THE UNIVERSITY OF CHICAGO

OPTIMAL INTERGENERATIONAL RISK-SHARING WHEN STOCK AND LABOR MARKETS ARE CO-INTEGRATED

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Aan mijn ouders, Robert en Carolina.

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ABSTRACT

A well-established belief in the pension industry is that collective pension funds with mandatory participation can take more stock market risk in comparison to a system with individual retirement accounts, since current risks may be shared with future generations. We setup a continuous time OLG model with labor income risk in the spirit of Benzoni et al. (2007) and show that this idea may be misguided. For the empirical range of parameter values reported by Benzoni et. al., we find that optimal risk-sharing actually implies that collective pension funds should take less stock market risk, not more. If stock and labor markets move together in the long run, it is no longer efficient to shift risk from current to future generations, because their human wealth becomes correlated with current financial shocks. Furthermore, we find that the potential welfare gains from intergenerational risk-sharing are significantly reduced.

CHAPTER 1

INTRODUCTION

In this paper we consider the portfolio choice problem of a collective pension scheme with mandatory participation in an environment in which stock and labor markets are potentially co-integrated. Mandatory participation makes the problem of the pension fund manager distinct from a regular fund managers problem. Since future generations are obliged to participate, the pension fund manager has the option to allocate today's financial market shocks not only to current participants, but also to future participants. Therefore, the fund manager has to decide not only how much risk to take on behalf of current generations, but also on behalf of future generations.

In theory, being able to take risk on behalf of future generations creates an opportunity to improve welfare. The intuition for this is that non-overlapping generations cannot trade with each other in the marketplace. There is a missing market. Hence, potentially benefical trades cannot take place and there is room for welfare improvement, see e.g. Diamond (1977), Merton (1983) and Gordon and Varian (1988), and more recently Smetters (2006), Teulings and De Vries (2006), Ball and Mankiw (2007), Bovenberg et al. (2007), Gollier (2008), Cui et al. (2011) and Bovenberg and Mehlkopf (2014).

Some of these papers suggest that collective pension schemes with mandatory participation could fill the gap of the missing intergenerational market (e.g. Teulings and De Vries (2006), Bovenberg et al. (2007), Gollier (2008) and Cui et al. (2011)). Such pension schemes may be found in multiple countries. Some examples are the Japan Government Pension Investment Fund, the Canada Pension Plan, the Government Pension Fund in Norway, the ATP funds in Denmark and the occupational pension funds in the Netherlands, Switzerland and Iceland [1]

The aforementioned papers argue that collective pension schemes with mandatory par-

1. Novy-Marx and Rauh (2014) describe how risk-sharing may also be relevant for the US public sector pension funds

ticipation should take more equity risk, compared to a situation in which risk cannot be shared with future generations. Doing so would lead to significant welfare gains. This is by now a well-established belief in the pension industry. For example, AON, a world-leading pension advisor, argues, based on this literature, that collective pension schemes could take more risk (AON, 2013).

This conclusion however is based on models that abstract from labor-income risk. In this paper we point out that the long-run dynamics of labor income risk are crucial for the welfare maximizing policy of the collective pension scheme. We setup a continuous time overlapping generations model (OLG) with labor income risk.

Our main goal is to better understand the efficient allocation of risk across generations. Therefore, we keep the model simple in those dimensions that are not crucial for the question at hand. We consider a defined contribution setting similar to the one in Gollier (2008). This means that we ignore the consumption-savings decision.

We will assume that both labor income and returns on the financial market follow stochastic processes that are exogonously given. The advantage of this approach is that it allows us to provide solutions in closed-form and hence the results are easier to understand and digest. The disadvantage, obviously, is that we will be ignoring general equilibrium effects. We should therefore think of the results as relevant for relatively small players in a larger economy (i.e. an individual person or a collective pension fund that is a modest player in the overall financial market).

Working in a partial equilibrium setting is common in this literature. What sets our analysis apart is that we will allow labor income and income from capital to be cointegrated. The idea that labor income and income from capital are cointegrated in a portfolio choice setting, was suggested by Benzoni et al. (2007) (BCG hereafter). The appeal of their model specification is that it allows labor income and market returns to appear independent in the short-run while moving together at long horizons.

While BCG do attempt to provide empirical support for their cointegration assumption, it is insurmountable that the empirical evidence is weak. The effect of cointegration can only be observed over the course of decades, which implies that we only have so many independent observations in the data. Yet, cointegration is very appealing from a theoretical perspective. If labor and capital income were not cointegrated it would imply that the labor and capital share of GDP could potentially go to zero or one in the long-run. As pointed out by Baxter and Jermann (1997), the form of most production functions used in macroeconomic theory imply that the long-run restriction that the factor shares of labor and capital are stationary.

The assumption that labor income and dividend flows are cointegrated can also be found elsewhere in the literature, see e.g. Baxter and Jermann (1997), Menzly et al. (2004), Santos and Veronesi (2006) and Geanakoplos and Zeldes (2010). Earlier studies that investigate the link between aggregate labor income and asset prices empirically include Mayers (1974), Fama and Schwert (1977), Black (1995), Jagannathan and Kocherlakota (1996), and Campbell (1996). In the study of Campbell (1996), a high correlation between human capital and market returns is found. Different from BCG, this correlation however, is not driven by cash-flow movements, but by variation the (common) discount factor.

Co-integration between labor income and dividend income causes the human capital of young investors to become strongly correlated with stock returns, which reduces their appetite to invest in the stock market directly. In contrast to other studies that ignore long-run labor income risk, BCG find that it can even be optimal for young investors to take a short position in stocks, as this provides a hedge against future labor income shocks.

We show that, this also has consequences for the risk-sharing problem of a collective pension scheme. Shifting risk to future generations may be sub-optimal, because their human capital is already correlated with current shocks. As a matter of fact, for all levels of labor-stock market cointegration within the parameter range reported by BCG, it is optimal for the collective pension scheme in our model to take *less* stock market risk than one would take in the absence of intergenerational risk-sharing. Not more.

In the absence of labor income risk, it is optimal that retirement income provided by the fund is less volatile than the funds portfolio return. Current shocks are only partially allocated to the current generation of retirees. When cointegration is strong enough, this reverses. It is optimal that retirement income is more volatile than the funds portfolio return. A policy of 'smoothing' returns, which was optimal in absence of labor income risk, is replaced with a policy of - so to say - 'amplification'. By 'amplifying' the volatility of retirement income, the fund facilitates that human capital risk of future generations is shared by current generations.

In earlier work, Mehlkopf (2011) also considers the problem of a pension fund with intergenerational risk-sharing in a world in which labor and stock markets are co-integrated. Mehlkopf determines the optimal recovery period, an important policy variable for collective pension schemes in the Netherlands, using a numerical approach[2].

This paper contributes to the existing literature in three ways. Firstly, in a continuous time setting, we provide a closed-form solution to the intergenerational risk-sharing problem of a pension scheme when labor income and capital income are cointegrated. As a matter of fact, a continuous time version of the discrete time model in Gollier (2008) will feature as a special case within our setting.

A second contribution is that we consider the model and parameter risk policy makers face. Allowing for long-run correlation between labor and capital income can completely reverse the optimal direction of intergenerational risk transfers between generations within a collective pension scheme. Since long-run correlations are hard to reliably estimate, policy makers face significant parameter risk. We perform a robustness check and show that in our model a minimax pension fund manager would decide not to shift risk to future generations.

A third contribution is that we relax the assumption made in BCG that all volatility in stock market returns is a reflection of long-term risk. We extend the model to allow for a short-term risk component in dividend volatility, such that not all volatility in stock prices is

2. See, in particular, Chapter 4. Unfortunately it turns out that the stock price process used in the analysis in this study was inconsistent with the pricing kernel, which means that the (numerical) results in the study are biased.

associated with structural changes in expected dividends. In line with van Binsbergen et al. (2012), who find that the risk-premium on short-run dividend risk is relatively higher compared to long-run dividend risk, we allow the risk-premium on short- and long-run dividend risk to differ.

We show how this model extension bridges the modeling framework of earlier studies in a specific way: the model of BCG corresponds to the extreme case where all stock price movements are associated with long-term risk and have a permanent impact on the real economy (future dividends and future labor income), while the models that abstract from labor income risk, represent the other extreme case where all stock price movements are associated with short-term risk and the risk-premium is mainly a compensation for short-term risk. This novelty in our modeling setup allows us to perform a robustness check on the impact of long-run labor income risk on the risk appetite of young and future participants. Our results show that the main qualitative conclusions in this paper are unaffected by the introduction of a short term risk component in the model.

There are two other papers worth mentioning that explore intergenerational risk-sharing in a setting in which stock and labor markets are subject to a common risk factor: van Hemert (2005) and Bohn (2009). In van Hemert (2005), labor income and capital returns are allowed to be correlated, but the Markovian structure implies that labor income is not risky in the long run. Bohn (2009) estimates 30-year correlations between labor productivity and capital returns and finds a positive correlation between 30% and 60%. In line with our findings, Bohn (2009) finds that workers bear systematically more risk than retirees. Optimal policies would therefore shift risk from future generations to retirees.

The structure of the remainder is as follows. In section 2 we first set out the general setting, such as demographic assumptions. After this we continue in our partial equilibrium setting and introduce the stochastic processes that describe labor income and the financial market. In section 3 we derive the optimal portfolio allocation for individual generations in the absence of intergenerational risk-sharing. After that, we solve the problem of a pension fund manager who can also allocate risk to future generations in section 4 and compare this to the results from the previous section. After that, we discuss the role of model- and parameter uncertainty for policy making (section 6) and provide an extension of the model in which we allow stock market risk to have both a short-run and a long-term component (section 7). Section 8 concludes.

CHAPTER 2
THE MODEL

2.1 Overview

We consider a partial equilibrium model of overlapping generations in continuous time. There is a continuum of generations with each generation being of the same size, which is normalized to one. Once a generation is born it supplies a constant and fixed amount of labor. A fixed portion of labor income is saved for retirement. All generation alive at a specific moment in time receive the same level of labor income, which is stochastic. Retirement savings can be invested either in a risky asset, 'the stock market', or in an asset that gives a safe rate of return. After n years, a generation retires and receives a one-time retirement benefit which is equal to the total value of their retirement savings. In our illustrations we will set $n = 40$.

We will be comparing two alternative institutional arrangements. First we consider the situation in which each generation has its own individual retirement account. The pension scheme manager solves the portfolio optimization problem for a single generation. We will refer to this as 'autarky'. Then, we will compare this to the case in which all generations contribute their retirement savings compulsory to a collective fund. The fund manager then solves the collective portfolio problem for all generations, jointly. We will refer to this as the solution with 'intergenerational risk-sharing'.

2.2 Financial market

We will be modeling the long-run relation between labor income risk and stock market risk by letting the labor-to-dividend income ratio follow a mean-reverting process. We will first introduce the dividend process and other financial market features. Subsequently we will introduce the labor income process and derive the sensitivity of human capital to stock market risk.

The financial market features two traded assets: a risk-less asset and a risky stock index. The risk-less asset earns a fixed rate of return, r. The stock pays a continuous flow of dividends, the level of which follows a geometric Brownian motion with drift:

$$\frac{dD(t)}{D(t)} = g_D dt + \sigma dz_D, \tag{2.1}$$

where g_d is the average growth rate of dividends. The market value of future dividend cash-flows is captured by an exogonously given stochastic discount factor:

$$\frac{dM(t)}{M(t)} = -rdt + \phi_D dz_D \tag{2.2}$$

where ϕ_D represents the unit price of dividend risk. The stock price, the price of a claim to the stream of dividends between now and infinity, can now be found as the expected discounted sum of dividends: $P(t) = \int_t^\infty \mathbb{E}_t \left[\frac{M(s)}{M(t)} D(s) \right] ds$. Working out the expectation gives:

$$P(t) = \frac{D(t)}{r - \phi_D \sigma - g_D} \tag{2.3}$$

Let $S(t)$ denote the value of a mutual fund that holds only stocks and reinvests any dividends received back into the stock index. The instantaneous return on the stocks-only fund can then be written as:

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma dz_D \tag{2.4}$$

where it follows from the definition of the stochastic discount factor that $\mu_S = r - \sigma\phi_D$. In our illustrations and numerical examples we will set these parameters as follows[1]: $r = 0.02$, $\sigma = 0.136$ and the risk premium $-\sigma\phi_D = 0.039$.

1. These figures are chosen in accordance with the return distribution used by Gollier (2008), for comparability.

2.3 Labor income

We model the cointegration of labor income and the stock market by letting the dividend-labor income ratio follow a mean-reverting process.

Define:

$$\omega(t) \equiv w(t) - d(t) - \mu_{wd} \tag{2.5}$$

where $w(t)$ is log labor income (being saved for retirement), $d(t)$ is log dividend income and μ_{wd} is the long-run mean log labor income to dividend ratio. So, $\omega(t)$ represents the de-meaned log labor income to dividend ratio. It is then assumed that $\omega(t)$ follows a standard mean reverting Ornstein-Uhlenbeck process with zero mean:

$$d\omega(t) = -\kappa\omega(t)dt + \nu_W dz_W(t) - \nu_D dz_D(t) \tag{2.6}$$

Here $dz_W(t)$ is another standard Brownian motion that captures the part of labor income risk that is uncorrelated to stock market risk. We add this term here to highlight that the setup can easily allow for (aggregate) labor income shocks that are uncorrelated with dividend risk. This shock would primarily impact the optimal consumption-savings decision, but have no impact on the optimal portfolio choice problem that is central in our analysis. It would imply though that we would have to solve the problem numerically in the setting with intergenerational risk-sharing[2]. Since our focus is on the optimal allocation of financial market risk over different generations we will set this shock to zero. Combining 2.1, 2.5 and 2.6, and setting $\nu_W = 0$ we find that the process for log labor income is:

$$dw(t) = \left(-\kappa\omega(t) + g_D + \mu_{wd} - \frac{1}{2}\sigma^2\right) dt + (\sigma - \nu_D)dz_D(t) \tag{2.7}$$

2. The individual generations' problem will be a terminal wealth problem. This terminal wealth problem can be solved in closed-form, even if it features non-tradable labor income risk. The fund managers' problem with intergenerational risk-sharing, however, is an infinite horizon problem with intermediate consumption (retirement benefit distribution). When the asset menu is incomplete this problem can only be solved numerically (see Liu (2007)).

We will assume that labor income is contemporaneously uncorrelated with stock market risk by setting $\nu_D = \sigma$. Notice that in the long-run labor income will still be correlated through the mean-reversion in $\omega(t)$. This can be seen clearly when we solve for labor income at time t conditional on time $s < t$ information:

$$\begin{aligned} W(t) = W(s)\exp\Big\{ & -\kappa B(t-s)\omega(s) + (g_D + \mu_{wd} - \frac{1}{2}\nu_D^2)(t-s) \\ & + \kappa\nu_D \int_s^t B(t-v)dz_D(v)\Big\} \end{aligned} \tag{2.8}$$

where $B(x) = \frac{1}{\kappa}\left(1 - e^{-\kappa(x)}\right)$. In our numerical illustrations, we will set $(g_d + \mu_{wd} - \frac{1}{2}\nu_D^2) = 0$, such that there is no expected real income growth in case $\kappa = 0$. By doing so, $\kappa = 0$ corresponds to the setting in Gollier (2008) that features risk-free labor income without income growth.

2.4 Human capital process

Now we have specified the labor income process, let us consider what the present value of labor income looks like. It will be particularly relevant to know what the correlation of human capital with stock market risk is.

In appendix B we show that the present value at time t of a claim to labor income at future date τ , denoted by $PV_W(\omega, t, \tau)$, is:

$$\begin{aligned} PV_W(\omega, t, \tau) &\equiv \mathbb{E}_t\left[\frac{M(\tau)}{M(t)}W(\tau)\right] \\ &= W(t)\exp\left\{A(\tau - t) - \kappa B(\tau - t)\omega(t)\right\} \end{aligned} \tag{2.9}$$

where $A(x)$ is a function of the horizon only (see (B6)) and $B(x)$ is as specified above. Hence, the present value of human capital for generation T is:

$$H(\omega, t, T) = W(t)\int_{\max[t, T-n]}^{T} \exp\left\{A(\tau) - \kappa B(\tau)\omega(t)\right\} d\tau \tag{2.10}$$

From 2.10 we can find the exposure of human capital to dz_D:

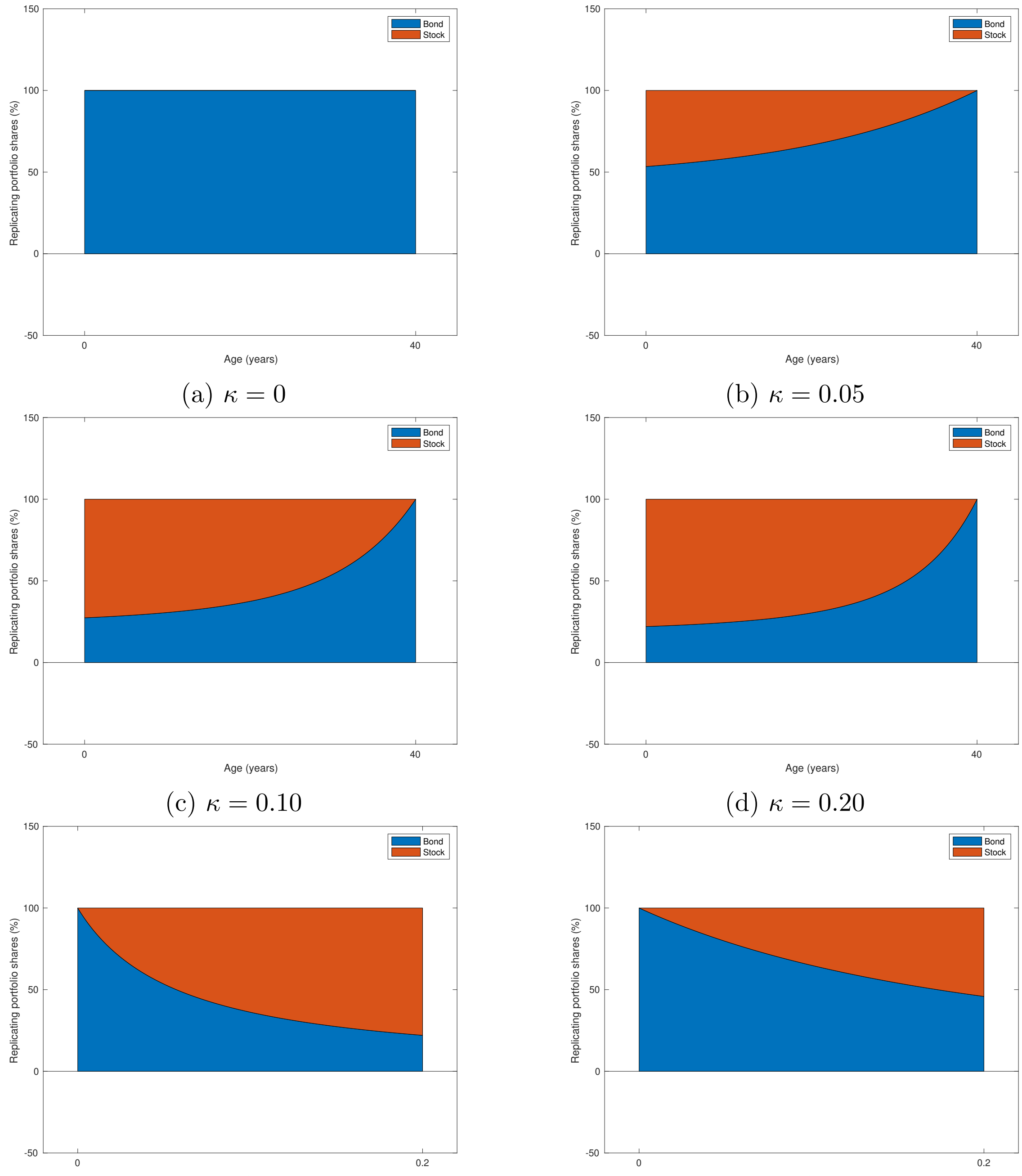
$$\frac{dH(\omega,t,T)}{H(\omega,t,T)} = (...)dt + \sigma_h(\omega,t,T)dz_D \tag{2.11}$$

with

$$\sigma_h(\omega,t,T) \equiv \bar{B}(\omega,t,T) = \frac{\int_{\max[t,T-n]}^{T} PV_W(\omega,t,\tau)B(\tau-t)d\tau}{\int_{\max[t,T-n]}^{T} PV_W(\omega,t,\tau)d\tau}\,\kappa\,\nu_D \tag{2.12}$$

Above formula captures the exposure of human capital to dividend shocks. The main determinants of this exposure are the strength of cointegration between dividends and labor income, measured by κ, and the remaining number of years in which labor income is earned. Figure 2.1 illustrates what the exposure looks like for different values of κ and how it changes over the life-cycle. In the figure we show the portfolio of stocks and bonds that perfectly replicates human capital for different levels of age and different values of κ.

When κ is zero, there is no co-integration. Human capital is risk-free and hence we see in the figure that the replicating portfolio is made up of bonds only for all ages. Human capital is ‘bond-like’. When κ is non-zero, human capital is exposed to dividend shocks. In this case, the replicating portfolio is a mixture of stocks and bonds. Only for the oldest age groups, human capital is still bond-like, since we assumed that dividend shocks do not have a direct impact on labor income. The younger the individual, the higher the exposure to dividend shocks becomes. So, when κ increases or when age decreases human capital becomes more ’stock-like’.



(a) $\kappa = 0$ (b) $\kappa = 0.05$ (c) $\kappa = 0.10$ (d) $\kappa = 0.20$ (e) Current age = 0 (f) Current age = 30

Figure 2.1: **Exposure of human capital to dividend shocks** The figure shows the portfolio of stocks and bonds that exactly replicates the value of human capital. Panel (a)-(d) show the portfolio weights of the replicating portfolio, as a function of age, for different levels of mean-reversion in the labor income - dividend ratio (κ). Panel (e) and (f) show the replicating portfolio as a function of κ, for an individual of age 0 and age 30 (a remaining life-time of 40 and 10 respectively).

CHAPTER 3

OPTIMAL RISK-TAKING IN AUTARKY

Having specified the financial market and the properties of human capital, we are now ready to consider the optimization problem for the individual generation in autarky.

3.1 Preferences

The preferences of each generation are represented by a CRRA utility function. Since we consider a defined contribution setting with a fixed rate of saving, utility during the contribution phase is irrelevant for the optimal investment problem. Therefore, we only consider utility defined over the retirement benefit $b(T)$ here. So, expected utility, $U(t,T)$, at time t of the generation that retires at date T is:

$$U(t,T) = \mathbb{E}_t \left[\frac{b(T)^{1-\gamma}}{1-\gamma} \right] \tag{3.1}$$

3.2 Wealth

Each generation has two sources of wealth: human capital and financial wealth, denoted $F(t,T)$. We will assume that each generation starts with zero financial wealth, $F(t,T) = 0$ for $t < (T-n)$. The financial wealth process follows:

$$\frac{dF(t,T)}{F(t,T)} = \big(r - \hat{x}_S \sigma \phi_D\big)dt + \hat{x}_S \sigma dz_D + \frac{W(t)}{F(t,T)}dt \qquad \forall\ (T-n) \leq t < T \tag{3.2}$$

where $\hat{x}_S(t,T)$ denotes the share of financial wealth invested in the stock index. The first two terms on the right-hand side reflect (the mean and variance of) the return on investment, while the third term reflects the inflow of new contributions. Note that, for notational simplicity, from this point onward, $W(t)$, will denote the wage income that is saved for retirement (instead of total wage income).

Likewise, when we speak of human capital, this refers only to the part of future income that will be saved for retirement. We will refer to the sum of human capital and financial wealth as total wealth, $V(t,T)$:

$$V(t,T) = F(t,T) + H(\omega,t,T) \tag{3.3}$$

The stochastic process for total wealth is given by:

$$\frac{dV(t,T)}{V(t,T)} = (...)dt + x_S(t,T)\sigma dz_D + h(t,T)\sigma_h(\omega,t,T)dz_D \qquad \forall\, t < T \tag{3.4}$$

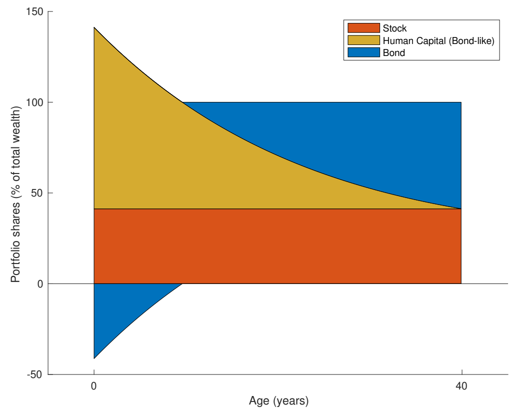
where we suppress the drift term for simplicity. $x_S(t,T)$ denotes the individuals investment in the stock index expressed as a percentage of total wealth and $h(t,T)$ is human capital as a percentage of total wealth. In autarky, we assume that individual generations cannot buy stocks before they enter the labor force. So $x_S(t,T) = 0\ \forall t < (T-n)$.

Once a generation enters, we will not impose a borrowing constraint. So, individual generations may borrow and take leverage. In absence of a borrowing constraint, the inefficiency in autarky comes solely from the fact that future generations are unable to get exposure to shocks that occur before their labor market entry.

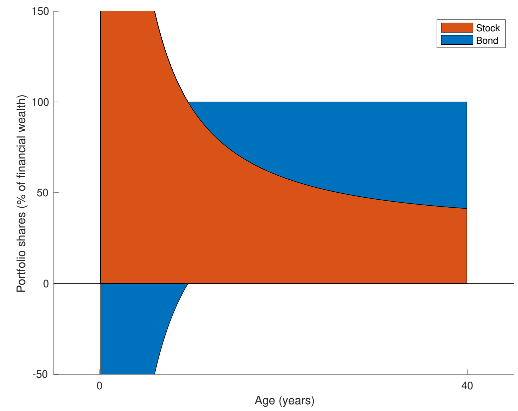
3.3 The individual generation's problem

For generations that are currently alive ($t > (T-n)$) the problem is:

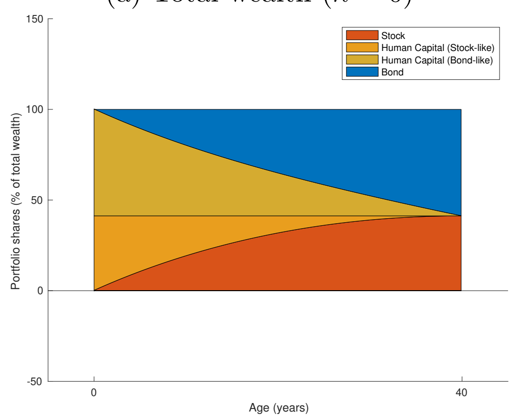
$$\begin{aligned}
&\max_{b(T)} \quad \mathbb{E}_t\left[\frac{b(T)^{1-\gamma}}{1-\gamma}\right] \\
&\text{s.t.} \quad \frac{dV(t,T)}{V(t,T)} = (...)dt + x_S(t,T)\sigma dz_D + h(t,T)\sigma_h(\omega,t,T)dz_D \qquad \forall\, t < T \\
&\qquad\quad b(T) = V(T,T)
\end{aligned} \tag{3.5}$$



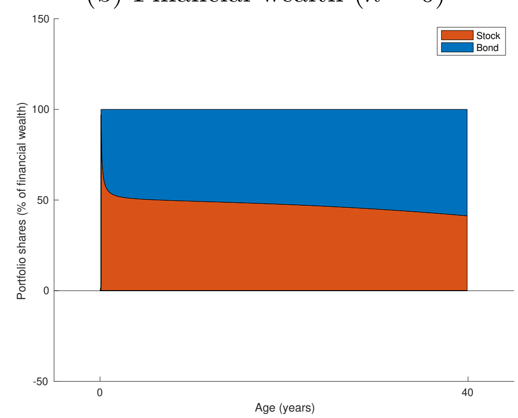
(a) Total wealth ($\kappa = 0$)



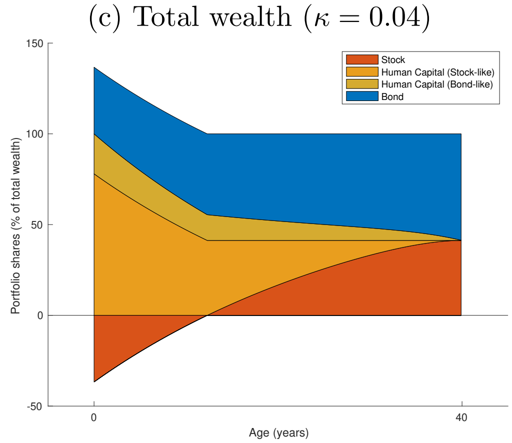
(b) Financial wealth ($\kappa = 0$)



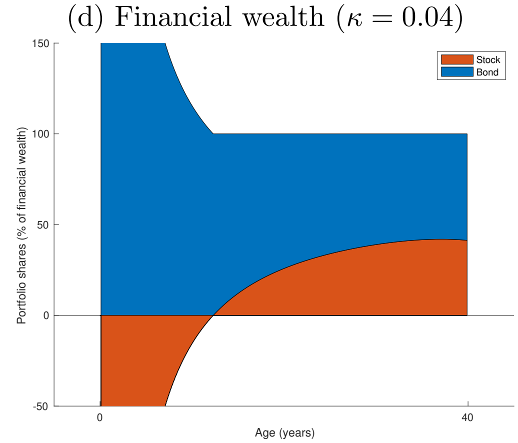
(c) Total wealth ($\kappa = 0.04$)



(d) Financial wealth ($\kappa = 0.04$)



(e) Total wealth ($\kappa = 0.2$)



(f) Financial wealth ($\kappa = 0.2$)

Figure 3.1: **Optimal portfolio allocation over the life-cycle** The figure illustrates the composition of total wealth and financial wealth over the life-cycle for different values of κ. The left panels show the optimal composition of total wealth. The right panels show the composition of optimal financial wealth. The graphs assume that $\gamma = 5$ and are based on the median scenario. To highlight the fact that the individual optimally chooses a constant relative exposure to dividend shocks, human capital is broken down into its 'stock-like' and 'bond-like' component.

Since there is only one source of risk, this problem is equivalent to the well-known Merton (1969) problem of a terminal wealth investor. Hence, the solution to this (unconstrained) portfolio choice problem is well known. The law of motion of the optimal total wealth process is (see appendix C for a derivation):

$$\frac{dV^*(t,T)}{V^*(t,T)} = \left(r + \frac{\phi^2}{\gamma}\right) dt - \frac{1}{\gamma}\phi_D dz_D, \tag{3.6}$$

Setting the optimal wealth process (3.6) equal to the actual wealth process in (3.4) we find that optimal investment in the stock market (as a percentage of total wealth) is given by:

$$x_S^*(t,T) = -\frac{1}{\gamma}\frac{\phi_D}{\sigma} - h(t,T)\frac{\sigma_h(\omega,t,T)}{\sigma} \tag{3.7}$$

The first term on the right hand side is the standard speculative demand for the risky asset that we know from Merton's problem. The second term is a term that compensates for the fact that labor income also provides exposure to dz_D. Since $\sigma_h(t,T)$ is a weighted average of strictly positive terms, the hedging term in (3.7) negatively influences the demand for stocks. So, the consequence of co-integration between labor income and dividend income, is that the optimal investment in the stock market is lower than in a setting without co-integration.

The magnitude of the impact of co-integration is driven by the level of κ, the strength of mean-reversion in the dividend-labor-income ratio. BCG argue that empirical estimates of κ are imprecise due to the limited availability of long-horizon data. Depending on the different sample periods they consider, they find levels of κ ranging from 0.05 to 0.2. They use $\kappa = 0.15$ as their baseline parameter estimate. In figure 3.1 we show the optimal equity exposure through the life-cycle for different values of κ. We show the limiting case where $\kappa = 0$ and the two boundaries of the range reported by BCG: $\kappa = 0.05$ and $\kappa = 0.2$.

In the limiting case where $\kappa = 0$, human capital is not correlated with dividend shocks ($\sigma_h(\omega,t,T) = 0$) as in Gollier (2008), Teulings and De Vries (2006) and Cui et al. (2011). Labor income is not exposed to dividend risk and hence we see the classic result that the optimal allocation to stocks is a fixed portion of total wealth (Merton (1969)). When we consider financial wealth only (right-hand panel), we see the standard glide path pattern (see i.e. Bodie et al. (1992)). In the early stage of the life-cycle, when financial wealth is still a small fraction of total wealth the individual optimally borrows to invest in the stock market.

When κ is non-zero, this result however can reverse. As κ is increased, the exposure of human capital to the dividend shock increases. For the range of κ reported by BCG, the life-cycle pattern is such that the individual actually goes short in the stock market during the early stage of life. By doing so, the individual hedges herself against the negative impact of shocks on her future income. For the parameter values we use in our illustrations, the optimal exposure to stocks in the beginning of life turns negative for values of κ above ≈ 0.04.

To highlight that the underlying optimal exposure to the dividend shock is still independent of age, we broke down human capital into its replicating portfolio: the bond-like part and the stock-like part. In panel 3.1c we see how the individual adjusts her stock-exposure to make sure that the stock exposure and her stock-like human capital exactly add up to a fixed share of her total wealth throughout the life-cycle. For the case where κ is larger than 0.04 this is also true. However, in this case, the human capital of young individuals is so stock-like that they have to go short in the stock to keep the overall exposure to dividend shocks at its optimal level (see panel 3.1e). The short position in stocks is such that it exactly offsets the overexposure to dividend shocks coming from their human capital.

Notice that the discontinuity in the slope of the human capital share is an artifact of the stock exposure going from negative to positive. Plotted in isolation, the slope of the human capital share would be perfectly smooth.

3.4 The certainty equivalent retirement benefit

The optimal retirement benefit is equal to terminal wealth, which, given the information at the time of labor market entry, is (see (C3)):

$$b^*(T) = H(T-n,T)\exp\left\{rn + \left(\frac{\phi_D^2}{\gamma} - \frac{1}{2}\frac{\phi_D^2}{\gamma^2}\right)n - \frac{\phi_D}{\gamma}\int_{T-n}^{T} dz_D(u)\right\} \tag{3.8}$$

When considering welfare implications, it will be useful to translate utility units into certainty equivalent units. Define the certainty equivalent at time t, $\text{CE}(t,T)$, to be the certain retirement benefit at time T that would yield the same expected utility as the stochastic benefit $b(T)$:

$$CE(t,T) \equiv \mathbb{E}_t\left[b(T)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \tag{3.9}$$

Then, in autarky, the certainty equivalent for each generation is given by:

$$\text{CE}(t,T) = \begin{cases} V(t,T)\exp\left\{\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)(T-t)\right\} & t \geq T-n \\ \mathbb{E}_t\left[H(T-n,T)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\exp\left\{\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)n\right\} & t < T-n \end{cases} \tag{3.10}$$

In the special case where $\kappa = 0$ the certainty equivalent simplifies to:

$$\text{CE}_{\kappa=0}(t,T) = V(t,T)\exp\left\{r(T-t) + \left(\frac{1}{2}\frac{\phi_D^2}{\gamma}\right)min[T-t,n]\right\} \tag{3.11}$$

The term $\frac{1}{2}\frac{\phi_D^2}{\gamma}$ captures the benefit, per unit of time, of being able to invest in the risky asset. For each generation, this is only possible during a maximum of n years. This special case helps to highlight where the inefficiency in autarky comes from: individual generations are not able to optimize their exposure to traded risk before they enter the model. If they were able to optimize their exposure also before labor market entry, the $min[T-t,n]$ term would become $(T-t)$.

If κ is zero, the actual exposure of future generations to dividend risk is zero, while the optimal exposure is $-\frac{1}{\gamma}\frac{\phi_D}{\sigma}$. If, instead, labor income is co-integrated and $\kappa > 0$, the optimal exposure is unchanged, but the actual exposure is no longer equal to zero. Future labor income now also provides exposure to shocks. It is therefore to be expected that the potential welfare gain will be smaller in this case. We will see that this is indeed the case in the next section where we consider the setting in which the pension fund manager can allocate risk to future generations.

CHAPTER 4

RISK-SHARING IN A COLLECTIVE PENSION FUND

We will now consider a setting in which there is a single collective pension scheme with mandatory participation for all generations. We will assume that the contribution level is unchanged. The pension scheme manager however is free to set both the collective asset allocation and the benefit distribution policy. Consequently, the manager can choose the collective exposure to risk and how this risk is internally allocated to the different generations.

In such a setting, the collective fund can act as a vehicle to transfer today's risk to future generations. This has the potential to create welfare gains. The pension scheme manager can alter future generations' exposure to contemporaneous shocks, by making the benefits of future generations contingent on today's returns. We will here derive what the welfare maximizing investment strategy and benefit distribution policy of the pension scheme are and determine what the welfare gain would be compared to the setting without intergenerational risk-sharing.

4.1 Social preferences

Let the social welfare function be a simple weighted sum of all individual generations utilities:

$$U(t) = \mathbb{E}_t \left[\int_t^{\infty} \beta(T) \frac{\tilde{b}(T)^{1-\gamma}}{1-\gamma} dT \right] \tag{4.1}$$

where $\beta(T)$ is the welfare weight the manager assigns to generation T and $\tilde{b}(T)$ is the retirement benefit the manager provides to generation T. We will be more specific about these weights in a moment. We will assume that the collective scheme is not borrowing constrained. Again, not imposing a constraint has the benefit that the optimal solution can be found analytically.

4.2 Collective wealth

In the collective setting there is no individual retirement wealth. Retirement savings are contributed to the collective fund. The value of the fund at time t is $\tilde{F}(t)$. The collective funds' wealth is subject to the following law of motion:

$$\frac{d\tilde{F}(t)}{\tilde{F}(t)} = \big(r - \tilde{\hat{x}}_S(t)\sigma\phi_D\big)dt + \tilde{\hat{x}}_S(t)\sigma dz_D + \frac{nW(t) - \tilde{b}(t)}{\tilde{F}(t)}dt \tag{4.2}$$

where $\tilde{\hat{x}}_S(t)$ is the share of the funds' wealth invested in stocks. The first two terms on the right-hand side reflect (the mean and variance of) the return on the funds investments, while the third term reflects the in- and outflow of the collective fund from new contributions and current retirement benefits. There are n generations paying a $W(t)$ into the fund and there is a continuous flow of generations retiring, that receive a benefit $\tilde{b}(t)$.

When optimizing social welfare, the fund manager will have to take into account future labor income. Aggregate human capital is the sum of human capital over all current and future generations. So,

$$\tilde{H}(\omega(t), t) \equiv \int_t^\infty H(\omega, t, T)dT = n\int_t^\infty PV_W(\omega, t, \tau)d\tau \tag{4.3}$$

The law of motion for total human capital can be written as:

$$\frac{d\tilde{H}(\omega(t), t)}{\tilde{H}(\omega(t), t)} = (...)dt + \tilde{\sigma}_h(\omega, t)dz_D \tag{4.4}$$

where we suppress the drift term for simplicity and

$$\tilde{\sigma}_h(\omega, t) \equiv \tilde{B}(\omega, t) = \frac{\int_t^\infty PV_W(\omega, t, \tau)B(\tau - t)d\tau}{\int_t^\infty PV_W(\omega, t, \tau)d\tau}\, \kappa\, \nu_D \tag{4.5}$$

Aggregate total wealth is the sum of aggregate financial wealth and aggregate human capital:

$$\tilde{V}(t) = \tilde{F}(t) + \tilde{H}(\omega(t), t) \tag{4.6}$$

The law of motion for aggregate total wealth can be written as:

$$\begin{aligned}\frac{d\tilde{V}(t)}{\tilde{V}(t)} =& \big(r - \big(\tilde{x}_S(t)\sigma + \tilde{h}(t)\tilde{\sigma}_h(\omega, t)\big)\phi_D\big)dt \\ &+ \big(\tilde{x}_S(t)\sigma + \tilde{h}(t)\tilde{\sigma}_h(\omega, t)\big)dz_D - \frac{\tilde{b}(T)}{\tilde{V}(t)}dt\end{aligned} \tag{4.7}$$

where $\tilde{x}_S(t)$ is the share of aggregate total wealth invested in the stock index and $\tilde{h}(t)$ aggregate human capital as a share of aggregate total wealth.

4.3 The fund managers problem

In order to maximize social welfare, the fund manager needs to solve:

$$\max_{\tilde{b}(T)} \; \mathbb{E}_t \left[\int_t^\infty \beta(T) \frac{\tilde{b}(T)^{1-\gamma}}{1-\gamma} dT \right]$$

$$\begin{aligned}\text{s.t.} \quad \frac{d\tilde{V}(t)}{\tilde{V}(t)} =& \big(r - \big(\tilde{x}_S(t)\sigma + \tilde{h}(t)\tilde{\sigma}_h(\omega, t)\big)\phi_D\big)dt - \frac{\tilde{b}(T)}{\tilde{V}(t)}dt + \big(\tilde{x}_S(t)\sigma + \tilde{h}(t)\tilde{\sigma}_h(\omega, t)\big)dz_D \\ 0 =& \lim_{\tau\to\infty} \mathbb{E}_t \left[\tilde{V}(\tau)\frac{M(\tau)}{M(t)} \right]\end{aligned} \tag{4.8}$$

Like the individual generations' problem, the collective problem is a variation of the well-known Merton problem, but now with intermediate consumption and an infinite horizon. The process for optimal total wealth in this case is given by (see appendix C for a derivation):

$$\frac{d\tilde{V}^*(t)}{\tilde{V}^*(t)} = \left(r + \frac{\phi_D^2}{\gamma} - \frac{\tilde{b}^*(t)}{\tilde{V}^*(t)} \right) dt - \frac{\phi_D}{\gamma} dz_D \tag{4.9}$$

Setting the law of motion of optimal total wealth (4.9) equal to the law of motion of actual total wealth (4.7) and solving for $\tilde{x}_S(t)$ gives:

$$\tilde{x}_S^*(t) = -\frac{1}{\gamma}\frac{\phi_D}{\sigma} - \tilde{h}(t)\frac{\tilde{\sigma}_h(\omega, t)}{\sigma} \tag{4.10}$$

The optimal fraction of total wealth invested in the stock index looks similar to the solution of the autarky problem. The important difference being that the pension fund manager now takes into account the human capital of future generations. Not also that the optimal stock exposure does not depend on our choice of the welfare weights.

CHAPTER 5

COMPARING RISK-SHARING AND AUTARKY

5.1 Stock market exposure at the aggregate level

We can now compare aggregate investment in the stock market in the autarky case with intergenerational risk-sharing in the collective fund. For a given initial distribution of financial wealth, we compare the aggregate of the optimal individual policies to the optimal policy in the collective setting. Denote the optimal value of aggregate stock holdings in autarky by $V_S^*(t)$, so:

$$\begin{aligned} V_S^*(t) &\equiv \int_t^{t+n} x_S^*(t,T)V(t,T)dT \\ &= -\frac{1}{\gamma}\frac{\phi_D}{\sigma}\int_t^{t+n} F(t,T)dT - \int_t^{t+n} H(\omega(t),t,T)\,\frac{\frac{1}{\gamma}\phi_D + \sigma_h(\omega,t,T)}{\sigma}dT \end{aligned} \tag{5.1}$$

Now, given that $\tilde{F}(t) = \int_t^{t+n} F(t,T)dT$, the optimal stock holdings in the collective setting with would be:

$$\begin{aligned} \tilde{V}_S^*(t) &\equiv \tilde{x}_S^*(t)\tilde{V}(t) \\ &= -\frac{1}{\gamma}\frac{\phi_D}{\sigma}\int_t^{t+n} F(t,T) - \int_t^{\infty} H(\omega(t),t,T)\,\frac{\frac{1}{\gamma}\phi_D + \sigma_h(\omega,t,T)}{\sigma}dT \end{aligned} \tag{5.2}$$

The difference between the two is:

$$\tilde{V}_S^*(t) - V_S^*(t) = \int_{t+n}^{\infty} H(\omega(t),t,T)\,\frac{-\frac{1}{\gamma}\phi_D - \sigma_h(\omega,t,T)}{\sigma}dT \tag{5.3}$$

Optimal stock investment in the collective setting is equal to aggregate optimal stock investment in autarky plus the difference between the optimal risk exposure of future generations and the actual risk-exposure embedded in their human capital.

Figure 5.1 shows how aggregate investment in the stock differs between autarky and the collective setting for different values of κ ranging from 0 to 0.2 . The figure assumes that aggregate financial wealth is equal to median aggregate financial wealth in autarky. In this case aggregate human capital and aggregate financial wealth each make up approximately 50 percent of total wealth. Note that figure shows aggregate investment in the stock market as a fraction of aggregate financial wealth.

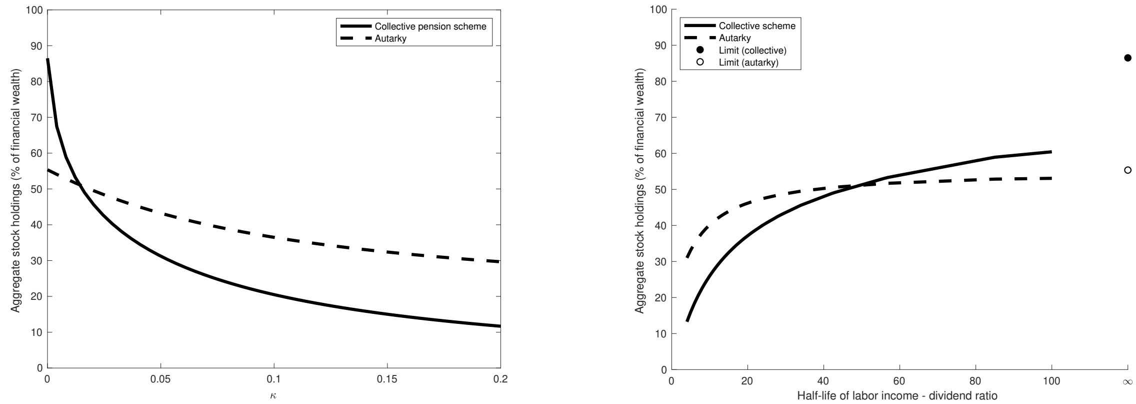
When there is no cointegration, $\kappa = 0$ and as a consequence $\sigma_h(\omega, t, T) = 0$. Since human capital is risk-less, future generations are not exposed to shocks in autarky. The optimal exposure to shocks is positive, hence the fund manager optimally takes additional risk on behalf of future generations. If $\kappa > 0$ however, human capital is risky. The higher κ the higher the exposure of future generations to current shocks through their human capital. The higher κ, the less stock market risk the manager would like to take on their behalf.

For our baseline case, in which γ is set to 5, we find that if κ is higher than 0.018, the exposure of future generations' human capital is so high that in the optimum the fund manager reduces risk compared to situation in autarky. For these values of κ the common wisdom, that the collective pension scheme with mandatory participation optimally takes more risk, reverses. The collective scheme should take less risk. The fund manager effectively replaces a part of the stock portfolio of current generations with a share in the stock-like human capital of future generations. By doing so, current generation still have the optimal exposure to current shocks, but the exposure of future generations is reduced.

Panel (b) shows the same information as panel (a) but now the x-axis does not represent values of κ, but the half-time of shocks to the log labor income to dividend ratio. The graph shows that, given our baseline parameters, the collective scheme would take less risk than the individual generations in autarky if the halftime is below approximately 50 years.

5.2 Stock market exposure at the individual level

Next, we will have a look at the implied exposure to stocks of individual participants in the collective scheme and compare this to the autarky setting. We determine the implied



(a) As function of κ (b) As function of half-time

Figure 5.1: **Optimal aggregate stock market exposure** Panel (a) shows the optimal share of aggregate financial wealth invested in stocks for different levels of mean-reversion in the labor income to dividend ratio, both in autarky and in the setting with a collective pension fund with intergenerational risk-sharing. Panel (b) shows the same information, but now as a function of the half-time of the labor income to dividend ratio. The figure assumes wealth is at the median autarky level, where aggregate human capital is approximately 50% of total wealth and $\omega(t) = 0$

exposure to stocks of an individual by deriving the portfolio that replicates the market consistent value of the retirement benefit.

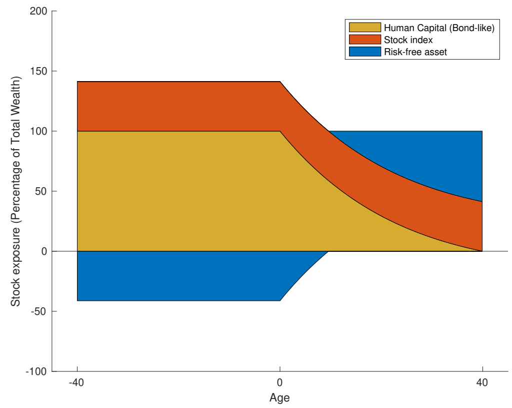
The optimal retirement benefit to generation T can be written as (see appendix C for details):

$$\tilde{b}^*(T) = \tilde{b}^*(0)\beta(T)^{\frac{1}{\gamma}} \exp\left\{\frac{1}{\gamma}\left(r + \frac{1}{2}\phi_D^2\right)T - \frac{1}{\gamma}\phi_D \int_0^T dz_D(u)\right\} \tag{5.4}$$

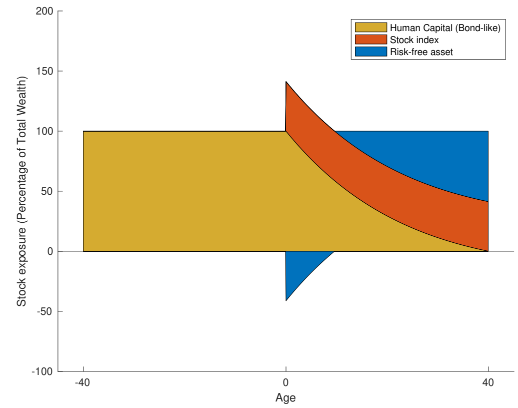
where we normalized $\beta(0)$ to one.

We determine a replicating portfolio of stocks, bonds and human capital that perfectly tracks the market consistent value of this individual benefit. The fact that both human capital and stocks provide exposure to the dividend shock means that we have a degree of freedom in picking the stock and human capital share in the replicating portfolio. We decide to set the human capital share in the collective setting equal to the human capital share in autarky for comparability[1].

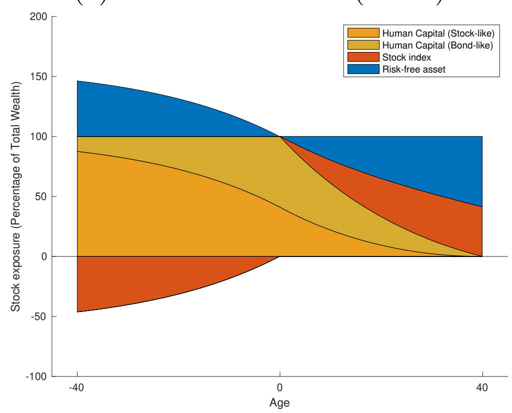
1. In practice, the market consistent value of the retirement benefit from the collective scheme will depend on the welfare weights of the social planner. So, figure 5.2 corresponds to a choice of welfare weights such that the market value of the retirement benefit of future generations is exactly equal to the value of their



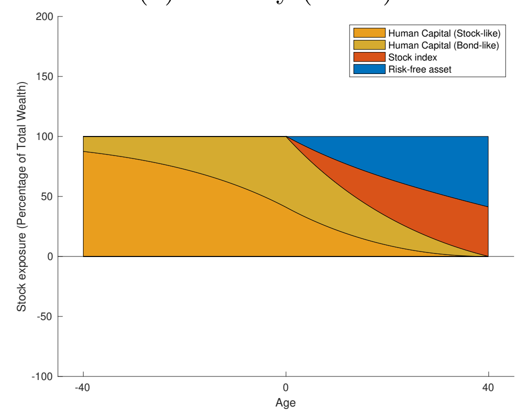
(a) Collective scheme ($\kappa = 0$)



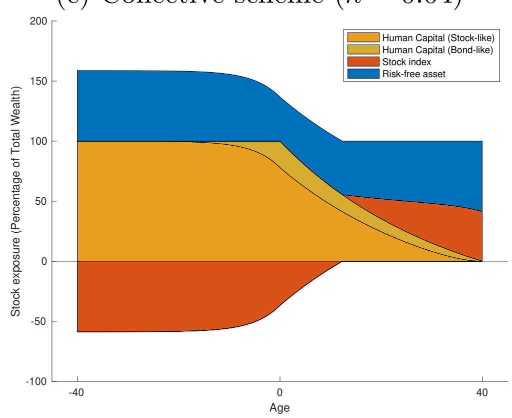
(b) Autarky ($\kappa = 0$)



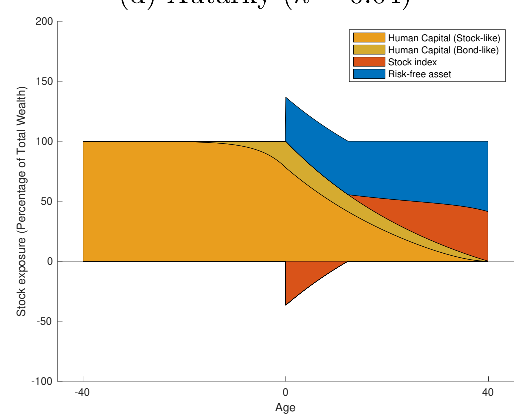
(c) Collective scheme ($\kappa = 0.04$)



(d) Autarky ($\kappa = 0.04$)



(e) Collective scheme ($\kappa = 0.2$)



(f) Autarky ($\kappa = 0.2$)

Figure 5.2: **Individual exposure under the optimal collective policy** This figure shows the optimal individual portfolio in autarky (right side) alongside the individual portfolio that replicates the exposure the individual faces in the collective scheme (left side). The panels on each row correspond to a different value of κ. The human capital share in the replicating portfolio is, for comparison, chosen such that it matches the human capital share in autarky.

We have plotted the replicating portfolio in the left panels of figure 5.2. The right hand panels show the indvidual portfolio composition in autarky (consistent with figure 3.1) for comparison. The horizontal axis represents the age of the individual and runs from minus 40 to plus 40. The vertical axis hows the portfolio weights in percentage terms.

Note that for the positive age levels, the graphs on the left and right are identical. When generations are alive, they are unrestricted in autarky. Hence, their exposure is already optimal and the manager of the collective scheme can not improve on their autarky exposure.

Future generations are the ones affected by the restriction that trade can only take place during when alive. For these generations, the planner of the collective scheme can improve the exposure to dividend risk. Depending on the value of κ, the collective manager optimizes the exposure for future generations by either implicitly shorting bonds and buying stock (panel (a)) or vice-versa (panel (c) and (e)). As we saw before, when κ is low, human capital is very much bond-like and it is optimal for the manager to buy additional stocks on behalf of future generations. When κ is high, human capital is more stock-like and the manager sells stocks on behalf of future generations to optimally reduce their exposure.

So, while the collective solution is usually referred to as "intergenerational risk-sharing", risk is not really shared or traded between generations in this model. Instead, the collective schemes' manager really trades risk on behalf of future generations on the financial market. So, it is actually more appropriate to speak of "trading on behalf of future generations" instead of "intergenerational risk-sharing".

human capital. More on the choice of the welfare weights in the next sub-section on welfare effects.

5.3 Welfare and redistribution

The certainty equivalent in the collective scheme for each generation can be found by pluging (5.4) into the definition of the certainty equivalent (3.9), to get:

$$\tilde{CE}(0,T) = \tilde{b}(0)\beta(T)^{\frac{1}{\gamma}} \exp\left\{ \frac{1}{\gamma} \left[\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma} \right) T \right] \right\} \tag{5.5}$$

where the welfare weights, $\beta(T)$, still need to be specified. The choice of $\beta(T)$ determines how collective wealth is distributed across generations. Of course, the choice of welfare weights is subjective, but let us consider three choices of $\beta(T)$ here that may be of particular interest.

One choice of $\beta(T)$ could be to set it such that the market consistent present value of the retirement benefit for each generation is unchanged compared to the autarky solution. This is a non-re-distributive welfare weight and we will label it $\beta_{\mathrm{NR}}(T)$.

Gollier (2008) suggests to choose welfare weights such that all generations obtain the same certainty equivalent. We will denote this choice for the welfare weights $\beta_{EQ}(T)$. As we will see in a moment, a downside of this choice for $\beta(T)$ is that it does not ensure that the introduction of risk-sharing is a Pareto improvement.

A third choice could be set the welfare weights such that all generations have the same gain compared to autarky in percentage terms. So, we could set $\beta(T)$ such that the certainty equivalent of all generations is multiplied by the same factor, Δ_{CE}. This choice also has the convenience that we can summarize the efficiency gain by a single percentage for all generations. We denote this choice $\beta_{\Delta}(T)$.

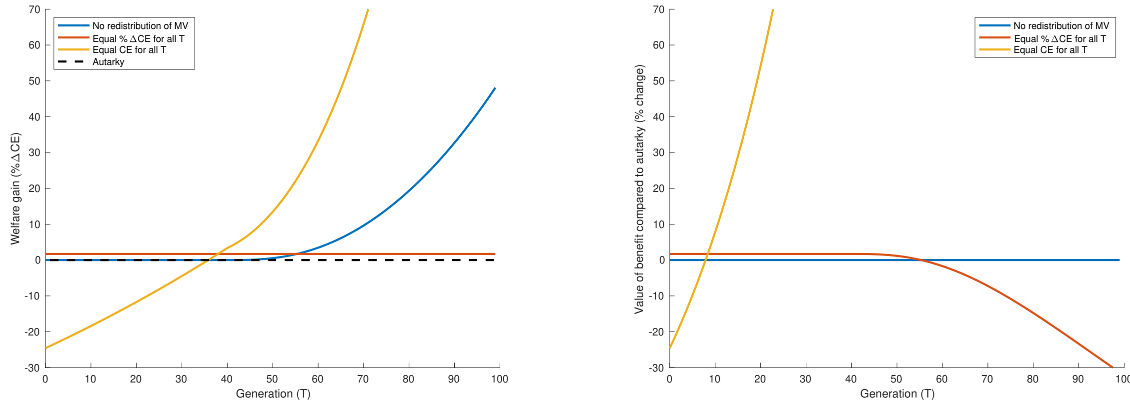
In appendix D we derive explicit expressions for the different choices of $\beta(T)$ and the certainty equivalents that follow from each choice. In addition, we derive the market consistent value for each generation of the retirement benefit they receive from the collective scheme. This allows us to determine to what extent the fund manager redistributes market value between generations for the different choices of the welfare weights.

Figure 5.3 plots both the welfare gain and redistribution of market value for all generation assuming $\kappa = 0.04$ and a relative risk aversion of 5. The left panel shows the change in the certainty equivalent retirement benefit. The right panel shows the change in the market value of the retirement benefit between autarky and the collective scheme.

If the fund manager chooses not to redistribute any market value, current generations ($T < 40$) are not affected by introduction of the collective scheme. These generations were already optimizing and since the fund manager does not redistribute any market value, the certainty equivalent for these generations is unchanged. For all future generations there is a welfare gain. Although the market value of the human capital of future generations is unchanged, the manager is able to optimize the risk exposure of future generations, leading to a welfare improvement. The welfare gain increases exponentially with the time until birth and increases without bound for generations in the far future.

If the fund manager chooses the welfare weights such that all generations have the same certainty equivalent, the oldest generations lose in welfare terms. We have assumed that these generations have earned a medium return on their reitrement saving in the past. Since they are almost retiring they do not face much uncertainty anymore at this point. Consequently their certainty equivalent consumption is relatively high compared to later generations. When equalizing the certainty equivalent, the fund manager redistributes market value from these generations to future generations. These future generations now benefit for two reasons: the collective scheme subsidizes them and allows them to take risk before they are born. Consequently, the welfare gain of future generations is even bigger than the case without market value redistribution.

Note that that manager only needs to take between 0 and 20 percent of market value from generations between $T = 0$ and $T = 10$ to greatly increase the market value of the benefit of all generation with a retirement 10 years and further into the future ($T > 10$). The reason is that the oldest generations own a relatively big share of total wealth. The generations retiring in the next 10 years own over 30 percent of total wealth.



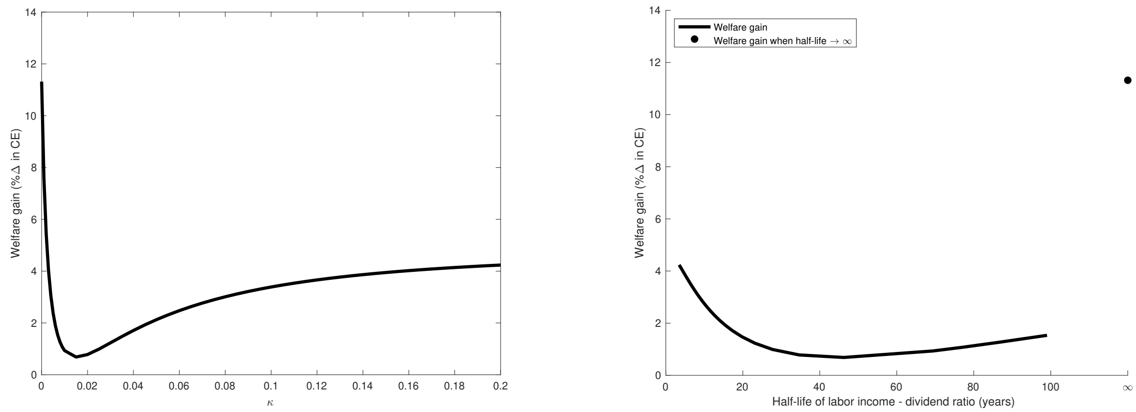
(a) Welfare gain (b) Market value redistribution

Figure 5.3: **Welfare weights, welfare gain and market value redistribution** This figure shows the impact of different choices for the welfare weights on welfare and the distribution of collective assets. The left panel shows the percentage increase in certainty equivalent retirement benefit between the optimal collective scheme and autarky for different generations. Each line represents a different choice for the welfare weights, as described in the main text. The right panel shows for the same welfare weights the percentage difference between the market value of the retirement benefit in the collective scheme and in autarky.

If the fund manager instead chooses the welfare weights such that all generations increase their certainty equivalent by the same percentage, the left hand graph is particularly simple. It is a flat line. In this case. When $\kappa = 0.04$, we find that the percentage increase in certainty equivalent retirement benefit is 1.7 percent. Compared to non-re-distributive welfare weights, the manager uses its re-distributive power to take a part of the welfare gain for future generations and transfer this also to current generations, so everybody benefits equally in percentage terms.

In order to illustrate the impact of labor income and dividend co-integration 5.4 shows the welfare gain for different levels of cointegration assuming that $\gamma = 5$ and considering welfare weights that equalize the percentage welfare gain accros generations ($\beta_\Delta(T)$). We show the welfare effect for the range of κ from 0 (risk-free human capital) to 0.2 (the highest estimate reported by BCG).

The figure shows that the boundary case, where kappa is 0, corresponds to the welfare gain of 11.3 percent. This is consistent with the high levels of welfare gain found in other



(a) As function of κ

(b) As function of half-time

Figure 5.4: **Welfare gain for different levels of co-integration** This figure shows the percentage increase in certainty equivalent retirement benefit between autarky and optimal risk-sharing. Panel (a) plots the welfare gain as a function of κ. Panel (b) shows the same information, but now the horizontal is re-scaled to represent the half-life of shocks to the labor income to dividend ratio. The welfare weights are chosen such that the percentage gain of all generations is the same. The figure assumes wealth levels are equal to the median autarky scenario, where aggregate human capital is approximately 50% of total wealth and $\omega(t) = 0$

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studies (Teulings and De Vries (2006), Gollier (2008) and Cui et al. (2011)). For values of kappa in the range reported by BCG (0.04 - 0.2), the welfare gain ranges from 1 to 4 percent. In this case the welfare gain is not only much smaller, it also comes from a different source. This welfare gain is no longer a benefit from extra overall risk-taking. It is a benefit from the fact that current generations now bear some human capital risk of future generations.

The welfare gain is minimized at approximately $\kappa = 0.015$. This roughly coincides with the point where the optimal aggregate portfolio allocation to the stock market is the same in autarky and under the optimal risk-sharing scheme. In this case, the optimal investment in the stock market on behalf of future generations on aggregate is exactly 0. While aggregate exposure of future generations is already at its optimum, even in autarky, the collective fund manager will still be able to achieve some welfare gain. The manager does so by optimally (re)allocating the aggregate human capital risk of future generations among these generations themselves.

Thus, we can conclude that not only does co-integration potentially reverse the optimal policy (taking less instead of more risk in the collective scheme) it also greatly affects the potential for welfare gain. In the absence of co-integration, there seems to be a significant welfare gain to be reaped. In the presence of co-integration however, this potential might be very limited.

CHAPTER 6

AMBIGUITY

6.1 Parameter uncertainty

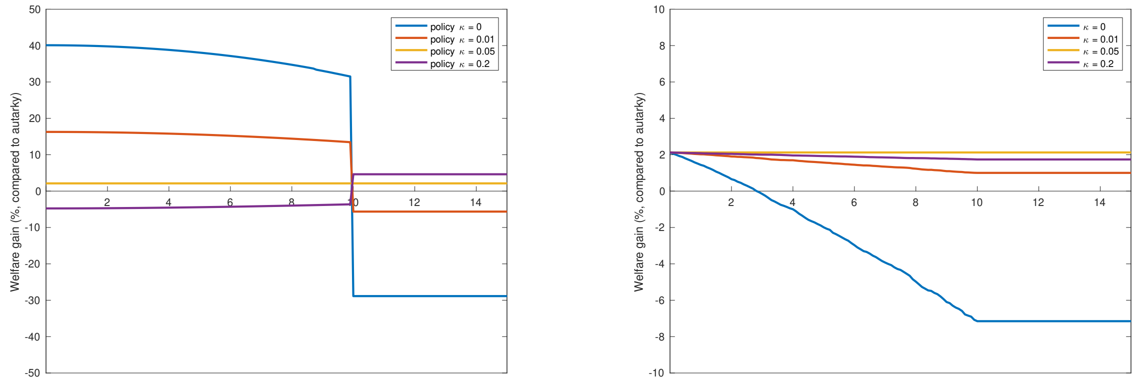
The possible presence of co-integration has a significant impact on the optimal policy of the collective fund manager. At the same time, it is hard to make precise statistical statements about the presence of co-integration. Therefore, we will now consider what the consequences are if the manager is ambiguous about the exact value of κ.

First we will illustrate what the impact on the welfare of different generations is if the manager follows a policy that is based on a wrong estimate of κ. We perform the following thought experiment. We assume that the true value of κ is 0.05. Next, we assume that during a period of 10 years the fund manager implements the optimal policy based on a wrong estimate of κ. After this period, the manager will revert to the optimal policy based on the correct κ. Figure 6.1 shows the welfare implications for the different generations.

If we first look at panel 6.1a, we see that the impact of picking the wrong κ are significant. The lines show the welfare gain of each generation compared to autarky. The autarky solution is based on the optimal policy using the actual κ. One interpretation of the lines is that this is the welfare gain or loss of an individual who believes $\kappa = 0.05$, but is facing a fund manager that imposes a collective policy based on an alternative level of κ.

Basing a policy on the wrong κ has two major effects. Firstly, it leads the collective fund manager to wrongly measure the riskiness of human capital. Consequently, the pension scheme manager chooses the wrong asset allocation. A second issue is that the manager will also value human capital incorrectly. A wrong estimate of total wealth implies that the manager sets a sub-optimal benefit level. This second effect turns out to have the biggest welfare effect.

For example, if the manager believes that $\kappa = 0$, human capital is risk-free and hence relatively valuable. Consequently, the manager is too optimistic and starts to distribute

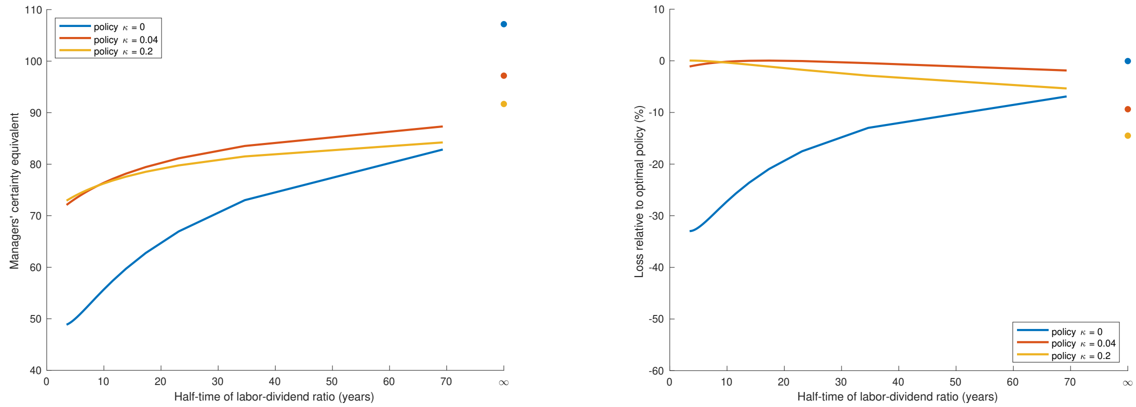


(a) Welfare impact of wrong policy κ (b) Excluding consumption timing effect

Figure 6.1: **Welfare under incorrect estimation of $\boldsymbol{\kappa}$** Panel (a) shows the welfare impact if the collective fund manager implements an optimal policy that is based on a wrong value of κ. The true κ is assumed to be 0.05 and the different lines illustrate the welfare gain under different policy choices for different generations. The figure assumes that after ten years the fund manager reverts to the optimal policy for the actual value of κ. In each scenario, the manager has the same time-preference ($\beta(T)$). Panel (b) isolates the effect of suboptimal allocation from the effect of sub-optimal collective asset draw-down. The lines represent the welfare gain/loss compared to autarky due to suboptimal risk-taking only.

benefits that are too high in the short-run. We see in the figure that initially the manager starts to pay out a retirement benefit that is 40 percent higher than under the optimal policy. Obviously this causes the collective assets to be drawn down at a sub-optimally high pace. Due to the fact that total wealth is not clearly observable and the long-run nature of the problem, the excessive distribution of benefits is not obvious in the short-run. Only in the long-run the collective fund is destined to completely run down its assets. In our thought experiment the manager returns to the true optimal policy before this happens. The graph clearly reveals the dramatic long-run consequences of following the wrong policy however: All future generations have lost 29 percent of their certainty equivalent compared to autarky.

In panel 6.1b we disentangle which part of this 29 percent is due to the sub-optimal asset allocation and which part is due to the sub-optimal draw down of collective assets. This graph shows the welfare effect in case only the asset allocation is based on the wrong κ, while the rate at which collective wealth is distributed to retirees is equal to the optimal policy under the true κ. We see that the 29 percent welfare loss for all $T > 10$ generations is for



(a) Managers' certainty equivalent

(b) Welfare loss relative to optimal policy

Figure 6.2: **Manager's utility under 'wrong' policies** The left panel shows the certainty equivalent benefit level of the manager from a policy based on a specific estimated level of κ. The horizontal axis shows different levels of the true κ, represented by the half-time of the labor income to dividend ratio. The right panel shows the same information, but now, at each level of the true κ, the managers' certainty equivalent is divided by the maximum achievable certainty equivalent for that level of the true κ.

roughly 7 percentage points due to the sub-optimal asset allocation and for 22 percentage points due to the managers' overly optimistic benefit distribution.

6.2 Robust policy making

In the illustration of the impact of a wrong policy choice, we simply assumed a true value of κ. In practice the true value of κ is hard to know. Therefore, fund managers may want to come up with a robust choice for the value of κ on which to base their policy. One such choice could be a minimax level of κ.

Suppose that the fund manager considers all values for κ between 0 and 0.2 to be realistic. In this case, our numerical calculations show that the minimax policy would be the policy based on $\kappa = 0.2$. The driver for this result is that high levels of kappa are associated with lower levels of utility, since the value of human capital is lower. Consequently, a minimax policy favors policies that are optimal in those scenarios where the value of κ is unfavorable. This implies that the fund manager that follows a minimax approach would prefer to be cautious and decide not to shift risk to future generations. We illustrate this in figure (6.2).

The left panel shows the utility of the manager (in certainty equivalent terms) for different policies over different levels of the true κ in the left panel of figure 6.2a. The horizontal axis represents the actual value of κ (here in half-time terms, so $\kappa = 0$ corresponds to ∞). When the half-time of the labor income to dividend ratio is low, the certainty equivalents are relatively low for all policies and hence the policy that works best in this scenario is favored by the minimax criterium.

An alternative approach could be to apply the minimax criterium not to the absolute welfare level the manager achieves, but to the relative impact of the policy mistake. So, instead of looking at the certainty equivalent of the manager in absolute terms, we could look at the percentage difference between the certainty equivalent under the actual policy and under the optimal policy for that true level of κ. This approach is illustrated by the right-hand panel of figure 6.2. Each line shows for a policy based on a given level of κ, what the relative loss of certainty equivalent is for the manager for different levels of the true κ. Note that the lines touch zero exactly at the point where the true κ is equal to the κ on which the policy is based. Applying the minimax criterion to the relative policy mistake would lead the manager to favor a policy based on $\kappa = 0.014$.

CHAPTER 7

SHORT-RUN VERSUS LONG-RUN RISK

In the model we have considered so far, all stock market risk was long-run risk. All shocks had a permanent impact on the level of the dividend cash-flow, on the stock price and in the long-run on the level of labor income. We saw that this reduces the appetite for stock market risk by young and future generations. van Binsbergen et al. (2012) show that the return on claims to short-run dividends are higher than the return on long-term dividends. A question that arises from this is, what if not all risk is long-run risk and the risk-premium is actually a reward for short-term risk? We will here illustrate that the results from the studies that treat labor income as risk-free, could also be thought of as an extreme case in a model with both long-run and short-run risk. Namely, it coincides with the case where only short-run risk earns a risk-premium.

7.1 Adding short-run risk

Assume that the dividend cash-flow is now the product of a short-run and a long-run component:

$$D(t) = D_{lr}(t) \times D_{sr}(t) \tag{7.1}$$

where $D_{lr}(t)$ is the long-run component of the dividend cash-flow and $D_{sr}(t)$ is the short-run component. Let the long-run component follow the same process we considered previously:

$$\frac{dD_{lr}(t)}{D_{lr}(t)} = g_D dt + \sigma_{lr} dz_{lr}, \tag{7.2}$$

and let the log of the short-run component follow:

$$d \log D_{sr}(t) = -\kappa_{sr} \log D_{sr}(t) + \sigma_{sr} dz_{sr} \tag{7.3}$$

So, the dividend process we used before now acts as a long term trend and we added a proportional temporary deviation from this trend. The long term trend is co-integrated with the labor income process, as before (cf. 2.5). Assume that the innovations to the short run deviation (dz_{sr}) and the innovations to the the long-run trend component (dz_{lr}) are uncorrelated.

Since we introduced a new source of risk, extend the pricing kernel as follows:

$$\frac{dM(t)}{M(t)} = -rdt + \phi_{lr} dz_{lr} + \phi_{sr} dz_{sr} \tag{7.4}$$

The return on the stock index (a claim on the stream of future dividends) then becomes:

$$\frac{dS(t)}{S(t)} = \left(r - \sigma_{S,sr}\ \phi_{sr} - \sigma_{lr}\phi_{lr}\right) dt + \sigma_{S,lr} dz_{lr} + \sigma_{S,sr}(D_{sr}(t)) dz_{sr} \tag{7.5}$$

Note that the sensitivity of the stock return to short run shocks, $\sigma_{S,sr}(D_{sr}(t))$, depends on the current level of the short-run dividend component (see appendix E for details).

7.2 Optimal allocation in autarky

In order to obtain the optimal exposure to both the long and short-run shocks, an investor will now need at least two assets that jointly span both dimensions of uncertainty. We will not explicitly specify these assets, but one could think of two different portfolios of stocks that each have different exposure to the long-run and short-run shock (i.e. growth and value stocks). If we assume that the financial market is complete with respect to these two shocks, the individual can obtain any combination of exposures to the two shocks and the optimal wealth process during life in autarky follows:

$$\frac{dV^*(t,T)}{V^*(t,T)} = \left(r + \frac{\phi_{sr}^2 + \phi_{lr}^2}{\gamma}\right) dt - \frac{1}{\gamma}\phi_{lr} dz_{lr} - \frac{1}{\gamma}\phi_{sr} dz_{sr}, \tag{7.6}$$

and the certainty equivalent is:

$$\mathrm{CE}(0,T) = \begin{cases} W(0,T)\exp\left\{\left(r + \frac{1}{2}\frac{\phi_{sr}^2+\phi_{lr}^2}{\gamma}\right)T\right\} & T \leq n \\ \mathbb{E}_0\left[H(T-n,T)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\exp\left\{\left(r + \frac{1}{2}\frac{\phi_{sr}^2+\phi_{lr}^2}{\gamma}\right)n\right\} & T > n \end{cases} \tag{7.7}$$

The optimal autarky solution is very similar to the solution we saw before. The only difference is that there are now two sources of risk and two relevant risk-premia.

7.3 Optimal risk-sharing

The solution to the manager's problem under risk-sharing, when the participation of future generations is mandatory, is very similar to the problem we saw before. The optimal process for total wealth in the manager's problem becomes:

$$\frac{d\tilde{V}(t)}{\tilde{V}(t)} = \left(r + \frac{\phi_{sr}^2 + \phi_{lr}^2}{\gamma}\right)dt - \frac{\tilde{b(t)}}{\tilde{V}(t)}dt - \frac{\phi_{sr}}{\gamma}dz_{sr} - \frac{\phi_{lr}}{\gamma}dz_{lr} \tag{7.8}$$

and the optimal retirement benefit:

$$\tilde{b}(T) = \tilde{b}(0)\beta(T)^{\frac{1}{\gamma}}\exp\left\{\frac{1}{\gamma}\left(r + \frac{1}{2}(\phi_{sr}^2 + \phi_{lr}^2)\right)T - \frac{1}{\gamma}\phi_{lr}\int_0^T dz_{lr}(u) - \frac{1}{\gamma}\phi_{sr}\int_0^T dz_{sr}(u)\right\} \tag{7.9}$$

We assume that the welfare weights are chosen such that all generations proportional gain the same proportion in certainty equivalent terms, so we have:

$$\beta(T) = \frac{CE(0,T)}{CE(0,0)}\exp\left\{-\left[\left(r + \frac{1}{2}\frac{\phi_{sr}^2 + \phi_{lr}^2}{\gamma}\right)T\right]\right\} \tag{7.10}$$

7.4 Welfare effects of risk-sharing with long and short-run risk

We will now illustrate how the potential welfare gain from risk-sharing depends on the source of risk, being short- or long-run risk. To connect with the previous sections, we will choose the parameters such that the overall utility gain from optimal risk-taking is unchanged ($\phi_{lr}^2 + \phi_{sr}^2 = \phi^2$) as is the overall variance (in the median scenario) of the price of a claim to all future dividends ($\sigma_{lr} + \sigma_{S,sr}(0) = \sigma$). We will vary how much risk and risk-premium is coming from long-run sources and how much from short run sources by varying a new parameter λ. We will set:

$$\sigma_{S,sr}(0) = (1-\lambda)\sigma$$
$$\sigma_{lr} = \lambda\sigma$$
$$\phi_{sr} = \sqrt{1-\lambda}\phi$$
$$\phi_{lr} = \sqrt{\lambda}\phi$$

and let λ vary between 0 (only short-run risk) and 1 (only long-run risk). Figure 7.1 shows the potential welfare gain from intergenerational risk-sharing as we vary λ, assuming $\kappa = 0.05$.

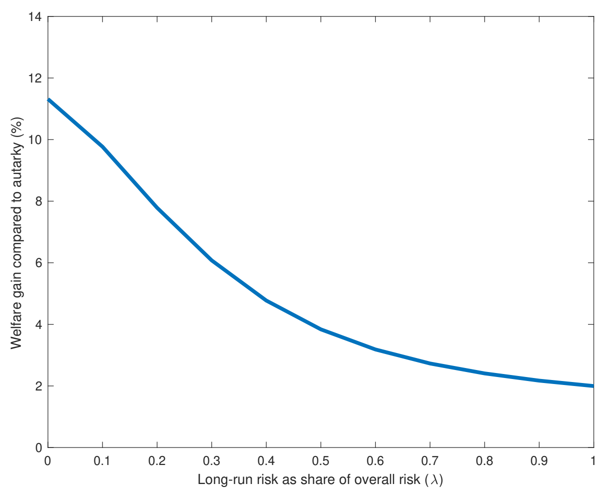


Figure 7.1: **Maximum welfare gain in presence of long- and short-run risk** This figure illustrates the maximum welfare gain the fund manager can achieve for different assumed combinations of long-term and short-term risk. The figure assumes that total volatility of the stock index is constant and equal to σ ($\sigma_{lr} + \sigma_{S,sr} = \sigma$) and the optimal gain from risk taking is unchanged ($\phi_{lr}^2 + \phi_{sr}^2 = \phi^2$). The figure assumes that wealth is at its median level, where aggregate human capital is approximately 50% of total wealth and $\omega(0) = 0$. The mean-reversion parameters are set to $\kappa = 0.05$ and $\kappa_{sr} = 0.25$

In the absence of long-term risk ($\lambda = 0$), we find the same welfare gain as we saw before in the case of risk-free labor income ($\kappa = 0$). The whole risk-premium is related to short-run risk sources. Long-term labor income is not correlated with these shocks. Hence, the fund manager can generate large gains by allowing future generations to gain access to this short-run risk-premium. When all risk comes from long-run sources, the potential welfare gains are lower and depend on parameter κ. As mentioned, van Binsbergen et al. (2012) show that, contrary to what standard models predict, the risk-premium for short-run dividend risk is higher than the risk-premium for long-term dividend risk. This does not necessarily imply however, that the equity risk-premium on aggregate is mostly compensation for short-run risk. Schorfheide et al. (2018) provide a decomposition of the equity risk-premium into short-run risk, long-term growth risk, volatility risk and preference risk and show that on average less than 5 percent of the equity risk-premium is a compensation for short-run risk, while over 60 percent is compensation for long-term growth risk. Our setup did not allow for volatility and preference risk (discount rate variation), which could be an interesting further extension to the model.

CHAPTER 8

CONCLUSION

We show that the potential presence of co-integration between labor income risk and stock market risk has a significant impact on collective portfolio choice and optimal risk-sharing. Our findings imply that the commonly held idea that collective pension plans with mandatory participation can and should take more risk, should be considered a boundary case. For the empirical parameter range reported by Benzoni et al. (2007), our model actually suggests that a collective fund should optimally take less stock market risk. This observation is in line with the findings of Bohn (2009), who also concludes that it is more likely that an efficient policy shifts risk from workers to retirees, instead of the other way around. We furthermore find that the welfare gains from risk-sharing turn out to be much lower than in the boundary case where labor income risk is completely uncorrelated with stock market risk.

If we accept the idea that stock market risk and labor income risk are co-integrated, we highlight that it still matters whether the risk-premium on stock market risk is associated with its short-run or long-run component. If the risk-premium is associated with the short-run component, the findings of Gollier (2008) still hold, even in the presence of co-integration. Schorfheide et al. (2018) however present evidence that suggests that the risk-premium is largely compensation for long-run risk.

Our results highlight a challenge for policy makers running collective pension plans. Policy makers will have to decide what the true model is and what the true parameter values within that model are. As we showed, these beliefs have a significant impact on the optimal portfolio allocation, the optimal distribution of collective risk and wealth and the potential welfare gains from risk-sharing. Not only does our analysis suggest that the potential welfare gains from collective risk-sharing are smaller. It also highlights that it is not easy for policy makers to reap these potential benefits. Picking the wrong model has significant implications for the optimal policy. Our model is, of course, no exception in this respect. Especially if we bear in mind that our analysis merely focused on one modeling dimension, albeit an important one.

In the presence of model risk, one approach is to consider which assumptions or policies are most robust. We therefore considered the optimal policy from the perspective of a maximin pension scheme manager. Worlds with high long-run correlations between labor income and stock market risk are worse from a welfare perspective than worlds with a low long-run correlation. Therefore, the maximin manager prefers to follow a policy that is designed to do well in the scenario in which human capital is stock-like. This implies that the manager would prefer a policy that transfers risk from future generation to current generations instead of the other way around.

An important dimension worth exploring further is the presence of discount rate variation. It is well-known that both interest rates and risk-premia vary in real-world financial markets (see i.e. Cochrane (2011)). Such variation is particularly relevant for long-horizon portfolio problems like the one considered here. Our continuous time setting could be extended to allow the interest rate to vary, while we would still be able to obtain closed-form solution.

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APPENDIX A

THE STOCK RETURN PROCESS

Note that equation (2.1) implies that:

$$D(s) = D(t) \exp\left\{ \left(g_D - \frac{1}{2}\sigma_D^2 \right)(s-t) + \sigma_D \int_t^s z_D(v) \right\} \tag{A1}$$

And (2.2) implies that:

$$\frac{M(s)}{M(t)} = \exp\left\{ -\left(r + \frac{1}{2}\phi_D^2 \right)(s-t) + \phi_D \int_t^s z_D(v) \right\} \tag{A2}$$

The present value of the dividend flow is the expectation of their product:

$$\mathbb{E}_t \left[\frac{M(s)}{M(t)} D(s) \right] = D(t) \exp\left\{ \, (g_D - r + \phi_D \sigma_D)(s-t) \right\} ds \tag{A3}$$

The stock price, $P(t)$, is the total present value of all future dividend flows:

$$\begin{aligned} P(t) &= \int_t^\infty \mathbb{E}_t \left[\frac{M(s)}{M(t)} D(s) \right] ds \\ &= D(t) \int_t^\infty \exp\left\{ \, (g_D - r + \phi_D \sigma_D)(s-t) \right\} ds \\ &= \frac{D(t)}{r - \phi_D \sigma_D - g_D} \end{aligned} \tag{A4}$$

where we impose that $r - \phi_D \sigma_D - g_D > 0$. The return on a stock-only fund is the sum of the dividend return and the relative price change:

$$\frac{dS(t)}{S(t)} = \frac{D(t)}{P(t)} dt + \frac{dP(t)}{P(t)} \tag{A5}$$

After noting that $\frac{dP(t)}{P(t)} = \frac{dD(t)}{D(t)}$ we find that

$$\frac{dS(t)}{S(t)} = (r - \phi_D \sigma_D)\, dt + \sigma_D dz_D \tag{A6}$$

which is equation (2.4) in the main text.

APPENDIX B

THE HUMAN CAPITAL PROCESS

B.1 Labor income

Integrating the log-labor income process (2.7) gives that labor income at time t as a function of information at time s:

$$W(t) = W(s)\exp\left\{-\kappa\int_s^t \omega(u)du + (g_D + \mu_{wd} - \frac{1}{2}\sigma^2))(t-s) + (\sigma - \nu_D)\int_s^t dz_D(u)\right\} \tag{B1}$$

First, let us express $\omega(t)$ as a function of time s information. Since $\omega(t)$ follows an Ornstein-Uhlenbeck process, this solution is well known (remember that we have set $\nu_W = 0$).

$$\omega(t) = \omega(s)e^{-\kappa(t-s)} - \nu_D\int_s^t e^{-\kappa(t-v)}dz_D(v) \tag{B2}$$

Substituting this into B1 gives:

$$\begin{aligned} W(t) = W(s)\exp\Big\{&\omega(s)(e^{-\kappa(t-s)} - 1) + (g_D + \mu_{wd} - \frac{1}{2}\sigma^2))(t-s) \\ &+\kappa\nu_D\int_s^t\int_s^u e^{-\kappa(u-v)}dz_D(v)du\Big\} \end{aligned} \tag{B3}$$

Which can be simplified into:

$$\begin{aligned} W(t) = W(s)\exp\Big\{&-\kappa B(t-s)\omega(s) + (g_D + \mu_{wd} - \frac{1}{2}\sigma^2)(t-s) \\ &+\kappa\nu_D\int_s^t B(t-v)dz_D(v)\Big\} \end{aligned} \tag{B4}$$

where $B(x) = \frac{1}{\kappa}\left(1 - e^{-\kappa(x)}\right)$

B.2 The present value of labor income

The present value of labor income is simply the sum of present values of all future labor income cash-flows. Remember that T denotes the retirement date and n denotes the time the individual is in the labor force:

$$H(\omega(t), t, T) = \int_{\max[t, T-n]}^{T} PV_W(\omega, t, \tau) d\tau$$

where

$$PV_W(\omega, t, \tau) = \mathbb{E}_t \left[\frac{M(\tau)}{M(t)} W(\tau) \right]$$

Substituting the expression for $W(\tau)$ and $\frac{M(\tau)}{M(t)}$ in gives:

$$PV_W(\omega, t, \tau) = W(t) \mathbb{E}_t \left[\exp \left\{ -\kappa B(\tau - t)\omega(u) + \bar{h}\,(\tau - t) \int_t^\tau (\phi_D + \nu_D \kappa B(\tau - v))\, dz_D(v) \right\} \right]$$

where $\bar{h} = (g_d + \mu_{wd} - \frac{1}{2}\sigma^2) - \left(r + \frac{1}{2}\phi_D^2\right)$. Note that the expectation and variance of the term inside the exponential are given by

$$\begin{aligned}
\mathbb{E}_t\,[...] &= -\kappa B(\tau - t)\omega(u) + \bar{h}\,(\tau - t) \\
Var_t\,[...] &= \mathbb{E}_s \left[\left(\int_t^\tau (\phi_D + \nu_D \kappa B(\tau - v))\, dz_D(v) \right)^2 \right] \\
&= (\phi_D + \nu_D)^2 (\tau - t) - 2(\phi_D + \nu_D)\nu_D B(\tau - t) + \nu_D^2 \frac{B(2(\tau - t))}{2}
\end{aligned}$$

So, we find that:

$$PV_D(\omega, t, \tau) = W(t) \exp\left\{ A(\tau - t) - \kappa B(\tau - t)\omega(t) \right\} \tag{B5}$$

where

$$A(\tau - t) = (g_D + \mu_{wd} + \nu_D \phi_D - r)(\tau - t) - (\phi_D + \nu_D)\nu_D B(\tau - t) + \nu_D^2 \frac{B(2(\tau - t))}{4} \tag{B6}$$

which brings us to (2.9) in the main text.

APPENDIX C

SOLUTION TO INDIVIDUAL AND COLLECTIVE PROBLEM

C.1 Individual problem

Since there is only one source of risk and this source of risk may be traded using the stock, we are facing a complete market problem. This means that the problem can be solved as a static problem using the so-called martingale approach (Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), and Cox and Huang (1991)). The static problem to be solved is:

$$\begin{aligned} \max_{b(T)} \quad & \mathbb{E}_t\left[\frac{V(T,T)^{1-\gamma}}{1-\gamma}\right] \\ \text{s.t.} \quad V(t,T) = \; & \mathbb{E}_t\left[V(T,T)\frac{M(T)}{M(t)}\right] \end{aligned} \tag{C1}$$

The first order condition to the problem is:

$$V(T,T)^{-\gamma} - \lambda \frac{M(T)}{M(t)} = 0 \tag{C2}$$

where λ is the Lagrange multiplier. The solution to the problem is:

$$\begin{aligned} V^*(T,T) &= V(t,T)\mathbb{E}_t\left[\left(\frac{M(T)}{M(t)}\right)^{\frac{\gamma-1}{\gamma}}\right]^{-1}\left(\frac{M(T)}{M(t)}\right)^{-\frac{1}{\gamma}} \\ &= V(t,T)\exp\left\{\left(r + \frac{\phi_D^2}{\gamma} - \frac{\phi_D^2}{2\gamma^2}\right)(T-t) - \frac{\phi_D}{\gamma}\int_t^T dz_D(s)\right\} \end{aligned} \tag{C3}$$

This equation gives us the optimal total wealth as a function of the dividend shocks. To find the optimal portfolio weight, we simply need to make sure that the portfolio weight is chosen such that actual wealth mimics optimal wealth. For actual total wealth we have:

$$V(T,T) = V(t,T)\exp\left\{(...)(T-t) + (x_S(t,T)\sigma + h(t,T)\sigma_h(\omega,t,T))\int_t^T dz_D(s)\right\} \tag{C4}$$

where we suppressed the drift term. Comparing (C3) to (C4) learns that, for actual wealth to mimic optimal wealth, we need:

$$-\frac{\phi_D}{\gamma} = x_S(t,T)\sigma + h(t,T)\sigma_h(\omega,t,T) \qquad \forall t \tag{C5}$$

Solving this equation for $x_S(t,T)$ gives (3.7) in the main text.

C.2 Manager's problem

Like the individual problem, the manager's problem in the setting with intergenerational risk-sharing can also be solved using the martingale approach. The static problem to be solved is:

$$\begin{aligned} \max_{b(T)} \quad & \mathbb{E}_t\left[\int_t^\infty \beta(T)\frac{\tilde{b}(T)^{1-\gamma}}{1-\gamma}dT\right] \\ \text{s.t.} \quad \tilde{V}(t) = \; & \mathbb{E}_t\left[\int_t^\infty \tilde{b}(T)\frac{M(T)}{M(t)}dT\right] \end{aligned} \tag{C6}$$

The first order condition to the problem is:

$$\beta(T)\tilde{b}(T)^{-\gamma} - \lambda\frac{M(T)}{M(t)} = 0 \tag{C7}$$

where λ is the Lagrange multiplier. Using the constraint to eliminate the Lagrange multiplier, we find:

$$\tilde{b}^*(T) = \tilde{V}(t)\tilde{A}(t)\beta(T)^{\frac{1}{\gamma}}\left(\frac{M(T)}{M(t)}\right)^{-\frac{1}{\gamma}} \tag{C8}$$

where $\tilde{A}(t)$ captures the pay-out rate from total wealth:

$$\tilde{A}(t) = \mathbb{E}_t\left[\int_t^\infty \beta(\tau)^{\frac{1}{\gamma}}\left(\frac{M(\tau)}{M(t)}\right)^{\frac{\gamma-1}{\gamma}} d\tau\right]^{-1} \tag{C9}$$

Dividing $\tilde{b}^*(T)$ by $\tilde{b}^*(0)$ and solving explicitly for the expectation term results in equation (5.4) in the main text.

Optimal wealth at time $s > t$ is simply the present value of the optimal future retirement benefits:

$$
\begin{aligned}
\tilde{V}^*(s) &= \mathbb{E}_s \left[\int_s^\infty \tilde{b}^*(\tau) \frac{M(T)}{M(s)} d\tau \right] \\
&= \tilde{V}(t)\tilde{A}(t)\mathbb{E}_s \left[\int_t^\infty \beta(\tau)^{\frac{1}{\gamma}} \left(\frac{M(\tau)}{M(s)} \right)^{\frac{\gamma-1}{\gamma}} d\tau \right] \left(\frac{M(s)}{M(t)} \right)^{-\frac{1}{\gamma}}
\end{aligned}
\tag{C10}
$$

where only the last term is stochastic from time s onwards. Writing this in differential form gives us the law of motion of optimal wealth:

$$
\frac{d\tilde{V}^*(s)}{\tilde{V}^*(s)} = (...)dt - \frac{\phi_D}{\gamma} dz_D(s) \tag{C11}
$$

where we suppress the drift term for simplicity. The optimal share of total wealth invested in the stock index is found by setting the volatility of optimal wealth equal to the volatility of actual total wealth (4.7). This gives:

$$
-\frac{\phi_D}{\gamma} = \tilde{x}_S^*(t)\sigma + \tilde{h}(t)\tilde{\sigma}_h(\omega, t) \tag{C12}
$$

Solving this equation for $x_S(t, T)$ gives (4.10) in the main text.

APPENDIX D

WELFARE AND REDISTRIBUTION

D.1 Certainty equivalent

We start by plugging the expression for the optimal collective benefit leven (5.4) into the definition of the certainty equivalent (3.9). Solving for the expectation gives:

$$\tilde{CE}(0,T) = \tilde{b}^*(0)\beta(T)^{\frac{1}{\gamma}} \exp\left\{\frac{1}{\gamma}\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\}$$

which is equation (5.5) in the main text.

D.2 Non-re-distributional weights

First consider a choice for the welfare weights that would leave the market consistent value of a generations' retirement benefit unchanged compared to autarky. We will denote these weights $\beta_{\text{NR}}(T)$. The expression for these weights is found by solving the following equation for $\beta(T)$:

$$V(0,T) = \mathbb{E}_0\left[\tilde{b}^*(T)\frac{M(T)}{M(0)}\right] \tag{D1}$$

The left hand side being the value of total wealth of generation T in autarky and the right hand side being the present value of the retirement benefit in the collective scheme. Substitute (5.4) for $\tilde{b}^*(T)$, get rid of the expectation and solve for $\beta(T)$, to find:

$$\beta_{\text{NR}}(T) = \frac{V(0,T)}{\tilde{b}^*(0)} \exp\left\{(\gamma - 1)\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\} \tag{D2}$$

Since there is no redistribution, it must also hold that $\tilde{b}^*(0) = b(0) = V(0,0)$, so we get

$$\beta_{NR}(T) = \frac{V(0,T)}{V(0,0)} \exp\left\{(\gamma - 1)\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\} \tag{D3}$$

The optimal benefit from the collective scheme can in this case be written as a simple function of initial individual wealth:

$$\tilde{b}^{*}_{NR}(T) = V(0,T)\exp\left\{\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\} \tag{D4}$$

D.3 Equal certainty equivalent

Gollier (2008) suggests setting the welfare weights such that all generations have the same certainty equivalent level of consumption. We will denote this choice of the welfare weights β_{EQ}. Equal certainty equivalents for all T means that we have:

$$CE(0,T) \equiv CE(0,0) = \tilde{b}^{*}(0) \qquad \forall T \tag{D5}$$

After substituting (5.5) into (D5) and solving for $\beta_{EQ}(T)$ we find:

$$\beta_{\mathrm{EQ}}(T) = \exp\left\{-\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\} \tag{D6}$$

Substituting this back into the eqpression for the collective benefit level (5.4) gives:

$$\tilde{b}^{*}_{EQ}(T) = \tilde{b}^{*}(0)\exp\left\{\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\phi_D^2}{\gamma}T - \frac{1}{\gamma}\phi_D\int_0^T dz_D(u)\right\} \tag{D7}$$

The market value of this benefit to generation T is:

$$\begin{aligned} V_{EQ}(0,T) &\equiv \mathbb{E}_0\left[\tilde{b}^{*}(T)\frac{M(T)}{M(0)}\right] \\ &= \tilde{b}^{*}(0)\exp\left\{-\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\} \end{aligned} \tag{D8}$$

The resource constraint requires that

$$\tilde{V}(0) = \int_0^\infty V_{EQ}(0,T)\, dT \tag{D9}$$

Substituting (D7) into the resource constraint gives:

$$\tilde{b}^*_{EQ}(0) = \tilde{V}(0)\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right) \tag{D10}$$

The general expression for $\tilde{b}^*_{EQ}(T)$ becomes:

$$\tilde{b}^*_{EQ}(T) = \tilde{V}(0)\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)\exp\left\{\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\phi_D^2}{\gamma}T - \frac{1}{\gamma}\phi_D\int_0^T dz_D(u)\right\} \tag{D11}$$

D.4 Equal proportional welfare gain

Now consider the case in which the manager wants to make sure every generation gets the same percentage increase in their certainty equivalent benefit compared to autarky. Let us denote the welfare weights for this case $\beta_\Delta(T)$. So, $\beta_\Delta(T)$ is such that:

$$\frac{\tilde{CE}(0,T)}{CE(0,T)} = \frac{\tilde{CE}(0,0)}{CE(0,0)} \equiv \Delta_{CE} \qquad \forall T$$

Substituting equation (5.5) for $CE(0,T)$ and $CE(0,0)$ and solving for $\beta(T)$ then gives:

$$\beta_\Delta(T) = \frac{CE(0,T)}{CE(0,0)}\exp\left\{-\left[\left(r + \frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right]\right\} \tag{D12}$$

The expression for the benefit level in this case becomes:

$$\tilde{b}^*_\Delta(T) = \tilde{b}^*(0)\frac{CE(0,T)}{CE(0,0)}\exp\left\{\frac{1}{2}\left(\frac{\gamma-1}{\gamma}\right)\frac{\phi_D^2}{\gamma}\,T - \frac{1}{\gamma}\phi_D\int_0^T dz_D(u)\right\}$$

The market value of the benefit to generation T is:

$$
\begin{aligned}
V_\Delta(0,T) &\equiv \mathbb{E}_0\left[\tilde{b}^*_\Delta(T)\frac{M(T)}{M(0)}\right] \\
&= \tilde{b}^*_\Delta(0)\frac{CE(0,T)}{CE(0,0)}\exp\left\{-\left(r+\frac{1}{2}\frac{\phi_D^2}{\gamma}\right)T\right\}
\end{aligned}
\tag{D13}
$$

We can now find $\tilde{b}^*(0)$ by substituting the equation into the resource constraint. This leave us with:

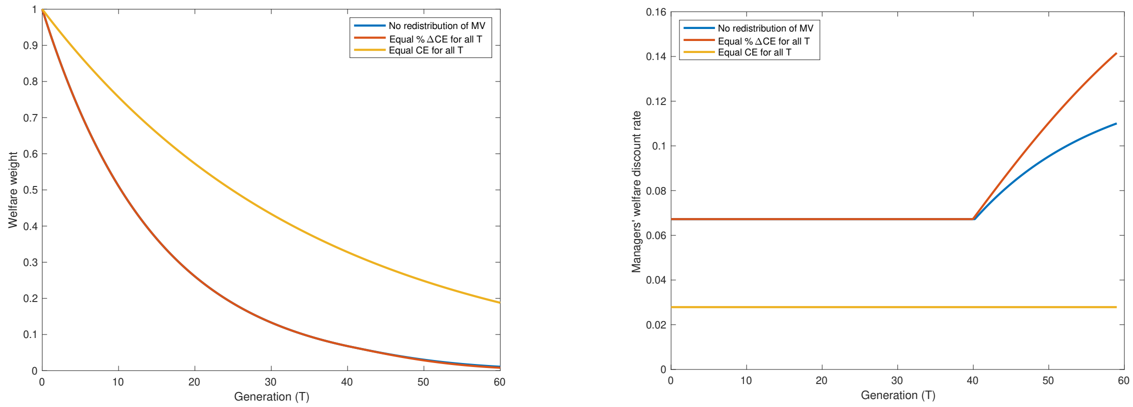
$$
\tilde{b}^*(0) = \tilde{V}(0)\left[\int_0^\infty \frac{CE(0,\tau)}{CE(0,0)}\exp\left\{-\left(r+\frac{1}{2}\frac{\phi_D^2}{\gamma}\right)\tau\right\}d\tau\right]^{-1} \tag{D14}
$$

In order to evaluate this expression, we need to evaluate the certainty equivalent in autarky (3.10). For all generations currently alive ($T < n$), this a simple closed form expression. For generations $T > n$, however, we have to evaluate the term $\mathbb{E}_0[H(T-n,T)^{1-\gamma}]$. This term has no simple closed-form expression. It is however straightforward to evaluate this expression numerically (i.e. through Monte-Carlo or Gaussian quadrature).

D.5 Comparing welfare weights

Figure D.1 compares the different welfare weights outlined above. The figure is based on our baseline setting with $\gamma = 5$, $\kappa = 0.05$ and assuming that the initial distribution of wealth is equal to a median autarky scenario.

In the left panel we see that, if the manager equalizes the certainty equivalent for all generations, she puts relatively much weight on future generations. This is consistent with the fact that current generations lose in welfare terms (see figure 5.3). We see that the non-re-distributional choice and the choice that equalizes the relative welfare gain are very similar. As a matter of fact, they are identical for current generations ($T < n$). The non-re-distributional choice puts a somewhat higher weight on future generations. This is consistent with the fact that under the weights that equalize the relative welfare gain for all generations, the collective manager transfers some welfare gain from future generations to



(a) Welfare weights

(b) Intergenerational discount rate

Figure D.1: **Welfare weights** The left panel shows the welfare weight ($\beta(T)$) put on different generations for the different welfare weight choices described in the text. The right panel shows the intergenerational discount rate $\left(-\frac{d\beta(T)}{dT}\frac{1}{\beta(T)}\right)$, implied by these weights.

current generations.

Note that the non-re-distributional welfare weights and the welfare weights that equalize the welfare gain imply a relatively high intergenerational discount rate (right panel). This is a result of the fact that future income is very uncertainty and hence has a low present value. Choosing a discount rate of around 3 percent implies that the collective manager would prefer to re-distribute significant value from current generations to future generations, to compensate for the high level of uncertainty in the long-run (see figure 5.3).

APPENDIX E

ADDING SHORT-RUN RISK

First note that $D(\tau)$ given time t information can be written as:

$$D(\tau) = D(t)\exp\left\{\left(g_D - \frac{1}{2}\sigma_{lr}^2\right)(\tau - t) + \sigma_{lr}\int_t^\tau dz_{lr}(s)\right\} \times \exp\left\{e^{-\kappa_{sr}(\tau-t)}\log D_{sr}(t) + \sigma_{sr}\int_t^\tau e^{-\kappa_{sr}(\tau-s)}dz_{sr}(s)\right\} \tag{E1}$$

For the pricing kernel at time τ given time t information we have:

$$\frac{M(\tau)}{M(t)} = \exp\left\{-\left(r + \frac{1}{2}\phi_{lr}^2 + \frac{1}{2}\phi_{sr}^2\right)(\tau - t) + \phi_{lr}\int_t^\tau dz_{lr} + \phi_{sr}\int_t^\tau dz_{sr}\right\} \tag{E2}$$

Hence, the present value at time t of the time τ dividend cash-flow is:

$$PV_t(D(\tau)) = \mathbb{E}_t\left[D(\tau)\frac{M(\tau)}{M(t)}\right] = D_{lr}(t)\exp\{(-r + g_D + \phi_{lr}\sigma_{lr})(\tau - t)\} \times \exp\left\{e^{-\kappa_{sr}(\tau-t)}\log D_{sr}(t) + \frac{\sigma_{sr}^2 B_{sr}(2(\tau-t))}{4} + \sigma_{sr}\phi_{sr}B_{sr}(\tau - t)\right\} \tag{E3}$$

where $B_{sr}(x) = \frac{1}{\kappa_{sr}}\left(1 - e^{-\kappa_{sr}x}\right)$.

The price of a claim on the dividend stream is:

$$S(t) = \int_t^\tau PV_t(D(\tau))d\tau \tag{E4}$$

And the return on this claim is:

$$\frac{dS(t)}{S(t)} = (...)dt + \frac{dD_{lr}(t)}{D_{lr}(t)} + \frac{\int_t^\tau PV_t(D(\tau))e^{-\kappa_{sr}(\tau-t)}d\tau}{\int_t^\tau PV_t(D(\tau))d\tau}d\log D_{sr}(t) \tag{E5}$$

Which can be written as:

$$\frac{dS(t)}{S(t)} = \left(r - \sigma_{S,sr}\ \phi_{sr} - \sigma_{lr}\phi_{lr}\right)dt + \sigma_{S,lr}dz_{lr} + \sigma_{S,sr}dz_{sr} \tag{E6}$$

where we should keep in mind that $\sigma_{S,sr}$ depends on the current level of the short-run dividend component:

$$\sigma_{S,sr}(D_{sr}(t)) = \frac{\int_t^T PV_t(D(\tau))e^{-\kappa_{sr}(\tau-t)}d\tau}{\int_t^T PV_t(D(\tau))d\tau}\sigma_{sr} \tag{E7}$$