THE UNIVERSITY OF CHICAGO

TAUT SUTURED HANDLEBODIES AS TWISTED HOMOLOGY PRODUCTS

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To Nick, with whom everything is awesome.

“People say nothing is impossible, but I do nothing every day.”

– Winnie-the-Pooh

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ABSTRACT

Friedl and Kim show any taut sutured manifold can be realized as a twisted homology product, but their proof gives no practical description of how complicated the realizing representation needs to be. In this thesis, we give a number of results illustrating the relationship between the topology of a taut sutured handlebody and the complexity of a representation realizing it as a homology product.

CHAPTER 1

INTRODUCTION

A *sutured* 3-manifold (M, γ) is a manifold with boundary marked by a set of sutures, γ, which consists of oriented curves dividing ∂M into oriented collections of components R_+ and R_-.[1]

Gabai [Gab84] introduced the notion of a *taut* sutured manifold (M, γ), which, roughly speaking, requires M to be irreducible and the boundary components $R_\pm$ to be of minimal complexity.

Under suitable hypotheses, a sutured manifold is taut if R_+ and R_- realize the Thurston norm of their (common) homology class. Here is an important example. Suppose K is a knot in S^3, and let R be a Seifert surface for K. Cutting S^3 open along R produces a sutured manifold M whose boundary decomposes along the knot K into two copies $R_\pm$ of R. This sutured manifold is taut precisely when R is of minimal genus. Thus the theory of sutured manifolds can be (and is) used to compute knot genus.

Suppose M is a sutured manifold, and $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ is a representation. Then α restricts to representations $\pi_1(R_\pm) \to \mathrm{GL}(V)$, and we can define the twisted homology groups $H_*(M; E_\alpha)$, $H_*(R_\pm; E_\alpha)$. We say that M is an *α-homology product* if the maps $H_*(R_\pm, E_\alpha) \to H_*(M, E_\alpha)$ induced by inclusion are all isomorphisms. If α is not specified, we say M is a *twisted homology product*.

This concept is important, because of

Theorem 1.0.1 (Friedl-Kim [FK13])**.** *If M is a twisted homology product, it is taut.*

Conversely, using Agol's Virtual Fibering Theorem ([Ago08]), they show

Theorem 1.0.2 (Friedl-Kim [FK13])**.** *If M is taut, it is a twisted homology product for some representation α.*

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[1] We note Gabai's original definition allowed sutures to consist of entire torus components of the boundary. Here we are interested in sutured handlebodies, and this aspect of the definition never arises.

We call such an α *certifying* for M. The result of Friedl and Kim is not effective, in the sense that it gives no upper or lower bounds for the complexity of a certifying representation. This potentially reduces the practical value of twisted homology as a tool. Therefore, the fundamental question we study in this thesis addresses precisely this issue:

Question 1.0.3. *If M is a taut sutured manifold, what is the simplest representation for which it is a twisted homology product, and what is the relationship of the complexity of the representation to the topology of M?*

For M a hyperbolic manifold, Agol and Dunfield found substantial computer evidence that M is a twisted homology product for the geometric representation $\pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ ([AD15]). They conjectured in general that every taut M has a 2-dimensional certifying representation, and proved this for a simple class of manifolds, namely books of I-bundles.

For a given M the search for a certifying representation falls into two parts: understanding the linear representations of $\pi_1(M)$, and understanding when such a representation is certifying. To simplify the discussion we restrict attention to the case that M is a handlebody, so that $\pi_1(M)$ is free.

This case is of practical importance, since it often happens that the complement of a minimal genus Seifert surface is a handlebody.

1.1 Sutured manifolds and the Thurston norm

Definition 1.1.1. A *sutured manifold* is a four-tuple $(M, R_\pm, \gamma)$ consisting of a compact 3-manifold M and a collection of pairwise disjoint, embedded curves $\gamma \subset \partial M$, which partition $\partial M - \gamma$ into oriented subsurfaces R_+ and R_-, such that the orientations induced on their common boundary γ agree.

Though this definition does not require it, we will always assume M is connected. Some sources define the sutures to be a collection of annuli; our definition as a collection of curves is equivalent, though we occasionally view the sutures as annuli when convenient for notational or conceptual purposes.

Example 1.1.2.

1. Given any compact surface S, the manifold $M = S \times I$ can be given a natural sutured structure, where $\gamma = \partial S \times I$, $R_+ = S \times 1$, and $R_- = S \times 0$.

2. Any Seifert surface S associated to a knot K, or more generally a link L, defines a sutured manifold $S^3 - N(S)$, with $\gamma = K$ (or L) and $R_\pm \cong S$. The knot (or link) is fibered by S exactly when this sutured manifold is a product.

We are particularly interested in *taut* sutured manifolds, which we define below. We recall first the *Thurston norm* on $H_2(M, \partial M)$. Given a connected embedded surface $(S, \partial S) \subseteq (M, \partial M)$, we define $\chi_-(S) = \max\{0, -\chi(S)\}$. For S not connected, $\chi_-(S) = \sum_{T \subseteq S} \chi_-(T)$, taken over connected components of S. Finally, the Thurston norm of $\sigma \in H_2(M, \partial M)$ is defined as

$$\|\sigma\| = \min_{[S]=\sigma} \chi_-(S).$$

This norm has the following properties. For any S, T (not necessarily disjoint), $\chi_-(S \cup T) = \chi_-(S) + \chi_-(T)$, so $\| \cdot \|$ is subadditive. Also, $\|n\sigma\| = n\|\sigma\|$ for any n. Extrapolating this second property, we can extend $\| \cdot \|$ to $H_2(M, \partial M; \mathbb{Q})$, which in turn extends to a unique continuous norm on $H_2(M, \partial M; \mathbb{R})$.

The Thurston norm defines a concrete and fundamental notion of complexity for a homology class. This norm captures a remarkable amount of data about M, for instance, giving a homological description of how fiberings of M arise. Thurston shows in his defining work on the norm that the associated unit ball is a convex, rational polyhedron. A face of this polyhedron defines a cone within $H_2(M, \partial M; \mathbb{R})$ consisting of those homology classes which projectively lie within the interior of that face. Associate to a homology class φ the face F_φ of the unit ball to which it projects. We say $\varphi \in H_2(M, \partial M)$ is a *fibered class* if φ has a representative F such that $(M, \partial M)$ fibers over S^1 with fiber F.

Theorem 1.1.3 (Thurston [Thu86]). *If φ is a fibered class of $H_2(M, \partial M)$, then any other $\psi \in H_2(M, \partial M)$ which lies in the cone over F_φ is also a fibered class.*

Phrased another way, the unit ball has *fibered faces*: any integral homology class associated to one of these faces is a fibered class.

This gives a reformulation of the question of whether a (cover of) a three-manifold M is fibered, asking whether the unit ball of (the cover of) M contains a fibered face. Moreover, this gives a practical method for verifying that M is not fibered, namely, by checking a single class within each of the finitely many cones.

Definition 1.1.4. A sutured manifold M is *taut* if it is irreducible and $R_\pm$ are *taut*, that is, they are incompressible and realize the Thurston norm of their homology class.

Definition 1.1.5. A sutured manifold M is *balanced* if it is irreducible and $\chi(R_+) = \chi(R_-)$, and moreover M is not a solid torus without sutures, and if any component of $R_\pm$ has positive Euler characteristic, then M is D^3 with a single suture.

Notice that a taut sutured manifold is necessarily balanced. We will often make use of this prerequisite, in particular that $\chi(R_+) = \chi(R_-)$ in the section and chapter to follow.

Example 1.1.6. A product sutured manifold, as in Example 1.1.2, has the form $M \cong S \times I$, with $R_\pm = S \times \partial I$ and $\gamma = \partial S \times I$. Any product sutured manifold is taut: as S is a deformation retract of M, any other representative of $[S]$ surjects onto S, and so cannot have lower Thurston norm.

1.2 Homology with twisted coefficients

Our method for understanding the property of tautness is to study a generalization of the usual homology and cohomology groups associated to M, which incorporates additional algebraic information. This takes the form of a choice of representation of the fundamental group $\alpha : \pi_1(M) \to \mathrm{GL}(V)$. These groups are called the *(co)homology groups with coefficients twisted by* α, or, more briefly, twisted (co)homology. We write these as $H_*(M; E_\alpha)$ and $H^*(M; E_\alpha)$.

We are primarily interested in the homology side of this picture, which we define carefully. We follow the geometrically-minded viewpoint, where the coefficients E_α may be thought of as the vector bundle

$$\begin{tikzcd} V \arrow[r] & E_\alpha \arrow[d] \\ & M \end{tikzcd}$$

with the action of $\pi_1(M)$ on V given by α. Note there is a more algebraic formulation of this same theory via modules (see Hatcher, Section 3.H [Hat02] for both treatments). One observation more evident from the module definition is that these homology groups are effectively the usual homology groups associated to an appropriate cover of M, with coefficients in the base field of V.

The chain groups $C_n(M; E_\alpha)$ consist of finite sums $\sum_i n_i\sigma_i$ with $\sigma_i : \Delta^n \to M$ an n-simplex and $n_i : \Delta^n \to E_\alpha$ a lift of σ_i to E_α. We make sense of the statement $n_i\sigma_i + m_i\sigma_i = (n_i + m_i)\sigma_i$ by interpreting the addition $n_i + m_i$ as pointwise within a single fiber V, giving a new section $n_i + m_i : \Delta_i \to E_\alpha$. The boundary map $\partial : C_n(M; E_\alpha) \to C_{n-1}(M; E_\alpha)$ is given by the usual formula $\partial\big(\sum_i n_i\sigma_i\big) = \sum_{i,j}(-1)^j n_i\sigma_i|_{[v_0,\ldots,\hat{v_j},\ldots,v_n]}$, with the caveat that the lift n_i on the right also restricts to $[v_0, \ldots, \hat{v_j}, \ldots, v_n]$. The usual argument shows $\partial^2 = 0$.

In order for a map $f : M' \to M$ to induce a map between twisted homology groups, we need an understanding of how the associated bundles $E'_{\alpha'}$ and E_α are related. For the scope of this thesis, we will be interested primarily in the situation where we are given the map

$f : M' \to M$, and choose to pull back a representation and bundle E_α to M':

$$\begin{tikzcd}
f^*E_\alpha \arrow[r, "\tilde{f}"] \arrow[d] & E_\alpha \arrow[d] \\
M' \arrow[r, "f"] & M
\end{tikzcd}$$

In particular, given our setting of sutured manifolds, we will often be interested in the map induced by the inclusions $i^\pm : R_\pm \hookrightarrow M$, which define representations $i^\pm_* \alpha : \pi_1(R_\pm) \to \mathrm{GL}(V)$, and in turn maps

$$H_*(R_\pm; (i^\pm_*)^* E_\alpha) \xrightarrow{i^\pm_*} H_*(M; E_\alpha),$$

and similarly on cohomology. We will generally elide the pullback notation, and just write $H_*(R_\pm; E_\alpha)$.

Definition 1.2.1. A sutured manifold M is an *α-homology product* for a representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ if the maps $i^\pm_*$ are all isomorphisms.

We can alternatively consider the *relative* twisted homology groups $H_*(M, R_\pm; E_\alpha)$, which are defined analogously to usual homology, and the pairs $(M, R_\pm)$ similarly satisfy a long exact sequence of their twisted homology groups. The above definition is then equivalent to requiring

$$H_*(M, R_\pm; E_\alpha) = 0.$$

Note that if R_+ (respectively, R_-) meets every component of M, then for any representation α, already $H_0(M, R_+; E_\alpha) = 0$ (respectively, $H_0(M, R_-; E_\alpha) = 0$).

Example 1.2.2. If $M \cong S \times I$ is a product sutured manifold, it is a twisted homology product for any representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$. Since M and S are homotopy equivalent, the groups $H_*(M, R_\pm; E_\alpha)$ all vanish.

We finish this section with a discussion of a handful of standard homological tools, and how they transfer to the setting of twisted coefficients.

Poincaré-Lefschetz duality and its relative version hold in this context, with exactly the same assumptions as for usual (co)homology. The cap product is defined analogously here, and assuming the existence of an appropriate fundamental class $[M] \in H_3(M; E_\alpha)$, capping with $[M]$ gives an isomorphism $\frown [M] : H^k(M; E_\alpha) \to H_{3-k}(M; E_\alpha)$. For sutured manifolds, the relative form gives isomorphisms $H^k(M, R_\pm; E_\alpha) \to H_{3-k}(M, R_\mp; E_\alpha)$.

The homological Euler characteristic $\chi(H_*(M; E_\alpha)) = \sum_k (-1)^k \mathrm{rk} H_k(M; E_\alpha)$ is not quite $\chi(M)$. The ranks of $H_k(M; E_\alpha)$ are $n = \dim V$ times that of $H_k(M)$, so in general $\chi(H_*(M; E_\alpha)) = n \cdot \chi(M)$. However, for M a 3-manifold, $\chi(M) = 0$, and these do agree.

Finally, we come to a standard tool which requires more care. While the universal coefficient theorem does translate to twisted coefficients, it yields the identifications

$$H^k(M; E_\alpha) \cong H_k(M; E_\alpha^*) \quad \text{and} \quad H^k(M, R; E_\alpha) \cong H_k(M, R; E_\alpha^*),$$

where E_α^* is the *dual bundle* to E_α. This bundle corresponds to the 'dual representation' α^* defined as the unique representation such that $\langle \alpha(g^{-1})v, w \rangle = \langle v, \alpha^*(g)w \rangle$ for all $v, w \in \mathrm{GL}(V)$ and $g \in \pi_1(M)$. See, for instance, Kirk-Livingston ([KL99]).

In general, $H_k(M; E_\alpha)$ and $H_k(M; E_\alpha^*)$ will not be isomorphic. In the case that they are isomorphic for all k, we say α is *homologically self-dual.* For example, this is true of any representation to $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{GL}_1(\mathbb{C})$.

1.3 An aside on taut foliations

Though we do not work directly with foliations in this thesis, they lie under the surface of many of the ideas under consideration. Gabai introduced sutured manifolds as a structure for studying taut foliations, and the topological condition of tautness we have given is equivalent to the sutured manifold admitting a taut foliation. Given the deep relationship here, a brief background on the theory of taut foliations is warranted.

Definition 1.3.1. A *codimension-one foliation* $\mathscr{F}$ of a three-manifold M is a choice of charts U_i with maps $\varphi_i : U_i \to \mathbb{R}^3$ so that the compositions $\varphi_i^{-1} \circ \varphi_j : \mathbb{R}^3 \to \mathbb{R}^3$ respect the splitting $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, namely, they can always be written

$$\varphi_i^{-1} \circ \varphi_j(x, y) = (\varphi_{ij}^1(x), \varphi_{ij}^2(x, y)).$$

This condition gives each U_i a product-like structure of a one-dimensional family of surfaces, which glue together preserving this structure. $\mathscr{F}$ can be thought of as filling out M with parallel (generally not closed) surfaces, called the *leaves* of $\mathscr{F}$.

In the case of a sutured manifold, we add additional restrictions at the boundary. Away from γ, the leaves must lie parallel to ∂M. At γ, the leaves should meet transverse to ∂M, with $\partial L \subseteq \gamma$ for any leaf L. $\mathscr{F}$ necessarily contains closed leaves, namely the boundary subsurfaces $R_\pm$.

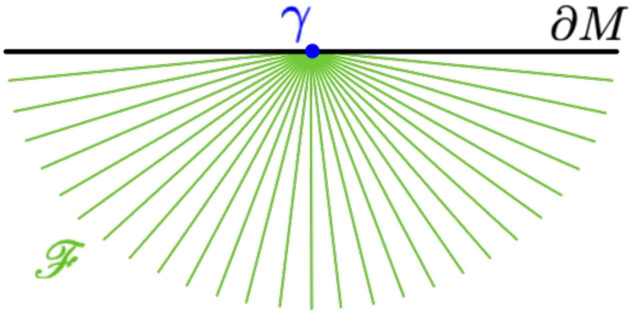


Figure 1.1: A cartoon of what a codimension-one foliation of M looks like near $\gamma \subset \partial M$.

Definition 1.3.2. A codimension-one foliation $\mathscr{F}$ of a sutured manifold M is *taut* if there exists a proper embedding of $I \to M$ transverse to $\mathscr{F}$, which intersects every leaf.

There are several equivalent conditions for $\mathscr{F}$ to be taut, see for instance [Cal07]. Though this definition appears different from our definition of a taut sutured manifold, in fact they are capturing the same idea.

Theorem 1.3.3 (Gabai [Gab83]). *An embedded surface $S \subset M$ representing a nonzero class $[S] \in H_2(M, \gamma)$ realizes the Thurston norm if and only if M admits a taut codimension-one* *foliation with S a compact leaf.*

This perspective at times provides a clearer picture of the structure of taut sutured manifolds. For example, since pull-backs of taut foliations are easily seen to still be taut,

Proposition 1.3.4. *A finite cover of a taut sutured manifold is taut.*

We will occasionally appeal to the foliation picture, to quickly justify facts which are cumbersome in our usual setting.

To conclude this section, we discuss one method we might try to use for constructing representations, and why it fails. Gabai introduced the concept of a *sutured manifold hierarchy* of a taut sutured manifold M in [Gab83]. This is a sequence of decompositions by surfaces S_k

$$M = M_0 \xrightarrow{S_1} M_1 \xrightarrow{S_2} M_2 \xrightarrow{S_3} \cdots \xrightarrow{S_n} M_n$$

such that each S_k meets the sutures of M_{k-1} transversally, each M_k is taut, and every embedded surface in M_n is separating. Gabai proved such hierarchies always exist, and moreover, that if a sequence of decompositions of an arbitrary sutured manifold M satisfies certain additional conditions, tautness of M_n implies M is taut as well.

As these hierarchies are often used in inductive arguments, one might hope that such a hierarchy can be used to inductively construct certifying representations. More precisely, if $M \xrightarrow{S} N$ is a decomposition, then N is a subspace of M, so a representation of M restricts to a representation of N. Suppose M and N are both taut, and that α is certifying for M. One might naïvely imagine that the restriction of α is certifying for N. This is not true, as the following example shows.

Example 1.3.5. The handlebodies M and N in Figure 1.2 are related by a decomposition along a disk meeting the sutures in M in four points. In this case, we may realize $\pi_1(M)$ as an HNN extension of $\pi_1(N) \cong F_2$, with $\pi_1(M) \cong F_3$ gaining a free generator z. The representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{C})$ defined by $\alpha(x) = -1$ and $\alpha(y) = \alpha(z) = 1$ is certifying for M, as can be verified via Proposition 3.2.1. However, when restricted to N, the representation α is no longer certifying: the locus of representations which fail to be certifying are those with $x \mapsto -1$.

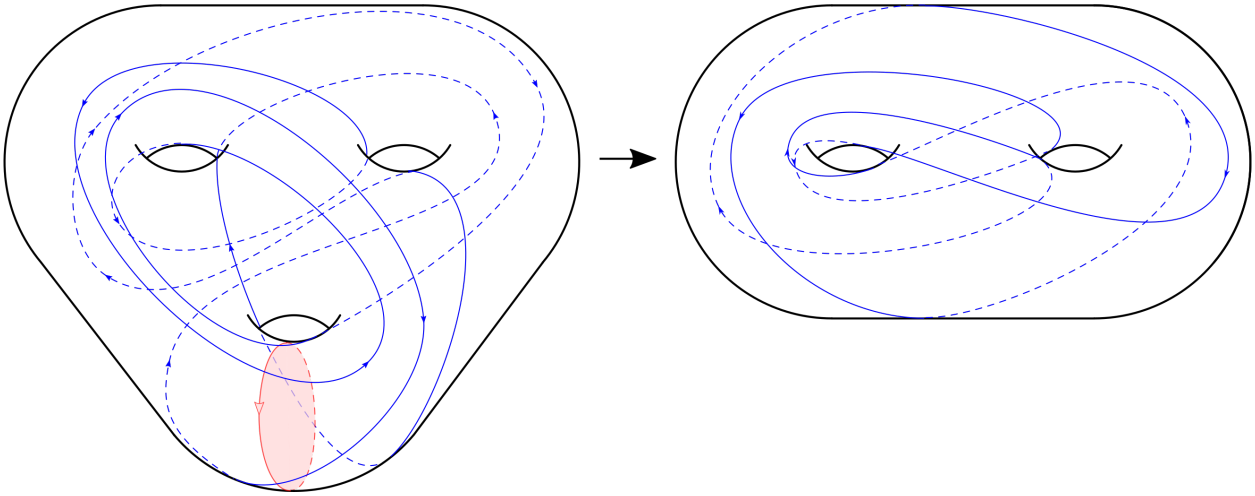


Figure 1.2: The decomposition of M (left) along the disk S to obtain N (right).

The reason for this is that there is part of the boundary of N which is not contained in the boundary of M. Understanding when this naïve guess fails requires analyzing how the suture structure changes with this new boundary, which is subtle in practice. However, this failure is isolated to the local situation of the decomposition. That is, if $S_\pm \subseteq N$ are the two copies of S in the boundary of N, there is still an injection $H_*(R_\pm - S_\pm; E_{\alpha|_N}) \hookrightarrow H_*(N; E_{\alpha|_N})$.

In this example, it is the case that both manifolds admit one-dimensional certifying representations. It is possible that this weaker condition of simply admitting some one-dimensional certifying representation behaves well in relation to a decomposition $M \xrightarrow{S} N$. Proving such a statement would require a deeper understanding of how a decomposition interacts with the algebraic condition of admitting such a certifying representation. (See Lemma 1.4.5 for details on this condition.)

As we will see in Lemma 3.3.1, in the special case that the surface S is a disk meeting the sutures of M exactly twice, a certifying representation for N can be extended to one which certifies M.

1.4 Statement of results

The results herein primarily take the form of lower bounds on the complexity of a certifying representation. Our first theorem demonstrates the sharpness of the bound conjectured by Agol and Dunfield.

Theorem 1.4.1. *For all $g \geq 2$, there are taut sutured handlebodies M_g of genus g which fail to be a twisted homology product for any one-dimensional representation.*

Our construction for genus $g \geq 3$ exploits a condition on how $\pi_1(R_\pm)$ sit inside $\pi_1(M)$ which prevents M from being a one-dimensional twisted homology product.

The genus 2 example was found by a computer search. This example has a suture set consisting of three curves. Note that in genus two, $R_\pm$ will either be pairs of pants, or once-punctured tori. A similar search has produced no examples with $R_\pm$ once-punctured tori, which leads us to the following conjecture.

Conjecture 1.4.2. *Let M be a taut sutured genus-two handlebody with a single connected suture. Then M is a twisted homology product for some representation $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$.*

Besides the computational evidence for this conjecture, allowing only a single suture imposes a stronger relationship between $\pi_1(R_+)$ and $\pi_1(R_-)$, which we expect simplifies the situation so that, roughly speaking, less can go wrong. For example, by Mayer-Vietoris, the twisted homology groups in this setting satisfy a splitting $H_1(\partial M; E_\alpha) = H_1(R_+; E_\alpha) \oplus H_1(R_-; E_\alpha)$, which does not occur with multiple suture curves.

One might ask if, within this simplest setting, twisted coefficients are even necessary; perhaps tautness is already detected by rational homology. This is not the case, as we illustrate in Example 3.1.2.

We generalize the obstruction from the proof of Theorem 1.4.1 to obstructions for admitting solvable representations of arbitrarily large derived length. We use this to prove the following strong negation of Agol and Dunfield's conjecture within the restricted setting of solvable representations.

Theorem 1.4.3. *There exist taut sutured manifolds* M_k *such that* M_k *is not a twisted homology product for any solvable representation* $\alpha : \pi_1(M_k) \to \mathrm{GL}_{\varphi(k)}(\mathbb{C})$*, where* $\varphi(k) \to \infty$ *with* k*.*

Dropping the requirement that the representation be solvable, these examples are certified by some two-dimensional representation.

Remark 1.4.4. The manifolds M_k are handlebodies, which have free, and therefore residually finite rationally solvable (RFRS), fundamental group. The representations produced by Friedl and Kim in their proof of Theorem 1.0.2 are in general virtually solvable, and in the case the fundamental group of the sutured manifold is RFRS, solvable on the nose. This theorem demonstrates the inherent weakness in their approach, if one hopes to find tight bounds on the minimal dimension of a certifying representation.

Finally, we take a broader perspective, and explore an algebraic condition to M admitting any certifying representation to $\mathrm{GL}_n(\mathbb{C})$. This utilizes the observation that, within the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C})$, certifying representations for a Zariski-open subspace, and for M a handlebody, this subspace is either empty or dense. We discuss a generalization $\det_{\mathscr{D}}$ of the usual determinant to skew fields, which allows us to make sense of the following lemma and pair of conjectures.

Lemma 1.4.5. *If* M *is taut sutured handlebody,*

$$\det{}_{\mathscr{D}} \left(\partial_{x_i} i_*(a_j)\right) \neq 0.$$

Conjecture 1.4.6. M *is taut if and only if*

$$\det{}_{\mathscr{D}} \left(\partial_{x_i} i_*(a_j)\right) \neq 0.$$

A potential argument proving the above conjecture would also prove

Conjecture 1.4.7. *There exists a uniform bound* n *such that any taut sutured handlebody* M *is certified by a representation* $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$.

This thesis is organized as follows. In Chapter 2 we review previous work of Friedl-Kim ([FK13]) and Agol-Dunfield ([AD15]). Chapter 3 contains the bulk of our new results. In Section 3.1, we review a couple of examples which fail to be homology products with respect to $H_*(\,\cdot\,;\mathbb{Q})$, illustrating the need to consider twisted coefficients. In Section 3.2 we present a condition for a representation to be certifying in Proposition 3.2.1 and a couple of consequences. We also return to the examples of Section 3.1 to see these admit one-dimensional certifying representations. We use these conditions in Section 3.3 to prove Theorem 1.4.1. In Section 3.4 we generalize the results of Sections 3.2 and 3.3 to prove Theorem 1.4.3. Lastly, Chapter 4 introduces the Dieudonné determinant, which allows us to prove Lemma 1.4.5 in Section 4.3.

CHAPTER 2

A THEOREM AND A CONJECTURE

In this chapter we discuss the theorem and the conjecture which motivate our work.

2.1 A theorem of Friedl and Kim

Our interest in twisted homology products is motivated by the following theorem of Friedl and Kim ([FK13]).

Theorem 2.1.1 (Friedl-Kim). *Let M be a balanced sutured manifold. Then M is taut if and only if M is an α-homology product for some $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$.*

In particular, the representation α may always be taken to be a unitary representation. This proves any taut sutured manifold can be realized as a twisted homology product, giving a novel method for verifying tautness of sutured manifolds. The proof of the 'if' direction uses standard homological techniques, especially exploiting the relationship between Euler characteristic and the ranks of (twisted) homology groups. However, their construction of the certifying representation uses in a key way Agol's virtual fibering ([Ago08]). Friedl and Kim find a finite cover of M, still taut, in which the pullback of $[R_+]$ is quasifibered, meaning it lies in the closure of the cone over a fibered face of the Thurston unit ball. These classes admit one-dimensional certifying representations, which in turn induce down to a certifying representation M. In particular, the dimension of the induced representation is exactly the degree of the finite cover.

The problem of certifying the tautness of M is thus reduced to producing a certifying representation. The method of Friedl and Kim's proof is impractical for actually producing a representation, and would yield (in general) a certifying representation of very large dimension. One then is lead to ask whether simple certifying representations always exist, or whether a bound on the dimension needed can be given based on more readily apparent topological or geometric information of the sutured manifold.

We will often be interested in representations which satisfy a homological generalization of the condition that E_α and E_α^*, its dual, be isomorphic.

Definition 2.1.2. A representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$ is *homologically self-dual* if, for any subspace $A \subseteq M$, there is an isomorphism $H_*(M, A; E_\alpha) \cong H^*(M, A; E_\alpha)$.

For example, any unitary representation is homologically self-dual, as is any representation to $SL_2(K)$, for any field K. This condition is of particular use because it greatly simplifies verifying M as a twisted homology product.

Proposition 2.1.3 (Agol-Dunfield, Proposition 3.1)**.** *Suppose M is a connected, balanced sutured manifold with $R_\pm$ nonempty. If α is homologically self-dual, then M is an α-homology product if and only if any one of the following vanish:*

$$H_k(M, R_\pm; E_\alpha),\ H^k(M, R_\pm; E_\alpha) \text{ for } k = 1, 2.$$

We give their proof to highlight a couple of facts which do not need the assumption of homological self-duality.

Proof. As $R_\pm$ are nonempty, we know $H_0(M, R_\pm; E_\alpha) = H^0(M, R_\pm; E_\alpha) = 0$. By Poincaré duality, also $H_3(M, R_\mp; E_\alpha) = 0$. Now suppose $H_1(M, R_-; E_\alpha) = 0$; the other cases are similar. Since M is balanced, we have $\chi(R_\pm) = \chi(M)$, so $\chi(H_*(M, R_-; E_\alpha)) = 0$. Then, since $H_k(M, R_-; E_\alpha) = 0$ for $k \neq 2$, we also have $H_2(M, R_-; E_\alpha)) = 0$. Poincaré duality now shows $H^*(M, R_+; E_\alpha) = 0$. Finally, as α is homologically self-dual, this gives $H_*(M, R_+; E_\alpha) = H^*(M, R_+; E_\alpha) = 0$. $\square$

We do not use self-duality until the last step. More generally, we can say

Corollary 2.1.4. *For $R = R_\pm$,*

$$H_1(M, R; E_\alpha) = 0 \quad \Longleftrightarrow \quad H_2(M, R; E_\alpha) = 0,$$

and

$$H^1(M, R; E_\alpha) = 0 \qquad \Longleftrightarrow \qquad H^2(M, R; E_\alpha) = 0.$$

Corollary 2.1.5. *M is an α-homology product if and only if*

$$H_1(M, R; E_\alpha) = H_1(M, R; E_\alpha^*) = 0$$

if and only if

$$H_2(M, R; E_\alpha) = H_2(M, R; E_\alpha^*) = 0$$

for either choice of $R = R_\pm$.

In particular, if M is an α-homology product, it is also an α^*-homology product.

2.2 A conjecture of Agol and Dunfield

In [AD15], Agol and Dunfield make the following conjecture.

Conjecture 2.2.1 (Agol-Dunfield). *For M a taut sutured manifold, there always exists $\alpha : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ for which M is an α-homology product.*

This is motivated by the analogous conjecture by Dunfield, Friedl, and Jackson regarding twisted Alexander polynomials ([DFJ12]), which asserts that, given a hyperbolic knot in S^3, a version of the polynomial stemming from a lift to $\mathrm{SL}_2(\mathbb{C})$ of the hyperbolic holonomy representation should determine the knot's genus. Both conjectures are backed by substantial experimental evidence.

One might further ask whether further the certifying representation in Agol-Dunfield's conjecture can be similarly chosen to be a lift of hyperbolic holonomy representation. Agol and Dunfield give an example where such a lift fails to yield a twisted homology product, but note that a different choice of lift works.

Beyond computed examples, there are only two classes of manifolds for which Conjecture 2.2.1 has been proved, both by Agol and Dunfield in [AD15].

Theorem 2.2.2 (Agol-Dunfield)**.** *For M a genus-two handlebody with a single suture, if M is taut and acylindrical, then there exists $\alpha : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ certifying M.*

Theorem 2.2.3 (Agol-Dunfield)**.** *If M is a taut sutured manifold with the structure of a book of I-bundles, then there exists $\alpha : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ certifying M.*

CHAPTER 3

BOUNDS ON COMPLEXITY OF CERTIFYING REPRESENTATIONS

3.1 Necessity of twisted coefficients

We give two examples of genus-two taut sutured handlebodies which fail to be rational homology products. This illustrates the necessity of twisted coefficients for certifying tautness, even in this topologically simple setting. Our first example captures the essential feature of this failure, that significant information may be lost in abelianizing an injection $\pi_1(R_\pm) \to \pi_1(M)$ to the induced map on homology $H_1(R_\pm;\mathbb{Q}) \to H_1(M;\mathbb{Q})$.

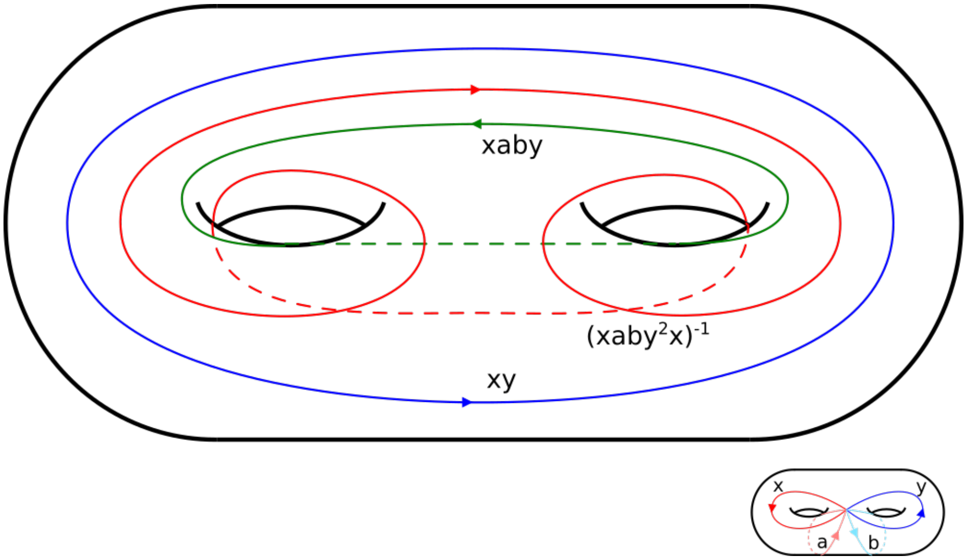


Figure 3.1: A simple example of a genus-2 taut handlebody.

Example 3.1.1. Let M be a genus-two handlebody, with suture γ consisting of the three curves shown in Figure 3.1. These correspond to the free homotopy classes yx, $xaby$, and $(xaby^2x)^{-1}$.

The boundary components $R_\pm$ are topological pants. Their fundamental groups, as subgroups of $\pi_1(\partial M)$, are both freely generated by yx and $xaby$. These inject into $\pi_1(M)$

as the subgroup $\langle xy, yx \rangle$. Abelianizing, we see this is not a rational homology product: the generators of the fundamental group map to the same cycle in $H_1(M;\mathbb{Q})$.

The suture set in the above example consists of three curves. We can also produce examples with only a single suture curve, though none as simple as the example above.

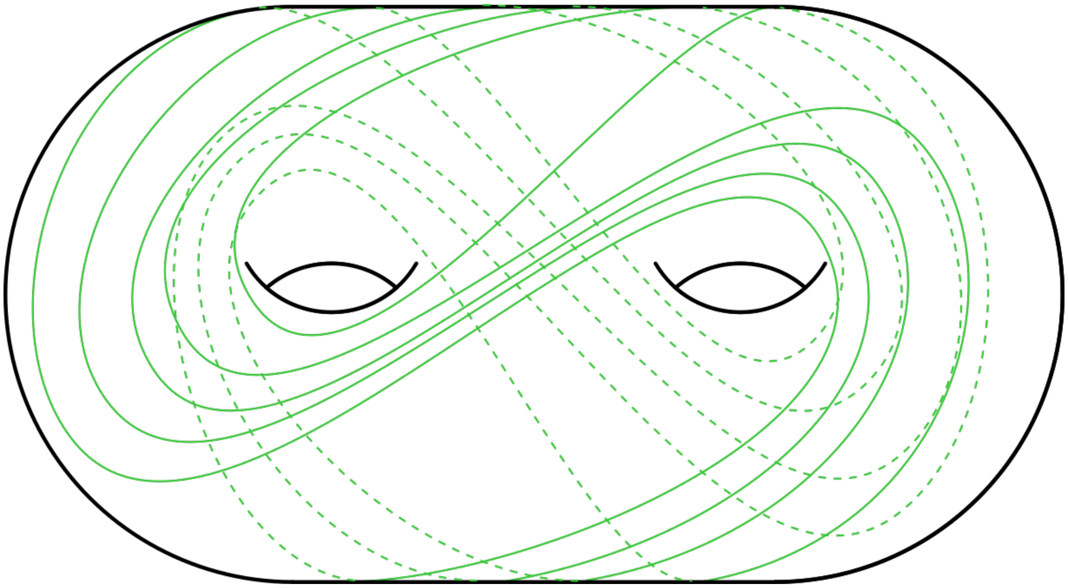


Figure 3.2: A genus-2 example with a single suture. The suture curve γ has image $[x,[x,y][x,y^{-1}]] \in \pi_1(M)$.

Example 3.1.2. Consider M as in Figure 3.2. The generators of R_+ map to x and $[x,y][x,y^{-1}]$ in $\pi_1(M)$. Under the map on homology $H_1(R_+;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ induced by the inclusion $i : R_+ \hookrightarrow M$, the second generator is killed. Thus its image has rank one, and so M cannot be a rational homology product.

We return to these examples in Section 3.2 to prove tautness using tools developed therein. Alternatively, they can both be seen to be taut by observing that if not, each suture component would need to bound a disk in M, but in both examples, the suture sets contain a disk-busting curve.

Remark 3.1.3. These examples can be extended to any higher example by attaching *sutured one-handles* (see Section 3.3 for a definition). By Lemma 3.3.1, this is still taut, and still fails to be a rational homology product.

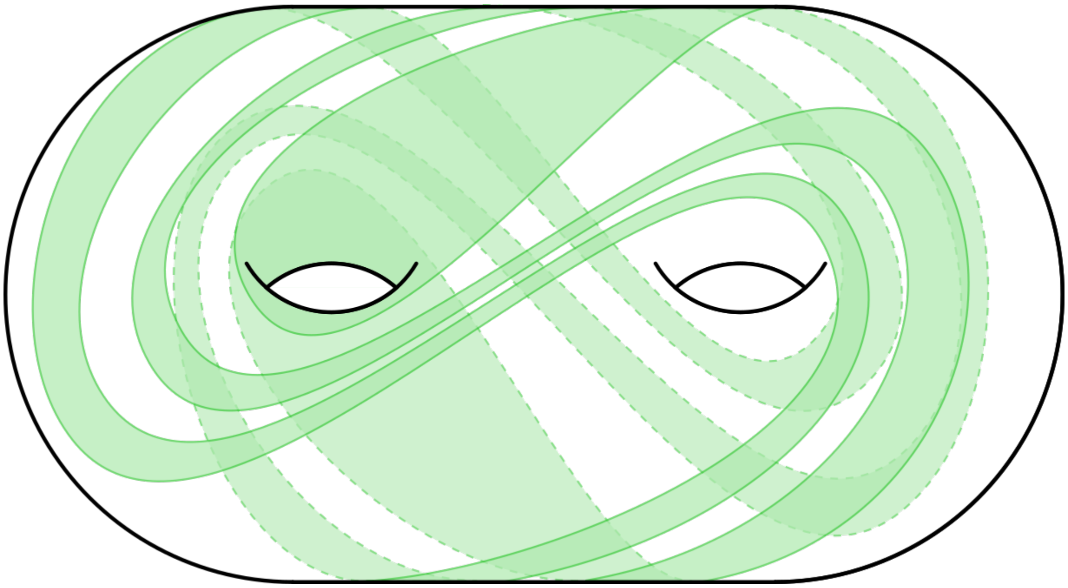


Figure 3.3: The subsurface R_+.

3.2 A condition for twisted products

The following Proposition is a straightforward generalization of Proposition 5.2 of [AD15], in which $g = n = 2$. Our proof follows analogously to that of Agol-Dunfield. Take M to be a balanced sutured handlebody of genus g, with R_+ connected. Then $\pi_1(M)$ and $\pi_1(R_+)$ are free groups of rank g.

Let $\pi_1(M) = \langle x_1, \ldots, x_g \rangle$ and $\pi_1(R_+) = \langle a_1, \ldots, a_g \rangle$, and let $i_* : \pi_1(R_+) \to \pi_1(M)$ be the map induced by the inclusion $i : R_+ \hookrightarrow M$. Given a word $w \in \pi_1(M)$, we write $\partial_{x_i} w$ for its Fox derivatives in $\mathbb{Z}[x_1, \ldots, x_g]$ ([Fox53]). Notice that any representation $\alpha : \pi_1 M \to \mathrm{GL}(V)$ extends naturally to a ring homomorphism $\alpha : \mathbb{Z}[x_1, \ldots, x_g] \to \mathrm{End}(V)$.

Proposition 3.2.1. *For a fixed representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$, with $\dim V = n$, if the sutured handlebody M is an α-homology product, then the $gn \times gn$ matrix*

$$\Big(\alpha\big(\partial_{x_i} i_*(a_j)\big)\Big)_{i,j}$$

has nonzero determinant.

Furthermore, when α is homologically self-dual, this condition is sufficient.

Proof. Let W be a two-complex with a single vertex v, $2g$ edges $e_{x_1}, \dots, e_{x_g}$ and $e_{a_1}, \dots, e_{a_g}$, and g faces $r_1, \dots, r_g$, which are attached to the edges according to $i_*(a_i)a_i^{-1}$, for each i. Set $B = \bigcup_i e_{a_i}$. Then there is a map $j : (W, B) \to (M, R_+)$ realizing the map $\pi_1(W) \to \pi_1(M)$ sending $[e_{x_i}] \mapsto x_i$ and $[e_{a_i}] \mapsto a_i$, and which is a homotopy equivalence of these pairs of spaces.

The map $j_* : \pi_1(W) \to \pi_1(M)$ pulls α back to a representation $\alpha \circ j_* : \pi_1(W) \to \mathrm{GL}(V)$. The map j then induces maps between the twisted homology and cohomology groups associated to α:

$$H_*(M; E_\alpha) \to H_*(W; E_{\alpha\circ j_*}), \qquad H^*(W; E_{\alpha\circ j_*}) \to H^*(M; E_\alpha),$$

$$H_*(R_+; E_\alpha) \to H_*(B; E_{\alpha\circ j_*}), \qquad H^*(B; E_{\alpha\circ j_*}) \to H^*(R_+; E_\alpha).$$

As j is a homotopy equivalence $M \to W$ and $R_+ \to B$, these maps are all isomorphisms. The isomorphisms on cohomology combine with the long exact sequence for pairs to give the following commutative diagram.

$$\begin{tikzcd}
H^{n-1}(W; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H^{n-1}(M; E_\alpha) \arrow[d, "i_*"'] \\
H^{n-1}(B; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "d"'] & H^{n-1}(R_+; E_\alpha) \arrow[d, "d"'] \\
H^{n}(W, B; E_{\alpha\circ j_*}) \arrow[r, dashed] \arrow[d] & H^{n}(M, R_+; E_\alpha) \arrow[d] \\
H^{n}(W; E_{\alpha\circ j_*}) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H^{n}(M; E_\alpha) \arrow[d, "i_*"'] \\
H^{n}(B; E_{\alpha\circ j_*}) \arrow[r, "j_*"] & H^{n}(R_+; E_\alpha)
\end{tikzcd}$$

By the five lemma, the induced map $j_* : H_*(M, R_+; E_\alpha) \to H_*(W, B; E_{\alpha\circ j_*})$ is also an isomorphism.

If M is an α-homology product, then $H_2(M, R_-; E_\alpha) \cong H^1(M, R_+; E_\alpha) = 0$, and also $H^1(W,B;E_{\alpha\circ j_*}) = 0$. To define $H^1(W,B;E_{\alpha\circ j_*})$, we begin with the chain complex $C_*(\widetilde{W};\mathbb{Z})$ of $\mathbb{Z}[\pi_1(M)]$-modules associated to the universal cover of W, then take $\mathbb{Z}[\pi_1(M)]$-module homomorphisms of this complex to V. Writing $\Lambda = \mathbb{Z}[\langle x_1,\ldots,x_g\rangle] = \mathbb{Z}[\pi_1(M)]$, the chain complex has the form

$$C_*(\widetilde{W};\mathbb{Z}): \quad 0 \to \bigoplus_i \Lambda r_{a_i} \xrightarrow{\partial_2} \bigoplus_i (\Lambda e_{x_i} \oplus \Lambda e_{a_i}) \xrightarrow{\partial_1} \Lambda v \to 0.$$

The left-module map ∂_i can be represented as a matrix, which act on an element of $C_i(\widetilde{W};\mathbb{Z})$, viewed as a row vector, by multiplication on their left and the vector's right, namely $\partial_i(u) = u\cdot\partial_i$. These matrices are

$$\partial_1 = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_g - 1 \\ i_*(a_1) - 1 \\ \vdots \\ i_*(a_g) - 1 \end{pmatrix} \qquad \partial_2 = \begin{pmatrix} \partial_{x_1} i_*(a_1) & \cdots & \partial_{x_g} i_*(a_1) & -1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ \partial_{x_1} i_*(a_g) & \cdots & \partial_{x_g} i_*(a_g) & & & -1 \end{pmatrix}$$

Note the left half of the second matrix consists of the Fox derivatives $\partial_{x_i} i_*(a_j)a_j^{-1} = \partial_{x_i} i_*(a_j)$; similarly, the entries in the right half are $\partial_{a_i} i_*(a_j)a_j^{-1} = -\delta_{i,j}$.

Applying $\mathrm{Hom}_\Lambda(\cdot, V)$ to $C_*(\widetilde{W};\mathbb{Z})$ gives the cochain complex $C^*(W;E_{\alpha\circ j_*})$. The effect of applying this functor replaces each Λ with a copy of V, and applying α (extended to a ring homomorphism) to each element of the matrices representing ∂_1 and ∂_2 to obtain the d^i maps. As matrices, the d^i act by multiplication on column vectors to their right. Then the cochain complex is

$$C^*(W;E_{\alpha\circ j_*}): \quad 0 \leftarrow V^g \xleftarrow{d^1} V^{2g} \xleftarrow{d^0} V \leftarrow 0.$$

We are interested, however, in $C^*(W, B; E_{\alpha\circ j_*})$. This consists of those cochains which vanish when restricted B, namely, which are supported away from B. As B is one-dimensional, this consists of all of $C^2(W; E_{\alpha\circ j_*})$, as well as the cochains supported on the e_{x_i} in $C^1(W; E_{\alpha\circ j_*})$. Thus the relative cochain complex is

$$C^*(W, B; E_{\alpha\circ j_*}): \quad 0 \leftarrow V^g \xleftarrow{d^1} V^g \leftarrow 0 \leftarrow 0.$$

Here d^1 restricts to the $\alpha(\partial_{x_i} i_*(a_j))$ half of the full matrix.

As M is taut, $H^1(W, B; E_{\alpha\circ j_*}) = 0$, so d^1 must have full rank, that is, the determinant in the statement of the proposition must be nonvanishing.

Lastly, if α is homologically self-dual and this determinant is nonzero, by Corollary 2.1.5, M is taut. $\square$

The condition of this determinant being nonzero corresponds exactly to $H^2(M, R_+; E_\alpha)$ (and therefore $H_1(M, R_-; E_\alpha)$) vanishing. In the case α is not homologically self-dual, we can still verify tautness by checking that neither this determinant nor that associated to α^* vanishes.

Example 3.2.2 (Examples 3.1.1 and 3.1.2). We can apply Proposition 3.2.1 to see the manifolds in our earlier examples are taut. First, for (M, γ) in Example 3.1.1, let $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ be any one-dimensional representation. By Proposition 3.2.1, M is an α-homology product when

$$\det \begin{pmatrix} \alpha\big(\partial_x(xy)\big) & \alpha\big(\partial_y(xy)\big) \\ \alpha\big(\partial_x(yx)\big) & \alpha\big(\partial_y(yx)\big) \end{pmatrix} \neq 0.$$

That is to say,

$$\det \begin{pmatrix} \alpha\big(\partial_x(xy)\big) & \alpha\big(\partial_y(xy)\big) \\ \alpha\big(\partial_x(yx)\big) & \alpha\big(\partial_y(yx)\big) \end{pmatrix} = \det \begin{pmatrix} \alpha(1) & \alpha(x) \\ \alpha(y) & \alpha(1) \end{pmatrix} = 1 - \alpha(xy) \neq 0.$$

Rephrasing what we saw in Example 3.1.1, this shows in particular we cannot take α to be the trivial representation, where $\alpha(x) = \alpha(y) = 1$. However, for any choice of α with $\alpha(xy) \neq 1$, this will not be 0.

We turn to (M, γ) from Example 3.1.2. For $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ fixed, M is an α-homology product when

$$\det \begin{pmatrix} \alpha\big(\partial_x(x)\big) & \alpha\big(\partial_y(x)\big) \\ \alpha\big(\partial_x([x,y][x,y^{-1}])\big) & \alpha\big(\partial_y([x,y][x,y^{-1}])\big) \end{pmatrix} \neq 0.$$

These Fox derivatives are

$$\begin{gathered} \partial_x(x) = 1, \qquad\qquad \partial_y(x) = 0, \\ \partial_x([x,y][x,y^{-1}]) = 1 - xyx^{-1} + [x,y] - [x,y]xy^{-1}x^{-1}, \\ \partial_y([x,y][x,y^{-1}]) = x - [x,y] - [x,y]xy^{-1} + [x,y]xy^{-1}x^{-1}. \end{gathered}$$

The image of α is abelian, so

$$\begin{aligned} &\det \begin{pmatrix} \alpha\big(\partial_x(x)\big) & \alpha\big(\partial_y(x)\big) \\ \alpha\big(\partial_x([x,y][x,y^{-1}])\big) & \alpha\big(\partial_y([x,y][x,y^{-1}])\big) \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 \\ \alpha(2 - y - y^{-1}) & \alpha(x - 1 - xy^{-1} + y^{-1}) \end{pmatrix} \\ &= \alpha\big((y^{-1} - 1)(1 - x)\big) \neq 0. \end{aligned}$$

This condition is non-vanishing – specifically, whenever $\alpha(x), \alpha(y) \neq 1$ – yielding a nonempty Zariski-open set of certifying α.

Returning to the setting of a genus-g sutured handlebody M, consider the case of a one-dimensional representation $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$. Here, we have an algebraic understanding of what it means to be a twisted homology product. For a word $w \in \pi_1(M)$, we write ∂w for the vector of Fox derivatives of w with respect to $x_1, \ldots, x_g$.

Proposition 3.2.3. *M is a one-dimensional twisted homology product if and only if the vectors of abelianized Fox derivatives* $\mathrm{ab}(\partial i_*(a_j))$ *are linearly independent.*

Proof. Consider the composition of maps

$$\pi_1(M) \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^g,$$

which takes a $a \in \pi_1(M)$ to the vector of the α-images of its g partial Fox derivatives $\partial_{x_i} a$. Since $\mathrm{GL}_1(\mathbb{C})$ is abelian, α factors through the abelianization

$$\pi_1(M) \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\mathrm{ab}} \mathbb{Z}[\mathbb{Z}^g]^g \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^g.$$

Similarly, the composition of maps

$$\pi_1(M)^g \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^{g^2} \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^{g^2} \xrightarrow{\det} \mathbb{C},$$

factors

$$\pi_1(M)^g \xrightarrow{\partial} \mathbb{Z}[\pi_1(M)]^{g^2} \xrightarrow{\mathrm{ab}} \mathbb{Z}[\mathbb{Z}^g]^{g^2} \xrightarrow{\alpha} (\mathrm{GL}_1(\mathbb{C}))^{g^2} \xrightarrow{\det} \mathbb{C}$$

We claim for any $w \in \mathbb{Z}[\mathbb{Z}^g] = \mathbb{Z}[\pi_1(M)^{\mathrm{ab}}]$, we can choose α to detect w, meaning $\alpha(w) \neq 0$. Order the monomial terms of w by setting $\prod x_i^{n_i} > \prod x_i^{m_i}$ if $n_i > m_i$ for the first i such that $n_i \neq m_i$. Then pick α so that $\alpha(x_1) \gg \alpha(x_2) \gg \cdots \gg \alpha(x_g)$; the leading term of w will dominate, and $\alpha(w) \neq 0$.

Because α is one-dimensional, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}^g]^{g^2} \arrow[r, "\alpha"] \arrow[d, "\det"] & (\mathrm{GL}_1(\mathbb{C}))^{g^2} \arrow[d, "\det"] \\
\mathbb{Z}[\mathbb{Z}^g] \arrow[r, "\alpha"] & \mathrm{GL}_1(\mathbb{C})
\end{tikzcd}$$

Thus $\det\big(\mathrm{ab}(\partial i_*(a_j))\big) = 0$ exactly when $\det\big(\alpha(\partial i_*(a_j))\big) = 0$ for all choices of α.

□

We end this section with a lemma which provides a condition for being a one-dimensional twisted product. It will prove useful for finding non-examples. Recall the *derived series* $G^{(k)}$ of G is defined by $G^{(0)} = G$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$.

Lemma 3.2.4. *If M is a one-dimensional twisted homology product, then*

$$\pi_1(R_\pm) \cap \pi_1(M)^{(2)} \subseteq \pi_1(R_\pm)^{(1)}.$$

Proof. Suppose M is an α-homology product for some $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$.

Recall that $H^1(M; E_\alpha)$ is the group of all twisted homomorphisms $f : \pi_1(M) \to \mathbb{C}$, modulo twisted homomorphisms of the form $\hat{z}(g) = \alpha(g) \cdot z - z$ for $z \in \mathbb{C}$. Any $f \in H^1(M; E_\alpha)$ necessarily vanishes on $\pi_1(M)^{(2)}$. We see this first by observing that

$$\begin{aligned} f([u,v]) &= f(u) + \alpha(u)f(v) + \alpha(uv)f(u^{-1}) - \alpha(uvu^{-1})f(v^{-1}) \\ &= f(u) + \alpha(u)f(v) - \alpha(uvu^{-1})f(u) - \alpha(uvu^{-1}v^{-1})f(v) \\ &= (1 - \alpha(v))f(u) - (1 - \alpha(u))f(v), \end{aligned}$$

since $\mathrm{GL}_1(\mathbb{C})$ is abelian. Now, this is zero when $\alpha(u) = \alpha(v) = 1$, for instance, for $u, v \in \pi_1(M)^{(1)}$. Such elements $[u, v]$ normally generate $\pi_1(M)^{(2)}$, so f must vanish on all of $\pi_1(M)^{(2)}$.

Consider now $H^1(R_\pm; E_\alpha)$. Any twisted homomorphism is determined by its values on the generators $a_1, \ldots, a_g$ of $\pi_1(R_\pm)$. Fix $w \in \pi_1(R_\pm) \cap \pi_1(M)^{(2)}$ and let $\#_{a_i} w$ denote the number of occurrences of a_i (counted with sign) in w. Notice $\#_{a_i} w = 0$ for all i is exactly the condition for $w \in \pi_1(R_\pm)^{(1)}$. Supposing $w \notin \pi_1(R_\pm)^{(1)}$, then some $\#_{a_i} w \neq 0$. Define $g \in H^1(R_\pm; E_\alpha)$ by $g(a_j) = \delta_{ij}$. By construction, $g(w) \neq 0$.

Consider the long exact sequence of cohomology groups

$$\cdots \to H^1(M; E_\alpha) \xrightarrow{i_*} H^1(R_\pm; E_\alpha) \xrightarrow{\delta} H^2(M, R_\pm; E_\alpha) \to \cdots .$$

As any $f \in H^1(M; E_\alpha)$ vanishes on w, the twisted homomorphism g constructed above does not lie in the image of i_*. But by exactness, g then is not in the kernel of δ, so $H^2(M, R_\pm; E_\alpha) \neq 0$. By Poincaré duality, then $H_1(M, R_\mp; E_\alpha) \neq 0$, which contradicts our assumption that M is an α-homology product. □

3.3 Examples which are not one-dimensional homology products

In this section, we give a family of handlebodies of all genus $g \geq 2$ which are not twisted homology products for any one-dimensional representation. We begin with a lemma which describes a way of increasing genus of a taut handlebody while preserving the set of certifying representations. This is used in conjunction with a genus-two example to prove the main result of the section. The genus-two example was found via computer search with SnapPy ([CDGW]). We follow this with an explicit construction an example of a genus-three handlebody, which better elucidates the obstruction to admitting a one-dimensional certifying representation.

For ease of notation, we treat γ as a collection of annuli instead of curves. Given a sutured manifold M, we can construct a new sutured manifold N by attaching a *sutured one-handle.* The one-handle $D^2 \times D^1$ is given a product sutured structure $I \times (D^1 \times D^1)$. It is attached to M along the disks $I \times (D^1 \times \partial D^1)$, which we require to meet γ in two strips so that $0 \times (D^1 \times \partial D^1) \subset R_-$ and $1 \times (D^1 \times \partial D^1) \subset R_+$. This construction is illustrated in Figure 3.4.

Lemma 3.3.1. *Suppose M is a taut sutured manifold. If $(N, R'_\pm, \gamma')$ is obtained by attaching a sutured one-handle to M, then N is also taut. Moreover, for any representation $\alpha : \pi_1(M) \to \mathrm{GL}(V)$, there is a representation $\alpha' : \pi_1(N) \to \mathrm{GL}(V)$ with $\alpha'|_{\pi_1(M)} = \alpha$ such*

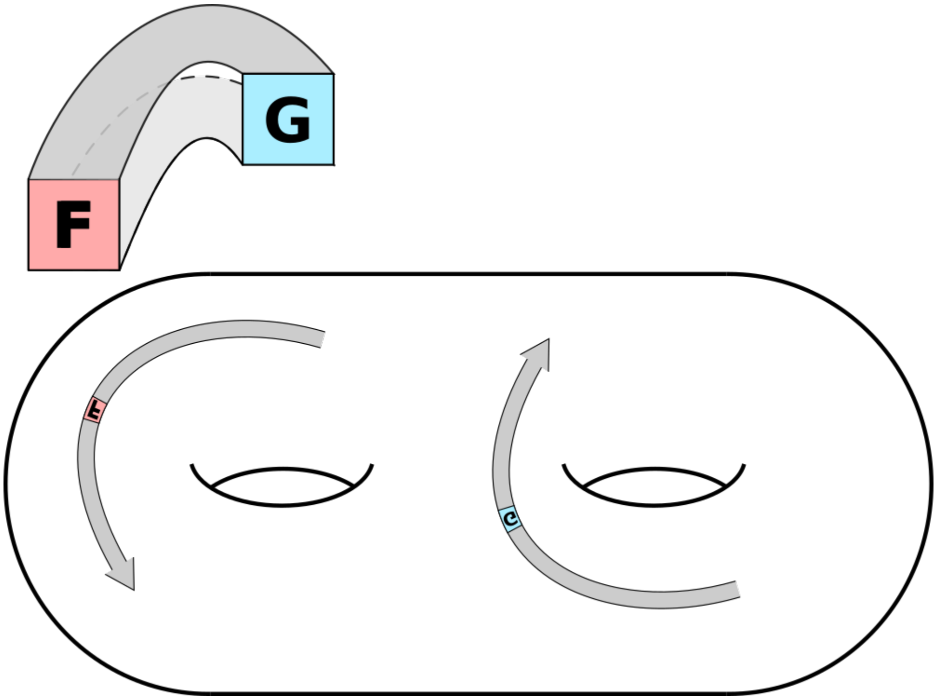


Figure 3.4: Attaching a sutured handle. The grey bands represent portions of the suture γ, viewed as annuli and oriented by the arrows drawn. Homeomorphisms F and G identify the corresponding disks. The letters F and G indicate the maps' respective orientations and restrictions to the boundary.

that M is an α-homology product if and only if N is an α'-homology product.

Proof. Note $\pi_1(N) = \pi_1(M) * \langle x \rangle$, where x is the core of the one-handle. Moreover, $\pi_1(R'_\pm) = \pi_1(R_\pm) * \langle x \rangle$. Define α' to agree with α on $\pi_1(M)$ and to map x to the identity.

Notice

$$\det\Big(\alpha'\big(\partial_{x'_i} i_*(a'_j)\big)\Big) = \det\begin{pmatrix} \alpha\big(\partial_{x_i} i_*(a_j)\big) & 0 \\ 0 & I \end{pmatrix} = \det\Big(\alpha\big(\partial_{x_i} i_*(a_j)\big)\Big).$$

The corresponding equality also holds for the dual representations. Thus the result follows from Proposition 3.2.1. □

Theorem 3.3.2 (Theorem 1.4.1)**.** *For every $g \geq 2$, there is a taut sutured handlebody M_g of genus g such that M_g is not an α-homology product for any representation $\alpha : \pi_1(M_g) \to \mathrm{GL}_1\mathbb{C}$.*

Proof. For $g = 2$, we construct $M = M_2$ as follows. Define a sutured structure on M by taking $R_+ \cong \Sigma_{0,3} \subset \partial M$ to be a tubular neighborhood of the curves illustrated in Figure 3.5; γ the boundary of this neighborhood; and $R_- = \partial M - R_+$. Fix x and y as generators of $\pi_1(M)$. The two boundary curves in the figure, which generate $\pi_1(R_+)$, map to $a = [y, x^{-1}][x, y^{-1}]$ and $b = [y^{-1}, x][y^{-1}, x^{-1}]$.

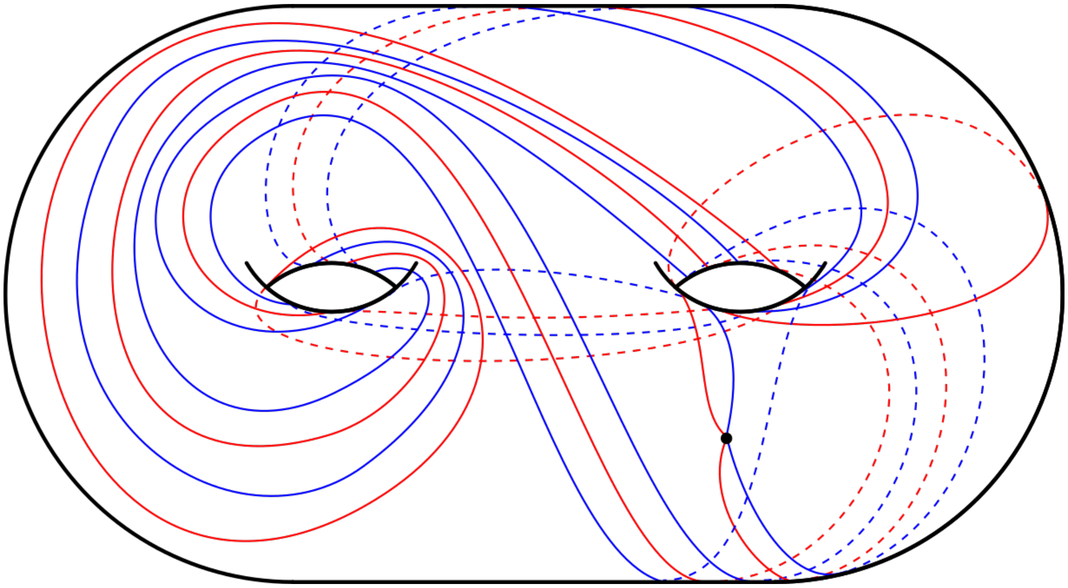


Figure 3.5: Example with no certifying 1-dimensional representation.

Let $\alpha : \pi_1(M) \to \mathrm{GL}_1(\mathbb{C})$ be any representation. Since α has abelian image, we can simplify the matrix A in Proposition 3.2.1 by replacing the entries with the abelianization of the Fox derivatives. This yields

$$A = \begin{pmatrix} \alpha(-x^{-1}y + x^{-1} + 1 - y^{-1}) & \alpha(1 - x^{-1} - xy^{-1} + y^{-1}) \\ \alpha(y^{-1} - 1 - x^{-1}y^{-1} + x^{-1}) & \alpha(-2y^{-1} + xy^{-1} + x^{-1}y^{-1}) \end{pmatrix}.$$

We leave it to the reader to verify the determinant of this matrix vanishes, independent of the choice of α.

We demonstrate that M is taut via the certifying representation $\beta : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, defined by

$$\beta(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As remarked earlier, representations to $\mathrm{SL}_2(\mathbb{C})$ are homologically self-dual, so Proposition 3.2.1 applies.

The (non-abelianized) Fox derivatives of a and b are

$$\begin{aligned}
\partial_x a &= -yx^{-1} + yx^{-1}y^{-1} + yx^{-1}y^{-1}x - yx^{-1}y^{-1}x^2y^{-1}x^{-1} \\
\partial_y a &= 1 - yx^{-1}y^{-1} - yx^{-1}y^{-1}x^2y^{-1} + yx^{-1}y^{-1}x^2y^{-1}x^{-1} \\
\partial_x b &= y^{-1} - y^{-1}xyx^{-1} - y^{-1}xyx^{-1}y^{-1}x^{-1} + y^{-1}xyx^{-1}y^{-1}x^{-1}y \\
\partial_y b &= -y^{-1} + y^{-1}x - y^{-1}xyx^{-1}y^{-1} + y^{-1}xyx^{-1}y^{-1}x^{-1}
\end{aligned}$$

Then

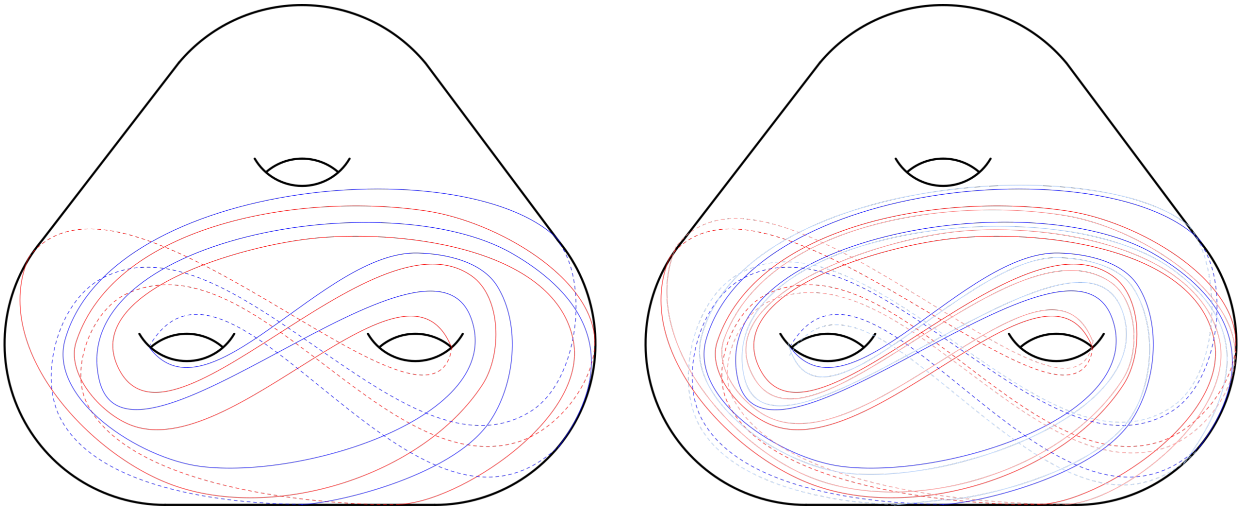
$$\det \begin{pmatrix} \beta(\partial_x a) & \beta(\partial_y a) \\ \beta(\partial_x b) & \beta(\partial_y b) \end{pmatrix} = \det \begin{pmatrix} 0 & 3 & 0 & -2 \\ 0 & 6 & -1 & -3 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} = 1,$$

so M is taut.

We may iteratively apply Lemma 3.3.1 to construct higher genus handlebodies from this example. The process in the Lemma gives a handlebody M_g for all $g > 2$ which is still a two-dimensional twisted homology product, and fails to admit a certifying one-dimensional representation. □

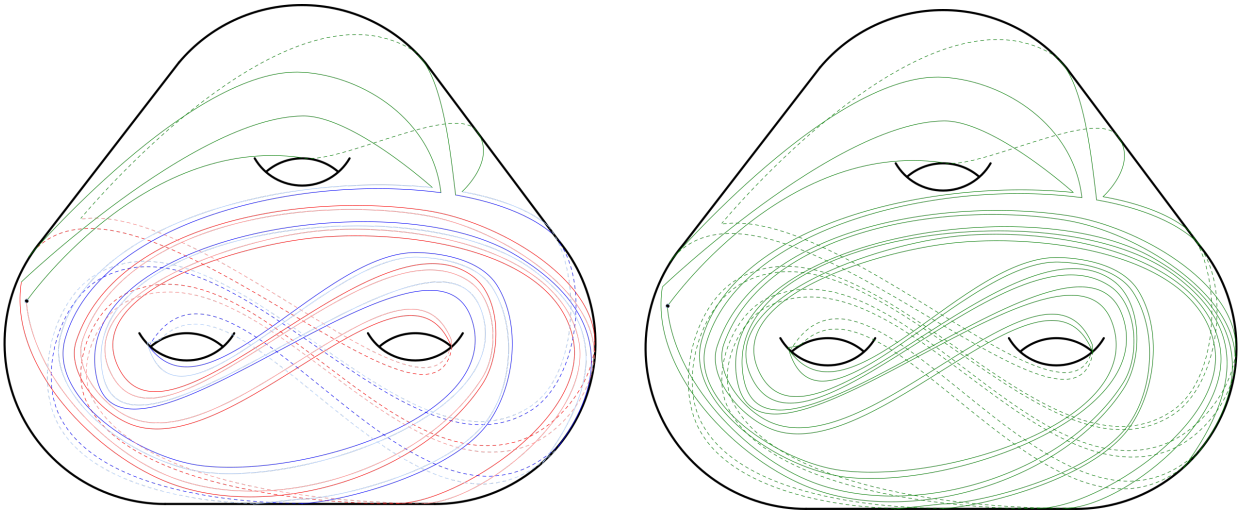
We now give an alternative, explicit construction of a genus-three example. This example puts to use Lemma 3.2.4, by building a curve which lies in $\pi_1(M)^{(2)}$. It is also a precursor to the construction within the proof of Theorem 3.4.1 in Section 3.4.

Example 3.3.3. We build a taut $(M, R_\pm, \gamma)$ with R_+ containing a curve whose image in $\pi_1(M)$ lies in $\pi_1(M)^{(2)}$. Let M be a genus-3 handlebody, with $\pi_1(M) = \langle x, y, z \rangle$.



(a) Disjoint curves A, B with image in $\pi_1(M)^{(1)}$.

(b) A, B doubled and pushed off themselves.



(c) Copies of A and B connected by segments z.

(d) The curve $a = [A, zBz^{-1}]$.

Figure 3.6: Construction of the curve a.

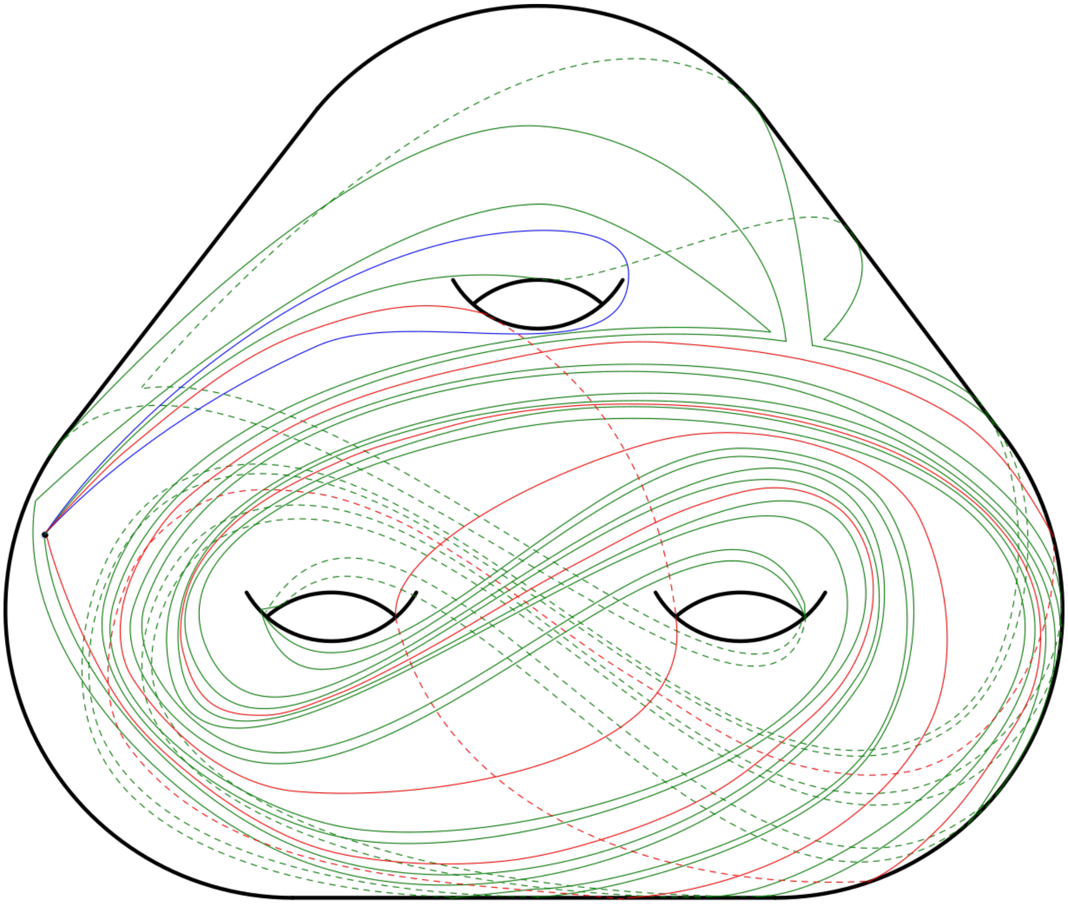


Figure 3.7: Curves a, b, and c.

To describe the sutured structure on M, we begin by constructing a simple closed curve a on the boundary of M which lives in $\pi_1(M)^{(2)}$. Figure 3.6 illustrates this process. First, we draw the curves A and B, which are disjoint and have image in $\pi_1(M)^{(1)}$. The curve a is constructed from A and B to have image $a = [A, zBz^{-1}] \in \pi_1(M)$. Figures 3.6b and 3.6c show this construction, by first taking two copies of each A and B, and then connecting them via arcs to yield a simple closed curve with the desired image. Picking a basepoint along a, we then find two more simple closed curves b and c on ∂M, disjoint away from the basepoint, as shown in Figure 3.7. This captures all the information we need to define $(M, R_\pm, \gamma)$: a neighborhood of this defines R_+, which is homeomorphic to $\Sigma_{1,2}$, its boundary γ, and its complement R_-. From the construction, we see

$$\operatorname{Im}(\pi_1(R_+)) = \left\langle \Big[[x,y][x^{-1},y], z[y^{-1},x][y,x]z^{-1}\Big], [x,y][y^{-1},x^{-1}], z \right\rangle.$$

We now check our example is taut. We do this by exhibiting a two-dimensional representation $\beta : \pi_1(M) \to \mathrm{GL}_2(\mathbb{C})$ which realizes M as a twisted homology product. Define β as follows:

$$\beta(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \beta(y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \beta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact β is a representation to $\mathrm{SL}_2(\mathbb{C})$, so the associated twisted homology is self-dual. Then we can apply Proposition 3.2.1. The relevant matrix is

$$\begin{pmatrix} \beta\big(\partial_x i_*(a)\big) & \beta\big(\partial_y i_*(a)\big) & \beta\big(\partial_z i_*(a)\big) \\ \beta\big(\partial_x i_*(b)\big) & \beta\big(\partial_y i_*(b)\big) & \beta\big(\partial_z i_*(b)\big) \\ \beta\big(\partial_x i_*(c)\big) & \beta\big(\partial_y i_*(c)\big) & \beta\big(\partial_z i_*(c)\big) \end{pmatrix} = \begin{pmatrix} -8 & -36 & 7 & 23 & 37 & 4 \\ -2 & -7 & 1 & 5 & 8 & 1 \\ 4 & 2 & -1 & -1 & 0 & 0 \\ 2 & 3 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is invertible with determinant 32, so M is indeed taut.

3.4 Restricting to solvable representations

In this section, we restrict to the setting of solvable representations. A group G is *solvable* if its derived series $G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$ has finite length. For G solvable, let K denote the length of this series, that is, the smallest index k such that $G^{(k)} = 1$. We then say G is solvable of *degree* K. This is equivalent to realizing G as a $(K-1)$-fold abelian extension of an abelian group. We say a representation $\alpha : G \to \mathrm{GL}(V)$ is *solvable* if it has solvable image, and similarly we define the *degree of solvability* of α to be the degree of solvability of its image.

We prove the following.

Theorem 3.4.1. *For any K, there is a taut sutured handlebody M_K which fails to be a twisted homology product for any solvable representation of degree less than K.*

Observe that Example 3.1.1 and Theorem 3.3.2 are examples of sutured handlebodies satisfying the $K = 1, 2$ cases of this theorem, respectively. In the setting of $\mathrm{GL}_n(\mathbb{C})$, Zassenhaus shows for a fixed n any solvable subgroup is of bounded degree of solvability ([Zas37]). Let $\varphi(K)$ denote the smallest n for which $\mathrm{GL}_n(\mathbb{C})$ admits a solvable subgroup of degree K.

Corollary 3.4.2 (Theorem 1.4.3)**.** *The handlebody M_K is not a twisted homology product for any solvable representation to $\mathrm{GL}_n(\mathbb{C})$ for $n < \varphi(K)$.*

In particular, the conjecture of Agol and Dunfield is false when restricted to the class of solvable representations.

The next lemma captures the connection between solvability of a representation and its behavior with respect to the Fox derivative.

Lemma 3.4.3. *If $\alpha : G \to \mathrm{GL}(V)$ is solvable of degree K, then $\alpha(\partial g) = 0$ for any $g \in G^{(K+1)}$.*

Proof. We show this holds for $g = [g_1, g_2]$ where $g_1, g_2 \in G^{(K)}$; as elements of this form generate $G^{(K+1)}$, this suffices. Recall

$$\partial g = \partial g_1 + g_1 \partial g_2 - g_1 g_2 g_1^{-1} \partial g_1 - g_1 g_2 g_1^{-1} g_2^{-1} \partial g_2.$$

As $\alpha(g_1) = \alpha(g_2) = 1$, thus

$$\alpha(\partial g) = \alpha(\partial g_1) + \alpha(\partial g_2) - \alpha(\partial g_1) - \alpha(\partial g_2) = 0. \qquad \square$$

The idea of the proof of Theorem 3.4.1 is to construct sutured manifolds which carry curves deeper and deeper in the derived series of the manifold's fundamental group, thereby allowing us to exploit this property of the Fox derivative. The construction of these curves follows the same "double-then-cut-and-paste" method we use in Example 3.3.3 to build a curve in $(\pi_1(M))^{(2)}$.

Proof of 3.4.1. We construct the manifolds M_K by induction on K. We make the following assumptions on M_{K-1}:

1. The suture set γ consists of a curve γ. We realize R_+ as a closed neighborhood of g simple closed curves $c_1, \ldots c_g$ disjoint away from a common basepoint;
2. Some curve c_i has image in $\pi_1(M_{K-1})^{(K-1)} \leq \pi_1(M_{K-1})$.

Let M_1 and M_2 be two copies of M_{K-1}, and let a_1 and a_2 denote the curves from condition (2). As the sutures are single curves, there is some c_i in each with geometric intersection $i(c_i, a_j) = 1$; denote these by b_1 and b_2. We first construct an intermediate handlebody M'_K, by joining M_1 and M_2 by a one-handle $H_1 = D^2 \times D^1$ such that the disks $D^2 \times \partial D^1$ are identified with disks disjoint from all the curves c_i. Then $\pi_1(M_K) = \pi_1(M_1) * \pi_1(M_2)$. Apply the procedure from the proof of Theorem 3.3.2 to a_1 and a_2, as illustrated in Figure 3.8, to construct a curve a whose image in $\pi_1(M'_K)$ is $[a_1, a_2]$, and therefore lies in $\pi_1(M'_K)^{(K)}$. We fix a basepoint along an arc of a within H_2.

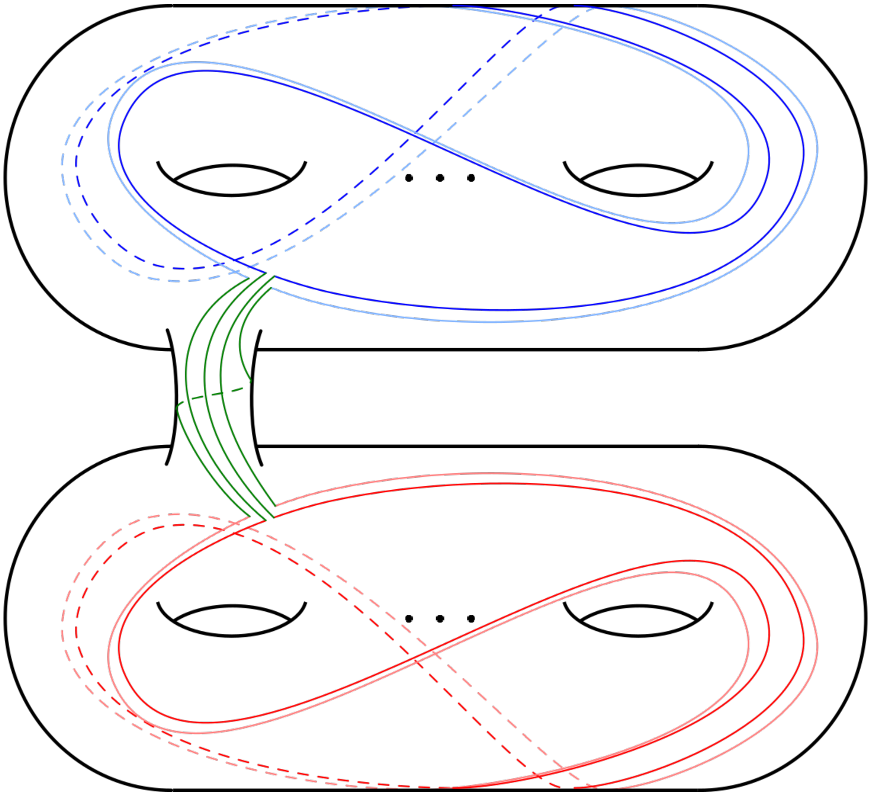


Figure 3.8: The curve a constructed from a_1 and a_2.

To obtain M_K, we add an additional one-handle $H_2 = D^2 \times D^1$ to M'_K by attaching the disks $D^2 \times \partial D^1$ within a small neighborhood of the basepoint, to either side of the locally separating arc of a.

To the collection of curves c_i in ∂M_K, we add a new curve c which runs around this second handle, parallel to its core, and intersecting a in exactly the basepoint. The remaining curves c_i may intersect a. We modify them as illustrated in Figure 3.10. This procedure alters the $\pi_1(M_K)$-image of a curve in one of the following ways:

$$c_i \mapsto c_i \qquad \text{(Figure 3.10a)}$$

$$c_i \mapsto a_j c_i a_j^{-1} \qquad \text{(Figure 3.10b)}$$

$$c_i \mapsto c_i a_j^{-1} \qquad \text{(Figure 3.10c)}$$

These curves are once more disjoint away from a basepoint, as Figure 3.10d suggests. While not all combinatorial arrangements of curves are shown, the remaining cases are similar.

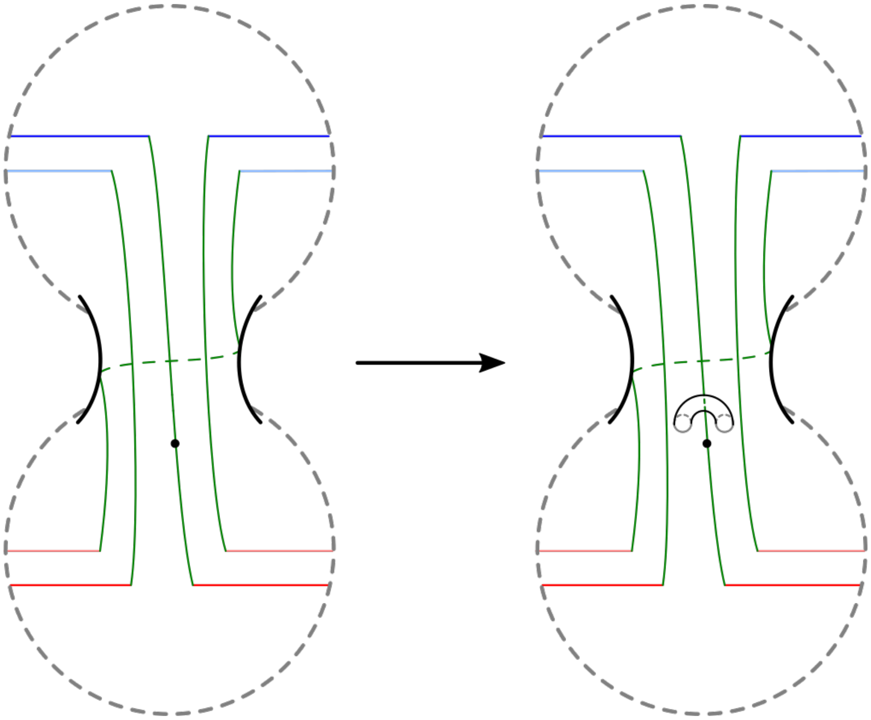


Figure 3.9: Adding the second one-handle H_2 to obtain M_K.

We add one final curve $b = a_1ca_2$, which is also included in Figure 3.10d, giving a total of $2g+1$ curves. Take a closed neighborhood of these as the new R_+ and its boundary as the suture set γ defining a sutured structure on M_K. This construction shows M_K satisfies the inductive conditions (1) and (2); in particular the curve c ensures γ is connected.

To verify M_K is taut, we exhibit a sutured manifold decomposition

$$M_K \xrightarrow{S_1} M \xrightarrow{S_2} M' \xrightarrow{S_3} M'' \cup M_2,$$

where M'' is another taut sutured handlebody of genus g.[1] This decomposition is illustrated in Figure 3.11, and described below.

The surface S_1 is the disk $D^2 \times \left\{\frac{1}{2}\right\} \subset H_2$. The decomposition kills c, and by choosing appropriate choice of orientation of S_1, the curve $b = a_1ca_2$ becomes a_1.

The surface S_2 is a once-punctured torus bound by the curve a. Topologically, it is

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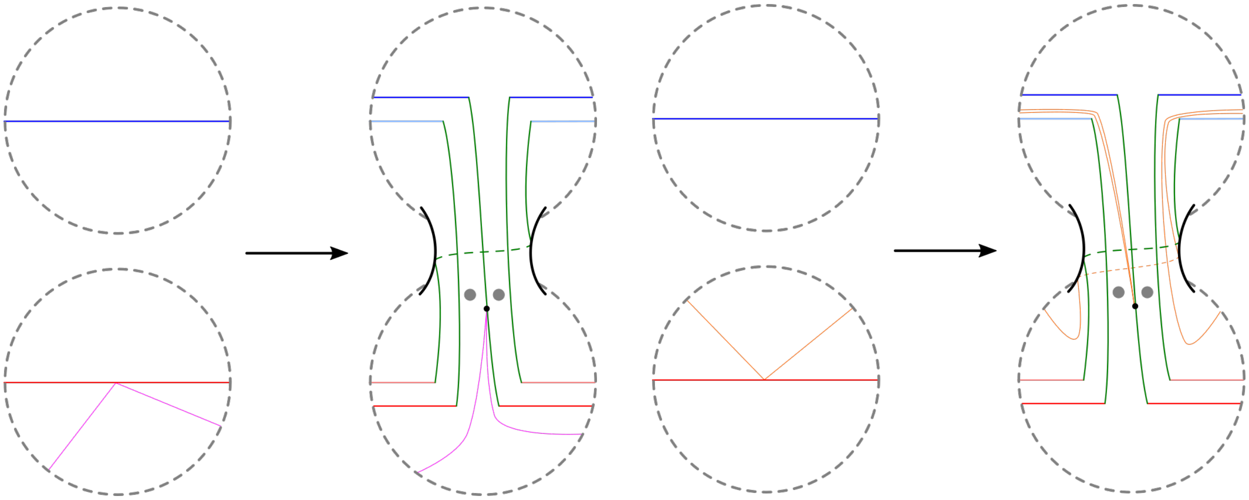
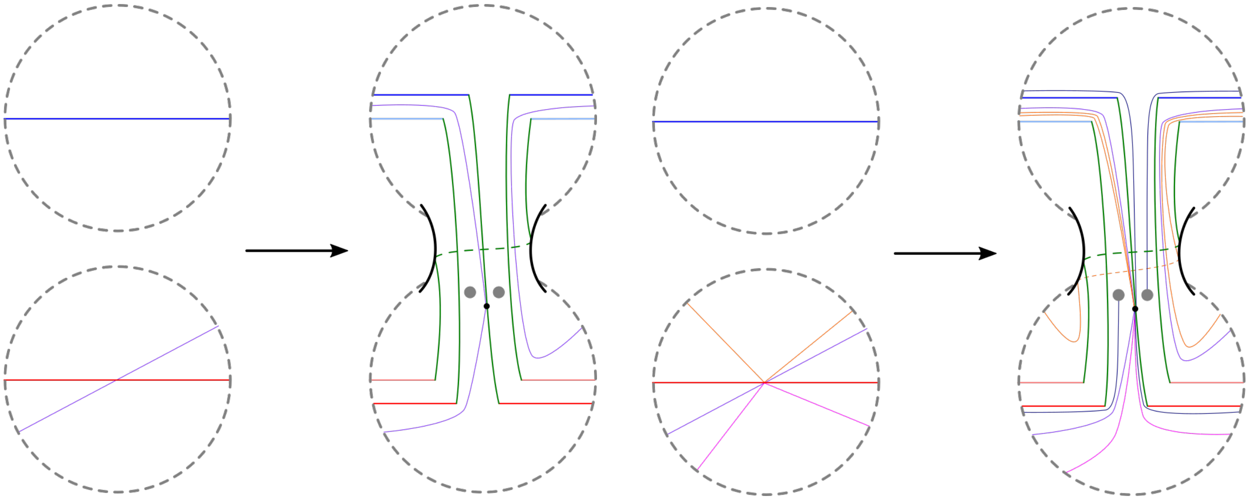
[1] In fact this shows the intermediate manifolds are also taut, in particular M, which retains the obstruction to admitting a certifying solvable representation of derived length K.



(a) $c_i \mapsto c_i$.

(b) $c_i \mapsto a_j c_i a_j^{-1}$.



(c) $c_i \mapsto c_i a_j^{-1}$.

(d)

Figure 3.10: Modifying the c_i on M_K. The handle H_2 is not shown, but is attached at the points shown.

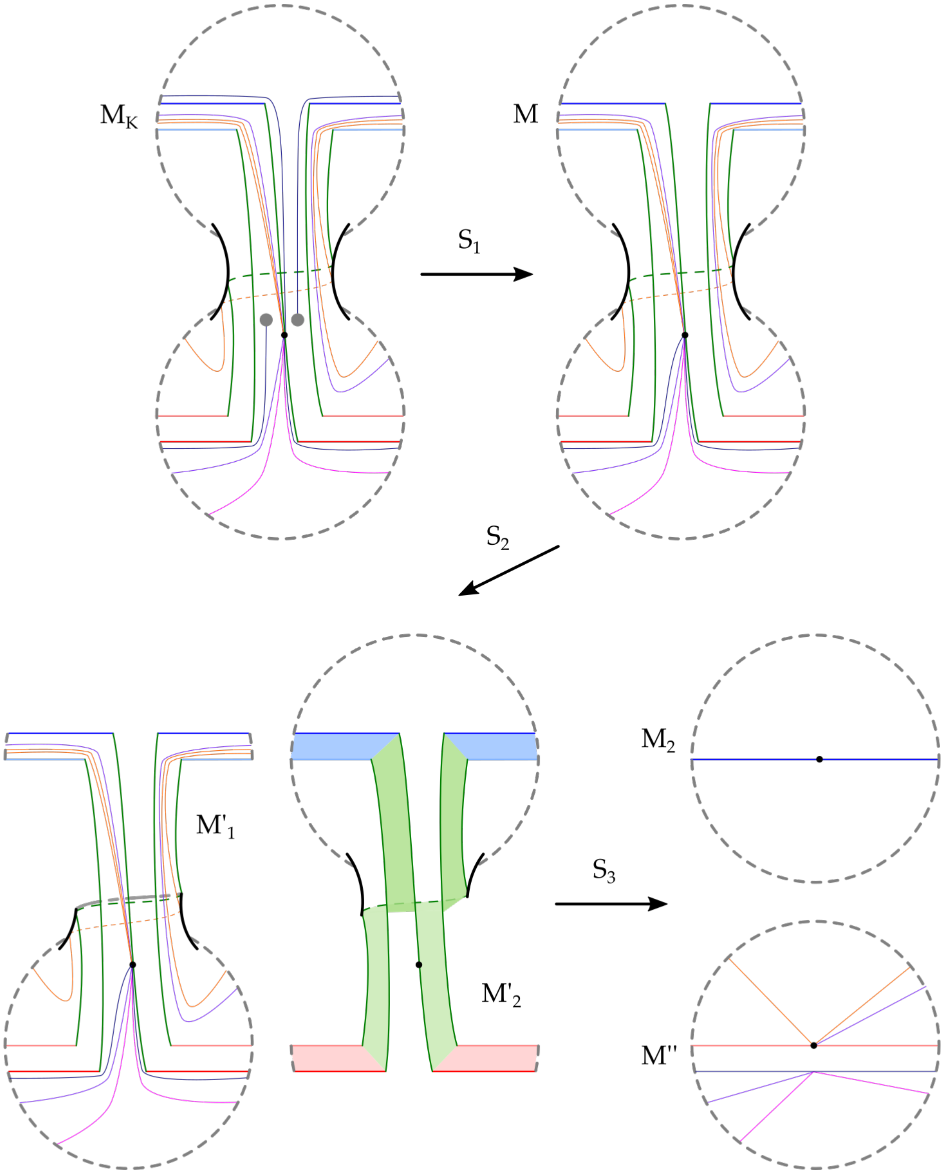


Figure 3.11: Decomposing M_K into two taut handlebodies of genus g.

the two strips between the two copies of a_1 and a_2 used to construct a, glued to the disk $D^2 \times \{\frac{1}{2}\} \subset H_1$, then pushed slightly into the handlebody. Orient S_2 so that M_2 lies on the positive side of this disk. This separates M into two genus-$(g+1)$ handlebodies M_1' and M_2'. Notice in M_2', the two copies of a_1 used to construct a are now parallel in R_+, and similarly the copies of a_2.

Finally, S_3 consists of two disks, each cutting one of the new handles created by the decomposition along S_2. Choose these disks to be oriented to agree with a_1 and a_2, respectively. Additionally, push them off the sutures where possible, to eliminate unnecessary intersections, by dragging the disks toward the basepoint.

In M_2', this results in a disk which intersects the suture in exactly two points, cutting the a_1-bands in $R_\pm$. The remainder of the c_i are unaffected, and so the resulting sutured manifold is M_2.

In M_1', the situation is more complicated. This decomposition results in a handlebody whose sutured structure is similar to, but not exactly that of M_1. The subsurface R_+ has fundamental group with generators $c_1, \ldots, c_g$, with the exception of any curve c_i with geometric intersection $i(c_i, a_1) = 1$, such as b_1. In this case, b_1 is replaced by $b_1 a_1 b_1^{-1}$, and other such c_i can be replaced by $c_i b_1^{-1}$. Notice that the existence of b_1 ensures that R_+ is connected. We observe, however, that this handlebody is taut exactly when M_1 is: on the level of Fox derivatives, this difference translates to

$$\alpha(\partial i_*(b_1)) \mapsto \alpha(\partial i_*(b_1 a_1 b_1^{-1})) = \alpha(1 - i_*(b_1 a_1 b_1^{-1}))\alpha(\partial i_*(b_1)) + \alpha(i_*(b_1))\alpha(\partial i_*(a_1)),$$
$$\alpha(\partial i_*(c_i)) \mapsto \alpha(\partial i_*(c_i b_1^{-1})) = \alpha(\partial i_*(c_i)) - \alpha(i_*(c_i b_1^{-1}))\alpha(\partial i_*(b_1)).$$

In the matrix given by Proposition 3.2.1, this demonstrates the matrix corresponding to M'' is obtained from that for M_1 via elementary row operations. This preserves invertibility, unless $\alpha(i_*(b_1 a_1 b_1^{-1})) = 1$; in such a situation α may be perturbed away from this locus, yielding a certifying representation for both M_1 and M''.

Since $a \in \pi_1(M_K)^{(K)}$, by Lemma 3.4.3, the determinant in Proposition 3.2.1 vanishes for any solvable representation of degree less than K. Therefore M_K is not a twisted homology product for any such representation. $\square$

CHAPTER 4

A PERSPECTIVE FROM REPRESENTATION VARIETIES AND SKEW FIELDS

In this chapter we explore the determinant condition given in Proposition 3.2.1. This gives insights into the nature of the structure of the set of certifying representations of dimension n within the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C}))$. Moreover, this condition gives a potential method for giving a uniform upper bound on the minimum dimension of a certifying representation for M a taut sutured handlebody.

4.1 Representation varieties

From the perspective of the representation variety $\mathrm{Hom}(\pi_1(M), \mathrm{GL}(V))$, the condition given by Proposition 3.2.1 determines a Zariski-open subspace of certifying representations. In the setting of M a handlebody, the representation variety is connected, and so such a subspace is either empty, or dense in the full variety.

On a practicable level, this is good news for certifying tautness. Supposing we knew an upper bound on minimal complexity of a certifying representation, we expect a 'random' representation of that complexity to in fact be certifying.

This is also a useful observation for an approach to prove the existence of a certifying representation, and one we will return to in Section 4.2. In particular, we can restrict to any dense subset of representations, which perhaps has some nice property we can analyze, and this subset will still contain certifying representations.

Proposition 3.2.1 specifically applies to sutured handlebodies, and does not immediately generalize outside of this setting. However, we expect this intuition for the space of certifying representation within the representation variety to generalize, and the certifying representations to similarly form a Zariski-open subspace. However, for M not a handlebody, this representation variety may not be connected. This creates difficulties, for instance, in generalizing the approach described in Section 4.3.

4.2 The Dieudonné determinant

We follow the exposition of Draxl ([Dra83]).

The Dieudonné determinant is a generalization of the usual determinant for vector spaces to vector fields defined over skew (i.e., non-commutative) fields, introduced by Dieudonné in [Die43]. Our motivation is to understand the algebraic condition in Proposition 3.2.1 independent of a choice of representation $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$.

Given $\mathbb{F}$ a skew field, define $\mathrm{GL}_n(\mathbb{F})$ to be those matrices $M \in M_n(\mathbb{F})$ which admit left and right inverses.

Proposition 4.2.1. *For $\mathbb{F}$ a skew field and $A \in M_n(\mathbb{F})$, the following are equivalent:*

1. *$A \in \mathrm{GL}_n(\mathbb{F})$,*
2. *A has a left inverse,*
3. *A has a right inverse,*
4. *The rows of A are left-linearly independent, and*
5. *The columns of A are right-linearly independent.*

As the last two properties illustrate, the notion of invertibility in $M_n(\mathbb{F})$ is nuanced. For instance, invertibility is not preserved under taking the transpose, as

$$A = \begin{pmatrix} 1 & x \\ y & xy \end{pmatrix}$$

is invertible – the rows are left-linearly independent – but

$$A^t = \begin{pmatrix} 1 & y \\ x & xy \end{pmatrix}$$

is not if $xy \neq yx$.

As a first step toward defining this generalized determinant, we describe a decomposition of A called the *Bruhat normal form*, a standard way any invertible M can be written.

Within $M_n(\mathbb{F})$, we define the following special classes of matrices. Let $\mathcal{L}_n(\mathbb{F})$ be the set of lower triangular $n \times n$ matrices with 1 on the diagonal; $\mathcal{U}_n(\mathbb{F})$ the set of upper triangular matrices with 1 on the diagonal; $\mathcal{D}_n(\mathbb{F})$ the set of diagonal matrices with no zeroes on the diagonal; and $\mathcal{P}_n(\mathbb{F})$ the set of permutation matrices, corresponding to elements of S_n.

Definition 4.2.2. A *Bruhat normal form* for $A \in M_n(\mathbb{F})$ is a decomposition

$$A = LDPU,$$

where $L \in \mathcal{L}_n(\mathbb{F})$, $D \in \mathcal{D}_n(\mathbb{F})$, $P \in \mathcal{P}_n(\mathbb{F})$, and $U \in \mathcal{U}_n(\mathbb{F})$.

One can always find a Bruhat normal form for $A \in \mathrm{GL}_n(\mathbb{F})$, by carrying out a fairly standard 'factorization' argument using elementary matrices, i.e., matrices with a single nonzero entry away from the diagonal, which has all 1 entries.

Observe in the case that $\mathbb{F}$ is commutative, $\det A = \det D = \prod_i d_i$, where d_i are the diagonal entries of D. As a generalization of the usual determinant, the Dieudonné determinant should also have this property, and in particular, the matrices L, P, and U all 'should' have determinant 1. In fact this is how we define the Dieudonné determinant.

Definition 4.2.3. For $A \in M_n(\mathbb{F})$, the *Dieudonné determinant* of A is

$$\det{}_{\mathscr{D}} = \begin{cases} \operatorname{sgn}(P) \prod_i d_i & \text{if } A \in \mathrm{GL}_n(\mathbb{F}) \text{ with Bruhat normal form } A = LDPU \\ 0 & \text{if } A \notin \mathrm{GL}_n(\mathbb{F}). \end{cases}$$

Hidden in this is the fact that this determinant is independent of the choice of decomposition. From this definition, we immediately see

Theorem 4.2.4. *$A \in M_n(\mathbb{F})$ is invertible if and only if* $\det_{\mathscr{D}} A \neq 0$.

4.3 Application to certifying tautness

We discuss the connection between the Dieudonné determinant and the algebraic condition for tautness given in Proposition 3.2.1. This perspective allows us to remove the necessity of referring to a specific representation to demonstrate tautness, by instead considering the matrix of Fox derivatives associated to a sutured handlebody

$$\Big(\partial_{x_i} i_*(a_j)\Big)_{i,j}.$$

We begin with a straightforward observation about this relationship.

Lemma 4.3.1 (Lemma 1.4.5)**.** *If M is taut sutured handlebody,*

$$\det{}_{\mathscr{D}}\big(\partial_{x_i} i_*(a_j)\big) \neq 0.$$

Proof. Suppose this determinant is zero. Then its rows, the Fox derivatives $\partial i_*(a_j)$, are left-linearly dependent. This property is preserved by any representation $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$, so the block matrix of the α images of the Fox derivatives

$$\Big(\alpha\big(\partial_{x_i} i_*(a_j)\big)\Big)_{i,j}$$

has linearly dependent rows. By Proposition 3.2.1, M is not an α-homology product. So M admits no certifying representation of any dimension, so by Friedl and Kim ([FK13]), M is not taut. □

In fact, we conjecture the converse holds as well.

Conjecture 4.3.2 (Conjecture 1.4.6)**.** *M is taut if and only if*

$$\det{}_{\mathscr{D}}\big(\partial_{x_i} i_*(a_j)\big) \neq 0.$$

A discussion of how we might try to prove the 'if' direction follows.

For $n \geq 2$, let $\mathcal{I}_n \subseteq \operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C}))$ consist of those representations $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$ which are injective and induce an injection $\alpha : \mathbb{Z}[\pi_1(M)] \to \operatorname{End}(\mathbb{C}^n)$ on the integral group ring of $\pi_1(M)$, with image in $\mathrm{GL}_n(\mathbb{C}) \cup \{0\}$.[1] These representations may be naturally extended to be defined on the formal inverses of elements of $\mathbb{Z}[\pi_1(M)]$. This locus may be realized by removing countably many subvarieties from $\operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C}))$:

$$\mathcal{I}_n = \operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C})) - \bigcup_{g \neq 1 \in \pi_1 M} \{\alpha : \alpha(g) = 1\} - \bigcup_{w \neq 0 \in \mathbb{Z}[\pi_1 M]} \{\alpha : \det \alpha(w) = 0\}.$$

Alternatively, it may be viewed as the countable intersection

$$\begin{aligned}\mathcal{I}_n = &\left(\bigcap_{g \neq 1} \operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C})) - \{\alpha : \alpha(g) = 1\}\right) \\ &\qquad\qquad \cap \left(\bigcap_{w \neq 0} \operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C})) - \{\alpha : \det \alpha(w) = 0\}\right).\end{aligned}$$

Each subset in the intersection is a proper Zariski-open subspace of the representation variety. In particular, if nonempty, each is dense. So if each subset is nonempty, by the Baire category theorem, $\mathcal{I}_n$ is open and dense in $\operatorname{Hom}(\pi_1(M), \mathrm{GL}_n(\mathbb{C}))$.

The 'if' direction of Conjecture 1.4.6 follows if we assume some $\mathcal{I}_n$ is non-empty: restricting to these representations,

$$\det \left(\alpha\big(\partial_{x_i} i_*(a_j)\big)\right)_{i,j} = \det \alpha \left(\det_{\mathscr{D}} \big(\partial_{x_i} i_*(a_j)\big)_{i,j}\right),$$

which is nonzero. Moreover, knowing that $\mathcal{I}_n$ is non-empty, this argument would immediately prove the following.

Conjecture 4.3.3 (Conjecture 1.4.7)**.** *There exists a uniform bound n such that any taut*

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[1] No such representation exists for $n = 1$, as $\mathrm{GL}_1(\mathbb{C})$ is abelian.

sutured handlebody M is certified by a representation $\alpha : \pi_1(M) \to \mathrm{GL}_n(\mathbb{C})$.

Remark 4.3.4. It should be noted that $\mathcal{I}_n$ will always be empty, as $\mathrm{End}(\mathbb{C}^n)$ bears so-called *polynomial identities*, which preclude any representation from inducing an injection on $\mathbb{Z}[\pi_1(M)]$. So as written, this argument cannot be completed. However, one might instead consider the representations which induce injections on the subring generated by the set of Fox derivatives of elements of $\pi_1(M)$, and ask whether this set of representations is nonempty.

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