

Phase Transitions in Nonreciprocal Driven-Dissipative Condensates

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We investigate the influence of boundaries and spatial nonreciprocity on nonequilibrium driven-dissipative phase transitions. We focus on a one-dimensional lattice of nonlinear bosons described by a Lindblad master equation, where the interplay between coherent and incoherent dynamics generates nonreciprocal interactions between sites. Using a mean-field approach, we analyze the phase diagram under both periodic and open boundary conditions. For periodic boundaries, the system always forms a condensate at nonzero momentum and frequency, resulting in a time-dependent traveling wave pattern. In contrast, open boundaries reveal a far richer phase diagram, featuring multiple static and dynamical phases, as well as exotic phase transitions, including the spontaneous breaking of particle-hole symmetry associated with a critical exceptional point and phases with distinct bulk and edge behavior. Our model does not require postselection and is experimentally realizable in platforms such as superconducting circuits.

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Introduction—Driven-dissipative many-body systems have emerged as a versatile platform for exploring novel nonequilibrium phenomena that are inaccessible in equilibrium. These systems exhibit a rich variety of exotic phases of matter, phase transitions, and unique dynamical behaviors [1]. Much of the interest is driven by the rise of programmable quantum simulators across various platforms, including exciton-polariton systems in semiconductors [2], ultracold atomic systems [3], photonic platforms [4,5], and superconducting circuits [6], where both coherent interactions and dissipation can be engineered.

Recent years have seen a surge of interest in nonreciprocally interacting many-body systems. Prototypical examples often consist of multispecies systems where the interactions between different species are asymmetric [7–14]. Nonreciprocal interactions have also been shown to give rise to exotic nonequilibrium phases of matter, such as long-range order in two spatial dimensions [15–17], time-crystalline order [8,10–12,14], nonequilibrium boundary modes [18], and chaotic phases [19]. These phenomena have attracted significant attention in diverse fields such as active matter [20], biology [21], and neural networks [22],

with explorations of analogous quantum systems also beginning to emerge [23–27].

Another example of nonreciprocity arises when a system exhibits preferred spatial directionality [28–33]. A paradigmatic example is the Hatano-Nelson model [34,35], a one-dimensional system governed by a non-Hermitian Hamiltonian with asymmetric left and right hopping rates. This asymmetry causes exponential localization of all eigenstates at one boundary under open boundary conditions (OBC), while under periodic boundary conditions (PBC), the eigenstates remain extended—a phenomenon known as the non-Hermitian skin effect (NHSE) [36–38]. While recent studies have explored nonlinear extensions of such models [27,31,39–45], the impact on phase transitions and the specific role of boundaries, particularly beyond the framework of non-Hermitian Hamiltonians, remains largely underexplored.

In this work, we investigate the interplay of nonreciprocity, interactions, and boundary conditions on nonequilibrium phase transitions in a minimal driven-dissipative condensate. Driven dissipative condensates have been extensively studied across a range of systems [2,46–56]. Recent theoretical [57] and experimental [58] works have begun to explore condensates in nonreciprocal settings, but the impact of nonreciprocity on phase transitions remains unexplored. We focus on a mean-field treatment of a one dimensional condensate described by a Lindblad master equation, and find that the steady state is highly sensitive to boundary conditions. While periodic boundary conditions lead to a single phase characterized by a finite momentum

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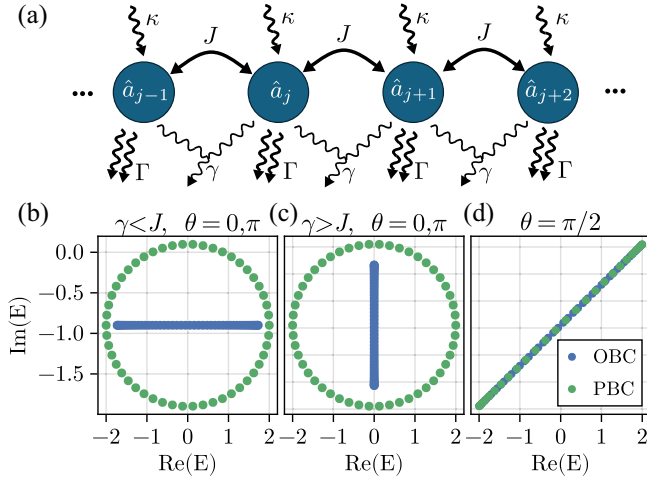


FIG. 1. (a) Schematic illustration of the system. (b), (c) Spectrum of the Hatano-Nelson Hamiltonian Eq. (1) in the regime of maximal nonreciprocity. (d) Spectrum in the reciprocal case.

and frequency condensate, open boundaries result in a rich variety of static and dynamical phases. These include exotic phase transitions associated with the breaking and dynamical restoration of particle-hole symmetry, boundary-induced critical exceptional points [9,12,13,24,25,59–63], and regimes with qualitatively distinct edge and bulk dynamics.

Model—We consider a chain (or a ring) of bosonic sites [see Fig. 1(a)] described by a quantum master equation [64] $\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_{\mu=1,2,3} \sum_j \mathcal{D}[\hat{L}_j^{(\mu)}] \hat{\rho}$ where the Hamiltonian, $\hat{H} = -J \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j)$ describes hoppings with rate J , where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the site- j creation and annihilation operators. The dissipators $\mathcal{D}[\hat{L}] \hat{\rho} = \hat{L}^\dagger \hat{\rho} \hat{L} - \frac{1}{2} \{ \hat{L}^\dagger \hat{L}, \hat{\rho} \}$ describe three types of incoherent processes: single particle pumping with rate κ ($\hat{L}_j^{(1)} = \sqrt{2\kappa} \hat{a}_j^\dagger$), two particle decay with rate Γ ($\hat{L}_j^{(2)} = \sqrt{2\Gamma} \hat{a}_j^2$), and correlated single particle decay with rate γ [$\hat{L}_j^{(3)} = \sqrt{2\gamma} (\hat{a}_j - ie^{i\theta} \hat{a}_{j+1})$].

At the quadratic level (i.e., $\Gamma = 0$), this realizes the dissipative version of the Hatano-Nelson model [65–67]. This can be seen, for example, from the Heisenberg equations of motion $i\partial_t \hat{a}_n = \sum_m (H_{\text{HN}})_{nm} \hat{a}_m$, where the (single-particle) Hatano-Nelson Hamiltonian H_{HN} is

$$H_{\text{HN}} = \sum_j (\Delta |j\rangle \langle j| + J_+ |j\rangle \langle j+1| + J_- |j\rangle \langle j-1|), \quad (1)$$

where $\Delta = i(\kappa - 2\gamma)$, $J_+ = -(J + \gamma e^{i\theta})$, $J_- = -(J - \gamma e^{-i\theta})$. The interplay between the coherent dynamics (J) and the correlated loss (γ) generates asymmetric hoppings rates $J_+ \neq J_-$, favoring left- (right-) propagating modes when $\theta = 0$ ($\theta = \pi$). The NHSE is reflected in the spectrum of Eq. (1), illustrated in Figs. 1(b)–1(d). The PBC spectrum is an ellipse in the complex plane, whereas for OBC the spectrum always falls on a line [Figs. 1(b) and 1(c)]. For

$\theta = \pm\pi/2$, the model is reciprocal (i.e., $|J_+| = |J_-|$), and the ellipse shrinks into a line signaling the absence of the NHSE [Fig. 1(d)].

The quadratic system ($\Gamma = 0$) can be dynamically unstable; with these interaction terms ($\Gamma \neq 0$) the system is always stable and becomes genuinely many-body. Here, we study the mean-field dynamics of the Lindbladian, described by the equation of motion

$$i\partial_t \alpha_j = i(\kappa - 2\gamma)\alpha_j - i\Gamma|\alpha_j|^2\alpha_j - (J + \gamma e^{i\theta})\alpha_{j+1} - (J - \gamma e^{-i\theta})\alpha_{j-1}, \quad (2)$$

where $\alpha_j \equiv \langle \hat{a}_j \rangle$ is the coherent state amplitude on each site.

Periodic boundary conditions—We begin with a study of the steady-state solutions of Eq. (2) with PBC, i.e., $\alpha_{N+1} = \alpha_1$. For large N , the vacuum state $\alpha_j = 0$ is unstable for any $\kappa > 0$. This can be seen from the spectrum of Eq. (1) which is diagonalized in momentum space $H_{\text{HN}} = \sum_q (i\kappa - i\gamma_q + \omega_q) |q\rangle \langle q|$, where

$$\gamma_q = 2\gamma[1 + \sin(q + \theta)], \quad \omega_q = -2J \cos(q), \quad (3)$$

are the decay rate and energy for momentum $q = 2\pi m/N$, $m = 0, \dots, N-1$. For any θ , the mode at $q + \theta = -\pi/2$ (a dark mode of all $\hat{L}_j^{(2)}$ dissipators) experiences no dissipation and can be excited with infinitesimal κ (for $N \rightarrow \infty$).

Despite the nonlinearity in Eq. (2) coupling these plane waves, the time-dependent traveling wave states (i.e., finite momentum condensates)

$$\alpha_j(t) = r_q e^{iqj - i\omega_q t}, \quad r_q = \sqrt{\frac{\kappa - \gamma_q}{\Gamma}}, \quad (4)$$

are solutions of Eq. (2) with PBC whenever $\gamma_q < \kappa$. Not all of these plane-wave solutions are stable: a momentum- q traveling wave can become unstable against creation of pairs at momenta $q \pm k$, for some k that depends on κ and γ [see Supplemental Material (SM) for details [68]]. Figure 2 shows the resulting stability diagram for $\theta = \pi$ and $\gamma/J = 0.5$. For $\kappa \rightarrow 0$, only the q^* mode such that $\gamma_{q^*} \rightarrow 0$ is stable ($q^* = \pi/2$ for $\theta = \pi$), as expected. As $\kappa \rightarrow \infty$, on the other hand, all the modes in a range of π centered about $q^* = -\pi/2 - \theta$ are stable (corresponding to all the right-moving modes for $\theta = \pi$). Numerically, these traveling waves appear to be the only possible behavior at long times. For PBC and $\theta = \pi$, we thus find that the system always condenses into a single positive, finite momentum mode (chosen by the initial conditions), with a nonzero frequency ω_q (unless $q = \pi/2$). Other choices of θ lead to qualitatively similar results, with everything in Fig. 2 shifted to $q^* = -\pi/2 - \theta$. Analogous traveling wave solutions also appear in the complex Landau-Ginzburg equation [69,70],

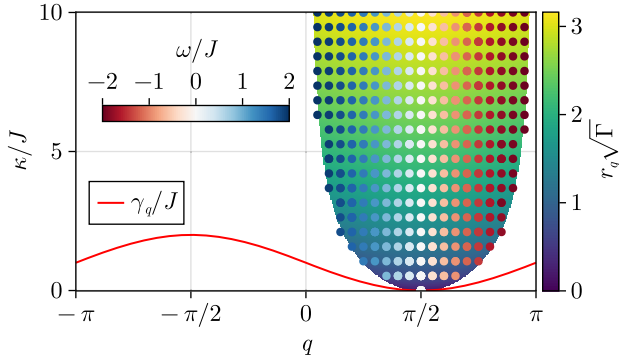


FIG. 2. Stability diagram of the PBC solutions in Eq. (4) for $\theta = \pi, \gamma/J = 0.5$. Colored region corresponds to the stable solutions for $N \rightarrow \infty$, with the color corresponding to the amplitude r_q . Dots correspond to stable solutions for a finite system of $N = 40$, with the color showing the oscillation frequency ω_q [Eq. (3)]. Red line is the momentum decay rate [Eq. (3)].

the continuum version of the reciprocal $\theta = -\pi/2$ limit of our model, where the stable possible condensates are centered about $q^* = 0$.

Open boundary conditions—For PBC, the angle θ (and hence the nonreciprocity $|J_+| - |J_-|$) does not play a significant role. In contrast, for OBC, the value of θ is crucial to the physics. At $\theta = 0, \pi$, the hopping is maximally nonreciprocal, and the full nonlinear system has an antilinear $\mathbb{Z}_2$ particle-hole $\mathcal{PH}$ symmetry. At the level of Eq. (2), this symmetry implies that if $\alpha_j(t)$ is a solution, then so is $\mathcal{PH}[\alpha_j(t)] = e^{i\pi j} \alpha_j^*(t)$. We will demonstrate below that this symmetry can be spontaneously broken for OBC. At the level of the linear $\Gamma = 0$ theory, this symmetry transforms the quadratic Hatano-Nelson Hamiltonian as $(\mathcal{PH})H_{\text{HN}}(\mathcal{PH})^{-1} = -H_{\text{HN}}$ [71]. In contrast, for more general $\theta \neq 0, \pi$, the $\mathcal{PH}$ symmetry is explicitly broken, making nontrivial steady states time dependent. Here, we focus on the $\mathcal{PH}$ -symmetric, maximally nonreciprocal case of $\theta = 0, \pi$, choosing $\theta = \pi$ for concreteness. The corresponding phase diagram is shown in Fig. 3(a).

The first intriguing feature of the OBC phase diagram is that the vacuum solution, $\alpha_j = 0$, becomes stable within a certain range of parameters, shown in yellow in Fig. 3(a). This is in sharp contrast to the PBC case, where the vacuum is always unstable for $\kappa > 0$. The linear stability of the OBC vacuum phase is determined by examining the imaginary parts of the eigenvalues of Eq. (1) with OBC. The spectrum of Eq. (1) differs qualitatively depending on whether $\gamma < J$ or $\gamma > J$ [Figs. 1(b) and 1(c)], with $\gamma = J$ marking an N th order exceptional point. The eigenvalues are given by [68]

$$E_m = \begin{cases} i(\kappa - 2\gamma) + 2\sqrt{J^2 - \gamma^2} \cos(q_m) & \text{for } \gamma < J, \\ i[\kappa - 2\gamma + 2\sqrt{\gamma^2 - J^2} \cos(q_m)] & \text{for } \gamma > J, \end{cases} \quad (5)$$

where $q_m = \pi m/(N+1)$, $m = 1, \dots, N$. For $\gamma < J$, all eigenstates have the same damping rate, and the phase transition occurs at $\kappa_{\text{crit}} = 2\gamma$ [black line in Fig. 3(a)] where all linear modes become unstable simultaneously. For $\gamma > J$, the $m = 1$ mode ($q \rightarrow 0$ in the thermodynamic limit) is the first to become unstable, yielding the critical value $\kappa_{\text{crit}} = 2\gamma - 2\sqrt{\gamma^2 - J^2}$.

For $\gamma > J$, as κ is increased above κ_{crit} , the system undergoes a continuous phase transition from the vacuum phase ($\langle \alpha_j \rangle_j = 0$) to a time-independent static condensate [blue region in Fig. 3(a)] with $\langle \alpha_j \rangle_j \neq 0$ [Fig. 3(c)], where $\langle o \rangle_j$ denotes the spatial average of o . Up to a spontaneously chosen global $U(1)$ phase, the complex amplitudes in the static phase are given by $\alpha_j^s = r_j^s e^{iq_s j}$, where $q_s = \pi/2(-\pi/2)$ for $\theta = \pi(\theta = 0)$. Interestingly, this phase order corresponds to the minimal-damping wave vector of the PBC system in Eq. (3). This finite-momentum condensate emerges as κ increases in the form of a kink that sweeps across the system from right to left [Fig. 3(b)]. Numerically, we find that the height of the kink is $\sqrt{\kappa/\Gamma}$. In fact, the uniform state $\alpha_j = \sqrt{\kappa/\Gamma} e^{i\pi j/2}$ solves Eq. (2) (with $\theta = \pi$) for all j except at the edges ($j = 1, N$). As $\kappa \rightarrow \infty$, the system approaches this uniform state.

Just above the phase transition, the kink first emerges at the right edge ($j = N$), reflecting the localization of linear eigenstates at this boundary for $\theta = \pi$ in the vacuum phase. As κ increases further, the kink shifts toward the left boundary, leading to a continuous increase in the average amplitude $\langle r_j \rangle_j$ [Fig. 3(c)]. Numerically, we find that the kink position scales as $(\kappa - \kappa_{\text{crit}})^\beta$ with $\beta = -0.5$ near the phase transition [inset in Fig. 3(c)]. This critical exponent can be understood as follows: nonreciprocity induces directional propagation toward the right (for $\theta = \pi$). This net current balances with the local pump and loss, saturating the right-most sites at an amplitude of $\sqrt{\kappa/\Gamma}$. Disregarding these saturated sites, the remaining sites with zero amplitude form an effective smaller system. The finite-size expression in Eq. (5) can then be used to determine the new, slightly larger κ required for the vacuum of this smaller system to become unstable. Alternatively, solving Eq. (5) for the effective system size—or, equivalently, the kink position—as a function of the distance from the phase transition $(\kappa - \kappa_{\text{crit}})$ yields precisely $\beta = -0.5$.

Next, we examine the green region of the phase diagram in Fig. 3(a), which corresponds to time-dependent (dynamical) phases. This region contains multiple attractors that encompass a variety of phases. We will primarily focus on the most prominent phase, which spans nearly the entire region. For fixed $\gamma/J < 1$, dynamical phases exist only within a finite range of κ : they transition to the vacuum phase at small κ and become unstable to the static condensate phase at large κ . To locate this upper phase boundary, we perform a numerical linear stability analysis

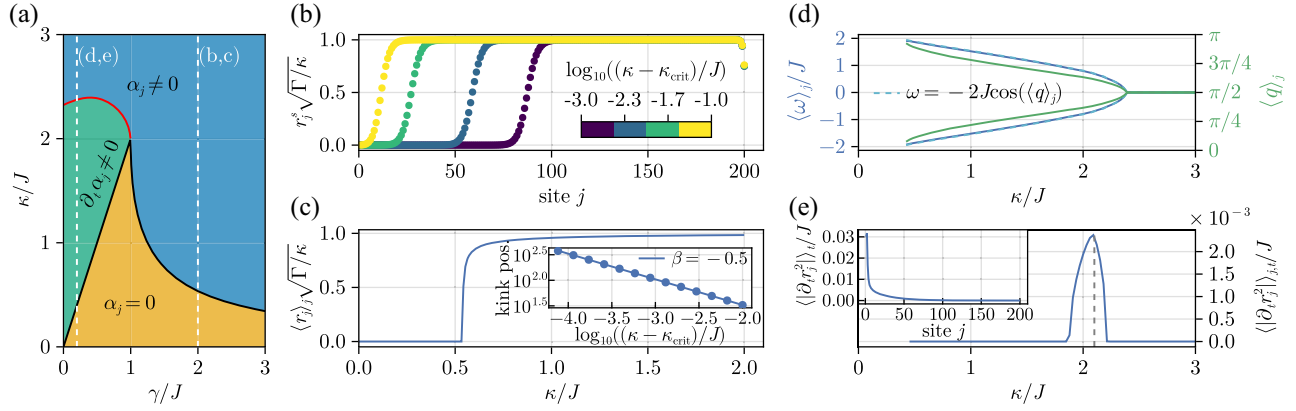


FIG. 3. (a) Steady-state OBC phase diagram for $\theta = 0, \pi$. Dashed lines at $\gamma/J = 0.2$ and $\gamma/J = 2$ correspond to line cuts shown in panels (d), (e) and (b), (c), respectively. (b) Spatial profile of the amplitude in the static condensate phase for different values of κ near the phase transition for $\gamma/J = 2, \theta = \pi$. (c) Average amplitude as a function of κ . The inset shows the kink position near the phase transition, with the solid line representing a power-law fit yielding a critical exponent $\beta = -0.5$. (d) Average frequency (left axis, blue) and average wave vector (right axis, green) as functions of κ/J for $\gamma/J = 0.2, \theta = \pi$. The dashed line (overlapping with the numerically obtained solid blue line) represents the PBC dispersion from Eq. (3). (e) Time and space average of the magnitude of the rate of change of density as a function of κ . The inset shows the spatial profile of the time-averaged density rate of change at $\kappa/J = 2.1$ (dashed line).

of the static condensate state, yielding the red line in Fig. 3(a). Notably, the static condensate state α_j^s is $\mathcal{PH}$ symmetric (i.e., $\mathcal{PH}[\alpha_j^s] = \alpha_j^s$).

Decreasing κ along the leftmost white dashed line in Fig. 3(a) towards the red line, the dynamical matrix describing fluctuations of the static condensate develops a second zero eigenvalue, whose eigenvector coalesces with the Goldstone mode [associated with the spontaneous $U(1)$ symmetry breaking]. We thus identify this portion of the red line as a second-order critical exceptional point [68]. Below the red line of Fig. 3(a), the system spontaneously breaks the $\mathcal{PH}$ symmetry, giving rise to two bistable time-dependent traveling wave states (wave vectors $q_{\pm} = q_s \pm \epsilon$) analogous to the PBC solutions in Eq. (4). The wave vector $q_{\pm}$ evolves continuously as κ is lowered, bifurcating from $q_{\pm} = q_s = \pi/2$ in the static phase ($\epsilon = 0$) to either $q_- = 0$ or $q_+ = \pi$ at the point where the system enters the vacuum phase ($\epsilon = \pi/2$). Similarly, as shown in Fig. 3(d), the frequency evolves continuously from zero, bifurcating into two branches corresponding to clockwise (CW) or counterclockwise (CCW) rotations (analogous to the “chiral” states in Ref. [8]). The $\mathcal{PH}$ symmetry maps one solution onto the other, transforming a CW rotating state into a CCW one (i.e., flipping the sign of ω), while simultaneously flipping the wave vector from $q_s + \epsilon$ to $q_s - \epsilon$, and vice versa. Remarkably, the PBC result for the frequency in Eq. (3) holds perfectly for these two states, using the numerically extracted average wave vector $\langle q \rangle_j$.

Edge and bulk phase transitions—So far, we have considered cases where only the phase of each site varied in time. In this section, we will briefly discuss some additional dynamical phases where the amplitudes also fluctuate. These fluctuations occur predominantly at the

left edge, leading to distinct bulk versus edge behavior [Fig. 4(a)]. In fact, Fig. 3(d) already contains such an example. This is illustrated in Fig. 3(e), which displays the time- and space-averaged absolute value of the rate of change of the magnitude, $\langle |\partial_t r_j^2| \rangle_{j,t}$. This quantity becomes nonzero at $\kappa/J \sim 1.9$ – 2.2 . The inset in Fig. 3(e) shows an example of the time-averaged spatial profile ($\langle |\partial_t r_j^2| \rangle_t$) of these amplitude fluctuations, revealing that they are peaked at the left edge. Away from the edge, this quantity quickly drops, and then exponentially decays into the bulk. Notably, these amplitude fluctuations do not affect the $\langle \omega \rangle_j$ order parameter shown in Fig. 3(d), as the dominant Fourier component, even for the left-most sites, remains at the same bulk value as the average value $\langle \omega \rangle_j$ depicted in Fig. 3(d).

The dynamical region ($\partial_t \alpha_j \neq 0$) includes additional phases with distinct edge and bulk behaviors, with some regions exhibiting multistability, indicated by stripes in Fig. 4(a). Phase IV, indicated in purple in Fig. 4(a), features amplitude fluctuations at the left edge and phase-only oscillations in the bulk. A representative example ($\gamma = 0.3, \kappa = 2.2$) of the amplitude and phase space-time dynamics is shown in Figs. 4(b)(i) and 4(b)(ii), respectively. Left of the black dashed line in Fig. 4(a), the two limit cycles of phase I—the CW and the CCW solutions—merge, resulting in more complex dynamics. The phases of the bulk sites act as pendula, involving both CW and CCW rotations [see Fig. 4(e)]. In some parts of this phase, particularly at smaller γ , the dynamics is nevertheless periodic. Notably, the $\mathbb{Z}_2$ symmetry is dynamically restored in this case, corresponding to $\mathcal{PH}$ accompanied by the breaking of continuous time translation symmetry [72,73]. Specifically, the $\mathcal{PH}$ operation acts as a translation by a half

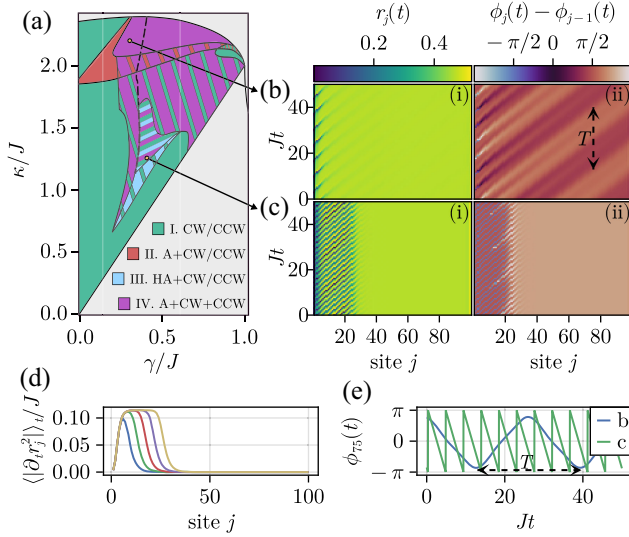


FIG. 4. (a) Enlarged view of the dynamical region of the steady-state OBC phase diagram from Fig. 3(a). Colored stripes indicate multistable regions with multiple phases with distinct dynamical phenomena. The dashed black line marks the approximate boundary where the bulk phase oscillations become chaotic (to the right of the line). (b), (c) Space-time dynamics of the amplitude (i) and phase differences (ii) for representative points in phases IV and III at $\gamma = 0.3, \kappa = 2.2$ (b) and $\gamma = 0.4, \kappa = 1.25$ (c). (d) Time-averaged magnitude of the rate of change of the density in five different steady states, corresponding to the parameters in (c). (e) Time dependence of the phase of oscillator 75 corresponding to panels (b) and (c).

period, and the steady state satisfies $\alpha_j(t) = \mathcal{PH}[\alpha_j(t + T/2)]$ for some period T [68]. Additional details on these phases and their symmetry classification are provided in the End Matter.

In Phase IV, the steady state alternates between exhibiting periodic and chaotic dynamics until a critical value of γ [black dashed line in Fig. 4(a)] beyond which only the chaotic dynamics persists. Surprisingly, we find that despite the onset of chaos, the dynamics at the edge remain regular [68]. We also find a region [phase III in Fig. 4(a)] with a hierarchy of steady states with varying edge lengths. A representative example ($\gamma = 0.4, \kappa = 1.25$) is shown in Fig. 4(c). The bulk dynamics is $\mathcal{PH}$ broken, with CW or CCW phase oscillations [Fig. 4(e)] as in phase I. We observe five steady states with progressively longer edge-amplitude fluctuations, shown in Fig. 4(d), although it remains unclear whether additional steady states exist. While the bulk dynamics in phases II and III are periodic, the edge exhibits quasiperiodicity, breaking time translational symmetry and introducing an additional zero mode (beyond the existing Goldstone mode [68]). Exploring these edge phase transitions in greater detail is left for future work.

Conclusions and outlook—We have demonstrated how the NHSE and spatial nonreciprocity can fundamentally

alter the phase diagram of driven-dissipative systems in a manner that strongly depends on boundary conditions. The remarkable differences between OBC and PBC seen here are in stark contrast to the fermionic and spin counterparts of our model [27,67], where excitations generically have finite lifetimes and decay before reaching the edge, leaving bulk properties unchanged. In contrast, our bosonic system can possess an undamped Goldstone mode, allowing excitations to propagate to the boundaries and making boundary effects significant.

In our model, OBC result in a particularly rich phase diagram, where even the condensation phase transition exhibits unconventional features compared to its reciprocal counterpart. This intriguing difference invites further exploration to understand whether they differ in their critical behavior. More broadly, the role of classical and quantum fluctuations beyond mean-field theory presents an exciting avenue for future research. Additionally, a deeper understanding of the observed edge phase transitions and their underlying mechanisms requires further investigation. Extending this work to higher dimensions [43] could uncover additional novel spatiotemporal patterns and exotic phases. Lastly, exploring the manifestation of these phenomena in active matter and other nonequilibrium systems remains an exciting direction for future studies.

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Data availability—The data that support the findings of this Letter are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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End Matter

Symmetry classification of OBC phases—We discuss here how each phase in our model can be elegantly classified in terms of distinct broken symmetries. There are three explicit symmetries of interest in the model. For all values of θ , Eq. (2) of the main text admits a $U(1) \cong SO(2)$ symmetry, as well as a time-translational symmetry. For specific values of θ , there is also a discrete particle-hole symmetry, where the particle-hole operator is defined as $\mathcal{PH}[\alpha_j] = e^{i\pi j} \alpha_j^*(t)$ which is an involution.

More explicitly, when $\theta = 0, \pi$, $\mathcal{PH}[\alpha_j(t)]$ obeys the same equation of motion as $\alpha_j(t)$, resulting in a $\mathbb{Z}_2$ symmetry via application of the $\mathcal{PH}$ operator. The full symmetry group for $\theta = 0, \pi$ becomes $O(2) \cong SO(2) \rtimes \mathbb{Z}_2$ allowing for a richer phase diagram and collective phenomena. The continuous time-translation symmetry of the model noted above can be broken to a discrete time translation symmetry, resulting in what is often termed a time-crystalline or limit-cycle phase (see, e.g., [72,73]). The remaining discrete time translational symmetry can be further broken, so that it only remains intact on a subset of measurements; this is known as *quasiperiodicity*. Time translational symmetry can also be completely broken, as is the case when one has chaotic dynamics.

Table I summarizes how each of the above symmetries are broken across the various condensed phases of our model. Notably, almost all of the finite number of allowed symmetry-breaking combinations are realized in this work (though a few combinations do not seem to be realized, e.g., a $\mathbb{Z}_2$ -symmetry broken chaotic phase).

In the static phase, the solutions takes the form $\alpha_j = r_j e^{i(\phi + j\pi/2)}$ with the phase ϕ determined by the spontaneous breaking of $U(1)$. $\mathcal{PH}$ is left unbroken as $\alpha_j \sim \mathcal{PH}[\alpha_j]$, where we use $\sim$ to mean equivalent up to a global rotation ($\alpha_j \rightarrow e^{i\varphi} \alpha_j$). In phase I, the long-time behavior admits two branches of possible solutions with the form $\alpha_j^\pm = r_j e^{i(\pm\omega t + j q_\pm + \phi)}$, where $q_\pm = \pi/2 \pm \epsilon$. The two branches of solutions are related by $\mathcal{PH}[\alpha_j^\pm] \sim \alpha_j^\mp$ implying that the $\mathcal{PH}$ symmetry is broken because there does not exist a fixed $U(1)$ transformation that allows $\mathcal{PH}[\alpha_j^\pm(t)]$ to equal $\alpha_j^\pm(t)$ for all times.

In the remaining phases, the quasiperiodic or chaotic breaking of continuous time translational symmetry can be determined via the numerical computation of the associated Lyapunov characteristic exponents (LCEs). In Fig. 5(a), we plot the four largest LCEs for a horizontal cut through the phase diagram of Fig. 4(a) for a fixed $\kappa = 2.2$. We divide phase IV into two parts (left and right), depending on whether the value of γ is to the left or right of the dashed line in Fig. 4(a). In phase I, there is a single zero mode [associated with the breaking of $U(1)$ symmetry], while in phase II and phase IV (left) the onset of edge amplitude dynamics coincides with the existence of two zero modes. Phase III [not shown in Fig. 5(a)] also admits two zero modes. The existence of multiple zero modes corresponds generically to quasiperiodic dynamics due to the modes oscillating at incommensurate frequencies.

For further insight, we plot in Fig. 5(b) phase trajectories of two sites near the left edge for various parameter regimes. For parameters corresponding to phases II and III, one clearly sees nonrepeating dynamics, whereas periodic motion is retained in trajectories corresponding to phases I and IV (Left). In phases II and III, the quasiperiodicity is primarily localized near the left edge of the chain (where both the phases and amplitudes are time dependent). Far from the edge, the dynamics becomes less distinguishable from periodic motion. The combination of amplitude and phase dynamics in phases II and III is similar to the “chiral +swap” phase observed in Ref. [8] in a different model.

Returning to Fig. 5(a), note that the region of phase IV (Left) plotted always has a single stable attractor. However, depending on the value of γ , the phase can be chaotic (i.e., there can be a single positive LCE). Phase IV (right) generically admits multiple positive Lyapunov exponents, a situation termed in dynamical systems theory as hyperchaos [75,76], implying that this phase admits a higher dimensional chaotic attractor than the chaos seen in phase IV (left).

In the nonchaotic portion of phase IV (left), the system retains periodicity despite the existence of two zero modes. The total phase variable $\Phi := \sum \phi_i / N$ is static in this phase, in contrast to the $\mathcal{PH}$ -broken phase I, where $\dot{\Phi} = \pm\omega$. The discrete time translation symmetry is restored because the Goldstone mode associated with Φ

TABLE I. Classification of OBC phases by the symmetry actions.

Phase	$U(1)$	$\mathbb{Z}_2$	Continuous time translation
Vacuum	Unbroken	Unbroken	Unbroken
Static	Broken	Unbroken	Unbroken
Phase I (Periodic pair)	Broken	Broken	Broken to discrete translations
Phase II, III (Quasiperiodic)	Broken	Broken	Broken (except certain measurements)
Phase IV Left (Periodic individual)	Broken	Dynamically Restored	Broken to discrete translations
Phase IV Right (Chaotic)	Broken	Restored on Attractor	Completely broken

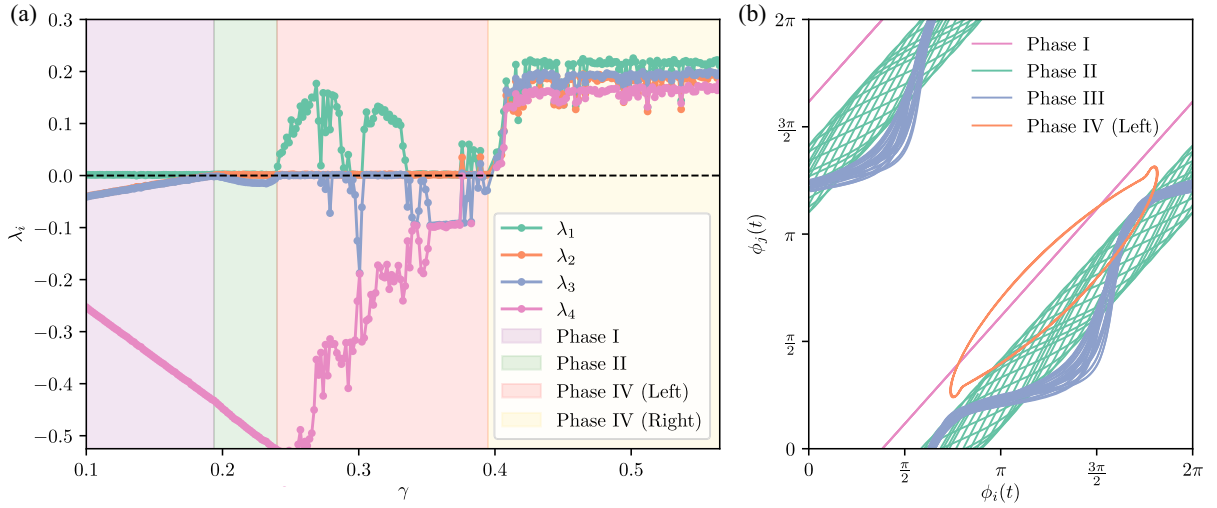


FIG. 5. (a) Sweep of the 4 largest Lyapunov characteristic exponents at $\kappa = 2.2$ and $N = 100$. The LCEs are computed using the Benettin *et al.* algorithm [74]. Phase IV (left) and (right) refer to the portion of phase IV on the corresponding side of the dotted black line in Fig. 4(a) of the main text. (b) A phase space plot of $\phi_j(t) := \arg[\alpha_j(t)]$ for two sites near the edge of the chain with the phase I trajectory at $\kappa = 2.2$, $\gamma = 0.1$ with $i = 1$, $j = 2$, phase II at $\kappa = 2$, $\gamma = 0.2$ with $i = 1$, $j = 2$, phase III at $\kappa = 1.25$, $\gamma = 0.4$ with $i = 11$, $j = 12$, and phase IV at $\kappa = 2.2$, $\gamma = 0.35$ with $i = 11$, $j = 12$.

is now at zero frequency, making the two zero modes commensurate. A $\mathbb{Z}_2$ symmetry restoration also takes place with the explicit form $\alpha_j(t) \sim \mathcal{PH}[\alpha_j(t + T/2)]$. In Fig. 6(a), we demonstrate the symmetry restoration at a periodic point of phase IV (left). By comparing (a)(i) and (a)(ii) in Fig. 6, one sees that the conjugated solution is simply a time translation of the original solution. The difference between the original and the $\mathcal{PH}$ -conjugate solution shifted by a half period is shown to be within numerical error of zero in Fig. 6(a)(iii).

In the periodic portion of phase IV, the $\mathbb{Z}_2$ symmetry restoration $\alpha_j(t) \sim \mathcal{PH}[\alpha_j(t + T/2)]$ implies that the $\mathcal{PH}$ operation maps the entire limit cycle trajectory back onto

itself. This symmetry appears to be retained as the periodic orbit bifurcates into a chaotic attractor such that the set of points on the chaotic attractor is now invariant under the $\mathcal{PH}$ operation. This is demonstrated in Fig. 6(b). In Fig. 6(b)(i), we use a delay coordinate representation to give a low dimensional projection of the chaotic attractor. The attractor generated by the $\mathcal{PH}$ -conjugated initial condition, shown in 6(b)(ii), is nearly identical to the original attractor, up to small deviations due to only plotting a finite sample of the attractor. This invariance of the attractor in the chaotic phase implies that a $\mathbb{Z}_2$ symmetry remains despite the state not retaining any subgroup of the time translational symmetry.

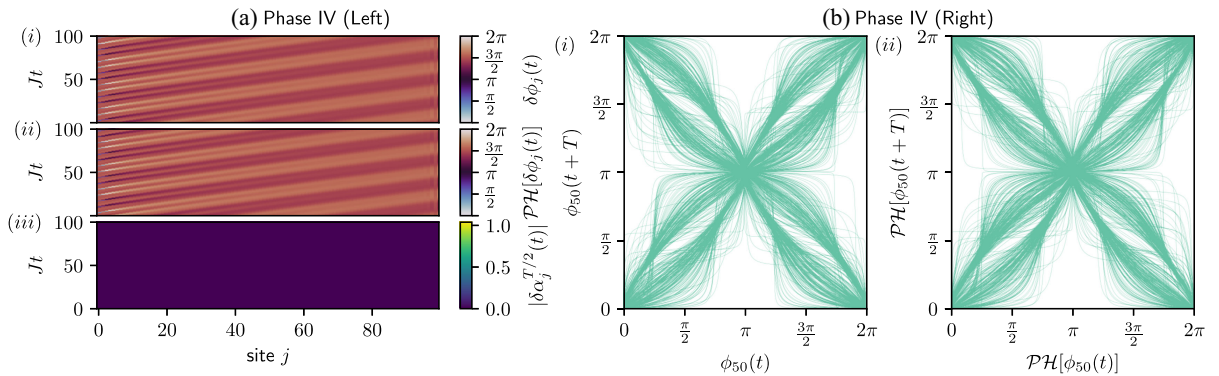


FIG. 6. (a) Demonstration of the $\mathcal{PH}$ restoration at $\kappa = 2.2$, $\gamma = 0.3$, corresponding to point (b) in Fig. 4(a) of the main text. (a)(i) Space-time dynamics of the phase differences, $\delta\phi_j(t) := \phi_{j+1}(t) - \phi_j(t)$. (a)(ii) Space-time dynamics of the phase differences of the $\mathcal{PH}$ -conjugated solution, i.e., $\mathcal{PH}[\delta\phi_j(t)] := \arg(\mathcal{PH}[\alpha_{j+1}(t)] - \arg(\mathcal{PH}[\alpha_j(t)]))$. (a)(iii) Magnitude of $\delta\alpha_j^{T/2}(t) := \alpha_j(t) - \mathcal{PH}[\alpha_j(t + T/2)]$, where $T \approx 26.66$ for these parameters. The global phase of $\alpha_j(t)$ is chosen so no additional rotation is needed for $\alpha_j(t) - \mathcal{PH}[\alpha_j(t + T/2)]$ to be zero. (b) Time-delay embedding of the chaotic attractor at site $j = 50$ for $\kappa = 2.2$, $\gamma = 0.5$, using phase values at times t and $t + T$ with delay $T = 14/J$, comparable to the return time. (i) Original attractor. (ii) $\mathcal{PH}$ -conjugate attractor, with $\mathcal{PH}[\phi_j(t)] := \arg(\mathcal{PH}[\alpha_j(t)])$.