mathcal{Z}_{s_0})/(\mathcal{B}^{\lambda_0}_{s_0})^r_M = d-rl.$$

For the bottom Gauss map, this follows from the fact that

$$p_{\lambda_0} : \mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M \to p_{\lambda_0}(\mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M)$$

is generically finite on its image for dimensional reasons.

Now, the key observation is that the upper right corner of diagram (3.8) is a specialization along $\sigma_\lambda(S_\lambda) \subset G$ of the following diagram defined over $G := \mathrm{Gr}(g-l, \mathscr{T})$:

$$\begin{array}{ccc} \mathcal{Z}'_G/(\mathcal{A}_G)^r_M & \overset{\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}\Big(d, \mathscr{T}^k_G/(\mathscr{T}_G)^r_M\Big) \\ & & \Big\downarrow {\scriptstyle \pi_{\mathcal{H}}} \\ & & \mathrm{Gr}\Big(d, \mathscr{T}^k_G/[\mathcal{H}^k + (\mathscr{T}_G)^r_M]\Big). \end{array} \tag{3.9}$$

Moreover, the composition of these rational maps is well-defined. Indeed, it is defined over $\sigma_{\lambda_0}(s_0) \in G$ since $p_{\lambda_0}|_{\mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M}$ is generically finite on its image and diagram (3.8) is commutative. In fact, the composition $\pi_{\mathcal{H}} \circ \mathscr{G}$ is generically finite on its image over $\sigma_{\lambda_0}(s_0) \in G$ since both $p_{\lambda_0}|_{\mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M}$ and the Gauss map of $p_{\lambda_0}(\mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M)$ are generically finite on their images. For the Gauss map, this follows from results of Griffiths and Harris (see (4.14) in [GH79]) along with the fact that

$$p_{\lambda_0}(\mathcal{Z}'_{s_0}/(\mathcal{A}_{s_0})^r_M) = p_{\lambda_0}(\mathcal{Z}_{s_0}/(\mathcal{A}_{s_0})^r_M) \subset (\mathcal{B}^{\lambda_0}_{s_0})^k/(\mathcal{B}^{\lambda_0}_{s_0})^r_M$$

is not stabilized by an abelian subvariety.

Hence, there is a non-empty open $U \subset G$ over which $\pi_{\mathcal{H}} \circ \mathscr{G}$ is a well-defined rational map which is generically finite on its image. We claim that for any $\lambda \in \Lambda_l$ we have the inclusion

$$S_\lambda \cap \sigma_\lambda^{-1}(U) \subset R_{st} \cap R_{gf}.$$

This implies that $\mathcal{Z}'/\mathcal{A}^r_M$ satisfies condition $(***)$ since the following subset is dense

$$U \cap \bigcup_{\lambda \in \Lambda_l} \sigma_\lambda(S_\lambda) \subset G.$$

Our claim follows from the fact that $\pi_{\mathcal{H}^\lambda} \circ \mathscr{G}$ is generically finite on its image over any s

such that $\sigma_\lambda(s) \in U$. Because diagram (3.8) is commutative, this implies both that

$$p_\lambda(\mathcal{Z}'_s/(\mathcal{A}_s)^r_M) \subset (\mathcal{B}^\lambda_s)^k/(\mathcal{B}^\lambda_s)^r_M$$

is not stabilized by an abelian subvariety and that

$$p_\lambda : \mathcal{Z}'_s/(\mathcal{A}_s)^r_M \to p_\lambda(\mathcal{Z}'_s/(\mathcal{A}_s)^r_M)$$

is generically finite. □

Lemma 3.1.5 allows us to reduce to the case where $\mathcal{Z} \subset \mathcal{A}^k$ satisfies condition $(***)$ to prove Proposition 3.1.1. Indeed, if $\lambda \in \Lambda_l$ and B in the family $\mathcal{B}^\lambda$ are such that the family

$$p_\lambda(\mathcal{Z}'_{S_\lambda(B)}/(\mathcal{A}_{S_\lambda(B)})^r_M) \subset (B^k/B^r_M)_{S_\lambda(B)}$$

gives rise to a finite morphism from $S_\lambda(B)$ to $\mathrm{Chow}_d(B^k/B^r_M)$, then the family

$$p_\lambda(\mathcal{Z}_{S_\lambda(B)}) \subset B^k_{S_\lambda(B)}$$

also gives rise to a finite morphism from $S_\lambda(B)$ to $\mathrm{Chow}_d(B^k)$. Hence, if the conclusion of Proposition 3.1.1 holds for $\mathcal{Z}'/\mathcal{A}^r_M$, then it also holds for $\mathcal{Z}$ itself. While everything we will do from now on is valid with $\mathcal{Z}'/\mathcal{A}^r_M$ satisfying $(***)$, we will keep writing $\mathcal{Z} \subset \mathcal{A}^k$ in an effort to simplify the notation.

3.1.6 Step (II): Birational factorization

In the previous section we saw that, given $\mathcal{Z} \subset \mathcal{A}^k$ satisfying condition $(***)$, we have a rational map

$$q := \pi_{\mathcal{H}} \circ \mathscr{G} : \mathcal{Z}_G \dashrightarrow \mathrm{Gr}\big(d, (\mathscr{T}_G/\mathcal{H})^k\big),$$

which is generically finite on its image, and, along each locus $\sigma_\lambda(S_\lambda)$, a factorization

$$\begin{array}{ccc} \mathcal{Z}_{\sigma_\lambda(S_\lambda)} \cong \mathcal{Z}_{S_\lambda} & \overset{q:=\pi_{\mathcal{H}}\circ\mathscr{G}}{\dashrightarrow} & \mathrm{Gr}\big(d, (\mathscr{T}_G/\mathcal{H})^k\big) \\ & {\scriptstyle p_\lambda}\searrow \qquad \nearrow{\scriptstyle \mathscr{G}} & \\ & p_\lambda(\mathcal{Z}_{S_\lambda}). & \end{array} \tag{3.10}$$

In this section our goal is to show that there exists a factorization of q over G that identifies birationally with p_λ along $\sigma_\lambda(S_\lambda)$ for many $\lambda \in \Lambda_l$. This is a key input in Step (III). While the content of this section is likely known to experts, we decided to spell it out at length because it plays a decisive role in the argument.

Lemma 3.1.6. *Consider $\mathcal{Z}/S$, a family with irreducible fibers and base, and $q : \mathcal{Z}/S \to \mathcal{X}/S$ such that $q_s : \mathcal{Z}_s \to \mathcal{X}_s$ is generically finite for each $s \in S$. Let $S' \subset S$ be a Zariski dense subset such that for each $s' \in S'$ we have a factorization of q over s' as follows:*

$$\begin{array}{ccc} \mathcal{Z}_{s'} & \xrightarrow{\quad q \quad} & \mathcal{X}_s \\ & {\scriptstyle f_{s'}}\searrow \qquad \nearrow{\scriptstyle g_{s'}} & \\ & f(\mathcal{Z}_{s'}). & \end{array}$$

Then there is a family $\mathcal{Z}'/S$, morphisms $p : \mathcal{Z} \to \mathcal{Z}'$ and $p' : \mathcal{Z}' \to \mathcal{X}$, and a Zariski dense subset $S'' \subset S'$ such that:

- *For any $s'' \in S''$, the varieties $p(\mathcal{Z}_{s''})$ and $f(\mathcal{Z}_{s''})$ are birational,*
- *p and p' induce the same morphism on function fields as $f_{s''}$ and $g_{s''}$ respectively over any $s'' \in S''$.*

Proof. We first restrict to a Zariski open subset of $\mathcal{X}$ (which we call $\mathcal{X}$ in keeping with Remark 3.0.2) over which q is finite étale and such that $\mathcal{X} \to S$ is smooth. By work of Hironaka, we can find a compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ with simple normal crossing divisors at infinity. Restricting to an open in the base, we can assume that $\overline{\mathcal{X}}_s \setminus \mathcal{X}_s$ is an snc divisor for any $s \in S$. One can use a version of Ehresmann's lemma allowing for an snc divisor at infinity to see that $\mathcal{X} \to S$ is a locally-trivial fibration in the category of smooth manifolds.

It follows that we get a covering

$$q : \mathcal{Z}/S \to \mathcal{X}/S.$$

Note that we renamed as $\mathcal{Z}$ an open subset of $\mathcal{Z}$ over which q is étale. To complete the proof of Lemma 3.1.6 we will need the following Lemma.

Lemma 3.1.7. *Consider a covering $q : \mathcal{Z}/S \to \mathcal{X}/S$, with $\mathcal{Z}_s$ connected for every $s \in S$, and a factorization of q over $s_0 \in S$:*

$$\begin{array}{ccc} \mathcal{Z}_{s_0} & \xrightarrow{\quad q \quad} & \mathcal{X}_{s_0} \\ & {\scriptstyle f_{s_0}} \searrow \quad \nearrow {\scriptstyle g_{s_0}} & \\ & f_{s_0}(\mathcal{Z}_{s_0}). & \end{array}$$

Then, there is a factorization

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\quad q \quad} & \mathcal{X} \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle g} & \\ & f(\mathcal{Z}), & \end{array}$$

which identifies with the original factorization over $s_0 \in S$.

Proof. Consider the Galois closure $\mathcal{Z}' \to \mathcal{X}$ of the covering $q : \mathcal{Z} \to \mathcal{X}$. Note that $\mathcal{Z}'_{s_0}$ is connected. Indeed, there is a normal subgroup of the Galois group of $\mathcal{Z}'/\mathcal{X}$ corresponding to deck transformations inducing the trivial permutation of the connected components of $\mathcal{Z}'_{s_0}$. This subgroup corresponds to a Galois cover of $\mathcal{Z}$ since $\mathcal{Z}_{s_0}$ is connected.

It follows from the fact that $\mathcal{Z}'_{s_0}$ is connected that the map $\mathrm{Gal}(\mathcal{Z}'/\mathcal{X}) \to \mathrm{Gal}(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0})$

is injective because a deck transformation which is the identity on the base and on fibers must be the identity. Since $\mathrm{Gal}(\mathcal{Z}'/\mathcal{X})$ has order

$$d := \deg(\mathcal{Z}'/\mathcal{X}) = \deg(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}),$$

and $\mathrm{Gal}(\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0})$ has order at most d, this restriction morphism must be an isomorphism and $\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}$ is thus also Galois. One then has an equivalence of categories between the poset of intermediate coverings of $\mathcal{Z}'/\mathcal{X}$ and that of $\mathcal{Z}'_{s_0}/\mathcal{X}_{s_0}$, and hence between the poset of intermediate coverings of $\mathcal{Z}/\mathcal{X}$ and that of $\mathcal{Z}_{s_0}/\mathcal{X}_{s_0}$. □

By the previous lemma, to each factorization $f_{s'}$ we can associate an intermediate cover $\mathcal{Z} \to \mathcal{Z}^{s'}$ of $\mathcal{Z} \to \mathcal{X}$ which agrees with $f_{s'}$ at s'. Since there are only finitely many intermediate covers of $\mathcal{Z} \to \mathcal{X}$, we get a partition of S' according to the isomorphism type of the cover $\mathcal{Z} \to \mathcal{Z}^{s'}$. One subset $S'' \subset S'$ of this partition must be dense in S. Let $f : \mathcal{Z} \to f(\mathcal{Z})$ be the corresponding intermediate cover. □

For the rest of the proof of Proposition 3.1.1 we are back in the situation of diagram (3.10).

Corollary 3.1.8. *There is a variety $\mathcal{Z}'/G$, a morphism $p : \mathcal{Z} \to \mathcal{Z}'$, and a subset $\Lambda_{l,0} \subset \Lambda_l$, such that:*

- $\bigcup_{\lambda \in \Lambda_{l,0}} \sigma_\lambda(S_\lambda) \subset G$ *is dense,*
- $p_\lambda(\mathcal{Z}_t)$ *and* $p(\mathcal{Z}_t)$ *are birational for any* $\lambda \in \Lambda_{l,0}$, $t \in \sigma_\lambda(S_\lambda)$,
- $p : \mathcal{Z}_t \to p(\mathcal{Z}_t)$ *and* $p_\lambda : \mathcal{Z}_t \to p_\lambda(\mathcal{Z}_t)$ *induce the same map on function fields, for any* $\lambda \in \Lambda_{l,0}$, $t \in \sigma_\lambda(S_\lambda)$.

Proof. This follows from the previous lemma and its proof once we observe that the intermediate covering of q (or rather of an appropriate étale restriction of q as above) associated to the factorization $p_\lambda : \mathcal{Z}_t \to p_\lambda(\mathcal{Z}_t)$ is independent of $t \in \sigma_\lambda(S_\lambda)$. □

3.1.7 Step (III): Final argument

Proof of Proposition 3.1.1. Using Step (I), we may assume that $\mathcal{Z} \subset \mathcal{A}^k$ satisfies condition $(***)$. By Step (II), we have a variety $\mathcal{Z}'_G/G$, a morphism $p : \mathcal{Z}_G \to \mathcal{Z}'_G$, and a subset $\Lambda_{l,0} \subset \Lambda_l$, such that:

- p identifies birationally with $p_\lambda : \mathcal{Z}_{S_\lambda} \to p_\lambda(\mathcal{Z}_{S_\lambda})$ along $\sigma_\lambda(S_\lambda) \subset G$, for all $\lambda \in \Lambda_{l,0}$,
- $\bigcup_{\lambda \in \Lambda_{l,0}} \sigma_\lambda(S_\lambda) \subset G$ is dense.

Up to shrinking G, we can consider desingularizations with smooth fibers:

$$\widetilde{p} : \widetilde{\mathcal{Z}_G}/G \to \widetilde{\mathcal{Z}'_G}/G.$$

We have the inclusion

$$j : \mathcal{Z}_G \to \mathcal{A}^k_G$$

as well as the map

$$\widetilde{j} := j \circ \rho : \widetilde{\mathcal{Z}_G} \to \mathcal{A}^k_G,$$

where $\rho : \widetilde{\mathcal{Z}_G} \to \mathcal{Z}_G$ is the natural map. The morphism $\widetilde{j}$ gives rise to a pullback map

$$\widetilde{j}^* : \mathrm{Pic}^0(\mathcal{A}_t) \to \mathrm{Pic}^0(\widetilde{\mathcal{Z}}_t).$$

Since $\widetilde{p}$ is generically finite on its image we can consider the composition

$$\widetilde{p}_* \circ \widetilde{j}^* : \mathrm{Pic}^0(\mathcal{A}^k_t) \to \mathrm{Pic}^0(\widetilde{\mathcal{Z}}_t) \to \mathrm{Pic}^0(\widetilde{\mathcal{Z}'_t}).$$

This is a morphism of abelian varieties defined for every $t \in G$.

We first show that it is non-zero along $\sigma_\lambda(S_\lambda)$ for any $\lambda \in \Lambda_{l,0}$. Consider $t \in \sigma_\lambda(S_\lambda)$.

The variety $\mathcal{A}_t$ is isogenous to $\mathcal{B}_t^\lambda \times \mathcal{E}_t^\lambda$ and we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_t & \xrightarrow{\ j\ } & \mathcal{A}_t^k \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle p_\lambda} \\ p_\lambda(\mathcal{Z}_t) & \xrightarrow{\ j'\ } & (\mathcal{B}_t^\lambda)^k. \end{array}$$

Consider a desingularization of $p_\lambda(\mathcal{Z}_t)$ and the induced map

$$\widetilde{j}' : \widetilde{p_\lambda(\mathcal{Z}_t)} \to (\mathcal{B}_t^\lambda)^k.$$

The fact that $\widetilde{p}$ identifies birationally to p_λ along $\sigma_\lambda(S_\lambda)$ implies that the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Pic}^0(\widetilde{\mathcal{Z}}_t) & \xleftarrow{\quad \widetilde{j}^* \quad} & \operatorname{Pic}^0(\mathcal{A}_t^k) \\ {\scriptstyle \widetilde{p}^*}\uparrow & & \uparrow{\scriptstyle p_\lambda^*} \\ \operatorname{Pic}^0(\widetilde{\mathcal{Z}}'_t) \cong \operatorname{Pic}^0\left(\widetilde{p_\lambda(\mathcal{Z}_t)}\right) & \xleftarrow{\ \widetilde{j}'^*\ } & \operatorname{Pic}^0\left((\mathcal{B}_t^\lambda)^k\right). \end{array}$$

It follows that

$$\widetilde{p}_* \circ (\widetilde{j}^* \circ {p_\lambda}^*) = \widetilde{p}_* \circ (\widetilde{p}^* \circ \widetilde{j}'^*) = (\widetilde{p}_* \circ \widetilde{p}^*) \circ \widetilde{j}'^* = [\deg(\widetilde{p})] \circ \widetilde{j}'^*.$$

Since $p_\lambda(\mathcal{Z}_t)$ is positive dimensional, the morphism $\widetilde{j}'^*$ is non-zero and so $\widetilde{p}_* \circ \widetilde{j}^*$ is non-zero.

We conclude that the connected component of the identity of the kernel of $\widetilde{p}_* \circ \widetilde{j}^*$ is an abelian subscheme of $\mathcal{A}^k$ which is not all of $\mathcal{A}^k$. For very general $t \in G$, the abelian variety $\mathcal{A}_t$ is simple. Therefore, for such a t, the abelian subvariety

$$\ker(\widetilde{p}_* \circ \widetilde{j}^*)_t^0 := \ker\left(\widetilde{p}_* \circ \widetilde{j}^* : \operatorname{Pic}^0(\mathcal{A}_t^k) \to \operatorname{Pic}^0(\widetilde{\mathcal{Z}'_t})\right)^0$$

is of the form $(\mathcal{A}_t)_M^r$, with $M \in M_{k\times r}(\mathbb{Z})$ of rank r, and $r \leq k-1$. Choosing M and r such that

$$\{t \in G : \ker(\widetilde{p}_* \circ \widetilde{j}^*)_t^0 = (\mathcal{A}_t)_M^r\} \subset G$$

is dense, and observing that this set is locally closed, we see that, shrinking G if needed, $\ker(\widetilde{p}_* \circ \widetilde{j}^*)_t^\circ = (\mathcal{A}_t)_M^r$ for all $t \in G$. In particular, for $t \in \sigma_\lambda(S_\lambda)$, the abelian variety

$$\mathrm{Pic}^0\big(\ker(p_\lambda)_t\big)/\ker(\widetilde{p}_* \circ \widetilde{j}^*)_t \cap \mathrm{Pic}^0\big(\ker(p_\lambda)_t\big)$$

is isogenous to the abelian variety

$$\mathrm{Pic}^0\big(\ker(p_\lambda)_t\big)/(\mathcal{A}_t)_M^r \cap \mathrm{Pic}^0\big(\ker(p_\lambda)_t\big) \neq 0.$$

Now consider $\lambda \in \Lambda_l$ such that $\sigma_\lambda(S_\lambda) \neq \emptyset$, namely such $\sigma_\lambda(S_\lambda)$ has survived the various base change by generically finite maps, and $B \in \mathcal{B}^\lambda$ such that $\sigma_\lambda(S_\lambda(B)) \neq \emptyset$. Suppose that there is a curve

$$C \subset \sigma_\lambda(S_\lambda(B)) \cong S_\lambda(B),$$

such that $p_\lambda(\widetilde{Z}_t) = p_\lambda(\widetilde{Z}_{t'})$ for any $t, t' \in C$, namely such that C is contracted by the morphism from $S_\lambda(B)$ to $\mathrm{Chow}_d(B^k)$ associated to the family $p_\lambda(\mathcal{Z}_{S_\lambda(B)}) \subset B^k_{S_\lambda(B)}$. Since the abelian variety

$$\mathrm{Pic}^0(\widetilde{\mathcal{Z}}'_t) \cong \mathrm{Pic}^0\Big(\widetilde{p_\lambda(\mathcal{Z}_t)}\Big)$$

does not depend on $t \in C$, it must admit an isogeny from each of the abelian varieties

$$\mathrm{Pic}^0\big(\ker(p_\lambda)_t\big)/(\mathcal{A}_t)_M^r \cap \mathrm{Pic}^0\big(\ker(p_\lambda)_t\big), \qquad t \in C.$$

Because an abelian variety cannot admit an isogeny from every member of a non-isotrivial family of abelian varieties, this provides the desired contradiction. This completes the proof of Proposition 3.1.1. $\square$

3.2 Salvaging generic finiteness and the Projection Lemma

In this section, we refine the results from the previous section in order to bypass assumption $(*)$ in the inductive application of Proposition 3.1.1. The idea is quite simple: In the last section we saw that we can specialize to abelian varieties $\mathcal{A}_s$ isogenous to a product $B \times E_s$, where E_s is an elliptic curve, in such a way that the image of $\mathcal{Z}_s \subset \mathcal{A}_s^k$ under the projection $\mathcal{A}_s^k \to B^k$ varies with s. In this section, we will specialize to abelian varieties $\mathcal{A}_s$ isogenous to a product $D \times F \times E_s$, where E_s is an elliptic curve and D an abelian variety of dimension at least 2.

Under suitable assumptions, the image of $\mathcal{Z}_s$ under the projections $\mathcal{A}_s^k \to D^k$ and $\mathcal{A}_s^k \to (D \times F)^k$ varies with s. Hence, if we consider in $(D \times F)^k$ and D^k the union of the image of these projections for every s, we get varieties $Z_1 \subset (D \times F)^k$ and $Z_2 \subset D^k$ of dimension $\dim_S \mathcal{Z} + 1$. Since the restriction of the projection $(D \times F)^k \to D^k$ to Z_1 has image Z_2 and $\dim Z_1 = \dim Z_2$, this restriction is generically finite on its image. This argument allows us to get condition $(*)$ for free when applying Proposition 3.1.1 inductively. We spend this section making this simple idea rigorous and presenting a proof of the Projection Lemma 3.2.6 to be used in the proof of Theorem 3.3.2.

3.2.1 Salvaging generic finiteness

The main result of this section is the following.

Proposition 3.2.1. *Suppose that $\mathcal{Z} \subset \mathcal{A}^k$ satisfies $(**)$ for $l' \geq 2$. There exists a $\lambda \in \Lambda_{(g-1)}$ such that*

$$\varphi_{\mathcal{B}^\lambda}\big(p_\lambda(\mathcal{Z}_{S_\lambda})\big) \subset \varphi_{\mathcal{B}^\lambda}(\mathcal{B}^\lambda)^k$$

satisfies $(*)$ for l' and has relative dimension $\dim_S \mathcal{Z} + 1$ over $\varphi_{\mathcal{B}^\lambda}(S_\lambda)$.

The proof of this proposition will rest on the following lemma:

Lemma 3.2.2. *Let $\mathcal{A}/S$ be a locally complete family of abelian g-folds and consider $\lambda \in \Lambda_{(g-1)}$ and $\mu \in \Lambda^{\lambda}_{l'}$, where $l' < g-1$. If $p_{\lambda,\mu}(\mathcal{Z}_t) \subset D^k$ varies with $t \in S_{\lambda,\mu}(D,F)$, then $\varphi_{\mathcal{B}^\lambda}\big(S_{\lambda,\mu}(D,F)\big)$ lies in $R_{gf} \subset \varphi_{\mathcal{B}^\lambda}(S_\lambda)$.*

Note that the curve $S_{\lambda,\mu}(D,F)$ is contracted by the map $\varphi_{\mathcal{B}^\lambda}$ from S_λ to the moduli stack of polarized abelian $(g-1)$-folds with an appropriate polarization type.

Proof. Given $\lambda \in \Lambda_{(g-1)}(\mathcal{A})$, the family $\varphi_{\mathcal{B}^\lambda}\big(\mathcal{B}^\lambda\big)$ is locally complete. For any $\mu \in \Lambda^{\lambda}_{l'}(\mathcal{A})$, there is a $\mu' \in \Lambda_{l'}\big(\varphi_{\mathcal{B}^\lambda}\big(\mathcal{B}^\lambda\big)\big)$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}^k_{S_{\lambda,\mu}} & \xrightarrow{\varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\lambda,\mu}} & \varphi_{\mathcal{D}^{\lambda,\mu}}(\mathcal{D}^{\lambda,\mu})^k \\ & {\scriptstyle \varphi_{\mathcal{B}^\lambda} \circ p_\lambda} \searrow \quad \nearrow {\scriptstyle \varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\mu'}} & \\ & \varphi_{\mathcal{B}^\lambda}\big(\mathcal{B}^\lambda_{S_{\lambda,\mu}}\big)^k. & \end{array}$$

Consider the restriction of this diagram to $S_{\lambda,\mu}(D,F)$, where D and F are members of the families $\mathcal{D}^{\lambda,\mu}$ and $\mathcal{F}^{\lambda,\mu}$:

$$\begin{array}{ccc} \mathcal{A}^k_{S_{\lambda,\mu}(D,F)} & \xrightarrow{\varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\lambda,\mu}} & D^k \\ & {\scriptstyle \varphi_{\mathcal{B}^\lambda} \circ p_\lambda} \searrow \quad \nearrow {\scriptstyle \varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\mu'}} & \\ & \varphi_{\mathcal{B}^\lambda}\left(\mathcal{B}^\lambda_{S_{\lambda,\mu}(D,F)}\right)^k. & \end{array} \tag{3.11}$$

If $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D,F)$, we have

$$\dim(\varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\lambda,\mu})\left(\mathcal{Z}_{S_{\lambda,\mu}(D,F)}\right) = \dim_S \mathcal{Z} + 1$$

and thus the restriction of $\varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\lambda,\mu}$ to the $(\dim_S \mathcal{Z} + 1)$-dimensional variety $\mathcal{Z}_{S_{\lambda,\mu}(D,F)} \subset \mathcal{A}^k_{S_{\lambda,\mu}(D,F)}$ is generically finite on its image. From the commutativity of diagram (3.11), we see that the restriction of $\varphi_{\mathcal{D}^{\lambda,\mu}} \circ p_{\mu'}$ to

$$(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)\left(\mathcal{Z}_{S_{\lambda,\mu}(D,F)}\right) \subset \varphi_{\mathcal{B}^\lambda}\left(\mathcal{B}^\lambda_{S_{\lambda,\mu}(D,F)}\right)^k$$

is generically finite on its image. It follows that $\varphi_{\mathcal{B}^\lambda}(S_{\lambda,\mu}(D,F)) \in R_{gf}$. $\square$

Remark 3.2.3. Note that if $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D,F)$ then $p_\lambda(\mathcal{Z})$ varies with $t \in S_{\lambda,\mu}(D,F)$. In particular, it follows that $(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda})$ has relative dimension $\dim_S \mathcal{Z} + 1$ over $\varphi_{\mathcal{B}^\lambda}(S_\lambda)$.

Proof of Proposition 3.2.1. In light of Remark 3.2.3 and Lemma 3.2.2, it suffices to show that there is a $\lambda_0 \in \Lambda_l$ and a subset $\Lambda^\lambda_{l',0} \subset \Lambda^\lambda_{l'}$ such that:

- $\bigcup_{\mu \in \Lambda^\lambda_{l',0}} \sigma_{\lambda,\mu}(S_{\lambda,\mu}) \subset G'_{S_\lambda}$ is dense,
- $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D,F)$ for every D, F.

Following the same argument as in the proof of Lemma 3.1.6, we get a partition

$$\bigcup_{\lambda \in \Lambda_{(g-1)}} \Lambda^\lambda_{l'} = T_1 \sqcup T_2 \sqcup \ldots \sqcup T_n$$

according to the isomorphism type of the étale covering associated to the map $p_{\lambda,\mu}$. Consider the following function:

$$\begin{aligned} D : \Lambda_{(g-1)} &\longrightarrow \{I \subset \{1,\ldots,n\} : I \neq \emptyset\} \\ \lambda &\mapsto \Big\{ i \in \{1,\ldots,n\} : \bigcup_{\mu \in T_i \cap \Lambda^\lambda_{l'}} \sigma_{\lambda,\mu}(S_{\lambda,\mu}) \subset G'_{S_{\lambda,\mu}} \text{ is dense} \Big\}. \end{aligned}$$

Its fibers make up a partition of $\Lambda_{(g-1)}$ and so there is a fiber $D^{-1}(I)$ such that

$$\bigcup_{\lambda \in D^{-1}(I)} \sigma_\lambda(S_\lambda) \subset G \text{ is dense.}$$

Pick $i_0 \in I$ and let $T = T_{i_0}$. By construction, the following subset is dense for any $\lambda \in D^{-1}(I)$:

$$\bigcup_{\mu \in T \cap \Lambda^\lambda_{l'}} \sigma_{\lambda,\mu}(S_{\lambda,\mu}) \subset G'_{S_{\lambda,\mu}}.$$

It follows that the following subset is also dense:

$$\bigcup_{\lambda\in D^{-1}(I)} \bigcup_{\mu\in T\cap\Lambda_{l'}^{\lambda}} \sigma_{\lambda,\mu}(S_{\lambda,\mu}) \subset G'.$$

We can carry out the proof of Proposition 3.1.1 for l' using the loci $S_{\lambda,\mu}$ with $\lambda \in D^{-1}(I)$ and $\mu \in T$ instead of the loci S_η with $\eta \in \Lambda_{l'}$. It follows that $p_{\lambda,\mu}(\mathcal{Z}_t)$ varies with $t \in S_{\lambda,\mu}(D,F)$ for every D and F if $\lambda \in D^{-1}(I)$ and $\mu \in T$. Indeed, the key ingredient of Proposition 3.1.1 is that $\mathcal{Z}$ satisfies $(**)$ along with the density statement of Lemma 3.1.2. Choosing $\lambda \in D^{-1}(I)$ and setting $\Lambda_{l',0}^{\lambda} := T \cap \Lambda_{l'}^{\lambda}$ completes the proof of this proposition. □

3.2.2 Condition $(*)$ for $l = 2$ and the Projection Lemma

In light of Proposition 3.1.1, Proposition 3.2.1, and Lemma 3.1.4, the key assumption to be able to apply the induction argument of Section 3.1.2 to a flat family of suborbits $\mathcal{Z} \subset \mathcal{A}^k/S$ is that condition $(*)$ be satisfied for $l = 2$. In this section, we show that this assumption is valid in all instances of interest for the study of measures of irrationality. These results can be found in [CMNP19] and were obtained in collaboration with E. Colombo, J. C. Naranjo, and G. P. Pirola.

Proposition 3.2.4 (Claim 2.4 in [CMNP19]). *A family $\mathcal{Z} \subset \mathcal{A}^{k,0}/S$ of suborbits of relative dimension d over S satisfies condition $(*)$ for $l = 2$ provided that either*

- *$g = \dim_S \mathcal{A}$ is even,*
- *$1 \leq d \leq g$ and $k \leq 2g - 2$.*

The same holds if $\mathcal{Z} \subset \mathcal{A}^{k,0}/S$ is a family of subvarieties foliated by suborbits of codimension one.

Given an abelian subvariety T of an abelian variety A, let $p_T : A \to A/T$ be the quotient map. Denote by $\mathrm{Sub}(A)$ the poset of positive dimensional abelian subvarieties of A under inclusion. The proof of Proposition 3.2.4 uses the following lemma:

Lemma 3.2.5 (Lemma 3.1 in [CMNP19])**.** *Let Z be a subvariety of an abelian variety A. There is a finite subset $S_Z \subset \mathrm{Sub}(A)$ satisfying the following: if $T \in \mathrm{Sub}(A)$ is such that $p_T|_Z$ is not generically finite on its image and $p_T(Z)$ is not covered by tori, then T contains an abelian subvariety in S_Z.*

Note that under the above assumptions $p_T(Z)$ is positive dimensional, else it would be covered by a 0-dimensional torus.

Proof. This result is consequence of the theory of generic vanishing. We easily reduce to the case $A = \mathrm{Alb}(\widetilde{Z})$, where $\widetilde{Z}$ smooth and Z is the image of $\widetilde{Z}$ under the albanese map

$$\mathrm{alb}_{\widetilde{Z}} : \widetilde{Z} \to A,$$

which is birational on its image. Let T be a subtorus of A such that $p_T|_Z$ is not generically finite on its image and $p_T(Z)$ is not covered by tori. Changing the desingularization $\widetilde{Z} \to Z$ if needed, denote by $Y \to p_T(Z)$ a desingularization of $p_T(Z)$ such that the rational map $\widetilde{Z} \dashrightarrow Y$ extends to a morphism $g : \widetilde{Z} \to Y$. The quotient $A \to A/T$ factorizes via $\mathrm{Alb}(Y)$. Note that $g^*\mathrm{Pic}^0(Y) \subset S^{\dim Y}(\widetilde{Z})$, where

$$S^k(\widetilde{Z}) := \{\alpha \in \mathrm{Pic}^0(\widetilde{Z}) : h^k(\widetilde{Z}, \alpha) > 0\}$$

are the cohomological support loci (see [GL91]). To see this, observe that by hypothesis $p_T(Z)$ is not covered by tori, hence the same holds for the albanese image of Y. In particular, $\chi(K_Y) > 0$ by Theorem 3 in [EL97]. Therefore, by generic vanishing, for any $\beta \in \mathrm{Pic}^0(Y)$ we have $h^{\dim Y}(Y, \beta) > 0$ and thus $h^{\dim Y}(\widetilde{Z}, g^*\beta) > 0$. Hence, $g^*\mathrm{Pic}^0(Y)$ is contained in an irreducible component W of some $S^k(\widetilde{Z})$ with $k < \dim Z$. Since all the irreducible components of $S^k(\widetilde{Z})$ are translates of subtori, W is an abelian variety. Dualizing we get the factorization

$$p_T : A \to W^* \to \mathrm{Alb}(Y) \to A/T.$$

The lemma then follows by the observation that the number of irreducible components W of $\bigcup_{k<\dim Z} S^k(\widetilde{Z})$ is finite. $\square$

Let A be an abelian variety and r, l, k be positive integers with $r \geq 2$ and $1 \leq l \leq r-1$. Given $M \in M_{r\times l}(\mathbb{Z})$, we consider the map $i_M : A^l \to A^r$ and the quotient map

$$p_M := p_{A^l_M} : [A^r]^k \to [A^r/A^l_M]^k.$$

We use the previous lemma to deduce the following:

Projection Lemma 3.2.6 (Projection Lemma 3.2 in [CMNP19])**.** *Let $Z \subset [A^r]^k$ be a subvariety which is not covered by tori. If $M \in M_{r\times l}(\mathbb{Z})$ is generic then $p_M(Z)$ is covered by tori or $p_M|_Z : Z \to [A^r/A^l_M]^k$ is generically finite on its image.*

Proof. By Lemma 3.2.5 there is a finite set S_Z of abelian subvarieties $F \subset [A^r]^k$ such that if $p_M(Z)$ is not covered by tori and $p_M|_Z$ is not generically finite on its image then $[A^l_M]^k$ contains some element of S_Z. It is then enough to show that for the generic $M \in M_{r\times l}(\mathbb{Z})$ the abelian subvariety $[A^l_M]^k \subset [A^r]^k$ does not contain any such subvariety.

This is almost obvious: $T_A \cong H_1(A, \mathbb{Z}) \otimes \mathbb{R}$ and a vector $\mathbf{w}^F \in T_F$ is identified with a vector $(\underline{w}^F_1, \dots, \underline{w}^F_k)$ with $\underline{w}^F_i \in H_1(A, \mathbb{Z})^r \otimes \mathbb{R}$. For each $F \in S_Z$, choose a non-zero vector $\mathbf{w}^F \in T_F$ such that its components are in $H_1(A, \mathbb{Z})^r \otimes \mathbb{Z}$ and a non-zero component $\underline{w}^F_{i_F} \in H_1(A, \mathbb{Z})^r$ of $\mathbf{w}^F$. Choosing an isomorphism $H_1(A, \mathbb{Z}) \cong \mathbb{Z}^{2\dim A}$, we get an isomorphism $H_1(A, \mathbb{Z})^r \cong [\mathbb{Z}^r]^{2\dim A}$ and for each F choose a non-zero component $\underline{u}_F \in \mathbb{Z}^r$ of $\underline{w}^F_{i_F} \in H_1(A, \mathbb{Z})^r$. It is enough to take $M \in M_{r\times l}(\mathbb{Z})$ such that $\underline{u}_F$ is not in the space generated by the columns of M for any $F \in S_Z$. The set of such M is Zariski open in $M_{r\times l}(\mathbb{Z})$ and is not all of $M_{r\times l}(\mathbb{Z})$. $\square$

Finally we prove Proposition 3.2.4:

Proof of Proposition 3.2.4. First observe that by Lemma 3.1.4, up to passing to a Zariski open in S, we can assume that $\mathcal{Z}_s$ is not covered by tori for any $s \in S$. We first reduce to the case where $\dim_S \mathcal{A} = g$ is even. It is in the course of this reduction that we will use the hypothesis that $1 \leq d \leq \dim A$ if g is odd.

Suppose that g is odd, we show that there is a $\lambda \in \Lambda_{g-1}$ such that $p_\lambda|_{\mathcal{Z}_{S_\lambda}} : \mathcal{Z}_{S_\lambda} \to (\mathcal{B}^\lambda)^k$ is generically finite on its image. Then the image of $\mathcal{Z}_{S_\lambda(E)}$ under this projection is a family of suborbits $\mathcal{Z}'/S'$ for a locally complete family of abelian $(g-1)$-folds $\varphi_{\mathcal{B}^\lambda}(\mathcal{B}^\lambda)/\varphi_{\mathcal{B}^\lambda}(S_\lambda)$. We are reduced in this way to the case of even-dimensional abelian varieties.

Consider $s \in S$ such that $\mathcal{A}_s$ is isogenous to E^g for some elliptic curve E. We contend that there is a choice of $M \in M_{g,1}(\mathbb{Z}) = \mathbb{Z}^g$ such that the composition

$$\mathcal{Z}_s \hookrightarrow \mathcal{A}_s^k \to [E^g]^k \to [E^g/i_M(E)]^k$$

of the inclusion, the isogeny, and the projection is generically finite on its image. This is obvious if $d < \dim A$ so it suffices to show it for $d = \dim A$.

If this is not the case, it is easy to see that for any y in the smooth locus of $\mathcal{Z}_s$ the tangent space to $\mathcal{Z}_s$ at y is of the form $(T_{\mathcal{A}_s})_M$ for some $M \in \mathbb{C}^k$. Indeed, this follows from the following lemma along with the fact that the following subset is dense:

$$\{\mathbb{C}^1_M : M \in \mathbb{Z}^g\} \subset \mathbb{P}(\mathbb{C}^k).$$

Lemma 3.2.7. *Consider a g-dimensional vector space V, a positive integer k, and a g-dimensional subspace $W \subset V^k$. If the restriction of the projection $\pi_L : V^k \to (V/L)^k$ to W is not an isomorphism for any $L \in \mathbb{P}(V)$, then $W = V_M$ for some $M \in \mathbb{C}^k$.*

□

Given $1 \leq q \leq g$ and $\omega \in H^0(\mathcal{A}_s, \Omega^q_{\mathcal{A}_s})$, the form $\omega_k := \sum_{i=1}^k \mathrm{pr}_1^* \omega$ restricts to zero on $\mathcal{Z}_s$ since it is a suborbit or it is foliated by normalized suborbits of codimension one. It follows that the image of the Gauss map of $\mathcal{Z}_s$ lies in the subvariety of the Grassmanian of g-dimensional subspaces of $T^k_{\mathcal{A}_s}$ parametrizing subspaces of the form $(T_{\mathcal{A}_s})_M$ with $M = (m_1, \ldots, m_k) \in \mathbb{C}^k$ satisfying

$$\sum_{i=1}^k m_i^q = 0 \qquad \text{for } i = 1, \ldots, q.$$

These are $q-1$ independent conditions on $M \in \mathbb{C}^k$ and so the image of the Gauss map of $\mathcal{Z}_s$ must be at most $k-(q-1)$-dimensional. As long as $k \leq 2g-2$, this show that the Gauss map of $\mathcal{Z}_s$ is not generically finite. Since $\mathcal{Z}_s$ is not covered by tori, this contradicts well-known results about non-degeneracy of the Gauss map of subvarieties of abelian varieties (see (4.14) in [GH79]).

In the case of abelian varieties of even dimension $g = 2n$, consider the locus $Y \subset S$ of abelian varieties isogenous to D^n, for some simple abelian surface D. Since Y is dense in S, to show that $(*)$ is satisfied it suffices to show that $Y \subset R_{fg}$. For every $y \in Y$, we can fix an isogeny $\mathcal{A}_y \sim D^n$. Then, for any $M \in M_{n\times(n-1)}(\mathbb{Z})$ of maximal rank, we get a projection $p_M : \mathcal{A}_y \to D^k$ obtained by composing the fixed isogeny with the projection map

$$[D^n]^k \to [D^n/i_M(D^{n-1})]^k.$$

The maps p_M are specializations of maps p_λ with $\lambda \in \Lambda_2$ to $y \in Y$. But for any $y \in Y \cap S_\lambda$, the variety $p_\lambda(\mathcal{Z}_y)$ is a suborbit or foliated by suborbits of codimension one and hence not covered by tori. Indeed, up to an isogeny, $p_\lambda(\mathcal{Z}_y)$ is a subvariety of D^k, and if $p_\lambda(\mathcal{Z}_y)$ were covered by tori it would be covered by tori of the form $D_M \subset D^k$, for some $M \in \mathbb{Z}^k$. This together with the fact that $p_\lambda(\mathcal{Z}_y)$ must be totally isotropic for the 2-form $\omega_k := \sum_{i=1}^k \mathrm{pr}_i^* \omega$

for any $\omega \in H^0(D, \Omega^2_D)$ would contradict Lemma 2.1.19. We are thus in a position to apply the Projection Lemma 3.2.6 to see that for any such y there is a $\lambda \in \Lambda_2$ such that $y \in S_\lambda$ and $p_\lambda|_{\mathcal{Z}_y}$ is generically finite on its image. It follows that $Y \subset R_{gf}$. $\square$

3.3 Proof of Voisin's conjecture

In this section we present the main results about rational equivalence of zero-cycles on abelian varieties which we obtain by applying the induction argument sketched in Section 3.1.2. We deduce several applications to the study of measures of irrationality for abelian varieties and their subvarieties. Moreover, we discuss existence results for positive dimensional orbits and families of such orbits. Finally, we present a strenghtening of Conjecture 1.0.3 which follows from one of the conjectures of Voisin in [Voi18].

3.3.1 Main results on rational equivalence of zero-cycles

Corollary 3.3.1. *Suppose that a very general abelian variety of dimension g has a d-dimensional orbit of degree k and let A be a very general abelian variety of dimension $(g-l) \geq 2$. If g is even of $d \leq g$, then $A^{k,0}$ contains an $(l+d)$-dimensional subvariety foliated by d-dimensional suborbits.*

If $l \geq 2$, then $A^{k,0}$ contains an $\left(d + \binom{l+1}{2}\right)$-dimensional subvariety foliated by d-dimensional suborbits or an $(l+d)$-dimensional subvariety foliated by $(d+1)$-dimensional suborbits.

Proof. Under the assumption of this corollary, we have a flat family of irreducible d-dimensional suborbits $\mathcal{Z} \subset \mathcal{A}^{k,0}/S$, where $\mathcal{A} \to S$ is a locally complete family of abelian g-folds. By Proposition 3.2.4, $\mathcal{Z}$ satisfies $(*)$ for $l = 2$. We can apply Proposition 3.2.1 inductively since condition $(**)$ follows from $(*)$ by Proposition 3.2.1.

For the second part, observe that by the first part there is a $\lambda \in \Lambda_2$ such that

$$(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda})/\varphi_{\mathcal{B}^\lambda}(S_\lambda)$$

has relative dimension at least $d + l$. This was obtained by successive specializations and projections. But the morphism

$$\begin{aligned} S_\lambda &\to \mathrm{Chow}_d\Big((\mathcal{B}^\lambda)^{k,0}\Big) \\ s &\mapsto \big[(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_s)\big] \end{aligned}$$

is an $\binom{l+1}{2}$-parameter family of d-dimensional suborbits lying in $(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda})$. Hence, if $(\varphi_\lambda \circ p_\lambda)(\mathcal{Z}_{S_\lambda})$ has dimension less than $\binom{l+1}{2} + d$, it must be foliated by suborbits of dimension at least $d + 1$. $\square$

Theorem 3.3.2 (Theorem 1.1 in [CMNP19])**.** *If A is a very general abelian variety of dimension at least* 3*, d is a positive integer, and $d \leq \dim A$ if $\dim A$ is odd, then all fibers of the map $\Sigma_k : A^k \to CH_0(A)$ have dimension less than d if*

$$k < d + \frac{\dim A + 1}{2}.$$

Proof. By Corollary 3.3.1, if a very general abelian variety of dimension $g \geq 3$ has a d-dimensional orbit of degree k, then for a very general abelian surface B either there is a subvariety of $B^{k,0}$ of dimension $d + \binom{g-1}{2}$ foliated by d-dimensional orbits or there is a subvariety of $B^{k,0}$ of dimension $d+g-2$ foliated by $(d+1)$-dimensional orbits. By Corollary 2.1.17, in the first case we get

$$d + \binom{g-1}{2} \leq 2(k-1) - d$$

or

$$k \geq d + 1 + \frac{1}{2}\binom{g-1}{2}.$$

In the second case, we get

$$d + g - 2 \leq 2(k-1) - (d+1)$$

or

$$k \geq (2d + g + 1)/2.$$

This gives the desired inequality. □

Corollary 3.3.3 (Corollary 4.7 in [Mar20])**.** *Conjecture 1.0.3 holds: a very general abelian variety of dimension $\geq 2k-2$ has no positive dimensional orbits of degree k.*

Corollary 3.3.4 (Corollary 1.2 in [CMNP19])**.** *If A is a very general abelian variety and $Z \subset A$ is a subvariety, then*

$$\mathrm{irr}(Z) \geq \mathrm{uni.irr}(Z) \geq \dim Z + \frac{\dim A + 1}{2}.$$

Equivalently,

$$\mathrm{uni.irr}_d(A) \geq d + \frac{\dim A + 1}{2}.$$

In particular,

$$\mathrm{uni.irr}(A) \geq \frac{3\dim A + 1}{2}.$$

Remark 3.3.5. The bound $\mathrm{irr}(A) \geq (3\dim A + 1)/2$ on the degree of irrationality of a very general abelian variety is an improvement of the Alzati-Pirola bound (Theorem 2.3.1). While it will be superseded for some polarization types in the next chapter, it remains the best bound which applies to all polarization types.

We can use the same argument to rule out the existence of subvarieties of $\mathcal{A}^{k,0}$ foliated by suborbits of codimension one.

Theorem 3.3.6 (Theorem 1.3 in [CMNP19])**.** *Let d be a positive integer and A a very general abelian variety of dimension at least* $\max(4, d)$*. If A admits a one-dimensional family of d-dimensional normalized orbits of degree k, then*

$$k \geq d + \dim A/2 + 1.$$

Note that this theorem follows trivially from Theorem 3.3.4 if $\dim A$ is even.

Proof. We can carry out the same argument as in Corollary 3.3.1. If a very general abelian variety A of dimension $g \geq 4$ contains a one-dimensional family of normalized d-dimensional orbits then the following holds: Given a very general abelian surface B, either there is a subvariety of $B^{k,0}$ of dimension $d + \binom{g-1}{2}$ foliated by d-dimensional orbits or there is a subvariety of $B^{k,0}$ of dimension $d+g-1$ foliated by $(d+1)$-dimensional orbits. By Corollary 2.1.17, in the first case we get

$$d + \binom{g-1}{2} \leq 2(k-1) - d$$

or

$$k \geq d + 1 + \frac{1}{2}\binom{g-1}{2}.$$

In the second case we get

$$d + g - 1 \leq 2(k-1) - (d+1)$$

or

$$k \geq d + g/2 + 1.$$

This gives the desired inequality. □

Our method yields stronger results if we only consider orbits of the form $|\sum_{i=1}^{k-l}\{a_i\} + l\{0_A\}|$. The following generalizes Theorem 1.4 (4) of [Voi18].

Theorem 3.3.7. *A very general abelian variety A of dimension at least $2k+3-l-d$ does not have a d-dimensional orbit of the form $|\sum_{i=1}^{k-l}\{a_i\}+l\{0_A\}|$. In particular, if A is a very general abelian variety of dimension at least $k+1$, the orbit $|k\{0_A\}|$ is countable.*

Proof. Suppose that a very general abelian variety of dimension $g \geq 2k+3-l-d$ has a d-dimensional orbit of the form $|\sum_{i=1}^{k-l}\{a_i\}+l\{0_A\}|$. We can find $\mathcal{A}/S$, a locally complete family of abelian $(2k+3-l-d)$-folds, and $\mathcal{Z} \subset \mathcal{A}^k$, a family of d-dimensional suborbits, such that

$$\mathcal{A}_s^{k-l} \times \{0_{\mathcal{A}_s}\}^l \cap \mathcal{Z}_s \neq \emptyset, \qquad \forall s \in S.$$

By Propositions 3.2.1, 3.1.4, and 3.2.4, there is a $\lambda \in \Lambda_2$ such that $(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda})$ has relative dimension $2k+1-l$. Given B in the family $\mathcal{B}^\lambda$ and $\underline{b} = (b_1, \ldots, b_{k-l}, 0_B, \ldots, 0_B) \in B^k$, consider

$$S_\lambda(B, \underline{b}) := \{s \in S_\lambda(B) : \underline{b} \in (\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_s)\}.$$

Clearly, $(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda(B,\underline{b})})$ is a suborbit. In particular, $(\varphi_{\mathcal{B}^\lambda} \circ p_\lambda)(\mathcal{Z}_{S_\lambda(B)})$ is foliated by suborbits of codimension at most $2(k-l)$. This contradicts Corollary 2.1.18. □

3.3.2 Existence results for positive dimensional orbits

When seeking lower bounds on measures of irrationality one is led to rule out existence of large dimensional orbits of small degree. Since the study of orbits for rational equivalence is interesting in its own right, one could instead seek existence results for subvarieties of A^k foliated by d-dimensional suborbits. Alzati and Pirola show in Examples 5.2 and 5.3 of [AP93] that for any abelian surface A there is a 2-dimensional orbit in $A^{3,0}$ and a threefold in $A^{3,0}$ foliated by suborbits of positive dimension. In particular, using the argument from Remark 2.1.16, we see that Corollary 2.1.18 is sharp for $d = 0, 1, 2$.

Example 3.3.8. *In [Lin16] Lin shows that Corollary 2.1.18 is sharp for every d.*

The methods of [Lin16] can be used to show the following:

Proposition 3.3.9. *Let A be an abelian g-fold which is a quotient of the Jacobian of a smooth genus g' curve C. For any $k \geq g'+d-1$, the variety $A^{k,0}$ contains a $(g(k+1-g'-d)+d)$-dimensional subvariety foliated by d-dimensional suborbits.*

Proof. To simplify notation we identify C with its image in $J(C)$. We can assume that $0_A \in C$. Recall that the summation map $\mathrm{Sym}^l C \to J(C)$ has $\mathbb{P}^{l-g'}$ as generic fiber for all $l \geq g'$. Moreover, if $(c_1, \ldots, c_l)$ and $(c'_1, \ldots, c'_l)$ are such that $\sum c_i = \sum c'_i \in J(C)$, then the zero-cycles $\sum\{c_i\}$ and $\sum\{c_i\}'$ are equal in $CH_0(C)$, and thus in $CH_0(A)$.

In light of Remark 2.1.16, it suffices to show that $A^{g'+d-1,0}$ contains a d-dimensional suborbit. Consider the following map:

$$\psi : C \times C^{g'+d-1} \to A^{g'+d-1}$$

$$(c_0, (c_1, \ldots, c_{g'+d-1})) \mapsto (c_1 - c_0, \ldots, c_{g'+d-1} - c_0).$$

This morphism is generically finite on its image since the restriction of the summation map $A^2 \to A$ to $C^2 \subset A^2$ is generically finite. The intersection of the image of ψ with $A^{g'+d-1,0}$ is a d-dimensional suborbit. Indeed, given

$$(c_1 - c_0, \ldots, c_{g'+d-1} - c_0) \in \mathrm{Im}(\psi) \cap A^{g'+d-1,0},$$

we have

$$\sum_{i=1}^{g'+d-1} c_i = (g'+d-1)c_0,$$

so that

$$\sum_{i=1}^{g'+d-1} \{c_i\} = (g'+d-1)\{c_0\} \in CH_0(C).$$

It follows that

$$\sum_{i=1}^{g'+d-1} \{c_i - c_0\} = (g'+d-1)\{0_A\} \in CH_0(A).$$

□

Remark 3.3.10. The previous proposition is by no means optimal. Indeed, it provides a one-dimensional suborbit of degree 3 for a very general abelian 3-fold while, as pointed out by a referee for [Mar20], such an abelian variety admits a one-dimensional family of one-dimensional suborbits. To see this, observe that a very general abelian 3-fold is isogenous to the Jacobian of a quartic plane curve C. Projecting from a point $c \in C$ gives a degree 3 rational map $\varphi_c : C \dashrightarrow \mathbb{P}^1$. Let $Z \subset C \times \mathrm{Sym}^3 C$ be the image of the rational map $C \times \mathbb{P}^1 \dashrightarrow C \times \mathrm{Sym}^3 C$ taking a generic point $(c,t) \in C \times \mathbb{P}^1$ to $(c, \varphi_c^{-1}(t))$. The image of Z under the map

$$C \times \mathrm{Sym}^3 C \longrightarrow \mathrm{Sym}^3 A$$
$$(c, x+y+z) \mapsto \{3x+c\} + \{3y+c\} + \{3z+c\}$$

is easily checked to be a surface foliated by positive dimensional suborbits and contained in a fiber of the summation map.

3.3.3 Further conjectures

We believe that Theorem 3.3.3 can be improved. In fact, though Conjecture 1.0.3 is the main conjecture of [Voi18], it is not the most ambitious. Voisin proposes to attack Conjecture 1.0.3 by studying the locus Z_A of positive dimensional normalized orbits of degree k:

$$Z_A := \Big\{ a_1 \in A : \exists\, a_2, \dots, a_{k-1} : \dim \Big| \{a_1\} + \ldots + \{a_{k-1}\} + \Big\{ -\sum_{i=1}^{k} a_i \Big\} \Big| > 0 \Big\}.$$

In particular, she suggests to deduce Conjecture 1.0.3 from the following:

Conjecture 3.3.11 (Voisin, Conj. 6.2 in [Voi18])**.** *If A is a very general abelian variety* *then*

$$\dim Z_A \leq k - 1.$$

While Voisin shows that this conjecture implies Conjecture 1.0.3, it in fact implies the following stronger conjecture:

Conjecture 3.3.12. *A very general abelian variety of dimension at least $k+1$ does not have a positive dimensional orbit of degree k.*

To see how Conjecture 3.3.12 follows from Conjecture 3.3.11, suppose that a very general abelian variety A of dimension k has a normalized positive dimensional orbit

$$|\{a_1\} + \ldots + \{a_{k-1}\}|.$$

By Remark 2.1.16, for any $a \in A$, the following normalized orbit is positive dimensional:

$$|\{(k-1)a\} + \{a_1 - a\} + \ldots + \{a_{k-1} - a\}|.$$

It follows that $Z_A = A$ and so $\dim Z_A = k > k - 1$.

CHAPTER 4

DEGREE OF IRRATIONALITY OF ABELIAN VARIETIES

In this chapter we present a cohomological obstruction to the existence of low degree dominant rational maps from an abelian variety to a variety which admits a cohomological decomposition of the diagonal. In Section 3.3.2, we saw that a very general abelian surface A has a 2-dimensional orbit of degree 3. Hence, once cannot rule out the existence of a dominant rational map of degree three $\varphi : A \dashrightarrow \mathbb{P}^2$ as naively as in Corollary 3.3.4. However, one can compute the cohomology class of the orbit $\mathrm{Fib}(\mathbb{P}^2)$ that would arise from such a φ as explained in Lemma 2.2.20. For most polarizations types, this contradicts Mumford's theorem 2.1.2. This cohomological obstruction can be generalized to give a new lower bound on the degree of irrationality of very general abelian varieties for most polarization types.

4.1 Fibers of a dominant rational map

Let X and Y be smooth projective n-folds, $\varphi : X \dashrightarrow Y$ a dominant rational map of degree k, and $U \subset X$ an open on which φ restricts to a degree k étale covering of its image. Consider,

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\widetilde{\varphi}} & \\ X & \overset{\varphi}{\dashrightarrow} & Y, \end{array}$$

a resolution of the indeterminacies of φ. In this section, we prove Proposition 4.1.1 which, in the case where Y admits a decomposition of the diagonal, determines the class of

$$Z_l = \overline{\{(x_1, \ldots, x_l) \in U^l : x_i \neq x_j, \varphi(x_i) = \varphi(x_j),\ \forall i, j \in [l], i \neq j\}},$$

in $CH_n(X^l)$ up to cycles which are supported on D^l for some closed proper subset $D \subset X$.

Given a positive integer l, we denote by $\Delta_X \subset X^l$ the small diagonal and for a subset

$I \subset [l]$ we let

$$\Delta_I := \mathrm{pr}_I^{-1}(\Delta_X) = \{(x_1, \ldots, x_l) : x_i = x_j,\ \forall i, j \in I\} \subset X^l.$$

Moreover, given cycles $W_1, \ldots, W_l$ on X, we denote by

$$W_1 \boxtimes \ldots \boxtimes W_l \in CH^\bullet(X^l)$$

the cycle $\prod_{i=1}^l \mathrm{pr}_i^*(W_i)$ and by $W_1^{\boxtimes l}$ the cycle

$$\underbrace{W_1 \boxtimes \ldots \boxtimes W_1}_{l \text{ times}}.$$

We also abuse notation and write $X \times W_1$ for $\{X\} \boxtimes W_1 \in CH^\bullet(X^2)$. Throughout this chapter, given a subvariety $W \subset X$, we will write W for the cycle $\{W\} \in CH^\bullet(X)$. Finally, for $\alpha \in \mathbb{Q}$, we let

$$\frac{\alpha!}{(\alpha - r)!} := \alpha \cdot (\alpha - 1) \cdots (\alpha - r + 1).$$

Proposition 4.1.1. *Suppose Y admits a decomposition of the diagonal $N\Delta_Y = \alpha \times Y + \Gamma$, where Γ is supported $Y \times D$ for some closed proper subset $D \subset Y$. We have the equality of cycles*

$$N^{2(l-1)} Z_l = N^{2(l-1)} \sum_{i=1}^{l} (-1)^{i+1} \frac{(1/N^2)!(i-1)!}{(1/N^2 - l + i)!} \sum_{I \subset [l], \#I = i} \Delta_I \cdot \mathrm{pr}_{[l]\setminus I}^* \Big(\varphi^*(\alpha)^{\boxtimes (l-i)}\Big) + \Gamma'$$
$$\in CH_n(X^l),$$

where Γ' is supported on D'^l for $D' = (\pi \circ \widetilde{\varphi}^{-1})(D) \cup (X \setminus U) \subset X$.

Suppose Y admits a cohomological decomposition of the diagonal $N[\Delta_Y] = N[0 \times Y] + [\Gamma]$, where Γ is supported $X \times D$ for some closed proper subset $D \subset Y$. We have the equality of *cycle classes*

$$N^{2(l-1)}[Z_l] = N^{2(l-1)} \sum_{i=1}^{l} (-1)^{i+1} \frac{k!(i-1)!}{(k-l+i)!} \sum_{I\subset[l],\#I=i} [\Delta_I]\cdot \mathrm{pr}^*_{[l]\setminus I}\left(\left[0^{\boxtimes(l-i)}\right]\right) + [\Gamma'] \in H^{2n(l-1)}(X^l, \mathbb{Z}),$$

where Γ' *is supported on* D'^l *for* $D' = (\pi \circ \widetilde{\varphi}^{-1})(D) \cup (X \setminus U) \subset X$.

Remark 4.1.2. As explained in the proof of Lemma 2.2.20, φ gives rise to an n-cycle

$$(q_k^{-1} \circ \mathrm{Fib})(Y) \subset X^k.$$

Recall that $q_k : X^k \to \mathrm{Sym}^k(X)$ is the quotient map and $\mathrm{Fib} : Y \dashrightarrow \mathrm{Sym}^k(X)$ takes a generic point of Y to the fiber of φ over it. Note that $Z_k = (q_k^{-1} \circ \mathrm{Fib})(Y)$, so that Proposition 4.1.1 determines the cycle $(q_k^{-1} \circ \mathrm{Fib})(Y)$ up to cycles which are supported on D^k for some closed proper subset $D \subset X$.

The first step towards proving Proposition 4.1.1 is to determine the rational equivalence class of Z_2 up to cycles supported on $D \times D$ for some closed proper subset $D \subset X$. In what follows, we fix a point $0 \in X$ and also denote by 0 the subvariety $\{0\} \subset X$.

Lemma 4.1.3. *Suppose* Y *admits a decomposition of the diagonal and has torsion order* N. *There is a degree* N^2 *zero-cycle* $\alpha \in CH_0(Y)$ *and an equality*

$$N^2\Delta_Y = \alpha \times Y + Y \times \alpha + \Gamma \in CH^n(Y \times Y),$$

where Γ *is supported on* $D \times D$ *for some closed proper subset* $D \subset Y$.

Suppose Y *admits a cohomological decomposition of the diagonal and has cohomological* *torsion order N. There is a degree N^2 zero-cycle $\alpha \in CH_0(Y)$ and an equality*

$$N^2[\Delta_Y] = N^2([0 \times Y] + [Y \times 0]) + [\Gamma] \in H^{2n}(Y \times Y, \mathbb{Z}),$$

where Γ is supported on $D \times D$ for some closed proper subset $D \subset Y$.

Proof. We treat the case where Y has a decomposition of the diagonal. The proof in the case where Y has a cohomological decomposition of the diagonal proceeds along the same lines. Since Y has torsion order N there is a degree N zero-cycle α' such that

$$N\Delta_Y = Y \times \alpha' + \Gamma' \in CH^n(Y \times Y),$$

where Γ' is supported on $D \times Y$ for some closed proper subset $D \subset Y$. Since $\Delta_Y^t = \Delta_Y$ and $\Delta_Y \circ \Delta_Y = \Delta_Y$, we have

$$N^2\Delta_Y = (Y \times \alpha' + \Gamma') \circ (\alpha' \times Y + \Gamma'^t) = N(\alpha' \times Y + Y \times \alpha') + \Gamma' \circ \Gamma'^t.$$

Since $\Gamma' \circ \Gamma'^t$ is supported on $D \times D$ we can take $\alpha := N\alpha'$ and $\Gamma := \Gamma' \circ \Gamma'^t$ to conclude. □

Lemma 4.1.4. *We have an equality*

$$Z_2 + \Delta_X = (\pi \times \pi)_* \circ (\widetilde{\varphi} \times \widetilde{\varphi})^*(\Delta_Y) + \Gamma \in CH_n(X \times X),$$

where Γ is supported on $(X \setminus U) \times (X \setminus U)$.

Proof. The restriction

$$(\varphi \times \varphi)|_{U \times U} : U \times U \to Y \times Y$$

is flat and so

$$(\varphi \times \varphi)|_{U \times U}^*(\Delta_Y) = (\varphi \times \varphi)|_{U \times U}^{-1}(\Delta_Y).$$

Moreover, the pullback of $Z_2 + \Delta_X$ under the inclusion $\iota : U \times U \to X \times X$ is clearly $(\varphi \times \varphi)|_{U\times U}^{-1}(\Delta_Y)$. The result now follows from the localization exact sequences and the commutativity of the following diagram:

$$\begin{array}{ccccc} & & CH_n(Y \times Y) & & \\ & \swarrow^{(\pi\times\pi)_* \circ (\widetilde{\varphi}\times\widetilde{\varphi})^*} & & \searrow^{(\varphi\times\varphi)|_{U\times U}^*} & \\ CH_n(X \times X) & & \xrightarrow{\quad \iota \quad} & & CH_n(U \times U). \end{array}$$

□

Corollary 4.1.5. *Suppose Y admits a decomposition of the diagonal $N\Delta_Y = X \times \alpha + \Gamma$, where Γ is supported on $Y \times D$ for some closed proper subset $D \subset Y$. There is a cycle $\Gamma' \in CH^n(X \times X)$ such that*

$$N^2(Z_2 + \Delta_X) = \varphi^*(\alpha) \times X + X \times \varphi^*(\alpha) + \Gamma' \in CH^n(X \times X),$$

and Γ' is supported on $(\pi \circ \widetilde{\varphi}^{-1})(D)^2$.

Suppose Y admits a cohomological decomposition of the diagonal $N[\Delta_Y] = N[X \times 0] + [\Gamma]$, where Γ is supported on $Y \times D$ for some closed proper subset $D \subset Y$. There a cycle $\Gamma' \in CH^n(X \times X)$ such that

$$N^2([Z_2] + [\Delta_X]) = Nk([0 \times X] + [X \times 0]) + [\Gamma'] \in H^{2n}(X \times X, \mathbb{Z}),$$

and Γ' is supported on $(\pi \circ \widetilde{\varphi}^{-1})(D)^2$.

Proof. Just pullback the decomposition of Lemma 4.1.3 by $\widetilde{\varphi} \times \widetilde{\varphi}$ and push it forward by $\pi \times \pi$. □

Now that we know what the class of Z_2 is up to cycles supported on $D \times D$ for some closed proper subset $D \subset X$, we can proceed by induction to determine the class of Z_l up to cycles which are supported on D'^l for some closed proper subset $D' \subset X$.

Proof of Proposition 4.1.1. Observe that for all $l \geq 2$ we have an equality

$$Z_{l+1} = \mathrm{pr}_{[l]}^*(Z_l) \cdot \left(\mathrm{pr}_{\{l,l+1\}}^*(Z_2) - \sum_{i=1}^{l-1} \Delta_{i,l+1} \right) + \Gamma_{l+1},$$

for some $\Gamma_{l+1} \in CH_n(X^l)$ supported on D'^{l+1}. Indeed, letting $\iota : U^{l+1} \to X^{l+1}$ denote the inclusion, the cycles

$$(\mathrm{pr}_{[l]} \circ \iota)^*(Z_l)$$

and

$$\iota^* \left(\mathrm{pr}_{\{l,l+1\}}^*(Z_2) - \sum_{i=1}^{l-1} \Delta_{i,l+1} \right)$$

meet transversally and their intersection is the sum of the irreducible components of the closed subset

$$\Big\{(x_1, \ldots, x_{l+1}) \in U^{l+1} : \varphi(x_1) = \ldots = \varphi(x_{l+1}), x_i \neq x_j, \forall i, j \in [l+1]\Big\} \subset U^{l+1}.$$

This coincides with the cycle $\iota^*(Z_{l+1})$. The localisation exact sequence allows us to conclude. The proof then follows by an induction argument which can be found in Appendix A. □

4.2 A cohomological obstruction

In this section, we show how Mumford's theorem 2.1.2 can be leveraged along with Proposition 4.1.1 in order to give a cohomological obstruction to the existence of low degree dominant rational maps from an abelian variety to a variety with a cohomological decomposition of the diagonal. We first focus on the simple case of abelian surfaces and then generalize the argument to abelian varieties of arbitrary dimension. In the process, we need to prove a non-degeneracy result about Hodge classes of A^{k-1} which are supported on a product of divisors.

Keeping the notation of the previous section, suppose that $X = A$ is an abelian g-fold. By composing with a translation, we can assume that $\varphi : A \dashrightarrow Y$ is such that $Z_k \subset A^{k,0} \subset A^k$. The map

$$\iota : A^{k-1} \to A^k$$
$$(a_1, \ldots, a_{k-1}) \mapsto (a_1, \ldots, a_{k-1}, a_1 + \ldots + a_{k-1})$$

then restricts to an isomorphism $\iota|_{Z_{k-1}} : Z_{k-1} \to Z_k$. By Remark 4.1.2 and Mumford's Theorem 2.1.2, given a generator $\omega \in H^0(A, \Omega^g_A)$, the form $\omega_k := \sum_{i=1}^k \mathrm{pr}_i^*(\omega) \in H^0(A^k, \Omega^g_{A^k})$ restricts to zero on Z_k. Hence,

$$\omega_k \cdot [Z_k] = \omega_k \cdot \iota_*([Z_{k-1}]) = 0 \in H^{(2k-1)g}(A^k, \mathbb{C}).$$

and so

$$\iota^*(\omega_k) \cdot [Z_{k-1}] = 0 \in H^{(2k-3)g}(A^{k-1}, \mathbb{C})$$

In particular,

$$\iota^*(\omega_k \wedge \overline{\omega}_k) \cdot [Z_{k-1}] = 0 \in H^{2(k-1)g}(A^{k-1}, \mathbb{C}) \cong \mathbb{Z}.$$

4.2.1 The case of abelian surfaces

We single out the case of abelian surface for which the obstruction we obtain is especially simple.

Theorem 4.2.1 (Theorem 1.1 in [Mar19])**.** *If A is an abelian surface such that the image of the intersection pairing $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ does not contain $12N^2$, then A does not admit a dominant rational map of degree less than 4 to a CH_0-trivial surface with cohomological torsion order N.*

In particular, if the image of the intersection pairing $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ *does not contain* 12*, then*

$$\mathrm{irr}(A) = 4.$$

Proof. In light of Chen's bound (Theorem 2.3.6), it suffices to show that under these assumptions A does not admit a dominant rational map of degree less than 4 to a CH_0-trivial surface X with cohomological torsion order N. Note that degrees 2 or less are ruled out by the Alzati-Pirola bound (Theorem 2.3.1).

Suppose $\varphi : A \dashrightarrow X$ is such a rational map of degree 3. We have a cohomological decomposition of the diagonal

$$N^2[Z_2] = N^2\left(3([A \times 0] + [0 \times A]) - [\Delta]\right) + [\Gamma] \in H^4(A^2, \mathbb{Z}),$$

where Γ is symmetric and supported on $D \times D$ for some closed proper subset $D \subset A$. In particular,

$$[\Gamma] \in \mathrm{Sym}^2 NS(A).$$

Recall that given $m_1, m_2, m_3 \in \mathbb{Z}$,

$$i_{(m_1,m_2,m_3)} : A \to A^3$$

is the morphism given by

$$a \mapsto (m_1 a, m_2 a, m_3 a),$$

and let $\omega \in H_0(A, \Omega_A^2)$ be such that $\omega \wedge \overline{\omega} = [0_A] \in H^4(A, \mathbb{Z})$.

We then have the equality

$$\begin{aligned}\int_{A^2}[A\times 0]\cdot\iota^*(\omega_3\wedge\overline{\omega_3}) &= \int_{A^3} f_{(1,0,-1)_*}([A])\cdot\omega_3\wedge\overline{\omega_3}\\ &= \int_A f_{(1,0,-1)}{}^*\left[(\mathrm{pr}_1^*\omega+\mathrm{pr}_2^*\omega+\mathrm{pr}_3^*\omega)\wedge(\mathrm{pr}_1^*\overline{\omega}+\mathrm{pr}_2^*\overline{\omega}+\mathrm{pr}_3^*\overline{\omega})\right]\\ &= 4\int_A\omega\wedge\overline{\omega}=4,\end{aligned}$$

and by symmetry

$$\int_{A\times A}[0_A\times A]\cdot\iota^*(\omega_3\wedge\overline{\omega_3})=4.$$

Similarly, we see that

$$\begin{aligned}\int_{A\times A}[\Delta_A]\cdot\iota^*(\omega_3\wedge\overline{\omega_3}) &= \int_{A^3} f_{(1,1,2)_*}([A])\cdot(\mathrm{pr}_1^*\omega+\mathrm{pr}_2^*\omega+\mathrm{pr}_3^*\omega)\wedge(\mathrm{pr}_1^*\overline{\omega}+\mathrm{pr}_2^*\overline{\omega}+\mathrm{pr}_3^*\overline{\omega})\\ &= \int_A f^*_{(1,1,2)}\left[(\mathrm{pr}_1^*\omega+\mathrm{pr}_2^*\omega+\mathrm{pr}_3^*\omega)\wedge(\mathrm{pr}_1^*\overline{\omega}+\mathrm{pr}_2^*\overline{\omega}+\mathrm{pr}_3^*\overline{\omega})\right]\\ &= 36\int_A\omega\wedge\overline{\omega}=36.\end{aligned}$$

Finally, we compute that

$$\begin{aligned}\int_{A\times A}\mathrm{pr}_1^*[C]\cdot\mathrm{pr}_2^*[C']\cdot\iota^*(\omega_3\wedge\overline{\omega_3}) &= \int_{A^2}\mathrm{pr}_1^*[C]\cdot\mathrm{pr}_2^*[C']\cdot\iota^*(\mathrm{pr}_3^*\omega\wedge\mathrm{pr}_3^*\overline{\omega}_3)\\ &= \int_{A^2}\mathrm{pr}_1^*[C]\cdot\mathrm{pr}_2^*[C']\cdot[\Delta_A]\\ &= (C,C'),\end{aligned}$$

so that

$$N^2[Z_2]\cdot\iota^*(\omega_3\wedge\overline{\omega_3}) = 3\cdot(4+4)N^2-36N^2+\Gamma\cdot\iota^*(\omega_3\wedge\overline{\omega_3})=0, \tag{4.1}$$

and

$$\Gamma\cdot\iota^*(\omega_3\wedge\overline{\omega_3})=12N^2.$$

Since $\Gamma \in \mathrm{Sym}^2 NS(A)$ and cupping with $\iota^*(\omega_3 \wedge \overline{\omega_3})$ gives the intersection product on $\mathrm{Sym}^2 NS(A)$, the result follows. □

Corollary 4.2.2. *A very general $(1,d)$-polarized abelian surface has degree of irrationality 4 if $d \nmid 6$.*

This corollary answers Question 1.0.5 affirmatively and provides the first examples of abelian surfaces with degree of irrationality 4.

Example 4.2.3. *A very general $(1,d)$-polarized abelian surface with $d \nmid 6$ has degree of irrationality 4 and is isogenous to a very general $(1,2)$-polarized abelian surface. Since such a surface has degree of irrationality 3, the degree of irrationality is not an isogeny invariant and Question 1.0.6 has a negative answer.*

4.2.2 The general case

The proof of Theorem 4.2.1 can be adapted to the case of higher-dimensional abelian varieties. Let A be an abelian g-fold and

$$V_l \subset \mathrm{Hdg}^l(A^{k-1}) \tag{4.2}$$

be the sublattice of classes supported on D^{k-1} for some closed proper subset $D \subset A$. Consider the function

$$M_g(k) := (k-1)! \sum_{j=0}^{k} (-1)^j (j-1) \left(1-(1-j)^{g-1}\right)^2 \binom{k}{j}.$$

Theorem 4.2.4. *If the image of*

$$\begin{aligned} V_{(k-2)g} &\longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z}) \cong \mathbb{Z} \\ \alpha &\longmapsto \alpha \cdot [A^{k-1,0}] \end{aligned}$$

does not contain $N^{2(k-2)}M_g^k$, then A does not admit a dominant rational map of degree k to a CH_0-trivial variety with cohomological torsion order N.

Proof. Suppose that $\varphi : A \dashrightarrow X$ is a dominant rational map of degree k to a CH_0-trivial g-fold X with torsion order N. Using the same notation as in Section 4.1, Proposition 4.1.1 gives the equality

$$N^{2(k-2)}Z_{k-1} = N^{2(k-2)}\sum_{i=1}^{k-1}(-1)^{i+1}\frac{k!}{i(i+1)}\sum_{I\subset[k-1],\#I=i}[\Delta_I]\cdot\mathrm{pr}^*_{[l]\setminus I}\left[0^{\boxtimes(l-i)}\right] + [\Gamma]$$
$$\in H^{2g(k-2)}(A^{k-1}),$$

where Γ is symmetric and supported on D^{k-1} for some closed proper subset $D \subset A$. Let $\omega \in H_0(A, \Omega^g_A)$ be such that $\omega \wedge \overline{\omega} = [0_A] \in H^{2g}(A, \mathbb{Z})$ and, given $I \subset [k-1]$ with $\#I = i$, let

$$i_I := i_{(\epsilon^I_1,\ldots,\epsilon^I_{k-1},-i)} : A \to A^k,$$

where ϵ^I_j is 1 if $j \in I$ and 0 otherwise.

We have the equality

$$\begin{aligned}\int_{A^{k-1}}[\Delta_I]\cdot\mathrm{pr}^*_{[l]\setminus I}\left(\left[0^{\boxtimes(l-i)}\right]\right)\cdot\iota^*(\omega_{k-1}\wedge\overline{\omega}_{k-1}) &= \int_{A^k} i_{I*}([A])\cdot\omega_{k-1}\wedge\overline{\omega}_{k-1}\\ &= \int_A i_I^*\left[\omega_{k-1}\wedge\overline{\omega}_{k-1}\right]\\ &= i^2(1-(-i)^{g-1})^2\int_A\omega\wedge\overline{\omega}\\ &= i^2(1-(-i)^{g-1})^2.\end{aligned}$$

Hence,

$$0 = N^{2(k-2)}\int_{A^{k-1}}[Z_{k-1}]\cdot\iota^*(\omega_k\wedge\overline{\omega}_k)$$

$$= N^{2(k-2)} \sum_{i=1}^{k-1} (-1)^{i+1} \frac{k!}{i(i+1)} \sum_{I \subset [k-1], \# I = i} i^2 (1-(-i)^{g-1})^2 + \int_{A^{k-1}} [\Gamma] \cdot \iota^*(\omega_k \wedge \overline{\omega}_k),$$

so that

$$\begin{aligned}
\int_{A^{k-1}} [\Gamma] \cdot \iota^*(\omega_k \wedge \overline{\omega}_k) &= -N^{2(k-2)} \sum_{i=1}^{k-1} (-1)^{i+1} \frac{k!}{i(i+1)} i^2 (1-(-i)^{g-1})^2 \binom{k-1}{i} \\
&= -N^{2(k-2)} (k-1)! \sum_{i=1}^{k-1} (-1)^{i+1} i (1-(-i)^{g-1})^2 \binom{k}{i+1} \\
&= -N^{2(k-2)} (k-1)! \sum_{j=2}^{k} (-1)^{j} (j-1) (1-(1-j)^{g-1})^2 \binom{k}{j} \\
&= -N^{2(k-2)} (k-1)! \sum_{j=0}^{k} (-1)^{j} (j-1) (1-(1-j)^{g-1})^2 \binom{k}{j}.
\end{aligned}$$

To conclude, it suffices to show that

$$\int_{A^{k-1}} [\Gamma] \cdot \iota^*(\omega_k \wedge \overline{\omega}_k)$$

lies in the image of the cup product with $[A^{k,0}] \in H^{2g}(A^{k-1})$ on V. This follows from the following equality:

$$\begin{aligned}
\int_{A^{k-1}} [\Gamma] \cdot \iota^*(\omega_k \wedge \overline{\omega}_k) &= \int_{A^k} \iota_*[\Gamma] \cdot (\omega_k \wedge \overline{\omega}_k) \\
&= \int_{A^k} \iota_*[\Gamma] \cdot \mathrm{pr}_k^*(\omega) \wedge \mathrm{pr}_k^*(\overline{\omega}) \\
&= \int_{A^{k-1}} [\Gamma] \cdot \iota^*[A^{k-1} \times 0] \\
&= \int_{A^{k-1}} [\Gamma] \cdot [A^{k-1,0}].
\end{aligned}$$

□

Remark 4.2.5. For any polynomial $p(t) \in \mathbb{Q}[t]$ with $\deg p(t) < n$, we have the equality

$$\sum_{t=0}^{n} (-1)^i p(t) \binom{n}{t} = 0.$$

Indeed, since $\binom{t}{0}, \ldots, \binom{t}{n-1}$ form a basis of the space of polynomials of degree less than n, it suffices to show this indentity for $p(t) = \binom{t}{d}$, where d is a non-negative integer smaller than n.

$$\sum_{t=0}^{n} (-1)^t \binom{t}{d} \binom{n}{t} = (-1)^d \binom{n}{d} \sum_{t=d}^{n} (-1)^{t-d} \binom{n-d}{t-d} = (-1)^d \binom{n}{d} (1-1)^{n-d} = 0.$$

It follows that $M_g(k)$ vanishes for $k \geq 2g$.

Next, we show that the obstruction in Proposition 4.2.4 does not vanish in the range of interest for the degree of irrationality.

Lemma 4.2.6. *The integer $M_g(k)$ does not vanish if $g + 1 \leq k \leq 2g - 1$.*

Proof. In light of Remark 4.2.5,

$$M_g(k) = (k-1)! \sum_{j=0}^{k} (-1)^j (j-1)^{2g-1} \binom{k}{j}$$

in this range. Observe that

$$\frac{(j-1)^{2g-1}}{(2g-1)!}$$

is the coefficient of z^{2g-1} in $e^{(j-1)z}$. Hence, $M_g(k)$ is the coefficient of z^{2g-1} in

$$\begin{aligned}(2g-1)!(k-1)! \sum_{j=0}^{k} (-1)^j e^{(j-1)z} \binom{k}{j} &= (2g-1)!(k-1)! e^{-z} (1-e^z)^k \\ &= (2g-1)!(k-1)!(e^z + e^{-z} - 2)(1-e^z)^{k-2}.\end{aligned}$$

The result then follows from the fact that the coefficient of z^j in $(1-e^z)^{k-2}$ is positive for all $j > 0$ and that the coefficient of z^j in $(e^z + e^{-z} - 2)$ is zero for odd j and positive for even $j > 0$. $\square$

4.2.3 A non-degeneracy result for Hodge classes on A^{k-1}

In order to use Theorem 4.2.4 to obtain lower bounds for the degree of irrationality of abelian varieties, we need to make sure that the image of

$$\begin{aligned} V_{(k-2)g} &\longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z}) \cong \mathbb{Z} \\ \alpha &\longmapsto \alpha \cdot [A^{k-1,0}] \end{aligned}$$

is a high index subgroup when A is a very general abelian variety with a high degree polarization. This is achieved by the following non-degeneracy result which is of independent interest.

Theorem 4.2.7. *Let A be a $(1, d_2, \ldots, d_g)$-polarized abelian g-fold with maximal special Mumford-Tate group and $B \subset A$ be an abelian subvariety of codimension rg. Then the image of*

$$\begin{aligned} V_{(k-1-r)g} &\longrightarrow H^{2(k-1)g}(A^{k-1}, \mathbb{Z}) \cong \mathbb{Z} \\ \alpha &\longmapsto \alpha \cdot [B] \end{aligned}$$

is contained in $d_g\mathbb{Z}$. Moreover, it is contained in $pd_g\mathbb{Z}$ if $g = p^l$ for some prime p and some $l \in \mathbb{Z}_{>0}$.

Proof. First, we can reduce to the case $r = 1$. Indeed, we can find abelian subvarieties $B', B'' \subset A^{k-1}$ such that $\operatorname{codim}(B') = g$, $\operatorname{codim}(B'') = (r-1)g$ and $B' \cap B'' = B$. Then, for any $\alpha \in V_{(k-1-r)g}$,

$$\alpha \cdot [B] = (\alpha \cdot [B'']) \cdot [B'].$$

Moreover, $\alpha \cdot [B'']$ is also supported on D^{k-1} for some closed proper subset $D \subset A$ so that $\alpha \cdot [B''] \in V_{(k-2)g}$.

It will be convenient to use the Fourier transform

$$\mathcal{F} : H^\bullet(A^{k-1}, \mathbb{Z}) \to H^{2(k-1)g-\bullet}(\widehat{A}^{k-1}, \mathbb{Z}).$$

See [Bea83] for the definition and properties of the Fourier transform. Note that the Fourier transform of the class of the abelian subvariety $B \subset A^{k-1}$ is up to sign the class of the abelian subvariety $\ker(\widehat{A} \to \widehat{B})$. Call this subvariety B' and denote by $\widehat{i}$ the inclusion $B' \to \widehat{A}^{k-1}$. Then, for any $\alpha \in V_{(k-2)g}$,

$$\int_{A^{k-1}} \alpha \cdot [B] = \pm \int_{\widehat{A}^{k-1}} \mathcal{F}(\alpha) \cdot \left[B'\right] = \pm \widehat{i}^* \mathcal{F}(\alpha). \tag{4.3}$$

Since A has a polarization of type $(1, d_2, \ldots, d_g)$, the abelian variety $\widehat{A}$ carries a dual polarization of type

$$(\widehat{d}_1, \ldots, \widehat{d}_g) := (1, d_g/d_{g-1}, \ldots, d_g/d_2, d_g).$$

Let $\widehat{A} = W/\Lambda$, where W is a g-dimensional complex vector space and $\Lambda \subset W$ is a cocompact lattice. Choose real coordinate functions $x_1, \ldots, x_g, y_1, \ldots, y_g$ on W with respect to a symplectic basis of Λ for the polarization. Denote by x_i^j and y_i^j, $i \in [k-1]$, $j \in [g]$, the real coordinate functions on W^{k-1} given by $x_i \circ \mathrm{pr}_j : W^{k-1} \to \mathbb{R}$ and $y_i \circ \mathrm{pr}_j : W^{k-1} \to \mathbb{R}$. Finally, consider the basis $\mathcal{B}^l$ for $H^l(\widehat{A}^{k-1}, \mathbb{Z})$ which consists (up to signs which we disregard) of wedge products of l elements of the basis $\{dx_i^j, dy_i^j : i \in [g], j \in [k-1]\}$ for $H^1(\widehat{A}^{k-1}, \mathbb{Z})$.

Recall that a very general $(1, \widehat{d}_2, \ldots, \widehat{d}_g)$-polarized abelian g-fold $\widehat{A}$ has special Mumford-Tate group equal to $\mathrm{Sp}(2g, \mathbb{Q})$ and thus that Hodge classes on $\widehat{A}^{k-1}$ lie in the subalgebra of $H^\bullet(\widehat{A}^{k-1}, \mathbb{Q})$ generated by divisors. The space of these divisors is $\binom{k}{2}$-dimensional and has as basis the classes

$$\theta_i := \mathrm{pr}_i^*\theta = -\sum_{n=1}^{g} \widehat{d}_n(dx_n^i \wedge dy_n^i), \qquad i = 1, \dots, k-1,$$

$$\lambda_{ij} := \mathrm{pr}_{ij}^*(\lambda) = -\sum_{n=1}^{g} \widehat{d}_n(dx_n^i \wedge dy_n^j + dx_n^j \wedge dy_n^i), \qquad 1 \leq i < j \leq k-1,$$

where θ is the class of the polarization and λ is the pullback of the Poincaré bundle on $\widehat{A} \times A$ by the isogeny $\widehat{A} \times \widehat{A} \to \widehat{A} \times A$ induced by the polarization on $\widehat{A}$. Hence, a basis for $\mathrm{Hdg}^g(\widehat{A}^{k-1})_{\mathbb{Q}}$ is given by

$$\mathcal{B}_{\mathrm{can}} = \left\{ v_{\mathbf{a},\mathbf{b}} : (\mathbf{a}, \mathbf{b}) \in T \right\},$$

where

$$v_{\mathbf{a},\mathbf{b}} := \prod_{i=1}^{k-1} \theta_i^{a_i} \prod_{1 \leq j < l \leq k-1} \lambda_{jl}^{b_{jl}},$$

and

$$T = \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_{\geq 0}^{k-1} \times \mathbb{Z}_{\geq 0}^{\binom{k-1}{2}} : \sum_{i=1}^{k} a_i + \sum_{1 \leq j < l \leq k-1} b_{jl} = g \right\}.$$

Moreover, the fact that classes in $V_{(k-2)g}$ are supported on D^k means that classes in $\mathcal{F}(V_{(k-2)g})$ have vanishing Künneth components in

$$\prod_{\substack{j_1+\ldots+j_{k-1}=g \\ j_i \geq 2g-1 \text{ for some } i}} H^{j_1}(\widehat{A}) \otimes \ldots \otimes H^{j_{k-1}}(\widehat{A}). \tag{4.4}$$

Observe that an element of $\langle \mathcal{B}_{\mathrm{can}} \rangle$ satisfying the Künneth condition (4.4) must be in $\langle \mathcal{B}'_{\mathrm{can}} \rangle$, where

$$\mathcal{B}'_{\mathrm{can}} = \left\{ v_{\mathbf{a},\mathbf{b}} : (\mathbf{a}, \mathbf{b}) \in T \text{ and } a_i + \sum_{j \neq i} b_{ij} < g, \forall i \in [k-1] \right\}.$$

Indeed, an element of $\mathcal{B}_{\mathrm{can}} \setminus \mathcal{B}'_{\mathrm{can}}$ is of the form $\theta_i^l \prod_{r=1}^{g-l} \lambda_{ij_r}$ and is the only element of $\mathcal{B}$

which has a non-zero coefficient for the term

$$dx_1^i \wedge \ldots \wedge dx_g^i \wedge dy_1^i \wedge \ldots \wedge dy_l^i \wedge dy_{l+1}^{j_1} \wedge \ldots dy_g^{j_g}$$

when expressed as a linear combination of elements of $\mathcal{B}^{2g}$. It follows that

$$\mathcal{F}(V_{(k-2)g}) \subset \langle \mathcal{B}'_{\text{can}} \rangle.$$

The next step is to study the divisibility of elements of $\langle \mathcal{B}'_{\text{can}} \rangle$. For any $w \in \mathcal{B}^{2g}$, let

$$\mathbf{r}(w) = (r_1(w), \ldots, r_g(w)) \in \mathbb{Q}^g,$$

where $r_i(w)$ is half the number of dx_i^j and dy_i^j with $j \in [k-1]$ in w. Note that for any w which appears with a non-zero coefficient in an element of $\langle \mathcal{B}_{\text{can}} \rangle$,

$$\mathbf{r}(w) \in \mathbb{Z}^g.$$

Moreover, the coefficient of w in $v_{\mathbf{a},\mathbf{b}}$ is a clearly a multiple of $\widehat{d}_1^{\,r_1(w)} \cdots \widehat{d}_g^{\,r_g(w)}$.

Since $w \cdot [B'] = 0$ unless $\mathbf{r}(\omega) = (1, \ldots, 1)$ by (4.3), the first claim in Theorem 4.2.7 follows from:

Lemma 4.2.8. *We use the notation which we have set up above. If $w \in \mathcal{B}^{2g}$ is such that $r(w) = (1, \ldots, 1)$ the coefficient of w in any element of*

$$\langle \mathcal{B}'_{\text{can}} \rangle \otimes \mathbb{Q} \cap H^{2g}(\widehat{A}^{k-1}, \mathbb{Z})$$

is divisible by $d_g = \widehat{d}_g$.

Proof. It suffice to show that for any

$$\alpha = \sum_{(\mathbf{a},\mathbf{b})} \alpha_{\mathbf{a},\mathbf{b}} v_{\mathbf{a},\mathbf{b}} \in \langle \mathcal{B}'_{\text{can}} \rangle \otimes \mathbb{Q} \cap H^{2g}(\widehat{A}^{k-1}, \mathbb{Z}),$$

the following rational number is an integer

$$(\widehat{d}_2 \cdots \widehat{d}_{g-1}) \cdot \Big(\prod_{i=1}^{k-1} a_i! \Big) \cdot \Big(\prod_{1 \le j < l \le k-1} b_{jl}! \Big) \cdot \alpha_{\mathbf{a},\mathbf{b}}.$$

To simplify notation we write

$$(\mathbf{a}, \mathbf{b})! := \Big(\prod_{i=1}^{k-1} a_i \Big) \cdot \prod_{1 \le j < l \le k-1} b_{jl}!.$$

Let

$$T' := \{(\mathbf{a}, \mathbf{b}) : (\widehat{d}_2 \cdots \widehat{d}_{g-1})(\mathbf{a}, \mathbf{b})! \alpha_{\mathbf{a},\mathbf{b}} \in \mathbb{Z}\} \subset T.$$

We show that $T = T'$.

First, we show that if $(\mathbf{a}, \mathbf{b})$ is such that there are distinct non-zero b_{jl}, then $(\mathbf{a}, \mathbf{b}) \in T'$. Without loss of generality, we can suppose that either

- $b_{12} > 0$ and $b_{13} > 0$,
- $b_{12} > 0$ and $b_{34} > 0$.

In the first case, consider

$$w' = dx_1^1 \wedge dy_1^2 \wedge dx_1^1 \wedge dy_1^3 \wedge \eta \in \mathcal{B}^{2g},$$

where η is any element of $\mathcal{B}^{2(g-2)}$ with $\mathbf{r}(\eta) = (0, 1, \ldots, 1, 0)$ such that w' appears in the expansion of $v_{\mathbf{a},\mathbf{b}}$. Clearly, w' will only appear in the expansion for $v_{\mathbf{a},\mathbf{b}}$ and its coefficient will be $\pm(\mathbf{a}, \mathbf{b})! \widehat{d}_2 \cdots \widehat{d}_{g-1}$. Hence, w' will appear in the expansion of α with coefficient

$\pm(\mathbf{a},\mathbf{b})!\widehat{d}_2 \cdots \widehat{d}_{g-1}\alpha_{\mathbf{a},\mathbf{b}}$. Since $\alpha \in H^{2g}(\widehat{A}^{k-1}, \mathbb{Z})$ this coefficient is an integer and $(\mathbf{a},\mathbf{b}) \in T'$. The second case is treated analogously with

$$w' = dx_1^1 \wedge dy_1^2 \wedge dx_1^3 \wedge dy_1^4 \wedge \eta \in \mathcal{B}^{2g}.$$

Next, we consider the case where $a_i > 0$ for some i and $b_{jl} \neq 0$ for some $j, l \neq i$. Without loss of generality we can suppose that

$$a_1 > 0 \text{ and } b_{23} > 0.$$

We deal with these cases as above by considering

$$w' = dx_1^1 \wedge dy_1^1 \wedge dx_1^2 \wedge dy_1^3 \wedge \eta \in \mathcal{B}^{2g}.$$

Now, suppose that all b_{jl} are zero and there are at least 3 distinct non-zero a_i, say $a_1, a_2, a_3 \neq 0$. Consider

$$w' = dx_1^1 \wedge dy_1^1 \wedge dx_1^2 \wedge dy_1^2 \wedge dx_1^3 \wedge dy_1^3 \wedge \eta \in \mathcal{B}^{2g},$$

where η is any element of $\mathcal{B}^{2(g-2)}$ with $\mathbf{r}(\eta) = (0, 1, \ldots, 1, 0, 0)$ such that w' appears in the expansion of $v_{\mathbf{a},\mathbf{b}}$. The coefficient of w' in $v_{\mathbf{a},\mathbf{b}}$ is $\pm(\mathbf{a},\mathbf{b})!\widehat{d}_2 \cdots \widehat{d}_{g-2}$ and, while w' might appear in the expansion of other basis elements $v_{\mathbf{a}',\mathbf{b}'}$, these basis elements must have either two non-zero b'_{jl} or a non-zero b'_{jl} and a non-zero a'_i with $i \neq j, l$. It follows from the previously treated cases and the fact that $\alpha \in H^{2g}(\widehat{A}^{k-1}, \mathbb{Z})$ that $\pm(\mathbf{a},\mathbf{b})!\widehat{d}_2 \cdots \widehat{d}_{g-2}\alpha_{\mathbf{a},\mathbf{b}} \in \mathbb{Z}$ and thus $(\mathbf{a},\mathbf{b}) \in T'$.

Finally, we have left to treat the case where there is at most one non-zero b_{jl}, say b_{12}, and $a_1, a_2 \neq 0$, namely $\mathbf{a} = (a_1, a_2, 0, \ldots, 0)$, $\mathbf{b} = (b_{12}, 0, \ldots, 0)$, and $a_1, a_2 > 0$. Then,

$$v_{\mathbf{a},\mathbf{b}} = \theta_1^{a_1}\theta_2^{a_2}\lambda_{12}^{g-a_1-a_2}.$$

We will denote such elements of $\mathcal{B}'_{\text{can}}$ by v'_{a_1,a_2}. Let m be the maximal value of $g - a_1 - a_2$ among the elements $v'_{a_1,a_2} \in T \setminus T'$ appearing in the expansion of α as a linear combination of elements of $\mathcal{B}'_{\text{can}}$, i.e.,

$$m = \max\left\{g - a_1 - a_2 : (\mathbf{a}, \mathbf{b}) = \big((a_1, a_2, \ldots, 0), (g - a_1 - a_2, 0, \ldots, 0)\big) \in T \setminus T'\right\}.$$

Let $v'_{p,g-p-m}$ be the basis element of the form v'_{a_1,a_2} with $g - a_1 - a_2 = m$ and minimal a_1 in $T \setminus T'$. Consider the term

$$\begin{aligned} w' =& (dx_1^1 \wedge dy_1^1 \wedge \ldots \wedge dx_p^1 \wedge dy_p^1) \wedge (dx_1^2 \wedge dy_1^2) \\ &\wedge (dx_{p+1}^2 \wedge dy_{p+1}^2 \wedge \ldots \wedge dx_{g-m-1}^2 \wedge dy_{g-m-1}^2) \\ &\wedge (dx_{g-m}^1 \wedge dy_{g-m}^2 \wedge \ldots \wedge dx_{g-1}^1 \wedge dy_{g-1}^2) \in \mathcal{B}^{2g}. \end{aligned}$$

This term can only appear in elements of $\mathcal{B}'_{\text{can}}$ of the form v'_{a_1,a_2} for integers $a_1, a_2 > 0$ satisfying

- $a_1 + a_2 \leq g$,
- $g - a_1 - a_2 \geq m$,
- $2a_1 + (g - a_1 - a_2) \geq 2p + m$.

Aside from possibly $v'_{p,g-p-m}$, all these terms contribute an integral multiple of w' to α by the choice of $v'_{p,g-p-m}$. It follows that the coefficient of w' in

$$\alpha_{(p,g-p-m,0,\ldots,0),(m,0,\ldots,0)}v'_{p,g-p-m}$$

is an integer. Since this coefficient is

$$p!(g-p-m)!m! \cdot (\widehat{d}_2 \cdots \widehat{d}_{g-1}) \cdot \alpha_{(p,g-p-m,0,\ldots,0),(m,0,\ldots,0)},$$

we see that

$$((p, g-p-m, 0, \ldots, 0), (m, 0, \ldots, 0)) \in T.$$

We conclude that all v'_{a_1,a_2} are in T'. Hence, $T = T'$ and the claim follows. □

Finally, we show the second claim in Theorem 4.2.7. We will abuse notation by using some of the notation introduced in the proof of 4.2.8 for the cohomology of A^{k-1} instead of $\widehat{A}^{k-1}$. Let

$$\mathcal{B}''_{\text{can}} = \left\{ v_{\mathbf{a},\mathbf{b}} : (\mathbf{a}, \mathbf{b}) \in T'' \right\},$$

where

$$v_{\mathbf{a},\mathbf{b}} := \prod_{i=1}^{k-1} \theta_i^{a_i} \prod_{1 \le j < l \le k-1} \lambda_{jl}^{b_{jl}} \in H^{2(k-2)g}(A^{k-1}, \mathbb{Z}),$$

and

$$T'' = \left\{ (\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_{\ge 0}^{k-1} \times \mathbb{Z}_{\ge 0}^{\binom{k-1}{2}} : \sum_{i=1}^{k} a_i + \sum_{1 \le j < l \le k-1} b_{jl} = (k-2)g \right\}.$$

There is an S_g action on $H^{\bullet}(A^{k-1}, \mathbb{Z})$ given by $\sigma(dx_i^j) = dx_{\sigma(i)}^j$. Now suppose that $\alpha \in \langle \mathcal{B}''_{\text{can}} \rangle$ is such that

$$\alpha \cdot [A^{k-1,0}] \neq 0.$$

Write α and $[A^{k,0}]$ as integral linear combination of elements of $\mathcal{B}^{2(k-2)g}$ and $\mathcal{B}^{2g}$. Let $\omega \in \mathcal{B}^{2(k-2)g}$ and $\omega' \in \mathcal{B}^{2g}$ be two elements which appear with non-zero coefficient $m, n \in \mathbb{Z}$ and such that

$$\omega \wedge \omega' \neq 0.$$

One checks that for any $\sigma \in S_g$ the elements $\sigma(\omega)$ and $\sigma(\omega')$ appear with non-zero coefficients in the expansions of α and $[A^{k,0}]$ and that

$$\omega \wedge \omega' = \sigma(\omega) \wedge \sigma(\omega').$$

Hence, the contribution of these classes to $\alpha \cdot [A^{k,0}]$ will be

$$mn \cdot \#\{\sigma(\omega') : \sigma \in S_g\} \cdot \omega \wedge \omega'.$$

If $g = p^l$ for some prime p and some $l \in \mathbb{Z}_{>0}$, the orbit of ω' has size divisible by p unless w' is stabilized by S_g. This can happen only if ω' is up to sign a product of two elements of the set

$$\left\{dx_1^j \wedge \dots dx_g^j, dy_1^j \wedge \dots dy_g^j : j \in [g]\right\}.$$

Since such an ω' satisfies $\theta_1 \cdots \theta_{k-1} \cdot \omega' = 0$ and α is supported on D^{k-1}, these terms cannot contribute to $\alpha \cdot [A^{k,0}]$. □

4.2.4 Lower bounds on the degree of irrationality

In what follows, we let

$$d_g' := \begin{cases} pd_g \text{ if } g = p^l \text{ for some prime } p, \\ d_g \text{ otherwise.} \end{cases}$$

Corollary 4.2.9. *If $d_g \nmid N^{2(k-2)} M_g^k$, then A does not admit a dominant rational map of degree k to a CH_0-trivial variety with cohomological torsion order N.*

Corollary 4.2.10. *This follows at once from Proposition 4.2.4 and Theorem 4.2.7.*

Theorem 4.2.11. *Let A be a very general $(1, d_2, \dots, d_g)$-polarized abelian g-fold. If*

$$d_g' \nmid \operatorname{lcm}\left\{N^{2(k-2)} M_g(k) : \frac{3g+1}{2} \leq k \leq 2g-1\right\}$$

then A does not admit a dominant rational map of degree less than $2g$ to a CH_0-trivial variety with torsion order N. In particular, if

$$d'_g \nmid \operatorname{lcm}\left\{M_g(k) : \frac{3g+1}{2} \leq k \leq 2g-1\right\},$$

then

$$\operatorname{irr}(A) \geq 2g.$$

See table B.2 for a list of the prime factorizations of lcm $\left\{M_k(g) : \frac{3g+1}{2} \leq k \leq 2g-1\right\}$ for $3 \leq g \leq 19$.

Proof. The theorem follows from Corollary 4.2.9. Note that we only have to consider the least common multiple of $N^{2(k-2)}M_k(g)$ for k between $(3g+1)/2$ and $2g-1$ in light of Theorem 3.3.2. □

We can get a more explicit result which applies to abelian varieties with maximal special Mumford-Tate group but only at the cost of replacing

$$\operatorname{lcm}\left\{M_g(k) : \frac{3g+1}{2} \leq k \leq 2g-1\right\},$$

with

$$\operatorname{lcm}\left\{M_k(g) : g+1 \leq k \leq 2g-1\right\}.$$

Indeed, we do not have any control about which very general abelian varieties satisfy the conclusion of Theorem 3.3.2. Table B.1 shows that the two numbers above can indeed differ.

Theorem 4.2.12. *Let A be a $(1, d_2, \ldots, d_g)$-polarized abelian g-fold with maximal special Mumford-Tate group. If*

$$d'_g \nmid \operatorname{lcm}\left\{N^{2(k-2)}M_g(k) : g+1 \leq k \leq 2g-1\right\}$$

then A does not admit a dominant rational map of degree less than $2g$ to a CH_0-trivial variety with torsion order N. In particular, if

$$d'_g \nmid \operatorname{lcm}\left\{M_g(k) : g+1 \leq k \leq 2g-1\right\},$$

then

$$\operatorname{irr}(A) \geq 2g.$$

See table B.3 for a list of the prime factorizations of $\operatorname{lcm}\{M_k(g) : g+1 \leq k \leq 2g-1\}$ for $3 \leq g \leq 19$.

Corollary 4.2.13. *There is at most finitely-many polarization types $(1, d_2, \ldots, d_g)$ such that a very general $(1, d_2, \ldots, d_g)$-polarized abelian g-fold has degree of irrationality less than $2g$.*

Proof. By Theorem 4.2.11, if a very general $(1, d_2, \ldots, d_g)$-polarized abelian surface has degree less than $2g$, then

$$d_g \,\Big|\, \operatorname{lcm}\left\{M_g(k) : \frac{3g+1}{2} \leq k \leq 2g-1\right\}.$$

□

Question 4.2.14. *Are there any abelian g-folds with degree of irrationality greater than $2g$? Are there polarization types for which a very general abelian g-fold has degree of irrationality greater (or smaller) than $2g$?*

APPENDIX A

PROOF OF PROPOSITION 4.1.1

Proof. Let's pick up where we left off in Section 4.1. We proceed by induction on l and give the proof in the case where Y has a decomposition of the diagonal. A similar argument can be used when Y has a cohomological decomposition of the diagonal. The base case $l = 2$ is provided by Corollary 4.1.5. Hence, one computes:

$$
\begin{aligned}
N^{2l} Z_{l+1} = & \operatorname{pr}_{[l]}^*(N^{2(l-1)} Z_l) \cdot \left(\operatorname{pr}_{\{l,l+1\}}^*(N^2 Z_2) - N^2 \sum_{j=1}^{l-1} \Delta_{j,l+1} \right) + N^{2l}\Gamma_{l+1} \\
= & N^{2(l-1)} \sum_{i=1}^{l} (-1)^{i+1} \frac{(1/N^2)!(i-1)!}{(1/N^2 - l + i)!} \sum_{I\subset[l],\#I=i} \Delta_I \cdot \operatorname{pr}_{[l]\setminus I}^* \left(\varphi^*(\alpha)^{\boxtimes(l-i)}\right) \\
& \cdot \left(X^l \times \varphi^*(\alpha) + X^{l-1} \times \varphi^*(\alpha) \times X - N^2 \sum_{j=1}^{l} \Delta_{j,l+1} \right) + N^{2l}\Gamma_{l+1} \\
= & N^{2(l-1)} \sum_{i=1}^{l} (-1)^{i+1} \frac{(1/N^2)!(i-1)!}{(1/N^2 - l + i)!} \Bigg[\sum_{J\subset[l+1],\#J=i,l+1\notin J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l+1-i)}\right) \\
& + \sum_{I\subset[l],\#I=i,l\in I} \varphi^*(\alpha)^{\boxtimes l} \times X - N^2 i \sum_{J\subset[l+1],\#J=i+1,l+1\in J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l-i)}\right) \\
& - N^2(l-i) \sum_{J\subset[l+1],\#J=i,l+1\notin J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l+1-i)}\right) \Bigg] + N^{2l}\Gamma_{l+1} \\
= & N^{2l} \sum_{i=1}^{l} (-1)^{i+1} \frac{(1/N^2)!(i-1)!}{(1/N^2 - l + i)!} \Bigg[\frac{1}{N^2} \sum_{I\subset[l],\#I=i,l\in I} \varphi^*(\alpha)^{\boxtimes l} \times X \\
& + (1/N^2 - l + i) \sum_{J\subset[l+1],\#J=i,l+1\notin J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l+1-i)}\right) \\
& - i \sum_{J\subset[l+1],\#J=i+1,l+1\in J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l-i)}\right) \Bigg] + N^{2l}\Gamma_{l+1} \\
= & N^{2l} \Bigg[\sum_{i=1}^{l} (-1)^{i+1} \frac{(1/N^2)!(i-1)!}{(1/N^2 - (l+1) + i)!} \sum_{J\subset[l+1],\#J=i,l+1\notin J} \Delta_J \cdot \operatorname{pr}_{[l+1]\setminus J}^* \left(\varphi^*(\alpha)^{\boxtimes(l+1-i)}\right)
\end{aligned}
$$

$$+\sum_{j=2}^{l+1}(-1)^{j+1}\frac{(1/N^2)!(j-1)!}{(1/N^2-(l+1)-j)!}\sum_{J\subset[l+1],\#J=j,l+1\in J}\Delta_J\cdot \mathrm{pr}^*_{[l+1]\setminus J}\left(\varphi^*(\alpha)^{\boxtimes(l+1-j)}\right)$$
$$\left.+\sum_{i=1}^{l}(-1)^{i+1}\frac{(1/N^2)!(i-1)!}{(1/N^2-l+i)!}\frac{1}{N^2}\sum_{I\subset[l],\#I=i,l\in I}\varphi^*(\alpha)^{\boxtimes l}\times X\right]+N^{2l}\Gamma_{l+1}.$$

It thus suffices to show that

$$\sum_{i=1}^{l}(-1)^{i+1}\frac{(1/N^2)!(i-1)!}{(1/N^2-l+i)!}\frac{1}{N^2}\sum_{I\subset[l],\#I=i,l\in I}1=\frac{(1/N^2)!}{(1/N^2-l)!}. \tag{A.1}$$

Clearly, this amounts to showing that

$$\sum_{i=1}^{l}(-1)^{i+1}\frac{(1/N^2)!(i-1)!}{(1/N^2-l+i)!}\binom{l-1}{i-1}=\frac{(1/N^2-1)!}{(1/N^2-l)!} \tag{A.2}$$

or

$$\begin{aligned}\sum_{i=1}^{l}(-1)^{i+1}&\frac{(1/N^2-l+1)!(l-1)!(i-1)!}{(1/N^2-l+i)!(l-i)!(i-1)!}\\ &=\sum_{j=0}^{l-1}(-1)^j\binom{l-1}{j}\binom{1/N^2-l+j+1}{j}^{-1}\\ &=\frac{(1/N^2-l+1)}{(1/N^2)}.\end{aligned}$$

This follows from an identity due to Frisch (see [GQ16]). □

APPENDIX B

TABLES OF OBSTRUCTIONS IN SMALL DEGREES AND DIMENSIONS

$k \backslash g$	2	3	4	5	6	7
1	0	0	0	0	0	0
2	2^2	0	2^2	0	2^2	0
3	$-2^2 \cdot 3$	$-2^2 \cdot 3^2$	$-2^2 \cdot 3 \cdot 5^2$	$-2^2 \cdot 3^2 \cdot 5^2$	$-2^2 \cdot 3 \cdot 19^2$	$-2^2 \cdot 3^4 \cdot 7^2$
4	0	$2^4 \cdot 3^2 \cdot 5$	$2^7 \cdot 3^4$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 29$	$2^7 \cdot 3^3 \cdot 5 \cdot 59$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 53$
5	0	$-2^6 \cdot 3^2 \cdot 5$	$-2^9 \cdot 3^2 \cdot 5 \cdot 7$	$-2^6 \cdot 3^2 \cdot 5^2 \cdot 281$	$-2^{10} \cdot 3^2 \cdot 5^2 \cdot 347$	$-2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 10069$
6	0	0	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$2^9 \cdot 3^3 \cdot 5^5 \cdot 73$	$2^9 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 1193$
7	0	0	$-2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	$-2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 41$	$-2^8 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 619$	$-2^9 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 127 \cdot 131$
8	0	0	0	$2^{11} \cdot 3^7 \cdot 5^2 \cdot 7^2$	$2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^2$	$2^{12} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 17$
9	0	0	0	$-2^{14} \cdot 3^6 \cdot 5^2 \cdot 7^2$	$-2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	$-2^{15} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 47$
10	0	0	0	0	$2^{17} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11$	$2^{17} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 37$
11	0	0	0	0	$-2^{16} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$	$-2^{16} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 127$
12	0	0	0	0	0	$2^{18} \cdot 3^9 \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13$
13	0	0	0	0	0	$-2^{20} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13$
14	0	0	0	0	0	0
15	0	0	0	0	0	0

Table B.1: Prime factorization of $M_g(k)$

The non-vanishing obstructions which are relevant for bounding below the degree of irrationality of very general abelian varieties (resp. abelian varieties with maximal special Mumford-Tate group) are in dark grey (resp. light and dark grey).

g	$\mathrm{lcm}\{M_g(k) : (3g+1)/2 \leq k \leq 2g-1\}$
3	$2^6 \cdot 3^2 \cdot 5$
4	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
5	$2^{14} \cdot 3^7 \cdot 5^2 \cdot 7^2$
6	$2^{17} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$
7	$2^{20} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 127$
8	$2^{22} \cdot 3^{12} \cdot 5^5 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 47$
9	$2^{30} \cdot 3^{12} \cdot 5^6 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 79$
10	$2^{34} \cdot 3^{16} \cdot 5^6 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 53 \cdot 173$
11	$2^{36} \cdot 3^{19} \cdot 5^8 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 137 \cdot 443 \cdot 54721$
12	$2^{38} \cdot 3^{18} \cdot 5^9 \cdot 7^6 \cdot 11^4 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 43 \cdot 29569$
13	$2^{44} \cdot 3^{20} \cdot 5^{11} \cdot 7^6 \cdot 11^5 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 211 \cdot 673 \cdot 3371 \cdot 47711$
14	$2^{47} \cdot 3^{24} \cdot 5^{12} \cdot 7^6 \cdot 11^5 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 83 \cdot 127 \cdot 859 \cdot 1069$
15	$2^{50} \cdot 3^{26} \cdot 5^{12} \cdot 7^8 \cdot 11^5 \cdot 13^5 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 43 \cdot 107 \cdot 181 \cdot 317 \cdot 4789 \cdot 39623$
16	$2^{52} \cdot 3^{28} \cdot 5^{14} \cdot 7^9 \cdot 11^5 \cdot 13^5 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 277 \cdot 677 \cdot 2069 \cdot 2243 \cdot 7321$
17	$2^{62} \cdot 3^{30} \cdot 5^{15} \cdot 7^9 \cdot 11^6 \cdot 13^5 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 89 \cdot 193 \cdot 283 \cdot 541 \cdot 997 \cdot 1013 \cdot 1277 \cdot 3803 \cdot 130869301$
18	$2^{67} \cdot 3^{34} \cdot 5^{16} \cdot 7^{10} \cdot 11^6 \cdot 13^5 \cdot 17^4 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 43 \cdot 127 \cdot 233 \cdot 673 \cdot 5441 \cdot 14389 \cdot 6322807$
19	$2^{68} \cdot 3^{34} \cdot 5^{16} \cdot 7^{10} \cdot 11^6 \cdot 13^5 \cdot 17^5 \cdot 19^2 \cdot 23^4 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 41 \cdot 127 \cdot 593 \cdot 18149 \cdot 2523877 \cdot 35763437 \cdot 139866423077$

Table B.2: Prime factorization of $\mathrm{lcm}\{M_g(k) : (3g+1)/2 \leq k \leq 2g-1\}$.

g	$\operatorname{lcm}\{M_g(k) : g+1 \leq k \leq 2g-1\}$
2	$2^2 \cdot 3$
3	$2^6 \cdot 3^2 \cdot 5$
4	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7$
5	$2^{14} \cdot 3^7 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 41$
6	$2^{17} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 619$
7	$2^{20} \cdot 3^{10} \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 37 \cdot 47 \cdot 127$
8	$2^{22} \cdot 3^{12} \cdot 5^6 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 47 \cdot 599 \cdot 3929 \cdot 13399$
9	$2^{30} \cdot 3^{13} \cdot 5^6 \cdot 7^5 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 29 \cdot 37 \cdot 59 \cdot 79 \cdot 593 \cdot 661 \cdot 11117$
10	$2^{34} \cdot 3^{16} \cdot 5^6 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 31 \cdot 53 \cdot 173 \cdot 383 \cdot 2083 \cdot 3037 \cdot 8224729$
11	$2^{36} \cdot 3^{19} \cdot 5^8 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 89 \cdot 137 \cdot 373 \cdot 443 \cdot 2777 \cdot 4547 \cdot 4637 \cdot 9241 \cdot 26387 \cdot 37813 \cdot 54721$
12	$2^{38} \cdot 3^{18} \cdot 5^9 \cdot 7^6 \cdot 11^4 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 41 \cdot 43 \cdot 727 \cdot 1123 \cdot 4679 \cdot 14197 \cdot 29569 \cdot 514271 \cdot 8708807 \cdot 9374831$
13	$2^{44} \cdot 3^{20} \cdot 5^{11} \cdot 7^6 \cdot 11^5 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 37 \cdot 59 \cdot 97 \cdot 211$
	$251 \cdot 389 \cdot 443 \cdot 673 \cdot 1327 \cdot 2267 \cdot 3371 \cdot 5281 \cdot 12527 \cdot 47711 \cdot 128437 \cdot 541711$
14	$2^{47} \cdot 3^{24} \cdot 5^{12} \cdot 7^6 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 83 \cdot 127$
	$157 \cdot 193 \cdot 443 \cdot 653 \cdot 787 \cdot 859 \cdot 1069 \cdot 1151 \cdot 130619 \cdot 237173 \cdot 262139 \cdot 537233 \cdot 2916523 \cdot 5228329$
15	$2^{50} \cdot 3^{26} \cdot 5^{12} \cdot 7^8 \cdot 11^5 \cdot 13^5 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 43 \cdot 47 \cdot 107 \cdot 137 \cdot 181 \cdot 317$
	$457 \cdot 607 \cdot 1979 \cdot 2153 \cdot 2333 \cdot 4789 \cdot 6299 \cdot 6599 \cdot 22853 \cdot 39623 \cdot 239851 \cdot 500909 \cdot 161539033 \cdot 771097253$

Table B.3: Prime factorization of $\operatorname{lcm}\{M_g(k) : g+1 \leq k \leq 2g-1\}$.

APPENDIX C

SUPPORT OF ZERO-CYCLES ON ABELIAN VARIETIES

In [Voi18] the author shows the following surprising proposition:

Proposition C.0.1. *Consider an abelian variety A and an effective zero-cycle $\sum_{i=1}^{k}\{x_i\}$ on A such that*

$$\sum_{i=1}^{k}\{x_i\} = k\{0_A\} \in CH_0(A)_{\mathbb{Q}}.$$

Then for $i = 1, \dots, k$

$$(\{x_i\} - \{0_A\})^{*k} = 0 \in CH_0(A)_{\mathbb{Q}},$$

where $*$ denotes the Pontryagin product.

Voisin defines a subset

$$A_k := \{a \in A : (\{a\} - \{0\})^{*k} = 0 \in CH_0(A)_{\mathbb{Q}}\} \subset A,$$

and shows that $\dim A_k \leq k - 1$. Given a smooth projective variety X and an effective zero-cycle $z = \sum_{i=1}^{k}\{x_i\} \in Z_0(X)$, the support of z is

$$\mathrm{supp}(z) = \{x_i : i = 1, \dots, k\} \subset X.$$

Similarly, we will call the k-support of z the following subset of X:

$$\mathrm{supp}_k(z) = \bigcup_{z' = \sum_{i=1}^{k}\{x'_i\} : z' \sim z} \mathrm{supp}(z').$$

The previous proposition is equivalent to the inclusion

$$\mathrm{supp}_k(k\{0_A\}) \subset A_k.$$

Here we present a generalization of this result. Given $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$, we let

$$A_{k,\mathbf{x}} := \{a \in A : (\{a\} - \{x_1\}) * \cdots * (\{a\} - \{x_k\}) = 0 \in CH_0(A)_{\mathbb{Q}}\}.$$

Using the same argument as Voisin, one shows easily that $\dim A_{k,\mathbf{x}} \leq k-1$.

Proposition C.0.2. *Consider an abelian variety A and effective zero-cycles $\sum_{i=1}^{k}\{x_i\}$, $\sum_{i=1}^{k}\{y_i\}$ on A such that*

$$\sum_{i=1}^{k}\{x_i\} = \sum_{i=1}^{k}\{y_i\} \in CH_0(A)_{\mathbb{Q}}.$$

Then for $i = 1, \ldots, k$

$$\prod_{j=1}^{k}(\{x_i\} - \{y_j\}) = 0 \in CH_0(A)_{\mathbb{Q}},$$

where the product is the Pontryagin product.

Upon presenting this result to Nori he recognized it as a reformulation of results of his from around 2005. They had been obtained in an attempt to understand work of Colombo-van Geemen [CvG93] but left unpublished for lack of applications. We present Nori's proof as it is more elegant than our original proof. This proof was also suggested to Voisin by Beauville in the context of Proposition C.0.1.

Let X be a smooth projective variety and consider the graded algebra

$$\bigoplus_{n=1}^{\infty} CH_0(X^n)_{\mathbb{Q}}.$$

The multiplication is given by extending by linearity the maps

$$X^m \times X^n \to X^{m+n}$$

$$\big((x_1, \ldots, x_m), (x'_1, \ldots, x'_n)\big) \mapsto (x_1, \ldots, x_m, x'_1, \ldots, x'_n)$$

to get a product

$$Z_0(X^m) \times Z_0(X^n) \to Z_0(X^{n+m}).$$

It is easy to see that the resulting product descends to rational equivalence. Let

$$R := \bigoplus_{n=1}^{\infty} CH_0(X^n)_{\mathbb{Q}}/(ab - ba) = \bigoplus_{i=1}^{\infty} R_n$$

be the abelianization of this algebra.

Lemma C.0.3. *(Nori c. 2005, unpublished) If* $z = \sum_{i=1}^{k} \{x_i\} \in Z_0(X)$ *and* $y \in \operatorname{supp}_k(z)$*, then*

$$(\{y\} - \{x_1\})(\{y\} - \{x_2\}) \cdots (\{y\} - \{x_k\}) = 0 \in R,$$

where the product is taken in R *and we regard* $\{y\} - \{x_i\}$ *as elements of* $R_1 \subset R$*.*

Proof. Since $y \in \operatorname{supp}_k(z)$, there is a $\mathbf{y} = (y_1 = y, y_2, \ldots, y_k) \in X^k$ such that $\sum_{i=1}^{k} \{y_i\} = \sum_{i=1}^{k} \{x_i\}$. Consider the diagonal embeddings

$$\Delta_{[l]} : X \to X^l.$$

These give linear maps

$$\Delta_{[l]*} : R_1 \to R_l$$

such that

$$\Delta_{[l]*}\left(\sum_{i=1}^{k} \{y_i\}\right) = \sum_{i=1}^{k} \{y_i\}^l \in R_l.$$

Since

$$\sum_{i=1}^{k} \{y_i\} = \sum_{i=1}^{k} \{x_i\}$$

we get

$$p_l(\{\mathbf{y}\}) = \sum_{i=1}^{k} \{y_i\}^l = \sum_{i=1}^{k} \{x_i\}^l = p_l(\{\mathbf{x}\}) \in R_l,$$

where p_l is the l^{th} Newton polynomial and $\{\mathbf{x}\} = (\{x_1\}, \ldots, \{x_k\})$, $\{\mathbf{y}\} = (\{y_1\}, \ldots, \{y_k\})$. On the other hand, we have

$$(\{y\}-\{x_1\})(\{y\}-\{x_2\})\ldots(\{y\}-\{x_k\}) = \{y\}^k - e_1(\{\mathbf{x}\})\{y\}^{k-1} + \ldots + (-1)^k e_k(\{\mathbf{x}\}) \in R_k,$$

where e_l is the l^{th} elementary symmetric polynomial. Since the elementary symmetric polynomials can be written as polynomials in the Newton polynomials and since

$$p_l(\{\mathbf{y}\}) = p_l(\{\mathbf{x}\}), \qquad \forall l \in \mathbb{Z}_{\geq 0},$$

we get

$$\begin{aligned}
\prod_{i=1}^{k}(\{y\} - \{x_i\}) &= \sum_{i=0}^{k}(-1)^i \{y\}^{k-i} e_i(\{\mathbf{x}\}) \\
&= \sum_{i=0}^{k}(-1)^i \{y\}^{k-i} e_i(\{\mathbf{y}\}) \\
&= \prod_{i=1}^{k}(\{y\} - \{y_i\}) = 0 \in R_k.
\end{aligned}$$

□

Proof of Proposition C.0.2. If $X = A$ is an abelian variety we have a summation morphism $A^l \to A$ inducing maps

$$CH_0(A^l)_{\mathbb{Q}} \to CH_0(A)_{\mathbb{Q}},$$

and so a map

$$\sigma : R \to CH_0(A)_{\mathbb{Q}},$$

such that

$$\sigma\left(\prod_{i=1}^{k}(\{y\} - \{x_i\})\right) = (\{y\} - \{x_1\}) * \ldots * (\{y\} - \{x_k\}) \in CH_0(A)_{\mathbb{Q}}.$$

□

Lemma C.0.3 in fact has many more interesting corollaries. Consider $p(t_1, \ldots, t_k) \in \mathbb{C}[t_1, \ldots, t_k]$ and the S_k action on $\mathbb{C}[t_1, \ldots, t_k]$ given by permutation of the variables. Let $H_p \subset S_k$ be the subgroup stabilizing p.

Corollary C.0.4. *Consider an abelian variety A and effective zero-cycles $\sum_{i=1}^k \{x_i\}$, $\sum_{i=1}^k \{y_i\}$ on A. Then*

$$\sum_{i=1}^k \{x_i\} = \sum_{i=1}^k \{y_i\} \in CH_0(A)_{\mathbb{Q}}$$

if and only if

$$\prod_{\sigma \in S_k/H_p} \big(p(\{y_1\}, \ldots, \{y_k\}) - (\sigma \cdot p)(\{x_1\}, \ldots, \{x_k\})\big) = 0 \in CH_0(A)_{\mathbb{Q}}$$

for every $p \in \mathbb{C}[t_1, \ldots, t_k]$. Here the product is the Pontryagin product.

Proof. The if direction follows trivially from considering $p(t_1, \ldots, t_k) = t_1 + \ldots + t_k$. The proof of Lemma C.0.3 completes the argument. □

The special case $p = t_1$ is Proposition C.0.2. Another corollary of Lemma C.0.3 is the following:

Corollary C.0.5 (Nori)**.** *Given an effective zero-cycle $z = \sum_{i=1}^k \{x_i\}$ on an abelian variety A and $y_1, \ldots, y_{k+1} \in \mathrm{supp}_k(z)$, the following identity is satisfied*

$$\prod_{i<j} (\{y_i\} - \{y_j\}) = 0 \in CH_0(A)_{\mathbb{Q}}.$$

Proof. Let e_l be the l^{th} elementary symmetric polynomial. By Lemma C.0.3 we have

$$\{y_i\}^k - e_1(\{\mathbf{x}\})\{y_i\}^{k-1} + \ldots + (-1)^k e_k(\{\mathbf{x}\}) = 0 \in R_k$$

for $i = 1, \ldots, k+1$, where $\{\mathbf{x}\} = (\{x_1\}, \ldots, \{x_k\})$. This gives a non-trivial linear relation between the rows of the Vandermonde matrix $(\{y_i\}^{j-1})_{1 \leq i,j \leq k+1}$. It follows that the Vandermonde determinant vanishes. Using the morphism $\sigma : R \to CH_0(A)_{\mathbb{Q}}$ from the proof of C.0.2 finishes the proof.

$\square$

GLOSSARY OF NOTATION

$Z_d(X), (Z^d(X))$	The free abelian group on (co)dimension d subvarieties of X.
$Z_\bullet(X)$, $Z^\bullet(X)$	The group $\sum_{d=0}^{\dim X} Z_d(X)$.
$\sim$	Rational equivalence of algebraic cycles.
$CH_d(X), (CH^d(X))$	Chow group of (co)dimension d cycles on X.
$CH^\bullet(X)$	The Chow ring of X.
$[Z]$	The cycle class of $Z \in Z^\bullet(X)$ or $Z \in CH^\bullet(X)$.
$\Sigma_k, \Sigma_{(k)}$	The maps $X^k \to CH_0(X)$ and $\mathrm{Sym}^k(X) \to CH_0(X)$ (2.1).
$\lvert z \rvert$	The fiber of Σ_k or $\Sigma_{(k)}$ that contains z.
$\{x\}$	The image of $x \in X$ under the map $\Sigma_1 : X \to CH_0(X)$.
$W_1 \boxtimes \ldots \boxtimes W_l$	The cycle $\prod_{i=1}^{l} \mathrm{pr}_i^*(W_i) \in CH^\bullet(X^l)$.
$W^{\boxtimes l}$	The cycle $\underbrace{W \boxtimes \ldots \boxtimes W}_{l \text{ times}} \in CH^\bullet(X^l)$.
$\mathrm{Chow}_d(X)$	The Chow variety parametrizing effective pure d-dimensional cycles with a given (unspecified) cycle class.
$[n]$	The set $\{1, \ldots, n\}$.
$\mathrm{pr}_I, \mathrm{pr}_i, \mathrm{pr}_{ij}$	Given a product $\prod_{i\in[n]} X_i$ and a subset $I \subset [n]$, we write pr_I for the projection to $\prod_{i\in I} X_i$. Moreover, we let $\mathrm{pr}_i := \mathrm{pr}_{\{i\}}$ and $\mathrm{pr}_{ij} := \mathrm{pr}_{\{i,j\}}$.
$\Delta_X \subset X^k$	The small diagonal $\{(x, \ldots, x) : x \in X\} \subset X^k$.
$\Delta_I, \Delta_{ij} := \Delta_{\{ij\}}$	Given $I \subset [k]$, the cycle $\Delta_I := \{(x_1, \ldots, x_k) : x_i = x_j \ \forall i, j \in I\} \subset X^k$.
ω_k	Given $\omega \in H^0(X, \Omega_X^l)$, we write $\omega_k := \sum_{i=1}^{k} \mathrm{pr}_i^* \omega$.

i_M	The map $A^r \to A^k$ or $\mathbb{C}^r \to \mathbb{C}^k$ given by $$(a_1, \ldots, a_r) \mapsto \Big(\sum_{j=1}^{r} m_{1j} a_j, \ldots, \sum_{j=1}^{r} m_{kj} a_j\Big).$$ where $M = (m_{ij}) \in M_{k\times r}(\mathbb{Z})$ (resp. $M \in M_{k\times r}(\mathbb{C})$).
$A^r_M, A_M := A^1_M$	The image of $i_M : A^r \to A^k$.
$\mathbb{C}^r_M, \mathbb{C}_M := \mathbb{C}^1_M$	The image of $i_M : \mathbb{C}^r \to \mathbb{C}^k$.
$A^{k,0}$	The subvariety $\Big\{(a_1, \ldots, a_k) : \sum_{i=1}^k a_i = 0\Big\} \subset A^k$.
$\varphi_{\mathcal{A}}$	Given a family of abelian varieties $\mathcal{A}/S$, the morphism from S to the moduli stack of abelian g-folds with appropriate polarization type or the k-fold cartesian product of the natural map between $\mathcal{A}$ and the universal family over this stack.
S_λ, $S_{\lambda,\mu}$	First paragraph of (3.1.1).
$\Lambda_l, \Lambda_l(\mathcal{A})$	(3.1.1).
$\Lambda^\lambda_{l'}, \Lambda^\lambda_{l'}(\mathcal{A})$	(3.1.1).
$S_\lambda(B), S_{\lambda,\mu}(D, F)$	(3.3).
$p_\lambda, p_\mu, p_{\lambda,\mu}$	(3.1) and (3.2).
R_{gf}, R_{ab}, R_{st}	(3.4).
$T_A := T_{A,0_A}$	The tangent space of an abelian variety A at the identity.
$\mathscr{T}, G, G', \sigma_\lambda, \sigma_{\lambda,\mu}$	(3.5), (3.6).
$\mathcal{H}, \mathcal{H}', \mathcal{H}^\lambda, \mathcal{H}'^{\lambda,\mu}$	The universal families of hyperplanes and the pullbacks of the universal families of hyperplanes by σ_λ and $\sigma_{\lambda,\mu}$.
$\mathscr{G}$	The Gauss map of some subvariety of an abelian variety.
V_l	The sublattice of (l, l) Hodge classes supported on products of divisors (4.2).
$\mathcal{F}$	The Fourier transform on the cohomology of an abelian variety.

References

[AP92] A. Alzati and G. P. Pirola, *On the holomorphic length of a complex projective variety*, Arch. Math. **59 (4)** (1992), 398–402.

[AP93] ______, *Rational orbits on three-symmetric products of abelian varieties*, Trans. Amer. Math. Soc. **337 (2)** (1993), 965–980.

[Bas12] F. Bastianelli, *On symmetric products of curves*, Trans. Amer. Math. Soc. **364 (5)** (2012), 2493–2519.

[BCFS19] F. Bastianelli, C. Ciliberto, F. Flamini, and P. Supino, *Gonality of curves on general hypersurfaces*, J. Math. Pures Appl. **125** (2019), 94–118.

[BCP14] F. Bastianelli, R. Cortini, and P. De Poi, *The gonality theorem of Noether for hypersurfaces*, J. Alg. Geom. **23** (2014), 313–339.

[BCV95] F. Bardelli, C. Ciliberto, and A. Verra, *Curves of minimal genus on a general abelian variety*, Compos. Math. **96** (1995), 115–147.

[BE78] I. Bernstein and A. L. Edmonds, *The degree and branch set of a branched covering*, Invent. Math. **45** (1978), 213–220.

[Bea83] A. Beauville, *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne.*, Algebraic Geometry, Proceedings of the Japan-France Conference held at Tokyo and Kyoto, 1982, Lecture Notes in Mathematics, vol. 1016, 1983, pp. 238–260.

[BKL76] S. Bloch, A. Kas, and D. Lieberman, *Zero cycles on surfaces with $p_g = 0$*, Compositio Mathematica **33 (2)** (1976), 135–145.

[BPE+17] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, and B. Ullery, *Measures of irrationality for hypersurfaces of large degree*, Compos. Math. **153 (11)** (2017), 2368–2393.

[BS83] S. Bloch and V. Srinivas, *Remarks on correspondences and algebraic cycles*, Amer. J. Math. **105 (5)** (1983), 1235–1253.

[Che19] N. Chen, *Degree of irrationality of very general abelian surfaces*, Algebra Number Theory **13 (9)** (2019), 2191–2198.

[Cil84] C. Ciliberto, *Alcune applicazioni di un classico procedimento di Castelnuovo*, Seminari di geometria, Bologna 1982-1983, 1984, pp. 17–43.

[CMNP19] E. Colombo, O. Martin, J. C. Naranjo, and G. P. Pirola, *Degree of irrationality of a very general abelian variety*, arXiv:1906.11309 (2019), 1–9.

[CS19] N. Chen and D. Stapleton, *Fano hypersurfaces with arbitrarily large degrees of irrationality*, arXiv:1908.02803 (2019), 8 p.

[CvG93] E. Colombo and B. van Geemen, *Note on curves in a Jacobian*, Compos. Math. **88 (3)** (1993), 333–353.

[Dol16] I. V. Dolgachev, *A brief introduction to Enriques surfaces*, Development of Moduli Theory - Kyoto 2013, Adv. Stud. Pure Math., vol. 69, 2016, pp. 1–32.

[EL97] L. Ein and R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), 243–251.

[Enr06] F. Enriques, *Sopra le superficie algebriche di bigenere uno*, Mem. Soc. Ital. delle Scienze **14** (1906), 327–352.

[GH79] P. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. Éc. Norm. Supér. **12 (3)** (1979), 355–452.

[GK19] F. Gounelas and A. Kouvidakis, *Measures of irrationality of the Fano surface of a cubic threefold*, Transactions of the American Mathematical Society **371** (2019), 7111–7133.

[GL91] M. Green and R. Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991), 87–103.

[GQ16] H. W. Gould and J. Quaintance, *On the binomial identities of Frisch and Klamkin*, J. Integer Seq. **19 (7)** (2016), 1–4.

[Har86] R. Hartshorne, *Generalized divisors on Gorenstein curves and a theorem of Noether*, J. Math. Kyoto Univ. **26 (3)** (1986), 375–386.

[HPT18] B. Hassett, A. Pirutka, and Y. Tschinkel, *Stable rationality of quadric surface bundles over surfaces*, Acta Math. **220** (2018), 341–365.

[Huy14] D. Huybrechts, *Curves and cycles on K3 surfaces*, Algebr. Geom. **1** (2014), 69–106.

[Jan94] U. Jannsen, *Motivic sheaves and filtrations on Chow groups*, Motives, Proceedings of Symposia in Pure Mathematics, vol. 55 (1), 1994, pp. 245–302.

[Keu90] J. H. Keum, *Every algebraic Kummer surface is the k3-cover of an Enriques surface*, Nagoya Math. J. **118** (1990), 99–110.

[KT19] M. Kontsevich and Y. Tschinkel, *Specialization of birational types*, Invent. Math. **217** (2019), 415–432.

[Lin16] H.-Y. Lin., *On the chow group of zero-cycles of a generalized Kummer variety*, Adv. Math. **298** (2016), 448–472.

[LP95] A. Lopez and G. P. Pirola, *On the curves through a general point on a smooth surface of* $\mathbb{P}^3$, Math. Z. **219** (1995), 93–106.

[Mar19] O. Martin, *The degree of irrationality of most abelian surfaces is 4*, arXiv:1911.00296 (2019), 1–5.

[Mar20] ———, *On a conjecture of Voisin on the gonality of very general abelian varieties*, Adv. Math. **369** (2020).

[MM83] S. Mori and S. Mukai, *The uniruledness of the moduli space of curves of genus 11*, Algebraic geometry (Tokyo/Kyoto, 1982), Lect. Notes in Math., vol. 1016, 1983, pp. 334–353.

[Mum69] D. Mumford, *Rational equivalence of zero cycles on surfaces*, J. Math. Kyoto Univ. **9** (1969), 195–204.

[Noe83] M. Noether, *Zur Grundlegung der Theorie der algebraischen Raumcurven*, Verl. d. Konig. Akad. d. Wiss., 1883.

[NS19] J. Nicaise and E. Shinder, *The motivic nearby fiber and degeneration of stable rationality*, Invent. Math. **217** (2019), 377–413.

[Pir89] G. P. Pirola, *Curves on generic Kummer varieties*, Duke Math. J. **59** (1989), 701–708.

[Roĭ71] A. A. Roĭtman, *γ-equivalence of zero-dimensional cycles (in Russian)*, Mat. Sb. **86 (128)** (1971), 557–570.

[Roĭ72] ———, *Rational equivalence of zero-cycles (in Russian)*, Mat. Sb. **89 (131)** (1972), 569–585.

[Roĭ80] ———, *The torsion of the group of 0-cycles modulo rational equivalence*, Ann. of Math. **111 (3)** (1980), 553–569.

[RS14] J. Rosen and A. Shnidman, *Néron-Severi groups of product abelian surfaces*, arXiv:1402.2233 **111 (3)** (2014), 1–28.

[Sta17] D. Stapleton, *The degree of irrationality of very general hypersurfaces in some homogeneous spaces*, Ph.D. thesis, Stony Brook University, 2017.

[TY95] H. Tokunaga and H. Yoshihara, *Degree of irrationality of abelian surfaces*, J. Algebra **174** (1995), 1111–1121.

[Voi02] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés, vol. 10, Société Mathématique de France, 2002.

[Voi14] ———, *Chow rings, decomposition of the diagonal, and the topology of families*, Annals of Mathematics Studies, vol. 187, Princeton University Press, 2014.

[Voi18] ———, *Chow ring and the gonality of general abelian varieties*, Ann. H. Lebesgue **1** (2018), 313–332.

[Yan19] R. Yang, *On irrationality of hypersurfaces in* $\mathbb{P}^{n+1}$, Proc. Amer. Math. Soc. **147** (2019), 971–976.

[Yos90] H. Yoshihara, *Double coverings of* $\mathbb{P}^2$, Proc. Japan Acad. **66** (1990), 233–236.

[Yos96] ______, *Degree of irrationality of a product of two elliptic curves*, Proc. Am. Math. Soc. **124 (5)** (1996), 1371–1375.

[Yos98] ______, *A note on the inequality of degrees of irrationalities of algebraic surfaces*, J. Algebra **207** (1998), 272–275.

[Yos00] ______, *Degree of irrationality of hyperelliptic surfaces*, Algebra Colloq. **7 (3)** (2000), 319–328.