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SEARCH AND (IN-)ELASTIC REST UNEMPLOYMENT: AN ANLYTICAL FRAMEWORK

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YING ZHOU

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Abstract

This paper develops a tractable quantitative framework for analyzing sectoral labor reallocation and unemployment. The framework features analytical solutions for sectoral wages, employment and unemployment dynamics, as well as aggregate unemployment, which allows fast model estimation from labor market transition data and convenient counterfactual exercises to quantify the impact of sectoral shocks and the relevant labor market institutions. In particular, the framework accommodates two important features of the data: (i) heterogeneous response of sectoral labor market dynamics to shocks; and (ii) persistent unemployment accompanied by persistently depressed wages for certain sectors. I apply the framework to test the sectoral shifts hypothesis and find that a 1% increase in sectoral shock dispersion would raise aggregate unemployment by 0.55%. The result is consistent with the observation of slow employment recovery post recent recessions with job polarization.

Key words: unemployment, sectoral shock dispersion, labor reallocation

1 Introduction

In the modern era of economic transformation, we see the rise and fall of sectors in the economy. The Rust Belt is one epitome. People have blamed automation and import competition for the fall of Detroit. However, as manufacturing employment underwent contraction in the Rust Belt, we also see it expanding in Sun Belt states such as Alabama and Texas. For example, on September 13, 2008, The Wall Street Journal comments:

"Yes, Michigan lost 83,000 auto manufacturing jobs during the past decade and a half, but more than 91,000 new auto manufacturing jobs sprung up in Alabama, Tennessee, Kentucky, Georgia, South Carolina, Virginia and Texas."

This naturally raises the questions of why regional labor markets respond to similar shocks differently, and what are the underlying sources of heterogeneity that lead to the differential responses of regional labor markets to shocks. Also, the manufacturing industry in the Rust Belt has been declining ever since the 1960s with import penetration from Japan, devaluation of the US dollars and later import competition from China. Yet, unemployment in the Rust Belt has been persistently high, even with persistently depressed wages. Why are some workers in such persistently depressed industries or regions so reluctant to reallocate to more productive ones, and would rather stay unemployed?

This paper builds a tractable quantitative framework to address such problems related to sectoral labor reallocation and unemployment. In particular, the framework captures two important features of the data: (1) heterogeneous responses of sectoral labor market dynamics to shocks; and (2) persistently depressed wages accompanied by persistent unemployment in certain sectors. In addition, the framework features analytical solutions for sectoral wages, employment and unemployment dynamics, as well as aggregate unemployment, so that one could conveniently estimate the model from labor market transition data and quickly perform counterfactuals to quantify the impact of sectoral shocks on labor market dynamics and the welfare implication of labor market institutions.

The framework is one variation of the island search model developed by Lucas and Prescott (1974) [41] and is directly built upon Alvarez and Shimer (2011) [3]. First, I relax the assumption of perfectly elastic intrasectoral labor supply (employment) in Alvarez and Shimer (2011) [3], so that sectoral wages are more informative of unobserved sectoral productivity. Second, imperfect and isoelastic within-sector employment elasticity also induces sectoral employment and different types of unemployment to comove with wages in a clean fashion. These two properties enable structural estimation of the model from labor market transition data, such as the Survey of Income and Program Participation (SIPP). Since the model features analytical solutions, one can quickly perform counterfactuals for quantitative analysis on sectoral labor reallocation and unemployment.

More specifically, two types of labor reallocation through unemployment happen in the model, one within sectors and the other across sectors. **Search unemployment** is the intersectoral labor reallocation that moves workers across sectors facing sectoral productivity differentials, yet is costly and irreversible, à la Lucas and Prescott (1974) [41]. By contrast, **rest unemployment** is the intrasectoral labor reallocation that allows an unemployed worker to go back to her previous sector, thus is costless and reversible, à la Jovanovic (1987) [37]. In Alvarez and Shimer (2011) [3], the sectoral labor force is fully employed when sectoral productivity is sufficiently high (call such productivity space full-employment region). However, cumulative adverse sectoral shocks would gradually push wages down, due to the competitive pressure from CES aggregation of sectoral labor outputs, to a reservation wage where further pressure on wages is relieved firstly by reallocating sectoral employment to rest unemployment (call the productivity space rest region), and eventually by dismissing some sectoral labor into search unemployment (call the productivity space search region). Within the rest region, sectoral shocks are perfectly offset by rest unemployment to keep sectoral wages at the reservation value; that is, rest unemployment (or intrasectoral labor reallocation) is perfectly elastic with respect to wage. But in my paper, intrasectoral labor reallocation is assumed to be imperfectly elastic, causing rest unemployment and wage to comove with each other and jointly absorb the sectoral shocks in the rest region. Such comovement separates the wage threshold of rest unemployment from that of search unemployment. These distinct wage thresholds and comovement patterns of wage, employment, and search and rest unemployment across the productivity spaces provide a foundation for structural estimation of the model from labor market transition data.

Not only does the assumption of imperfect elasticity of intrasectoral labor reallocation facilitate the model estimation, but it also accommodates the data much better. One important empirical prediction of Alvarez and Shimer (2011) [3] is the relationship between sectoral wage persistence and the relative portion of search and rest unemployment. Since sectoral wage is very persistent, nearly unit-root in the data, Alvarez and Shimer (2011) [3] model the unobserved sectoral productivity process as permanent (Brownian motion). Among the two types of unemployment, only search unemployment can adjust the sectoral labor force to permanently alleviate wage pressure from accumulated negative productivity shocks, cause wages to bounce off the reservation value and decrease wage persistence, whereas rest unemployment has little impact on sectoral wage persistence. Given the observed wage persistence, Alvarez and Shimer (2011) [3] conclude that rest unemployment must consist of a large fraction of total unemployment (3/4 according to their calibration). However, rest unemployment accounts for less than half of the total unemployment in the SIPP data, which is rather consistent with the assumption of imperfect elasticity of intrasectoral labor reallocation: Since wages share part of the adverse shocks with rest unemployment, wage persistence can still be justified with less rest unemployment present. In fact, the counterfactual exercise in section 6.1 shows that matching the share of rest unemployment and the wage persistence in the data at the the same time is impossible under the assumption of perfect elastic rest unemployment as in Alvarez and Shimer (2011) [3].

Based on the estimated model, I examine the impact of sectoral shock dispersion on aggregate unemployment by varying the volatility of sectoral productivity growth in the model. I find that a 1% increase in sectoral shock dispersion would raise aggregate unemployment by 0.554%, which fits the observation of slow employment recovery post recent recessions with job polarization as in Jaimovich and Siu (2018) [36]. However, the underlying mechanism is very different from that in Jaimovich and Siu (2018) [36], who argue that aggregate unemployment rises due to the accelerated worker reallocation from routine to non-routine occupations during downturns, that is, an increase in reallocation across sectors. In my model, most of the increase comes from rest unemployment, which rises from 3.037% to 3.064% (0.90% increase), while search unemployment moves much less (from 3.612% to 3.622%). The strong response of rest unemployment and the weak reaction of search unemployment mitigate the discrepancy between the observation in Jaimovich and Siu (2018) [36] and the conclusion in Pilossoph (2014) [45] that sectoral shock dispersion has no impact on (search) unemployment. I discuss this point in more detail in section 7.1.

2 Literature

The impact of productivity shocks on labor market dynamics, especially unemployment fluctuation, is an everlasting subject. Standard devices, such as the vintage DMP model (Diamond (1982) [27], Mortensen and Pissarides (1994) [43]), allow economists to investigate the cyclical behavior of unemployment in a concise manner, as in Shimer (2005) [48], Hall and Milgrom (2008) [31], Hagedorn and Manovskii (2009) [30], etc. This line of research concentrates on the impact of aggregate rather than sectoral shocks, since the quantitative tractability of DMP models dissolves quickly with multiple sectors. One exception is Jaimovich and Siu (2018) [36], who first associate the slow recovery of employment in recent recessions with job polarization.

With the contemporary structural change due to episodes such as skill-biased technical change (e.g., Acemoglu (2002) [2]), job polarization (e.g., Autor and Dorn (2013) [10]) and uneven impact of trade across industries (e.g., Autor et al. (2016) [11]), more research has started to investigate the impact of sectoral shocks on labor market dynamics. Research along this line mostly adapts two types of models to characterize frictional labor reallocation across sectors. One is the Lucas and Prescott (1974) [41] island search model; the other is the Roy (1951) [47] model of occupational choice. Each has its strengths and weaknesses per se in application. Some applications also integrate the two models (sometimes the DMP model as well) to exploit their desired features in explaining the empirical moments of the data.

The island search model intuitively interprets unemployment as the search process to reallocate workers across sectors due to sectoral productivity differentials. Thus, one needs to track the joint distribution of both productivity and labor force across sectors over time when solving the equilibrium, the complexity of which grows exponentially with the number of sectors and productivity states. Projecting the model into micro moments of sectoral labor market dynamics is even harder, as there are limited analytical results for that purpose. Previous efforts mainly focus on calibration exercises with a small number of sectors to maintain tractability of the model (e.g., Garin et al. (2013) [29]).

The Roy model was originally designed to think about occupational choice of workers given demand for occupations and skill heterogeneity of workers. After labor economists structure the Roy model as a discrete choice problem[1] with later specification of idiosyncratic preference shocks as drawn from the Gumbel (T1EV) distribution, the model gained both analytical and quantitative tractability with little compromise between theory and data: The model moments can be precisely mapped into empirical micro moments to estimate the deep parameters, as the Lucas critique emphasizes. In the past decade, a series of macro labor literature has borrowed the tool from there to investigate the impact of sectoral shocks on intersectoral labor reallocation.[2] However, from the very beginning, the Roy model was designed to explain the relationship between sectoral wage differentials and sectoral employment, rather than unem-

[1]Borjas (1987) [16] is the first to formalize the Roy model mathematically. Heckman and Sedlacek (1985) [33] first structure the Roy model into a discrete choice problem as analyzed by Domencich and McFadden (1975) [28].

[2]Artuc et al. (2010) [6], Traiberman (2018) [51] and Caliendo et al. (2019) [20] are some of the excellent examples.

ployment. It is difficult to distinguish between unemployment (which mainly involves labor reallocation through frictional search process) and non-employment (which contains both unemployment and non-participation in the labor market and does not necessarily involve labor reallocation activities) through the lens of the Roy model.

In other words, one has to return to the island search model to study the impact of sectoral shocks on unemployment. One compromising solution is to integrate the Roy model with the island search model, maintain empirical tractability by modeling a limited number of sectors, such as in Pilossoph (2014) [45], or give up empirical tractability of the model in general, but rather strengthen the quantitative analysis with rigorous reduced-form evidence, as in Chodorow-Reich and Wieland (2018) [23].

The perfect solution would be a framework that allows us to think about the model and the data in a coalescing manner, as researchers have structured the Roy model. We would like to capture the micro moments in the data through the lens of our models in a clean fashion, which helps us pin down the deep parameters in the model with econometric exercises in turn. To this end, I find the specification of the sectoral productivity process as Brownian moition useful, which allows Alvarez and Shimer (2011) [3] to obtain analytical equilibrium solutions. Yet their model is still not empirically tractable, as its sectoral dynamics depend on unobserved sectoral productivity. By introducing intrasectoral employment elasticity, my paper gains empirical tractability for structural estimation, reserves theoretical tractability for fast counterfactuals, and enriches sectoral labor market dynamics to better fit the data. To the best of my knowledge, this is the first paper that could conveniently estimate the island search model with more than two sectors.

As mentioned before, on the theoretical side, this paper belongs to the class of island search models with later development by Alvarez and Shimer (2011) [3].[3] Recently, researchers have started to pay attention to the role of search and rest unemployment in labor market dynam-

[3] Other theoretical development includes Rogerson (2005) [46] and Chang (2011) [22].

ics, as in Alvarez and Shimer (2012, 2014) [4] [5] and Carrillo-Tudela and Visschers (2013) [21]. Jovanovic (1987) [37] is the first paper to distinguish between search and rest unemployment. Alvarez and Shimer (2011) [3] specify the sectoral shock process as Brownian motion to obtain closed-form equilibrium solutions and find that both types of unemployment are important for explaining sectoral wage persistence. In their later papers in 2012 and 2014, they also introduce more realistic features such as human capital and unions into the framework. Carrillo-Tudela and Visschers (2013) [21] discuss the role of search and rest unemployment in the cyclicality of unemployment. This paper builds directly upon Alvarez and Shimer (2011) [3] with the relaxed assumption on intrasectoral labor reallocation elasticity. Compared to Alvarez and Shimer (2011) [3], where rest unemployment perfectly offsets adverse sectoral shocks to keep sectoral wage constant at the lower end of the productivity distribution, both wages and rest unemployment absorb part of the adverse sectoral shock in my paper, which generates realistic comovement of wages, employment and rest unemployment at the sectoral level that could be directly mapped into the data for model estimation.

On the empirical side, my paper is related to the emerging literature on structual estimation of analytical frameworks of labor supply, such as Keane and Rogerson (2011) [38], Heathcote, Storesletten and Violante (2014) [32] and Attanasio et al (2017) [9]. Similar to these papers, my paper exploits analytical moment conditions out of the model for structural estimation, thus creating a strong tie between theory and empirics. With the estimated model and the analytical equilibrium solutions, I can quickly perform counterfactuals to quantify the impact of sectoral shocks as well as welfare consequences of labor market institutions.

One counterfactual exercise in this paper tests the sectoral shifts hypothesis proposed by Lilien (1982) [39] that aggregate unemployment would increase with sectoral shock dispersion, as workers are more incentivized to reallocate across sectors with the increase in sectoral productivity differentials. Abraham and Katz (1986) [1] later argue that the aggregate demand shock is more important. Over the years, researchers have provided various reduced-form evidence[4] and calibration exercises[5], mostly in favor of the hypothesis. The discussion has been revived recently, as job polarization has dramatically impacted the US labor market. Jaimovich and Siu (2018) [36] are the first to connect job polarization to the slow recovery of employment post recent recessions. Yet Pilossoph (2014) [45] brings up an interesting angle that if the gross flows of labor reallocation (due to both sectoral and idiosyncratic shocks) are large enough, the net flows from sectoral productivity differentials could be well accommodated by the gross flows, thus creating no more unemployment upon sectoral shock dispersion. In this paper, gross flows also exceed net flows, but the counterfactual based on the estimated model still indicates that unemployment increases with sectoral shock dispersion. The reason is that the gross flows in my model come from the endogenous labor reallocation both within and across sectors, in response to sectoral shocks, whereas in Pilossoph (2014) [45], gross flows come from exogenous separation of workers from firms, thus are much less responsive to sectoral shock dispersion. An interesting observation though, related to the intuition of Pilossoph (2014) [45], is that the impact of sectoral shock dispersion would diminish when the exogenous separation rate of workers from sectors rises.

The paper will proceed as follows. Section 3 introduces a stylized two-period model to illustrate the main intuition of the full model. Section 4 elaborates the full continuous-time model with sectoral shock process specified as Brownian motion, as well as the analytical equilibrium solutions. Section 5 estimates the model. Section 6 validates the model with untargeted moments. Section 7 conducts counterfactual exercises based on the estimated model. Section 8 concludes.

[4] Examples include Blanchard and Diamond (1989) [13], Davis Haltiwanger (1990, 1992) [25] [26], Brainard and Cutler (1993) [17], Loungani and Rogerson (1989) [40], Mills et al. (1995) , Shin (1997) [49] and and Cortes et al. (2016) [24].

[5] For example, Garin et al. (2013) [29].

3 Stylized Model

In this section, I present a two-period stylized model to illustrate the intrasectoral and intersectoral labor reallocation problems as well as the key intuition for the continuous-time model in section 4.

3.1 Setup

A unit measure of workers indexed by i is located across a continuum of islands $j \in [0,1]$. Each island produces a heterogeneous good with linear technology[6] in labor and idiosyncratic labor productivity x_j, i.e.

$$Y_j = x_j E_j$$

where Y_j is the output of island j, and E_j is the employment on the island. One can think of the islands as any reasonable partition of the economy that involves frictional labor reallocation in between, such as firms, industries, occupations, geographic locations or some combination of them.

A final good producer aggregates island outputs into final output Y in a CES fashion

$$Y = \left(\int_0^1 Y_j^{(\theta-1)/\theta} dj \right)^{\theta/(\theta-1)}$$

Constant θ is the elasticity of substitution between island outputs y_j. The price for island good p_j is decided competitively

$$p_j = (Y/y_j)^{1/\theta}$$

[6] The linear technology of labor production is for convenience and could be easily adapted with some returns to scale.

Assume a competitive labor market within each island. Thus, island wage w_j is

$$w_j = (Y/E_j)^{1/\theta} x_j^{1-1/\theta}$$

3.2 Intra-island labor allocation problem

First consider just one period. At the beginning of the period, each island j is endowed with labor force L_j and hit by i.i.d. labor productivity shock $x_j \sim f(\cdot)$. Call $\log x_j$ the **primitive shock** throughout the paper. Next, each worker i is hit by i.i.d. preference shock $d_i \sim U[0,1]$, which generates leisure utility $b_R d_i^{1/\eta}$. Constants $b_R > 0$, $\eta > 0$. Last, workers choose either to work and get paid wage w_j or to enjoy leisure, that is,

$$\max\left(w_j, b_R d_i^{1/\eta}\right)$$

Call the leisure activity here **rest unemployment** to differentiate it from the leisure activity due to inter-island labor reallocation (**search unemployment**) to be illustrated later. Note that rest unemployment does not necessarily reflect the absolute preference for leisure in reality. It could also be micro-founded with heterogeneous productivity of workers or hours availability, causing some workers to choose leisure over work when wage is not sufficiently high.

In equilibrium, workers must either strictly prefer working (if their wages are above the maximal possible value from leisure), or they are indifferent between employment and rest unemployment. Denote island employment rate $e_j \equiv E_j/L_j$. The work-rest decision implies the following labor supply within island j

$$\log e_{jt} = \begin{cases} 0 & \text{, if } w_{jt} \geq b_R, \\ \eta(\log w_{jt} - \log b_R) & \text{, if } w_{jt} < b_R. \end{cases} \tag{1}$$

Proposition 1: Island wage and employment rate only depend on $\omega_j \equiv [\log Y + (\theta - 1)\log x_j - \log L_j]/\theta$, the potential log wage if all island labor is employed.

Proof: The competitive wage determination (labor demand) from CES aggregation is

$$\log w_j = \omega_j - \log e_j/\theta. \tag{2}$$

Putting (1) and (2) together,

$$\begin{cases} \log e_j = 0,\ \log w_j = \omega_j, & \text{if } \omega_j \geq \log b_R, \\ \log e_j = \frac{\theta\eta(\omega_j - \log b_R)}{\theta+\eta},\ \log w_j = \frac{\theta\omega_j + \eta \log b_R}{\theta+\eta}, & \text{if } \omega_j < \log b_R. \end{cases} \tag{3}$$

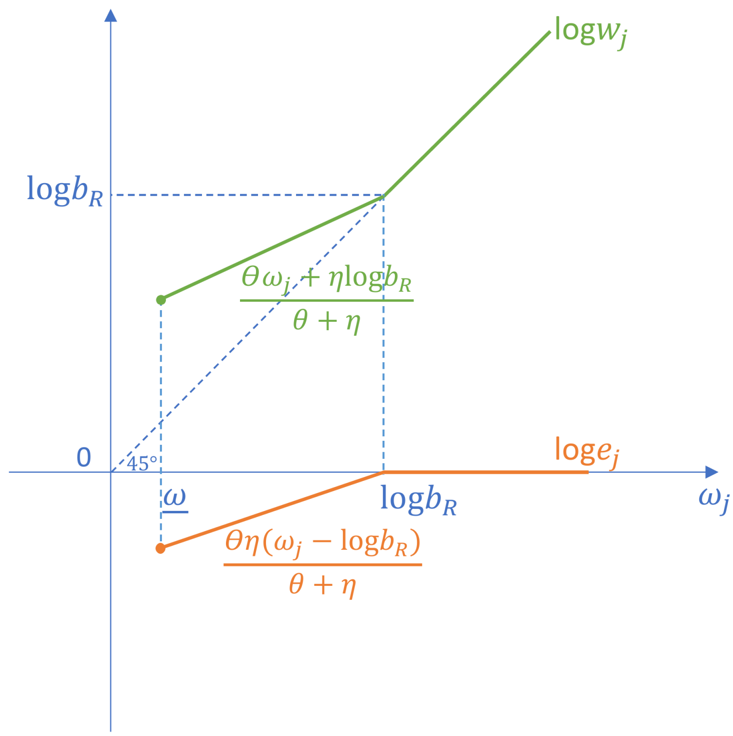
Q.E.D.

Throughout the paper, call the full-employment log wage ω_j the **adjusted productivity**, which accounts for both the primitive labor productivity shock and the island labor force, and summarizes the wage pressure on the island from CES aggregation. The fluctuation in the adjusted productivity induces intra-island labor reallocation between employment and rest unemployment when the island is hit by labor productivity shocks in the dynamic model. I will show that the adjusted productivity alone could determine both intra- and inter-island labor reallocation, thus is the only state variable that matters for islanders in the dynamic model.

Figure 1 illustrates the island wage and employment schedule (equation (3)). Island j absorbs the adjusted productivity shock ω_j through wage and island employment e_j. When the adjusted productivity is sufficiently high (above $\log b_R$), the island exhibits full employment and the adjusted productivity shock passes on to wage alone. If the adjusted productivity is below $\log b_R$, some workers with relatively high taste for leisure switch to rest unemployment. And the adjusted productivity shock is absorbed by wage and employment jointly. Therefore, $\log b_R$

is the rest-unemployment threshold of the adjusted productivity, below which rest unemployment occurs. θ captures the demand elasticity for sectoral employment and η captures the supply elasticity of sectoral employment (in the short run). Together they determine the share of sectoral shocks absorbed by wage or employment when rest unemployment is present.

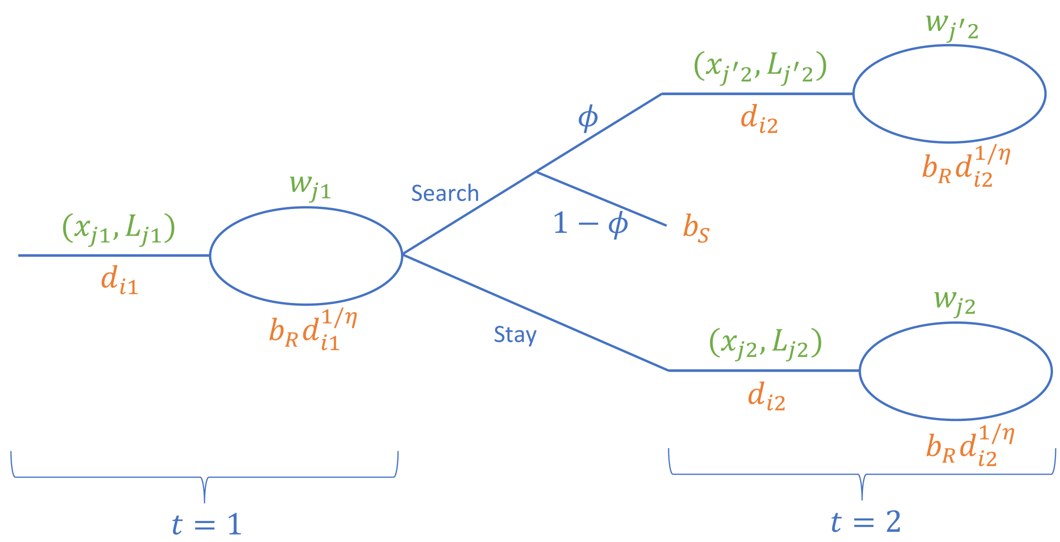
Figure 1. Island wage and employment schedule



3.3 Inter-island labor reallocation problem

Now let the economy go on for two periods. Within each period is the intra-island labor reallocation problem, whereas between periods, an inter-island labor reallocation problem, in the spirit of Lucas and Prescott (1974) [41], needs to be solved as well. Figure 2 illustrates the timeline of the problem.

Figure 2. Timeline of inter-island labor reallocation problem



In period 1, each island j is endowed with labor force $L_{j1} \sim g_L(\cdot)$ and hit by i.i.d. labor productivity $x_{j1} \sim g_x(\cdot)$, which leads to the density of adjusted productivity $\omega \sim f(\cdot)$ across workers. Assume $f(\cdot)$ is continuous. An island worker i is hit by a preference shock for leisure d_{i1}, i.i.d. $\sim U[0,1]$. Last, workers choose between employment and rest unemployment.

Between period 1 and period 2, an islander chooses to stay on her island or leave the island to search for a new one. Call the leisure activity generated by such inter-island labor reallocation **search unemployment**. Search is an irreversible decision. Once the islander leaves her island, she cannot come back in period 2. [7]With exogenous probability ϕ, a searcher relocate to a new island with expected utility $\bar{U}$, which is endogeneously determined in equilibrium. Assume $\bar{U}$ is high enough so that some workers choose search over stay.

In period 2, unsuccessful searchers obtain leisure value b_S. Assume b_S is low enough so that some workers choose stay over search. After the inter-island labor reallocation, island j has new labor force L_{j2} and is hit by labor productivity shock x_{j2}, $\log x_{j2} = \log x_{j1} + \epsilon_{j2}$, ϵ_{j2} i.i.d.

[7]The chance of going back to her previous island through search is zero under random search, since her island is but one in the continuum of islands; in the case of directed search, a worker would only leave a low-productivity island to search for the most productive islands.

$\sim N(\mu_x, \sigma_x^2)$. That is, primitive shock $\log x_j$ follows a random walk with drift μ_x and standard deviation σ_x[8]. Next, islander i is hit by idiosyncratic preference shock $d_{i2} \sim U[0,1]$, $d_{i2} \perp d_{i1}$. Last, islanders again choose between employment and rest unemployment.

Lemma 1: Expected utility in period 2 only depends on the adjusted productivity ω_{j2} (upon realization of ω_{j2}).

Proof: Solution to the intra-island labor reallocation problem (3) implies both the wage rate and employment decision depend only on ω_{j2}. With idiosyncratic preference shocks for leisure, the expected utility in period 2 must also depend only on ω_{j2}.

Q.E.D.

The adjusted productivity summarizes the wage pressure from CES aggregation in the short run. If adjusted productivity ω_{jt} is sufficiently high (exceeding $\log b_R$), the island with wage equal to ω_{jt} will have full employment. Otherwise, intra-island labor reallocation (rest unemployment) would share some of the wage pressure.

Now consider a partial equilibrium where demand for final output Y is given and stays constant across the two periods, since in the dynamic model, we would only look for the stationary equilibrium where aggregate quantities are held constant. [9] Also, only consider directed search for the moment; that is, searchers only look for islands with the highest expected utility in period 2. In equilibrium, some workers from less productive islands may want to leave their island, relieving the wage pressure, whereas high productivity islands may see inflow of workers pushing down their wages.

Proposition 2: Suppose the demand for final output Y stays constant across the two periods. Then, inter-island labor reallocation only depends on period 1 adjusted productivity ω_{j1}.

[8]One can interpret μ_x as the average growth rate of island labor productivity and σ_x as the volatility in that growth.

[9]Another way to think about the constant output is that the inter-island reallocation problem only happens among a small fraction of the islands, so that aggregate output is unaffected in the general equilibrium.

Proof: See Appendix A.

Q.E.D.

Intuition for the proposition is as follows. First, according to Lemma 1, intra-island labor reallocation problem only depends on the adjusted productivity in each period. Next, the inter-island labor reallocation problem is

$$\max\left\{\mathbb{E}[U(\omega_{j2})|x_{j1},L_{j1};\underline{U},\bar{U}],\phi\bar{U}+(1-\phi)b_s\right\}$$

s.t.

$$\omega_{j2}=\omega_{j1}+\frac{\theta-1}{\theta}\epsilon_{j2}-\frac{1}{\theta}\Delta\log L_{j2}.$$

$\bar{U}$ is the expected utility in period 2 for the most productive islands, which is endogenously decided in equilibrium. For islands with expected utility in period 2 above $\bar{U}$ absent from inter-island labor adjustment, they will see inflow of search-unemployed workers pushing down the expected utility to $\bar{U}$. So $\bar{U}$ is also the expected utility in period 2 for successful searchers. $\underline{U}$ is the expected utility in period 2 for the most unproductive islands, also endogenously decided in equilibrium. For islands with expected utility in period 2 below $\underline{U}$ absent from inter-island labor adjustment, they will see outflow of workers into search unemployment, which relieves the wage pressure and raises expected utility to $\underline{U}$. So $\underline{U}$ is also the expected utility in period 2 for the marginal stayers.

We need to prove the expected utility in period 2 depends on period 1 adjusted productivity ω_{j1} alone, rather than both x_{j1} and L_{j1}, i.e. $\mathbb{E}[U(\omega_{j2})|x_{j1},L_{j1};\underline{U},\bar{U}]\equiv\mathbb{E}[U(\omega_{j2})|\omega_{j1};\underline{U},\bar{U}]$. I prove by guess and verify. Guess ∃ search and inflow thresholds $(\underline{\omega}, \bar{\omega})$ and the labor adjustment rule

$$\Delta \log L_{j2} = \begin{cases} \theta(\omega_{j1} - \bar{\omega}) & \text{, if } \omega_{j1} > \bar{\omega}, \\ 0 & \text{, if } \omega_{j1} \in [\underline{\omega}, \bar{\omega}], \\ -\theta(\underline{\omega} - \omega_{j1}) & \text{, if } \omega_{j1} < \underline{\omega}. \end{cases} \tag{4}$$

Then, the law of motion of the adjusted productivity $\omega_{j2} = \omega_{j1} + \frac{\theta-1}{\theta}\epsilon_{j2} - \frac{1}{\theta}\Delta \log L_{j2}$ depends only on ω_{j1}. Therefore $\mathbb{E}[U(\omega_{j2})|x_{j1}, L_{j1}; \underline{U}, \bar{U}] \equiv \mathbb{E}[U(\omega_{j2})|\omega_{j1}; \underline{\omega}, \bar{\omega}]$. Then verify that each worker alone would not benefit from deviating from such a policy. Therefore, inter-island labor reallocation (search unemployment) also only depends on the adjusted productivity in equilibrium. The trick here is that any endogenous labor force adjustment must be isoelastic to the adjusted productivity shock, so that the random walk of the primitive shock translate into a random walk of the adjusted productivity. And I no longer need to track the primitive shock and the island labor force separately. Both the homotheticity of the island output aggregation and the random walk of the primitive shock are important to obtain the result. The intuition extends to the continuous-time model with Brownian motion, which is essentially a random walk in continuous time.

4 Continuous-time Model with Brownian Motion

The full model is directly built upon Alvarez and Shimer (2011) [3], which renders the analytical results for the model estimation and the counterfactual exercises in the later sections.

Time is continuous with an infinite horizon. Each island produces a heterogeneous good with linear technology in labor. Islands are continuously hit by idiosyncratic labor productivity shock

$x_j(t)$. Primitive shock $\log x_j(t)$ follows Brownian motion:

$$d \log x_j(t) = \mu_x dt + \sigma_x dZ_j(t),$$

with constant drift μ_x and standard deviation $\sigma_x > 0$. $Z_j(t)$ is a standard Wiener process. An island exits the economy at exogenous rate δ; a new island emerges in place with exogenous initial productivity x_0 (and endogenous initial labor force L_0). Once an island exits the economy, all its workers need to search for new islands. [10]

Fluctuation in island labor productivity potentially has two impacts on its labor market. First, it induces workers within islands to reallocate between employment and rest unemployment. Second, it causes productivity differentials across islands which may incentivize workers to reallocate across islands through search unemployment.

The final good producer aggregates island outputs according to

$$Y(t) = \left(\int_0^1 y_j(t)^{(\theta-1)/\theta} dj \right)^{\theta/(\theta-1)}.$$

Prices and wages are decided competitively.

4.1 Labor market transition

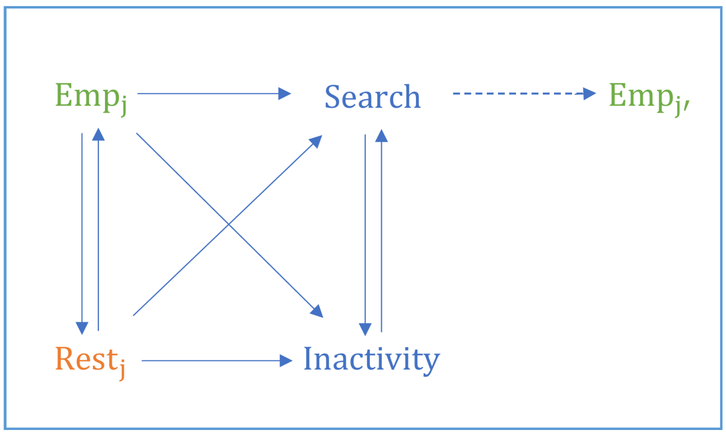
There are four labor market statuses in the economy: employment, rest unemployment, search unemployment and inactivity.[11] Figure 3 illustrates transition across the four statuses. On any island j, workers are either employed or rest-unemployed. They can move freely between the two statuses. They can also choose to leave their islands without incurring any cost. But once

[10]The exit and emergence of islands keep the distribution of island productivity constant over time in the stationary equilibrium. This set up is mainly for tractability, but is also empirically supported.

[11]Inactivity is not necessary in the intra- and inter-island labor reallocation problems, but it is useful for pinning down the value functions in an intuitive manner, as we will see in section 4.3.

they leave an island, workers become memoryless and do not come back. Off island, workers can transit freely between inactivity and search unemployment, but they can only relocate to a new island j' through a frictional search process (search unemployment), which could be either random or directed.[12]

Figure 3. Transition across labor market statuses



4.2 Household composition and preference

A continuum of representative households, each with a unit measure of workers, are located across the four labor market statuses and across the islands. Among them,

- $L(t)$ workers are located on one of the islands, with $L_j(t)$ workers on island j;
- $E(t)$ workers are employed, with $E_j(t)$ workers employed on island j;
- $R(t) \equiv L(t) - E(t)$ workers are rest-unemployed, with $R_j(t) \equiv L_j(t) - E_j(t)$ workers rest-unemployed on island j;
- $S(t)$ workers are search-unemployed and looking for a new island; they are not affiliated with any island;

[12]Search friction here can account for physical search activities or skill acquisition to enter new sectors in reality.

- Lastly, workers of measure $I(t) \equiv 1 - L(t) - S(t)$ are inactive and out of the labor force.

The preference of a representative household is given by

$$\int_0^\infty e^{-\rho t} \left\{ u(C(t)) + b_I I(t) + \int_0^1 b_{Rj}(t) R_j(t) dj + b_S S(t) \right\} dt$$

s.t.

$$C(t) = \int_0^1 w_j(t) E_j(t) dj$$

$$\int_0^1 [E_j(t) + R_j(t)] dt + S(t) + I(t) = 1.$$

ρ is the discount rate. $C(t)$ is the household consumption. Constant b_I is the flow utility of inactivity and constant b_S that of search unemployment. $b_{Rj}(t)$ is the flow utility of rest unemployment on island j at time t. It depends on the island labor market condition at the time, which will be specified in detail in section 4.4. $w_j(t)$ is the wage rate on island j.

The large household structure allows full risk-sharing. Labor income from all its workers across the island is pooled together to finance the household's consumption. In this way, I can separate workers' labor reallocation problem from their consumption-savings choice, which are not so relevant in general.

For the rest of the paper, I will only consider the stationary equilibrium where aggregate quantities are held constant.

4.3 Value of workers off islands

Given flow utility of inactivity b_I and search unemployment b_S, the value of a worker outside of islands is decided by the following Hamilton–Jacobi–Bellman (HJB) equation

$$\rho \underline{V} = \max\left\{b_I, b_S + \phi(\bar{V} - \underline{V})\right\}, \tag{5}$$

where b_I is the flow utility of inactivity and b_S that of search unemployment. $\bar{V}$ is the expected value on a new island for the search-unemployed workers. With exogenous job finding rate ϕ, a worker gets the option value of landing a job on a new island. If search is directed, $\bar{V}$ is the highest value for island workers; if search is random,[13] $\bar{V}$ is the mean value across island workers. Notice that $\underline{V}$ bounds worker value from below. Since workers outside of islands can choose between inactivity and search unemployment freely, in equilibrium, workers must be indifferent between the two. Therefore, $\underline{V} \equiv b_I/\rho$. And the indifference condition $b_I = b_S + \phi(\bar{V} - \underline{V})$ pins down $\bar{V}$.

4.4 Value of workers on islands

The value of a worker on island j is given by

$$\begin{aligned}\rho V(\omega_j(t)) =& \underbrace{\max(w_j(t), b_R e_j(t)^{1/\eta})}_{\text{intratemporal optimization}} + \underbrace{(\delta + q)[\underline{V} - V(\omega_j(t))]}_{\text{exogenous separation}} \\ &+ \underbrace{\mu V^{'}(\omega_j(t)) + \frac{1}{2}\sigma^2 V^{''}(\omega_j(t))}_{\text{productivity fluctuation}},\end{aligned}$$

which has three components: flow value from the intratemporal optimization between work

[13] Random search as in Burdett and Vishwanath (1988) [19] with the inflow of workers proportionate to the island labor force.

and rest unemployment, option value from the exogenous separation due to either exit of islands at rate δ or quit of workers at rate q, and the option value due to productivity fluctuation.

First is the intratemporal component. $w_j(t)$ is the wage rate (measured in the marginal utility of consumption) of island j at time t. $e_j(t) = E_j(t)/L_j(t)$ is the employment rate of island j. b_R captures the level of utility from rest unemployment. η captures the externality of island employment on the utility of rest. $\eta > 0$ so that rest is more attractive when fewer peers are resting. [14] At anytime, workers on island j choose freely between work with payoff $w_j(t)$ and rest unemployment with utility $b_R e_j(t)^{1/\eta}$. The externality assumption is the main deviation of the dynamic model here from that in Alvarez and Shimer (2011) [3]. But it brings much richer island labor market dynamics and accommodates the data much better. I will illustrate this point in the quantitative exercises in section 5.

CES aggregation of island outputs and linear production of labor imply

$$\log w_j(t) = \omega_j(t) - \log e_j(t)/\theta,$$

where $\omega_j(t) \equiv [\log Y + (\theta-1)\log x_j(t) - \log L_j(t)]/\theta + \log u^{'}(C)$[15] is the potential log wage if all island workers are employed, measured in utils. Again, call $\omega_j(t)$ the adjusted productivity of island j. In stationary equilibrium, adjusted productivity $\omega_j(t)$ is the only state variable that matters for workers' reallocation problems. The intuition is similar to the stylized model. Analogous to the primitive shock $\log x_j(t)$, $\omega_j(t)$ follows Brownian motion:

$$d\log\omega_j(t) = \mu dt + \sigma dZ_j(t)$$

[14]This assumption is just to micro-found the intra-island employment elasticity; in reality, the responsiveness of employment on the island to wages could have other explanations, such as idiosyncratic preference shock for leisure as in section 3. I model this way for tractability (instead of tracking distribution of idiosyncratic preference of workers for each island across time).

[15]Note that in stationary equilibrium, aggregate output Y and consumption C stay constant, while island primitive shock $x_j(t)$ and island labor force $L_j(t)$ fluctuate over time. I will omit the time argument t later wherever there is no confusion.

with drift μ and standard deviation σ, if no endogenous adjustment of island labor force $L_j(t)$ occurs.

For directed search,

$$\mu = \mu_d = \frac{(\theta-1)\mu_x + q}{\theta}, \sigma = \frac{|\theta-1|}{\theta}\sigma_x.$$

For random search,

$$\mu = \mu_r(i_r) = \frac{(\theta-1)\mu_x + q - i_r}{\theta}, \sigma = \frac{|\theta-1|}{\theta}\sigma_x.$$

One can see that the adjusted productivity process is simply a linear transformation of the primitive shock process. Exogenous quit rate q affects $\omega_j(t)$ through relieving the wage pressure from final aggregation of island outputs. For random search, workers flow into any island at an endogenous rate i_r, determined by the equilibrium flow balance of workers.

Lastly, the Brownian motion of $\omega_j(t)$ renders the clean expression of the value function for island workers with the impact of the fluctuation of the adjusted productivity summarized by the last two terms $\mu V^{'}(\omega_j(t)) + \frac{1}{2}\sigma^2 V^{''}(\omega_j(t))$ from Ito's Lemma.

Omit the time variable t moving forward wherever there is no confusion.

4.5 Imperfectly elastic rest unemployment

As in the stylized model, the following wage and employment schedule holds.[16]

$$\begin{cases} \log e_j = 0, \ \log w_j = \omega_j, & \text{if } \omega_j \geq \log b_R, \\ \log e_j = \frac{\theta\eta(\omega_j - \log b_R)}{\theta+\eta}, \ \log w_j = \frac{\theta\omega_j + \eta \log b_R}{\theta+\eta}, & \text{if } \omega_j < \log b_R. \end{cases}$$

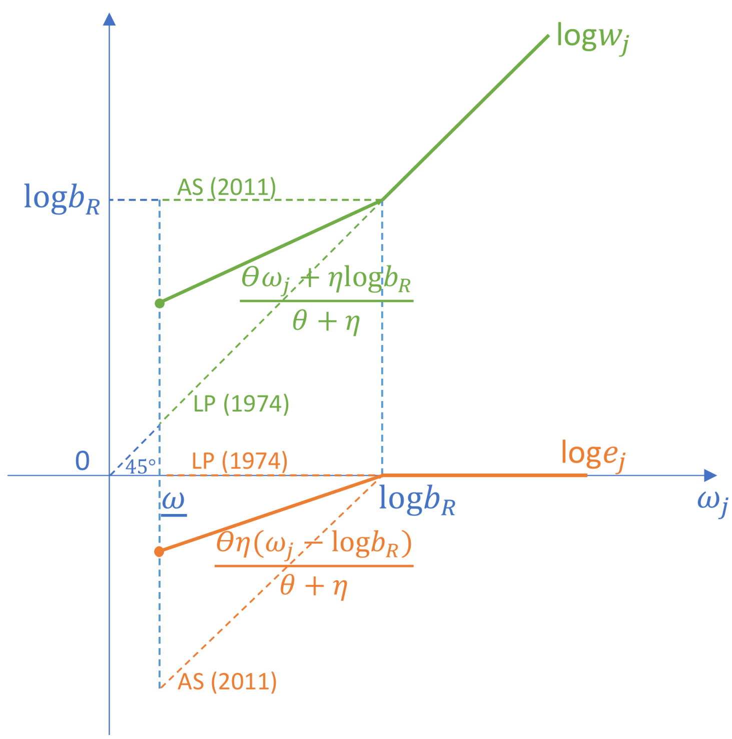
Essentially, employment and rest unemployment on an island consist an internal labor market that incurs little labor reallocation cost and determines the intratemporal island labor supply,

[16] One can also introduce a "natural rate of rest unemployment" e_0 to account for things like seasonal recall and demean $\log e_j$ with $\log e_0$. The estimation procedure of the model could be adjusted accordingly.

whereas the search unemployment pool serves as the external labor market for all the islands in the economy, which incurs costly labor reallocation in the form of search friction and affects the intertemporal island labor supply. Parameter η measures the labor supply elasticity of the internal labor market for an island j when it adjusted productivity $\omega_j \le \log b_R$. [17] If $\eta \to +\infty$, log employment rate $\log e_j \to \theta(\omega_j - \log b_R)$ and log wage $\log w_j \to \log b_R$. Rest unemployment perfectly offsets the primitive shock to keep wage constant, which boils down to Alvarez and Shimer (2011) [3]. If $\eta \to 0$, log employment rate $\log e_j \to 0$ and log wage $\log w_j \to \omega_j$; that is, there is always full employment on the island, which is the case in Lucas and Prescott (1974) [41]. In summary, η decides how adjusted productivity shock ω_j is absorbed by employment and wage in the short run and thus affects island labor market dynamics. Figure 4 below illustrates these ideas. The imperfect employment elasticity is the key to the structural estimation of the model, as I will explain in detail in section 5.

[17] The key assumption of the paper is the imperfect employment elasticity but not the employment externality. One could also model the elasticity through seniority rules of hiring and layoff (e.g., Alvarez and Shimer (2014) [45]), hours requirement (e.g., Keane and Rogerson (2011) [38]), or idiosyncratic labor productivity shocks as in the stylized model, etc. Here I model this way for tractability.

Figure 4. Lucas and Prescott (1974) versus Alvarez and Shimer (2011)



Now, the value function of island workers can be written as

$$(\rho+\delta+q)V(\omega_j) = \exp\left[\max\left(\omega_j, \frac{\theta\omega_j+\eta\log b_R}{\theta+\eta}\right)\right] + (\delta+q)\underline{V} + \mu V^{'}(\omega_j) + \frac{1}{2}\sigma^2 V^{''}(\omega_j) \tag{6}$$

The adjusted productivity ω_j is clearly the only state variable of workers on island j in the stationary equilibrium, since the flow utility and option value of island workers only depends on ω_j. In equilibrium, $V(\underline{\omega}) = \underline{V}$ determines the search unemployment threshold $\underline{\omega}$, below which island workers are better off leaving the island and searching for new ones. In fact, the endogenous exit of workers into search unemployment keep the adjusted productivity exactly at the boundary $\underline{\omega}$. Similarly, $V(\bar{\omega}) = \bar{V}$ determines the inflow unemployment threshold $\bar{\omega}$, above which search-unemployed workers would flow into the island and push down the adjusted productivity to $\bar{\omega}$.

4.6 Stationary equilibrium with directed search

Denote the density function of stationary distribution of workers across adjusted productivity $f(\omega)$, $\omega \in [\underline{\omega}, \bar{\omega}]$.

Definition 1: A stationary equilibrium with directed search is characterized by the aggregates $(Y, C, L, E, R, S, L_0, f(\omega), \underline{\omega}, \bar{\omega})$ and the value functions of workers (5) - (6), s.t.

1. *flow balance of workers across adjusted productivity* [18]

$$(q+\delta)f(\omega) = -\mu f^{'}(\omega) + \frac{\sigma^2}{2} f^{"}(\omega), \text{ if } \omega \in (\underline{\omega}, \bar{\omega}),$$

$$\frac{\sigma^2}{2} f^{'}(\underline{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\underline{\omega}) = 0,$$

$$\frac{\sigma^2}{2} f^{'}(\bar{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\bar{\omega}) = \delta \frac{L_0}{L},$$

2. *property of density function*

$$\int_{\underline{\omega}}^{\bar{\omega}} f(\omega) = 1,$$

3. *flow balance of search unemployment*

$$[q + \delta + \frac{\theta\sigma^2}{2} f(\underline{\omega})] L = \phi S, \tag{7}$$

[18]These equations are also called Kolmogorov forward equations.

4. *aggregation of rest unemployment and output*

$$\frac{R}{L} = \int_{\underline{\omega}}^{\log b_R} \left[1 - \exp\left(\frac{\theta\eta(\omega - \log b_R)}{\theta + \eta}\right)\right] f(\omega) d\omega, \tag{8}$$

$$\frac{Y}{L} = \frac{1}{u'(Y)} \left\{\int_{\underline{\omega}}^{\log b_R} \exp\left[\frac{\theta(1+\eta)\omega_j + (1-\theta)\eta \log b_R}{\theta + \eta}\right] f(\omega) d\omega + \int_{\log b_R}^{\bar{\omega}} \exp(\omega) f(\omega) d\omega\right\},$$

5. *goods and labor market clearing*

$$Y = C, \tag{9}$$

$$E + R = L, \tag{10}$$

$$L + S = 1, \tag{11}$$

6. *new-born island labor force L_0 decided by*

$$\bar{\omega} = [\log Y - (\theta - 1) \log x_0 - \log L_0]/\theta + \ln u'(C),$$

7. *value-matching and smooth-pasting of $V(\omega)$ at the boundaries $(\underline{\omega}, \bar{\omega})$*

$$V(\underline{\omega}) = \underline{V}, V'(\underline{\omega}) = 0$$

$$V(\bar{\omega}) = \bar{V}, V'(\bar{\omega}) = 0$$

To solve for the equilibrium, first solve for the stationary distribution of adjusted productivity across workers

$$f(\omega) = \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \exp(\lambda_i(\omega - \underline{\omega}))}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}, \quad (12)$$

where $\lambda_1 < 0 < \lambda_2$ solve $q + \delta = -\mu\lambda + \sigma^2\lambda^2/2$.[19] See appendix B for how to solve the value functions and the search and inflow thresholds $\underline{\omega}$ and $\bar{\omega}$. With the stationary distribution of adjusted productivity and the thresholds, it is easy to derive the aggregate quantities from the equilibrium conditions analytically, which ensures the tractability of the model in the quantitative exercises.

4.7 Stationary equilibrium with random search

Definition 2: A stationary equilibrium with random search is characterized by aggregates $(Y, C, L, E, R, S, L_0, i_r, f(\omega), \underline{\omega}, \omega_0)$ and value functions of workers (5) - (6) s.t.

1. *flow balance of workers across adjusted productivity*

$$(q + \delta - i_r) f(\omega) = -\mu f^{'}(\omega) + \frac{\sigma^2}{2} f^{"}(\omega), \text{ if } \omega > \underline{\omega},$$

2. *density around initial adjusted productivity ω_0 such that inital labor force L_0 is*

$$f^{'}_{-}(\omega_0) - f^{'}_{+}(\omega_0) = \frac{2\delta L_0}{\sigma^2 L},$$

3. *property of density function*

$$\int_{\underline{\omega}}^{+\infty} f(\omega) = 1,$$

4. *flow balance of search unemployment (7),*

[19]See Alvarez and Shimer (2011) [3] for detailed proof.

5. *aggregation of rest unemployment (8) and output*

$$\frac{Y}{L} = \frac{1}{u^{'}(Y)} \left\{ \int_{\underline{\omega}}^{\log b_R} \exp\left[\frac{\theta(1+\eta)\omega_j + (1-\theta)\eta \log b_R}{\theta+\eta}\right] f(\omega) d\omega + \int_{\log b_R}^{\infty} \exp(\omega) f(\omega) d\omega \right\},$$

6. *goods and labor market clearing (9) - (11),*

7. *new-born island labor force L_0 decided by*

$$\omega_0 = [\log Y - (\theta-1)\log x_0 - \log L_0]/\theta + \ln u'(C),$$

and value at new islands

$$V(\omega_0) = \bar{V},$$

8. *value-matching and smooth-pasting of $V(\omega)$ at the boundary $\underline{\omega}$*

$$V(\underline{\omega}) = \underline{V}, V^{'}(\underline{\omega}) = 0,$$

$$\bar{V} = \int_{\underline{\omega}}^{+\infty} V(\omega) f(\omega) d\omega,$$

Stationary density of workers across adjusted productivity is

$$f(\omega) = \begin{cases} \frac{(\lambda_1\lambda_2 + \frac{2\delta L_0}{\sigma^2 L}) \sum_{i=1}^{2} |\lambda_i + \theta| \exp(\lambda_i(\omega - \underline{\omega}))}{\theta(\lambda_2 - \lambda_1)} & \text{, if } \omega \in [\underline{\omega}, \omega_0], \\ \frac{(\lambda_1\lambda_2 + \frac{2\delta L_0}{\sigma^2 L}) \sum_{i=1}^{2} |\lambda_i + \theta| \exp(\lambda_i(\omega - \underline{\omega}))}{\theta(\lambda_2 - \lambda_1)} & \\ + \frac{\frac{2\delta L_0}{\sigma^2 L}[\exp(\lambda_1(\omega-\omega_0)) - \exp(\lambda_2(\omega - \omega_0))]}{\lambda_2 - \lambda_1} & \text{, if } \omega > \omega_0, \end{cases}$$

where $\lambda_1 < 0 < \lambda_2$ solve $q + \delta - i_r = -\mu_r(i_r)\lambda + \sigma^2\lambda^2/2$. [20] See appendix C for how to solve the value functions and the search threshold $\underline{\omega}$.

[20] See Alvarez and Shimer (2011) [3] for detailed proof.

5 Estimation

5.1 Discrete-time representation of the model

In the following quantitative exercises, I will translate the continuous-time model into discrete time to match the data. The imperfectly elastic rest unemployment assumption is useful as I can translate the adjusted productivity thresholds into wage thresholds, since wage is monotonically increasing in adjusted productivity[21]. In this way, the model can be estimated based on observed sectoral wages instead of unobserved adjusted productivity. For the directed search model, I will exploit the following analytical results to estimate the model:

$$\log w_j = [\log Y + (\theta - 1)\log x_j - \log E_j]/\theta \tag{13}$$

$$\log e_{jt} = \begin{cases} 0 & , \text{if } w_{jt} \geq b_R \\ \eta(\log w_{jt} - \log b_R) & , \text{if } w_{jt} < b_R \end{cases} \tag{14}$$

$$\Delta \log L_{jt+1} = \begin{cases} \theta(\log w_{jt} - \log \bar{w}) & , \text{if } w_{jt} > \bar{w} \\ -q & , \text{if } w_{jt} \in [\underline{w}, \bar{w}] \\ -(\theta + \eta)(\log \underline{w} - \log w_{jt}) & , \text{if } w_{jt} < \underline{w} \end{cases} \tag{15}$$

For random search, equation (15) is instead

$$\Delta \log L_{jt+1} = \begin{cases} i_r - q & , \text{if } w_{jt} \geq \underline{w} \\ -(\theta + \eta)(\log \underline{w} - \log w_{jt}) & , \text{if } w_{jt} < \underline{w} \end{cases} \tag{16}$$

[21] In Alvarez and Shimer (2011) [3], rest unemployment and search unemployment perfectly offset the productivity shocks so that wage is constant whenever endogenous unemployment occurs. Therefore it is hard to identify their model from observables. Instead, they use indirect inference to examine their model by exploring the relationship between sectoral wage persistence and share of search and rest unemployment. I show in Section 6.1 that their conclusion about such relationship no longer holds with the relaxed assumption on the elasticity of rest unemployment and is not empirically attractive either.

Equation (13) describes the relationship between wage, employment and primitive labor productivity shocks. This equation simply comes out of the CES aggregation of island outputs and the linear production of labor. It mainly reflects the demand for sectoral employment. In particular, θ captures the demand elasticity for sectoral employment given the primitive labor productivity.

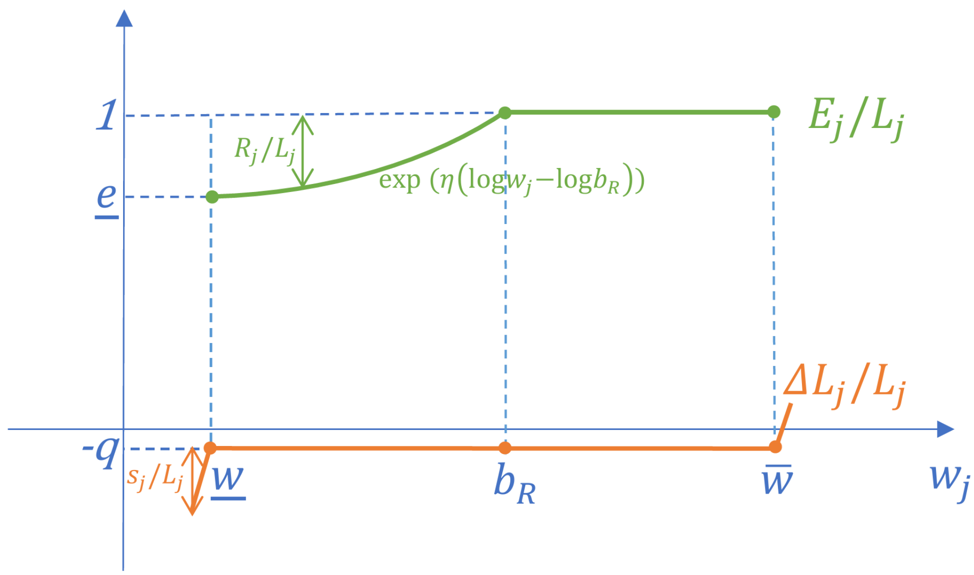
Equation (14) captures the intrasectoral labor reallocation. This equation comes from the intratemporal optimization of sectoral workers. It mainly reflects the supply of sectoral employment. In particular, sectoral employment rate $\log e_{jt}$ responds to wage $\log w_{jt}$ with elasticity η when wage falls below the rest threshold b_R[22]; it always equals 0 in the full-employment region.

Equations (15) and (16) describe the intersectoral labor reallocation, which is similar to equation (4) in the stylized model. They come from the intertemporal labor adjustment of sectors. For directed search, within the inaction region of sectors (full-employment or rest-unemployment regions), there is exogenous quit of workers from their sectors at rate q. Upon hitting the search and inflow thresholds (or the barriers), island labor would adjust to bring the adjusted productivity and the wage back to their corresponding barriers.[23] Figure 5 below is an illustration of the sectoral search- and rest-unemployment schedule for directed search.

[22]I ignore the unit of marginal utility of consumption in b_R here. But the result is isomorphic.

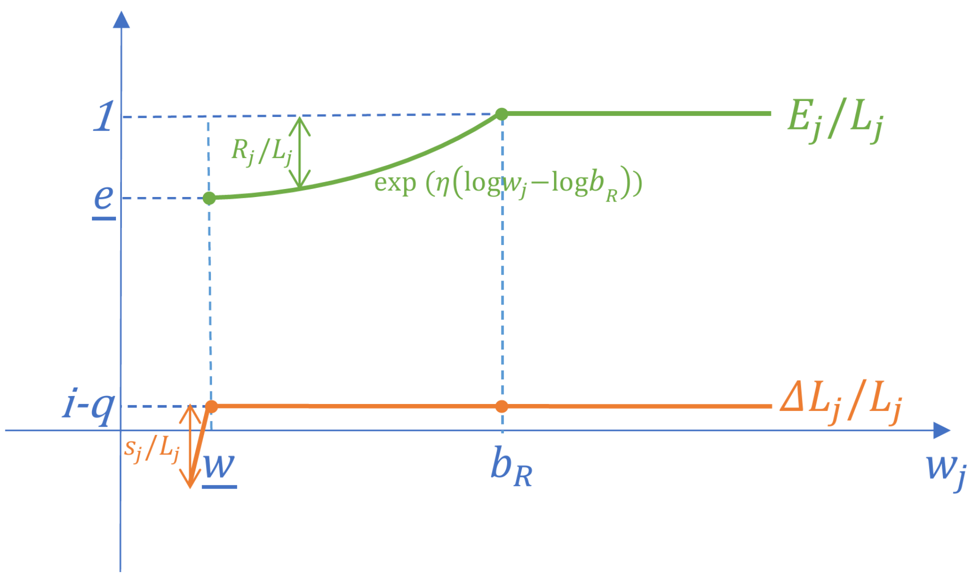
[23]The process of the adjusted productivity is thus named Brownian motion with reflected barriers.

Figure 5. Search- and rest-unemployment schedule (directed search)



For random search, the search- and rest-unemployment schedule is in Figure 6. Both exogenous inflow and outflow of workers occur in the inaction region (above the search threshold). But inflow should exceed outflow in the full-employment region due to the flow balance of search unemployment. When negative shocks of the same size hit two sectors, one in the full-employment region, and the other in the rest-unemployment region, the full-employment sector could still see employment growth while the rest unemployment sector would see more unemployment (and the flow into unemployment could be massive if the sector is near the search unemployment threshold). Now think of the sectors as an industry in two geographic locations such as the automobile industry in the Rust Belt and the Sun Belt. When import competition comes, the automobile industry in the Rust Belt may have already accumulated a lot of rest unemployment. Therefore, its ability to absorb the trade shock through intrasectoral labor reallocation is limited, and the industry see a massive flow into search unemployment. Yet the industry in the Sun Belt states could be still in the full-employment region, so that the shock could be easily absorbed through wages alone, without incuring any endogenous unemployment. Such heterogeneous responses across sectoral labor markets add to the empirical relevance of the model.

Figure 6. Search- and rest-unemployment schedule (random search)



Before moving on, let us put the intratemporal and intertemporal margins of sectoral employment together. For directed search,

$$\Delta \log E_{jt+1} \equiv \Delta \log e_{jt+1} + \Delta \log L_{jt+1}$$
$$= \begin{cases} \theta(\log w_{jt} - \log \bar{w}) & \text{, if } w_{jt} > \bar{w}, \\ -q & \text{, if } w_{jt} \in [b_R, \bar{w}), \\ \eta \Delta \log w_{jt+1} - q & \text{, if } w_{jt} \in [\underline{w}, b_R), \\ \eta \Delta \log w_{jt+1} - (\theta+\eta)(\underline{w} - \log w_{jt}) & \text{, if } w_{jt} < \underline{w}. \end{cases} \tag{17}$$

For random search, equation (17) is instead

$$\Delta \log E_{jt+1} = \begin{cases} i_r - q & \text{, if } w_{jt} \geq b_R, \\ \eta \Delta \log w_{jt+1} + i_r - q & \text{, if } w_{jt} \in [\underline{w}, b_R), \\ \eta \Delta \log w_{jt+1} - (\theta+\eta)(\underline{w} - \log w_{jt}) & \text{, if } w_{jt} < \underline{w}. \end{cases}$$

Obviously, the comovement of sectoral wage and employment is heterogeneous across different wage quantiles. In the full-employment region, there is constant sectoral employment growth or decline depending on the search technology. In the rest-unemployment region, sectoral employment growth is positively correlated with wage growth. But near the search-unemployment threshold, the employment growth could go in any direction with respect to wage growth. Therefore, estimates of the elasticity of sectoral employment to wages from OLS would be largely biased according to the model.

Moving forward, I specify sectors as 2-digit industries.

5.2 Estimate primitives

In this subsection, I utilize the statistical property of Brownian motion to estimate the primitive shock process and the sectoral labor demand elasticity θ. Remember that CES aggregation

$$\log w_{jt} = [\log Y + (\theta - 1)\log x_{jt} - \log E_{jt}]/\theta$$

implies

$$\Delta \log E_{jt} = -\theta \Delta \log w_{jt} + (\theta - 1)\Delta \log x_{jt}. \tag{18}$$

Without $\Delta \log x_{jt} \perp \Delta \log w_{jt}$, OLS estimate of θ from equation (18) is biased.

Proposition 3. Sectoral labor demand elasticity θ is identified with the moment condition

$$\theta = -\frac{\text{Cov}(\Delta \log E_{jt}, \log w_{js})}{\text{Cov}(\Delta \log w_{jt}, \log w_{js})},$$

where $s < t$ and s is close to t.

Proof. See appendix A.

Q.E.D.

Essentially, I instrument current wage growth with lagged wages. This instrument works for two reasons: First, since lagged wage is informative about the underlying productivity space, it has predictive power about current wage growth according to the model[24] (IV is relevant); second, unobserved productivity growth $d\log x_{jt}$ is independent of lagged wages (IV satisfies the exclusion restriction). For the IV to be relevant enough, lagged period s should be close to current period t.[25]

Once the demand elasticity θ is known, (μ_x, σ_x^2) can be easily pinned down from

$$\Delta \log x_{jt} = \frac{\Delta \log E_{jt} + \theta \Delta \log w_{jt}}{\theta - 1}.$$

I use the Current Population Survey (CPS) monthly data (1976-2019) to estimate $(\theta, \mu_x, \sigma_x^2)$ following the procedure. To control for individual human capital of workers, I use residual log wage from education, age, age^2 and sex. The underlying assumption is that workers' reallocation behavior is incentivized by their wage premium across sectors given their observed human capital level (rather than the absolute wage levels across sectors). I then aggregate the residual wage of workers to 2-digit industry level. Industry employment is in terms of employment share among all industries. Details of the empirical specification and regression results are in Appendix D. Below are the estimates.

Table 1. Estimated primitives

Parameter	Target	Estimate
Labor demand elasticity θ	Emp growth w.r.t. wage growth	0.0925
Primitive shock (μ_x, σ_x)	Mean and s.d. of wage growth	$(-0.00348\%, 0.0386)$

[24]Note that sectoral employment on the other hand is not informative about the adjusted productivity in the model. Therefore we cannot use it as an instrument.

[25]In practice, I find one-period lag works very well; more lags lead to insignificant estimate but with similar magnitude. I report the 1st stage and estimation results in Appendix D.

Note that the demand elasticity θ for sectoral labor in the model should not be interpreted the same as the CES elasticity in the traditional trade literature such as Broda and Weinstein (2006) [18]. Here, θ captures both the substitution between sectoral labor outputs (which is already different from the actual sectoral outputs) and the returns to scale in sectoral production.[26] Here, I simply estimate this synthesized elasticity directly from the data, while staying agnostic about its composition. The estimation result shows that θ does deviate from the estimate of Broda and Weinstein (2006) [18]. But considering the fact that industry employment share is insensitive to wage fluctuation, the magnitude of the estimate is reasonable. [27]

For the primitive shock process, one could see that the mean primitive labor productivity growth μ_x is very small compared to the disturbance in growth σ_x. So most of the productivity fluctuation would come from the random disturbance σ_x instead of the constant drift μ_x.

Next, I externally calibrate two parameters. For the discount rate ρ, I target the average annual fed funds rate during the sample period of my main data (Survey of Income and Program Participation (SIPP)), 3.36%. For the industry exit rate, I target for two out of 16 industries being replaced during the SIPP sample period, according to the industry cross-walk for SIPP from the Census Bureau. The two replaced industries are Business and Repair Services and Personal Services. Although some workers are still in the two industries, their numbers are not large enough to sustain 2-digit industries. Instead, another two industries, Information and Other Professional and Related Services arise, which is roughly consistent with the setting of the model. Also note that the magnitude of the exit rate calibrated here is too small to have a significant impact on the analysis of the overall labor market dynamics.

[26]The model assumes linear production in labor, which implies unit-elastic sectoral labor supply at the external margin. One can easily introduce increasing/decreasing returns to scale of sectoral labor and obtain an isomorphic model in terms of labor market dynamics.

[27]Some recent trade and macro labor literature also estimates a CES elasticity smaller than 1, e.g., Atalay (2017) [8], Baqaee and Farhi (2019) [12], Boehm et al. (2015) [15] and Herrendorf et al. (2013) [34].

Table 2. Calibrated parameters

Parameter	Target	Value
Discount rate ρ	Annual fed funds rate 3.36%	0.28%
Industry exit rate δ	2/16 industries replaced	0.045%

5.3 Labor market transition data

The main data I use to estimate the model is the labor market transition data from the Survey of Income and Participation Program (SIPP). It contains a series of panels starting from 1990-1993, 1996, 2001, 2004 and 2008. Each panel consists of 14,000 to 52,000 households observed at monthly frequency over 2.5 to 5 years. Together they roughly cover years 1990 to 2013.[28] The variables of interest are employment statuses, industries and residual wages of workers over time. I aggregate worker level information to 2-digit industry for each month. I truncated each panel by the first and last 6 months to avoid sample selection. [29] This leaves me 2,960 industry-month observations for estimation.

The main technical difficulty here is that rest unemployment is not directly observed. What I observe is the industry of the unemployed before and after her unemployment spell. Define an unemployed worker as a mover if she changes her industry in the next UE transition[30], or a stayer if she retains her industry in the UE transition. [31]Unemployed stayers are closely related

[28]Many of the panels are overlapping in time. In that case I would weight the statistics with the sample sizes of those panels.

[29]For example, I would not be able to observe long-term unemployment at the beginning of the panel and I would be less likely to see unemployed workers reallocate to new industries near the end of the panel. Also, SIPP consists of 4 monthly rotation groups, so the sample in the first and last 3 months is unbalanced.

[30]Keep this information missing for IE transition

[31]Keep this information missing if her last job was more than 18 months ago. That is, the maximum unemployment duration in my data would be less or equal to18 months. Since some of the panels only sustain for 2.5 years, and I truncate 6 months at the beginning and the end, truncating the unemployment duration at18 months avoids non-comparability across panels. And keep the information as missing if she is not observed in any month between the two jobs.

to the rest unemployment in the model but not exactly equal.

Figure 7 below compares the data structure to the model concepts. An unemployed stayer returns to her last industry, so she must be rest-unemployed during the entire unemployment spell, as in case (1) in Figure 6. An unemployed mover changes her industry after the unemployment spell, so it is possible that she is search-unemployed during the entire unemployment spell, as in case (2). The discrepancy between search unemployment and unemployed movers occurs in case (3), that is, some unemployed movers may begin their unemployment spell as rest-unemployed, but persistently depressed industry wage may force them to become search-unemployed. Therefore, some imputation method is needed to correctly capture the search and rest unemployment in the data.

Figure 7. Model concepts and data structure

$$...E_{j_1}\overbrace{\underbrace{U...U}_{R}}^{\text{Stayer (1)}} E_{j_1}...E_{j_1}\overbrace{\underbrace{U...U}_{S}}^{\text{Mover (2)}} E_{j_2\neq j_1}...E_{j_2}\overbrace{\underbrace{U...U}_{R}\underbrace{U...U}_{S}}^{\text{Mover (3)}}E_{j_3\neq j_2}...$$

5.4 Inventory method to impute rest unemployment

Since the model implies clear sectoral labor market dynamics, I could impute rest unemployment from industry-level unemployed movers/stayers with the following inventory method.

To begin with, notice that Figure 7 implies $Stayers_{jt} \leq R_{jt} \leq Unemp_{jt} = Stayers_{jt} + Movers_{jt}$, since stayers are always rest-unemployed, while movers could be either search- or rest-unemployed. So I impute $R_{jt} = Stayers_{jt}$ whenever $Movers_{jt} \to 0$. [32] This gives me a starting point to impute rest unemployment.

[32] In practice, there is almost always movers among the unemployed for each industry (one explanation through the lens of the model is the exogenous quit of workers from their industries); I thus impute $R_{jt} = Stayers_{jt}$ when the share of stayers in unemployment out of an industry exceeds the 95th percentile across all observations, which is around 0.7.

The model also implies the following law of motion for rest unemployment

$$R_{jt}\begin{cases} = 0 & , \text{if } w_{jt} \geq b_R \\ = R_{jt-1} + \underbrace{Inflow_{jt} - qL_{jt-1}}_{\Delta L_{jt}} - \Delta E_{jt} & , \text{if } w_{jt} \in (\underline{w}, b_R) \\ \approx \frac{R_{jt-1}}{E_{jt-1}} E_{jt} & , \text{if } w_{jt} \leq \underline{w} \end{cases}$$

If industry j is in full-employment region ($w_{jt} \geq b_R$), there will be no rest unemployment; if it is in rest unemployment region ($w_{jt} \in (\underline{w}, b_R)$), change in rest unemployment should equal to the difference between the change in industry labor force (which comes from observable inflow of workers and the exogenous quit of workers at rate), and the change in employment (which is observed). For an industry in search unemployment region ($w_{jt} \leq \underline{w}$), it would be hard to quantify the actual outflow of workers from the industry. But since intrasectoral labor reallocation could no longer absorb shocks near the bottom, the employment rate within the industry should stay relatively constant near the bottom (exactly constant in the model). So I could use last period employment rate to approximate current employment rate and impute rest unemployment from last period employment rate and current employment. Reverse-engineering, I could also impute rest unemployment from next period information.

Last but not least, impose the data restriction $Stayers_{jt} \leq R_{jt} \leq Stayers_{jt} + Movers_{jt}$, so that the rest unemployment of an industry would never fall below the stock of unemployed stayers or exceed the total unemployment stock out of the industry.

Note that three reduced-form model parameters are required in the imputation procedure: exogenous quit rate q and the wage thresholds of search and rest unemployment ($\underline{w}, b_R$). In the following estimation procedure, I will nest the imputation procedure in the estimation procedure, and look for the stationary equilibrium parameterization that is most representative of the data.

5.5 Estimate the rest of the model

Besides primitives $(\rho, \delta, \theta, \mu_x, \sigma_x)$, the remaining parameters in the model to be estimated are $(q, \phi, \eta, b_I, b_S, b_R)$. Utilizing the statistical property of the stationary equilibrium, I develop the following algorithm to estimate the rest of the model:

1. Guess the frequency of industries in full employment p_F and rest unemployment p_R;

2. Partition data into full-employment (with p_F), rest-unemployment (with p_R) and barrier-hitting regions (with $1 - p_F - p_R$) by wage;

3. Estimate exogenous quit rate $q = EU_{jt}/E_{jt-1}$ on full-employment region; for random search, also estimate inflow rate $i_r = SE_t/E_{t-1}$ (SE is the UE transitions with industry change, i.e. transition from search unemployment into employment);

4. Impute rest unemployment in each industry R_{jt} given quit rate q and the data partition following the procedure in section 5.4; impute search unemployment $S_t = U_t - \sum_j R_{jt}$[33];

5. Estimate employment elasticity η from $\Delta \log e_{jt} = \eta \Delta \log w_{jt}$ in the rest-unemployment region;

6. Estimate job finding rate $\phi = SE_t/S_{t-1}$;

7. Compute the theoretical search and rest unemployment $(\tilde{S}, \tilde{R})$ with the estimated parameters;

8. Repeat 1-7 to find (p_F, p_R) which minimize the following objective function

$$\sqrt{(\tilde{S} - \frac{1}{T}\sum_t S_t)^2 + (\tilde{R} - \frac{1}{T}\sum_t R_t)^2}$$

[33] In practice, I sometimes could not observe (impute) the unemployment composition of industries with small employment share, such as mining, because I observe few unemployed workers in such industries. In that case, I rescale the total observed (imputed) rest unemployment with the total employment shares of the industries where rest unemployment information is available.

so that imputed search and rest unemployment $\{S_t, R_t\}_{t=1}^{T}$ are closest to the model-implied ones $(\tilde{S}, \tilde{R})$.

This algorithm fully utilizes the statistical property of the adjusted productivity process to identify the set of parameters that renders the most representative underlying stationary equilibrium of the data. It is in the spirit of the Method of Simulated Moments (SMM), but is much faster than the regular SMM. I only need to simulate and target two data moments: search and rest unemployment; and I obtain all the other parameters for free by reduced-form estimation from the data directly. Therefore, the estimated model also fits the data perfectly in the directly estimated moments.

Notice that instead of estimating the "deep" parameters (b_I, b_R, b_S), I estimate the probabilities of industry observations in full-employment and rest-unemployment regions (p_F, p_R). First, I can estimate two parameters instead of three due to the proportionality property of the model[34]. That is, what matters for workers' reallocation behavior is the ratios between (b_I, b_R, b_S), but not their absolute levels. Second, I choose to estimate (p_F, p_R) so that I can group the observations into different regions (adjusted productivity spaces) by wage quantiles and directly pin down the moments in the corresponding regions from the data.[35]

To compute (p_F, p_R), it is necessary to derive the adjusted productivity distribution across industries (rather than across workers).

Proposition 4: The stationary density of adjusted productivity across islands in the directed search model is

$$g(\omega) = \frac{\sum_{i=1}^{2} |\kappa_i + 1| \exp(\kappa_i(\omega - \underline{\omega}))}{\sum_{i=1}^{2} |\kappa_i + 1| [\exp(\kappa_i(\bar{\omega} - \underline{\omega})) - 1]/\kappa_i},$$

[34] Since preferences and production functions are all homothetic in the model, proportionately changing (b_I, b_R, b_S) would not affect the equilibrium quantities, ceteris paribus. See Alvarez and Shimer (2011) [3] for detailed explanation.

[35] A more straightforward alternative is to estimate the wage thresholds $(\underline{w}, b_R, \bar{w})$, but it is less efficient since only the ratios of these thresholds matters, ceteris paribus.

where $\kappa_1 < 0 < \kappa_2$ solve $\delta = -\mu\kappa + \sigma^2\kappa^2/2$. And the probabilities of islands in the full-employment region and the rest-unemployment region are

$$p_F = \int_{\log b_R}^{\bar{\omega}} g(\omega)d\omega - \frac{\sigma^2}{2} g(\bar{\omega}) \tag{19}$$

and

$$p_R = \int_{\underline{\omega}}^{\log b_R} g(\omega)d\omega - \frac{\sigma^2}{2} g(\underline{\omega}). \tag{20}$$

Proof: See Appendix A.

Q.E.D.

The second terms in equations (19) and (20) are the barrier-hitting probabilities. Notice that the density across islands is similar to that across workers (equation (12)). Essentially, when an island hits the barriers, its worker force adjusts at rate θ w.r.t. the adjusted productivity difference to the barriers (equation (4)), whereas the adjustment rate for the island is simply 1. Additionally, islands are only subject to the exogenous exit shock δ for islands but not the quit shock q for workers.

5.6 Estimation results

Table 3 below presents the estimation results for the directed search model.

Table 3. Estimated parameters (directed search)

Parameter	Target	Estimate
Full-emp prob p_F	Search and rest unemployment	0.235
Rest prob p_R		0.733
Exogenous quit rate q	EU rate on full-emp region	0.424%
Labor supply elasticity η	Emp elasticity on rest region	0.0852
Job finding rate ϕ	UE transition into new industries	13.4%

To begin with, full-employment and rest-unemployment probabilities almost add up to 1 (with tiny probability of hitting the search and inflow thresholds). The result is similar to Alvarez and Shimer (2011) [3]. For the 16 industries in my sample, there is roughly one industry hitting either barrier at a certain month in each year. But whenever the industry hits the threshold, the triggered endogenous separation into search unemployment or inflow of workers from search unemployment is going to be large (around 37.9% of the industry labor force[36]). This is an extreme characterization of the empirical search unemployment. One would expect endogenous separation of workers from industries to occur more often, but with smaller magnitude. Future work could be done to finely characterize the worker flows into search unemployment.

The exogenous quit rate q is 0.424%, which is significantly smaller than the average EU transition rate. The reason is that the exogenous quit rate is estimated from industries in full employment. Their productivity and wages are high. Thurs, their workers have little incentive to leave the industries. The exogenous quit rate here accounts for events that separate workers from industries but are unrelated to the industry productivity, such as idiosyncratic preference shocks of workers.

The sectoral labor supply elasticity η is also small, but is similar in magnitude to the sectoral labor demand elasticity θ. Such result is expected, because in general, employment is not very

[36] Remember that the island labor adjusts at rate θ w.r.t. the offsetted adjusted productivity difference to the barriers when hitting the barriers, which is roughly $\theta\sigma = |\theta - 1|\sigma_x = 37.9\%$.

responsive to wage fluctuation. Importantly, the two elasticities must be of similar magnitude, so that the shares of search and rest unemployment generated by the model are empirically relevant to the data.

Lastly, the job finding rate ϕ is 13.4%. Note that the job finding rate here is only for search-unemployed workers. The empirical job finding rate is higher, since the rest-unemployed workers usually return to work at a faster rate. See Table 6 in section 6.2 for a comparison of the unemployment spell distribution of unemployed stayers and movers in the data: The unemployment spells of stayers are significantly shorter than those of movers.

Table 4 below is the comparison of the targeted data moments with the model-implied ones. Notice that I exactly matched the data moments other than search and rest unemployment: the exogenous quit rate on the full-employment region, labor supply elasticity on the rest unemployment region and job finding rate for the search-unemployed workers. For the remaining two moments, search-unemployment and rest unemployment, the fit is still good: The deviation of rest unemployment is negligible, and the deviation of search unemployment is insignificant. But because I could pin down some of the parameters directly from the data, instead of simulating the model to target corresponding data moments, the estimation procedure is very efficient.

Table 4. Goodness of fit

Moment	Data	Model
Exogenous quit rate q (%)	0.424	
Sectoral labor supply elas η	0.0852	
Job finding rate ϕ (%)	13.4	
Search unemp (%)	3.55	3.61
Rest unemp (%)	3.01	3.04
Total unemp (%)	6.56	6.65

6 Validation

I validate the model with two untargeted moments. First is the sectoral wage persistence; the other is the distribution of unemployment spells for unemployed stayers and movers. The validation shows that the estimated model captures well these dimensions of the data.

6.1 Sectoral wage persistence

The most important empirical prediction in Alvarez and Shimer (2011) [3] is the relationship between the share of rest unemployment and sectoral wage persistence. Sectoral wage persistence would contribute to wage inequality across sectors in the long run and translate into wage inequality across workers if workers could not reallocate across sectors freely. In Alvarez and Shimer (2011) [3], sectoral shocks only pass onto search and rest unemployment, but not wage, in low-productivity spaces with endogenous unemployment. Among the two types of unemployment, wage persistence is only preserved with rest unemployment, since search unemployment reallocates workers across sectors and thus relieve the wage pressure on the previous sectors. Therefore, Alvarez and Shimer (2011) [3] infer that rest unemployment must consist of a large fraction of unemployment, given the observed wage persistence at the industry level; they calibrate the fraction to be around 3/4 according to the literature[37]. Given the calibrated share of rest unemployment, they simulate the model to obtain the most wage persistence that the model could produce. They use the the following auxiliary statistical model:

$$\log w_{jt} = \beta_w \log w_{jt-1} + (1-\beta_w) \log w_{j\cdot} + \varepsilon_{jt}$$

[37] Murphy and Topel (1987) [44] using March CPS and Loungani and Rogerson (1989) [40] using the Panel Study of Income Dynamics (PSID) both estimate the share of industry movers to be around one quarter. However, Carrillo-Tudela and Visschers (2014) find that around half of the unemployed workers change their occupations in their next jobs using 1986-2011 SIPP data, which is consistent with my observation at the industry level. I suspect the discrepancy mainly comes from the time periods. One would expect workers to change their sectors more often in recent decades.

assuming industry log wage follows an AR(1) process with persistence β_w and innovation $\varepsilon \sim N(0,\sigma_w^2)$. To estimate β_w, they use Exact Maximum Likelihood. That is, for each industry, assume the first observation is drawn from an ergodic distribution, i.e. $\log w_{j0} \sim N(\log w_{j\cdot}, \sigma_w^2/(1-\beta_w^2))$, and subsequent observation $\log w_{jt+1} \sim N(\beta_w \log w_{jt} + (1-\beta_w)\log w_{j\cdot}, \sigma_w^2)$. Details of the estimation procedure can be found in Appendix D. I execute the same procedure to estimate industry wage persistence with simulated data from the model and obtain the following results.

Table 5. Validation on sectoral wage persistence

Moment		Data	Model	AS	C'factual
Target	Search (%)	3.55	3.61	1.3	3.61
	Rest (%)	3.01	3.04	4.2	3.04
	Unemp (%)	6.56	6.65	5.5	6.65
	Emp elas η	0.085	0.085	∞	∞
Non-target	Persistence	0.954	0.918	0.837	0.710

The wage persistence in the data (0.954) is taken from Alvarez and Shimer (2011) [3] using the Quarterly Census of Employment and Wages (QCEW)[38] for 2-digit industries.

Different from Alvarez and Shimer (2011) [3], with imperfectly elastic rest unemployment, sectoral shocks pass onto both rest unemployment and wage before search unemployment occurs. Thus, even with a smaller fraction of unemployment being rest unemployment (45.71% according to the imputed data), sectoral wage persistence is still justified.

The facts in SIPP shows that the conclusion is robust to the imputation procedure. First, more than half (57.76%) of UE transition comes with change in industry. Second, the unemployment

[38]The frequency of the data (monthly or quarterly) should not affect estimated wage persistence, since in the model, wage persistence simply corresponds to the frequency of islands hitting barriers (which would be the same in monthly and quarterly data).

spell of an unemployed mover is significantly longer than that of a stayer (see Table 6 in section 6.2). Putting the two pieces of evidence together, rest unemployment should account for less than half of total unemployment, which is only consistent with the imperfect elastic rest unemployment assumption.

6.2 Distribution of unemployment spells

To show that the model captures well sectoral labor reallocation and unemployment in the data, I simulate the unemployment spells of stayers and movers with the estimated model. In particular, the estimated model successfully captures the non-trivial fraction of long-term unemployment in the data.

I constructed the distribution of unemployment spells in the data as follows. First, I drop the first six months for each SIPP panel to reduce selection in the duration of unemployment spells.[39] In total, I observe 41,674 unemployment spells of various duration with information on both previous and subsequent industries. I allow workers to be inactive at some time during their unemployment spells. But I do not allow the workers to enter reemployment directly from inactivity. That is, all the unemployment spells I account for here lead to UE transitions. Next, I group these unemployment spells by duration of 1-3, 4-6, 7-12, 13-18 and 18+ months. Note that columns (2) - (5) in Table 7 report the shares of the unemployment spells with duration of 1-3, 4-6, 7-12 and 13-18 months among the unemployment spells within an 18-month duration, whereas the last column reports the share of unemployment spells above 18 months in the total unemployment spells. Because some of the SIPP panels only lasts for 2.5 years (and I truncate the first 6 months), duration of the unemployment spell is self-selected to be shorter than that period. And I only count the unemployment spells with previous and future industry information available. So the sample is self-selected to have less long-term

[39] I do not need to drop the last six months for this exercise, since observed unemployment spells at the end of the panels are not subject to the selection problem.

unemployment. Therefore, I only calculate the shares of unemployment spells among those within 18 months. Even with the sample selection, I still observe a non-trivial fraction of unemployment spells above 18 months. Thus, persistent unemployment does exist in the US labor market, which fits our perception about certain geographic areas such as the Rust Belt.

I then simulate the distribution of unemployment spells with the estimated model. Since the model only tells the stock of rest unemployment in each industry but not the actual labor market transitions, I assume the most effortless labor market transition to hit those rest unemployment stocks. That is, if the adjusted productivity of an industry drifts down and increases rest unemployment, employed workers are randomly assigned to the increment in rest unemployment; if industry adjusted productivity drifts up and decreases rest unemployment, rest unemployed workers are randomly assigned to the increment of employment.

The simulation results and the statistics from the data are presented in Table 6 below. The model successfully reproduces two features of the empirical unemployment spell distribution. First, unemployment spells of stayers are significantly shorter than those of movers. Second, there is persistent unemployment for both stayers and movers. The deviation from the data is that unemployment spells of stayers in the data are often shorter than those in the model. One way to fix the result is to assume a "natural rate of rest unemployment" to account for some short-run rest unemployment due to, for example, seasonal recall. One can easily implement the idea in the simulation with some random fraction of short-run transition between employment and rest unemployment.

Table 6. Validation on distribution of unemployment spells

(1)		(2)	(3)	(4)	(5)	(6)
Unemp months		1-3	4-6	7-12	13-18	18+
Data	Stayers (%)	26.05	9.45	5.89	1.41	0.93
	Movers (%)	26.72	15.77	11.28	3.44	2.53
Model	Stayers (%)	17.69	7.31	6.06	2.13	1.26
	Movers (%)	24.40	16.29	18.02	8.08	6.03

7 Applications

7.1 Sectoral shifts hypothesis

As mentioned in the introduction, sectoral shifts hypothesis used to be in a prolonged and important debate for macroeconomists. With structural shifts profoundly changing the US economy and potentially the labor market as well, economists have started to revisit the question. Jaimovich and Siu (2018)[36] are the first to associate job polarization with the recent slow recovery of employment post recessions. The argument is similar to Lilien (1982) [39], that sectoral productivity differentials induce workers to reallocate towards better sectors, through unemployment. When such productivity differentials widen, as in the case of job polarization, the need to reallocate increases, and so does unemployment; especially so during downturns according to Jaimovich and Siu (2018). However, Pilossoph (2014) [45] makes an interesting point that sectoral shock dispersion does not necessarily change aggregate unemployment. In her model, sectoral productivity differentials only lead to net flows of labor reallocation through unemployment across sectors, yet the gross flows of labor reallocation through unemployment are the combined result of both sectoral shocks and idiosyncratic preference shocks. Thus, gross flows always exceed net flows. With sectoral shock dispersion, more reallocation flows towards relatively more productive sectors will occur, but also fewer reallocation flows towards relatively less productive sectors. The increase in net flows, as long as it is not too large, would be well-accommodated by the gross flows. She estimates the island search model with two sectors (construction and non-construction) using CPS data, runs counterfactual sectoral shock dispersion, and finds that it is indeed the case. So she concludes that sectoral shifts will not affect aggregate unemployment.

Performing the same counterfactual of sectoral shock dispersion in my framework is easy. According to the estimated model, a 1% increase in sectoral shock dispersion σ raises unemployment by 0.554% (Case 1 in Table 7). Why is the effect so large compared to Pilossoph (2014) [45]? Note there are two important components in her model. First, gross flows should be relatively constant in response to sectoral shock dispersion; second, change in net flows in response to sectoral shock dispersion would be relatively small in order to be accomodated in the gross flows after the dispersion shock. So I decompose the change in unemployment in my model. I find that search unemployment moves little (3.612% to 3.622%), while rest unemployment rises from 3.037% to 3.064% (0.90% increase). Therefore, most of the increase comes from rest unemployment or intrasectoral labor reallocation. Similar to Pilossoph (2014) [45], my model captures that the response of search unemployment or intersectoral labor reallocation to sectoral shock dispersion is limited; however the total unemployment do, indeed, respond significantly to sectoral shock dispersion. The reason Pilossoph (2014) [45] misses the responsiveness of total unemployment to the sectoral shock dispersion is that she assumes exogenous separation between workers and firms, which turns into the gross flows or total unemployment in her model after workers' stay-move decision. However, in my model, separation of workers from firms is endogenously, and separation into rest unemployment strongly responds to sectoral shock dispersion. This is because sectoral shock dispersion raises the option value of rest unemployment compared to search unemployment near the bottom of the productivity distribution across sectors. Larger shock dispersion means workers are more likely to hit by larger shocks, either positive or negative. When hit by larger negative shocks, workers always have the option to stay rest-unemployed and enjoy similar value from leisure; but when hit by larger positive shocks, workers' value from returning to work would increase a lot. Therefore, the convexity of worker value near the bottom productivity space due to the option of rest unemployment generates "risk-loving" behavior, and leaves more workers choosing rest unemployment to wait for the sectoral productivity to resume in face of shock dispersion. In this sense, the share of rest unemployment is crucial for the responsiveness of unemployment to sectoral shock dispersion. If the exogenous separation rate q increases by 1% from 0.424% to 0.428% (so that more workers are forced to become search unemployed and reallocate towards other sectors, which replaces part of the endogenous rest unemployment), the impact of a 1% increase in sectoral shock dispersion on aggregate unemployment will decrease to +0.534% (from Case 2 to Case 3 in Table 8). Interestingly, as mentioned before, exogenous separation in my model accounts for workers' idiosyncratic preference to leave a sector that is unrelated to sectoral productivity. It effectively alleviates the responsiveness of aggregate unemployment to sectoral shock dispersion, just as the idiosyncratic preference shock does in Pilossoph (2014) [45]. Therefore, although my model specification deviates from that in Pilossoph (2014) [45] in many details, the two models do share some similar features.

Table 7. Counterfactual unemployment with sectoral shock dispersion

Moment	Case 0	Case 1	Case 2	Case 3
Dispersion σ_x	0.0386	0.0390	0.0386	0.0390
Quit rate q (%)	0.424		0.428	
Search (%)	3.612	3.622	3.637	3.647
Rest (%)	3.037	3.064	3.016	3.044
Unemp (%)	6.649	6.686	6.653	6.691
$\triangle$Unemp	+0.554%		+0.534%	

7.2 Counterfactual Rust Belt labor market dynamics

In this section, I use the framework to understand why Rust Belt labor market is persistently depressed with two counterfactual exercises. The first one recovers the role of industry composition on the Rust Belt labor market dynamics. The second one examines the impact of unionization on the Rust Belt labor market.

For the first exercise, I construct Bartik-like Rust Belt labor market dynamics by aggregating the predicted industrial labor market dynamics at the national level from the estimated model, weighted by the industry composition in the Rust Belt. The purpose of the exercise is to see how much Rust Belt labor market dynamics could be explained straightforwardly by its industry composition, especially its concentration in manufacturing.

For the second exercise, I examine the impact of unionization in the Rust Belt labor market. Intuitively, unionization raises wages of the unionized members above the equilibrium level otherwise, restricts the ability of sectoral wages to absorb shocks, and thus induces more rest unemployment in the short run, and potentially more search unemployment in the long run if the sector is continuously hit by adverse shocks.

8 Conclusion

This paper develops a tractable quantitative framework for analyzing sectoral labor reallocation and unemployment. The framework features analytical sectoral wages, employment and unemployment dynamics and analytical stationary equilibrium, which facilitates model estimation from labor market transition data and counterfactual exercises to quantify the impact of sectoral shocks and the relevant labor market institutions. By exploiting the statistical property of the model, I estimate the model from labor market transition data such as SIPP with computationally efficient algorithm. I then validate the model with two untargeted but important data moments. First is the sectoral wage persistence, which has important implication for wage inequality in the long run. I show that the estimated model successfully reproduce the wage persistence in the data, while closely tracking the share of rest unemployment in the data, which is in contrast to Alvarez and Shimer (2011). Second is the distribution of unemployment spells. The estimated model successfully captures the difference in the length of unemployment spells for the unemployed stayers and movers, as well as the non-trivial fraction of long-term unemployment in the data. Counterfactual exercises based on the estimated model indicate that sectoral shocks have a sizable impact on sectoral labor reallocation and aggregate unemployment. In particular, a 1% increase in sectoral shock dispersion would increase aggregate unemployment by 0.55%. Future research would be enriching the framework by incorporating other realistic features of the data and examining the role of other sectoral shocks, such as uneven impact of trade across sectors, and other labor market institutions, such as unemployment benefits and job training programs in sectoral labor reallocation and unemployment fluctuations.

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Appendix A: Proof of propositions

Proposition 2: Suppose the demand for final output Y stays constant across the two periods. There exists an equilibrium where inter-island labor reallocation only depends on period 1 adjusted productivity ω_{j1}.

Proof: According to Lemma 1, intra-island labor reallocation problem only depends on the adjusted productivity ω_{jt}, $t = 1, 2$.

The stay-search problem is

$$\max\{\mathbb{E}[U(\omega_{j2})|x_{j1}, L_{j1}; \underline{U}, \bar{U}], \phi\bar{U} + (1-\phi)b_s\}$$

s.t.

$$\omega_{j2} = \omega_{j1} + \frac{\theta-1}{\theta}\epsilon_{j2} - \frac{1}{\theta}\Delta\log L_{j2}$$

Guess $\exists$ search and inflow thresholds $(\underline{\omega}, \bar{\omega})$ s.t.

$$\Delta\log L_{j2} = \begin{cases} \theta(\omega_{j1} - \bar{\omega}) & \text{, if } \omega_{j1} > \bar{\omega} \\ 0 & \text{, if } \omega_{j1} \in [\underline{\omega}, \bar{\omega}] \\ -\theta(\underline{\omega} - \omega_{j1}) & \text{, if } \omega_{j1} < \underline{\omega} \end{cases}$$

The law of motion of the adjusted productivity $\omega_{j2} = \omega_{j1} + \frac{\theta-1}{\theta}\epsilon_{j2} - \frac{1}{\theta}\Delta\log L_{j2}$ implies

$$\omega_{j2}|\omega_{j1} \sim N\left(\min\left[\max(\omega_{j1}, \underline{\omega}), \bar{\omega}\right] + \mu, \sigma\right)$$

$$\mu = \frac{\theta-1}{\theta}\mu_x, \sigma = \frac{|\theta-1|}{\theta}\sigma_x.$$

Then $\mathbb{E}[U(\omega_{j2})|x_{j1}, L_{j1}; \underline{U}, \bar{U}] = \mathbb{E}[U(\omega_{j2})|\omega_{j1}; \underline{\omega}, \bar{\omega}]$ is satisfied. I.e. the stay-search decision only depends on ω_{j1}.

Next, check that workers cannot benefit from deviating from the labor adjustment rule above.

To complete the proof, solve search and inflow thresholds $(\underline{\omega}, \bar{\omega})$ and corresponding expected utility thresholds from

$$\bar{U} = \mathbb{E}[U(\omega_{j2})|\bar{\omega}]$$

$$\underline{U} = \mathbb{E}[U(\omega_{j2})|\underline{\omega}]$$

$$\underline{U} = (1-\phi) b_s + \phi \bar{U}$$

$$\underbrace{\phi \int_{\omega_{j1} < \underline{\omega}} \theta(\underline{\omega} - \omega_{j1}) f(\omega_{j1}) d\omega_{j1}}_{\text{outflow from worst sectors}} = \underbrace{\int_{\omega_{j1} > \bar{\omega}} \theta(\omega_{j1} - \bar{\omega}) f(\omega_{j1}) d\omega_{j1}}_{\text{inflow to best sectors}}$$

Notice that I actually approximate labor force growth rate with $\Delta \log L_{j2}$, which requires labor adjustment across periods to be relatively small to Period 1 labor forces of the sectors near the thresholds. This concern relieves once we move to the continuous-time model where labor adjustment in infinitesimal amount of time is always small.

Therefore, inter-island problem only depends on period 1 adjusted productivity ω_{j1}.

Q.E.D.

Proposition 3. Sectoral labor demand elasticity θ is identified with the moment condition

$$\theta = -\frac{\text{Cov}(\Delta \log E_{jt}, \log w_{js})}{\text{Cov}(\Delta \log w_{jt}, \log w_{js})},$$

where $s < t$ and s is close to t.

Proof. Note that $\Delta \log x_{jt} \perp \log w_{js}$, $\forall s < t$. Therefore,

$$
\begin{aligned}
0 &= \text{Cov}((\theta - 1)\Delta \log x_{jt}, \log w_{js}) \\
&= \text{Cov}(\Delta \log E_{jt} + \theta \Delta \log w_{jt}, \log w_{js}) \\
&= \text{Cov}(\Delta \log E_{jt}, \log w_{js}) + \theta \text{Cov}(\Delta \log w_{jt}, \log w_{js}),
\end{aligned}
$$

so that

$$\theta = -\frac{\text{Cov}(\Delta \log E_{jt}, \log w_{js})}{\text{Cov}(\Delta \log w_{jt}, \log w_{js})},$$

which pins down θ from industry wage and employment data. [40]

Q.E.D.

Proposition 4: The stationary distribution of adjusted productivity across islands in the directed search model is

$$g(\omega) = \frac{\sum_{i=1}^{2} |\kappa_i + 1| \exp(\kappa_i(\omega - \underline{\omega}))}{\sum_{i=1}^{2} |\kappa_i + 1| [\exp(\kappa_i(\bar{\omega} - \underline{\omega})) - 1]/\kappa_i}$$

where $\kappa_1 < 0 < \kappa_2$ solve $\delta = -\mu\kappa + \sigma^2\kappa^2/2$. And the probabilities of industries in full-employment region and rest unemployment region are

$$p_F = \int_{\log b_R}^{\bar{\omega}} g(\omega)d\omega - \frac{\sigma^2}{2} g(\bar{\omega})$$

and

$$p_R = \int_{\underline{\omega}}^{\log b_R} g(\omega)d\omega - \frac{\sigma^2}{2} g(\underline{\omega})$$

[40] The estimation procedure here is similar to the earnings process estimation in the income and consumption dynamics literature, such as Blundell et al. (2008) [14].

Proof: The Kolmogorov forward equations for islands are

$$\delta g(\omega) = -\mu g^{'}(\omega) + \frac{\sigma^2}{2} g^{"}(\omega), \forall \omega \in (\underline{\omega}, \bar{\omega})$$

$$\frac{\sigma^2}{2} g^{'}(\underline{\omega}) - (\mu + \frac{\sigma^2}{2}) g(\underline{\omega}) = 0$$

$$\frac{\sigma^2}{2} g^{'}(\bar{\omega}) - (\mu + \frac{\sigma^2}{2}) g(\underline{\omega}) = \delta$$

One can verify that equation (43) satisfies these Kolmogorov forward equations.

Q.E.D.

Appendix B: Solution with directed search

Lemma 2: The general solution to an ODE of the form

$$kV(\omega) = a + \exp(m\omega + (1-m)b) + \mu V^{'}(\omega) + \frac{1}{2}\sigma^2 V^{"}(\omega)$$

is

$$V(\omega) = \frac{a}{k} + \frac{\exp(m\omega + (1-m)b)}{-\frac{\sigma^2}{2}(r_1 - m)(r_2 - m)} + \sum_{i=1}^{2} C_i \exp(r_i\omega)$$

where r_i, $i = 1,2$, are roots of $\frac{\sigma^2}{2}r^2 + \mu r = k$ and C_i , $i = 1,2$, are constants of integration.

Proof: It is not hard to get the general solution with guess and verify. Notice that I use the fact that $r_1 + r_2 = -2\mu/\sigma^2$ and $r_1 r_2 = -2k/\sigma^2$ to substitute

$$k - m\mu - m^2\frac{\sigma^2}{2} = -\frac{\sigma^2}{2}(r_1 - m)(r_2 - m)$$

Lemma 3: The solution to an ODE of the form

$$kV(\omega) = a + \max(\exp(m_1\omega + (1-m_1)b), \exp(m_2\omega + (1-m_2)b)) + \mu V^{'}(\omega) + \frac{1}{2}\sigma^2 V^{"}(\omega)$$

with boundary conditions

$$V(\underline{\omega}) = \underline{V} < \bar{V} = V(\bar{\omega})$$

$$V^{'}(\underline{\omega}) = 0 = V^{'}(\bar{\omega})$$

, $0 \leq m_1 < m_2 \leq 1$, $\underline{V} < \frac{a+\exp(b)}{k} < \bar{V}$, is

$$V(\omega) = \begin{cases} \frac{a}{k} + \frac{\exp(m_1\omega + (1-m_1)b)}{-\frac{\sigma^2}{2}(r_1 - m_1)(r_2 - m_1)} + \sum_{i=1}^{2} \underline{C}_i \exp(r_i\omega) & \text{if } \omega < b \\ \frac{a}{k} + \frac{\exp(m_2\omega + (1-m_2)b)}{-\frac{\sigma^2}{2}(r_1 - m_2)(r_2 - m_2)} + \sum_{i=1}^{2} \bar{C}_i \exp(r_i\omega) & \text{if } \omega \geq b \end{cases}$$

where r_i, $i = 1, 2$, are roots of

$$\frac{\sigma^2}{2} r^2 + \mu r = k,$$

constants of integration are

$$\underline{C}_1 = \frac{r_2(\underline{V} - \frac{a}{k}) - \frac{\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_1-m_1)}}{(r_2 - r_1)\exp(r_1\underline{\omega})}$$

$$\underline{C}_2 = \frac{r_1(\underline{V} - \frac{a}{k}) - \frac{\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_2-m_1)}}{(r_1 - r_2)\exp(r_2\underline{\omega})}$$

$$\bar{C}_1 = \frac{r_2(\bar{V} - \frac{a}{k}) - \frac{\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_1-m_2)}}{(r_2 - r_1)\exp(r_1\underline{\omega})}$$

$$\bar{C}_2 = \frac{r_1(\bar{V} - \frac{a}{k}) - \frac{\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_2-m_2)}}{(r_1 - r_2)\exp(r_2\underline{\omega})}$$

and boundaries solve

$$\frac{\exp((m_1 - r_1)\hat{\underline{\omega}}) - 1}{(r_1 - m_1)} + \frac{\frac{\sigma^2}{2} r_2(\underline{V} - \frac{a}{k})}{\exp(r_1\hat{\underline{\omega}} + b)} = \frac{\exp((m_2 - r_1)\hat{\bar{\omega}}) - 1}{(r_1 - m_2)} + \frac{\frac{\sigma^2}{2} r_2(\bar{V} - \frac{a}{k})}{\exp(r_1\hat{\bar{\omega}} + b)}$$

$$\frac{\exp((m_1 - r_2)\hat{\underline{\omega}}) - 1}{(r_2 - m_1)} + \frac{\frac{\sigma^2}{2} r_1(\underline{V} - \frac{a}{k})}{\exp(r_2\hat{\underline{\omega}} + b)} = \frac{\exp((m_2 - r_2)\hat{\bar{\omega}}) - 1}{(r_2 - m_2)} + \frac{\frac{\sigma^2}{2} r_1(\bar{V} - \frac{a}{k})}{\exp(r_2\hat{\bar{\omega}} + b)}$$

where $\hat{\bar{\omega}} = \bar{\omega} - b$ and $\hat{\underline{\omega}} = \underline{\omega} - b$.

Proof: First, $\max(\exp(m_1\omega + (1-m_1)b), \exp(m_2\omega + (1-m_2)b)) = \begin{cases} \exp(m_1\omega + (1 - m_1)b) & \text{if } \omega < b \\ \exp(m_2\omega + (1 - m_2)b) & \text{if } \omega \geq b \end{cases}.$

Applying Lemma 1, I get the general solution

$$V(\omega)=\begin{cases}\frac{a}{k}+\frac{\exp(m_1\omega+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_1-m_1)(r_2-m_1)}+\sum_{i=1}^{2}\underline{C}_i\exp(r_i\omega) & \text{if } \omega<b\\ \frac{a}{k}+\frac{\exp(m_2\omega+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_1-m_2)(r_2-m_2)}+\sum_{i=1}^{2}\bar{C}_i\exp(r_i\omega) & \text{if } \omega\geq b\end{cases}.$$

Substitute in the general solution into the boudary conditions

$$\underline{V}=\frac{a}{k}+\frac{\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_1-m_1)(r_2-m_1)}+\sum_{i=1}^{2}\underline{C}_i\exp(r_i\underline{\omega})$$

$$\bar{V}=\frac{a}{k}+\frac{\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_1-m_2)(r_2-m_2)}+\sum_{i=1}^{2}\bar{C}_i\exp(r_i\bar{\omega})$$

$$0=\frac{m_1\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_1-m_1)(r_2-m_1)}+\sum_{i=1}^{2}\underline{C}_i r_i\exp(r_i\underline{\omega})$$

$$0=\frac{m_2\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_1-m_2)(r_2-m_2)}+\sum_{i=1}^{2}\bar{C}_i r_i\exp(r_i\bar{\omega})$$

and solve for the constants of integration as functions of the boundaries

$$\underline{C}_1=\frac{r_2(\underline{V}-\frac{a}{k})-\frac{\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_1-m_1)}}{(r_2-r_1)\exp(r_1\underline{\omega})}$$

$$\underline{C}_2=\frac{r_1(\underline{V}-\frac{a}{k})-\frac{\exp(m_1\underline{\omega}+(1-m_1)b)}{-\frac{\sigma^2}{2}(r_2-m_1)}}{(r_1-r_2)\exp(r_2\underline{\omega})}$$

$$\bar{C}_1=\frac{r_2(\bar{V}-\frac{a}{k})-\frac{\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_1-m_2)}}{(r_2-r_1)\exp(r_1\bar{\omega})}$$

$$\bar{C}_2=\frac{r_1(\bar{V}-\frac{a}{k})-\frac{\exp(m_2\bar{\omega}+(1-m_2)b)}{-\frac{\sigma^2}{2}(r_2-m_2)}}{(r_1-r_2)\exp(r_2\bar{\omega})}$$

Continuity and differentiability of $V(\omega)$ at the kink b imply

$$\frac{\exp(b)}{-\frac{\sigma^2}{2}(r_1-m_1)(r_2-m_1)}+\sum_{i=1}^{2}\underline{C}_i\exp(r_i b)=\frac{\exp(b)}{-\frac{\sigma^2}{2}(r_1-m_2)(r_2-m_2)}+\sum_{i=1}^{2}\bar{C}_i\exp(r_i b)$$

$$\frac{m_1\exp(b)}{-\frac{\sigma^2}{2}(r_1-m_1)(r_2-m_1)}+\sum_{i=1}^{2}\underline{C}_i r_i\exp(r_i b)=\frac{m_2\exp(b)}{-\frac{\sigma^2}{2}(r_1-m_2)(r_2-m_2)}+\sum_{i=1}^{2}\bar{C}_i r_i\exp(r_i b)$$

Substituting in $\underline{C}_i$, $\bar{C}_i$, $i=1,2$, they boil down to

$$\frac{\exp((m_1-r_1)\underline{\hat{\omega}})-1}{(r_1-m_1)}+\frac{\frac{\sigma^2}{2}r_2(\underline{V}-\frac{a}{k})}{\exp(r_1\underline{\hat{\omega}}+b)}=\frac{\exp((m_2-r_1)\hat{\bar{\omega}})-1}{(r_1-m_2)}+\frac{\frac{\sigma^2}{2}r_2(\bar{V}-\frac{a}{k})}{\exp(r_1\hat{\bar{\omega}}+b)}$$

$$\frac{\exp((m_1-r_2)\underline{\hat{\omega}})-1}{(r_2-m_1)}+\frac{\frac{\sigma^2}{2}r_1(\underline{V}-\frac{a}{k})}{\exp(r_2\underline{\hat{\omega}}+b)}=\frac{\exp((m_2-r_2)\hat{\bar{\omega}})-1}{(r_2-m_2)}+\frac{\frac{\sigma^2}{2}r_1(\bar{V}-\frac{a}{k})}{\exp(r_2\hat{\bar{\omega}}+b)}$$

I.e.

$$\exp(r_1(\hat{\bar{\omega}}-\underline{\hat{\omega}}))[\frac{\sigma^2}{2}\frac{r_2(\underline{V}-\frac{a}{k})}{\exp(b)}-\frac{\exp(r_1\underline{\hat{\omega}})-\exp(m_1\underline{\hat{\omega}})}{r_1-m_1}]=\frac{\sigma^2}{2}\frac{r_2(\bar{V}-\frac{a}{k})}{\exp(b)}-\frac{\exp(r_1\hat{\bar{\omega}})-\exp(m_2\hat{\bar{\omega}})}{r_1-m_2}$$

$$\exp(r_2(\hat{\bar{\omega}}-\underline{\hat{\omega}}))[\frac{\sigma^2}{2}\frac{r_1(\underline{V}-\frac{a}{k})}{\exp(b)}-\frac{\exp(r_2\underline{\hat{\omega}})-\exp(m_1\underline{\hat{\omega}})}{r_2-m_1}]=\frac{\sigma^2}{2}\frac{r_1(\bar{V}-\frac{a}{k})}{\exp(b)}-\frac{\exp(r_2\hat{\bar{\omega}})-\exp(m_2\hat{\bar{\omega}})}{r_2-m_2}$$

where $\hat{\bar{\omega}}=\bar{\omega}-b$ and $\underline{\hat{\omega}}=\underline{\omega}-b$.

Q.E.D.

Proposition 4: solution to the ODE of the value function of island workers

$$(\rho+\delta+q)V(\omega)=\exp[\max(\omega,\frac{\theta\omega+\eta\tilde{b}_R}{\theta+\eta})]+(\delta+q)\underline{V}+\mu V^{'}(\omega)+\frac{1}{2}\sigma^2V^{''}(\omega)$$

with value-matching conditions

$$V(\underline{\omega}) = \underline{V} < \bar{V} = V(\bar{\omega})$$

and smooth-pasting conditions

$$V^{'}(\underline{\omega}) = 0 = V^{'}(\bar{\omega})$$

is

$$V(\omega) = \begin{cases} (1-\hat{\rho})\underline{V} + \frac{\exp(\omega)}{-\frac{\sigma^2}{2}(\xi_1-1)(\xi_2-1)} + \sum_{i=1}^{2} \bar{C}_i \exp(\xi_i \omega) & \text{if } \omega \geq \tilde{b}_R \\ (1-\hat{\rho})\underline{V} + \frac{\exp(\hat{\theta}\omega+(1-\hat{\theta})\tilde{b}_R)}{-\frac{\sigma^2}{2}(\xi_1-\hat{\theta})(\xi_2-\hat{\theta})} + \sum_{i=1}^{2} \underline{C}_i \exp(\xi_i \omega) & \text{if } \omega < \tilde{b}_R \end{cases}$$

where $\xi_1 < 0 < \xi_2$ solve $\frac{\sigma^2}{2}\xi^2 + \mu\xi = \rho + \delta + q$, $\hat{\rho} \equiv \frac{\rho}{\rho+\delta+q}$ and $\hat{\theta} \equiv \frac{\theta}{\theta+\eta}$, with constants of integration

$$\bar{C}_1 = \frac{[\bar{V} - (1-\hat{\rho})\underline{V}]\xi_2 + \frac{\exp(\bar{\omega})}{\frac{\sigma^2}{2}(\xi_1-1)}}{(\xi_2-\xi_1)\exp(\xi_1\bar{\omega})}$$

$$\bar{C}_2 = \frac{[\bar{V} - (1-\hat{\rho})\underline{V}]\xi_1 + \frac{\exp(\bar{\omega})}{\frac{\sigma^2}{2}(\xi_2-1)}}{(\xi_1-\xi_2)\exp(\xi_2\bar{\omega})}$$

$$\underline{C}_1 = \frac{\hat{\rho}\underline{V}\xi_2 + \frac{\exp(\hat{\theta}\underline{\omega}+(1-\hat{\theta})\tilde{b}_R)}{\frac{\sigma^2}{2}(\xi_1-\hat{\theta})}}{(\xi_2-\xi_1)\exp(\xi_1\underline{\omega})}$$

$$\underline{C}_2 = \frac{\hat{\rho}\underline{V}\xi_1 + \frac{\exp(\hat{\theta}\underline{\omega}+(1-\hat{\theta})\tilde{b}_R)}{\frac{\sigma^2}{2}(\xi_2-\hat{\theta})}}{(\xi_1-\xi_2)\exp(\xi_2\underline{\omega})}$$

and boundaries solving

$$\exp(\xi_1(\hat{\bar{\omega}} - \underline{\hat{\omega}}))[\frac{\sigma^2}{2}\frac{\xi_2\hat{\rho}\underline{V}}{b_R} - \frac{\exp(\xi_1\underline{\hat{\omega}}) - \exp(\hat{\theta}\underline{\hat{\omega}})}{\xi_1-\hat{\theta}}] = \frac{\sigma^2}{2}\frac{\xi_2(\bar{V} - \underline{V} + \hat{\rho}\underline{V})}{b_R} - \frac{\exp(\xi_1\hat{\bar{\omega}}) - \exp(\hat{\bar{\omega}})}{\xi_1 - 1}$$

$$\exp(\xi_2(\hat{\bar{\omega}} - \underline{\hat{\omega}}))[\frac{\sigma^2}{2}\frac{\xi_1\hat{\rho}\underline{V}}{b_R} - \frac{\exp(\xi_2\underline{\hat{\omega}}) - \exp(\hat{\theta}\underline{\hat{\omega}})}{\xi_2-\hat{\theta}}] = \frac{\sigma^2}{2}\frac{\xi_1(\bar{V} - \underline{V} + \hat{\rho}\underline{V})}{b_R} - \frac{\exp(\xi_2\hat{\bar{\omega}}) - \exp(\hat{\bar{\omega}})}{\xi_2 - 1}$$

where $\hat{\bar{\omega}} = \bar{\omega} - \tilde{b}_R$ and $\underline{\hat{\omega}} = \underline{\omega} - \tilde{b}_R$. So

$$\exp(\xi_1\hat{\bar{\omega}})[A2 - \frac{\exp(\hat{\theta}\underline{\hat{\omega}}) - \exp(\xi_1\underline{\hat{\omega}})}{\hat{\theta} - \xi_1}] = \exp(\xi_1\underline{\hat{\omega}})[B2 - \frac{\exp(\hat{\bar{\omega}}) - \exp(\xi_1\hat{\bar{\omega}})}{1 - \xi_1}]$$

$$\exp(\xi_2\hat{\bar{\omega}})[A1 - \frac{\exp(\xi_2\underline{\hat{\omega}}) - \exp(\hat{\theta}\underline{\hat{\omega}})}{\xi_2 - \hat{\theta}}] = \exp(\xi_2\underline{\hat{\omega}})[B1 - \frac{\exp(\xi_2\hat{\bar{\omega}}) - \exp(\hat{\bar{\omega}})}{\xi_2 - 1}]$$

where

$$A1 = \frac{\sigma^2}{2}\frac{\xi_1\hat{\rho}\underline{V}}{b_R}$$

$$A2 = \frac{\sigma^2}{2}\frac{\xi_2\hat{\rho}\underline{V}}{b_R}$$

$$B1 = A1 + \frac{\sigma^2}{2}\xi_1\frac{\bar{V} - \underline{V}}{b_R}$$

$$B2 = A2 + \frac{\sigma^2}{2}\xi_2\frac{\bar{V} - \underline{V}}{b_R}$$

If $\hat{\theta} = 0$ as in AS (2011),

$$B2 - \frac{\exp(\xi_1\hat{\bar{\omega}}) - \exp(\hat{\bar{\omega}})}{\xi_1 - 1} = \exp(\xi_1(\hat{\bar{\omega}} - \underline{\hat{\omega}}))[A2 - \frac{\exp(\xi_1\underline{\hat{\omega}}) - 1}{\xi_1}]$$

$$B1 - \frac{\exp(\xi_2\hat{\bar{\omega}}) - \exp(\hat{\bar{\omega}})}{\xi_2 - 1} = \exp(\xi_2(\hat{\bar{\omega}} - \underline{\hat{\omega}}))[A1 - \frac{\exp(\xi_2\underline{\hat{\omega}}) - 1}{\xi_2}]$$

Proof: Value-matching and smooth-pasting conditions serve as the boundary conditions in Lemma 2. Directly adapting Lemma 2 renders the result.

Q.E.D.

Lemma 4: For an integral of the form $X = \int_{b_0}^{b_1} \exp(m\omega) f(\omega) d\omega$, where

$$f(\omega) = \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \exp(\lambda_i(\omega - \underline{\omega}))}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}$$

and $\lambda_1 < 0 < \lambda_2$ solve $q + \delta = -\mu\lambda + \sigma^2\lambda^2/2$, the integration renders

$$\begin{aligned} X &= \int_{b_0}^{b_1} \exp(m\omega) f(\omega) d\omega \\ &= \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \frac{\exp(mb_1 + \lambda_i(b_1 - \underline{\omega})) - \exp(mb_0 + \lambda_i(b_0 - \underline{\omega}))}{m + \lambda_i}}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i} \end{aligned}$$

Proposition 5: Stationary output is

$$\frac{Y}{L} = \frac{1}{u'(Y)} \{ \int_{\tilde{b}_R}^{\bar{\omega}} \exp(\omega) f(\omega) d\omega + \int_{\underline{\omega}}^{\tilde{b}_R} \exp[\hat{\theta}(1+\eta)\omega_j + \hat{\eta}(1-\theta)\tilde{b}_R] f(\omega) d\omega \}$$

$$\begin{aligned} \frac{u'(Y)Y}{L} =& b_R \sum_{i=1}^{2} |\lambda_i + \theta| \{ \frac{\exp(\lambda_i(\tilde{b}_R - \underline{\omega})) - \exp(\hat{\theta}(1+\eta)(\underline{\omega} - \tilde{b}_R)}{\lambda_i + \hat{\theta}(1+\eta)} \\ & - \exp(\lambda_i(\bar{\omega} - \underline{\omega})) \frac{\exp(\lambda_i(\tilde{b}_R - \bar{\omega})) - \exp(\bar{\omega} - \tilde{b}_R)}{\lambda_i + 1} \} / [\sum_{i=1}^{2} \frac{|\lambda_i + \theta|}{\lambda_i} (\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1)] \end{aligned}$$

$$\begin{aligned} \frac{R}{L} &= \int_{\underline{\omega}}^{\tilde{b}_R} [1 - \exp(\frac{\theta\eta(\omega - \tilde{b}_R)}{\theta + \eta})] f(\omega) d\omega \\ &= \frac{\sum_{i=1}^{2} \frac{|\lambda_i + \theta|}{\lambda_i} [\frac{\lambda_i \exp(\hat{\eta}(\underline{\omega} - \tilde{b}_R)) + \theta\hat{\eta} \exp(\lambda_i(\tilde{b}_R - \underline{\omega}))}{\lambda_i + \theta\hat{\eta}} - 1]}{\sum_{i=1}^{2} \frac{|\lambda_i + \theta|}{\lambda_i} [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]} \end{aligned}$$

where $\hat{\eta} = \frac{\eta}{\theta+\eta}$. If $\eta \to +\infty$, $\hat{\eta} \to 1$. Then

$$\frac{R}{L} = \theta \frac{\frac{\exp(\lambda_2(\tilde{b}_R - \underline{\omega}))-1}{\lambda_2} - \frac{\exp(\lambda_1(\tilde{b}_R - \underline{\omega}))-1}{\lambda_1}}{\sum_{i=1}^{2} \frac{|\lambda_i+\theta|}{\lambda_i}[\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]}$$

as in Alvarez and Shimer (2011).

Denote N_s the measure of workers leaving the islands, either due to exogenous separation shock $q+\delta$, or endogenous separation due to hitting the barrier

$$\frac{S}{L} = (q + \delta + \frac{\theta\sigma^2}{2} f(\underline{\omega}))/\phi$$

$$E + R = L$$

, new-born industry labor force L_0 s.t.

$$\begin{aligned}
\bar{\omega} &= [\tilde{Y} + (\theta-1)\tilde{x}_0 - \tilde{L}_0]/\theta + \log u'(Y) \\
&= [\tilde{Y} + (\theta-1)\tilde{x}_0 - \tilde{L}_0]/\theta - \gamma\tilde{Y} \\
&= \frac{(1-\theta\gamma)\tilde{Y} + (\theta-1)\tilde{x}_0 - \tilde{L}_0}{\theta}
\end{aligned}$$

$$\frac{\sigma^2}{2} f'(\bar{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\bar{\omega}) = \delta\frac{L_0}{L}$$

So

$$\begin{aligned}
L &= \delta L_0 / [\frac{\sigma^2}{2} f'(\bar{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\bar{\omega})] \\
&= \frac{\delta \exp((1-\theta\gamma)\tilde{Y} + (\theta-1)\tilde{x}_0 - \theta\bar{\omega})}{\frac{\sigma^2}{2} f'(\bar{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\bar{\omega})}
\end{aligned}$$

So

$$Y^{(\theta-1)\gamma} = \frac{\delta b_R x_0^{\theta-1}}{\exp(\theta\bar{\omega})} \sum_{i=1}^{2} |\lambda_i + \theta| \{ \frac{\exp(\lambda_i(\tilde{b}_R - \underline{\omega})) - \exp(\hat{\theta}(1+\eta)(\underline{\omega} - \tilde{b}_R)}{\lambda_i + \hat{\theta}(1+\eta)}$$
$$- \exp(\lambda_i(\bar{\omega} - \underline{\omega})) \frac{\exp(\lambda_i(\tilde{b}_R - \bar{\omega})) - \exp(\bar{\omega} - \tilde{b}_R)}{\lambda_i + 1} \}$$
$$/\{[\frac{\sigma^2}{2} f^{'}(\bar{\omega}) - (\mu + \frac{\theta\sigma^2}{2}) f(\bar{\omega})] \sum_{i=1}^{2} \frac{|\lambda_i + \theta|}{\lambda_i} (\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1)\}$$

For an integral of the form $X = \int_{b_0}^{b_1} \exp(m\omega) f(\omega) d\omega$, where

$$f(\omega) = \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \exp(\lambda_i(\omega - \underline{\omega}))}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}$$

and $\lambda_1 < 0 < \lambda_2$ solve $q + \delta = -\mu\lambda + \sigma^2\lambda^2/2$, the integration renders

$$X = \int_{b_0}^{b_1} \exp(m\omega) f(\omega) d\omega$$
$$= \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \frac{\exp(mb_1 + \lambda_i(b_1 - \underline{\omega})) - \exp(mb_0 + \lambda_i(b_0 - \underline{\omega}))}{m + \lambda_i}}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}$$

So average wage rate is

$$E[w] = \int_{\hat{\omega}}^{\bar{\omega}} \exp(\omega) f(\omega) d\omega + \int_{\underline{\omega}}^{\hat{\omega}} \exp(\hat{\theta}\omega + \hat{\eta}\hat{\omega}) f(\omega) d\omega$$
$$= \exp(\underline{\omega}) \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \frac{\exp((1+\lambda_i)(\bar{\omega} - \underline{\omega})) - \exp((1+\lambda_i)(\hat{\omega} - \underline{\omega}))}{1 + \lambda_i}}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}$$
$$+ \exp(\hat{\theta}\underline{\omega} + \hat{\eta}\hat{\omega}) \frac{\sum_{i=1}^{2} |\lambda_i + \theta| \frac{\exp(\hat{\theta}(\hat{\omega} - \underline{\omega}) + \lambda_i(\hat{\omega} - \underline{\omega})) - 1}{\hat{\theta} + \lambda_i}}{\sum_{i=1}^{2} |\lambda_i + \theta| [\exp(\lambda_i(\bar{\omega} - \underline{\omega})) - 1]/\lambda_i}$$

Appendix C: Solution with random search

Lemma 5: the entry point ω_0 is larger than the rest unemployment threshold $\tilde{b}_R$.

Proposition 6: Solution to the ODE of the value function of island workers

$$(\rho+\delta+q)V(\omega) = \exp[\max(\omega, \frac{\theta\omega+\eta\tilde{b}_R}{\theta+\eta})] + (\delta+q)\underline{V} + \mu V^{'}(\omega) + \frac{1}{2}\sigma^2 V^{''}(\omega)$$

with value-matching condition

$$V(\underline{\omega}) = \underline{V}$$

$$\int_{\underline{\omega}}^{\infty} V(\omega) f(\omega) d\omega = \bar{V}$$

$$f(\omega) = \begin{cases} \frac{(\lambda_1\lambda_2+\frac{2\delta L_0}{\sigma^2 L})\sum_{i=1}^{2}|\lambda_i+\theta|\exp(\lambda_i(\omega-\underline{\omega}))}{\theta(\lambda_2-\lambda_1)} & \text{, if } \omega \in [\underline{\omega}, \omega_0] \\ \frac{(\lambda_1\lambda_2+\frac{2\delta L_0}{\sigma^2 L})\sum_{i=1}^{2}|\lambda_i+\theta|\exp(\lambda_i(\omega-\underline{\omega}))}{\theta(\lambda_2-\lambda_1)} & \\ +\frac{\frac{2\delta L_0}{\sigma^2 L}[\exp(\lambda_1(\omega-\omega_0))-\exp(\lambda_2(\omega-\omega_0))]}{\lambda_2-\lambda_1} & \text{, if } \omega > \omega_0 \end{cases}$$

and smooth-pasting condition

$$V^{'}(\underline{\omega}) = 0$$

Appendix D: Details on reduced-form estimations

Instrumental Variable estimation of the demand elasticity of sectoral employment

First, I normalize workers' monthly earnings to the national average. Then I take the residual of monthly earning of workers from their observed human capital using the following specification

$$\log w_{it} = \beta_0 + \beta_1 highsch_i + \beta_2 col_i + \beta_3 age_i + \beta_4 age_i^2 + \beta_5 sex_i$$

Next, I aggregate workers' residual wage to 2-digit industry level. Lastly, I control for seasonal movement of industries by removing monthly fixed effects. I use bootstrapping to compute the standard error to ensure the estimate is robust to the underlying data generating process. Below are the 1st stage and IV regression results:

Table D1. 1st stage of IV estimation of θ[41]

	$d\log w_{jt}$	$d\log w_{jt}$
$\log w_{jt-1}$	-0.00456^{**}	-0.0199^{**}
	(-4.29)	(-9.00)
month FE	N	Y
Bootstrap	N	N
Obs	7248	7248

Table D2. IV Estimation results of θ[42]

[41] t statistics in parentheses; Data source: Current Population Survey monthly data 1976-2019; $^{*} p < 0.05, ^{**} p < 0.01, ^{***} p < 0.001$

[42] Instrument $d\log w_{jt}$ by $\log w_{jt-1}$; t statistics in parentheses; Data source: Current Population Survey monthly data 1976-2019; $^{*} p < 0.05, ^{**} p < 0.01, ^{***} p < 0.001$

	$d\log E_{jt}$	$d\log E_{jt}$	$d\log E_{jt}$	$d\log E_{jt}$
$d\log w_{jt}$	-0.0307	-0.0925^{**}	-0.0307	-0.0925^{**}
	(-0.46)	(-2.83)	(-0.40)	(-2.75)
month FE	N	Y	N	Y
Bootstrap	N	N	Y	Y
Obs	7248	7248	7248	7248

Exact maximum likelihood estimation of wage persistence

Denote log wage for industry j at time t as $\tilde{w}_{jt}$. Following Alvarez and Shimer (2011), estimate equation

$$\tilde{w}_{jt} = \beta_w \tilde{w}_{jt-1} + (1-\beta_w)\tilde{w}_{j\cdot} + \varepsilon_{jt}$$

with exact maximum likelihood assuming industry log wage follows AR(1) process with persistence β_w and innovation ε ~N(0,σ_w^2) For each industry, assume the first observation draws from an ergodic distribution, i.e. $\tilde{w}_{j0} \sim N(\tilde{w}_{j\cdot}, \sigma_w^2/(1-\beta_w^2))$, and subsequent observation $\tilde{w}_{jt} \sim N(\beta_w \tilde{w}_{jt} + (1-\beta_w)\tilde{w}_{j\cdot}, \sigma_w^2)$. The log likelihood function is

$$-\frac{1}{2\sigma_w^2}\sum_{j=1}^{J}\{\sum_{t=1}^{T}[\tilde{w}_{jt} - \beta_w \tilde{w}_{jt-1} - (1-\beta_w)\tilde{w}_{j\cdot}]^2 + (1-\beta_w^2)(\tilde{w}_{j0} - \tilde{w}_{j\cdot})^2\}$$
$$+J\log(1-\beta_w^2) - J(T+1)\log\sigma_w - \frac{1}{2}J(T+1)\log(2\pi)$$

F.o.c.

$$2J\beta_w\sigma_w^2 = (1-\beta_w^2)\{\beta_w\sum_{j=1}^{J}(\tilde{w}_{j0} - \tilde{w}_{j\cdot})^2 + \sum_{j=1}^{J}\sum_{t=1}^{T}(\tilde{w}_{jt-1} - \tilde{w}_{j\cdot})[\tilde{w}_{jt} - \beta_w\tilde{w}_{jt-1} - (1-\beta_w)\tilde{w}_{j\cdot}]\}$$

$[\tilde{w}_{j\cdot}]$ $\tilde{w}_{j\cdot} = [(1+\beta_w)\tilde{w}_{j0} + \sum_{t=1}^{T}(\tilde{w}_{jt} - \beta_w\tilde{w}_{jt-1})]/[1+\beta_w + T(1-\beta_w)]$

$[\sigma_w]$ $\sigma_w^2 = \{\sum_{j=1}^{J}\sum_{t=1}^{T}[\tilde{w}_{jt} - \beta_w \tilde{w}_{jt-1} - (1-\beta_w)\tilde{w}_{j\cdot}]^2 + (1-\beta_w^2)\sum_{j=1}^{J}(\tilde{w}_{j0} - \tilde{w}_{j\cdot})^2\}/(J(T+1))$

$[\beta_w]$ $(1-\beta_w^2)\{\beta_w \sum_{j=1}^{J}(\tilde{w}_{j0}-\tilde{w}_{j\cdot})^2 + \sum_{j=1}^{J}\sum_{t=1}^{T}(\tilde{w}_{jt-1}-\tilde{w}_{j\cdot})[\tilde{w}_{jt}-\beta_w\tilde{w}_{jt-1}-(1-\beta_w)\tilde{w}_{j\cdot}]\} = 2J\beta_w\sigma_w^2$

This system of equations could be solved numerically. Starting with an initial guess of β_w, use f.o.c. $[\tilde{w}_{j\cdot}]$ and f.o.c. $[\sigma_w]$ to calculate $\tilde{w}_{j\cdot}$ and σ_w^2. Then update β_w by solving f.o.c. $[\beta_w]$ with the calculated $\tilde{w}_{j\cdot}$ and σ_w. Repeat the process till the updated β_w converges. Notice that f.o.c. $[\beta_w]$ has unique solution on $(-1,1)$ though it is in cubic form.