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TRANSITIONS IN LONG RUN ECONOMIC GROWTH

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SUMMARY OF THE THESIS

The dissertation is a collection of three essays that share a common theme: understanding long run economic changes that do not exhibit balanced growth. In Chapter 1, I analyze the mechanisms causing economies to transition from slow preindustrial rates of growth to modern growth. In Chapter 2, I consider why growth is recently slowing, and what it will look like in the future. In Chapter 3, I examine the military implications of structural change during growth's transition. Balanced growth characterizes the past two centuries, in the richest countries, at the most aggregate level. But to understand how it operates and how it will evolve, we must understand why growth varies so significantly across time and space, and at disaggregated levels. Below, I outline how the three chapters contribute to this understanding:

Chapter 1: Urbanization, Long-Run Growth, and the Demographic Transition

Advanced economies undergo three transitions during their development: 1. They transition from a rural to an urban economy. 2. They transition from low income growth to high income growth. 3. Their demographics transition from initially high fertility and mortality rates to low modern levels. The timings of these transitions are correlated in the historical development of most advanced economies. I unify complementary theories of the transitions into a nonlinear model of endogenous long run economic and demographic change. The model reproduces the timing and magnitude of the transitions. Because the model captures the interactions between all three transitions, it is able to explain three additional empirical patterns: a declining urban-rural wage gap, a declining rural-urban family size ratio, and most surprisingly, that early urbanization slows development.

Chapter 2: Decreasing Returns to R&D and Declining Growth Rates

Why is growth slowing? Two facts are documented: 1. Richer countries spend a greater share of their income on research and development, and 2. Countries with high spending on research and development grow slower. These facts are evident in both the US time series and in the cross-section of countries. This paper proposes a model that explains these two facts, driven by declining returns to research and development. Crucially, the model is tractable enough to easily analyze an economy in transition towards the long run balanced growth path. This is because the size distribution of firms can be expressed analytically as a function of the aggregate technology level. As technology advances, it costs a greater share of output to increase at the same rate; innovators compensate by spending more in R&D, but cannot compensate fully. In the long run, the R&D share of output asymptotes to 3.0-3.9%, and the per capita GDP growth rate declines to 1.0-1.5%.

Growth and the Rise and Fall of Warfare

For a thousand years, income growth was associated with a rising military employment share. But this share peaked in the early 20th century, after which military employment shares fell with income growth. This paper presents evidence that the rising military shares were driven by structural change out of agriculture, and the recent declines are driven by a substitution effect from soldiers towards military goods. The substitution effect is supported in the data, as the ratio of the military expenditure share to the military employment share rises over time. A game theoretic model of growth and warfare is calibrated to micro data on war returns. It reproduces the time-series patterns of military expenditure and employment. The model correctly predicts that the time series pattern does not hold in the current cross-section of countries, where military employment shares are increasing in income. Finally, when the war decision is endogenous, the frequency of wars falls as income grows.

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CHAPTER 1

URBANIZATION, LONG-RUN GROWTH, AND THE DEMOGRAPHIC TRANSITION

1.1 Introduction

Why do economies transition from milennia of near-zero income growth to modern income growth rates? Leading theories of long-run growth attempt to understand development through one of two mechanisms. A literature following Becker et al. (1990) and Galor and Weil (2000) theorize that the central mechanism is substitution of child quantity to child quality, and jointly explain the growth transition and the demographic transition. Simultaneously, a literature following Hansen and Prescott (2002) and Lucas (2004) theorize that the central mechanism is structural transformation, and jointly explain the growth transition and urbanization.

But these mechanisms are not substitutes. The incentives for quantity-quality substitution differ between urban and rural areas, and structural transformation alone cannot explain the rapid acceleration of economic growth. I propose a unifying theory which features both mechanisms and endogenously reproduces the timing and magnitude of the three transitions. Only by considering growth, urbanization, and demographics jointly can this theory predict the three following observations: a declining urban-rural wage gap, a declining rural-urban family size ratio, and early urbanization slows development. The third prediction, that urbanization is not a panacea for growth, is a result of high urban child mortality and is novel in this literature.

1.1.1 Model Description

The model economy has two sectors.[1] Human capital growth drives production to shift out of the rural sector, which has diminishing returns to scale.[2] The higher returns to scale of the urban sector increases income growth associated with any level of human capital growth.

Households choose how much time to work in the market, how much time to spend raising children, and how much time to spend investing in their children's human capital, in the spirit of Becker (1960). As the child mortality rate improves, the household can afford higher quantity and quality of children. Increasing the number of children increases the cost of investing a unit of human capital in each child (as in Becker and Lewis (1973)), so parents reduce fertility and spend more time on human capital investment. At high mortality levels, households have more net children as they become less costly. But as child mortality falls further, the income effect dominates the substitution effect, so households shift from child quantity to child quality.[3] As families choose fewer children and more investment per child, per capita human capital grows faster and faster. Per capita income growth rises from near-stagnation to modern levels

Urban households suffer higher child mortality than rural households, so the relative wage in urban areas is high, because households must be compensated for moving to the deadly city. As human capital grows, increased knowledge reduces mortality. Declines in the difference between urban and rural mortality reduces the wage premium needed to induce

1. Trade is missing from this theory, which is not a trivial omission. Stokey (1996) shows that openness to trade can speed a country's human capital accumulation with capital-skill complementarity, and Stokey (2001) shows that trade accelerated England's transition. ORourke and Williamson (2005) also shows trade's large effect on the English transition, demonstrating that increased trade openness explained a much of the increase in the ratio of wages to land rents. Galor and Mountford (2008) adds trade to a unified growth model, and shows that an early transition increases demand for the human capital-intensive sector through trade, accelerating the growth and demographic transitions.

2. This dominance is similar to the results of Ngai and Pissarides (2007) or Acemoglu and Guerrieri (2008), where the sector spending the least on a fixed factor dominates in the long run if the elasticity of substitution among sectors is greater than one.

3. This Giffen property of child quantity is not new. See for example Willis (1973), or (Becker, 1981, Chapter 5) for the effect of child mortality declines in particular. Soares (2005) features the Giffen property and shows that child mortality declines can contribute to escape from a Malthusian trap. This driver of quantity-quality substitution is distinct from the precautionary motive in Kalemli-Ozcan (2002)

households to live in an urban area, enabling further urbanization.

1.1.2 Related Research

A large branch of the unified growth literature considers the quantity-quality trade-off to be the central mechanism behind the growth transition. The motivation for this hypothesis is generally the correlation between the growth transition and the demographic transition. Becker et al. (1990) first analyze the quantity-quality trade-off in the context of an endogenous growth model; Lucas (2002) considers introducing land as a fixed factor, allowing for either a Malthusian or modern growth outcome. Galor and Weil (2000) model fertility increasing as workers escape their subsistence consumption constraint and work fewer hours, then substitute to quality as returns to education rise. Galor and Moav (2004) introduce physical capital to the framework and study inequality during the transition. Doepke (2004) consider a two sector model with a child quantity-quality decision, where education subsidies and especially child labor regulation can influence a country's transition timing. Empirical evidence supports the quantity-quality substitution during industrialization, for example in Prussia (Becker et al., 2010) and in the American South (Bleakley and Lange, 2009).

The quantity-quality decision is governed by the return to human capital, which changes over the transition period. Some authors hypothesize that this return changes due to level effects in technology or growth. For example: Galor and Moav (2002) assumes a complementarity between education and the technological growth rate, while Doepke (2004) assumes that an increase in the level of skill-intensive technology increases the return. Other hypotheses include capital-skill complementarity; Fernandez-Villaverde (2001) finds the capital-specific technological change can explain more than 50% of England's growth and demographic transitions.

I assume a different channel: declining child mortality increases the return to human capital investment, driving the quantity-quality substitution. (Galor, 2011, Chapter 4) rejects this channel on theoretical grounds.[4] Using a static model of consumption and fertility choice, he shows that declines in child mortality rates should not affect fertility and will just increase surviving children, if the household has balanced growth compatible preferences. Doepke (2005) reaches a similar conclusion. The model described in Section 1.3 rejects this conclusion when preferences are dynastic, and households invest in each child's human capital, even with balanced growth compatibility.

A second set of theories focus on structural transformation as the cause of the growth transition, rather than the quantity-quality trade-off. The motivation for this hypothesis is generally the correlation between the growth transition and urbanization. Hansen and Prescott (2002) consider an economy where only one sector uses land as an input and is perfectly substitutable with a constant returns sector. Given exogenous population and technological growth, the economy transitions from a Malthusian regime where only the land-intensive sector operates, to a modern regime where both operate. Lucas (2004) examines an endogenous growth model in which urban locations have increasing returns to scale in human capital as workers exchange ideas and learn from each other. Growth drives structural transformation out of agriculture due to the presence of a fixed factor, land.[5] Agriculture makes up the majority of employment in pre-industrial Europe (Allen, 2000) so structural transformation out of agriculture leads to urbanization if agriculture is not entirely substituted for rural non-agricultural industries. Economic growth can lead to both

4. Galor also rejects the child mortality channel on empirical grounds, given that the mortality in England declined significantly during the 18th century, over a hundred years prior to the demographic transition, without an associated decline in fertility. But this is only true of the crude death rate, when the relevant measure is the child mortality rate, which Wrigley and Schofield (1983) document as not declining significantly over the same period (Figure 1.1). A large literature suggests child mortality improvements are central to fertility declines. For example, Eckstein et al. (1999), Kalemli-Ozcan (2002), Lagerlof (2003), or Hazan and Zoabi (2006). Some theories such as Meltzer (1992) and Kalemli-Ozcan et al. (2000) suggest that the relevant mortality improvements for growth is adult mortality. This is supported in some empirical analysis (Lorentzen et al., 2008) but not others (Acemoglu and Johnson, 2007).

5. A bevy of papers follow this basic approach, for example: Gollin et al. (2007) use a two-sector model of structural transformation to consider the impact of different agricultural productivity processes on countries' growth transitions. And, Michaels et al. (2012) directly relate technology-driven structural transformation to urbanization during the American transition. Strulik and Weisdorf (2008) build a two-sector model of the industrial population boom, where population growth drives productivity growth, creating an simultaneous income boom.

technological or preference-driven structural transformation, but the formal model in this paper considers technological structural transformation, motivated by evidence from Kuznets (1966), Maddison (1980), and Baumol et al. (1985), among many others.[6]

The remainder of this paper is organized as follows: Section 1.2 describes the empirical transitions, Section 1.3 describes the model environment, Section 1.4 defines equilibrium and characterizes several properties, Section 1.5 outlines the calibration procedure and simulation results, Section 1.6 considers the model under alternative calibrations and examines the empirical implications, and Section 1.7 concludes.

1.2 Empirical Patterns

Figure 1.1 plots the three transitions in England from 1295 CE. Before the industrial revolution, real income growth is consistently less than 1%. The urban share of people is less than 10%. Fertility and mortality rates are high. Then, since 1800, all of these series transition to modern values. This joint transition is an empirical regularity: among large countries with a thousand years of urbanization and income estimates, there is no evidence of a sustained transition for income growth, urbanization, fertility, or mortality before 1800.[7]

Moreover, these transitions occur around the same time within a country. To illustrate, I calculate the first year that each country surpasses a benchmark level for each series: (a) twenty-five years of 1% annual income growth, (b) 50% urban, (c) total fertility rate below 3, and (d) child mortality below 5%. Table 1.1 reports the correlation table for these transition years.[8] Countries that experience an early growth transition also tend to urbanize early, and have fertility and mortality fall early. This correlation is also observable in the current

6. Recent research from Herrendorf et al. (2013) and Comin et al. (2015) find that when considered together, both technology and preferences have driven structural transformation, so a more complete model of structural transformation would incorporate income effects as well.

7. Except for Belgium and the Netherlands, which had urban shares near 30% in 1500 CE.

8. The set of countries with one thousand years of data for income and urban population share, defined as having 50% of the population living in cities of at least 5,000 people. 18 countries are in this dataset: China, India, and 16 European countries. Historical estimates for these datasets corresponds to the modern states' current geographic area whenever possible.

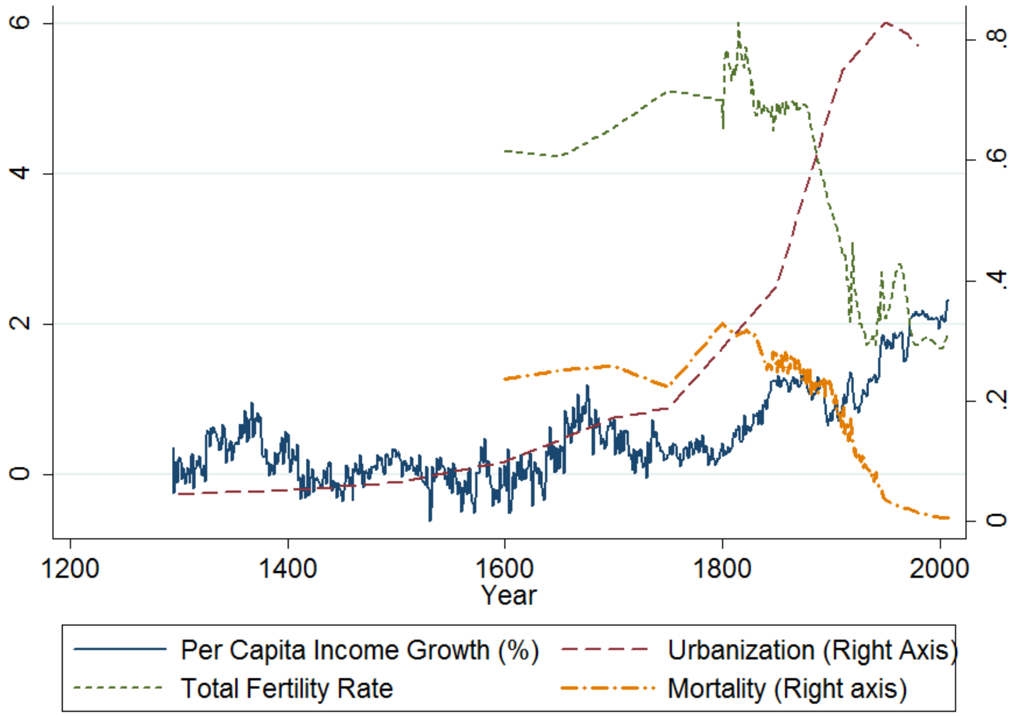

Figure 1.1: Transitions in England

Notes: GDP per capita is from Bolt and van Zanden (2013) and Broadberry et al. (2010). Urbanization data are from Bairoch (1991). TFR and Mortality are from Ajus (2015) and Johansson et al. (2015) after 1800. Before 1800, they are from Wrigley and Schofield (1983).

cross-section. Table 1.2 reports the percentage of countries surpassing the urbanization and demographic benchmarks for two income groups. Countries with 2012 GDP per capita of at least $10,000 are broadly urban with low fertility and low mortality. Countries with GDP per capita less than $1,000 tend to be rural with high fertility and high mortality.

	Income Growth	Urbanization	Fertility	Mortality
Income Growth	1			
Urbanization	0.518	1		
Fertility	0.542	0.393	1	
Mortality	0.608	0.467	0.881	1

Table 1.1: Correlation of Transition Years

Only by studying growth, urbanization, and demographics together is the theory able to

	Urban > 50%	TFR < 3	Child Mort. < 5%
Income > \$10K	93%	96%	97%
Income < \$1K	7%	10%	13%

Table 1.2: Transitioned Percentage of Countries by Income in 2012

speak to the following patterns.

1.2.1 Early Urbanization Predicts Later Transition

The prediction is unique in distinguishing this theory from other models of urbanization and long-run growth. Theories such as Hansen and Prescott (2002) or Lucas (2004) feature an urban sector with strictly greater returns than the rural sector. In such a model, an economy that is parameterized to choose a higher urbanization level for a given income level will grow faster.

The model presented in section 1.3 also has higher urban returns, but features a trade-off: high child mortality. This reduces the household budget set, decreasing the return to human capital investment, which delays the income growth transition. Then, over the following transition, growth and urbanization are highly correlated o accommodate this trade-off and explain the empirical pattern, theories of long-run growth and urbanization must also account for the incentives that govern the demographic transition.

Section 1.6 describes this pattern in the data. Early urbanization only predicts a later transition when controlling for income. Across time, space, and theory, urbanization is generally associated with higher income. Indeed, that's why the prediction is surprising.

1.2.2 Urban-Rural Differences

The model also produces two other facts observed in the English transition: a declining urban-rural wage premium, and a declining rural-urban family size ratio. We focus on

England, because of the quality of its long-run macroeconomic time series, and availability of historical urban and rural data on fertility, mortality, and wages. The model is calibrated to English data in Section 1.5.1.

The urban-rural wage gap declines over time.[9] In the 1830's, Williamson (1987) calculates a nominal wage gap for unskilled workers of 73 %, and a real wage gap of 46%; he estimates that the majority of the gap was compensating for high urban mortality. In contrast, DCosta and Overman (2013) estimates an unconditional wage gap of 14 % in Britain from 1998-2008. Conditioning on observables such as occupation and skill further reduces the gap to 2 %, in line with estimates for other countries.[10]

The rural-urban family size ratio declines over time. Clark (2009) estimates gross fertilities for the 15th-18th century that are 27% higher on farms than in London, and 12% higher in other non-farm households than in London. Mortality differences led farm-dwelling fathers to have over twice as many surviving children than a Londoner. And other non-farm fathers had 70% more surviving children than a Londoner. By the turn of the 20th Century, (Szreter and Hardy, 2001, Table 20.6) estimates that rural fertilities were only 3-5% larger than in urban areas. In modern European countries with available data, rural crude birth rates average 98% of urban rates (United Nations Statistics Divison, 2012, Table 9). And in 2007 England, London's crude birth rate is now 30% higher than the country as a whole (Office for National Statistics, 2008, Table 6.2), although due to demographic differences its total fertility rate is lower Kulu and Washbrook (2014). This pattern is documented in many countries.[11]

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9. Specifically, the wage gap controlling for worker skill. Income differences between urban and rural workers may be very large if urban workers accumulate much more human capital, as in Lucas (2004). In the cross-section, Lagakos and Waugh (2013) and Young (2013) use a worker-selection model to estimate that most of the productivity gap in poor countries is due to sorting on skill.

10. Additionally, there is cross-sectional evidence that the urban-rural productivity gap is declining in income, and nearly disappears in rich countries (Gollin et al., 2013).

11. For example: Germany (Knodel et al., 1974, Chapter 3), Italy (Bacci, 1977), or the United States (Kiser, 1960)

1.3 Model

The model economy contains two production sectors: an urban sector where the only input is human capital, and a rural sector with human capital and land inputs. Land is in fixed supply, but human capital grows endogenously, and is the only source of growth in the model. Households have overlapping generations, and parents decide the quantity and quality of their children.

1.3.1 Production

The rural production sector, denoted with the subscript R, combines human capital and land to produce output. Its production function is:

$$F_R(\tilde{h}, \tilde{l}) = \tilde{h}^{\theta} \tilde{l}^{1-\theta} \tag{1.1}$$

The rural firms are land intensive, such as a farm, a mine, or a logger. An individual rural firm chooses human capital $\tilde{h}$ and land $\tilde{l}$.

The urban production sector, denoted with the subscript U. It uses only human capital to linearly produce output. Its production function is

$$F_U(\tilde{h}) = \tilde{h} \tag{1.2}$$

Urban firms are relatively less land intensive than farms, which characterizes most of the nonagricultural sector of the economy. An urban firm might be a factory, a craftsman, or a service firm. An urban firm chooses only human capital $\tilde{h}$.

The unique final good is produced competitively by combining the output of the urban and rural sectors, with elasticity of substitution ϵ

$$F(\tilde{x_R}, \tilde{x_U}) = A(\zeta \tilde{x_U}^{\frac{\epsilon-1}{\epsilon}} + (1-\zeta)\tilde{x_R}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}} \tag{1.3}$$

Final goods firms choose rural goods $\tilde{x_R}$ and urban goods $\tilde{x_U}$ as inputs.

Firms in all sectors are small and competitive, so they take prices as given. Let p_R denote the intermediate rural good's price, and p_U the intermediate urban good's price. Normalize the price of the final output good to one. Also, let r denote the rental rate of land, w_R the rural wage rate per unit of human capital, and w_U the urban wage rate per unit of human capital. Then, a rural firm solves:

$$\max_{\tilde{h},\tilde{l}} p_R \tilde{h}^{\theta} \tilde{l}^{1-\theta} - w_R \tilde{h} - r\tilde{l} \tag{1.4}$$

An urban firm solves:

$$\max_{\tilde{h}} p_U \tilde{h} - w_U \tilde{h} \tag{1.5}$$

A final goods firm solves:

$$\max_{\tilde{x_R},\tilde{x_U}} A(\zeta \tilde{x_U}^{\frac{\epsilon-1}{\epsilon}} + (1-\zeta)\tilde{x_R}^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}} - p_R \tilde{x_R} - p_U \tilde{x_u} \tag{1.6}$$

1.3.2 Households

Agents live for two periods: in their first period of life they are children, and in the second period they are parents.[12] Generations overlap within a household: each household consists of one parent and a number of children. The parent makes all of the household's choices, choosing consumption, the number of children, and education spending. The parent must also choose whether to live in an urban or rural area, and how much time to dedicate to market work. Households do not own land; similar to Galor and Weil (2000), we suppose that an infinitesimally small fraction of the population holds all the land, and has a negligible impact on aggregate human capital or demographics.

Utility is dynastic. Parents enjoy present consumption c, their number of surviving

12. Because adults all live to the same age, all mortality improvements are to child mortality. By construction this ignores any impact on transition dynamics from changes to adult mortality, which Lorentzen et al. (2008) suggest affects the quantity-quality trade-off, even when controlling for child mortality.

children n, and their dynasty's discounted future utility. A parent discounting by β has utility:

$$V_t = u(c_t, n_t) + \beta V_{t+1} \tag{1.7}$$

where $u(c_t, n_t)$ is the period utility function, V_t is the parent's dynastic utility, and V_{t+1} is the dynastic utility of the next generation. Parents' preference for quantity of children is driven by their period utility, $u(c_t, n_t)$, because V_{t+1} is each child's future utility, not the total utility of the next generation.[13]

The period utility function $u(c, n)$ is increasing in both arguments and must be balanced growth compatible, so that as the time cost of raising children rises, it is offset by an income effect. When necessary, I assume the functional form from Barro and Sala-i Martin (2004):

$$u(c, n) \equiv \frac{(cn^{\phi})^{\sigma}}{\sigma} \tag{1.8}$$

where $\phi > 0$, $\sigma < 1$ and $\phi\sigma < 1$. ϕ controls the preference for consumption relative to children, while σ controls substitutability across generations: $\frac{1}{1-\sigma}$ is the elasticity of intergenerational substitution.

Parents choose how to allocate their time to three activities: market work (τ_c), producing children (τ_n), and educating children (τ_h). They have one unit of time to allocate to these activities:

$$\tau_c + \tau_n + \tau_h = 1 \tag{1.9}$$

Households in sector $j \in U, R$ earn wage w_j per unit of human capital, per unit of time worked. Income is spent on consumption, so a parent with human capital h working time τ_c consumes:

$$c = w_j h \tau_c \tag{1.10}$$

13. This formulation is a simplification of Becker and Barro (1988), in which the discount factor is a concave function of n_t. I eliminate the dependence on n_t for tractability and parsimony. The first order condition for children is simpler and will yield a constant share of time spent working in the market with Cobb-Douglas utility. Eliminating the dependence on n_t also reduces the number of parameters to be calibrated.

A household choosing time τ_n produces n surviving children by:

$$n = S_j \alpha \tau_n \tag{1.11}$$

where parameter α is the productivity for producing children. S_j is the fraction of newborns that survive to adulthood in sector j. S_j is exogenous from the perspective of the household, but will depend on aggregate human capital, so it may vary over time. Child production is time intensive, so productivity is not improved by parental human capital.

All children are endowed with their parents' human capital, but parents can spend time educating their children to increase their human capital further. A household with human capital h choosing education time τ_h produces human capital h' for their n children linearly:

$$(h' - h)n = \xi \tau_h h \tag{1.12}$$

The number of surviving children n enters this equation because parents must spend time educating each of their children. All child mortality resolves before parents start to invest in their human capital.[14] The parent's ability to impart human capital is increasing in their own human capital, h, and proportional to the parameter ξ.

Combining equations (1.9), (1.10), (1.11) and (1.12) yield the combined budget constraint:

$$c + \frac{w_j(h' - h)n}{\xi} + \frac{w_j hn}{\alpha S_j} = w_j h \tag{1.13}$$

The household's time is used for consumption, human capital investment, or producing children. The total value in numeraire of the household's time is $w_j h$. The value of time spent in the market is what they earn and spend on consumption c. The value of time spent investing $(h' - h)n$ units of human capital is $\frac{w_j(h'-h)n}{\xi}$, and the value of time spent

14. Tamura (2006) considers an alternative framework where some human capital investment may be lost due to child mortality risk. Reductions in child mortality increase the return to human capital even more strongly in such an environment.

producing n children is $\frac{w_j hn}{\alpha S_j}$.

The Household's Problem

The household's problem is to choose consumption c, children n, their children's future human capital h', and location j to maximize dynastic utility. Let Λ denote the aggregate state of the economy; then the household's Bellman equation is

$$V(h;\Lambda) = \max_{c,n,h',j\in J} u(c,n) + \beta V(h';\Lambda') \tag{1.14}$$

subject to the budget constraint (1.13), location choice set $j \in U, R$, and non-negativity constraints:

$$c \geq 0 \;\; n \geq 0 \;\; h' \geq h \tag{1.15}$$

Solving the household's problem yields the first order conditions:

$$u_n(c,n) = u_c(c,n)(\frac{w_j h'}{\xi} + \frac{w_j h}{\alpha S_j}) \tag{1.16}$$

$$u_c(c,n) w_j n = \xi \beta V'(h';\Lambda') \tag{1.17}$$

and envelope condition:

$$V'(h;\Lambda) = u_c(c,n) w_j (1 + \frac{n}{\xi} - \frac{n}{\alpha S_j}) \tag{1.18}$$

When the preferences in (1.8) are applied to first order condition (1.16), consumption is a constant share of income:

$$\frac{c}{w_j h} = \frac{1}{1+\phi} \tag{1.19}$$

This also implies that $\tau_c = \frac{1}{1+\phi}$ is constant for all households. This result is due to the marginal cost of children being proportional to total income, and the balanced growth compatibility of preferences. As total income $w_j h$ rises, the income effect exactly offsets the substitution effect, and households spend the same amount of time $\tau_n + \tau_h$ on children, although they may reallocate their time between child quantity and human capital investment.

Different children of the same parent might choose different locations, so a household does not have a single Euler equation. Rather, the Euler equation for child k with the balanced growth preferences is:

$$(\frac{c_k'}{c})^{1-\sigma} = (\frac{n_k'}{n})^{\phi\sigma+1}\frac{w_k'}{w_j}\xi\beta(\frac{1}{n_k'} + \frac{1}{\xi} - \frac{1}{\alpha S_k'}) \tag{1.20}$$

Denote human capital growth by $1+g \equiv \frac{h'}{h}$. Then the Euler equation can be rewritten using the budget constraint and consumption share in terms of fertilities, human capital growth, and wages:

$$(1+g)^{1-\sigma}(\frac{n}{n_k'})^{\phi\sigma}(\frac{w_j}{w_k'})^{\sigma} = \beta\frac{\xi}{n}(\tau_c + n_k'\frac{1+g_k'}{\xi}) \tag{1.21}$$

The left hand side of equation (1.21) is marginal utility growth across generations. On the right hand side, $\tau_c + n_k'\frac{1+g_k'}{\xi}$ is the return to human capital investment, and $\frac{\xi}{n}$ is the productivity of parental time at producing human capital for each child. As n rises, it costs more parental time to give each child a unit of human capital, so the return on parental time falls. Depending on marginal utility growth, this force creates the potential for child mortality declines to induce substitution from child quantity to child quality.

1.3.3 Aggregates and Laws of Motion

The state of the economy is determined by the function $\lambda(h)$, which denotes the number households with human capital h.

The total population in the economy N is:

$$N = \sum_h \lambda(h) \tag{1.22}$$

The number of households with h in sector j is denoted by $\lambda(h, j)$; note that this is an equilibrium object because sector j is a choice. All households work τ_c units of time, so aggregate human capital inputs in the economy are:

$$H_j = \sum_h \tau_c h \lambda(h, j) \tag{1.23}$$

and aggregate land is L, a fixed value. Given factor prices w_U, w_R, r, total income in the economy is:

$$Y = w_U H_U + w_R H_R + rL \tag{1.24}$$

Let n_j denote the fertility choice of a household in sector j. Let $h(h', j)$ denote the human capital of a household in sector j that would choose h' for their children. The distribution of households evolves by:

$$\lambda(h') = \sum_j n_j \lambda(h(h', j), j) \tag{1.25}$$

which simply says that the number of households with h' equals the number of households that chose h' for their children, times the number of surviving children per household n_j.

Child survival $S_j(\bar{h})$ is a function of location j and average human capital, $\bar{h}$:

$$\bar{h} = \sum_h \frac{h\lambda(h)}{N} \tag{1.26}$$

The dependence on location captures differences in child mortality across urban and rural areas. The dependence on average human capital captures the impact of the technology level on child mortality. This may come in the form of beneficial technological improvements such as clean water, food safety, and medicine.[15]

15. See for example Preston (1996)'s overview, Szreter (1988)'s examination of the U.K.'s decline in particular, or Deaton (2006)'s review of Fogel (2004)'s conflicting findings. Empirically, income growth also allows for household investments in child survival, such as improved nutrition, which research such as McKeown (1976) and Fogel (2004) emphasize. But $S_j(\bar{h})$ only captures the impact of the technology level.

Assume the function $S_j(\bar{h})$ is increasing in $\bar{h}$ and has common limit for all j:

$$\lim_{\bar{h}\to\infty} S_j(\bar{h}) = \bar{S} \tag{1.27}$$

It must also be that $S_j(\bar{h}) \in [0,1]$ for all $\bar{h} > 0$. A particular form will be estimated in Section 1.5.

Finally, to determine the population distribution, an assumption must be made about how households allocate themselves. *Assume there is no reverse migration*: children stay where they are born unless some migration is needed from their birth location to the other location; no one leaves their birth location if net migration is flowing into it. Without this assumption, optimality conditions and constraints will only determine the allocation of aggregate human capital, but not of people, who might have differing human capital levels. In equilibrium, this assumption implies that dynasties move from rural to urban areas, and never return.[16]

1.4 Equilibrium

1.4.1 Definition

A competitive equilibrium in this economy consists of sequences for $t \geq 0$ of prices, p_R, p_U, w_R, w_U, r; aggregate allocations, Y, x_U, x_R, H_U, H_R, Z; distribution of household human capital $\lambda(h,j)$; and household allocations, $c(h,j), d(h,j), n(h,j), \tau(h,j)$; given initial distribution of human capital $\lambda(h)_0$ and the aggregate quantity of land L, such that:

1. The firm allocations solve (1.4), (1.5), and (1.6).

2. The household allocations maximize (1.14) subject to (1.13) and (1.15).

3. All households choose the location that maximizes their utility.

16. This is not a perfect assumption. Young (2013) shows that most urban-rural migration is from the countryside to the city, but there is still a reverse flow of workers returning to rural areas.

4. Markets clear: $Y = F(x_U, x_R)$, $X_U = F_U(H_U)$, $X_R = F_R(H_R, L)$

5. The law of motion (1.25) holds for all human capital levels.

6. Household aggregates add up, satisfying equations (1.22), (1.23), (1.24), and (1.26), and there is no reverse migration.

1.4.2 Equilibrium Prices

The firms' profit maximization (equations (1.4), (1.5), and (1.6)) implies that equilibrium prices must relate to equilibrium factors by:

$$w_U = p_U \qquad w_R = p_R\theta(H_R)^{\theta-1}L^{1-\theta} \qquad r = p_R(1-\theta)(H_R)^{\theta}L^{-\theta} \tag{1.28}$$

$$p_U = A^{\frac{\epsilon-1}{\epsilon}}\zeta\left(\frac{Y}{x_U}\right)^{\frac{1}{\epsilon}} \qquad p_R = A^{\frac{\epsilon-1}{\epsilon}}(1-\zeta)\left(\frac{Y}{x_R}\right)^{\frac{1}{\epsilon}} \tag{1.29}$$

1.4.3 Equilibrium Location Choice

Households choose the location that gives them the highest utility. As usual, the household's value function is the maximum of the value of choosing each location. In most models this upper envelope is not differentiable at the point of indifference. But in this model, the value function is differentiable for indifferent households.

Proposition 1 *If households are indifferent between urban and rural locations in equilibrium, then their marginal value of human capital is equal in both locations.*

Proposition 1 is proved in Appendix A.1.1. Marginal value equalization implies a convenient equilibrium condition for the wage premium. Setting the envelope condition (1.18) equal in both locations, and substituting for consumption by equation (1.19) yields:

$$w_R^{\sigma}n_R^{\sigma\phi+1}\left(\frac{1}{n_R}+\frac{1}{\xi}-\frac{1}{\alpha S_R}\right) = w_U^{\sigma}n_U^{\sigma\phi+1}\left(\frac{1}{n_U}+\frac{1}{\xi}-\frac{1}{\alpha S_U}\right) \tag{1.30}$$

The wage premium is a compensating differential for mortality differences. If urban child survival S_U is lower than rural survival, then all else equal equation (1.30) will imply $w_U > w_R$. But in equilibrium all else is not equal, and urban households will change their child rearing decision n_U to partially compensate for a lower survival rate.

1.4.4 Equilibrium in the Limit

In this section I derive the asymptotic behavior of the economy. I show that the urban share approaches one, and the urban-rural wage, growth, and fertility gaps disappear.

The following propositions are proved in Appendix A.1.

Proposition 2 *If* $\lim_{t\to\infty} \bar{h} = \infty$, *then the limiting urban-rural wage premium is* $\frac{w_U}{w_R} \to 1$.

Proposition 3 *If* $\lim_{t\to\infty} \bar{h} = \infty$, $\lim_{t\to\infty} n \geq 1$ *and* $\epsilon > 1$, *then the long-run urban share converges to* 1.

Proposition 4 *If* $\lim_{t\to\infty} \bar{h} = \infty$, $\lim_{t\to\infty} n \geq 1$ *and* $\epsilon > 1$, *then the the limit of both urban and rural wages is* $\bar{w} \equiv A\zeta^{\frac{\epsilon}{\epsilon-1}}$.

Proposition 4 implies that wages are not growing or falling in the limit, so long run human capital growth $\bar{g}$ and children $\bar{n}$ are determined in the limit by the long run budget constraint and the long run steady state Euler equation:

$$\tau_c + \frac{\bar{g}\bar{n}}{\xi} + \frac{\bar{n}}{\alpha \bar{S}} = 1 \tag{1.31}$$

$$(1+\bar{g})^{1-\sigma} = \beta(\frac{\xi \tau_c}{\bar{n}} + 1 + \bar{g}) \tag{1.32}$$

1.5 Quantitative Analysis

Parameter values are chosen to match key features of the data, an initial condition is chosen to look like England in year 1500 C.E., and the economy is simulated in transition to modern growth.

1.5.1 Calibration

Ten parameters must be calibrated: production parameters A, θ, ζ, and ϵ; preference parameters ϕ, σ, and β; and household parameters α and ξ. Initial conditions must be chosen: land L and population N_0 are normalized to one. All households are initialized with $h = 1$. The two technology functions $S_U(Z)$ and $S_R(Z)$ must also be characterized. Finally, assume one model period is 25 years. Calibrated values appear in Table 1.3.

	Parameter	**Value**	**Interpretation**
(i)	θ	0.74	Labor Share in Rural Sector
(ii)	β	0.36	Discount Factor
(iii)	σ	0.49	Utility Curvature
(iv)	ϕ	0.74	Child Preference
(v)	α	3.74	Childrearing Productivity
(vi)	ξ	3.29	Education Productivity
(vii)	A	3.68	Total Factor Productivity
(viii)	ζ	0.36	Urban Goods Weight
(ix)	ϵ	4.50	Urban-Rural Substitution Elasticity
(x)	υ	0.35	Technology Effect on Survival

Table 1.3: Calibrated Parameters

The rural production parameter θ is set to 0.74 so that the land share of farm income is 26%, the value for England in 1500 C.E. estimated by Clark (2010).

To calibrate the parameters $(A, \zeta, \epsilon, \alpha, \xi, \phi, \sigma, \beta)$, we target several empirical moments.

First, the initial urban share is targeted to 0.064, estimated by Bairoch et al. (1988) for England in 1500. Initial human capital growth is targeted to 1.3%, the smoothed 25-year income growth at 1500 CE, in the Broadberry et al. (2010) data. Long run human capital growth $\bar{g}$ is targeted to 52%, England's 25-year real income growth rate since 1950.

Initial fertility and mortality rates are targeted to estimates from Clark (2009) for England in 1500-1800. Initial urban and rural probabilities of surviving to age 25 are $S_{U,0} = 0.59$ and $S_{R,0} = 0.68$. The ratio of urban to rural surviving children per adult is targeted to 0.77, the ratio estimated by Clark (2009). However, the estimates in levels are too high to map directly to the model, because the data is from wills and does not account for people who choose not to have children. So $n_{R,0}$ and $n_{U,0}$ are chosen to target an initial population growth rate of 8.5% per 25 years, which matches the growth rate for England from 1400-1600 estimated by Broadberry et al. (2010). The long run population growth is targeted to 0%, implying $\bar{n} = 1$.

Five preference and household parameters $(\phi, \sigma, \beta, \xi, \alpha)$ can be solved for jointly given targets for human capital growth, fertility and mortality, and a target long run 5% annual rate of return on human capital investment. The five parameters are identified by five equations: the long run and initial rural budget constraints, long run and initial steady state Euler equations, and the return to human capital investment. The initial rural Euler equation is not identical to the steady steady Euler equation because there are small movements in wages and net fertilities initially, so the equilibrium value of $n_{R,0}$ and $g_{R,0}$ will not exactly match the targets.

The initial urban-rural wage premium is implied by the indifference equation (1.30). Chosen empirical targets imply an initial premium of $\frac{w_{U,0}}{w_{R,0}} = 1.23$. The initial urban share, normalization of $h = 1$, and market time of $\tau_c = \frac{1}{1+\phi}$ imply initial supplies of human capital $H_{R,0}$ and $H_{U,0}$. Setting the ratio of marginal products equal to the initial wage premium identifies the weighting parameter ζ in the production function, conditional on a choice of the elasticity of substitution ϵ. Targeting long run wage $\bar{w} = 1$ then implies a value for TFP

A.

The child survival function $S_j(\bar{h})$ requires a functional form. This function should have four properties: $S(\bar{h}) \in (0,1)$ for all $\bar{h} \geq 0$, $S_j(\bar{h}_0)$ matches the target for $S_{j,0}$, $S'(\bar{h}) > 0$ for all $\bar{h} \geq 0$, and $S_j(\infty) = \bar{S}$ so that in the long run, survival approaches a chosen limit. A form satisfying these properties is:

$$S_j(\bar{h}) = \bar{S} - (\bar{S} - S_{j,0}) \frac{1 + \upsilon \bar{h}_0}{1 + \upsilon \bar{h}} \tag{1.33}$$

This is a transformed logistic CDF, which is chosen for parsimony as it is governed by only one free parameter υ, and also for having a positive limit as $\bar{h} \to 0$. It satisfies the other desired conditions: when $\bar{h} = \bar{h}_0$, then $S_j(\bar{h}) = S_{j,0}$; $S_j'(\bar{h}) > 0$; and in the limit as $\bar{h} \to \infty$, then $S_j(\bar{h}) \to \bar{S}$.

The function is estimated on England's child mortality time series, given the targets for $S_{j,0}$ and $\bar{S}$. Appendix A.2 describes this estimation.

The final parameter to calibrate is the elasticity of substitution ϵ. The elasticity of substitution controls the speed of urbanization as aggregate human capital grows. Figure 1.2 plots the transition year for urbanization and for income growth. A higher value of ϵ speeds the urbanization transition by making urban and rural sectors more substitutable: given a decline in the wage premium, more human capital will shift into the urban sector. But a higher value of ϵ also decreases growth: there are more urban households, which face lower child survival rates and spend less time investing in human capital for their children (see Section 1.5.2). The dashed lines are the empirical transition years. The elasticity of substitution is selected to minimize the mean squared error between the model and empirical transition years.

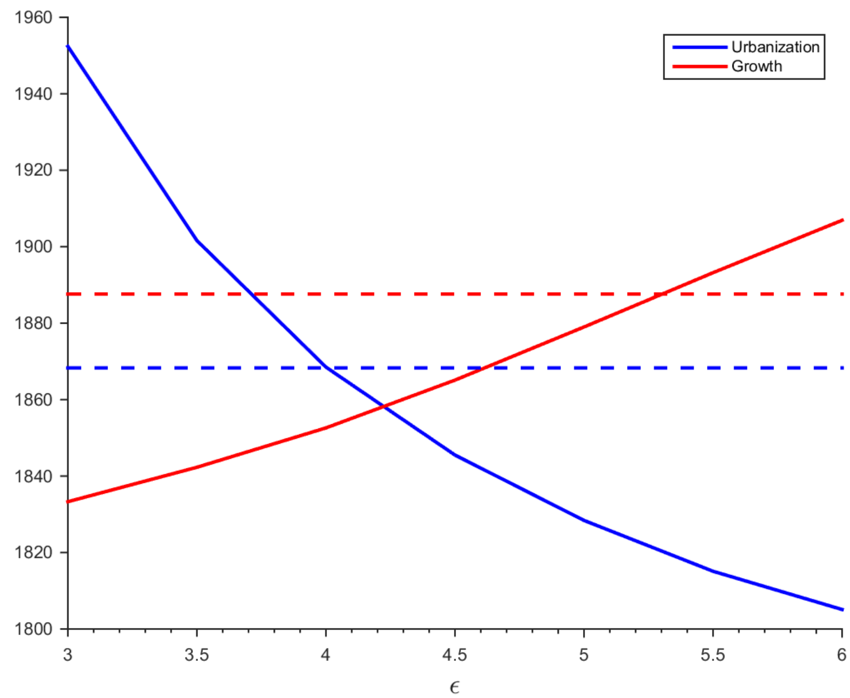

Figure 1.2: Elasticities of Substitution and Transition Years

Notes: Urbanization transition is urban share > 50%. Growth transition is annual income growth > 1%.

1.5.2 Results

The economy is initialized in 1500 and is run 21 periods to 2000. The economy begins with most of the population in the rural sector. As the population grows and human capital accumulates, households move to the urban sector (Figure 1.3). The simulated urban share surpasses 50% in year 1846, versus the empirical urban share which reached 50% around 1863. In the long run, the population fully urbanizes.

As mortality falls, surviving children become cheaper. But increasing the number of children increases the cost of investing a unit of human capital in each child. So parents reduce fertility and spend more time on human capital investment. Quantitatively, fertility falls more than one for one with the decrease in cost for unconstrained households, so surviving children fall and households substitute from quantity to quality. Income per household grows

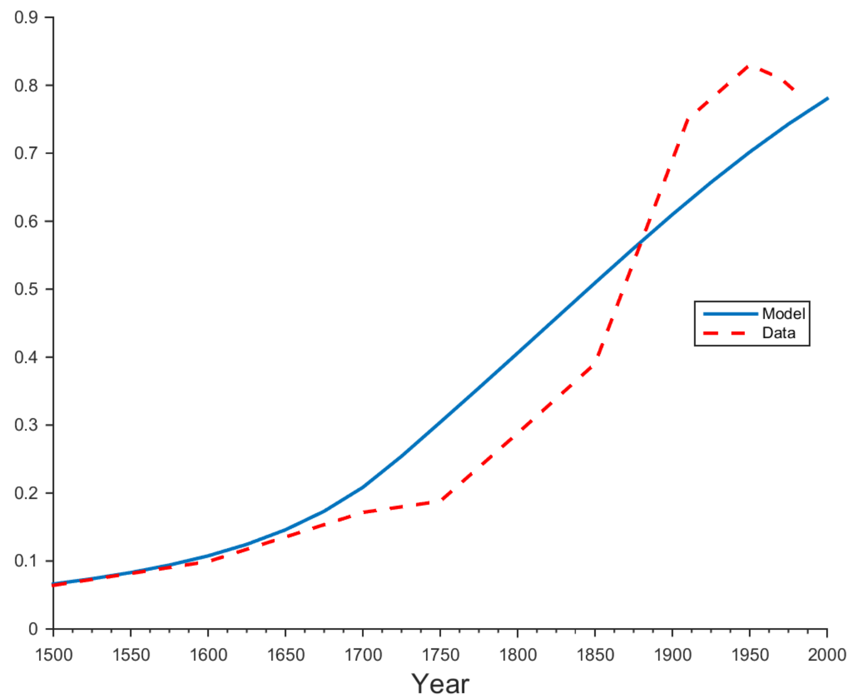

Figure 1.3: Simulation: Urban Share

Notes: Data from Bairoch (1991) and Bairoch et al. (1988)

slowly at first, but eventually rises, asymptoting to the long run value (Figure 1.4).

To understand the dynamics of the two sectors, Figure 1.5 plots the Euler equation in (1.21) in a steady state:

$$(1+g_{ss})^{1-\sigma} = \beta \frac{\xi}{n_{ss}}(\tau_c + n_{ss}\frac{1+g_{ss}}{\xi}) \tag{1.34}$$

For the steady state Euler equation, children choose the same location as their parent. $\tau_c + n_{ss}\frac{1+g_{ss}}{\xi}$ is the return on human capital, and $\frac{\xi}{n_{ss}}$ is the productivity of parental time in producing a unit of human capital for each child. With calibrated paramter values, the steady state Euler equation implies that g_{ss} is decreasing in n_{ss} for $g \in (0, \bar{g}]$.[17] In this region,

17. The steady Euler Equation gives n as a decreasing function of g for $(1-\sigma)(1+g)^{-\sigma} > \beta$ which always holds when $\sigma \leq 0$, i.e. when the intergenerational elasticity of substitution is less than 1. However, the calibration gives $\sigma > 0$, so n is decreasing in g for all $g < 14.5$, which is well above the long run steady state.

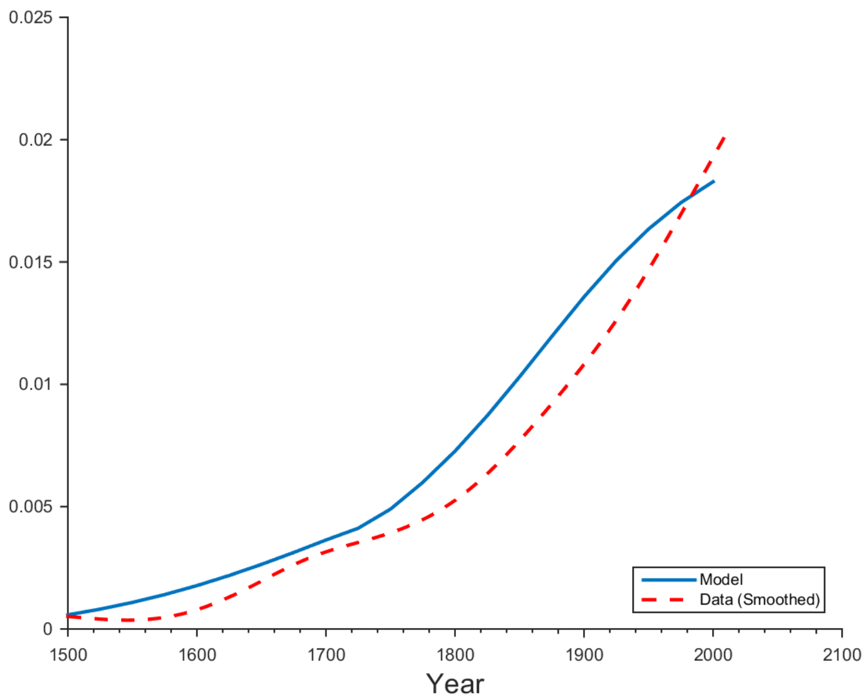

Figure 1.4: Simulation: Income Growth

Notes: Data from Broadberry et al. (2010) and Bolt and van Zanden (2013), smoothed with an HP filter.

households will always trade-off child quantity for quality, and never increase both. Thus an expansion in the household's budget set caused by declines in child mortality will induce substitution form quantity to quality even though quantity has become cheaper cheaper.

To understand this effect, Figure 1.5 also plots the normalized budget constraint, which divides the budget constraint (1.13) by total income:

$$\tau_c + \frac{gn}{\xi} + \frac{n}{\alpha S_j} = 1 \tag{1.35}$$

This budget constraint is plotted for three different survival levels: $\bar{S}$, $S_{R,0}$, and $S_{U,0}$. The steady state Euler equation differs slightly from the equilibrium Euler equation for initial urban or rural households, but this figure is a useful approximation for understanding the dynamics. As the rural survival rate improves, the rural budget constraint shifts towards

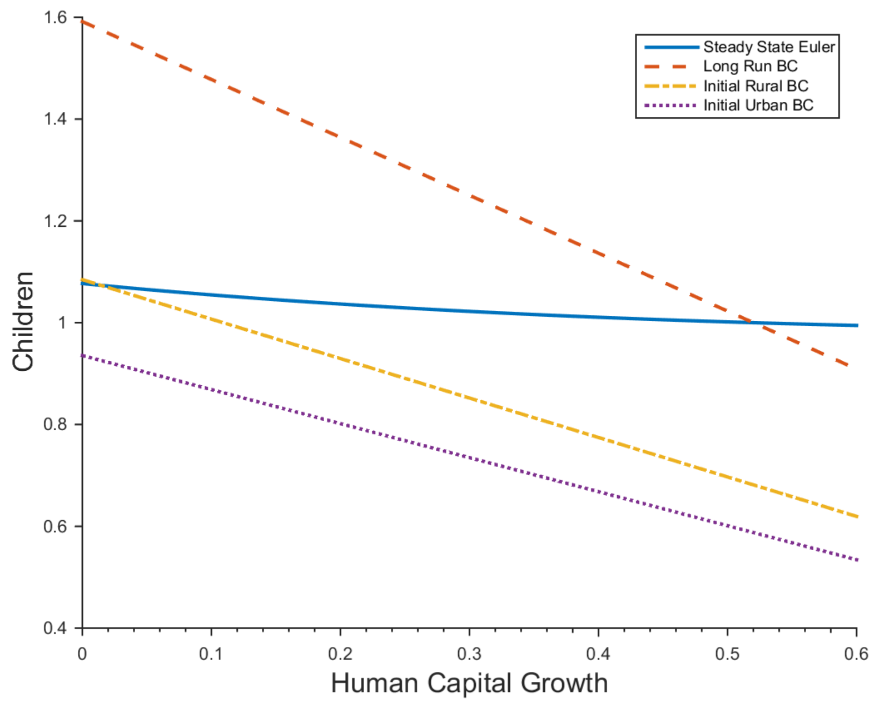

Figure 1.5: Quantity-Quality Substitution

the long run budget constraint, and the rural allocation moves down the Euler equation, shifting from quantity towards quality. The initial urban budget constraint does not intersect the Euler equation: urban households are constrained at $g = 0$ so that the non-negativity constraint 1.15 is satisfied. As the urban survival rate improves, the urban budget constraint shifts towards the rural budget constraint, and children increase. When the survival rate has improved sufficiently to unconstrain urban households, they follow the rural households and substitute from quantity to quality.

Figure 1.6 plots the ratio of urban to rural values for three quantities: wages, children and survival, exhibiting the predictions from Section 1.2.2. As human capital grows, urban and rural survival rates both grow towards the same limit, so the ratio rises to one. The urban-rural wage ratio is the compensating differential for mortality differences. Williamson (1987) estimates this ratio is 1.46 in the early 1800s, versus 1.05 in the model in 1800 and 1.22 in 1500. Survival is initially lower in urban areas, so a high wage premium is necessary to make households indifferent between locations. As the survival ratio rises to one, wage ratio falls to one, and the compensating differential disappears in the limit. Urban households initially choose fewer children than rural households because they are constrained at $g = 0$ and urban children are very expensive due to their low survival rate. As the survival rate improves, the urban-rural child ratio rises as urban households have more children and rural households substitute from quantity to quality. Eventually the urban households become unconstrained and also substitute towards quality. The ratio approaches one in the long run, as the survival differential disappears.

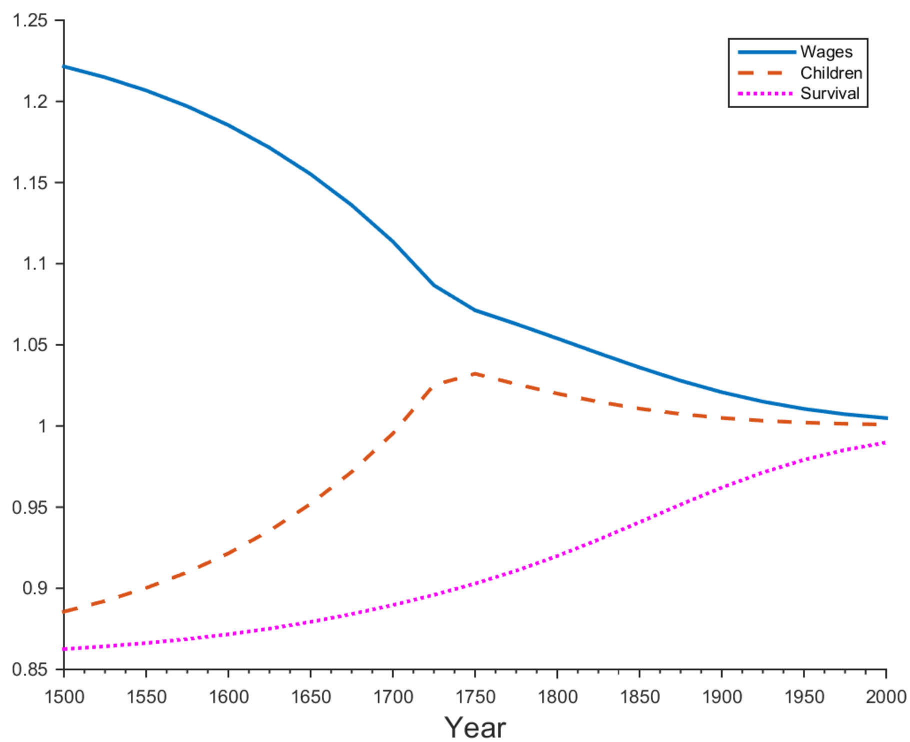

Figure 1.6: Simulation: Urban/Rural Ratios

While the urban-rural family size ratio increases from the initial period to the long run, fitting the empirical pattern in Section 1.2.2, it is not monotonic over the whole sample, which isn't true in the data. This is because urban households choose higher fertilities than rural households, to compensate for high child mortalities. This is true empirically in the modern day, but not during the 19th or early 20th centuries. To explain the fertility ratio over this period, the theory needs other urban-rural differences, such as the cost of raising children in the city, or higher urban returns to human capital (Becker, 1981, Chapter 5).

In the aggregate, fertility and mortality fall as the economy urbanizes and transitions to modern growth. Figure 1.7 plots births, deaths, and the difference: net population growth. Births are calculated before accounting for the fraction S_j that do not survive to adulthood. Total births start rising briefly because declining mortality makes production of surviving children cheaper, but soon start to decline as fewer newborns are necessary to produce a given surviving child. In the long run, the birth rate falls to the limiting population growth rate, because child mortality disappears. Similarly, the death rate falls to one in the long run - all adults die every period, and all children live. The difference is the population growth rate which falls to zero in the long run.

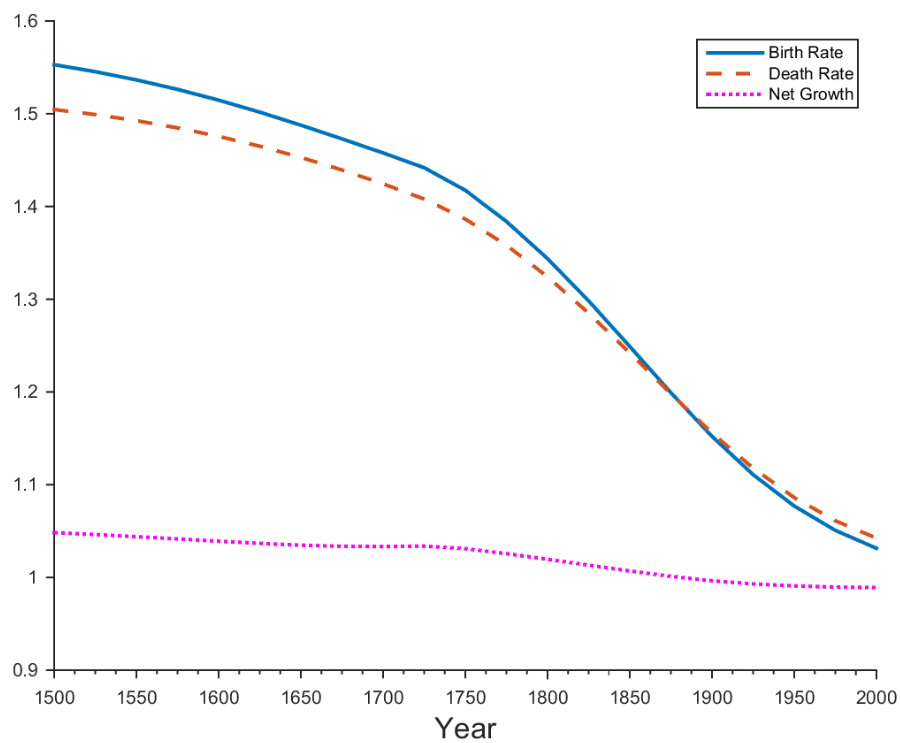

Figure 1.7: Simulation: Demographic Transition

1.6 Cross-Country Analysis

What impacts do the initial conditions have on the equilibrium dynamics? Subsection 1.6.1 considers the effect of changing initial calibration targets on the transition timing. Subsection 1.6.2 examines the empirical implications.

1.6.1 Model Sensitivity

The transition timing is sensitive to the initial calibration targets. In particular, three targets have large effects: the initial urban share, the initial human capital growth rate, and the initial population growth rate.

First, I vary the initial urban share target while holding constant the other targets. Varying the initial urban share chiefly operates through production parameters. In general, a change to a calibration target will not have an effect on just a subset of parameters. But the urban share's effects on calibration are relatively straightforward. Raising the initial urban share requires increasing ζ, the weight on urban goods in the final production sector, and decreasing TFP A, to keep the long run marginal productivity of human capital constant. Note that the elasticity of substitution ϵ is kept constant, for this parameter is identified off of the transition timing. There are small changes to household parameters, which must be adjusted to keep initial population growth at the target level, but these changes are small because $s_{U,0}$ is small.

Figure 1.8 plots the year that the model economy surpasses 1% income growth against the initial urban share. All other calibration targets are baseline values. As the initial urban share increases, the growth transition is delayed. Because the economy is more urban, and urban parents choose lower human capital growth for their children, the economy grows more slowly for many centuries. In the long run, the economy catches up to the baseline long run growth target as urban mortality improves.

Next, I vary only the initial human capital growth target, which primarily affects house-

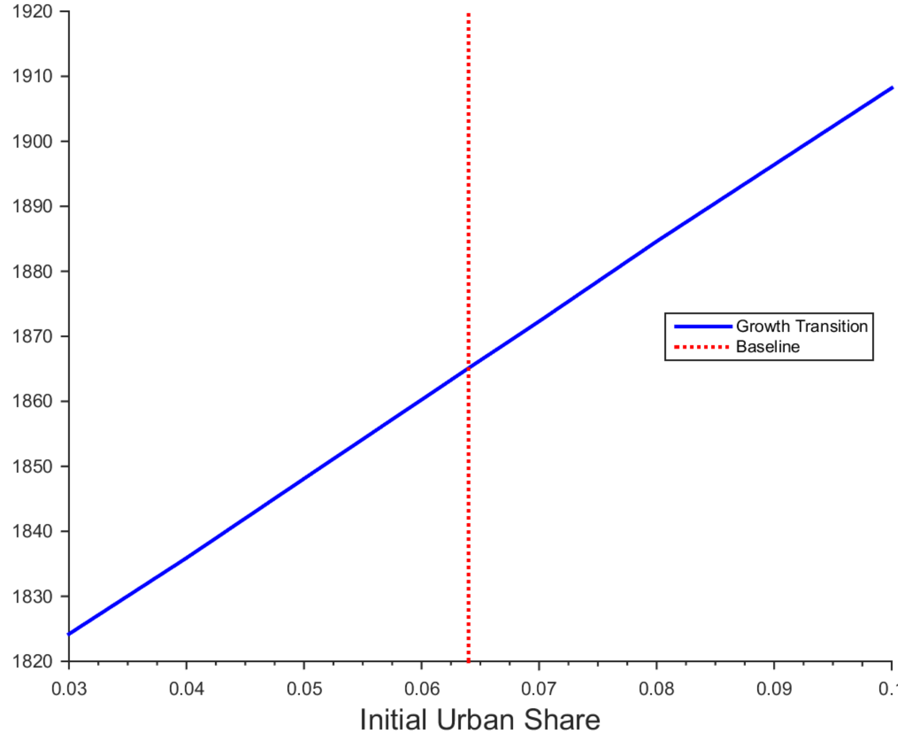

Figure 1.8: Transition Years and Initial Urban Share

hold parameters. Increasing the initial growth target increases the necessary household productivity of human capital investment ξ, and decreases the productivity of child-rearing α. Intuitively, increasing ξ makes the household richer, but decreasing α raises the relative price of children quantity versus quality. Thus the initial period household chooses the same initial population growth, but a higher rate of human capital growth. Of course, other parameters must have small adjustments to maintain the long run calibration targets.

Figure 1.9 plots the year that the model economy surpasses 1% income growth against the initial income growth rate. Other calibration targets are unchanged from the baseline. The transition timing is very sensitive to the initial growth rate. An economy with low initial growth has poor productivity of human capital investment. This decreases the growth rate along the transition, and the economy takes longer to converge to the long run limit. Lower human capital investment has some secondary effects: urbanization is slowed, which increases income growth by shifting the population composition towards the lower mortality rural sector, but the mortality transition is also slowed for both sectors, reducing income growth.

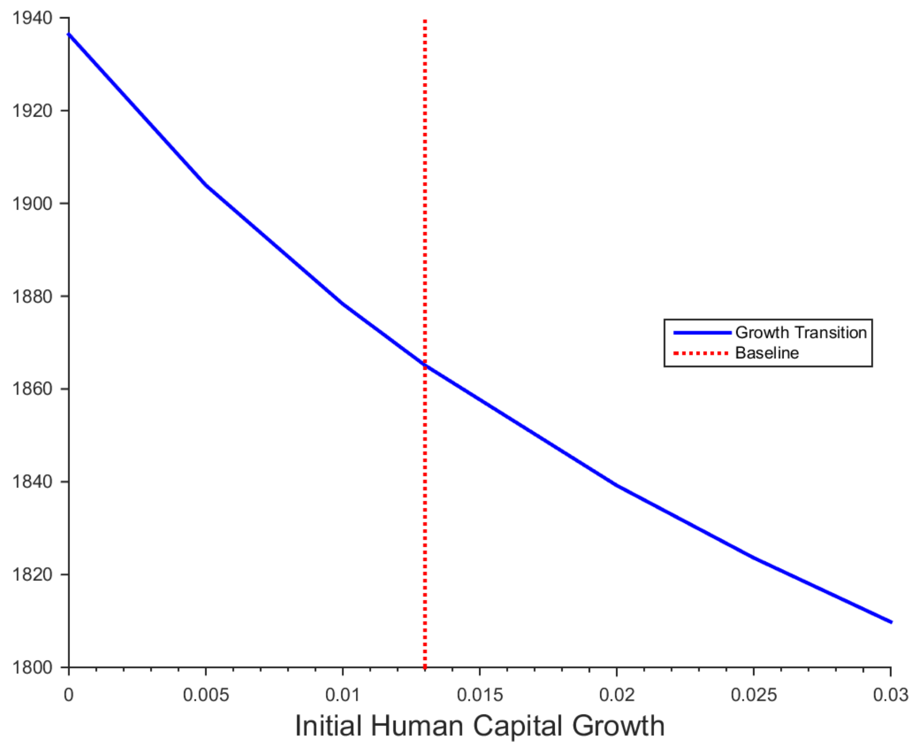

Figure 1.9: Transition Years and Initial Human Capital Growth

Lastly, increasing the initial population growth target speeds the economy's transition. Higher population growth is mainly achieved by increasing the productivity of childrearing α, but with a decrease in child preference ϕ to maintain the long run population growth. Because the initial urban households are constrained at $g = 0$ thanks to the high child mortality, they spend all of their non-market income producing children. So an increase in α disproportionately increases initial urban children relative to rural children. It takes less time for urban households to become unconstrained, and and to start substituting from child quantity to quality. The income growth transition year is plotted against the initial population growth rate, all else equal, in Figure 1.10. A higher population growth rate with the same household human capital growth rate speeds the income growth transition as households substitute to child quality earlier.

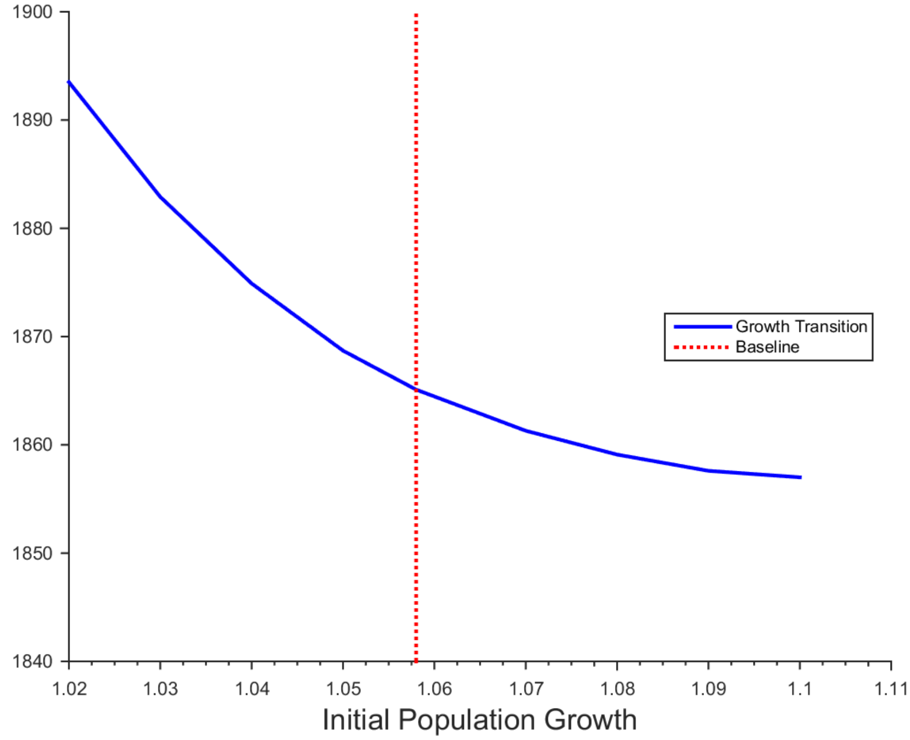

Figure 1.10: Transition Years and Initial Population Growth

1.6.2 Cross-Country Empirics

The analysis in section 1.6.1 suggest that, all else equal, a country will have a faster growth transition if it has: 1. a lower initial urban share, 2. a higher initial income growth rate, or 3. a higher initial population growth rate.

To test these relationships in the data, I construct transition years for 29 countries for which I have the relevant data in year 1500. Then I regress the transitions years T_j against country characteristics in year 1500:

$$T_j = \beta_0 + \beta_1 s_{U,0,j} + \beta_2 \Delta y_{0,j} + \beta_3 n_{0,j} + \boldsymbol{\alpha}' \mathbf{D}_j \tag{1.36}$$

where $s_{U,0,j}$ is country j's initial urbanization rate, $\Delta y_{0,j}$ is their initial per capita real income growth, $n_{0,j}$ is their initial population growth, and $\mathbf{D}_j$ is a vector of country characteristics for some regression specifications. Income and population data are from Bolt and van Zanden (2013). For comparability, England's data is also from this source, instead of the superior data used in section 1.5.1. Before 1820, income and population data are centennial, so in a given year (e.g. 1500) growth is the annualized rate over the preceding century. After 1820, income data is annual, and the transition year is defined as the first time that 25-year moving average of income growth exceeds 1%. Finally, urbanization data is from Bairoch et al. (1988) and The Clio Infra Project (2016).

Table 1.4 reports the baseline results in the first column. The growth transition year is regressed on the initial urbanization rate, income growth rate, and population growth rate. As predicted by the model, initial urbanization predicts a later transition, while higher income and population growth predict an earlier transition. The coefficient on initial urbanization implies that an addition 10 percentage points of urbanization should delay the growth transition by 28 years, all else equal. Both the urbanization and income growth rate coefficients are significant at the 5% level, but population growth is not, which is the case for almost every specification of these regressions.

Table 1.4 also reports the results of several robustness checks. The second regression uses population density as a proxy for urbanization, in case mismeasurement of the historical urbanization rates correlated with transition. But population density also predicts a later transition, and the effect is significant at the 1% level. The third regression includes a vector of geographic controls[18] considered by Ashraf and Galor (2011). The effect of urbanization is strengthened in this regression, and is significant at the 1% level. The fourth regression includes continent fixed effects, which weakens the relationship, and the fifth regression includes dummies identifying colonies, which has little effect on the estimates.

18. Absolute latitude, percentage of arable land, percentage of land within 100 km. of a coast or river, percentage of land in temperate zones, and percentage of land in tropical or subtropical zones.

	Baseline	Proxy	Geo. Controls	Continent FE	Colony FE
Urban Share	283.3**		377.1***	151.7	282.5**
	(2.67)		(3.32)	(1.55)	(2.43)
Income Growth	-36160.9**	-61019.8***	-28379.7	-26879.2	-36329.0*
	(-2.23)	(-3.23)	(-1.28)	(-1.32)	(-1.96)
Population Growth	-5731.0	-6818.4	-4403.9	4348.0	-5717.0
	(-0.80)	(-1.03)	(-0.56)	(0.64)	(-0.78)
Log Pop. Density		14.68***			
		(3.24)			
Observations	29	29	29	29	29

t statistics in parentheses

* $p < .1$, ** $p < .05$, *** $p < .01$

Table 1.4: Effects of 1500 CE Conditions on Growth Transition Year

The year 1500 CE is used to initialize the baseline calibration in Section 1.5.1 because it is the earliest period on which urban-rural differences in mortality and fertility are available. But the empirical effects of urbanization and income growth on transition timing can be examined for other years. Table 1.8 reports the baseline regression for many initial years. Income growth predicts an earlier transition in all cases except 1000 CE, when the data is especially poor. Urbanization slows the transition for all years, although it is not always significant, particularly in 1800 CE. As a country approaches its transition date, a higher urbanization rate no longer predicts a later transition even when conditioned on the growth rate.

In the context of the model, high initial urban shares are interpreted as reflecting high urban productivity relative to rural productivity.[19] In equilibrium, this results in a higher level of urbanization at every income level, although it may not be higher at every point in time. To illustrate, Figure 1.11 plots urbanization and income level for the baseline calibration, and for an alternative with China's initial urban share of 0.12. At every level

19. Ashraf and Galor (2011) estimate that countries in 1500 CE with high agricultural productivity have greater population density, particularly China and India. It must be that these countries are initially urban because their urban productivity is especially high.

of income, the alternative has higher urbanization. Why? The urban-rural wage premium is the compensating differential for the urban-rural mortality ratio. And the mortality ratio falls as the country's human capital rises. Because the urban sector is more productive relative to the rural sector in the alternative calibration, more households must be urban for a given wage premium.

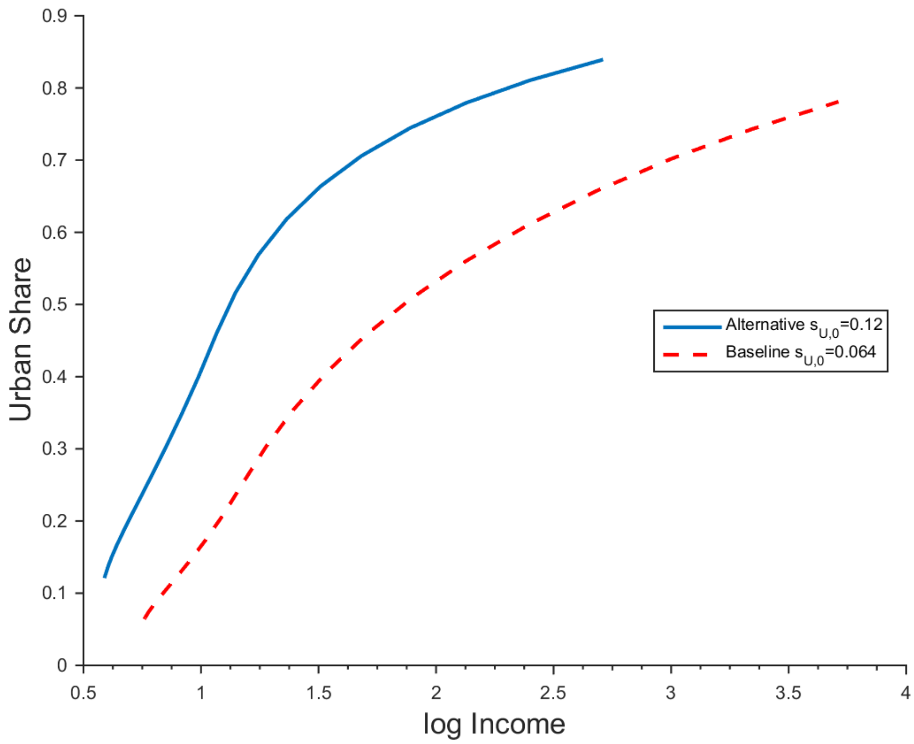

Figure 1.11: Urbanization and Income Levels: Model

Empirically, some countries have consistently higher urbanization rates for every level of income. Figure 1.12 compares this relationship for China and England. I use a two-stage regression approach to test generally if countries with high rates of urbanization relative to income have later growth transitions, as predicted by the model. First, I run the following panel regression, for country j in year t:

$$s_{U,t,j} = \gamma \log y_{j,t} + d_j + \kappa + \varepsilon_{j,t} \tag{1.37}$$

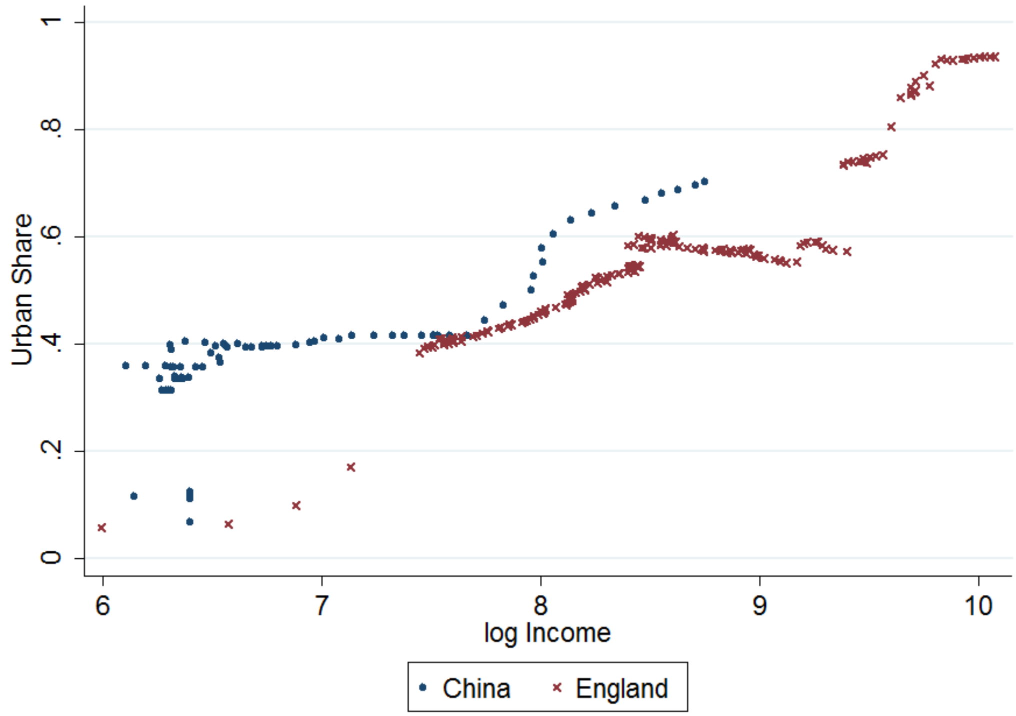

Figure 1.12: Urbanization and Income Levels: China and England

This is a regression of urban share on log income with country fixed effects. Next, the I regress the transition year T_j on the estimated fixed effects:

$$T_j = \psi \hat{d}_j + \varkappa + \varphi_j \tag{1.38}$$

Table 1.5 summarizes the 1st stage estimated country fixed effects. There are 57 countries with urbanization data before their income growth transition, and 6,491 total year-country observations. The regressor in the second stage is an estimate and analytical standard errors will be incorrect, so standard errors are calculated by boostrapping. Table 1.6 reports the results of the second stage regression. Countries that have a higher level urbanization conditional on their income transition much later.

Figure 1.13 plots countries' first-stage estimated fixed effects versus their transition year, and the second state regression line. Geographic patterns emerge. In the lower left are many

	mean	sd	min	max
Country Fixed Effects	-.0096558	.058645	-.1051271	.1711157
Observations	6491			

Table 1.5: Summary of Estimated Urbanization Fixed Effects

Country Fixed Effects	496.5
	(<0.001)
Constant	1890.1
	(<0.001)
Observations	57

p-values in parentheses

Table 1.6: Impact of Estimated Urbanization Fixed Effects on Transition Timing

Notes: Standard Errors calculated by bootstrapping 500 times over 6,491 first stage observations.

Western and Central European Powers and their colonies, which were initially very rural and transitioned early. In the upper right are many Asian countries, including China and India, which were urban early in their development, but transitioned later.

Both regression approaches suggest that countries relatively predisposed towards urbanization will transition to modern growth later, despite the general correlation of urbanization and income growth over time.

1.7 Concluding Remarks

This paper has developed a unified endogenous growth model producing three simultaneous transitions: the growth transition, urbanization, and the demographic transition. The model quantitatively reproduces the timing and magnitude of England's transitions. Because the model considers growth, urbanization, and demographics jointly, it also generates three additional empirical observations: a declining urban-rural wage gap, a declining rural/urban family size ratio, and that early urbanization delays a country's transition.

The relationship between early urbanization and transition timing is an identifying fea-

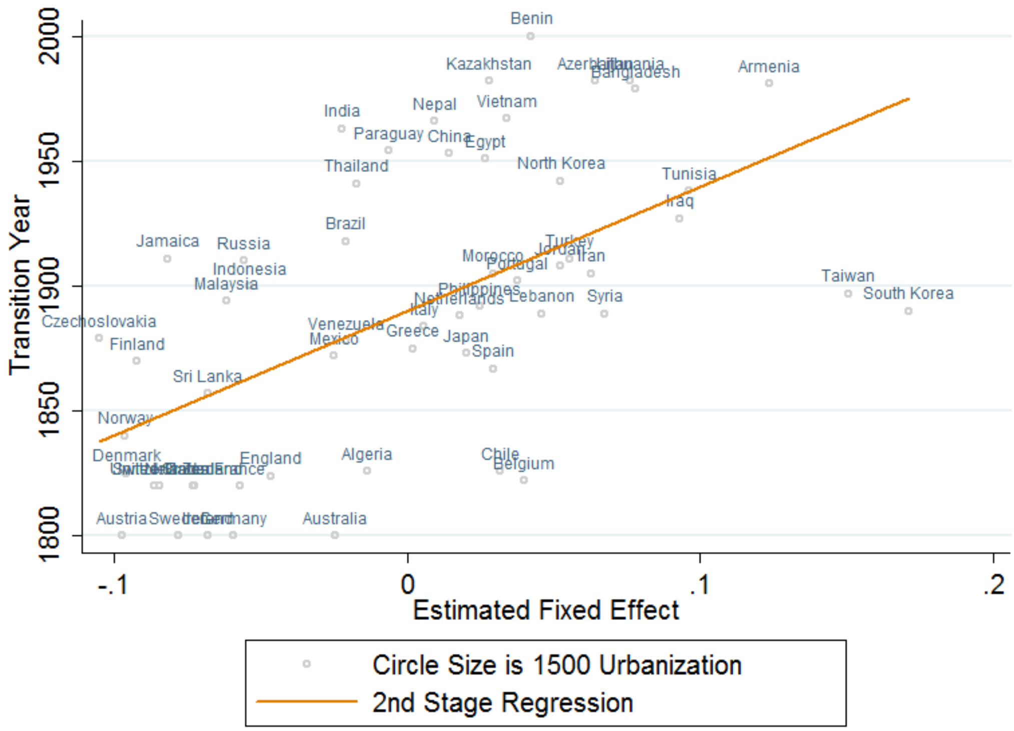

Figure 1.13: Estimated Country Effects and Transition Years

ture of the model which distinguishes it from other theories of urbanization and long run growth. I use several estimation strategies to show that the relationship between early urbanization and transition timing is robust in the historical experiences of many countries. This empirical fact raises further research questions. I have identified one plausible channel driving this relationship; could there be others? And does this channel apply to current low income countries? Future work can address these questions by applying and expanding on the theory in this paper.

	Target	Empirical Value	Source
(i)	Land Share in Agriculture	0.260	Clark (2010)
(ii)	Initial Survival Probabilities	$S_{R,0} = 0.681, S_{U,0} = 0.543$	Clark (2009)
(iii)	Urban/Rural Surviving Child Ratio	$n_{U,0}/n_{R,0} = 0.771$	Clark (2009)
(iv)	Initial 25-year Population Growth	1.085	Bolt and van Zanden (2013)
(v)	Initial 25-year Human Capital Growth	1.013	Bolt and van Zanden (2013)
(vi)	Long-Run 25-year Human Capital Growth	1.520	Bolt and van Zanden (2013)
(vii)	Initial Urban Share	0.064	Bairoch et al. (1988)

Table 1.7: Empirical Targets

	1000	1200	1300	1400	1500	1600	1700	1800
Urban Share	225.1	571.3**	556.5***	151.3	283.3**	251.0**	226.0**	69.69
	(1.03)	(2.72)	(3.08)	(1.32)	(2.67)	(2.43)	(2.38)	(0.72)
Income Growth	85115.3	-70168.6***	-76611.3***	-71015.8**	-36160.9**	-15482.4*	-21955.4**	-17511.8**
	(0.76)	(-3.02)	(-3.38)	(-2.46)	(-2.23)	(-1.93)	(-2.68)	(-2.45)
Population Growth	-43442.5	-6018.8	-3400.9	-10093.5	-5731.0	1053.8	-857.9	2152.7
	(-1.01)	(-0.58)	(-0.34)	(-0.82)	(-0.80)	(0.34)	(-0.27)	(0.76)
Constant	1865.1***	1903.9***	1892.1***	1922.2***	1865.6***	1851.1***	1863.5***	1872.9***
	(71.04)	(72.27)	(72.54)	(63.94)	(123.95)	(138.48)	(134.21)	(108.31)
Observations	18	18	18	18	29	29	32	35

t statistics in parentheses

* $p < .1$, ** $p < .05$, *** $p < .01$

Table 1.8: Effects of Urbanization and Growth on Transition Timing: Many Initial Years

CHAPTER 2

DECREASING RETURNS TO R&D AND DECLINING GROWTH RATES

2.1 Introduction

Why is growth slowing? One source of productivity growth is science, but scientific research and development has a fickle relationship with growth. Empirically, it exhibits two facts in both the cross-section of countries and the time series:

- **Fact 1:** High income countries spend a greater share of their income on research and development.
- **Fact 2:** Countries with low growth spend a greater share of their income on research and development.

These two facts are documented for the cross-section in Figures 2.1 and 2.2 respectively. The GDP data is from the World Bank, and is real and PPP-adjusted. The cross-country research and development data are from UNESCO.[1] The figures are generally robust to the observation period: 2009 is used for *Figure 1* because it is the most recent year with many available observations.

These facts are also generally true in the time series. Figure 2.3 plots per capita GDP growth against the research and development share of income for the United States. As the economy has grown, it has spent a greater share of income on research and development, despite declining growth rates. Moreover, in the US, research share of GDP has increased despite the post-Cold War decline of federal funding (Figure 2.4). Declining growth and rising research shares is also observed in the time series of Japan, France, and Germany, and

1. Figures 2.1 and 2.2 use all available observations, unweighted, except for 5 small outliers: Malta, Kuwait, Macao, Hong Kong, and Luxembourg. These are small states whose economies are very specialized, and thus look very dissimilar from large heterogeneous countries.

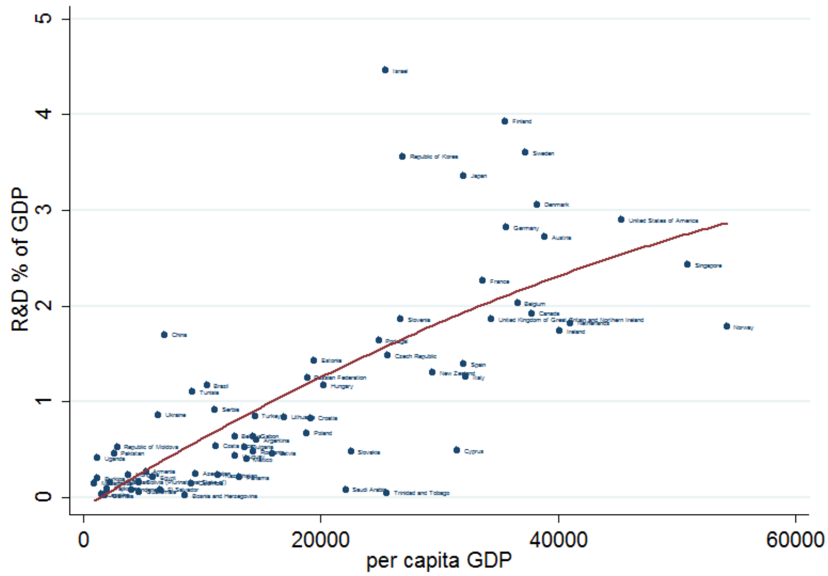

Figure 2.1: R&D and Income in the Cross-Section

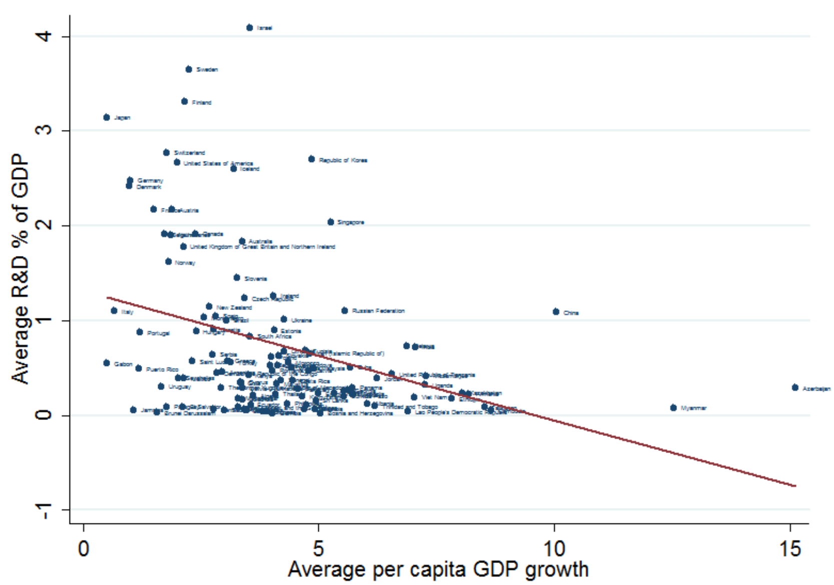

Figure 2.2: R&D and Growth in the Cross-Section

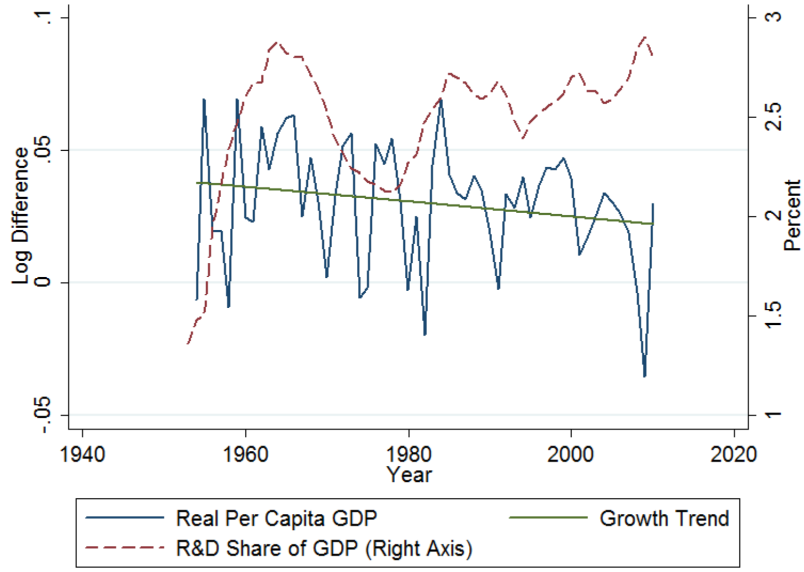

Figure 2.3: R&D and Growth in the US Time Series

to some extent South Korea, whose long run growth rate is more stable. Of countries with available data, the only anomaly is China, which has exhibited a rising research share, but also rising growth rates. Time series R&D data is from the NSF.

These joint facts are difficult to reconcile with existing models of R&D growth, which follow in the footsteps of seminal works such as Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992). Such models either feature: R&D shares decreasing with income, imposing that constant levels of research lead to constant productivity growth; or growth rates increasing with research and development shares.

These difficulties are not entirely new realizations. For example, Jones (1995) documents that the observed upward trend in research and development has not been associated with an upward trend in productivity. Kortum (1997) and Segerstrom (1998) identify this pattern in the US time series, and employs models of research patenting to connect rising research employment with constant patent growth.

The observed decline in growth rates is of renewed concern. Gordon (2012) projects that growth rates will continue to decline, and cites slowing technological innovation as one of the causes. This paper is empathetic to slowing technological innovation, and offers declining

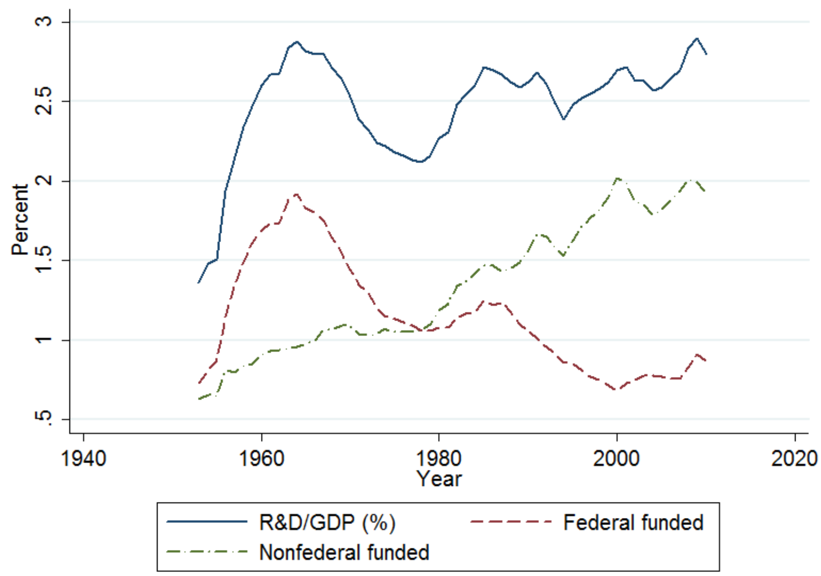

Figure 2.4: United States R&D Composition

returns to research and development as a mechanism, and a projection of the future long run economic growth rate.

A third empirical fact to match will be the size distribution of firms, known to be approximately Pareto distributed with tail parameter close to one - Zipf's Law. This fact is less of a novel outcome of the model than the first two, but is necessary to motivate the distribution of research innovations. Several recent papers have examined the relationship between the firm size distribution and growth, such as Luttmer (2007) which micro-founds the Pareto distribution, and Klette and Kortum (2004) which connect several moments of the firm size distribution with firm R&D decisions.

This paper presents a model of research and development with growth, that matches Facts 1 and 2 and the size distribution of firms without Boserupian effects. Firms produce in a continuum of sectors, and if they choose to spend researching a new technology in a sector, they obtain a temporary monopoly. Patent protection motivates firms to invest in research, but caries the cost that the provided monopoly suppresses output. As a commonly discussed and used policy tool, the model allows a policymaker to pick a tradeoff between income growth and income levels. Aggregate productivity grows as many firms innovate their technologies yielding a Pareto stationary distribution, but such innovation becomes increasingly costly as the technology frontier advances. More resources must be spent in research and development to keep up the growth rate of the economy. In equilibrium, countries spend little on R&D when their income is low, and spend more as they grow, despite their decreasing growth rates.

The calibrated model suggest that in the long run, research and development will asymptote to 3.4% of GDP, and long run GDP per capita growth will fall to 1.3%. Finally, Monte Carlo moment-matching is used to generate a distribution of growth projections, by assigning prior distributions to targeted moments and parameters.

The paper proceeds by laying out the model, describing equilibrium, exploring dynamics, calibrating parameters, checking robustness, and concluding.

2.2 Model

2.2.1 Households

Households in the model have CRRA preferences over current and future consumption

$$\int_0^\infty e^{-\rho t} \frac{\theta}{\theta - 1} c(t)^{\frac{\theta-1}{\theta}} dt$$

where $c(t)$ is per capita consumption at time t, $\rho > 0$ is the rate of time preference, and $\theta > 0$ is the intertemporal elasticity of substitution. Households supply labor inelastically, equal to the population which we normalize to one $L = 1$.

The households earn labor income $w(t)$ and hold assets $a(t)$, which pay a rate of return $r(t)$. Letting $\dot{a}(t)$ denote the rate of change in a household's asset holdings, the household budget constraint is

$$\dot{a}(t) + c(t) = r(t)a(t) + w(t)$$

where consumption goods are the numeraire.

The household's current value Hamiltonian is

$$H(a,c) = \frac{\theta}{\theta - 1} c(t)^{\frac{\theta-1}{\theta}} + \mu(t)(w(t) + r(t)a(t) - c(t))$$

implying the Euler equation[2]

$$r(t) = \rho + \frac{1}{\theta}\frac{\dot{c}(t)}{c(t)} \tag{2.1}$$

2.2.2 Firms

Competitive Firms

There are several types of firms.

First, a perfectly competitive final goods sector combines a continuous set J of specialized inputs, indexed by j, into the final output good by a CES production function

$$F(\{x_j\}_{j\in J}; A) = A(\int_{j\in J} x_j^{\frac{\sigma-1}{\sigma}} dj)^{\frac{\sigma}{\sigma-1}}$$

where A is an exogenous total factor productivity term. This TFP will vary over time, but for simplicity, the time scripts are suppressed for firms with no dynamic decisions.

The price of input x_j is p_j, and the price of the final output good is one. Final goods producers solve

$$\max_{\{x_j\}_{j\in J}} A(\int x_j^{\frac{\sigma-1}{\sigma}} dj)^{\frac{\sigma}{\sigma-1}} - \int p_j x_j dj$$

The first order condition for input j is

$$A^{1-\frac{1}{\sigma}} F(\{x_j\}_{j\in J}; A)^{\frac{1}{\sigma}} x_j^{-\frac{1}{\sigma}} = p_j \ \ \forall j$$

With Y denoting aggregate final output, this first order condition implies the demand func-

2. Derivation in Appendix B.1

tion for each input

$$p_j = A^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} x_j^{-\frac{1}{\sigma}} \tag{2.2}$$

Second, inputs are produced with production function

$$f(l_j; z_j) = z_j l_j$$

linear in the labor l_j employed by producer j, and with j-specific productivity z_j. A subset $J_C \subset J$ of these goods are produced competitively. These firms solve

$$\max_{l_j} p_j z_j l_j - w l_j$$

which implies

$$p_j = \frac{w}{z_j} \;\; \forall j \in J_C \tag{2.3}$$

Monopolistic Firms

The complementary subset $J_M \subset J$ of specialized goods are produced monopolistically. Recognizing that they control the price by equation (2), the monopolistic producer of good j solves

$$\max_{l_j, x_j} A^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} x_j^{\frac{\sigma-1}{\sigma}} - w l_j$$

$$s.t. \;\; x_j = z_j l_j$$

which has first order condition

$$\frac{\sigma-1}{\sigma} A^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} x_j^{-\frac{1}{\sigma}} = \frac{w}{z_j} \;\; \forall j \in J_M \tag{2.4}$$

Combined with equation (2), the price in monopolistic sectors must be

$$p_j = \frac{\sigma}{\sigma - 1}\frac{w}{z_j} \ \ \forall j \in J_M$$

The salient characteristics of the monopolist will be important in matching the firm size distribution, so we derive them now. The monopolist's total revenue is

$$p_j x_j = A^{\sigma-1} Y (\frac{\sigma - 1}{\sigma}\frac{z_j}{w})^{\sigma-1}$$

its labor demanded is

$$l_j = A^{\sigma-1} Y (\frac{\sigma - 1}{\sigma w})^{\sigma} z_j^{\sigma-1}$$

and its profit Π_j is

$$\Pi_j = p_j x_j - w l_j$$

$$\Pi_j = Y (\frac{A z_j}{w})^{\sigma-1} (\frac{\sigma - 1}{\sigma})^{\sigma-1} \frac{1}{\sigma}$$

which is derived in Appendix B.2.

Empirically, revenues and employment are distributed close to Pareto in the cross-section, a pattern that has been well-documented since at least Simon and Bonini (1958). Firm size in the competitive sectors is undetermined in this model. But the firm size distribution in monopolist sectors can be matched, if a monopolist is assumed to be a single firm. The only firm-specific term in a monopolist's revenue or labor is its productivity z_j, so we will look for z_j to be distributed Pareto, because if z_j is Pareto with tail parameter λ, then $z_j^{\sigma-1}$ is Pareto with tail parameter $\frac{\lambda}{\sigma-1}$. Revenue and labor are each proportional to $z_j^{\sigma-1}$, so they are also Pareto (Appendix B.4). We will require that $\frac{\lambda}{\sigma-1} > 1$ so that the first moment exists.

To this point, firm decisions have been static. But there are dynamics that govern both productivities z_j, and whether j is a monopolistic sector, or a competitive sector. We now characterize these dynamics such that z_j is Pareto.

If a firm has access to productivity z_j, it can spend on research and development to increase the productivity. Its new productivity will be $z_j' = Xz_j$ where X is a Pareto random variable on $[1, \infty)$ with tail parameter λ. When a firm innovates and increases its productivity z_j, it receives a monopoly on producing good j. For simplicity, we assume this monopoly is a patent, protected by law until the patent is lost, at which point the technology becomes available to anyone.

Patents have a probability of expiring, which for tractability is independent of how long the patent has survived: the measure of all patents expires at rate $\delta(t)$. Next, patents can also become obsolete if they are surpassed by aggregate technological progress. We say this occurs for technology z_j if it falls at or below the lower bound of the productivity distribution, i.e. if $z_j \leq \min(z_i)\ \forall i \in J$.

It remains to be shown that with these dynamics, the distribution of z_j is Pareto. We now conjecture that at any moment, only the minimum productivity technologies z_m are improved. This is a similar evolution for technology as in Perla and Tonetti (2014). Once the technology for innovation is described, we can confirm the conjecture.

Theorem 1: If only the minimum productivity technologies z_m are improved, then the stationary distribution of $\frac{z}{z_m}$ is Pareto. Suppose that monopolistic productivities and competitive productivities are both distributed Pareto with minimum z_m and parameter λ.

1. If all monopolistic technologies have a probability patent expiry independent of their productivity, then the distribution of productivities changing from monopolistic to competitive will be Pareto with minimum z_m and parameter λ. Existing competitive technologies share this distribution, so the resulting distribution of competitive productivities is unchanged.

2. If all technologies (competitive and monopolistic) with minimum productivity z_m are improved, then the resulting productivities Xz_m are distributed Pareto with minimum z_m and parameter λ. X is Pareto with parameter λ and minimum 1, and any

such Pareto times a scalar ϵ becomes Pareto with parameter λ and minimum ϵ (Appendix B.4). The non-innovated monopolistic productivities are distributed Pareto with productivities are distributed Pareto with infimum z_m and parameter λ, so the total distribution of monopolistic productivities remains Pareto with tail parameter λ.

Innovating Firms

The third type of firm is the innovator. The innovator makes research and development decisions, and owns patents to the technologies that it invents. It could rent these patents out, but for simplicity assume it directly owns production firms that use its technology. The innovator returns profits to households as dividends. The innovator makes the R&D decision, so their choices will govern the dynamics of technology. Innovators have free entry, so on the margin, the present value of new innovations must be zero, net of research cost.

Innovators are competitive, big enough to own a representative set of firms (i.e. earn the average return), but small enough to be price takers. This assumption is crucial: innovation in this economy has an externality, because after yielding monopoly profits to the researcher, an innovation is dispersed to anyone who cares to use it. Whomever is making an investment decision in this economy cannot be so big that they price the externality.

Consider the decision of a small innovator owning mass $k(t)$ of a representative set of monopolistic firms - that is, its firms have productivities distributed Pareto with minimum $z_m(t)$ and tail parameter λ. How does $k(t)$ evolve? Patents expire at rate $\delta(t)$, so the share of the mass of monopolistic firms that do not lose their patents to expiry after Δ time (supposing $\delta(t)$ is invariant over this interval) is $e^{-\delta(t)\Delta}$. Secondly, if the minimum productivity in the aggregate grows from $z_m(t)$ to $z_m(t+\Delta)$, the share of firms that do not lose their patents to obsolescence is one minus the Pareto CDF:

$$1 - G(z_m(t+\Delta); \lambda, z_m(t)) = (\frac{z_m(t)}{z_m(t+\Delta)})^{\lambda}$$

Because the patent expiration is independent of productivity, the two losses are multiplicative, so the share that survives both expiration and obsolescence over Δ time is $(\frac{z_m(t)}{z_m(t+\Delta)})^\lambda e^{-\delta(t)\Delta}$. Then the law of motion for $k(t)$ over Δ time is

$$k(t+\Delta) = k(t)(\frac{z_m(t)}{z_m(t+\Delta)})^\lambda e^{-\delta(t)\Delta}$$

and the instantaneous law of motion as $\Delta \to 0$ is[3]

$$\frac{\dot{k}(t)}{k(t)} = -\delta(t) - \lambda \frac{\dot{z_m}(t)}{z_m(t)}$$

Next, we introduce the innovation technology. The returns to innovation are fixed by assumption: $E[Xz] = \frac{\lambda z}{\lambda - 1}$, thanks to the the Pareto distribution of X. Spending on research and development will determine the rate at which firms innovate. $I(b; z, g)$ is this production function, where z is the productivity of the technologies to be innovated, g is the density of technologies to be innovated, and b is a flow of research and development expenditure (in output goods). I is increasing in b, so that the more firms spend on research, the faster they innovate. I is decreasing in z so that innovation becomes more expensive as the economy advances, and proportionally decreasing in g, so that it is invariant to scale. Specifically, we adopt the form

$$I(b; z, g) = \frac{\lambda b}{g z^{\eta+1}}$$

The curvature parameter η determines how costly additional innovations become as the technology level grows. The innovation function is constant returns with respect to the the R&D input, but a higher η implies more severely decreasing returns to research and development at the economy-wide level.

A innovator with mass k of technologies has density $k\frac{\lambda}{z_m}$ of productivities at the lower

3. Derivation in Appendix B.5

bound of its distribution. If these are the only technologies that are innovated (a conjecture we have yet to confirm) then the law of motion for its minimum productivity $z_m(t)$ is

$$\dot{z_m}(t) = \frac{b(t)}{k(t)z(t)_m^\eta}$$

The innovator's net income is the profits from the firms it owns, less its research investment $b(t)$. Recall that the period profit for a firm with productivity z_j is $\Pi_j(t) = \frac{Y(t)}{\sigma}(\frac{\sigma-1}{w(t)\sigma})^{\sigma-1}z_j^{\sigma-1}$, and for a innovator $z^{\sigma-1}$ is Pareto distributed with average $\frac{\lambda}{\lambda+1-\sigma}z_m^{\sigma-1}$. Thus it's net income is

$$k(t)z_m^{\sigma-1}(t)d(t) - b(t)$$

where $d(t) \equiv \frac{Y(t)}{\sigma}(\frac{\sigma-1}{w(t)\sigma})^{\sigma-1}\frac{\lambda}{\lambda+1-\sigma}$

The innovator's objective is to maximize present value dividends and discounts by the interest rate $r(t)$ to ensure no arbitrage, so its current-value Hamiltonian is

$$H(b, z_m) = k(t)z_m(t)^{\sigma-1}d(t) - b(t) + \mu(t)\frac{b(t)}{k(t)z_m(t)^\eta}$$

with equilibrium condition[4]

$$(\sigma-1)d(t)z_m(t)^{\sigma-2-\eta} = r(t) + \delta(t) + (\lambda+1-\eta)\frac{\dot{z_m}(t)}{z_m(t)} \tag{2.5}$$

This Euler equation holds with equality, so the firm is exactly indifferent to innovating. If it strictly preferred to innovate a technology, it would already have done so! Recall that innovators are small and have free entry, so they must be indifferent to innovating on the margin. Finally, it prefers to innovate its lowest productivity technologies over its higher productivity technologies because the cost would be higher with no greater gain, assuming that $\sigma - 2 - \eta < 0$.

4. Derivation in Appendix B.3

2.2.3 Equilibrium

A dynamic equilibrium in this economy is an allocation of prices $\{w(t), r(t)\}$, aggregates $\{Y(t), C(t), B(t), Z(t)\}$, monopoly share of technologies $\{M(t)\}$ and growth rates $\{\dot{Z}(t), \dot{C}(t), M(t)\}$ for all t such that

1. Households maximize utility, satisfying (1)
2. Production firms maximize profits, satisfying (2) (3) and (4)
3. Innovators make investments to maximize their present value, satisfying (5)
4. The representative household consumes aggregate consumption $C(t)$, the net supply of assets is $a(t) = 1$, the representative innovator makes aggregate investment and holds the aggregate portfolio of technologies $b(t) = B(t)$, $k(t) = 1$, $z_m(t) = Z(t)$
5. Market clearing $Y(t) = B(t) + C(t)$ is satisfied

To characterize aggregate output, let M denote the share of sectors that are monopolistic. Aggregate output can be expressed as a function of M and the equilibrium outputs in each type of sector:

$$Y = A\left(M \int_{j\in J_m} x_j^{\frac{\sigma-1}{\sigma}} dj + (1-M) \int_{j\in J_c} x_j^{\frac{\sigma-1}{\sigma}} dj\right)^{\frac{\sigma}{\sigma-1}}$$

with some rearranging and substitution of the equilibrium demand for input j we find that wage, aggregate output, and total monopolist profit are proportional to minimum productivity $Z(t)$ and TFP $A(t)$ by[5]

$$w(t) = \omega Z(t)A(t) \tag{2.6}$$

$$\omega \equiv \left(\frac{\lambda}{\lambda+1-\sigma}\right)^{\frac{1}{\sigma-1}} \left(M\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} + (1-M)\right)^{\frac{1}{\sigma-1}}$$

5. Derivation in Appendix B.6

$$Y(t) = \psi Z(t) A(t) \tag{2.7}$$

$$\psi \equiv \frac{\omega}{1 - (\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M}$$

$$\Pi(t) = \pi Z(t) A(t) \tag{2.8}$$

$$\pi \equiv \psi(\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M$$

Notice that the output growth rate is

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{Z}(t)}{Z(t)}$$

and $\frac{\dot{Z}(t)}{Z(t)}$ also the rate at which no technologies are innovated, so the endogenous productivity growth rate and the patent rate are proportional, as documented by Kortum (1997) and Segerstrom (1998).

As expected, the level of output is decreasing in M: $(\frac{\sigma-1}{\sigma})^{\sigma-1} < 1$ so $\frac{d\omega}{dM} < 0$, and $(\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+\sigma-1} > 0$ so $\frac{d\psi}{dM} < 0$. Increasing the share of sectors that are monopolistic suppresses output, because monopolists markup above the competitive price. This is one side of the trade-off between income levels and income growth.

Next, the monopolistic share of sectors M is determined by the patent expiry rate δ_t and the rate of obsolescence. At any moment, the density of technologies at the lower productivity bound Z is $\frac{\lambda}{Z}$. Share M of these technologies are held by monopolists, who immediately innovate upon obsolescence, and remain monopolists. Share $1 - M$ are public, produced competitively. These sectors are innovating at speed $\dot{Z}$, and become monopolist. Therefore the change to the monopolist share is

$$\dot{M} = \frac{\lambda}{Z}\dot{Z}(1-M) - \delta_t M$$

To produce a constant monopolist share, assume the expiration rate is determined by

$$\delta_t \equiv \delta \frac{\dot{Z}}{Z}$$

This simplifies the economy by keeping M constant, so that it is not a state variable. M is interpreted as patent policy, which chooses the share of technologies to be protected. The parameter δ is the lever of patent policy by which a policy maker might choose M. The parameter δ determines this share by

$$\dot{M} = 0 \Longrightarrow \lambda(1-M)\frac{\dot{Z}}{Z} = \delta \frac{\dot{Z}}{Z} M$$

$$M = \frac{\lambda}{\lambda + \delta}$$

Thus the policy maker's decision is to choose the monopolist share of the economy, instead of choosing expiration rates. Observe that the monopolist share is decreasing in δ: increasing the expiration rate of patents reduces the share of firms sectors operating under patent protection.

With aggregate variables, the innovator's Euler equation becomes

$$\frac{(\sigma-1)\pi A(t)}{Z(t)^{\eta}} = r(t) + \alpha \frac{\dot{B}(t)}{B(t)} + (\delta + \lambda + 1 - \eta)\frac{\dot{Z}(t)}{Z(t)}$$

and the household's Euler equation becomes

$$r(t) = \rho + \frac{1}{\theta}\frac{\dot{C}(t)}{C(t)}$$

If we define the research share of output as $S_r(t) \equiv \frac{B(t)}{Y(t)} = \frac{B(t)}{\psi A(t) Z(t)}$, it relates to consumption by $C(t) = (1 - S_r(t))Y(t) = (1 - S_r(t))\psi A(t) Z(t)$. Then the growth rates of consumption and research are

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{Z}(t)}{Z(t)} - \frac{\dot{S}_r(t)}{1 - S_r(t)}$$

$$\frac{\dot{B}(t)}{B(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{Z}(t)}{Z(t)} + \frac{\dot{S}_r(t)}{S_r(t)}$$

The law of motion for $Z(t)$ is $\dot{Z}(t) = \frac{B(t)}{Z(t)^\eta}$ which can be rewritten with $S_r(t)$ as

$$\frac{\dot{Z}(t)}{Z(t)} = \psi \frac{S_r(t)A(t)}{Z(t)^\eta}$$

and combining the two Euler equations yields

$$(\sigma - 1)\pi S_r(t)\frac{A(t)}{Z(t)^\eta} = \rho + \frac{1}{\theta}\frac{\dot{A}(t)}{A(t)} - \frac{1}{\theta}\frac{\dot{S}_r(t)}{1 - S_r(t)} + (\delta + \lambda + 1 - \eta + \frac{1}{\theta})\frac{\dot{Z}(t)}{Z(t)} \tag{2.9}$$

Together, these two equations govern the endogenous dynamics of the economy! Researched productivity $Z(t)$ is an endogenous state variable, TFP $A(t)$ is an exogenous state variable, and the research share $S_r(t)$ is the control. But this is not an autonomous system yet - that will require a normalization.

2.2.4 Balanced Growth Path

If $A(t)$, the exogenous TFP coefficient, grows at a constant rate, then the economy features a balanced growth path. Denote a constant TFP growth rate by

$$\frac{\dot{A}(t)}{A(t)} \equiv \gamma_A$$

A balanced growth path features constant growth rate $\frac{\dot{Z}(t)}{Z(t)} = g_Z$ in researched productivity and a constant research share $S_r(t) = \overline{S_r}$. On the balanced growth path, equation (9) becomes

$$(\sigma-1)\pi\frac{A(t)}{Z(t)^{\eta}}=\rho+\frac{1}{\theta}\gamma_A+(\lambda+\delta+1-\eta+\frac{1}{\theta})g_z$$

For $\gamma_A>0$, it must be that $A(t)$ and $Z(t)^{\eta}$ are in constant proportion, so that

$$g_Z=\frac{\gamma_A}{\eta}$$

Define the normalization $K(t)\equiv\frac{A(t)}{Z(t)^{\eta}}$, in order to express the model as an autonomous system. $K(t)$ has the growth rate $\frac{\dot{K}(t)}{K(t)}=\gamma_A-\eta\frac{\dot{Z}(t)}{Z(t)}$. Then, using $g_z=\psi\frac{S_r(t)A(t)}{Z(t)^{\eta}}$, the normalized Euler equation and normalized law of motion are

$$(\sigma-1)\pi K=\rho+\frac{1}{\theta}\gamma_A-\frac{1}{\theta}\frac{\dot{S}_r}{1-S_r}+(\delta+\lambda+1-\eta+\frac{1}{\theta})\psi S_r K \tag{2.10}$$

$$\gamma_A-\frac{\dot{K}}{K}=\eta\psi S_r K \tag{2.11}$$

which are the two differential equations governing the dynamics of the economy. Notice that the dynamic system is reduced to one state and one control variable: $K(t)$ and $S_r(t)$. For ease of reading, we can rewrite the normalized Euler as

$$\frac{1}{\theta}\frac{\dot{S}_r}{1-S_r}=\phi_1-\phi_2 K+\phi_3 S_r K$$

$$\phi_1\equiv\rho+\frac{1}{\theta}\gamma_A$$

$$\phi_2\equiv(\sigma-1)\pi$$

$$\phi_3\equiv(\delta+\lambda+1-\eta+\frac{1}{\theta})\psi$$

Setting $\dot{S}_r = 0$ in the Euler equation provides one locus of the system

$$[\dot{S}_r = 0] \;\; K = \frac{\phi_1}{\phi_2 - \phi_3 S_r} \tag{2.12}$$

and setting $\dot{K} = 0$ in the law of motion provides the other

$$[\dot{K} = 0] \;\; K = \frac{\gamma_A}{\eta \psi S_r} \tag{2.13}$$

On the balanced growth path, $K(t) = \overline{K}$ and $S_r(t) = \overline{S_r}$ With this normalization, the balanced growth variables $\overline{K}$ and $\overline{S_r}$ are the solution to two equations (12) and (13) in two unknowns and parameters. The solution is

$$\overline{S}_r = \frac{\phi_2 \gamma_A}{\phi_1 \eta \psi + \gamma_A \phi_3} = \frac{(\sigma - 1)\pi \gamma_A}{(\rho + \frac{1}{\theta}\gamma_A)\eta\psi + \gamma_A(\delta + \lambda + 1 - \eta + \frac{1}{\theta})\psi} \tag{2.14}$$

$$\overline{K} = \frac{\phi_1 \eta \psi + \gamma_A \phi_3}{\phi_2 \eta \psi} = \frac{(\rho + \frac{1}{\theta}\gamma_A)\eta + \gamma_A(\delta + \lambda + 1 - \eta + \frac{1}{\theta})}{(\sigma - 1)\pi\eta} \tag{2.15}$$

What affects the balanced growth path? The long run research share $\overline{S}_r$ is increasing in $\frac{\pi}{\psi}$, the profit share of the economy. $\frac{\pi}{\psi}$ is increasing in the monopolist share M, so higher patent protection increases the expenditure devoted to research. At the same time, $\overline{K}$ is decreasing in π, which is increasing in M. Higher K corresponds to lower levels of output, because $A(t)$ is exogenous so higher K means $Z(t)$ is lower at any point in time. Thus, increased patent protection produces the traditional trade-off in the long run: the output level is bolstered, but a greater share of expenditure must be spent to keep it growing at the long run.

How important is the exogenous TFP growth? $\overline{S}_r$ is increasing in the TFP growth rate γ_A, because exogenous and endogenous productivity are complementary. If there were no exogenous TFP growth, the long run research share of expenditure would be zero, and $\overline{K}$ would determine the finite long run value of productivity. It is crucial that research and development is not the only source of growth.

2.2.5 Dynamics

If K is above the $\dot{K}=0$ locus (i.e. $K > \frac{\gamma_A}{\eta\psi S_r}$), then $\dot{K} < 0$ because $\frac{\dot{K}}{K}$ is decreasing in K. If K is above the $\dot{S}_r = 0$ locus (i.e. $K > \frac{\phi_1}{\phi_2 - \phi_3 S_r}$), the result depends on the parametrization: For $\phi_3 \overline{S}_r - \phi_2$ positive (negative) $\frac{\dot{S}_r}{1-S_r}$ is increasing (decreasing) in K near the steady state. The increasing case is necessary to achieve a single equilibrium path in which S_r rises and K falls. Declining K is consistent with declining growth: as K falls to its steady state value, $\frac{\dot{K}}{K}$ rises to zero, and g_Y is decreasing in $\frac{\dot{K}}{K}$ for $\eta > 0$.

Figure 5 plots the loci for the calibrated economy, with directional arrows away from the steady state. The system of differential equations features a saddle path steady state, and the saddle path is exactly the $\dot{S}_r = 0$ locus. *Figure 6* plots the phase portrait with calibrated vectors.

Figure 7 plots the equilibrium path, with initial condition calibrated to the US in 1955. GDP growth slows as new innovations become more expensive to research. The share of expenditure devoted to research rises to compensate, but cannot compensate fully. GDP growth falls towards its balanced growth path value, and the research expenditure share asymptotes up towards its balanced growth path value. In the long run, growth is permanently low. The dynamics are consistent with the two empirical facts observed in the time series and cross-section:

- **Fact 1:** Output growth is always positive, and S_r is increasing to $\overline{S}_r$ in transition, so **high incomes correspond to high R&D shares.**

- **Fact 2:** The income growth rate is decreasing as technology advances. Because S_r is increasing to $\overline{S}_r$ in transition, **low income growth rates correspond to high R&D shares**.

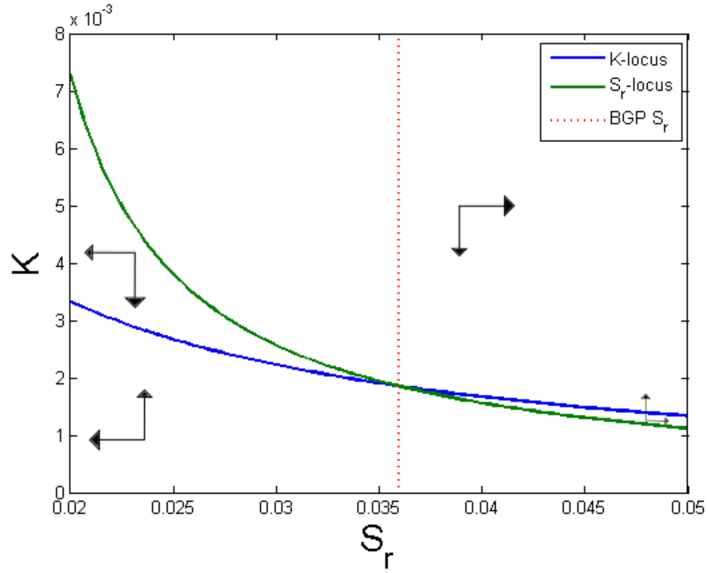

Figure 2.5: Loci

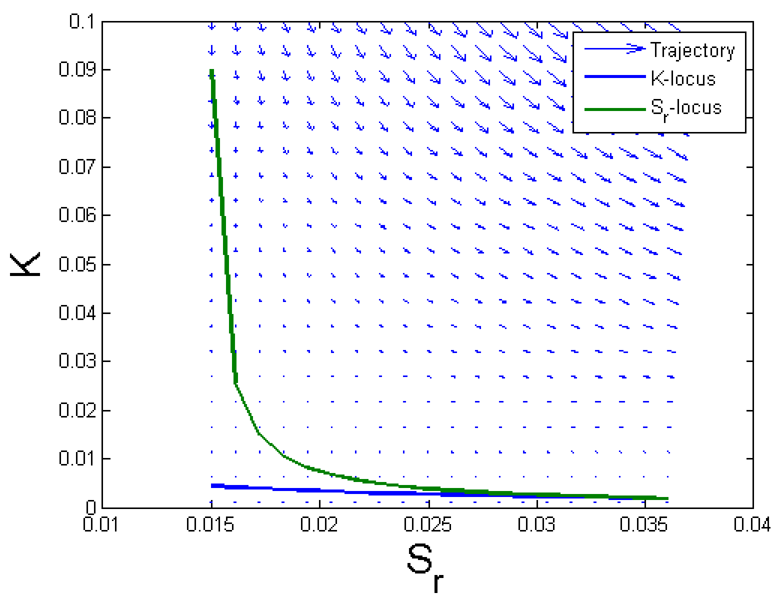

Figure 2.6: Phase Portrait

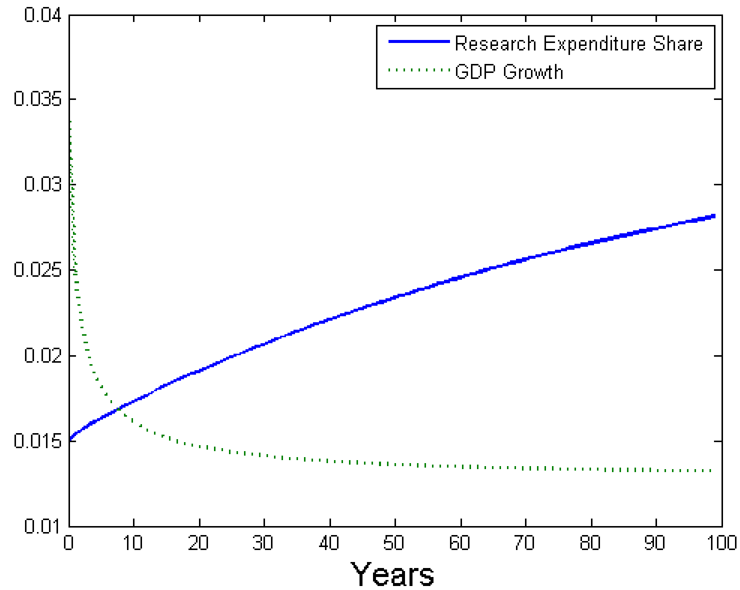

Figure 2.7: Equilibrium Path

2.3 Quantitative Exercise

2.3.1 Calibration

The model economy is characterized by 7 parameters: σ, λ, δ, η, θ, ρ, and γ_A, and one initial value $K(0)$.

$\sigma = 2$ is chosen as the production elasticity of substitution. This parameter is well identified in the literature, beyond the need for $\sigma > 1$, but the value 2 is chosen because it considerably simplifies some algebra (particularly in Appendix B.6, which features several $\sigma - 1$ powers.)

The elasticity of intertemporal substitution is set to $\theta = .5$, the minimum value considered by Lucas (1990) to be plausible. One moment to target is the initial growth rate, set to $g_Y(0) = .035$, the average value for per capita GDP growth in the US 1950s and 1960s. Second, the initial interest rate is targeted to $r = .045$, the long-run US after-tax return on wealth estimated by Saez and Zucman (2014). Together, these values imply $\rho = -.025$ by the household's Euler equation.

The tail parameter λ governing returns to innovation is set to target the the size distribution of firms. Zipf's Law suggests $\frac{\lambda}{\sigma-1}$ is close to one. Specifically, Luttmer (2007)'s tail parameter estimate implies $\lambda = 1.06$, given $\sigma = 2$.

Papers such as Rotemberg and Woodford (1996) or Basu and Fernald (1997) estimate economy profit shares to be about 3%. In this model, the profit share of the economy is $\frac{\pi}{\psi} = (\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M$. With the current parametrization, ω reduces to $\omega = \frac{\lambda}{\lambda-1}(1-\frac{M}{2})$. Substituting in ω, $M = \frac{\lambda}{\lambda+\delta}$, and parameter values, we find

$$\lambda\frac{\frac{\psi}{\pi}-2}{4} = \delta$$

implying patent expiration parameter $\delta = 8.3$.

The last two parameters to calibrate are η, which governs the decreasing returns to research, and γ_A, the growth rate of exogenous TFP. They are jointly identified by targeting initial and long run research shares. Recall that the growth rate is related to K and S_r by

$$g_y = \gamma_A + \psi S_r K$$

on the $\dot{S}_r = 0$ locus, growth can be expressed just in terms of the research share:

$$g_y = \gamma_A + \frac{(\rho + \frac{1}{\theta}\gamma_A)\psi S_r}{(\sigma-1)\pi - (\delta+\lambda+1-\eta+\frac{1}{\theta})\psi S_r}$$

This equation evaluated at $(g_Y(0), S_r(0))$, in combination with the long run research share equation 2.14, provide a system of two equations and two unknowns: η and γ_A. The initial research share is targeted to $S_r(0) = .015$, the US research share in 1955. The long run research share is targeted to $\overline{S}_r = .036$, which is calculated by estimating a nonlinear trend for the US research expenditure share. The statistical model is

$$S_r = b_0 + b_1 b_2^t + \varepsilon$$

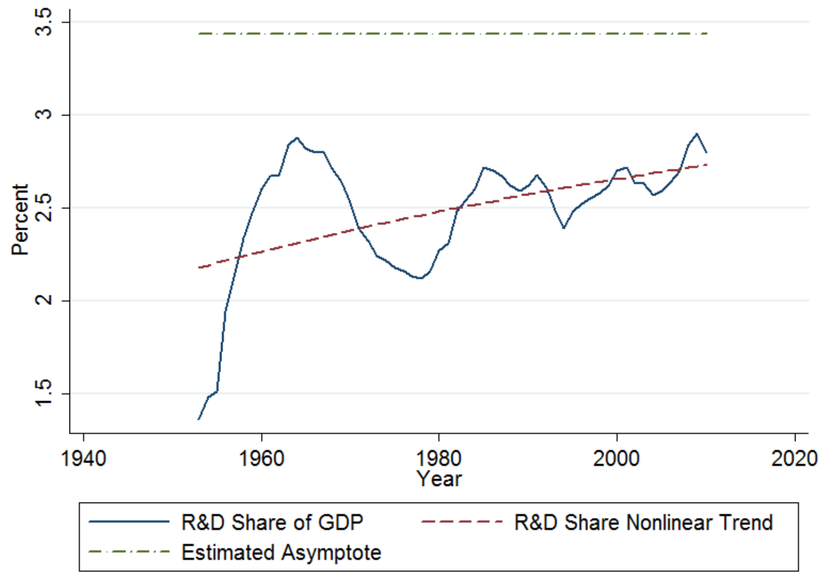

Figure 2.8: R&D Nonlinear Trend and Asymptote

Parameter	Value
ρ	$-.025$
θ	.5
σ	2
λ	1.06
δ	8.3
η	10.3
γ_A	.012

Table 2.1: Calibrated Values

which is estimated using nonlinear least squares. The estimated asymptote is $\hat{b}_0 = .036$, and the nonlinear trend is plotted in Figure 2.8.

The initial and long run research targets imply parameter values $\eta = 10.3$ and $\gamma_A = .012$. These parameter values and initial targets imply the initial condition $K(0) = .09$. The long run growth rate in the economy is $\overline{g}_y = .013$. Thus, exogenous TFP growth accounts for 91% of growth in the very long run, despite accounting for only 34% of growth in the initial period.

2.3.2 Monte Carlo Moment Matching

It can be difficult to ascertain how much confidence to put in the predictions of models such as this one, where all parameters are taken from outside literature or chosen to analytically match estimated moments. The Monte Carlos Moment Matching (MCMM) approach allows a degree of confidence to be placed in the predictions of a structural model. Distributions are assumed for parameters or moments whose values are well established. Then, random draws are sampled from the moment distributions, and the model is solved for each draw. The resulting distribution of model predictions offers information on how robust the benchmark prediction is to its inputs.

4 moments are sampled from: the tail parameter for firm size $\frac{\lambda}{\sigma-1}$, the profit share of income $\frac{\pi}{\psi}$, the long run research and development share $\overline{S}_r$, and the initial interest rate r. The tail parameter for firm size $\frac{\lambda}{\sigma-1}$ is known to be close to one, and Luttmer (2007) estimates it to be 1.06. It is assigned a uniform prior over $[1, 1.12]$. The pure profit share of the economy is typically estimated close to zero (Basu and Fernald (1997)) but an exact estimate is difficult, so $\frac{\pi}{\psi}$ is assigned a uniform distribution over $[.01, .05]$. $\overline{S}_r$ is estimated with 90% confidence between 2.96 and 3.91, so it's assign a truncated normal prior over this interval with standard error .237. Lastly The initial interest rate is relatively stable over the 20th century. It is assigned a uniform distribution over $[.4, .5]$.

Each moment is sampled from independently 20,000 times. The resulting distribution of the long run growth rate g_Y is plotted in *Figure 9*. The distribution is largely centered around the benchmark estimate .013, but not exactly. 95% of the mass is below .015 and 5% is below .010, so there remains considerable uncertainty over the long run nature of the economy.

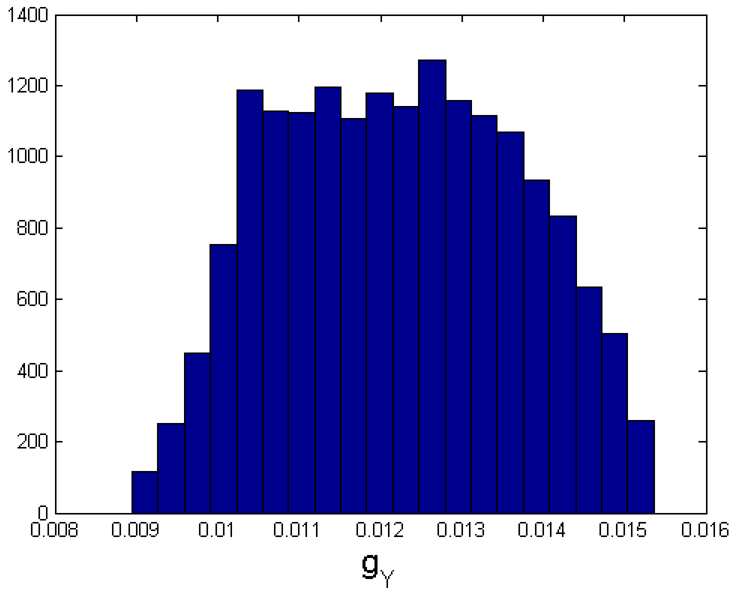

Figure 2.9: MCMM Distribution of Long Run Growth Rates

2.4 Conclusion

This model jointly explains two empirical facts about research and development that are salient in both the time series and cross section: high research and development shares of spending are associated with high incomes; and high research shares are simultaneously associated with lower income growth rates. Thirdly, firms' research decisions are microfounded, and their innovation process is such that the firm size distribution fits Zipf's Law, as observed in the data. Firms' research decisions are motivated by their patent protection, a policy tool which simultaneously increases income growth but suppresses income levels.

The share of income spend on research and development is predicted to continue climbing and approach 3.4%. Per capita GDP growth is expected to fall to around 1.3% in the long run.

CHAPTER 3

GROWTH AND THE RISE AND FALL OF WARFARE

3.1 Introduction

Over one thousand years, warfare has exhibited a hump-shaped pattern. The military employment share of populations rose steady as countries became richer, peaking in the early 20th century, and falling over the past hundred years. This pattern is driven by economic factors, and this paper presents a theory that explains this long-run rise and decline of warfare. The parsimonious model is able to explain variety of time series: the hump shaped military employment shares, the rising military expenditure/employment ratio, and the declining frequency of wars. Most strikingly, the employment and expenditure patterns have the opposite sign in the cross section, which is predicted by the theory.

The fundamental driver of the time series patterns is income growth. When countries are poor, they must spend most of their resources in agriculture to feed the population. As incomes grow, countries are able to spend more income on everything else, including warfare: military employment and expenditure shares both rise. Military personnel and equipment are substitutes in the production of military power, so as productivity rises, countries shift workers to the goods producing sector, raising the military expenditure to employment ratio. In the long run, the military employment share asymptotes to zero, while the expenditure share stabilizes.

Figure 3.1 plots the time series for the military employment to population ratio for an unbalanced panel of 12 European countries over 1,000 years. The data are presented in 25-year bins, using estimates on historical army sizes from Sorokin (1937) and more recent military data from the National Military Capacity Database v4.0.[1]. The time series documents a 900 year rise, followed by a 100 decline in the military employment share. Figure 3.2 plots the share for the same set of countries using annual data since 1900 CE.

1. The 4th version is an update of the original Singer et al. (1972)

The annual data reveal considerable year to year variation, but also the general decline since the 1918 CE peak.

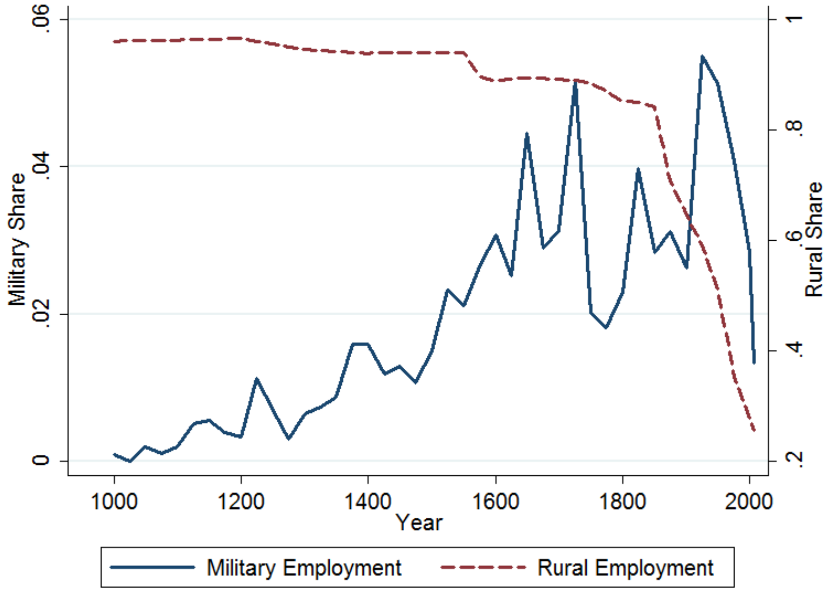

Figure 3.1: Military and Rural Employment Shares

Figure 3.1 also plots the time series for the rural population share of the 12 countries over 1,000 years. The data are estimates from Bairoch et al. (1988) and Bairoch (1991). Countries are predominantly rural and agrian for centuries. As they develop, they urbanize and shift out of agriculture and into other sectors. A large literature on structural change examines this pattern and its relationship with economic growth[2].

Countries have also seen the ratio of their military expenditure share to employment share rise regularly over the past century. For example, Figure 3.3 plots France's ratio since 1914 CE. If soldiers' wages relative to the price of military equipment rise with income, then an increasing ratio of expenditure to employment shares reflect substitution from soldiers towards equipment. This is the central economic force driving the long run decline in military employment in the model.

2. See for example Kuznets (1966), Maddison (1980), Baumol et al. (1985), and Herrendorf et al. (2013) among many others.

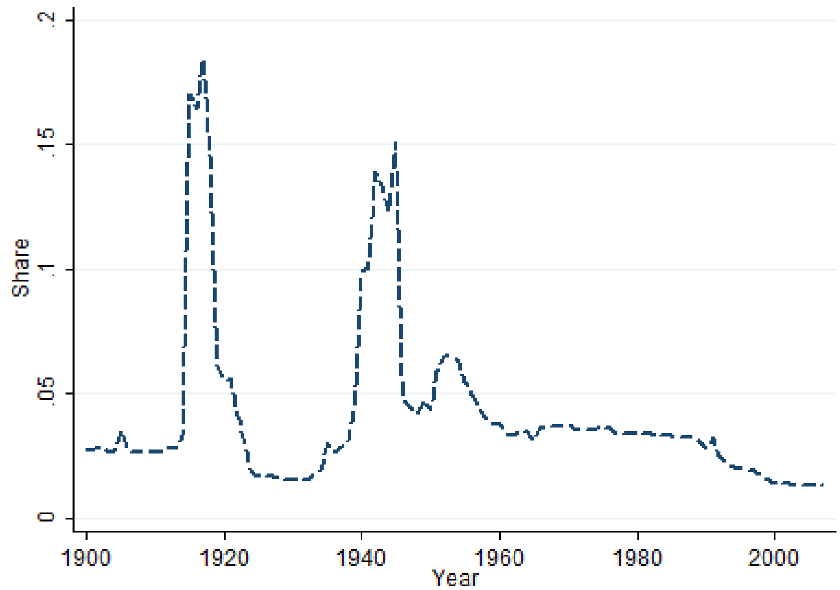

Figure 3.2: 20th Century Military Employment Share

Recent research has made considerable progress on understanding the economic forces behind conflict and warfare. This paper falls in two of these literatures. First, it contribute to the theoretical literature describing the relationship between economic factors and conflict (e.g. Skaperdas (1992), Powell (1993), Yared (2010), and Acemoglu et al. (2012)). Second, it joins a long literature researching economic determinants and consequences of very long run trends in conflict and warfare (e.g. Findlay and O'Rourke (2007), Dincecco and Prado (2012), Arbatli et al. (2015)). Finally, it joins the broader literature of economic determinants of conflict, which Collier and Hoeffler (2007) and Blattman and Miguel (2010) survey.

The theory omits some common ingredients. There is no role for bargaining or transfers between countries to prevent war, which in rational models (e.g. De Mesquita (1985)) can often prevent war. Decision makers make choices in the best interest of their countries, which is not generally realistic. Jackson and Morelli (2007) consider how the political bias of decision makers affects their decisions to lead a country into war, and how this provide incentives for countries to choose biased leaders to gain bargaining position. There is no uncertainty over the state of game, which can increase the prevalence of war, as in Gartzke

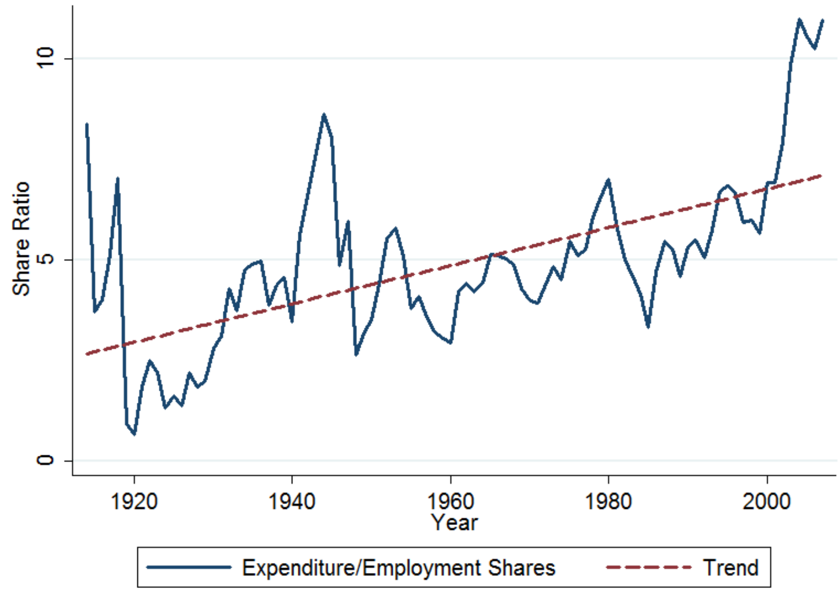

Figure 3.3: French Military Substitution

(1999).

The remainder of this paper is organized as follows: Section 3.2 describes the model environment, Section 3.3 examines the cross-sectional patterns and analyzes asymmetric equilibria, Section 3.4 considers an extension allowing for endogenous warfare and characterizes the implication for the frequency of war, and Section 3.5 concludes.

3.2 Model

In the baseline model, two countries are endowed with productivity and population and always go to war. They solve a static game, which has a Nash equilibrium in pure strategies.

3.2.1 Preferences and Technology

There are two countries, indexed by $j \in 1, 2$. Country j has population N_j and productivity Z_j, which are exogenous. Each country's decisions are made by a government, which maximizes the utility of its surviving citizens. Consumption C_j is distributed equally to all citizens of country j, but D_j citizens die in warfare, so the utility function over C_j and D_j

is

$$(N_j - D_j)u(\frac{C_j}{N_j}) \tag{3.1}$$

The utility function $u(\cdot)$ is strictly increasing and concave.

Citizens serve one of two roles. They can be soldiers or they can be workers. $N_{X,j}$ denotes the number of soldiers and $N_{Y,j}$ denotes the number of workers, which must add up to the population:

$$N_{X,j} + N_{Y,j} = N_j \tag{3.2}$$

Workers have productivity Z_j and produce three types of goods: agriculture A_j, guns G_j, and surplus Y_j. The resource constraint for goods is:

$$Z_j N_{Y,j} = A_j + G_j + Y_j \tag{3.3}$$

Including a different marginal cost for guns will be useful for determining the long run share of spending on guns.

Agriculture is used to feed the population. Each person consumes ν units of agriculture, so the agricultural goods constraint is:

$$A_j = \nu N_j \tag{3.4}$$

As a result, ν behaves like a subsistence constraint.

Guns and soldiers are used to produce military power, X_j, with a production function $f(N_X, G)$. Assume this production function is quasi-convex and has constant returns to scale:

$$X_j = f(N_X, G) \tag{3.5}$$

The substitutability of guns and soldiers in this production function will be crucial for determining the long-run behavior of the economy.

3.2.2 War

The two countries go to war, where military power is used to compete over a share θ of the joint surplus of the two countries. Without loss of generality, label the countries 1 and 2. The contestable share of the joint surplus of the two countries is $\theta(Y_1+Y_2)$. The war function $\Gamma(X_1, X_2)$ determines the share of this surplus that accrues to country 1. $\Gamma(\cdot,\cdot)$ is increasing in the first argument, decreasing in the second argument, bounded by $\Gamma(\cdot,\cdot) \in [0,1]$, and is symmetric so that $\Gamma(X_1, X_2) = 1 - \Gamma(X_2, X_1)$.

A country uses its surplus for consumption, net of war gains or losses. Then the consumption for country 1 is determined by:

$$C_1 = \Gamma(X_1, X_2)\theta(Y_1 + Y_2) + (1 - \theta)Y_1 \tag{3.6}$$

The country faces a trade-off in turning goods into consumption. Spending more on Y_1 increases the total surplus, but decreases the resources that can be spent on X_1, reducing the share of total surplus that is retained.

Lastly, a fraction δ of soldiers die, so that

$$D_j = \delta N_{X,j} \tag{3.7}$$

3.2.3 Equilibrium Definition

A Nash equilibrium in this economy consists of labor allocations $N_{X,j}, N_{Y,j}$; goods allocations A_j, G_j, Y_j, C_j; military powers X_j; and deaths D_j; given populations and productivities Z_j, N_j, such that each country $j \in 1, 2$:

1. Satisfies goods constraints (3.3) and (3.4)

2. Satisfies population constraints (3.2) and (3.7)

3. Satisfies military constraints (3.5) and (3.6)

4. Maximizes its utility (3.1) as the best response to the other country's choices X_i, Y_i, $i \neq j$

3.2.4 Equilibrium Conditions

The country's decision can be rewritten as an unconstrained maximization problem in two variables - guns and soldiers - here expressed from the perspective of country 1:

$$\max_{N_{X,1},G_1} (N_1 - \delta N_{X,1})\times$$
$$u(\frac{(\Gamma(f(N_{X,1},G_1),X_2)\theta + 1 - \theta)(Z_1(N_1 - N_{X,1}) - \nu N_1 - G_1) + \Gamma(f(N_{X,1},G_1),X_2)\theta Y_2}{N_1}))$$

The optimality conditions of this maximization problem are lengthy, but can be written in an intuitive form by substituting in some definitions. I also omit for readability the arguments of the function Γ, its partial derivative with respect to the first argument Γ_1, and marginal return to military from increasing soldiers f_N. The equilibrium condition for soldiers is

$$\Gamma_1 f_N \theta(Y_1 + Y_2) - (\Gamma\theta + 1 - \theta)Z_1 = \delta \frac{u(\frac{C_1}{N_1})}{(1 - \delta\frac{N_{X,1}}{N_1})u'(\frac{C_1}{N_1})} \tag{3.8}$$

On the left hand side is the marginal increase in consumption from increased military power, less the marginal consumption lost from shifting soldier to workers. On the right hand side is the marginal rate of substitution between reducing deaths and increasing consumption. The right hand side is positive, so at the margin, the government could increase consumption by increasing soldiers, but this would entail too costly a marginal increase in casualties.

The equilibrium condition for guns is

$$\Gamma_1 f_G \theta(Y_1 + Y_2) = \Gamma\theta + 1 - \theta \tag{3.9}$$

On the left hand side is the marginal increase in consumption, from an increase in military power, due to an increase in guns. On the right hand side is the marginal consumption given up by shifting output from surplus to guns. The utility function does not appear because neither the costs nor benefits of increasing guns affects soldier deaths.

3.2.5 Functional Forms

Some functional forms are useful to further characterize the baseline economy. The production function for military power is a CES aggregator of guns and soldiers:

$$f(N_X, G) \equiv \left(\alpha N_{X,j}^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) G_j^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} \tag{3.10}$$

The elasticity of substitution ϵ will be a crucial parameter determining the relationship between income growth and warfare.

A useful utility function satisfying the conditions in section 3.2.1 is

$$u(\frac{C}{N}) = \frac{C}{N}^{\sigma} \tag{3.11}$$

The utility function enters the equilibrium conditions through the ratio of utility to marginal utility, so the curvature parameter σ enters only as a constant coefficient in $\frac{u(C/N)}{u'(C/N)} = \frac{C/N}{\sigma}$. This characteristic of power utility constrains curvature in the equilibrium conditions. Alternative utility functions would allow for more flexibility.

Lastly, a functional form is needed for the war function $\Gamma(X_1, X_2)$, satisfying the conditions in section 3.2.2. One such function is that countries receive a share of surplus equal to their share of military power:

$$\Gamma(X_1, X_2) = \frac{X_1}{X_1 + X_2} \tag{3.12}$$

This has the convenient but restrictive property of homogeneity of degree one.

3.2.6 Symmetric Equilibrium

I first examine symmetric equilibria, where $Z_1 = Z_2$ and $N_1 = N_2$. Section 3.3 considers the implications of asymmetric equilibria.

With the assumed functional forms, the equilibrium conditions (3.8) and (3.9) become

$$\frac{2\delta}{(2-\theta)\sigma}\frac{c}{(1-\delta n_X)} = \frac{\alpha}{1-\alpha}(\frac{g}{n_X})^{\frac{1}{\epsilon}} - Z \tag{3.13}$$

$$(1-\alpha)\frac{c}{g} = \frac{\theta + 2(1-\theta)}{\theta}(\frac{x}{g})^{\frac{\epsilon-1}{\epsilon}} \tag{3.14}$$

where lower case letters denote per capita variables, e.g. $c \equiv \frac{C}{N}$. Substituting in $x = f(n_X, g)$, equation (3.14) becomes

$$(1-\alpha)\frac{c}{g} = \frac{2-\theta}{\theta}\left(\alpha(\frac{n_X}{g})^{\frac{\epsilon-1}{\epsilon}} + 1 - \alpha\right) \tag{3.15}$$

Then given productivity Z, the symmetric equilibrium is characterized by three variables (n_X, g, c) and three equations: (3.13), (3.14), and the per capita symmetric budget constraint,

$$Z = \nu + g + c \tag{3.16}$$

3.2.7 Equilibrium in the Limit

To understand how productivity growth affects the economy, it is useful to start by considering equilibrium in the limit as Z becomes large. Define shares of total income $s_C \equiv \frac{c}{Z}$ and $s_G \equiv \frac{g}{Z}$. Shares s_C, s_G, and n_X are all bounded by 0 and 1. Let $(\overline{s_C}, \overline{s_G}, \overline{n_X})$ denote their limits, if they exist.

Express the condition (3.13) in terms of shares and rearrange to get

$$\left(1 + \frac{2\delta}{(2-\theta)\sigma}\frac{s_C}{(1-\delta n_X)}\right)\frac{1-\alpha}{\alpha} = (\frac{Z^{\epsilon-1}s_G}{n_X})^{\frac{1}{\epsilon}} \tag{3.17}$$

If $\epsilon > 1$ and if s_C and s_G are finite and strictly positive in the limit, then for $\lim_{Z\to\infty}(\frac{Z^{\epsilon-1}s_G}{n_X})^{\frac{1}{\epsilon}}$ to be finite and strictly positive, it must be that $\overline{n_X} = 0$. This is why it central that guns and soldiers are substitutes to imply a declining share of soldiers as income grows. Guns become cheap relative to soldiers as productivity grows. If the two inputs are substitutes in the military function, then this price change implies a quantity shift from soldiers towards guns.

Given that $\overline{n_X} = 0$, in the limit condition (3.15) implies $\overline{s_C} = \overline{s_G}\frac{2-\theta}{\theta}$. And from the budget constraint (3.16), we know that $1 = \frac{\nu}{Z} + s_G + s_C$, so in the limit as $Z \to \infty$,

$$1 = \overline{s_G} + \overline{s_C} \tag{3.18}$$

This also implies that as productivity rises, the share of total income spent on agriculture, $\frac{\nu N}{ZN}$ goes to zero. Solving for each share yields $\overline{s_C} = 1 - \frac{\theta}{2}$ and $\overline{s_G} = \frac{\theta}{2}$.

Figure 3.4 plots the symmetric equilibrium for various levels of productivity, to show how the equilibrium evolves as productivity grows from $Z = 1$ to high levels. In this example $\nu = .9$, so initially agriculture is 90% of total income Z. As income grows and a lower share needs to be paid to agriculture, more is spent on consumption, which rises to its' long run share $\overline{s_C}$, which in this example is .5 because $\theta = 1$. Spending on both military inputs rise as a share of income initially as the economy transitions out of agriculture. But eventually the substitution effect starts to dominate, so the soldier share falls to $\overline{n_X} = 0$ while the guns share rises to $\overline{s_G}$.

3.3 Asymmetric Equilibrium

In this section, I relax the assumption of symmetry to explain two facts characterizing the current cross section of countries: military employment shares are increasing in income, while military expenditure shares are constant in income.

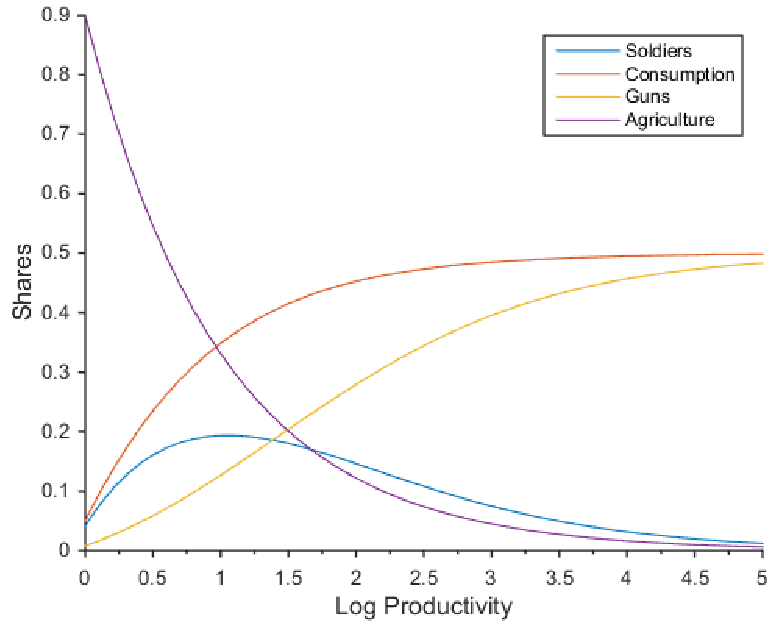

Figure 3.4: Example Symmetric Equilibria as Income Grows

To document these facts, I use data from the National Military Capacity Database v4.0.[3] and the World Development Indicators (World Bank, 2013). Figure 3.5 plots average military employment shares of the population over the past 25 years of available data against average log real per capita income over the same period, for 157 countries. The upward sloping relationship is opposite the time series relationship, where military employment shares fall as income rises, a characteristic of both the model and the data. Figure 3.6 plots military expenditure shares against log real per capita income for the same time period and set of countries. This relationship is constant or decreasing.

Why would the cross-sectional relationship between military shares and income differ from the time series relationship? And why would employment be positively correlated with income but not expenditure? These patterns exist because of the game theoretic nature of the model. Countries incentives are quantitatively different when they are richer or poorer than their neighbors.

3. The 4th version is an update of the original Singer et al. (1972) dataset through 2007, and with more complete coverage.

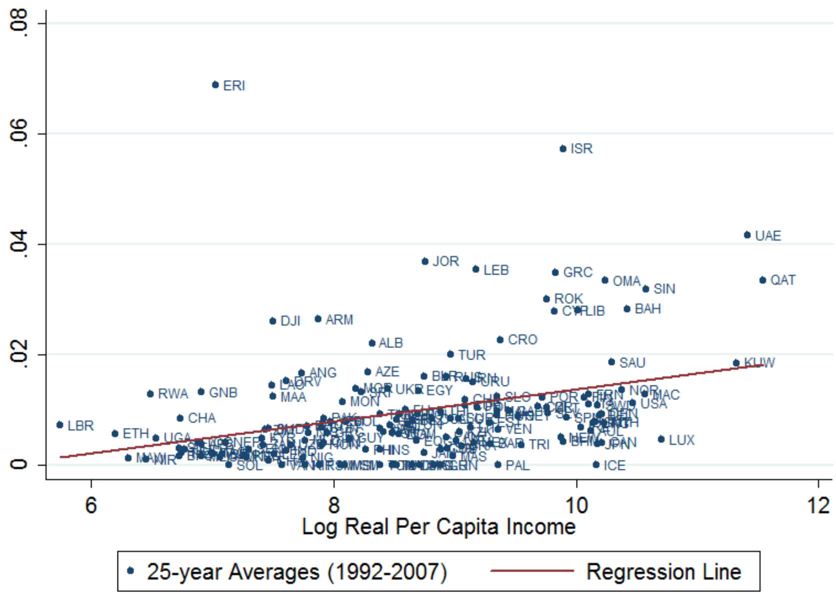

Figure 3.5: Cross-country Military Employment Shares

To characterize the properties of asymmetric equilibria, I approximate the model near the long-run symmetric equilibrium. In general, the asymmetric equilibrium must be calculated numerically, but I can analytically characterize asymmetric behavior for the approximation. This is a reasonable approximation, because most countries in the current cross-section have small[4] employment shares, and are much richer than most points in the historical time series when military employment shares were large.

The key implication of the $N_X \approx 0$ approximation is that it implies $f(N_X, G) = (1-\alpha)^{\frac{\epsilon}{\epsilon-1}} G$. Maintaining the assumption of population symmetry, functional forms, and choosing $\theta = 1$, the equilibrium condition for guns (3.9) is becomes

$$\frac{Z_1 + Z_2}{2} - \nu = G_1 + G_2 \tag{3.19}$$

Rewrite the guns expenditures as shares of total income $G_1 = s_{G,1} Z_1$, assume $\frac{\nu}{Z_1} \approx 0$ because this approximation is near the limiting equilibrium, and define relative income of

4. The sample mean is less than 1%.

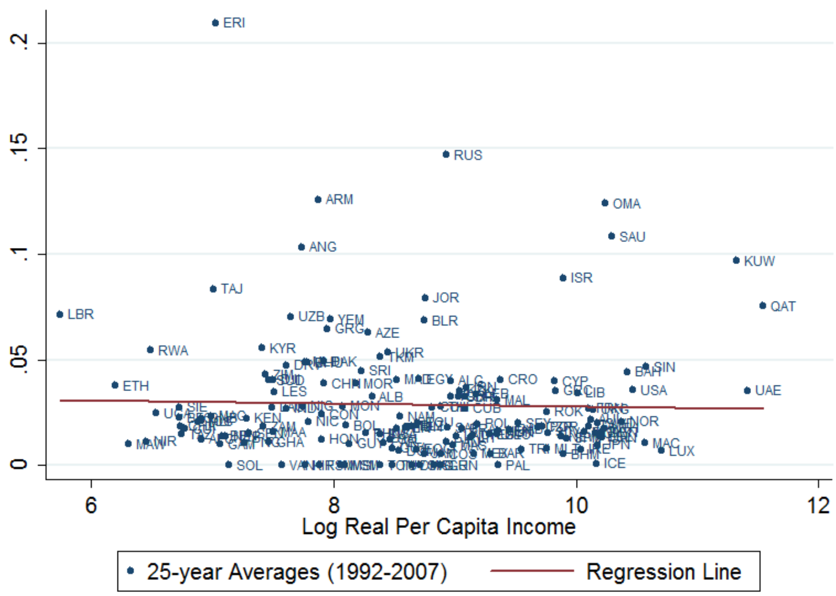

Figure 3.6: Cross-country Military Expenditure Shares

country 2 $z_2 \equiv \frac{Z_2}{Z_1}$ get an expression for country 1's best response function $g(s_{G,2}, z_2)$ to country 2's guns share and relative income:

$$s_{G,1} = g(s_{G,2}, z_2) = \frac{1}{2} + z_2(\frac{1}{2} - s_{G,2}) \tag{3.20}$$

In the symmetric equilibrium with $\theta = 1$, the limiting guns share $\overline{s_G}$ was $\frac{1}{2}$. The best response function confirms that this is the symmetric Nash equilibrium, because the unique share s_G satisfying $s_G = g(s_G, 1)$ is $s_G = \frac{1}{2}$.

More importantly, $s_G = \frac{1}{2}$ is the *only* guns share that is satisfied in this asymmetric equilibrium. This is because the line $s_{G,1} = g(s_{G,2}, z_2)$ intersects the point $(s_{G,1}, s_{G,2}) = (\frac{1}{2}, \frac{1}{2})$ for any positive z_2. No matter the income difference, both countries choose the same guns expenditure share. This result depends on the assumption of $\theta = 1$. More generally, the best response function is

$$s_{G,1} = g(s_{G,2}, z_2; \theta) = \frac{1 + z_2 - 2\frac{1}{\theta}s_{G,2}z_2 + \sqrt{(1 + z_2 - 2\frac{1}{\theta}s_{G,2}z_2)^2 - 4\frac{(1-\theta^2)}{\theta^2}s_{G,2}^2 z_2^2}}{2(\frac{1}{\theta} + 1)} \tag{3.21}$$

The asymmetric Nash Equilibrium then satisfies two equations: $s_{G,1} = g(s_{G,2}, z_2; \theta)$ and $s_{G,2} = g(s_{G,1}, \frac{1}{z_2}; \theta)$. For z_2 near 1, this solution is close to $(s_{G,1}, s_{G,2}) = (\frac{1}{2}, \frac{1}{2})$. But when $z_2 < 1$ (i.e. country 1 is richer), then $s_{G,1} < s_{G,2}$. This case is plotted in figure 3.7, for $\theta = .5$ and a 20% income difference. The richer country chooses a lower expenditure share, suggesting that the cross-sectional relationship could be downward sloping for small enough θ.

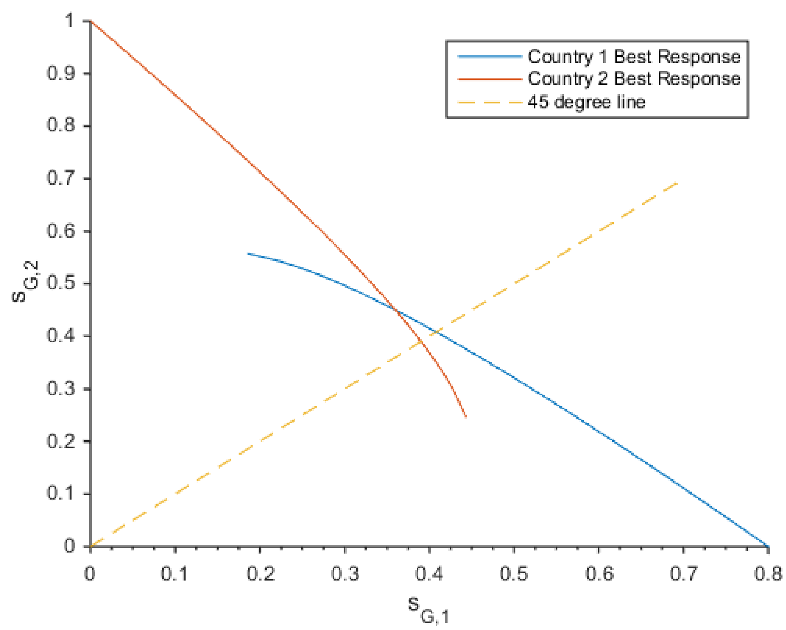

Figure 3.7: Asymmetric Best Response Functions

The employment-income relationship is now analytically characterizable given that $\theta = 1$, because guns expenditure shares are unchanging in income differences. The equilibrium condition for soldiers (3.8) becomes

$$(\frac{\alpha}{1-\alpha}(\frac{1}{2})^{\frac{1}{\epsilon}}(\frac{Z_1}{n_{X,1}})^{\frac{1}{\epsilon}} - Z_1) = \frac{1}{2}\frac{\delta}{\sigma}(Z_1 + Z_2) \tag{3.22}$$

Notice that foreign income Z_2 only appears in this equation on the right hand side, through the marginal rate of substitution between consumption and deaths. Without the utility trade-off, the cross-country employment-income pattern would not appear! Rearrange and substitute for relative income $z_2 \equiv \frac{Z_2}{Z_1}$ to get an expression for soldier share depending on country 1's own income and country 2's relative income:

$$n_{X,1} = \frac{(\frac{\alpha}{1-\alpha})^{\epsilon}\frac{1}{2}Z_1^{1-\epsilon}}{(\frac{1}{2}\frac{\delta}{\sigma} + 1 + \frac{1}{2}\frac{\delta}{\sigma}z_2)^{\epsilon}} \tag{3.23}$$

The soldier share is decreasing in income for $\epsilon > 1$ as in the symmetric equilibrium. But now, the soldier share is also increasing in the other country's relative income z_2. When country 1 is poor relative to country 2, they choose a lower employment share. When they are richer, they choose a higher employment share. This is cross-country relationship in the data, and is opposite the time series relationship, which the model predicts.

3.4 Endogenous Warfare

I now allow governments to decide whether or not they go to war. If at least one government chooses to go to war, a war occurs as in section BASELINE. If neither government decides to go to war, there is peace: no casualties occur, and countries consume their entire surplus. Governments make production decisions after observing whether or not a war will occur, so if there is peace governments have no incentive to build guns or allocate soldiers.

Country j 's utility when there is peace is $N_j u(\frac{Z_j N_j - \nu N_j}{N_j})$. When there is war, it's utility is $(N_j - \delta N^*_{X,j})u(\frac{C^*_j}{N_j}) + N_j\varepsilon_j$ where $N^*_{X,j}$ and C^*_j are the solutions to the country's problem conditional on a war occurring. ε_j is a stochastic utility term that capture's a country's preference for war for reasons exogenous to the model. It is multiplied by N_j so that the exogenous preference for war scales with population like the rest of the utility function.

War occurs if for either county j , $u(Z_j - \nu) < (1 - \delta n^*_{X,j})u(\frac{C^*_j}{N_j}) + \varepsilon_j$. Going to war strictly dominates peace for this country. But, if $u(\frac{Z_j - \nu N_j}{N_j}) \geq (1 - \delta n^*_{X,j})u(\frac{C^*_j}{N_j}) + \varepsilon_j$ for both countries $j \in \{1, 2\}$, then choosing not to go to war is the weakly dominant strategy. In this case, there are two Nash equilibria: both countries choose peace and both countries choose war. But only peace is a trembling hand equilibrium, so I ignore the possibility of war when both countries prefer peace.

Let ε be distributed with cumulative distribution function $F(\varepsilon)$. The probability that country j prefers peace is

$$P_{Peace,j} = F(u(Z_j - \nu) - (1 - \delta n^*_{X,j})u(\frac{C^*_j}{N_j})) \tag{3.24}$$

And if ε_j is distributed independently for each country j , the probability of a war occurring is strictly decreasing in $P_{Peace,1}$ and $P_{Peace,2}$.

There are two forces that lead a country to prefer peace to war, conditional on ε. First, countries have higher utility from consumption in peacetime, as $u(Z_j - \nu) - u(\frac{C^*_j}{N_j})$ is strictly greater than zero. Second, countries don't lose utility from casualties in peacetime, avoiding the utility cost of $\delta n^*_{X,j} u(\frac{C^*_j}{N_j})$.

As income grows, countries' probability of preferring peace increases too. Per capita consumption approaches $\overline{s_C} Z$. So in the limit, the probability of preferring peace approaches one:

$$\lim_{Z_j \to \infty} P_{Peace,j} = \lim_{Z_j \to \infty} F(Z_j^\sigma - (1 - \delta \overline{n_X})(\overline{s_C} Z_j)^\sigma) = 1 \tag{3.25}$$

Both countries almost always prefer peace in the long run, so the probability of a war occurring goes to zero.

3.5 Concluding Remarks

This paper has presented a theory of warfare and economic growth that explains the very long run rise and decline in the military employment share. This pattern is driven by income growth, which incentivizes countries to substitute from military employment towards military equipment. The model predicts that the time series correlations of these values with income are opposite the cross-sectional correlations, and this is confirmed in the data.

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APPENDIX A

URBANIZATION, LONG-RUN GROWTH, AND THE DEMOGRAPHIC TRANSITION

A.1 Proofs

A.1.1 Proof of Proposition 1

In this section I prove that if households are indifferent between urban and rural locations in equilibrium, then their marginal value of human capital is equal in both locations.

Proof. Dynastic utility (1.7) can be expanded into the discounted sum:

$$V_t = \sum_{k=0}^{\infty} \beta^k \frac{(c_{t+k} n_{t+k}^{\phi})^{\sigma}}{\sigma} \tag{A.1}$$

Let $\mathcal{J}$ denote a sequence of location choices, where $\mathcal{J}(t)$ is the sector chosen in period t. Substituting for the household's consumption choice, dynastic utility becomes:

$$V_t = \sum_{k=0}^{\infty} \beta^k \frac{(\tau_c w_{t+k,\mathcal{J}(t+k)} h_{t+k} n_{t+k}^{\phi})^{\sigma}}{\sigma}$$

$$= h_t^{\sigma} \sum_{k=0}^{\infty} \beta^k \frac{(\tau_c w_{t+k,\mathcal{J}(t+k)} \frac{h_{t+k}}{h_t} n_{t+k}^{\phi})^{\sigma}}{\sigma} \tag{A.2}$$

Normalized human capital $\frac{h_{t+k}}{h_t}$ can be expressed in terms of growth rates:

$$\frac{h_{t+k}}{h_t} = \prod_{s=t}^{t+k-1} (1 + g_s) \;\; k \geq 1$$

substituting this expression into (A.2) gives V_t in terms of sequences of wages, locations, choices of n and g, and h_t. Lemma 5 (proved below) says that choices of n and g are independent of h_t. So given these sequences, the utility for a location sequence $\mathcal{J}$ is a function of h, proportional to current human capital to a power:

$$V_{\mathcal{J}}(h) \propto h^{\sigma} \tag{A.3}$$

Now consider two different location sequences $\mathcal{J}$ and $\mathcal{J}'$. Because of the proportionality in (A.3), it is true that:

- If a household is indifferent for some $\hat{h}$, then

$$V_{\mathcal{J}}(h) = V_{\mathcal{J}'}(h) \quad \forall h > 0 \tag{A.4}$$

- If a household strictly prefers $\mathcal{J}$ for some $\hat{h}$, then

$$V_{\mathcal{J}}(h) > V_{\mathcal{J}'}(h) \quad \forall h > 0 \tag{A.5}$$

In equilibrium, households must be indifferent between urban and rural locations for some $\hat{h}$. This follows from the equilibrium property that all households cannot strictly prefer one location, and that the household utility given a particular location decision (equation A.3) is continuous in h.

Let $\mathcal{J}_U$ and $\mathcal{J}_R$ denote optimal location sequences for a household with $\hat{h}$ given a current period choice of urban or rural location respectively. The household is indifferent by definition of $\hat{h}$, so $V_{\mathcal{J}_U}(\hat{h}) = V_{\mathcal{J}_R}(\hat{h})$. Then it follows from (A.4) and (A.5) that households are indifferent between $\mathcal{J}_U$ and $\mathcal{J}_R$ for all $\forall h > 0$, and there is no other sequence of locations that any household strictly prefers.

This sequence indifference implies that for any $\mathcal{J} \in \{\mathcal{J}_U, \mathcal{J}_R\}$:

$$V_{\mathcal{J}}(h_t) = h_t^{\sigma} \sum_{k=0}^{\infty} \beta^k \frac{(\tau_c w_{t+k,\mathcal{J}(t+k)} \prod_{s=t}^{t+k-1} (1+g_s) n_{t+k}^{\phi})^{\sigma}}{\sigma} \tag{A.6}$$

$$\equiv h_t^{\sigma} \mathcal{V} \tag{A.7}$$

Thus the marginal value of human capital is equalized in both locations:

$$V'_{\mathcal{J}}(h_t) = \sigma h_t^{\sigma-1}\mathcal{V} \quad \forall \mathcal{J} \in \{\mathcal{J}_U, \mathcal{J}_R\} \tag{A.8}$$

■

Lemma 5 *Given a series of wages $w_{t,j}$, survival rates $S_{t,j}$, and sequence of locations $\mathcal{J}(t)$, a dynasty's choice of children n_t and human capital growth g_t is independent of its level of human capital h_t.*

The central assumption driving this result is the homotheticity of the balanced growth compatible preferences.

Proof. The combined budget constraint (1.13) and equilibrium choice of consumption $c = \tau_c w_j h$ imply that the budget constraint can be normalized by dividing by $w_j h$:

$$\tau_c + \frac{gn}{\xi} + \frac{n}{\alpha S_j} = 1$$

and recall that $\tau_c = \frac{1}{1+\phi}$ is constant. This normalized budget constraint and the Euler equation (1.21) jointly characterize the household's equilibrium behavior, and neither depends on the level of h. ■

A.1.2 Proof of Proposition 2

In this section I prove that if $\lim_{t\to\infty} \bar{h} = \infty$, then the limiting urban-rural wage premium is $\frac{w_U}{w_R} \to 1$.

Proof.

Suppose that $S_R = S_U$ but $w_R < w_U$. Consider the optimal rural allocations (c_R, n_R, h'_R) given w_R and S_R. A household could choose to live in the urban area and, per the combined budget constraint (1.13), would be able to afford the allocation $(\tilde{c_U}, n_R, h'_R)$ where $\tilde{c_U} > c_R$. Thus they would strictly prefer the urban location and this could not be an equilibrium.

Similarly, if $w_R > w_U$ then an urban household could switch to a rural location and be strictly better off. The only possible equilibrium given $S_R = S_U$ must have $w_R = w_U$.

By assumption $\lim_{\bar{h}\to\infty} S_j(\bar{h}) = \bar{S}$ for all j. So in the limit, it must be that $\frac{w_U}{w_R} \to 1$. ■

A.1.3 Proof of Proposition 3

In this section I prove that if $\lim_{t\to\infty} \bar{h} = \infty$, $\lim_{t\to\infty} n \geq 1$ and $\epsilon > 1$, then the long-run urban share converges to 1.

Proof.

The limits for $\bar{h}$ and n imply that aggregate human capital $H = N\bar{h}$ is growing in the long run: $\lim_{t\to\infty} H = \infty$.

Use the equilibrium prices in equations (1.28) and (1.29) to express the wage premium as:

$$\frac{w_U}{w_R} = \frac{H_R^{1-\theta}\zeta x_R^{\frac{1}{\epsilon}}}{\theta L^{1-\theta}(1-\zeta)x_U^{\frac{1}{\epsilon}}}$$

Then substitute with the sectoral production functions to express the wage premium in terms of human capital inputs:

$$\frac{w_U}{w_R} = \frac{H_R^{1-\theta(1-\frac{1}{\epsilon})}\zeta}{\theta L^{(1-\theta)(1-\frac{1}{\epsilon})}(1-\zeta)H_U^{\frac{1}{\epsilon}}}$$

Aggregate human capital supplied is $\tau_c H$. The urban share of aggregate human capital is s_U. Substituting and rearranging gives:

$$\frac{w_U}{w_R}(\tau_c H)^{(\frac{1}{\epsilon}-1)(1-\theta)} = \frac{(1-s_U)^{1-\theta(1-\frac{1}{\epsilon})}\zeta}{s_U^{\frac{1}{\epsilon}}\theta L^{(1-\theta)(1-\frac{1}{\epsilon})}(1-\zeta)}$$

The agricultural labor share θ is between 0 and 1 by assumption, so if $\epsilon > 1$ then the left hand side of this equation is decreasing in H, and the right hand side is decreasing in s_U. Proposition 2 says that in the limit $w_U = w_R$, so if $H \to \infty$, the limit of the left hand side of this equation is zero. The right hand side is positive and decreasing in the urban share for $s_U \in (0,1)$, and

$$\lim_{s_U \to 1-} \frac{(1-s_U)^{1-\theta(1-\frac{1}{\epsilon})}\zeta}{s_U^{\frac{1}{\epsilon}}\theta L^{(1-\theta)(1-\frac{1}{\epsilon})}(1-\zeta)} = 0$$

So it must be that $s_U \to 1$. ■

A.1.4 Proof of Proposition 4

In this section I prove that if $\lim_{t\to\infty} \bar{h} = \infty$, $\lim_{t\to\infty} n \geq 1$ and $\epsilon > 1$, then the the limit of both urban and rural wages is $\bar{w} \equiv A\zeta^{\frac{\epsilon}{\epsilon-1}}$.

Proof. Use the final good production function (1.3) and equilibrium prices in equations (1.28) and (1.29) to express the equilibrium urban wage as:

$$w_U = A^{\frac{\epsilon-1}{\epsilon}}\zeta \left(\frac{A(\zeta x_U^{\frac{\epsilon-1}{\epsilon}} + (1-\zeta)x_R^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}}}{x_U} \right)^{\frac{1}{\epsilon}}$$

Substitute for intermediate inputs and express human capital inputs in terms of aggregate human capital and the urban share s_U:

$$w_U = A^{\frac{\epsilon-1}{\epsilon}}\zeta \left(\frac{A(\zeta(\tau_c s_U H)^{\frac{\epsilon-1}{\epsilon}} + (1-\zeta)((\tau_c(1-s_U)H)^{\theta}L^{1-\theta})^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}}}{\tau_c s_U H} \right)^{\frac{1}{\epsilon}}$$

Take the limit, given that the limits for $\bar{h}$ and n imply $H \to \infty$ and Proposition 3 implies $s_U \to 1$:

$$\lim_{t\to\infty} w_U = \lim_{t\to\infty} A^{\frac{\epsilon-1}{\epsilon}}\zeta \left(\frac{A(\zeta(\tau_c s_U H)^{\frac{\epsilon-1}{\epsilon}} + (1-\zeta)((\tau_c(1-s_U)H)^{\theta}L^{1-\theta})^{\frac{\epsilon-1}{\epsilon}})^{\frac{\epsilon}{\epsilon-1}}}{\tau_c s_U H} \right)^{\frac{1}{\epsilon}}$$

$$= A\zeta^{\frac{\epsilon}{\epsilon-1}} \equiv \bar{w}$$

■

A.2 Survival Function

In this section I describe the estimation of the survival function. The one parameter version specification of the survival function is a transformed logistic cdf:

$$S_j(\bar{h}) = \bar{S} - (\bar{S} - S_{j,0})\frac{1 + \upsilon\bar{h}_0}{1 + \upsilon\bar{h}}$$

This function is able to hit both the initial target $S_{j,0}$ and the long run limit $\bar{S}$. It has all the desired properties: it is strictly increasing in $\bar{h}$, bounded by $[0, \bar{S}]$, and has finite limits as $\bar{h} \to 0$ and $\bar{h} \to \infty$.

The targets for $S_{R,0}$ and $S_{U,0}$ are from Clark (2009). I estimate the survival equation using nonlinear least squares. Child mortality data is from Johansson et al. (2015), and average income is used to approximate average human capital. Non-linear least squares gives $\upsilon = 0.35$ when $\bar{h}_0$ is normalized to one. Figure A.1 plots England's mortality data, income, and the fitted survival function given the year's income level.

A.3 Computation

In this section I describe my method of calculating the equilibrium. The strategy is to express the equilibrium allocation for each period t as a function of the rural choice of children $n_{R,t}$, and express the next period's choice $n_{R,t+1}$ as a function of period t variables. Then, an initial guess for $n_{R,0}$ is chosen, and a shooting algorithm is used to find the equilibrium value of $n_{R,0}$ and the following equilibrium allocations for all t.

First, it is useful to rewrite the location indifference condition (1.30) in terms of allocations instead of wages. This equation says that the right hand side of the Euler equation for urban and rural households is equal. Thsi implies that the left hand side is also equal, so substituting with equation (1.21) implies:

$$w_R^{\sigma} n_R^{\sigma\phi+1}(1 + g_R)^{1-\sigma} = w_U^{\sigma} n_U^{\sigma\phi+1}(1 + g_U)^{1-\sigma} \tag{A.9}$$

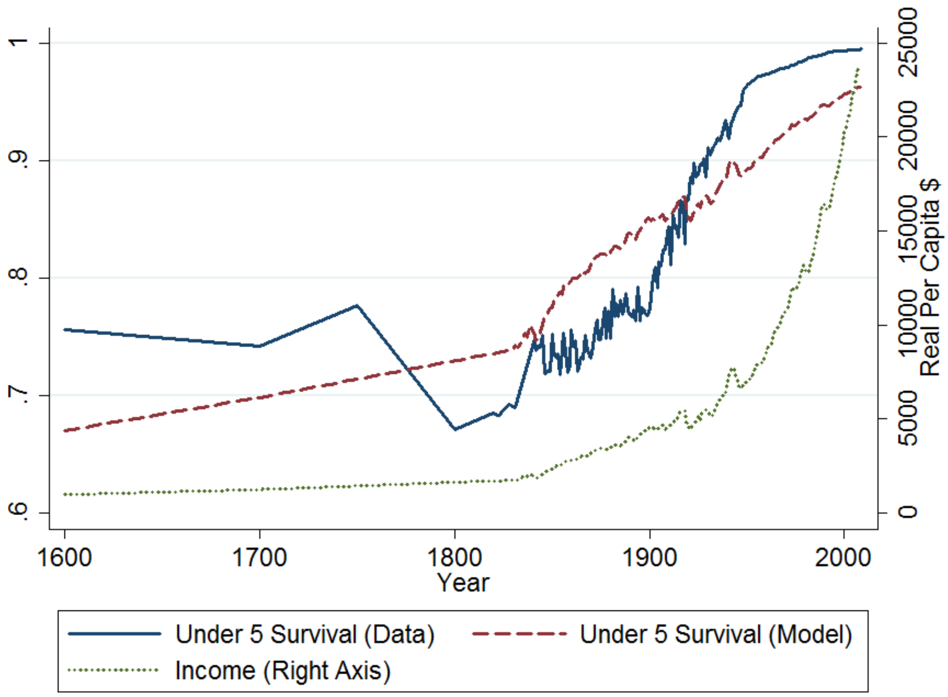

Figure A.1: Empirical and Estimated Survival Rates

Then dividing equation (1.30) by equation (A.9) yields:

$$\frac{\frac{1}{n_R} + \frac{1}{\xi} - \frac{1}{\alpha S_R}}{(1+g_R)^{1-\sigma}} = \frac{\frac{1}{n_U} + \frac{1}{\xi} - \frac{1}{\alpha S_U}}{(1+g_U)^{1-\sigma}} \tag{A.10}$$

Next, combine equation (A.10) with the normalized budget constraint (1.35) to yield an equation relating n_U, n_R, S_U, S_R, and parameters:

$$\frac{\frac{1}{n_R} + \frac{1}{\xi} - \frac{1}{\alpha S_R}}{(1+\xi\frac{1-\tau_c}{n_R} - \frac{\xi}{\alpha S_R})^{1-\sigma}} = \frac{\frac{1}{n_U} + \frac{1}{\xi} - \frac{1}{\alpha S_U}}{(1+\xi\frac{1-\tau_c}{n_U} - \frac{\xi}{\alpha S_U})^{1-\sigma}} \tag{A.11}$$

The shooting algorithm proceeds as follows. Guess a value of $n_{R,0}$. In period t, $n_{R,t}$, $S_{R,t}$, $S_{U,t}$, and the distribution of human capital Λ_t are known. In period $t = 0$, $n_{R,0}$ is a guess, and $S_{R,0}$ and $S_{U,0}$ are calculated from the initial condition for Λ_0.

1. Numerically solve equation (A.11) for $n_{U,t}$. If the implied value of n_U is infeasible, the urban households must be constrained and their Euler equation doesn't hold, so set

$n_U = (1 - \tau_c)\alpha S_{U,t}$

2. Analytically solve the normalized budget constraints (1.35) for $g_{R,t}$ and $g_{U,t}$.

3. Calculate the wage premium $\frac{w_{U,t}}{w_{R,t}}$ from the indifference condition (1.30).

4. Numerically calculate the aggregate human capitals supplied $H_{R,t}$ and $H_{U,t}$ that are consistent with the wage ratio and the aggregate human capital supplied implied by Λ_t.

5. Analytically calculate the wages $w_{R,t}$ and $w_{U,t}$ implied by $H_{R,t}$ and $H_{U,t}$ using the equations for equilibrium prices (1.28) and (1.29).

6. Calculate next period's distribution of human capital Λ_{t+1} from the law of motion (1.25).

7. Use Λ_{t+1} to calculate next period's average human capital level and find $S_{R,t+1}$ and $S_{U,t+1}$ from equation (1.33).

8. Solve numerically for $n_{R,t+1}$:

   (a) Express the next period's wage in location j as a function of $n_{j,t+1}$ through the Euler equation (1.21)

   (b) Express next period's human capitals supplied $H_{R,t+1}$ and $H_{U,t+1}$ as functions of $n_{R,t+1}$ and $n_{U,t+1}$, using the equations for equilibrium prices (1.28) and (1.29).

   (c) Numerically find the values of $n_{R,t+1}$ and $n_{U,t+1}$ that imply values of $H_{R,t+1}$ and $H_{U,t+1}$ that are consistent with Λ_{t+1}.

   (d) If $n_{U,t+1}$ is infeasible, urban households must be constrained, so repeat steps (b) and (c) assuming $n_{U,t+1} = (1 - \tau_c)\alpha S_{U,t+1}$.

9. Return to step 1. for period $t + 1 \leq T$.

Period T approximates the long run. If the calculated long run rural children $n_{R,T}$ is within tolerance ε to the equilibrium long run value $\bar{n}$, consider the equilibrium solved. Otherwise, for $n_{R,T} > \bar{n} + \varepsilon$ revise the initial guess downwards, and for $n_{R,T} < \bar{n} + \varepsilon$ revise the initial guess upwards.

APPENDIX B

DECREASING RETURNS TO R&D AND DECLINING GROWTH RATES

B.1 Solving the Household's Problem

The household's Hamiltonian

$$H(a,c) = u(c(t)) + \mu(t)(w(t) + r(t)a(t) - c(t))$$

$$H_c : \;\; e^{-\rho t}u'(c(t)) = \mu(t)$$

$$H_a : \;\; \mu(t)r(t) + \dot{\mu}(t) = \rho\mu(t)$$

$$\dot{a}(t) = w(t) + r(t)a(t) - c(t)$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) - \rho$$

$$r(t) = \rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t)$$

with $u(c) = \frac{\theta}{\theta-1}c^{\frac{\theta-1}{\theta}}$ we get $u'(c) = c^{-\frac{1}{\theta}}$ and $u''(c) = -\frac{1}{\theta}c^{-\frac{1}{\theta}-1}$. With this specification, the Euler equation becomes

$$r(t) = \rho + \frac{1}{\theta}\frac{\dot{c}(t)}{c(t)}$$

B.2 Solving the Monopolist's Problem

Monopolist Labor Demand:

Rearranging the monopolist's FOC, we get

$$A^{\sigma-1}Y(\frac{\sigma-1}{\sigma}\frac{z_j}{w})^{\sigma} = x_j$$

substituting the production function $x_j = z_j l_j$ we obtain

$$l_j = A^{\sigma-1} Y (\frac{\sigma-1}{\sigma w})^{\sigma} z_j^{\sigma-1}$$

Monopolists' Profit:

$$p_j x_j = A^{\sigma-1} Y (\frac{\sigma-1}{\sigma} \frac{z_j}{w})^{\sigma-1}$$

$$l_j = A^{\sigma-1} Y (\frac{\sigma-1}{\sigma w})^{\sigma} z_j^{\sigma-1}$$

$$p_j x_j - w l_j = A^{\sigma-1} Y (\frac{\sigma-1}{\sigma} \frac{z_j}{w})^{\sigma-1} - w A^{\sigma-1} Y (\frac{\sigma-1}{\sigma w})^{\sigma} z_j^{\sigma-1}$$

$$= A^{\sigma-1} Y (\frac{z_j}{w})^{\sigma-1} ((\frac{\sigma-1}{\sigma})^{\sigma-1} - (\frac{\sigma-1}{\sigma})^{\sigma})$$

$$= A^{\sigma-1} Y (\frac{z_j}{w})^{\sigma-1} (\frac{\sigma-1}{\sigma})^{\sigma-1} (1 - \frac{\sigma-1}{\sigma})$$

$$\Pi_j = A^{\sigma-1} Y (\frac{z_j}{w})^{\sigma-1} (\frac{\sigma-1}{\sigma})^{\sigma-1} \frac{1}{\sigma}$$

B.3 Solving the Innovator's Problem

The innovator's objective is to maximize present value dividends and discounts by the interest rate $r(t)$ to ensure no arbitrage, so its current-value Hamiltonian is

$$H(b, z_m) = k(t) z_m(t)^{\sigma-1} d(t) - b(t) + \mu(t) \frac{b(t)}{k(t) z_m(t)^{\eta}}$$

with first order conditions

$$H_b : \quad k(t) z_m^{\eta}(t) = \mu(t)$$

$$H_{z_m} : \quad (\sigma-1) z_m^{\sigma-2}(t) k(t) d(t) - \mu(t) k(t) \eta \frac{b(t)}{z(t)_m^{\eta+1}} = r(t)\mu(t) - \dot{\mu(t)}$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = \eta \frac{\dot{z_m}(t)}{z_m(t)} - \frac{\dot{k}(t)}{k(t)}$$

$$(\sigma-1)d(t)z_m(t)^{\sigma-2-\eta} - \frac{\dot{z_m}(t)}{z_m(t)} = r(t) + \frac{\dot{k}(t)}{k(t)} - \eta\frac{\dot{z_m}(t)}{z_m(t)}$$

and using the law of motion $\frac{\dot{k}(t)}{k(t)} = -\delta(t) - \lambda\frac{\dot{z_m}(t)}{z_m(t)}$ yields

$$(\sigma-1)d(t)z_m(t)^{\sigma-2-\eta} = r(t) + \delta(t) + (\lambda+1-\eta)\frac{\dot{z_m}(t)}{z_m(t)}$$

B.4 The Pareto Distributions

The CDF of Pareto random variable z with minimum z_m and tail parameter λ is

$$G(z; z_m, \lambda) = 1 - (\frac{z_m}{z})^{\lambda}$$

Taking the term in parentheses to the powers $\sigma - 1$ and $\frac{1}{\sigma-1}$ we get

$$G(z; z_m, \lambda) = 1 - (\frac{z_m^{\sigma-1}}{z^{\sigma-1}})^{\frac{\lambda}{\sigma-1}}$$

So $z^{\sigma-1}$ is Pareto with minimum $z_m^{\sigma-1}$ and tail parameter $\frac{\lambda}{\sigma-1}$. The expected value is

$$E[z^{\sigma-1}] = \frac{\frac{\lambda}{\sigma-1}}{\frac{\lambda}{\sigma-1} - 1} z_m^{\sigma-1}$$

$$= \frac{\lambda}{\lambda+1-\sigma} z_m^{\sigma-1}$$

Multiplying the numerator and denominator by a positive scalar ϵ, we get

$$G(z; z_m, \lambda) = 1 - (\frac{\epsilon z_m^{\sigma-1}}{\epsilon z^{\sigma-1}})^{\frac{\lambda}{\sigma-1}}$$

So $\epsilon z^{\sigma-1}$ is Pareto with minimum $\epsilon z_m^{\sigma-1}$ and tail parameter $\frac{\lambda}{\sigma-1}$.

Accordingly, if z_j is Pareto with minimum z_m and tail parameter λ, then revenue

$$p_j x_j = Y(\frac{\sigma-1}{\sigma}\frac{z_j}{w})^{\sigma-1}$$

is Pareto with minimum $Y(\frac{\sigma-1}{\sigma}\frac{z_m}{w})^{\sigma-1}$ and tail parameter $\frac{\lambda}{\sigma-1}$, and labor demanded

$$l_j = wY(\frac{\sigma-1}{\sigma w})^{\sigma} z_j^{\sigma-1}$$

is Pareto with minimum $Y(\frac{\sigma-1}{\sigma})^{\sigma}(\frac{z_m}{w})^{\sigma-1}$ and tail parameter $\frac{\lambda}{\sigma-1}$.

B.5 Deriving the Law of Motion for a Measure of Technologies

The law of motion for $k(t)$ over Δ time is

$$k(t+\Delta) = k(t)(\frac{z_m(t)}{z_m(t+\Delta)})^{\lambda} e^{-\delta(t)\Delta}$$

Taking logs, we get

$$\ln k(t+\Delta) - \ln k(t) = -\delta(t)\Delta - \lambda(\ln z_m(t+\Delta) - \ln z_m(t))$$

$$\frac{\ln k_{t+\Delta} - \ln k_t}{\Delta} = -\delta(t) - \lambda \frac{\ln z_m(t+\Delta) - \ln z_m(t)}{\Delta}$$

then taking the limit as $\Delta \to 0$:

$$\frac{d \ln k(t)}{dt} = \frac{\dot{k}(t)}{k(t)} = -\delta(t) - \lambda \frac{\dot{z_m}(t)}{z_m(t)}$$

B.6 Calculating Some Aggregates

$$Y = A\left(\int x_j^{\frac{\sigma-1}{\sigma}} dj\right)^{\frac{\sigma}{\sigma-1}} = A\left(M \int_{j \in J_M} x_j^{\frac{\sigma-1}{\sigma}} dj + (1-M) \int_{j \in J_C} x_j^{\frac{\sigma-1}{\sigma}} dj\right)^{\frac{\sigma}{\sigma-1}}$$

$$= A\left(M\int_{j\in J_M}(A^{\sigma-1}Y(\frac{\sigma-1}{\sigma}\frac{z_j}{w})^{\sigma})^{\frac{\sigma-1}{\sigma}}dj + (1-M)\int_{j\in J_C}(A^{\sigma-1}Y(\frac{z_j}{w})^{\sigma})^{\frac{\sigma-1}{\sigma}}dj\right)^{\frac{\sigma}{\sigma-1}}$$

$$= A^{\sigma}\left(MY^{\frac{\sigma-1}{\sigma}}(\frac{\sigma-1}{\sigma})^{\sigma-1}w^{1-\sigma}\int_{j\in J_M} z_j^{\sigma-1}dj + (1-M)Y^{\frac{\sigma-1}{\sigma}}w^{1-\sigma}\int_{j\in J_C} z_j^{\sigma-1}dj\right)^{\frac{\sigma}{\sigma-1}}$$

$$1 = A^{\sigma}\left(M(\frac{\sigma-1}{\sigma})^{\sigma-1}w^{1-\sigma}\int_{j\in J_M} z_j^{\sigma-1}dj + (1-M)w^{1-\sigma}\int_{j\in J_C} z_j^{\sigma-1}dj\right)^{\frac{\sigma}{\sigma-1}}$$

using that $\int_{j\in J_M} z_j^{\sigma-1}dj = \int_{j\in J_C} z_j^{\sigma-1}dj = \frac{\lambda}{\lambda+1-\sigma}Z^{\sigma-1}$ where $Z \equiv \min(z_j)$ we get

$$1 = A^{\sigma}\left(M(\frac{\sigma-1}{\sigma})^{\sigma-1}w^{1-\sigma}\frac{\lambda}{\lambda+1-\sigma}Z^{\sigma-1} + (1-M)w^{1-\sigma}\frac{\lambda}{\lambda+1-\sigma}Z^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}}$$

$$1 = (\frac{AZ}{w})^{\sigma}(\frac{\lambda}{\lambda+1-\sigma})^{\frac{\sigma}{\sigma-1}}\left(M(\frac{\sigma-1}{\sigma})^{\sigma-1} + (1-M)\right)^{\frac{\sigma}{\sigma-1}}$$

$$w = ZA(\frac{\lambda}{\lambda+1-\sigma})^{\frac{1}{\sigma-1}}\left(M(\frac{\sigma-1}{\sigma})^{\sigma-1} + (1-M)\right)^{\frac{1}{\sigma-1}} \equiv \omega ZA$$

Total monopolist profits are

$$\Pi(t) = \int_{j\in J_M}\Pi_j dj = \int_{j\in J_M} Y(\frac{Az_j}{w})^{\sigma-1}(\frac{\sigma-1}{\sigma})^{\sigma-1}\frac{1}{\sigma}dj$$

$$= Y(t)(\frac{A(t)}{w(t)}\frac{\sigma-1}{\sigma})^{\sigma-1}\frac{1}{\sigma}\int_{j\in J_M} z_j^{\sigma-1}dj$$

$$= Y(t)(\frac{A(t)}{w(t)}\frac{\sigma-1}{\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}MZ(t)^{\sigma-1}$$

substituting with the wage equation:

$$= Y(t)(\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M$$

Aggregate labor supply is one, so by Walras' Law

$$Y(t) = w(t) + \Pi(t) = w(t) + Y(t)(\frac{\sigma - 1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda + 1 - \sigma}M$$

$$Y(t) = \frac{w(t)}{1 - (\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M}$$

$$= \frac{\omega Z(t)A(t)}{1 - (\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M}$$

$$Y(t) = \psi Z(t)A(t)$$

$$\psi \equiv \frac{\omega}{1 - (\frac{\sigma-1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda+1-\sigma}M}$$

and substituting this back into the profit equation

$$\Pi(t) = \pi Z(t)A(t)$$

$$\pi \equiv \psi(\frac{\sigma - 1}{\omega\sigma})^{\sigma-1}\frac{1}{\sigma}\frac{\lambda}{\lambda + 1 - \sigma}M$$