

THE UNIVERSITY OF CHICAGO

A TOY MODEL OF SHTUKAS

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# TABLE OF CONTENTS

ACKNOWLEDGMENTS . . . . .	vi
ABSTRACT . . . . .	vii
1 INTRODUCTION . . . . .	1
1.1 Background and motivation . . . . .	1
1.2 Approach . . . . .	2
1.3 Contents . . . . .	2
1.3.1 Drinfeld shtukas . . . . .	2
1.3.2 Toy shtukas . . . . .	2
1.3.3 Tate toy shtukas . . . . .	3
1.3.4 Relation between shtukas and toy shtukas . . . . .	4
1.4 Notation and conventions . . . . .	5
2 DRINFELD SHTUKAS . . . . .	7
2.1 Review of Drinfeld shtukas . . . . .	7
2.1.1 Definition of Drinfeld shtukas . . . . .	7
2.1.2 General constructions for shtukas . . . . .	8
2.1.3 The action of $GL_d(\mathbb{A})/k^\times$ on $\text{Sht}_{\text{all}}^d$ . . . . .	10
2.1.4 Partial Frobenius . . . . .	11
2.2 Reducible shtukas over a field . . . . .	11
2.2.1 Definitions . . . . .	11
2.2.2 Maximal trivial sub and maximal trivial quotient . . . . .	12
3 TOY SHTUKAS . . . . .	15
3.1 A toy model of shtukas . . . . .	15
3.1.1 Definition of toy shtukas . . . . .	15
3.2 The schemes of toy shtukas . . . . .	16
3.2.1 Definitions of the schemes of toy shtukas . . . . .	16
3.2.2 Explicit local description of the scheme of toy shtukas . . . . .	17
3.2.3 Basic properties of $\text{ToySht}$ . . . . .	19
3.2.4 The scheme of nontrivial toy shtukas . . . . .	20
3.2.5 Partial Frobeniuses . . . . .	21
3.2.6 Irreducibility of the scheme of toy shtukas . . . . .	22
3.3 Schubert divisors of the schemes of toy shtukas . . . . .	23
3.3.1 A lemma for toy shtukas . . . . .	23
3.3.2 Notation and conventions . . . . .	23
3.3.3 Definition of Schubert divisors . . . . .	24
3.3.4 Description of Schubert divisors . . . . .	25
3.3.5 An open subscheme of $\text{Schub}_V^W \cap \text{ToySht}_V^n$ . . . . .	26
3.3.6 Transversality . . . . .	26
3.4 Toy Horospherical Divisors . . . . .	27
3.4.1 Notation . . . . .	27

3.4.2	Short complexes related with principal toy horospherical divisors	28
3.4.3	Partial Frobeniuses and toy horospherical divisors . . . . .	29
3.4.4	Degree of partial Frobeniuses . . . . .	32
3.5	Recollections of linear algebra . . . . .	33
4	TATE TOY SHTUKAS . . . . .	36
4.1	Tate toy shtukas . . . . .	36
4.1.1	Tate linear algebra . . . . .	36
4.1.2	Sato Grassmannians . . . . .	38
4.1.3	Determinant of a family of d-lattices relative to a c-lattice . .	40
4.1.4	Definition of Tate toy shtukas . . . . .	41
4.2	Open charts of the scheme of Tate toy shtukas . . . . .	41
4.2.1	Notation . . . . .	41
4.2.2	Admissible pairs of c-lattices . . . . .	42
4.2.3	${}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$ as a projective limit . . . . .	43
4.2.4	Transition maps are affine . . . . .	43
4.2.5	Transition maps are smooth . . . . .	44
4.3	Functorial properties of toy horospherical divisors . . . . .	47
4.3.1	Notation . . . . .	47
4.3.2	Pullbacks of toy horospherical divisors under transition maps .	48
4.3.3	Functoriality for non-horospherical toy shtukas . . . . .	50
4.4	Tate toy horospherical subschemes . . . . .	53
4.4.1	Basic properties of Tate toy horospherical subschemes . . . . .	53
4.4.2	The group of divisors supported on the union of Tate toy horospherical subschemes . . . . .	56
4.5	Schubert divisors of ${}^{\circ}\text{ToySht}_T^n$ . . . . .	61
4.5.1	Schubert divisors of Sato Grassmannians . . . . .	61
4.5.2	Schubert divisors of the scheme of Tate toy shtukas . . . . .	62
4.6	Principal toy horospherical $\mathbb{Z}[\frac{1}{p}]$ -divisors of ${}^{\circ}\text{ToySht}_T^n$ . . . . .	63
4.6.1	Formulation of the main result . . . . .	63
4.6.2	Descent of principal divisors . . . . .	65
4.6.3	Support of extension of pullback of principal toy horospherical divisors . . . . .	65
4.6.4	Extension of pullback of principal toy horospherical divisors .	67
4.6.5	Proof of Theorem 4.6.2 . . . . .	68
4.7	Partial Frobeniuses for Tate toy shtukas . . . . .	70
4.7.1	Definition of left/right Tate toy shtukas . . . . .	71
4.7.2	Partial Frobeniuses . . . . .	71
4.7.3	Identification of $L\mathfrak{D}_T^n$ and $R\mathfrak{D}_T^n$ with $\mathfrak{D}_T^n$ . . . . .	71
4.7.4	Pullback of Tate toy horospherical subschemes under partial Frobeniuses . . . . .	74
4.8	A canonical subgroup of $\text{Pic}({}^{\circ}\text{ToySht}_T^n)$ . . . . .	75
4.8.1	Definition . . . . .	75
4.8.2	Some divisors in the classes of $\ell_{T,n,a}$ and $\ell_{T,n,b}$ . . . . .	77

4.8.3	The preimage of $\text{Pic}_{\text{can}}({}^{\circ}\text{ToySht}_T^n) \otimes \mathbb{Z}[\frac{1}{p}]$ in $\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$ . . .	78
5	RELATION BETWEEN SHTUKAS AND TOY SHTUKAS . . . . .	81
5.1	Relation between shtukas and toy shtukas . . . . .	81
5.1.1	Right shtukas . . . . .	81
5.1.2	Left shtukas . . . . .	83
5.1.3	Shtukas . . . . .	83
5.2	The morphism from the moduli scheme of shtukas with structures of all levels to the moduli scheme of Tate toy shtukas . . . . .	84
5.2.1	Construction of the morphism $\theta$ . . . . .	84
5.2.2	$GL_d(\mathbb{A})$ -equivariance of the morphism $\theta$ . . . . .	86
5.3	Review of horospherical divisors . . . . .	86
5.3.1	Trivial shtukas . . . . .	87
5.3.2	Definition of horospherical cycles . . . . .	89
5.3.3	Basic properties of horospherical cycles . . . . .	93
5.3.4	Irreducibility of the scheme of reducible shtukas . . . . .	97
5.3.5	Criteria for a Tate toy horospherical subscheme to contain the image of a horospherical divisor . . . . .	98
5.3.6	Irreducible components of horospherical divisors . . . . .	101
5.4	Pullback of Tate toy horospherical divisors under $\theta$ . . . . .	104
5.4.1	$(\theta_{\eta}^{d,0})^*$ as a direct sum . . . . .	104
5.4.2	Definition of the averaging maps . . . . .	106
5.4.3	Formula for pullback of horospherical divisors under $\theta$ . . . . .	108
5.4.4	A subspace of principal horospherical $\mathbb{Z}[\frac{1}{p}]$ -divisors . . . . .	108
5.4.5	Multiplicity one for pullback of toy horospherical divisors . . .	109
5.4.6	Description of $(\theta_{\eta}^{d,0,\Pi})^*$ . . . . .	112
	REFERENCES . . . . .	115

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# ABSTRACT

In this thesis, we introduce the notion of toy shtukas. The motivation comes from the question of constructing certain rational functions on the moduli stack of Drinfeld shtukas. These rational functions are somewhat similar to modular units, with their divisors supported on the horospherical locus. Analogously to horospherical divisors on the moduli stack of Drinfeld shtukas, there are toy horospherical divisors on the moduli scheme of toy shtukas. We describe the space of principal toy horospherical divisors. There is a canonical morphism from the moduli stack of Drinfeld shtukas to the moduli scheme of toy shtukas. The main result of this thesis is a description of the space of principal horospherical divisors obtained from the pullback.

# CHAPTER 1

## INTRODUCTION

### 1.1 Background and motivation

The notion of a shtuka (originally called an  $F$ -sheaf) was introduced by Drinfeld in [4, 6], and was used to prove the Langlands correspondence over function fields by Drinfeld in the case of  $GL_2$  and by L. Lafforgue in the case of  $GL_d$  for arbitrary  $d \geq 2$ . More recently, V. Lafforgue proved the automorphic-to-Galois direction of the Langlands correspondence for arbitrary reductive groups over function fields using a more general notion of shtukas.

In [5, Section 6], Drinfeld calculated the intersection pairing between horospherical divisors of the compactified moduli scheme of shtukas of rank 2 with all level structures, and showed that its kernel is equal to the space of horospherical divisors that are homologically equivalent to zero. Since  $H^1$  of the moduli scheme vanishes as shown in [7], this kernel equals the space of those horospherical divisors that are principal up to torsion. Its description can be thought of as a function field analogue of the Manin-Drinfeld theorem on cusps of modular curves. In [5, Section 6, Remark (2)], Drinfeld asked to explicitly construct the rational functions corresponding to those principal horospherical divisors.

This thesis is motivated by the above question by Drinfeld. We construct certain rational functions on the moduli stack of Drinfeld shtukas of rank  $d \geq 2$ . When  $d = 2$ , the construction answers the question of Drinfeld. When  $d > 2$ , we give a description of the linear relations among horospherical divisors generated by these rational functions, and we hope that they are all of them.



## 1.2 Approach

In order to obtain rational functions and principal horospherical divisors on the moduli stack of shtukas  $\text{Sht}$ , we introduce the notion of toy shtukas. Their moduli space  $\text{ToySht}$  is a closed subscheme of the Grassmannian. Analogously to horospherical divisors on  $\text{Sht}$ , we have toy horospherical divisors on  $\text{ToySht}$ . Using partial Frobeniuses, we are able to calculate the space of principal toy horospherical divisors. It is generated by principal Schubert divisors of  $\text{ToySht}$ , and one can find the corresponding rational functions explicitly. There is a natural morphism  $\theta$  from  $\text{Sht}$  to  $\text{ToySht}$ . Pullback along  $\theta$  gives rational functions and principal horospherical divisors on  $\text{Sht}$ .

## 1.3 Contents

### 1.3.1 Drinfeld shtukas

In the first half of Chapter 2, we give a review of Drinfeld shtukas. We recall the action of the adelic group. We also review the notion of partial Frobeniuses, which plays an important role in the theory of shtukas.

In the second part of Chapter 2, we study reducible shtukas over a field. When the zero and the pole satisfy condition (+) in Section 1.4, the structure of such a shtuka is described in Theorem 2.2.6. This result is crucial to the relation between horospherical divisors and toy horospherical divisors.

### 1.3.2 Toy shtukas

In Chapter 3, we start to study toy shtukas.

The definition of toy shtukas is given in Section 3.1. A toy shtuka over a field is defined as follows. Let  $V$  be a finite dimensional vector space over a finite field  $\mathbb{F}_q$ . Let

$\text{Grass}_V^n$  be the Grassmannian for subspaces of  $V$  of dimension  $n$ . We define a closed subscheme  $\text{ToySht}_V^n \subset \text{Grass}_V^n$  whose  $\overline{\mathbb{F}}_q$ -points consist of subspaces  $L \subset V \otimes \overline{\mathbb{F}}_q$  such that  $L \cap \text{Fr}^* L$  has codimension at most 1 in  $L$ . (See Definition 3.1.1 for the description of  $\text{ToySht}_V^n(S)$  for any scheme  $S$  over  $\mathbb{F}_q$ .) A point of  $\text{ToySht}_V^n$  is called a *toy shtuka* for  $V$  of dimension  $n$ . Analogously to Drinfeld shtukas, we have the notion of left and right toy shtukas, and there are partial Frobeniuses relating them. See Section 3.1 and Section 3.2.5 for more details.

In Section 3.2, we prove basic properties of  $\text{ToySht}_V^n$ .

Analogously to horospherical divisors on the stack of shtukas, we have toy horospherical divisors on the scheme of toy shtukas. They are studied in Section 3.4. We give more details here. Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$  with  $\dim_{\mathbb{F}_q} V \geq 3$ . Let  $1 \leq n \leq N - 1$ . For a one-dimensional subspace  $J \subset V$ , those toy shtukas for  $V$  of dimension  $n$  which contain  $J$  form a codimension 1 subscheme of  $\text{ToySht}_V^n$ . There is a similar statement for any hyperplane  $H \subset V$ . When restricted to the smooth locus of  $\text{ToySht}_V^n$ , these codimension 1 subschemes are called *toy horospherical divisors*. We give  $\mathbb{Z}[\frac{1}{p}]$ -linear relations between the classes of toy horospherical divisors in Theorem 3.4.6.

Restrictions of Schubert divisors of  $\text{Grass}_V^n$  give divisors on the nonsingular locus of  $\text{ToySht}_V^n$ . In Section 3.3, we show that these divisors are toy horospherical divisors, and we give an explicit description of them. These divisors are related to the Knudsen-Mumford divisors on the stack of Drinfeld shtukas.

### 1.3.3 Tate toy shtukas

When dealing with shtukas, to get the action of the adelic group, we consider shtukas with structures of all levels. Correspondingly, we need to study toy shtukas for Tate spaces, which is the content of Chapter 4.

In Section 4.1.1, we give a brief survey of Tate spaces. Eventually the Tate space

will be  $\mathbb{A}^d$ , where  $\mathbb{A}$  is the ring of adeles of some function field, and  $d$  is the rank of the shtukas.

Let  $T$  be a Tate space over  $\mathbb{F}_q$ . Roughly speaking, a Tate toy shtuka for  $T$  is a discrete lattice of  $T$  which is almost preserved by Frobenius. The moduli problem of Tate toy shtukas for  $T$  is representable by a scheme  $\text{ToySht}_T^n$ , which is a closed subschemes of the Sato Grassmannian  $\text{Grass}_T^n$ . Let  ${}^\circ\text{ToySht}_T^n$  be the nontrivial locus of  $\text{ToySht}_T^n$ , i.e., the locus where the discrete lattice is not preserved by Frobenius. When  $T$  is nondiscrete and noncompact,  ${}^\circ\text{ToySht}_T^n$  is not locally Noetherian. Therefore, one cannot hope to write a Cartier divisor of  ${}^\circ\text{ToySht}_T^n$  as a sum of irreducible ones.

In Theorem 4.4.12, we identify the (partially) ordered abelian group of Tate toy horospherical divisors of  ${}^\circ\text{ToySht}_T^n$  as a certain class of locally constant functions on the totally disconnected topological space  $(T - \{0\}) \coprod (T^* - \{0\})$ . Here we are seriously working with Cartier divisors of schemes which are not locally Noetherian. This seems to be a novel feature of our work.

In Theorem 4.6.2, we describe the (partially) ordered abelian group of those principal  $\mathbb{Z}[\frac{1}{p}]$ -divisors of  ${}^\circ\text{ToySht}_T^n$  supported on the union of Tate toy horospherical subschemes. The appearance of Fourier transform there gives a link to the intertwining operator for Eisenstein series, which plays an important role in the theory of horospherical divisors on the stack of Drinfeld shtukas.

### 1.3.4 *Relation between shtukas and toy shtukas*

In Chapter 5, we pullback principal toy horospherical divisors along the canonical morphism  $\text{Sht} \rightarrow \text{ToySht}$  to obtain principal horospherical divisors on the moduli scheme of shtukas equipped with all level structures.

We relate Drinfeld shtukas with toy shtukas in Section 5.1. In Propositions 5.1.1, 5.1.2 and 5.1.3, to a shtuka equipped with level structures satisfying certain vanishing

conditions on its cohomology, we functorially associate a toy shtuka. In Section 5.2, we obtain the canonical morphism  $\theta$  from the moduli scheme of shtukas with structures of all levels to the moduli scheme of Tate toy shtukas.

In Section 5.4, we calculate the pullback of a Tate toy horospherical divisor under the morphism  $\theta$ . The pullback turns out to be an averaging operator, as shown in Theorem 5.4.6. The main result of this thesis is Theorem 5.4.7, which gives a subspace of principal horospherical  $\mathbb{Z}[\frac{1}{p}]$ -divisors of the moduli scheme of shtukas with structures of all levels. We hope that this subspace actually equals the whole space.

## 1.4 Notation and conventions

The following notation and conventions will be used throughout the article.

Fix a prime number  $p$  and fix a finite field  $\mathbb{F}_q$  of characteristic  $p$  with  $q$  elements.

For any vector space  $V$  over  $\mathbb{F}_q$ , we denote  $\mathbf{P}_{V^*}$  to be the set of codimension 1 subspaces of  $V$ , and we denote  $\mathbf{P}_V$  to be the set of dimension 1 subspaces of  $V$ .

Let  $X$  be a smooth projective geometrically connected curve over  $\mathbb{F}_q$ . Let  $k$  be the field of rational functions on  $X$ . Let  $\mathbb{A}$  be the ring of adeles of  $k$ . Let  $\mathcal{O}$  be the subring of integral adeles in  $\mathbb{A}$ .

For a scheme  $S$  over  $\mathbb{F}_q$ , we denote  $\mathrm{Fr}_S$  to be its Frobenius endomorphism relative to  $\mathbb{F}_q$ . We denote  $\Phi_S = \mathrm{Id}_X \times \mathrm{Fr}_S : X \times S \rightarrow X \times S$ , and let  $\pi_S : X \times S \rightarrow S$  be the projection. We sometimes write  $\Phi$  and  $\pi$  instead of  $\Phi_S$  and  $\pi_S$  when there is no ambiguity about  $S$ .

For two schemes  $S_1$  and  $S_2$  over  $\mathbb{F}_q$ ,  $S_1 \times S_2$  denotes the product of  $S_1$  and  $S_2$  over  $\mathbb{F}_q$ , and by a morphism  $S_1 \rightarrow S_2$  we mean a morphism over  $\mathbb{F}_q$ .

For a scheme  $S$  over  $\mathbb{F}_q$  and two morphisms  $\alpha, \beta : S \rightarrow X$ , we say that they satisfy

condition  $(*)$  if

$$\Gamma_{\mathrm{Fr}_X^i \circ \alpha}, \Gamma_{\mathrm{Fr}_X^j \circ \beta}, (i, j \in \mathbb{Z}_{\geq 0}) \text{ are mutually disjoint subsets of } X \times S. \quad (*)$$

We say that they satisfy condition  $(+)$  if

$$\Gamma_{\mathrm{Fr}_X^i \circ \alpha} \cap \Gamma_{\mathrm{Fr}_X^j \circ \beta} = \emptyset \text{ for all } i, j \in \mathbb{Z}_{\geq 0}. \quad (+)$$

Condition  $(*)$  is equivalent to the combination of  $(+)$  and the condition that  $\alpha$  and  $\beta$  map  $S$  to the generic point of  $X$ .

For a morphism  $\alpha : S \rightarrow X$ , we sometimes write  $\alpha$  instead of its graph  $\Gamma_\alpha$ .

## CHAPTER 2

### DRINFELD SHTUKAS

#### 2.1 Review of Drinfeld shtukas

##### 2.1.1 Definition of Drinfeld shtukas

Let  $S$  be a scheme over  $\mathbb{F}_q$ . Recall that we denote  $\Phi = \text{Id}_X \times \text{Fr}_S : X \times S \rightarrow X \times S$ .

**Definition 2.1.1** (Drinfeld). A *left shtuka* of rank  $d$  over  $S$  is a diagram

$$\Phi^* \mathcal{F} \xleftarrow{j} \mathcal{F}' \xrightarrow{i} \mathcal{F}$$

of locally free sheaves of rank  $d$  on  $X \times S$ , where  $i$  and  $j$  are injective morphisms, the cokernel of  $i$  is an invertible sheaf on the graph  $\Gamma_\alpha$  of some morphism  $\alpha : S \rightarrow X$ , the cokernel of  $j$  is an invertible sheaf on the graph  $\Gamma_\beta$  of some morphism  $\beta : S \rightarrow X$ .

A *right shtuka* of rank  $d$  over  $S$  is a diagram

$$\Phi^* \mathcal{F} \xrightarrow{f} \mathcal{F}' \xleftarrow{g} \mathcal{F}$$

of locally free sheaves of rank  $d$  on  $X \times S$ , where  $f$  and  $g$  are injective morphisms, the cokernel of  $f$  is an invertible sheaf on the graph  $\Gamma_\alpha$  of some morphism  $\alpha : S \rightarrow X$ , the cokernel of  $g$  is an invertible sheaf on the graph  $\Gamma_\beta$  of some morphism  $\beta : S \rightarrow X$ .

We say that  $\alpha$  is the zero of the shtuka and  $\beta$  is the pole of the shtuka.

**Definition 2.1.2** (Drinfeld). For two morphisms  $\alpha, \beta : S \rightarrow X$ , a *shtuka* of rank  $d$  over  $S$  with zero  $\alpha$  and pole  $\beta$  is a locally free sheaf  $\mathcal{F}$  of rank  $d$  on  $X \times S$  equipped with an injective morphism  $\Phi^* \mathcal{F} \rightarrow \mathcal{F}(\Gamma_\beta)$  inducing an isomorphism  $\Phi^* \det \mathcal{F} \xrightarrow{\sim} (\det \mathcal{F})(\Gamma_\beta - \Gamma_\alpha)$ , such that the image of the composition  $\Phi^* \mathcal{F} \rightarrow \mathcal{F}(\Gamma_\beta) \rightarrow \mathcal{F}(\Gamma_\beta)/\mathcal{F}$  has rank at most 1 at  $\Gamma_\beta$ .

*Remark 2.1.3.* When the zero and the pole have disjoint graph, Construction A in Section 2.1.2 shows that there is no difference between a shtuka, a left shtuka and a right shtuka. This remains true after applying any powers of partial Frobeniuses (see Section 3.2.5) if we impose condition (+) on the zero and the pole.

From now on, when the zero and the pole satisfy condition (+), we do not distinguish left and right shtukas, and simply call them shtukas.

**Lemma 2.1.4.** *Let  $\mathcal{M}$  be a quasi-coherent sheaf on  $X$  and let  $\widetilde{\mathcal{M}}$  be the pullback of  $\mathcal{M}$  under the projection  $X \times S \rightarrow X$ . Then we have a canonical isomorphism  $\Phi^* \widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}$ .  $\square$*

**Definition 2.1.5** (Drinfeld). Let  $\mathcal{F}$  be a shtuka (resp. a left shtuka, resp. a right shtuka) of rank  $d$  over  $S$  with zero  $\alpha$  and pole  $\beta$ . Let  $D$  be a finite subscheme of  $X$  such that  $\alpha$  and  $\beta$  map to  $X - D$ . A structure of level  $D$  on  $\mathcal{F}$  is an isomorphism  $\iota : \mathcal{F} \otimes \mathcal{O}_{D \times S} \xrightarrow{\sim} \mathcal{O}_{D \times S}^d$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} \otimes \mathcal{O}_{D \times S} & \xrightarrow[\sim]{\iota} & \mathcal{O}_{D \times S}^d \\ \sim \uparrow & & \sim \uparrow \\ \Phi^* \mathcal{F} \otimes \mathcal{O}_{D \times S} & \xrightarrow[\sim]{\Phi^* \iota} & \Phi^* \mathcal{O}_{D \times S}^d \end{array}$$

### 2.1.2 General constructions for shtukas

We have the following constructions for shtukas, which induce morphisms between moduli stacks of shtukas.

#### Construction A:

(i) Let  $\Phi^* \mathcal{G} \xleftarrow{j} \mathcal{F} \xrightarrow{i} \mathcal{G}$  be a left shtuka with zero  $\alpha$  and pole  $\beta$  such that  $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ . We form the pushout diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{g} & \mathcal{H} \\ i \uparrow & & f \uparrow \\ \mathcal{F} & \xrightarrow{j} & \Phi^* \mathcal{G} \end{array}$$

Then  $\Phi^*\mathcal{G} \xrightarrow{f} \mathcal{H} \xleftarrow{g} \mathcal{G}$  is a right shtuka of the same rank, with the same zero and pole.

(ii) Let  $\Phi^*\mathcal{G} \xrightarrow{f} \mathcal{F} \xleftarrow{g} \mathcal{G}$  be a right shtuka with zero  $\alpha$  and pole  $\beta$  such that  $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ . We form the pullback diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{g} & \mathcal{F} \\ i \uparrow & & \uparrow f \\ \mathcal{H} & \xrightarrow{j} & \Phi^*\mathcal{G} \end{array}$$

Then  $\Phi^*\mathcal{G} \xleftarrow{j} \mathcal{H} \xrightarrow{i} \mathcal{G}$  is a left shtuka of the same rank, with the same zero and pole.

**Construction B:**

(i) For a left shtuka  $\Phi^*\mathcal{G} \xleftarrow{j} \mathcal{F} \xrightarrow{i} \mathcal{G}$  with zero  $\alpha$  and pole  $\beta$ , we can construct a right shtuka  $\Phi^*\mathcal{F} \xrightarrow{\Phi^*i} \Phi^*\mathcal{G} \xleftarrow{j} \mathcal{F}$  of the same rank, with zero  $\text{Fr}_X \circ \alpha$  and pole  $\beta$ .

(ii) For a right shtuka  $\Phi^*\mathcal{F} \xrightarrow{f} \mathcal{G} \xleftarrow{g} \mathcal{F}$  with zero  $\alpha$  and pole  $\beta$ , we can construct a left shtuka  $\Phi^*\mathcal{G} \xleftarrow{\Phi^*g} \Phi^*\mathcal{F} \xrightarrow{f} \mathcal{G}$  of the same rank, with zero  $\alpha$  and pole  $\text{Fr}_X \circ \beta$ .

**Construction C:** For a shtuka  $\mathcal{F}$  with zero  $\alpha$  and pole  $\beta$  satisfying  $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ , its dual  $\mathcal{F}^\vee$  is a shtuka of the same rank, with zero  $\beta$  and pole  $\alpha$ .

**Construction D:** Let  $\mathcal{F}$  be a shtuka over  $S$ . Let  $\mathcal{L}$  be an invertible sheaf on  $X$ , and let  $\widetilde{\mathcal{L}}$  be the pullback of  $\mathcal{L}$  under the projection  $X \times S \rightarrow X$ . then  $\mathcal{F} \otimes \widetilde{\mathcal{L}}$  is a shtuka of the same rank, with the same zero and pole.

**Construction D':** Suppose in Construction D the shtuka  $\mathcal{F}$  is equipped with a structure of level  $D$  and the invertible sheaf  $\mathcal{L}$  is trivialized at  $D$ . Then the shtuka  $\mathcal{F} \otimes \widetilde{\mathcal{L}}$  is naturally equipped with a structure of level  $D$ .

**Construction E:** Let  $\mathcal{F}$  be a shtuka of rank  $d$  equipped with a structure of level  $D$ . Suppose we are given an  $\mathcal{O}_D$ -submodule  $\mathcal{R} \subset \mathcal{O}_D^d$ . Let  $\mathcal{F}'$  be the kernel of the composition  $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times S}} \mathcal{O}_{D \times S} \xrightarrow{\sim} \mathcal{O}_{D \times S}^d \rightarrow \mathcal{O}_{D \times S}^d / (\mathcal{R} \boxtimes \mathcal{O}_S)$ . Then  $\mathcal{F}'$  is a shtuka of the same rank, with the same zero and pole.



**Construction E’:** Suppose in Construction E we are given a surjective morphism of  $\mathcal{O}_D$ -modules  $\mathcal{R} \rightarrow \mathcal{O}_{D'}^d$ , where  $D'$  is a subscheme of  $D$ . Then the composition  $\mathcal{F}' \rightarrow \mathcal{R} \boxtimes \mathcal{O}_S \rightarrow \mathcal{O}_{D' \times S}^d$  defines a structure of level  $D'$  on  $\mathcal{F}'$ .

### 2.1.3 The action of $GL_d(\mathbb{A})/k^\times$ on $\text{Sht}_{\text{all}}^d$

Let  $\text{Sht}_{\text{all}}^d$  be the moduli scheme of shtukas equipped with structures of all levels compatible with each other.

We recall the (left) action of  $GL_d(\mathbb{A})/k^\times$  on  $\text{Sht}_{\text{all}}^d$  defined in [6, Section 3].

We first define an action of the semigroup  $GL_d^-(\mathbb{A}) = \{g \in GL_d(\mathbb{A}) | g^{-1} \in \text{Mat}_{d \times d}(O)\}$ .

Let  $\mathcal{F} \in \text{Sht}_{\text{all}}^d(S)$  and let  $g \in GL_d^-(\mathbb{A})$ . We have  $g^{-1}O^d \subset O^d$ . Choose an open ideal  $I \subset O$  such that  $I \cdot O^d \subset g^{-1}O^d$ . Let  $D \subset X$  be the subscheme corresponding to  $I$ , and let  $\mathcal{R} \subset \mathcal{O}_D^d$  be the  $\mathcal{O}_D$ -submodule corresponding to the submodule  $g^{-1}O^d/I \cdot O^d \subset (O/I)^d$ . Applying Construction E in Section 2.1.2 to  $\mathcal{F}$  and  $\mathcal{R}$ , we obtain a right shtuka  $\mathcal{F}'$ , which does not depend on the choice of  $I$ .

Let  $D'$  be a finite subscheme of  $X$  and let  $J \subset O$  be the ideal corresponding to  $D'$ . In order to equip  $\mathcal{F}$  with a structure of level  $D'$ , we choose  $I$  such that  $I \cdot O^d \subset J \cdot h^{-1}O^d$ . Then the composition  $g^{-1}O^d/I \cdot O^d \rightarrow g^{-1}O^d/J \cdot g^{-1}O^d \xrightarrow{g} (O/J)^d$  is surjective. Applying Construction E’, we obtain a structure of level  $D'$  on  $\mathcal{F}'$ .

It is easy to see that the above construction induces an action of the semigroup  $GL_d^-(\mathbb{A})$  on  $\text{Sht}_{(\alpha, \beta), \text{all}}^d$ .

On the other hand, we have a canonical isomorphism  $\mathbb{A}^\times/k^\times = \varprojlim_D \text{Pic}_D$ . Construction D’ gives the action of  $\text{Pic}_D(X)$  on  $\text{Sht}_{\text{all}}^d$ . Passing to projective limit, we get an action of  $\mathbb{A}^\times/k^\times$  on  $\text{Sht}_{\text{all}}^d$ . One can check that this action on  $\mathbb{A}^\times \cap GL_d^-(\mathbb{A})$  coincides with the restriction of the action of  $GL_d^-(\mathbb{A})$ .

Since  $GL_d(\mathbb{A}) = \mathbb{A}^\times \cdot GL_d^-(\mathbb{A})$ , we obtain an action of  $GL_d(\mathbb{A})/k^\times$  on  $\text{Sht}_{\text{all}}^d$ .

**Lemma 2.1.6.** *The action of  $g \in GL_d(\mathbb{A})$  increases the Euler characteristic of a shtuka by  $\deg g$ .*  $\square$

### 2.1.4 Partial Frobenius

**Definition 2.1.7.** Let  $F_1$  be the construction of first applying A(ii) and then applying B(i). Let  $F_2$  be the construction of first applying B(ii) and then applying A(i). They are called partial Frobenius.

**Proposition 2.1.8.** *For a shtuka  $\mathcal{F}$  over  $S$  with zero and pole satisfying condition (+), we have natural isomorphisms  $F_1 F_2 \mathcal{F} \cong F_2 F_1 \mathcal{F} \cong \text{Fr}_S^* \mathcal{F}$ .*  $\square$

*Remark 2.1.9.* If a shtuka  $\mathcal{F}$  over  $S$  has zero  $\alpha$  and pole  $\beta$ , then  $F_1 \mathcal{F}$  has zero  $\alpha \circ \text{Fr}_S = \text{Fr}_X \circ \alpha$  and pole  $\beta$ , and  $F_2 \mathcal{F}$  has zero  $\alpha$  and pole  $\beta \circ \text{Fr}_S = \text{Fr}_X \circ \beta$ .

## 2.2 Reducible shtukas over a field

In this section, we fix a field  $E$  over  $\mathbb{F}_q$ . Recall that we denote  $\Phi = \text{Id}_X \otimes \text{Fr}_E : X \otimes E \rightarrow X \otimes E$ .

### 2.2.1 Definitions

**Definition 2.2.1.** A shtuka  $\mathcal{F}$  over  $\text{Spec } E$  of rank  $d$  with zero  $\alpha$  and pole  $\beta$  is said to be *reducible* if  $\mathcal{F}$  contains a nonzero subsheaf  $\mathcal{E}$  of rank  $< d$  such that the image of  $\Phi^* \mathcal{E}$  in  $\mathcal{F}(\beta)$  is contained in  $\mathcal{E}(\beta)$ .

**Lemma 2.2.2.** *With the same notation as in Definition 2.2.1, if  $\mathcal{G}$  is the saturation of  $\mathcal{E}$  in  $\mathcal{F}$ , then the image of  $\Phi^* \mathcal{G}$  in  $\mathcal{F}(\beta)$  is contained in  $\mathcal{G}(\beta)$ ,  $(\Phi^* \mathcal{G} + \mathcal{G})/\mathcal{G}$  is a torsion sheaf at  $\beta$  of length at most one, and  $(\Phi^* \mathcal{G} + \mathcal{G})/\Phi^* \mathcal{G}$  is a torsion sheaf at  $\alpha$  of length at most one.*

*Proof.* Let  $\eta = \operatorname{Spec} E'$  be the generic point of  $X \otimes E$ . Recall that  $k$  is the field of rational functions on  $X$ . The injective homomorphism  $\operatorname{Id}_k \otimes \operatorname{Fr}_E : k \otimes E \rightarrow k \otimes E$  induces a homomorphism  $\psi : E' \rightarrow E'$ . The morphism  $\Phi^* \mathcal{F} \hookrightarrow \mathcal{F}(\beta)$  induces a linear map of  $E'$ -vectors spaces  $\phi : \psi^* \mathcal{F}_\eta \rightarrow \mathcal{F}_\eta$ . The condition  $\Phi^* \mathcal{E} \subset \mathcal{E}(\beta)$  implies that  $\phi(\psi^* \mathcal{E}_\eta) = \mathcal{E}_\eta$ . Note that  $\mathcal{F}_\eta = (\Phi^* \mathcal{F})_\eta$ . Thus the two subbundles  $\mathcal{G} \subset \mathcal{F}$  and  $\Phi^* \mathcal{G} \subset \Phi^* \mathcal{F}$  correspond to the same  $E'$ -subspace  $\mathcal{E}_\eta \subset \mathcal{F}_\eta$ . Since  $\mathcal{G}(\beta)$  is saturated in  $\mathcal{F}(\beta)$ , we have  $\Phi^* \mathcal{G} \subset \mathcal{G}(\beta)$ . The definition of a shtuka implies that  $(\Phi^* \mathcal{F} + \mathcal{F})/\mathcal{F}$  is a torsion sheaf at  $\beta$  of length at most one, hence is  $(\Phi^* \mathcal{G} + \mathcal{G})/\mathcal{G}$ . The proof for  $(\Phi^* \mathcal{G} + \mathcal{G})/\Phi^* \mathcal{G}$  is similar.  $\square$

*Remark 2.2.3.* Let the notation be the same as in Definition 2.2.1. Let  $\mathcal{G}$  be the saturation of  $\mathcal{E}$  in  $\mathcal{F}$ . We have an exact sequence of locally free sheaves on  $X \otimes E$

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0.$$

Assume  $\alpha \neq \beta$ . From Lemma 2.2.2 we see that one (and only one) of the following two possibilities holds.

- (1)  $\mathcal{G}$  is a shtuka with zero  $\alpha$  and pole  $\beta$ , and the morphism  $\Phi^* \mathcal{F} \hookrightarrow \mathcal{F}(\beta)$  induces an isomorphism  $\Phi^* \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ .
- (2)  $\mathcal{H}$  is a shtuka with zero  $\alpha$  and pole  $\beta$ , and the morphism  $\Phi^* \mathcal{F} \hookrightarrow \mathcal{F}(\beta)$  induces an isomorphism  $\Phi^* \mathcal{G} \xrightarrow{\sim} \mathcal{G}$ .

### 2.2.2 Maximal trivial sub and maximal trivial quotient

**Lemma 2.2.4.** *Let  $\alpha, \beta : \operatorname{Spec} E \rightarrow X$  be two morphisms. If an effective divisor  $D$  of  $X \otimes E$  satisfies  $\Phi^* D + \alpha = D + \beta$ , then  $\beta = \operatorname{Fr}_X^n \circ \alpha$  for some  $n \geq 0$ .*

*Proof.* We prove by induction on the degree of  $D$ . When  $D = 0$ , we get  $\alpha = \beta$ . When  $D > 0$  and  $\alpha \neq \beta$ , set  $D' = D - \alpha$ . Then  $D'$  is effective,  $\deg D' < \deg D$ , and

$\Phi^*D' - D' = \beta - \text{Fr}_X \circ \alpha$ . The induction hypothesis shows that  $\beta = \text{Fr}_X^m \circ \text{Fr}_X \circ \alpha = \text{Fr}_X^{m+1} \circ \alpha$  for some  $m \geq 0$ .  $\square$

**Proposition 2.2.5.** *Let  $\mathcal{G}$  be a shtuka over  $\text{Spec } E$  with zero  $\alpha$  and pole  $\beta$ . Assume that there exists a subsheaf  $\mathcal{E} \subset \mathcal{G}$  satisfying*

- (i)  $\text{rank } \mathcal{E} = \text{rank } \mathcal{G}$ ;
- (ii) *the image of  $\Phi^*\mathcal{E}$  in  $\mathcal{G}(\beta)$  is contained in  $\mathcal{E}$ .*

*Then  $\beta = \text{Fr}_X^n \circ \alpha$  for some  $n \geq 0$ .*

*Proof.* The  $d$ -th exterior power of a shtuka of rank  $d$  is a shtuka of rank 1 with the same zero and pole. So we can apply the  $d$ -th exterior power to all sheaves involved to reduce the problem to the case  $d = 1$ .

Now we have  $\mathcal{G} = \mathcal{E}(D)$ , where  $D$  is an effective divisor of  $X \otimes E$ . We see that  $\Phi^*\mathcal{G} = (\Phi^*\mathcal{E})(\Phi^*D)$ . Since  $\chi(\Phi^*\mathcal{E}) = \chi(\mathcal{E})$ , the assumption  $f(\Phi^*\mathcal{E}) \subset g(\mathcal{E})$  implies  $f(\Phi^*\mathcal{E}) = g(\mathcal{E})$ . Hence  $\Phi^*\mathcal{G} \cong \mathcal{E}(\Phi^*D)$ . Since the shtuka has rank 1, we have  $\Phi^*\mathcal{G} \cong \mathcal{G}(\beta - \alpha) = \mathcal{E}(D + \beta - \alpha)$ . Thus we get  $\Phi^*D = D + \beta - \alpha$ . The statement now follows from Lemma 2.2.4.  $\square$

**Theorem 2.2.6.** *Let  $\mathcal{F}$  be a shtuka over  $\text{Spec } E$  with zero  $\alpha$  and pole  $\beta$  satisfying condition (+). Let  $S_1$  (resp.  $S_2$ ) be the poset of all subsheaves  $\mathcal{E} \subset \mathcal{F}$  satisfying the following condition (1) (resp. (2)).*

(1) *The image of  $\Phi^*\mathcal{E}$  in  $\mathcal{F}(\beta)$  is contained in  $\mathcal{E}(\beta)$ , the sheaf  $\mathcal{F}/\mathcal{E}$  is locally free, and the morphism  $\Phi^*(\mathcal{F}/\mathcal{E}) \rightarrow (\mathcal{F}/\mathcal{E})(\beta)$  induces an isomorphism  $\Phi^*(\mathcal{F}/\mathcal{E}) \xrightarrow{\sim} (\mathcal{F}/\mathcal{E})$ .*

(2) *The image of  $\Phi^*\mathcal{E}$  in  $\mathcal{F}(\beta)$  is  $\mathcal{E}$ .*

*Then the poset  $S_1$  has a least element, denoted by  $\mathcal{F}^{\text{I}}$ . The poset  $S_2$  has a greatest element, denoted by  $\mathcal{F}^{\text{II}}$ , and  $\mathcal{F}/\mathcal{F}^{\text{II}}$  is locally free.*

*Proof.* The set  $S_2$  is nonempty since it contains the zero subsheaf. Let  $\mathcal{F}^{\text{II}} = \sum_{\mathcal{E} \in S_2} \mathcal{E}$ . We see that the image of  $\Phi^*\mathcal{F}^{\text{II}}$  in  $\mathcal{F}(\beta)$  is contained in  $\mathcal{F}^{\text{II}}$ . Since

$\chi(\Phi^* \mathcal{F}^{\text{II}}) = \chi(\mathcal{F}^{\text{II}})$ , the image of  $\Phi^* \mathcal{F}^{\text{II}}$  in  $\mathcal{F}(\beta)$  is equal to  $\mathcal{F}^{\text{II}}$ . Thus  $\mathcal{F}^{\text{II}}$  is the greatest element of  $S_2$ .

Suppose that  $\mathcal{F}/\mathcal{F}^{\text{II}}$  is not locally free. Let  $\mathcal{G}$  be the saturation of  $\mathcal{F}^{\text{II}}$  in  $\mathcal{F}$ . We have  $\mathcal{F}^{\text{II}} \subsetneq \mathcal{G}$ . Hence the image of  $\Phi^* \mathcal{G}$  in  $\mathcal{F}(\beta)$  is not contained in  $\mathcal{G}$ . Now Remark 2.2.3 implies that  $\mathcal{G}$  is a shtuka with zero  $\alpha$  and pole  $\beta$ . Applying Proposition 2.2.5 to  $\mathcal{G}$  and its subsheaf  $\mathcal{F}^{\text{II}}$ , we deduce that  $\beta = \text{Fr}_X^n \circ \alpha$  for some  $n \geq 0$ , contradicting condition (+).

Let  $\mathcal{F}^\vee$  be the dual of the given shtuka  $\mathcal{F}$ . It has zero  $\beta$  and pole  $\alpha$  which also satisfy condition (+). Therefore, there is a unique maximal subsheaf  $(\mathcal{F}^\vee)^{\text{II}} \subset \mathcal{F}^\vee$  satisfying condition (2), and  $\mathcal{F}^\vee/(\mathcal{F}^\vee)^{\text{II}}$  is locally free. Then  $\mathcal{F}^{\text{I}} = (\mathcal{F}^\vee/(\mathcal{F}^\vee)^{\text{II}})^\vee$  is a least element of  $S_1$ .  $\square$

*Remark 2.2.7.* Suppose  $\mathcal{F}$  is a right shtuka over  $\text{Spec } E$  with zero and pole satisfying condition (+). Then  $\mathcal{F}$  is irreducible if and only if  $\mathcal{F}^{\text{I}} = \mathcal{F}$  and  $\mathcal{F}^{\text{II}} = 0$ .

**Corollary 2.2.8.** *Let  $\mathcal{F}, \mathcal{F}^{\text{I}}, \mathcal{F}^{\text{II}}$  be as in Theorem 2.2.6. Let  $\mathcal{M}$  be an invertible sheaf on  $X$  and let  $\mathcal{G}$  be the shtuka  $\mathcal{F} \otimes (\mathcal{M} \otimes E)$  obtained by Construction E in Section 2.1.2. Then  $\mathcal{G}^{\text{I}} = \mathcal{F}^{\text{I}} \otimes (\mathcal{M} \otimes E)$  and  $\mathcal{G}^{\text{II}} = \mathcal{F}^{\text{II}} \otimes (\mathcal{M} \otimes E)$ .  $\square$*

## CHAPTER 3

### TOY SHTUKAS

#### 3.1 A toy model of shtukas

Fix a vector space  $V$  over  $\mathbb{F}_q$  with  $\dim_{\mathbb{F}_q} V = N < \infty$ . Let  $S$  be a scheme over  $\mathbb{F}_q$ . For an  $\mathcal{O}_S$ -module  $\mathcal{F}$  and a point  $s \in S$ , we denote  $\mathcal{F}_s$  to be the pullback of  $\mathcal{F}$  to  $s$ .

##### 3.1.1 Definition of toy shtukas

**Definition 3.1.1.** A *toy shtuka* for  $V$  over  $S$  is an  $\mathcal{O}_S$ -submodule  $\mathcal{L} \subset V \otimes \mathcal{O}_S$  such that  $(V \otimes \mathcal{O}_S)/\mathcal{L}$  is locally free and the composition

$$\mathrm{Fr}_S^* \mathcal{L} \hookrightarrow (\mathrm{Fr}_S^* V) \otimes \mathcal{O}_S = V \otimes \mathcal{O}_S \twoheadrightarrow (V \otimes \mathcal{O}_S)/\mathcal{L}$$

has rank at most 1. (In other words, the corresponding morphism  $\bigwedge^2 \mathrm{Fr}_S^* \mathcal{L} \rightarrow \bigwedge^2 ((V \otimes \mathcal{O}_S)/\mathcal{L})$  is zero.)

**Definition 3.1.2.** A *right toy shtuka* for  $V$  over  $S$  is a pair of  $\mathcal{O}_S$ -submodules  $\mathcal{L}, \mathcal{L}' \subset V \otimes \mathcal{O}_S$  such that  $(V \otimes \mathcal{O}_S)/\mathcal{L}$  and  $(V \otimes \mathcal{O}_S)/\mathcal{L}'$  are locally free,  $\mathcal{L} \subset \mathcal{L}'$ ,  $\mathrm{Fr}_S^* \mathcal{L} \subset \mathcal{L}'$ , and  $\dim \mathcal{L}'_s - \dim \mathcal{L}_s = 1$  for every  $s \in S$ .

*Remark 3.1.3.* In the exact sequence

$$0 \longrightarrow \mathcal{L}'/\mathcal{L} \longrightarrow (V \otimes \mathcal{O}_S)/\mathcal{L} \longrightarrow (V \otimes \mathcal{O}_S)/\mathcal{L}' \longrightarrow 0,$$

both  $(V \otimes \mathcal{O}_S)/\mathcal{L}$  and  $(V \otimes \mathcal{O}_S)/\mathcal{L}'$  are locally free. So  $\mathcal{L}'/\mathcal{L}$  is also locally free. Hence the condition  $\dim \mathcal{L}'_s - \dim \mathcal{L}_s = 1$  for every  $s \in S$  implies that  $\mathcal{L}'/\mathcal{L}$  is invertible. Similarly  $\mathcal{L}'/\mathrm{Fr}_S^* \mathcal{L}$  is invertible.

**Definition 3.1.4.** A *left toy shtuka* for  $V$  over  $S$  is a pair of  $\mathcal{O}_S$ -submodules  $\mathcal{L}, \mathcal{L}' \subset V \otimes \mathcal{O}_S$  such that  $(V \otimes \mathcal{O}_S)/\mathcal{L}$  and  $(V \otimes \mathcal{O}_S)/\mathcal{L}'$  are locally free,  $\mathcal{L}' \subset \mathcal{L}$ ,  $\mathcal{L}' \subset \mathrm{Fr}_S^* \mathcal{L}$ , and  $\dim \mathcal{L}_s - \dim \mathcal{L}'_s = 1$  for every  $s \in S$ .

*Remark 3.1.5.* Similarly to the above remark,  $\mathcal{L}/\mathcal{L}'$  and  $(\mathrm{Fr}_S^* \mathcal{L})/\mathcal{L}'$  are invertible.

## 3.2 The schemes of toy shtukas

In this section, schemes (e.g. Grassmannians, the schemes of matrices) are defined over  $\mathbb{F}_q$ .

Fix an vector space  $V$  over  $\mathbb{F}_q$  with  $\dim_{\mathbb{F}_q} V = N < \infty$ .

For  $0 \leq n \leq N$ , let  $\mathrm{Grass}_V^n$  denote the Grassmannian of  $n$ -dimensional subspaces of  $V$ .

### 3.2.1 Definitions of the schemes of toy shtukas

**Definition 3.2.1.** Let  $\mathrm{ToySht}_V^n$  (resp.  $\mathrm{LToySht}_V^n$ , resp.  $\mathrm{RToySht}_V^n$ ) be the functor which associates to each  $\mathbb{F}_q$ -scheme  $S$  the set of isomorphism classes of toy shtukas  $\mathcal{L} \subset V \otimes \mathcal{O}_S$  (resp. left toy shtukas  $\mathcal{L}' \subset \mathcal{L} \subset V \otimes \mathcal{O}_S$ , resp. right toy shtukas  $\mathcal{L} \subset \mathcal{L}' \subset V \otimes \mathcal{O}_S$ ) such that  $\mathrm{rank} \mathcal{L} = n$ .

*Remark 3.2.2.* From the definition of toy shtukas, we see that the functor  $\mathrm{ToySht}_V^n$  (resp.  $\mathrm{LToySht}_V^n$ , resp.  $\mathrm{RToySht}_V^n$ ) is representable by a subschemes of  $\mathrm{Grass}_V^n$  (resp.  $\mathrm{Grass}_V^{n-1} \times \mathrm{Grass}_V^n$ , resp.  $\mathrm{Grass}_V^n \times \mathrm{Grass}_V^{n+1}$ ). See Section 3.2.2 for explicit local description of these schemes.

*Remark 3.2.3.* There is a natural morphism  $\mathrm{LToySht}_V^n \rightarrow \mathrm{ToySht}_V^n$  which maps a left toy shtuka  $\mathcal{L}' \subset \mathcal{L}$  to  $\mathcal{L}$ , and there is a natural morphism  $\mathrm{RToySht}_V^n \rightarrow \mathrm{ToySht}_V^n$  which maps a right toy shtuka  $\mathcal{L} \subset \mathcal{L}'$  to  $\mathcal{L}$ .

### 3.2.2 Explicit local description of the scheme of toy shtukas

#### Affine open charts of Grassmannians

For a finite dimensional vector space  $M$  over  $\mathbb{F}_q$ , we denote  $\underline{M} = \text{Spec Sym } M^*$ . If  $M = \text{Hom}(W', W)$ , we write  $\underline{\text{Hom}}(W', W)$  instead of  $\underline{\text{Hom}}(W', W)$ .

Fix an  $(N - n)$ -dimensional subspace  $W$  of  $V$ . Denote  $U_W$  to be the open subscheme of  $\text{Grass}_V^n$  parameterizing those  $n$ -dimensional subspaces of  $V$  which are transversal to  $W$ . Fix an  $n$ -dimensional subspace  $W'$  of  $V$  such that  $V = W \oplus W'$ . Then we have an identification  $U_W = \underline{\text{Hom}}(W', W)$ . For an  $\mathbb{F}_q$ -algebra  $R$ , an  $R$ -point of  $U_W$  is the graph of an  $R$ -linear map  $W' \otimes R \rightarrow W \otimes R$ .

Define the Artin-Schreier morphism  $AS_{W', W} : \underline{\text{Hom}}(W', W) \rightarrow \underline{\text{Hom}}(W', W)$  by  $AS_{W', W} = \text{Id} - \text{Fr}$ . We know that  $AS_{W', W}$  is surjective finite étale of degree  $q^{n(N-n)}$ .

Define a closed subscheme  $U_W^{\leq 1} := \underline{\text{Hom}}(W', W)^{\text{rank} \leq 1} \subset \underline{\text{Hom}}(W', W) = U_W$  whose  $R$ -points are  $R$ -linear maps  $A : W' \otimes R \rightarrow W \otimes R$  such that  $\bigwedge^2 A = 0$ .

#### Explicit local description of ToySht

We consider  $\text{ToySht}_V^n$  as a subscheme of  $\text{Grass}_V^n$ .

**Lemma 3.2.4.** *When  $1 \leq n \leq N - 1$ ,  $\text{ToySht}_V^n \cap U_W$  is the inverse image of  $U_W^{\leq 1}$  under  $AS_{W', W}$ . In particular  $\text{ToySht}_V^n$  is a closed subscheme of  $\text{Grass}_V^n$ .*

*Proof.* The statement follows from the definition of toy shtukas.  $\square$

#### Explicit local description of LToySht

For  $0 \leq i \leq j \leq N$ , let  $\text{Flag}_V^{i,j}$  be the closed subscheme of  $\text{Grass}_V^i \times \text{Grass}_V^j$  which consists of pairs  $(M_1, M_2)$  such that  $M_1 \subset M_2$ .

We consider  $\text{LToySht}_V^n$  as a subscheme of  $\text{Flag}_V^{n-1, n}$ .



Let  $R$  be an  $\mathbb{F}_q$ -algebra. Let  $A : W' \otimes R \rightarrow W \otimes R$  be an  $R$ -linear morphism and  $\Gamma_A$  be its graph. Projection from  $\Gamma_A$  to  $W' \otimes R$  induces a bijective correspondence between those submodules of  $\Gamma_A$  whose quotient is locally free and those submodules of  $W' \otimes R$  whose quotient is locally free. This bijection is functorial in  $R$ . In this way we get an identification  $\text{Flag}_V^{n-1,n} \cap (\text{Grass}_V^{n-1} \times U_W) = \text{Grass}_{W'}^{n-1} \times U_W$ .

Define a closed subscheme  $C_{W,W'}^b \subset \text{Grass}_{W'}^{n-1} \times U_W = \text{Grass}_{W'}^{n-1} \times \underline{\text{Hom}}(W', W)$  consisting of pairs  $(H, A)$  such that  $H \subset \ker A$ .

**Lemma 3.2.5.** *When  $1 \leq n \leq N - 1$ ,  $\text{LToySht}_V^n \cap (\text{Grass}_V^{n-1} \times U_W)$  is the inverse image of  $C_{W,W'}^b$  under  $\text{Id}_{\text{Grass}_{W'}^{n-1}} \times AS_{W',W}$ . In particular,  $\text{LToySht}_V^n$  is a closed subscheme of  $\text{Flag}_V^{n-1,n}$ .*

*Proof.* The statement follows from the definition of left toy shtukas.  $\square$

**Lemma 3.2.6.** *When  $1 \leq n \leq N - 1$ ,  $\text{LToySht}_V^n$  is smooth of pure dimension  $N - 1$  over  $\mathbb{F}_q$ .*

*Proof.* We observe that  $C_{W,W'}^b$  is a  $(N - n)$ -dimensional vector bundle over  $\text{Grass}_{W'}^{n-1}$ . Hence it is smooth of pure dimension  $N - 1$ . Since  $\text{Id}_{\text{Grass}_{W'}^{n-1}} \times AS_{W',W}$  is étale, the statement follows from Lemma 3.2.5.  $\square$

## Explicit local description of $\text{RToySht}$

We consider  $\text{RToySht}_V^n$  as a subscheme of  $\text{Flag}_V^{n,n+1}$ .

Let  $R$  be an  $\mathbb{F}_q$ -algebra. Let  $A : W' \otimes R \rightarrow W \otimes R$  be an  $R$ -linear morphism and  $\Gamma_A$  be its graph. Intersection with  $W \otimes R$  induces a bijective correspondence between those submodules  $M \subset V \otimes R$  containing  $\Gamma_A$  such that  $(V \otimes R)/M$  is locally free and those submodules of  $W \otimes R$  whose quotient is locally free. This bijection is functorial in  $R$ . In this way we get an identification  $\text{Flag}_V^{n,n+1} \cap (U_W \times \text{Grass}_V^{n+1}) = U_W \times \text{Grass}_W^1$ .

Define a closed subscheme  $C_{W,W'}^\sharp \subset U_W \times \text{Grass}_W^1 = \underline{\text{Hom}}(W', W) \times \text{Grass}_W^1$  consisting of pairs  $(A, H)$  such that  $\text{im } A \subset H$ .

**Lemma 3.2.7.** *When  $1 \leq n \leq N - 1$ ,  $\text{RToySht}_V^n \cap (U_W \times \text{Grass}_V^{n+1})$  is the inverse image of  $C_{W,W'}^\#$  under  $AS_{W',W} \times \text{Id}_{\text{Grass}_W^1}$ . In particular,  $\text{RToySht}_V^n$  is a closed subscheme of  $\text{Flag}_V^{n,n+1}$ .*

*Proof.* The statement follows from the definition of right toy shtukas.  $\square$

**Lemma 3.2.8.** *When  $1 \leq n \leq N - 1$ ,  $\text{RToySht}_V^n$  is smooth of pure dimension  $N - 1$  over  $\mathbb{F}_q$ .*

*Proof.* We observe that  $C_{W,W'}^\#$  is an  $n$ -dimensional vector bundle over  $\text{Grass}_W^1$ . Hence it is smooth of pure dimension  $N - 1$ . Since  $AS_{W',W} \times \text{Id}_{\text{Grass}_W^1}$  is étale, the statement follows from Lemma 3.2.7.  $\square$

### 3.2.3 Basic properties of ToySht

Determinantal varieties are Cohen-Macaulay [11]. See [1, Chapter 2, Section 3] for a review of basic properties of determinantal varieties.

We know that for  $1 \leq n \leq N - 1$ ,  $\text{Mat}_{n \times (N-n)}^{\text{rank} \leq 1}$  is the affine cone over  $(\mathbb{P}^\vee)^{N-n-1} \times \mathbb{P}^{n-1}$ . In particular, it has pure dimension  $N - 1$  and it is reduced.

**Lemma 3.2.9.** *For  $1 \leq n \leq N - 1$ ,  $\text{ToySht}_V^n$  is reduced and has pure dimension  $N - 1$ . We have  $\text{ToySht}_V^1 = \text{Grass}_V^1$ ,  $\text{ToySht}_V^{N-1} = \text{Grass}_V^{N-1}$ . For  $n \in \{1, N - 1\}$ ,  $\text{ToySht}_V^n$  is smooth. For  $2 \leq n \leq N - 2$ ,  $\text{ToySht}_V^n$  is smooth outside  $\text{Grass}_V^n(\mathbb{F}_q)$  and the singularity at each point of  $\text{Grass}_V^n(\mathbb{F}_q)$  is étale locally the vertex of the cone over  $(\mathbb{P}^\vee)^{N-n-1} \times \mathbb{P}^{n-1}$ .*

*Proof.* Choose bases of  $W$  and  $W'$  to identify  $\underline{\text{Hom}}(W', W)^{\text{rank} \leq 1}$  with  $\text{Mat}_{n \times (N-n)}^{\text{rank} \leq 1}$ . Since the Artin-Schreier morphism is finite étale, the statement follows from the corresponding properties of  $\text{Mat}_{n \times (N-n)}^{\text{rank} \leq 1}$ .  $\square$

### 3.2.4 The scheme of nontrivial toy shtukas

**Definition 3.2.10.** For a toy shtuka (resp. a right toy shtuka, resp. a left toy shtuka)  $\mathcal{L}$  over  $S$ , we say that it is *nontrivial at  $s \in S$*  if  $\mathrm{Fr}_s^* \mathcal{L}_s \neq \mathcal{L}_s$ , where  $\mathcal{L}_s$  is the pullback of  $\mathcal{L}$  to  $s$ .

*Remark 3.2.11.* A left/right toy shtuka is nontrivial at  $s$  if and only if it is as a toy shtuka.

*Remark 3.2.12.* A pointwise nontrivial left/right toy shtuka is the same as a pointwise nontrivial toy shtuka. See Corollary 3.2.17 for a more precise statement.

*Remark 3.2.13.* We know that the trivial locus of  $\mathrm{ToySht}_V^n$  is equal to the Frobenius fixed points, or equivalently it is the intersection of the graph of Frobenius morphism and the diagonal. So the trivial locus of  $\mathrm{ToySht}_V^n$  is  $\mathrm{Grass}_V^n(\mathbb{F}_q)$ , a reduced 0-dimensional closed subscheme of  $\mathrm{Grass}_V^n$ .

**Definition 3.2.14.** Let  ${}^\circ\mathrm{ToySht}_V^n$  (resp.  ${}^\circ\mathrm{LToySht}_V^n$ , resp.  ${}^\circ\mathrm{RToySht}_V^n$ ) be the nontrivial locus of  $\mathrm{ToySht}_V^n$  (resp.  $\mathrm{RToySht}_V^n$ , resp.  $\mathrm{LToySht}_V^n$ ).

*Remark 3.2.15.* Since nontrivialness is an open condition for toy shtukas (resp. left toy shtukas, resp. right toy shtukas),  ${}^\circ\mathrm{ToySht}_V^n$  (resp.  ${}^\circ\mathrm{LToySht}_V^n$ , resp.  ${}^\circ\mathrm{RToySht}_V^n$ ) is an open subscheme of  $\mathrm{ToySht}_V^n$  (resp.  $\mathrm{RToySht}_V^n$ , resp.  $\mathrm{LToySht}_V^n$ ). We have  ${}^\circ\mathrm{ToySht}_V^n = \mathrm{ToySht}_V^n - \mathrm{Grass}_V^n(\mathbb{F}_q)$  from Remark 3.2.13. In particular, we know that  ${}^\circ\mathrm{ToySht}_V^n$  is smooth over  $\mathbb{F}_q$ . We also see that  ${}^\circ\mathrm{ToySht}_V^n$  is nonempty if and only if  $1 \leq n \leq N - 1$ .

**Lemma 3.2.16.** *Let  $\mathcal{L}$  be a toy shtuka over  $S$  which is nontrivial at each  $s \in S$ . Then  $\mathcal{L} + \mathrm{Fr}_S^* \mathcal{L}$  and  $\mathcal{L} \cap \mathrm{Fr}_S^* \mathcal{L}$  are subbundles of  $\mathfrak{p}_S^* V$ . Also,  $(\mathcal{L} + \mathrm{Fr}_S^* \mathcal{L})/\mathcal{L}$  and  $\mathcal{L}/(\mathcal{L} \cap \mathrm{Fr}_S^* \mathcal{L})$  are invertible sheaves.*

*Proof.* By the definition of toy shtukas, the morphism  $\mathrm{Fr}_S^* \mathcal{L} \rightarrow \mathfrak{p}_S^* V/\mathcal{L}$  has rank at most 1. It has strictly constant rank 1 since  $\mathcal{L}$  is nontrivial at each  $s \in S$ . The statements now follow from Corollary 3.5.5.  $\square$

**Corollary 3.2.17.** *For  $1 \leq n \leq N$ , the natural morphisms  $\text{LToySht}_V^n \rightarrow \text{ToySht}_V^n$  and  $\text{RTToySht}_V^n \rightarrow \text{ToySht}_V^n$  in Remark 3.2.3 induce isomorphisms  ${}^\circ\text{LToySht}_V^n \rightarrow {}^\circ\text{ToySht}_V^n$  and  ${}^\circ\text{RTToySht}_V^n \rightarrow {}^\circ\text{ToySht}_V^n$ .*

### 3.2.5 Partial Frobeniuses

We have the following constructions for left/right toy shtukas:

- (i) For a left toy shtuka  $\mathcal{L}' \subset \mathcal{L}$  over an  $\mathbb{F}_q$ -scheme  $S$ , the pair  $\mathcal{L}' \subset \text{Fr}_S^* \mathcal{L}$  forms a right toy shtuka over  $S$ .
- (ii) For a right toy shtuka  $\mathcal{L} \subset \mathcal{L}'$  over an  $\mathbb{F}_q$ -scheme  $S$ , the pair  $\text{Fr}_S^* \mathcal{L} \subset \mathcal{L}'$  forms a left toy shtuka over  $S$ .

**Definition 3.2.18.** We define partial Frobeniuses  $F_{V,n}^- : \text{LToySht}_V^n \rightarrow \text{RTToySht}_V^{n-1}$ , ( $1 \leq n \leq N$ ) and  $F_{V,n}^+ : \text{RTToySht}_V^n \rightarrow \text{LToySht}_V^{n+1}$ , ( $0 \leq n \leq N-1$ ) induced by the above constructions.

**Lemma 3.2.19.** *Let  $f_1 : Y_1 \rightarrow Y_2, f_2 : Y_2 \rightarrow Y_3$  be two morphisms of schemes. If  $f_1$  is surjective and  $f_2 \circ f_1$  is a universal homeomorphism, then  $f_2$  is also a universal homeomorphism.*

*Proof.* This is [9, Proposition 3.8.2(iv)]. □

**Lemma 3.2.20.** *For two morphisms  $f_1 : Y_1 \rightarrow Y_2, f_2 : Y_2 \rightarrow Y_1$  between two schemes  $Y_1$  and  $Y_2$ , if  $f_2 \circ f_1$  and  $f_1 \circ f_2$  are universal homeomorphisms, then  $f_1$  and  $f_2$  are universal homeomorphisms.*

*Proof.* Since  $f_1 \circ f_2$  is a homeomorphism,  $f_1$  is surjective. Thus  $f_2$  is a universal homeomorphism by Lemma 3.2.19. Similarly,  $f_1$  is also a universal homeomorphism. □

**Lemma 3.2.21.** *Assume  $0 \leq n \leq N-1$ . We have  $F_{V,n+1}^- \circ F_{V,n}^+ = \text{Fr}_{\text{RTToySht}_V^n}$ ,  $F_{V,n}^+ \circ F_{V,n+1}^- = \text{Fr}_{\text{LToySht}_V^{n+1}}$ . In particular,  $F_{V,n+1}^-$  and  $F_{V,n}^+$  induce universal homeomorphisms between  $\text{RTToySht}_V^n$  and  $\text{LToySht}_V^{n+1}$ .*

*Proof.* The first statement follows from definition of partial Frobeniuses. The second statement follows from Lemma 3.2.20.  $\square$

### 3.2.6 Irreducibility of the scheme of toy shtukas

Recall that we have  ${}^\circ\text{ToySht}_V^n = {}^\circ\text{LToySht}_V^n = {}^\circ\text{RToySht}_V^n$  for  $1 \leq n \leq N$  by Corollary 3.2.17.

**Lemma 3.2.22.** *For  $1 \leq n \leq N - 1$ ,  ${}^\circ\text{ToySht}_V^n$  is dense in  $\text{ToySht}_V^n$ ,  $\text{LToySht}_V^n$  and  $\text{RToySht}_V^n$ .*

*Proof.* Since  $\text{ToySht}_V^n$  has pure dimension  $N - 1$  by Lemma 3.2.9, we see from Remark 3.2.15 that  ${}^\circ\text{ToySht}_V^n = \text{ToySht}_V^n - \text{Grass}_V^n(\mathbb{F}_q)$  is dense in  $\text{ToySht}_V^n$ .

Consider the morphism  $\text{LToySht}_V^n \rightarrow \text{ToySht}_V^n$ . It induces an isomorphism between  ${}^\circ\text{LToySht}_V^n$  and  ${}^\circ\text{ToySht}_V^n$ . For any  $M \in \text{Grass}_V^n(\mathbb{F}_q)$ , we see that the inverse image of  $M$  in  $\text{LToySht}_V^n$  is isomorphic to  $\text{Grass}_M^{n-1}$ , and we have  $\dim \text{Grass}_M^{n-1} = n - 1 < N - 1$ . Since  $\text{LToySht}_V^n$  has pure dimension  $N - 1$  by Lemma 3.2.6,  ${}^\circ\text{LToySht}_V^n$  is dense in  $\text{LToySht}_V^n$ .

The proof for  $\text{RToySht}_V^n$  is similar, with  $\text{Grass}_M^{n-1}$  replaced by  $\text{Grass}_{V/M}^1$ .  $\square$

**Proposition 3.2.23.** *For  $1 \leq n \leq N - 1$ ,  $\text{ToySht}_V^n$ ,  $\text{LToySht}_V^n$  and  $\text{RToySht}_V^n$  are geometrically irreducible.*

*Proof.* The three schemes in question all contain a dense subscheme  ${}^\circ\text{ToySht}_V^n$  by Lemma 3.2.22. This is still true after base change to  $\overline{\mathbb{F}}_q$  since the morphism  $\text{Spec } \overline{\mathbb{F}}_q \rightarrow \text{Spec } \mathbb{F}_q$  is universally open. Now Lemma 3.2.21 and Corollary 3.2.17 reduce the question to the case  $n = 1$ . We know that  $\text{ToySht}_V^1 = \text{Grass}_V^1$  is geometrically irreducible.  $\square$

### 3.3 Schubert divisors of the schemes of toy shtukas

#### 3.3.1 A lemma for toy shtukas

Let  $E$  be a field over  $\mathbb{F}_q$  and let  $L \in \text{ToySht}_V(E)$ ; in other words,  $L$  is a subspace of  $V \otimes E$  is a subspace such that  $\dim L - \dim((\text{Fr}_E^* L) \cap L) = \dim \text{Fr}_E^* L - \dim((\text{Fr}_E^* L) \cap L) \leq 1$ .

Let  $W$  be a subspace of  $V$ .

Denote  $L' = L \cap (W \otimes E)$ ,  $L'' = \text{im}(L \rightarrow (V/W) \otimes E)$ . We have  $\text{Fr}_E^* L' = (\text{Fr}_E^* L) \cap (W \otimes E)$ ,  $\text{Fr}_E^* L'' = \text{im}(\text{Fr}_E^* L \rightarrow (V/W) \otimes E)$ .

**Lemma 3.3.1.** *At least one of the following two statements holds.*

- (i)  $\text{Fr}_E^* L' = L'$ ;
- (ii)  $\text{Fr}_E^* L'' = L''$ .

*Proof.* First note that  $\dim L' = \dim \text{Fr}_E^* L'$ ,  $\dim L'' = \dim \text{Fr}_E^* L''$ . Suppose there exists  $v \in \text{Fr}_E^* L'$  such that  $v \notin L'$ . Then  $\text{Fr}_E^* L$  is contained in the linear span of  $L$  and  $v$ . Since  $v$  maps to 0 in  $(V/W) \otimes E$ , we have  $\text{Fr}_E^* L'' \subset L''$ . Hence  $\text{Fr}_E^* L'' = L''$  by dimension comparison.  $\square$

#### 3.3.2 Notation and conventions

The following notation and conventions will be used in the rest of this section.

Fix a finite dimensional vector space  $V$  over  $\mathbb{F}_q$  with  $\dim_{\mathbb{F}_q} V = N \geq 3$ .

Fix an integer  $n$  such that  $1 \leq n \leq N - 1$ , and a subspace  $W \subset V$  of codimension  $n$ .

As in Section 3.2.2, for a codimension  $n$  subspace  $M$  in  $V$ , let  $U_M$  be the affine chart of  $\text{Grass}_V^n$  parameterizing those  $n$ -dimensional subspaces of  $V$  which are transversal to  $M$ .

### 3.3.3 Definition of Schubert divisors

Define the closed subscheme  $\text{Schub}_V^W \subset \text{Grass}_V^n$  by the following equation:

$$\text{Schub}_V^W := \{L \in \text{Grass}_V^n \mid \det(L \rightarrow V/W) = 0\}.$$

It is easy to prove and well-known that  $\text{Schub}_V^W \subset \text{Grass}_V^n$  is an irreducible divisor (a Schubert variety).

**Lemma 3.3.2.**  $\text{Schub}_V^W \cap \text{ToySht}_V^n$  is a Cartier divisor in  $\text{ToySht}_V^n$ .

*Proof.* We know that  $\text{ToySht}_V^n$  is irreducible and reduced, and  $\text{Schub}_V^W$  is a Cartier divisor of  $\text{Grass}_V^n$ . To prove the statement, it suffices to show that  $\text{ToySht}_V^n$  is not contained in  $\text{Schub}_V^W$ . We have  $U_W = \text{Grass}_V^n - \text{Schub}_V^W$ , and we see in Section 3.2.2 that  $\text{ToySht}_V^n \cap U_W$  is nonempty.  $\square$

We have a perfect complex

$$\mathcal{S}_{V,W}^\bullet = (\mathcal{S}_{V,W}^{-1} \rightarrow \mathcal{S}_{V,W}^0)$$

on  $\text{Grass}_V^n$ , where  $\mathcal{S}_{V,W}^{-1} \subset V \otimes \mathcal{O}_{\text{Grass}_V^n}$  is the universal locally free sheaf on  $\text{Grass}_V^n$ ,  $\mathcal{S}_{V,W}^0 = (V/W) \otimes \mathcal{O}_{\text{Grass}_V^n}$ , and the morphism  $\mathcal{S}_{V,W}^{-1} \rightarrow \mathcal{S}_{V,W}^0$  is the natural one.

Since  $W$  has codimension  $n$  in  $V$ , the morphism  $\mathcal{S}_{V,W}^{-1} \rightarrow \mathcal{S}_{V,W}^0$  is an isomorphism on the dense open subscheme  $U_W \subset \text{Grass}_V^n$ . So the complex  $\mathcal{S}_{V,W}^\bullet$  is good in the sense of Knudsen-Mumford.

*Remark 3.3.3.* We have  $\text{Schub}_V^W = \text{Div}(\mathcal{S}_{V,W}^\bullet)$ . (See [14, Chapter II] for the definition of  $\text{Div}$ .)

We call  $\text{Schub}_V^W \cap \text{ToySht}_V^n$  (resp.  $\text{Schub}_V^W \cap {}^\circ\text{ToySht}_V^n$ ) the *Schubert divisor* of  $\text{ToySht}_V^n$  (resp.  ${}^\circ\text{ToySht}_V^n$ ) for  $W$ .

### 3.3.4 Description of Schubert divisors

For any subspace  $H \subset V$ , we have a natural closed immersion  $\text{ToySht}_H^n \rightarrow \text{ToySht}_V^n$ . We consider  $\text{ToySht}_H^n$  as a subscheme of  $\text{ToySht}_V^n$ . It is a Weil divisor if  $H$  has codimension 1 in  $V$ . If  $H$  contains  $W$ , then  $\text{ToySht}_H^n \subset \text{Schub}_V^W \cap \text{ToySht}_V^n$ .

For a subspace  $J \subset V$ , there is a natural bijection between subspaces of  $V/J$  and subspaces of  $V$  containing  $J$ . So we get a natural closed immersion  $\text{ToySht}_{V/J}^{n-\dim J} \rightarrow \text{ToySht}_V^n$ . We consider  $\text{ToySht}_{V/J}^{n-\dim J}$  as a subscheme of  $\text{ToySht}_V^n$ . It is a Weil divisor if  $J$  has dimension 1. If  $J$  is contained in  $W$ ,  $\text{ToySht}_{V/J}^{n-\dim J}$  is contained in  $\text{Schub}_V^W \cap \text{ToySht}_V^n$ .

The following statement describes the Schubert divisors. Note that by Proposition 3.2.23,  $\text{ToySht}_H^n$  is irreducible when  $n < \dim H$ , and  $\text{ToySht}_{V/J}^{n-\dim J}$  is irreducible when  $n > \dim J$ . Also note that by Remark 3.2.15,  ${}^\circ\text{ToySht}_H^n$  is empty when  $n = \dim H$ , and  ${}^\circ\text{ToySht}_{V/J}^{n-\dim J}$  is empty when  $n = \dim J$ .

Recall that we denote  $\mathbf{P}_{V^*}$  to be the set of codimension 1 subspaces of  $V$  and we denote  $\mathbf{P}_V$  to be the set of dimension 1 subspaces of  $V$ .

**Theorem 3.3.4.** *We have an equality of Cartier divisors of  ${}^\circ\text{ToySht}_V^n$*

$$\text{Schub}_V^W \cap {}^\circ\text{ToySht}_V^n = \sum_{\substack{H \in \mathbf{P}_{V^*} \\ H \supset W}} {}^\circ\text{ToySht}_H^n + \sum_{\substack{J \in \mathbf{P}_V \\ J \subset W}} {}^\circ\text{ToySht}_{V/J}^{n-1}.$$

*Proof.* Put  $Z_V^W = \text{Schub}_V^W \cap {}^\circ\text{ToySht}_V^n$ .

From the above discussion we know that  ${}^\circ\text{ToySht}_H^n$  and  ${}^\circ\text{ToySht}_{V/J}^{n-1}$  are Weil divisors of  ${}^\circ\text{ToySht}_V^n$ , hence Cartier divisors of  ${}^\circ\text{ToySht}_V^n$  since  ${}^\circ\text{ToySht}_V^n$  is smooth. We also see that  ${}^\circ\text{ToySht}_H^n \subset Z_V^W$  if  $H \supset W$  and  ${}^\circ\text{ToySht}_{V/J}^{n-1} \subset Z_V^W$  if  $J \subset W$ .

On the other hand, if  $s \in Z_V^W$ , then Lemma 3.3.1 implies that  $s \in \text{ToySht}_H^n$  for some  $H \in \mathbf{P}_{V^*}, H \supset W$  or  $s \in \text{ToySht}_{V/J}^{n-1}$  for some  $J \in \mathbf{P}_V, J \subset W$  (or both). This shows that the statement is set-theoretically true.



It remains to show that the multiplicity of  $Z_V^W$  at each  ${}^\circ\text{ToySht}_H^n$  or  ${}^\circ\text{ToySht}_{V/J}^{n-1}$  is one. This follows from Lemma 3.3.10.  $\square$

### 3.3.5 An open subscheme of $\text{Schub}_V^W \cap \text{ToySht}_V^n$

Define a closed subset  $(\text{Schub}_V^W)^{\geq 2} \subset \text{Schub}_V^W$  by the condition

$$(\text{Schub}_V^W)^{\geq 2} := \{L \in \text{Grass}_V^n \mid \text{rank}(L \rightarrow V/W) \leq n-2\}.$$

Let  $(\text{Schub}_V^W)^1 = \text{Schub}_V^W - (\text{Schub}_V^W)^{\geq 2}$  be the open subscheme of  $\text{Schub}_V^W$ .

*Remark 3.3.5.* We know that  $(\text{Schub}_V^W)^1 = \text{Schub}_V^W \cap (\cup U_M)$  where  $M$  runs through all codimension  $n$  subspaces of  $V$  whose intersection with  $W$  has codimension  $n+1$  in  $V$ .

*Remark 3.3.6.* Fix one such  $M$  as in Remark 3.3.5 and choose an  $n$ -dimensional subspace  $M'$  of  $V$  such that  $V = M \oplus M'$  and  $M' \cap W \neq 0$ . Then  $\dim(M' \cap W) = 1$ . We identify  $U_M$  with  $\underline{\text{Hom}}(M', M)$  as in Section 3.2.2. We see that  $\text{Schub}_V^W \cap U_M = (\text{Schub}_V^W)^1 \cap U_M$  as a subscheme of  $U_M$  is defined by the condition that the induced morphism  $M' \cap W \rightarrow M/(M \cap W)$  is zero.

**Lemma 3.3.7.**  $(\text{Schub}_V^W)^{\geq 2} \cap \text{ToySht}_V^n$  has codimension at least 2 in  $\text{ToySht}_V^n$ .

*Proof.* From Lemma 3.3.1 we see that  $s \in (\text{Schub}_V^W)^{\geq 2} \cap \text{ToySht}_V^n$  implies  $s \in \text{ToySht}_{V/J}^{n-2}$  for some 2-dimensional subspace  $J \subset V$  or  $s \in \text{ToySht}_H^n$  for some codimension 2 subspace  $H \subset V$  (or both). The statement follows from the result about dimensions of  $\text{ToySht}$  in Lemma 3.2.9.  $\square$

### 3.3.6 Transversality

For  $1 \leq a \leq s, 1 \leq b \leq t$ , let  $\text{Mat}_{s \times t}^{(a,b)=0}$  denote the subscheme of  $\text{Mat}_{s \times t}$  consisting of matrices whose  $(a, b)$ -entry is zero.

**Lemma 3.3.8.** *The locus of  $\text{Mat}_{s \times t}^{\text{rank} \leq 1}$  where it does not meet  $\text{Mat}_{s \times t}^{(a,b)=0}$  transversally is*

$$\text{Mat}_{s \times t}^{\text{rank} \leq 1} \cap \left( \bigcap_{i=1}^s \text{Mat}_{s \times t}^{(i,b)=0} \right) \cap \left( \bigcap_{j=1}^t \text{Mat}_{s \times t}^{(a,j)=0} \right),$$

*i.e., the locus where the  $a$ -th row and the  $b$ -th column is zero. In particular, this locus is isomorphic to  $\text{Mat}_{(s-1) \times (t-1)}^{\text{rank} \leq 1}$  and it has codimension 2 in  $\text{Mat}_{s \times t}^{\text{rank} \leq 1}$ .  $\square$*

**Lemma 3.3.9.** *Let  $AS = \text{Id} - \text{Fr} : \text{Mat}_{s \times t} \rightarrow \text{Mat}_{s \times t}$  be the Artin-Schreier morphism. Then  $\text{Mat}_{s \times t}^{(a,b)=0} \cap AS^{-1}(\text{Mat}_{s \times t}^{\text{rank} \leq 1} - \{0\})$  has multiplicity one at each irreducible component as a divisor of  $AS^{-1}(\text{Mat}_{s \times t}^{\text{rank} \leq 1} - \{0\})$ .*

*Proof.* If we identify the tangent spaces at each point of  $\text{Mat}_{s \times t}$  with the vector space  $\text{Mat}_{s \times t}$  itself, then  $AS$  induces the identity on the tangent spaces. Now the statement follows from Lemma 3.3.8 and the fact that  $AS$  is finite.  $\square$

**Lemma 3.3.10.**  $\text{Schub}_V^W \cap {}^\circ\text{ToySht}_V^n$  *is a reduced scheme.*

*Proof.* By Lemma 3.3.7, it suffices to show that  $(\text{Schub}_V^W)^1 \cap {}^\circ\text{ToySht}_V^n$  as a divisor of  ${}^\circ\text{ToySht}_V^n - (\text{Schub}_V^W)^{\geq 2}$  has multiplicity one at each irreducible component. By Remark 3.3.5, it suffices to show that  $(\text{Schub}_V^W)^1 \cap {}^\circ\text{ToySht}_V^n \cap U_M$  as a divisor of  ${}^\circ\text{ToySht}_V^n \cap U_M$  has multiplicity one at each irreducible component, where  $M$  is any codimension  $n$  subspace of  $V$  whose intersection with  $W$  has codimension  $n + 1$  in  $V$ . From the description of  $(\text{Schub}_V^W)^1 \cap U_M$  in Remark 3.3.6, the statement follows from Lemma 3.2.4 and Lemma 3.3.9.  $\square$

## 3.4 Toy Horospherical Divisors

### 3.4.1 Notation

Fix a finite dimensional vector space  $V$  over  $\mathbb{F}_q$  with  $\dim_{\mathbb{F}_q} V = N \geq 3$ . Fix  $n \in \mathbb{Z}$  such that  $1 \leq n \leq N - 1$ .

As in Section 3.3.4, for any  $H \in \mathbf{P}_{V^*}$ , we consider  ${}^\circ\text{ToySht}_H^n$  (resp.  $\text{LToySht}_H^n$ , resp.  $\text{RToySht}_H^n$ ) as a Cartier divisor of  ${}^\circ\text{ToySht}_V^n$  (resp.  $\text{LToySht}_V^n$ , resp.  $\text{RToySht}_V^n$ ). For any  $J \in \mathbf{P}_V$ , we consider  ${}^\circ\text{ToySht}_{V/J}^{n-1}$  (resp.  $\text{LToySht}_{V/J}^{n-1}$ , resp.  $\text{RToySht}_{V/J}^{n-1}$ ) as a Cartier divisor of  ${}^\circ\text{ToySht}_V^n$  (resp.  $\text{LToySht}_V^n$ , resp.  $\text{RToySht}_V^n$ ). These divisors are called *toy horospherical divisors* of  ${}^\circ\text{ToySht}_V^n$  (resp.  $\text{LToySht}_V^n$ , resp.  $\text{RToySht}_V^n$ ). As in Definition 3.2.18, we have partial Frobeniuses  $F_{V,n}^- : \text{LToySht}_V^n \rightarrow \text{RToySht}_V^{n-1}$ , ( $1 \leq n \leq N$ ) and  $F_{V,n}^+ : \text{RToySht}_V^n \rightarrow \text{LToySht}_V^{n+1}$ , ( $0 \leq n \leq N-1$ ).

### 3.4.2 Short complexes related with principal toy horospherical divisors

Let  $Y$  be a smooth scheme over  $\mathbb{F}_q$  and  $U \subset Y$  be a dense open subscheme. Then we have a short complex

$$0 \longrightarrow C^0(Y, U) \xrightarrow{d} C^1(Y, U) \longrightarrow 0,$$

where  $C^0(Y, U) := H^0(U, \mathcal{O}_U^\times)$ ,  $C^1(Y, U) := \{\text{divisors on } Y \text{ with zero restriction to } U\}$ .

The complex  $C^\bullet(Y, U)$  is a contravariant functor in  $(Y, U)$ .

*Remark 3.4.1.* Let  $Z, Z' \subset Y$  be closed subsets of codimension at least 2 such that  $U - Z' \subset Y - Z'$ . Then the map  $C^\bullet(Y, U) \rightarrow C^\bullet(Y - Z, U - Z')$  is an isomorphism.

*Remark 3.4.2.* The endomorphism of  $C^\bullet(Y, U)$  corresponding to  $\text{Fr} : (Y, U) \rightarrow (Y, U)$  is multiplication by  $q$ .

We denote

$${}^{\circ\circ}\text{ToySht}_V^n := \text{ToySht}_V^n - \left( \bigcup_{H \in \mathbf{P}_{V^*}} \text{ToySht}_H^n \right) \cup \left( \bigcup_{J \in \mathbf{P}_V} \text{ToySht}_{V/J}^{n-1} \right)$$

to be the open subscheme of  $\text{ToySht}_V^n$ . We have  ${}^{\circ\circ}\text{ToySht}_V^n \subset {}^\circ\text{ToySht}_V^n$ .

When  $2 \leq n \leq N - 2$ , the complement of  ${}^\circ\text{ToySht}_V^n$  has codimension at least 2 in  $\text{LToySht}_V^n$  or  $\text{RToySht}_V^n$ . Remark 3.4.1 implies that the natural maps of complexes

$$C^\bullet(\text{LToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n) \rightarrow C^\bullet({}^\circ\text{ToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n),$$

$$C^\bullet(\text{RToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n) \rightarrow C^\bullet({}^\circ\text{ToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n)$$

are isomorphisms. We denote

$$C_{V,n}^\bullet = C^\bullet(Y, {}^{\circ\circ}\text{ToySht}_V^n)$$

for  $Y = {}^\circ\text{ToySht}_V^n, \text{LToySht}_V^n, \text{RToySht}_V^n$ .

We define

$$C_{V,1}^\bullet = C^\bullet(\text{RToySht}_V^1, {}^{\circ\circ}\text{ToySht}_V^n),$$

$$C_{V,N-1}^\bullet = C^\bullet(\text{LToySht}_V^{N-1}, {}^{\circ\circ}\text{ToySht}_V^n).$$

We see that for  $1 \leq n \leq N - 2$ ,  $C_{V,n}^1 = C^1(\text{RToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n)$  is freely generated by  $\text{RToySht}_H^n (H \in \mathbf{P}_{V^*})$  and  $\text{RToySht}_{V/J}^{n-1} (J \in \mathbf{P}_V)$ . For  $2 \leq n \leq N - 1$ ,  $C_{V,n}^1 = C^1(\text{LToySht}_V^n, {}^{\circ\circ}\text{ToySht}_V^n)$  is freely generated by  $\text{LToySht}_H^n (H \in \mathbf{P}_{V^*})$  and  $\text{LToySht}_{V/J}^{n-1} (J \in \mathbf{P}_V)$ .

### 3.4.3 Partial Frobeniuses and toy horospherical divisors

Partial Frobeniuses induce morphisms of complexes

$$C_{V,1}^\bullet \xrightleftharpoons[(F_{V,1}^+)^*]{(F_{V,2}^-)^*} C_{V,2}^\bullet \xrightleftharpoons[(F_{V,2}^+)^*]{(F_{V,3}^-)^*} \cdots \xrightleftharpoons C_{V,N-1}^\bullet$$

**Lemma 3.4.3.** For  $1 \leq n \leq N - 2$  and  $H \in \mathbf{P}_{V^*}$ ,  $(F_{V,n}^+)^* \text{LToySht}_H^{n+1} = \text{RToySht}_H^n$ .

For  $2 \leq n \leq N - 1$  and  $J \in \mathbf{P}_V$ ,  $(F_{V,n}^-)^* \text{RToySht}_{V/J}^{n-2} = \text{LToySht}_{V/J}^{n-1}$ .

*Proof.* By Lemma 3.2.6 and Lemma 3.2.8,  $\text{LToySht}_H^{n+1}$  and  $\text{RToySht}_H^n$  are reduced and irreducible. To prove the first statement, it suffices to show that  $\text{RToySht}_H^n$  contains  $(F_{V,n}^+)^{-1} \text{LToySht}_H^{n+1}$ .

We have a Cartesian diagram

$$\begin{array}{ccc} (F_{V,n}^+)^{-1} \text{LToySht}_H^{n+1} & \longrightarrow & \text{RToySht}_V^n \\ \downarrow & & \downarrow F_{V,n}^+ \\ \text{LToySht}_H^{n+1} & \longrightarrow & \text{LToySht}_V^{n+1} \end{array}$$

Suppose we have a morphism  $S \rightarrow (F_{V,n}^+)^{-1} \text{LToySht}_H^{n+1}$ . From the above Cartesian diagram, we get a right toy shtuka  $\mathcal{L} \subset \mathcal{L}'$  of rank  $n$  over  $S$ , such that  $\mathcal{L}' \subset H \otimes \mathcal{O}_S$ . This shows that the morphism  $S \rightarrow (F_{V,n}^+)^{-1} \text{LToySht}_H^{n+1}$  factors through  $\text{RToySht}_H^n$ . Hence  $(F_{V,n}^+)^{-1} \text{LToySht}_H^{n+1} \subset \text{RToySht}_H^n$ .

The proof of the second statement is similar.  $\square$

**Lemma 3.4.4.** *For  $1 \leq n \leq N-2$  and  $J \in \mathbf{P}_V$ ,  $(F_{V,n}^+)^* \text{LToySht}_{V/J}^n = q \cdot \text{RToySht}_{V/J}^{n-1}$ . For  $2 \leq n \leq N-1$  and  $H \in \mathbf{P}_{V^*}$ ,  $(F_{V,n}^-)^* \text{RToySht}_H^{n-1} = q \cdot \text{LToySht}_H^n$ .*

*Proof.* Since the  $F_{V,n}^+ : \text{RToySht}_V^n \rightarrow \text{LToySht}_V^{n+1}$  and  $F_{V,n+1}^- : \text{LToySht}_V^{n+1} \rightarrow \text{RToySht}_V^n$  are universal homeomorphisms and their composition is the Frobenius morphism, the statements follow from Lemma 3.4.3.  $\square$

The following statement about finite Radon transform is standard.

**Lemma 3.4.5.** *For  $1 \leq n \leq N-1$ , The following two conditions for the two sets of numbers  $\{\lambda_H\}_{H \in \mathbf{P}_{V^*}}, \{\mu_J\}_{J \in \mathbf{P}_V} \subset \mathbb{Z}[\frac{1}{p}]$  are equivalent.*

$$\begin{aligned} (i_n) \quad & \sum_{J \in \mathbf{P}_V} \mu_J = 0 \text{ and } \lambda_H = q^{n-(N-1)} \sum_{\substack{J \in \mathbf{P}_V \\ J \subset H}} \mu_J \text{ for any } H \in \mathbf{P}_{V^*}. \\ (ii_n) \quad & \sum_{H \in \mathbf{P}_{V^*}} \lambda_H = 0 \text{ and } \mu_J = q^{1-n} \sum_{\substack{H \in \mathbf{P}_{V^*} \\ H \supset J}} \lambda_H \text{ for any } J \in \mathbf{P}_V. \end{aligned} \quad \square$$

For  $J \in \mathbf{P}_V, H \in \mathbf{P}_{V^*}$ , we denote

$$Z_{V,1,H} = \text{RToySht}_H^1 \in C_{V,1}^1, \quad Z_{V,1,J} = \text{RToySht}_{V/J}^0 \in C_{V,1}^1,$$

$$Z_{V,N-1,H} = \text{LToySht}_H^{N-1} \in C_{V,N-1}^1, \quad Z_{V,N-1,J} = \text{LToySht}_{V/J}^{N-2} \in C_{V,N-1}^1.$$

When  $2 \leq n \leq N-2$ , we denote

$$Z_{V,n,H} = \text{RToySht}_H^n \in C_{V,n}^1, \text{ or equivalently, } Z_{V,n,H} = \text{LToySht}_H^n \in C_{V,n}^1,$$

$$Z_{V,n,J} = \text{RToySht}_{V/J}^{n-1} \in C_{V,n}^1, \text{ or equivalently, } Z_{V,n,J} = \text{LToySht}_{V/J}^{n-1} \in C_{V,n}^1.$$

**Theorem 3.4.6.** *The element  $\sum_{H \in \mathbf{P}_{V^*}} \lambda_H \cdot Z_{V,n,H} + \sum_{J \in \mathbf{P}_V} \mu_J \cdot Z_{V,n,J}$  belongs to  $\text{im}(C_{V,n}^0 \rightarrow C_{V,n}^1) \otimes \mathbb{Z}[\frac{1}{p}]$  if and only if  $\{\lambda_H\}_{H \in \mathbf{P}_{V^*}}, \{\mu_J\}_{J \in \mathbf{P}_V}$  satisfy condition  $(i_n)$  (or equivalently condition  $(ii_n)$ ) in Lemma 3.4.5.*

*Proof.* Lemmas 3.4.3 and 3.4.4 show that

$$(F_{V,n}^+)^* Z_{V,n+1,H} = Z_{V,n,H}, \quad (F_{V,n}^+)^* Z_{V,n+1,J} = q \cdot Z_{V,n,J}$$

for  $1 \leq n \leq N-2, H \in \mathbf{P}_{V^*}, J \in \mathbf{P}_V$ . In view of Remark 3.4.2, it suffices to prove the statement in the case  $n = N-1$ .

We know that the morphism  $\pi_{L,N-1} : \text{LToySht}_V^{N-1} \rightarrow \text{ToySht}_V^{N-1}$  is the blow-up at the points of  $\text{ToySht}_V^{N-1}(\mathbb{F}_q)$ . For  $H \in \text{ToySht}_V^{N-1}(\mathbb{F}_q) = \mathbf{P}_{V^*}$ , the exceptional divisor with center  $H$  is  $\text{LToySht}_H^{N-1}$ . Thus for  $J \in \mathbf{P}_V$ , we have an identity of divisors of  $\text{LToySht}_V^{N-1}$

$$(\pi_{L,N-1})^* \text{ToySht}_{V/J}^{N-2} = \text{LToySht}_{V/J}^{N-2} + \sum_{\substack{H \in \mathbf{P}_{V^*} \\ H \supset J}} \text{LToySht}_H^{N-1}.$$

Also note that  $C_{V,N-1}^0$  consists of rational functions on  $\text{ToySht}_V^{N-1} = \mathbb{P}^V(V)$  with

zeros and poles supported on the hyperplanes  $\text{ToySht}_{V/J}^{N-2}(J \in \mathbf{P}_V)$ . Hence the statement is true in the case  $n = N - 1$ .  $\square$

### 3.4.4 Degree of partial Frobeniuses

Recall that we denote  $\text{Flag}_V^{i,j}$  be the closed subscheme of  $\text{Grass}_V^i \times \text{Grass}_V^j$  which consists of pairs  $(M, M')$  such that  $M \subset M'$ .

For  $0 \leq n \leq N - 1$ , we define a scheme  $\text{Flag}_{V,\{2\}}^{n,n+1}$  by the following Cartesian diagram

$$\begin{array}{ccc} \text{Flag}_{V,\{2\}}^{n,n+1} & \longrightarrow & \text{Flag}_V^{n,n+1} \\ \downarrow & & \downarrow \\ \text{Grass}_V^{n+1} & \xrightarrow{\text{Fr}} & \text{Grass}_V^{n+1} \end{array} \quad (3.1)$$

where the right vertical arrow is the projection. So  $\text{Flag}_{V,\{2\}}^{n,n+1}$  parameterizes pairs  $(M, M')$  such that  $M \subset \text{Fr}^* M'$ .

The commutative diagram

$$\begin{array}{ccc} \text{Flag}_V^{n,n+1} & \xrightarrow{\text{Fr}} & \text{Flag}_V^{n,n+1} \\ \downarrow & & \downarrow \\ \text{Grass}_V^{n+1} & \xrightarrow{\text{Fr}} & \text{Grass}_V^{n+1} \end{array}$$

induces a morphism  $f_n : \text{Flag}_V^{n,n+1} \rightarrow \text{Flag}_{V,\{2\}}^{n,n+1}$ , which is finite.

**Lemma 3.4.7.** *The morphism  $f_n : \text{Flag}_V^{n,n+1} \rightarrow \text{Flag}_{V,\{2\}}^{n,n+1}$  has degree  $q^n$*

*Proof.* In commutative diagram (3.1), denote  $g : \text{Flag}_{V,\{2\}}^{n,n+1} \rightarrow \text{Flag}_V^{n,n+1}$ . We have  $g \circ f_n = \text{Fr}_{\text{Flag}_V^{n,n+1}}$ . Since  $\text{Fr}_{\text{Grass}_V^{n+1}}$  is finite and flat, we have  $\deg g = \deg \text{Fr}_{\text{Grass}_V^{n+1}}$ . Thus

$$\deg f_n = \frac{\deg \text{Fr}_{\text{Flag}_V^{n,n+1}}}{\deg g} = \frac{\deg \text{Fr}_{\text{Flag}_V^{n,n+1}}}{\deg \text{Fr}_{\text{Grass}_V^{n+1}}} = q^{\dim \text{Flag}_V^{n,n+1} - \dim \text{Grass}_V^{n+1}} = q^n$$

□

**Proposition 3.4.8.** *For  $0 \leq n \leq N - 1$ , we have  $\deg F_{V,n}^+ = q^n$ . For  $1 \leq n \leq N$ , we have  $\deg F_{V,n}^- = q^{N-n}$ .*

*Proof.* The first statement follows from Lemma 3.4.7 and the Cartesian diagram

$$\begin{array}{ccc} \mathrm{RToySht}_V^n & \longrightarrow & \mathrm{Flag}_V^{n,n+1} \\ \downarrow F_{V,n}^+ & & \downarrow f_n \\ \mathrm{LToySht}_V^{n+1} & \longrightarrow & \mathrm{Flag}_{V,\{2\}}^{n,n+1} \end{array}$$

where the morphism  $\mathrm{RToySht}_V^n \rightarrow \mathrm{Flag}_V^{n,n+1}$  sends a right toy shtuka  $\mathrm{Fr}^* L \subset L' \supset L$  to the pair  $(L, L')$ , and the morphism  $\mathrm{LToySht}_V^{n+1} \rightarrow \mathrm{Flag}_{V,\{2\}}^{n,n+1}$  sends a left toy shtuka  $\mathrm{Fr}^* M \supset M' \subset M$  to the pair  $(M', M)$ .

The second statement follows from duality. □

### 3.5 Recollections of linear algebra

**Definition 3.5.1.** For two coherent locally free sheaves  $\mathcal{F}, \mathcal{G}$  over a scheme  $S$ , we say that a morphism  $\psi : \mathcal{F} \rightarrow \mathcal{G}$  has rank at most  $r$  if the induced morphism  $\bigwedge^{r+1} \psi : \bigwedge^{r+1} \mathcal{F} \rightarrow \bigwedge^{r+1} \mathcal{G}$  is zero. In this case, we say that  $\psi$  has strictly constant rank  $r$  if the induced morphism  $\bigwedge^r \psi : \bigwedge^r \mathcal{F} \rightarrow \bigwedge^r \mathcal{G}$  is nonzero at any point  $s \in S$ .

**Lemma 3.5.2.** *Let  $\psi : M \rightarrow N$  be a morphism between finitely generated free modules over a local ring  $A$ . Put  $e = \mathrm{rank} N$ . Then  $\psi$  has strictly constant rank  $r$  if and only if  $\mathrm{coker} \psi$  is free of rank  $e - r$ .*

*Proof.* If  $\psi$  has strictly constant rank  $r$ , then the matrix for  $\psi$  under two bases of  $M, N$  has an  $(r \times r)$ -minor with determinant being a unit of  $A$ . Hence we can choose



bases of  $M, N$  so that  $\psi$  has matrix

$$\begin{pmatrix} \text{Id}_{r \times r} & 0 \\ 0 & B \end{pmatrix}.$$

Since  $\psi$  has strictly constant rank  $r$ , we get  $B = 0$ . Hence  $\text{coker } \psi$  is free of rank  $e - r$ .

If  $\text{coker } \psi$  is free of rank  $(e - r)$ , then  $\text{im } \psi$  is free of rank  $r$ , so  $\psi$  has rank at most  $r$ . Let  $E$  be the residue field of  $A$ . Then  $\text{coker}(\psi \otimes E) = (\text{coker } \psi) \otimes E$  has dimension  $(e - r)$  over  $E$ . Hence  $\psi \otimes E$  has rank  $r$ . Therefore  $\psi$  has strictly constant rank  $r$ .  $\square$

**Lemma 3.5.3.** *Let  $S$  be a scheme. Let  $\psi : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of coherent locally free sheaves on  $S$ . Put  $g = \text{rank } \mathcal{G}$ . Then  $\psi$  has strictly constant rank  $r$  if and only if  $\text{coker } \psi$  is locally free of rank  $g - r$ .*

*Proof.* Since  $\text{coker } \psi$  is a finitely presented  $\mathcal{O}_S$ -module, it suffices to prove the statements for each stalks. This follows from Lemma 3.5.2.  $\square$

**Corollary 3.5.4.** *Let  $S, \psi, \mathcal{F}, \mathcal{G}$  be as in Lemma 3.5.3, then the following two exact sequences split locally.*

$$0 \longrightarrow \ker \psi \longrightarrow \mathcal{F} \longrightarrow \text{im } \psi \longrightarrow 0,$$

$$0 \longrightarrow \text{im } \psi \longrightarrow \mathcal{G} \longrightarrow \text{coker } \psi \longrightarrow 0.$$

*In particular,  $\ker \psi$  and  $\text{im } \psi$  are locally free.*

**Corollary 3.5.5.** *Let  $\mathcal{L}_1, \mathcal{L}_2$  be subbundles of a coherent locally free sheaf  $\mathcal{F}$  over a scheme  $S$ . Assume  $\mathcal{L}_1, \mathcal{L}_2$  have rank  $n$ . Then the following conditions are equivalent:*

- (i) *the morphism  $\mathcal{L}_1 \rightarrow \mathcal{F}/\mathcal{L}_2$  has strictly constant rank  $r$ ;*
- (ii) *the morphism  $\mathcal{L}_2 \rightarrow \mathcal{F}/\mathcal{L}_1$  has strictly constant rank  $r$ ;*
- (iii) *the morphism  $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{F}$  has strictly constant rank  $n + r$ .*

When the above conditions hold,  $\mathcal{L}_1 + \mathcal{L}_2$  and  $\mathcal{L}_1 \cap \mathcal{L}_2$  are subbundles of  $\mathcal{F}$ , and the four sheaves  $(\mathcal{L}_1 + \mathcal{L}_2)/\mathcal{L}_1$ ,  $(\mathcal{L}_1 + \mathcal{L}_2)/\mathcal{L}_2$ ,  $\mathcal{L}_1/(\mathcal{L}_1 \cap \mathcal{L}_2)$ ,  $\mathcal{L}_2/(\mathcal{L}_1 \cap \mathcal{L}_2)$  are locally free of rank  $r$ .

# CHAPTER 4

## TATE TOY SHTUKAS

### 4.1 Tate toy shtukas

#### 4.1.1 Tate linear algebra

We recall some basic definitions and facts in Tate linear algebra. See [16, Chapter 2, Section 6], [3, Section 2.7.7], [8, Section 3.1] and [12, Section 1] for more details.

We consider topological vector spaces over a discrete field  $E$ .

For a topological vector space  $M$ , its dual  $M^*$  is by definition the space of all continuous linear functionals  $M \rightarrow E$ .

For a discrete topological vector space  $Q$ , the topology on its dual  $Q^*$  is the weakest one such that the linear functional  $\langle v, - \rangle : Q^* \rightarrow E$  is continuous for all  $v \in Q$ .

**Definition 4.1.1.** A *Tate space* is a topological vector space isomorphic to  $P \oplus Q^*$ , where  $P$  and  $Q$  are discrete.

*Remark 4.1.2.* The topology on a Tate space is Hausdorff.

**Definition 4.1.3.** Let  $T$  be a Tate space. A linear subspace  $\Lambda \subset T$  is said to be *linearly compact* (resp. *linearly cocompact*) if it is closed and for any open vector subspace  $U \subset T$  one has  $\dim \Lambda/(\Lambda \cap U) < \infty$  (resp.  $\dim T/(\Lambda + U) < \infty$ ).

For a Tate space  $T$ , we equip its dual  $T^*$  with a topology as follows. We require the topology on  $T^*$  to be linear, i.e., its open linear subspaces form a basis of open neighborhoods of 0. A linear subspace of  $T^*$  is open if and only if it is the orthogonal complement of a linearly compact linear subspace of  $T$ .

Note that when  $T$  is discrete, the topology on  $T^*$  agrees with the one given before.

*Remark 4.1.4.* For a Tate space  $T$ , its dual  $T^*$  is again a Tate space. The canonical map  $T \rightarrow T^{**}$  is an isomorphism. Any discrete vector space is a Tate space. Any linearly compact topological vector space is a Tate space. Duality interchanges discrete and linearly compact Tate spaces.

*Remark 4.1.5.* A Tate space over a finite field is locally compact. A linearly compact Tate space over a finite field is compact.

**Definition 4.1.6.** A linear subspace of a Tate space is called a *c-lattice* if it is open and linearly compact. A linear subspace of a Tate space is called a *d-lattice* if it is discrete and linearly cocompact.

*Remark 4.1.7.* Suppose  $T = P \oplus Q^*$  is a Tate space, where  $P$  and  $Q$  are discrete. Then  $P$  is a d-lattice of  $T$ , and  $Q^*$  is a c-lattice of  $T$ . Thus there exist c-lattices and d-lattices in every Tate space.

*Remark 4.1.8.* c-lattices of a Tate space form a basis of neighborhood of 0.

**Definition 4.1.9.** For a Tate space  $T$  and two c-lattices  $L_1, L_2$  of  $T$ , we define the *relative dimension*  $d_{L_1}^{L_2} := \dim(L_2/L_1 \cap L_2) - \dim(L_1/L_1 \cap L_2) \in \mathbb{Z}$ .

**Definition 4.1.10.** A *dimension theory* on a Tate space  $T$  is a function

$$d : \{\text{c-lattices of } T\} \rightarrow \mathbb{Z}$$

such that  $d(L_2) - d(L_1) = d_{L_1}^{L_2}$  for any  $L_1, L_2$ .

A dimension theory exists and is unique up to adding an integer. So dimension theories on a Tate space  $T$  form a  $\mathbb{Z}$ -torsor, denoted by  $\text{Dim}_T$ .

For a d-lattice  $I$  and a c-lattice  $L$  of a Tate space  $T$ , we denote

$$\chi(I, L) = \dim(I \cap L) - \dim(T/(I + L)).$$

For any d-lattice  $I \subset T$ , the function  $L \mapsto \chi(I, L)$  is a dimension theory.

**Definition 4.1.11.** For a Tate space  $T$ , we define a function

$$\begin{aligned} \dim : \{\text{d-lattices of } T\} &\rightarrow \text{Dim}_T \\ I &\mapsto (L \mapsto \chi(I, L)) \end{aligned}$$

We call  $\dim(I)$  the *dimension* of  $I$ .

We borrow the definition of relative determinant from [2, Section 4].

**Definition 4.1.12.** For a Tate space  $T$  and two c-lattices  $L_1, L_2$  of  $T$ , we define their *relative determinant* to be the one-dimensional vector space

$$\det_{L_1}^{L_2} := \det(L_1/L_1 \cap L_2)^* \otimes \det(L_2/L_1 \cap L_2).$$

#### 4.1.2 Sato Grassmannians

Let  $E$  be a discrete field.

For an  $E$ -vector space  $V$  and a Tate space  $M$  over  $E$ , we denote

$$M \widehat{\otimes} V := \varprojlim_{\Lambda} (M/\Lambda) \otimes V,$$

where the projective limit is taken over all c-lattices  $\Lambda$  of  $M$ .

For an  $E$ -scheme  $S$  and a quasi-coherent sheaf  $\mathcal{G}$  on  $S$ , we denote  $M \widehat{\otimes} \mathcal{G}$  to be the sheaf of  $\mathcal{O}_S$ -modules such that

$$(M \widehat{\otimes} \mathcal{G})(U) = \varprojlim_{\Lambda} (M/\Lambda) \otimes \mathcal{G}(U)$$

for all open subset  $U \subset S$ , where the projective limit is taken over all c-lattices  $\Lambda$  of  $M$ .

Let  $T$  be a Tate space over  $E$  and fix  $n \in \text{Dim}_T$ .

**Definition 4.1.13.** We define a functor  $\text{Grass}_T^n$  from the category of  $E$ -schemes to the category of sets. For an  $E$ -algebra  $R$ , an  $R$ -point of  $\text{Grass}_T^n$  is an  $R$ -submodule  $L \subset T \hat{\otimes} R$  such that there exists a c-lattice  $\Lambda'$  of  $T$  such that the morphism  $L \rightarrow (T/\Lambda') \otimes R$  is injective and its cokernel is a finitely generated projective  $R$ -module of rank  $-n(\Lambda')$ .

The definition below is due to Kashiwara [13, Section 2]. Definition 4.1.13 and Definition 4.1.14 are equivalent by Lemma 4.1.15.

**Definition 4.1.14.** For an  $E$ -scheme  $S$ , an  $S$ -point of  $\text{Grass}_T^n$  is an  $\mathcal{O}_S$ -submodule  $\mathcal{F} \subset T \hat{\otimes} \mathcal{O}_S$  such that locally in the Zariski topology there exists a c-lattice  $\Lambda \subset T$  such that  $n(\Lambda) = 0$  and  $\mathcal{F} \rightarrow (T/\Lambda) \otimes \mathcal{O}_S$  is an isomorphism.

**Lemma 4.1.15.** *Let  $S = \text{Spec } R$ , where  $R$  is an  $E$ -algebra. Let  $G_1$  (resp.  $G_2$ ) be the set  $\text{Grass}_T^n(S)$ , where  $\text{Grass}_T^n$  is the functor in Definition 4.1.13 (resp. Definition 4.1.14). Then we have a canonical bijection  $G_2 \rightarrow G_1$ .*

*Proof.* Since  $S$  is quasi-compact, we get a map  $G_2 \rightarrow G_1$ .

For  $L \in G_1$  and  $s \in S$ , we can find a c-lattice  $\Lambda$  containing  $\Lambda'$  such that  $n(\Lambda) = 0$  and  $L \otimes k(s) \rightarrow (T/\Lambda) \otimes k(s)$  is an isomorphism of vector spaces, where  $k(s)$  is the residue field of  $s$ . Let  $\mathcal{F} = L \hat{\otimes}_R \mathcal{O}_S$ . Since  $((T/\Lambda') \otimes R)/L$  is projective, we see that for the fixed c-lattice  $\Lambda \subset T$  and the fixed  $\mathcal{O}_S$ -submodule  $\mathcal{F}$  the condition in Definition 4.1.14 is open on  $S$ . Hence there exists a Zariski neighborhood of  $s$  in which  $\mathcal{F} \rightarrow (T/\Lambda) \otimes \mathcal{O}_S$  is an isomorphism. This shows that  $\mathcal{F} \in G_1$ . So we get a map  $G_1 \rightarrow G_2$ .

It is easy to see that the above two maps are inverse of each other.  $\square$

**Proposition 4.1.16.**  $\text{Grass}_T^n$  is representable by a separated scheme over  $E$ .

*Proof.* This is [13, Proposition 2.2.1].  $\square$

*Remark 4.1.17.* If  $R$  is a field over  $E$ , then  $\text{Grass}_T^n(R)$  is the set of d-lattices of dimension  $n$  of the Tate space  $T \hat{\otimes} R$  over  $R$ .

### 4.1.3 Determinant of a family of d-lattices relative to a c-lattice

Let  $T$  be a Tate space over a field  $E$ .

We denote

$$\text{Grass}_T = \coprod_{n \in \text{Dim}_T} \text{Grass}_T^n.$$

We see that  $\text{Grass}_T$  parameterizes d-lattices of  $T$ .

**Definition 4.1.18.** Let  $S$  be a scheme over  $E$ . For  $\mathcal{L} \in \text{Grass}_T(S)$  and a c-lattice  $W$  of  $T$ , the *determinant* of  $\mathcal{L}$  relative to  $W$ , denoted by  $\det(\mathcal{L}, W)$ , is defined to be the invertible sheaf  $\det(\mathcal{L} \rightarrow (T/W) \otimes \mathcal{O}_S)$  on  $S$ , where  $\mathcal{L}$  is in degree 0.

*Remark 4.1.19.* The above definition commutes with base change, i.e., for a morphism of schemes  $f : S_1 \rightarrow S_2$  over  $E$ , an element  $\mathcal{L} \in \text{Grass}_T(S_2)$  and a c-lattice  $W$  of  $T$ , we have a canonical isomorphism  $\det(f^*\mathcal{L}, W) \cong f^*\det(\mathcal{L}, W)$

*Remark 4.1.20.* Let  $S$  be a scheme over  $E$ . For  $\mathcal{L} \in \text{Grass}_T(S)$  and two c-lattice  $W_1, W_2$  of  $T$ , we have a canonical isomorphism  $\det(\mathcal{L}, W_1) \otimes \det_{W_1}^{W_2} \cong \det(\mathcal{L}, W_2)$ . In particular, the two invertible sheaves  $\det(\mathcal{L}, W_1)$  and  $\det(\mathcal{L}, W_2)$  are isomorphic.

**Lemma 4.1.21.** *Let  $S$  be a scheme over  $E$  and let  $W$  be a c-lattice of  $T$ . For  $\mathcal{L}_1, \mathcal{L}_2 \in \text{Grass}_T(S)$  such that  $\mathcal{L}_1 \subset \mathcal{L}_2$ , we have a canonical isomorphism*

$$\det(\mathcal{L}_1, W) \otimes \det(\mathcal{L}_2/\mathcal{L}_1) \cong \det(\mathcal{L}_2, W)$$

*which commutes with base change.* □

**Lemma 4.1.22.** *Suppose the base field  $E$  is  $\mathbb{F}_q$ , and  $S$  is a scheme over  $E$ . For  $\mathcal{L} \in \text{Grass}_T(S)$  and a c-lattice  $W$  of  $T$ , we have a canonical isomorphism*

$$\det(\text{Fr}_S^* \mathcal{L}, W) \cong \det(\mathcal{L}, W)^{\otimes q}$$

*which commutes with base change.* □

#### 4.1.4 Definition of Tate toy shtukas

Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ . Let  $\text{Dim}_T$  denote the dimension torsor of  $T$ .

**Definition 4.1.23.** A *Tate toy shtuka* for  $T$  over an  $\mathbb{F}_q$ -scheme  $S$  of dimension  $n \in \text{Dim}_T$  is an element  $\mathcal{L} \in \text{Grass}_T^n(S)$  such that the composition

$$\text{Fr}_S^* \mathcal{L} \hookrightarrow \text{Fr}_S^*(T \widehat{\otimes} \mathcal{O}_S) = T \widehat{\otimes} \mathcal{O}_S \twoheadrightarrow (T \widehat{\otimes} \mathcal{O}_S) / \mathcal{L}$$

has rank at most 1. (In other words, the corresponding morphism  $\bigwedge^2 \text{Fr}_S^* \mathcal{L} \rightarrow \bigwedge^2 ((T \widehat{\otimes} \mathcal{O}_S) / \mathcal{L})$  is zero.)

For  $n \in \text{Dim}_T$ , let  $\text{ToySht}_T^n$  be the functor which associates to each  $\mathbb{F}_q$ -scheme  $S$  the set of isomorphism classes of Tate toy shtukas for  $T$  over  $S$  of dimension  $n$ .

As in the finite dimensional cases,  $\text{ToySht}_T^n$  is representable by a closed subscheme of  $\text{Grass}_T^n$ .

We denote  ${}^\circ\text{ToySht}_T^n := \text{ToySht}_T^n - \text{Grass}_T^n(\mathbb{F}_q)$ . As in Remark 3.2.15 we know that  ${}^\circ\text{ToySht}_T^n$  is an open subscheme of  $\text{ToySht}_T^n$ .

## 4.2 Open charts of the scheme of Tate toy shtukas

### 4.2.1 Notation

For a finite dimensional vector space  $V$  over  $\mathbb{F}_q$  and two subspaces  $V' \subset V''$  of  $V$ , denote  $\text{Grass}_V^{n, V', V''}$  to be the open subscheme of  $\text{Grass}_V^n$ , such that for any  $\mathbb{F}_q$ -algebra  $R$ ,  $\text{Grass}_V^{n, V', V''}(R)$  consists of  $L \in \text{Grass}_V^n(R)$  satisfying the following two conditions:

- (i) the morphism  $L \rightarrow (V/V') \otimes R$  is injective and its cokernel is projective;
- (ii) the morphism  $L \rightarrow (V/V'') \otimes R$  is surjective.



The morphism  $\text{Grass}_V^{n,V',V''} \rightarrow \text{Grass}_{V''/V'}^{n-\dim V/V''}$  maps  $L \in \text{Grass}_V^{n,V',V''}(R)$  to  $\text{im}(L'' \rightarrow (V''/V') \otimes R) \in \text{Grass}_{V''/V'}^{n-\dim V/V''}(R)$ , where  $L'' = \ker(L \rightarrow (V/V'') \otimes R)$ .

Let  ${}^\circ\text{Grass}_V^{n,V',V''}$  be the fiber product

$$\begin{array}{ccc} {}^\circ\text{Grass}_V^{n,V',V''} & \longrightarrow & \text{Grass}_V^{n,V',V''} \\ \downarrow f & & \downarrow \\ \text{Grass}_{V''/V'}^{n-\dim V/V''} - \text{Grass}_{V''/V'}^{n-\dim V/V''}(\mathbb{F}_q) & \longrightarrow & \text{Grass}_{V''/V'}^{n-\dim V/V''} \end{array} \quad (4.1)$$

We denote  ${}^\circ\text{ToySht}_V^{n,V',V''} = {}^\circ\text{ToySht}_V^n \cap {}^\circ\text{Grass}_V^{n,V',V''}$ . We have a morphism  ${}^\circ\text{ToySht}_V^{n,V',V''} \rightarrow {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  induced by  $f$ .

**Lemma 4.2.1.** *For a finite dimensional vector space  $V$  over  $\mathbb{F}_q$  and two subspaces  $V' \subset V''$  of  $V$ , the morphism  $\text{Grass}_V^{n,V',V''} \rightarrow \text{Grass}_{V''/V'}^{n-\dim V/V''}$  is affine.  $\square$*

**Lemma 4.2.2.** *For finite dimensional  $\mathbb{F}_q$ -vectors spaces  $V'_2 \subset V'_1 \subset V'_0 \subset V''_0 \subset V''_1 \subset V''_2$ , the morphism  ${}^\circ\text{Grass}_{V''_2/V'_2}^{n,V'_0/V'_2,V''_0/V'_2} \rightarrow {}^\circ\text{Grass}_{V''_1/V'_1}^{n-\dim V''_2/V''_1,V'_0/V'_1,V''_0/V'_1}$  is affine.  $\square$*

For a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$ , two c-lattices  $\Lambda' \subset \Lambda''$  of  $T$  and  $n \in \text{Dim}_T$ , we define the notation  $\text{Grass}_T^{n,\Lambda',\Lambda''}$ ,  ${}^\circ\text{Grass}_T^{n,\Lambda',\Lambda''}$  and  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''}$  similarly.

#### 4.2.2 Admissible pairs of c-lattices

Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ .

For two pairs of c-lattices  $(\tilde{\Lambda}', \tilde{\Lambda}'')$  and  $(\Lambda', \Lambda'')$  of  $T$ , we say that  $(\tilde{\Lambda}', \tilde{\Lambda}'')$  is *greater than*  $(\Lambda', \Lambda'')$  (denoted by  $(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ ), if  $\tilde{\Lambda}' \subset \Lambda'$  and  $\Lambda'' \subset \tilde{\Lambda}''$ . All pairs of c-lattices of  $T$  form a directed set under this partial order.

When  $(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ , we have  $\text{Grass}_T^{n,\Lambda',\Lambda''} \subset \text{Grass}_T^{n,\tilde{\Lambda}',\tilde{\Lambda}''}$ ,  ${}^\circ\text{Grass}_T^{n,\Lambda',\Lambda''} \subset {}^\circ\text{Grass}_T^{n,\tilde{\Lambda}',\tilde{\Lambda}''}$  and  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''} \subset {}^\circ\text{ToySht}_T^{n,\tilde{\Lambda}',\tilde{\Lambda}''}$ .

A pair of c-lattices  $(\Lambda', \Lambda'')$  of  $T$  is said to be *admissible with respect to*  $n \in \text{Dim}_T$  if  $n(\Lambda') \leq -2, n(\Lambda'') \geq 2$ , and  $\Lambda' \subset \Lambda''$ . Let  $AP_n(T)$  denote the set of pairs of c-lattices of  $T$  that are admissible with respect to  $n$ . It is a directed set with respect to the partial order above. We see that  ${}^\circ\text{ToySht}_T^n$  is covered by the union of  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  for all  $(\Lambda', \Lambda'') \in AP_n(T)$ .

### 4.2.3 ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$ as a projective limit

Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ . Let  $n \in \text{Dim}_T$  and  $(\Lambda', \Lambda'') \in AP_n(T)$ .

In this subsection we describe the open subscheme  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  of  ${}^\circ\text{ToySht}_T^n$  as a projective limit of schemes.

For any  $(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ , we denote  $U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''} = {}^\circ\text{ToySht}_{\tilde{\Lambda}''/\tilde{\Lambda}'}^{n(\tilde{\Lambda}''), \Lambda'/\tilde{\Lambda}', \Lambda''/\tilde{\Lambda}'}$ . In particular,  $U_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''} = {}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda''), \Lambda'/\Lambda', \Lambda''/\Lambda'}$ . We have a morphism  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''} \rightarrow U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  induced by the morphism  ${}^\circ\text{Grass}_T^{n, \Lambda', \Lambda''} \rightarrow {}^\circ\text{Grass}_{\tilde{\Lambda}''/\tilde{\Lambda}'}^{n(\tilde{\Lambda}''), \Lambda'/\tilde{\Lambda}', \Lambda''/\tilde{\Lambda}'}$ . For any  $(\Lambda'_2, \Lambda''_2) \succ (\Lambda'_1, \Lambda''_1) \succ (\Lambda', \Lambda'')$ , we have a transition map  $U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda', \Lambda''} \rightarrow U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda', \Lambda''}$ , which is affine by Lemma 4.2.6.

**Lemma 4.2.3.** *The morphisms  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''} \rightarrow U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  induce an isomorphism*

$${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}.$$

### 4.2.4 Transition maps are affine

**Lemma 4.2.4.** *Let  $f : Y_1 \rightarrow Y_2$  be a morphism of schemes over a scheme  $S$ . If  $Y_1$  is affine over  $S$  and  $Y_2$  is separated, then  $f$  is affine.*  $\square$

**Lemma 4.2.5.** *Let  $V'_2 \subset V'_1 \subset V'_0 \subset V''_0 \subset V''_1 \subset V''_2$  be finite dimensional vector*

spaces over  $\mathbb{F}_q$ . Then the morphism

$$\circ\text{ToySht}_{V_2''/V_2'}^{n, V_0'/V_2', V_0''/V_2'} \rightarrow \circ\text{ToySht}_{V_1''/V_1'}^{n-\dim V_2''/V_1'', V_0'/V_1', V_0''/V_1'}$$

is affine.

*Proof.* The morphism  $\circ\text{ToySht}_{V_2''/V_2'}^{n, V_0'/V_2', V_0''/V_2'} \rightarrow \circ\text{Grass}_{V_2''/V_2'}^{n, V_0'/V_2', V_0''/V_2'}$  is affine since it is a closed immersion. Lemma 4.2.2 implies that the morphism  $\circ\text{Grass}_{V_2''/V_2'}^{n, V_0'/V_2', V_0''/V_2'} \rightarrow \circ\text{Grass}_{V_1''/V_1'}^{n-\dim V_2''/V_1'', V_0'/V_1', V_0''/V_1'}$  is affine. Now we apply Lemma 4.2.4 with

$$Y_1 = \circ\text{ToySht}_{V_2''/V_2'}^{n, V_0'/V_2', V_0''/V_2'},$$

$$Y_2 = \circ\text{ToySht}_{V_1''/V_1'}^{n-\dim V_2''/V_1'', V_0'/V_1', V_0''/V_1'},$$

$$S = \circ\text{Grass}_{V_1''/V_1'}^{n-\dim V_2''/V_1'', V_0'/V_1', V_0''/V_1'},$$

the statement follows.  $\square$

**Lemma 4.2.6.** *Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ . Let  $n \in \text{Dim}_T$  and  $(\Lambda'_0, \Lambda''_0) \in AP_n(T)$ . Let  $(\Lambda'_1, \Lambda''_1), (\Lambda'_2, \Lambda''_2)$  be two  $c$ -lattices such that  $(\Lambda'_2, \Lambda''_2) \succ (\Lambda'_1, \Lambda''_1) \succ (\Lambda'_0, \Lambda''_0)$ . Then the morphism  $U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0} \rightarrow U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}$  is affine.*

*Proof.* The statement follows from Lemma 4.2.5.  $\square$

#### 4.2.5 Transition maps are smooth

**Lemma 4.2.7.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$ . Let  $W$  be a subspace of  $V$ . Then the morphism  $\circ\text{ToySht}_V^{n, 0, W} \rightarrow \circ\text{ToySht}_W^{n-\dim V/W}$  is smooth.*

*Proof.*  $\text{Grass}_W^{n-\dim V/W}$  is covered by  $\text{Grass}_W^{n-\dim V/W, W', W'}$  when  $W'$  runs through

all subspaces of  $W$  of dimension  $(\dim V - n)$ . So in view of the Cartesian diagram

$$\begin{array}{ccc} \text{Grass}_V^{n,W',W'} & \longrightarrow & \text{Grass}_V^{n,0,W} \\ \downarrow & & \downarrow \\ \text{Grass}_W^{n-\dim V/W,W',W'} & \longrightarrow & \text{Grass}_W^{n-\dim V} \end{array}$$

it suffices to show that the morphism

$$f : {}^\circ\text{ToySht}_V^{n,0,W} \cap \text{Grass}_V^{n,W',W'} \rightarrow {}^\circ\text{ToySht}_W^{n-\dim V/W} \cap \text{Grass}_W^{n-\dim V/W,W',W'}$$

is smooth for each subspace  $W' \subset W$  of dimension  $(\dim V - n)$ .

Fix one such  $W'$ . Choose splittings  $W = W' \oplus W'', V = W \oplus V'$ .

We define  $\mathbb{F}_q$ -schemes  $M, M', M''$  where

$$M = \{a \in \underline{\text{Hom}}(W'' \oplus V', W') \mid \text{rank } a = \text{rank}(W'' \xrightarrow{a|_{W''}} W') = 1\}$$

$$M' = \underline{\text{Hom}}(V', W'), \quad M'' = \underline{\text{Hom}}^{\text{rank}=1}(W'', W')$$

So  $M$  is a locally closed subscheme of  $\underline{\text{Hom}}(W'' \oplus V', W')$  and  $M''$  is a locally closed subscheme of  $\underline{\text{Hom}}(W'', W')$ .

We denote Artin-Schreier maps

$$AS = \text{Id} - \text{Fr} : \underline{\text{Hom}}(W'' \oplus V', W') \rightarrow \underline{\text{Hom}}(W'' \oplus V', W'),$$

$$AS' = \text{Id} - \text{Fr} : \underline{\text{Hom}}(V', W') \rightarrow \underline{\text{Hom}}(V', W'),$$

$$AS'' = \text{Id} - \text{Fr} : \underline{\text{Hom}}(W'', W') \rightarrow \underline{\text{Hom}}(W'', W').$$

From the explicit local description of  $\text{ToySht}$  in Section 3.2.2 and the Cartesian dia-

gram (4.1), we know that

$$\circ\text{ToySht}_V^{n,0,W} \cap \text{Grass}_V^{n,W',W'} = AS^{-1}(M),$$

$$\circ\text{ToySht}_W^{n-\dim V/W} \cap \text{Grass}_W^{n-\dim V/W,W',W'} = (AS'')^{-1}(M''),$$

and that the morphism  $f : AS^{-1}(M) \rightarrow (AS'')^{-1}(M'')$  is induced by the projection  $\underline{\text{Hom}}(W'' \oplus V', W') \rightarrow \underline{\text{Hom}}(W'', W')$ .

We denote

$$\mathbb{P}_{W'} \star M' = \{(L, A) \in \mathbb{P}_{W'} \times M' \mid L \supset \text{im } A\}$$

to be the closed subscheme of  $\mathbb{P}_{W'} \times M'$ .

We have to prove that the morphism  $f : AS^{-1}(M) \rightarrow (AS'')^{-1}(M'')$  is smooth.

To this end, we will construct a Cartesian diagram

$$\begin{array}{ccc} AS^{-1}(M) & \xrightarrow{g} & \mathbb{P}_{W'} \star M' \\ \downarrow f & & \downarrow h \\ (AS'')^{-1}(M'') & \xrightarrow{u} & \mathbb{P}_{W'} \end{array}$$

and prove that  $h$  is smooth.

The maps in the diagram are as follows. The morphism  $h$  is the composition  $\mathbb{P}_{W'} \star M' \xrightarrow{\text{Id} \times AS'} \mathbb{P}_{W'} \star M' \xrightarrow{\text{pr}} \mathbb{P}_{W'}$ . The composition  $AS^{-1}(M) \hookrightarrow \underline{\text{Hom}}(W'' \oplus V', W') \xrightarrow{\text{res}_{W''}} M'' \xrightarrow{\text{im}} \mathbb{P}_{W'}$  and the projection  $AS^{-1}(M) \rightarrow M'$  induce a morphism  $AS^{-1}(M) \rightarrow \mathbb{P}_{W'} \times M'$  which factors through  $\mathbb{P}_{W'} \star M'$ . This gives  $g$ . The morphism  $u$  is the composition  $(AS'')^{-1}(M'') \xrightarrow{AS''} M'' \xrightarrow{\text{im}} \mathbb{P}_{W'}$ .

One can check that the diagram is commutative and Cartesian.

The morphism  $\mathbb{P}_{W'} \star M' \xrightarrow{\text{Id} \times AS'} \mathbb{P}_{W'} \star M'$  is smooth since  $AS'$  is étale. The morphism  $\mathbb{P}_{W'} \star M' \xrightarrow{\text{pr}} \mathbb{P}_{W'}$  is smooth since it is the projection morphism for a vector bundle. So  $h$  is smooth. Hence is  $f$ .  $\square$

The following statement is dual to Lemma 4.2.7.

**Lemma 4.2.8.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$ . Let  $W$  be a subspace of  $V$ . Then the morphism  ${}^\circ\text{ToySht}_{V/W}^{n,W,V} \rightarrow {}^\circ\text{ToySht}_{V/W}^n$  is smooth.  $\square$*

**Proposition 4.2.9.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Then the morphism  ${}^\circ\text{ToySht}_V^{n,V',V''} \rightarrow {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V'}$  is smooth.  $\square$*

*Proof.* The statement follows from Lemma 4.2.7 and Lemma 4.2.8.  $\square$

**Corollary 4.2.10.** *With the same notation and assumptions of Lemma 4.2.6, the morphism  $U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0} \rightarrow U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}$  is smooth.*

*Proof.* The statement follows from Proposition 4.2.9.  $\square$

### 4.3 Functorial properties of toy horospherical divisors

In this section, we use the notation of Sections 4.2.1 and 4.2.2.

#### 4.3.1 Notation

Suppose  $M$  is a vector space over  $\mathbb{F}_q$  such that  $3 \leq \dim M < \infty$ .

For  $J \in \mathbf{P}_M$ , we consider  $\text{ToySht}_{M/J}^{n-1}$  as a closed subscheme of  $\text{ToySht}_M^n$  as before. Denote  $\Delta_{M,J}^n = \text{ToySht}_{M/J}^{n-1} \cap {}^\circ\text{ToySht}_M^n$ . For two subspaces  $M' \subset M''$  of  $M$ , denote  $\Delta_{M,J}^{n,M',M''} = \Delta_{M,J}^n \cap {}^\circ\text{Grass}_M^{n,M',M''}$ .

For  $H \in \mathbf{P}_{M^*}$ , we consider  $\text{ToySht}_H^n$  as a closed subscheme of  $\text{ToySht}_M^n$  as before. Denote  $\Delta_{M,H}^n = \text{ToySht}_H^n \cap {}^\circ\text{ToySht}_M^n$ . For two subspaces  $M' \subset M''$  of  $M$ , denote  $\Delta_{M,H}^{n,M',M''} = \Delta_{M,H}^n \cap {}^\circ\text{Grass}_M^{n,M',M''}$ .

We denote  $\Delta_M^n = (\bigcup_{H \in \mathbf{P}_{M^*}} \Delta_{M,H}^n) \cup (\bigcup_{J \in \mathbf{P}_M} \Delta_{M,J}^n)$  and  $\Delta_M^{n,M',M''} = \Delta_M^n \cap {}^\circ\text{Grass}_M^{n,M',M''}$ . They are the union of toy horospherical divisors of  ${}^\circ\text{ToySht}_M^n$  and  ${}^\circ\text{ToySht}_M^{n,M',M''}$  respectively.

By Proposition 3.2.23, each  $\Delta_{M,J}^n, \Delta_{M,J}^{n,M',M''}, \Delta_{M,H}^n, \Delta_{M,H}^{n,M',M''}$  is reduced and irreducible if it is nonempty.

### 4.3.2 Pullbacks of toy horospherical divisors under transition maps

In this subsection, we calculate the pullback of a toy horospherical divisor under transition maps. (The fact that this pullback is well-defined is clear from Corollary 4.2.10.)

**Lemma 4.3.1.** *Let  $W \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Let  $J' \in \mathbf{P}_{V/W}$  and denote  $\mathfrak{J} = \{J \in \mathbf{P}_V \mid \text{im}(J \rightarrow (V/W)) = J'\}$ . Let  $E$  be a field over  $\mathbb{F}_q$ . Suppose  $L \in {}^\circ\text{ToySht}_V^{n,W,V}(E)$  satisfies  $\text{im}(L \rightarrow ((V/W) \otimes E)) \supset J' \otimes E$ . Then  $L \supset J \otimes E$  for some  $J \in \mathfrak{J}$ .*

*Proof.* Let  $J''$  be the unique subspace of  $V$  containing  $W$  such that  $J''/W = J'$ . Denote  $L' = L \cap ((W + J'') \otimes E)$ ,  $L'' = \text{im}(L \rightarrow (V/(W + J'')) \otimes E)$ ,  $L''' = \text{im}(L \rightarrow (V/W) \otimes E)$ . Since  $L'''$  is a nontrivial toy shtuka and  $L''' \supset J' \otimes E$ ,  $L''$  is also a nontrivial toy shtuka. Applying Lemma 3.3.1 to  $L$  and  $W + J''$ , we get  $\text{Fr}_E^* L' = L'$ . Since  $L \in {}^\circ\text{ToySht}_V^{n,W,V}(E)$ , we have  $L \cap (W \otimes E) = 0$ . Hence  $\dim_E L' = 1$ . Thus  $L' = J \otimes E$  for some  $J \in \mathfrak{J}$ .  $\square$

**Lemma 4.3.2.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$ . Let  $W$  be a subspace of  $V$  such that  $\dim(V/W) \geq 3$ . Let  $u : {}^\circ\text{ToySht}_V^{n,W,V} \rightarrow {}^\circ\text{ToySht}_{V/W}^n$  be the transition map. Let  $H' \in \mathbf{P}_{(V/W)^*}$  and let  $H$  be the hyperplane of  $V$  such that  $H \supset W$  and  $H/W = H'$ . Let  $J' \in \mathbf{P}_{V/W}$  and denote  $\mathfrak{J} = \{J \in \mathbf{P}_V \mid \text{im}(J \rightarrow (V/W)) = J'\}$ .*

*When  $1 \leq n \leq \dim(V/W) - 2$ , we have an equality of divisors  $u^*(\Delta_{V/W,H'}^n) = \Delta_{V,H}^{n,W,V}$ . When  $2 \leq n \leq \dim(V/W) - 1$ , we have an equality of divisors  $u^*(\Delta_{V/W,J'}^n) = \sum_{J \in \mathfrak{J}} \Delta_{V,J}^{n,W,V}$ .*

*Proof.* The morphism  $u$  is smooth by Corollary 4.2.10. The schemes  $\Delta_{V/W,H'}^n, \Delta_{V,H}^{n,W,V}, \Delta_{V/W,J'}^n, \Delta_{V,J}^{n,W,V}$ , ( $J \in \mathfrak{J}$ ) are reduced by Lemma 3.2.9, it suffices to prove the statements set-theoretically.

When  $1 \leq n \leq \dim(V/W) - 2$ , it is obvious that  $u^{-1}(\Delta_{V/W,H'}^n) = \Delta_{V,H}^{n,W,V}$  as subsets of  ${}^\circ\text{ToySht}_V^{n,W,V}$ .

When  $2 \leq n \leq \dim(V/W) - 1$ , we have  $u(\Delta_{V,J}^{n,W,V}) \subset \Delta_{V/W,J'}^n$  for  $J \in \mathfrak{J}$ , and the inclusion  $u^{-1}(\Delta_{V/W,J'}^n) \subset \bigcup_{J \in \mathfrak{J}} \Delta_{V,J}^{n,W,V}$  is set-theoretically true by Lemma 4.3.1.  $\square$

**Lemma 4.3.3.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$ . Let  $W$  be a subspace of  $V$  such that  $\dim W \geq 3$ . Let  $u : {}^\circ\text{ToySht}_V^{n,0,W} \rightarrow {}^\circ\text{ToySht}_W^{n-\dim(V/W)}$  be the transition map. Let  $J \in \mathbf{P}_W$ ,  $H' \in \mathbf{P}_{W^*}$  and denote  $\mathfrak{H} = \{H \in \mathbf{P}_{V^*} | H \cap W = H'\}$ .*

*When  $\dim V/W + 2 \leq n \leq \dim V - 1$ , we have an equality of divisors between  $u^*(\Delta_{W,J}^{n-\dim V/W})$  and  $\Delta_{V,J}^{n,0,W}$ . When  $\dim V/W + 1 \leq n \leq \dim V - 2$ , we have an equality of divisors  $u^*(\Delta_{W,H'}^{n-\dim V/W}) = \sum_{H \in \mathfrak{H}} \Delta_{V,H}^{n,0,W}$ .*

*Proof.* The statement is dual to Lemma 4.3.2.  $\square$

**Proposition 4.3.4.** *Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ . Let  $n \in \text{Dim}_T$  and let  $(\Lambda'_0, \Lambda''_0) \in AP_n(T)$ . Let  $(\Lambda'_1, \Lambda''_1), (\Lambda'_2, \Lambda''_2)$  be two pairs of c-lattices of  $T$  such that  $(\Lambda'_2, \Lambda''_2) \succ (\Lambda'_1, \Lambda''_1) \succ (\Lambda'_0, \Lambda''_0)$ . Let  $u : U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0} \rightarrow U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}$  be the transition map as in Section 4.2.3.*

*For  $J_1 \in \mathbf{P}_{\Lambda''_1/\Lambda'_1}$  such that  $J_1 \not\subset \Lambda'_0/\Lambda'_1$ , put  $\mathfrak{J}_2 = \{J_2 \in \mathbf{P}_{\Lambda''_2/\Lambda'_2} | \text{im}(J_2 \cap \Lambda''_1 \rightarrow \Lambda''_2/\Lambda'_2) = J_1\}$ . For  $H_1 \in \mathbf{P}_{(\Lambda''_1/\Lambda'_1)^*}$  such that  $H_1 \not\supset \Lambda''_0/\Lambda'_1$ , put  $\mathfrak{H}_2 = \{H_2 \in \mathbf{P}_{(\Lambda''_2/\Lambda'_2)^*} | \text{im}(H_2 \cap \Lambda''_1 \rightarrow \Lambda''_2/\Lambda'_2) = H_1\}$ .*

*Then we have equalities of divisors*

$$u^*(\Delta_{\Lambda''_1/\Lambda'_1, J_1}^{n(\Lambda''_1)} \cap U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}) = \sum_{J_2 \in \mathfrak{J}_2} \Delta_{\Lambda''_2/\Lambda'_2, J_2}^{n(\Lambda''_2)} \cap U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0},$$

$$u^*(\Delta_{\Lambda''_1/\Lambda'_1, H_1}^{n(\Lambda''_1)} \cap U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}) = \sum_{H_2 \in \mathfrak{H}_2} \Delta_{\Lambda''_2/\Lambda'_2, H_2}^{n(\Lambda''_2)} \cap U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0}.$$

*Proof.* The statement follows from Lemmas 4.3.2, 4.3.3 and 4.3.5.  $\square$

**Lemma 4.3.5.** *With the same notation as in Proposition 4.3.4, the following commutative diagram is Cartesian.*  $\square$



$$\begin{array}{ccc}
U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0} & \longrightarrow & {}^\circ\text{ToySht}_{\Lambda''_2/\Lambda'_2}^{n(\Lambda''_2), \Lambda'_2/\Lambda'_1, \Lambda''_2/\Lambda'_1} \\
\downarrow & & \downarrow \\
U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0} & \longrightarrow & {}^\circ\text{ToySht}_{\Lambda''_1/\Lambda'_1}^{n(\Lambda''_1)}
\end{array}$$

**Lemma 4.3.6.** *Use the same notation of Proposition 4.3.4. Let  $J_2 \in \mathfrak{J}_2$ . Put  $Y_1 = \Delta_{\Lambda''_1/\Lambda'_1, J_1}^{n(\Lambda''_1)} \cap U_{\Lambda'_1, \Lambda''_1}^{n, \Lambda'_0, \Lambda''_0}$  and  $Y_2 = \Delta_{\Lambda''_2/\Lambda'_2, J_2}^{n(\Lambda''_2)} \cap U_{\Lambda'_2, \Lambda''_2}^{n, \Lambda'_0, \Lambda''_0}$ . Then we have  $u(Y_2) \subset Y_1$ , and the morphism  $Y_2 \rightarrow Y_1$  induced by  $u$  is dominant.*

*Proof.* It is clear that  $u(Y_2) \subset Y_1$ . Since  $(\Lambda'_0, \Lambda''_0) \in AP_n(T)$ , both  $Y_1$  and  $Y_2$  are nonempty. Hence they are irreducible by Proposition 3.2.23. Then Proposition 4.3.4 shows that  $Y_2$  is an irreducible component of  $u^{-1}(Y_1)$ . Now the statement follows from Corollary 4.2.10.  $\square$

### 4.3.3 Functoriality for non-horospherical toy shtukas

**Lemma 4.3.7.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Then the morphism  ${}^\circ\text{ToySht}_V^{n, V', V''} \rightarrow {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  is surjective.*

*Proof.* Fix a splitting  $V = V' \oplus V''/V' \oplus V/V''$ . Let  $L \in {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V''}(E)$ , where  $E$  is a field over  $\mathbb{F}_q$ . Then  $\tilde{L} = L \oplus ((V/V'') \otimes E) \subset V \otimes E$  is an  $E$ -point of  ${}^\circ\text{ToySht}_V^{n, V', V''}$  which maps to  $L$ .  $\square$

**Lemma 4.3.8.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$  such that  $\dim V''/V' \geq 3$ . Suppose  $J \in \mathbf{P}_V$  satisfies  $J \not\subset V''$ . Then the composition  $\Delta_{V, J}^{n, V', V''} \hookrightarrow {}^\circ\text{ToySht}_V^{n, V', V''} \rightarrow {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  is smooth.*

*Proof.* We choose a subspace  $V''' \subset V$  such that  $V''' \supset V''$  and  $V''' \oplus J = V$ . Then we get an isomorphism  ${}^\circ\text{ToySht}_{V'''}^{n-1, V', V''} \xrightarrow{\sim} \Delta_{V, J}^{n, V', V''}$ . The composition  ${}^\circ\text{ToySht}_{V'''}^{n-1, V', V''} \xrightarrow{\sim} \Delta_{V, J}^{n, V', V''} \hookrightarrow {}^\circ\text{ToySht}_V^{n, V', V''} \rightarrow {}^\circ\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  is the natural one, and is smooth by Proposition 4.2.9. The statement follows.  $\square$

Recall that the notation  ${}^{\circ\circ}\text{ToySht}_V^n$  is defined in Section 3.4.2.

**Lemma 4.3.9.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$  such that  $\dim V''/V' \geq 3$ . Assume  $\dim V/V'' + 1 \leq n \leq \dim V/V' - 1$ . Then  ${}^{\circ\circ}\text{ToySht}_V^n$  is contained in  ${}^{\circ}\text{ToySht}_V^{n,V',V''}$ .*

*Proof.* Let  $L \in {}^{\circ\circ}\text{ToySht}_V^n(E)$ , where  $E$  is a field over  $\mathbb{F}_q$ . From the description of Schubert divisors in Theorem 3.3.4, we see that  $L \cap (W \otimes E) = 0$  for any subspace  $W \subset V$  of codimension  $n$ . Thus  $\dim_E(V \cap (U \otimes E)) = \max\{0, n - \dim V/U\}$  for any subspace  $U \subset V$ . In particular, we have  $L \in \text{Grass}_V^{n,V',V''}(E)$ .

Let  $\bar{L} = \text{im}(L \cap (V'' \otimes E) \rightarrow (V''/V') \otimes E)$ . Then

$$\dim_E(\bar{L} \cap (M \otimes E)) = \max\{0, n - \dim V/V' + \dim M\} \quad (4.2)$$

for any subspace  $M \subset V''/V'$ . Suppose  $\bar{L} = M \otimes E$  for some subspace  $M \subset V''/V'$ . Then  $\dim_E(\bar{L} \cap (M \otimes E)) = \dim_E \bar{L} = n - \dim V/V''$ . On the other hand,  $\dim_E(\bar{L} \cap (M \otimes E)) = \dim M$ . We get a contradiction to (4.2) from the assumption  $\dim V/V'' + 1 \leq n \leq \dim V/V' - 1$ . Thus  $\bar{L}$  is a nontrivial toy shtuka for  $V''/V'$ .  $\square$

**Corollary 4.3.10.** *For  $(\Lambda', \Lambda''), (\tilde{\Lambda}', \tilde{\Lambda}'') \in AP_n(T)$  such that  $(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ ,  ${}^{\circ\circ}\text{ToySht}_{\tilde{\Lambda}''/\tilde{\Lambda}'}^{n(\tilde{\Lambda}'')}$  is contained in  $U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ .*  $\square$

Recall that  $\Delta_V^n$  denotes the union of all toy horospherical divisors of  ${}^{\circ}\text{ToySht}_V^n$ .

**Lemma 4.3.11.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$  such that  $\dim V''/V' \geq 3$ . Assume  $\dim V/V'' + 1 \leq n \leq \dim V/V' - 1$ . Then the inverse image of  $\Delta_{V''/V'}^{n - \dim V/V''}$  under the morphism  ${}^{\circ}\text{ToySht}_V^{n,V',V''} \rightarrow {}^{\circ}\text{ToySht}_{V''/V'}^{n - \dim V/V''}$  is set-theoretically contained in  $\Delta_V^n$ .*

*Proof.* The statement follows from Lemma 4.3.1 and the dual of it.  $\square$

**Remark 4.3.12.** Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$  such that  $\dim V''/V' \geq 3$ . Assume  $\dim V/V'' + 1 \leq n \leq \dim V/V' - 1$ . Then Lemma 4.3.9

and Lemma 4.3.11 show that the morphism  ${}^{\circ}\text{ToySht}_V^{n,V',V''} \rightarrow {}^{\circ}\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  induces a morphism  ${}^{\circ\circ}\text{ToySht}_V^n \rightarrow {}^{\circ\circ}\text{ToySht}_{V''/V'}^{n-\dim V/V''}$ .

**Lemma 4.3.13.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$  such that  $\dim V''/V' \geq 3$ . Assume  $\dim V/V'' + 1 \leq n \leq \dim V/V' - 1$ . Then the morphism  ${}^{\circ\circ}\text{ToySht}_V^n \rightarrow {}^{\circ\circ}\text{ToySht}_{V''/V'}^{n-\dim V/V''}$  is affine, smooth and surjective.*

*Proof.* The morphism is affine by Lemma 4.2.5 and is smooth by Proposition 4.2.9.

Now we prove that the morphism is surjective.

Denote  $u : {}^{\circ}\text{ToySht}_V^{n,V',V''} \rightarrow {}^{\circ}\text{ToySht}_{V''/V'}^{n-\dim V/V''}$ . Let  $x \in {}^{\circ\circ}\text{ToySht}_{V''/V'}^{n-\dim V/V''}$ .

We show that  $u^{-1}(x) \cap {}^{\circ\circ}\text{ToySht}_V^n$  is nonempty.

The morphism  $u$  is surjective by Lemma 4.3.7. Hence  $u^{-1}(x)$  is nonempty.

We have

$${}^{\circ\circ}\text{ToySht}_V^n = {}^{\circ}\text{ToySht}_V^{n,V',V''} - \left( \bigcup_{\substack{H \in \mathbf{P}_{V^*} \\ H \not\supset V''}} \Delta_{V,H}^{n,V',V''} \right) \cup \left( \bigcup_{\substack{J \in \mathbf{P}_V \\ J \not\subset V'}} \Delta_{V,J}^{n,V',V''} \right).$$

For  $J \in \mathbf{P}_V$  such that  $J \subset V''$  and  $J \not\subset V'$ , we have  $u(\Delta_{V,J}^{n,V',V''}) \subset \Delta_{V''/V',\bar{J}}^{n-\dim V/V''}$ , where  $\bar{J} = \text{im}(J \rightarrow V''/V')$ . So  $u^{-1}(x) \cap \Delta_{V,J}^{n,V',V''}$  is empty. For  $J \in \mathbf{P}_V$  such that  $J \not\subset V''$ , Lemma 4.3.8 implies that  $u^{-1}(x) \cap \Delta_{V,J}^{n,V',V''}$  has codimension 1 in  $u^{-1}(x)$ . Thus  $u^{-1}(x) \cap \left( \bigcup_{\substack{J \in \mathbf{P}_V \\ J \not\subset V'}} \Delta_{V,J}^{n,V',V''} \right)$  has codimension 1 in  $u^{-1}(x)$ .

Similarly,  $u^{-1}(x) \cap \left( \bigcup_{\substack{H \in \mathbf{P}_{V^*} \\ H \not\supset V''}} \Delta_{V,H}^{n,V',V''} \right)$  has codimension 1 in  $u^{-1}(x)$ .

The statement follows.  $\square$

Let  $T$  be a nondiscrete noncompact Tate space over  $\mathbb{F}_q$  and  $n \in \text{Dim}_T$ . We denote

$${}^{\circ\circ}\text{ToySht}_T^n = \varprojlim_{(\Lambda', \Lambda'') \in AP_n(T)} {}^{\circ\circ}\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$$

**Lemma 4.3.14.** *We have*

$${}^{\circ\circ}\mathrm{ToySht}_T^n = {}^{\circ}\mathrm{ToySht}_T^n - \left( \bigcup_{H \in \mathbf{P}_{T^*}} \Delta_{T,H}^n \right) \cup \left( \bigcup_{J \in \mathbf{P}_T} \Delta_{T,J}^n \right).$$

*Proof.* Let  $J \in \mathbf{P}_T$  and  $x \in \Delta_{T,J}^n$ . Choose  $(\Lambda', \Lambda'') \in AP_n(T)$  such that  $J \not\subset \Lambda'$ ,  $J \subset \Lambda''$  and  $x \in {}^{\circ}\mathrm{ToySht}_T^{n, \Lambda', \Lambda''}$ . Then the image of  $x$  under the morphism  ${}^{\circ}\mathrm{ToySht}_T^{n, \Lambda', \Lambda''} \rightarrow {}^{\circ}\mathrm{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  is contained in  $\Delta_{\Lambda''/\Lambda'}^{n(\Lambda'')}$ . There is a similar statement for  $y \in \Delta_{T,H}^n$ ,  $H \in \mathbf{P}_{T^*}$ .

On the other hand, Lemmas 4.2.3 and 4.3.11 imply that for any  $(\Lambda', \Lambda'') \in AP_n(T)$  the inverse image of  $\Delta_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  under the morphism  ${}^{\circ}\mathrm{ToySht}_T^{n, \Lambda', \Lambda''} \rightarrow {}^{\circ}\mathrm{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  is set-theoretically contained in the union of  $\Delta_{T,H}^n$  for  $H \in \mathbf{P}_{T^*}$  and  $\Delta_{T,J}^n$  for  $J \in \mathbf{P}_T$ .

The statement follows.  $\square$

**Lemma 4.3.15.** *For  $(\Lambda', \Lambda'') \in AP_n(T)$ ,  ${}^{\circ\circ}\mathrm{ToySht}_T^n$  is contained in  ${}^{\circ}\mathrm{ToySht}_T^{n, \Lambda', \Lambda''}$ .*

*Proof.* The proof is similar to that of Lemma 4.3.9.  $\square$

**Lemma 4.3.16.** *For  $(\Lambda', \Lambda'') \in AP_n(T)$ , the morphism  ${}^{\circ\circ}\mathrm{ToySht}_T^n \rightarrow {}^{\circ\circ}\mathrm{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  is surjective.*

*Proof.* The statement follows from Lemma 4.3.13.  $\square$

## 4.4 Tate toy horospherical subschemes

Fix a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$  and fix  $n \in \mathrm{Dim}_T$ .

In this section, we use the notation of Sections 4.2.1 and 4.2.2. We also frequently use Lemma 4.2.3 to describe open charts of  ${}^{\circ}\mathrm{ToySht}_T^n$  as a projective limit.

### 4.4.1 Basic properties of Tate toy horospherical subschemes

**Proposition 4.4.1.**  *${}^{\circ}\mathrm{ToySht}_T^n$  is irreducible.*

*Proof.* For  $(\Lambda', \Lambda''), (\widetilde{\Lambda}', \widetilde{\Lambda}'') \in AP_n(T)$  satisfying  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ ,  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  is a nonempty open subscheme of  ${}^\circ\text{ToySht}_{\widetilde{\Lambda}'/\widetilde{\Lambda}''}^{n(\widetilde{\Lambda}'')}$ , hence is irreducible by Proposition 3.2.23. So  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  is irreducible as the projective limit. Note that  ${}^\circ\text{ToySht}_T^n$  is the union of  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  for all  $(\Lambda', \Lambda'') \in AP_n(T)$ , and  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$  implies  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''} \subset {}^\circ\text{ToySht}_T^{n, \widetilde{\Lambda}', \widetilde{\Lambda}''}$ . Thus  ${}^\circ\text{ToySht}_T^n$  is irreducible.  $\square$

For  $J \in \mathbf{P}_T$ , denote  $\Delta_{T,J}^n = {}^\circ\text{ToySht}_T^n \cap \text{ToySht}_{T/J}$ . It is called a *Tate toy horospherical subscheme* of  ${}^\circ\text{ToySht}_T^n$ . Since  $T/J$  is a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ , Proposition 4.4.1 implies that  $\Delta_{T,J}^n$  is irreducible. Let  $\eta_J^n$  be the generic point of  $\Delta_{T,J}^n$ . For two c-lattices  $\Lambda' \subset \Lambda''$  of  $T$ , denote  $\Delta_{T,J}^{n, \Lambda', \Lambda''} = \Delta_{T,J}^n \cap {}^\circ\text{Grass}_T^{n, \Lambda', \Lambda''}$ .

For  $H \in \mathbf{P}_{T^*}$ , denote  $\Delta_{T,H}^n = {}^\circ\text{ToySht}_T^n \cap \text{ToySht}_H$ . It is called a *Tate toy horospherical subscheme* of  ${}^\circ\text{ToySht}_T^n$ . Since  $H$  is a nondiscrete noncompact Tate space over  $\mathbb{F}_q$ , Proposition 4.4.1 implies that  $\Delta_{T,H}^n$  is irreducible. Let  $\eta_H^n$  be the generic point of  $\Delta_{T,H}^n$ . For two c-lattices  $\Lambda' \subset \Lambda''$  of  $T$ , denote  $\Delta_{T,H}^{n, \Lambda', \Lambda''} = \Delta_{T,H}^n \cap {}^\circ\text{Grass}_T^{n, \Lambda', \Lambda''}$ .

**Lemma 4.4.2.** *Let  $E$  be a field over  $\mathbb{F}_q$ . Let  $P$  be a  $d$ -lattice of the Tate space  $T \widehat{\otimes} E$  over  $E$ . Let  $W$  be a finite dimensional subspace of  $T$  such that  $P \cap (W \otimes E) = 0$ . Then there exists a c-lattice  $L$  of  $T$  such that  $L \supset W$  and  $P \cap (L \widehat{\otimes} E) = 0$ .  $\square$*

**Lemma 4.4.3.** *Let  $J \in \mathbf{P}_T$  and  $(\Lambda', \Lambda'') \in AP_n(T)$ . For any  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$  such that  $J \subset \widetilde{\Lambda}''$ , we denote  $J_{(\widetilde{\Lambda}', \widetilde{\Lambda}'')} = \text{im}(J \rightarrow \widetilde{\Lambda}''/\widetilde{\Lambda}') \in \mathbf{P}_{\widetilde{\Lambda}''/\widetilde{\Lambda}'}$  and  $Z_{(\widetilde{\Lambda}', \widetilde{\Lambda}''), J} = \Delta_{\widetilde{\Lambda}''/\widetilde{\Lambda}', J_{(\widetilde{\Lambda}', \widetilde{\Lambda}'')}}^{n(\widetilde{\Lambda}'')}$ . Then the isomorphism in Lemma 4.2.3 induces an isomorphism*

$$\Delta_{T,J}^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \varprojlim_{(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')} Z_{(\widetilde{\Lambda}', \widetilde{\Lambda}''), J}.$$

*Proof.* It is clear that the image of  $\Delta_{T,J}^{n, \Lambda', \Lambda''}$  is contained in  $Z_{(\widetilde{\Lambda}', \widetilde{\Lambda}''), J}$  under each

morphism  ${}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''} \rightarrow U_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''}$ .

Suppose  $x \in {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}(E)$  but  $x \notin \Delta_{T,J}^{n,\Lambda',\Lambda''}(E)$ , where  $E$  is a field over  $\mathbb{F}_q$ . So  $x$  corresponds to a d-lattice  $P$  of the Tate space  $T \widehat{\otimes} E$  over  $E$  such that  $P \cap (J \otimes E) = 0$ . Then Lemma 4.4.2 shows that there exists a c-lattice  $\Lambda'_0$  of  $T$  such that  $P \cap (\Lambda'_0 \widehat{\otimes} E) = 0$ . Let  $\widetilde{\Lambda}' = \Lambda' \cap \Lambda'_0$  and  $\widetilde{\Lambda}'' = \Lambda''$ . Then the image of  $x$  in  $U_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''}$  is not contained in  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$ . Therefore, the intersection of the preimages of  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  in  ${}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$  for all  $(\widetilde{\Lambda'},\widetilde{\Lambda'') \succ (\Lambda',\Lambda'')$  is set-theoretically equal to  $\Delta_{T,J}^{n,\Lambda',\Lambda''}$ .

Since  $\Delta_{T,J}^{n,\Lambda',\Lambda''}$  and all  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  are reduced, the statement follows.  $\square$

**Lemma 4.4.4.** *For  $J \in \mathbf{P}_T$  and  $(\Lambda', \Lambda'') \in AP_n(T)$ ,  $\eta_J^n \in {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$  if and only if  $J \not\subset \Lambda'$ . For  $H \in \mathbf{P}_{T^*}$  and  $(\Lambda', \Lambda'') \in AP_n(T)$ ,  $\eta_H^n \in {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$  if and only if  $H \not\supset \Lambda''$ .*

*Proof.* If  $J \subset \Lambda'$ , then  $\text{ToySht}_{T/J} \cap {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$  is empty, so  $\eta_J^n \notin {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$ .

Suppose  $J \not\subset \Lambda'$ . For any  $(\widetilde{\Lambda'}, \widetilde{\Lambda'') \succ (\Lambda', \Lambda'')$  such that  $J \subset \widetilde{\Lambda''}$ , let  $J_{(\widetilde{\Lambda'},\widetilde{\Lambda''})}$  and  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  be as in Lemma 4.4.3. Since  $(\Lambda', \Lambda'') \in AP_n(T)$ , we have  $n(\widetilde{\Lambda'') \geq 2$ , so  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  is nonempty. Lemmas 4.4.3 and 4.3.6 imply that  $\Delta_{T,J}^{n,\Lambda',\Lambda''}$  is nonempty. Since  ${}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$  is open in  ${}^{\circ}\text{ToySht}_T^n$ , we deduce that  $\eta_J^n \in {}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$ .

The proof about  $\eta_H^n$  is similar.  $\square$

**Lemma 4.4.5.** *A filtered inductive limit of discrete valuation rings with common uniformizer is a discrete valuation ring with the same uniformizer.*  $\square$

**Lemma 4.4.6.** *For  $J \in \mathbf{P}_T$ , the local ring of  $\eta_J^n$  viewed as a point of  ${}^{\circ}\text{ToySht}_T^n$  is a discrete valuation ring.*

*Proof.* Choose  $(\Lambda', \Lambda'') \in AP_n(T)$  such that  $J \not\subset \Lambda'$  and  $J \subset \Lambda''$ . By Lemma 4.4.4,  $\eta_J^n$  is in  ${}^{\circ}\text{ToySht}_T^{n,\Lambda',\Lambda''}$ . For any  $(\widetilde{\Lambda'}, \widetilde{\Lambda'') \succ (\Lambda', \Lambda'')$  such that  $J \subset \widetilde{\Lambda''}$ , let  $J_{(\widetilde{\Lambda'},\widetilde{\Lambda''})}$  and  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  be as in Lemma 4.4.3. Since  $J \subset \Lambda''$ , each  $Z_{(\widetilde{\Lambda'},\widetilde{\Lambda''}),J}$  is nonempty,

hence irreducible by Proposition 3.2.23. Let  $A_{(\tilde{\Lambda}', \tilde{\Lambda}''), J}$  denote the local ring of  $Z_{(\tilde{\Lambda}', \tilde{\Lambda}''), J}$  in  $U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ . It is a discrete valuation ring since each  $Z_{(\tilde{\Lambda}', \tilde{\Lambda}''), J}$  has codimension 1 in  $U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ . Let  $A_J$  be the local ring of  $\eta_J^n$  viewed as a point of  ${}^\circ\text{ToySht}_T^n$ . We see that  $A_J$  is the local ring of  $\Delta_{T, J}^{n, \Lambda', \Lambda''}$  in  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$ . Then the projective limit  $\Delta_{T, J}^{n, \Lambda', \Lambda''} = \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} Z_{(\tilde{\Lambda}', \tilde{\Lambda}''), J}$  gives an isomorphism  $A_J = \varinjlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} A_{(\tilde{\Lambda}', \tilde{\Lambda}''), J}$ . Proposition 4.3.4 shows that for any  $(\Lambda'_2, \Lambda''_2) \succ (\Lambda'_1, \Lambda''_1) \succ (\Lambda', \Lambda'')$ , the morphism  $A_{(\Lambda'_1, \Lambda''_1), J} \rightarrow A_{(\Lambda'_2, \Lambda''_2), J}$  maps the uniformizer to the uniformizer. So the statement follows from Lemma 4.4.5.  $\square$

The following statement is dual to Lemma 4.4.6.

**Lemma 4.4.7.** *For  $H \in \mathbf{P}_{T^*}$ , the local ring of  $\eta_H^n$  viewed as a point of  ${}^\circ\text{ToySht}_T^n$  is a discrete valuation ring.*

*Remark 4.4.8.* Although  $\Delta_{T, J}^n (J \in \mathbf{P}_T)$  and  $\Delta_{T, H}^n (H \in \mathbf{P}_{T^*})$  have codimension 1 in  ${}^\circ\text{ToySht}_T^n$ , they are not Cartier divisors. (See Theorem 4.4.12.)

#### 4.4.2 The group of divisors supported on the union of Tate toy horospherical subschemes

Formulation of the main result

For an open subset  $U$  of a Tate space  $M$ , we denote  $C^\infty(U)$  to be the (partially) ordered abelian group of locally constant  $\mathbb{Z}$ -valued functions on  $U$ , and denote  $C_c^\infty(U)$  to be the (partially) ordered abelian group of locally constant  $\mathbb{Z}$ -valued functions on  $U$  with compact support. We define

$$C_+(U) := \{f \in C^\infty(U) \mid \text{supp } f \text{ is contained in some compact subset of } M\}$$

**Lemma 4.4.9.** *We have an isomorphism of ordered abelian groups  $C_+(M - \{0\}) \cong$*

$\varprojlim C_c^\infty(M - \Lambda)$ , where the projective limit is taken with respect to the directed set of all c-lattices  $\Lambda$  of  $M$ .

*Proof.* Remark 4.1.2 and Remark 4.1.8 imply that  $(M - \{0\})$  is the union of  $(M - \Lambda)$  for all c-lattices  $\Lambda$  of  $M$ . The statement follows.  $\square$

**Definition 4.4.10.** Let  $\mathfrak{D}_T^n$  denote the ordered abelian group of those Cartier divisors of  ${}^\circ\text{ToySht}_T^n$  whose restrictions to  ${}^\circ\text{ToySht}_T^n$  are zero. Elements in  $\mathfrak{D}_T^n$  are called *(Tate) toy horospherical divisors* of  ${}^\circ\text{ToySht}_T^n$ .

*Remark 4.4.11.* Lemma 4.3.14 shows that a Cartier divisor of  ${}^\circ\text{ToySht}_T^n$  is an element of  $\mathfrak{D}_T^n$  if and only if its support<sup>1</sup> is contained in the union of  $\Delta_{T,H}^n$  and  $\Delta_{T,J}^n$  for all  $H \in \mathbf{P}_{T^*}$  and  $J \in \mathbf{P}_T$ .

From Lemmas 4.4.6 and 4.4.7, we get an ordered homomorphism from  $\mathfrak{D}_T^n$  to  $\text{Maps}(\mathbf{P}_{T^*} \coprod \mathbf{P}_T, \mathbb{Z})$ , where  $\text{Maps}(\mathbf{P}_{T^*} \coprod \mathbf{P}_T, \mathbb{Z})$  is the ordered abelian group of  $\mathbb{Z}$ -valued functions on  $\mathbf{P}_{T^*} \coprod \mathbf{P}_T$ .

The goal of this section is to prove the following explicit description of  $\mathfrak{D}_T^n$ .

**Theorem 4.4.12.** *The homomorphism  $\mathfrak{D}_T^n \rightarrow \text{Maps}(\mathbf{P}_{T^*} \coprod \mathbf{P}_T, \mathbb{Z})$  induces an isomorphism of ordered abelian groups  $\mathfrak{D}_T^n \cong C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times}$ .*

### Reduction of Theorem 4.4.12 to Lemma 4.4.13

For two c-lattices  $\Lambda' \subset \Lambda''$  of  $T$ , let  $\mathfrak{D}_T^{n,\Lambda',\Lambda''}$  denote the (partially) ordered abelian group of those Cartier divisors of  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''}$  whose restrictions to  ${}^\circ\text{ToySht}_T^n$  are zero.

From Lemmas 4.4.4, 4.4.6 and 4.4.7, we get an ordered homomorphism  $\mathfrak{D}_T^{n,\Lambda',\Lambda''} \rightarrow \text{Maps}((T^* - (\Lambda'')^\perp) \coprod (T - \Lambda'), \mathbb{Z})$ .

---

1. The *support* of a divisor is the smallest closed subset such that the restriction of the divisor to its complement is zero.



The isomorphism in Lemma 4.2.3 induces an isomorphism of ordered abelian groups

$$\mathfrak{D}_T^n \xrightarrow{\sim} \varprojlim_{(\Lambda', \Lambda'') \in AP_n(T)} \mathfrak{D}_T^{n, \Lambda', \Lambda''}.$$

So Theorem 4.4.12 follows from Lemma 4.4.13.

**Lemma 4.4.13.** *For  $(\Lambda', \Lambda'') \in AP_n(T)$ , the homomorphism  $\mathfrak{D}_T^{n, \Lambda', \Lambda''} \rightarrow \text{Maps}((T^* - (\Lambda'')^\perp) \amalg (T - \Lambda'), \mathbb{Z})$  induces an isomorphism of ordered abelian groups  $\mathfrak{D}_T^{n, \Lambda', \Lambda''} \cong C_c^\infty(T^* - (\Lambda'')^\perp)^{\mathbb{F}_q^\times} \oplus C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}$ .*

Reduction of Lemma 4.4.13 to Lemmas 4.4.14, 4.4.15 and 4.4.16

Let  $\Lambda', \Lambda'', \widetilde{\Lambda}', \widetilde{\Lambda}''$  be c-lattices of  $T$  such that  $\widetilde{\Lambda}' \subset \Lambda' \subset \Lambda'' \subset \widetilde{\Lambda}''$ . We have injective homomorphisms of ordered abelian groups

$$C^\infty((\widetilde{\Lambda}''/\widetilde{\Lambda}') - (\Lambda'/\widetilde{\Lambda}'))^{\mathbb{F}_q^\times} \rightarrow C_c^\infty(\widetilde{\Lambda}'' - \Lambda')^{\mathbb{F}_q^\times} \rightarrow C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}, \quad (4.3)$$

where the first homomorphism is induced by the map  $(\widetilde{\Lambda}'' - \Lambda') \rightarrow ((\widetilde{\Lambda}''/\widetilde{\Lambda}') - (\Lambda'/\widetilde{\Lambda}'))$ , the second homomorphism is extension by zero from  $(\widetilde{\Lambda}'' - \Lambda')$  to  $(T - \Lambda')$ . We consider  $C^\infty((\widetilde{\Lambda}''/\widetilde{\Lambda}') - (\Lambda'/\widetilde{\Lambda}'))^{\mathbb{F}_q^\times}$  as an ordered subgroup of  $C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}$ .

Similarly, we have injective homomorphisms of ordered abelian groups

$$C^\infty((\widetilde{\Lambda}''/\widetilde{\Lambda}')^* - (\Lambda''/\widetilde{\Lambda}')^\perp)^{\mathbb{F}_q^\times} \rightarrow C_c^\infty((\widetilde{\Lambda}')^\perp - (\Lambda'')^\perp)^{\mathbb{F}_q^\times} \rightarrow C_c^\infty(T^* - (\Lambda'')^\perp)^{\mathbb{F}_q^\times}. \quad (4.4)$$

We consider  $C^\infty((\widetilde{\Lambda}''/\widetilde{\Lambda}')^* - (\Lambda''/\widetilde{\Lambda}')^\perp)^{\mathbb{F}_q^\times}$  as an ordered subgroup of  $C_c^\infty(T^* - (\Lambda'')^\perp)^{\mathbb{F}_q^\times}$ .

For  $(\Lambda', \Lambda'') \in AP_n(T)$  and  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ , we define  $\mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  to be the ordered abelian group of those Cartier divisors of  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  whose restrictions to  ${}^{\circ\circ}\text{ToySht}_{\widetilde{\Lambda}''/\widetilde{\Lambda}'}^{n(\widetilde{\Lambda}'')}$  are zero.

Now Lemma 4.4.13 follows from Lemmas 4.4.14, 4.4.15, and 4.4.16.

**Lemma 4.4.14.** *For  $(\Lambda', \Lambda'') \in AP_n(T)$ , the isomorphism*

$${}^{\circ}\text{ToySht}_T^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} U_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''}$$

*in Lemma 4.2.3 induces an isomorphism of ordered abelian groups*

$$\xi : \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} \mathfrak{D}_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \mathfrak{D}_T^{n, \Lambda', \Lambda''}.$$

**Lemma 4.4.15.** *Let  $\Lambda', \Lambda''$  be two  $c$ -lattices of  $T$  such that  $\Lambda' \subset \Lambda''$ . Then the map  $\phi$  induced by (4.3) and the map  $\psi$  induced by (4.4) are isomorphisms of ordered abelian groups.*

$$\phi : \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} C^\infty((\tilde{\Lambda}''/\tilde{\Lambda}') - (\Lambda'/\tilde{\Lambda}'))^{\mathbb{F}_q^\times} \xrightarrow{\sim} C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}$$

$$\psi : \varprojlim_{(\tilde{\Lambda}', \tilde{\Lambda}'') \succ (\Lambda', \Lambda'')} C^\infty((\tilde{\Lambda}''/\tilde{\Lambda}')^* - (\Lambda''/\tilde{\Lambda}')^\perp)^{\mathbb{F}_q^\times} \xrightarrow{\sim} C_c^\infty(T^* - (\Lambda'')^\perp)^{\mathbb{F}_q^\times}$$

**Lemma 4.4.16.** *For  $(\Lambda', \Lambda'') \in AP_n(T)$ , the composition  $\mathfrak{D}_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''} \rightarrow \mathfrak{D}_T^{n, \Lambda', \Lambda''} \rightarrow \text{Maps}((T^* - (\Lambda'')^\perp) \amalg (T - \Lambda'), \mathbb{Z})$  induces an isomorphism of ordered abelian groups*

$$\mathfrak{D}_{\tilde{\Lambda}', \tilde{\Lambda}''}^{n, \Lambda', \Lambda''} \xrightarrow{\sim} C^\infty((\tilde{\Lambda}''/\tilde{\Lambda}')^* - (\Lambda''/\tilde{\Lambda}')^\perp)^{\mathbb{F}_q^\times} \oplus C^\infty((\tilde{\Lambda}''/\tilde{\Lambda}') - (\Lambda'/\tilde{\Lambda}'))^{\mathbb{F}_q^\times}.$$

Proofs of Lemma 4.4.15 and Lemma 4.4.16

*Proof of Lemma 4.4.15.* It is obvious that each homomorphism

$$\phi_{(\tilde{\Lambda}', \tilde{\Lambda}'')} : C^\infty((\tilde{\Lambda}''/\tilde{\Lambda}') - (\Lambda'/\tilde{\Lambda}'))^{\mathbb{F}_q^\times} \rightarrow C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}$$

is injective, and  $\phi_{(\tilde{\Lambda}', \tilde{\Lambda}'')}(f) \leq \phi_{(\tilde{\Lambda}', \tilde{\Lambda}'')}(g)$  if and only if  $f \leq g$ .

Now to prove that  $\phi$  is an isomorphism of ordered abelian groups, it suffices to show that  $\phi$  is surjective.

Let  $f \in C_c^\infty(T - \Lambda')^{\mathbb{F}_q^\times}$ .

Since  $f$  is compactly supported, its support is contained in some c-lattice  $\widetilde{\Lambda}''$ .

Since  $f$  is locally constant, from Remark 4.1.8 we see that for every  $x \in (T - \Lambda')$  there is a c-lattice  $\Lambda_x$  of  $T$  such that  $f$  is constant on  $x + \Lambda_x$ . Since  $f$  is compactly supported, we can find finitely many  $x_1, \dots, x_r \in T$  and c-lattices  $\Lambda_1, \dots, \Lambda_r$  of  $T$  such that  $\text{supp } f \subset \bigcup_{i=1}^r (x_i + \Lambda_i)$  and  $f$  is constant on each  $x_i + \Lambda_i$  ( $1 \leq i \leq r$ ). We get a c-lattice  $\widetilde{\Lambda}' = \bigcap_{i=1}^r \Lambda_i$ .

Thus  $f$  is in the image of  $C^\infty((\widetilde{\Lambda}''/\widetilde{\Lambda}') - (\Lambda'/\widetilde{\Lambda}'))^{\mathbb{F}_q^\times}$ . This shows that  $\phi$  is surjective.

The proof for  $\psi$  is similar. □

*Proof of Lemma 4.4.16.* The statement follows from Lemma 4.4.17. □

**Lemma 4.4.17.** *Let  $(\Lambda', \Lambda'') \in AP_n(T)$  and  $Z \in \mathfrak{D}_T^{n, \Lambda', \Lambda''}$ . Assume that  $Z$  equals the pullback of a Cartier divisor  $\widetilde{Z}$  on  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  for some  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ .*

*Suppose  $J \in \mathbf{P}_T$  satisfies  $J \not\subset \Lambda'$ . If  $J \not\subset \widetilde{\Lambda}''$ , the multiplicity of  $Z$  at  $\Delta_{T, J}^{n, \Lambda', \Lambda''}$  is zero. If  $J \in \widetilde{\Lambda}''$ , the multiplicity of  $Z$  at  $\Delta_{T, J}^{n, \Lambda', \Lambda''}$  equals the multiplicity of  $\widetilde{Z}$  at  $\Delta_{\widetilde{\Lambda}''/\widetilde{\Lambda}', \widetilde{J}}^{n(\widetilde{\Lambda}'')}$   $\cap U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ , where  $\widetilde{J} = \text{im}(J \rightarrow \widetilde{\Lambda}''/\widetilde{\Lambda}')$ .*

*Suppose  $H \in \mathbf{P}_{T^*}$  satisfies  $H \not\supset \Lambda''$ . If  $H \not\supset \widetilde{\Lambda}'$ , then the multiplicity of  $Z$  at  $\Delta_{T, H}^{n, \Lambda', \Lambda''}$  is zero. If  $H \supset \widetilde{\Lambda}'$ , then the multiplicity of  $Z$  at  $\Delta_{T, H}^{n, \Lambda', \Lambda''}$  equals the multiplicity of  $\widetilde{Z}$  at  $\Delta_{\widetilde{\Lambda}''/\widetilde{\Lambda}', \widetilde{H}}^{n(\widetilde{\Lambda}'')}$   $\cap U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ , where  $\widetilde{H} = \text{im}(H \cap \widetilde{\Lambda}'' \rightarrow \widetilde{\Lambda}''/\widetilde{\Lambda}')$ .*

*Proof.* The statement follows from Proposition 4.3.4, Lemma 4.4.3 and Lemma 4.4.5. □

Proof of Lemma 4.4.14

**Lemma 4.4.18.** *Let  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ . Suppose  $D$  is a Cartier divisor of  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  whose pullback to  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  is an element of  $\mathfrak{D}_T^{n, \Lambda', \Lambda''}$ . Then  $D$  is an element of  $\mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ .*

*Proof.* The statement follows from Lemmas 4.3.14 and 4.3.16.  $\square$

*Proof of Lemma 4.4.14.* By Lemma 4.4.17, each homomorphism  $\xi_{(\widetilde{\Lambda}', \widetilde{\Lambda}'')}^{n, \Lambda', \Lambda''} : \mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''} \rightarrow \mathfrak{D}_T^{n, \Lambda', \Lambda''}$  is injective, and  $\xi_{(\widetilde{\Lambda}', \widetilde{\Lambda}'')}^{n, \Lambda', \Lambda''}(f) \leq \xi_{(\widetilde{\Lambda}', \widetilde{\Lambda}'')}^{n, \Lambda', \Lambda''}(g)$  if and only if  $f \leq g$ .

Now it suffices to show that  $\xi$  is surjective. Let  $D_0 \in \mathfrak{D}_T^{n, \Lambda', \Lambda''}$ . By Lemma 4.2.3, there exists  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$  and a Cartier divisor  $D$  of  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  such that  $D_0$  equals the pullback of  $D$ . Now Lemma 4.4.18 implies that  $D \in \mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ .  $\square$

## 4.5 Schubert divisors of ${}^\circ\text{ToySht}_T^n$

In this section, we fix a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$  and we fix an element  $n \in \text{Dim}_T$ .

### 4.5.1 Schubert divisors of Sato Grassmannians

Let  $W$  be a c-lattice of  $T$  such that  $n(W) = 0$ . We have a perfect complex

$$\mathcal{S}_W^\bullet = (\mathcal{S}_W^{-1} \rightarrow \mathcal{S}_W^0)$$

on  $\text{Grass}_T^n$ , where  $\mathcal{S}_W^{-1} \subset T \widehat{\otimes} \mathcal{O}_{\text{Grass}_T^n}$  is the universal locally free sheaf on  $\text{Grass}_T^n$ ,  $\mathcal{S}_W^0 = (T/W) \otimes \mathcal{O}_{\text{Grass}_T^n}$ , and the map  $\mathcal{S}_W^{-1} \rightarrow \mathcal{S}_W^0$  is the natural one.

Since  $n(W) = 0$ ,  $\mathcal{S}_W^{-1} \rightarrow \mathcal{S}_W^0$  is an isomorphism on the big cell  $\text{Grass}_T^{n, W, W}$ , which is an open dense subscheme of  $\text{Grass}_T^n$ . So the complex  $\mathcal{S}_W^\bullet$  is good in the sense of Knudsen-Mumford.

We define the *Schubert divisor* of  $\text{Grass}_T^n$  for  $W$  to be

$$\text{Schub}_T^W := \text{Div}(\mathcal{S}_W^\bullet).$$

**Lemma 4.5.1.**  $\text{Schub}_T^W \cap {}^\circ\text{ToySht}_T^n$  is a Cartier divisor of  ${}^\circ\text{ToySht}_T^n$ .

*Proof.* We know that  ${}^\circ\text{ToySht}_T^n$  is irreducible and reduced, and  $\text{Schub}_T^W$  is a Cartier divisor of  $\text{Grass}_T^n$ . To prove the statement, it suffices to show that  ${}^\circ\text{ToySht}_T^n$  is not contained in  $\text{Schub}_T^W$ . We have  ${}^\circ\text{ToySht}_T^n - \text{Schub}_T^W = {}^\circ\text{ToySht}_T^{n,W,W}$ . Choose  $(\Lambda', \Lambda'') \in AP_n(T)$  such that  $\Lambda' \subset W \subset \Lambda''$ . Then  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''} \subset {}^\circ\text{ToySht}_T^{n,W,W}$ , and  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''}$  is nonempty by Lemma 4.2.3 and the nonemptiness of each  $U_{\Lambda',\Lambda''}^{n,\Lambda',\Lambda''}$  in the projective limit there.  $\square$

We call  $\text{Schub}_T^W \cap {}^\circ\text{ToySht}_T^n$  the *Schubert divisor* of  ${}^\circ\text{ToySht}_T^n$  for  $W$ .

#### 4.5.2 Schubert divisors of the scheme of Tate toy shtukas

**Theorem 4.5.2.** For a  $c$ -lattice  $W$  of  $T$ , the Schubert divisor  $\text{Schub}_T^W \cap {}^\circ\text{ToySht}_T^n$  corresponds to the function  $(\mathbb{1}_{W^\perp - \{0\}}, \mathbb{1}_{W - \{0\}})$  via Theorem 4.4.12.

*Proof.* Let  $\mathcal{L}$  be the universal Tate toy shtuka on  ${}^\circ\text{ToySht}_T^n$ . Choose  $(\Lambda', \Lambda'') \in AP_n(T)$  such that  $\Lambda' \subset W \subset \Lambda''$ .

On  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''}$ , the complex  $\mathcal{S}_W^\bullet$  is quasi-isomorphic to its subcomplex

$$\mathcal{L} \cap (\Lambda'' \widehat{\otimes} \mathcal{O}_{{}^\circ\text{ToySht}_T^n}) \rightarrow (\Lambda''/W) \otimes \mathcal{O}_{{}^\circ\text{ToySht}_T^n}. \quad (4.5)$$

From the definition of the projection morphism  $u : {}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''} \rightarrow {}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  in Section 4.2.1, we see that the complex (4.5) on  ${}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''}$  is isomorphic to the pullback of the complex  $\mathcal{S}_{\Lambda''/\Lambda', W/\Lambda'}^\bullet$  on  ${}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  by  $u$ . Then Remark 3.3.3

implies that

$$\text{Schub}_T^W \cap {}^\circ\text{ToySht}_T^{n,\Lambda',\Lambda''} = u^*(\text{Schub}_{\Lambda''/\Lambda'}^{W/\Lambda'} \cap {}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}).$$

Since  ${}^\circ\text{ToySht}_T^n$  is covered by the union of  ${}^\circ\text{ToySht}_T^{\Lambda',\Lambda''}$  for  $(\Lambda', \Lambda'') \in AP_n(T)$  such that  $\Lambda' \subset W \subset \Lambda''$ , the statement now follows from Lemma 4.4.17 and Theorem 3.3.4.  $\square$

**Proposition 4.5.3.** *For two  $c$ -lattices  $W_1, W_2$  of  $T$  satisfying  $n(W_1) = n(W_2) = 0$ , the two divisors  $\text{Schub}_T^{W_1}$  and  $\text{Schub}_T^{W_2}$  of  $\text{Grass}_T^n$  are linearly equivalent.*

*Proof.* The statement follows from [17, Theorem 3.3].  $\square$

**Corollary 4.5.4.** *For two  $c$ -lattices  $W_1, W_2$  of  $T$  satisfying  $n(W_1) = n(W_2) = 0$ , the two divisors  $\text{Schub}_T^{W_1} \cap {}^\circ\text{ToySht}_T^n$  and  $\text{Schub}_T^{W_2} \cap {}^\circ\text{ToySht}_T^n$  of  ${}^\circ\text{ToySht}_T^n$  are linearly equivalent.*

*Proof.* The statement follows from Theorem 4.5.2 and Proposition 4.5.3.  $\square$

## 4.6 Principal toy horospherical $\mathbb{Z}[\frac{1}{p}]$ -divisors of ${}^\circ\text{ToySht}_T^n$

Fix a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$  and fix  $n \in \text{Dim}_T$ .

We normalize the Haar measure on  $T$  by the condition that the measure of any  $c$ -lattice  $\Lambda$  equals  $q^{n(\Lambda)}$ .

Recall that the notation  $\mathfrak{D}_{\widetilde{\Lambda'}, \widetilde{\Lambda''}}^{n,\Lambda',\Lambda''}$  is defined in Section 4.4.2. In particular, for  $(\Lambda', \Lambda'') \in AP_n(T)$ ,  $\mathfrak{D}_{\Lambda', \Lambda''}^{n,\Lambda',\Lambda''}$  is the ordered abelian group of horospherical divisors of  ${}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$ .

### 4.6.1 Formulation of the main result

Recall that  $\mathfrak{D}_T^n$  is defined in Section 4.4.2.

Let  $\mathfrak{R}_T^n$  be the subgroup of  $\mathfrak{D}_T^n$  generated by principal divisors.

Fix a nontrivial additive character  $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ . We define the Fourier transform

$$\text{Four}_\psi : C_c^\infty(T; \mathbb{C}) \rightarrow C_c^\infty(T^*; \mathbb{C})$$

such that for any  $f \in C_c^\infty(T; \mathbb{C})$ ,  $\omega \in T^*$ , we have

$$\text{Four}_\psi(f)(\omega) = \int_T f(v) \psi(\omega(v)) dv. \quad (4.6)$$

Recall the Haar measure on  $T$  is normalized by the condition that the measure of any c-lattice  $\Lambda$  equals  $q^{n(\Lambda)}$ .

When  $f \in C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ , we have  $\text{Four}_\psi(f) \in C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ , and  $\text{Four}_\psi(f)$  does not depend on the choice of  $\psi$ .

For an open subset  $U \subset T$ , we define  $C_0^\infty(U) := \{f \in C_c^\infty(U) \mid \int_U f(v) dv = 0\}$ .

Note that the definition does not depend on the choice of the Haar measure on  $T$ .

**Lemma 4.6.1.**  $\text{im}(C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \xrightarrow{\text{Four}_\psi} C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}) \subset C_0^\infty(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ .  $\square$

We make identifications of ordered abelian groups via Theorem 4.4.12

$$\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}] \cong C_+(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}.$$

The goal of this section is to prove the following statement.

**Theorem 4.6.2.** *An element  $(f_1, f_2)$  of  $C_+(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \cong \mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$  is contained in  $\mathfrak{R}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$  if and only if  $f_2 \in C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$  and  $f_1 = \text{Four}_\psi(f_2)$ .*

### 4.6.2 Descent of principal divisors

Recall that the notation  $U_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''}$  is defined in Section 4.2.3.

**Lemma 4.6.3.** *Let  $h$  be a nonzero rational function on  ${}^\circ\text{ToySht}_T^n$  such that  $\text{Div}(h) \in \mathfrak{D}_T^n$ . Then there exists  $(\Lambda', \Lambda'') \in AP_n(T)$  and a rational function  $f$  on  ${}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$ , such that  $\text{Div}(f) \in \mathfrak{D}_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''}$  and the pullback of  $f$  to  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  equals  $h$ .*

*Proof.* Choose  $(\Lambda'_0, \Lambda''_0) \in AP_n(T)$ . By Lemma 4.2.3, there exists  $(\Lambda', \Lambda'') \succ (\Lambda'_0, \Lambda''_0)$  and a rational function  $f_0$  on  $U_{\Lambda', \Lambda''}^{n, \Lambda'_0, \Lambda''_0}$  such that the pullback of  $f_0$  to  ${}^\circ\text{ToySht}_T^{n, \Lambda'_0, \Lambda''_0}$  equals  $h$ . Let  $f$  be the (unique) rational function on  ${}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  extending  $f_0$ . The pullback of  $f$  to  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  equals  $h$  since  ${}^\circ\text{ToySht}_T^{n, \Lambda'_0, \Lambda''_0}$  is dense in  ${}^\circ\text{ToySht}_T^n$ . Now Lemma 4.4.18 shows that  $\text{Div}(f) \in \mathfrak{D}_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''}$ .  $\square$

### 4.6.3 Support of extension of pullback of principal toy horospherical divisors

Recall that in Section 4.3.1, we defined the notation  $\Delta_{V,H}^m, \Delta_{V,J}^m$  for  $H \in \mathbf{P}_{V^*}, J \in \mathbf{P}_V$ . Also we denoted  $\Delta_{V,H}^{m,W,V} := \text{ToySht}_H^m \cap {}^\circ\text{Grass}_V^{m,W,V}$ .

**Lemma 4.6.4.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{F}_q$  such that  $\dim V \geq 3$ . Let  $W$  be a subspace of  $V$ . When  $1 \leq m \leq \dim V/W - 1$ ,  $\Delta_{V,H}^{m,W,V}$  is nonempty for any  $H \in \mathbf{P}_V$ .*

*Proof.* Note that  $\Delta_{V,H}^{m,W,V} = {}^\circ\text{ToySht}_H^{m,W \cap H, H}$ . From the assumptions we know that  $\dim H \geq 2$  and  $\dim(W \cap H) + m \leq \dim H$ . The statement follows.  $\square$

**Lemma 4.6.5.** *Let  $V' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Assume  $2 \leq m \leq \dim V/V' - 2$ . Let  $f$  be a nonzero rational function on  ${}^\circ\text{ToySht}_{V/V'}^m$  such that the restriction of  $\text{Div}(f)$  to  ${}^\circ\text{ToySht}_{V/V'}^n$  is zero. Let  $g_0$  be the pullback of  $f$  under the morphism  ${}^\circ\text{ToySht}_V^{m,V',V} \rightarrow {}^\circ\text{ToySht}_{V/V'}^m$ . Let  $g$  be the (unique) rational*



function on  ${}^{\circ}\text{ToySht}_V^m$  extending  $g_0$ . Then  $\text{Div}(g)$  is supported on the union of  $\Delta_{V,H}^m$  for  $H \in \mathbf{P}_{V^*}, H \supset V'$  and  $\Delta_{V,J}^m$  for  $J \in \mathbf{P}_V, J \not\subset V'$ .

*Proof.* Lemma 4.3.9 implies that  $\text{Div}(g)$  is supported on  $\Delta_V^m$ . So it suffices to show that the multiplicities of  $\text{Div}(g)$  are zero at  $\Delta_{V,H}^m$  for  $H \in \mathbf{P}_{V^*}, H \not\supset V'$  and  $\Delta_{V,J}^m$  for  $J \in \mathbf{P}_V, J \subset V'$ .

Let  $H \in \mathbf{P}_{V^*}$ . Recall that  $\Delta_{V,H}^m$  is irreducible by Proposition 3.2.23. We denote  $\lambda_H$  to be the multiplicity of  $\text{Div}(g)$  at  $\Delta_{V,H}^m$ , and we denote  $\lambda'_{H/V'}$  to be the multiplicity of  $\text{Div}(f)$  at  $\Delta_{V/V',H/V'}^m$ . Then Lemma 4.6.4 and Lemma 4.3.2 show that  $\lambda_H = \lambda'_{H/V'}$  if  $H \supset V'$ , and  $\lambda_H = 0$  if  $H \not\supset V'$ .

Let  $J \in \mathbf{P}_V$ . Denote  $\mu_J$  to be the multiplicity of  $\text{Div}(g)$  at  $\Delta_{V,J}^m$ . Applying Theorem 3.4.6 to  $V$ , we get

$$\mu_J = q^{1-m} \sum_{\substack{H \in \mathbf{P}_{V^*} \\ H \supset J}} \lambda_H.$$

Suppose  $J \subset V'$ . Then the above result about  $\lambda_H$  shows that

$$\mu_J = q^{1-m} \sum_{H' \in \mathbf{P}_{(V/V')^*}} \lambda'_{H'}.$$

Applying Theorem 3.4.6 to  $V/V'$ , we have

$$\sum_{H' \in \mathbf{P}_{(V/V')^*}} \lambda'_{H'} = 0.$$

The statement follows. □

The following statement is dual to Lemma 4.6.5.

**Lemma 4.6.6.** *Let  $V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Assume that  $2 + \dim V/V'' \leq m \leq \dim V - 2$ . Let  $f$  be a nonzero rational function on*

${}^{\circ}\text{ToySht}_{V''}^{m-\dim V/V''}$  such that the restriction of  $\text{Div}(f)$  to  ${}^{\circ\circ}\text{ToySht}_{V''/V'}^{m-\dim V/V''}$  is zero. Let  $g_0$  be the pullback of  $f$  under  ${}^{\circ}\text{ToySht}_V^{m,0,V''} \rightarrow {}^{\circ}\text{ToySht}_{V''}^{m-\dim V/V''}$ . Let  $g$  be the (unique) rational function on  ${}^{\circ}\text{ToySht}_V^m$  extending  $g_0$ . Then  $\text{Div}(g)$  is supported on the union of  $\Delta_{V,H}^m$  for  $H \in \mathbf{P}_{V^*}, H \not\supset V''$  and  $\Delta_{V,J}^m$  for  $J \in \mathbf{P}_V, J \subset V''$ .  $\square$

**Lemma 4.6.7.** *Let  $V' \subset V'' \subset V$  be finite dimensional vector spaces over  $\mathbb{F}_q$ . Assume  $\dim V/V'' + 2 \leq m \leq \dim V/V' - 2$ . Let  $f$  be a nonzero rational function on  ${}^{\circ}\text{ToySht}_{V''/V'}^{m-\dim V/V''}$  such that the restriction of  $\text{Div}(f)$  to  ${}^{\circ\circ}\text{ToySht}_{V''/V'}^{m-\dim V/V''}$  is zero. Let  $g_0$  be the pullback of  $f$  under  ${}^{\circ}\text{ToySht}_V^{m,V',V''} \rightarrow {}^{\circ}\text{ToySht}_{V''/V'}^{m-\dim V/V''}$ . Let  $g$  be the (unique) rational function on  ${}^{\circ}\text{ToySht}_V^m$  extending  $g_0$ . Then  $\text{Div}(g)$  is supported on the union of  $\Delta_{V,H}^m$  for  $H \in \mathbf{P}_{V^*}, H \supset V', H \not\supset V''$  and  $\Delta_{V,J}^m$  for  $J \in \mathbf{P}_V, J \not\subset V', J \subset V''$ .*

*Proof.* The statement follows from Lemmas 4.6.5 and 4.6.6.  $\square$

#### 4.6.4 Extension of pullback of principal toy horospherical divisors

For  $(\Lambda', \Lambda'') \in AP_n(T)$ , we define  $\varepsilon_{\Lambda', \Lambda''}$  to be the composition of homomorphism of ordered abelian groups

$$\varepsilon_{\Lambda', \Lambda''} : C(\mathbf{P}_{\Lambda''/\Lambda'}) \rightarrow C^\infty(\Lambda'' - \Lambda') \rightarrow C_c^\infty(T - \{0\}),$$

where the first homomorphism is induced by the map  $(\Lambda'' - \Lambda') \rightarrow \mathbf{P}_{\Lambda''/\Lambda'}$  and the second homomorphism is extension by zero.

Similarly, we define

$$\varepsilon_{\Lambda', \Lambda''}^* : C(\mathbf{P}_{(\Lambda''/\Lambda')^*}) \rightarrow C^\infty((\Lambda')^\perp - (\Lambda'')^\perp) \rightarrow C_c^\infty(T^* - \{0\})$$

For  $(\Lambda', \Lambda'') \in AP_n(T)$ , we identify of ordered abelian groups

$$\mathfrak{D}_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''} \cong C(\mathbf{P}_{(\Lambda''/\Lambda')^*}) \oplus C(\mathbf{P}_{\Lambda''/\Lambda'}).$$

As before, we make an identification via Theorem 4.6.2

$$\mathfrak{D}_T^n \cong C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times}.$$

We view  $\varepsilon_{\Lambda', \Lambda''}^* \oplus \varepsilon_{\Lambda', \Lambda''}$  as a homomorphism  $\mathfrak{D}_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''} \rightarrow \mathfrak{D}_T^n$ .

**Lemma 4.6.8.** *Let  $(\Lambda', \Lambda'') \in AP_n(T)$  and let  $f$  be a rational function on  ${}^\circ\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  such that  $\text{Div}(f) \in \mathfrak{D}_{\Lambda', \Lambda''}^{n, \Lambda', \Lambda''}$ . Let  $h_0$  be the pullback of  $f$  to  ${}^\circ\text{ToySht}_T^{n, \Lambda', \Lambda''}$  and let  $h$  be the (unique) extension of  $h_0$  to  ${}^\circ\text{ToySht}_T^n$ . Then  $\text{Div}(h) \in \mathfrak{D}_T^n$  and  $\text{Div}(h) = (\varepsilon_{\Lambda', \Lambda''}^* \oplus \varepsilon_{\Lambda', \Lambda''})(\text{Div}(f))$ .*

*Proof.* Lemma 4.3.15 shows that  $\text{Div}(h) \in \mathfrak{D}_T^n$ .

From Lemma 4.6.7 we know that  $\text{Div}(h)$  is supported on the union of  $\Delta_{T, H}^n$  for  $H \in \mathbf{P}_{T^*}, H \supset \Lambda', H \not\supset \Lambda''$  and  $\Delta_{T, J}^n$  for  $J \in \mathbf{P}_T, J \not\subset \Lambda', J \subset \Lambda''$ . So  $\text{Div}(h)$  is supported on  $((\Lambda')^\perp - (\Lambda'')^\perp) \amalg (\Lambda'' - \Lambda')$ . For  $v \in (\Lambda'' - \Lambda')$ , we have  $\text{Div}(h)(v) = \text{Div}(f)(\bar{v})$  by Lemma 4.4.17, where  $\bar{v}$  is the image of  $v$  in  $\mathbf{P}_{\Lambda''/\Lambda'}$ . There is a similar statement for  $\omega \in ((\Lambda')^\perp - (\Lambda'')^\perp)$ . The statement follows.  $\square$

#### 4.6.5 Proof of Theorem 4.6.2

For a finite set  $S$ , let  $C_0(S)$  denote the (partially) ordered abelian group of  $\mathbb{Z}$ -valued functions on  $S$  whose sum over  $S$  equals 0.

For  $(\Lambda', \Lambda'') \in AP_n(T)$ , we define a homomorphism (Radon transform)

$$R_{\Lambda', \Lambda''}^n : C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}]) \rightarrow C_0(\mathbf{P}_{(\Lambda''/\Lambda')^*}; \mathbb{Z}[\frac{1}{p}])$$

such that for any  $f \in C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}])$ ,  $H \in \mathbf{P}_{(\Lambda''/\Lambda')^*}$ , we have

$$R_{\Lambda', \Lambda''}^n(f)(H) = q^{n(\Lambda') + 1} \sum_{\substack{J \in \mathbf{P}_{\Lambda''/\Lambda'} \\ J \subset H}} f(J).$$

*Remark 4.6.9.* Since we have  $n(\Lambda'') - (\dim(\Lambda''/\Lambda') - 1) = n(\Lambda') + 1$ , Theorem 3.4.6 shows that  $(f_1, f_2) \in C_0(\mathbf{P}_{(\Lambda''/\Lambda')^*}; \mathbb{Z}[\frac{1}{p}]) \oplus C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}])$  corresponds to a principal toy horospherical  $\mathbb{Z}[\frac{1}{p}]$ -divisor of  ${}^{\circ}\text{ToySht}_{\Lambda''/\Lambda'}^{n(\Lambda'')}$  if and only if  $f_1 = R_{\Lambda', \Lambda''}^n(f_2)$ .

**Lemma 4.6.10.**  $\text{im} \left( C_0(\mathbf{P}_{\Lambda''/\Lambda'}) \xrightarrow{\varepsilon_{\Lambda', \Lambda''}} C_c^\infty(T - \{0\})^{\mathbb{F}_q^\times} \right) \subset C_0^\infty(T - \{0\})^{\mathbb{F}_q^\times}. \quad \square$

**Lemma 4.6.11.** *The following diagram is commutative.*

$$\begin{array}{ccc} C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}]) & \xrightarrow{R_{\Lambda', \Lambda''}^n} & C_0(\mathbf{P}_{(\Lambda''/\Lambda')^*}; \mathbb{Z}[\frac{1}{p}]) \\ \downarrow \varepsilon_{\Lambda', \Lambda''} \otimes \mathbb{Z}[\frac{1}{p}] & & \downarrow \varepsilon_{\Lambda', \Lambda''}^* \otimes \mathbb{Z}[\frac{1}{p}] \\ C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} & \xrightarrow{\text{Four}_\psi} & C_0^\infty(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \end{array}$$

*Proof.* Fix  $f \in C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}])$ . Let  $g = (\varepsilon_{\Lambda', \Lambda''} \otimes \mathbb{Z}[\frac{1}{p}])(f)$ ,  $h_1 = (\text{Four}_\psi \circ (\varepsilon_{\Lambda', \Lambda''} \otimes \mathbb{Z}[\frac{1}{p}]))(f)$ , and  $h_2 = ((\varepsilon_{\Lambda', \Lambda''}^* \otimes \mathbb{Z}[\frac{1}{p}]) \circ R_{\Lambda', \Lambda''}^n)(f)$ .

Let  $\omega \in (T^* - \{0\})$ .

Since  $\psi$  is nontrivial and  $g(v) = g(v + x)$  for any  $v \in (T - \{0\})$  and  $x \in \Lambda'$ , we see that  $h_1(\omega) = 0$  when  $\omega \notin (\Lambda')^\perp$ . Since  $\text{supp } g \subset \Lambda''$  and  $\sum_{J \in \mathbf{P}_{\Lambda''/\Lambda'}} f(J) = 0$ , we see that  $h_1(\omega) = 0$  when  $\omega \in (\Lambda'')^\perp$ .

From the definition of  $\varepsilon_{\Lambda', \Lambda''}^*$  we see that  $h_2(\omega) = 0$  when  $\omega \notin (\Lambda')^\perp$  or  $\omega \in (\Lambda'')^\perp$ .

Now we assume  $\omega \in ((\Lambda')^\perp - (\Lambda'')^\perp)$ .

Denote  $H_\omega = \{v \in T \mid \omega(v) = 0\}$  and  $\mathbf{P}_{\Lambda''/\Lambda'}^\omega = \{J \in \mathbf{P}_{\Lambda''/\Lambda'} \mid J \subset (H_\omega \cap \Lambda'')/\Lambda'\}$ .

From the definitions of  $R_{\Lambda', \Lambda''}^n$  and  $\varepsilon_{\Lambda', \Lambda''}^*$  we see that

$$h_2(\omega) = q^{n(\Lambda') + 1} \sum_{J \in \mathbf{P}_{\Lambda''/\Lambda'}^\omega} f(J).$$

On the other hand, we have

$$h_1(\omega) = \int_T g(v)\psi(\omega(v))dv = \mu(\Lambda') \sum_{z \in ((\Lambda''/\Lambda') - \{0\})} f(z)\psi(\omega(z)).$$

Since  $\psi$  is nontrivial, we get

$$h_1(\omega) = \mu(\Lambda') \left( \sum_{J \in (\mathbf{P}_{\Lambda''/\Lambda'} - \mathbf{P}_{\Lambda''/\Lambda'}^\omega)} -f(J) + \sum_{J \in \mathbf{P}_{\Lambda''/\Lambda'}^\omega} (q-1)f(J) \right).$$

Since  $\sum_{J \in \mathbf{P}_{\Lambda''/\Lambda'}} f(J) = 0$ , we get

$$h_1(\omega) = q^{n(\Lambda')} \cdot q \cdot \sum_{J \in \mathbf{P}_{\Lambda''/\Lambda'}^\omega} f(J).$$

(Recall the Haar measure on  $T$  is normalized by the condition that the measure of any c-lattice  $\Lambda$  equals  $q^{n(\Lambda)}$ .) Therefore, we have  $h_1 = h_2$ .  $\square$

**Lemma 4.6.12.** *We have*

$$\begin{aligned} & C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \\ &= \bigcup_{(\Lambda', \Lambda'') \in AP_n(T)} \text{im}(C_0(\mathbf{P}_{\Lambda''/\Lambda'}; \mathbb{Z}[\frac{1}{p}]) \xrightarrow{\varepsilon_{\Lambda', \Lambda''} \otimes \mathbb{Z}[\frac{1}{p}]} C_c^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}). \quad \square \end{aligned}$$

*Proof of Theorem 4.6.2.* The statement follows from Lemma 4.6.3, Lemma 4.6.8, Remark 4.6.9, Lemma 4.6.11, and Lemma 4.6.12.  $\square$

## 4.7 Partial Frobeniuses for Tate toy shtukas

In this section, we fix a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$ . Let  $\text{Dim}_T$  denote the dimension torsor of  $T$ .

#### 4.7.1 Definition of left/right Tate toy shtukas

**Definition 4.7.1.** A *right Tate toy shtuka* for  $T$  over an  $\mathbb{F}_q$ -scheme  $S$  of dimension  $n \in \text{Dim}_T$  is a pair  $\mathcal{L} \in \text{Grass}_T^n(S)$ ,  $\mathcal{L}' \in \text{Grass}_T^{n+1}(S)$ , such that  $\mathcal{L} \subset \mathcal{L}'$ ,  $\text{Fr}_S^* \mathcal{L} \subset \mathcal{L}'$  with  $\mathcal{L}'/\mathcal{L}$  and  $\mathcal{L}'/\text{Fr}_S^* \mathcal{L}$  being invertible.

**Definition 4.7.2.** A *left Tate toy shtuka* for  $T$  over an  $\mathbb{F}_q$ -scheme  $S$  of dimension  $n \in \text{Dim}_T$  is a pair  $\mathcal{L} \in \text{Grass}_T^n(S)$ ,  $\mathcal{L}' \in \text{Grass}_T^{n-1}(S)$ , such that  $\mathcal{L}' \subset \mathcal{L}$ ,  $\mathcal{L}' \subset \text{Fr}_S^* \mathcal{L}$  with  $\mathcal{L}/\mathcal{L}'$  and  $\text{Fr}_S^* \mathcal{L}/\mathcal{L}'$  being invertible.

For  $n \in \text{Dim}_T$ , let  $\text{LToySht}_T^n$  (resp.  $\text{RTToySht}_T^n$ ) be the functor which associates to each  $\mathbb{F}_q$ -scheme  $S$  the set of isomorphism classes of left (resp. right) Tate toy shtukas over  $S$  of dimension  $n$ .

As in the finite dimensional case,  $\text{LToySht}_T^n$  (resp.  $\text{RTToySht}_T^n$ ) is representable by a closed subscheme of  $\text{Grass}_T^n \times \text{Grass}_T^{n-1}$  (resp.  $\text{Grass}_T^n \times \text{Grass}_T^{n+1}$ ).

#### 4.7.2 Partial Frobeniuses

We have the following constructions for left/right Tate toy shtukas:

- (i) For a left Tate toy shtuka  $\mathcal{L}' \subset \mathcal{L}$  over an  $\mathbb{F}_q$ -scheme  $S$ , the pair  $\mathcal{L}' \subset \text{Fr}_S^* \mathcal{L}$  forms a right Tate toy shtuka over  $S$ .
- (ii) For a right Tate toy shtuka  $\mathcal{L} \subset \mathcal{L}'$  over an  $\mathbb{F}_q$ -scheme  $S$ , the pair  $\text{Fr}_S^* \mathcal{L} \subset \mathcal{L}'$  forms a left Tate toy shtuka over  $S$ .

For  $n \in \text{Dim}_T$ , we define partial Frobeniuses  $F_{T,n}^- : \text{LToySht}_T^n \rightarrow \text{RTToySht}_T^{n-1}$  and  $F_{T,n}^+ : \text{RTToySht}_T^n \rightarrow \text{LToySht}_T^{n+1}$  induced by the above constructions.

We have  $F_{T,n-1}^+ \circ F_{T,n}^- = \text{Fr}_{\text{LToySht}_T^n}$ ,  $F_{T,n+1}^- \circ F_{T,n}^+ = \text{Fr}_{\text{RTToySht}_T^n}$ .

#### 4.7.3 Identification of $L\mathfrak{D}_T^n$ and $R\mathfrak{D}_T^n$ with $\mathfrak{D}_T^n$

As in the finite dimensional case,  ${}^\circ\text{ToySht}_T^n$  is an open subscheme of  $\text{LToySht}_T^n$  and  $\text{RTToySht}_T^n$ .

Let  $L\mathfrak{D}_T^n$  (resp.  $R\mathfrak{D}_T^n$ ) be the (partially) ordered abelian group of Cartier divisors of  $\mathrm{LToySht}_T^n$  (resp.  $\mathrm{RToySht}_T^n$ ) whose restrictions to  ${}^{\circ}\mathrm{ToySht}_T^n$  are zero.

The goal of this subsection is to prove the following statement.

**Lemma 4.7.3.** *The open immersion  ${}^{\circ}\mathrm{ToySht}_T^n \hookrightarrow \mathrm{LToySht}_T^n$  (resp.  ${}^{\circ}\mathrm{ToySht}_T^n \hookrightarrow \mathrm{RToySht}_T^n$ ) induces an isomorphism of ordered abelian groups  $L\mathfrak{D}_T^n \xrightarrow{\sim} \mathfrak{D}_T^n$  (resp.  $R\mathfrak{D}_T^n \xrightarrow{\sim} \mathfrak{D}_T^n$ ).*

In the following we only prove the statement for  $\mathrm{LToySht}_T^n$ , since the statements for  $\mathrm{LToySht}_T^n$  and  $\mathrm{RToySht}_T^n$  are dual to each other.

**Lemma 4.7.4.** *Let  $E$  be a field over  $\mathbb{F}_q$  and let  $L \in {}^{\circ}\mathrm{Grass}_T^{n,\Lambda',\Lambda''}(E)$  for two  $c$ -lattices  $\Lambda' \subset \Lambda''$  of  $T$ . Put  $L' = L \cap \mathrm{Fr}_E^* L$ . Then  $L' \in \mathrm{Grass}_T^{n-1,\Lambda',\Lambda''}(E)$ .*

*Proof.* For any subspace  $M$  of  $T$ , we denote  $M_E := M \widehat{\otimes} E$ .

Since  $L \in {}^{\circ}\mathrm{Grass}_T^{n,\Lambda',\Lambda''}(E)$ , we have  $L \cap \Lambda_E' = 0$ . Hence  $L' \cap \Lambda_E' = 0$ . So it suffices to show that  $L' + \Lambda_E'' = T_E$ .

We have  $L + \Lambda_E'' = T_E$ . Hence  $\dim(L \cap \Lambda_E'') = \dim(L)(\Lambda_E'') = n(\Lambda'')$ . (See Definition 4.1.11 for the definition of the dimension of a d-lattice.) From the definition of  ${}^{\circ}\mathrm{Grass}$  in diagram (4.1) we know that  $L \cap \Lambda_E''$  is a nontrivial toy shtuka over  $\mathrm{Spec} E$ . Thus  $\dim(L' \cap \Lambda_E'') = \dim(L \cap \Lambda_E'') - 1 = n(\Lambda'') - 1$ . Therefore, in the Tate space  $T_E$  over  $E$ , the d-lattice  $L'$  and the  $c$ -lattice  $\Lambda_E''$  satisfy  $\dim(L')(\Lambda_E'') = \dim(L' \cap \Lambda_E'')$ . This implies that  $L' + \Lambda_E'' = T_E$ .  $\square$

For  $(\Lambda', \Lambda'') \in AP_n(T)$ , denote  $\mathrm{LToySht}_T^{n,\Lambda',\Lambda''} = (\mathrm{Grass}_T^{n,\Lambda',\Lambda''} \times \mathrm{Grass}_T^{n-1,\Lambda',\Lambda''}) \cap \mathrm{LToySht}_T^n$ . Lemma 4.7.4 shows that  ${}^{\circ}\mathrm{ToySht}_T^{n,\Lambda',\Lambda''} \subset \mathrm{LToySht}_T^{n,\Lambda',\Lambda''}$ .

For  $(\Lambda', \Lambda'') \in AP_n(T)$ , let  $L\mathfrak{D}_T^{n,\Lambda',\Lambda''}$  be the (partially) ordered abelian group of Cartier divisors of  $\mathrm{LToySht}_T^{n,\Lambda',\Lambda''}$  whose restrictions to  ${}^{\circ}\mathrm{ToySht}_T^n$  are zero.

The isomorphisms

$${}^{\circ}\mathrm{ToySht}_T^n = \varinjlim_{(\Lambda', \Lambda'') \in AP_n(T)} {}^{\circ}\mathrm{ToySht}_T^{n,\Lambda',\Lambda''},$$

$$\mathrm{LToySht}_T^n = \varinjlim_{(\Lambda', \Lambda'') \in AP_n(T)} \mathrm{LToySht}_T^{n, \Lambda', \Lambda''}$$

give isomorphisms of ordered abelian groups

$$\mathfrak{D}_T^n \xrightarrow{\sim} \varprojlim_{(\Lambda', \Lambda'') \in AP_n(T)} \mathfrak{D}_T^{n, \Lambda', \Lambda''},$$

$$L\mathfrak{D}_T^n \xrightarrow{\sim} \varprojlim_{(\Lambda', \Lambda'') \in AP_n(T)} L\mathfrak{D}_T^{n, \Lambda', \Lambda''}.$$

The open immersions  ${}^\circ\mathrm{ToySht}_T^{n, \Lambda', \Lambda''} \hookrightarrow \mathrm{LToySht}_T^{n, \Lambda', \Lambda''}$  for different  $(\Lambda', \Lambda'') \in AP_n(T)$  induce ordered homomorphisms  $L\mathfrak{D}_T^{n, \Lambda', \Lambda''} \rightarrow \mathfrak{D}_T^{n, \Lambda', \Lambda''}$  which are compatible with transition homomorphisms. Thus Lemma 4.7.3 follows from Lemma 4.7.5.

**Lemma 4.7.5.** *The open immersion  ${}^\circ\mathrm{ToySht}_T^{n, \Lambda', \Lambda''} \hookrightarrow \mathrm{LToySht}_T^{n, \Lambda', \Lambda''}$  induces an isomorphism of ordered abelian groups  $L\mathfrak{D}_T^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \mathfrak{D}_T^{n, \Lambda', \Lambda''}$  for  $(\Lambda', \Lambda'') \in AP_n(T)$ .*

*Proof.* For  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')$ , denote  $LU_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''} = \mathrm{LToySht}_{\widetilde{\Lambda}'/\widetilde{\Lambda}'}^{n(\widetilde{\Lambda}''), \Lambda'/\widetilde{\Lambda}', \Lambda''/\widetilde{\Lambda}'}$ , and denote  $L\mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  to be the ordered abelian group of Cartier divisors of  $LU_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  whose restrictions to  ${}^\circ\mathrm{ToySht}_{\widetilde{\Lambda}'/\widetilde{\Lambda}'}^{n(\widetilde{\Lambda}''), \Lambda'/\widetilde{\Lambda}', \Lambda''/\widetilde{\Lambda}'}$  are zero.

The complement of  ${}^\circ\mathrm{ToySht}_{\widetilde{\Lambda}'/\widetilde{\Lambda}'}^{n(\widetilde{\Lambda}''), \Lambda'/\widetilde{\Lambda}', \Lambda''/\widetilde{\Lambda}'}$  in  $LU_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  has codimension  $-n(\widetilde{\Lambda}') \geq 2$  since  $(\widetilde{\Lambda}', \widetilde{\Lambda}'') \in AP_n(T)$ . Hence the complement of  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  in  $LU_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  has codimension at least 2. Therefore, the open immersion of regular schemes  $U_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''} \hookrightarrow LU_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$  induces an isomorphism of ordered abelian groups  $L\mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}$ . Such isomorphisms are compatible with transition homomorphisms for different pairs  $(\widetilde{\Lambda}', \widetilde{\Lambda}'')$ , and thus give rise to an isomorphism of ordered abelian groups

$$\lambda : \varinjlim_{(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')} L\mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''} \xrightarrow{\sim} \varinjlim_{(\widetilde{\Lambda}', \widetilde{\Lambda}'') \succ (\Lambda', \Lambda'')} \mathfrak{D}_{\widetilde{\Lambda}', \widetilde{\Lambda}''}^{n, \Lambda', \Lambda''}.$$



Similarly to Lemma 4.2.3, we have an isomorphism

$$\mathrm{LTToySht}_T^{n,\Lambda',\Lambda''} \xrightarrow{\sim} \varprojlim_{(\widetilde{\Lambda'},\widetilde{\Lambda''}) \succ (\Lambda',\Lambda'')} LU_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''}.$$

It induces an ordered homomorphism

$$L\xi : \varprojlim_{(\widetilde{\Lambda'},\widetilde{\Lambda''}) \succ (\Lambda',\Lambda'')} L\mathfrak{D}_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''} \rightarrow L\mathfrak{D}_T^{n,\Lambda',\Lambda''}.$$

Similarly to the proof of Lemma 4.4.14 in Section 4.4.2, one can show that  $L\xi$  is surjective using Lemma 4.3.16.

We get a commutative diagram of ordered abelian groups.

$$\begin{array}{ccc} \varprojlim_{(\widetilde{\Lambda'},\widetilde{\Lambda''}) \succ (\Lambda',\Lambda'')} L\mathfrak{D}_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''} & \xrightarrow{L\xi} & L\mathfrak{D}_T^{n,\Lambda',\Lambda''} \\ \downarrow \lambda & & \downarrow \lambda' \\ \varprojlim_{(\widetilde{\Lambda'},\widetilde{\Lambda''}) \succ (\Lambda',\Lambda'')} \mathfrak{D}_{\widetilde{\Lambda'},\widetilde{\Lambda''}}^{n,\Lambda',\Lambda''} & \xrightarrow{\xi} & \mathfrak{D}_T^{n,\Lambda',\Lambda''} \end{array}$$

We proved that  $\lambda$  is an isomorphism of ordered abelian groups. So is  $\xi$  by Lemma 4.4.14.

Both  $L\xi$  and  $\lambda'$  are ordered homomorphisms, and we proved that  $L\xi$  is surjective.

Hence  $\lambda'$  is an isomorphism of ordered abelian groups.  $\square$

#### 4.7.4 Pullback of Tate toy horospherical subschemes under partial

##### *Frobeniuses*

We identify  $L\mathfrak{D}_T^n$  and  $R\mathfrak{D}_T^n$  with  $\mathfrak{D}_T^n$  by Lemma 4.7.3.

Partial Frobeniuses induce homomorphisms between ordered abelian groups

$$\dots \longleftrightarrow \mathfrak{D}_T^{n-1} \xrightleftharpoons[(F_{T,n-1}^+)^*]{(F_{T,n}^-)^*} \mathfrak{D}_T^n \xrightleftharpoons[(F_{T,n}^+)^*]{(F_{T,n+1}^-)^*} \mathfrak{D}_T^{n+1} \longleftrightarrow \dots$$

Similarly to Lemmas 3.4.3 and 3.4.4, we have the following statement.

**Lemma 4.7.6.** *Identifying  $\mathfrak{D}_T^{n-1}, \mathfrak{D}_T^n, \mathfrak{D}_T^{n+1}$  with  $C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times}$  via Theorem 4.4.12, the two homomorphisms*

$$\begin{aligned} (F_{T,n}^-)^* &: \mathfrak{D}_T^{n-1} \rightarrow \mathfrak{D}_T^n, \\ (F_{T,n}^+)^* &: \mathfrak{D}_T^{n+1} \rightarrow \mathfrak{D}_T^n \end{aligned}$$

are given by

$$\begin{aligned} (F_{T,n}^-)^* &: C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times} \rightarrow C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times} \\ &(\lambda_1, \lambda_2) \mapsto (q\lambda_1, \lambda_2) \\ (F_{T,n}^+)^* &: C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times} \rightarrow C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times} \\ &(\lambda_1, \lambda_2) \mapsto (\lambda_1, q\lambda_2) \end{aligned}$$

## 4.8 A canonical subgroup of $\text{Pic}({}^\circ\text{ToySht}_T^n)$

In this section, we fix a nondiscrete noncompact Tate space  $T$  over  $\mathbb{F}_q$  and we fix an element  $n \in \text{Dim}_T$ .

### 4.8.1 Definition

Let  $\mathcal{L}_{T,n}$  be the universal Tate toy shtuka over  ${}^\circ\text{ToySht}_T^n$ . Let  $\widetilde{\mathcal{L}}_{T,n}^L \subset \mathcal{L}_{T,n}^L$  (resp.  $\mathcal{L}_{T,n}^R \subset \widetilde{\mathcal{L}}_{T,n}^R$ ) be the universal left (resp. right) Tate toy shtuka over  $\text{LToySht}_T^n$  (resp.  $\text{RToySht}_T^n$ ).

Denote  $\mathcal{L}'_{T,n} = \mathcal{L}_{T,n} \cap \text{Fr}^*_{\circ\text{ToySht}_T^n} \mathcal{L}_{T,n}$ ,  $\mathcal{L}''_{T,n} = \mathcal{L}_{T,n} + \text{Fr}^*_{\circ\text{ToySht}_T^n} \mathcal{L}_{T,n}$ .

*Remark 4.8.1.* We see that  $\mathcal{L}'_{T,n}$  is the pullback of  $\mathcal{L}^R_{T,n-1}$  under the composition  $\circ\text{ToySht}_T^n \hookrightarrow \text{LToySht}_T^n \xrightarrow{F_{T,n}^-} \text{RTToySht}_T^{n-1}$ , and  $\mathcal{L}''_{T,n}$  is the pullback of  $\mathcal{L}^L_{T,n+1}$  under the composition  $\circ\text{ToySht}_T^n \hookrightarrow \text{RTToySht}_T^n \xrightarrow{F_{T,n}^+} \text{LToySht}_T^{n+1}$ .

**Definition 4.8.2.** We define two invertible sheaves  $\ell_{T,n,a} := \mathcal{L}_{T,n}/\mathcal{L}'_{T,n}$ ,  $\ell_{T,n,b} := \mathcal{L}''_{T,n}/\mathcal{L}_{T,n}$  on  $\circ\text{ToySht}_T^n$ .

**Definition 4.8.3.** For a c-lattice  $W$  of  $T$ , we define an invertible sheaf  $\ell^W_{T,n,\det} := \det(\mathcal{L}_{T,n}, W)$ . (See Definition 4.1.18 for the definition of  $\det(\mathcal{L}_{T,n}, W)$ .)

*Remark 4.8.4.* For two c-lattices  $W_1, W_2$  of  $T$ , Remark 4.1.20 shows that there is a canonical isomorphism  $\ell^W_{T,n,\det} \otimes \det^{W_2}_{W_1} \cong \ell^{W_2}_{T,n,\det}$ , and the two invertible sheaves  $\ell^{W_1}_{T,n,\det}$  and  $\ell^{W_2}_{T,n,\det}$  are isomorphic. Also,  $(\ell^{W_1}_{T,n,\det})^{\otimes(q-1)}$  and  $(\ell^{W_2}_{T,n,\det})^{\otimes(q-1)}$  are canonically isomorphic since  $(\det^{W_2}_{W_1})^{\otimes(q-1)}$  is canonically isomorphic to  $\mathbb{F}_q$ .

**Lemma 4.8.5.** *Let  $W$  be a c-lattice of  $T$ . We have a canonical isomorphism  $\ell_{T,n,b} \otimes \ell_{T,n,a}^{-1} \cong (\ell^W_{T,n,\det})^{\otimes(q-1)}$ .*

*Proof.* For any  $\mathcal{O}_{\circ\text{ToySht}_T^n}$ -module  $\mathcal{E}$ , denote  ${}^\tau\mathcal{E} = \text{Fr}^*_{\circ\text{ToySht}_T^n} \mathcal{E}$ .

The canonical isomorphism  $\mathcal{L}_{T,n}/\mathcal{L}'_{T,n} \cong \mathcal{L}''_{T,n}/{}^\tau\mathcal{L}_{T,n}$  induces a canonical isomorphism  $\det(\mathcal{L}_{T,n}, W) \otimes \det(\mathcal{L}'_{T,n}, W)^{-1} \cong \det(\mathcal{L}''_{T,n}, W) \otimes \det({}^\tau\mathcal{L}_{T,n}, W)^{-1}$  by Lemma 4.1.21.

Lemma 4.1.22 gives a canonical isomorphism  $\det({}^\tau\mathcal{L}_{T,n}, W) \cong \det(\mathcal{L}_{T,n}, W)^{\otimes q}$ .

Lemma 4.1.21 gives canonical isomorphisms

$$\ell_{T,n,a} \cong \det(\mathcal{L}_{T,n}, W) \otimes \det(\mathcal{L}'_{T,n}, W)^{-1},$$

$$\ell_{T,n,b} \cong \det(\mathcal{L}''_{T,n}, W) \otimes \det(\mathcal{L}_{T,n}, W)^{-1}.$$

The statement follows. □

#### 4.8.2 Some divisors in the classes of $\ell_{T,n,a}$ and $\ell_{T,n,b}$

We have an inclusion of ordered abelian groups

$$C_c^\infty(T^*)^{\mathbb{F}_q^\times} \oplus C_c^\infty(T)^{\mathbb{F}_q^\times} \subset C_+(T^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\})^{\mathbb{F}_q^\times}.$$

For an invertible sheaf  $\ell$  on  ${}^\circ\text{ToySht}_T^n$  and an element  $(f_1, f_2) \in C_c^\infty(T^*)^{\mathbb{F}_q^\times} \oplus C_c^\infty(T)^{\mathbb{F}_q^\times}$ , we write  $\ell \sim (f_1, f_2)$  if  $[\ell] = [\mathcal{O}^{\circ\text{ToySht}_T^n}(D_{(f_1, f_2)})]$  in  $\text{Pic}({}^\circ\text{ToySht}_T^n)$ , where  $D_{(f_1, f_2)}$  is the Cartier divisor of  ${}^\circ\text{ToySht}_T^n$  corresponding to  $(f_1, f_2)$  via Theorem 4.4.12.

**Lemma 4.8.6.** *Let  $W_{-1}, W_0, W_1$  be  $c$ -lattices of  $T$  such that  $W_{-1} \subset W_0 \subset W_1$  and  $n(W_i) = i$ , ( $i = -1, 0, 1$ ). Then we have*

$$\begin{aligned} \ell_{T,n,a} &\sim (q \cdot \mathbb{1}_{W_1^\perp} - \mathbb{1}_{W_0^\perp}, \mathbb{1}_{W_1 - W_0}), \\ \ell_{T,n,b} &\sim (-\mathbb{1}_{W_{-1}^\perp - W_0^\perp}, -q \cdot \mathbb{1}_{W_{-1}} + \mathbb{1}_{W_0}). \end{aligned}$$

*Proof.* Recall that we denoted  $\mathcal{L}'_{T,n} = \mathcal{L}_{T,n} \cap \text{Fr}^*_{\circ\text{ToySht}_T^n} \mathcal{L}_{T,n}$ ,  $\mathcal{L}''_{T,n} = \mathcal{L}_{T,n} + \text{Fr}^*_{\circ\text{ToySht}_T^n} \mathcal{L}_{T,n}$ .

Recall that  $\ell_{T,n,a} = \mathcal{L}_{T,n} / \mathcal{L}'_{T,n}$ . So Lemma 4.1.21 gives an isomorphism  $\ell_{T,n,a} \cong \det(\mathcal{L}_{T,n}, W_0) \otimes \det(\mathcal{L}'_{T,n}, W_0)^{-1}$ . Hence  $\ell_{T,n,a} \cong \det(\mathcal{L}_{T,n}, W_0) \otimes \det(\mathcal{L}'_{T,n}, W_1)^{-1}$  by Remark 4.1.20.

Theorem 4.5.2 implies that  $\det(\mathcal{L}_{T,n}, W_0) \sim (-\mathbb{1}_{W_0^\perp}, -\mathbb{1}_{W_0})$ .

Theorem 4.5.2 and Lemma 4.7.3 imply that  $\det(\mathcal{L}_{T,n-1}^R, W_1) \sim (-\mathbb{1}_{W_1^\perp}, -\mathbb{1}_{W_1})$ . Remarks 4.8.1 and 4.1.19 show that  $\det(\mathcal{L}'_{T,n}, W_1)$  is the pullback of  $\det(\mathcal{L}_{T,n-1}^R, W_1)$  under the composition  ${}^\circ\text{ToySht}_T^n \hookrightarrow \text{LToySht}_T^n \xrightarrow{F_{T,n}^-} \text{RTToySht}_T^{n-1}$ . Then Lemma 4.7.6 implies that  $\det(\mathcal{L}'_{T,n}, W_1) \sim (-q \cdot \mathbb{1}_{W_1^\perp}, -\mathbb{1}_{W_1})$ .

Therefore, we get

$$\ell_{T,n,a} \sim (-\mathbf{1}_{W_0^\perp}, -\mathbf{1}_{W_0}) - (-q \cdot \mathbf{1}_{W_1^\perp}, -\mathbf{1}_{W_1}) = (q \cdot \mathbf{1}_{W_1^\perp} - \mathbf{1}_{W_0^\perp}, \mathbf{1}_{W_1 - W_0}).$$

The proof of the statement about  $\ell_{T,n,b}$  is similar.  $\square$

### 4.8.3 The preimage of $\text{Pic}_{\text{can}}({}^\circ\text{ToySht}_T^n) \otimes \mathbb{Z}[\frac{1}{p}]$ in $\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$

We normalize the Haar measure on  $T$  by the condition that the measure of any  $\mathfrak{c}$ -lattice  $\Lambda$  equals  $q^{n(\Lambda)}$ .

Fix a nontrivial additive character  $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ .

The Fourier transform  $\text{Four}_\psi$  is defined by the equation (4.6).

We have a homomorphism

$$C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \xrightarrow{\text{Four}_\psi \oplus 1} C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$$

and an inclusion of abelian groups

$$C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \subset C_+(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}.$$

Theorem 4.4.12 gives a homomorphism

$$C_+(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \rightarrow \text{Pic}({}^\circ\text{ToySht}_T^n) \otimes \mathbb{Z}[\frac{1}{p}].$$

The composition of above homomorphisms gives rise to a homomorphism

$$\gamma : C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \rightarrow \text{Pic}({}^\circ\text{ToySht}_T^n) \otimes \mathbb{Z}[\frac{1}{p}].$$

*Remark 4.8.7.* Theorem 4.6.2 implies that  $\ker \gamma = C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ . Also note that  $C_0^\infty(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} = \{f \in C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \mid f(0) = 0, \int_T f(v) dv = 0\}$ .

**Proposition 4.8.8.** For  $f \in C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ , let  $a(f) = \int_T f(v) dv$  and  $b(f) = f(0)$ . Then

$$(q-1)\gamma(f) = a(f)[\ell_{T,n,a}] - b(f)[\ell_{T,n,b}].$$

*Remark 4.8.9.* In the definition of  $a(f)$ , we normalize the Haar measure on  $T$  by the condition that the measure of any c-lattice  $\Lambda$  equals  $q^{n(\Lambda)}$ .

*Proof of Proposition 4.8.8.* The statement follows from Lemma 4.8.6 and the facts that

$$\begin{aligned} a(\mathbb{1}_{W_1-W_0}) &= q-1, & b(\mathbb{1}_{W_1-W_0}) &= 0, \\ a(-q \cdot \mathbb{1}_{W_{-1}} + \mathbb{1}_{W_0}) &= 0, & b(-q \cdot \mathbb{1}_{W_{-1}} + \mathbb{1}_{W_0}) &= -(q-1), \\ q \cdot \mathbb{1}_{W_1^\perp} - \mathbb{1}_{W_0^\perp} &= \text{Four}_\psi(\mathbb{1}_{W_1-W_0}), & -\mathbb{1}_{W_{-1}^\perp} - \mathbb{1}_{W_0^\perp} &= \text{Four}_\psi(-q \cdot \mathbb{1}_{W_{-1}} + \mathbb{1}_{W_0}). \end{aligned}$$

□

**Definition 4.8.10.** Let  $\text{Pic}_{\text{can}}({}^\circ\text{ToySht}_T^n)$  be the subgroup of  $\text{Pic}({}^\circ\text{ToySht}_T^n)$  generated by  $[\ell_{T,n,a}]$ ,  $[\ell_{T,n,b}]$  and  $[\ell_{T,n,\det}^W]$ , where  $W$  is a c-lattice of  $T$ .

*Remark 4.8.11.* Remark 4.8.4 implies that the class of  $\ell_{T,n,\det}^W$  is independent of the choice of the c-lattice  $W$ . Hence  $\text{Pic}_{\text{can}}({}^\circ\text{ToySht}_T^n)$  is well-defined.

*Remark 4.8.12.* Lemma 4.8.6, Theorem 4.5.2 and Theorem 4.4.12 show that the image of  $\mathfrak{D}_T^n$  in  $\text{Pic}({}^\circ\text{ToySht}_T^n)$  contains  $\text{Pic}_{\text{can}}({}^\circ\text{ToySht}_T^n)$ .

As before, we identify  $\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$  with  $C_+(T^* - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_+(T - \{0\}; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$  via Theorem 4.4.12, and we consider  $C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$  as an ordered subgroup of  $\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$ .

**Theorem 4.8.13.** The preimage of  $\text{Pic}_{\text{can}}({}^\circ\text{ToySht}_T^n) \otimes \mathbb{Z}[\frac{1}{p}]$  in  $\mathfrak{D}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$  is the ordered  $\mathbb{Z}[\frac{1}{p}]$ -submodule

$$\{(f_1, f_2) \in C_c^\infty(T^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \oplus C_c^\infty(T; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times} \mid f_1 = \text{Four}_\psi(f_2)\}.$$

*Proof.* Let  $W_0, W_1, W_{-1}$  be as in Lemma 4.8.6. By Lemma 4.8.6 and Theorem 4.5.2, the preimage is the  $\mathbb{Z}[\frac{1}{p}]$ -span of the three elements

$$(\mathbb{1}_{W_0^\perp}, \mathbb{1}_{W_0}), (q \cdot \mathbb{1}_{W_1^\perp} - \mathbb{1}_{W_0^\perp}, \mathbb{1}_{W_1 \setminus W_0}), (-\mathbb{1}_{W_{-1}^\perp \setminus W_0^\perp}, -q \cdot \mathbb{1}_{W_{-1}} + \mathbb{1}_{W_0})$$

and the  $\mathbb{Z}[\frac{1}{p}]$ -module  $\mathfrak{R}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$ . Note that the three elements are contained in the above  $\mathbb{Z}[\frac{1}{p}]$ -submodule of  $\mathfrak{O}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$ . So the statement follows from the description of  $\mathfrak{R}_T^n \otimes \mathbb{Z}[\frac{1}{p}]$  in Theorem 4.6.2.  $\square$

## CHAPTER 5

### RELATION BETWEEN SHTUKAS AND TOY SHTUKAS

#### 5.1 Relation between shtukas and toy shtukas

Let  $S$  be a scheme over  $\mathbb{F}_q$ . For an  $\mathcal{O}_{X \times S}$ -module  $\mathcal{F}$  and a point  $s \in S$ , we denote  $\mathcal{F}_s$  to be the pullback of  $\mathcal{F}$  to  $X \times s$ . Put  $\Phi = \text{Id}_X \times \text{Fr}_S : X \times S \rightarrow X \times S$ . Let  $\pi : X \times S \rightarrow S$  be the projection.

##### 5.1.1 Right shtukas

Let  $\Phi^* \mathcal{F} \hookrightarrow \mathcal{F}' \hookleftarrow \mathcal{F}$  be a right shtuka of rank  $d$  over  $S$  equipped with a structure of level  $D$ . Suppose that  $D$  viewed as an effective divisor is represented as  $D'' - D'$  so that for every point  $s \in S$  one has

$$H^0(X \times s, \mathcal{F}'_s(D' \times s)) = 0, \quad (5.1)$$

$$H^1(X \times s, \mathcal{F}_s(D'' \times s)) = 0. \quad (5.2)$$

Let  $V = H^0(X, (\mathcal{O}_X(D'')/\mathcal{O}_X(D'))^d)$ ,  $\mathcal{L} = \pi_* \mathcal{F}(D'' \times S)$ ,  $\mathcal{L}' = \pi_* \mathcal{F}'(D'' \times S)$ .

**Proposition 5.1.1.** *The pair  $\mathcal{L}, \mathcal{L}'$  forms a right toy shtuka for  $V$  over  $S$ .*

*Proof.* Consider the composition

$$\mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_{D \times S} \xrightarrow{\sim} \mathcal{O}_{D \times S}^d$$

where the second morphism is the isomorphism from the structure of level  $D$ . Tensoring with  $\mathcal{O}_{X \times S}(D'' \times S)$ , we get a composition

$$\mathcal{F}(D'' \times S) \rightarrow \mathcal{F}(D'' \times S)/\mathcal{F}(D' \times S) \xrightarrow{\sim} (\mathcal{O}_{X \times S}(D'' \times S)/\mathcal{O}_{X \times S}(D' \times S))^d$$



This induces a morphism  $\mathcal{L} \rightarrow V \otimes \mathcal{O}_S$ .

Similarly one gets a morphism  $\mathcal{L}' \rightarrow V \otimes \mathcal{O}_S$ .

Consider the exact sequence

$$0 \longrightarrow \mathcal{F}'(D' \times S) \longrightarrow \mathcal{F}'(D'' \times S) \longrightarrow \mathcal{F}'(D'' \times S)/\mathcal{F}'(D' \times S) \longrightarrow 0.$$

Assumption (5.1) and base change for cohomology imply that  $\pi_* \mathcal{F}'(D' \times S) = 0$ .

Hence the morphism  $\mathcal{L}' \rightarrow V \otimes \mathcal{O}_S$  is injective.

Since  $\mathcal{F}'/\mathcal{F}$  is torsion, assumption (5.2) implies  $H^1(X \times s, \mathcal{F}'_s(D'' \times s)) = 0$  for every point  $s \in S$ . From base change for cohomology we get  $R^1 \pi_* \mathcal{F}'(D'' \times S) = 0$ . So  $(V \otimes \mathcal{O}_S)/\mathcal{L}' = R^1 \pi_* \mathcal{F}'(D' \times S)$ . For any point  $s \in S$ , we have  $H^0(X \times s, \mathcal{F}'_s(D' \times s)) = 0$  by assumption (5.1), so in particular the base change morphism  $k(s) \otimes \pi_* \mathcal{F}'(D' \times s) \rightarrow H^0(X \times s, \mathcal{F}'_s(D' \times s))$  is surjective. By [10, Chapter 3, Theorem 12.11(b)],  $(V \otimes \mathcal{O}_S)/\mathcal{L}' = R^1 \pi_* \mathcal{F}'(D' \times S)$  is locally free.

A similar argument shows that  $(V \otimes \mathcal{O}_S)/\mathcal{L}$  is locally free.

Since  $\mathcal{F} \subset \mathcal{F}'$ , we have  $\mathcal{L} \subset \mathcal{L}'$ . Since  $\Phi^* \mathcal{F} \subset \mathcal{F}'$ , we have  $\text{Fr}_S^* \mathcal{L} \subset \mathcal{L}'$ .

Assumption (5.2) and base change for cohomology imply that  $R^1 \pi_* \mathcal{F}(D'' \times S) = 0$ . Since  $\Gamma_\beta \cap (D'' \times S) = \emptyset$ , we have an isomorphism  $\mathcal{F}'(D'' \times S)/\mathcal{F}(D'' \times S) \cong \mathcal{F}'/\mathcal{F}$ . So we have an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}' \longrightarrow \pi_*(\mathcal{F}'/\mathcal{F}) \longrightarrow 0.$$

Since  $\mathcal{F}'/\mathcal{F}$  is an invertible sheaf on  $\Gamma_\beta$ , and  $\pi : X \times S \rightarrow S$  induces an isomorphism  $\Gamma_\beta \rightarrow S$ , we see that  $\mathcal{L}'/\mathcal{L}$  is invertible.

Similarly,  $\mathcal{L}'/\text{Fr}_S^* \mathcal{L}$  is also invertible. □

### 5.1.2 Left shtukas

Let  $\Phi^* \mathcal{F} \leftarrow \mathcal{F}' \hookrightarrow \mathcal{F}$  be a left shtuka of rank  $d$  over  $S$  equipped with a structure of level  $D$ . Suppose that  $D$  viewed as an effective divisor is represented as  $D'' - D'$  so that for every point  $s \in S$  one has

$$H^0(X \times s, \mathcal{F}_s(D' \times s)) = 0, \quad (5.3)$$

$$H^1(X \times s, \mathcal{F}'_s(D'' \times s)) = 0. \quad (5.4)$$

Let  $V = H^0(X, (\mathcal{O}_X(D'')/\mathcal{O}_X(D'))^d)$ ,  $\mathcal{L} = \pi_* \mathcal{F}(D'' \times S)$ ,  $\mathcal{L}' = \pi_* \mathcal{F}'(D'' \times S)$ .

Similarly to the case of a right shtuka, one can prove the following statement.

**Proposition 5.1.2.** *The pair  $\mathcal{L}, \mathcal{L}'$  forms a left toy shtuka for  $V$  over  $S$ .  $\square$*

### 5.1.3 Shtukas

Let  $\mathcal{F}$  be a shtuka of rank  $d$  over  $S$  equipped with a structure of level  $D$ . Suppose that  $D$  viewed as an effective divisor is represented as  $D'' - D'$  so that for every point  $s \in S$  one has

$$H^0(X \times s, \mathcal{F}_s(D' \times s)) = 0, \quad (5.5)$$

$$H^1(X \times s, \mathcal{F}_s(D'' \times s)) = 0. \quad (5.6)$$

Let  $V = H^0(X, (\mathcal{O}_X(D'')/\mathcal{O}_X(D'))^d)$ ,  $\mathcal{L} = \pi_* \mathcal{F}(D'' \times S)$ .

Similarly to the case of a right shtuka, one can prove the following statement.

**Proposition 5.1.3.**  *$\mathcal{L}$  forms a toy shtuka for  $V$  over  $S$ .  $\square$*

## 5.2 The morphism from the moduli scheme of shtukas with structures of all levels to the moduli scheme of Tate toy shtukas

Fix a field  $E$  over  $\mathbb{F}_q$  and two morphisms  $\alpha, \beta : \text{Spec } E \rightarrow X$  satisfying condition (+). Let  $d$  be a positive integer.

Let  $\text{Sht}_{E,\text{all}}^d$  denote the moduli scheme of shtukas over  $\text{Spec } E$  with zero  $\alpha$  and pole  $\beta$  equipped with structures of all levels compatible with each other. Let  $\text{Sht}_{E,\text{all}}^{d,\chi}$  be the components of  $\text{Sht}_{E,\text{all}}^d$  on which the shtuka  $\mathcal{F}$  has Euler characteristic  $\chi$ .

For any integer  $\chi$ , let  $n_\chi \in \text{Dim}_{\mathbb{A}^d}$  be such that  $n_\chi(O^d) = \chi$ .

For a divisor  $D$  of  $X$ , we denote  $O_D$  to be the c-lattice

$$\varprojlim_{D'} H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X(D'))$$

of  $\mathbb{A}$ , where  $D'$  runs through all divisors of  $X$  such that  $D' \leq D$ . In other words,  $O_D$  consists of those adeles with poles bounded by  $D$ . For two divisors  $D', D''$  of  $X$  such that  $D' \leq D''$ , we have

$$O_{D''}/O_{D'} = H^0(X, \mathcal{O}_X(D'')/\mathcal{O}_X(D')).$$

We have an isomorphism

$$\varinjlim_{D''} \varprojlim_{D' \leq D''} O_{D''}/O_{D'} \xrightarrow{\sim} \mathbb{A}.$$

### 5.2.1 Construction of the morphism $\theta$

We first construct the morphism from  $\text{Sht}_{E,\text{all}}^{d,\chi}$  to  $\text{ToySht}_{\mathbb{A}^d}^{n_\chi}$ .

Let  $S$  be a scheme over  $\text{Spec } E$  and let  $\mathcal{F} \in \text{Sht}_{E,\text{all}}^{d,\chi}(S)$ . Let  $\pi : X \times S \rightarrow S$  be

the projection.

Let  $S^{D', D''}$  be the open subscheme of  $S$  such that all  $s \in S^{D', D''}$  satisfy conditions (5.5) and (5.6). Proposition 5.1.3 shows that  $\pi_* \mathcal{F}(D'')$  is a toy shtuka over  $S$  for  $O_{D''}^d / O_{D'}^d$ . Moreover, conditions (5.5) and (5.6) imply that  $\pi_* \mathcal{F}(D'')$  has rank  $\chi + d \cdot \deg D''$ .

For divisors  $\widetilde{D}', \widetilde{D}''$  such that  $\widetilde{D}' \leq D' \leq D'' \leq \widetilde{D}''$ , we have  $S^{D', D''} \subset S^{\widetilde{D}', \widetilde{D}''}$ , and the composition

$$S^{\widetilde{D}', \widetilde{D}''} \rightarrow \mathrm{ToySht}_{O_{D'}^d / O_{D''}^d}^{\chi + d \cdot \deg \widetilde{D}''} \rightarrow \mathrm{ToySht}_{O_{D''}^d / O_{D'}^d}^{\chi + d \cdot \deg D''}$$

when restricted to  $S^{D', D''}$  coincides with the morphism  $S^{D', D''} \rightarrow \mathrm{ToySht}_{O_{D''}^d / O_{D'}^d}^{\chi + d \cdot \deg D''}$ .

For each  $s \in S$ , there exists a pair of divisors  $D' \leq D''$  such that  $s \in S^{D', D''}$ .

Passing to the double limit, we see that

$$\mathcal{L} = \varinjlim_{D''} \pi_* \mathcal{F}(D'')$$

is a Tate toy shtuka over  $S$  of dimension  $n_\chi$  for  $\mathbb{A}^d$ .

**Proposition 5.2.1.** *For each  $\chi \in \mathbb{Z}$ , the above construction induces a morphism*

$$\theta_E^{d, \chi} : \mathrm{Sht}_{E, \mathrm{all}}^{d, \chi} \rightarrow {}^\circ \mathrm{ToySht}_{\mathbb{A}^d}^{n_\chi}.$$

*Proof.* The above construction induces a morphism  $\mathrm{Sht}_{E, \mathrm{all}}^{d, \chi} \rightarrow \mathrm{ToySht}_{\mathbb{A}^d}^{n_\chi}$ .

Put  $\mathcal{M} = \mathrm{Sht}_{E, \mathrm{all}}^d$ . Let  $\mathcal{F}$  be the universal right shtuka over  $\mathcal{M}$ . Denote  $\Phi = \mathrm{Id}_X \times \mathrm{Fr}_{\mathcal{M}} : X \times \mathcal{M} \rightarrow X \times \mathcal{M}$ .

For any point  $s \in \mathcal{M}$ , there exists a divisor  $D'' \subset X$  such that  $\mathcal{F}_s(D'' \times s)$  and  $\Phi_s^* \mathcal{F}_s(D'' \times s)$  are generated by their global sections. Since  $\alpha \neq \beta$ , we have  $\mathcal{F}_s \neq \Phi_s^* \mathcal{F}_s$ . Hence  $\pi_{s*} \mathcal{F}_s(D'' \times s) \neq \pi_{s*} \Phi_s^* \mathcal{F}_s(D'' \times s) = \mathrm{Fr}_s^* \pi_{s*} \mathcal{F}_s(D'' \times s)$ . This shows that the image of  $s$  in  $\mathrm{ToySht}_{\mathbb{A}^d}^{n_\chi}$  is contained in  ${}^\circ \mathrm{ToySht}_{\mathbb{A}^d}^{n_\chi}$ .  $\square$

### 5.2.2 $GL_d(\mathbb{A})$ -equivariance of the morphism $\theta$

For  $g \in GL_d(\mathbb{A})$ ,  $L \in \text{Grass}_{\mathbb{A}^d}(R)$ , where  $R$  is an  $\mathbb{F}_q$ -algebra, we define  $g \cdot L$  to be the image of  $L \hookrightarrow \mathbb{A}^d \widehat{\otimes} R \xrightarrow{g \otimes 1} \mathbb{A}^d \widehat{\otimes} R$ . In this way we get a (left) action of  $GL_d(\mathbb{A})$  on  $\text{Grass}_{\mathbb{A}^d}$ . We see that this action preserves  ${}^\circ\text{ToySht}_{\mathbb{A}^d}$ .

**Proposition 5.2.2.** *The morphism  $\theta_E^d : \text{Sht}_{E, \text{all}}^d \rightarrow {}^\circ\text{ToySht}_{\mathbb{A}^d}$  is  $GL_d(\mathbb{A})$ -equivariant.*

*Proof.* Let  $S$  be a scheme over  $E$ . Let  $\mathcal{F} \in \text{Sht}_{E, \text{all}}^d(S)$ .

For any  $g \in GL_d(\mathbb{A})$ , the definition of the action of  $g$  in [6, Section 3] implies that the following diagram commutes.

$$\begin{array}{ccc}
\pi_* \left( \varinjlim_{D''} \varprojlim_{D' \leq D''} (g^* \mathcal{F})(D'' \times S) / (g^* \mathcal{F})(D' \times S) \right) & \longrightarrow & \pi_* \left( \varinjlim_{D''} \varprojlim_{D' \leq D''} \mathcal{F}(D'' \times S) / \mathcal{F}(D' \times S) \right) \\
\downarrow \sim & & \downarrow \sim \\
\pi_* \left( \varinjlim_{D''} \varprojlim_{D' \leq D''} (\mathcal{O}_{X \times S}(D'' \times S) / \mathcal{O}_{X \times S}(D' \times S))^d \right) & \longrightarrow & \pi_* \left( \varinjlim_{D''} \varprojlim_{D' \leq D''} (\mathcal{O}_{X \times S}(D'' \times S) / \mathcal{O}_{X \times S}(D' \times S))^d \right) \\
\downarrow = & & \downarrow = \\
\mathbb{A}^d \widehat{\otimes} R & \xrightarrow{g^{-1} \otimes 1} & \mathbb{A}^d \widehat{\otimes} R
\end{array}$$

The natural morphism

$$\varinjlim_{D''} (g^* \mathcal{F})(D'' \times S) \rightarrow \varinjlim_{D''} \mathcal{F}(D'' \times S)$$

induces an isomorphism

$$\varinjlim_{D''} \pi_*((g^* \mathcal{F})(D'' \times S)) \xrightarrow{\sim} \varinjlim_{D''} \pi_*(\mathcal{F}(D'' \times S))$$

The statement follows. □

## 5.3 Review of horospherical divisors

The goal of Sections 5.3 and 5.4 is to prove Theorem 5.4.6, which relates horospherical divisors on the moduli scheme of shtukas with Tate toy horospherical

divisors on the moduli scheme of Tate toy shtukas, and reduces algebraic geometry to representation theory.

Let  $\eta$  denote the generic point of  $X \times X$ . Let  $\alpha, \beta : \eta \rightarrow X$  be the first and second projection. In this section, all shtukas will have zero  $\alpha$  and pole  $\beta$ .

For a finite subscheme  $D \subset X$ , let  $\text{Sht}_{\eta, D}^d$  denote the moduli stack which to each scheme  $S$  over  $\eta$  associates the groupoid of shtukas over  $S$  with zero  $\alpha$  and pole  $\beta$  equipped with a structure of level  $D$ .

Let  $\text{Sht}_{\eta, \text{all}}^d$  denote the moduli scheme which to each scheme  $S$  over  $\eta$  associates the set of isomorphism classes of shtukas over  $S$  with zero  $\alpha$  and pole  $\beta$  equipped with structures of all levels compatible with each other.

For  $\chi \in \mathbb{Z}$ , we denote  $n_\chi$  to be the element of  $\text{Dim}_{\mathbb{A}^d}$  such that  $n_\chi(O^d) = \chi$ .

Let  $\mathfrak{Vect}_{X, D}^d$  (resp.  $\mathfrak{Vect}_{X, \text{all}}^d$ ) denote the set of isomorphism classes of locally free sheaves of rank  $d$  on  $X$  equipped with a structure of level  $D$  (resp. equipped with structures of all levels compatible with each other).

### 5.3.1 Trivial shtukas

**Definition 5.3.1.** For  $d \geq 1$  and an effective divisor  $D \subset X$ , let  $\text{TrSht}_D^d$  denote the moduli stack which to each scheme  $S$  over  $\mathbb{F}_q$  associates the groupoid of locally free sheaves  $\mathcal{M}$  on  $X \times S$  equipped with the following data:

- (i) a structure of level  $D$ , i.e., an isomorphism  $\gamma : \mathcal{M} \otimes \mathcal{O}_{D \times S} \xrightarrow{\sim} \mathcal{O}_{D \times S}^d$ ;
- (ii) an isomorphisms  $\Phi_S^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}$  such that the diagram

$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{O}_{D \times S} & \xrightarrow[\sim]{\gamma} & \mathcal{O}_{D \times S}^d \\ \sim \uparrow & & \sim \uparrow \\ \Phi_S^* \mathcal{M} \otimes \mathcal{O}_{D \times S} & \xrightarrow[\sim]{\Phi_S^* \gamma} & \Phi_S^* \mathcal{O}_{D \times S}^d \end{array}$$

commutes.

An element of  $\text{TrSht}_D(S)$  is called a *trivial shtuka* over  $S$  equipped with a structure

of level  $D$ .

For  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^d$ , let  $\mathrm{TrSht}_{\mathcal{E},D}$  denote the quotient stack  $[\mathrm{Spec} \mathbb{F}_q / \mathrm{Aut} \mathcal{E}]$ .

The following statement is [15, Chapter I, Section 3, Theorem 2].

**Proposition 5.3.2.** *The stack  $\mathrm{TrSht}_D^d$  is a disjoint union*

$$\mathrm{TrSht}_D^d = \coprod_{\mathcal{E} \in \mathfrak{Vect}_D^d} \mathrm{TrSht}_{\mathcal{E},D}.$$

**Proposition 5.3.3.** *Let  $S$  be a projective scheme over  $\mathbb{F}_q$ . Let  $E$  be an separably closed field over  $\mathbb{F}_q$ . Then the functor  $\mathcal{F} \mapsto \mathcal{F} \otimes E$  is an equivalence between the category of coherent sheaves  $\mathcal{F}$  on  $S$  and the category of coherent sheaves  $\mathcal{M}$  on  $S \otimes E$  equipped with an isomorphism  $(\mathrm{Id}_S \otimes \mathrm{Fr}_E)^* \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ .  $\square$*

*Remark 5.3.4.* The above statement is [6, Proposition 1.1] when  $E$  is algebraically closed. The same proof applies when  $E$  is separably closed.

**Corollary 5.3.5.** *Suppose  $E$  is separably closed and  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^d$ . For any  $\mathcal{M} \in \mathrm{TrSht}_{\mathcal{E},D}(E)$ , we can find an isomorphism  $\mathcal{M} \xrightarrow{\sim} \mathcal{E} \otimes E$  compatible with the structure of level  $D$ .*

**Lemma 5.3.6.** *Let  $E$  be a field over  $\mathbb{F}_q$ . Let  $\mathcal{F}$  be a shtuka over  $\mathrm{Spec} E$  with zero  $\alpha$  and pole  $\beta$  satisfying condition (\*). Let  $\mathcal{G}$  be a subshtuka of  $\mathcal{F}$  of the same rank with the same zero and pole. Then  $\Phi_E^* \mathcal{G} \not\subset \mathcal{F}$  and  $\mathcal{G} \not\subset \Phi_E^* \mathcal{F}$ .*

*Proof.* We apply  $d$ -th exterior power to all sheaves involved to reduce the problem to the case  $d = 1$ .

Suppose  $\Phi_E^* \mathcal{G} \subset \mathcal{F}$ . Then we have  $\mathcal{F} = \mathcal{G}(\beta + W)$  for some effective divisor  $W$  of  $X \otimes E$ . From the isomorphisms  $\Phi_E^* \mathcal{G} \cong \mathcal{G}(\beta - \alpha)$  and  $\Phi_E^* \mathcal{F} \cong \mathcal{F}(\beta - \alpha)$  we deduce that  $\beta - \alpha + \mathrm{Fr}_X \circ \beta + \Phi^* W = \beta + W + \beta - \alpha$ . Hence  $\beta + W = \mathrm{Fr}_X \circ \beta + \Phi^* W$ . Applying Lemma 2.2.4 to the two morphisms  $\beta, \mathrm{Fr}_X \circ \beta : \mathrm{Spec} E \rightarrow X$ , we see that  $\beta = \mathrm{Fr}_X^i \circ \beta$  for some  $i \geq 1$ , a contradiction to condition (\*).

The proof of the second statement is similar.  $\square$

**Lemma 5.3.7.** *Let  $E$  be an algebraically closed field. Let  $\mathcal{F}$  be a shtuka of rank  $d$  over  $\text{Spec } E$  with zero and pole satisfying condition  $(*)$ . Let  $\mathcal{G}$  be a subshtuka of  $\mathcal{F}$  of the same rank with the same zero and pole. Then  $\mathcal{F}/\mathcal{G}$  is supported on  $D \otimes E$  for some finite subscheme  $D \subset X$ . Moreover, for any structure of level  $D$  on  $\mathcal{F}$ ,  $\mathcal{G}$  is obtained from  $\mathcal{F}$  by applying Construction E in Section 2.1.2 with respect to that level structure and an  $\mathcal{O}_D$ -submodule  $\mathcal{R} \subset \mathcal{O}_D^d$ .*

*Proof.* Let  $\mathcal{F}' = \Phi_E^* \mathcal{F} + \mathcal{F}$ ,  $\mathcal{G}' = \Phi_E^* \mathcal{G} + \mathcal{G}$ . Lemma 5.3.6 shows that  $\mathcal{F} \cap \mathcal{G}' = \mathcal{G}$  and  $\Phi_E^* \mathcal{F} \cap \mathcal{G}' = \Phi_E^* \mathcal{G}$ . Thus the morphisms  $\mathcal{F}/\mathcal{G} \rightarrow \mathcal{F}'/\mathcal{G}'$  and  $\Phi_E^*(\mathcal{F}/\mathcal{G}) \rightarrow \mathcal{F}'/\mathcal{G}'$  are injective. The sheaves  $\mathcal{F}/\mathcal{G}$ ,  $\mathcal{F}'/\mathcal{G}'$ ,  $\Phi_E^*(\mathcal{F}/\mathcal{G})$  are torsion sheaves on  $X \otimes E$ , and we have  $h^0(\mathcal{F}/\mathcal{G}) = h^0(\mathcal{F}'/\mathcal{G}') = h^0(\Phi_E^*(\mathcal{F}/\mathcal{G}))$ . Hence the morphisms  $\mathcal{F}/\mathcal{G} \rightarrow \mathcal{F}'/\mathcal{G}'$  and  $\Phi_E^*(\mathcal{F}/\mathcal{G}) \rightarrow \mathcal{F}'/\mathcal{G}'$  are isomorphisms. So we have  $\Phi_E^*(\mathcal{F}/\mathcal{G}) \cong \mathcal{F}/\mathcal{G}$ . Proposition 5.3.3 gives an isomorphism  $\mathcal{F}/\mathcal{G} \cong \mathcal{M} \otimes E$  for some coherent sheaf  $\mathcal{M}$  on  $X$ . Since  $\mathcal{F}$  and  $\mathcal{G}$  have the same rank,  $\mathcal{M}$  is supported on a finite subscheme  $D \subset X$ .

Equip  $\mathcal{F}$  with a structure of level  $D$ . Let  $\mathcal{P} = \mathcal{G}/\mathcal{F}(-D \otimes E) \subset \mathcal{F}/\mathcal{F}(-D \otimes E) \cong \mathcal{O}_{D \otimes E}^d$ . Since  $\mathcal{F}(-D \otimes E)$  is a subshtuka of  $\mathcal{G}$ , we get an isomorphism  $\Phi_E^* \mathcal{P} \xrightarrow{\sim} \mathcal{P}$  which is compatible with the natural isomorphism  $\Phi_E^* \mathcal{O}_{D \otimes E}^d \xrightarrow{\sim} \mathcal{O}_{D \otimes E}^d$ . Proposition 5.3.3 gives an isomorphism  $\mathcal{P} \cong \mathcal{R} \otimes E$  for some  $\mathcal{O}_D$ -submodule  $\mathcal{R} \subset \mathcal{O}_D^d$ . We see that  $\mathcal{G}$  is obtained from  $\mathcal{F}$  by applying Construction E with respect to  $\mathcal{R}$ .  $\square$

### 5.3.2 Definition of horospherical cycles

For  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ , denote  $\text{TrSht}_{\eta,\mathcal{E},D}$  to be the base change of  $\text{TrSht}_{\mathcal{E},D}$  from  $\text{Spec } \mathbb{F}_q$  to  $\eta$ .

**Definition 5.3.8.** Given  $d \geq 2$ ,  $1 \leq i \leq d-1$  and  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ , we denote  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\text{I}}$  to be the moduli stack which to each scheme  $S$  over  $\eta$  associates the



groupoid of exact sequences

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{B} \longrightarrow 0,$$

where

- (i)  $\mathcal{A} \in \text{Sht}_{\eta,D}^{d-i}(S)$ ,  $\mathcal{F} \in \text{Sht}_{\eta,D}^{d,\chi=0}(S)$ ,  $\mathcal{B} \in \text{TrSht}_{\eta,\mathcal{E},D}(S)$ ;
- (ii) the morphisms  $\mathcal{A} \rightarrow \mathcal{F}$  and  $\mathcal{F} \rightarrow \mathcal{B}$  are morphisms of shtukas with structures of level  $D$ ;
- (iii) the structures of level  $D$  give the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A} \otimes \mathcal{O}_{D \times S} & \longrightarrow & \mathcal{F} \otimes \mathcal{O}_{D \times S} & \longrightarrow & \mathcal{B} \otimes \mathcal{O}_{D \times S} \longrightarrow 0 \\ & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ 0 & \longrightarrow & \mathcal{O}_{D \times S}^{d-i} & \longrightarrow & \mathcal{O}_{D \times S}^d = \mathcal{O}_{D \times S}^{d-i} \oplus \mathcal{O}_{D \times S}^i & \longrightarrow & \mathcal{O}_{D \times S}^i \longrightarrow 0 \end{array}$$

where the lower exact sequence is the standard one.

**Definition 5.3.9.** Given  $d \geq 2$ ,  $1 \leq i \leq d-1$  and  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ , we denote  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\Pi}$  to be the moduli stack which to each scheme  $S$  over  $\eta$  associates the groupoid of exact sequences

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{B} \longrightarrow 0,$$

where

- (i)  $\mathcal{A} \in \text{TrSht}_{\eta,\mathcal{E},D}(S)$ ,  $\mathcal{F} \in \text{Sht}_{\eta,D}^{d,\chi=0}(S)$ ,  $\mathcal{B} \in \text{Sht}_{\eta,D}^{d-i}(S)$ ;
- (ii) the morphisms  $\mathcal{A} \rightarrow \mathcal{F}$  and  $\mathcal{F} \rightarrow \mathcal{B}$  are morphisms of shtukas with structures of level  $D$ ;

(iii) the structures of level  $D$  give the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{A} \otimes \mathcal{O}_{D \times S} & \longrightarrow & \mathcal{F} \otimes \mathcal{O}_{D \times S} & \longrightarrow & \mathcal{B} \otimes \mathcal{O}_{D \times S} \longrightarrow 0 \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
0 & \longrightarrow & \mathcal{O}_{D \times S}^i & \longrightarrow & \mathcal{O}_{D \times S}^d = \mathcal{O}_{D \times S}^i \oplus \mathcal{O}_{D \times S}^{d-i} & \longrightarrow & \mathcal{O}_{D \times S}^{d-i} \longrightarrow 0
\end{array}$$

where the lower exact sequence is the standard one.

**Definition 5.3.10.** Given  $d \geq 2$ ,  $1 \leq i \leq d-1$ , and  $\mathcal{E} \in \mathfrak{Vect}_{X, \text{all}}^i$ , we define

$$\text{RedSht}_{\eta, \mathcal{E}, \text{all}}^{d, i, \text{I}} := \varprojlim_D \text{RedSht}_{\eta, \mathcal{E}, D}^{d, i, \text{I}},$$

$$\text{RedSht}_{\eta, \mathcal{E}, \text{all}}^{d, i, \text{II}} := \varprojlim_D \text{RedSht}_{\eta, \mathcal{E}, D}^{d, i, \text{II}},$$

where  $D$  runs through all finite subschemes of  $X$ .

**Definition 5.3.11.** Given  $d \geq 2$ ,  $i, j \geq 1$ ,  $i+j \leq d$  and  $\mathcal{A} \in \mathfrak{Vect}_{X, D}^i$ ,  $\mathcal{B} \in \mathfrak{Vect}_{X, D}^j$ , we define  $\text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D}^{d, i, j, \text{I} \wedge \text{II}}$  to be the moduli stack which to each scheme  $S$  over  $\eta$  associates the groupoid of the following data:

(i) an exact sequence of shtukas with structures of level  $D$

$$0 \longrightarrow \mathcal{A}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow 0,$$

where  $\mathcal{A}' \in \text{TrSht}_{\eta, \mathcal{A}, D}(S)$ ,  $\mathcal{F} \in \text{Sht}_{\eta, D}^{d, \chi=0}(S)$ ,  $\mathcal{N} \in \text{Sht}_{\eta, D}^{d-i}(S)$ , such that the structures of level  $D$  induce the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{D \times S}^i \longrightarrow \mathcal{O}_{D \times S}^d \longrightarrow \mathcal{O}_{D \times S}^{d-i} \longrightarrow 0.$$

(ii) an exact sequence of shtukas with structures of level  $D$

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{B}' \longrightarrow 0,$$

where  $\mathcal{M} \in \text{Sht}_{\eta,D}^{d-i-j}(S)$ ,  $\mathcal{B}' \in \text{TrSht}_{\eta,\mathcal{B},D}(S)$ ,  $\mathcal{N}$  is as in (i), such that the structures of level  $D$  induce the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{D \times S}^{d-i-j} \longrightarrow \mathcal{O}_{D \times S}^{d-i} \longrightarrow \mathcal{O}_{D \times S}^j \longrightarrow 0.$$

The above data (i), (ii) are equivalent to the following data:

(i') an exact sequence of shtukas with structures of level  $D$

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{B}' \longrightarrow 0,$$

where  $\mathcal{L} \in \text{Sht}_{\eta,D}^{d-j}(S)$ ,  $\mathcal{F} \in \text{Sht}_{\eta,D}^{d,\chi=0}(S)$ ,  $\mathcal{B}' \in \text{TrSht}_{\eta,\mathcal{B},D}(S)$ , such that the structures of level  $D$  induce the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{D \times S}^{d-j} \longrightarrow \mathcal{O}_{D \times S}^d \longrightarrow \mathcal{O}_{D \times S}^j \longrightarrow 0.$$

(ii') an exact sequence of shtukas with structures of level  $D$

$$0 \longrightarrow \mathcal{A}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where  $\mathcal{A}' \in \text{TrSht}_{\eta,\mathcal{A},D}(S)$ ,  $\mathcal{M} \in \text{Sht}_{\eta,D}^{d-i-j}(S)$ ,  $\mathcal{L}$  is as in (i'), such that the structures of level  $D$  induce the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{D \times S}^i \longrightarrow \mathcal{O}_{D \times S}^{d-j} \longrightarrow \mathcal{O}_{D \times S}^{d-i-j} \longrightarrow 0.$$

*Remark 5.3.12.* We have two Cartesian diagrams

$$\begin{array}{ccc}
\text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D}^{d, i, j, \text{I} \wedge \text{II}} & \longrightarrow & \text{RedSht}_{\eta, \mathcal{B}, D}^{d-i, j, \text{I}} \\
\downarrow & & \downarrow \\
\text{RedSht}_{\eta, \mathcal{A}, D}^{d, i, \text{II}} & \longrightarrow & \text{Sht}_{\eta, D}^{d-i}
\end{array}
\quad
\begin{array}{ccc}
\text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D}^{d, i, j, \text{I} \wedge \text{II}} & \longrightarrow & \text{RedSht}_{\eta, \mathcal{A}, D}^{d-j, i, \text{II}} \\
\downarrow & & \downarrow \\
\text{RedSht}_{\eta, \mathcal{B}, D}^{d, j, \text{I}} & \longrightarrow & \text{Sht}_{\eta, D}^{d-j}
\end{array}
\tag{5.7}$$

**Definition 5.3.13.** For  $\mathcal{A} \in \mathfrak{Vect}_{X, \text{all}}^i$ ,  $\mathcal{B} \in \mathfrak{Vect}_{X, \text{all}}^j$ , we define

$$\text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, \text{all}}^{d, i, j, \text{I} \wedge \text{II}} := \varprojlim_D \text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D}^{d, i, j, \text{I} \wedge \text{II}},$$

where  $D$  runs through all finite subschemes of  $X$ .

### 5.3.3 Basic properties of horospherical cycles

The following statement follows from [15, Chapter I, Section 3, Corollary 6].

**Proposition 5.3.14.** *Let  $D$  be a finite subscheme of  $X$ . Then  $\text{Sht}_{\eta, D}^d$  is a Deligne-Mumford stack and it is separated over  $\eta$ .*

The following statement follows from [15, Chapter I, Section 2, Theorem 9].

**Proposition 5.3.15.** *Let  $D$  be a finite subscheme of  $X$ . The natural morphism  $\text{Sht}_{\eta, D}^d \rightarrow \eta$  is smooth of pure relative dimension  $(2d - 2)$ .  $\square$*

The following statement is [15, Chapter I, Section 3, Proposition 5].

**Proposition 5.3.16.** *For two finite subschemes  $D_1 \subset D_2 \subset X$ , the natural morphism  $\text{Sht}_{\eta, D_2}^d \rightarrow \text{Sht}_{\eta, D_1}^d$  is representable, finite, étale and Galois.*

The following statement is a consequence of [18, Proposition 2.16(a)].

**Proposition 5.3.17.** *For every finite subscheme  $D_1 \subset X$  and every quasi-compact open substack  $U \subset \text{Sht}_{\eta, D_1}^d$ , there exists an integer  $N$  such that  $U \times_{\text{Sht}_{\eta, D_1}^d} \text{Sht}_{\eta, D_2}^d$  is a scheme for all finite subschemes  $D_2 \subset X$  satisfying  $D_2 \geq D_1$  and  $\deg D_2 \geq N$ .*

Propositions 5.3.15 and 5.3.16 imply the following statement.

**Proposition 5.3.18.** *The scheme  $\mathrm{Sht}_{\eta, \mathrm{all}}^d$  has pure dimension  $(2d - 2)$ .  $\square$*

The following statement follows from [15, Chapter II, Section 1, Corollary 10].

**Proposition 5.3.19.** *Let  $D$  be a finite subscheme of  $X$  and let  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ . Then  $\mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{I}}$  (resp.  $\mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{II}}$ ) is a Deligne-Mumford stack and it is separated and locally of finite type over  $\eta$ .*

The following statement follows from [15, Chapter II, Section 1, Theorem 5].

**Proposition 5.3.20.** *Let  $D$  be a finite subscheme of  $X$  and let  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ . Then the natural morphism*

$$\begin{aligned} \mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{I}} &\rightarrow \mathrm{Sht}_{\eta, D}^d \\ (\text{resp. } \mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{II}} &\rightarrow \mathrm{Sht}_{\eta, D}^d) \end{aligned}$$

*is representable, quasi-finite,  $G$ -unramified<sup>1</sup> and separated.  $\square$*

The following statement is [15, Chapter II, Section 1, Theorem 11].

**Proposition 5.3.21.** *Let  $D$  be a finite subscheme of  $X$  and let  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ . Then the natural morphism*

$$\begin{aligned} \mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{I}} &\rightarrow \mathrm{Sht}_{\eta, D}^{d-i} \times \mathrm{TrSht}_{\eta, \mathcal{E}, D} \\ (\text{resp. } \mathrm{RedSht}_{\eta, \mathcal{E}, D}^{d,i,\mathrm{II}} &\rightarrow \mathrm{TrSht}_{\eta, \mathcal{E}, D} \times \mathrm{Sht}_{\eta, D}^{d-i}) \end{aligned}$$

*is of finite type and smooth of pure relative dimension  $i$ .  $\square$*

The following statement follows from Propositions 5.3.15 and 5.3.21.

---

1. We use the definition from Stack Project. A morphism is unramified (resp.  $G$ -unramified) if and only if it is locally of finite type (resp. locally of finite presentation) and formally unramified.

**Proposition 5.3.22.** *Let  $D$  be a finite subscheme of  $X$  and let  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ . Then the natural morphisms  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\text{I}} \rightarrow \eta$  and  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\text{II}} \rightarrow \eta$  are smooth of pure dimension  $(2d - i)$ .  $\square$*

The following statement is [15, Chapter II, Section 1, Proposition 4].

**Proposition 5.3.23.** *Let  $D_1 \subset D_2$  be two finite subschemes of  $X$ . Suppose  $\mathcal{E}_2 \in \mathfrak{Vect}_{X,D_2}^i$  and let  $\mathcal{E}_1 \in \mathfrak{Vect}_{X,D_1}^i$  be the image of  $\mathcal{E}_2$  under the natural map  $\mathfrak{Vect}_{X,D_2} \rightarrow \mathfrak{Vect}_{X,D_1}$ . Then the natural morphism*

$$\text{RedSht}_{\eta,\mathcal{E}_2,D_2}^{d,i,\text{I}} \rightarrow \text{RedSht}_{\eta,\mathcal{E}_1,D_1}^{d,i,\text{I}} \quad (\text{resp. } \text{RedSht}_{\eta,\mathcal{E}_2,D_2}^{d,i,\text{II}} \rightarrow \text{RedSht}_{\eta,\mathcal{E}_1,D_1}^{d,i,\text{II}})$$

*is representable, finite, étale and Galois.  $\square$*

Propositions 5.3.22 and 5.3.23 imply the following statement.

**Proposition 5.3.24.** *For  $\mathcal{E} \in \mathfrak{Vect}_{X,\text{all}}^i$ , the schemes  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,i,\text{I}}$  and  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,i,\text{II}}$  are reduced and of pure dimension  $(2d - i)$ .  $\square$*

Propositions 5.3.20 and 5.3.22 imply the following statement.

**Proposition 5.3.25.** *Let  $D$  be a finite subscheme of  $X$  and let  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^i$ . Then the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\text{I}} \rightarrow \text{Sht}_{\eta,D}^d$  (resp.  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,i,\text{II}} \rightarrow \text{Sht}_{\eta,D}^d$ ) is reduced and has pure dimension  $(2d - i - 2)$ .  $\square$*

**Lemma 5.3.26.** *Let  $I$  be a directed set. Let  $(X_i)_{i \in I}, (Y_i)_{i \in I}$  be two projective systems of schemes with affine surjective transition maps. Let  $(X_i \rightarrow Y_i)_{i \in I}$  be morphisms compatible with transition maps. Denote  $Z_i$  to be the closure of the image of the morphism  $X_i \rightarrow Y_i$ . Then  $\varprojlim_i Z_i$  equals the closure of the image of the morphism  $\varprojlim_i X_i \rightarrow \varprojlim_i Y_i$ .  $\square$*

**Proposition 5.3.27.** *For  $\mathcal{E} \in \mathfrak{Vect}_{X,\text{all}}^i$ , the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,i,\text{I}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$  (resp.  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,i,\text{II}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$ ) is reduced and has pure dimension  $(2d - i - 2)$ .  $\square$*

*Proof.* Let  $\mathcal{Y}_{\text{all}}$  denote the closure of the image of the morphism  $\text{RedSht}_{\eta, \mathcal{E}, \text{all}}^{d, i, \text{I}} \rightarrow \text{Sht}_{\eta, \text{all}}^d$ . Pick an irreducible component  $\mathcal{W}_{\text{all}}$  of  $\mathcal{Y}_{\text{all}}$ .

For a finite subscheme  $D \subset X$ , let  $\mathcal{Y}_D$  denote the closure of the image of the morphism  $\text{RedSht}_{\eta, \mathcal{E}, D}^{d, i, \text{I}} \rightarrow \text{Sht}_{\eta, D}^d$ , and let  $S_D$  denote the set of irreducible components of  $\mathcal{Y}_D$  that contain the image of  $\mathcal{W}_{\text{all}}$ . We see that the transition map  $S_{D_2} \rightarrow S_{D_1}$  is surjective for any  $D_1 \subset D_2$ . Since the set of finite subschemes of  $X$  is countable,  $\varprojlim_D S_D$  is nonempty. Thus we can find an irreducible component  $\mathcal{W}_D \subset \mathcal{Y}_D$  for each  $D$  such that  $\mathcal{W}_{\text{all}} \subset \varprojlim_D \mathcal{W}_D$ . Since each  $\mathcal{W}_D$  is irreducible,  $\varprojlim_D \mathcal{W}_D$  is also irreducible. Proposition 5.3.17 and Lemma 5.3.26 show that  $\varprojlim_D \mathcal{Y}_D = \mathcal{Y}_{\text{all}}$ . So  $\varprojlim_D \mathcal{W}_D$  is an irreducible component of  $\mathcal{Y}_{\text{all}}$ . Hence we have  $\mathcal{W}_{\text{all}} = \varprojlim_D \mathcal{W}_D$ .

For two finite subschemes  $D_1 \subset D_2 \subset X$ , the transition map  $\mathcal{W}_{D_2} \rightarrow \mathcal{W}_{D_1}$  is finite by Proposition 5.3.16. Both  $\mathcal{W}_{D_2}$  and  $\mathcal{W}_{D_1}$  have dimension  $(2d - i - 2)$  by Proposition 5.3.25. Thus the transition map  $\mathcal{W}_{D_2} \rightarrow \mathcal{W}_{D_1}$  is finite and surjective. Then we see that the morphism  $\mathcal{W}_{\text{all}} \rightarrow \mathcal{W}_\emptyset$  is integral and surjective. Therefore,  $\dim \mathcal{W}_{\text{all}} = \dim \mathcal{W}_\emptyset = 2d - i - 2$ . This shows that  $\mathcal{Y}_{\text{all}}$  has pure dimension  $(2d - i - 2)$ .

Reducedness of  $\mathcal{Y}_{\text{all}}$  follows from reducedness of  $\text{RedSht}_{\eta, \mathcal{E}, \text{all}}^{d, i, \text{I}}$  by Proposition 5.3.24.

The statement for the morphism  $\text{RedSht}_{\eta, \mathcal{E}, \text{all}}^{d, i, \text{II}} \rightarrow \text{Sht}_{\eta, \text{all}}^d$  follows from duality.  $\square$

The following statement follows from Cartesian diagrams (5.7) and Propositions 5.3.15, 5.3.21 and 5.3.22.

**Proposition 5.3.28.** *For  $\mathcal{A} \in \mathfrak{Vect}_{X, D}^i$  and  $\mathcal{B} \in \mathfrak{Vect}_{X, D}^j$ , the morphism from  $\text{RedSht}_{\eta, \mathcal{E}_1, \mathcal{E}_2, D}^{d, i, j, \text{I} \wedge \text{II}}$  to  $\eta$  is smooth of pure dimension  $(2d - i - j - 2)$ .  $\square$*

The following statement follows from Cartesian diagrams (5.7) and Propositions 5.3.16 and 5.3.23.

**Proposition 5.3.29.** *For two finite subschemes  $D_1 \subset D_2 \subset X$  and  $\mathcal{A} \in \mathfrak{Vect}_{X, D_2}^i$ ,  $\mathcal{B} \in \mathfrak{Vect}_{X, D_2}^j$ , the natural morphism  $\text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D_2}^{d, i, j, \text{I} \wedge \text{II}} \rightarrow \text{RedSht}_{\eta, \mathcal{A}, \mathcal{B}, D_1}^{d, i, j, \text{I} \wedge \text{II}}$  is representable, finite, étale and Galois.*

Propositions 5.3.28 and 5.3.29 imply the following statement.

**Proposition 5.3.30.** *For  $\mathcal{A} \in \mathfrak{Vect}_{X,\text{all}}^i$ ,  $\mathcal{B} \in \mathfrak{Vect}_{X,\text{all}}^j$ , the scheme  $\text{RedSht}_{\eta,\mathcal{A},\mathcal{B},\text{all}}^{d,i,j,\text{I} \wedge \text{II}}$  has pure dimension  $(2d - i - j - 2)$ .  $\square$*

#### 5.3.4 Irreducibility of the scheme of reducible shtukas

**Theorem 5.3.31.** *Given  $d \geq 2$ , a finite subscheme  $D \subset X$  and  $\mathcal{E} \in \mathfrak{Vect}_{X,D}^1$ , the stacks  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,1,\text{I}}$  and  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,1,\text{II}}$  are irreducible.*

**Theorem 5.3.32.** *For  $d \geq 2$  and  $\mathcal{E} \in \mathfrak{Vect}_{X,\text{all}}^1$ , the schemes  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,1,\text{I}}$  and  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,1,\text{II}}$  are irreducible.*

**Proposition 5.3.33.** *Let  $\mathcal{E} \in \mathfrak{Vect}_{X,\text{all}}^1$ . Denote  $\mathcal{Y}_{\text{all}}^{\text{I}}$  (resp.  $\mathcal{Y}_{\text{all}}^{\text{II}}$ ) to be the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,1,\text{I}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$  (resp.  $\text{RedSht}_{\eta,\mathcal{E},\text{all}}^{d,1,\text{II}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$ ). Then the local ring of  $\mathcal{Y}_{\text{all}}^{\text{I}}$  (resp.  $\mathcal{Y}_{\text{all}}^{\text{II}}$ ) in  $\text{Sht}_{\eta,\text{all}}^d$  is a discrete valuation ring.*

*Proof.* For a finite subscheme  $D \subset X$ , denote  $\mathcal{Y}_D^{\text{I}}$  to be the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{E},D}^{d,1,\text{I}} \rightarrow \text{Sht}_{\eta,D}^d$ . Theorem 5.3.31 implies that  $\mathcal{Y}_D^{\text{I}}$  is irreducible. Choose a quasi-compact substack  $U_\emptyset \subset \text{Sht}_{\eta,\emptyset}^d$  such that  $U_\emptyset \cap \mathcal{Y}_\emptyset^{\text{I}}$  is dense in  $\mathcal{Y}_\emptyset^{\text{I}}$ . Denote  $U_D = U_\emptyset \times_{\text{Sht}_{\eta,\emptyset}^d} \text{Sht}_{\eta,D}^d$ . Proposition 5.3.23 shows that the morphism  $\mathcal{Y}_D^{\text{I}} \rightarrow \mathcal{Y}_\emptyset^{\text{I}}$  is dominant. Thus  $U_D \cap \mathcal{Y}_D^{\text{I}}$  is dense in  $\mathcal{Y}_D^{\text{I}}$  for all  $D$ . By Proposition 5.3.17, there exists an integer  $N$  such that  $U_D$  is a scheme when  $\deg D \geq N$ . Let  $A_D$  be the local ring of  $U_D \cap \mathcal{Y}_D^{\text{I}}$  in  $U_D$  for those  $D$  satisfying  $\deg D \geq N$ . Let  $A_{\text{all}}$  be the local ring of  $\mathcal{Y}_{\text{all}}^{\text{I}}$  in  $\text{Sht}_{\eta,\text{all}}^d$ . Then Lemma 5.3.26 implies that  $A_{\text{all}} = \varinjlim_{\deg D \geq N} A_D$ . For each  $D$ ,  $\text{Sht}_{\eta,D}^d$  is smooth over  $\eta$  by Proposition 5.3.15, and  $\mathcal{Y}_D^{\text{I}}$  has codimension 1 in  $\text{Sht}_{\eta,D}^d$  by Propositions 5.3.15 and 5.3.25. Hence  $A_D$  is a discrete valuation. For two finite subschemes  $D_1 \subset D_2 \subset X$ , the transition map  $U_{D_2} \rightarrow U_{D_1}$  is smooth by Proposition 5.3.16. Hence the transition homomorphism  $A_{D_1} \rightarrow A_{D_2}$  sends uniformizer to uniformizer. Now Lemma 4.4.5 implies that  $A_{\text{all}}$  is a discrete valuation ring.

The statement for  $\mathcal{Y}_{\text{all}}^{\text{II}}$  follows from duality.  $\square$



### 5.3.5 Criteria for a Tate toy horospherical subscheme to contain the image of a horospherical divisor

Fix an integer  $d \geq 2$ . For a shtuka  $\mathcal{G}$  over a perfect field with zero and pole satisfying condition (+), the notation  $\mathcal{G}^I$  and  $\mathcal{G}^{II}$  is defined in Theorem 2.2.6.

**Definition 5.3.34.** Let  $\mathcal{Z}_{\eta,1}^{d,I}$  (resp.  $\mathcal{Z}_{\eta,1}^{d,II}$ ) denote the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{O}_X,\text{all}}^{d,1,I} \rightarrow \text{Sht}_{\eta,\text{all}}^{d,\chi=0}$  (resp.  $\text{RedSht}_{\eta,\mathcal{O}_X,\text{all}}^{d,1,II} \rightarrow \text{Sht}_{\eta,\text{all}}^{d,\chi=0}$ ), where  $\mathcal{O}_X \in \mathfrak{Vect}_{X,\text{all}}^1$  is equipped with the standard structures of all levels.

The following statement follows from Theorem 5.3.32 and Proposition 5.3.24.

**Theorem 5.3.35.**  $\mathcal{Z}_{\eta,1}^{d,I}$  and  $\mathcal{Z}_{\eta,1}^{d,II}$  are reduced and irreducible.

Let  $\xi_{\eta,1}^{d,I}$  (resp.  $\xi_{\eta,1}^{d,II}$ ) denote the generic point of  $\mathcal{Z}_{\eta,1}^{d,I}$  (resp.  $\mathcal{Z}_{\eta,1}^{d,II}$ ).

**Lemma 5.3.36.** Let  $\xi'$  be a geometric generic point of  $\mathcal{Z}_{\eta,1}^{d,I}$ . Let  $\mathcal{F}$  be the shtuka over  $\xi'$ . Then  $\mathcal{F}^{II} = 0$ .

*Proof.* Denote  $\mathcal{A}' = \mathcal{F}^{II}$  and  $i = \text{rank } \mathcal{A}'$ . By Proposition 5.3.3, we can find an isomorphism  $\mathcal{A}' \cong \mathcal{A} \otimes \xi'$  for some  $\mathcal{A} \in \mathfrak{Vect}_{X,\emptyset}^{\text{rank } \mathcal{A}'}$ . Suppose  $\mathcal{A} \neq 0$ .

For a finite subscheme  $D \subset X$ , denote  $\mathcal{L}'_D$  to be the image of the composition  $\mathcal{A} \otimes \mathcal{O}_D \otimes \xi' \rightarrow \mathcal{F} \otimes \mathcal{O}_{D \otimes \xi'}$ . We see that there exists an  $\mathcal{O}_D$ -submodule  $\mathcal{L}_D \subset \mathcal{O}_D^d$  such that  $\mathcal{L}'_D = \mathcal{L}_D \otimes \xi'$ . Moreover, for two subschemes  $D_1 \subset D_2 \subset X$ , we have  $\mathcal{L}_{D_1} = \mathcal{L}_{D_2} \otimes \mathcal{O}_{D_1}$ . Thus we can find  $g \in GL_d(\mathcal{O})$  such that  $g(\mathcal{O}_D^i \oplus 0) = \mathcal{L}_D$  for all finite subschemes  $D \subset X$ . We equip  $\mathcal{A}$  with structures of all levels compatible with each other using the standard structures of all levels on  $\mathcal{O}_X^i$ . Then the image of  $g \cdot \xi'$  is contained in the image of the morphism  $\text{RedSht}_{\eta,\mathcal{A},\text{all}}^{d,i,II} \rightarrow \text{Sht}_{\eta,\text{all}}^d$ . Proposition 5.3.27 implies that  $i = 1$  for dimensional reasons.

From the definition of  $\text{RedSht}_{\eta,\mathcal{O}_X,\text{all}}^{d,1,I}$  we obtain an exact sequence of shtukas

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{F} \longrightarrow \mathcal{B}' \longrightarrow 0$$

where  $\mathcal{B}' \in \text{TrSht}_{\eta, \mathcal{O}_X, \text{all}}(\xi')$ ,  $\mathcal{S} \in \text{Sht}_{\eta, \text{all}}^{d-1}(\xi')$

From the definition of  $\mathcal{F}^{\text{II}}$  in Theorem 2.2.6 we know that  $\mathcal{A}'$  is saturated in  $\mathcal{F}$ .

If the composition  $\mathcal{A}' \rightarrow \mathcal{F} \rightarrow \mathcal{B}'$  is zero, we have an exact sequence of shtukas

$$0 \longrightarrow \mathcal{A}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Then one can find  $h \in GL_d(O)$  such that the image of  $h \cdot \xi'$  is contained in the image of the morphism  $\text{RedSht}_{\eta, \mathcal{A}, \mathcal{O}_X, \text{all}}^{d, 1, 1, \text{I} \wedge \text{II}} \rightarrow \text{Sht}_{\eta, \text{all}}^d$ . This is a contradiction to Propositions 5.3.27 and 5.3.30 for dimensional reasons. Thus the composition  $\mathcal{A}' \rightarrow \mathcal{F} \rightarrow \mathcal{B}'$  is nonzero, and it is injective since  $\text{rank } \mathcal{A}' = 1$ . Therefore, the morphism  $\mathcal{A}' \oplus \mathcal{S} \rightarrow \mathcal{F}$  is injective.

Lemma 5.3.7 implies that  $\mathcal{A}' \oplus \mathcal{S}$  is obtained from  $\mathcal{F}$  by Construction E. Hence there exists  $w \in GL_d(\mathbb{A})$  such that the image of  $w \cdot \xi'$  is contained in the image of the morphism  $\text{Sht}_{\eta, \text{all}}^{d-1} \rightarrow \text{Sht}_{\eta, \text{all}}^d$  which sends a shtuka  $\mathcal{S}$  of rank  $(d-1)$  to the shtuka  $\mathcal{A}' \oplus \mathcal{S}$  of rank  $d$ . By Propositions 5.3.18 and 5.3.27 we get a contradiction for dimensional reasons.  $\square$

As in Section 4.4.1, we denote  $\Delta_{\mathbb{A}^d, J}^n = {}^\circ\text{ToySht}_{\mathbb{A}^d}^n \cap \text{ToySht}_{\mathbb{A}^d/J}$  for  $J \in \mathbf{P}_{\mathbb{A}^d}$ , and we denote  $\Delta_{\mathbb{A}^d, H}^n = {}^\circ\text{ToySht}_{\mathbb{A}^d}^n \cap \text{ToySht}_H$  for  $H \in \mathbf{P}_{(\mathbb{A}^d)^*}$ .

Denote  $GL_d(\mathbb{A})_0 = \{g \in GL_d(\mathbb{A}) \mid \deg g = 0\}$ .

Recall that for  $\chi \in \mathbb{Z}$ , we denote  $n_\chi$  to be the element of  $\text{Dim}_{\mathbb{A}^d}$  such that  $n_\chi(O^d) = \chi$ . In particular, we have  $n_0 \in \text{Dim}_{\mathbb{A}^d}$  corresponding to  $\chi = 0$ .

Let  $\theta_\eta^d : \text{Sht}_{\eta, \text{all}}^d \rightarrow {}^\circ\text{ToySht}_{\mathbb{A}^d}$  be the morphism defined in Proposition 5.2.1.

**Lemma 5.3.37.** *For all  $g \in GL_d(\mathbb{A})_0$  and  $J \in \mathbf{P}_{\mathbb{A}^d}$ , we have  $\theta_\eta^d(g \cdot \xi_{\eta, 1}^{d, \text{I}}) \notin \Delta_{\mathbb{A}^d, J}^{n_0}$ .*

*Proof.* Since  $\theta_\eta^d$  is  $GL_d(\mathbb{A})_0$ -equivariant, it suffices to prove the statement in the case  $g = 1$ . Let  $\xi'$  be a geometric generic point of  $\mathcal{Z}_{\eta, 1}^{d, \text{I}}$ . Let  $\mathcal{F}$  be the shtuka over  $\xi'$ . Suppose  $\theta_\eta^d(\xi_{\eta, 1}^{d, \text{I}}) \in \Delta_{\mathbb{A}^d, J}^{n_0}$  for some  $J \in \mathbf{P}_{\mathbb{A}^d}$ . Then there exists a divisor  $D''$  of  $X$  and a nonzero element  $z \in H^0(X \times \xi', \mathcal{F}(D'' \times \xi'))$  such that  $\Phi_{\xi'}^* z = z$ . Let  $\mathcal{G}$  be

the subsheaf of  $\mathcal{F}(D'' \times \xi')$  generated by  $z$ . We see that  $\mathcal{G} \neq 0$  and  $\Phi_{\xi'}^* \mathcal{G} = \mathcal{G}$ . Thus  $(\mathcal{F}(D'' \times \xi'))^{\Pi} \neq 0$ . Then Corollary 2.2.8 shows that  $\mathcal{F}^{\Pi} \neq 0$ . This is a contradiction to Lemma 5.3.36.  $\square$

**Lemma 5.3.38.** *Let  $E$  be a separably closed field over  $\eta$ . Let  $\xi'' : \text{Spec } E \rightarrow \mathcal{Z}_{\eta,1}^{d,\Pi}$  be a morphism over  $\eta$  whose image lands in the generic point of  $\mathcal{Z}_{\eta,1}^{d,\Pi}$ . Let  $\mathcal{F}$  be the shtuka over  $\xi''$ . Then there exists an isomorphism  $\mathcal{O}_{X \times \xi''} \xrightarrow{\sim} \mathcal{F}^{\Pi}$ , such that for every finite subscheme  $D \subset X$ , the composition*

$$\mathcal{O}_{X \times \xi''} \xrightarrow{\sim} \mathcal{F}^{\Pi} \rightarrow \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{O}_{D \times \xi''} \xrightarrow{\sim} \mathcal{O}_{D \times \xi''}^d$$

is the following standard morphism

$$\mathcal{O}_{X \times \xi''} \twoheadrightarrow \mathcal{O}_{D \times \xi''} \hookrightarrow \mathcal{O}_{D \times \xi''} \oplus \mathcal{O}_{D \times \xi''}^{d-1} = \mathcal{O}_{X \times \xi''}^d.$$

*Proof.* From the definition of  $\text{RedSht}_{\eta, \text{all}}^{d,1,\Pi}$  and Corollary 5.3.5 we see that there exists an injective morphism  $\mathcal{O}_{X \times \xi''} \hookrightarrow \mathcal{F}^{\Pi}$  satisfying the required conditions. Moreover,  $\mathcal{O}_{X \times \xi''}$  is saturated in  $\mathcal{F}$ . Thus it suffices to show that  $\text{rank } \mathcal{F}^{\Pi} = 1$ .

Suppose  $\text{rank } \mathcal{F}^{\Pi} \geq 2$ . Then we can find  $g \in GL_d(O)$  such that the image of  $g \cdot \xi''$  is contained in the image of the morphism  $\text{RedSht}_{\eta, \mathcal{A}, \text{all}}^{d,2,\Pi} \rightarrow \text{Sht}_{\eta, \text{all}}^d$  for some  $\mathcal{A} \in \mathfrak{Vect}_{X, \text{all}}^2$ . We get a contradiction by Proposition 5.3.27 for dimensional reasons.  $\square$

Recall that  $k$  denotes the field of rational functions on  $X$ .

**Proposition 5.3.39.** *For  $J \in \mathbf{P}_{\mathbb{A}^d}$ ,  $\theta_{\eta}^d(\mathcal{Z}_{\eta,1}^{d,\Pi}) \subset \Delta_{\mathbb{A}^d, J}^{n_0}$  if and only if  $J = \mathbb{F}_q \cdot (a, 0, \dots, 0)^t$  for some  $a \in k^{\times}$ .*

*Proof.* Recall that  $\mathcal{Z}_{\eta,1}^{d,\Pi}$  is reduced and irreducible by Theorem 5.3.35. Let  $\xi''$  be a geometric generic point of  $\mathcal{Z}_{\eta,1}^{d,\Pi}$ . Thus  $\theta_{\eta}^d(\mathcal{Z}_{\eta,1}^{d,\Pi}) \subset \Delta_{\mathbb{A}^d, J}^{n_0}$  if and only if the image of  $\theta_{\eta}^d \circ \xi''$  is contained in  $\Delta_{\mathbb{A}^d, J}^{n_0}$ .

Let  $\mathcal{F}$  be the shtuka over  $\xi''$  and let  $L \subset \mathbb{A}^d \otimes \xi''$  be the corresponding Tate toy shtuka. Definition 5.3.9 gives a subshtuka  $\mathcal{A} \subset \mathcal{F}$ . Here  $\mathcal{A} \in \text{TrSht}_{\eta, \mathcal{O}_X, \text{all}}^1(\xi'')$  and  $\mathcal{O}_X$  is equipped with the standard structure of all levels. Moreover, the morphism  $\mathcal{A} \otimes \mathcal{O}_{D \times \xi''} \rightarrow \mathcal{F} \otimes \mathcal{O}_{D \times \xi''}$  induces the standard inclusion of the first entry  $\mathcal{O}_{D \times \xi''} \rightarrow \mathcal{O}_{D \times \xi''}^d$  for all finite subscheme  $D \subset X$ . Proposition 5.3.3 shows that  $\mathcal{A} \cong \mathcal{O}_{X \times \xi''}$ . Thus the constant function  $1 \in H^0(X \times \xi'', \mathcal{O}_{X \times \xi''})$  gives an element  $(1, 0, \dots, 0)^t \in L$ . Hence the image of  $\theta_\eta^d \circ \xi''$  is contained in  $\Delta_{\mathbb{A}^d, J_1}^{n_0}$ , where  $J_1 = \mathbb{F}_q \cdot (1, 0, \dots, 0)^t \in \mathbf{P}_{\mathbb{A}^d}$ . Since the  $k^\times$ -action on  $\text{Sht}_{\eta, \text{all}}^{d, \chi=0}$  is trivial and the morphism  $\theta_\eta^d$  is  $k^\times$ -equivariant, we deduce that  $\theta_\eta^d(\mathcal{Z}_{\eta, 1}^{d, \text{II}}) \subset \Delta_{\mathbb{A}^d, J_a}^{n_0}$  for all  $a \in k^\times$ , where  $J_a = \mathbb{F}_q \cdot (a, 0, \dots, 0)^t \in \mathbf{P}_{\mathbb{A}^d}$ .

Suppose that the image of  $\theta_\eta^d \circ \xi''$  is contained in  $\Delta_{\mathbb{A}^d, J}^{n_0}$  for  $J \in \mathbf{P}_{\mathbb{A}^d}$ . From the definition of  $\theta_\eta^d$  we see that

$$J \otimes \xi'' \subset H^0(X \times \xi'', \mathcal{F}(D'' \times \xi''))$$

for some divisor  $D''$  of  $X$ . Let  $\mathcal{F}^{\text{II}}$  be as in Theorem 2.2.6. Now Corollary 2.2.8 shows that

$$J \otimes \xi'' \subset H^0(X \times \xi'', \mathcal{F}^{\text{II}}(D'' \times \xi'')).$$

Applying Lemma 5.3.38, we deduce that  $J = \mathbb{F}_q \cdot (a, 0, \dots, 0)^t$  for some  $a \in k^\times$ .  $\square$

### 5.3.6 Irreducible components of horospherical divisors

We introduce two subgroups  $P_d^{\text{I}} = \begin{pmatrix} GL_{d-1}^* & \\ & 1 \end{pmatrix} \subset GL_d$  and  $P_d^{\text{II}} = \begin{pmatrix} 1 & \\ & GL_{d-1}^* \end{pmatrix} \subset GL_d$ .

Let  $P_d^{\text{I}}(\mathbb{A})_0 = \{g \in P_d^{\text{I}}(\mathbb{A}) \mid \deg g = 0\}$ ,  $P_d^{\text{II}}(\mathbb{A})_0 = \{g \in P_d^{\text{II}}(\mathbb{A}) \mid \deg g = 0\}$ .

**Lemma 5.3.40.** *For  $g \in GL_d(\mathbb{A})$ ,  $g \cdot \mathcal{Z}_{\eta, 1}^{d, \text{I}} = \mathcal{Z}_{\eta, 1}^{d, \text{I}}$  if and only if  $g \in k^\times \cdot P_d^{\text{I}}(\mathbb{A})_0$ ,  $g \cdot \mathcal{Z}_{\eta, 1}^{d, \text{II}} = \mathcal{Z}_{\eta, 1}^{d, \text{II}}$  if and only if  $g \in k^\times \cdot P_d^{\text{II}}(\mathbb{A})_0$ .*

*Proof.* Recall that the  $k^\times$ -action on  $\text{Sht}_{\eta, \text{all}}^d$  is trivial.

Let  $g \in P_d^{\text{II}}(\mathbb{A})_0$ . Let  $\mathcal{O}_X \in \mathfrak{Vect}_{X,\text{all}}^1$  be equipped with the standard structures of all levels. Recall that  $\mathcal{Z}_{\eta,1}^{d,\text{II}}$  is irreducible. Let  $\xi$  be the generic point of  $\mathcal{Z}_{\eta,1}^{d,\text{II}}$ . Then  $\xi$  is contained in the image of the morphism  $\text{RedSht}_{\eta,\mathcal{O}_X,\text{all}}^{d,1,\text{II}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$ . Let  $\mathcal{F}$  be the shtuka over  $\xi$ . We have an inclusion of shtukas  $\mathcal{A} \subset \mathcal{F}$ , where  $\mathcal{A} \in \text{TrSht}_{\eta,\mathcal{O}_X,\text{all}}$ . Moreover, for all finite subscheme  $D \subset X$ , the induced morphism  $\mathcal{A} \otimes \mathcal{O}_{D \times \xi} \rightarrow \mathcal{F} \otimes \mathcal{O}_{D \times \xi}$  gives the standard inclusion of the first entry  $\mathcal{O}_{D \times \xi} \rightarrow \mathcal{O}_{D \times \xi}^d$ . From the construction of the  $GL_d(\mathbb{A})$ -action on  $\text{Sht}_{\eta,\text{all}}^d$ , we see that  $\mathcal{A} \subset g \cdot \mathcal{F}$ , and for all finite subscheme  $D \subset X$ , the induced morphism  $\mathcal{A} \otimes \mathcal{O}_{D \times \xi} \rightarrow (g \cdot \mathcal{F}) \otimes \mathcal{O}_{D \times \xi}$  gives the standard inclusion of the first entry  $\mathcal{O}_{D \times \xi} \rightarrow \mathcal{O}_{D \times \xi}^d$ . This shows that  $g \cdot \xi$  is contained in the image of the morphism  $\text{RedSht}_{\eta,\mathcal{O}_X,\text{all}}^{d,1,\text{II}} \rightarrow \text{Sht}_{\eta,\text{all}}^d$ , hence contained in  $\mathcal{Z}_{\eta,1}^{d,\text{I}}$ .

Suppose  $g \cdot \mathcal{Z}_{\eta,1}^{d,\text{II}} = \mathcal{Z}_{\eta,1}^{d,\text{II}}$ . Lemma 2.1.6 shows that  $g \in GL_d(\mathbb{A})_0$ . Let  $\mathbf{J}_1 = \{J \in \mathbf{P}_{\mathbb{A}^d} \mid J = \mathbb{F}_q \cdot (a, 0, \dots, 0)^t, a \in k^\times\}$ . For  $J \in \mathbf{P}_{\mathbb{A}^d}$ , Proposition 5.3.39 shows that  $\theta_\eta^d(\mathcal{Z}_{\eta,1}^{d,\text{II}}) \in \Delta_{\mathbb{A}^d,J}^{n_0}$  if and only if  $J \in \mathbf{J}_1$ . Now  $g \cdot \mathcal{Z}_{\eta,1}^{d,\text{II}} = \mathcal{Z}_{\eta,1}^{d,\text{II}}$  implies that  $g \cdot \mathbf{J}_1 = \mathbf{J}_1$ . Hence  $g \in k^\times \cdot P_d^{\text{II}}(\mathbb{A})_0$ .

The second statement follows from duality.  $\square$

Denote

$${}^{\circ\circ}\text{Sht}_{\eta,\text{all}}^d := \text{Sht}_{\eta,\text{all}}^d - \bigcup_{g \in GL_d(\mathbb{A})} g \cdot \mathcal{Z}_{\eta,1}^{d,\text{I}} - \bigcup_{g \in GL_d(\mathbb{A})} g \cdot \mathcal{Z}_{\eta,1}^{d,\text{II}}.$$

We denote  $\Omega_\eta^{d,\text{I}} = GL_d(\mathbb{A})_0 / (k^\times \cdot P_d^{\text{I}}(\mathbb{A})_0)$ ,  $\Omega_\eta^{d,\text{II}} = GL_d(\mathbb{A})_0 / (k^\times \cdot P_d^{\text{II}}(\mathbb{A})_0)$ .

**Definition 5.3.41.** A Cartier divisor of  $\text{Sht}_{\eta,\text{all}}^d$  is called a *horospherical divisor* if its support has empty intersection with  ${}^{\circ\circ}\text{Sht}_{\eta,\text{all}}^d$ .

The following statement is a corollary of Lemma 5.3.40.

**Proposition 5.3.42.** *The set of irreducible components of horospherical divisors of  $\text{Sht}_{\eta,\text{all}}^{d,\chi=0}$  is  $\Omega_\eta^{d,\text{I}} \amalg \Omega_\eta^{d,\text{II}}$ .*

**Lemma 5.3.43.** *The  $GL_d(\mathbb{A})_0$ -action on  $\Omega_\eta^{d,I} \coprod \Omega_\eta^{d,II}$  is continuous.*

*Proof.* The statement follows the construction of the  $GL_d(\mathbb{A})$ -action on  $\text{Sht}_{\eta,\text{all}}^d$ .  $\square$

**Lemma 5.3.44.** *Let  $E$  be an algebraically closed field over  $\eta$ . Suppose  $\mathcal{F} \in \text{Sht}_{\eta,\text{all}}^d(E)$  satisfies  $\mathcal{F}^{II} \neq 0$ . Then there exists  $g \in GL_d(\mathbb{A})$  such that  $g \cdot \mathcal{F} \in \mathcal{Z}_{\eta,1}^{d,II}$ .*

*Proof.* By Proposition 5.3.3, we have an isomorphism  $\mathcal{F}^{II} \cong \mathcal{M} \otimes E$  for some  $\mathcal{M} \in \mathfrak{Vect}_{X,\emptyset}^{\text{rank } \mathcal{F}}$ . Choose an invertible subsheaf  $\mathcal{A} \subset \mathcal{M}$  on  $X$  which is saturated in  $\mathcal{M}$ . Then we have an exact sequence of shtukas

$$0 \longrightarrow \mathcal{A} \otimes E \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

For a finite subscheme  $D \subset X$ , Let  $\mathcal{L}'_D$  be the image of the composition  $\mathcal{A} \otimes \mathcal{O}_{D \otimes E} \rightarrow \mathcal{F} \otimes \mathcal{O}_{D \otimes E} \xrightarrow{\sim} \mathcal{O}_{D \otimes E}^d$ . We see that  $\mathcal{L}'_D$  is invariant under the natural isomorphism  $\Phi_E^* \mathcal{O}_{D \otimes E}^d \xrightarrow{\sim} \mathcal{O}_{D \otimes E}^d$ . By Proposition 5.3.3, we have  $\mathcal{L}'_D = \mathcal{L}_D \otimes E$  for some  $\mathcal{O}_D$ -submodule  $\mathcal{L}_D \subset \mathcal{O}_D^d$ . Moreover, for two finite subschemes  $D_1 \subset D_2 \subset X$ , we have  $\mathcal{L}_{D_1} = \mathcal{L}_{D_2} \otimes \mathcal{O}_{D_1}$ . Thus we can choose  $g_1 \in GL_d(O)$  such that  $g_1(\mathcal{O}_D \oplus 0) = \mathcal{L}_D$  for all finite subschemes  $D \subset X$ . We equip  $\mathcal{A}$  with structures of all levels compatible with each other using the standard structures of all levels on  $\mathcal{O}_X$ .

Since  $\mathbb{A}^\times$  acts transitively on  $\mathfrak{Vect}_{X,\text{all}}^1$ , we can choose  $a \in \mathbb{A}^\times$  such that  $a \cdot \mathcal{A} = \mathcal{O}_X$ , where  $\mathcal{O}_X$  is equipped with the standard structures of all levels. Pick  $B \in GL_{d-1}(\mathbb{A})$  such that  $\deg a + \deg \det B + \chi(\mathcal{F}) = 0$ . Let  $g_2 = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}$ .

Then we see that  $g_2 g_1 \cdot \mathcal{F} \in \mathcal{Z}_{\eta,1}^{d,II}$ .  $\square$

The following statement is dual to Lemma 5.3.44.

**Lemma 5.3.45.** *Let  $E$  be an algebraically closed field over  $\eta$ . Suppose  $\mathcal{F} \in \text{Sht}_{\eta,\text{all}}^d(E)$  satisfies  $\mathcal{F}^I \neq 0$ . Then there exists  $g \in GL_d(\mathbb{A})$  such that  $g \cdot \mathcal{F} \in \mathcal{Z}_{\eta,1}^{d,I}$ .  $\square$*

**Lemma 5.3.46.** *Let  $s$  be a geometric point of  $\mathrm{Sht}_{\eta,\mathrm{all}}^d$  such that  $s \notin g \cdot \mathcal{Z}_{\eta,1}^{d,\mathrm{I}}, s \notin g \cdot \mathcal{Z}_{\eta,1}^{d,\mathrm{II}}$  for all  $g \in GL_d(\mathbb{A})$ . Then the shtuka over  $s$  is irreducible.*

*Proof.* The statement follows from Lemmas 5.3.44 and 5.3.45.  $\square$

**Lemma 5.3.47.** *For any point  $s \in {}^{\circ\circ}\mathrm{Sht}_{\eta,\mathrm{all}}^{d,\chi}$ , we have  $\theta_{\eta}^d(s) \in {}^{\circ\circ}\mathrm{ToySht}_{\mathbb{A}^d}^{n_{\chi}}$ .*

*Proof.* Proposition 5.2.1 shows that  $\theta_{\eta}^d(s) \in {}^{\circ}\mathrm{ToySht}_{\mathbb{A}^d}^{n_{\chi}}$ .

We may assume that  $s$  is a geometric point of  ${}^{\circ\circ}\mathrm{Sht}_{\eta,\mathrm{all}}^{d,\chi}$ . Let  $\mathcal{F}$  be the shtuka over  $s$ . Then  $\mathcal{F}$  is irreducible by Lemma 5.3.46. Hence  $\mathcal{F}^{\mathrm{II}} = 0$  by Theorem 2.2.6. Now Corollary 2.2.8 shows that  $\mathcal{F}(D'' \times s)^{\mathrm{II}} = 0$  for any divisor  $D''$  of  $X$ . Thus  $\theta_{\eta}^d(s) \notin \Delta_{\mathbb{A}^d,J}^{n_{\chi}}$  for any  $J \in \mathbf{P}_{\mathbb{A}^d}$ . By duality,  $\theta_{\eta}^d(s) \notin \Delta_{\mathbb{A}^d,H}^{n_{\chi}}$  for any  $H \in \mathbf{P}_{(\mathbb{A}^d)^*}$ . The statement follows.  $\square$

## 5.4 Pullback of Tate toy horospherical divisors under $\theta$

In this section, we fix an integer  $d \geq 2$ .

Let  $\theta_{\eta}^d : \mathrm{Sht}_{\eta,\mathrm{all}}^d \rightarrow {}^{\circ}\mathrm{ToySht}_{\mathbb{A}^d}$  be the morphism defined in Proposition 5.2.1.

For  $\chi \in \mathbb{Z}$ , we denote  $n_{\chi}$  to be the element of  $\mathrm{Dim}_{\mathbb{A}^d}$  such that  $n_{\chi}(O^d) = \chi$ . In particular, we have  $n_0 \in \mathrm{Dim}_{\mathbb{A}^d}$  corresponding to  $\chi = 0$ .

Denote  $\theta_{\eta}^{d,0} : \mathrm{Sht}_{\eta,\mathrm{all}}^{d,\chi=0} \rightarrow {}^{\circ}\mathrm{ToySht}_{\mathbb{A}^d}^{n_0}$ .

### 5.4.1 $(\theta_{\eta}^{d,0})^*$ as a direct sum

Recall that a Cartier divisor of  ${}^{\circ}\mathrm{ToySht}_{\mathbb{A}^d}$  is called a *(Tate) toy horospherical divisor* if its restriction to  ${}^{\circ\circ}\mathrm{ToySht}_{\mathbb{A}^d}$  is zero. Lemma 5.3.47 shows that it makes sense to pullback Tate toy horospherical divisors under  $\theta_{\eta}^d$ , and the pullback is a horospherical divisor of  $\mathrm{Sht}_{\eta,\mathrm{all}}^d$ .

Recall that  $\mathfrak{D}_{\mathbb{A}^d}^{n_0}$  denotes the space of Tate toy horospherical divisors on  ${}^{\circ}\mathrm{ToySht}_{\mathbb{A}^d}^{n_0}$ .

Theorem 4.4.12 gives an isomorphism

$$\mathfrak{D}_{\mathbb{A}^d}^{n_0} \cong C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times}.$$

Proposition 5.3.42 shows that the set of irreducible components of horospherical divisors of  $\text{Sht}_{\eta, \text{all}}^{d, \chi=0}$  is  $\Omega_\eta^{d, \text{I}} \amalg \Omega_\eta^{d, \text{II}}$

Considering the multiplicity of pullback of Tate toy horospherical divisors under the morphism  $\theta_\eta^{d, 0} : \text{Sht}_{\eta, \text{all}}^{d, \chi=0} \rightarrow {}^\circ\text{ToySht}_{\mathbb{A}^d}^{n_0}$ , we get a homomorphism of (partially) ordered abelian groups

$$(\theta_\eta^{d, 0})^* : C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d, \text{I}}) \oplus C^\infty(\Omega_\eta^{d, \text{II}}).$$

**Lemma 5.4.1.** *The homomorphisms*

$$C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d, \text{II}}),$$

$$C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d, \text{I}})$$

*induced by  $(\theta_\eta^{d, 0})^*$  are zero.*

*Proof.* The first homomorphism is zero by Lemma 5.3.37. By duality, the second one is also zero.  $\square$

The homomorphism  $(\theta_\eta^{d, 0})^*$  induces two maps

$$(\theta_\eta^{d, 0, \text{I}})^* : C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d, \text{I}}),$$

$$(\theta_\eta^{d, 0, \text{II}})^* : C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d, \text{II}}).$$

The following statement is a corollary of Lemma 5.4.1.

**Lemma 5.4.2.**  $(\theta_\eta^{d, 0})^* = (\theta_\eta^{d, 0, \text{I}})^* \oplus (\theta_\eta^{d, 0, \text{II}})^*.$   $\square$



### 5.4.2 Definition of the averaging maps

Recall that  $k$  denotes the field of rational functions on  $X$ . We introduced two subgroups  $P_d^I = \begin{pmatrix} GL_{d-1}^* & \\ & 1 \end{pmatrix} \subset GL_d$  and  $P_d^{II} = \begin{pmatrix} 1 & \\ & GL_{d-1}^* \end{pmatrix} \subset GL_d$  in Section 5.3.6. We also denoted  $\Omega_\eta^{d,I} = GL_d(\mathbb{A})_0 / (k^\times \cdot P_d^I(\mathbb{A})_0)$ ,  $\Omega_\eta^{d,II} = GL_d(\mathbb{A})_0 / (k^\times \cdot P_d^{II}(\mathbb{A})_0)$ .

We introduce two varieties over  $\mathbb{F}_q$ .

$$Y_d^I := GL_d / P_d^I, \quad Y_d^{II} := GL_d / P_d^{II}.$$

We equip  $Y_d^I(\mathbb{A})$  and  $Y_d^{II}(\mathbb{A})$  with topologies as homogeneous spaces of  $GL_d(\mathbb{A})$ .

We identify  $\Omega_\eta^{d,I}$  (resp.  $\Omega_\eta^{d,II}$ ) with  $Y_d^I(\mathbb{A})/k^\times$  (resp.  $Y_d^{II}(\mathbb{A})/k^\times$ ).

Define a map

$$\begin{aligned} \iota_d^{II} : Y_d^{II}(\mathbb{A}) &\rightarrow \mathbb{A}^d - \{0\} \\ g &\mapsto g \cdot (1, 0, \dots, 0)^t \end{aligned}$$

We see that  $\iota_d^{II}$  is well-defined, injective and continuous, but the topology on  $Y_d^{II}(\mathbb{A})$  is different from the subset topology.

Define an averaging map

$$\text{Av}_d^{II} : C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d,II})$$

by the composition

$$C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} = C_+(\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times \xrightarrow{f_1} C_+(Y_d^{II}(\mathbb{A})/\mathbb{F}_q^\times) \xrightarrow{f_2} C^\infty(Y_d^{II}(\mathbb{A})/k^\times) = C^\infty(\Omega_\eta^{d,II})$$

where  $f_1$  and  $f_2$  are as follows. We identify  $C^\infty(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times}$  with  $C^\infty((\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times)$ , and  $C_+(\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times$  is the subgroup of  $C^\infty((\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times)$  corresponding to  $C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times}$ . The inclusion  $\iota_d^{II}$  gives a pullback  $C^\infty((\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times) \rightarrow$

$C^\infty(Y_d^\Pi(\mathbb{A})/\mathbb{F}_q^\times)$ . Let  $C_+(Y_d^\Pi(\mathbb{A})/\mathbb{F}_q^\times)$  be the image of  $C_+((\mathbb{A}^d - \{0\})/\mathbb{F}_q^\times)$ . This gives  $f_1$ . The map  $f_2$  is the pushforward, i.e., for  $\varphi \in C_+(Y_d^\Pi(\mathbb{A})/\mathbb{F}_q^\times)$ ,  $f_2(\varphi)$  is defined by

$$(f_2(\varphi))(x) = \sum_{a \in k^\times / \mathbb{F}_q^\times} \varphi(ax), \quad (x \in Y_d^\Pi(\mathbb{A}))$$

From the definition of  $C_+$  in Section 4.4.2 we see that all but finitely many summands are zero.

For a rational differential  $\omega$  on  $X$  and an element  $a \in \mathbb{A}$ , we denote  $\text{Res}_\omega(a)$  to be the sum of residues of  $a\omega$  at all closed points of  $X$ .

For a rational differential  $\omega$  on  $X$ , we define a map

$$\begin{aligned} \iota_{d,\omega}^I : Y_d^I(\mathbb{A}) &\rightarrow (\mathbb{A}^d)^* - \{0\} \\ g &\mapsto (v \mapsto \text{Res}_\omega((0, \dots, 0, 1) \cdot g^{-1}v)), \quad (v \in \mathbb{A}^d) \end{aligned}$$

We see that  $\iota_{d,\omega}^I$  is well-defined, injective and continuous, but the topology on  $Y_d^I(\mathbb{A})$  is different from the subset topology.

Similarly to  $\text{Av}_d^\Pi$ , we define an averaging map

$$\text{Av}_d^I : C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d,I})$$

by the composition

$$C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} = C_+(((\mathbb{A}^d)^* - \{0\})/\mathbb{F}_q^\times) \xrightarrow{h_1} C_+(Y_d^I(\mathbb{A})/\mathbb{F}_q^\times) \xrightarrow{h_2} C^\infty(Y_d^I(\mathbb{A})/k^\times) = C^\infty(\Omega_\eta^{d,I}).$$

Here  $h_2$  is the pullback along  $\iota_{d,\omega}^I$  for a nonzero rational differential  $\omega$  on  $X$ . We see that the composition does not depend on the choice of  $\omega$ .

### 5.4.3 Formula for pullback of horospherical divisors under $\theta$

Recall that the homomorphism

$$(\theta_\eta^{d,0})^* : C_+((\mathbb{A}^d)^* - \{0\})^{\mathbb{F}_q^\times} \oplus C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times} \rightarrow C^\infty(\Omega_\eta^{d,I}) \oplus C^\infty(\Omega_\eta^{d,II})$$

is defined in Section 5.4.1, and we have  $(\theta_\eta^{d,0})^* = (\theta_\eta^{d,0,I})^* \oplus (\theta_\eta^{d,0,II})^*$  by Lemma 5.4.2.

**Proposition 5.4.3.**  $(\theta_\eta^{d,0,II})^* = \text{Av}_d^{II}$ . □

The proof of Proposition 5.4.3 will be given in Section 5.4.6.

**Lemma 5.4.4.** *An element  $a \in \mathbb{A}$  is contained in  $k$  if and only if  $\text{Res}_\omega(a) = 0$  for all rational differentials  $\omega$  of  $X$ .* □

In view of Lemma 5.4.4, the following statement is dual to Proposition 5.4.3.

**Proposition 5.4.5.**  $(\theta_\eta^{d,0,I})^* = \text{Av}_d^I$ . □

Now we obtain the main theorem of this section.

**Theorem 5.4.6.**  $(\theta_\eta^{d,0})^* = \text{Av}_d^I \oplus \text{Av}_d^{II}$ .

*Proof.* The statement follows from Propositions 5.4.3 and 5.4.5. □

### 5.4.4 A subspace of principal horospherical $\mathbb{Z}[\frac{1}{p}]$ -divisors

We normalize the Haar measure on  $\mathbb{A}^d$  such that  $O^d$  has measure 1.

Fix a nontrivial additive character  $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ . We define the Fourier transform

$$\text{Four}_\psi : C_c^\infty(\mathbb{A}^d; \mathbb{C}) \rightarrow C_c^\infty((\mathbb{A}^d)^*; \mathbb{C})$$

such that for any  $f \in C_c^\infty(\mathbb{A}^d; \mathbb{C})$ ,  $\omega \in (\mathbb{A}^d)^*$ , we have

$$\text{Four}_\psi(f)(\omega) = \int_{\mathbb{A}^d} f(v) \psi(\omega(v)) dv.$$

When  $f \in C_c^\infty(\mathbb{A}^d; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ , we have  $\text{Four}_\psi(f) \in C_c^\infty((\mathbb{A}^d)^*; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}$ , and  $\text{Four}_\psi(f)$  does not depend on the choice of  $\psi$ .

Denote  $C_0^\infty(\mathbb{A}^d) = \{f \in C_c^\infty(\mathbb{A}^d) | f(0) = \int_{\mathbb{A}^d} f dv = 0\}$ .

Combining Theorems 4.6.2 and 5.4.6, we get the following statement.

**Theorem 5.4.7.** *For  $d \geq 2$ , the (partially) ordered abelian group of principal horospherical  $\mathbb{Z}[\frac{1}{p}]$ -divisors of  $\text{Sht}_{\eta, \text{all}}^{d, \chi=0}$  contains the following subgroup*

$$\{(\text{Av}_d^{\text{I}}(\text{Four}_\psi(f)), \text{Av}_d^{\text{II}}(f)) \in C^\infty(\Omega_\eta^{d, \text{I}}; \mathbb{Z}[\frac{1}{p}]) \oplus C^\infty(\Omega_\eta^{d, \text{II}}; \mathbb{Z}[\frac{1}{p}]) | f \in C_0^\infty(\mathbb{A}^d; \mathbb{Z}[\frac{1}{p}])^{\mathbb{F}_q^\times}\}.$$

#### 5.4.5 Multiplicity one for pullback of toy horospherical divisors

Recall that the notation  $O_D$  for a divisor  $D$  of  $X$  is defined in Section 5.2. It is the c-lattice of  $\mathbb{A}$  which consists of those adeles with poles bounded by  $D$ .

Let  $\chi \in \mathbb{Z}$ . Let  $D', D''$  be two divisors of  $X$  such that  $(O_{D''}^d, O_{D'}^d) \in AP_{n_\chi}(\mathbb{A}^d)$ . (See Section 4.2.2 for the definition of the notation  $AP$ .)

Let  $E$  be a separably closed field over  $\mathbb{F}_q$  and fix two morphisms  $\alpha, \beta : \text{Spec } E \rightarrow X$  satisfying condition (\*).

Let  $\text{Sht}_{E, D}^{d, \chi, D', D''}$  be the moduli stack which to each scheme  $S$  over  $\text{Spec } E$  associates the groupoid of shtukas  $\mathcal{F}$  over  $S$  of rank  $d$  with zero  $\alpha \circ \mathbf{p}_S$  and pole  $\beta \circ \mathbf{p}_S$  equipped with a structure of level  $D$ , satisfying the following conditions:

- (i)  $\chi(\mathcal{F}) = \chi$ .
- (ii) For every  $s \in S$  one has  $H^0(X \times s, \mathcal{F}_s(D' \times s)) = 0$  and  $H^1(X \times s, \mathcal{F}_s(D'' \times s)) = 0$ .

Proposition 5.1.3 gives a morphism  $\text{Sht}_{E, D}^{d, \chi, D', D''} \rightarrow \text{ToySht}_{O_{D''}^d/O_{D'}^d}^{\chi+d \cdot \deg D''} \otimes E$ . The image of the morphism lands in  ${}^\circ\text{ToySht}_{O_{D''}^d/O_{D'}^d}^{\chi+d \cdot \deg D''}$  since we have  $\alpha \neq \beta$  by condition (\*).

Define  $\text{Art}_E$  to be the category of local Artinian rings whose residue fields are identified with  $E$ . Define  $\text{Art}_E^{(n)}$  to be the full subcategory of  $\text{Art}_E$  whose objects

satisfy the condition that  $x^{q^n} = 0$  for all  $x$  in the maximal ideal.

For a shtuka  $\mathcal{G}$  over a perfect field, the notation  $\mathcal{G}^I$  and  $\mathcal{G}^{II}$  is defined in Theorem 2.2.6.

**Lemma 5.4.8.** *Let  $\mathcal{G} \in \text{Sht}_{E,D}^{d,\chi,D',D''}(E)$ . Suppose that  $\mathcal{G}^{II} = \mathcal{A} \otimes E$ , where  $\mathcal{A}$  is an invertible sheaf on  $X$ . Let  $A \in \text{Art}_E^{(1)}$ . Let  $\tilde{\mathcal{G}}$  be a shtuka over  $\text{Spec } A$  extending the one over  $\text{Spec } E$ . Let  $\tilde{L} = \pi_{A*}(\tilde{\mathcal{G}}(D'' \otimes A)) \in {}^\circ\text{ToySht}_{O_{D''}^d/O_{D'}^d}^{\chi+d \cdot \deg D''}(A)$  be the associated toy shtuka over  $\text{Spec } A$ . Assume that  $\tilde{L} \supset J \otimes A$  for some  $J \in \mathbf{P}_{O_{D''}^d/O_{D'}^d}$ . Then  $\tilde{\mathcal{G}}$  contains  $\mathcal{A} \otimes A$ , and  $\tilde{\mathcal{G}}/(\mathcal{A} \otimes A)$  is locally free.*

*Remark 5.4.9.* Since  $A \in \text{Art}_E^{(1)}$ , we have  $\Phi_A^* \tilde{\mathcal{G}} = (\Phi_E^* \mathcal{G}) \otimes_E A$ . We view  $\tilde{\mathcal{G}}$  as a subsheaf of  $(\Phi_A^* \tilde{\mathcal{G}})(\Gamma_{\tilde{\alpha}})$ , where  $\tilde{\alpha} : \text{Spec } A \rightarrow X$  is the zero of  $\mathcal{G}$ . We also have  $\Phi_E^* \mathcal{G} \supset \mathcal{A} \otimes E$ . So  $\mathcal{A} \otimes A$  in the above lemma is viewed as a subsheaf of  $\Phi_A^* \tilde{\mathcal{G}}$ .

*Proof of Lemma 5.4.8.* We have  $J \otimes E \subset H^0(X \otimes E, \mathcal{G}(D'' \otimes E))$ . Since  $\text{Fr}_E^*(J \otimes E) = J \otimes E$ , Theorem 2.2.6 shows that  $J \otimes E \subset H^0(X \otimes E, (\mathcal{G}(D'' \otimes E))^{II})$ . Since  $\mathcal{G}^{II} = \mathcal{A} \otimes E$ , Corollary 2.2.8 implies that  $(\mathcal{G}(D'' \otimes E))^{II} = \mathcal{A}(D'') \otimes E$ . Together with the fact that  $J$  is defined over  $\mathbb{F}_q$ , we deduce that  $J \subset H^0(X, \mathcal{A}(D''))$ . Let  $\mathcal{J}_0$  be the subsheaf of  $\mathcal{A}(D'')$  generated by  $J$ . Then  $\mathcal{J}_0 = \mathcal{A}(D'' - D_0)$ , where  $D_0$  is an effective divisor of  $X$ . The assumption  $\tilde{L} \supset J \otimes A$  implies that  $\tilde{\mathcal{G}}(D'' \otimes A) \supset \mathcal{J}_0 \otimes A$ . Hence

$$\tilde{\mathcal{G}} \supset (\mathcal{A} \otimes A)(-D_0 \otimes A). \quad (5.8)$$

Since  $A \in \text{Art}_E^{(1)}$ , we have  $\Phi_A^* \tilde{\mathcal{G}} = (\Phi_E^* \mathcal{G}) \otimes_E A$ . From the definition of shtuka we see that  $\tilde{\mathcal{G}} \supset ((\Phi_E^* \mathcal{G}) \otimes_E A)(-\Gamma_{\tilde{\beta}})$ , where  $\tilde{\beta} : \text{Spec } A \rightarrow X$  is the pole of  $\tilde{\mathcal{G}}$ . We also have  $(\Phi_E^* \mathcal{G}) \otimes_E A \supset \mathcal{A} \otimes A$ . Thus

$$\tilde{\mathcal{G}} \supset (\mathcal{A} \otimes A)(-\Gamma_{\tilde{\beta}}). \quad (5.9)$$

Condition  $(*)$  implies that image of  $\beta : \text{Spec } E \rightarrow X$  is the generic point of  $X$ .

Hence  $D_0 \otimes A$  and  $\Gamma_{\tilde{\beta}}$  are disjoint. Now (5.8) and (5.9) imply that  $\tilde{\mathcal{G}} \supset \mathcal{A} \otimes A$ .

The base change of the inclusion  $\mathcal{A} \otimes A \hookrightarrow \tilde{\mathcal{G}}$  to  $X \otimes E$  is the inclusion  $\mathcal{A} \otimes E \hookrightarrow \mathcal{G}$ , which is injective at each point according to the definition of  $\mathcal{G}^{\text{II}} = \mathcal{A} \otimes E$  in Theorem 2.2.6. Since  $A \in \text{Art}_E^{(1)}$ , the morphism  $X \otimes E \rightarrow X \otimes A$  is bijective on points. Thus the inclusion  $\mathcal{A} \otimes A \hookrightarrow \tilde{\mathcal{G}}$  has strictly constant rank 1. Now Lemma 3.5.3 implies that  $\tilde{\mathcal{G}}/(\mathcal{A} \otimes A)$  is locally free.  $\square$

Recall that  $\mathcal{Z}_{\eta,1}^{d,\text{II}}$  is defined in Definition 5.3.34.

Now we consider the case  $\chi = 0$ . Denote  $\mu : \text{Sht}_{\eta,\text{all}}^{d,\chi=0,D',D''} \rightarrow {}^{\circ}\text{ToySht}_{O_{D''}^d/O_{D'}^d}^{d,\deg D''}$  to be the morphism given by Proposition 5.1.3.

**Proposition 5.4.10.** *Let  $J \in \mathbf{P}_{O_{D''}^d/O_{D'}^d}$ . Let  $\Delta = \Delta_{O_{D''}^d/O_{D'}^d,J}^{d,\deg D''}$  be a toy horospherical divisor of  ${}^{\circ}\text{ToySht}_{O_{D''}^d/O_{D'}^d}^{d,\deg D''}$ . Then the multiplicity of  $\mu^{-1}(\Delta)$  at  $(g \cdot \mathcal{Z}_{\eta,1}^{d,\text{II}}) \cap \text{Sht}_{\eta,\text{all}}^{d,D',D''}$  is  $\leq 1$  for all  $g \in GL_d(\mathbb{A})_0$ .*

*Proof.* Let  $\xi = \text{Spec } L$  be the generic point of  $(g \cdot \mathcal{Z}_{\eta,1}^{d,\text{II}}) \cap \text{Sht}_{\eta,\text{all}}^{d,D',D''}$ . Let  $R$  be the local ring of  $\xi$  in  $\text{Sht}_{\eta,\text{all}}^{d,D',D''}$ . Let  $S = \mu^{-1}(\Delta) \cap \text{Spec } R$ .

Let  $E$  be a separable closure of  $L$ . Let  $\mathcal{F}$  be the shtuka over  $\text{Spec } E$ . Lemma 5.3.38 shows that  $(g^{-1} \cdot \mathcal{F})^{\text{II}} \cong \mathcal{O}_X$ . Hence  $\mathcal{F}^{\text{II}} = \mathcal{A} \otimes E$  for some invertible sheaf  $\mathcal{A}$  on  $X$ .

Let  $\mathcal{Y}$  be the closure of the image of the morphism  $\text{RedSht}_{\eta,\mathcal{A},\emptyset}^{d,1,\text{II}} \rightarrow \text{Sht}_{\eta,\emptyset}^d$ . Proposition 5.3.22 shows that  $\mathcal{Y}$  is reduced. Proposition 5.3.23 shows that the inverse image of  $\mathcal{Y}$  under the morphism  $\text{Spec } R \rightarrow \text{Sht}_{\eta,\emptyset}^d$  is equal to  $\xi$ .

Lemma 5.4.8 shows that for any  $A \in \text{Art}_E^{(1)}$ , a morphism  $\text{Spec } A \rightarrow S$  factors through  $\xi$  as long as the composition  $\text{Spec } E \rightarrow \text{Spec } A \rightarrow S$  equals the composition  $\text{Spec } E \rightarrow \text{Spec } L \rightarrow S$ . Hence  $S = \xi$ .  $\square$

### 5.4.6 Description of $(\theta_\eta^{d,0,\Pi})^*$

In this subsection, we describe  $(\theta_\eta^{d,0,\Pi})^*$  in terms of whether a Tate toy horospherical divisor of  ${}^\circ\text{ToySht}_{\mathbb{A}^d}^{n_0}$  contains the image of a horospherical divisor of  $\text{Sht}_{\eta,\text{all}}^{d,\chi=0}$  (See Lemma 5.4.11). Then we finish the proof of Proposition 5.4.3 to obtain a formula for  $(\theta_\eta^{d,0,\Pi})^*$ .

Let  $x \in \Omega_\eta^{d,\Pi}$ . Denote  $\mathcal{Z}_{\eta,x}^{d,\Pi} = x \cdot \mathcal{Z}_{\eta,1}^{d,\Pi}$ . Let  $\xi_{\eta,x}^{d,\Pi}$  denote the generic point of  $\mathcal{Z}_{\eta,x}^{d,\Pi}$ .

We denote

$$\mathbf{J}_x = \{J \in \mathbf{P}_{\mathbb{A}^d} \mid \theta_\eta^d(\xi_{\eta,x}^{d,\Pi}) \in \Delta_{\mathbb{A}^d,J}^{n_0}\}.$$

**Lemma 5.4.11.** *Let  $f \in C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times}$ . Then the set*

$$\mathbf{J}_{x,f} := \{J \in \mathbf{J}_x \mid f(J) \neq 0\}$$

*is finite, and we have*

$$((\theta_\eta^{d,0,\Pi})^* f)(x) = \sum_{J \in \mathbf{J}_x} f(J).$$

□

*Proof of Proposition 5.4.3.* The group  $GL_d(\mathbb{A})_0$  acts transitively on  $\Omega_\eta^{d,\Pi}$ . The morphism  $\theta_\eta^{d,0}$  is  $GL_d(\mathbb{A})_0$ -equivariant. Hence it suffices to show that  $((\theta_\eta^{d,0,\Pi})^* f)(1) = (\text{Av}_d^\Pi f)(1)$  for any  $f \in C_+(\mathbb{A}^d - \{0\})^{\mathbb{F}_q^\times}$ . This follows from Lemma 5.4.11 and Proposition 5.3.39. □

Recall that the notation  $O_D$  for a divisor  $D$  of  $X$  is defined in Section 5.2. Choose two divisors  $D', D''$  of  $X$  such that  $(O_{D'}^d, O_{D''}^d) \in AP_{n_0}(\mathbb{A}^d)$  and  $\xi_{\eta,x}^{d,\Pi} \in \text{Sht}_{\eta,\text{all}}^{d,\chi=0,D',D''}$ .

For divisors  $\widetilde{D}' \leq D' \leq D'' \leq \widetilde{D}''$  of  $X$ , we denote

$$U_{\widetilde{D}', \widetilde{D}''}^{n_0, D', D''} = {}^\circ\text{ToySht}_{O_{\widetilde{D}''}^d / O_{\widetilde{D}'}^d}^{d \cdot \deg \widetilde{D}'', O_{D'}^d / O_{D''}^d, O_{D''}^d / O_{D'}^d},$$

$$\mathbf{J}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''} = \{J \in \mathbf{J}_x \mid J \not\subset O_{D'}^d, J \subset O_{D''}^d\}.$$

For  $\bar{J} \in \mathbf{P}_{O_{\widetilde{D''}}^d/O_{\widetilde{D'}}^d}$ , we denote  $Z_{\widetilde{D'}, \widetilde{D''}, \bar{J}}^{D', D''} = (\Delta_{O_{\widetilde{D''}}^d/O_{\widetilde{D'}}^d, \bar{J}}^{d, \deg \widetilde{D''}} \cap U_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''})$ . We denote

$$\bar{\mathbf{J}}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''} = \{\bar{J} \in \mathbf{P}_{O_{\widetilde{D''}}^d/O_{\widetilde{D'}}^d} \mid (u_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''} \circ \theta_{\eta}^d)(\xi_{\eta, x}^{d, \Pi}) \in Z_{\widetilde{D'}, \widetilde{D''}, \bar{J}}^{D', D''}\}$$

where  $u_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''} : {}^{\circ}\text{ToySht}_{\mathbb{A}^d}^{n_0, D', D''} \rightarrow U_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''}$  is the projection.

For  $J \in \mathbf{J}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''}$ , we have  $u_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''}(\Delta_{\mathbb{A}^d, J}^{n_0, D', D''}) \subset Z_{\widetilde{D'}, \widetilde{D''}, \bar{J}}^{D', D''}$ , where  $\bar{J} = \text{im}(J \cap O_{\widetilde{D''}}^d \rightarrow O_{\widetilde{D''}}^d/O_{\widetilde{D'}}^d)$ . Thus we get a map

$$\begin{aligned} \mathbf{J}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''} &\rightarrow \bar{\mathbf{J}}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''} \\ J &\mapsto \bar{J} \end{aligned}$$

**Lemma 5.4.12.** *The map  $\mathbf{J}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''} \rightarrow \bar{\mathbf{J}}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''}$  is bijective.*

*Proof.* Let  $\bar{J} \in \bar{\mathbf{J}}_{x, \widetilde{D'}, \widetilde{D''}}^{D', D''}$ . By Lemma 4.4.3, it suffices to show that for any two divisors  $\mathring{D'}, \mathring{D''}$  of  $X$  such that  $\mathring{D'} \leq \widetilde{D'} \leq \widetilde{D''} \leq \mathring{D''}$ , there is a unique  $\mathring{J} \in \mathring{\mathfrak{J}}$  such that  $(u_{\mathring{D'}, \mathring{D''}}^{n_0, D', D''} \circ \theta_{\eta}^d)(\xi_{\eta, x}^{d, \Pi}) \in Z_{\mathring{D'}, \mathring{D''}, \mathring{J}}^{D', D''}$ , where the set  $\mathring{\mathfrak{J}}$  is defined by

$$\mathring{\mathfrak{J}} = \{J \in \mathbf{P}_{O_{\mathring{D''}}^d/O_{\mathring{D'}}^d} \mid \text{im}(J \cap O_{\mathring{D''}}^d \rightarrow O_{\mathring{D''}}^d/O_{\mathring{D'}}^d) = \bar{J}\}.$$

Let  $\bar{\Delta} = Z_{\widetilde{D'}, \widetilde{D''}, \bar{J}}^{D', D''}$ . Denote  $u : U_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''} \rightarrow U_{\mathring{D'}, \mathring{D''}}^{n_0, D', D''}$  to be the projection.

Consider the composition of morphisms

$$\text{Sht}_{\eta, \text{all}}^{d, \chi=0, D', D''} \xrightarrow{\theta_{\eta}^d} {}^{\circ}\text{ToySht}_{\mathbb{A}^d}^{n_0, D', D''} \xrightarrow{u_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''}} U_{\mathring{D'}, \mathring{D''}}^{n_0, D', D''} \xrightarrow{u} U_{\widetilde{D'}, \widetilde{D''}}^{n_0, D', D''}.$$



Proposition 4.3.4 implies that

$$u^*(\overline{\Delta}) = \sum_{J \in \mathfrak{J}} Z_{\mathring{D}', \mathring{D}'', J}^{D', D''}. \quad (5.10)$$

Corollary 4.2.10 shows that the morphism  $u$  is smooth. Hence

$$u^{-1}(\overline{\Delta}) = \bigcup_{J \in \mathfrak{J}} Z_{\mathring{D}', \mathring{D}'', J}^{D', D''}.$$

The assumption  $\overline{J} \in \overline{\mathbf{J}}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$  implies that  $(u \circ u_{\mathring{D}', \mathring{D}''}^{n_0, D', D''} \circ \theta_\eta^d)(\xi_{\eta, x}^{d, \Pi}) \in \overline{\Delta}$ . Hence  $(u_{\mathring{D}', \mathring{D}''}^{n_0, D', D''} \circ \theta_\eta^d)(\xi_{\eta, x}^{d, \Pi}) \in Z_{\mathring{D}', \mathring{D}'', \mathring{J}}^{D', D''}$  for some  $\mathring{J} \in \mathfrak{J}$ . This shows the existence of  $\mathring{J}$ .

Proposition 5.4.10 shows that the pullback of  $\overline{\Delta}$  to  $\text{Sht}_{\eta, \text{all}}^{d, \chi=0, D', D''}$  has multiplicity  $\leq 1$  at  $\mathcal{Z}_{\eta, x}^{d, \Pi}$ . From formula (5.10), we see that there is at most one  $\mathring{J} \in \mathfrak{J}$  such that  $(u_{\mathring{D}', \mathring{D}''}^{n_0, D', D''} \circ \theta_\eta^d)(\xi_{\eta, x}^{d, \Pi}) \in Z_{\mathring{D}', \mathring{D}'', \mathring{J}}^{D', D''}$ . We obtain the uniqueness of  $\mathring{J}$ .  $\square$

*Proof of Lemma 5.4.11.* Let  $Q \in \mathfrak{D}_{\mathbb{A}^d}^{n_0}$  be the divisor of  ${}^\circ\text{ToySht}_{\mathbb{A}^d}^{n_0}$  corresponding to  $0 \oplus f$ . Let  $Q^{D', D''} \in \mathfrak{D}_{\mathbb{A}^d}^{n_0, D', D''}$  be the restriction of  $Z$  to  ${}^\circ\text{ToySht}_{\mathbb{A}^d}^{n_0, D', D''}$ . From Lemma 4.2.3 we see that there exist two divisors  $\widetilde{D}', \widetilde{D}''$  of  $X$  such that  $\widetilde{D}' \leq D' \leq D'' \leq \widetilde{D}''$  and that  $Q^{D', D''}$  equals the pullback of a toy horospherical divisor  $\overline{Z}$  of  $U_{\widetilde{D}', \widetilde{D}''}^{n_0, D', D''}$ . Thus we have an inclusion  $\mathbf{J}_{x, f} \subset \mathbf{J}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$ . Since  $\overline{\mathbf{J}}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$  is finite,  $\mathbf{J}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$  is also finite by Lemma 5.4.12, hence is  $\mathbf{J}_{x, f}$ .

Proposition 5.4.10 implies that

$$((\theta_\eta^{d, 0, \Pi})^* f)(x) = \sum_{\overline{J} \in \overline{\mathbf{J}}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}} m(f, \overline{J})$$

where  $m(f, \overline{J})$  is the multiplicity of  $\overline{Z}$  at  $Z_{\widetilde{D}', \widetilde{D}'', \overline{J}}^{D', D''}$ . For any  $\overline{J} \in \overline{\mathbf{J}}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$ , Lemma 5.4.12 shows that there is a unique  $J \in \mathbf{J}_{x, \widetilde{D}', \widetilde{D}''}^{D', D''}$  such that  $\overline{J} = \text{im}(J \cap O_{\widetilde{D}''}^d \rightarrow O_{\widetilde{D}''}^d / O_{\widetilde{D}'}^d)$ . Lemma 4.4.17 shows that  $m(f, \overline{J}) = f(J)$ . The statement follows.  $\square$

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