THE UNIVERSITY OF CHICAGO

RANDOM GROUPS AT DENSITY $D < 3/14$ ACT NON-TRIVIALLY ON A CAT(0)

CUBE COMPLEX

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For my mom, who knew I could do it.

When I am an old woman I shall wear purple

With a red hat which doesn't go, and doesn't suit me.

And I shall spend my pension on brandy and summer gloves

And satin sandals, and say we've no money for butter.

I shall sit down on the pavement when I'm tired

And gobble up samples in shops and press alarm bells

And run my stick along the public railings

And make up for the sobriety of my youth.

I shall go out in my slippers in the rain

And pick flowers in other people's gardens

And learn to spit.

But maybe I ought to practise a little now?

So people who know me are not too shocked and surprised

When suddenly I am old, and start to wear purple.

<div style="text-align: right;">

From "Warning" by Jenny Joseph, included

in the anthology *Tools of the Trade: Poems*

for new doctors

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ABSTRACT

For random groups in the Gromov density model at $d < 3/14$, we construct walls in the Cayley complex X which give rise to a non-trivial action by isometries on a CAT(0) cube complex. This extends results of Ollivier-Wise and Mackay-Przytycki at densities $d < 1/5$ and $d < 5/24$, respectively. We are able to overcome one of the main combinatorial challenges remaining from the work of Mackay-Przytycki, and we give a construction that plausibly works at any density $d < 1/4$.

CHAPTER 1

INTRODUCTION

1.1   Hyperbolic Groups

Geometric group theory is the study of groups using geometric techniques. The primary benefit of this approach that an *a priori* entirely abstract concept is given a physical, geometric realization (in fact, more than one!). By investigating these geometric objects, properties of the group can be uncovered.

The first such object many mathematicians learn about is the presentation complex of a group: Given a group presentation one builds a simplicial complex with a single 0-cell, a 1-cell for each generator, and a 2-cell for each relator, glued such that the boundary path of the 2-cell corresponding to relator r is the word r. The universal cover of this complex, called the *Cayley complex*, is another such object. The action of G on itself by left-multiplication gives an action on the Cayley complex of G which is properly discontinuous and cocompact. In fact, if G has no 2-torsion this action is also free.

In general, one might expect that this object will depend greatly on the choice of presentation for the group. However, the power of the geometric approach comes from the following result, due to Schwartz: If a group G acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space, then that metric space must be quasi-isometric to a Cayley complex of G. In particular, all Cayley complexes associated to G are quasi-isometric to each other.

This allows us to identify a group with the quasi-isometry class of its Cayley complex. Moreover, if a space X is δ-hyperbolic (meaning that for any geodesic triangle, the δ-neighborhood of any two edges contains the third edge), then every space quasi-isometric to X is also δ'-hyperbolic, for some δ'. In particular, we can say that a group G is *hyperbolic* if at least one (and therefore all) of its Cayley complexes is δ-hyperbolic for some δ.

Hyperbolic groups are an especially popular class of groups, both in the sense that they are well-studied and the sense that they are ubiquitous. A commonly heard statement among group theorists is that 'most' groups are hyperbolic, a claim that will be explained further in Section 1.2.2.

These groups satisfy several nice properties, including (importantly for this thesis) an isoperimetry property. In particular, consider a *disk diagram D* (a contractible 2-complex with a fixed planar embedding) in the Cayley complex X. Its *boundary path ∂D* is the attaching map of the cell at infinity. In a hyperbolic group, any disk diagram must satisfy a linear isoperimetric inequality:

$$|\partial D| \geq k|D|,$$

for some constant k where $|D|$ denotes the number of 2-cells in D. In fact, this is a combinatorial version of a more general statement about all embedded disks in any space on which G acts geometrically.

Another special property of δ-hyperbolic spaces is the *local-to-global property of geodesics*: in particular, given a curve which is geodesic on a local scale (the necessary size of this scale depends on δ), this curve must be globally geodesic. In other words, the global property of minimizing distance can be certified on a finite scale, where that scale is not dependent on the choice of geodesic.

1.2   Random Groups

1.2.1   History and Models of Random Groups

The study of random groups is one way of answering the question, 'What does a *typical* group look like?' To answer this, one needs a *model* of a random group. In particular, one needs some way of listing and choosing groups randomly. The standard method is to choose a group *presentation* at random, subject to some constraints on the relators. Given a model,

you can then make statements about the probability of random groups satisfying certain properties.

These probablistic statements usually take one of two forms: either the probability is 1, in which case you might say that 'most' groups satisfy this property; or the probability is non-zero, in which case you know that at least one group satisfies this property. Thus the study of random groups can answer questions about 'most' groups, or it can answer questions about exceptional groups. In this paper, we'll concentrate on the former situation.

There have been many models proposed - and fruitfully used - to this end. The model used in this paper is the *Gromov density model*, as described in the following sub-section. Other models include the *few relators model*, the *random graphs model*, and the *n-gonal model*.

As evidence of the robustness of the theory of random groups, results in one model are often replicated in another. In particular, the famous statement that 'most' groups are hyperbolic has been shown repeatedly in a variety of models. Results in one model can also be pushed forward into other models in some cases. For example, results the n-gonal model can be pushed into the Gromov density model. This method was used by Zuk ([Ż03]) and Kotowski-Kotowski ([KK13]) to show that random groups in the Gromov density model satisfy Property (T) with overwhelming probability.

1.2.2   Gromov Density Model

One of the most fruitful models of random groups was developed by Gromov (see [Gro93, Oll05]). Heuristically, one chooses a group presentation with n generators at random by selecting relators of a fixed length ℓ. The number of relators is controlled by the *density* of the group. In particular, we have the following definition.

Definition 1.2.1. A *random group $G(n, d, \ell)$ in the Gromov density model* with density $d \in (0, 1)$ is a group with presentation $G = G(n, d, \ell) = \langle S | R \rangle$, where S is a generating set

of size $n \geq 2$ and R is a collection of $(2n-1)^{d\ell}$ cyclically reduced words in $S \cup S^{-1}$ of length ℓ chosen uniformly at random from the set of all such words. A random group at density d satisfies property P *with overwhelming probability*, abbreviated w.o.p., if the probability of G satisfying P tends to 1 as $\ell \to \infty$.

Roughly speaking, there are around $(2n-1)^{\ell}$ cyclically reduced relators of length ℓ on the alphabet $S \cup S^{-1}$. So the number of relators, $(2n-1)^{d\ell}$, is an 'exponential fraction' of the total number of relators.

These groups satisfy several properties, including the following:

- For densities $d > 1/2$, a random group is w.o.p. trivial or $\mathbb{Z}/2\mathbb{Z}$ [Gro93, Oll05].

- For densities $d < 1/2$, a random groups is w.o.p. infinite hyperbolic, torsion-free, with contractible Cayley complex [Gro93, Oll04].

- For densities $d > 1/3$, w.o.p. a random group satisfies Property (T) [Ż03, KK13].

- For densities $d < 5/24$, w.o.p. a random group acts nontrivially on a CAT(0) cube complex [OW11, MP15]. Furthermore, [OW11] showed that this action is proper for densities $d < 1/6$.

In particular, since satisfying Property (T) and acting non-trivially on a CAT(0) cube complex are mutually exclusive properties, this raises the question: *What happens at densities between 5/24 and 1/3?* This thesis begins to answer that question.

1.2.3   Disk Diagrams and Isoperimetry

While one might expect that working in random groups would involve a certain amount of probability theory, in fact much of that work is swept under the rug in favor of geometric methods. In particular, the primary object of interest is the Cayley complex of a random group. The following Proposition was first written by Gromov, and the proof completed by Ollivier.

Proposition 1.2.2 ([Oll07] Theorem 2). *For each $\varepsilon > 0$, w.o.p. there is no disk diagram D fulfulling R and satisfying*

$$\mathrm{Cancel}(D) > (d + \varepsilon)|D|\ell.$$

The main idea of the proof is a counting argument. Given a disk diagram, one can calculate the probability that two adjacent cells share a path of length I. Using this, one can calculate the cancellation of the diagram and the result follows (up to dealing with the case that two adjacent cells correspond to the same relator).

This condition can be interpreted as a generalization of a small-cancellation property. Indeed, while small-cancellation provides restrictions on the amount that two 2-cells in a Cayley complex can overlap, this gives a restriction on the amount that *many* cells in a Cayley complex can 'overlap.' By working only in the realm of groups which satisfy this restriction, one can prove results about random groups.

For example, if the density of a random group is less than $1/2$, then by this Proposition w.o.p. every disk diagram must satisfy a linear isoperimetric inequality, and therefore must be hyperbolic. Thus random groups in the Gromov density model with density $d < 1/2$ are w.o.p. hyperbolic. Notice that this proof does not require a probablistic argument beyond appealing to Proposition 1.2.2.

1.3 Cube Complexes

1.3.1 Cubes, Hyperplanes, and CAT(0) Structures

An *n-cube* is a copy of $[-\frac{1}{2}, \frac{1}{2}]^n$. A *face* of an n-cube is a restriction of some of its coordinates to $\pm\frac{1}{2}$. A *cube complex* is a cell complex obtained by gluing together n-cubes isometrically along their faces. One can put a metric on this space by using the Euclidean metric within each cube and 'glueing' these metrics together along the faces.

Definition 1.3.1. A cube complex is CAT(0) if it is CAT(0) as a metric space. It is

nonpositively curved, or NPC, if its universal cover is CAT(0).

This is not always easy to identify. In fact, there is a nice combinatorial condition which is equivalent to a cube complex being NPC. A *flag complex* is a complex in which every complete subgraph spans a simplex. Note that a graph is flag if and only if the length of its shortest cycle is ≥ 4. Gromov [Gro87] showed that a cube complex is NPC if and only if the link of every vertex is a flag complex.

A CAT(0) cube complex may seem at first glance like a clunky construction, without the freedom of a simplicial complex. In fact, those restrictions provide a structure for groups which has been found to be both robust and surpisingly applicable, as shown by Agol.

Theorem 1.3.2 ([Ago13]). *If a group G is hyperbolic and acts cocompactly and properly discontinuously on a CAT(0) cube complex, then G is virtually special (and hence G is, e.g., residually finite and a-T-menable).*

In particular, by showing that a group G acts sufficiently nicely on a CAT(0) cube complex, one also shows that G satisfies nice properties. This inspired the following definition.

Definition 1.3.3. A group G is *cubulated* if it acts non-trivially cocompactly and properly discontinuously on a CAT(0) cube complex.

Thus proving residual finiteness or a-T-menability is translated to a question of showing that G acts nicely on a CAT(0) cube complex. In this paper, we will show that our groups act non-trivially cocompactly, though not necessarily properly discontinuously, on a CAT(0) cube complex. However, showing that a hyperbolic group acts non-trivially and cocompactly is sufficient to show that the group fails to satisfy Property (T).

1.3.2   Sageev's Construction

The traditional way of showing that a group acts on a CAT(0) cube complex is to build such a complex, via a construction of Sageev [Sag95]. This construction can be defined both

algebraically and geometrically. I will present the geometric interpretation here.

For the purposes of this thesis, I will define the construction in the context that it will be used, namely defining walls as graphs within the Cayley complex of a random group G. For a more general construction, see Chapters 6 and 7 of [Wis12].

To begin, let X be the Cayley complex of a random group G. A *wall* of X is an immersed graph W in X which is permuted by the action of G on X and which divides X into two essential complementary pieces, $\overleftarrow{W}$ and $\overrightarrow{W}$, called the *half-spaces of W*.

The *wallspace* $(X, \mathcal{W})$ is the space of X along with a collection of walls, $\mathcal{W}$, which follow the following finiteness restrictions:

1. For any two points $p, q \in X$, the number of walls *separating* p, q (meaning the number of walls for which p, q lie in distinct complementary components) is finite, and

2. For any $p \in X$, p lies on finitely many walls.

One can build a CAT(0) cube complex which is dual to the wallspace, according to the following construction. Let the 0-cubes of C be given by a choice of half-space for each wall, satisfying the following consistency conditions:

1. No two chosen half-spaces have empty intersection, and

2. For any point $p \in X$, p lies in all but finitely many of the half-spaces.

A 1-cube joins two 0-cubes exactly when they differ on one choice of half-space, and n-cubes are attached inductively when their $(n-1)$-skeleton exists.

This cube complex is CAT(0) and is naturally acted on non-trivially by G due to the action of G on each wall. In general, this action need not be particularly nice. However, given certain conditions on the walls one can guarantee that this action is both cocompact and properly discontinuous. In particular, if the walls are *quasi-convex*, then the action of G on C must be cocompact. If, additionally, there are 'enough' walls, then the action is properly discontinuous.

1.4 Statement of Main Theorem and Rough Proof Outline

Putting all of this together, the goal of this thesis is to prove the following theorem:

Theorem 1.4.1. *At density $d < 3/14$, a random group in the Gromov density model w.o.p. acts non-trivially cocompactly on a CAT(0) cube complex.*

The method of proof will be to construct G-equivariant quasi-convex walls in the Cayley complex X which separate X essentially and satisfy the given finiteness properties. Given those walls, the construction of Sageev will provide a CAT(0) cube complex on which G acts non-trivially and cocompactly.

Building these walls will involve finding quasi-convex embedded trees in X. We will build these trees inductively, and show that they are embedded via a local-to-global argument. In particular, we will show that these graphs are locally embedded trees, and then show that this implies that they must be embedded trees on a global scale.

This method of proof is motivated by the work of Olliver-Wise and Mackay-Przytycki, as described in the following chapter.

CHAPTER 2

MOTIVATIONS

2.1 Ollivier-Wise

Ollivier-Wise [OW11] proved that random groups at densities $d < 1/5$ act non-trivially co-compactly on a CAT(0) cube complex (and at $d < 1/6$ this action is properly discontinuous). Their method is similar to that developed by Ollivier in [Oll07] to prove that small cancellation groups are cubulated. They construct a graph Γ in the Cayley complex X of a random group via the following construction: the vertices of Γ are the midpoints of the 1-cells in X, and there is an edge connecting two vertices of Γ if and only if they are antipodal in some 2-cell of X. Each connected component of Γ is then a wall in X.

These walls are immersed graphs, which *a priori* may contain loops or self-intersections in X. However, Olliver-Wise showed that they are in fact embedded trees. To do this, they consider the possibility that some wall W has a self-intersection. Then some minimal path in W bounds a disk diagram in X. They show that this gives a contradiction using the Isoperimetric Inequality (Proposition 1.2.2). They then go on to show that each wall is quasi-convex and separates X into two unbounded components, and apply the construction of Sageev.

However, for densities $d > 1/5$ these 'walls' are no longer embedded trees. In fact, Ollivier-Wise show that there is only *one* connected component of Γ, and it is (massively) self-intersecting!

2.2 Mackay-Przytycki

Mackay-Przytycki resolved this problem by finding a more subtle construction of a wall system. Rather than construct an immersed graph one 2-cell at a time, they examine small complexes in X which exhibit (combinatorial) positive curvature, called *tiles.* Note that
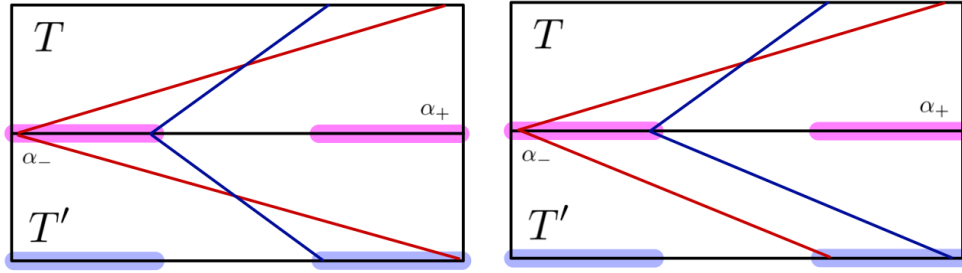
Figure 2.1: The construction from Mackay-Przytycki. The left tile shows walls before they are 'unbent,' and the right tile shows the resulting walls after 're-wiring.' The red wall is sharply bent prior to re-wiring, and unbent afterwards.

while the Cayley complex of a random group G is globally hyperbolic, there can exist small pockets of (local) positive curvature. For example, if two 2-cells C and D intersect along a 'long' path, their union is a pocket of positive (local) curvature. As the density of G increases, these pockets can become larger; in our example, the intersection of C and D can become larger. In this case, some of the wall-paths that result from concatenation are sharply bent; in particular, the paths which pass through edge midpoints near the endpoints of $C \cap D$. Mackay-Przytycki 'unbend' these walls according to the following rule:

Given a path $C \cap D$ which is longer than $\frac{\ell}{4}$, let $\alpha_{\pm}$ be the subpaths of $C \cap D$ which are the complement of the $\frac{\ell}{4}$-neighborhoods of each endpoint. Let $s_{\pm}$ be the symmetry of $\alpha_{\pm}$ which swaps its endpoints. Then for any wall in C which has an endpoint $x \in \alpha_{\pm}$, replace that wall with one connected to $s_{\alpha_{\pm}}(x)$ (see Figure 2.1.) Mackay-Przytycki show that the endpoints of the resulting wall in $C \cup D$ are separated by at least

$$\text{Bal}(C \cup D) = \frac{\ell}{4}(|C \cup D| + 1) - \text{Cancel}(C \cup D),$$

in which case we call the wall *balanced*.

They use this to make a local-to-global argument similar to that of Olliver-Wise: If some wall is not an embedded tree, it must bound a disk diagram. Using a generalization

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of the Isoperimetric Inequality for non-planar 2-complexes, they show that this leads to a contradiction.

In general, this construction produces balanced walls whenever the intersection of two tiles is a path or a tripod. However, as the density increases, the size of the tiles increases and the intersection of two tiles need not be a path or tripod. In this case, it is not at all obvious how to generalize the constrction to densities $d > 5/24$ where the intersections of tiles are more complex.

2.3   Statement of Results and a (More Precise) Proof Strategy

In this thesis I give one such construction. It is well-defined and has desirable properties at all densities $d < 3/14$. In other words:

Theorem 2.3.1. *At density $d < 3/14$, random groups admit a non-trivial action by isometries on a CAT(0) cube complex.*

The construction in this thesis plausibly applies to all densities below $1/4$. This is a natural threshold for several reasons, the most significant of which is that tiles may not be finite in densities above $1/4$. Thus one might hope that density $1/4$ represents a sharp bound for random groups acting on a CAT(0) cube complex.

The problematic pockets of positive curvature that arise in a random group at densities between $1/5$ and $1/4$ are called *tiles*; to a first approximation, they are built inductively from simpler tiles, and there is an *a priori* bound on the number of steps in this inductive process depending on d. Let T and T' be two simple tiles that are being glued into a more complicated tile $T \cup T'$. The crucial thing when constructing walls which are not sharply bent is to understand which edges in $T \cap T'$ give rise to a bent wall in $T \cup T'$. Mackay-Przytycki proved that the intersection $T \cap T'$ is necessarily a tree (see Lemma 3.2.3), and the key is to distinguish between two cases: whether this tree is *long* (e.g. the diameter of the tree is

large with respect to the total size of the tree) or *round*. One can show the concatenation of tile walls in T and T' results in unbalanced walls in $T \cup T'$ only when $T \cap T'$ is a long tree.

In this case, we can use a (in fact, any) diameter of $T \cap T'$ to identify the regions $\alpha_\pm$ through which any unbalanced wall in $T \cup T'$ must pass. When $T \cap T'$ is long, these regions are well-defined, regardless of the choice of diameter.

We then consider the paths $\alpha_\pm^i = \alpha_\pm \cap C_i$ for each 2-cell $C_i \in T'$, and replace any wall path in C_i which connects to a point $x \in \alpha_\pm^i$ with one connecting to $s_{\alpha_\pm^i}(x)$. To prove that the resulting walls in $T \cup T'$ are not bent, we check several cases, depending on the ways that a wall-path γ might lie in $T \cup T'$. For two of these cases, the proof relies on the inductive structures of T and T'. Thus, while this construction resolves the combinatorial problem of how to unbend a wall when the intersection of two tiles is complex, it raises a new question: *How can one unbend walls when each tile is itself complex?* To extend the result of Theorem 2.3.1 to all densities $d < 1/4$, one would need to complete the proof for arbitrary sized tiles in these two cases.

2.4   Organization

The rest of this thesis is organized as follows: In Chapter 3, we review the Isoperimetric Inequality for random groups and a key generalization to non-planar diagrams and explicitly construct tiles. In Chapter 4 we define walls in each tile and prove metric properties about them (namely that they are not sharply bent). We show that these balanced tile-walls give rise to walls in X which are embedded trees. In Chapter 5 we show that this gives rise to a non-trivial action of G on a CAT(0) cube complex.

CHAPTER 3

BUILDING TILES

3.1 Background and Definitions

From now on, let $G = \langle S|R \rangle$ be a random group of density d with word length ℓ, and let X be the Cayley 2-complex of G. By subdividing edges in X, we may assume that ℓ is a multiple of 4.

Definition 3.1.1. Let Y be a 2-complex. The *size* of Y, denoted $|Y|$, is the number of 2-cells in Y. The *cancellation* of Y is

$$\mathrm{Cancel}(Y) = \sum_{\substack{e \text{ a 1-cell} \\ \text{of } Y}} (\deg(e) - 1).$$

Notice that if $\{Y_i\} \subset X$ are subcomplexes equal to the closure of their 2-cells and sharing no 2-cells, then

$$\mathrm{Cancel}\left(\bigcup_i Y_i\right) \geq \sum_i \left(\mathrm{Cancel}(Y_i) + \frac{1}{2}\left|Y_i \cap \bigcup_{j \neq i} Y_j\right|\right)$$

Further, this is an equality if $i \leq 2$, and an inequality if there exists an edge in $\bigcup_i Y_i$ which lies in at least three distinct Y_i.

We can think of $\mathrm{Cancel}(Y)$ as a combinatorial proxy for the curvature of Y. For example, if Y is a disc diagram with large cancellation, then $|\partial Y|$ is, roughly speaking, small with relation to $|Y|$; in other words, Y has positive curvature. This relationship is made explicit in Proposition 1.2.2, and is generalized in Proposition 3.1.4.

A 2-complex Y is *fulfilled* by a set of relators R if there is a combinatorial map from Y to the presentation complex X/G that is locally injective around edges (but not necessarily around vertices). In particular, every subcomplex of X is fulfilled by R.

In the context of random groups, a conquence of the Isoperimetric Inequality (Proposition 1.2.2) is the following:

Corollary 3.1.2 ([MP15] Lemma 2.3, Corollary 2.5). *Let $d < 1/4$. Then w.o.p. there is no embedded closed path of length $< \ell$, and every embedded path of length $\leq \ell/2$ is geodesic in $X^{(1)}$.*

The following Proposition generalizes Proposition 1.2.2 in the situation that Y is non-planar. A 2-complex Y is (K, K')-*bounded* if $|Y| \leq K$ and Y is obtained from the disjoint union of its 2-cells by gluing them along $\leq K'$ subpaths of their boundary paths.

Remark 3.1.3. As is pointed out in [MP15] Remark 3.3, Corollary 3.1.2 implies that when $d < 1/4$, if $Y \subset X$ with $|Y| \leq K$ then Y is $(K, \frac{1}{2}K(K-1))$-bounded.

Proposition 3.1.4 ([MP15] Proposition 2.8, [Odr14] Theorem 1.5). *For any K, K' and $\epsilon > 0$, w.o.p. there is no (K, K')-bounded 2-complex Y fulfilling R and satisfying*

$$\mathrm{Cancel}(Y) > (d + \epsilon)|Y|l.$$

3.2 Potiles, Tiles and Tile Assignments

Ollivier-Wise defined a system of walls in the Cayley complex of a random group using only the antipodal relationship on edges of 2-cells [OW11]. In [MP15], Mackay-Przytycki broadened their focus from 2-cells to small complexes of relatively large combinatorial curvature, which they called *tiles*. We will use a similar definition, and show that these 2-complexes satisfy several nice properties.

Definition 3.2.1. A *potential tile*, or *potile*, is a non-empty connected 2-complex T equal to the closure of it 2-cells which satisfies the property

$$\mathrm{Cancel}(T) \geq \frac{\ell}{4}(|T| - 1).$$
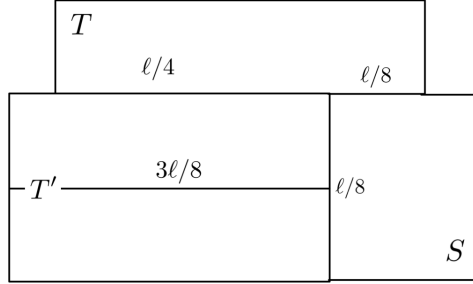
Figure 3.1: The 2-potile T' is shown, along with an adjacent 1-potiles S and T. The union $S \cup T'$ is a potile, even though $|S \cap T'| < \frac{\ell}{4}$, but $S \cup T$ is not. Also, $|T \cap T'| \geq \frac{\ell}{4}$, so $T \cup S \cup T'$ is also a potile.

To specify the size of a potile, we say T is an n-potile if it is composed of n 2-cells.

Remark 3.2.2. This definition is a generalization of Definition 3.1 in [MP15]; Mackay-Przytycki define a *tile* as an inductively built 2-complex which satisfies the strict version of the above inequality.

Not every complex Y is a potile. This is illustrated in Figure 3.1. For example, two 2-cells T, S which have an overlap of $|T \cap S| < \frac{\ell}{4}$ is not a potile, since $\text{Cancel}(T \cup S) < \frac{\ell}{4}$. However, if two potiles T, T' which do not share 2-cells have $|T \cap T'| \geq \frac{\ell}{4}$, then their union is a potile. Furthermore, if there is a third potile S so that $T' \cup S$ is a potile, then $T \cup T' \cup S$ must be a potile as well. On the other hand, if S, T' are potiles and T' is not a 2-cell, then just because $S \cup T'$ is a potile does not necessarily imply that $|S \cap T'| \geq \frac{\ell}{4}$.

One of the nice properties of potiles is that they have controlled intersections. In particular, one can see that the intersection of two distinct 2-cells in X must be a connected path of length $\leq \frac{\ell}{2}$ (See [OW11], Cor 1.11). One can extend this to get a similar result for the intersection of two potiles.

Lemma 3.2.3 ([MP15] Lemma 3.4, Remark 2.5)**.** *If T and T' are potiles sharing no 2-cells, then $T \cap T'$ is a connected tree and $|T \cap T'| \leq \frac{\ell}{2}$.*

Furthermore, the size of a potile is controlled as well. Heuristically, this is because X is globally negatively curved, and as the density d increases the hyperbolicity constant δ

increases. This allows pockets of local positive curvature to increase in size as well. This is made explicit in the following remark:

Remark 3.2.4. As shown in [MP15] Remark 3.3, potiles in random groups of density $d \leq \frac{N}{4(N+1)}$ have a maximal size of N.

In this paper, we will construct potiles in X via an inductive process.

Definition 3.2.5. A *tile collection* $\mathcal{T}$ is a set of potiles in X which satisfy the following properties:

1. $\mathcal{T}$ is invariant under the action of G on X,

2. If $T, T' \in \mathcal{T}$ share 2-cells, then the closure of the union of those 2-cells is a union of tiles in $\mathcal{T}$, and

3. The union of the elements in $\mathcal{T}$ is all of X.

A potile contained in a tile collection is a *tile*. Given a tile $T \in \mathcal{T}$, S is a *subtile of* T if S is a subcomplex of T and $S \in \mathcal{T}$.

Example 3.2.6. The set of single 2-cells, $\mathcal{T}^0$, is a tile collection.

In general, a tile collection may contain tiles that share 2-cells. This is not the case in $\mathcal{T}^0$. However, we will use an inductive process to build a more complicated tile collection in which this can happen. At each stage in the process, we will obtain a tile collection $\mathcal{T}^i$. Each $\mathcal{T}^i$ is partitioned into the disjoint union of three sets, denoted $\mathcal{T}^i_1, \mathcal{T}^i_c, \mathcal{T}^i_n$. The set $\mathcal{T}^i_1$ is given by $\mathcal{T}^0 \cap \mathcal{T}^i$. In other words, this is the set of tiles in $\mathcal{T}^i$ composed of a single 2-cell. The sets $\mathcal{T}^i_c$ and $\mathcal{T}^i_n$, called *core tiles* and *non-core tiles*, respectively, will be defined in the construction. We set $\mathcal{T}^0_c = \mathcal{T}^0_n = \emptyset$.

While $\mathcal{T}^i$ will be more complicated than $\mathcal{T}^0$, the tiles in $\mathcal{T}^i$ will have controlled overlaps, as described in the following Proposition.

Proposition 3.2.7. *The tile collections $\mathcal{T}^i = \mathcal{T}_1^i \sqcup \mathcal{T}_c^i \sqcup \mathcal{T}_n^i$ satisfy the following properties:*

1. *If $T, T' \in \mathcal{T}^i - \mathcal{T}_n^i$, then T and T' share no 2-cells, and T, T' contain no proper sub-tiles.*

2. *Every tile $T \in \mathcal{T}_n^i$ contains at least one tile $T_c \in \mathcal{T}_c^i$.*

3. *If $T, T' \in \mathcal{T}^i$ share 2-cells, then $\overline{T \cap T'}$ and $\overline{T - T'}$ are unions of potiles, all of which are tiles of some tile collection T_i.*

We will prove this proposition immediately after describing the construction of $\mathcal{T}^i$.

3.3 The Tile Construction

The construction of the tile-assignment is described here in words, explained in a flow chart in Figure 3.2, and illustrated in an example in Figure 3.3. The rough idea is to build tiles iteratively by glueing tiles together when they share no 2-cells and their union is a potile. When the size of the intersection of our two tiles is at least $\frac{\ell}{4}$, we do so in such a way that we maximize first the size of the resulting potile, and then the size of the intersection. When that process stops, we still may find a pair of tiles who share no 2-cells and have union a potile, but the size of their intersection is less than $\frac{\ell}{4}$. In this case, we add the union of these two tiles to our tile collection, but do not get rid of the constituent tiles. We then go back to looking for large intersections.

Since this process is done G-equivariantly, there are finitely many G-orbits of 2-cells, and potiles in X have a uniformly bounded size, this process eventually terminates.

Remark 3.3.1. It is reasonable to worry that we may run into a situation in which two tiles in the same G-orbit are glued together. However, if T, T' are tiles and $|T \cap T'| \geq \frac{\ell}{4}$, this is impossible. Indeed, suppose $T' \cap gT$ contains 2-cells. Consider the 2-complex Y obtained by identifying the edges in $T \cap T'$ with their pre-image under the action of g. This complex is realized by R, but $\mathrm{Cancel}(Y) \geq \mathrm{Cancel}(T) + |T \cap T'| \geq \frac{\ell}{4}|T|$, which violates Proposition 3.1.4.
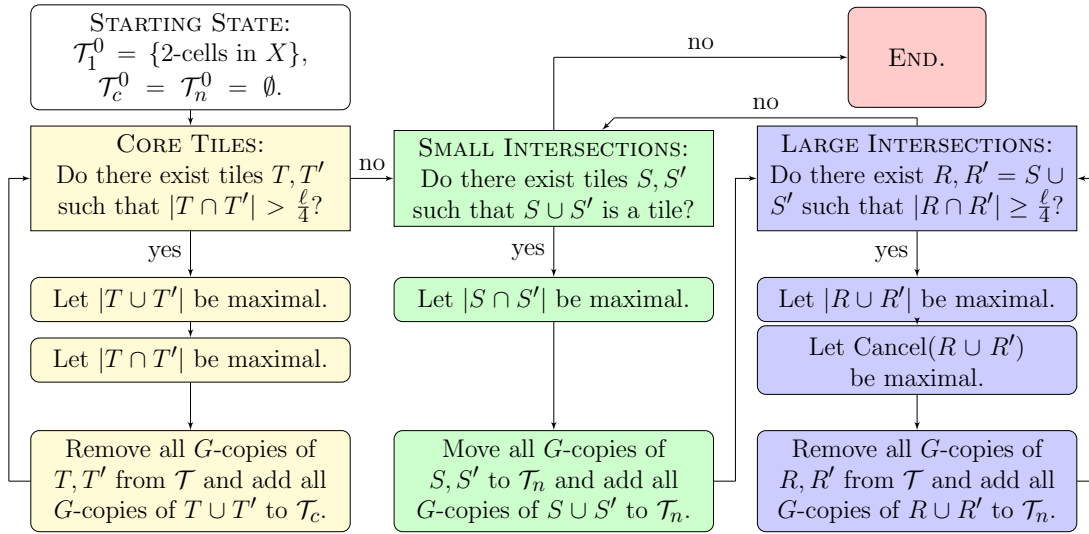
Figure 3.2: How to construct tiles, as in Construction 3.3.2.

Flowchart contents:

STARTING STATE:
$\mathcal{T}_1^0 = \{2\text{-cells in } X\}$,
$\mathcal{T}_c^0 = \mathcal{T}_n^0 = \emptyset$.

CORE TILES:
Do there exist tiles T, T' such that $|T \cap T'| > \frac{\ell}{4}$? — no → SMALL INTERSECTIONS

SMALL INTERSECTIONS:
Do there exist tiles S, S' such that $S \cup S'$ is a tile? — no → LARGE INTERSECTIONS

no → END.

LARGE INTERSECTIONS:
Do there exist $R, R' = S \cup S'$ such that $|R \cap R'| \geq \frac{\ell}{4}$?

CORE TILES — yes:
Let $|T \cup T'|$ be maximal.
Let $|T \cap T'|$ be maximal.
Remove all G-copies of T, T' from $\mathcal{T}$ and add all G-copies of $T \cup T'$ to $\mathcal{T}_c$.

SMALL INTERSECTIONS — yes:
Let $|S \cap S'|$ be maximal.
Move all G-copies of S, S' to $\mathcal{T}_n$ and add all G-copies of $S \cup S'$ to $\mathcal{T}_n$.

LARGE INTERSECTIONS — yes:
Let $|R \cup R'|$ be maximal.
Let $\mathrm{Cancel}(R \cup R')$ be maximal.
Remove all G-copies of R, R' from $\mathcal{T}$ and add all G-copies of $R \cup R'$ to $\mathcal{T}_n$.

Figure 3.3: This image shows a possible 'ancestry' of a 5-potile. Due to the maximality constraints, we know that $|A \cap B| \geq |D \cap E|$. Additionally, $|C \cap (A \cup B)| < \frac{\ell}{4}$. since $A \cup B$ and C remain in the tile collection. On the other hand, $|((A \cup B) \cap C) \cap (D \cup)| \geq \frac{\ell}{4}$ since $A \cup B \cup C$ and $D \cup E$ are removed from the tile collection. In Step 3, the tile $A \cup B$ is older than $D \cup E$ and C is younger than both of them.

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It is possible that if two tiles T, T' have overlap $|T \cap T'| < \frac{\ell}{4}$ that they could lie in the same G-orbit, but this will not be a problem.

In this thesis, we will limit ourselves to tiles of size ≤ 5. However, this construction applies to tiles of unlimited size.

Construction 3.3.2 (Tile Collections $\mathcal{T}^i$). The construction follows three steps, and is inspired by the process in [MP15]. The starting state is $\mathcal{T}^0 = \mathcal{T}_1 \sqcup \mathcal{T}_c \sqcup \mathcal{T}_n$, where $\mathcal{T}_1$ is the set of 2-cells, and $\mathcal{T}_c = \mathcal{T}_n = \emptyset$.

Step 1: (Core Intersections) If there exists a pair of tiles T, T' in $\mathcal{T}^i$ so that $|T \cap T'| > \frac{\ell}{4}$ and $T \cap T'| \leq 5$, choose such a pair so that $|T \cup T'|$ is maximal and $|T \cap T'|$ is maximal, in that order. Then

$$\mathcal{T}^{i+1} = \mathcal{T}^i - \{gT, gT' \mid g \in G\} \sqcup \{gT \cup gT' \mid g \in G\},$$

and assign each $gT \cup gT' \in \mathcal{T}_c^{i+1}$. Repeat Step 1 until there are no such pairs.

Since tiles are uniformly bounded in size and there are finitely many G-orbits of 2-cells, this process must eventually stop. At the end of this, we will have a set $\mathcal{T}_c$ which we will call the *core tiles of* $\mathcal{T}$.

Step 2: (Small Intersections) If there exist tiles $S, S' \in \mathcal{T}^i$ which do not share 2-cells so that $S \cup S'$ is a potile and $|S \cap S'| \leq 5$, choose such a pair which maximizes the size of $|S \cap S'|$. Notice that $|S \cap S'| \leq \frac{\ell}{4}$. Then define

$$\mathcal{T}^{i+1} = \mathcal{T}^i \sqcup \{gS \cup gS' \mid g \in G\},$$

and assign each $gS, gS', gS \cup gS' \in \mathcal{T}_n^{i+1}$. Move on to Step 3. Note that in this situation, we can not rule out the possibility that $S = gS'$ for some $g \in G$.

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Step 3: (Large Intersections) If there is a pair R, R' so that $|R \cap R'| > \frac{\ell}{4}$, $|R \cap R'| \leq 5$, and R does not share 2-cells with R', then one of these tiles, say R', must contain $S \cup S'$ from Step 2. Choose such a pair which maximizes $|R \cup R'|$ and $\mathrm{Cancel}(R \cup R')$, in that order. Define

$$\mathcal{T}^{i+1} = \mathcal{T}^i - \{gR, gR' \mid g \in G\} \sqcup \{gR \cup gR' \mid g \in G\},$$

and assign each $gR \cup gR' \in \mathcal{T}_n^{i+1}$. Repeat Step 3 until it terminates. Then return to Step 2.

From now on, the tile assignment $\mathcal{T} = \mathcal{T}_1 \sqcup \mathcal{T}_c \sqcup T_n$ will refer to a terminal tile assignment constructed in this manner. Note that by construction, $\mathcal{T}_1, \mathcal{T}_c, \mathcal{T}_n$ are all pairwise disjoint.

Definition 3.3.3. If T and T' are tiles that appear at some stage of the inductive process after the base-case, then T' is *younger* than T if T' appeared at a later step than T. By convention, we will always declare a tile from the starting tile collection to be the youngest tile.

Example 3.3.4. In the 5-tile construction in Figure 3.3, at Step 3 the tile $A \cup B$ is older than $D \cup E$, and C is younger than both of these 2-tiles.

Proof of Proposition 3.2.7. .

1. Since tiles in $\mathcal{T}_c$ are made by glueing non-overlapping tiles, the only way this could occur is if $T \cap gT'$ contained 2-cells for some $g \in G$. But by Remark 3.3.1, this can not happen.

2. This is immediate following the construction.

3. This is immediate following the construction.

$\square$

Remark 3.3.5. If $T, T' \in \mathcal{T}_c$, $|T|, |T'| \leq 3$, and T' is younger than T, then T must have been completed before the first 2-tile in T was formed. In particular, if D, D' are the first 2-cells glued together in T, T', respectively, then

$$\mathrm{Cancel}(D) \geq \mathrm{Cancel}(D').$$

Note that this need not be true when $|T| > 3$.

CHAPTER 4

BUILDING WALLS

4.1 Tile-Walls, Balance, and Some Inequalities

As in [OW11] and [MP15], we will find an action of G on a CAT(0) cube complex using the method of Sageev [Sag95]. This requires that we find subspaces of X which are quasi-convex, permuted by the action of G on X, and subdivide X into two essential components. We will build these inductively by first creating trees in each tile, called *tile-walls*, and then glueing these tile-walls along the intersection of the tiles to form an embedded tree in X. In this section, we will describe the construction of the tile-walls and establish some of their metric properties.

Definition 4.1.1. A *tile-wall* Γ_T is an immersed, connected graph in a tile T with vertices given by a subset of the edge mid-points in T, such that each 2-cell of T contains at most two vertices of Γ_T. There is an edge (v, w) connecting two vertices v, w if and only if v, w lie in a single 2-cell.

A path γ in Γ_T is a *wall path*.

Example 4.1.2. Given any 2-cell C with even boundary length, we can lay an edge from each vertex midpoint to its antipodal vertex midpoint. These are tile-walls in C. If we have two such decorated 2-cells, C_1 and C_2, and $|C_1 \cap C_2| \geq \frac{\ell}{4}$, then their union is a potile, and the immersed graphs generated by concatening wall paths which share endpoints are also tile-walls, as illustrated in Figure 4.1.

However, if we consider a 3-potile as in Figure 4.2, we see that the tile-walls obtained by the antipodal relationship do not necessarily produce a tile-wall structure.

One way to understand why the antipodal relationship does not always give a tile-wall structure is to consider what happens when two tiles are glued along a long path. For a
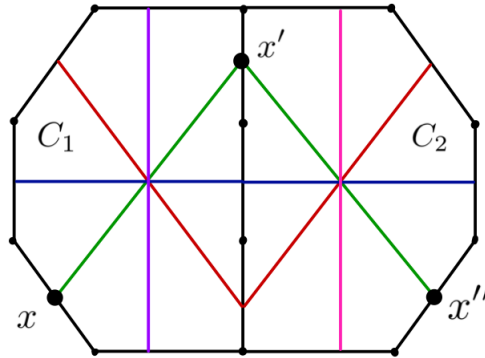
Figure 4.1: Each color denotes a different tile-wall. The edge midpoints x, x', x'' all lie in the same tile-wall, given by the antipodal relationship. In this example, every tile wall is also a wall-path.
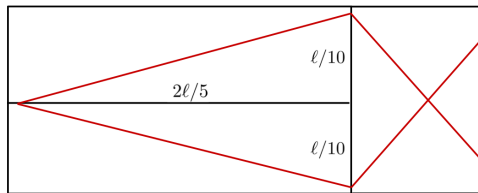
Figure 4.2: This 3-potile demonstrates that using the antipodal relationship to does not always produce tile-walls. The marked graph, connecting antipodal edges, is sharply bent as it passes through the path of length $2\ell/5$.

wall-path passing through an edge of this intersection near one of its endpoints, after glueing the wall is sharply 'bent', meaning endpoints of the wall are close together. In this instance, a third tile might be glued so that it intersects both endpoints of the same wall-path. Thus the primary objective is to 'unbend' walls just enough during the gluing process in Steps 1 and 3 of Construction 3.3.2 so that any two endpoints are 'separated' – in particular, so that no single tile can intersect two endpoints of the same tile-wall.

We quantify this in the following definition:

Definition 4.1.3. ([MP15], Definition 4.2) The *balance* of a tile T is given by

$$\text{Bal}(T) = \frac{\ell}{4}(|T| + 1) - \text{Cancel}(T).$$

In particular, we have the following result, illustrated in Figure 4.3.

Lemma 4.1.4 (See [MP15] Lemma 4.4). *If T, T' are potiles with no shared 2-cells, then*

$$|T \cap T'| \leq \min\{\text{Bal}(T), \text{Bal}(T')\}.$$

Proof. Suppose we have two such potiles T, T', and $|T \cap T'| > \text{Bal}(T)$. Then

$$\text{Cancel}(T \cup T') = \text{Cancel}(T) + \text{Cancel}(T') + |T \cap T'|$$
$$\geq \text{Cancel}(T) + \left(\frac{\ell}{4}|T'| - \frac{\ell}{4}\right) + \text{Bal}(T)$$
$$= \frac{\ell}{4}|T| + \frac{\ell}{4} + \left(\frac{\ell}{4}|T'| - \frac{\ell}{4}\right) = \frac{\ell}{4}(|T \cup T'|),$$

which contradicts Proposition 3.1.4. □

If the endpoints of every wall-path in a tile T are separated by at least $\text{Bal}(T)$, then there is no potile T' which contains two vertices of the same tile-wall in T.
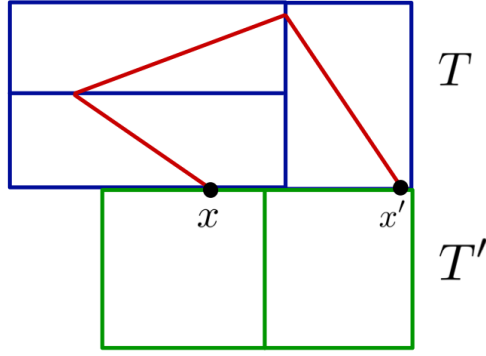
Figure 4.3: If T, T' are tiles and the red curve represents a tile-wall, then the endpoints of the red curve must have distance less than $\mathrm{Bal}(T)$ and $\mathrm{Bal}(T')$.

Remark 4.1.5. If T is a potile, then $\frac{\ell}{4} \leq \mathrm{Bal}(T) \leq \frac{\ell}{2}$. However, if T is not a potile, then $\mathrm{Bal}(T) > \frac{\ell}{2}$. This follows directly from the definitions of potiles and balance.

The ultimate goal is to construct walls which are embedded trees. To verify this, we will associate a tile to each wall path γ in T. The following definition is inspired by the definition of an *augmented tile* in [MP15], Definition 4.11, but differs sufficiently that I have used a new term to describe them.

Definition 4.1.6. For a tile-wall Γ_T in $T \in \mathcal{T}^i$ and a wall path γ in Γ_T with endpoints x, x', the *shard* associated to γ in T, written $\mathrm{Sh}_T(\gamma)$, is the subtile of T containing γ defined inductively as follows:

- In T^0, $\mathrm{Sh}_T(\gamma) = T$ for every wall path γ and tile T.

- If a tile $T \cup T'$ is made in Step 1, then $\mathrm{Sh}_{T \cup T'}(\gamma) = T \cup T'$ for every wall path γ.

- If a tile $S \cup S'$ is made in Step 2, and $\gamma \subset S$ and $\mathrm{Bal}(S) < \mathrm{Bal}(S \cup S')$, then $\mathrm{Sh}_{S \cup S'}(\gamma) = S$. Similarly for $\gamma \subset S'$. Otherwise, $\mathrm{Sh}_{S \cup S'}(\gamma) = S \cup S'$.

- If a tile $R \cup R'$ is made in Step 3, if $\gamma \subset R' = S \cup S'$ then $\mathrm{Sh}_{R \cup R'}(\gamma) = \mathrm{Sh}_{R'}(\gamma)$. Otherwise, $\mathrm{Sh}_{R \cup R'}(\gamma) = R \cup R'$.

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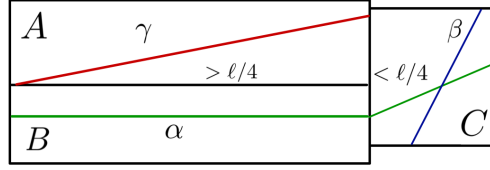
Figure 4.4: For tile $T = A \cup B \cup C$ with intersection sizes as shown, the shards of the illustrated wall-paths are $\mathrm{Sh}_T(\gamma) = A \cup B, \mathrm{Sh}_T(\alpha) = T, \mathrm{Sh}_T(\beta) = C$.

Example 4.1.7. Suppose that T is a tile composed as a union of three 2-cells A, B, C where $|A \cap B| \geq \frac{\ell}{4}$ and $|(A \cup B) \cap C| < \frac{\ell}{4}$ (see Figure 4.4). Note that the proper sub-tiles of T are C and $A \cup B$, but not A or B (since they were removed from $\mathcal{T}$). If γ is a wall-path contained in A, then $\mathrm{Sh}_T(\gamma) = A \cup B$. For a wall-path α with edges in B and C, $\mathrm{Sh}_T(\alpha) = T$. A wall-path β in C has $\mathrm{Sh}_T(\beta) = C$.

Definition 4.1.8. A wall-path γ with endpoints x, x' in a tile T is *balanced with respect to* T if

$$|x, x'|_T \geq \mathrm{Bal}(T).$$

Alternatively, given a tile T and wall-path $\gamma \subset T$ with endpoints x, x', T is γ-*balanced* if

$$|x, x'|_T \geq \mathrm{Bal}(\mathrm{Sh}_T(\gamma)).$$

We say T is *balanced* if it is γ-balanced for every $\gamma \in T$.

Example 4.1.9. For a 1-tile T, the only balanced tile-wall is the graph which connects antipodal edge midpoints. (See Figure 4.1.) Indeed, for any antipodal edge midpoints x, x', we have $|x, x'|_T = \frac{\ell}{2} = \mathrm{Bal}(T)$.

In general, the tile-wall generated by identifying antipodal edge midpoints will not give a balanced tile-wall structure. As we saw in Example 4.1.2, it may not even give a tile-wall structure at all. However, by making a small adjustment to the tile-walls when we glue two tiles together, we can obtain a balanced tile-wall structure. The geometric idea behind this

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is that if that two tiles have a long overlap, the result of glueing them together is a sharply bent tile-wall, with vertices that are close together. To resolve this, we slightly 'unbend' the tile-walls near the ends of these large intersections.

The following example, taken from [MP15], also gives the flavor of how we will prove that our tile-walls are balanced. In particular, we will consider the possible ways that a wall-path γ can lie in a tile $T \cup T'$, and show that in each of these situations, $\mathrm{Sh}_{T \cup T'}(\gamma)$ is γ-balanced. This example is illustrated in Figure 4.5.

Example 4.1.10 (Bending Tile Walls). Suppose T, T' are 1-tiles with intersection $\frac{\ell}{2} \geq |T \cap T'| \geq \frac{\ell}{4}$, each decorated with antipodal tile-walls. By Lemma 3.2.3, $T \cap T'$ is a geodesic path. Let the endpoints be called u_+ and u_-, and label the (unique) points in $|T \cap T'|$ at distance $\frac{\ell}{4}$ from $u_\pm$ with $v_\mp$. Let $\alpha_\pm$ denote the path from $u_\pm$ to $v_\pm$, and let $s_\pm$ be the symmetry of $\alpha_\pm$ which swaps its endpoints. For any tile-wall Γ in T' connecting x to x', if $x \in \alpha_\pm$ we replace that edge with one connecting x' to $s_\pm(x)$. Otherwise, we leave the tile-walls as they are. Then the tile-wall structure on $T \cup T'$ generated by the tile-wall structure on T and the adjusted tile-wall structure on T' is balanced. We verify this claim below.

Consider a wall-path γ in $T \cup T'$ with endpoints x, x'. Then $\mathrm{Sh}_{T \cup T'}(\gamma) = T \cup T'$. If γ lies entirely in T or T', then at most one of x, x' lies in $T \cap T'$ (by Lemma 4.1.4). Furthermore, suppose x, x' are antipodal. Then $|x, x'| = \frac{\ell}{2} \geq \mathrm{Bal}(T \cup T')$, by Remark 4.1.5. If $x \in T \cap T'$ and $x' \in T'$, then $|x, x'| = |s(x), x'| - |s(x), x| \geq \frac{\ell}{2} - (|T \cap T'| - \frac{\ell}{4}) = \mathrm{Bal}(T \cup T')$. If γ traverses both T and T', then γ has a midpoint y in $T \cap T'$. Let z, z' be the nearest point projections in $(T \cup T')^{(1)}$ of x, x', respectively, to $T \cap T'$. Then $|x, x'| \geq |x, z| + |x', z'|$. If $y \notin \alpha_\pm$, then y is antipodal to both x and x', and y is within $\frac{\ell}{4}$ of z and z', so $|x, x'| \geq |x, y| - |y, z| + |x', y| - |y, z'| \geq \ell - \frac{\ell}{4} - \frac{\ell}{4} = \frac{\ell}{2} \geq \mathrm{Bal}(T \cup T')$. If, on the other hand, y is in $\alpha_\pm$, then $y = s(y')$ for some point y' in $T \cap T'$ which is antipodal to x', and $|x, x'| \geq |x, y| - |y, z| + |x', y'| - |y', z'| \geq \ell - (2|T \cap T'| - (|T \cap T'| - \frac{\ell}{4})) = \frac{3\ell}{4} - |T \cap T'| = \mathrm{Bal}(T)$.
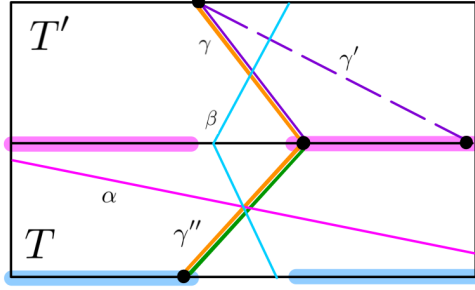
Figure 4.5: The 2-cell T is on the top, and T' on the bottom. The overlap of these two tiles is larger than $\frac{\ell}{4}$. The paths $\alpha_{\pm}$ are indicated in red. The paths $\alpha, \beta.\gamma, \gamma''$ represent wall paths in $T \cup T'$. The path γ' is the wall in T' which is adjusted to give the wall γ in $T \cup T'$. (Refer to Example 4.1.10.)

This is the motivating example for the method of balancing our tile-walls, as presented in the following section. While in general the intersection of two tiles is not a path but an embedded tree, we will see that when the tile-walls in $T \cup T'$ generated by glueing tile-walls in T and T' along their intersections are not balanced, then the tree $T \cap T'$ is 'almost' a path, and a similar construction will result in balanced tile-walls.

The rest of this section is devoted to statements and proofs of lemmas which will be useful in the following section. Where referenced, these are 'translations' of lemmas from [MP15] into the language of this paper. Proofs are included of all lemmas for completeness.

Lemma 4.1.11 (See [MP15] Lemma 4.4). *Let* $T = \mathrm{Sh}_S(\gamma), T'$ *be shards in* X *that do not share 2-cells, and suppose that* S *is* γ-*balanced. Then at most one of the endpoints of* γ *lies in* T'.

Proof. This is a restatement of Lemma 4.1.4 in terms of balanced tile-walls. □

Lemma 4.1.12 (See [MP15] Lemma 4.5). *Let* $T = \mathrm{Sh}_S(\gamma), T'$ *be shards in* X *that do not share 2-cells. Suppose that* S *is* γ-*balanced. Then the endpoints* x, x' *of* γ *satisfy*

$$|x, x'|_{T \cup T'} \geq \mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4},$$

and if $|T \cap T'| \geq \frac{\ell}{4}$, *then* $|x, x'| \geq \mathrm{Bal}(T \cup T')$.
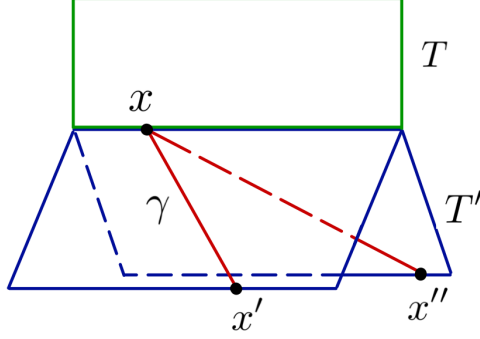
Figure 4.6: By Lemma 4.1.12, if the tile T' is γ-balanced and $|T \cap T'| \geq \frac{\ell}{4}$, then $|x, x'| \geq \text{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4}$. Similarly for $|x, x''|$ and $|x', x''|$.

Refer to Figure 4.6.

Proof. By Lemma 3.2.3, $|T \cap T'| \leq \frac{\ell}{2}$. This is geodesic by Corollary 3.1.2, so

$$|x, x'|_{T \cup T'} = |x, x'|_T > \text{Bal}(T).$$

We also know that

$$\begin{aligned}
\text{Bal}(T \cup T') &= \frac{\ell}{4}|T \cup T'| + \frac{\ell}{4} - \text{Cancel}(T \cup T') \\
&= \frac{\ell}{4}(|T| + 1) + \frac{\ell}{4}|T'| - \text{Cancel}(T) - \text{Cancel}(T') - |T \cap T'| \\
&\leq \text{Bal}(T) - |T \cap T'| + \frac{\ell}{4}.
\end{aligned}$$

If $|T \cap T'| > \frac{\ell}{4}$, then $\text{Bal}(T \cup T') \leq \text{Bal}(T) \leq |x, x'|_{T \cup T'}$. $\qquad\square$

Lemma 4.1.13 (See [MP15] Lemma 4.6)**.** *Let $T = \text{Sh}_S(\gamma)$ and $T' = \text{Sh}_{S'}(\gamma')$ be shards of S, S' which are γ, γ'-balanced, respectively. Suppose T, T' do not share 2-cells, and let the endpoints of γ be called x'', y and the endpoints of γ' be called x, x'. Suppose $|T \cap T'| > \frac{\ell}{4}$ and there is a path $\alpha \subset T \cap T'$ such that $T \cap T' \subset N_{\ell/4}(\alpha)$. Let s_α be the symmetry of α*
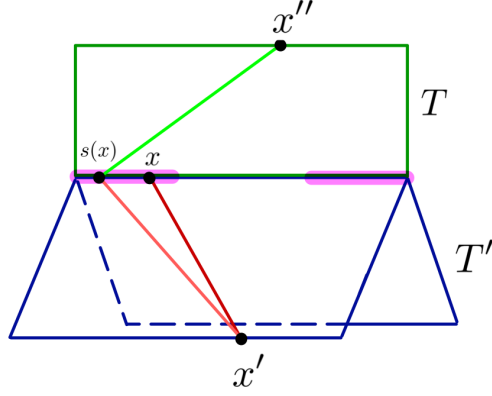
Figure 4.7: The tile T is γ-balanced, T' is γ'-balanced, and $|T \cap T'| \geq \frac{\ell}{4}$. If $|T \cap T'|$ is contained in the $\frac{\ell}{4}$-neighborhood of the path α_+, then by Lemma 4.1.13, the tile-wall in $T \cup T'$ connecting $x', s(x), x''$ is balanced.

which swaps its endpoints. If $s_\alpha(x) = y$, then

$$|x', x''|_{T \cup T'} \geq \mathrm{Bal}(T \cup T').$$

Refer to Figure 4.7.

The proof of this lemma uses the following:

Lemma 4.1.14 ([MP15] Sublemma 4.7). *Let A be a tree, $\alpha \subset A$ a path such that A is contained in the q-neighborhood of α. Let s be the symmetry of α exchanging its endpoints. Then for any points $z, z' \in A$ and $y \in \alpha$ we have*

$$|y, z|_A + |s(y), z'|_A \leq |A| + \max\{|\alpha|, q\}.$$

Proof of Lemma 4.1.13. Apply Lemma 4.1.14 with $A = T \cap T'$ and $q = \frac{\ell}{4}$. Let z, z' be the nearest point projections in $T \cup T'^{(1)}$ of x', x'', respectively, to $T \cap T'$. Then $|y, z| + |x, z'| \leq$

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$|T \cap T'| + \frac{\ell}{4}$, so

$$|x', x''|_{T \cup T'} \geq |x', z|_T + |x'', z'|_{T'}$$

$$\geq |x', x|_T - |x, z|_T + |x'', y|_{T'} - |y, z'|_{T'}$$

$$\geq \frac{\ell}{4}(|T| + 1) - \text{Cancel}(T) + \frac{\ell}{4}(|T'| + 1) - \text{Cancel}(T') - (|T \cap T'| + \frac{\ell}{4})$$

$$= \text{Bal}(T \cup T'),$$

as desired. $\qquad\square$

Remark 4.1.15. The proof of the previous lemma does not require that T, T' are potiles; it merely requires that $T \cap T'$ is a connected tree of size at most $\frac{\ell}{2}$.

As a special case of Lemma 4.1.13 when $|T \cap T'| < \frac{\ell}{4}$, we have the following:

Lemma 4.1.16 (See [MP15] Corollary 4.8)**.** *Let* $T = \text{Sh}_S(\gamma), T' = \text{Sh}_{S'}(\gamma')$ *be shards in* S, S' *which are* γ, γ'-*balanced, respectively, and do not share* 2-*cells. Let the endpoints of* γ *be* x, y *and the endpoints of* γ' *be* y, x', *where* $y \in T \cap T'$ *is an edge midpoint such that* $T \cap T'$ *is contained in the* $\frac{\ell}{4}$-*neighborhood of* y. *Then*

$$|x, x'|_{T \cup T'} \geq \text{Bal}(T \cup T').$$

Proof. Apply Lemma 4.1.13 with $\alpha = \{y\}$, and $q = \frac{\ell}{4}$. $\qquad\square$

Lemma 4.1.17. *If* T, T' *are tiles in* $\mathcal{T}_c$ *and* T' *is younger than* T, *then for any* 2-*cell* $C \in T'$, $|T \cap C| < \frac{\ell}{4}$.

Proof. Suppose not. Then T' is composed of two tiles S, S', where (without loss of generality) S contains C. Prior to the step in which T' was formed, we must have had S, S', T tiles in our collection. Then $|T \cap S| \geq |T \cap C| \geq \frac{\ell}{4}$, so by the maximality condition in Step 1 of Construction 3.3.2 we should have created $T \cup S$, rather than T'. $\qquad\square$

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Lemma 4.1.18. *If $T, T', T \cup T' \in \mathcal{T}_c$ share no 2-cells and $|T| \leq |T'| = 3$, then*

$$\mathrm{Bal}(T \cup T') \leq \frac{5\ell}{4} - 2\,\mathrm{Cancel}(D') - |T \cap T'|$$

where D, D' are as in Remark 3.3.5.

Proof. By Remark 3.3.5, T can be written as the union of a 2-tile D and a 2-cell C, and similarly $T' = D' \cup C'$, where $\mathrm{Cancel}(D) \geq \mathrm{Cancel}(D')$. Let $\beta = D \cap C$ and $\beta' = D' \cap C'$. Notice that if $|T| = 2$, then $|\beta| = 0$ and $\frac{\ell}{4}|T| - |\beta| = \frac{\ell}{2}$. If $|T| = 3$, then $|\beta| \geq \frac{\ell}{4}$, and $\frac{\ell}{4}|T| - \beta \leq \frac{3\ell}{4} - \frac{\ell}{4} = \frac{\ell}{2}$. Additionally, $|\beta'| \geq \frac{\ell}{4}$. Putting this all together, we get

$$
\begin{aligned}
\mathrm{Bal}(T \cup T') &= \frac{\ell}{4}(|T| + |T'| + 1) - \mathrm{Cancel}(T \cup T') \\
&= \frac{\ell}{4}|T| + \frac{\ell}{4}|T'| + \frac{\ell}{4} - \mathrm{Cancel}(T) - \mathrm{Cancel}(T') - |T \cap T'| \\
&= \frac{\ell}{4}|T| + \frac{3\ell}{4} + \frac{\ell}{4} - (|\beta| + \mathrm{Cancel}(D)) - (|\beta'| + \mathrm{Cancel}(D')) - |T \cap T'| \\
&\leq \frac{\ell}{2} + \frac{3\ell}{4} + \frac{\ell}{4} - \frac{\ell}{4} - 2\,\mathrm{Cancel}(D') - |T \cap T'| \\
&= \frac{5\ell}{4} - 2\,\mathrm{Cancel}(D') - |T \cap T'|.
\end{aligned}
$$

$\square$

4.2   Construction of Tile Walls

In this section, we build the tile-walls and prove that they are balanced. The construction of tile-walls will parallel the iterative process of constructing tiles. The proof that the resulting tiles are balanced reduces to checking several cases, based on the possible arrangements of tile-walls in $T \cup T'$.

The proof of each case is a finite problem which can be reduced to a linear programming

problem, where each variable is given by the size of the intersection of each pair of 2-cells, and the inequalities are given by (1) maximality conditions in the construction, (2) the Isoperimetric Inequality as given in Lemma 3.1.4, and (3) an assumption that at least one wall is not balanced. Enumeration of the cases is complicated by the geometric requirement that the intersection of two potiles is a connected tree. Tiles are certified as balanced if the space of potential solutions is empty. I implemented this code in the case that my tiles are both size 3, which demonstrated that this process produced balanced walls. One can also prove this 'by hand,' as demonstrated below.

As we saw in Example 4.1.2, when two tiles are glued together along a sufficiently large intersection, the tile-walls obtained by concatenating tile-walls in each constituent tile may result in highly 'bent' tile-walls, which have two endpoints that are close together. Glueing this to another tile so that the intersection contains endpoints could result in a self-intersection. In Example 4.1.10, we identified the edges of the path $T \cap T'$ for which concatenated tile-walls are sharply bent in $T \cup T'$; specifically, these are the edges which are near the endpoints of $T \cap T'$. In general, the intersection of two potiles is not a path, but tree. Identifying the edges which may lead to a bent tile-wall is thus a more subtle problem.

Definition 4.2.1. A tree A with $\frac{\ell}{2} \geq |A| \geq \frac{\ell}{4}$ is *long* if $\frac{1}{2}(|A| + \frac{\ell}{4})) < \operatorname{diam} A$. If A is not long, then it is *round*.

Figure 4.8 illustrates this definition, and the following lemma.

Lemma 4.2.2. *If A is a long tree, there exist regions $\alpha_\pm$ so that for any diameter D of A with endpoints $u_\pm$, α_+ is the complement of the $\frac{\ell}{4}$-neghborhood of u_-, and analogously for α_-.*

Proof. If A admits only one diameter D, let $u_\pm$ be the endpoints of D. If that is not the case, let D, D' be two distinct diameters of A. Then

$$|D| + |D'| - |D \cap D'| = 2\operatorname{diam}(A) - |D \cap D'| \leq |A|.$$

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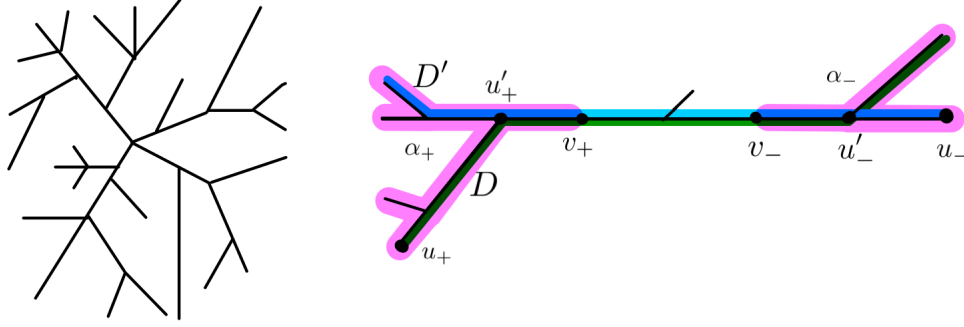
Figure 4.8: At left, we have a 'round' tree. The right image shows a 'long' tree along with potential diameters D, D' in blue and green; points $u'_\pm, u_\pm, v_\pm$; and regions $\alpha_\pm$ highlighted in pink.

Since A is a long tree, $2\operatorname{diam}(A) > |A| + \frac{\ell}{4}$, so

$$|D \cap D'| \geq \frac{\ell}{4}.$$

Let $u'_\pm$ be the end points of $D \cap D'$. The two components of $A - (D \cap D')$ containing $u'_\pm$ are the *ends* of A. Let $u_\pm$ be any point in the end containing $u'_\pm$ which is furthest from $u'_\pm$. Note that if $u'_\pm$ not an leaf of A, then it is a branching point of A, otherwise $D \cap D'$ could be extended. So either $u_\pm = u'_\pm$, or there are at least two options for $u_\pm$. Note that the path from u_+ to u_- is a diameter, and any diameter can be chosen in this way.

Let $\alpha_\mp$ be the complement of $N_{\ell/4}(u_\pm)$ in $T \cap T'$. Notice that $\alpha_\pm$ contain the ends of $T \cap T'$, and no edge is contained in both ends. Importantly, the assignment of $\alpha_\pm$ does not depend on the choice of $D, D', u_\pm$. $\qquad\square$

This only identifies these regions in the case that $T \cap T'$ is a long tree. If $T \cap T'$ is a round tree, then the tile-walls which result from concatenation are balanced.

Lemma 4.2.3. *Let T, T' be tiles which share no 2-cells. If $|T \cap T'| \geq \frac{\ell}{4}$ and $T \cap T'$ is a round tree, and λ, λ' are balanced tile-walls in T, T', respectively, then*

1. *The tile-walls λ, λ' are balanced in $T \cup T'$, and*

2. *If λ, λ' share an endpoint $y \in T \cap T'$, then the concatenation of these two tile-walls is a balanced tile-wall in $T \cup T'$.*

Proof. Consider the first situation. Let the endpoints of λ, λ' be x, y and x', y, respectively. Let z (resp. z') be the nearest point projection of x (resp. x') to $T \cap T'$. Then

$$
\begin{aligned}
|x, x'|_{T \cup T'} &\geq |x, z|_T + |x', z'|_{T'} \\
&\geq |x, y|_T - |y, z|_T + |x', y|_{T'} - |y, z'|_{T'} \\
&\geq \mathrm{Bal}(T) + \mathrm{Bal}(T') - (|y, z|_{T \cap T'} + |y, z'|_{T \cap T'}) \\
&\geq \mathrm{Bal}(T) + \mathrm{Bal}(T') - 2\,\mathrm{diam}(T \cap T') \\
&= \mathrm{Bal}(T \cup T') + |T \cap T'| + \frac{\ell}{4} - 2\,\mathrm{diam}(T \cap T') \\
&\geq \mathrm{Bal}(T \cup T') + |T \cap T'| + \frac{\ell}{4} - (|T \cap T'| + \frac{\ell}{4}) \\
&= \mathrm{Bal}(T \cup T').
\end{aligned}
$$

Now, if λ is a wall path in T or T' with endpoints x, x', then $|x, x'| \geq \mathrm{Bal}(T)$ (or $\mathrm{Bal}(T')$, respectively.) By Lemma 4.1.12,

$$
|x, x'|_{T \cup T'} \geq \mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4} \geq \mathrm{Bal}(T \cup T'),
$$

which is what we wanted to prove. $\qquad\square$

This means that when glueing two tiles together, the tile-walls resulting from concatenation will only be unbalanced in the case that $T \cap T'$ is a long tree. Therefore, this is the only case in which we will alter tile-walls:

Construction 4.2.4 (Tile-Walls Γ_i)**.** We begin with tile-walls in $\mathcal{T}^0$ given by laying an edge between any antipodal edge midpoints in T.

Step 1 : (Core Intersections) We have two tiles T, T' such that $|T \cap T'| \geq \frac{\ell}{4}$. There are two

cases, depending on whether $T \cap T'$ is a round tree or a long tree.

(a) *Round Trees* Suppose $\operatorname{diam}(T \cap T') \leq \frac{1}{2}(|T \cap T'| + \frac{\ell}{4})$.

The tile-wall structure is generated by connecting walls in T to walls in T' along identified edges.

(b) *Long Trees* Suppose that $\frac{1}{2}(|T \cap T'| + \frac{\ell}{4}) < \operatorname{diam}(T \cap T')$. This situation is illustrated in Figure 4.9.

Let T be older than T'. For each 2-cell C_i in T', let $\alpha_i^{\pm} = C_i \cap \alpha^{\pm}$. For any path β, let s_β be the symmetry of β that swaps its endpoints. When it is clear which path is being altered, we will write s_β as s. Define a tile-wall structure on $S = T \cup T'$ generated by the following rule: For any 2-cell $C_i \in T'$ adjacent to $T \cap T'$ and for any edge midpoint $x \in \alpha_i^{\pm}$, replace the edge connecting x to x' with one connecting $s_{\alpha_i^{\pm}}(x)$ to x'. Then, concatenate adjacent tile-walls as in Step 1(a).

Step 2 : (Small Intersections) We have two tiles T, S so that $|S \cap T| < \frac{\ell}{4}$. As in Step 1(a), the tile-walls of $S \cup T$ are generated by connecting walls in S to walls in T along identified edges.

Step 3 : (Large Intersections) We have two tiles, R and R' where R' contains the tile $S \cup T$ from the most recent interation of Step 2. We do not adjust walls.

Remark 4.2.5. Note that in Step 3, for R to be younger than R' it must be a 1-tile. Otherwise, R is a 2-tile and R' is a 3-tile, so R' must be the younger tile. Indeed, a tile made in Step 2 must contain at least three 2-cells, and $R \cup R'$ is (at most) a 5-tile.

As in Example 4.1.10, to prove that the resulting tile-walls are balanced, we will check several cases. The following lemma enumerates the potential cases.
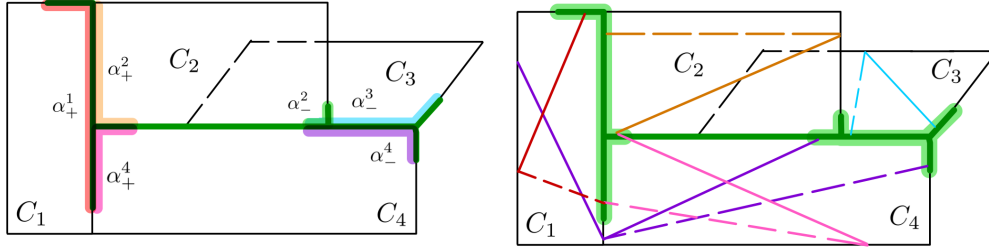
Figure 4.9: The left image shows a non-planar tile T' and a potential long tree $T \cap T'$, along with each $\alpha_i^{\pm}$ indicated for each 2-cell. At right, the old and new tile-walls are illustrated. Each colored solid line indicates a tile-wall in $T \cup T'$, and the dashed lines of the same color indicates the corresponding wall-path in T'.

Lemma 4.2.6. *If T, T' are balanced tiles which share no 2-cells and $T \cup T'$ is a potile, then the immersed graphs as constructed in Construction 4.2.4 give a tile-wall structure. Furthermore, there are 7 ways that a wall path γ of $T \cup T'$ can lie in $T \cup T'$, as listed here:*

1. *γ lies entirely in one of T or T' and does not intersect $T \cap T'$, or*

2. *γ lies entirely in one of T or T' and has a single endpoint in $T \cap T'$ and*

    (a) *one of the endpoints of γ lies in $\alpha^{\pm}$ or*

    (b) *neither endpoint of γ lies in $\alpha^{\pm}$, or*

3. *γ lies entirely in one of T or T' and $y = \gamma \cap (T \cap T')$ is a single interior vertex of γ and*

    (a) *the interior vertex y lies in $\alpha^{\pm}$ or*

    (b) *no vertex of γ lies in $\alpha^{\pm}$, or*

4. *γ does not lie entirely in one of T or T', and $y = \gamma \cap (T \cap T')$ is a single interior vertex of γ and*

    (a) *the interior vertex y lies in $\alpha^{\pm}$ or*

    (b) *no vertex of γ lies in $\alpha^{\pm}$.*

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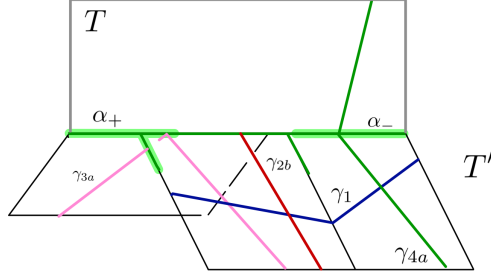Figure 4.10: In tile T', the paths $\alpha_\pm$ are highlighted in light green. The three 2-cells of T' are labelled. The tile-walls in $T \cup T$ given by $\gamma_1, \gamma_{2b}, \gamma_{3a}$, and γ_4 illustrate the corresponding cases in Lemma 4.2.6.

Some of the possible situations in this lemma are illustrated in Figure 4.10.

Proof. By induction, since the antipodal relationship gives a tile-wall structure on $\mathcal{T}^0$ it suffices to guarantee that each tile-wall has at most one edge in any 2-cell. This follows from Lemma 4.1.11. The seven cases are clear. $\qquad\square$

Theorem 4.2.7. *For tiles $T, T' \in \mathcal{T}^i$, if T' is younger than T and $|T'| \leq 3$, then the tile walls constructed in Construction 4.2.4 are balanced.*

Proof. We will prove this by considering each Step in Construction 3.3.2 and each Case in Lemma 4.2.6. In Steps 1(a) and 1(b), the shard associated to any wall-path γ in $T \cup T'$ is $T \cup T'$.

Step 1(a). Suppose that $T \cap T'$ is a round tree. Let γ a wall path in $T \cup T'$. By Lemma 4.2.3, γ is balanced in $T \cup T'$.

Step 1(b). Now suppose that $T \cap T'$ is a long tree. We will check each case in Lemma 4.2.6.

CASES 1, 2(b), 3(b). Let γ be a wall-path in $T \cup T'$, which lies in a single tile. In these cases, γ is the same as a wall-path in T or T'. This is also true in Cases 2(a), 3(a) if we assume γ lies entirely in T'. In any of these cases, since γ is not adjusted we get the desired result by Lemma 4.1.12.
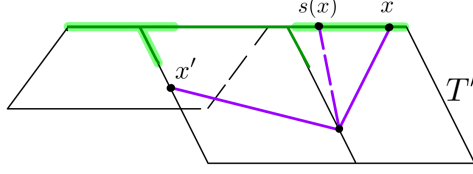
Figure 4.11: This figure illustrates Step 1(b), Case 2(a) in Theorem 4.2.7. The solid purple line indicates a tile-wall in T', and the dotted purple line shows part of the corresponding tile-wall in $T \cup T'$.

CASE 2(a). Now suppose that γ is a wall-path in T' with endpoints x and x', where x is the endpoint of γ which lies in $\alpha^{\pm}$. This case is illustrated in Figure 4.11.

Then the neighborhood of x in γ lies in some 2-cell C_i. Let $s(x) = s_{\alpha_i^{\pm}}(x)$, so that x' and $s(x)$ were the endpoints of the wall-path in T' which gave rise to γ. By Lemma 4.1.12, $|x', s(x)|_{T \cup T'} \geq \mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4}$. Notice that $|s(x), x|_S \leq |\alpha_i^{\pm}| \leq |T \cap T'| - \frac{\ell}{4}$, since $\alpha_i^{\pm}$ is contained in $\alpha^{\pm}$, which is the complement of the $\frac{\ell}{4}$-neighborhood of a point. Therefore

$$
\begin{aligned}
|x, x'|_S &\geq |x', s(x)|_S - |s(x), x|_S \\
&\geq \left(\mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4} \right) - \left(|T \cap T'| - \frac{\ell}{4} \right) \\
&= \mathrm{Bal}(T \cup T').
\end{aligned}
$$

This concludes the proof for Case 2(a). .

CASE 3(a). Suppose now that γ is a wall-path in $T \cup T'$ which lies entirely in T', with endpoints x, x' and a midpoint $y \in \alpha^{\pm}$. *A priori*, it seems that there may be many situations in which this case arises. However, we will show that there are only two types of tile (illustrated in Figure 4.12) which can give rise to this case, and then show that in each, γ is balanced.

Note first that y must be adjacent to at least two 2-cells, C_1 and C_2 which are traversed by γ. By Construction 4.2.4, $x \in C_1, x' \in C_2$ were connected by wall-paths in T' to points $z = s_1(y), z' = s_2(y)$, respectively. If $z = z'$, then $|x, x'| \geq \mathrm{Bal}(T') \geq \mathrm{Bal}(T \cup T')$, by Lemma

4.1.12. Assume that this is not the case.

Notice that u_- is not in C_1, since this would imply $|C_1 \cap T| > \frac{\ell}{4}$, which contradicts Lemma 4.1.17. Similarly, $u_- \notin C_2$. Therefore there must be a distinct 2-cell C_3 in T'. In particular, $|T'| = 3$.

Since T' is the union of two smaller tiles and $|T'| \leq 3$, it must be the case that $T' = S \cup S'$, where $|S| = 2$ and $|S'| = 1$. If $C_3 \in S$, then $T \cap S$ would contain a path from u_- to α_+, which must have length at least $\frac{\ell}{4}$. This contradicts the maximality of the construction. So $S = C_1 \cup C_2$, and $S' = C_3$. Furthermore $\mathrm{Cancel}(S) \geq \frac{\ell}{4}$ and $|S \cap C_3| \geq \frac{\ell}{4}$.

By the maximality condition of Construction 3.3.2, $|T \cap S| < \frac{\ell}{4}$, so at most one endpoint of $C_1 \cap C_2$ lies in $T \cap T'$. We will call this endpoint a, and the other b. Since $s_1(y) \neq s_2(y)$, there must be a non-trivial path in $T \cap (C_1 - C_2)$, without loss of generality, and furthermore this path must have a as one of its endpoints. Notice that there may also be a non-trivial path in $(C_2 - C_1) \cap T$, and if so it also has a as an endpoint.

Let v_+ be the point on the path between x and u_- which is $\frac{\ell}{4}$ away from u_-. There are two situations, both illustrated in Figure 4.12; either v_+ lies in $C_1 - C_2$ (without loss of generality), or v_+ lies in $C_1 \cap C_2$. Both of these situations are illustrated in Figure 4.12. Notice that $v_+ \notin C_3$, since this would imply that $|C_3 \cap T| \geq \frac{\ell}{4}$. In the first situation, $C_3 \cap (C_1 \cap C_2) \neq \emptyset$, and in particular it must be a path in either C_1 or C_2 of length at least $\frac{\ell}{4}$ which does not contain v_+, and might contain b. In the second situation, T' is planar, and furthermore, since $|C_3 \cap S| \geq \frac{\ell}{4}$ and $v_+ \notin C_3$, $C_3 \cap C_2 = \emptyset$. In either situation, we will prove that there are no points x, x' as given with $|x, x'| \leq \mathrm{Bal}(T \cup T')$.

Now we will show that in either case, the tile-wall γ is balanced. Let λ be a geodesic path in the edges of T connecting x to x'. Notice that λ must pass through $C_1 \cap C_2$.

If z is on the path λ, then $|\lambda| > |x, z| > \mathrm{Bal}(T \cup T')$ by Lemma 4.1.12. So we may assume that $z \notin \lambda$, and similarly that $z' \notin \lambda$.

Claim 4.2.8. If λ contains a, then $|x, x'| \geq \mathrm{Bal}(T \cup T')$.
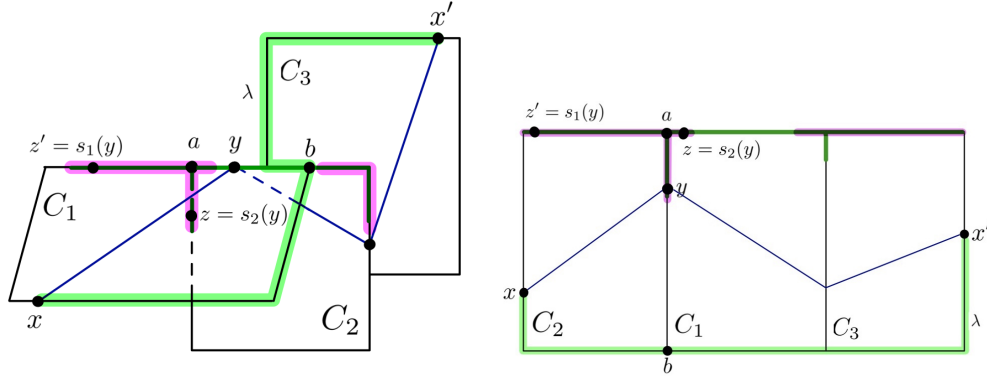
Figure 4.12: These images illustrate the two possible situations in which a tile-wall in Step 1(b), Case 3(a) could produce an unbalanced wall after adjusting according to Construction 4.2.4. In both illustrations, a possible geodesic connecting x to x' is highlighted in bright green.

Proof. Consider the subpath of λ connecting x to a. We have

$$|\lambda| \geq |x,a| \geq (|x,a| + |a,z|) - |z,a|$$

$$\geq |x,z| - |z,a|$$

$$> \mathrm{Bal}(T') - |z,a|$$

$$> \mathrm{Bal}(T \cup T') + |T \cap T'| - \frac{\ell}{4} + |z,a|$$

Since $z, a \in \alpha_+$, the path $|z,a| \leq |\alpha_+| \leq |T \cap T'| - \frac{\ell}{4}$, so $|\lambda| \geq \mathrm{Bal}(T \cup T')$. $\qquad\square$

Consider the case that λ contains b but not a.

Claim 4.2.9. If $|x,z| \neq \frac{\ell}{2}$, then $|x,b| \geq \frac{\ell}{2} - |C_1 \cap C_2|$, and similarly for x' and z'.

Proof. If $|x,z| \neq \frac{\ell}{2}$, then the wall from x to z must have been altered at some earlier point in the construction. In particular, $z \in C_1 \cap C_2$ and x is antipodal to some point x'' in $C_1 \cap C_2$,

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so $|x, b| + |b, a| > \frac{\ell}{2}$. Therefore

$$\begin{aligned} |x, b| &> \frac{\ell}{2} - (|a, b|) \\ &\geq \frac{\ell}{2} - |C_1 \cap C_2| \end{aligned}$$

$\square$

Claim 4.2.10. If $|x, z| = \frac{\ell}{2}$, then $|x, b| \geq \frac{\ell}{2} - |C_1 \cap C_2| - |z, b|$, and similarly for x', z'.

Proof. First, suppose that $z \in C_1 \cap C_2$. Then $|x, b| = \frac{\ell}{2} - |z, b|$, where z lies in the $\frac{\ell}{4}$-neighborhood of b, so $|x, b| \geq \frac{\ell}{2} - \frac{\ell}{4} \geq \frac{\ell}{2} - |C_1 \cap C_2|$.

On the other hand, if $z \notin C_1 \cap C_2$, then $|x, b| = |x, z| - (|z, a| + |a, b|) = \frac{\ell}{2} - |C_1 \cap C_2| - |z, a|$. $\square$

To finish showing that γ must be balanced, recall that by Lemma 4.1.18,

$$\begin{aligned} \mathrm{Bal}(T \cup T') &\leq \frac{5\ell}{4} - 2|C_1 \cap C_2| - |T \cap T'| \\ &= \frac{5\ell}{4} - 2|C_1 \cap C_2| - (|\alpha_+| + |v_+, u_-|) \\ &= \ell - 2|C_1 \cap C_2| - |\alpha_+| \\ &= 2(\frac{\ell}{2} - |C_1 \cap C_2|) - |\alpha_+|. \end{aligned}$$

The paths connecting z and z' to b are subpaths of λ and they share only an endpoint, namely b. Therefore $|z, b| + |z', b| = |z, z'| \leq |\alpha_+|$.

This concludes the proof that γ is balanced in Case 3(a).

CASE 4(a). Suppose now that γ has endpoints $x \in T - T'$, $x' \in T' - T$, and a midpoint $y \in (T \cap T')$. If $y \in \alpha^\pm$, then for the sake of notation, let $y \in \alpha_1^+$, where γ traverses the 2-cell $C_1 \in T$.

By Lemma 4.1.13, if $T \cap T' \subset N_{\ell/4}(C_1 \cap T)$, then this wall is balanced. Assume that

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this is not the case. Let v_+ denote the point on the path between x and u_- which is $\frac{\ell}{4}$ away from u_-. Then there is some other 2-cell in T' containing v_+; call this 2-cell C_2. Neither C_1 nor C_2 can contain u_-, since their intersection with T would then be $\geq \frac{\ell}{4}$. So there must be some other 2-cell C_3 which contains u_-. Therefore $|T'| = 3$, so $T' = S \cup S'$ where S is not a 1-potile. Notice that $(C_2 \cup C_3) \cap T$ contains u_- and v_+, so $|(C_2 \cup C_3) \cap T| \geq \frac{\ell}{4}$. Therefore by the maximality condition of Construction 3.3.2, C_2, C_3 are not both in S. For the same reason, we know C_1, C_3 are not both in S. So, without loss of generality, we can say $C_1 \cup C_2 = S$ and $C_3 = S'$, $|C_1 \cap C_2| \geq \frac{\ell}{4}$. By maximality, $|S \cap T| < \frac{\ell}{4}$, so at most one endpoint of $C_1 \cap C_2$ lies in T. Since $v_+ \in C_2$ and $C_3 \cap S$ is a path of length at least $\frac{\ell}{4}$ which does not contain v_+, $C_3 \cap C_1 = \emptyset$ and T' is planar.

Let ξ in $(T \cup T')^{(1)}$ be a geodesic joining x to x'. Since $|T \cap T'| < \frac{\ell}{2}$, it is a geodesic tree by Lemma 3.1.2 and ξ must enter and exit $T \cap T'$ at most once. Let z be the point where ξ enters $T \cap T'$, and let z' be the point where ξ exits $T \cap T'$. Then

$$
\begin{aligned}
|x, x'|_S &\geq |x, z|_S + |z', x'|_S \\
&\geq |x, y|_T - |y, z|_{T \cap T'} - |x', s(y)|_{T'} - |s(y), z'|_{T \cap T'} \\
&\geq \mathrm{Bal}(T) + \mathrm{Bal}(T') - |y, z|_{T \cap T'} - |s(y), z'|_{T \cap T'} \\
&= \mathrm{Bal}(T \cup T') + |T \cap T'| + \frac{\ell}{4} - |y, z|_{T \cap T'} - |s(y), z'|_{T \cap T'} \\
&\geq \mathrm{Bal}(T \cup T') + \frac{\ell}{4} - |y, z|_{T \cap T'}.
\end{aligned}
$$

Thus it suffices to prove that $\frac{\ell}{4} - |y, z|_{T \cap T'} \geq 0$.

If $z \in C_1$, then $|y, z| \leq |C_1 \cap T| < \frac{\ell}{4}$ and this is true. If z is not in C_1, then γ must have a sub-path which has one endpoint in $C_1 \cap T$ and the other in $C_1 \cap C_2$. But then there must be a point antipodal to x which lies in $C_1 \cap C_2$, and $|C_1 \cap (T \cup C_2 \cup C_3)| > \frac{\ell}{2}$. But $C_2 \cup C_3$ is a potile and $C_2 \cup C_3$ contains both v_+ and u_-, so $P = T \cup C_2 \cup C_3$ is a potile. But $|P \cap C_1| > \frac{\ell}{2}$, which contradicts Lemma 3.2.3.

CASE 4(b). If, instead, we have that y does not lie in $\alpha_\pm$, then $u_\pm$ are both contained in the $\frac{\ell}{4}$-neighborhood of y. Since the path connecting u_- to u_+ is a diameter, all of $T \cap T'$ must be contained in the $\frac{\ell}{4}$-neighborhood of y, and by Lemma 4.1.16, $|x, x'| \geq \mathrm{Bal}(T \cup T')$.

Step 2. Suppose we have two tiles T, S such that $T \cup S$ is a potile and $|T \cap S| < \frac{\ell}{4}$. If a wall-path lies in both T and S, then by Lemma 4.1.16 the resulting (concatenated) tile-wall is balanced. Otherwise, the tile-wall is balanced (with respect to its shard) by the inductive assumption.

Step 3. Suppose now that R, R' are two tiles with $|R \cap R'|$, and R' was made after Step 2. Let γ be a tile-wall in $R \cup R'$. As in Step 1, we will analyze each case to show that γ is balanced with respect to its shard.

If γ lies entirely in R or R', then the shard of γ is either $R \cup R'$ or it is the same as it was in the previous step. In the latter case, γ is balanced with respect to its shard by the inductive hypothesis. In the former, γ is balanced with respect to $R \cup R'$ by Lemma 4.1.12 and the fact that $|R \cap R'| > \frac{\ell}{4}$.

This covers Cases 1 - 3(b).

Now suppose that γ traverses 2-cells in both R and R', as in Case 4. Let γ', with endpoints x', y, be the restriction of γ to R'. If S is a shard contained in R' and γ' traverses S, then $|x', y| \geq \mathrm{Bal}(S)$. Indeed, this is true if $\mathrm{Sh}_{R'}(\gamma') = S$ by the inductive hypothesis. If, however, $\mathrm{Sh}_{R'}(\gamma') \neq S$, then by the construction of shards, it must be the case that $\mathrm{Bal}(\mathrm{Sh}_{R'}(\gamma')) > \mathrm{Bal}(S)$.

If, as in Case 4(a), γ traverses $|R \cap R'|$ with a midpoint $y \in \alpha^\pm$, then we can see that

$$
\begin{aligned}
|x, x'| &\geq \mathrm{Bal}(S) + \mathrm{Bal}(R) - (|R \cap R'| - \frac{\ell}{4}) \\
&\geq \frac{\ell}{4} + \frac{\ell}{4} + (|R \cap R'| - \frac{\ell}{4}) \\
&= \frac{3\ell}{4} - |R \cap R'|.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
\mathrm{Bal}(R \cup R') &= \frac{\ell}{4}|R \cup R'| + \frac{\ell}{4} - \mathrm{Cancel}(R \cup R') \\
&= \frac{\ell}{4}|R \cup R'| + \frac{\ell}{4} - \mathrm{Cancel}(R) - \mathrm{Cancel}(R') - |R \cap R'| \\
&\leq \frac{\ell}{4}|R \cup R'| + \frac{\ell}{4} - (\frac{\ell}{4}|R| - \frac{\ell}{4}) - (\frac{\ell}{4}|R'| - \frac{\ell}{4}) - |R \cap R'| \\
&= \frac{3\ell}{4} - |R \cap R'| \\
&\leq |x, x'|.
\end{aligned}$$

Finally, if, as in Case 4(b), γ traverses $R \cap R'$ such that there is no midpoint $y \in \alpha^{\pm}$, then note that since S was not glued to R in Step 2, then $|S \cap S'| \geq |S \cap R|$. Therefore, we get:

$$\begin{aligned}
|x, x'| &\geq \mathrm{Bal}(S) + \mathrm{Bal}(R) - 2|S \cap R| \\
&> \mathrm{Bal}(S) + \mathrm{Bal}(R) - |S \cap R| - |S \cap S'| \\
&= \mathrm{Bal}(S \cup R) + \frac{\ell}{4} - |S \cap S'| \\
&= \mathrm{Bal}(S \cup R) + \frac{\ell}{2} - \frac{\ell}{4} - |S \cap S'| \\
&> \mathrm{Bal}(S \cup R) + \mathrm{Bal}(S') - \frac{\ell}{4} - |S' \cap (S \cup R)| \\
&= \mathrm{Bal}(R \cup R').
\end{aligned}$$

$\square$

It should be noted here the constructions of tiles and walls require only that $d < 1/4$. Indeed, it is only the proof that the resulting walls are balanced that causes potential problems for tiles of size larger than 3, and even then only in Cases 3(a) and 4(a). I am not aware of any 2-complexes in higher densities which do not allow the construction of balanced walls.

4.3 Walls are Embedded Trees

By concatenating tile-walls across identified edges in the Cayley complex, we obtain a potential wallspace structure on the Cayley complex. A connected component, Γ, of the resulting immersed graph is a *wall*.

We prove that the walls constructed in Construction 3.3.2 are embedded trees in two steps. This is essentially a local-to-global argument, though we have yet to claim that walls are quasi-isometrically embedded.

Definition 4.3.1. A *decomposition of length n* of a path $[x, x']$ in wall Γ is a concatenation of wall-paths $\gamma_1 \cdots \gamma_n = [x, \ldots, x']$ and assignment of tiles T_i such that for each i, $\gamma_i \subset T_i \subset \mathrm{Sh}_T(\gamma_i)$ for some tile $T \supset T_i$

A decomposition is *reduced* if for any pair of adjacent tiles T_i, T_{i+1}, their union $T_i \cup T_{i+1}$ is not a tile, and no tiles T_i, T_j for $j > i+1$ share 2-cells.

Lemma 4.3.2. *If a reduced decomposition $\gamma_1 \cdots \gamma_n$ is of minimal length and $|T_i \cup T_{i+1}| \leq 5$, then $T_i \cup T_{i+1}$ is not a tile. Furthermore, if $T_i \cap T_{i+1}$ contains 2-cells, then it is a potile, as is $\overline{T_{i+1} - T_i}$.*

Proof. If $T_i \cup T_{i+1}$ were a tile, then it must have been glued at some point in the construction. Then the shards associated to the paths γ_i, γ_{i+1} would contain $T_i \cup T_{i+1}$. By replacing γ_i, γ_{i+1} with the concatenation of these two paths, we would reduce the length of the decomposition. But the decomposition is said to be of minimal length, so this is a contradiction.

Finally, if $T_i \cap T_{i+1}$ contains 2-cells then it must be the union of subtiles in T_i and T_{i+1}, by Proposition 3.2.7. Without loss of generality, suppose T_{i+1} is younger than T_i. The step in which T_{i+1} was formed created an overlapping pair of tiles, so it was not Step 1. If it occured in Step 2, then the overlap $T_i \cap T_{i+1}$ must have been one of the two tiles which formed T_{i+1}, and in particular both $T_{i+1} - T_i$ and $\overline{T_{i+1} - T_i}$ are potiles. If T_{i+1} was formed in Step 3, then either $T_i \cap T_{i+1}$ was one of the two constituent tiles of T_{i+1} (as in Step 2),
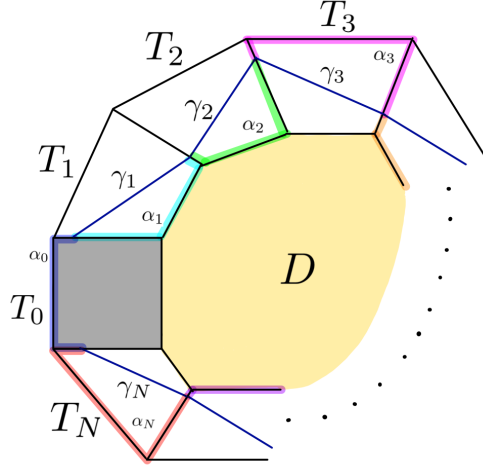
Figure 4.13: A returning decomposition of the wall-path γ, bounding a disk diagram D.

or T_{i+1} was formed as the union of two tiles which already contained overlapping subtiles. In this situation, the T_i and T_{i+1} are the same age, and their overlap could be a non-trivial union of potiles. However, in this case T_i and T_{i+1} are subtiles of a larger tile, T, and $\text{Sh}_T(\gamma) \supset T_i \cup T_{i+1}$, so by replacing T_i, T_{i+1} with $\text{Sh}_T(\gamma)$, we see that the decomposition is not minimal. □

Definition 4.3.3. Suppose a decomposition $\gamma = \gamma_1 \cdots \gamma_n$ has endpoints x_0, x_n. Then γ *returns at T_0 for a tile $T_0 \in \mathcal{T}$ if there is no T_i which contains $\bigcup_j T_j$ and $x_0, x_n \in T_0$.*

This is illustrated in Figure 4.13.

Theorem 4.3.4. *Given a tile collection as built in Construction 3.3.2, with balanced tile walls as built in Construction 4.2.4, with overwhelming probability, for each $N > 0$ there is no wall segment of length $< N$ which returns at a tile T.*

The proof of this will occupy most of this section, and closely follows the proof of Proposition 5.6 in [MP15]. However, the ways that two tiles can share 2-cells is more complicated than in [MP15]. We first show that it suffices to prove this for reduced decompositions.

Lemma 4.3.5. *Any hypergraph segment γ of length $\leq N$ admits a reduced decomposition of length $\leq N$, up to taking a subpath of γ. Furthermore, if γ is returning at some tile T_0, then*

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up to taking a subpath of γ, we may assume that the reduced decomposition is also returning (possibly at a different tile).

Proof. For any minimal length decomposition, $T_i \cup T_{i+1}$ is not a potile by Lemma 4.3.2.

Suppose that two non-adjacent tiles T_i, T_j share 2-cells. If $\gamma \cap (T_i \cap T_j) \neq \emptyset$, then we may look at the sub-path of γ through $T_i, \ldots, T_j$, which must be returning at either T_i or T_j.

On the other hand, suppose that $\gamma \cap (T_i \cap T_j) = \emptyset$. Consider when this intersection $T_i \cap T_j$ arose. If it arose during Step 2 of the construction, then $\text{Sh}(\gamma) \subset T_j - (T_i \cap T_j)$, which contradicts the definition of a decomposition. If the intersection arose during Step 3, then there must be some tile T containing both T_i and T_j, and we can consider the subpath of γ through $T_{i+1}, \ldots, T_{j-1}$, which is returning at T. $\square$

By choosing arbitrary paths α_i connecting x_{i-1} to x_i (modulo n), one can see that every returning decomposition bounds a disk diagram D. Note that the α_i connect edge-midpoints, so they are not full edge paths in X.

Proof of Theorem 4.3.4. It suffices to show that there is no reduced decomposition $\gamma_1 \cdots \gamma_n$ returning at a tile $T_0 \in \mathcal{T}$, where $n \leq N$. Suppose for contradiction that there is such a decomposition.

Then $\{T_i\}$ bounds a disk diagram D, where each α_i is chosen at random. By passing to a subdiagram, we may assume that no 2-cell in D mapped to T_i is adjacent to α_i.

Let $Y \subset X$ be the union of $T_0, T_1, \ldots, T_n$ and the image of D. Let E be the 2-cells of Y which do not lie in $T_0, \ldots, T_n$.

Let $T'_{i+1} = \overline{T_{i+1} - T_i}$. By Lemma 4.3.2, T'_{i+1} must be a potile, rather than just a union of potiles. By pushing the endpoints x_i to the right, we may assume that $\gamma_{i+1} \subset T'_{i+1}$. Notice that since $T_{i+1} \in \mathcal{T}_n$, the walls in T'_{i+1} were not adjusted. Therefore $|\alpha_{i+1}| \geq \text{Bal}(T'_{i+1})$ by the inductive hypothesis.

From now on, by an abuse of notation we will use T_i to refer to T'_i.

Claim 4.3.6. We have $|E| = 0$, $|Y| \leq 6$, and $|\alpha_0| \leq \frac{\ell}{2}$.

Proof. Thinking of $Y = \{T_0\} \cup \{T_i\} \cup E$, we can bound Cancel$(Y)$ by:

$$
\begin{aligned}
\text{Cancel}(Y) &\geq \text{Cancel}(T_0) + \sum_{i=1}^{n} \text{Cancel}(T_i) + \frac{1}{2} \left(\sum_{i=1}^{n} |\alpha_i| + |E|\ell + |\alpha_0| \right) \\
&\geq \text{Cancel}(T_0) + \frac{\ell}{4} |\bigcup T_i| + \frac{1}{2}|E|\ell + \frac{1}{2}|\alpha_0| \\
&\geq \frac{\ell}{4}(|T_0| - 1) + \frac{\ell}{4}|\bigcup T_i| + \frac{1}{2}|E|\ell + \frac{1}{2}|\alpha_0| \\
&= \frac{\ell}{4}(|Y| - 1 + 2|E|) + \frac{1}{2}|\alpha_0| \\
&> \frac{\ell}{4}(|Y| - 1 + 2|E|).
\end{aligned}
$$

Since $n \leq N$ and tiles of $\mathcal{T}$ have size ≤ 6, $|\partial D|/\ell$ is uniformly bounded. By Theorem 1.2.2, $|D|$ is uniformly bounded, so Y is as well. Thus by Proposition 3.1.4, $|E| = 0$ and therefore $|\alpha_0| < \frac{\ell}{2}$. By Remark 3.1.3, since Y is a potile we have $|Y| \leq 6$. $\qquad \square$

Claim 4.3.7. We have $|D| = 0$, so D is a tree.

Proof. Suppose this is not the case. Let $T \subset D$ be a connected component in the pre-image of some T_i. In the above calculation, we can replace α_i with the image of ∂T in Y. By Corollary 3.1.2, $|\partial D| \geq \ell$ so Cancel$(T_i) + \frac{1}{2}|\alpha_i| \geq \frac{\ell}{4}(|T_i| + 1)$. But this gives an extra $\frac{\ell}{4}$ in the calculation above, which contradicts Proposition 3.1.4. $\qquad \square$

Claim 4.3.8. We have $n > 2$. In particular, for every pair of tiles $|T_i \cup T_j| \leq 5$.

Proof. By Lemma 4.1.11, $n > 1$. If $n = 2$, then D is a tripod. If $|T_1 \cap T_2| \geq \frac{\ell}{4}$, since $|T_1| + |T_2| \leq 6 - |T_0| \leq 5$, then they would have been glued together in Construction 3.3.2,

which is a contradiction. Therefore:

$$
\begin{aligned}
\mathrm{Cancel}(Y) &\geq \mathrm{Cancel}(T_0) + \sum_{i=1}^{n} \mathrm{Cancel}(T_i) + \sum_{i=1}^{2} |\alpha_i| - \frac{\ell}{4} \\
&\geq \frac{\ell}{4}(|T_0| - 1) + \frac{\ell}{4} \sum_{i=1}^{2}(|T_i| + 1) - \frac{\ell}{4} \\
&= \frac{\ell}{4}|Y|,
\end{aligned}
$$

which contradicts Lemma 3.1.4. $\qquad\square$

Claim 4.3.9. There is some $1 \leq i \leq n$ and $j = i \pm i$ (modulo $n + 1$) such that $T_i \cup T_j$ is a potile.

Proof. Since $|Y - T_0| \leq 5$, and $n > 2$, the maximal size of $|T_i \cup T_j| \leq 5$. Since D is a tree, there is some α_i' is contained in $\alpha_{i-1}' \cup \alpha_{i+1}'$. Choose $j \in \{i-1, i+1\}$ to maximize $|\alpha_i' \cap \alpha_j'|$. So $|\alpha_i' \cap \alpha_j'| \geq \frac{1}{2}\mathrm{Bal}(T_i)$, and $T_i' \cup T_j'$ is a tile. Indeed,

$$
\begin{aligned}
\mathrm{Cancel}(T_i \cup T_j) &\geq \mathrm{Cancel}(T_j) + \frac{1}{2}\mathrm{Bal}(T_i) + \mathrm{Cancel}(T_i) \\
&\geq \frac{\ell}{4}(|T_i| - 1) + \frac{\ell}{8}(|T_i| + 1) + \frac{1}{2}\mathrm{Cancel}(T_j) \\
&\geq \frac{\ell}{4}(|T_j| - 1) + \frac{\ell}{4}|T_i|.
\end{aligned}
$$

$\qquad\square$

Finally, since $T_i \cup T_j$ is a tile and it has size at most 5, this is a contradiction of Construction 3.3.2. $\qquad\square$

As an immediate consequence, we get the following:

Corollary 4.3.10. *At density $d < 3/14$, with overwhelming probability, for every $N > 0$ there is no returning decomposition of length $< N$.*

CHAPTER 5

PUTTING IT ALL TOGETHER: ACTING ON A CAT(0) CUBE COMPLEX

Theorem 5.0.1. *There exist constants* Λ, c, *such that w.o.p. the map from the vertex set* V *of any hypergraph segment to* $X^{(1)}$ *is a* (Λ, c)*-quasi-isometric embedding.*

The proof of this is identical to the proof of [MP15] Theorem 6.1. While they gave their proof in the specific case that $d < 5/24$, it in fact holds for any tile and balanced tile-wall construction which admits reduced decompositions in which the distance between endpoints of γ_i are at least $\text{Bal}(T_i)$.

Proof Sketch. The Cayley graph $X^{(1)}$ of a random group at a fixed density $d < 1/2$ is w.o.p. hyperbolic, with hyperbolicity constant linear in ℓ. By [GdlH90] Theorem 5.21, it suffices to find λ such that for some sufficiently large $N = N(\lambda)$, the map to $X^{(1)}$ from any V of cardinality $\leq N$ is bilipschitz. Choose $\lambda = \frac{1}{1-4d}$. $\qquad\square$

Theorem 5.0.2. *At density* $d < 3/14$, *all walls* Γ *as constructed in Construction 3.3.2 are embedded trees.*

Proof. Suppose a hypergraph segment γ self intersects. Then it contains a subpath with endpoints x, x' which are in the same 2-cell, so $|x, x'| \leq \frac{1}{2}\ell$. Let n be the number of 2-cells traversed by the subpath of γ from x to x'. Then by Theorem 5.0.1,

$$|x, x'|_\Gamma \geq \frac{1}{\Lambda}\left(\frac{n}{2}\ell\right) - c\ell.$$

Therefore it suffices to take $N = (2c + 1)\Lambda$ in Theorem 4.3.4. $\qquad\square$

Lemma 5.0.3. *There is a wall* Γ *and an element* $g \in \text{Stab}(\Gamma)$ *which swaps complementary components of* Γ *in* $\widetilde{X(\Gamma)}$.

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Proof. The proof is identical to [MP15] Lemma 6.2. □

Lemma 5.0.4. *There is a wall Γ which has essential complementary components in $\widetilde{X(G)}$.*

Proof. Choose a wall Γ from the walls constructed in 4.2.4. Then the complementary components of Γ are either both essential or both non-essential. Suppose that the complementary components are not essential. Since Γ is an embedded tree, there is some constant $R > 0$ so that $\widetilde{X(G)} \subset N_R(\Gamma)$, so G is quasi-isometric to a tree. However, G is 1-ended by [Gro93], so it is not free and this is impossible. □

We are now ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Let H be the index 2 subgroup of $\mathrm{Stab}(\Gamma)$ which preserves the components of $X - \Gamma$. The number of relative ends of H is greater than 1, so by [Sag95], there is a CAT(0) (finite dimensional) cube complex on which G acts non-trivially cocompactly by isometries. □

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