

THE UNIVERSITY OF CHICAGO

GRAPH-STRUCTURED NONLINEAR OPTIMIZATION AND STOCHASTIC
OPTIMIZATION FOR (PARTIALLY)OBSERVABLE LINEAR SYSTEM

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To my parents. Especially to my mom.

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ABSTRACT

In this thesis, we investigate a class of optimization problems and introduce a novel family of optimization algorithms. We begin by studying large-scale deterministic optimization problems formulated within a graph-based framework. To overcome the limitations of existing decomposition methods—namely, the lack of global convergence guarantees and slow convergence rates—we propose a new decomposition algorithm. The proposed method ensures both local and global convergence, and we theoretically establish that its local convergence rate improves exponentially with the size of the overlap between subdomains. Experimental results on problems constrained by partial differential equations (PDEs) further validate our theoretical findings.

We further extend our investigation from deterministic to stochastic settings by studying the nonstationary Linear Quadratic Regulator (LQR) problem. To address challenges related to sample complexity and computational efficiency, we propose a scalable framework for estimating the unknown system parameters and approximately solving the resulting control problem. The proposed algorithm achieves significant reductions in sample complexity, estimation error, and computational cost, thereby enhancing its applicability to large-scale systems.

Building on the results obtained for the fully observed LQR system, we extend our investigation to the partially observed LQR problem under the stationary dynamics. To address the challenges of limited data and computational resources, we develop a scalable strategy for parameter estimation tailored to this setting. The proposed algorithm demonstrates strong suitability for large-scale systems, and its performance is supported by a complete theoretical analysis. In addition, we introduce a candidate parameter estimation algorithm with the potential to generalize to a broader class of problems. A full theoretical analysis and comprehensive empirical validation of this method are left for future work.

CHAPTER 1

INTRODUCTION

Optimization is a mathematical process of finding the best solution by maximizing or minimizing an objective function, subject to a set of constraints that define the feasible solution set. It is a fundamental tool with a long-standing history and widespread applications across diverse fields, including biology, chemistry, finance, and physics. As real-world problems continue to grow in scale and complexity, developing efficient methods to solve large-scale optimization problems with low computational cost and high accuracy has become an increasingly popular research focus.

Nonlinear optimization addresses problems in which the objective function or at least one of the constraints is nonlinear. These problems, known as nonlinear programs (NLPs), can often be represented using a graph-based structure. In such a representation, decision variables are assigned to nodes, and edges are drawn between nodes to capture dependencies or interactions among variables. This graph representation provides structural insights that can be exploited to design more efficient solution algorithms. In particular, graph-structured nonlinear programs (gsNLPs) [62] are defined as:

$$\min_{\{\mathbf{x}_i \in \mathbb{R}^{r_i}\}_{i \in \mathcal{V}}} \sum_{i \in \mathcal{V}} f_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]}), \quad (1.1a)$$

$$\text{s.t.} \quad \mathbf{c}_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]}) = \mathbf{0}, \quad i \in \mathcal{V}, \quad (\boldsymbol{\lambda}_i). \quad (1.1b)$$

Here, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected graph, $\mathcal{V}$ is the strictly ordered node set, $\mathcal{E} \subseteq \{\{i, j\} : i, j \in \mathcal{V}, i \neq j\}$ is the edge set, and $N_{\mathcal{G}}[i] := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\} \cup \{i\}$ is the closed neighborhood of $i \in \mathcal{V}$ on $\mathcal{G}$ visualized in Figure 1.1. For each node i , we have the following associated quantities: $\mathbf{x}_i \in \mathbb{R}^{r_i}$ is the primal variable, $f_i : \prod_{j \in N_{\mathcal{G}}[i]} \mathbb{R}^{r_j} \rightarrow \mathbb{R}$ is the objective function, $\mathbf{c}_i : \prod_{j \in N_{\mathcal{G}}[i]} \mathbb{R}^{r_j} \rightarrow \mathbb{R}^{m_i}$ is the equality constraint function, and $\boldsymbol{\lambda}_i \in \mathbb{R}^{m_i}$ is the dual variable associated with the constraint $\mathbf{c}_i$. GsNLPs can be adapted to many different

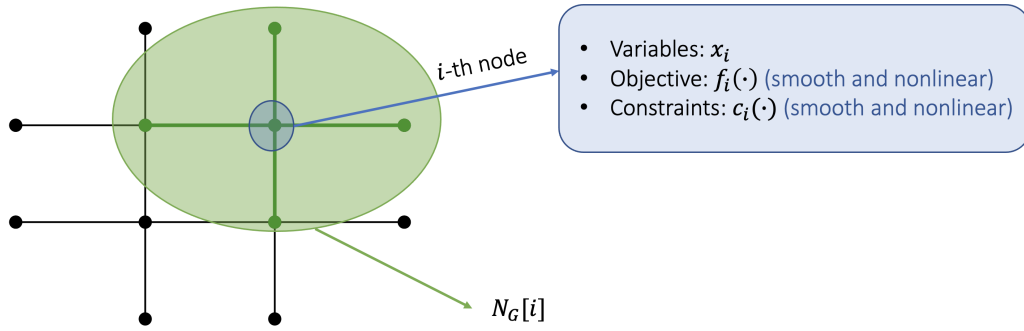


Figure 1.1: Illustration of f_i , c_i , and $N_G[i]$ under a two-dimensional graph.

graphs which can be applied in optimal control [7], partial differential equation (PDE)-constrained problems [8], multistage stochastic optimization [56], and network optimization [80], of which illustrations are given in [62].

In this paper, we study a scalable solution method for the large-scale instances of (1.1). The complexity of these problems arises from their substantial number of variables and constraints, which can render centralized solution methods intractable. To mitigate this, decomposition methods are employed to break down the intractable full problem into a set of smaller, manageable subproblems. These subproblems are designed to be individually tractable, and decomposition algorithms employ an iterative scheme that facilitates the convergence of the iterates towards the full solution. The convergence can be achieved through the exchange of solution information across the subproblems in each iteration step. Examples of decomposition algorithms include temporal decomposition [27, 77], Lagrangian decomposition [27, 3], Jacobi/Gauss-Seidel methods [79], and alternating direction method of multipliers (ADMM) [20]. Decomposition algorithms offer several advantages over centralized methods, including computational scalability and flexibility in implementation. First, by breaking down the full problem into smaller subproblems, decomposition methods can handle significantly larger problem instances compared to centralized approaches. Additionally, these methods provide flexibility in implementation since the problem data, such as Hessian and Jacobian matrices, does not need to be stored in a single shared memory.

However, it is worth noting that decomposition algorithms are known to exhibit slower convergence rates compared to centralized methods [31]. This slower convergence can be attributed to the iterative nature of decomposition schemes and limited amount of information being shared across the subproblems. As demonstrated in the benchmark study [31], the practically observe convergence rate of conventional decomposition schemes is often too slow to be practically useful. Such limitation of the existing decomposition algorithms motivates the study of new decomposition algorithms that can overcome the slow convergence issue of the conventional decomposition methods, while preserving its desired scalability.

In the recent line of work on *overlapping decomposition*, the *overlap* has been exploited to improve the convergence behavior of decomposition algorithms [61, 60, 50, 48]. This method decomposes the original problem with a set of *overlapping subproblems*, and these subproblems are iteratively solved to promote the convergence to the full solution. Due to the presence of overlap, it is typically required to solve a set of slightly larger subproblems compared to the conventional decomposition methods, but it has been shown that even with a small overlap, the convergence behavior can be dramatically improved. *Overlapping Schwarz method* is one of the particular algorithms that utilize this type of overlapping decomposition strategy. The origin of this method traces back to the classical domain decomposition methods, which was originally developed for solving second-order elliptic partial differential equations [14, 13]. Recently, this scheme was adapted for structured optimization problems, and its convergence properties have been studied in different settings, including convex unconstrained quadratic programs [61], constrained quadratic programs [60], nonlinear optimal control [48], and general nonlinear optimization problems [59]. In these works, authors have established not only the convergence of these algorithms, but also the relation between the convergence rate and the size of overlap. Specifically, they have shown that the (local) convergence rate improves exponentially with respect to the size of overlap, meaning that one can address the slow convergence issue of the decomposition methods by slightly increas-

ing the subproblem complexity. Thus, overlapping decomposition can directly address the main disadvantage of the traditional decomposition algorithms mentioned above.

While the Schwarz scheme resolves the main issue of the conventional decomposition schemes, the Schwarz method does not employ sophisticated globalization strategies, such as merit function or filter, to enforce the global convergence. In particular, the previous works on the Schwarz schemes only guarantee the local convergence, meaning that convergence is only guaranteed when the initial guess is sufficiently close to the local solutions. This limitation motivates a design of an enhanced algorithm, equipped with a globalization strategy to enforce the global convergence, while preserving the desired fast local convergence of the Schwarz schemes. This has motivated the study of applying the overlapping decomposition within a well-established nonlinear optimization algorithm, in particular, within the sequential quadratic programming (SQP) method [9] in Chapter 2, where the algorithm is presented in Section 2.3. In a recent work [50], authors have designed a so-called *fast overlapping temporal decomposition procedure* (FOTD) for the nonlinear equality-constrained dynamic problems (NLDPs). This method embeds the decomposition procedure into the SQP framework with exact augmented Lagrangian merit function. They have shown that this algorithm enjoys the global convergence while preserving the desired fast local convergence of the Schwarz schemes.

Given the promising results of FOTD for dynamic programs, we aim to extend this result to a more general class of nonlinear programs. We propose the *Fast Overlapping Graph Decomposition* (FOGD) method to handle gsNLPs. To achieve global convergence, we leverage the SQP framework with a similar flavor to [50]. Since computing the exact Newton direction at every iteration is usually a computational challenge for the SQP methods, we apply an overlapping graph decomposition to compute an approximate Newton direction. We begin by showing that computing the exact Newton direction is equivalent to solving an optimization problem. Based on this formulation on the full domain, we then carefully

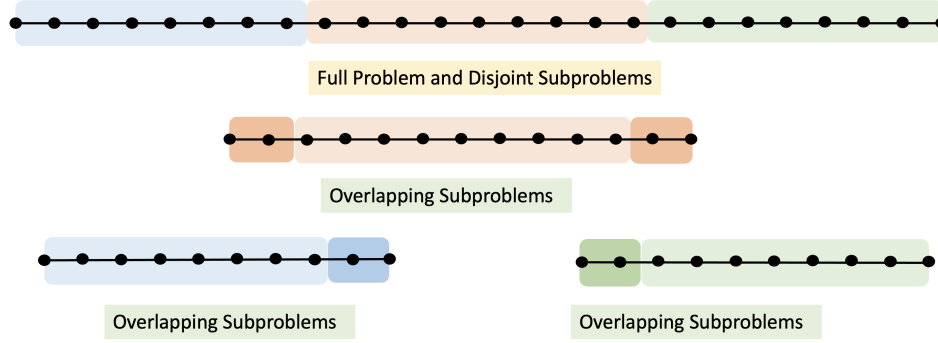


Figure 1.2: Illustration of overlapping graph decomposition with the overlapping size set as 2 for 1-D case. We first decompose a full line into three disjoint intervals. Then, we expand each interval with the size of overlap 2.

design subproblems on each subdomain. Finally, we solve these subproblems to construct the approximate Newton direction. The 1-D case is used to illustrate how to use the overlapping graph decomposition to obtain the subdomains in Figure 1.2. After obtaining the approximate Newton direction, we select a proper stepsize by conducting the line search on the exact augmented Lagrangian function. Under the regularity conditions, we show that the approximate Newton direction is also a descent direction of the exact augmented Lagrangian function by careful design of the subproblems and boundary specification. Our main technical contribution lies in demonstrating the global and local convergence of the FOGD method and proving that the local convergence rate of the FOGD method increases exponentially with respect to the size of overlap. To the best of our knowledge, FOGD is the first overlapping decomposition scheme that can solve gsNLPs with a global convergence guarantee while maintaining a fast local convergence rate.

Graph-structured nonlinear optimization studied in Chapter 2 focuses on deterministic problems that exclude randomness in their formulation. However, many real-world applications inherently involve uncertainty, which can render purely deterministic models intractable or insufficient. As a result, stochastic programming with control has emerged as another key topic in the field of optimization. In Chapter 3, we study finite-horizon control for discrete-time, non-stationary linear systems with unknown but structured dynamics. Specifically, we

consider the stochastic linear quadratic regulator (LQR) with horizon T , where the dynamics are given by

$$\begin{aligned} \min_{\substack{\{x_t\}_{t \in [T+1]} \\ \{u_t\}_{t \in [T]}}} & \frac{1}{2} x_T^T Q x_T + \sum_{t=0}^{T-1} \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t, \\ \text{s.t. } & x_0 = \bar{x}_0, \quad x_{t+1} = A_t x_t + B_t u_t + w_t, \quad t \in [T]. \end{aligned}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^d$, the matrices $A_t \in \mathbb{R}^{n \times n}$ and $B_t \in \mathbb{R}^{n \times d}$ are unknown and vary with time, and $\bar{x}_0$ is the initial state. We assume that A_t and B_t admit a known parametric form:

$$A_t = \sum_{j=1}^{n_A} \alpha_{t,j} \bar{A}_j, \quad B_t = \sum_{k=1}^{n_B} \beta_{t,k} \bar{B}_k, \quad (1.2)$$

where the basis matrices $\{\bar{A}_j\}$ and $\{\bar{B}_k\}$ are known, and the coefficients $\alpha_{t,j}, \beta_{t,k}$ are unknown and time-varying. This structure captures many real-world systems in which the dynamics evolve within a low-dimensional subspace. Examples include distributed control, PDE-constrained systems [8], and large-scale networked systems[62], where n is large but the number of free parameters $n_\theta = n_A + n_B$ is small.

Our goal is to design a computationally efficient algorithm that learns and controls such systems with provably low dynamic regret. The main challenges arise from (i) the time-varying nature of the dynamics, and (ii) the need to learn the parameters $\alpha_{t,j}, \beta_{t,k}$ from data without prior knowledge of their evolution.

We develop a two-stage algorithm that alternates between model estimation and control. The time horizon T is partitioned into N intervals. At the beginning of each interval, the system undergoes an *exploration stage* during which a randomized stabilizing controller is used to collect data. The unknown parameters are then estimated via ordinary least squares (OLS). In the subsequent *control stage*, we apply a receding-horizon controller computed from a model predictive control (MPC) formulation using the estimated dynamics. This

approach avoids online change-point detection by following a fixed re-identification schedule, simplifying both implementation and analysis.

Our method builds on the framework of [42], which analyzes regret for non-stationary LQR in the absence of structural assumptions. That work achieves a dynamic regret of $\tilde{O}(n^3 V_T^{2/5} T^{3/5})$, where V_T is the total variation in the system parameters. However, it requires $O(n)$ samples per interval for accurate estimation and $O(n^3)$ computation per step due to Riccati solves. In contrast, our algorithm leverages the parametric structure to reduce the required samples to $O(n_\theta)$ and achieves per-step computational complexity $O(n \log T)$ by solving truncated-horizon MPC problems instead of full-horizon Riccati equations.

We make the following contributions: (1) We propose a new algorithm for controlling non-stationary LQR systems with parametric dynamics, combining OLS-based identification with truncated-horizon MPC. (2) We show that the estimation error depends only on n_θ , improving over the $O(n)$ scaling in prior work [42]. (3) We establish a high-probability dynamic regret bound of $\tilde{O}(n_\theta^2 n V_T^{2/5} T^{3/5})$, strictly improving the $\tilde{O}(n^3 V_T^{2/5} T^{3/5})$ bound of [42] when $n_\theta \ll n$. (4) We demonstrate that our method significantly reduces computational cost, requiring only $O(\log T)$ per-step MPC computations.

Linear quadratic regulator (LQR) problems are foundational in control theory and have been widely applied in traffic systems [74, 73], robotics [38], and macroeconomic policy [15]. Time-varying and non-stationary LQR formulations also play a critical role in approximating nonlinear control problems. For example, Chapter 8 of [69] highlights how trajectory-linearized LQR is used to design feedback controllers around nominal nonlinear trajectories. Recent work has focused on learning and controlling LQR systems with unknown and non-stationary dynamics. A notable example is [42], which analyzes regret in settings with arbitrary time variation in A_t and B_t , and proposes a re-identification strategy that alternates between exploration and control. While that approach achieves dynamic regret $\tilde{O}(n^3 V_T^{2/5} T^{3/5})$, it requires $O(n)$ samples per interval and incurs $O(n^3)$ computational

cost per step due to full-horizon Riccati solves.

Our work introduces a parametric model in which A_t and B_t are linear combinations of known basis matrices. This structure captures scenarios where the system evolves in a low-dimensional subspace and arises naturally in PDE-constrained optimization [8] and large-scale networks [62]. To the best of our knowledge, regret guarantees for non-stationary LQR under such parametric dynamics—where only the coefficients vary over time—have not been previously established. By leveraging this structure, our algorithm reduces the estimation sample complexity from $O(n)$ to $O(n_\theta)$ and significantly lowers the per-step computation via truncated-horizon MPC. Finally, our control strategy draws on the literature on model predictive control (MPC) for time-varying and nonlinear systems. Prior work has studied the stability and performance of MPC in dynamic settings [40, 70, 41, 78] and regret analysis of the model with the known parameters [39, 63], but these results typically do not provide regret guarantees in the learning-from-data setting we consider. Our analysis contributes to this literature by quantifying regret for a learning-based MPC approach in non-stationary parametric systems.

The Linear Quadratic Regulator (LQR) problem with fully observed state variables in Chapter 3 has been extensively studied. A natural and practically important extension is the setting of partially observed linear dynamical systems studied in Chapter 4, where the state variables are not directly accessible and must be estimated from noisy observations. The Linear Quadratic Regulator (LQR) problem under partial observability has become a central topic in reinforcement learning and control theory [36], with applications in areas such as robotic systems with partially observed feedback [75], systems biology [10], electric power grid [58], etc. One key reason for the long-standing importance and broad applicability of LQR is its ability to approximate nonlinear dynamical systems. Specifically, nonlinear constraints can be approximated to nonstationary linear dynamics by linearization. Letting

$\bar{x}_t = x_t - x_{t,0}$ and $\bar{u}_t := u_t - u_{t,0}$, we have

$$\begin{aligned} E[\bar{x}_{t+1}] &= x_t - x_{t,0} + f(x_t, u_t) - f(x_{t,0}, u_{t,0}) \\ &= \bar{x}_t + \frac{\partial f(x_{t,0}, u_{t,0})}{\partial x_{t,0}} \bar{x}_t + \frac{\partial f(x_{t,0}, u_{t,0})}{\partial u_{t,0}} \bar{u}_t = A_t \bar{x}_t + B_t \bar{u}_t. \end{aligned}$$

It also makes the partially observed LQR setting particularly relevant when some state variables are unobservable, making the fully observed LQR inapplicable. In the fully observed setting, it is standard practice to compute the control input directly based on the current state. However, in the partially observed setting, selecting the control based on the unobservable state introduces strong dependencies between future observations and past inputs/outputs. This coupling complicates the estimation of unknown system parameters and makes the learning problem intractable using standard techniques.

In recent years, the problem of unknown parameter estimation in partially observed LQR systems has attracted increasing attention. To mitigate the correlation between state estimation and past input/output data—and to make parameter estimation tractable—a common strategy is to excite the system by applying control inputs that are independent and identically distributed (i.i.d.) Gaussian random variables with zero mean [54, 72, 34, 68]. Specifically, the control inputs are chosen as $u_t \stackrel{i.i.d.}{\sim} \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$ where d is the dimension of the control variable. Then, one collects a single trajectory consisting of the input control variables and corresponding output observation variables. Once a sufficiently long trajectory has been gathered, the data is used to estimate the Markov parameter matrix, from which the unknown system parameters can be recovered, up to a similarity transformation, using the Ho-Kalman algorithm, a singular value decomposition (SVD) based algorithm. Unfortunately, these methods suffer from similar limitations as those encountered in the fully observed large-scale setting, including high sample complexity, large estimation error, and substantial computational cost. Moreover, because the system parameters are recovered from the estimated Markov parameter matrix, the resulting estimates are not unique, due to

the inherent ambiguity of the Ho-Kalman realization algorithm. These limitations motivate the development of a more efficient algorithm that is better suited for large-scale systems and practical implementation.

Inspired by the promising results of the parametric approach used in the fully observed case, we study the scalable partially observed linear quadratic regulator problem with stationary dynamics:

$$\begin{aligned} \min_{\substack{\{x_t\}_{t \in [T+1]} \\ \{u_t\}_{t \in [T]}}} & \frac{1}{2} x_T^T Q x_T + \sum_{t=0}^{T-1} \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t, \\ \text{s.t. } & x_0 = \bar{x}_0, \\ & x_{t+1} = A x_t + B u_t + w_t, \quad t \in [T], \\ & y_t = C x_t + D u_t + v_t, \quad t \in [T+1]. \end{aligned}$$

where T is the length of horizon, $\bar{x}_0 \sim \mathcal{N}(\mathbf{0}, \Sigma_0)$, $x_t \in \mathbb{R}^n$ is the state variable which is not observed, $u_t \in \mathbb{R}^d$ is the control variable, $y_t \in \mathbb{R}^m$ is the observed variable, and $w_t \sim \mathcal{N}(\mathbf{0}, \sigma_x^2 I_n)$ and $v_t \sim \mathcal{N}(\mathbf{0}, \sigma_y^2 I_m)$ are random noise. We assume that C and D are known sparse matrices, but A and B are unknown sparse matrices. In addition, A and B exhibit a *parametric* structure of the following form:

$$\begin{aligned} A &= \sum_{j=1}^{n_A} \alpha_j A_j, \quad \alpha_j \in \mathbb{R}, \quad A_j \in \mathbb{R}^{n \times n}, \quad \alpha^T = \begin{bmatrix} \alpha_1 & \dots & \alpha_{n_A} \end{bmatrix} \quad \alpha \in \mathbb{R}^{n_A}, \\ B &= \sum_{k=1}^{n_B} \beta_k B_k, \quad \beta_k \in \mathbb{R}, \quad B_k \in \mathbb{R}^{n \times d}, \quad \beta^T = \begin{bmatrix} \beta_1 & \dots & \beta_{n_B} \end{bmatrix} \quad \beta \in \mathbb{R}^{n_B}. \end{aligned}$$

Here, we assume that α_j and β_k are unknown, while the basis matrices $\{A_j\}_{j=1}^{n_A}$ and $\{B_j\}_{j=1}^{n_B}$ are known constant sparse matrices. Therefore, we learn unknown vectors α and β instead of learning the whole matrices A and B . This system structure is especially effective in large-scale settings, where the dimension of x_t is much larger than the number of fixed

known matrices ($n \gg n_\theta = n_A + n_B$). PDEs serve as a representative example in which numerous discretization schemes can be formulated as linear combinations of a fixed set of sparse matrices. Additionally, the number of such basis matrices remains independent of the dimensionality of the system.

In our algorithm, we input the control variables $u_t \stackrel{i.i.d}{\sim} \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$ and collect a sufficiently long trajectory with input (u_t) and output y_t . Since y_t can be expressed as follows,

$$\begin{aligned} y_t &= Cx_t + Du_t + v_t \\ &= C \left(Ax_{t-1} + \left(\sum_{i=1}^{n_B} \beta_i B_i \right) u_{t-1} + w_{t-1} \right) + Du_t + v_t \\ &= Du_t + \left(\sum_{i=1}^{n_B} \beta_i C B_i \right) u_{t-1} + e_{t,B} + Cw_{t-1} + v_t \end{aligned} \tag{1.3}$$

where $e_{t,B} = CAx_{t-1}$ captures the error induced by the effect of the state at time $t-1$, and $e_{t,B}$, Cw_t , and v_t are treated as the additive noise. We first apply the OLS method to get the estimate of β and the estimated B . Then, using the fact that y_t can be further written as

$$\begin{aligned} y_t &= Cx_t + Du_t + v_t \\ &= C(Ax_{t-1} + Bu_{t-1} + w_{t-1}) + Du_t + v_t \\ &= C(A(Ax_{t-2} + Bu_{t-2} + w_{t-2}) + Bu_{t-1} + w_{t-1}) + Du_t + v_t \\ &= Du_t + CBu_{t-1} + \sum_{i=1}^{n_A} \alpha_i CA_i Bu_{t-2} + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t \end{aligned} \tag{1.4}$$

where $e_{t,A} = CA^2x_{t-2}$ captures the error induced by the effect of the state at time $t-2$, and $e_{t,A}$, Cw_t , CAw_{t-2} , and v_t are treated as the additive noise. With the estimated B , we apply the OLS method again to get the estimate of α and the estimated A .

Generally, the proposed algorithm has four contributions summarized as follows: (1)

Improved Sample Complexity: The required sample complexity of the proposed algorithm scales linearly with $\log(n)$, whereas the sample complexity in [54] is at least linear in n , where n denotes the dimension of the state variables. (2) Dimension-Independent Estimation Error: The estimation error of the proposed algorithm is independent of n , in contrast to the classical approach, which exhibits an estimation error on the order of $\sqrt{n}$. (3) Significantly Low Computational Cost: The computational cost of the proposed algorithm scales linearly with n . In comparison, the algorithm in [54] needs the cost of $O(n^3)$ in the OLS estimation stage and an additional $O(n^3)$ for the Ho-Kalman realization, which includes a singular value decomposition (SVD) step. Therefore, the total computational cost of that method is $O(n^3)$ per time step. (4) Unique Parameter Estimates: The proposed algorithm yields unique parameter estimates, in contrast to the method in [54], where the parameters recovered from the Ho-Kalman algorithm are only identifiable up to a similarity transformation and therefore not unique.

However, one main drawback of the OLS approach for the unknown parameter estimation is its stability assumption on A , excluding large classes of problems with unstable eigenvalues. At present, we do not see a practical way to relax this stability requirement within the OLS framework. Therefore, we want to try different algorithms. Motivated by the classical EM algorithm, we propose a scalable EM-like algorithm, which is a candidate to be extended to more general classes of problems-reducing the restriction on A . This algorithm is still in progress, and this chapter presents the current version of the algorithm along with its underlying framework. At the beginning of the algorithm, we generate input sequences $\{u_t\}_{t=1}^N$ and $\{y_t\}_{t=1}^N$. At every iteration, in the E-step, the algorithm obtains the estimated unobserved state variable based on the current estimator using the approximated Kalman gain instead of the exact Kalman gain. Then, in the M-step, treating the estimated unobserved state variable as the observed data, we get the updated estimator by performing OLS. This iterative process continues until convergence. The classical EM algorithm requires comput-

ing the exact Kalman gain and performing a full Kalman smoothing procedure, which incurs a computational cost of $O(n^3)$ per time step. In contrast, the computational cost of the proposed algorithm scales linearly with $n \cdot W$, where n is the state dimension and W is the truncated MPC horizon. We aim to show that $W = O(\log T)$, which ensures scalability and computational efficiency, particularly for problems with long time horizons. However, we acknowledge that the current version of the algorithm still relies on the stability assumption of A , and we offer preliminary insights into potential directions for relaxing this requirement in future work.

CHAPTER 2

DISTRIBUTED SQP WITH GRAPH DECOMPOSITION AND AUGMENTED LAGRANGIAN

2.1 Introduction

In this chapter, we address the challenge of solving large-scale graph-structured nonlinear programs (gsNLPs) in a scalable manner. GsNLPs are problems in which the objective and constraint functions are associated with nodes on a graph and depend on the variables of adjacent nodes. This graph-structured formulation encompasses various specific instances, such as dynamic optimization, PDE-constrained optimization, multistage stochastic optimization, and general network optimization. By leveraging the sequential quadratic programming (SQP) framework, we propose a globally convergent overlapping graph decomposition method to solve large-scale gsNLPs under standard mild regularity conditions on the graph topology. In each iteration, we perform an overlapping graph decomposition to compute an approximate Newton direction in a parallel environment. Then, we select a suitable stepsize and update the primal-dual iterate by performing a backtracking line search on an exact augmented Lagrangian merit function. Built on the exponential decay of sensitivity of gsNLPs, we show that the approximate Newton direction is a descent direction of the augmented Lagrangian, which leads to global convergence with local linear convergence rate. In particular, global convergence is achieved for sufficiently large overlaps, and the local linear convergence rate improves exponentially in terms of the overlap size. Our results match existing state-of-the-art guarantees established for dynamic programs (which simply correspond to linear graphs). We validate the theory on a semilinear elliptic PDE-constrained problem.

2.2 Background

We consider *graph-structured nonlinear programs* (gsNLPs) [62]:

$$\min_{\{\mathbf{x}_i \in \mathbb{R}^{r_i}\}_{i \in \mathcal{V}}} \sum_{i \in \mathcal{V}} f_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]}), \quad (2.1a)$$

$$\text{s.t.} \quad \mathbf{c}_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]}) = \mathbf{0}, \quad i \in \mathcal{V}, \quad (\boldsymbol{\lambda}_i). \quad (2.1b)$$

Here, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected graph, $\mathcal{V}$ is the strictly ordered node set, $\mathcal{E} \subseteq \{\{i, j\} : i, j \in \mathcal{V}, i \neq j\}$ is the edge set, and $N_{\mathcal{G}}[i] := \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\} \cup \{i\}$ is the closed neighborhood of $i \in \mathcal{V}$ on $\mathcal{G}$. For each node i , we have the following associated quantities: $\mathbf{x}_i \in \mathbb{R}^{r_i}$ is the primal variable, $f_i : \Pi_{j \in N_{\mathcal{G}}[i]} \mathbb{R}^{r_j} \rightarrow \mathbb{R}$ is the objective function, $\mathbf{c}_i : \Pi_{j \in N_{\mathcal{G}}[i]} \mathbb{R}^{r_j} \rightarrow \mathbb{R}^{m_i}$ is the equality constraint function, and $\boldsymbol{\lambda}_i \in \mathbb{R}^{m_i}$ is the dual variable associated with the constraint $\mathbf{c}_i$. Problem (2.1) arises in various applications, including optimal control [7], partial differential equation (PDE)-constrained problems [8], multistage stochastic optimization [56], and network optimization [80].

In this chapter, we develop a scalable method for solving large-scale problems of form eq. (2.1). The complexity of these problems stems from the large size of the underlying graph $\mathcal{G}$, which can make many centralized methods intractable. To address this issue, decomposition-based methods are of particular interest. In these methods, the intractable full problem is decomposed into a set of smaller subproblems, each designed to be individually tractable on the local machine. Decomposition methods apply an iterative manner to ensure the convergence of the iterates toward the full solution; the convergence is achieved by exchanging solution information among (adjacent) subproblems in each iteration step. Examples of decomposition methods include Lagrangian decomposition [27, 3], temporal decomposition [27, 77], Jacobi or Gauss–Seidel methods [79], and alternating direction method of multipliers (ADMM) [20].

Although decomposition methods offer several advantages over centralized methods, such

as computational scalability and flexibility in implementation, they may exhibit slower convergence rates compared to centralized methods (e.g., Ipopt) [31]. The slow convergence can be attributed to their iterative manner and limited amount of information shared across the subproblems. As investigated in the benchmark study [31], the practically observed convergence rates of existing decomposition schemes are often too slow to make those schemes practically useful. Such limitations motivate the design of a new decomposition paradigm, which aims to resolve the issue of slow convergence while preserving the desired scalability.

Unlike the majority of decomposition methods that exclusively decompose the full problem, there is a fast growing literature studying the *overlapping decomposition* [61, 60, 50, 48]. This type of methods decomposes the full problem into a set of *overlapping* subproblems. The overlap can facilitate exchanging more information between (adjacent) subproblems in each iteration, hereby significantly improving the convergence behavior compared to the methods that perform exclusive decomposition. One particular instance of overlapping decomposition is the *overlapping Schwarz method*, whose origin can be traced back to classical domain decomposition methods proposed for solving second-order elliptic PDEs [14, 13]. The overlapping Schwarz method has been applied to various structured problems, and its convergence properties have been investigated under different settings, including unconstrained quadratic programs [61], constrained quadratic programs [60], and nonlinear optimal control [48]. This series of literature has established not only the *local convergence* of the Schwarz method, but also the relationship between its local convergence rate and the overlap size. Specifically, the authors have shown that *the local linear convergence rate of the Schwarz method improves exponentially with respect to the overlap size*, meaning that one can address the issue of slow convergence in classical (non-overlapping) decomposition methods by slightly increasing the subproblem complexity.

While the overlapping Schwarz method may enjoy some benefits over classical methods, it faces two major concerns. First, the subproblems in each iteration of the Schwarz method

are nonlinear and nonconvex, and the method solves *all* subproblems to *optimality*, albeit in parallel, leading to significant per-iteration computational costs on each local machine. Second, the Schwarz method lacks the utilization of sophisticated globalization strategies, such as backtracking line search and trust region, to enforce global convergence. In particular, the aforementioned works on the Schwarz method only provided a local convergence guarantee, meaning that the convergence is guaranteed to occur only when the initial guess is sufficiently close to the local solution. The above concerns have motivated a recent enhanced algorithm design to reduce per-iteration computational costs and incorporate a globalization strategy into the Schwarz method, while still preserving its desired fast local linear convergence rate [50]. In that work, the authors designed a *Fast Overlapping Temporal Decomposition* (FOTD) procedure for solving nonlinear dynamic programs. The method embeds overlapping temporal decomposition into the Sequential Quadratic Programming (SQP) framework, where the decomposition is applied to the (large-scale) quadratic program in each SQP step, making all subproblems easily-solvable quadratic programs. Additionally, FOTD performs backtracking line search on a differentiable exact augmented Lagrangian merit function. Although the exact augmented Lagrangian serves as an alternative to non-differentiable ℓ_1/ℓ_2 merit functions in SQP methods (see [47, 26] and references therein), recent research has demonstrated that it may offer particular benefits when applied to dynamic programs under different settings [44, 50]. Specifically, with such a globalization strategy, [50] showed that FOTD enjoys global convergence while preserving the local linear convergence rate of the Schwarz method.

Inspired by the promising results of FOTD in dynamic programs, this chapter generalizes that method to a much broader class of nonlinear programs. In particular, we design the *Fast Overlapping Graph Decomposition* (FOGD) method to handle gsNLPs in the form of eq. (2.1), so that dynamic programs simply correspond to linear graphs. To achieve global convergence, we leverage the SQP framework with a similar flavor to [50]. Specifically, we apply an overlapping graph decomposition to compute an approximate Newton direction, which is often re-

garded as the main computational bottleneck of the SQP methods. Then, we select a proper stepsize by conducting line search on the exact augmented Lagrangian function. Our main technical contribution lies in demonstrating the global and local convergence of the FOGD method. We propose mild regularity conditions on the topology of the graph $\mathcal{G}$ to restrict its growth rate. (These conditions exclude tree graphs, which grow exponentially as seen in multistage stochastic optimization. Intuitively, accumulating exponentially small decomposition errors for an exponential number of nodes may not be well controlled.) Under the regularity conditions, we decompose the full quadratic program in each SQP iteration into overlapping subproblems while carefully specifying the boundary loss and constraints. Through our subproblem formulation, we establish the error between the approximate Newton direction and the exact Newton direction. Ultimately, we show that the approximate Newton direction is a descent direction of the augmented Lagrangian, leading to global convergence with a local linear convergence rate. To the best of our knowledge, FOGD is the first overlapping decomposition scheme that can solve gsNLPs with a global convergence guarantee while matching the state-of-the-art local linear convergence rate of the Schwarz method.

Structure of the chapter The remainder of this chapter is organized as follows. In Section 2.3, we introduce SQP and OGD and propose the FOGD procedure. In Section 2.4, we establish the error bound of the approximate Newton direction. Then, we prove the global convergence of FOGD in Section 2.5, and prove the local linear convergence and show the relationship between the linear rate and the overlap size in Section 2.6. We present numerical experiments to demonstrate our theoretical findings in Section 2.7, and conclude with future directions in Section 2.8. Some of the proofs are deferred to the appendix of this chapter to make the main chapter compact.

Notation Throughout the chapter, all vectors $\mathbf{x}_i$ are assumed to be column vectors. For any integer n , we let $[n] := \{1, \dots, n\}$. For any set $\mathcal{I}$, we denote $|\mathcal{I}|$ as the number of elements

in $\mathcal{I}$. For any two sets $\mathcal{I}$ and $\mathcal{J}$, we let $\mathcal{I} \setminus \mathcal{J} := \{i : i \in \mathcal{I}, i \notin \mathcal{J}\}$. For a strictly ordered node set $\mathcal{I} = \{i_1, i_2, \dots, i_{|\mathcal{I}|}\}$ with $i_1 < i_2 < \dots < i_{|\mathcal{I}|}$, we define $\{\mathbf{x}_i\}_{i \in \mathcal{I}} := (\mathbf{x}_{i_1}; \dots; \mathbf{x}_{i_{|\mathcal{I}|}})$ as the vector that stacks all variables of $\mathcal{I}$. For a (one- or) two-way indexed array M , $M[\mathcal{I}][\mathcal{J}]$ represents the subarray where the first index lies in the (strictly ordered node) set $\mathcal{I}$, and the second index lies in the set $\mathcal{J}$. For the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a node set $\mathcal{W} \subseteq \mathcal{V}$, we let $N_{\mathcal{G}}^b[\mathcal{W}] := \{i \in \mathcal{V} : d_{\mathcal{G}}(i, \mathcal{W}) \leq b\}$ be the b -hop neighborhood of $\mathcal{W}$, where $d_{\mathcal{G}}(i, \mathcal{W})$ is the number of edges in the shortest path connecting the node i and the set $\mathcal{W}$ on graph $\mathcal{G}$. We omit the superscript b when $b = 1$, i.e., $N_{\mathcal{G}}[\mathcal{W}] = N_{\mathcal{G}}^1[\mathcal{W}]$. We also let $N_{\mathcal{G}}(\mathcal{W}) := N_{\mathcal{G}}[\mathcal{W}] \setminus \mathcal{W}$ be the neighborhood of $\mathcal{W}$ excluding itself (i.e., the open neighborhood).

2.3 Fast Overlapping Graph Decomposition

We present FOGD in three steps: (1) introduce the Newton system in each SQP iteration, (2) apply OGD to approximately solve the Newton system, and (3) select a proper stepsize by performing line search on the exact augmented Lagrangian merit function. We summarize the full method in algorithm 1.

Step 1: the Newton system in each SQP iteration We define the Lagrangian function of eq. (2.1) as $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{x})$, where $f(\mathbf{x}) = \sum_{i \in \mathcal{V}} f_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]})$ is the objective function, $\mathbf{c}(\mathbf{x}) = \{\mathbf{c}_i(\{\mathbf{x}_j\}_{j \in N_{\mathcal{G}}[i]})\}_{i \in \mathcal{V}}$ are the equality constraints, and $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_i\}_{i \in \mathcal{V}}$ are the dual variables associated with the constraints. We apply Newton's method to the Karush-Kuhn-Tucker (KKT) condition $\nabla \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$. In particular, given the τ -th iterate $(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau)$, the search direction $(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau)$ is computed by solving the following Newton system:

$$\begin{bmatrix} \hat{H}^\tau & (G^\tau)^T \\ G^\tau & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^\tau \\ \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} = - \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}^\tau \end{bmatrix}, \quad (2.2)$$

where $\nabla \mathcal{L}^\tau = \nabla \mathcal{L}(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau)$ is the Lagrangian gradient, $G^\tau = \nabla \mathbf{c}(\mathbf{x}^\tau)$ is the constraints Jacobian, and $\hat{H}^\tau$ is a modification of the Lagrangian Hessian $H^\tau = \nabla_{\mathbf{x}}^2 \mathcal{L}^\tau$. The modification is to ensure that $\hat{H}^\tau$ is positive definite in the null space $\{\boldsymbol{\omega} : G^\tau \boldsymbol{\omega} = \mathbf{0}\}$, which, together with the full row-rankness of G^τ , ensures the well-definedness of the system (2.2) [53, Lemma 16.1].

Solving the Newton system eq. (2.2) is equivalent to solving the quadratic program:

$$\min_{\boldsymbol{\omega}} \quad \frac{1}{2} \boldsymbol{\omega}^T \hat{H}^\tau \boldsymbol{\omega} + \boldsymbol{\omega}^T \nabla_{\mathbf{x}} \mathcal{L}^\tau, \quad \text{s.t.} \quad \mathbf{c}^\tau + G^\tau \boldsymbol{\omega} = \mathbf{0}, \quad (\boldsymbol{\zeta}), \quad (2.3)$$

where $\mathbf{c}^\tau = \mathbf{c}(\mathbf{x}^\tau)$ and $\boldsymbol{\zeta}$ is the dual vector. Since the solution of eq. (2.2) is the unique global solution of Problem eq. (2.3), we let $((\boldsymbol{\omega}^\tau)^*, (\boldsymbol{\zeta}^\tau)^*)$ be the solution of eq. (2.3) and have $(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau) = ((\boldsymbol{\omega}^\tau)^*, (\boldsymbol{\zeta}^\tau)^*)$. To ease the notation, we may occasionally suppress the iteration index τ when referring to the variables in a generic iteration.

Step 2: OGD on the Newton system For large-scale gsNLPs, solving eq. (2.3) exactly can be computationally intractable. To address this issue, we apply OGD to eq. (2.3) to compute an approximate solution in a parallel environment. The OGD procedure is summarized as follows. First, we decompose the entire problem domain $\mathcal{V}$ into a set of overlapping subdomains. Next, we formulate a subproblem for each subdomain by considering only the associated objectives and constraints within the subdomain. Then, we solve the subproblems in parallel; the solutions from the exclusive parts of each subdomain are composed to form the final approximate Newton direction.

In particular, let us consider a disjoint partition $\{\mathcal{V}_\ell\}_{\ell \in [M]}$ of the full node set $\mathcal{V}$, and extend each set $\mathcal{V}_\ell$ to its b -hop neighborhood, where $b \geq 1$ is a user-specified parameter (called the overlap size):

$$\mathcal{V} = \bigcup_{\ell=1}^M \mathcal{V}_\ell, \quad \mathcal{V}_\ell \text{ disjoint with each other,} \quad N_{\mathcal{G}}^b[\mathcal{V}_\ell] \subseteq \mathcal{W}_\ell, \quad \forall \ell \in [M].$$

We may just let $\mathcal{W}_\ell = N_{\mathcal{G}}^b[\mathcal{V}_\ell]$ in the simple case. With the above overlapping decomposition $\{\mathcal{W}_\ell\}_{\ell \in [M]}$, we define the subproblem $\mathcal{SP}_\ell^{\tau, \mu}$ for the subdomain $\mathcal{W}_\ell$ in the τ -th iteration as:

$$\min_{\tilde{\omega}_\ell} \frac{1}{2} \tilde{\omega}_\ell^T \hat{H}^\tau[\mathcal{W}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell + \tilde{\omega}_\ell^T \nabla_{\mathbf{x}} \mathcal{L}^\tau[\mathcal{W}_\ell] + \frac{\mu}{2} \left\| \mathbf{c}^\tau[\hat{\mathcal{W}}_\ell] + G^\tau[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell \right\|^2, \quad (2.4a)$$

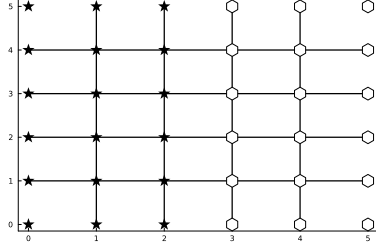
$$\text{s.t. } \mathbf{c}^\tau[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] + G^\tau[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell = \mathbf{0}, \quad (\tilde{\zeta}_\ell), \quad (2.4b)$$

where $\tilde{\mathcal{W}}_\ell := N_{\mathcal{G}}(N_{\mathcal{G}}(\mathcal{W}_\ell)) \cap \mathcal{W}_\ell$ is the set of internal boundary nodes of $\mathcal{W}_\ell$, $\hat{\mathcal{W}}_\ell := N_{\mathcal{G}}(\mathcal{W}_\ell) \cup \tilde{\mathcal{W}}_\ell$ is the set of internal and external boundary nodes of $\mathcal{W}_\ell$, and $\tilde{\omega}_\ell = \omega[\mathcal{W}_\ell]$ and $\tilde{\zeta}_\ell = \zeta[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell]$ are the primal and dual (sub-)variables. The solution of Problem (2.4) is denoted as $((\tilde{\omega}_\ell^\tau)^*, (\tilde{\zeta}_\ell^\tau)^*)$, though again we may omit the index τ to ease the notation. An illustration of all the sets $\mathcal{V}_\ell$, $\mathcal{W}_\ell$, $N_{\mathcal{G}}(\mathcal{W}_\ell)$, $\tilde{\mathcal{W}}_\ell$, and $\hat{\mathcal{W}}_\ell$ is shown in fig. 2.1.

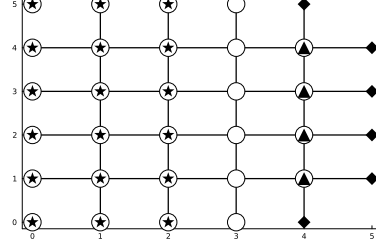
The subproblem (2.4) is essentially the truncation of the full problem (2.3) onto the set $\mathcal{W}_\ell$. However, there is a key difference: only the *internal constraints* — those that fully depend on the variables within $\mathcal{W}_\ell$ — are enforced in (2.4b), while the *coupling constraints* — those that partially depend on the variables outside $\mathcal{W}_\ell$ — are incorporated as an augmented Lagrangian-like regularization term in (2.4a). As will be shown in lemma 2.4.1, this formulation ensures the subproblems to satisfy the desired regularity conditions, provided that the conditions are satisfied by the full problem (2.3) and the parameter $\mu > 0$ is large enough.

In each SQP iteration, we solve M overlapping subproblems $\{\mathcal{SP}_\ell^{\mu}\}_{\ell \in [M]}$ in parallel, and for each subproblem ℓ , we retain only the solution $(\tilde{\omega}_\ell^*[\mathcal{V}_\ell], \tilde{\zeta}_\ell^*[\mathcal{V}_\ell])$ within the disjoint set $\mathcal{V}_\ell$. Then, we compose all solution pieces to obtain the approximate Newton direction $(\tilde{\Delta}\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda})$ (in the τ -th iteration). In particular, we introduce the following definition. For a subdomain variable $\tilde{\omega}_\ell$ (similar for $\tilde{\zeta}_\ell$), we let $\tilde{\omega}_{\ell,k} := \tilde{\omega}_\ell[\{k\}]$ be the variable associated with the node $k \in \mathcal{W}_\ell$. This notation aims to distinguish $\tilde{\omega}_{\ell_1,k}$ and $\tilde{\omega}_{\ell_2,k}$ when both $k \in \mathcal{W}_{\ell_1}$ and $k \in \mathcal{W}_{\ell_2}$.

Definition 2.3.1 (Composition and Decomposition). *Given M subdomain variables $\{(\tilde{\omega}_\ell, \tilde{\zeta}_\ell)\}_{\ell=1}^M$, we define the composition operator as $\mathcal{C}(\{(\tilde{\omega}_\ell, \tilde{\zeta}_\ell)\}_{\ell=1}^M) = (\omega, \zeta)$, where for*



(a) $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$



(b) $\mathcal{V}_1$, $\mathcal{W}_1$, $N_G(\mathcal{W}_1)$, $\tilde{\mathcal{W}}_1$, and $\hat{\mathcal{W}}_1$

Figure 2.1: Illustration of $\mathcal{V}_i$, $\mathcal{W}_i$, $N_G(\mathcal{W}_i)$, $\tilde{\mathcal{W}}_i$, and $\hat{\mathcal{W}}_i$. We consider a two-dimensional graph divided into two subgraphs with the overlap size $b = 2$. In fig. 2.1a, star markers represent $\mathcal{V}_1$ and hexagon markers represent $\mathcal{V}_2$. In fig. 2.1b, star markers represent $\mathcal{V}_1$, circle markers represent $\mathcal{W}_1$, diamond markers represent $N_G(\mathcal{W}_1)$, triangle markers represent $\tilde{\mathcal{W}}_1$, and the combination of diamond and triangle markers represents $\hat{\mathcal{W}}_1$.

any node $k \in \mathcal{V}$, $(\omega_k, \zeta_k) := (\tilde{\omega}_{\ell,k}, \tilde{\zeta}_{\ell,k})$ for (the unique) ℓ such that $k \in \mathcal{V}_\ell$. Conversely, given a full-domain variable (ω, ζ) , we define the decomposition operator as $\mathcal{D}(\omega, \zeta) = \{\mathcal{D}_\ell(\omega, \zeta)\}_{\ell=1}^M$ with $\mathcal{D}_\ell(\omega, \zeta) = (\tilde{\omega}_\ell, \tilde{\zeta}_\ell) = (\omega[\mathcal{W}_\ell], \zeta[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell])$.

By definition 2.3.1, at the τ -th iteration, we have $(\tilde{\Delta}\mathbf{x}^\tau, \tilde{\Delta}\boldsymbol{\lambda}^\tau) = \mathcal{C}(\{((\tilde{\omega}_\ell^\tau)^*, (\tilde{\zeta}_\ell^\tau)^*)\}_{\ell=1}^M)$.

Step 3: stepsize selection with exact augmented Lagrangian After obtaining the approximate Newton direction $(\tilde{\Delta}\mathbf{x}^\tau, \tilde{\Delta}\boldsymbol{\lambda}^\tau)$, we then select a proper stepsize α_τ by performing backtracking line search on a differentiable exact augmented Lagrangian merit function. Particularly, we update the primal-dual iterate as

$$\begin{bmatrix} \mathbf{x}^{\tau+1} \\ \boldsymbol{\lambda}^{\tau+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^\tau \\ \boldsymbol{\lambda}^\tau \end{bmatrix} + \alpha_\tau \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix}, \quad (2.5)$$

where α_τ is selected to satisfy the Armijo condition for a parameter $\beta \in (0, 0.5)$ as

$$\mathcal{L}_\eta^{\tau+1} \leq \mathcal{L}_\eta^\tau + \beta \alpha_\tau \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix}, \quad (2.6)$$

and $\mathcal{L}_\eta^\tau = \mathcal{L}_\eta(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau)$ (similar for $\mathcal{L}_\eta^{\tau+1}$, $\nabla \mathcal{L}_\eta^\tau$) is the exact augmented Lagrangian, defined as

$$\mathcal{L}_\eta(\mathbf{x}, \boldsymbol{\lambda}) = \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) + \frac{\eta_1}{2} \|\nabla_{\boldsymbol{\lambda}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\|^2 + \frac{\eta_2}{2} \|\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})\|^2 \quad (2.7)$$

with the penalty parameters $\eta = (\eta_1, \eta_2)$. Here, the first penalty biases the feasibility residual while the second penalty biases the optimality residual.

Although there are many alternative non-differentiable ℓ_1/ℓ_2 merit functions [53], recent research has shown that the exact augmented Lagrangian (2.7) may offer certain benefits when applied to optimal control problems [50, 44], as well as to more general graph-structured problems as demonstrated in this chapter. Roughly speaking, we desire a merit function that involves the dual variables to endure an approximation error in the dual component. Furthermore, we also desire a differentiable merit function that can overcome the Maratos effect and achieve fast local convergence (by accepting the unit stepsize) without the need of advanced enhancements of the method (e.g., second-order correction), as required by non-differentiable merit functions. The enhancements like second-order correction have to be decentralized in the distributed setting, significantly complicating the algorithm applicability. The detailed discussion regarding the choice of the merit function is deferred to lemma 2.5.2 and lemma 2.6.1.

We finally summarize the complete FOGD method in algorithm 1.

Algorithm 1 A Fast Overlapping Graph Decomposition Method

- 1: **Input:** initialization $(\mathbf{x}^0, \boldsymbol{\lambda}^0)$, number of subproblems M , scalars $\mu, \eta > 0$, $\beta \in (0, 0.5)$;
 - 2: **for** $\tau = 0, 1, 2, \dots$ **do**
 - 3: Compute $\hat{H}^\tau, G^\tau$, and $\nabla \mathcal{L}^\tau$;
 - 4: **for** $\ell = 1, 2, \dots, M$ (**in parallel**) **do**
 - 5: Solve $\mathcal{SP}_\ell^{\tau, \mu}$ in (2.4) to obtain the solution $((\tilde{\omega}_\ell^\tau)^*, (\tilde{\zeta}_\ell^\tau)^*)$;
 - 6: **end for**
 - 7: Let $(\tilde{\Delta} \mathbf{x}^\tau, \tilde{\Delta} \boldsymbol{\lambda}^\tau) = \mathcal{C}(\{((\tilde{\omega}_\ell^\tau)^*, (\tilde{\zeta}_\ell^\tau)^*)\}_{\ell=1}^M)$;
 - 8: Update the iterate as (2.5) by selecting α_τ via (2.6);
 - 9: **end for**
-

2.4 Error Analysis of OGD

We have introduced how to approximately solve the Newton system in each SQP iteration by applying overlapping graph decomposition. Then, a natural question is how close the approximate direction is to the exact direction. In this section, we answer this question by providing an upper bound for the approximation error $(\tilde{\Delta}\mathbf{x}^\tau - \Delta\mathbf{x}^\tau, \tilde{\Delta}\boldsymbol{\lambda}^\tau - \Delta\boldsymbol{\lambda}^\tau)$. We will finally show in theorem 2.4.1 that, under standard mild regularity conditions, the error decays exponentially fast in terms of the overlap size b .

To study the approximation error, we leverage the sensitivity analysis of graph-structured linear-quadratic programs. In particular, as seen in lemma 2.4.3, the truncated exact direction onto the node set $\mathcal{W}_\ell$, that is $\mathcal{D}_\ell(\Delta\mathbf{x}^\tau, \Delta\boldsymbol{\lambda}^\tau) = \mathcal{D}_\ell((\boldsymbol{\omega}^\tau)^*, (\boldsymbol{\zeta}^\tau)^*)$, is the solution of a linear-quadratic program in a form similar to (2.4). The only difference lies in the formulation for boundary nodes. We hereby introduce a parameterization trick such that the linear-quadratic program for the truncated exact direction corresponds to one parameter, while the subproblem (2.4) for the approximate direction corresponds to the other. By sensitivity analysis, the approximation error will be fully characterized by the perturbation of these two parameters. Moreover, since parameter perturbations only occur at boundary nodes, and we retain only the solution for the internal, exclusive part of each overlapping node set, recent studies on the exponential decay of sensitivity [45, 62] are then applied (contingent on checking several conditions for subproblems) to establish our final error bound.

Following the above logic, let us first introduce a parameterized version of Problem (2.4),

denoted as $\mathcal{SP}_\ell^{\tau,\mu}(\mathbf{d}_\ell)$:

$$\min_{\tilde{\boldsymbol{\omega}}_\ell} \frac{1}{2} \begin{bmatrix} \tilde{\boldsymbol{\omega}}_\ell \\ \bar{\boldsymbol{\omega}}_\ell \end{bmatrix}^T \begin{bmatrix} \hat{H}^\tau[\mathcal{W}_\ell][\mathcal{W}_\ell] & \hat{H}^\tau[\mathcal{W}_\ell][\bar{\mathcal{W}}_\ell] \\ \hat{H}^\tau[\bar{\mathcal{W}}_\ell][\mathcal{W}_\ell] & \hat{H}^\tau[\bar{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell] \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\omega}}_\ell \\ \bar{\boldsymbol{\omega}}_\ell \end{bmatrix} + \tilde{\boldsymbol{\omega}}_\ell^T \nabla_{\mathbf{x}} \mathcal{L}^\tau[\mathcal{W}_\ell] \quad (2.8a)$$

$$+ \bar{\boldsymbol{\zeta}}_\ell^T G^\tau[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]\tilde{\boldsymbol{\omega}}_\ell + \frac{\mu}{2} \left\| \mathbf{c}^\tau[\hat{\mathcal{W}}_\ell] + G^\tau[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]\tilde{\boldsymbol{\omega}}_\ell + G^\tau[\hat{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell]\bar{\boldsymbol{\omega}}_\ell \right\|^2,$$

$$\text{s.t. } \mathbf{c}^\tau[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] + G^\tau[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell]\tilde{\boldsymbol{\omega}}_\ell = \mathbf{0}, \quad (\tilde{\boldsymbol{\zeta}}_\ell), \quad (2.8b)$$

where $\tilde{\mathcal{W}}_\ell, \hat{\mathcal{W}}_\ell$ are defined in (2.4), $\bar{\mathcal{W}}_\ell := N_{\mathcal{G}}(\mathcal{W}_\ell) \cup N_{\mathcal{G}}(N_{\mathcal{G}}[\mathcal{W}_\ell])$ is the set of external boundary nodes of $\mathcal{W}_\ell$ within depth two, and $\mathbf{d}_\ell = (\bar{\boldsymbol{\omega}}_\ell, \bar{\boldsymbol{\zeta}}_\ell) = (\{\bar{\boldsymbol{\omega}}_{\ell,j}\}_{j \in \bar{\mathcal{W}}_\ell}, \{\bar{\boldsymbol{\zeta}}_{\ell,j}\}_{j \in \hat{\mathcal{W}}_\ell})$ are primal-dual boundary parameters specified to the problem. Clearly, the solution of $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ depends on the parameters $\mathbf{d}_\ell$, which we denote as $(\tilde{\boldsymbol{\omega}}_\ell^*(\mathbf{d}_\ell), \tilde{\boldsymbol{\zeta}}_\ell^*(\mathbf{d}_\ell))$ (the iteration index τ is omitted). It is worth noting that when $\mathbf{d}_\ell = \mathbf{0}$, $\mathcal{SP}_\ell^\mu(\mathbf{0})$ reduces to $\mathcal{SP}_\ell^\mu$ in (2.4).

We then introduce some preliminary assumptions.

Assumption 2.4.1 (Hessian modification). *For any $\tau \geq 0$, we assume the modified Hessian $\hat{H}^\tau$ satisfies $\boldsymbol{\omega}^T \hat{H}^\tau \boldsymbol{\omega} \geq \gamma_H \|\boldsymbol{\omega}\|^2$ for any $\boldsymbol{\omega} \in \{\boldsymbol{\omega} : G^\tau \boldsymbol{\omega} = \mathbf{0}\}$ for a constant $\gamma_H \in (0, 1]$.*

Assumption 2.4.2 (Linear independence constraint qualification). *For any $\tau \geq 0$, we assume $G^\tau (G^\tau)^T \succeq \gamma_G I$ for a constant $\gamma_G \in (0, 1]$.*

Assumption 2.4.3 (Uniform boundedness). *For any $\tau \geq 0$ and any nodes $i, j \in \mathcal{V}$, we assume $\max\{\|\hat{H}_{i,j}^\tau\|, \|H_{i,j}^\tau\|, \|G_{i,j}^\tau\|\} \leq \Upsilon$ for a constant $\Upsilon \geq 1$, where $\hat{H}_{i,j}^\tau, H_{i,j}^\tau$, and $G_{i,j}^\tau$ are the (i, j) -block of $\hat{H}^\tau, H^\tau$, and G^τ , respectively.*

Assumption 2.4.4 (Polynomial graph). *There are two polynomial functions $\text{poly}_{\mathcal{W}}(\cdot)$*

and $\text{poly}_{\mathcal{G}}(\cdot)$ such that

$$d_{\mathcal{G}}(i, j) \leq \text{poly}_{\mathcal{W}}(b), \quad \forall i, j \in \mathcal{W}_{\ell}, \ell \in [M], \quad (2.9a)$$

$$|\{j \in \mathcal{V} : d_{\mathcal{G}}(i, j) \leq d\}| \leq \text{poly}_{\mathcal{G}}(d), \quad \forall i \in \mathcal{V}, \quad (2.9b)$$

where b is the overlap size (cf. Section 2.3) and d is any integer.

Assumptions 2.4.1, 2.4.2, and 2.4.3 are standard assumptions in SQP literature [5, 22] and are crucial for sensitivity analysis of graph-structured problems [62]. In particular, Assumptions 2.4.1 and 2.4.2 ensure that the quadratic program (2.3) has a unique global solution [53, Lemma 16.1], and Assumption 2.4.3 implies a boundedness condition on the full matrices $(\hat{H}^{\tau}, H^{\tau}, G^{\tau})$. We note that the conditions on the upper and lower bound constants, $\Upsilon \geq 1 \geq \max(\gamma_H, \gamma_G)$, are inessential (similar for other constants later), which are only used to simplify the presentation. Without such conditions, our results still hold by replacing $\Upsilon, \gamma_H, \gamma_G$ with $\max(\Upsilon, 1), \min(\gamma_H, 1), \min(\gamma_G, 1)$, respectively.

To satisfy Assumption 2.4.1, we employ a structure-preserving modification procedure for the Lagrangian Hessian $H^{\tau} = \nabla_{\mathbf{x}}^2 \mathcal{L}^{\tau}$. One simple example is the Levenberg-type regularization [17]: $\hat{H}^{\tau} = H^{\tau} + (\gamma_H + \|H^{\tau}\|)I$. Assumption 2.4.2 is equivalent to assuming G^{τ} to have full row rank with the least singular value (uniformly) bounded away from zero. Furthermore, we should note that Assumption 2.4.3 actually only needs to be imposed on blocks with adjacent nodes, as both the Hessian H^{τ} and the Jacobian G^{τ} from Problem (2.1) exhibit a nice banded structure (so does $\hat{H}^{\tau}$):

$$H_{i,j}^{\tau} = \mathbf{0} \quad \text{if } d_{\mathcal{G}}(i, j) > 2, \quad G_{i,j}^{\tau} = \mathbf{0} \quad \text{if } d_{\mathcal{G}}(i, j) > 1.$$

With the above banded structure, we know that the block-wise boundedness implies the full matrix boundedness (with a bound independent of the graph size). Specifically, we apply As-

sumptions 2.4.3, 2.4.4 and [62, Lemma 5.10], and have for any $\tau \geq 0$,

$$\max\{\|\hat{H}^\tau\|, \|H^\tau\|, \|G^\tau\|\} \leq \Upsilon \cdot \text{poly}_{\mathcal{G}}(1)^2 =: \hat{\Upsilon}. \quad (2.10)$$

Assumption 2.4.4 imposes mild conditions on the graph topology. In particular, (2.9a) ensures that the diameter of each overlapping subdomain $\mathcal{W}_\ell$ is bounded by a polynomial of the overlap size, and (2.9b) ensures that the graph $\mathcal{G}$ can only grow polynomially. Combining (2.9a) and (2.9b), we know for any $\ell \in [M]$,

$$|\mathcal{W}_\ell| \leq \text{poly}_{\mathcal{G}}(\text{poly}_{\mathcal{W}}(b)) =: \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b). \quad (2.11)$$

We mention that (2.9b) excludes tree graphs, which grow exponentially as seen in multistage stochastic programs. Decomposing (scenario) tree structures that can break their curse of dimensionality is a challenging topic. The difficulty lies in the fact that the error induced by decomposition perturbations decays only linearly along the tree branches, which cannot govern the speed of their growth rate. A similar bottleneck has been observed in [64] and we defer the inclusion of tree structures to future work. That being said, it's important to note that all our established bounds are independent of the graph size $|\mathcal{V}|$ and the number of subproblems M .

Given Assumptions 2.4.1–2.4.4, we are able to establish the existence and uniqueness of the solution to subproblem $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ in the following lemma, which holds for any iteration $\tau \geq 0$. (The iteration index τ is omitted here and there.)

Lemma 2.4.1. *If Assumptions 2.4.1–2.4.4 hold for the full Problem (2.3), then for any $\ell \in [M]$ and any boundary variables $\mathbf{d}_\ell$, the subproblem $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ with $\mu \geq \hat{\mu} := 4\hat{\Upsilon}^2/(\gamma_G \gamma_H)$ will also satisfy these assumptions. In particular,*

(a) *the Jacobian $G_\ell := G^\tau[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell]$ satisfies $G_\ell G_\ell^T \succeq \gamma_G I$.*

(b) the Hessian $\hat{H}_\ell^\mu := \hat{H}^\tau[\mathcal{W}_\ell][\mathcal{W}_\ell] + \mu \cdot (G^\tau[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T G^\tau[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]$ satisfies

$$\tilde{\omega}_\ell^T \hat{H}_\ell^\mu \tilde{\omega}_\ell \geq \gamma_H \|\tilde{\omega}_\ell\|^2 / 2, \quad \text{for any } \tilde{\omega}_\ell \in \{\tilde{\omega}_\ell : G_\ell \tilde{\omega}_\ell = \mathbf{0}\}.$$

This implies that $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ has a unique global solution.

Proof. See section 2.9.1. □

lemma 2.4.1 shows that the subproblems enjoy the same regularity conditions as those of the full problem. In fact, Assumptions 2.4.1–2.4.4 are critical since they imply an exponential decay structure for the KKT matrix inverse, which plays a key role when analyzing the approximation error of Newton directions. We note that [46, Lemma 2] established the decay result for optimal control problems (i.e., linear graphs), while we study here a more general graph. We first introduce the KKT matrix of $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ and its inverse partition as follows:

$$K_\ell^\mu := \begin{bmatrix} \hat{H}_\ell^\mu & G_\ell^T \\ G_\ell & \mathbf{0} \end{bmatrix}, \quad (K_\ell^\mu)^{-1} := \begin{bmatrix} (K_\ell^\mu)_1^{-1} & \{(K_\ell^\mu)_2^{-1}\}^T \\ (K_\ell^\mu)_2^{-1} & (K_\ell^\mu)_3^{-1} \end{bmatrix}. \quad (2.12)$$

For any nodes $i, j \in \mathcal{W}_\ell$ and $k, v \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$, we let $[(K_\ell^\mu)_1^{-1}]_{i,j}$, $[(K_\ell^\mu)_2^{-1}]_{k,j}$, and $[(K_\ell^\mu)_3^{-1}]_{k,v}$ denote the (i, j) , (k, j) , and (k, v) -blocks of the submatrices $(K_\ell^\mu)_1^{-1}$, $(K_\ell^\mu)_2^{-1}$, and $(K_\ell^\mu)_3^{-1}$, respectively. With the above partition, we present the structure of $(K_\ell^\mu)^{-1}$ in the next lemma.

Lemma 2.4.2 (Structure of the KKT inverse). *Suppose Assumptions 2.4.1–2.4.4 hold for the full Problem (2.3) and $\mu \geq \hat{\mu}$, then for any nodes $i, j \in \mathcal{W}_\ell$ and $k, v \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$, we have*

$$\begin{aligned} \|[(K_\ell^\mu)_1^{-1}]_{i,j}\| &\leq \Upsilon_{\mu\rho_\mu}^{d_{\mathcal{G}}(i,j)}, & \|[(K_\ell^\mu)_2^{-1}]_{k,j}\| &\leq \Upsilon_{\mu\rho_\mu}^{d_{\mathcal{G}}(k,j)}, \\ \|[(K_\ell^\mu)_3^{-1}]_{k,v}\| &\leq \Upsilon_{\mu\rho_\mu}^{d_{\mathcal{G}}(k,v)}, \end{aligned} \quad (2.13)$$

where $\Upsilon_\mu = \frac{\Upsilon_K}{\gamma_K^2 \rho_\mu^2}$ and $\rho_\mu = \left(\frac{\Upsilon_K^2 - \gamma_K^2}{\Upsilon_K^2 + \gamma_K^2} \right)^{\frac{1}{4}} \in (0, 1)$ with $\Upsilon_K = 1.5\mu\hat{\Upsilon}^2$ and $\gamma_K = \frac{\gamma_H^3 \gamma_G^2}{75\Upsilon_K^6}$.

Proof. By (2.10), lemma 2.4.1, and the facts that $\mu \geq 4$ and $\hat{\Upsilon} \geq 1$, we know

$$\|K_\ell^\mu\| \leq \|\hat{H}_\ell^\mu\| + \|G_\ell\| \leq 2\hat{\Upsilon} + \mu\hat{\Upsilon}^2 \leq 1.5\mu\hat{\Upsilon}^2 =: \Upsilon_K. \quad (2.14)$$

By [64, Theorem A.3], we know the least singular value of K_ℓ^μ , denoted by $\sigma_{\min}(K_\ell^\mu)$, satisfies

$$\sigma_{\min}(K_\ell^\mu) \geq \left(\frac{4}{\gamma_H} + \left(1 + \frac{8\Upsilon_K}{\gamma_H} + \frac{16\Upsilon_K^2}{\gamma_H^2} \right) \frac{\Upsilon_K(1 + \nu\Upsilon_K)}{\gamma_G} + \nu \right)^{-1} \quad (2.15)$$

with $\nu = (8\Upsilon_K^2/\gamma_H + \gamma_H + 2\Upsilon_K)/(2\gamma_G)$. Since $\gamma_H \leq 1 \leq \Upsilon_K^2/6^2$ and $\Upsilon_K \leq \Upsilon_K^2/6$, we know $\nu \leq 4.2\Upsilon_K^2/(\gamma_H\gamma_G)$. Plugging it into (2.15) and using the facts that

$$\begin{aligned} 1 + \frac{8\Upsilon_K}{\gamma_H} + \frac{16\Upsilon_K^2}{\gamma_H^2} &\leq \left(\frac{1}{36} + \frac{4}{3} + 16 \right) \frac{\Upsilon_K^2}{\gamma_H^2} \leq 17.4 \frac{\Upsilon_K^2}{\gamma_H^2}, \\ \frac{\Upsilon_K(1 + \nu\Upsilon_K)}{\gamma_G} &\leq \frac{\Upsilon_K}{\gamma_G} + \frac{4.2\Upsilon_K^4}{\gamma_H\gamma_G^2} \leq \left(\frac{1}{6^3} + 4.2 \right) \frac{\Upsilon_K^4}{\gamma_H\gamma_G^2} \leq 4.21 \frac{\Upsilon_K^4}{\gamma_H\gamma_G^2}, \end{aligned}$$

we can obtain

$$\begin{aligned} \sigma_{\min}(K_\ell^\mu) &\geq \left(\frac{4}{\gamma_H} + 17.4 \cdot 4.21 \frac{\Upsilon_K^6}{\gamma_H^3 \gamma_G^2} + \frac{4.2\Upsilon_K^2}{\gamma_H\gamma_G} \right)^{-1} \\ &\geq \left\{ \left(\frac{4}{6^6} + 17.4 \cdot 4.21 + \frac{4.2}{6^4} \right) \frac{\Upsilon_K^6}{\gamma_H^3 \gamma_G^2} \right\}^{-1} \geq \frac{\gamma_H^3 \gamma_G^2}{75\Upsilon_K^6} =: \gamma_K. \end{aligned} \quad (2.16)$$

With the bounds in (2.14) and (2.16), the result is obtained from [62, Theorem 3.6] by noting that K_ℓ^μ has a bandwidth 2. We complete the proof. $\square$

Lemma 2.4.2 suggests that for any nodes $i, j \in \mathcal{W}_\ell$, the (i, j) -block of the KKT matrix inverse (of the subproblem) has an exponentially decaying magnitude in terms of the distance $d_{\mathcal{G}}(i, j)$ on the graph. We will rely on this property to study the difference between two solu-

tions with different boundary variables, $(\tilde{\omega}_\ell^*(\mathbf{d}_\ell), \tilde{\zeta}_\ell^*(\mathbf{d}_\ell)) - (\tilde{\omega}_\ell^*(\mathbf{d}'_\ell), \tilde{\zeta}_\ell^*(\mathbf{d}'_\ell))$. Before that, we show in the next lemma that if the boundary variables are set to the components of the exact Newton direction $(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$, i.e., $\mathbf{d}_\ell^* := (\{\Delta\mathbf{x}_j\}_{j \in \bar{\mathcal{W}}_\ell}, \{\Delta\boldsymbol{\lambda}_j\}_{j \in \hat{\mathcal{W}}_\ell})$, then the subproblem $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell^*)$ will recover the truncated exact Newton direction $\mathcal{D}_\ell(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$.

Lemma 2.4.3. *Suppose Assumptions 2.4.1–2.4.4 hold for the full Problem (2.3) and $\mu \geq \hat{\mu}$. Let $(\boldsymbol{\omega}^*, \boldsymbol{\zeta}^*) = (\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$ be the solution of (2.3). Then, for any $\ell \in [M]$, $\mathcal{D}_\ell(\boldsymbol{\omega}^*, \boldsymbol{\zeta}^*)$ is the solution of $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell^*)$ in (2.8) with $\mathbf{d}_\ell^* = (\boldsymbol{\omega}^*[\bar{\mathcal{W}}_\ell], \boldsymbol{\zeta}^*[\hat{\mathcal{W}}_\ell])$.*

Proof. By lemma 2.4.1, $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell^*)$ has a unique global solution. Thus, it suffices to show that $\mathcal{D}_\ell(\boldsymbol{\omega}^*, \boldsymbol{\zeta}^*)$ satisfies the KKT conditions of $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell^*)$. Noting that (2.2) is the KKT conditions of Problem (2.3), we can analogously obtain the KKT conditions of (2.8) as follows:

$$\begin{cases} \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell]\bar{\omega}_\ell + \hat{H}[\mathcal{W}_\ell][\bar{\mathcal{W}}_\ell]\bar{\omega}_\ell + (G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T \bar{\zeta}_\ell + (G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T \tilde{\zeta}_\ell \\ \quad + \mu(G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T (\mathbf{c}[\hat{\mathcal{W}}_\ell] + G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]\bar{\omega}_\ell + G[\hat{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell]\bar{\omega}_\ell) = -\nabla_{\mathbf{x}} \mathcal{L}[\mathcal{W}_\ell], \\ G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell]\bar{\omega}_\ell = -\mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell], \end{cases} \quad (2.17)$$

where $\bar{\omega}_\ell = \boldsymbol{\omega}^*[\bar{\mathcal{W}}_\ell]$ and $\bar{\zeta}_\ell = \boldsymbol{\zeta}^*[\hat{\mathcal{W}}_\ell]$. Since $\mathbf{c} + G\boldsymbol{\omega}^* = \mathbf{0}$, we have for the rows of $\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$ that

$$\begin{aligned} \mathbf{0} &= \mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] + G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{V}]\boldsymbol{\omega}^* \\ &= \mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] + G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{W}_\ell] + G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{V} \setminus \mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{V} \setminus \mathcal{W}_\ell] \\ &= \mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] + G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{W}_\ell], \end{aligned}$$

where the last equality is due to the facts that $N_{\mathcal{G}}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] \subseteq \mathcal{W}_\ell$ and for any nodes $i, j \in \mathcal{V}$, $G_{i,j} = \mathbf{0}$ if $j \notin N_{\mathcal{G}}[i]$ (i.e., $d_{\mathcal{G}}(i, j) > 1$), so that $G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{V} \setminus \mathcal{W}_\ell] = \mathbf{0}$. This verifies the sec-

ond condition in (2.17). Following the same reasoning, we consider the rows of $\hat{\mathcal{W}}_\ell$ and have

$$\begin{aligned}
\mathbf{0} &= \mathbf{c}[\hat{\mathcal{W}}_\ell] + G[\hat{\mathcal{W}}_\ell][\mathcal{V}]\boldsymbol{\omega}^* = \mathbf{c}[\hat{\mathcal{W}}_\ell] + G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{W}_\ell] + G[\hat{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell]\boldsymbol{\omega}^*[\bar{\mathcal{W}}_\ell] \\
&\quad + G[\hat{\mathcal{W}}_\ell][\mathcal{V} \setminus (\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell)]\boldsymbol{\omega}^*[\mathcal{V} \setminus (\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell)] \\
&= \mathbf{c}[\hat{\mathcal{W}}_\ell] + G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{W}_\ell] + G[\hat{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell]\boldsymbol{\omega}^*[\bar{\mathcal{W}}_\ell],
\end{aligned} \tag{2.18}$$

where the last equality is due to $G[\hat{\mathcal{W}}_\ell][\mathcal{V} \setminus (\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell)] = \mathbf{0}$. Furthermore, we consider the rows of $\mathcal{W}_\ell$ of the equality $\hat{H}\boldsymbol{\omega}^* + G^T\boldsymbol{\zeta}^* = -\nabla_{\mathbf{x}}\mathcal{L}$, and have

$$\begin{aligned}
-\nabla_{\mathbf{x}}\mathcal{L}[\mathcal{W}_\ell] &= \hat{H}[\mathcal{W}_\ell][\mathcal{V}]\boldsymbol{\omega}^* + (G[\mathcal{V}][\mathcal{W}_\ell])^T\boldsymbol{\zeta}^* \\
&= \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell]\boldsymbol{\omega}^*[\mathcal{W}_\ell] + \hat{H}[\mathcal{W}_\ell][\bar{\mathcal{W}}_\ell]\boldsymbol{\omega}^*[\bar{\mathcal{W}}_\ell] + (G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T\boldsymbol{\zeta}^*[\hat{\mathcal{W}}_\ell] \\
&\quad + (G[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T\boldsymbol{\zeta}^*[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell].
\end{aligned} \tag{2.19}$$

The second equality here is due to $\hat{H}[\mathcal{W}_\ell][\mathcal{V} \setminus (\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell)] = \mathbf{0}$ and $G[\mathcal{V} \setminus (\hat{\mathcal{W}}_\ell \cup (\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell))][\mathcal{W}_\ell] = \mathbf{0}$. Combining (2.18) and (2.19), we see that the first condition in (2.17) holds. Thus, we know $\mathcal{D}_\ell(\boldsymbol{\omega}^*, \boldsymbol{\zeta}^*) = (\boldsymbol{\omega}^*[\mathcal{W}_\ell], \boldsymbol{\zeta}^*[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell])$ satisfies (2.17) and we complete the proof. $\square$

We now provide an upper bound of the solution difference with different boundary variables, $\|(\tilde{\boldsymbol{\omega}}_\ell^*(\mathbf{d}_\ell), \tilde{\boldsymbol{\zeta}}_\ell^*(\mathbf{d}_\ell)) - (\tilde{\boldsymbol{\omega}}_\ell^*(\mathbf{d}'_\ell), \tilde{\boldsymbol{\zeta}}_\ell^*(\mathbf{d}'_\ell))\|$. Recall (see above definition 2.3.1) that for any node $k \in \mathcal{W}_\ell$, $\tilde{\boldsymbol{\omega}}_{\ell,k}^*$ denotes the variable associated with this node (similar for $\tilde{\boldsymbol{\zeta}}_{\ell,k}^*$).

Lemma 2.4.4. *Suppose Assumptions 2.4.1–2.4.4 hold for the full Problem (2.3) and $\mu \geq \hat{\mu}$. Then, for any $\ell \in [M]$ and any nodes $k \in \mathcal{W}_\ell$ and $j \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$, we have*

$$\|\tilde{\boldsymbol{\omega}}_{\ell,k}^*(\mathbf{d}_\ell) - \tilde{\boldsymbol{\omega}}_{\ell,k}^*(\mathbf{d}'_\ell)\| \leq \Upsilon_\mu \Upsilon_K \cdot \{\text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)\}^{1/2} \cdot \rho_\mu^{d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 2} \|\mathbf{d}_\ell - \mathbf{d}'_\ell\|, \tag{2.20a}$$

$$\|\tilde{\boldsymbol{\zeta}}_{\ell,j}^*(\mathbf{d}_\ell) - \tilde{\boldsymbol{\zeta}}_{\ell,j}^*(\mathbf{d}'_\ell)\| \leq \Upsilon_\mu \Upsilon_K \cdot \{\text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)\}^{1/2} \cdot \rho_\mu^{d_{\mathcal{G}}(j, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 2} \|\mathbf{d}_\ell - \mathbf{d}'_\ell\|, \tag{2.20b}$$

where $\Upsilon_\mu, \Upsilon_K, \rho_\mu$ are defined in lemma 2.4.2, and $\text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)$ is defined in (2.11).

Proof. The main idea of the proof is utilizing the exponential decay structure of the KKT matrix inverse in lemma 2.4.2. First, we note that the KKT conditions of (2.8) are given by

$$\begin{bmatrix} \hat{H}_\ell^\mu & G_\ell^T \\ G_\ell & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\omega}_\ell^*(\mathbf{d}_\ell) \\ \tilde{\zeta}_\ell^*(\mathbf{d}_\ell) \end{bmatrix} = \begin{bmatrix} -F_\ell \mathbf{d}_\ell - \mu(G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T \mathbf{c}[\hat{\mathcal{W}}_\ell] - \nabla_{\mathbf{x}} \mathcal{L}[\mathcal{W}_\ell] \\ -\mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] \end{bmatrix}, \quad (2.21)$$

where F_ℓ is given as follows

$$F_\ell = \begin{pmatrix} \hat{H}[\mathcal{W}_\ell][\tilde{\mathcal{W}}_\ell] + \mu(G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T G[\hat{\mathcal{W}}_\ell][\tilde{\mathcal{W}}_\ell] & (G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T \end{pmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \tilde{\omega}_\ell^*(\mathbf{d}_\ell) - \tilde{\omega}_\ell^*(\mathbf{d}'_\ell) \\ \tilde{\zeta}_\ell^*(\mathbf{d}_\ell) - \tilde{\zeta}_\ell^*(\mathbf{d}'_\ell) \end{bmatrix} = \begin{bmatrix} (K_\ell^\mu)_1^{-1} & \{(K_\ell^\mu)_2^{-1}\}^T \\ (K_\ell^\mu)_2^{-1} & (K_\ell^\mu)_3^{-1} \end{bmatrix} \begin{bmatrix} -F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell) \\ \mathbf{0} \end{bmatrix}.$$

With the above equality, we define $\tilde{\mathcal{W}}_\ell := (N_{\mathcal{G}}(\tilde{\mathcal{W}}_\ell) \cap \mathcal{W}_\ell) \cup \tilde{\mathcal{W}}_\ell$ to be the set of internal boundary nodes of $\mathcal{W}_\ell$ within depth two. Then, for any $k \in \mathcal{W}_\ell$, we apply lemma 2.4.2 and have

$$\begin{aligned} \|\tilde{\omega}_{\ell,k}^*(\mathbf{d}_\ell) - \tilde{\omega}_{\ell,k}^*(\mathbf{d}'_\ell)\| &= \left\| \sum_{j \in \mathcal{W}_\ell} [(K_\ell^\mu)_1^{-1}]_{k,j} [F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j \right\| \\ &= \left\| \sum_{j \in \tilde{\mathcal{W}}_\ell} [(K_\ell^\mu)_1^{-1}]_{k,j} [F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j \right\| \leq \sum_{j \in \tilde{\mathcal{W}}_\ell} \Upsilon_\mu \rho_\mu^{d_{\mathcal{G}}(k,j)} \cdot \|[F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j\| \\ &\leq \Upsilon_\mu \sqrt{|\tilde{\mathcal{W}}_\ell|} \left\{ \sum_{j \in \tilde{\mathcal{W}}_\ell} \rho_\mu^{2d_{\mathcal{G}}(k,j)} \cdot \|[F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j\|^2 \right\}^{1/2} \\ &\leq \Upsilon_\mu \sqrt{|\tilde{\mathcal{W}}_\ell|} \left\{ \rho_\mu^{2(d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 2)} \sum_{j \in \tilde{\mathcal{W}}_\ell} \|[F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j\|^2 \right\}^{1/2} \\ &= \Upsilon_\mu \sqrt{|\tilde{\mathcal{W}}_\ell|} \rho_\mu^{d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 2} \cdot \|F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)\| \\ &\leq \Upsilon_\mu \sqrt{|\tilde{\mathcal{W}}_\ell|} \rho_\mu^{d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 2} \cdot (2\hat{\Upsilon} + \mu\hat{\Upsilon}^2) \|\mathbf{d}_\ell - \mathbf{d}'_\ell\|, \end{aligned} \quad (2.22)$$

where the second equality is due to the facts that $\hat{H}[j][\bar{\mathcal{W}}_\ell] = \mathbf{0}$ and $G[\hat{\mathcal{W}}_\ell][j] = \mathbf{0}$ for any $j \in \mathcal{W}_\ell \setminus \check{\mathcal{W}}_\ell$, so that $[F_\ell(\mathbf{d}_\ell - \mathbf{d}'_\ell)]_j = \mathbf{0}$ for $j \in \mathcal{W}_\ell \setminus \check{\mathcal{W}}_\ell$; the fourth inequality is by Cauchy-Schwarz inequality; the fifth inequality is due to the fact that $d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) \leq d_{\mathcal{G}}(k, j) + 2$ for any $j \in \check{\mathcal{W}}_\ell$; the last inequality is due to (2.10). Noting that $\mu \geq 4$ and $|\check{\mathcal{W}}_\ell| \leq \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)$ by (2.11), we complete the proof of (2.20a). For (2.20b), we follow the derivation of (2.22) by replacing $(K_\ell^\mu)_1^{-1}$ with $(K_\ell^\mu)_2^{-1}$, and obtain the same result. This completes the proof. $\square$

We now establish the approximation error of the Newton direction computed by overlapping graph decomposition. We note that the approximate direction $(\tilde{\Delta}\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda})$ is composed by $(\tilde{\omega}_\ell^*(\mathbf{0}), \tilde{\zeta}_\ell^*(\mathbf{0}))$, while the exact direction $(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$ is composed by $(\tilde{\omega}_\ell^*(\mathbf{d}_\ell^*), \tilde{\zeta}_\ell^*(\mathbf{d}_\ell^*))$ (refer to lemma 2.4.3). We remind the reader that all our results hold for any iteration $\tau \geq 0$.

Theorem 2.4.1 (Approximation error of OGD). *Suppose Assumptions 2.4.1–2.4.4 hold for the full Problem (2.3) and $\mu \geq \hat{\mu}$. Then, the approximate direction $(\tilde{\Delta}\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda})$ computed by OGD with the overlap size b satisfies*

$$\|(\tilde{\Delta}\mathbf{x} - \Delta\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda} - \Delta\boldsymbol{\lambda})\| \leq \Upsilon_{\mu,b} \cdot \rho_\mu^b \|(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})\|, \quad (2.23)$$

where $\Upsilon_\mu, \Upsilon_K, \rho_\mu$ are defined in lemma 2.4.2 and

$$\Upsilon_{\mu,b} := \frac{\sqrt{2}\Upsilon_\mu\Upsilon_K}{\rho_\mu\sqrt{1-\rho_\mu^2}} \cdot \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b) \{\text{poly}_{\mathcal{G}}(b+2)\}^{1/2}.$$

Proof. By definition 2.3.1, lemma 2.4.3, and lemma 2.4.4, we know that for any node $k \in \mathcal{V}_\ell$,

$$\|\tilde{\Delta}\mathbf{x}_k - \Delta\mathbf{x}_k\|^2 = \|\tilde{\omega}_{\ell,k}^*(\mathbf{0}) - \tilde{\omega}_{\ell,k}^*(\mathbf{d}_\ell^*)\|^2 \leq \Upsilon_1 \rho_\mu^{2d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 4} \|\mathbf{d}_\ell^*\|^2,$$

where $\Upsilon_1 := \Upsilon_\mu^2 \Upsilon_K^2 \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)$, and the above bound holds for $\|\tilde{\Delta}\boldsymbol{\lambda}_k - \Delta\boldsymbol{\lambda}_k\|^2$ as well. With

the above bound, we have

$$\begin{aligned}
\|\tilde{\Delta}\mathbf{x} - \Delta\mathbf{x}\|^2 &= \sum_{\ell=1}^M \sum_{k \in \mathcal{V}_\ell} \|\tilde{\Delta}\mathbf{x}_k - \Delta\mathbf{x}_k\|^2 \leq \Upsilon_1 \sum_{\ell=1}^M \sum_{k \in \mathcal{V}_\ell} \rho_\mu^{2d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) - 4} \|\mathbf{d}_\ell^*\|^2 \\
&= \Upsilon_1 \sum_{\ell=1}^M \|\mathbf{d}_\ell^*\|^2 \sum_{d=b+1}^{\infty} |\{k \in \mathcal{V}_\ell : d_{\mathcal{G}}(k, N_{\mathcal{G}}(\mathcal{W}_\ell)) = d\}| \cdot \rho_\mu^{2d-4} \\
&\leq \Upsilon_1 \sum_{\ell=1}^M \|\mathbf{d}_\ell^*\|^2 |\mathcal{V}_\ell| \sum_{d=b+1}^{\infty} \rho_\mu^{2d-4} \stackrel{(2.11)}{\leq} \Upsilon_{1\text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)} \frac{\rho_\mu^{2b-2}}{1 - \rho_\mu^2} \sum_{\ell=1}^M \|\mathbf{d}_\ell^*\|^2.
\end{aligned}$$

We note that the boundary variables of two subproblems, $\mathbf{d}_{\ell_1}^*$ and $\mathbf{d}_{\ell_2}^*$ with $\ell_1 \neq \ell_2$, may have overlaps so that they share the same variables of $(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$. Below we give an upper bound for the number of subproblems (i.e., subgraphs) that can share one boundary node. For any $\ell \in [M]$, we know $\mathbf{d}_\ell^*$ is associated with the boundary nodes $\bar{\mathcal{W}}_\ell \cup \hat{\mathcal{W}}_\ell$. Thus, for any node $k \in \bar{\mathcal{W}}_\ell \cup \hat{\mathcal{W}}_\ell$, if it is the boundary of the ℓ' -th subgraph, then $d_{\mathcal{G}}(k, \mathcal{V}_{\ell'}) \leq b+2$. Since $\{\mathcal{V}_\ell\}_{\ell \in [M]}$ are mutually disjoint and by Assumption 2.4.4, $|\{j \in \mathcal{V} : d(k, j) \leq b+2\}| = \text{poly}_{\mathcal{G}}(b+2)$, one boundary node is shared by at most $\text{poly}_{\mathcal{G}}(b+2)$ subgraphs. With this fact, we have

$$\sum_{\ell=1}^M \|\mathbf{d}_\ell^*\|^2 \leq \sum_{j \in \mathcal{V}} \text{poly}_{\mathcal{G}}(b+2) \|(\Delta\mathbf{x}_j, \Delta\boldsymbol{\lambda}_j)\|^2 = \text{poly}_{\mathcal{G}}(b+2) \|(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})\|^2.$$

Combining the above two displays, we obtain

$$\|\tilde{\Delta}\mathbf{x} - \Delta\mathbf{x}\|^2 \leq \Upsilon_{1\text{poly}_{\mathcal{G} \circ \mathcal{W}}(b)} \cdot \text{poly}_{\mathcal{G}}(b+2) \frac{\rho_\mu^{2b-2}}{1 - \rho_\mu^2} \|(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})\|^2.$$

The result of $\|\tilde{\Delta}\boldsymbol{\lambda} - \Delta\boldsymbol{\lambda}\|^2$ follows the same derivations above. This completes the proof. $\square$

theorem 2.4.1 indicates that the approximate Newton direction $(\tilde{\Delta}\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda})$ approaches the exact Newton direction $(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})$ exponentially fast with the increasing overlap size b . We note that the scaling factor $\Upsilon_{\mu,b}$ grows only polynomially in b ; thus, the approximation error is eventually dominated by the exponential rate in b and can be arbitrarily small as long as b is large enough. It is worth mentioning that the constant $\Upsilon_{\mu,b}$ is conservative; in our experiments (cf. section 2.7), we observe that the method may still work even for small b (e.g., 1 or

2), although larger b can improve the convergence rate.

2.5 Global Convergence of FOGD

In this section, we prove the global convergence of FOGD. A key step is to show that the approximate Newton direction is a descent direction of the exact augmented Lagrangian merit function (2.7), so that FOGD can keep decreasing the merit function over iterates. To achieve this, we rely on the approximation error established in theorem 2.4.1. For simplicity of the presentation, let us define $\delta_{\mu,b} := \Upsilon_{\mu,b} \rho_{\mu}^b$, and we always suppose a fixed $\mu \geq \hat{\mu}$ (cf. lemma 2.4.1) without repeating it.

We need the following assumption in the global convergence analysis, which is commonly used in the SQP literature [5, Proposition 4.15]. It enhances Assumption 2.4.3 by imposing it on a compact set.

Assumption 2.5.1 (Compactness). *There exists a compact set $\mathcal{X} \times \Lambda$ such that $(\mathbf{x}^\tau + \alpha \tilde{\Delta} \mathbf{x}^\tau, \boldsymbol{\lambda}^\tau + \alpha \tilde{\Delta} \boldsymbol{\lambda}^\tau) \in \mathcal{X} \times \Lambda$, $\forall \tau \geq 0$, $\alpha \in [0, 1]$; and we assume that $\{f_i, \mathbf{c}_i\}_{i \in \mathcal{V}}$ are thrice continuously differentiable in $\mathcal{X} \times \Lambda$, and for a constant $\Upsilon \geq 1$,*

$$\sup_{\mathbf{x} \in \mathcal{X}, \boldsymbol{\lambda} \in \Lambda} \|H_{i,j}(\mathbf{x}, \boldsymbol{\lambda})\| \leq \Upsilon \quad \text{and} \quad \sup_{\mathbf{x} \in \mathcal{X}} \|G_{i,j}(\mathbf{x})\| \leq \Upsilon, \quad \forall i, j \in \mathcal{V}. \quad (2.24)$$

Here, we abuse the notation Υ in Assumption 2.4.3 to denote the upper bound of the Hessian and Jacobian matrices in the set $\mathcal{X} \times \Lambda$. Again, $H_{i,j}$ and $G_{i,j}$ actually have banded structures with distance measured on the graph $\mathcal{G}$; thus, (2.24) is only required for $d_{\mathcal{G}}(i, j) \leq 2$.

The next lemma is a direct result of Assumptions 2.4.1–2.4.4 and 2.5.1.

Lemma 2.5.1. *Under Assumptions 2.4.4 and 2.5.1, we have for the constant $\hat{\Upsilon}$ in (2.10) that*

$$\sup_{\mathbf{x} \in \mathcal{X}, \boldsymbol{\lambda} \in \Lambda} \|H(\mathbf{x}, \boldsymbol{\lambda})\| \leq \hat{\Upsilon} \quad \text{and} \quad \sup_{\mathbf{x} \in \mathcal{X}} \|G(\mathbf{x})\| \leq \hat{\Upsilon}. \quad (2.25)$$

Furthermore, if Assumptions 2.4.1–2.4.4 hold, then the KKT inverse satisfies

$$\left\| \begin{bmatrix} \hat{H}^\tau & (G^\tau)^T \\ G^\tau & \mathbf{0} \end{bmatrix}^{-1} \right\| \leq \frac{4\hat{\Upsilon}^2}{\gamma_H \gamma_G}, \quad \forall \tau \geq 0. \quad (2.26)$$

Proof. See section 2.9.2. □

We are now able to introduce the sufficient conditions on the merit function parameters η_1, η_2 and the approximation error $\delta_{\mu,b}$ so that $(\tilde{\Delta}\mathbf{x}, \tilde{\Delta}\boldsymbol{\lambda})$ is a descent direction of the exact augmented Lagrangian (2.7). We omit the iteration index τ .

Lemma 2.5.2. *Suppose Assumptions 2.4.1–2.4.4 and 2.5.1 hold. If*

$$\eta_1 \geq \frac{9}{\eta_2 \gamma_G}, \quad \eta_2 \leq \frac{\gamma_H}{7\hat{\Upsilon}^2}, \quad \delta_{\mu,b} \leq \frac{\eta_2 \gamma_G}{17\eta_1 \hat{\Upsilon}^2}, \quad (2.27)$$

then we have

$$\begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x} \\ \tilde{\Delta}\boldsymbol{\lambda} \end{bmatrix} \leq -\frac{\eta_2}{8} \left\| \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L} \end{bmatrix} \right\|^2 - \frac{\eta_2 \gamma_G}{16} \left\| \begin{bmatrix} \Delta\mathbf{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} \right\|^2.$$

Proof. We note that

$$\begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x} \\ \tilde{\Delta}\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta \end{bmatrix}^T \begin{bmatrix} \Delta\mathbf{x} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x} - \Delta\mathbf{x} \\ \tilde{\Delta}\boldsymbol{\lambda} - \Delta\boldsymbol{\lambda} \end{bmatrix} := \mathcal{I}_1 + \mathcal{I}_2, \quad (2.28)$$

where by direct calculation,

$$\begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta \end{bmatrix} = \begin{bmatrix} I + \eta_2 H & \eta_1 G^T \\ \eta_2 G & I \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L} \end{bmatrix}. \quad (2.29)$$

For the term $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 &\stackrel{(2.29)}{=} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} I + \eta_2 H & \eta_1 G^T \\ \eta_2 G & I \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L} \end{bmatrix} \\
&\stackrel{(2.2)}{=} - \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} I + \eta_2 H & \eta_1 G^T \\ \eta_2 G & I \end{bmatrix} \begin{bmatrix} \hat{H} & G^T \\ G & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} \\
&= - \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} \hat{H} + \eta_2 H \hat{H} + \eta_1 G^T G & G^T + \eta_2 H G^T \\ G + \eta_2 G \hat{H} & \eta_2 G G^T \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} \\
&\stackrel{(2.10)}{\leq} -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + \eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \eta_1 \|G \Delta \mathbf{x}\|^2 - 2\Delta \boldsymbol{\lambda}^T G \Delta \mathbf{x} \\
&\quad + 2\eta_2 \hat{\Upsilon} \|\Delta \mathbf{x}\| \|G^T \Delta \boldsymbol{\lambda}\| - \eta_2 \|G^T \Delta \boldsymbol{\lambda}\|^2 \\
&\stackrel{(2.2)}{\leq} -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + \eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \eta_1 \|\mathbf{c}\|^2 + 2\mathbf{c}^T \Delta \boldsymbol{\lambda} + 2\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \frac{\eta_2}{2} \|G^T \Delta \boldsymbol{\lambda}\|^2 \\
&\leq -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + 3\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \eta_1 \|\mathbf{c}\|^2 + \frac{8}{\eta_2 \gamma_G} \|\mathbf{c}\|^2 + \frac{\eta_2 \gamma_G}{8} \|\Delta \boldsymbol{\lambda}\|^2 - \frac{\eta_2 \gamma_G}{4} \|\Delta \boldsymbol{\lambda}\|^2 - \frac{\eta_2}{4} \|G^T \Delta \boldsymbol{\lambda}\|^2 \\
&\stackrel{(2.2)}{=} -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + 3\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \left(\eta_1 - \frac{8}{\eta_2 \gamma_G} \right) \|\mathbf{c}\|^2 - \frac{\eta_2 \gamma_G}{8} \|\Delta \boldsymbol{\lambda}\|^2 - \frac{\eta_2}{4} \|\hat{H} \Delta \mathbf{x} + \nabla_{\mathbf{x}} \mathcal{L}\|^2 \\
&\stackrel{(2.10)}{\leq} -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + 3\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \left(\eta_1 - \frac{8}{\eta_2 \gamma_G} \right) \|\mathbf{c}\|^2 - \frac{\eta_2 \gamma_G}{8} \|\Delta \boldsymbol{\lambda}\|^2 - \frac{\eta_2}{8} \|\nabla_{\mathbf{x}} \mathcal{L}\|^2 + \frac{\eta_2 \hat{\Upsilon}^2}{4} \|\Delta \mathbf{x}\|^2 \\
&= -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + 3.25\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 - \left(\eta_1 - \frac{8}{\eta_2 \gamma_G} \right) \|\mathbf{c}\|^2 - \frac{\eta_2 \gamma_G}{8} \|\Delta \boldsymbol{\lambda}\|^2 - \frac{\eta_2}{8} \|\nabla_{\mathbf{x}} \mathcal{L}\|^2,
\end{aligned}$$

where the sixth inequality is due to Assumption 2.4.2. To simplify the above bound, we de-

compose the step $\Delta \mathbf{x}$ as $\Delta \mathbf{x} = \Delta \mathbf{x}_1 + \Delta \mathbf{x}_2$ with $\Delta \mathbf{x}_1 \in \text{span}(G^T)$ and $G\Delta \mathbf{x}_2 = \mathbf{0}$. Then, we have

$$\begin{aligned}
& -\Delta \mathbf{x}^T \hat{H} \Delta \mathbf{x} + 3.25\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 \\
& = -\Delta \mathbf{x}_1^T \hat{H} \Delta \mathbf{x}_1 - 2\Delta \mathbf{x}_1^T \hat{H} \Delta \mathbf{x}_2 - \Delta \mathbf{x}_2^T \hat{H} \Delta \mathbf{x}_2 + 3.25\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 \\
& \stackrel{(2.10)}{\leq} \hat{\Upsilon} \|\Delta \mathbf{x}_1\|^2 + 2\hat{\Upsilon} \|\Delta \mathbf{x}_1\| \|\Delta \mathbf{x}_2\| - \gamma_H \|\Delta \mathbf{x}_2\|^2 + 3.25\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 \\
& \leq \left(\hat{\Upsilon} + \frac{2\hat{\Upsilon}^2}{\gamma_H} \right) \|\Delta \mathbf{x}_1\|^2 - \frac{\gamma_H}{2} \|\Delta \mathbf{x}_2\|^2 + 3.25\eta_2 \hat{\Upsilon}^2 \|\Delta \mathbf{x}\|^2 \\
& = \left(\hat{\Upsilon} + \frac{2\hat{\Upsilon}^2}{\gamma_H} + \frac{\gamma_H}{2} \right) \|\Delta \mathbf{x}_1\|^2 - \left(\frac{\gamma_H}{2} - 3.25\eta_2 \hat{\Upsilon}^2 \right) \|\Delta \mathbf{x}\|^2 \\
& \leq \left(\hat{\Upsilon} + \frac{2\hat{\Upsilon}^2}{\gamma_H} + \frac{\gamma_H}{2} \right) \frac{1}{\gamma_G} \|\mathbf{c}\|^2 - \left(\frac{\gamma_H}{2} - 3.25\eta_2 \hat{\Upsilon}^2 \right) \|\Delta \mathbf{x}\|^2 \\
& \leq \frac{3.5\hat{\Upsilon}^2}{\gamma_H \gamma_G} \|\mathbf{c}\|^2 - \left(\frac{\gamma_H}{2} - 3.25\eta_2 \hat{\Upsilon}^2 \right) \|\Delta \mathbf{x}\|^2,
\end{aligned}$$

where the fifth inequality is due to the fact that $\|\mathbf{c}\|^2 = \|G\Delta \mathbf{x}\|^2 = \|G\Delta \mathbf{x}_1\|^2 \geq \gamma_G \|\Delta \mathbf{x}_1\|^2$, and the sixth inequality is due to $\hat{\Upsilon} \geq 1 \geq \gamma_H$. Combining the above two displays, we obtain

$$\begin{aligned}
\mathcal{I}_1 \leq & -\frac{\eta_2}{8} \left\| \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L} \\ \nabla_{\lambda} \mathcal{L} \end{bmatrix} \right\|^2 - \frac{\eta_2 \gamma_G}{8} \left\| \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \lambda \end{bmatrix} \right\|^2 - \left(\eta_1 - \frac{8}{\eta_2 \gamma_G} - \frac{3.5\hat{\Upsilon}^2}{\gamma_H \gamma_G} - \frac{\eta_2}{8} \right) \|\mathbf{c}\|^2 \\
& - \left(\frac{\gamma_H}{2} - 3.25\eta_2 \hat{\Upsilon}^2 - \frac{\eta_2 \gamma_G}{8} \right) \|\Delta \mathbf{x}\|^2.
\end{aligned}$$

Noting that

$$\begin{aligned}
\frac{\gamma_H}{2} - 3.25\eta_2 \hat{\Upsilon}^2 - \frac{\eta_2 \gamma_G}{8} \geq 0 & \iff \frac{\gamma_H}{2} - 3.375\eta_2 \hat{\Upsilon}^2 \geq 0 \iff \eta_2 \leq \frac{\gamma_H}{7\hat{\Upsilon}^2}, \\
\eta_1 - \frac{8}{\eta_2 \gamma_G} - \frac{3.5\hat{\Upsilon}^2}{\gamma_H \gamma_G} - \frac{\eta_2}{8} \geq 0 & \iff \eta_1 - \frac{8.5}{\eta_2 \gamma_G} - \frac{\eta_2}{8} \geq 0 \iff \eta_1 \geq \frac{9}{\eta_2 \gamma_G},
\end{aligned}$$

we know under the condition (2.27),

$$\mathcal{I}_1 \leq -\frac{\eta_2}{8} \left\| \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L} \end{bmatrix} \right\|^2 - \frac{\eta_2 \gamma_G}{8} \left\| \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} \right\|^2. \quad (2.30)$$

For the term $\mathcal{I}_2$, we have

$$\begin{aligned} \mathcal{I}_2 &\stackrel{(2.2),(2.29)}{=} - \begin{bmatrix} \tilde{\Delta} \mathbf{x} - \Delta \mathbf{x} \\ \tilde{\Delta} \boldsymbol{\lambda} - \Delta \boldsymbol{\lambda} \end{bmatrix}^T \begin{bmatrix} \hat{H} + \eta_2 H \hat{H} + \eta_1 G^T G & G^T + \eta_2 H G^T \\ G + \eta_2 G \hat{H} & \eta_2 G G^T \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x} \\ \Delta \boldsymbol{\lambda} \end{bmatrix} \\ &\stackrel{(2.23),(2.10)}{\leq} \delta_{\mu,b} \|(\Delta \mathbf{x}, \Delta \boldsymbol{\lambda})\|^2 \left\{ 2\hat{\Upsilon} + 3\eta_2 \hat{\Upsilon}^2 + \eta_1 \hat{\Upsilon}^2 \right\} \leq 1.05\eta_1 \hat{\Upsilon}^2 \delta_{\mu,b} \|(\Delta \mathbf{x}, \Delta \boldsymbol{\lambda})\|^2, \end{aligned} \quad (2.31)$$

where the last inequality uses the fact that $2\hat{\Upsilon} + 3\eta_2 \hat{\Upsilon}^2 \leq 2\hat{\Upsilon} + 3/7 \leq 2.5\hat{\Upsilon}^2 \leq 0.05\eta_1 \hat{\Upsilon}^2$ (since $\eta_1 \geq 63$ by $\hat{\Upsilon} \geq 1 \geq \max\{\gamma_H, \gamma_G\}$). Combining (2.31) with (2.30), (2.28), and (2.27), we complete the proof. $\square$

lemma 2.5.2 shows that the approximate Newton direction $(\tilde{\Delta} \mathbf{x}, \tilde{\Delta} \boldsymbol{\lambda})$ is a descent direction of the exact augmented Lagrangian merit function (2.7), ensuring sufficient reduction in each step, provided η_1 is large enough and η_2 and $\delta_{\mu,b}$ are small enough. The choices of η_1 , η_2 , and $\delta_{\mu,b}$ are independent of τ and M , and the negative inner product between $(\nabla_{\mathbf{x}} \mathcal{L}_\eta, \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta)$ and $(\tilde{\Delta} \mathbf{x}, \tilde{\Delta} \boldsymbol{\lambda})$ guarantees the existence of a stepsize α_τ to satisfy the Armijo condition (2.6) (by the mean value theorem).

The result in lemma 2.5.2 illustrates the significance of utilizing the exact augmented Lagrangian merit function. We observe that the left-hand side of (2.28) consists of two terms: $\mathcal{I}_1$ that measures the reduction of the exact Newton direction on the augmented Lagrangian, and $\mathcal{I}_2$ that characterizes the approximation error of OGD. By (2.30), $\mathcal{I}_1$ is further upper bounded by two terms. The first term is proportional to the KKT residual $\|\nabla \mathcal{L}\|^2$, which is crucial for accumulating reductions across steps and achieving global convergence. The second term

is proportional to the squared magnitude of the exact Newton direction $\|(\Delta \mathbf{x}, \Delta \boldsymbol{\lambda})\|^2$, which is the key to tolerate the approximation error of OGD in $\mathcal{I}_2$. As elucidated in lemma 2.4.4 and theorem 2.4.1, the approximation error (even for the primal variable $\|\tilde{\Delta} \mathbf{x} - \Delta \mathbf{x}\|$) depends not only on $\Delta \mathbf{x}$ but also on $\Delta \boldsymbol{\lambda}$ in the boundary variables. Therefore, a merit function that depends on both primal and dual variables, capable of tolerating the dual approximation error, is particularly preferable for our algorithm and analysis.

With theorem 2.4.1 and lemma 2.5.2, we are able to prove the global convergence of FOGD in the next theorem.

Theorem 2.5.1 (Global convergence of FOGD). *Suppose Assumptions 2.4.1–2.4.4, 2.5.1 hold and the parameters $(\eta_1, \eta_2, \delta_{\mu,b})$ satisfy (2.27). Then, algorithm 1 satisfies $\|\nabla \mathcal{L}^\tau\| \rightarrow 0$ as $\tau \rightarrow \infty$.*

Proof. By twice differentiability of $\{f_i, c_i\}_{i \in \mathcal{V}}$, we know $\nabla \mathcal{L}_\eta$ is Lipschitz continuous in the set $\mathcal{X} \times \Lambda$. Let us denote its Lipschitz constant as Υ_η . Then, we have

$$\begin{aligned}
\mathcal{L}_\eta^{\tau+1} &\leq \mathcal{L}_\eta^\tau + \alpha_\tau \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + \frac{\Upsilon_\eta \alpha_\tau^2}{2} \left\| \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} \right\|^2 \\
&\stackrel{(2.23)}{\leq} \mathcal{L}_\eta^\tau + \alpha_\tau \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + \frac{\Upsilon_\eta \alpha_\tau^2 (1 + \delta_{\mu,b})^2}{2} \left\| \begin{bmatrix} \Delta \mathbf{x}^\tau \\ \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} \right\|^2 \\
&\stackrel{(2.2)}{\leq} \mathcal{L}_\eta^\tau + \alpha_\tau \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + \frac{32 \hat{\Upsilon}^4 \Upsilon_\eta \alpha_\tau^2}{\gamma_H^2 \gamma_G^2} \|\nabla \mathcal{L}^\tau\|^2 \quad (\text{by lemma 2.5.1 and } \delta_{\mu,b} \leq 1) \\
&\leq \mathcal{L}_\eta^\tau + \alpha_\tau \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} - \frac{256 \hat{\Upsilon}^4 \Upsilon_\eta \alpha_\tau^2}{\gamma_H^2 \gamma_G^2 \eta_2} \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} \quad (\text{by lemma 2.5.2}) \\
&= \mathcal{L}_\eta^\tau + \alpha_\tau \left\{ 1 - \frac{256 \hat{\Upsilon}^4 \Upsilon_\eta}{\gamma_H^2 \gamma_G^2 \eta_2} \alpha_\tau \right\} \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}.
\end{aligned}$$

Thus, the Armijo condition (2.6) is satisfied if $\alpha_\tau \leq (1-\beta)\gamma_H^2\gamma_G^2\eta_2/(256\hat{\Upsilon}^4\Upsilon_\eta)$, which implies the existence of a lower bound of α_τ , i.e., $\alpha_\tau \geq \bar{\alpha} > 0$, $\forall \tau \geq 0$. Applying lemma 2.5.2 to (2.6), we obtain

$$\mathcal{L}_\eta^{\tau+1} \leq \mathcal{L}_\eta^\tau - \frac{\eta_2\alpha_\tau\beta}{8}\|\nabla\mathcal{L}^\tau\|^2 \leq \mathcal{L}_\eta^\tau - \frac{\eta_2\bar{\alpha}\beta}{8}\|\nabla\mathcal{L}^\tau\|^2.$$

Accumulating the above reduction from $\tau = 0$ to ∞ , we obtain $\sum_{\tau=0}^{\infty} \|\nabla\mathcal{L}^\tau\|^2 \leq \frac{8}{\eta_2\bar{\alpha}\beta}(\mathcal{L}_\eta^0 - \min_{\mathcal{X} \times \Lambda} \mathcal{L}_\eta(\mathbf{x}, \boldsymbol{\lambda})) < \infty$. This completes the proof. $\square$

In the next section, we will show that FOGD exhibits local linear convergence. Enjoying global convergence and preserving fast local linear rate of the Schwarz method makes FOGD an appealing algorithm compared to other decomposition methods.

2.6 Local Convergence of FOGD

To segue into local analysis, we assume in this section that the iterate $(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau)$ converges to a strict local solution $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ as $\tau \rightarrow \infty$. We rely on the following assumptions regarding the vanishing modification and the local Lipschitz continuity of the Lagrangian Hessian in local analysis.

Assumption 2.6.1 (Vanishing modification). *We assume $\hat{H}^\tau \rightarrow H^\tau$ as $\tau \rightarrow \infty$, where $\hat{H}^\tau$ is a modification of the Lagrangian Hessian $H^\tau = \nabla_{\mathbf{x}}^2\mathcal{L}^\tau$.*

We mention that the Hessian modification is intended to enforce Assumption 2.4.1 to have a well-defined quadratic program (2.3). Since $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a strict local solution, by the second-order sufficient condition, Assumption 2.4.1 directly applies to the exact Hessian H^τ for large τ , suggesting that no modifications may be needed locally and, thus, Assumption 2.6.1 holds naturally. We also need the local Lipschitz continuity condition.

Assumption 2.6.2 (Local Lipschitz continuity). *There exists a constant $\Upsilon_L \geq 1$ such that for any two points $(\mathbf{x}, \boldsymbol{\lambda})$ and $(\mathbf{x}', \boldsymbol{\lambda}')$ sufficiently close to $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ and any nodes $i, j \in$*

$\mathcal{V}$,

$$\begin{aligned}\|H_{i,j}(\mathbf{x}, \boldsymbol{\lambda}) - H_{i,j}(\mathbf{x}', \boldsymbol{\lambda}')\| &\leq \Upsilon_L \max_{k \in N_{\mathcal{G}}^2[i]} \|(\mathbf{x}_k, \boldsymbol{\lambda}_k) - (\mathbf{x}'_k, \boldsymbol{\lambda}'_k)\|, \\ \|G_{i,j}(\mathbf{x}, \boldsymbol{\lambda}) - G_{i,j}(\mathbf{x}', \boldsymbol{\lambda}')\| &\leq \Upsilon_L \max_{k \in N_{\mathcal{G}}[i]} \|(\mathbf{x}_k, \boldsymbol{\lambda}_k) - (\mathbf{x}'_k, \boldsymbol{\lambda}'_k)\|.\end{aligned}$$

We note that $H_{i,j}$ only depends on variables $\{(\mathbf{x}_k, \boldsymbol{\lambda}_k)\}_{k \in N_{\mathcal{G}}^2[i]}$ and $G_{i,j}$ only depends on variables $\{(\mathbf{x}_k, \boldsymbol{\lambda}_k)\}_{k \in N_{\mathcal{G}}[i]}$. The next result shows that the unit stepsize is locally admissible to satisfy the Armijo condition (2.6).

Lemma 2.6.1. *Suppose Assumptions 2.4.1–2.4.4, 2.5.1, 2.6.1–2.6.2 hold and the parameters $(\eta_1, \eta_2, \delta_{\mu,b})$ satisfy*

$$\eta_1 \geq \frac{9}{\eta_2 \gamma_G}, \quad \eta_2 \leq \frac{\gamma_H}{7 \hat{\Upsilon}^2}, \quad \delta_{\mu,b} \leq \frac{0.5 - \beta}{1.5 - \beta} \cdot \frac{\eta_2 \gamma_G}{8.5 \eta_1 \hat{\Upsilon}^2}, \quad (2.32)$$

then $\alpha_\tau = 1$ for all sufficiently large τ .

Proof. See section 2.9.3. □

lemma 2.6.1 retains the same conditions on η_1 and η_2 as in lemma 2.5.2. However, since $(0.5 - \beta)/(1.5 - \beta) \leq 1/3$, it imposes a stronger condition on $\delta_{\mu,b}$. lemma 2.6.1 suggests that the exact augmented Lagrangian merit function effectively overcomes the Maratos effect and admits the unit stepsize locally, provided that η_2 and $\delta_{\mu,b}$ are sufficiently small and η_1 is sufficiently large. Admitting the unit stepsize is crucial for achieving a fast local convergence rate. We may need to enhance the design by employing other techniques, such as second-order correction, to overcome the Maratos effect when using other non-differentiable merit functions. Those techniques would need to be decentralized in our distributed setting, significantly complicating the applicability of the method.

We then show the local linear convergence of FOGD in the next theorem.

Theorem 2.6.1 (Uniform local linear convergence of FOGD). *Let us define $\Psi^\tau := \max_{k \in \mathcal{V}} \Psi_k^\tau := \max_{k \in \mathcal{V}} \|(\mathbf{x}_k^\tau, \boldsymbol{\lambda}_k^\tau) - (\mathbf{x}_k^*, \boldsymbol{\lambda}_k^*)\|$. Suppose Assumptions 2.4.1–2.4.4, 2.5.1, 2.6.1–2.6.2 hold and the parameters $(\eta_1, \eta_2, \delta_{\mu,b})$ satisfy (2.32). Then, for sufficiently large τ , we have*

$$\Psi^{\tau+1} \leq 2\{\Upsilon_{\mu} \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4)\}^2 \rho_\mu^b \cdot \Psi^\tau,$$

where (Υ_μ, ρ_μ) are defined in lemma 2.4.2 and b is the overlap size.

Proof. To ease notation, we drop the iteration index τ in the proof, and by lemma 2.6.1 we assume τ is large enough so that $\alpha_\tau = 1$. We use $(\mathbf{x}, \boldsymbol{\lambda})$ and $(\mathbf{x}', \boldsymbol{\lambda}')$ to denote two successive iterates, and let $(\tilde{\mathbf{x}}_\ell, \tilde{\boldsymbol{\lambda}}_\ell) := \mathcal{D}_\ell(\mathbf{x}, \boldsymbol{\lambda})$, $(\tilde{\mathbf{x}}'_\ell, \tilde{\boldsymbol{\lambda}}'_\ell) := \mathcal{D}_\ell(\mathbf{x}', \boldsymbol{\lambda}')$, $(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*) := \mathcal{D}_\ell(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ be truncated iterates and solution of the subproblem ℓ (cf. definition 2.3.1). With the above notation, for any $k \in \mathcal{V}$, we consider the subproblem ℓ such that $k \in \mathcal{V}_\ell$. Then, we have

$$\begin{aligned} \mathbb{I}_{\mathcal{V}_\ell} \begin{bmatrix} \tilde{\mathbf{x}}'_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}'_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} &= \mathbb{I}_{\mathcal{V}_\ell} \left\{ \begin{bmatrix} \tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\omega}}_\ell^*(0) \\ \tilde{\boldsymbol{\zeta}}_\ell^*(0) \end{bmatrix} \right\} \\ &\stackrel{(2.21)}{=} \mathbb{I}_{\mathcal{V}_\ell} \left\{ \begin{bmatrix} \tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} - \underbrace{\begin{bmatrix} \hat{H}_\ell^\mu & G_\ell^T \\ G_\ell & \mathbf{0} \end{bmatrix}^{-1}}_{K_\ell^\mu} \underbrace{\begin{bmatrix} \mu(G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell])^T \mathbf{c}[\hat{\mathcal{W}}_\ell] + \nabla_{\mathbf{x}} \mathcal{L}[\mathcal{W}_\ell] \\ \mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell] \end{bmatrix}}_{\mathbf{g}_\ell^\mu} \right\}. \end{aligned} \quad (2.33)$$

Here, $\mathbb{I}_{\mathcal{V}_\ell}$ is a block-diagonal matrix with two blocks of sizes $|\mathcal{V}_\ell| \times |\mathcal{W}_\ell|$ and $|\mathcal{V}_\ell| \times |\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell|$. For each block, only the entries (k, k) with $k \in \mathcal{V}_\ell$ are 1, while other entries are 0. (Note that the matrices here are graph-indexed). This means (2.33) holds only for nodes in $\mathcal{V}_\ell$, because we discard the variables on the overlap region. Furthermore, K_ℓ^μ and $\mathbf{g}_\ell^\mu$ are evaluated at $(\tilde{\mathbf{x}}_\ell, \tilde{\boldsymbol{\lambda}}_\ell, \tilde{\mathbf{d}}_\ell)$ with boundary variables $\tilde{\mathbf{d}}_\ell = (\mathbf{x}[\bar{\mathcal{W}}_\ell], \boldsymbol{\lambda}[\hat{\mathcal{W}}_\ell])$ (the evaluations do not solely depend on the truncated iterate $(\tilde{\mathbf{x}}_\ell, \tilde{\boldsymbol{\lambda}}_\ell) = (\mathbf{x}[\mathcal{W}_\ell], \boldsymbol{\lambda}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell])$; see (2.4) and (2.8) for the definitions of $\tilde{\mathcal{W}}_\ell$, $\hat{\mathcal{W}}_\ell$, and $\bar{\mathcal{W}}_\ell$). We aim to apply Taylor expansion for the vector $\mathbf{g}_\ell^\mu$. Let

us define two segments

$$\tilde{\mathbf{x}}_\ell(\phi) = \tilde{\mathbf{x}}_\ell^* + \phi(\tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_\ell^*) \quad \text{and} \quad \tilde{\boldsymbol{\lambda}}_\ell(\phi) = \tilde{\boldsymbol{\lambda}}_\ell^* + \phi(\tilde{\boldsymbol{\lambda}}_\ell - \tilde{\boldsymbol{\lambda}}_\ell^*), \quad \text{for } \phi \in [0, 1].$$

Then, we denote $(\mathbf{g}_\ell^\mu)^* = ((\mathbf{g}_\ell^\mu)_1^*, (\mathbf{g}_\ell^\mu)_2^*)$ to be the vector evaluated at $(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell)$ and denote $H_\ell^\mu(\phi)$ and $G_\ell(\phi)$ to be the matrices evaluated at $(\tilde{\mathbf{x}}_\ell(\phi), \tilde{\boldsymbol{\lambda}}_\ell(\phi), \tilde{\mathbf{d}}_\ell)$. With these notation, we further have

$$\begin{aligned} \begin{bmatrix} (\mathbf{g}_\ell^\mu)_1 \\ (\mathbf{g}_\ell^\mu)_2 \end{bmatrix} &= \begin{bmatrix} (\mathbf{g}_\ell^\mu)_1^* \\ (\mathbf{g}_\ell^\mu)_2^* \end{bmatrix} + \begin{bmatrix} (\mathbf{g}_\ell^\mu)_1 - (\mathbf{g}_\ell^\mu)_1^* \\ (\mathbf{g}_\ell^\mu)_2 - (\mathbf{g}_\ell^\mu)_2^* \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{g}_\ell^\mu)_1^* \\ (\mathbf{g}_\ell^\mu)_2^* \end{bmatrix} + \int_0^1 \begin{bmatrix} H_\ell^\mu(\phi) + \mu \langle \nabla_{\mathbf{x}[\mathcal{W}_\ell]}^2 \mathbf{c}[\hat{\mathcal{W}}_\ell](\phi), \mathbf{c}[\hat{\mathcal{W}}_\ell] \rangle & G_\ell^T(\phi) \\ G_\ell(\phi) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} d\phi, \end{aligned} \quad (2.34)$$

where

$$\langle \nabla_{\mathbf{x}[\mathcal{W}_\ell]}^2 \mathbf{c}[\hat{\mathcal{W}}_\ell](\phi), \mathbf{c}[\hat{\mathcal{W}}_\ell] \rangle := \sum_{i \in \hat{\mathcal{W}}_\ell} \mathbf{c}_i(\mathbf{x}[\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell]) \cdot \nabla_{\mathbf{x}[\mathcal{W}_\ell]}^2 \mathbf{c}_i(\{\tilde{\mathbf{x}}_\ell(\phi), \mathbf{x}[\bar{\mathcal{W}}_\ell]\}).$$

Combining (2.33) and (2.34) together, we have

$$\begin{aligned} \mathbb{I}_{\mathcal{V}_\ell} \begin{bmatrix} \tilde{\mathbf{x}}'_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}'_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} &= \mathbb{I}_{\mathcal{V}_\ell} \left\{ - \begin{bmatrix} \hat{H}_\ell^\mu & G_\ell^T \\ G_\ell & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} (\mathbf{g}_\ell^\mu)_1^* \\ (\mathbf{g}_\ell^\mu)_2^* \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \hat{H}_\ell^\mu & G_\ell^T \\ G_\ell & \mathbf{0} \end{bmatrix}^{-1} \int_0^1 \begin{bmatrix} \hat{H}_\ell^\mu - H_\ell^\mu(\phi) - \mu \langle \nabla_{\mathbf{x}[\mathcal{W}_\ell]}^2 \mathbf{c}[\hat{\mathcal{W}}_\ell](\phi), \mathbf{c}[\hat{\mathcal{W}}_\ell] \rangle & G_\ell^T - G_\ell^T(\phi) \\ G_\ell - G_\ell(\phi) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_\ell^* \\ \tilde{\boldsymbol{\lambda}}_\ell - \tilde{\boldsymbol{\lambda}}_\ell^* \end{bmatrix} d\phi \right\} \\ &:= \mathbb{I}_{\mathcal{V}_\ell}(\mathcal{J}_1 + \mathcal{J}_2). \end{aligned}$$

We obtain the nodewise upper bounds for the terms $\mathcal{J}_1$ and $\mathcal{J}_2$ separately as follows. We first deal with $\mathcal{J}_1$. Recall from the proof of lemma 2.4.4 that $\tilde{\mathcal{W}}_\ell := (N_{\mathcal{G}}(\tilde{\mathcal{W}}_\ell) \cap \mathcal{W}_\ell) \cup \tilde{\mathcal{W}}_\ell$ is

the set of internal boundary nodes of $\mathcal{W}_\ell$ within the depth of two. For notational clarity, we spell out the evaluation point. For $(\mathbf{g}_\ell^\mu)_1^*$ and any node $j \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$,

$$\begin{aligned} [(\mathbf{g}_\ell^\mu)_1^*]_j &= [(\mathbf{g}_\ell^\mu)_1]_j(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) \\ &= \mu \{ (G[\hat{\mathcal{W}}_\ell][j])^T \mathbf{c}[\hat{\mathcal{W}}_\ell] \}(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) + \nabla_{\mathbf{x}} \mathcal{L}[j](\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) \\ &= \nabla_{\mathbf{x}} \mathcal{L}[j](\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*) = \mathbf{0}, \end{aligned}$$

where the third equality is because $G[\hat{\mathcal{W}}_\ell][j] = \mathbf{0}$ at any evaluation point (since $j \notin N_{\mathcal{G}}[\hat{\mathcal{W}}_\ell]$) and the evaluation of $\nabla_{\mathbf{x}} \mathcal{L}[j]$ does not depend on $\tilde{\mathbf{d}}_\ell$. On the other hand, for any node $j \in \tilde{\mathcal{W}}_\ell$, we have

$$\begin{aligned} \|[(\mathbf{g}_\ell^\mu)_1^*]_j\| &= \|[(\mathbf{g}_\ell^\mu)_1]_j(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell)\| = \|[(\mathbf{g}_\ell^\mu)_1]_j(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) - [(\mathbf{g}_\ell^\mu)_1]_j(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell^*)\| \\ &\leq \|\nabla_{\mathbf{x}_j} \mathcal{L}(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) - \nabla_{\mathbf{x}_j} \mathcal{L}(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell^*)\| \\ &\quad + \mu \|\{ (G[\hat{\mathcal{W}}_\ell][j])^T \mathbf{c}[\hat{\mathcal{W}}_\ell] \}(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) - \{ (G[\hat{\mathcal{W}}_\ell][j])^T \mathbf{c}[\hat{\mathcal{W}}_\ell] \}(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell^*)\| \\ &\stackrel{(2.25)}{\leq} \hat{\Upsilon} \|\tilde{\mathbf{d}}_\ell - \tilde{\mathbf{d}}_\ell^*\| + \mu \int_0^1 \left\| \{ (G[\hat{\mathcal{W}}_\ell][j])^T G[\hat{\mathcal{W}}_\ell][\bar{\mathcal{W}}_\ell] \}(\tilde{\mathbf{x}}_\ell^*, \mathbf{x}[\bar{\mathcal{W}}_\ell](\phi)) \right\| \cdot \|\mathbf{x}[\bar{\mathcal{W}}_\ell] - \mathbf{x}^*[\bar{\mathcal{W}}_\ell]\| d\phi \\ &\quad + \mu \int_0^1 \left\| \{ \langle \nabla_{\mathbf{x}_j \mathbf{x}[\bar{\mathcal{W}}_\ell]}^2 \mathbf{c}[\hat{\mathcal{W}}_\ell], \mathbf{c}[\hat{\mathcal{W}}_\ell] \rangle \}(\tilde{\mathbf{x}}_\ell^*, \mathbf{x}[\bar{\mathcal{W}}_\ell](\phi)) \right\| \cdot \|\mathbf{x}[\bar{\mathcal{W}}_\ell] - \mathbf{x}^*[\bar{\mathcal{W}}_\ell]\| d\phi \\ &\leq 1.3\mu \hat{\Upsilon}^2 \|\tilde{\mathbf{d}}_\ell - \tilde{\mathbf{d}}_\ell^*\| \leq 1.3\mu \hat{\Upsilon}^2 \cdot |\hat{\mathcal{W}}_\ell \cup \bar{\mathcal{W}}_\ell| \cdot \Psi \leq 1.3\mu \hat{\Upsilon}^2 \cdot |\mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell| \cdot \Psi \\ &\leq 1.3\mu \hat{\Upsilon}^2 \cdot \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4) \cdot \Psi, \end{aligned} \tag{2.35}$$

where the fifth inequality is due to (2.25) and the facts that $\mu \geq 4$, $\|\mathbf{x}[\bar{\mathcal{W}}_\ell] - \mathbf{x}^*[\bar{\mathcal{W}}_\ell]\| \leq \|\tilde{\mathbf{d}}_\ell - \tilde{\mathbf{d}}_\ell^*\|$, and $\|\mathbf{c}[\hat{\mathcal{W}}_\ell](\tilde{\mathbf{x}}_\ell^*, \mathbf{x}[\bar{\mathcal{W}}_\ell](\phi))\|$ is arbitrarily small when $\mathbf{x}$ is sufficiently close to $\mathbf{x}^*$ (i.e., τ is large enough); the sixth inequality is by the definition of $\Psi (= \Psi^\tau)$; and the last inequality is due to Assumption 2.4.4 (note that $d_{\mathcal{G}}(j_1, j_2) \leq \text{poly}_{\mathcal{W}}(b) + 4 \leq \text{poly}_{\mathcal{W}}(b+4)$ for any nodes $j_1, j_2 \in \mathcal{W}_\ell \cup \bar{\mathcal{W}}_\ell$). We analyze $(\mathbf{g}_\ell^\mu)_2^*$ in a similar way. For any node $j \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$,

$$[(\mathbf{g}_\ell^\mu)_2^*]_j = [(\mathbf{g}_\ell^\mu)_2]_j(\tilde{\mathbf{x}}_\ell^*, \tilde{\boldsymbol{\lambda}}_\ell^*, \tilde{\mathbf{d}}_\ell) = \mathbf{c}[\mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell](\tilde{\mathbf{x}}_\ell^*) = \mathbf{0}.$$

Next, we deal with the term $\mathcal{J}_2$ and show that the nodewise upper bound of the integral term is only $o(\Psi)$, where $o(\cdot)$ denotes the small “o” order in the usual sense. Specifically, by Assumptions 2.6.1, 2.6.2, (2.25), and the fact that $\mathbf{c}[\hat{\mathcal{W}}_\ell](\tilde{\mathbf{x}}_\ell, \tilde{\mathbf{d}}_\ell)$ is arbitrarily small when $\mathbf{x}$ is close to $\mathbf{x}^*$, we know

$$\begin{bmatrix} \hat{H}_\ell^\mu - H_\ell^\mu(\phi) - \mu \langle \nabla_{\mathbf{x}[\mathcal{W}_\ell]}^2 \mathbf{c}[\hat{\mathcal{W}}_\ell](\phi), \mathbf{c}[\hat{\mathcal{W}}_\ell] \rangle & G_\ell^T - G_\ell^T(\phi) \\ G_\ell - G_\ell(\phi) & \mathbf{0} \end{bmatrix} = o(1).$$

Combining $\mathcal{J}_1$ and $\mathcal{J}_2$ in the above four displays together and applying lemma 2.4.2, we have

$$\begin{aligned} \|\tilde{\mathbf{x}}'_{\ell,k} - \tilde{\mathbf{x}}^*_{\ell,k}\| &\leq \left\| \sum_{j \in \mathcal{W}_\ell} [(K_\ell^\mu)_1^{-1}]_{k,j} [(\mathbf{g}_\ell^\mu)_1^*]_j + \sum_{j \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell} [(K_\ell^\mu)_2^{-1}]_{k,j} [(\mathbf{g}_\ell^\mu)_2^*]_j \right\| + o(\Psi) \\ &= \left\| \sum_{j \in \tilde{\mathcal{W}}_\ell} [(K_\ell^\mu)_1^{-1}]_{k,j} [(\mathbf{g}_\ell^\mu)_1^*]_j \right\| + o(\Psi) \stackrel{(2.35)}{\leq} 1.5\mu\hat{\Upsilon}^2 \cdot \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4) \cdot \Psi \cdot \Upsilon_\mu \sum_{j \in \tilde{\mathcal{W}}_\ell} \rho_\mu^{d_{\mathcal{G}}(k,j)} \\ &\leq 1.5\mu\hat{\Upsilon}^2 \cdot \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4) \cdot \Psi \cdot \Upsilon_\mu \rho_\mu^{b-1} |\tilde{\mathcal{W}}_\ell| \\ &\stackrel{(2.11)}{\leq} \Upsilon_K \cdot \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4) \cdot \Psi \cdot \Upsilon_\mu \rho_\mu^{b-1} \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b) \leq \{\Upsilon_\mu \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4)\}^2 \rho_\mu^b \Psi, \end{aligned}$$

where the third inequality applies lemma 2.4.2, the fourth inequality is because $d_{\mathcal{G}}(k, j) \geq b-1$ for any $k \in \mathcal{V}_\ell$ and $j \in \tilde{\mathcal{W}}_\ell$, and the fifth and sixth inequalities are because of the definition in lemma 2.4.2. The above derivations also hold for $\|\tilde{\boldsymbol{\lambda}}'_{\ell,k} - \tilde{\boldsymbol{\lambda}}^*_{\ell,k}\|$. We complete the proof by noting that $\|(\tilde{\mathbf{x}}'_{\ell,k}, \tilde{\boldsymbol{\lambda}}'_{\ell,k}) - (\tilde{\mathbf{x}}^*_{\ell,k}, \tilde{\boldsymbol{\lambda}}^*_{\ell,k})\| \leq \|\tilde{\mathbf{x}}'_{\ell,k} - \tilde{\mathbf{x}}^*_{\ell,k}\| + \|\tilde{\boldsymbol{\lambda}}'_{\ell,k} - \tilde{\boldsymbol{\lambda}}^*_{\ell,k}\|$. $\square$

From the proof of theorem 2.6.1, we observe that the error $(\mathbf{x}_k^\tau - \mathbf{x}_k^*, \boldsymbol{\lambda}_k^\tau - \boldsymbol{\lambda}_k^*)$ consists of two terms. The first term, $\mathcal{J}_1$, represents a nodewise linear convergence rate achieved by the overlapping decomposition procedure, while the second term, $\mathcal{J}_2$, is of the order $o(\Psi^\tau)$ and represents a superlinear convergence rate achieved by the SQP framework. The linear rate of the decomposition procedure dominates the convergence rate of FOGD for sufficiently large τ , and improves exponentially in terms of the overlap size. We note that the uniform linear

convergence also matches that of the Schwarz method [48]. We finally summarize the global and local convergence results of FOGD in the next theorem.

Theorem 2.6.2 (Convergence of FOGD). *Consider algorithm 1 under Assumptions 2.4.1–2.4.4 and 2.5.1, and suppose the parameters $(\eta_1, \eta_2, \delta_{\mu,b})$ satisfy (2.32). Then, $\|\nabla \mathcal{L}^\tau\| \rightarrow 0$ as $\tau \rightarrow \infty$. Moreover, if Assumptions 2.6.1–2.6.2 hold locally, then for sufficiently large τ , $\Psi^{\tau+1} \leq 2\{\Upsilon_\mu \text{poly}_{\mathcal{G} \circ \mathcal{W}}(b+4)\}^2 \rho_\mu^b \cdot \Psi^\tau$, where the nodewise error Ψ^τ is defined in theorem 2.6.1, the constants (Υ_μ, ρ_μ) are defined in lemma 2.4.2, and b is the overlap size.*

Proof. We combine theorem 2.5.1 with theorem 2.6.1 and complete the proof. $\square$

2.7 Numerical Experiment

We verify our theoretical results and illustrate the performance of FOGD by conducting a numerical experiment on a semilinear elliptic PDE problem [1]. We let $\Omega \subseteq \mathbb{R}^2$ be a two-dimensional domain and consider the problem

$$\min_{\mathbf{u}, \mathbf{z}} f(\mathbf{u}, \mathbf{z}) = \int_{\omega \in \Omega} \left\{ (\mathbf{u}(\omega) - u_d)^2 + \alpha \cdot \mathbf{z}^2(\omega) \right\} d\omega, \quad (2.36a)$$

$$\text{s.t. } g(\mathbf{u}, \mathbf{z}) = \begin{cases} -\Delta \mathbf{u} + \mathbf{u}^p - \mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.36b)$$

where $\mathbf{u}, \mathbf{z} : \Omega \rightarrow \mathbb{R}$ are the state and control mappings, u_d is the desired state, and Δ is the Laplace operator of form $\Delta \mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial \omega_1^2} + \frac{\partial^2 \mathbf{u}}{\partial \omega_2^2}$. We let $\Omega = (0, 39)^2$, $u_d = -5$, $\alpha = 0.5$, and $p = 4$. To solve Problem (2.36), we discretize Ω by a 40×40 mesh grid with evenly spaced nodes. We then decompose the grid into five disjoint subgraphs, given by $\{(0, 7), (8, 15), \dots, (32, 39)\} \times (0, 39)$. The overlapping graph decomposition is illustrated in fig. 2.2.

We consider five different overlap sizes $b \in \{1, 2, 4, 6, 8\}$ and set $\eta_1 = 5$, $\eta_2 = 0.1$, $\beta = 0.1$, and $\mu = 1$. For backtracking line search, we first initialize the stepsize α_τ to 1 and continue to

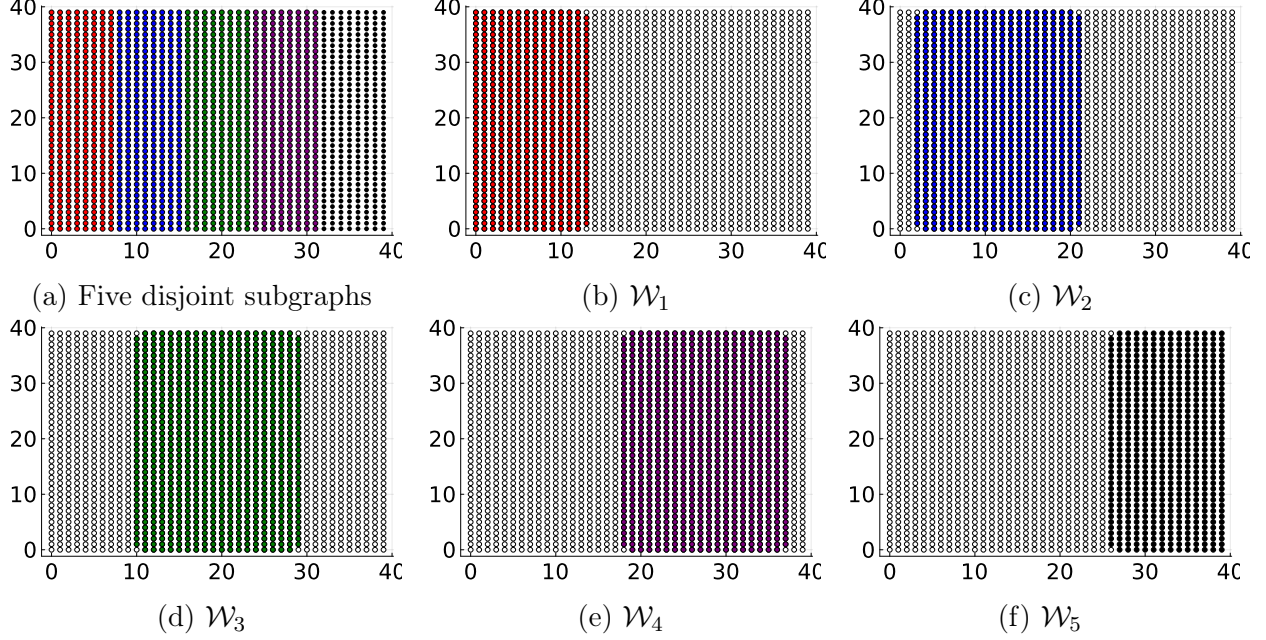


Figure 2.2: Illustration of the graph decomposition. The full graph is 40×40 mesh grid. fig. 2.2a shows one way to decompose the full graph into five disjoint subgraphs, where different colors represent different subgraphs. fig. 2.2b–fig. 2.2f show the five overlapping subgraphs with the overlap size $b = 6$.

decrease the stepsize by $\alpha_\tau \leftarrow 0.9\alpha_\tau$ until the Armijo condition (2.6) is satisfied. To demonstrate the global convergence, we generate five initial iterates randomly from $(\mathbf{u}^0, \mathbf{z}^0, \boldsymbol{\lambda}^0) \stackrel{\text{iid}}{\sim} \text{Uniform}(-10^2, 10^2)$. Then, we apply different values of b to solve (2.36) for each initialization. To demonstrate the local convergence rate, we set $(\mathbf{u}^0, \mathbf{z}^0, \boldsymbol{\lambda}^0) = (-10, -10, 0)$ to make the initial iterate closer to the local solution. The reference solution is solved by the Julia/JuMP package [18] with the Ipopt solver [76].

Our results are summarized in fig. 2.3. As seen in fig. 2.3a–2.3e, FOGD with different overlap sizes converges for all the five random initializations, validating our global convergence result (cf. theorem 2.5.1). By fig. 2.3f, we observe that the nodewise error Ψ^τ decays at least linearly with respect to τ , and the convergence rate increases as the overlap size b increases. This observation is consistent with our local convergence result (cf. theorem 2.6.1).

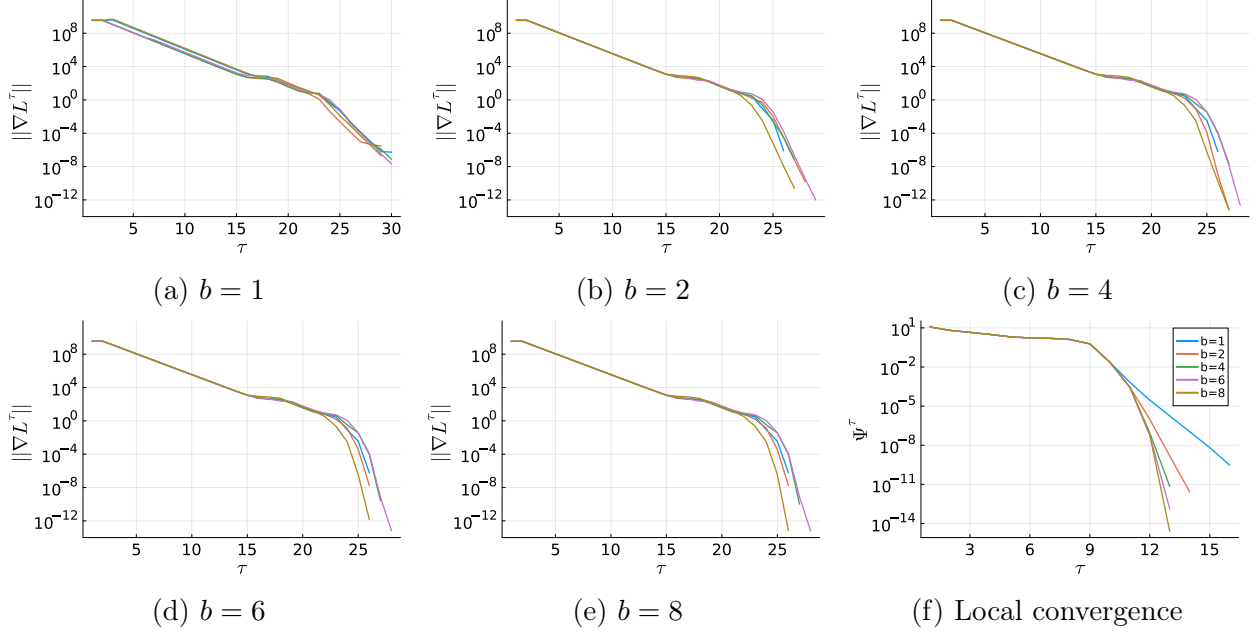


Figure 2.3: Global and local convergence plots. The global convergence plots are displayed in fig. 2.3a–2.3e, where we plot the KKT residual of each iteration. In each figure, five lines with different colors represent five initializations, and they all converge, indicating that FOGD converges for any initializations. The local convergence plot is displayed in fig. 2.3f, where five lines represent five overlap sizes. We observe that Ψ^τ decays at least linearly with respect to τ , and a larger b leads to a faster convergence rate.

2.8 Conclusion

In this chapter, we introduced a fast overlapping graph decomposition (FOGD) method for solving graph-structured nonlinear programs. We embedded the OGD technique into the SQP framework to compute approximate Newton directions in a parallel environment, and selected suitable stepsizes by performing line search on an exact augmented Lagrangian merit function. We established the global and local linear convergence guarantees for FOGD, and showed that the linear rate improves exponentially in terms of the overlap size. FOGD balances the flexibility and efficiency of centralized optimization solvers, addresses the slow convergence disadvantage of existing distributed methods with exclusive decomposition, and complements the overlapping Schwarz schemes that have expensive computational costs on local machines and lack global convergence guarantees. Numerical results on a semilinear PDE-constrained problem verified our theoretical findings.

Further extensions of FOGD are a promising research direction, inspired by empirical evidence that the Schwarz schemes have exhibited superior performance compared to the popular distributed solver, the alternating direction method of multipliers (ADMM) [61]. It would be worthwhile to incorporate OGD techniques into ADMM and other advanced SQP frameworks to enhance their performance under exclusive graph decomposition.

2.9 Appendix

2.9.1 Proof of Lemma 2.4.1

Throughout the proof, we omit the iteration index τ for the simplicity of notation. We first prove (a). We note that G_ℓ is a submatrix of $G = \nabla \mathbf{c}$, and for any nodes $i \in \mathcal{W}_\ell \setminus \tilde{\mathcal{W}}_\ell$ and $j \in \mathcal{V} \setminus \mathcal{W}_\ell$, $[G_\ell]_{i,j} = G_{i,j} = \mathbf{0}$ since $d_G(i, j) > 1$. Thus, we permute G as

$$G_P := \begin{bmatrix} G_\ell & \mathbf{0} \\ J_1 & J_2 \end{bmatrix},$$

where J_1 and J_2 are nonzero matrices. Since $G_\ell G_\ell^T$ is the diagonal submatrix of $G_P G_P^T$, and $G_P G_P^T \succeq \gamma_G I$ by Assumption 2.4.2, we complete the proof of (a). For the proof of (b), we apply Assumptions 2.4.1–2.4.4, (2.10), and [62, Lemma 5.6], and have

$$\hat{H} + \mu G^T G \succeq \frac{\gamma_H}{2} I \quad \text{provided} \quad \mu \geq \frac{2\hat{\Upsilon}^2 / \gamma_H + \gamma_H + \hat{\Upsilon}}{\gamma_G}. \quad (2.37)$$

The above condition on μ is implied by the statement of the lemma. Because $\hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell] + \mu(G[\mathcal{V}][\mathcal{W}_\ell])^T G[\mathcal{V}][\mathcal{W}_\ell]$ is the diagonal submatrix of $\hat{H} + \mu G^T G$, we know (2.37) holds for this

diagonal submatrix as well. Furthermore, for any $\tilde{\omega}_\ell$ such that $G_\ell \tilde{\omega}_\ell = \mathbf{0}$, we have

$$\begin{aligned}
\tilde{\omega}_\ell^T \hat{H}_\ell^\mu \tilde{\omega}_\ell &= \tilde{\omega}_\ell^T \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell + \mu \|G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell\|^2 \\
&= \tilde{\omega}_\ell^T \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell + \mu \|G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell\|^2 + \mu \|G_\ell \tilde{\omega}_\ell\|^2 \\
&= \tilde{\omega}_\ell^T \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell + \mu \|G[\hat{\mathcal{W}}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell\|^2 + \mu \|G_\ell \tilde{\omega}_\ell\|^2 + \mu \|G[\mathcal{V} \setminus N_{\mathcal{G}}[\mathcal{W}_\ell]][\mathcal{W}_\ell] \tilde{\omega}_\ell\|^2 \\
&= \tilde{\omega}_\ell^T \hat{H}[\mathcal{W}_\ell][\mathcal{W}_\ell] \tilde{\omega}_\ell + \mu \|G[\mathcal{V}][\mathcal{W}_\ell] \tilde{\omega}_\ell\|^2 \geq \gamma_H \|\tilde{\omega}_\ell\|^2 / 2,
\end{aligned}$$

where the third equality is due to the fact that $G[\mathcal{V} \setminus N_{\mathcal{G}}[\mathcal{W}_\ell]][\mathcal{W}_\ell] = \mathbf{0}$. This completes the proof of (b). With (a), (b), and [53, Theorem 16.2], we know $\mathcal{SP}_\ell^\mu(\mathbf{d}_\ell)$ has a unique global solution for any boundary variables $\mathbf{d}_\ell$. We complete the proof.

2.9.2 Proof of Lemma 2.5.1

By Assumptions 2.4.4, 2.5.1, and [62, Lemma 5.10], we know for the constant $\hat{\Upsilon}$ in (2.10) that

$$\|H(\mathbf{x}, \boldsymbol{\lambda})\| \leq \hat{\Upsilon} \quad \text{and} \quad \|G(\mathbf{x})\| \leq \hat{\Upsilon}, \quad \forall \mathbf{x} \in \mathcal{X}, \boldsymbol{\lambda} \in \Lambda.$$

This proves (2.25). Furthermore, let Z be the matrix whose columns are orthonormal and span the null space $\{\omega : G\omega = \mathbf{0}\}$. Then, we can verify that

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} \hat{H} & G^T \\ G & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2^T \\ \mathcal{B}_2 & \mathcal{B}_3 \end{bmatrix}, \\
\mathcal{B}_1 &= Z(Z^T \hat{H} Z)^{-1} Z^T, \quad \mathcal{B}_2 = (GG^T)^{-1} G(I - \hat{H} Z(Z^T \hat{H} Z)^{-1} Z^T), \\
\mathcal{B}_3 &= (GG^T)^{-1} G(\hat{H} Z(Z^T \hat{H} Z)^{-1} Z^T \hat{H} - \hat{H}) G^T (GG^T)^{-1}.
\end{aligned}$$

By Assumptions 2.4.1–2.4.4 and (2.10), we know

$$\|\mathcal{B}_1\| \leq \frac{1}{\gamma_H}, \quad \|\mathcal{B}_2\| \leq \frac{1}{\sqrt{\gamma_G}} \left(1 + \frac{\hat{\Upsilon}}{\gamma_H}\right), \quad \|\mathcal{B}_3\| \leq \frac{1}{\gamma_G} \left(\hat{\Upsilon} + \frac{\hat{\Upsilon}^2}{\gamma_H}\right).$$

Therefore, $\|\mathcal{B}\| \leq \max\{\|\mathcal{B}_1\|, \|\mathcal{B}_3\|\} + \|\mathcal{B}_2\| \leq \frac{2\hat{\Upsilon}^2}{\gamma_G\gamma_H} + \frac{2\hat{\Upsilon}}{\sqrt{\gamma_G}\gamma_H} \leq \frac{4\hat{\Upsilon}^2}{\gamma_H\gamma_G}$. This proves (2.26).

2.9.3 Proof of Lemma 2.6.1

For two positive sequences $\{a_\tau\}$ and $\{b_\tau\}$, we denote $a_\tau = o(b_\tau)$ if $a_\tau/b_\tau \rightarrow 0$. By (2.6), it suffices to show for sufficiently large τ that

$$\mathcal{L}_\eta(\mathbf{x}^\tau + \tilde{\Delta}\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau + \tilde{\Delta}\boldsymbol{\lambda}^\tau) \leq \mathcal{L}_\eta(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau) + \beta \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix}.$$

By thrice differentiability of $\{f_i, \mathbf{c}_i\}_{i \in \mathcal{V}}$, we know $\nabla^2 \mathcal{L}_\eta$ is continuously differentiable. Therefore, by direct calculation, we apply Taylor expansion and have

$$\begin{aligned} \mathcal{L}_\eta(\mathbf{x}^\tau + \tilde{\Delta}\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau + \tilde{\Delta}\boldsymbol{\lambda}^\tau) &\leq \mathcal{L}_\eta(\mathbf{x}^\tau, \boldsymbol{\lambda}^\tau) + \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix}^T \mathcal{H}^\tau \begin{bmatrix} \tilde{\Delta}\mathbf{x}^\tau \\ \tilde{\Delta}\boldsymbol{\lambda}^\tau \end{bmatrix} + o(\|(\tilde{\Delta}\mathbf{x}^\tau, \tilde{\Delta}\boldsymbol{\lambda}^\tau)\|^2), \end{aligned}$$

where

$$\mathcal{H}^\tau := \begin{bmatrix} H^\tau + \eta_1(G^\tau)^T G^\tau + \eta_2(H^\tau)^2 & (I + \eta_2 H^\tau)(G^\tau)^T \\ G^\tau(I + \eta_2 H^\tau) & \eta_2 G^\tau (G^\tau)^T \end{bmatrix}.$$

Combining the above three displays, we know $\alpha_\tau = 1$ as long as

$$(1 - \beta) \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \mathcal{H}^\tau \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + o(\|(\tilde{\Delta} \mathbf{x}^\tau, \tilde{\Delta} \boldsymbol{\lambda}^\tau)\|^2) \leq 0. \quad (2.38)$$

We note that

$$\begin{aligned} & \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_\eta^\tau \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_\eta^\tau \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} + \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \mathcal{H}^\tau \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix} \\ & \stackrel{(2.2)}{=} \stackrel{(2.29)}{=} \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \left\{ \mathcal{H}^\tau - \begin{bmatrix} \hat{H}^\tau + \eta_1 (G^\tau)^T G^\tau + \eta_2 H^\tau \hat{H}^\tau & (I + \eta_2 H^\tau)(G^\tau)^T \\ G^\tau (I + \eta_2 \hat{H}^\tau) & \eta_2 G^\tau (G^\tau)^T \end{bmatrix} \right\} \begin{bmatrix} \Delta \mathbf{x}^\tau \\ \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} \\ & \quad + \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \mathcal{H}^\tau \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau - \Delta \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau - \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} \\ & = \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \begin{bmatrix} (I + \eta_2 H^\tau)(H^\tau - \hat{H}^\tau) & \mathbf{0} \\ \eta_2 G^\tau (H^\tau - \hat{H}^\tau) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}^\tau \\ \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} + \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau \end{bmatrix}^T \mathcal{H}^\tau \begin{bmatrix} \tilde{\Delta} \mathbf{x}^\tau - \Delta \mathbf{x}^\tau \\ \tilde{\Delta} \boldsymbol{\lambda}^\tau - \Delta \boldsymbol{\lambda}^\tau \end{bmatrix} \\ & \leq o(\|(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau)\|^2) + (1 + \delta_{\mu,b}) \delta_{\mu,b} \|\mathcal{H}^\tau\| \cdot \|(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau)\|^2 \\ & \leq o(\|(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau)\|^2) + 2\delta_{\mu,b} \{2\hat{\Upsilon} + (\eta_1 + 2\eta_2)\hat{\Upsilon}^2\} \|(\Delta \mathbf{x}^\tau, \Delta \boldsymbol{\lambda}^\tau)\|^2, \end{aligned}$$

where the second inequality from the end is due to Assumption 2.6.1 and (2.10), and the last inequality is due to $\|\mathcal{H}^\tau\| \leq \max(\hat{\Upsilon} + (\eta_1 + \eta_2)\hat{\Upsilon}^2, \eta_2 \hat{\Upsilon}^2) + (\hat{\Upsilon} + \eta_2 \hat{\Upsilon}^2)$ and $\delta_{\mu,b} \leq 1$.

Plugging the above result into (2.38), we obtain

$$\begin{aligned}
& (1 - \beta) \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_{\eta}^{\tau} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_{\eta}^{\tau} \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^{\tau} \\ \tilde{\Delta} \boldsymbol{\lambda}^{\tau} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \tilde{\Delta} \mathbf{x}^{\tau} \\ \tilde{\Delta} \boldsymbol{\lambda}^{\tau} \end{bmatrix}^T \mathcal{H}^{\tau} \begin{bmatrix} \tilde{\Delta} \mathbf{x}^{\tau} \\ \tilde{\Delta} \boldsymbol{\lambda}^{\tau} \end{bmatrix} + o(\|(\tilde{\Delta} \mathbf{x}^{\tau}, \tilde{\Delta} \boldsymbol{\lambda}^{\tau})\|^2) \\
& \leq (0.5 - \beta) \begin{bmatrix} \nabla_{\mathbf{x}} \mathcal{L}_{\eta}^{\tau} \\ \nabla_{\boldsymbol{\lambda}} \mathcal{L}_{\eta}^{\tau} \end{bmatrix}^T \begin{bmatrix} \tilde{\Delta} \mathbf{x}^{\tau} \\ \tilde{\Delta} \boldsymbol{\lambda}^{\tau} \end{bmatrix} + \delta_{\mu,b} \{2\hat{\Upsilon} + (\eta_1 + 2\eta_2)\hat{\Upsilon}^2\} \|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2 + o(\|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2) \\
& \leq \left\{ (0.5 - \beta) \left\{ -\frac{\eta_2 \gamma G}{8} + 1.05\eta_1 \hat{\Upsilon}^2 \delta_{\mu,b} \right\} + 1.05\eta_1 \hat{\Upsilon}^2 \delta_{\mu,b} \right\} \|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2 + o(\|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2) \\
& = \left\{ -\frac{(0.5 - \beta)\eta_2 \gamma G}{8} + (1.5 - \beta)1.05\eta_1 \hat{\Upsilon}^2 \delta_{\mu,b} \right\} \|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2 + o(\|(\Delta \mathbf{x}^{\tau}, \Delta \boldsymbol{\lambda}^{\tau})\|^2),
\end{aligned}$$

where the second inequality is due to (2.28), (2.30), (2.31), and the fact that $2\hat{\Upsilon} + (\eta_1 + 2\eta_2)\hat{\Upsilon}^2 \leq 2\hat{\Upsilon} + 2/7 + \eta_1 \hat{\Upsilon}^2 \leq 2.5\hat{\Upsilon}^2 + \eta_1 \hat{\Upsilon}^2 \leq 1.05\eta_1 \hat{\Upsilon}^2$. By (2.32) and noting that the condition of $\delta_{\mu,b}$ implies the condition (2.27), we complete the proof.

CHAPTER 3

SAMPLE-EFFICIENT AND SCALABLE ALGORITHMS FOR THE IDENTIFICATION AND CONTROL OF UNKNOWN, NON-STATIONARY, PARAMETRIC LINEAR SYSTEMS

3.1 Introduction

We investigate a finite-horizon control problem for unknown, parametric, sparse, non-stationary linear systems. We propose a scalable approach to estimate the unknown parameters and solve the estimated system approximately. Under some assumptions, our algorithm achieves: (1) the computational cost of the parameter estimation is $O(n)$ per time step and the policy computation cost is $O(n \log T)$ per time step where n is the dimension of the system state and T is the number of the time step. This is a net improvement over classical results obtaining the policy by solving Ricatti equations which needs $O(n^3)$ computational cost. (2) the sample complexity is $O(\log n)$; (3) the estimation error is $O(n_\theta)$, where n_θ is the number of parameters in the parametric linear system; (4) the dynamic regret of $\tilde{O}\left(n_\theta^2 n V_T^{2/5} T^{3/5}\right)$ with high probability enhancing the state-of-the-art result of $\tilde{O}\left(n^3 V_T^{2/5} T^{3/5}\right)$ in [42] when $n \gg n_\theta$.

(Specifically, we assume that the system dynamics, characterized by matrices A_t and B_t , can be expressed as linear combinations of a set of known basis matrices. The coefficients of these combinations are unknown and vary over time, with the total variation bounded by $V_T = o(T)$. We analyze the regret of a control algorithm that integrates classical system identification with model predictive control, tailored for non-stationary environments. Our proposed algorithm leverages the parametric structure to achieve enhanced sample complexity. To address non-stationarity, we divide the time horizon into multiple intervals and perform parameter re-identification at scheduled intervals, enabling adaptive learning of the unknown parameters and approximate solutions to the control problem. We demonstrate

that exploiting the parametric structure reduces the estimation error from $O(n)$ to $O(n_\theta)$, where n is the state dimension and n_θ is the number of parameters in the parametric linear system. Furthermore, with improved sample and computational complexities, our algorithm attains a dynamic regret of $\tilde{O}\left(n_\theta^2 n V_T^{2/5} T^{3/5}\right)$ with high probability, enhancing the state-of-the-art result of $\tilde{O}\left(n^3 V_T^{2/5} T^{3/5}\right)$ in [42] when $n \gg n_\theta$. Additionally, by employing model predictive control with a truncated horizon, we achieve a computational complexity of $O(\log(T))$ per stage, as opposed to the $O(T)$ complexity required for solving the full-horizon control problem.

3.2 Background

We consider the discrete-time finite-horizon linear quadratic regulator (LQR) problem with non-stationary dynamics:

$$\begin{aligned} \min_{\substack{\{x_t\}_{t \in [T+1]} \\ \{u_t\}_{t \in [T]}}} & \frac{1}{2} x_T^T Q x_T + \sum_{t=0}^{T-1} \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t, \\ \text{s.t. } & x_0 = \bar{x}_0, \\ & x_{t+1} = A_t x_t + B_t u_t + w_t, \quad t \in [T]. \end{aligned} \tag{3.1}$$

where T is the length of horizon, $x_t \in \mathbb{R}^n$ is the state variable, $u_t \in \mathbb{R}^d$ is the control variable, and $w_t \sim \mathcal{N}(\mathbf{0}, \psi^2 I_n)$ is the random noise.

It is assumed that the dynamics A_t and B_t are unknown, but they exhibit a *parametric* structure of the following form:

$$\begin{aligned} A_t &= \sum_{j=1}^{n_A} \alpha_{t,j} \bar{A}_j, \quad \alpha_{t,j} \in \mathbb{R}, \quad \bar{A}_j \in \mathbb{R}^{n \times n}, \\ B_t &= \sum_{k=1}^{n_B} \beta_{t,k} \bar{B}_k, \quad \beta_{t,k} \in \mathbb{R}, \quad \bar{B}_k \in \mathbb{R}^{n \times d}. \end{aligned} \tag{3.2}$$

Here, we assume that $\alpha_{t,j}$ and $\beta_{t,k}$ are unknown and vary over time $t = 0, 1, \dots$, while the basis matrices $\bar{A}_j$ and $\bar{B}_k$ are known constant matrices. System (3.2) is of particular interest because, for a wide range of applications, the number of unknown parameters in the dynamics is much smaller than the full system size (i.e., $n_\theta := n_A + n_B \ll n$). Furthermore, in many practical applications, the system dynamics are non-stationary, and it is crucial to adapt to the changing environment to achieve good control performance. These problems often arise in large-scale control applications, such as control of distributed parameter systems or multi-agent/network systems, where the state dimension n is large.

Therefore, in this chapter, we aim to study algorithms that can (1) exploit such parametric structures and enjoy better sample complexity and overall regret, (2) handle non-stationary dynamics, and (3) have desirable computational scalability so that the algorithm can be applied to problems with large state dimensions. With rigorous sample complexity, regret, and computational complexity analyses, we aim to provide a comprehensive understanding of the proposed algorithm.

To handle non-stationarity, our algorithm divides the full horizon into multiple sub-intervals of equal size. Each interval comprises an *exploration stage* and a *control stage*. Since the dynamical system is non-stationary and accurate parameter estimation is crucial for effective control performance, we perform re-identification at the beginning of each interval to update the model. During the exploration stage, we estimate the system parameters using the ordinary least squares (OLS) method. The duration of the exploration stage is selected to ensure that sufficient samples are collected, allowing the estimated parameters to be sufficiently close to the true dynamic parameters within each interval. This approach minimizes the impact of implementing a suboptimal control policy induced by randomized exploration noise. Such re-identification approach is largely inspired by [42], which investigated the sample complexity and regret for learning non-stationary LQR parameters in a non-parametric setting and a full-horizon LQR policy. However, instead of employing online

state monitoring, we adopt a fixed re-identification schedule, which simplifies the analysis while maintaining the same regret bounds with high probability.

In each control stage, we formulate stationary LQR with the estimated parameters obtained from the exploration stage. In the control literature, it is well-known that the optimal control policy takes the form of linear state feedback $u_t = K_t x_t$ for some matrices $\{K_t\}$ [4]. While the original problem is stochastic control problem in nature, one can obtain the optimal control policy by considering deterministic equivalence due to the certainty equivalency [71]. Accordingly, the calculation of $\{K_t\}$ is typically performed by solving the finite-horizon Riccati equations. However, finding the exact solution to the finite horizon Riccati equation is computationally expensive, especially for long-horizon (large T) and large-scale systems (large n). In this work, instead of solving Riccati equations which requires an iterative solver, we solve MPC problems with the truncated horizon where we only need to solve a linear system. We approximately solve the problem by considering MPC framework: in each stage, we formulate the control problem with truncated horizon and we apply the receding-horizon control policy. Due to the well-known computational complexity result for MPC problems, the computational cost of solving the truncated MPC problem is only linear in T and cubic in n [29]. We further numerically demonstrate that if additional sparsity structure in A_t and B_t is exploited, the computational cost can be further reduced to linear in n . The intuition behind the truncated horizon MPC problem is that the approximated policy obtained by solving the truncated horizon MPC problem contains sufficient information on the optimal policy so that the regret error will decrease as the truncated horizon length increases [63]. Multiple works study nonlinear multi-stage MPC problems [40, 41, 70, 78] where authors obtain the approximated solutions to the full horizon MPC problem by solving the truncated ones sequentially. Specifically, in every time step, an MPC problem with the truncated horizon is solved to obtain the control variable of that time step and generate the state variable of the next time step. Then, the truncated MPC problem will be shifted one time step to

obtain the control variable of the next time step. Repeatedly, the approximated solutions to the full-horizon MPC problem are obtained by a sequence of truncated MPC problems. This method has widespread popularity among many applications such as stationary battery systems [32], central HVAC plants [33], islanded microgrids [23], etc.

Related Work LQR problems have been studied in the context of various applications, such as traffic networks [74], traffic control [73], robot control [38], economic policy [15], etc. Furthermore, non-stationary LQR has been extensively studied to obtain insights for oftentimes more complicated real-world sequential decision-making problems. As said in Chapter 8 of [69], “one of the most powerful applications of time-varying LQR involves linearizing around a nominal trajectory of a nonlinear system and using LQR to provide a trajectory controller.” Specifically, nonlinear dynamic constraints can be reduced to non-stationary LQR problems by linearization:

$$E[x_{t+1}] = x_t + f(x_t, u_t) \quad (3.3)$$

By defining $\bar{x}_t = x_t - x_{t,0}$ and $\bar{u}_t = u_t - u_{t,0}$, we can rewrite eq. (3.3) as

$$\begin{aligned} E[\bar{x}_{t+1}] &= x_t - x_{t,0} + f(x_t, u_t) - f(x_{t,0}, u_{t,0}) \\ &= \bar{x}_t + \frac{\partial f(x_{t,0}, u_{t,0})}{\partial x_{t,0}} \bar{x}_t + \frac{\partial f(x_{t,0}, u_{t,0})}{\partial u_{t,0}} \bar{u}_t = A_t \bar{x}_t + B_t \bar{u}_t \end{aligned}$$

In LQR problems, the governing dynamical systems are oftentimes unknown and they need to be identified from the data. The state-of-the-art practice is to estimate unknown parameters of the non-stationary LQR using observed data (e.g., by performing system identification), while solving the control problem (e.g., model predictive control; MPC) with the estimated parameters. Luo et al. [42] has studied learning of non-stationary dynamics and regret analysis of LQR, focusing on unstructured A_t and B_t . The authors have shown that by employing

the re-identification procedure, the sample complexity of learning the non-stationary LQR parameters can become $O(n)$ and the regret of the control algorithm is $\tilde{O}\left(n^3 V_T^{2/5} T^{3/5}\right)$, where V_T is the total variation of the unknown parameters over the time horizon T . However, to the best of our knowledge, there has not been work studying A_t and B_t taking parametric forms. It is beneficial to consider such a structure since it can reduce the sample complexity and estimation error if $n \gg n_\theta$. Furthermore, if the basis matrices are sparse, the computational cost of solving the control problem can be significantly reduced. For example, in partial differential equations (PDEs) constrained optimization problems, the finite difference scheme of PDEs can be represented as sparse matrices [8]. In network systems [62], sparse matrices can represent large graphs since most nodes are not directly connected. By exploiting the sparsity of sparse matrices, we can reduce sample complexity to learn unknown dynamics, reduce estimation error, optimize computational cost, and save data storage.

Contributions : This chapter makes three primary contributions. First, we significantly reduce the sample complexity required to learn non-stationary LQR parameters, achieving the estimation error proportional to n_θ . In contrast, the approach in [42] necessitates a sample complexity that scales linearly with n , making our method substantially more efficient for large-scale systems with $n \gg n_\theta$. Second, we demonstrate that our algorithm attains the same optimal regret bound as presented in [42] while reducing the computational cost to be linear in $n \log T$, where n is the state dimension and T is the time horizon. Unlike [42], which solves the Riccati equation using an iterative solver with a computational cost of $O(n^3)$, our approach approximates the LQR solution by solving a truncated horizon MPC problem.

Structure of the chapter : The remaining chapter will be organized as follows. In Section 3.3, we will present the algorithm to be analyzed (estimation and control over divided subintervals). In Section 3.4, we will analyze the required sample complexity (estimation

error) to learn the non-stationary LQR parameter for each interval. In Section 3.5, we provide the end-to-end regret analysis of the full algorithm. In particular, we will present the upper bound of the regret of our algorithm and compare the results with the results in [42]. Finally, we propose future directions in Section 3.6.

Notations : Without additional descriptions, all vectors are all column vectors. For any vector, $\|\cdot\|$ represents 2-norm and $\|\cdot\|_1$ represents 1-norm. For any matrix, $\|\cdot\|$ represents 2-norm and $\|\cdot\|_F$ represents Frobenius norm. I_k denotes the identity matrix with the dimension k . For any integer N , $[N] = \{0, 1, \dots, N-2, N-1\}$ and for any two integer N_1 and N_2 , $[N_1, N_2] = \{N_1, N_1+1, \dots, N_2-2, N_2-1\}$.

3.3 Algorithm

We present our algorithm *MPC for non-stationary systems* in Algorithm 2. Generally, we divide the full horizon T into equal size intervals $\mathcal{I}_i$ where $i \in [N]$. Every interval $\mathcal{I}_i$ consists of two stages, the exploration stage $\mathcal{I}_i^e$ and control stage $\mathcal{I}_i^r$. In the exploration stage $\mathcal{I}_i^e$, we define the control variable $u_t = K_{stab}x_t + \nu\eta_t$ where $\eta_t \sim \mathcal{N}(\mathbf{0}, I_d)$ are i.i.d random variables and K_{stab} is a constant matrix which stabilizes A_t and B_t . Since $\begin{bmatrix} x_t^T & u_t^T \end{bmatrix}^T$ is a $(n+d)$ dimension vector and the vector $\begin{bmatrix} x_t & u_t \end{bmatrix} = \begin{bmatrix} I_n & K_{stab} \end{bmatrix} x_t$ only lies in n dimensional space, it is important to add extra noise $\nu\eta_t$ to u_t to prevent the singularity when estimating dynamics parameters. At the end of the exploration stage, we estimate $\{\alpha_{t,k}\}_{k=1}^{n_A}$ and $\{\beta_{t,l}\}_{l=1}^{n_B}$ instead of A_t and B_t . We obtain the estimated vectors $\hat{\alpha}_i^T = [\hat{\alpha}_{i,1} \ \dots \ \hat{\alpha}_{i,n_A}]$ and $\hat{\beta}_i^T = [\hat{\beta}_{i,1} \ \dots \ \hat{\beta}_{i,n_B}]$ for every interval, which gives the estimated parameters $\hat{A}_i = \sum_{k=1}^{n_A} \hat{\alpha}_{i,k} \bar{A}_k$ and $\hat{B}_i = \sum_{l=1}^{n_B} \hat{\beta}_{i,l} \bar{B}_l$ where $\bar{A}_k$ and $\bar{B}_l$ are defined in (3.2). In the control stage $\mathcal{I}_i^r$, we formulate the MPC problem with the estimated $\hat{A}_i$ and $\hat{B}_i$ and truncated time horizon length W defined in (3.4), which yields the control policy of the following form: $u_t = \hat{K}_t x_t$.

Algorithm 2 System identification and control for unknown, parametric, non-stationary systems

Input: K_{stab} - Stabilizing controller; ν - the magnitude of random noise η_t ; W - truncated MPC horizon; T - the total horizon; N - the number of interval; $L_{\mathcal{I}} = T/N$ - every interval length;

for $i = 0$ **to** $N - 1$ **do**

for $t \in \mathcal{I}_i^e := [i L_{\mathcal{I}}, i L_{\mathcal{I}} + L_{\mathcal{I}}^{1/2}]$ **do**

$u_t = K_{stab}x_t + \nu\eta_t$;

 observe the next state $x_{t+1} = A_t x_t + B_t u_t + w_t$;

end for

 estimate $\hat{A}_i$ and $\hat{B}_i$ based on x_τ and u_τ for $\tau \in \mathcal{I}_i^e$;

for $t \in \mathcal{I}_i^r := [i L_{\mathcal{I}} + L_{\mathcal{I}}^{1/2}, (i + 1)L_{\mathcal{I}}]$ **do**

 solve MPC problem (3.4) with the estimated $\hat{A}_i$ and $\hat{B}_i$ to obtain $\hat{K}_i^W$ and $u_t = \hat{K}_i^W x_t$;

 observe the next state $x_{t+1} = A_t x_t + B_t u_t + w_t$;

end for

end for

$$\begin{aligned} \min_{\substack{\{x_t\}_{\tau \in [t, t'+1]} \\ \{u_t\}_{\tau \in [t, t']}}} & \quad \frac{1}{2} x_{t'}^T Q x_{t'} + \sum_{\tau=t}^{t'-1} \frac{1}{2} x_\tau^T Q x_\tau + \frac{1}{2} u_\tau^T R u_\tau, \\ \text{s.t. } & \quad t' = t + W, \end{aligned} \tag{3.4}$$

x_t = observed given x_{t-1} and u_{t-1} ,

$x_{\tau+1} = \hat{A}_i x_\tau + \hat{B}_i u_\tau, \tau \in [t, t']$.

3.4 Sample Complexity Analysis

The system identification for each interval is a critical part of our algorithm. One of the challenges is that the OLS estimator is a biased estimator for the non-stationary systems. Therefore, we must be careful about trade-offs between the bias and variance terms. Additionally, sample complexity and estimation error in the state-of-the-art sample complexity bound [42] is highly dependent on n , which is not desirable for large-scale LQR problems.

Therefore, we aim to take advantage of the structure of A_t and B_t defined in eq. (3.2) to achieve better sample complexity and tighter estimation error bounds.

The formulation of the nonstationary LQR model with unknown dynamics is defined in (3.1). We denote $\alpha_t = [\alpha_{t,1}, \dots, \alpha_{t,n_A}]^T$, $\alpha = [\alpha_1, \dots, \alpha_{n_A}]^T$, $\beta_t = [\beta_{t,1}, \dots, \beta_{t,n_B}]^T$, $\beta = [\beta_1, \dots, \beta_{n_B}]^T$, $\theta = [\alpha^T, \beta^T]^T$, $\theta_t = [\alpha_t^T, \beta_t^T]^T$, and write OLS estimator and its loss function as

$$\hat{\theta}_i = \arg \min_{\theta} \mathcal{L}_i(\theta), \quad \mathcal{L}_i(\theta) = \sum_{t \in \mathcal{I}_i^e} \|x_{t+1} - Q_t^T \theta\|^2 \quad (3.5)$$

where

$$\begin{aligned} x_{t+1} &= \left(\sum_{j=1}^{n_A} \alpha_{t,j}^* \bar{A}_j \right) x_t + \left(\sum_{j=1}^{n_B} \beta_{t,j}^* \bar{B}_j \right) u_t + w_t \\ &= Q_t^T \theta_t^* + w_t, \quad Q_t \in \mathbb{R}^{(n_\theta) \times n}, \\ Q_t^T &= [\bar{A}_1 x_t, \dots, \bar{A}_{n_A} x_t, \bar{B}_1 u_t, \dots, \bar{B}_{n_B} u_t]. \end{aligned}$$

The first-order condition of eq. (3.5) is expressed as

$$\frac{\partial \mathcal{L}_i(\theta)}{\partial \theta} = \sum_{t \in \mathcal{I}_i^e} Q_t \left(Q_t^T \theta_t^* - Q_t^T \theta + w_t \right) = 0$$

Therefore, the ordinary least square estimator $\hat{\theta}_i$ is obtained

$$\hat{\theta}_i = \Gamma_{\mathcal{I}_i^e}^{-1} \left(\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \theta_t^* \right) + \Gamma_{\mathcal{I}_i^e}^{-1} \sum_{t \in \mathcal{I}_i^e} Q_t w_t,$$

where $\Gamma_{\mathcal{I}_i^e} = \sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T$.

Now one can express the estimation error $\Delta \hat{\theta}_{i,\tau} = \hat{\theta}_i - \theta_\tau^*$ for each $\tau \in \mathcal{I}_i$ in the following

form:

$$\Delta\hat{\theta}_{i,\tau} = \Gamma_{\mathcal{I}_i^e}^{-1} \left(\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \Delta\theta_{t,\tau}^* \right) + \Gamma_{\mathcal{I}_i^e}^{-1} \sum_{t \in \mathcal{I}_i^e} Q_t w_t,$$

where $\Delta\theta_{t,\tau}^* = \theta_t^* - \theta_\tau^*$. By applying the triangle inequality, we can bound the estimation error as:

$$\|\Delta\hat{\theta}_{i,\tau}\| \leq \|\Gamma_{\mathcal{I}_i^e}^{-1}\| \left(\max_{t \in \mathcal{I}_i^e} \|\Delta\theta_{t,\tau}^*\| \|\Gamma_{\mathcal{I}_i^e}\| + \left\| \sum_{t \in \mathcal{I}_i^e} Q_t w_t \right\| \right).$$

Before we present the main theorem of the parameter estimation, we first introduce two assumptions.

Assumption 3.4.1. *We assume there exists a constant number $\lambda_{\min,M}$ such that*

$$\lambda_{\min} \left(\sum_{i=1}^n \tilde{M}_i \tilde{M}_i^T \right) \geq n \cdot \lambda_{\min,M}, \quad (3.6)$$

where

$$\begin{aligned} \bar{A}_i &= \begin{bmatrix} \bar{A}_{1,i} \\ \vdots \\ \bar{A}_{n,i} \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} \bar{B}_{1,k} \\ \vdots \\ \bar{B}_{n,k} \end{bmatrix}, \quad \tilde{A}_j = \begin{bmatrix} \bar{A}_{j,1} \\ \vdots \\ \bar{A}_{j,n_A} \end{bmatrix}, \\ \tilde{B}_j &= \begin{bmatrix} \bar{B}_{j,1} \\ \vdots \\ \bar{B}_{j,n_B} \end{bmatrix}, \quad \tilde{M}_j = \begin{bmatrix} \tilde{A}_j & \\ & \tilde{B}_j \end{bmatrix}, \end{aligned} \quad (3.7)$$

for $i \in [1, n_A + 1]$, $k \in [1, n_B + 1]$, $j \in [1, n + 1]$, $\tilde{A}_j \in \mathbb{R}^{n_A \times n}$, $\tilde{B}_j \in \mathbb{R}^{n_B \times d}$, and $\tilde{M}_j \in \mathbb{R}^{(n_\theta) \times (n+d)}$.

Since $\lambda_{\min} \left(\sum_{i=1}^n \tilde{M}_i \tilde{M}_i^T \right) \geq \sum_{i=1}^n \lambda_{\min} \left(\tilde{M}_i \tilde{M}_i^T \right)$, the assumption will be fulfilled if

there exists a uniform lower bound of $\lambda_{\min}(\tilde{M}_i \tilde{M}_i^T)$ for at least $O(n)$ number of i 's.

This assumption is satisfied for the following example:

$$\bar{A}_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \end{bmatrix}, \quad (3.8)$$

$$\bar{A}_3 = \begin{bmatrix} & & & \\ 1 & & & \\ & \ddots & & \\ & & & 1 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \quad (3.9)$$

One can see that for $j = 2, \dots, n_A - 1$, $\tilde{A}_j \tilde{A}_j^T = I$, which indicates that for $n - 2$ number of j 's, $\lambda_{\min}(\tilde{A}_j \tilde{A}_j^T) = 1$. Therefore, Assumption 3.4.1 is satisfied for this example. While (3.8) is a simple example, similar structures arise in many practical applications, such as discretization of PDEs.

We now introduced the second assumption, which is a standard assumption widely used in many LQR literature [42, 64].

Assumption 3.4.2. *There exists constants $k_T \geq 1$, $K_u, L \geq 1$, $\alpha \in (0, 1)$, and $\gamma \in (0, 1]$ such that*

1. $\max_{t \in [T]} \|A_t\|, \max_{t \in [T]} \|B_t\|, \|Q\| \leq L$, and $\|R\|, \|K_{stab}\|, \|x_0\|, \max_{\substack{1 \leq i \leq n_A \\ 1 \leq k \leq n_B}} (\|\bar{A}_i\|, \|\bar{B}_k\|) \leq L; \|K_{stab}\|, \|\hat{K}_i^W\| \leq K_u$,
2. $Q \succeq 0$ and $R \succeq \gamma I$ for $\gamma > 0$;
3. $(\{A_t\}_{t \in [T]}, \{B_t\}_{t \in [T]})$ is (L, α) -stabilizable: there exists a K_t such that $\|\Pi_{t=t'+1}^{t''}(A_t - B_t K_t)\| \leq L\alpha^{t''-t'}$ where $\|K_t\| \leq K_u$;

4. $(\{A_t\}_{t \in [T]}, Q^{1/2})$ is (L, α) -detectable: there exists K_t such that $\|\Pi_{t=t'+1}^{t''} (A_t - K_t Q^{1/2})\| \leq L\alpha^{t''-t'}$ where $\|K_t\| \leq K_u$;
5. $\sum_{t \in [T]} \|\theta_{t+1}^* - \theta_t^*\| \leq V_T = o(T)$ and $\sum_{t \in \mathcal{I}_i} \|\theta_{t+1}^* - \theta_t^*\| \leq k_T \cdot V_T/N$.

Note that we use the same assumption in [42] regarding the total variation of the dynamic parameter bounded by $V_T = o(T)$. Additionally, it is reasonable to assume the variation of parameters in each equal-size interval is proportional to the total variation V_T .

We are now ready to state the main result on the sample complexity and estimation error of the OLS estimator.

Theorem 3.4.1. *Under Assumption 3.4.1-3.4.2, if we have*

$$|\mathcal{I}_i^e| \geq \frac{2000 \cdot n_\theta}{9} \left(2 \log \left(\frac{1}{\delta} \right) + \log \left(\frac{(n+d)\bar{z}}{\bar{C}_1} \right) \right)$$

for $\delta \leq 3/100$, $\Delta \hat{\theta}_{i,\tau} = \hat{\theta}_i - \theta_\tau^*$, and $\Delta \theta_{t,\tau}^* = \theta_t^* - \theta_\tau^*$ where $\tau \in \mathcal{I}_i$, it holds with probability at least $1 - 8(n_\theta)nN\delta$ that

$$\|\Delta \hat{\theta}_{i,\tau}\| \leq \frac{6400C_Q^2 \cdot n_\theta}{9\bar{C}_1\lambda_{\min,M}} \max_{t \in \mathcal{I}_i} \|\Delta \theta_{t,\tau}^*\| + \frac{3200C_Q\lambda_{\max,M}^\psi \cdot n_\theta \sqrt{\log \frac{1}{\delta}}}{9\sqrt{|\mathcal{I}_i^e|}\bar{C}_1\lambda_{\min,M}} \quad (3.10)$$

where $\lambda_{\min, M}$ is defined in Assumption 3.4.1 and

$$\begin{aligned}
\bar{C}_1 &= \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\}, \quad \bar{z} = \bar{x}^2 (1 + 2K_u^2) + 2\nu^2, \\
C_Q &= \frac{7L^3 K_u (\nu + 1) (\psi + 1) (\bar{x}_m + 1)}{1 - \alpha}, \\
\bar{x}^2 &= \frac{10L^2 L_m^2 (\psi^2 + 1)}{(1 - \rho_m)^2}, \quad \lambda_{\max, M} = \max_{1 \leq i \leq n} \lambda_{\max}^{1/2} \left(\tilde{M}_i^T \tilde{M}_i \right), \\
\rho_m &= \max \left(\alpha, \rho, \hat{\rho}^{1/4} \right), \quad L_m = \max \left(L, \hat{L}, c_1, \hat{c}_1 \right), \\
\rho_L &= \max \left(L \alpha^{L^{1/2}}, L_m \rho_m^{L^{1/2}} \right), \quad \bar{x}_m = \|x_0\| + \frac{C_L}{1 - \rho_L}, \\
C_L &= \max \left(\frac{L^2 (\psi + \nu)}{1 - \alpha}, \frac{L_m L (\psi + \nu)}{1 - \rho_m} \right), \\
\rho &= \left(\frac{L_H^2 - \gamma_H^2}{L_H^2 + \gamma_H^2} \right)^{\frac{1}{2}}, \quad \hat{\rho} = \left(\frac{\hat{L}_H^2 - \hat{\gamma}_H^2}{\hat{L}_H^2 + \hat{\gamma}_H^2} \right)^{\frac{1}{2}}, \quad \hat{L} = L + \Delta, \\
c_1 &= \frac{L_H}{\gamma_H^2 \rho}, \quad L_H = 2L + 1, \quad \hat{c}_1 = \frac{\hat{L}_H}{\hat{\gamma}_H^2 \hat{\rho}}, \quad \hat{L}_H := 2\hat{L} + 1, \\
\gamma_F &= \frac{(1 - \alpha)^2}{(1 + L)^2 L^2}, \quad \hat{\gamma}_F = \frac{(1 - \alpha^{1/2})^2}{(1 + \hat{L})^2 \hat{L}^2}, \quad \Delta = \frac{\alpha^{1/2} - \alpha}{L}, \\
\gamma_G &= \frac{\gamma(1 - \alpha)^2}{2(1 + L)^2 L^4}, \quad \hat{\gamma}_G := \frac{\hat{\gamma}(1 - \alpha^{1/2})^2}{2(1 + \hat{L})^2 \hat{L}^4}, \\
\gamma_H^{-1} &= \left(\frac{2}{\gamma_G} + \left(1 + \frac{4L_H}{\gamma_G} + \frac{4L_H^2}{\gamma_G^2} \right) \frac{L_H(1 + \bar{\mu}L_H)}{\gamma_F} + \bar{\mu} \right), \\
\hat{\gamma}_H^{-1} &= \left(\frac{2}{\hat{\gamma}_G} + \left(1 + \frac{4\hat{L}_H}{\hat{\gamma}_G} + \frac{4\hat{L}_H^2}{\hat{\gamma}_G^2} \right) \frac{\hat{L}_H(1 + \hat{\mu}\hat{L}_H)}{\hat{\gamma}_F} + \hat{\mu} \right), \\
\bar{\mu} &= \frac{2L_H^2/\gamma_G + \gamma_G + L_H}{\gamma_F}, \quad \hat{\mu} = \frac{2\hat{L}_H^2/\hat{\gamma}_G + \hat{\gamma}_G + \hat{L}_H}{\hat{\gamma}_F}.
\end{aligned}$$

where we note that all constants defined above are independent of n .

Proof. The proof can be divided into three steps:

1. Obtain an upper bound of $\sum_{t \in \mathcal{I}} \|Q_t\|^2$ in Lemma 3.8.12.
2. Obtain a lower bound of the minimum eigenvalues of $\sum_{t \in \mathcal{I}} Q_t Q_t^T$ in Lemma 3.8.14.

3. Obtain an upper bound of $\|\sum_{t \in \mathcal{I}} Q_t w_t\|$ in Lemma 3.8.15.

We provide a sketch of proof on how we achieved the above three steps. Step 1: we first express $\sum_{t \in \mathcal{I}} \|Q_t\|^2$ in term of n_θ and $\|z_{\mathcal{I}_i^e}\|^2$ where $z_{\mathcal{I}_i^e}^T = \begin{bmatrix} x_{\mathcal{I}_i^e}^T & u_{\mathcal{I}_i^e}^T \end{bmatrix} P_{\mathcal{I}_i^e}^T$ where $P_{\mathcal{I}_i^e}$ is a permutation matrix, $x_{\mathcal{I}_i^e}$ is the stack of x_t for $t \in \mathcal{I}_i^e$, and $u_{\mathcal{I}_i^e}$ is the stack of u_t for $t \in \mathcal{I}_i^e$. Then, we rewrite $\begin{bmatrix} x_{\mathcal{I}_i^e}^T & u_{\mathcal{I}_i^e}^T \end{bmatrix}$ as

$$\begin{bmatrix} x_{\mathcal{I}_i^e}^T \\ u_{\mathcal{I}_i^e}^T \end{bmatrix} = M_{S, \mathcal{I}_i^e} \cdot \begin{bmatrix} w_{\mathcal{I}_i^e}^T \\ \eta_{\mathcal{I}_i^e}^T \end{bmatrix} + M_{C, \mathcal{I}_i^e} \cdot x_{iL_{\mathcal{I}}}$$

where $w_{\mathcal{I}_i^e}$ is the stack of w_t for $t \in \mathcal{I}_i^e$, and $\eta_{\mathcal{I}_i^e}$ is the stack of η_t for $t \in \mathcal{I}_i^e$. We use the property of K_{stab} being a stabilizing control to give the upper bound of coefficient matrices norms $\|M_{S, \mathcal{I}_i^e}\|$ and $\|M_{C, \mathcal{I}_i^e}\|$, use the Laurent-Massart bound to give the high probability upper bound of $\left\| \begin{bmatrix} w_{\mathcal{I}_i^e}^T & \eta_{\mathcal{I}_i^e}^T \end{bmatrix}^T \right\|$, and use Lemma 3.8.1 to give the high probability upper bound of $\|x_{t_1}\|$. Finally, we conclude the proof by the triangle inequality.

Step 2: we extend the upper bound of $\lambda_{\min} \left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T \right)$ to a more general case $\lambda_{\min} \left(\sum_{t \in \mathcal{I}_i^e} q_t q_t^T \right)$ in Lemma 3.8.13 where $q_t = \tilde{A} \cdot z_t$ with the full row rank $\tilde{A}$. Then, we rewrite $Q_t Q_t^T = \sum_{k=1}^n q_{k,t} q_{k,t}^T$ where $q_{k,t}$ has the similar structure as q_t . Finally, we complete the proof by applying Lemma 3.8.13 to $q_{k,t}$ for $1 \leq k \leq n$ respectively.

Step 3: we first give the upper bound of $\|\sum_{t \in \mathcal{I}_i^e} Q_t w_t\|$ by $\sum_{i=1}^n \sum_{j=1}^{n_\theta} |\sum_{t \in \mathcal{I}_i^e} s_{j,t}|$. Then, we complete the proof by applying [67, Lemma 4.2] to $|\sum_{t \in \mathcal{I}_i^e} s_{j,t}|$. $\square$

Theorem 3.4.1 shows that the sample complexity of our proposed algorithm is only of $O((n_\theta) \log n) = O(\log n)$ since $(n_\theta) \ll n$. Therefore, compared with the sample complexity in [42] where the sample complexity is of $O(n)$, we significantly reduced the sample complexity to learn the unknown dynamic parameter by taking advantage of the structure defined in (3.2). Additionally, according to (3.10), the parameter estimation error of our algorithm is independent of n while the result in [42] is proportional to $\sqrt{n}$, which shows that our algo-

rithm scales more favorably with respect to n . Thus, our algorithm can learn the unknown parameter of the large-scale non-stationary LQR with much fewer samples and maintain high accuracy. Especially, our algorithm will be particularly suitable for the large-scale LQR with high-dimensional state variables. In the next section, we will present the regret analysis result for the control stage.

3.5 Regret Analysis

To obtain the regret bound, we first establish the regret estimate of each interval. As the first step, we analyze the performance difference between two policies $\{K_{t,1}\}$ and $\{K_{t,2}\}$ by using $\max_{t \in \mathcal{I}_i} \|K_{t,1} - K_{t,2}\|$.

We first formally define the performance metric. For any interval $\mathcal{I} = [t_1, t_2]$, we define the general performance metric $J_{\mathcal{I}}$ as follows

$$J_{\mathcal{I}}(K_{\mathcal{I}}, \bar{\nu}) = E \left[\sum_{\tau=t_1}^{t_2} \frac{1}{2} x_{\tau}^T Q x_{\tau} + \frac{1}{2} u_{\tau}^T R u_{\tau} \right],$$

$$x_{t_1} = \text{observed}, \quad u_{\tau} = K_{\tau} x_{\tau} + \bar{\nu} \eta_t,$$

$$x_{\tau+1} = A_{\tau} x_{\tau} + B_{\tau} u_{\tau} + w_{\tau}, \quad \tau \in [t_1, t_2],$$

where $K_{\mathcal{I}} = \{K_t\}_{t \in [t_1, t_2]}$. We omit $\bar{\nu}$ in $J_{\mathcal{I}}(K_{\mathcal{I}}, \bar{\nu})$ if $\bar{\nu} = 0$. We define the optimal performance metric in $\mathcal{I}$ as $J_{\mathcal{I}}^* = J_{\mathcal{I}}(K_{\mathcal{I}}^*, 0)$ where $\{K_t^*\}_{t \in [t_1, t_2]}$ is the optimal policy for (3.1). Then, the optimal performance metric over the interval $\mathcal{I}_i$ is $J_{\mathcal{I}_i}^*(K_{\mathcal{I}_i})$.

We define $\hat{J}_{\mathcal{I}_i^e}^e = J_{\mathcal{I}_i^e}(K_{\mathcal{I}_i^e}, \nu)$ as the performance metric in the exploration stage $\mathcal{I}_i^e$ where $K_t = K_{stab} \quad \forall t \in \mathcal{I}_i^e$.

We define $\hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e) = J_{\mathcal{I}_i^r}(\hat{K}_{\mathcal{I}_i^r}^W, 0)$ as the performance metric in the control stage $\mathcal{I}_i^r$ where we obtained $\{\hat{K}_t^W(\mathcal{I}_i^e)\}_{t \in \mathcal{I}_i^r}$ by solving the MPC with the truncated horizon length W defined in eq. (3.4) and dynamic parameters $\hat{A}_i$ and $\hat{B}_i$ estimated from $\mathcal{I}_i^e$.

With the above definition, we are ready to give two key theorems to establish regret

analysis for the interval $\mathcal{I}_i$.

Lemma 3.5.1. *In the interval $\mathcal{I} = [t_1, t_2 + 1]$, we define $J_{\mathcal{I}}(K)$ with $x_{t_1} = \bar{x}_0$ and $\tilde{J}_{\mathcal{I}}(\tilde{K})$ with $\tilde{x}_{t_1} = \bar{x}'_0$. We assume for $\forall t', t'', t' \geq t''$, there exist $\tilde{\alpha} \in (0, 1)$ and $\tilde{L} > 0$ such that:*

$$\left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t) \right\|, \left\| \prod_{t=t'+1}^{t''} (A_t + B_t \tilde{K}_t) \right\| \leq \tilde{L} \tilde{\alpha}^{t''-t'}.$$

Then, if

$$\begin{aligned} \Delta \tilde{K}_{\mathcal{I}} &\leq \epsilon, \quad \Delta \tilde{K}_{\mathcal{I}} = \max_{t \in \mathcal{I}} \|\Delta \tilde{K}_t\|, \quad E[\bar{x}_0^2], E[\bar{x}'_0^2] \leq \bar{C}_{\bar{x}_0} \cdot n, \\ \Delta \tilde{K}_t &= K_t - \tilde{K}_t, \quad \max_{t \in [T]} \|K_t\|, \|\tilde{K}_t\| \leq \tilde{K}_u, \quad \tilde{K}_u \geq 1, \end{aligned}$$

there exist constants such that

$$\left| E[J_{\mathcal{I}}(K) - \tilde{J}_{\mathcal{I}}(\tilde{K})] \right| \leq \bar{C}_{\bar{x}_0} \frac{4L\tilde{L}^2\tilde{K}_u^2}{(1-\tilde{\alpha})^2} \cdot n + \frac{8L^2\tilde{K}_u^2\tilde{L}^3\psi^2}{\tilde{\alpha}(1-\tilde{\alpha}^2)^3} \cdot n|\mathcal{I}|\epsilon, \quad (3.11)$$

Additionally, if $\Delta \tilde{K}_t \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-t}$ with $C_{\tilde{\rho}} > 1$, there exist a constant such that

$$\left| E[J_{\mathcal{I}}(K) - \tilde{J}_{\mathcal{I}}(\tilde{K})] \right| \leq (\bar{C}_{\bar{x}_0} + 1) \frac{12L^2\tilde{L}^3\tilde{K}_u^2C_{\tilde{\rho}}}{\tilde{\alpha}(1-\tilde{\alpha}^2)^3(1-\tilde{\rho})} n.$$

The proof of Lemma 3.5.1 is presented in Section 3.8.6. The lemma states that when two performance metrics of the same interval are both non-optimal without the same initial states, their absolute difference consists of two terms, the linear term of n and the linear term of $|\mathcal{I}| \cdot \epsilon$ where $|\mathcal{I}|$ is the length of interval $\mathcal{I}$ and ϵ quantifies the maximum difference between two sequences of control policies. The lemma suggests that the decay rate of the regret with respect to ϵ is linear and we will have lower regret if we can have two closer sequences of control policies. Note that when the initial states of two performance metrics are the same, the linear term of n will disappear.

Lemma 3.5.2. *In the interval $\mathcal{I} = [t_1, t_2 + 1]$, we define $J_{\mathcal{I}}(K)$ with $x_{t_1} = \bar{x}_0$ and $J_{\mathcal{I}}^*(K^*)$ with $x_{t_1}^* = \bar{x}_0^*$. We assume for $\forall t', t'', t' \geq t''$, there exist $\alpha_1 \in (0, 1)$ and $L_1 > 0$ such that:*

$$\left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t) \right\|, \left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t^*) \right\| \leq L_1 \alpha_1^{t''-t'}.$$

Under the stability assumption of K , if $\Delta K_{\mathcal{I}} := \max_{t \in \mathcal{I}} \|K_t - K_t^*\| \leq \epsilon$; $\max_{t \in [T]} \|K_t\|, \|K_t^*\| \leq \tilde{K}_u$ with $\tilde{K}_u \geq 1$; and $E[\bar{x}_0^2], E[(\bar{x}_0^*)^2] \leq \bar{C}_{\bar{x}_0} \cdot n$, there exist constants such that

$$\left| E[J_{\mathcal{I}}(K) - J_{\mathcal{I}}^*(K^*)] \right| \leq \bar{C}_{\bar{x}_0} \frac{2LL_1^2 \tilde{K}_u^2}{1 - \alpha_1^2} n + \frac{20\psi^2 L^3 L_1^4 \tilde{K}_u^2}{(1 - \alpha_1^2)^3} \cdot |\mathcal{I}| n \epsilon^2,$$

The proof of Lemma 3.5.1 is presented in Section 3.8.6. Compared to Lemma 3.5.1, the only difference in Lemma 3.5.2 is that one of the performance metrics is optimal. Lemma 3.5.2 suggests that when one of the performance metrics is optimal, the decay rate of the regret with respect to ϵ increases from the linear rate to the quadratic rate, which suggests a much faster decay of the regret with the decrease of ϵ . Note that when the initial states of two performance metrics are the same, the linear term of n will disappear and the regret can become indefinitely close to 0 if two control policy sequences are close enough.

3.5.1 Regret Analysis of the Exploration Stage

We extend Lemmas 3.5.1 and 3.5.2 to the regret of the exploration stage. The theorem is given as follows.

Theorem 3.5.1. *In the interval $\mathcal{I} = [t_1, t_2 + 1]$, we define $J_{\mathcal{I}}^e(K, \nu)$ with $x_{t_1} = \bar{x}_0^e$ and $J_{\mathcal{I}}^*(K^*)$ with $x_{t_1}^* = \bar{x}_0^*$. We assume for $\forall t', t'', t' \geq t''$, there exists an α_2 and L_2*

$$\left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t) \right\|, \left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t^*) \right\| \leq L_2 \alpha_2^{t''-t'},$$

Under the stability assumption of K , if

$$\begin{aligned}\Delta K_{\mathcal{I}} &:= \max_{t \in \mathcal{I}} \|K_t - K_t^*\| \leq \epsilon, \quad \max_{t \in [T]} \|K_t\|, \|K_t^*\| \leq \tilde{K}_u, \\ E[(\bar{x}_0^e)^2], E[(\bar{x}_0^*)^2] &\leq \bar{C}_{\bar{x}_0} \cdot n,\end{aligned}$$

there exist constants such that

$$\left| E[J_{\mathcal{I}}^e(K, \nu) - J_{\mathcal{I}}^*(K^*)] \right| \leq \nu^2 \frac{5L^3 L_2^2 \tilde{K}_u^2 (\bar{C}_{\bar{x}_0} + 1)}{1 - \alpha_2^2} \cdot n|\mathcal{I}| + \frac{20\psi^2 L^3 L_2^4 \tilde{K}_u^2}{(1 - \alpha_2^2)^3} n|\mathcal{I}| \epsilon^2$$

For the exploration stage, since $\|K_{stab} - K_t^*\| \leq 2K_u$, $\epsilon = 2K_u$. By setting $\epsilon = 2K_u$, $\bar{C}_{\bar{x}_0} = 2\bar{x}^2$, $\alpha_2 = \max(\alpha^{1/2}, \rho, \hat{\rho}^{1/4})$, $C_{\rho_1}^*$, $L_2 = \max(2L, c_1, \hat{c}_1)$, and $\tilde{K}_u = K_u$, we obtain the regret of the exploration stage from Theorem 3.5.1. The proof of Theorem 3.5.1 is presented in Section 3.8.6 and its proof is very similar to Lemma 3.5.2. Theorem 3.5.1 suggests that the regret is also proportional to $n \cdot |\mathcal{I}|$, which is reasonable since we use the fixed control policy K_{stab} in the exploration stage and K_{stab} is not close to the optimal policy. But we will show later how we choose the length of the exploration stage to balance this linear term.

3.5.2 Regret Analysis of the control stage

After obtaining the estimated parameters from the exploration stage, we formulate the stationary full-horizon LQR with the estimated parameters and solve it approximately by the MPC defined in (3.4) with the truncated horizon length W . Here, we aim to show that we have more accurate solutions to approximate full-horizon LQR with the increase of W .

The theorem of regret in the control stage is also extended from Lemmas 3.5.1 and 3.5.2, which is given as follows

Theorem 3.5.2. *Under Assumption 3.4.2, for $\mathcal{I} = \mathcal{I}_i^r$ and $\tilde{\mathcal{I}} = \mathcal{I}_i^e$ with $i \in [N]$, if*

$$\begin{aligned} & \left\| \max_{t \in \tilde{\mathcal{I}}} \left(\Delta A_t(\tilde{\mathcal{I}}), \Delta B_t(\tilde{\mathcal{I}}) \right) \right\| \leq \epsilon, \\ & \epsilon \leq \min(\Delta, \hat{\Delta}, \tilde{\Delta})/2K_u, \\ & W \geq \frac{1}{2} \log_{1/\hat{\rho}} \left(2\hat{c}_1^2 \hat{L}^2 / \tilde{\Delta} \right), \end{aligned}$$

we obtain

$$|J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| \leq \tilde{a}n + \frac{2}{\hat{\gamma}_H} |\tilde{\mathcal{I}}| \cdot n \cdot (\hat{a}\epsilon^2 + \hat{b}\hat{\rho}^W) \quad (3.12)$$

with

$$\begin{aligned} \tilde{a} &= \left(2\bar{x}^2 + 1 \right) \frac{30L^2 (c_1^*)^3 K_u^2 C_{\rho}^*}{\rho_1^* (1 - \rho_1^*)^4}, \quad \hat{a} = \frac{20\psi^2 L^3 (c_1^*)^4 K_u^2}{(1 - \rho_1^*)^3 \gamma_H^2 \hat{\gamma}_H^2}, \\ \hat{b} &= \frac{8L^2 K_u^2 (c_1^*)^3 \psi^2 C_{\rho_1}^*}{(1 - \rho_1^*)^3}, \quad \hat{\Delta} = (\hat{\rho}^{1/2} - \hat{\rho})/\hat{c}_1, \\ \tilde{\Delta} &:= (\hat{\rho}^{1/4} - \hat{\rho}^{1/2})/\hat{c}_1, \quad c_1^* = \max(2L, c_1, \hat{c}_1), \\ \rho_1^* &= \max\left(\alpha^{1/2}, \rho, \hat{\rho}^{1/4}\right), \quad C_{\rho_1}^* = \max\left(\frac{c_1 L_H}{\gamma_H}, \frac{\hat{c}_1 \hat{L}_H}{\hat{\gamma}_H}\right). \end{aligned}$$

where K_u is defined in Assumption 3.4.2; α, L are defined in Assumption 3.4.2; and $\Delta, c_1, \rho, L_H, \gamma_H, \hat{c}_1, \hat{\rho}, \hat{L}_H$, and $\hat{\gamma}_H$ are defined in Theorem 3.4.1. Note that $\tilde{a}, \hat{a}$ and $\hat{b}$ are only expressed by the constants in Assumption 3.4.2.

The sketch of proof is as follows. We can rewrite $J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})$ as the summation of four terms $j_1 := J_{\mathcal{I}}^* - J_{\mathcal{I}}^{\mathcal{I},*}$, $j_2 := J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})$, $j_3 := \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}})$, and $j_4 := \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})$.

Then, we have

$$\left| J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}}) \right| = \left| \sum_{k=1}^4 j_k \right| \leq \sum_{k=1}^4 |j_k|$$

where $J_{\mathcal{I}}^*$ and $\hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})$ are defined in (3.11). For $J_{\mathcal{I}}^{\mathcal{I},*}, \hat{J}_{\mathcal{I}}^{\mathcal{I},*} = J_{\mathcal{I}}^{\mathcal{I},*}(K_{\mathcal{I}}^{\mathcal{I},*})$ where $K_{\mathcal{I}}^{\mathcal{I},*} = \{K_t^{\mathcal{I},*}\}_{t \in \mathcal{I}}$ is the optimal control policy of the truncated horizon LQR (3.1) whose horizon

range is truncated from $[T]$ to $\mathcal{I}$. Similarly, $\hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) = J_{\mathcal{I}}(\hat{K}_{\mathcal{I}}(\tilde{\mathcal{I}}))$ and $\hat{K}_{\mathcal{I}}(\tilde{\mathcal{I}}) = \{\hat{K}_t(\tilde{\mathcal{I}})\}_{t \in \mathcal{I}}$ is the optimal control policy of the truncated horizon MPC (3.4) whose horizon range is changed from $[t, t + W]$ to $[t, T]$. Also, $\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) = J_{\mathcal{I}}(\hat{K}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}))$ and $\hat{K}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})$ is the optimal control policy of the truncated horizon MPC (3.4) whose horizon is changed from $[t, t + W]$ to $\mathcal{I}$. The detailed definition of these terms is given in (3.57)-(3.59) in the appendix. For $\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\|$, it is clear that $\hat{K}_t(\tilde{\mathcal{I}})$ and $\hat{K}_t^W(\tilde{\mathcal{I}})$ are submatrices of two KKT matrices inverse respectively (we denote these KKT matrices as H_1 and H_2 here) where we can show that H_2 is a submatrix of H_1 . Then, we can obtain the upper bound of $\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\|$ by the exponential decay property of H_1 (Lemmas 3.8.10 and 3.8.11). For $\|\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{K}_t(\tilde{\mathcal{I}})\|$ and $\|K_t^{\mathcal{I},*} - K_t^*\|$, we can rewrite $\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}}) = \hat{K}_t^{\bar{W}_t}(\tilde{\mathcal{I}})$ and $K_t^{\mathcal{I},*} = K_t^{\bar{W}_t,*}$, consider them as the truncated MPC approach with the truncated horizon length $\bar{W}_t = t_2 - t + 1$, and follow the similar proof of $\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\|$. For $\|K_t^{|\mathcal{I}|,*} - \hat{K}_t^{|\mathcal{I}|}(\tilde{\mathcal{I}})\|$, we can obtain their upper bound by the parameter estimation theorem and the property of KKT matrices (Lemmas 3.8.10 and 3.8.11). We complete the proof by applying the first regret result of Lemma 3.5.1 to j_4 , the second regret result to j_1 and j_3 , and applying Lemma 3.5.2 to j_2 .

The bound in Theorem 3.5.2 consists of $\hat{\rho}^W$ term, which represents how the MPC approach with the truncated horizon length W contributes to the regret. We can see that as W increases, this term decays to 0 exponentially, which validates the efficiency and accuracy of our truncated horizon MPC approach. In the next subsection, we will combine the sample complexity results in Section 3.4 with the regret results in the previous subsection to analyze the sample complexity and total regret of the full horizon LQR.

3.5.3 Main Theorem of Full Horizon

In this subsection, we show that by choosing the proper value of every interval length $L_{\mathcal{I}}$ where $N = V_T^{4/5} T^{1/5}$ and $L_{\mathcal{I}} = T/N = V_T^{-4/5} T^{4/5}$. This result matches with the regret bound reported in [42].

Theorem 3.5.3 (Main Theorem). *Under Assumption 3.4.1-3.4.2, $N = V_T^{4/5} T^{1/5}$, and $L_{\mathcal{I}} = T/N = V_T^{-4/5} T^{4/5}$, if*

$$T \geq T_1 := \frac{32L^5(\tilde{C}_e k_T + C_e)^5 V_T K_u^5}{(\Delta^*)^5},$$

with the probability at least $1-\delta$, we obtain

$$|J^* - \hat{J}^W| \leq C_M^* \cdot n \cdot T^{3/5} \cdot V_T^{2/5}.$$

where

$$\begin{aligned} J^* &= \sum_{i=0}^{N-1} J_{\mathcal{I}_i^e}^* + J_{\mathcal{I}_i^r}^*, \quad \hat{J}^W = \sum_{i=0}^{N-1} J_{\mathcal{I}_i^e}(K_{stab}) + \hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e), \\ C_M^* &= \frac{85L^3 (c_1^*)^4 K_u^4 (2\bar{x}^2 + 1)(\nu^2 + \psi^2)}{(1 - \rho_1^*)^3} + \tilde{C}_J, \\ \tilde{C}_J &= \tilde{a} + \hat{a} C_E^2 (\tilde{C}_e k_T + C_e)^2 + \hat{b}, \\ T_2 &= \frac{2000(n_\theta)}{9} \left(2 \log \frac{1}{\delta^*} + \log \frac{(n+d)\bar{z}}{\bar{C}_1} \right), \quad T_3 = \frac{2\hat{c}_1^2 (L^*)^2}{\Delta^*}, \\ \delta^* &= \frac{\delta}{8N^2 n(n_\theta)}, \quad L_{\mathcal{I}} \geq \max(T_2^2, T_3), \quad \tilde{C}_e = \frac{6400C_Q^2(n_\theta)}{9\bar{C}_1 \lambda_{\min, M}}, \\ C_e &= \frac{3200C_Q \lambda_{\max, M} \psi(n_\theta) \sqrt{\log \frac{1}{\delta^*}}}{9\bar{C}_1 \lambda_{\min, M}}, \quad \Delta^* = \min(\Delta, \hat{\Delta}, \tilde{\Delta}), \\ L^* &= \max(2L, K_u), \quad W \geq \frac{1}{2} \log_{\hat{\rho}^{-1}} L_{\mathcal{I}}. \end{aligned}$$

Here, α , K_u , and L are defined in Assumption 3.4.2; $\bar{C}_1$, $\bar{z}$, $\bar{x}^2$, c_1 , ρ , L_H , γ_H , $\hat{c}_1$, $\hat{\rho}$, $\hat{L}_H$, and $\hat{\gamma}_H$ are defined in Theorem 3.4.1; $\hat{\Delta}$, $\tilde{\Delta}$, c_1^* , ρ_1^* , $C_{\rho_1}^*$, $\tilde{a}$, $\hat{a}$, and $\hat{b}$ are defined in Theorem 3.5.2. Note that $\tilde{a}$, $\hat{a}$, and $\hat{b}$ are only expressed by the constants in Assumption 3.4.2.

The proof of Theorem 3.5.3 is in Section 3.8.6. Theorem 3.5.3 is the direct result of Theorems 3.4.1, 3.5.1 and 3.5.2. By requiring $T \geq T_1$, we want the maximum estimation error

of A_t and B_t to be bounded by $\Delta^*/2K_u$ such that we can use Theorem 3.5.2. Requiring $T \geq T_2^2$ is equivalent to $|\mathcal{I}_i^e| > T_2$ such that we can apply Theorem 3.4.1. By requiring $T \geq T_3$, we want to fulfill the requirement of W in Theorem 3.5.2. By Theorem 3.5.1, we can obtain the regret of every exploration stage. By combining Theorem 3.4.1 with Theorem 3.5.2, we can obtain the regret of every control stage. Finally, by the summation of the regret of every interval, we obtain the total regret. Theorem 3.5.3 emphasizes three points. First, the sample complexity for the algorithm is proportional to $\log n$; this result improves $O(n)$ result in [42] for non-parameterized systems. Second, in the control stage, instead of solving the Riccati equation, which is equivalent to solving the infinite dimension LQR problem and requires an iterative solve, we solve the truncated MPC with the horizon length W where $W = O(\log T)$. Solving MPC problems with truncated horizons is computationally less expensive compared to solving the full-horizon problem. Thus, we achieve the same regret result in [42] but with less sample complexity and computational cost. In Section 3.6, we shall use numerically demonstrate that the computational cost of our algorithm scales linearly with respect to W .

3.6 Numerical Experiment

In this section, we aim to numerically demonstrate the convergence of our MPC approach and the computational cost. We first verify the convergence and accuracy of the truncated horizon MPC method. Since we replace the Riccati equation with the truncated horizon MPC, we want to know how well the control obtained from the truncated horizon MPC approximates the optimal control obtained by solving the Riccati equation solution. By Lemmas 3.5.1 and 3.5.2, we only need to verify that $\|K_t^W - K_t^*\|$ is decreasing exponentially with the increase of the truncated horizon length W . We consider the stationary dynamics with $n = d = 10$; $\nu = \psi = 0.01$; Q , R , and B are the identity matrix, and A is defined as

follows

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & \end{bmatrix}, \quad (3.13)$$

where $K_t^* = K^*$ is obtained by solving the Riccati equation; $K_t^W = K^W$ is obtained by solving the MPC with the truncated horizon length W , which is equivalent to solving the KKT condition with the linear solver; we set $u^* = K^*x_0$ and $u^W = K^Wx_0$ where $x_0 = \hat{x}_0/\|\hat{x}_0\|$ and $\hat{x}_0$ is a vector with element set as 1. We give the result in Figure 3.1 where we can see that $\log \|u^W - u^*\|$ is proportional to the horizon length W and it suggests that $\|K^W - K^*\|$ decays exponentially with the increase of W , which validates the accuracy and efficiency of our truncated horizon approach.

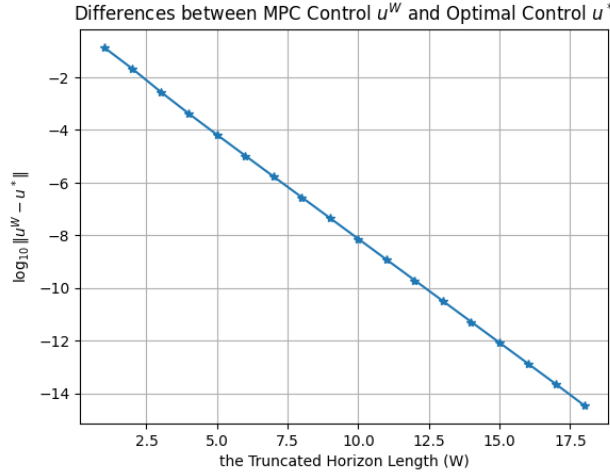


Figure 3.1: The correctness of the truncated horizon MPC approach. We obtain $K_t^* = K^*$ by solving the Riccati equation and $K_t^W = K^W$ by solving the MPC with the horizon length W . We set $u^* = K^*x_0$ and $u^W = K^Wx_0$. The figure shows that $\log \|u^W - u^*\|$ is proportional to the horizon length W and suggests that $\|K^W - K^*\|$ decays exponentially with the increase of W .

Then, we want to discuss the computational cost of our algorithm. For the exploration

stage, the cost per time step to generate x_t and u_t is $O(n)$ by the sparsity structure of A_t and B_t . For parameter estimation by OLS, computing $Q_t Q_t^T$ requires $O((n_\theta)^2 n)$, computing $\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T$ requires $O(|\mathcal{I}_i^e| (n_\theta)^2 n)$, and computing $(\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T)^{-1}$ requires $O((n_\theta)^3)$. Therefore, the total cost of estimating parameters per time step is $O((n_\theta)^2 n) = O(n)$. Combined with the cost of generating x_t and u_t , the cost per time step in the exploration stage is $O(n)$. In the control stage, instead of solving infinite horizon LQR, we solve the LQR with the estimated parameters approximately by the truncated MPC problem eq. (3.4), which is equivalent to solving a linear system with sparse matrices. By exploiting the sparsity of matrices, the numerical experiment shows that the cost of solving the truncated MPC per time step is $O(W(n+d))$ where W is the horizon of the truncated horizon MPC problem, and d is the dimension of the control variable u_t . Combined with the cost of generating x_t , the cost per time step in the control stage is $O(nW + n)$.

We validate the computational cost by the toy example where $n_A = 3$, $n_B = 1$, A_1 , B_1 , Q , and R are identity matrices, A_2 and A_3 are defined as

$$A_2 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \end{bmatrix}.$$

We define $\alpha_{t,1} = 2 \left(1 + \frac{\sin(t+1)}{t+1}\right)$, $\alpha_{t,2} = -\frac{1}{2}\alpha_{t,1}$, $\alpha_{t,3} = \alpha_{t,2}$, and $\beta_{t,1} = 1$ where it is not hard to show that this example fulfills the total variation assumption in Assumption 3.4.2. We will validate our results with two experiments. First, we fix every parameter and only change n . We choose $\nu = \psi = 0.01$, $T = 5070$, $N = 30$, $L = 169$, $W = 24$, and $x_0 = \underline{x}_0 / \|\underline{x}_0\|$ where $\underline{x}_0 \sim \mathcal{N}(\mathbf{0}, I_n)$. We obtain K_{stab} by solving the Riccati equation with A and B . The plot of the computational time and the list of n is presented in Figure 3.2. The plot suggests the linear relationship between the computational cost and n when everything else

is fixed. Second, we fix every parameter and only change W . We choose $n = 8$ and the

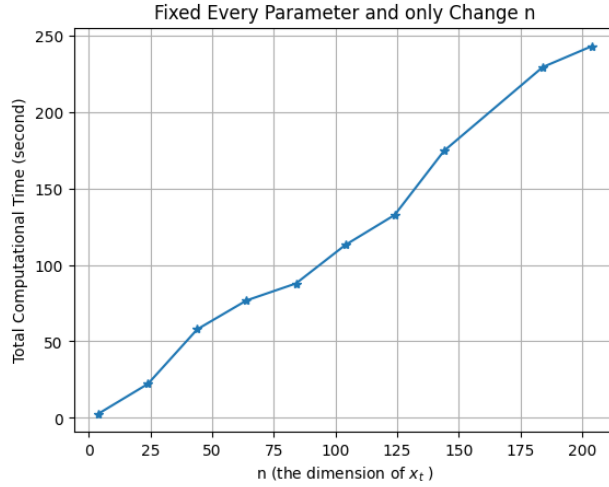


Figure 3.2: The plot of the computational time for n the dimension of the state variable x_t . We set $T = 5070$, $N = 30$, $L = 169$, $W = 24$, and $x_0 = \underline{x}_0 / \|\underline{x}_0\|$ where $\underline{x}_0 \sim \mathcal{N}(\mathbf{0}, I_n)$. We obtain K_{stab} by solving the Riccati equation with A and B defined in eq. (3.13). The curve is almost a straight line and suggests the linear relationship between the computational cost and n when everything else is fixed.

other unchanged parameters are the same as Figure 3.2. The plot of the computational time and the list of W is presented in Figure 3.3. The plot suggests the linear relationship between the computational cost and W when everything else is fixed. We verify that the computational cost of our algorithm is $O(nW)$ per iteration. Since we choose $W = O(\log T)$, the computational cost of our algorithm is $O(n \log T)$ per iteration.

3.7 Conclusions

We have presented the regret analysis of the MPC control policy with identified system models for parameterized, non-stationary dynamical systems. Our results provide sharper sample complexity bound for the systems with small number of unknown parameters. Specifically, by exploiting the structure of A_t and B_t , we show that our algorithm can achieve the same regret result in [42] but with lower sample complexity required by the parameter estimation.

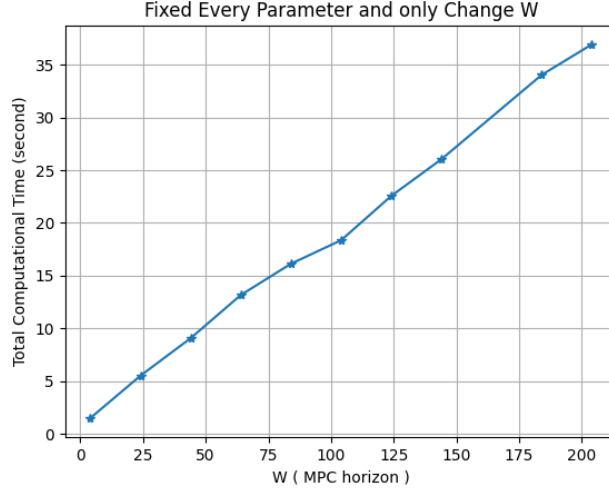


Figure 3.3: The plot of the computational time for W the truncated horizon length of MPC. We choose $n = 8$ and the other unchanged parameters are the same as Figure 3.2. The curve is almost a straight line and suggests the linear relationship between the computational cost and W when everything else is fixed.

Our sample complexity is proportional to $\log n$, much smaller than the sample complexity proved in [42]. For numerical experiments, we validate the exponential decay error rate of the truncated horizon MPC approach and show that our algorithm's computational cost is $O(n \log T)$ per iteration while authors in [42] solve the Ricatti equation which requires $O(n^3)$ every time.

3.8 Appendix

3.8.1 Help Definitions

Definition 3.8.1 (Definition 2.1 in [67]). (*Definition of BMSB for Real Values*) Let $\{\mathcal{F}_t\}_{t \geq 1}$ be a filtration and $(W_t)_{t \geq 1}$ be a random process of real value adapted to $\{\mathcal{F}_t\}_{t \geq 1}$. If we have $\frac{1}{l} \sum_{i=1}^l \mathbb{P}(|W_{i+k}| \geq \mu | \mathcal{F}_k) \geq q$ held for any $k \geq 0$ almost surely, we can say $(W_t)_{t \geq 1}$ satisfies the (l, μ, q) - block martingale small-ball (BMSB) condition.

(*Definition of BMSB for Real Vectors*) Let $(Y_t)_{t \geq 1}$ be a real vector process where

$Y_t \in \mathbb{R}^d$ for every t . If the real valued process $W_t := \langle w, Y_t \rangle$ satisfies $(l, \sqrt{w^T \Gamma_{sb} w}, q)$ -BMSB for every fixed $w \in \mathcal{S}^{d-1}$ where $\Gamma_{sb} \succ 0$, we are able to say that $(Y_t)_{t \geq 1}$ satisfies the (l, Γ_{sb}, q) -BMSB condition.

Definition 3.8.2 (Definition of ϵ -net and covering number). For $\epsilon > 0$, if for any $\psi \in \mathcal{S}_{\Gamma_{\min}}$, there exists a $\psi' \in \mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})$, such that $\|\psi - \psi'\|_{\Gamma_{\max}^{1/2}} \leq \epsilon$, where $\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max}) \subseteq \mathcal{S}_{\Gamma_{\min}}$ is an ϵ -net over the set $\mathcal{S}_{\Gamma_{\min}} \subseteq \mathbb{R}^{nq}$ in $\|\Gamma_{\max}^{1/2}(\cdot)\|_2$. Then, we call the size of the ϵ -net with the smallest size $|\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})|$ as the covering number.

3.8.2 Help Lemmas and Propositions

Lemma 3.8.1. For any interval $\mathcal{I} \subseteq [T]$ and $t \in \mathcal{I}$, we have

$$E \left[\|x_t\|^2 \right] \leq \bar{x}^2(n+d), \quad \sum_{t \in \mathcal{I}} E \left[\|x_t\|^2 \right] \leq \bar{x}^2(n+d)|\mathcal{I}|, \quad \bar{x}^2 = \frac{10L^2 L_m^2 (\psi^2 + 1)}{(1 - \rho_m)^2}. \quad (3.14)$$

where $\rho, c_1, \hat{\rho}, \hat{L}$, and $\hat{c}_1$ are defined in Theorem 3.4.1, and

$$\rho_m = \max \left(\alpha, \rho, \hat{\rho}^{1/4} \right), \quad L_m = \max(L, \hat{L}, c_1, \hat{c}_1).$$

Additionally, when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/n$, for $\delta \in (0, 1)$ and $k \in [N]$, we have

$$\mathbb{P} \left(\|x_{kL_{\mathcal{I}}}\| \geq \sqrt{2(n+d)|\mathcal{I}_i^e| \bar{x}_m} \right) \leq 2N\delta, \quad \bar{x}_m = \|x_0\| + \frac{C_L}{1 - \rho_L}. \quad (3.15)$$

where $L_{\mathcal{I}}$ is the length of every interval, $x_{kL_{\mathcal{I}}}$ is the initial point of every interval, and

$$C_L = \max \left(\frac{L^2(\psi + \nu)}{1 - \alpha}, \frac{L_m L(\psi + \nu)}{1 - \rho_m} \right), \quad \rho_L = \max \left(L\alpha^{L_{\mathcal{I}}^{1/2}}, L_m \rho_m^{L_{\mathcal{I}}^{1/2}} \right).$$

Proof. We know

$$\begin{aligned}
x_t &= \tilde{\Phi}_{0,t-1}x_0 + \sum_{m=0}^{t-1} \tilde{\Phi}_{m+1,t-1}w_m + \sum_{m=0}^{t-1} \nu_m \tilde{\Phi}_{m+1,t-1}B_m\eta_m, \\
\tilde{\Phi}_{t_1,t_2} &= \Pi_{t=t_1}^{t_2}(A_t + B_t\tilde{K}_t), \quad \tilde{\Phi}_{t',t'-1} = I, \\
\|x_t\| &\leq L_m\|x_0\| + \sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1}\| \|w_m\| + \nu L \sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1}\| \|\eta_m\|.
\end{aligned} \tag{3.16}$$

where $\tilde{K}_t = K_{stab}$ and $\nu_t = \nu$ when $t \in \mathcal{I}_i^e$, and $\nu_t = 0$ when $t \in \mathcal{I}_i^r$. Then, we know

$$\begin{aligned}
\text{Var} \left(\sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1} \cdot \psi\| \|w_m/\psi\| \right) &\leq \frac{nL_m^2\psi^2}{(1-\rho_m)^2}, \\
\text{Var} \left(\sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1}\| \|\eta_m\| \right) &\leq \frac{dL_m^2}{(1-\rho_m)^2}, \\
E \left[\sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1} \cdot \psi\| \|w_m/\psi\| \right] &\leq \frac{\sqrt{n}L_m\psi}{1-\rho_m}, \\
E \left[\sum_{m=0}^{t-1} \|\tilde{\Phi}_{m+1,t-1}\| \|\eta_m\| \right] &\leq \frac{\sqrt{d}L_m}{1-\rho_m}.
\end{aligned}$$

Therefore, we have

$$E \left[\|x_t\|^2 \right] = \text{Var} [\|x_t\|] + E [\|x_t\|]^2 \leq (n+d) \cdot \frac{10L^2L_m^2(\psi^2+1)}{(1-\rho_m)^2}. \tag{3.17}$$

Then, we conclude the proof of the first conclusion. Next, we want to analyze the upper bound of $\|x_{kL_{\mathcal{I}}}\|$ where $kL_{\mathcal{I}}$ is the initial time of each interval $\mathcal{I}_k$. We look at $x_{(i+1)L_{\mathcal{I}}}$ and write x_t for $t \in \mathcal{I}_i^e$ as follows

$$x_t = \underbrace{\tilde{\Phi}_{iL_{\mathcal{I}},t-1}x_{iL_{\mathcal{I}}} + \sum_{m=iL_{\mathcal{I}}}^{t-1} \tilde{\Phi}_{m+1,t-1}w_m + \sum_{m=iL_{\mathcal{I}}}^{t-1} \nu \tilde{\Phi}_{m+1,t-1}B_m\eta_m}_{x_{t,1}},$$

where

$$x_{t,1} \sim \mathcal{N}(\mathbf{0}, \text{Var}(x_{t,1})), \quad \|\text{Var}(x_{t,1})\| \leq \frac{L^2(\psi^2 + \nu^2 L^2)}{(1 - \alpha)^2},$$

$$\text{Var}(x_{t,1}) = \psi^2 \sum_{m=iL_{\mathcal{I}}}^{t-1} \tilde{\Phi}_{m+1,t-1} \tilde{\Phi}_{m+1,t-1}^T + \sum_{m=iL_{\mathcal{I}}}^{t-1} \nu^2 \tilde{\Phi}_{m+1,t-1} B_m B_m^T \tilde{\Phi}_{m+1,t-1}^T.$$

Since $x_{t,1} = \text{Var}(x_{t,1})^{1/2} \cdot y_{t,1}$ where $y_{t,1} \sim \mathcal{N}(\mathbf{0}, I_n)$ and $\|y_{t,1}\|^2$ is a chi-square distribution with n degree of freedom. By Laurent-Massart bound[35], when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/n$, we have

$$\mathbb{P}\left(\|y_{t,1}\|^2 \geq 2n \cdot |\mathcal{I}_i^e|\right) < \delta,$$

$$\mathbb{P}\left(\|x_t\| \geq L\alpha^{t-iL_{\mathcal{I}}}\|x_{iL_{\mathcal{I}}}\| + \frac{L(\psi^2 + \nu^2 L^2)^{1/2}}{1 - \alpha} \sqrt{2n \cdot |\mathcal{I}_i^e|}\right) < \delta,$$

Therefore, at the end of the exploration stage, we have

$$\mathbb{P}\left(\|x_{iL_{\mathcal{I}}+L_{\mathcal{I}}^{1/2}}\| \geq L\alpha^{L_{\mathcal{I}}^{1/2}}\|x_{iL_{\mathcal{I}}}\| + \frac{L(\psi^2 + \nu^2 L^2)^{1/2}}{1 - \alpha} \sqrt{2n \cdot |\mathcal{I}_i^e|}\right) < \delta$$

Similarly, at the end of the control stage, we have

$$\mathbb{P}\left(\|x_{(i+1)L_{\mathcal{I}}}\| \geq L_m \rho_m^{L_{\mathcal{I}}^{1/2}} \|x_{iL_{\mathcal{I}}+L_{\mathcal{I}}^{1/2}}\| + \frac{L_m(\psi^2 + \nu^2 L^2)^{1/2}}{1 - \rho_m} \sqrt{2n \cdot |\mathcal{I}_i^e|}\right) < \delta$$

We denote $C_L = \max\left(\frac{L^2(\psi+\nu)}{1-\alpha}, \frac{L_m L(\psi+\nu)}{1-\rho_m}\right)$ where $\max\left(\frac{L(\psi^2+\nu^2 L^2)^{1/2}}{1-\alpha}, \frac{L_m(\psi^2+\nu^2 L^2)^{1/2}}{1-\rho_m}\right) \leq C_L$ and $\rho_L = \max\left(L\alpha^{L_{\mathcal{I}}^{1/2}}, L_m \rho_m^{L_{\mathcal{I}}^{1/2}}\right)$ where we note that since α, L, ρ_m, L_m are independent of T, n , and d , we can choose T such that $\rho_L < 1$. Therefore, by combining the above two inequalities, we have

$$\mathbb{P}\left(\|x_{(i+1)L_{\mathcal{I}}}\| \geq \rho_L^2 \|x_{iL_{\mathcal{I}}}\| + (\rho_L + 1)C_L \sqrt{2n \cdot |\mathcal{I}_i^e|}\right) < 2\delta$$

Iteratively, for $\forall k \in [N]$, we have

$$\begin{aligned} \mathbb{P} \left(\|x_{kL_{\mathcal{I}}}\| \geq \rho_L^{2k} \|x_0\| + \left(\sum_{l=0}^{2k-1} \rho_L^l \right) C_L \sqrt{2n \cdot |\mathcal{I}_i^e|} \right) &< 2N\delta \\ \mathbb{P} \left(\|x_{kL_{\mathcal{I}}}\| \geq \|x_0\| + \frac{C_L}{1 - \rho_L} \sqrt{2n \cdot |\mathcal{I}_i^e|} \right) &< 2N\delta \end{aligned} \quad (3.18)$$

□

Lemma 3.8.2 (Lemma F.6 in [16]). *For any real value ν , we have that*

$$\lambda_{\min} \left(\begin{bmatrix} \Sigma & \Sigma K^T \\ K \Sigma & K \Sigma K^T + \nu^2 I \end{bmatrix} \right) \geq \nu^2 \min \left(\frac{1}{2}, \frac{\lambda_{\min}(\Sigma)}{2\|K \Sigma K^T\| + \nu^2} \right).$$

where $\Sigma \succ \mathbf{0} \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{p \times n}$.

Lemma 3.8.3 (Markov's inequality of the matrix by Stephen Tu). *Let $A \succ 0$, and let X be a random matrix such that $X \succeq 0$ almost surely. Then,*

$$\mathbb{P}(X \not\preceq A) \leq \text{Tr}(E[X] A^{-1})$$

Lemma 3.8.4 (Lemma D.1 in [67]). *If $0 \prec \Gamma_{\min} \preceq \Gamma_{\max} \in \mathbb{R}^{n \times n}$ and define $\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})$ as a minimal $\epsilon \leq 1/2$ -net of $\mathcal{S}_{\Gamma_{\min}}$ in the metric of $\|\Gamma_{\max}^{1/2}(\cdot)\|_2$, then*

$$\log |\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})| \leq n \log \left(1 + \frac{2}{\epsilon} \right) + \log \det(\Gamma_{\max} \Gamma_{\min}^{-1}).$$

Lemma 3.8.5 (Lemma 4.1 in [67]). *For a matrix $U \in \mathbb{R}^{T \times d}$, and $\Gamma_{\min}, \Gamma_{\max} \in \mathbb{R}^{d \times d}$, if*

$$\mathbf{0} \prec \Gamma_{\min} \preceq \Gamma_{\max}, \quad \inf_{w \in \mathcal{T}_{1/4}(\Gamma_{\min}; \Gamma_{\max})} w^T U^T U w \geq 1, \quad U^T U \preceq \Gamma_{\max},$$

then

$$U^T U \succeq \Gamma_{\min}/2.$$

where $\mathcal{T}_{1/4}(\Gamma_{\min}; \Gamma_{\max})$ is a $1/4$ -net of $\mathcal{S}_{\Gamma_{\min}}$ in $\|\Gamma_{\max}^{1/2}(\cdot)\|_2$.

Lemma 3.8.6. *Under Assumption 3.4.2, for any $t'' > t'$, we have*

$$\left\| \prod_{t=t'+1}^{t''} (A_t + B_t K_t^*) \right\| \leq c_1 \rho^{t''-t'}.$$

where K_t^* is the optimal policy of eq. (3.1) at time t ; ρ , γ_F , γ_G , c_1 , γ_H , L_H , and $\bar{\mu}$ are defined in Theorem 3.4.1.

Proof. The lemma can be obtained directly from Theorem 6 in [63]. $\square$

Note that the results of Lemma 3.8.6 are independent of the choice of interval $\mathcal{I}$.

Lemma 3.8.7. *Under Assumption 3.4.2, when $\|A_t - \hat{A}_{\mathcal{I}}\| \leq \Delta/2$ and $\|B_t - \hat{B}_{\mathcal{I}}\| \leq \Delta/2K_u$ where $\hat{A}_{\mathcal{I}}$ and $\hat{B}_{\mathcal{I}}$ are estimated from interval $\mathcal{I}$, we obtain*

1. $\|\hat{A}_{\mathcal{I}}\|, \|\hat{B}_{\mathcal{I}}\| \leq \hat{L} := L + \Delta$
2. $(\hat{A}_{\mathcal{I}}, \hat{B}_{\mathcal{I}})$ is $(\hat{L}, \alpha^{1/2})$ stabilizable.
3. $(\hat{A}_{\mathcal{I}}, Q^{1/2})$ is $(\hat{L}, \alpha^{1/2})$ detectable.
4. for any $t'' > t'$, we have

$$\left\| \prod_{t=t'+1}^{t''} (\hat{A}_{\mathcal{I}} + \hat{B}_{\mathcal{I}} \hat{K}_t) \right\| \leq \hat{c}_1 \hat{\rho}^{t''-t'}.$$

where $\hat{K}_t$ is obtained by solving eq. (3.58); and Δ , $\hat{\rho}$, $\hat{\gamma}_F$, $\hat{\gamma}_G$, $\hat{c}_1$, $\hat{\gamma}_H$, $\hat{L}_H$, and $\hat{\mu}$ are defined in Theorem 3.4.1.

Proof. Conclusions 1-3 can be obtained by the similar proof of Proposition 1 in [63], and conclusion 4 can be obtained directly from Theorem 6 in [63]. $\square$

Lemma 3.8.8. *Under Assumption 3.4.2, if $\|A_t - \hat{A}_{\mathcal{I}}\| \leq \min(\Delta, \hat{\Delta})/2$, $\|B_t - \hat{B}_{\mathcal{I}}\| \leq \min(\Delta, \hat{\Delta})/2K_u$, and $\|\hat{K}_t\| \leq K_u$, then for any $t'' > t'$, we have*

$$\prod_{t=t'+1}^{t''} (A_t + B_t \hat{K}_t) \leq \hat{c}_1 \hat{\rho}^{(t''-t')/2} \quad (3.19)$$

where $\hat{A}_{\mathcal{I}}$ and $\hat{B}_{\mathcal{I}}$ are estimated from interval $\mathcal{I}$; Δ and $\hat{\Delta}$ are defined in Theorem 3.4.1; K_u is defined in Assumption 3.4.2.

Proof. From Lemma 3.8.7, we have that $\{(\hat{A}_{\mathcal{I}} + \hat{B}_{\mathcal{I}} \hat{K}_t)\}_{t \in \mathcal{I}}$ is $(\hat{c}_1, \hat{\rho})$ -stable. Then, we analyze $\|(\hat{A}_{\mathcal{I}} + \hat{B}_{\mathcal{I}} \hat{K}_t) - (A_t + B_t \hat{K}_t)\|$ as follows

$$\begin{aligned} \|(\hat{A}_{\mathcal{I}} + \hat{B}_{\mathcal{I}} \hat{K}_t) - (A_t + B_t \hat{K}_t)\| &= \|(\hat{A}_{\mathcal{I}} - A_t) + (\hat{B}_{\mathcal{I}} - B_t) \hat{K}_t\| \\ &\leq \|\hat{A}_{\mathcal{I}} - A_t\| + \|\hat{K}_t\| \|\hat{B}_{\mathcal{I}} - B_t\| \leq \|\hat{A} - A_t\| + K_u \|\hat{B}_{\mathcal{I}} - B_t\| \leq \hat{\Delta}. \end{aligned}$$

Finally, the proof is completed by Lemma 4 in [63]. $\square$

Lemma 3.8.9. *Under Assumption 3.4.2, if*

$$\begin{aligned} \|A_t - \hat{A}_{\mathcal{I}}\| &\leq \min(\Delta, \hat{\Delta}, \tilde{\Delta})/2, \quad \|B_t - \hat{B}_{\mathcal{I}}\| \leq \min(\Delta, \hat{\Delta}, \tilde{\Delta})/2K_u, \\ \|\hat{K}_t\| &\leq K_u, \quad W \geq \frac{1}{2} \log_{\hat{\rho}} \left(\frac{\tilde{\Delta}}{2\hat{c}_1^2 \hat{L} L} \right), \end{aligned}$$

where $\tilde{\Delta} := (\hat{\rho}^{1/4} - \hat{\rho}^{1/2})/\hat{c}_1$, then we obtain

$$\left\| \prod_{t=t'+1}^{t''} (A_t + B_t \hat{K}_t^W) \right\| \leq \hat{c}_1 \hat{\rho}^{(t''-t')/4}.$$

where $\hat{K}_t^W$ is obtained by solving (3.4); Δ , $\hat{\Delta}$, and $\tilde{\Delta}$ are defined in Theorem 3.4.1; K_u is

defined in Assumption 3.4.2.

Proof. By Lemma 3.8.7 and Lemma 9 in [63], we can get

$$\|\hat{K}_t^W - \hat{K}_t\| \leq 2\hat{c}_1^2 \hat{L} \hat{\rho}^{2W}$$

If $W \geq \frac{1}{2} \log_{\hat{\rho}} \left(\frac{\tilde{\Delta}}{2\hat{c}_1^2 \hat{L} \hat{L}} \right)$, we have $\|(A_t + B_t \hat{K}_t) - (A_t + B_t \hat{K}_t^W)\| \leq \tilde{\Delta}$. Since $\{(A_t + B_t \hat{K}_t)\}_{t \in [T]}$ is $(\hat{c}_1, \hat{\rho}^{1/2})$ -stable, we complete the proof by Lemma 3.8.8 and Lemma 4 in [63]. $\square$

Lemma 3.8.10. Under Assumption 3.4.2 and for the KKT matrix of (3.57) $H_{\mathcal{I}} = \begin{bmatrix} G_{\mathcal{I}} & F_{\mathcal{I}}^T \\ F_{\mathcal{I}} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_{\mathcal{I}} \times n_{\mathcal{I}}}$ where $\mathcal{I}$ is an interval from $t = t_1$ to $t = t_2$ and

$$G_{\mathcal{I}} = \begin{bmatrix} Q & & & & \\ & R & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & R \\ & & & & & Q \end{bmatrix},$$

$$F_{\mathcal{I}} = \begin{bmatrix} I & & & & & \\ -A_{t_1} & -B_{t_1} & & I & & \\ & & -A_{t_1+1} & -B_{t_1+1} & I & \\ & & & \ddots & & \\ & & & & -A_{t_2-2} & -B_{t_2-2} & I \\ & & & & & -A_{t_2-1} & -B_{t_2-1} & I \end{bmatrix},$$

the following statements hold with constants,

1. $\gamma_H \leq \sigma(H_{\mathcal{I}}) \leq L_H$;

$$2. F_{\mathcal{I}} F_{\mathcal{I}}^T \succeq \gamma_F I;$$

$$3. \text{Re}H(G_{\mathcal{I}}, F_{\mathcal{I}}) \geq \gamma_G I;$$

4. We consider a partition $\{I_i\}_{i=0}^{t_2-t_1}$ of $\mathbb{I}_{[1, n_{\mathcal{I}}]}$. If $H_{\mathcal{I}}[I_i, I_j] = \mathbf{0}$ for any $|i - j| > 1$, then

$$\left\| (H_{\mathcal{I}})^{-1} [I_i, I_j] \right\| \leq c_1 \rho^{|i-j|}, \quad \forall i, j \in \mathbb{I}_{[0, t_2-t_1]};$$

where $\text{Re}H(G_{\mathcal{I}}, F_{\mathcal{I}}) = Z^T G_{\mathcal{I}} Z$ with Z is a null space of $F_{\mathcal{I}}$; and L_H , γ_F , γ_G , γ_H , c_1 , and ρ are defined in Theorem 3.4.1.

Proof. The proof of the lower bound of $\sigma(H_{\mathcal{I}})$ is similar to the proof of Theorem A.3 in [64] and we omit it here. The remaining part of the lemma is obtained directly from Lemma 3.8.6, and Lemma 2 and Theorem 5 in [63]. $\square$

Note that the results of Lemma 3.8.10 are independent of the choice of $\mathcal{I}$ as well.

Lemma 3.8.11. Under Assumption 3.4.2, if $\|A_t - \hat{A}_{\tilde{\mathcal{I}}}\| \leq \Delta/2$ and $\|B_t - \hat{B}_{\tilde{\mathcal{I}}}\| \leq \Delta/2K_u$, for the KKT matrix of eq. (3.59) $\hat{H}_{\mathcal{I}}(\tilde{\mathcal{I}}) = \begin{bmatrix} G_{\mathcal{I}} & \hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}})^T \\ \hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}}) & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{n_{\mathcal{I}} \times n_{\mathcal{I}}}$ where we consider the regret over the interval $\mathcal{I}$, we estimate $\hat{A}_{\tilde{\mathcal{I}}}$ and $\hat{B}_{\tilde{\mathcal{I}}}$ from the interval $\tilde{\mathcal{I}}$, and

$$\hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}}) = \begin{bmatrix} I & & & & & \\ -\hat{A}_{\tilde{\mathcal{I}}} & -\hat{B}_{\tilde{\mathcal{I}}} & I & & & \\ & & -\hat{A}_{\tilde{\mathcal{I}}} & -\hat{B}_{\tilde{\mathcal{I}}} & I & \\ & & & \ddots & & \\ & & & & -\hat{A}_{\tilde{\mathcal{I}}} & -\hat{B}_{\tilde{\mathcal{I}}} & I \\ & & & & & -\hat{A}_{\tilde{\mathcal{I}}} & -\hat{B}_{\tilde{\mathcal{I}}} & I \end{bmatrix},$$

the following statements hold with constants,

1. $\hat{\gamma}_H \leq \sigma \left(\hat{H}_{\mathcal{I}}(\tilde{\mathcal{I}}) \right) \leq \hat{L}_H$;
2. $\hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}}) \hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}})^T \succeq \hat{\gamma}_F I$;
3. $\text{Re} H(G_{\mathcal{I}}, \hat{F}_{\mathcal{I}}(\tilde{\mathcal{I}})) \geq \hat{\gamma}_G I$;
4. We consider a partition $\{I_i\}_{i=0}^{t_2-t_1}$ of $\mathbb{I}_{[1, n_{\mathcal{I}}]}$. If $\hat{H}_{\mathcal{I}}(\tilde{\mathcal{I}})[I_i, I_j] = \mathbf{0}$ for any $|i - j| > 1$, then

$$\left\| \left(\hat{H}_{\mathcal{I}}(\tilde{\mathcal{I}}) \right)^{-1} [I_i, I_j] \right\| \leq \hat{c}_1 \hat{\rho}^{|i-j|}, \quad \forall i, j \in \mathbb{I}_{[0, t_2-t_1]};$$

where $\hat{L}_H$, $\hat{\gamma}_F$, $\hat{\gamma}_G$, $\hat{\gamma}_H$, $\hat{c}_1$, and $\hat{\rho}$ are defined in Theorem 3.4.1.

Proof. The proof of the lower bound of $\sigma \left(\hat{H}_{\mathcal{I}}(\tilde{\mathcal{I}}) \right)$ is similar to the proof of Theorem A.3 in [64]. The remaining part of the lemma is obtained from Lemma 3.8.7 and Lemma 2, Theorem 5 in [63]. $\square$

Note that the results of Lemma 3.8.11 are independent of the choice of $\mathcal{I}$ as well.

Proposition 3.8.1 (Proposition 2.5 in [67]). *If the real valued random process $\{w_t\}_{t=1}^T$ satisfies the (l, ν, q) -BMSB condition, then we have*

$$\mathbb{P} \left[\sum_{i=1}^T w_i^2 \leq \frac{\nu^2 q^2}{8} l \lfloor T/l \rfloor \right] \leq e^{-\frac{\lfloor T/l \rfloor q^2}{8}}$$

3.8.3 Upper bound of $\sum_{t \in \mathcal{I}} \|Q_t\|^2$

As we discuss below Theorem 3.4.1, we need three main steps to prove the estimation theorem. We first give the result for $\sum_{t \in \mathcal{I}} \|Q_t\|^2$ that with high probability, $\sum_{t \in \mathcal{I}} \|Q_t\|^2$ is upper bounded.

Lemma 3.8.12. *For a $\delta \in (0, 1)$, an interval $\mathcal{I}_i^e = [t_1, t_2]$ with $|\mathcal{I}_i^e| > 16 \log \frac{1}{\delta} / n$, it holds*

with probability at least $1 - (2N + 1)\delta$ that

$$\left\| \sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \right\| \leq 2C_Q^2 \cdot (n_\theta) \cdot (n + d) \cdot |\mathcal{I}_i^e| \quad (3.20)$$

where C_Q is defined in Theorem 3.4.1 and $\tilde{M}_i$ is defined in Assumption 3.4.1.

Proof. We know

$$\begin{aligned} \left\| \sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \right\| &\leq \sum_{t \in \mathcal{I}_i^e} \|Q_t Q_t^T\| \leq \sum_{t \in \mathcal{I}_i^e} \sum_{i=1}^{n_A} x_t^T A_i^T A_i x_t + \sum_{j=1}^{n_B} u_t^T B_j^T B_j u_t \\ &\leq n_A L^2 \sum_{t \in \mathcal{I}_i^e} \|x_t\|^2 + n_B L^2 \sum_{t \in \mathcal{I}_i^e} \|u_t\|^2 \leq (n_\theta) L^2 \sum_{t \in \mathcal{I}_i^e} \|z_t\|^2 = (n_\theta) L^2 \|z_{\mathcal{I}_i^e}\|^2 \end{aligned} \quad (3.21)$$

where $z_{\mathcal{I}_i^e}$ is the stack of $z_t = [x_t^T, u_t^T]^T$ for $t \in \mathcal{I}_i^e$. With $u_t = K_{stab}x_t + \nu\eta_t$ and (eq. (3.16)),

we have

$$\begin{aligned} x_t &= \tilde{\Phi}_{t_1, t-1} x_{t_1} + \sum_{m=t_1}^{t-1} \tilde{\Phi}_{m+1, t-1} w_m + \nu \sum_{m=t_1}^{t-1} \tilde{\Phi}_{m+1, t-1} B_m \eta_m, \\ u_t &= \tilde{\Psi}_{t_1, t-1} x_{t_1} + \sum_{m=t_1}^{t-1} \tilde{\Psi}_{m+1, t-1} w_m + \nu \sum_{m=t_1}^{t-1} \tilde{\Psi}_{m+1, t-1} B_m \eta_m + \nu \eta_t, \\ \tilde{\Phi}_{t_1, t_2} &= \Pi_{t=t_1}^{t_2} (A_t + B_t K_{stab}), \quad \tilde{\Phi}_{t', t'-1} = I, \quad \tilde{\Psi}_{t_1, t_2} = K_{stab} \tilde{\Phi}_{t_1, t_2}. \end{aligned} \quad (3.22)$$

Then, we can write them as

$$\begin{aligned} x_{\mathcal{I}_i^e} &= S_{\mathcal{I}_i^e, x} w_{\mathcal{I}_i^e} + \nu S_{\mathcal{I}_i^e, B, x} \eta_{\mathcal{I}_i^e} + C_{\mathcal{I}_i^e, x} x_{t_1}, \\ u_{\mathcal{I}_i^e} &= S_{\mathcal{I}_i^e, u} w_{\mathcal{I}_i^e} + \nu S_{\mathcal{I}_i^e, B, u} \eta_{\mathcal{I}_i^e} + C_{\mathcal{I}_i^e, u} x_{t_1}, \end{aligned} \quad (3.23)$$

where

$$S_{\mathcal{I}_i^e, x} = \begin{bmatrix} \mathbf{0} & & & & \\ & \tilde{\Phi}_{t_1+1, t_1} & & & \\ & \tilde{\Phi}_{t_1+1, t_1+1} & \tilde{\Phi}_{t_1+2, t_1+1} & & \\ & \vdots & \ddots & \ddots & \\ \tilde{\Phi}_{t_1+1, t_2-1} & \cdots & \tilde{\Phi}_{t_2-1, t_2-1} & \tilde{\Phi}_{t_2, t_2-1} & \end{bmatrix}, \quad C_{\mathcal{I}_i^e, x} = \begin{bmatrix} \Phi_{t_1, t_1-1} \\ \Phi_{t_1, t_1} \\ \vdots \\ \Phi_{t_1, t_2-1} \end{bmatrix},$$

$$S_{\mathcal{I}_i^e, B, x} = S_{\mathcal{I}_i^e, x} \begin{bmatrix} B_{t_1} & & & \\ \mathbf{0} & B_{t_1+1} & & \\ \vdots & \ddots & \ddots & \\ \mathbf{0} & \cdots & \mathbf{0} & B_{t_2-1} \end{bmatrix}, \quad C_{\mathcal{I}_i^e, u} = \begin{bmatrix} \tilde{\Psi}_{t_1, t_1-1} \\ \tilde{\Psi}_{t_1, t_1} \\ \vdots \\ \tilde{\Psi}_{t_1, t_2-1} \end{bmatrix},$$

$$S_{\mathcal{I}_i^e, u} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \tilde{\Psi}_{t_1+1, t_1} & \mathbf{0} & & & \\ \tilde{\Psi}_{t_1+1, t_1+1} & \tilde{\Psi}_{t_1+2, t_1+1} & \mathbf{0} & & \\ \vdots & \ddots & \ddots & \ddots & \\ \tilde{\Psi}_{t_1+1, t_2-2} & \cdots & \tilde{\Psi}_{t_2-2, t_2-2} & \tilde{\Psi}_{t_2-1, t_2-2} & \mathbf{0} \\ \tilde{\Psi}_{t_1+1, t_2-2} & \cdots & \tilde{\Psi}_{t_2-2, t_2-2} & \tilde{\Psi}_{t_2-1, t_2-2} & \tilde{\Psi}_{t_2, t_2-1} \end{bmatrix},$$

$$S_{\mathcal{I}_i^e, B, u} = \begin{bmatrix} I & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \tilde{\Psi}_{t_1+1, t_1} B_{t_1} & I & & & \\ \tilde{\Psi}_{t_1+1, t_1+1} B_{t_1} & \tilde{\Psi}_{t_1+2, t_1+1} B_{t_1+1} & I & & \\ \vdots & \ddots & \ddots & \ddots & \\ \tilde{\Psi}_{t_1+1, t_2-2} B_{t_1} & \cdots & \tilde{\Psi}_{t_2-2, t_2-2} B_{t_2-3} & \tilde{\Psi}_{t_2, t_2-1} B_{t_2-1} & I \end{bmatrix},$$

$$w_{\mathcal{I}_i^e} = \begin{bmatrix} w_{t_1} \\ w_{t_1+1} \\ \vdots \\ w_{t_2-1} \end{bmatrix}, \quad \eta_{\mathcal{I}_i^e} = \begin{bmatrix} \eta_{t_1} \\ \eta_{t_1+1} \\ \vdots \\ \eta_{t_2} \end{bmatrix}, \quad x_{\mathcal{I}_i^e} = \begin{bmatrix} x_{t_1} \\ x_{t_1+1} \\ \vdots \\ x_{t_2} \end{bmatrix}, \quad u_{\mathcal{I}_i^e} = \begin{bmatrix} u_{t_1} \\ u_{t_1+1} \\ \vdots \\ u_{t_2}^T \end{bmatrix}.$$

By permutation, we can obtain $z_{\mathcal{I}_i^e}^T = \begin{bmatrix} x_{\mathcal{I}_i^e}^T & u_{\mathcal{I}_i^e}^T \end{bmatrix} P_{\mathcal{I}_i^e}^T$ where $P_{\mathcal{I}_i^e}$ is a permutation matrix, $x_{\mathcal{I}_i^e}$ is the stack of x_t for $t \in \mathcal{I}_i^e$, and $u_{\mathcal{I}_i^e}$ is the stack of u_t for $t \in \mathcal{I}_i^e$. We have

$$\begin{bmatrix} x_{\mathcal{I}_i^e} \\ u_{\mathcal{I}_i^e} \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{S}_{\mathcal{I}_i^e, x} & \nu \tilde{S}_{\mathcal{I}_i^e, B, x} \\ \tilde{S}_{\mathcal{I}_i^e, u} & \nu \tilde{S}_{\mathcal{I}_i^e, B, u} \end{bmatrix}}_{M_{S, \mathcal{I}_i^e}} \underbrace{\begin{bmatrix} \psi I_{n \cdot |\mathcal{I}_i^e|} & \mathbf{0} \\ \mathbf{0} & I_{d \cdot |\mathcal{I}_i^e|} \end{bmatrix}}_{M_{z, \mathcal{I}_i^e}} M_{z, \mathcal{I}_i^e}^{-1} \begin{bmatrix} w_{\mathcal{I}_i^e} \\ \eta_{\mathcal{I}_i^e} \end{bmatrix} + \begin{bmatrix} \tilde{C}_{\mathcal{I}_i^e, x} \\ \tilde{C}_{\mathcal{I}_i^e, u} \end{bmatrix} x_{t_1} \quad (3.24)$$

To have an upper bound of $M_{S, \mathcal{I}_i^e}$, we give the upper bound of every block. We first look at $\tilde{S}_{\mathcal{I}_i^e, x}$ which can be written as

$$\tilde{S}_{\mathcal{I}_i^e, x} = \begin{bmatrix} \mathbf{0} & & & \\ \tilde{\Phi}_{t_1+1, t_1} & & & \\ & \tilde{\Phi}_{t_1+2, t_1+1} & & \\ \vdots & \ddots & \ddots & \\ & \cdots & \tilde{\Phi}_{t_2, t_2-1} & \end{bmatrix} + \begin{bmatrix} \mathbf{0} & & & \\ \mathbf{0} & & & \\ \tilde{\Phi}_{t_1+1, t_1+1} & \mathbf{0} & & \\ \vdots & \ddots & \ddots & \\ \mathbf{0} & \cdots & \tilde{\Phi}_{t_2-1, t_2-1} & \mathbf{0} \end{bmatrix} + \cdots$$

Therefore, we have $\|\tilde{S}_{\mathcal{I}_i^e, x}\| \leq \frac{L}{1-\alpha}$. Similarly, we have

$$\begin{aligned} \|\tilde{S}_{\mathcal{I}_i^e, x}\| &\leq \frac{L}{1-\alpha}, \quad \|\tilde{S}_{\mathcal{I}_i^e, B, x}\| \leq \frac{L^2}{1-\alpha}, \quad \|\tilde{C}_{\mathcal{I}_i^e, x}\| \leq \frac{L}{1-\alpha}, \\ \|\tilde{S}_{\mathcal{I}_i^e, u}\| &\leq \frac{L \cdot K_u}{1-\alpha}, \quad \|\tilde{S}_{\mathcal{I}_i^e, B, u}\| \leq \frac{2L^2 \cdot K_u}{1-\alpha}, \quad \|\tilde{C}_{\mathcal{I}_i^e, u}\| \leq \frac{L \cdot K_u}{1-\alpha}, \\ \|M_{S, \mathcal{I}_i^e}\| &\leq \|\tilde{S}_{\mathcal{I}_i^e, x}\| + \nu \|\tilde{S}_{\mathcal{I}_i^e, u}\| + \|\tilde{S}_{\mathcal{I}_i^e, B, x}\| + \nu \|\tilde{S}_{\mathcal{I}_i^e, B, u}\| \leq \frac{5L^2 \cdot K_u \cdot \max(\nu, 1)}{1-\alpha}. \end{aligned} \quad (3.25)$$

We know $\|M_{z, \mathcal{I}_i^e}^{-1} \begin{bmatrix} w_{\mathcal{I}_i^e}^T & \eta_{\mathcal{I}_i^e}^T \end{bmatrix}^T\|^2$ is a chi-square distribution and by Laurent-Massart bound[35], we have

$$\mathbb{P} \left[\|M_{z, \mathcal{I}_i^e}^{-1} \begin{bmatrix} w_{\mathcal{I}_i^e}^T & \eta_{\mathcal{I}_i^e}^T \end{bmatrix}^T\|^2 \geq |a|_1 + 2|a|_2 \sqrt{\log \left(\frac{1}{\delta} \right)} + 2|a|_\infty \log \left(\frac{1}{\delta} \right) \right] \leq \delta,$$

$$|a|_1 := (n+d)|\mathcal{I}_i^e|, \quad |a|_2 := \sqrt{|\mathcal{I}_i^e|(n+d)}, \quad |a|_\infty := 1.$$

Since $(n+d)|\mathcal{I}_i^e| + 2\sqrt{(n+d)|\mathcal{I}_i^e| \log(\frac{1}{\delta})} + 2\log(\frac{1}{\delta}) \leq 2(n+d)|\mathcal{I}_i^e|$ when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/(n+d)$, we have

$$\mathbb{P} \left[\|M_{z, \mathcal{I}_i^e}^{-1} \begin{bmatrix} w_{\mathcal{I}_i^e}^T & \eta_{\mathcal{I}_i^e}^T \end{bmatrix}^T\|^2 \geq 2(n+d)|\mathcal{I}_i^e| \right] \leq \delta,$$

By Lemma 3.8.1, when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/n$, we have

$$\mathbb{P} \left(\|x_{t_1}\| \geq \sqrt{2(n+d)|\mathcal{I}_i^e| \bar{x}_m} \right) \leq 2N\delta,$$

Then, we can obtain when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/n$,

$$\mathbb{P} \left[\|z_{\mathcal{I}_i^e}\| \geq C_Q/L \cdot \sqrt{2(n+d)|\mathcal{I}_i^e|} \right] \leq (2N+1)\delta,$$

Therefore, when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})/n$, we have

$$\mathbb{P} \left[\left\| \sum_{t \in \mathcal{I}} Q_t Q_t^T \right\| \geq 2C_Q^2 \cdot (n_\theta) \cdot (n+d) \cdot |\mathcal{I}_i^e| \right] \leq (2N+1)\delta.$$

□

3.8.4 Lower Bound of $\lambda_{\min}(\Gamma_{\mathcal{I}_i^e}) = \lambda_{\min} \left(\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \right)$

We first prove a helper lemma before proving the main lemma.

Lemma 3.8.13. Suppose $q_t = \tilde{A} \cdot z_t$ where $q_t \in \mathbb{R}^{n_q}$, $\tilde{A} \in \mathbb{R}^{n_q \times (n+d)}$, and $z_t^T = \begin{bmatrix} x_t^T & u_t^T \end{bmatrix}$. We can show $z_t | \mathcal{F}_{t-1} \sim \mathcal{N}(\bar{z}_t, \Sigma_t)$ and $\Sigma_t \succeq \hat{\Sigma} := \nu^2 \min \left\{ \frac{1}{2}, \frac{\psi^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I \succ \mathbf{0}$ where $\{\mathcal{F}_t\}_{t \in [T]}$ is the filtration generated by $\{w_1, w_2, \dots, w_T\}$, and $\bar{z}_t, \Sigma_t$ are $\mathcal{F}_{t-1}$ -measurable. Therefore, we have $q_t | \mathcal{F}_{t-1} \sim \mathcal{N}(\tilde{A}\bar{z}_t, \tilde{A}\Sigma_t\tilde{A}^T)$. Let

$$\mathcal{E} \text{ be any event for which } \bar{\Lambda}_T := E \left[\left(\sum_{t=1}^T q_t q_t^T \right) \mathbb{I}_{\mathcal{E}} \right] \text{ satisfies } \text{Tr}(\bar{\Lambda}_T) \leq JT \text{ for some constant } J \geq 0 \quad (3.26)$$

Then for

$$T \geq \frac{2000}{9} \left(2n_q \log \left(\frac{100}{3} \right) + n_q \log \frac{J}{\lambda_{\min}(\tilde{A}\hat{\Sigma}\tilde{A}^T)} \right), \quad (3.27)$$

it holds that, for $\Lambda_0(\tilde{A}) := \frac{9T}{1600} \tilde{A}\hat{\Sigma}\tilde{A}^T$,

$$\mathbb{P} \left[\left\{ \sum_{i=1}^T q_i q_i^T \not\preceq \Lambda_0(\tilde{A}) \right\} \cap \mathcal{E} \right] \leq 2 \exp \left(-\frac{9}{2000(n_q + 1)} T \right). \quad (3.28)$$

Proof. Let $M = \begin{bmatrix} I_n \\ K_t \end{bmatrix}$ and we will use notations

$$y_t = Mx_t, \text{ and, } \xi_t = \nu \tilde{\eta}_t = \nu \begin{bmatrix} \mathbf{0}_{n \times d} \\ I_d \end{bmatrix} \eta_t, \quad \text{so that:}$$

$$z_t = \begin{bmatrix} x_t \\ u_t \end{bmatrix} = y_t + \xi_t, \quad \text{and} \quad x_{t+1} = \begin{bmatrix} A_t & B_t \end{bmatrix} z_t + w_t,$$

Therefore, we have

$$z_t = Mx_t + \xi_t = M \begin{bmatrix} A_t & B_t \end{bmatrix} z_{t-1} + Mw_t + \xi_t.$$

where we recall that $w_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \psi^2 I_n)$. Then, we obtain $z_t | \mathcal{F}_{t-1} \sim \mathcal{N}(\bar{z}_t, \Sigma_t)$ where

$$\Sigma_t = \psi^2 M M^T + \nu^2 \begin{bmatrix} \mathbf{0}_{n \times d} \\ I_d \end{bmatrix} \begin{bmatrix} \mathbf{0}_{n \times d} & I_d \end{bmatrix}^T = \begin{bmatrix} \psi^2 I_n & \psi^2 K_t^T \\ \psi^2 K_t & \psi^2 K_t K_t^T + \nu^2 I_d \end{bmatrix}. \quad (3.29)$$

By Lemma 3.8.2. In our case (3.29), $\Sigma = \psi^2 I_n$, $\sigma_u = \nu$, and $K = K_t$, we have

$$\lambda_{\min}(\Sigma_t) \geq \nu^2 \min \left\{ \frac{1}{2}, \frac{\psi^2}{\nu^2 + 2\psi^2 \|K_t\|^2} \right\} \geq \nu^2 \min \left\{ \frac{1}{2}, \frac{\psi^2}{\nu^2 + 2\psi^2 K_u^2} \right\}, \quad (3.30)$$

where K_u is defined in Assumption 3.4.2. We know that

$$\begin{aligned} z_t | \mathcal{F}_{t-1} &\sim \mathcal{N}(\bar{z}_t, \Sigma_t), \quad q_t | \mathcal{F}_{t-1} \sim \mathcal{N}(\tilde{A} \bar{z}_t, \tilde{A} \Sigma_t \tilde{A}^T), \\ \Sigma_t &\succeq \hat{\Sigma} := \nu^2 \min \left\{ \frac{1}{2}, \frac{\psi^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I_{n+d}. \end{aligned}$$

For $\forall w \in \mathbb{R}^{n_q}$, we have

$$\begin{aligned} w^T q_t | \mathcal{F}_{t-1} &\sim \mathcal{N}(w^T \tilde{A} \bar{z}, w^T \tilde{A} \Sigma_t \tilde{A}^T w), \\ \mathbb{P} \left(|w^T q_t| \geq \sqrt{w^T \tilde{A} \Sigma_t \tilde{A} w} | \mathcal{F}_{t-1} \right) &\geq \mathbb{P} \left(|w^T q_t| \geq \sqrt{w^T \tilde{A} \Sigma_t \tilde{A} w} | \mathcal{F}_{t-1} \right) \geq \frac{3}{10}. \end{aligned}$$

where the first inequality uses the fact that $\Sigma_t \succeq \Sigma$, and the second inequality follows from the Paley-Zygmund lower bound as follows

$$\forall t \in \mathbb{R}, \quad \mathbb{P}_{z \sim \mathcal{N}(0, \sigma^2)}(|t + z| \geq \sigma) \geq \mathbb{P}(|z| \geq \sigma) \geq \frac{3}{10}.$$

Since for $\forall w \in \mathcal{S}^{n_q-1}$ which denotes the unit sphere in $\mathbb{R}^{n_q}$, the process $\langle w, q_t \rangle$ satisfies $\left(1, \sqrt{w^T \tilde{A} \Sigma_t \tilde{A} w}, \frac{3}{10}\right) - BMSB$ where $BMSB$ is defined in Definition 3.8.1. Therefore, we say q_t satisfies $\left(1, \tilde{A} \hat{\Sigma} \tilde{A}^T, \frac{3}{10}\right)$. We denote $\tilde{Q}^T = \begin{bmatrix} q_1 & q_2 & \cdots & q_T \end{bmatrix}$. Then, $\forall w \in \mathbb{R}^{n_q}$, $\tilde{Q}w = \begin{bmatrix} q_1^T w, q_2^T w, \dots, q_T^T w \end{bmatrix}^T$. By Definition 3.8.1, we have $\left(q_1^T w, q_2^T w, \dots, q_T^T w\right)$ satis-

fies $\left(1, \sqrt{w^T \tilde{A} \Sigma \tilde{A}^T w}, \frac{3}{10}\right) - BMSB$. We will establish the upper bound for $\|\tilde{Q}w\|^2$ by Proposition 3.8.1. Since $\forall w \in \mathbb{R}^{n_q}$, $\tilde{Q}w$ satisfies $\left(1, \sqrt{w^T \tilde{A} \Sigma \tilde{A}^T w}, \frac{3}{10}\right) - BMSB$, we have $\forall w \in \mathbb{R}^{n_q}$,

$$\mathbb{P} \left(\|\tilde{Q}w\|^2 \leq \frac{T \left(\frac{3}{10}\right)^2 w^T \tilde{A} \Sigma \tilde{A}^T w}{8} \right) \leq \exp \left(-\frac{T \left(\frac{3}{10}\right)^2}{8} \right) \quad (3.31)$$

In order to establish $\mathbb{P} \left(\left\{ \left\{ \tilde{Q}^T \tilde{Q} \not\leq \frac{(\frac{3}{10})^2 \cdot T \cdot \tilde{A} \Sigma \tilde{A}^T}{16} \right\} \cap \left\{ \tilde{Q}^T \tilde{Q} \preceq \Lambda_+ \right\} \right\} \right)$, we need the definition of ϵ -net in Definition 3.8.2, Lemma 3.8.4, and Lemma 3.8.5. We denote $\Gamma_{\min} = \frac{(\frac{3}{10})^2 T \tilde{A} \Sigma \tilde{A}^T}{8}$ and $\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})$ as a minimal $\epsilon \leq 1/2$ -net of $\mathcal{S}_{\Gamma_{\min}}$ in the norm $\|\Gamma_{\max}^{1/2}(\cdot)\|$ where $\mathcal{T}_\epsilon(\cdot; \cdot)$ is a function of $\Gamma_{\min}$ and $\Gamma_{\max}$, and $\mathcal{S}_{\Gamma_{\min}} := \{w : w^T \Gamma_{\min} w = 1\}$ is a unit sphere measured by $\Gamma_{\min}$. Then, by Lemma 3.8.4, $\log |\mathcal{T}_\epsilon(\Gamma_{\min}; \Gamma_{\max})| \leq n_q \log \left(1 + \frac{2}{\epsilon}\right) + \log \det \left(\Gamma_{\max} \Gamma_{\min}^{-1}\right)$. When $\epsilon = \frac{1}{4}$, we have

$$\begin{aligned} \log |\mathcal{T}_{1/4}(\Gamma_{\min}; \Gamma_{\max})| &\leq n_q \log \left(1 + \frac{2}{\epsilon}\right) + \log \det \left(\Gamma_{\max} \Gamma_{\min}^{-1}\right) \\ &\leq n_q \log 9 + n_q \log \left(\frac{\|\Gamma_{\max}\|}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) + n_q \log \left(\frac{800}{9} \right) \quad (3.32) \\ &\leq 2n_q \log \left(\frac{100}{3} \right) + n_q \log \left(\frac{\|\Gamma_{\max}\|}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \end{aligned}$$

By Lemma 3.8.5 and assuming $\Gamma_{\max} = \Lambda_+$, we know that

$$\left\{ \underbrace{\left\{ \tilde{Q}^T \tilde{Q} \succeq \frac{\left(\frac{3}{10}\right)^2 \cdot T \cdot \tilde{A} \hat{\Sigma} \tilde{A}^T}{16} \right\}}_{\mathcal{E}_1} \cap \underbrace{\left\{ \tilde{Q}^T \tilde{Q} \preceq \Lambda_+ \right\}}_{\mathcal{E}_3} \right\} \supseteq \left\{ \underbrace{\left\{ \inf_{w \in \mathcal{T}_{1/4}(\Gamma_{\min}; \Lambda_+)} w^T \tilde{Q}^T \tilde{Q} w \geq 1 \right\}}_{\mathcal{E}_2} \cap \underbrace{\left\{ \tilde{Q}^T \tilde{Q} \preceq \Lambda_+ \right\}}_{\mathcal{E}_3} \right\}$$

Therefore, we have

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_3) \geq \mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}_3), \quad \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) \leq \mathbb{P}(\mathcal{E}_2^c \cap \mathcal{E}_3). \quad (3.33)$$

where $\mathcal{E}_i^c$ is the complement of $\mathcal{E}_i$ for $i = 1, 2, 3$. Then, we conclude by Lemma 3.8.5 that for $\forall \Gamma_{\max} := \Lambda_+$

$$\begin{aligned} & \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) \\ (3.33) \mathbb{P} & \leq \left(\underbrace{\left\{ \exists w \in \mathcal{T}_{1/4}(\Gamma_{\min}; \Lambda_+) : \|\tilde{Q}w\|^2 < 1 := w^T \Gamma_{\min} w = \frac{\left(\frac{3}{10}\right)^2 \cdot T \cdot w^T \tilde{A} \hat{\Sigma} \tilde{A}^T w}{8} \right\}}_{\mathcal{E}_2^c} \cap \mathcal{E}_3 \right) \\ & \leq \sum_{w \in \mathcal{T}_{1/4}(\Gamma_{\min}; \Lambda_+)} \mathbb{P} \left(\|\tilde{Q}w\|^2 \leq \frac{\left(\frac{3}{10}\right)^2 \cdot T \cdot w^T \tilde{A} \hat{\Sigma} \tilde{A}^T w}{8} \right) \end{aligned}$$

Since for $\forall w \in \mathbb{R}^{n_q}$, we have (3.31). We can have an upper bound as follows

$$\begin{aligned}
& \leq \mathbb{P} \left(\|\tilde{Q}w\|^2 \leq \frac{T \left(\frac{3}{10}\right)^2 w^T \tilde{A} \hat{\Sigma} \tilde{A}^T w}{8} \right) \cdot |\mathcal{T}_{1/4}(\Lambda_+)| \\
& \stackrel{(3.31)}{\leq} \exp \left(-\frac{T \cdot \left(\frac{3}{10}\right)^2}{8} \right) \cdot \exp \left(2n_q \log \left(\frac{100}{3} \right) + n_q \log \left(\frac{\|\Lambda_+\|}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \right) \quad (3.34) \\
& \leq \exp \left(-\frac{9T}{1000} + 2n_q \log \left(\frac{100}{3} \right) + n_q \log \left(\frac{\|\Lambda_+\|}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \right)
\end{aligned}$$

We select $\Lambda_+ = \frac{\text{Tr}(\bar{\Lambda}_T)}{\delta} I$ and apply Lemma 3.8.3 to $\tilde{Q}^T \tilde{Q}$ and have

$$\mathbb{P} \left(\underbrace{\left\{ \tilde{Q}^T \tilde{Q} \notin \Lambda_+ \right\}}_{\mathcal{E}_3^c} \cap \mathcal{E} \right) \leq \text{Tr} \left(E \left[\tilde{Q}^T \tilde{Q} \mathbb{I}_{\mathcal{E}} \right] \frac{\delta}{\text{Tr}(\bar{\Lambda}_T)} \right) \stackrel{(3.26)}{=} \text{Tr} \left(\bar{\Lambda}_T \frac{\delta}{\text{Tr}(\bar{\Lambda}_T)} \right) = \delta \quad (3.35)$$

Also, we observe that $\mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) = \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3 \cap \mathcal{E}) + \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3 \cap \mathcal{E}^c)$, $\mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E}) = \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E} \cap \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E} \cap \mathcal{E}_1^c)$, and $\mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}) = \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E} \cap \mathcal{E}_3) + \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E} \cap \mathcal{E}_3^c)$. Then, we have that $\mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E}) - \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}) = \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3 \cap \mathcal{E}^c) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E} \cap \mathcal{E}_1) \geq 0$. Therefore,

we obtain $\mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E}) \geq \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E})$ for $\forall \delta > 0$. Hence, we have

$$\begin{aligned}
& \mathbb{P} \left(\left\{ \tilde{Q}^T \tilde{Q} = \sum_{t=1}^T q_t q_t^T \not\geq \frac{\left(\frac{3}{10}\right)^2 \cdot T \cdot \tilde{A} \hat{\Sigma} \tilde{A}^T}{16} \right\} \cap \mathcal{E} \right) \\
&= \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}) \leq \inf_{\delta > 0} \mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}_3) + \mathbb{P}(\mathcal{E}_3^c \cap \mathcal{E}) \\
&\stackrel{(3.34)}{\underset{(3.35)}{\leq}} \inf_{\delta > 0} \exp \left(-\frac{9T}{1000} + 2n_q \log \left(\frac{100}{3} \right) + n_q \log \left(\frac{\text{Tr}(\bar{\Lambda}_T)}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T) \delta} \right) \right) + \delta \\
&\leq \underbrace{\inf_{\delta > 0} \delta^{-n_q} \exp \left(-\frac{9T}{1000} + 2n_q \log \left(\frac{100}{3} \right) + n_q \log \left(\frac{\text{Tr}(\bar{\Lambda}_T)}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \right)}_a + \delta
\end{aligned}$$

Note that balancing $a\delta^{-n_q} = \delta$ selects $\delta = a^{1/(n_q+1)}$, giving that the above is at most

$$2 \exp \left(-\frac{1}{n_q + 1} \left(\frac{9T}{1000} - 2n_q \log \left(\frac{100}{3} \right) - n_q \log \frac{\text{Tr}(\bar{\Lambda}_T)}{T \cdot \lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \right)$$

Since $\text{Tr}(\bar{\Lambda}_T) \leq JT$, we have

$$\mathbb{P}(\mathcal{E}_1^c \cap \mathcal{E}) \leq 2 \exp \left(-\frac{1}{n_q + 1} \left(\frac{9T}{1000} - 2n_q \log \left(\frac{100}{3} \right) - n_q \log \frac{J}{\lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right) \right)$$

When

$$T \geq \frac{2000}{9} \left(2n_q \log \left(\frac{100}{3} \right) + n_q \log \frac{J}{\lambda_{\min}(\tilde{A} \hat{\Sigma} \tilde{A}^T)} \right),$$

we have

$$\mathbb{P} \left[\left\{ \sum_{i=1}^T q_i q_i^T \not\geq \Lambda_0(\tilde{A}) := \frac{9T}{1600} \tilde{A} \hat{\Sigma} \tilde{A}^T \right\} \cap \mathcal{E} \right] \leq 2 \exp \left(-\frac{9}{2000(n_q + 1)} T \right)$$

as expected where the later part of the proof follows the proof of Lemma E.4 in [66]. $\square$

Now we are ready to give our main lemma for step 1.

Lemma 3.8.14. *For the exploration interval $\mathcal{I}_i^e$, recall $\Gamma_{\mathcal{I}_i^e}^Q := \sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T$, if we have*

$$|\mathcal{I}_i^e| \geq \frac{2000}{9} \left(2(n_\theta) \log \left(\frac{1}{\delta} \right) + (n_\theta) \log \left(\frac{(n+d)\bar{z}}{\min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\}} \right) \right), \quad (3.36)$$

then for $\delta \leq 3/100$, it holds with probability at least $1 - 2n\delta$ that

$$\lambda_{\min} \left(\Gamma_{\mathcal{I}_i^e}^Q \right) \geq \frac{9n|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} \lambda_{\min, M}. \quad (3.37)$$

where $\lambda_{\min, M}$ is defined in Assumption 3.4.1, $\bar{z}$ is defined in Theorem 3.4.1, and K_u is defined in Assumption 3.4.2.

Proof. Recall $\bar{A}_k$ for $1 \leq k \leq n_A$ and $\bar{B}_l$ for $1 \leq l \leq n_B$ defined in Assumption 3.4.1, and rewrite Q_t^T as follows

$$Q_t^T = \begin{bmatrix} \bar{A}_{1,1}x_t & \cdots & \bar{A}_{1,n_A}x_t & \bar{B}_{1,1}u_t & \cdots & \bar{B}_{1,n_B}u_t \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \bar{A}_{n,1}x_t & \cdots & \bar{A}_{n,n_A}x_t & \bar{B}_{n,1}u_t & \cdots & \bar{B}_{n,n_B}u_t \end{bmatrix},$$

If we denote $q_{k,t}^T = \begin{bmatrix} \bar{A}_{k,1}x_t & \cdots & \bar{A}_{k,n_A}x_t & \bar{B}_{k,1}u_t & \cdots & \bar{B}_{k,n_B}u_t \end{bmatrix}$ for $k = 1, \dots, n$, then we have

$$Q_t Q_t^T = \sum_{k=1}^n q_{k,t} q_{k,t}^T, \quad q_{k,t} = \begin{bmatrix} \tilde{A}_k x_t \\ \tilde{B}_k u_t \end{bmatrix} = \begin{bmatrix} \tilde{A}_k \\ \tilde{B}_k \end{bmatrix} z_t = \tilde{M}_k z_t, \quad k = 1, 2, \dots, n.$$

where $\tilde{M}_k$ is defined in Assumption 3.4.1. By singular value decomposition, there exist $U_{M,k}$ and $V_{M,k}$ such that

$$\tilde{M}_k = U_{M,k}^T \Sigma_{M,k} V_{M,k}, \quad k = 1, 2, \dots, n.$$

where $V_{M,k}V_{M,k}^T = I_{n_{M,k}}$. We consider vector $\tilde{q}_{t,k} = V_{M,k}z_t$ and first want to prove $\text{Tr}\left(E\left(\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T\right)\right) = \text{Tr}\left(V_{M,k}E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)V_{M,k}^T\right) = \text{Tr}\left(V_{M,k}^T V_{M,k}E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \leq \text{Tr}\left(E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right)$.

$$\begin{aligned}
& \text{Tr}\left(\left(I_{n+d} - V_{M,k}^T V_{M,k}\right)E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \\
&= \text{Tr}\left(U_{M,I,k} \Lambda_{M,I,k}^{\frac{1}{2}} \Lambda_{M,I,k}^{\frac{1}{2}} U_{M,I,k}^T E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \\
&= \text{Tr}\left(P_{M,k} P_{M,k}^T E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) = \text{Tr}\left(P_{M,k}^T E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right) P_{M,k}\right) \\
&= \sum_{l=1}^{n+d} e_l^T P_{M,k}^T E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right) P_{M,k} e_l \geq 0
\end{aligned}$$

where $I_{n+d} - V_{M,k}^T V_{M,k} = U_{M,I,k} \Lambda_{M,I,k} U_{M,I,k}^T$ by singular value decomposition with the unitary matrix $U_{M,I,k}$, the diagonal matrix $\Lambda_{M,I,k}$, and $P = U_{M,I,k} \Lambda_{M,I,k}^{\frac{1}{2}}$. Since

$$\begin{aligned}
& \text{Tr}\left(\left(I_{n+d} - V_{M,k}^T V_{M,k}\right)E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \geq 0, \\
& \text{Tr}\left(V_{M,k}^T V_{M,k} E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \leq \text{Tr}\left(E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right),
\end{aligned}$$

we have

$$\begin{aligned}
\text{Tr}\left(E\left(\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T\right)\right) &\leq \text{Tr}\left(E\left(\sum_{t \in \mathcal{I}_i^e} z_t z_t^T\right)\right) \leq E\left[\sum_{t \in \mathcal{I}_i^e} \|x_t\|^2 + \|u_t\|^2\right] \\
&\leq E\left[\sum_{t \in \mathcal{I}_i^e} \|x_t\|^2 + 2\|K_t\|^2 \|x_t\|^2 + 2\nu^2 \|\eta_t\|^2\right] \\
&\leq |\mathcal{I}_i^e|(n+d) \left(\bar{x}^2 (1 + 2K_u^2) + 2\nu^2\right),
\end{aligned} \tag{3.38}$$

where the third inequality is because of $\|u_t\|^2 \leq (\|K_t\|\|x_t\| + \nu\|\eta_t\|)^2 \leq 2(\|K_t\|^2\|x_t\|^2 + \nu^2\|\eta_t\|^2)$; last inequality is because of Lemma 3.8.1 and $E[\|\eta_t\|^2] = d$. Setting $\mathcal{E} = \Omega$ (the probability space) and

$$\Gamma_{0,i,k} = \frac{9|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I_{n_{M,k}},$$

Since $V_{M,k} V_{M,k}^T = I_{n_{M,k}}$ and $\hat{\Sigma} = \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I$, we have

$$V_{M,k} \hat{\Sigma} V_{M,k}^T = \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I_{n_{M,k}},$$

Let $\delta \leq 3/100$ and let

$$\begin{aligned} |\mathcal{I}_i^e| &\geq \frac{2000}{9} \left(2n_{M,k} \log \left(\frac{1}{\delta} \right) + n_{M,k} \log \frac{(n+d)\bar{z}}{\min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\}} \right) \\ &\geq \frac{2000}{9} \left(2n_{M,k} \log \left(\frac{100}{3} \right) + n_{M,k} \log \frac{(n+d)\bar{z}}{\min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\}} \right), \end{aligned}$$

By Lemma 3.8.13, we have

$$\begin{aligned} \mathbb{P} \left[\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T \not\preceq \Gamma_{0,i,k} \right] &\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot |\mathcal{I}_i^e| \right) \\ &\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot \frac{2000}{9} \left(2n_{M,k} \log \frac{1}{\delta} \right) \right) \\ &\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot \frac{2000}{9} \left((n_{M,k} + 1) \log \frac{1}{\delta} \right) \right) = 2\delta, \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T &= \sum_{t \in \mathcal{I}_i^e} \sum_{k=1}^n q_{k,t} q_{k,t}^T = \sum_{t \in \mathcal{I}_i^e} \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \tilde{q}_{k,t} \tilde{q}_{k,t}^T \Sigma_{M,k} U_{M,k} \\
&= \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \left(\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{k,t} \tilde{q}_{k,t}^T \right) \Sigma_{M,k}^T U_{M,k} \\
&\preceq \frac{9|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \Sigma_{M,k}^T U_{M,k} \\
&= \frac{9|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} \sum_{k=1}^n \tilde{M}_k \tilde{M}_k^T \\
&\preceq \frac{9|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} n \lambda_{\min, M} I_{n_\theta}.
\end{aligned}$$

where the last inequality holds because of Assumption 3.4.1. Therefore, we have

$$\mathbb{P} \left[\sum_{t \in \mathcal{I}_i^e} Q_t Q_t^T \not\geq \frac{9n|\mathcal{I}_i^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} \lambda_{\min, M} I_{n_\theta} \right] \leq 2n\delta.$$

We complete the proof. □

3.8.5 Upper bound of $\left\| \sum_{t \in \mathcal{I}} Q_t w_t \right\|$

Lemma 3.8.15. *For a $\delta \in (0, 1)$ and an interval $\mathcal{I}_i^e = [t_1, t_2]$ with $|\mathcal{I}_i^e| > 16 \log \frac{1}{\delta}$, it holds with probability at least $1 - 4(n_\theta)nN\delta$ that*

$$\left\| \sum_{t \in \mathcal{I}_i^e} Q_t w_t \right\| \leq \psi \cdot 2C_Q \lambda_{\max, M} \cdot (n_\theta) \cdot n \cdot \sqrt{|\mathcal{I}_i^e| \log \frac{1}{\delta}}. \quad (3.39)$$

where C_Q and $\lambda_{\max, M}$ are defined in Theorem 3.4.1.

Proof. We can rewrite $\sum_{t \in \mathcal{I}_i^e} Q_t w_t$ as

$$\begin{aligned} \sum_{t \in \mathcal{I}_i^e} Q_t w_t &= \sum_{t \in \mathcal{I}_i^e} \sum_{i=1}^n w_{t,i} \cdot q_{i,t} = \sum_{t \in \mathcal{I}_i^e} \sum_{i=1}^n w_{t,i} \cdot \tilde{M}_i z_t, \quad \tilde{M}_i z_t = \begin{bmatrix} \langle \tilde{M}_{i,1}, z_t \rangle \\ \vdots \\ \langle \tilde{M}_{i,n_\theta}, z_t \rangle \end{bmatrix}. \\ \sum_{t \in \mathcal{I}_i^e} Q_t w_t &= \sum_{i=1}^n \begin{bmatrix} \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,1}, z_t \rangle \\ \vdots \\ \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,n_\theta}, z_t \rangle \end{bmatrix}, \end{aligned}$$

where $\tilde{M}_{i,j}$ is the j -th row of $\tilde{M}_i$. By [67, Lemma 4.2], we know

$$\mathbb{P} \left(\left| \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,1}, z_t \rangle \right| \geq \psi \sqrt{2 \log \frac{1}{\delta} \sum_{t \in \mathcal{I}_i^e} \langle \tilde{M}_{i,1}, z_t \rangle^2} \right) \leq 2\delta, \quad (3.40)$$

We also know that when $|\mathcal{I}_i^e| \geq 16 \log(\frac{1}{\delta})$,

$$\begin{aligned} \mathbb{P} \left(\sum_{t \in \mathcal{I}_i^e} \langle \tilde{M}_{i,1}, z_t \rangle^2 \geq 2C_Q^2 \lambda_{\max, M}^2 |\mathcal{I}_i^e| / L^2 \right) &\leq 2N\delta, \\ \mathbb{P} \left(\left| \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,1}, z_t \rangle \right| \geq \psi \sqrt{|\mathcal{I}_i^e| \log \frac{1}{\delta}} 2C_Q \lambda_{\max, M} / L \right) &\leq 2(N+1)\delta, \end{aligned} \quad (3.41)$$

the proof is similar to what we proved before and we omit it here. Then

$$\left\| \sum_{t \in \mathcal{I}_i^e} Q_t w_t \right\| \leq \left\| \sum_{t \in \mathcal{I}_i^e} Q_t w_t \right\|_1 \leq \sum_{i=1}^n \sum_{j=1}^{n_\theta} \left| \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,j}, z_t \rangle \right|,$$

We have the probability at least $1 - 4Nn(n_\theta)\delta$ that

$$\sum_{i=1}^n \sum_{j=1}^{n_\theta} \left| \sum_{t \in \mathcal{I}_i^e} w_{t,i} \langle \tilde{M}_{i,j}, z_t \rangle \right| \leq \frac{\psi}{L} \cdot 2C_Q \lambda_{\max, M} \cdot (n_\theta) \cdot n \cdot \sqrt{|\mathcal{I}_i^e| \log \frac{1}{\delta}}.$$

Furthermore, with the probability at least $1 - 4Nn(n_\theta)\delta$ that

$$\left\| \sum_{t \in \mathcal{I}_i^e} Q_t w_t \right\| \leq \psi \cdot 2C_Q \lambda_{\max, M} \cdot (n_\theta) \cdot n \cdot \sqrt{|\mathcal{I}_i^e| \log \frac{1}{\delta}}.$$

□

3.8.6 Proof of Theorems in Regret Analysis

Proof of Lemma 3.5.1 We first give several definitions of the summations to simplify the analysis below.

$$\begin{aligned} U(n_1, \alpha) &= \sum_{i=0}^{n_1} \alpha^{2i} = \frac{1 - \alpha^{2n_1+2}}{1 - \alpha^2} \leq \frac{1}{1 - \alpha^2}, \\ V(n_1, \alpha) &= \sum_{i=0}^{n_1} i \alpha^{2i} = \frac{\alpha^2 (1 - \alpha^{2n_1})}{(1 - \alpha^2)^2} - \frac{n_1 \alpha^{n_1+1}}{1 - \alpha^2} \leq \frac{\alpha^2}{(1 - \alpha^2)^2}, \\ W(n_1, \alpha) &= \sum_{i=0}^{n_1} i^2 \alpha^{2i} = \frac{2V(n_1)}{1 - \alpha^2} - \frac{\alpha^2 (1 - \alpha^{2n_1})}{1 - \alpha^2} - \frac{n_1^2 \alpha^{n_1+1}}{1 - \alpha^2} \leq \frac{2\alpha^2}{(1 - \alpha^2)^3}. \end{aligned} \tag{3.42}$$

We can write $\sum_{t=t_1+1}^{t_2} \frac{1}{2} x_t^T Q x_t$ and $\sum_{t=t_1+1}^{t_2-1} \frac{1}{2} u_t^T R u_t$ as

$$\sum_{t=t_1+1}^{t_2} \frac{1}{2} x_t^T Q x_t = \frac{1}{2} \mathbf{x}^T D_Q \mathbf{x}, \quad \sum_{t=t_1+1}^{t_2-1} \frac{1}{2} u_t^T R u_t = \frac{1}{2} \mathbf{u}^T D_R \mathbf{u}. \tag{3.43}$$

where $\mathbf{x}$ is the stack vector of x_t for $t_1 + 1 \leq t \leq t_2$, $\mathbf{u}$ is the stack vector of u_t for $t_1 + 1 \leq t \leq t_2 - 1$, D_Q is the block diagonal matrix setting diagonal matrix as Q , and D_R is the block diagonal matrix setting diagonal matrix as R . Denote $\Phi_t = A_t + B_t K_t$. Since

$u_t = K_t x_t$, we obtain

$$\begin{aligned}
x_t &= (A_{t-1} + B_{t-1}K_{t-1})x_{t-1} + w_{t-1} = \cdots \\
&= w_{t-1} + \sum_{m=1}^{t-t_1-1} \left(\prod_{l=t-m}^{t-1} \Phi_l \right) w_{t-m-1} + \left(\prod_{m=t_1}^{t-1} \Phi_m \right) x_{t_1}, \\
u_t &= K_t w_{t-1} + \sum_{m=1}^{t-t_1-1} K_t \left(\prod_{l=t-m}^{t-1} \Phi_l \right) w_{t-m-1} + K_t \left(\prod_{m=t_1}^{t-1} \Phi_m \right) x_{t_1},
\end{aligned} \tag{3.44}$$

Denote $\left(\prod_{l=t-m}^{t-1} \Phi_l \right)$ as $\Phi_{t-m,t-1}$, $\Psi_{t-m,t-1} = K_t \Phi_{t-m,t-1}$, and define $\Phi_{t,t-1} = I$. We rewrite (3.44) as

$$\begin{aligned}
x_t &= \sum_{m=0}^{t-t_1-1} \Phi_{t-m,t-1} w_{t-m-1} + \Phi_{t_1,t-1} x_{t_1}, \\
u_t &= \sum_{m=0}^{t-t_1-1} \Psi_{t-m,t-1} w_{t-m-1} + \Psi_{t_1,t-1} x_{t_1},
\end{aligned} \tag{3.45}$$

Then we can write $\mathbf{x}$ and $\mathbf{u}$ as

$$\begin{aligned}
\mathbf{x} &= S_x \mathbf{w}_x + C_x \bar{x}_0, \quad S_x = \begin{bmatrix} \Phi_{t_1+1,t_1} & & & \\ \Phi_{t_1+1,t_1+1} & \Phi_{t_1+2,t_1+1} & & \\ \vdots & \ddots & \ddots & \\ \Phi_{t_1,t_2-1} & \cdots & \Phi_{t_2-1,t_2-1} & \Phi_{t_2,t_2-1} \end{bmatrix}, \\
\mathbf{u} &= S_u \mathbf{w}_u + C_u \bar{x}_0, \quad S_u = \begin{bmatrix} \Psi_{t_1+1,t_1} & & & \\ \Psi_{t_1+1,t_1+1} & \Psi_{t_1+2,t_1+1} & & \\ \vdots & \ddots & \ddots & \\ \Psi_{t_1+1,t_2-2} & \cdots & \Psi_{t_2-2,t_2-2} & \Psi_{t_2-1,t_2-2} \end{bmatrix}, \\
C_x^T &= \begin{bmatrix} \Phi_{t_1,t_1}^T & \Phi_{t_1,t_1+1}^T & \cdots & \Phi_{t_1,t_2-1}^T \end{bmatrix}, \quad \mathbf{w}_x^T = \begin{bmatrix} w_{t_1}^T & w_{t_1+1}^T & \cdots & w_{t_2-1}^T \end{bmatrix}, \\
C_u^T &= \begin{bmatrix} \Psi_{t_1,t_1}^T & \Psi_{t_1,t_1+1}^T & \cdots & \Psi_{t_1,t_2-2}^T \end{bmatrix}, \quad \mathbf{w}_u = \begin{bmatrix} w_{t_1} & w_{t_1+1} & \cdots & w_{t_2-2} \end{bmatrix}.
\end{aligned} \tag{3.46}$$

Let us look at $\sum_{t=t_1+1}^{t_2} x_t^T Q x_t$ first. Since $E[\mathbf{w}_x] = \mathbf{0}$, we obtain

$$\begin{aligned} E \left[\sum_{t=t_1+1}^{t_2} x_t^T Q x_t \right] &= E \left[E \left[(S_x \mathbf{w}_x + C_x \bar{x}_0)^T D_Q (S_x \mathbf{w}_x + C_x \bar{x}_0) \mid \bar{x}_0 \right] \right] \\ &= E \left[\mathbf{w}_x^T S_x^T D_Q S_x \mathbf{w}_x \right] + E \left[\bar{x}_0^T C_x^T D_Q C_x \bar{x}_0 \right], \end{aligned}$$

Similarly, we can get

$$E \left[\sum_{t=t_1+1}^{t_2} \tilde{x}_t^T Q \tilde{x}_t \right] = E \left[\mathbf{w}_x^T \tilde{S}_x^T D_Q \tilde{S}_x \mathbf{w}_x \right] + E \left[\bar{x}_0'^T \tilde{C}_x^T D_Q \tilde{C}_x \bar{x}_0' \right].$$

where the definition of $\tilde{S}_x$ and $\tilde{C}_x$ are similar to the definition of S_x and C_x . Since Q is symmetric and positive semi-definite, there exists a unitary matrix U_Q and a diagonal matrix Λ_Q such that

$$Q = U_Q \Lambda_Q U_Q^T = U_Q (\Lambda_Q)^{\frac{1}{2}} \left(U_Q (\Lambda_Q)^{\frac{1}{2}} \right)^T = Q_1^T Q_1, \quad Q_1 = \left(U_Q (\Lambda_Q)^{\frac{1}{2}} \right)^T. \quad (3.47)$$

where $\|Q_1\| \leq L^{\frac{1}{2}}$. Therefore, we can write D_Q as $D_Q = D_{Q_1}^T D_{Q_1}$ where D_{Q_1} is the block diagonal matrix setting diagonal matrix as Q_1 . By Assumption 3.4.2 and $\mathbf{w}_x \sim N(\mathbf{0}, \psi^2 I_{n \cdot (t_2 - t_1)})$, we obtain

$$\begin{aligned} &E \left[\mathbf{w}_x^T S_x^T D_Q S_x \mathbf{w}_x - \mathbf{w}_x^T \tilde{S}_x^T D_Q \tilde{S}_x \mathbf{w}_x \right] \\ &= \psi^2 \left(\text{Tr} \left((D_{Q_1} S_x)^T (D_{Q_1} S_x) \right) - \text{Tr} \left((D_{Q_1} \tilde{S}_x)^T (D_{Q_1} \tilde{S}_x) \right) \right) \\ &= \psi^2 \left(\|D_{Q_1} S_x\|_F^2 - \|D_{Q_1} \tilde{S}_x\|_F^2 \right). \end{aligned}$$

We first give the bound for $\|\Phi_{k,l} - \tilde{\Phi}_{k,l}\|$ as follows.

$$\begin{aligned}
& \|\Phi_{k,l} - \tilde{\Phi}_{k,l}\| = \|\Pi_{i=k}^l \Phi_i - \Pi_{i=k}^l \tilde{\Phi}_i\| \\
& = \|(\Phi_k - \tilde{\Phi}_k)(\Pi_{j=k+1}^l \tilde{\Phi}_j) + (\Pi_{j=k}^{l-1} \Phi_j)(\Phi_l - \tilde{\Phi}_l) + \sum_{i=k+1}^{l-1} (\Pi_{j=k}^{i-1} \Phi_j)(\Phi_i - \tilde{\Phi}_i)(\Pi_{j=i+1}^l \tilde{\Phi}_j)\| \\
& \leq \|B_k \Delta \tilde{K}_k\| \|\Pi_{j=k+1}^l \tilde{\Phi}_j\| + \|B_l \Delta \tilde{K}_l\| \|\Pi_{j=k}^{l-1} \Phi_j\| + \sum_{i=k+1}^{l-1} \|\Pi_{j=k}^{i-1} \Phi_j\| \|B_i \Delta \tilde{K}_i\| \|\Pi_{j=i+1}^l \tilde{\Phi}_j\| \\
& \leq \max_{t \in [T]} \|B_t\| \Delta \tilde{K}_{\mathcal{I}} \left(\|\Pi_{j=k+1}^l \tilde{\Phi}_j\| + \|\Pi_{j=k}^{l-1} \Phi_j\| + \sum_{i=k+1}^{l-1} \|\Pi_{j=k}^{i-1} \Phi_j\| \|\Pi_{j=i+1}^l \tilde{\Phi}_j\| \right) \\
& \leq L \Delta \tilde{K}_{\mathcal{I}} \tilde{L}^2 \left(2\tilde{\alpha}^{l-k} + \sum_{i=k+1}^{l-1} \tilde{\alpha}^{i-k} \tilde{\alpha}^{l-i} \right) = \Delta \tilde{K}_{\mathcal{I}} L \tilde{L}^2 (l - k + 1) \tilde{\alpha}^{l-k}.
\end{aligned} \tag{3.48}$$

Since $\|D_{Q_1} S_x\|_F^2 - \|D_{Q_1} \tilde{S}_x\|_F^2$ consists of $\|Q_1 \Phi_{k,l}\|_F^2 - \|Q_1 \tilde{\Phi}_{k,l}\|_F^2$ where $k \leq l$, $k, l = t_1 + 1, \dots, t_2 - 1$, for each term, we have

$$\begin{aligned}
& \left| \|Q_1 \Phi_{k,l}\|_F^2 - \|Q_1 \tilde{\Phi}_{k,l}\|_F^2 \right| = \left| \|Q_1 \Phi_{k,l}\|_F - \|Q_1 \tilde{\Phi}_{k,l}\|_F \right| (\|Q_1 \Phi_{k,l}\|_F + \|Q_1 \tilde{\Phi}_{k,l}\|_F) \\
& \leq n \left| \|Q_1 \Phi_{k,l}\| - \|Q_1 \tilde{\Phi}_{k,l}\| \right| (\|Q_1 \Phi_{k,l}\| + \|Q_1 \tilde{\Phi}_{k,l}\|) \leq nL \|\Phi_{k,l} - \tilde{\Phi}_{k,l}\| (\|\Phi_{k,l}\| + \|\tilde{\Phi}_{k,l}\|) \\
& \leq nL^2 \tilde{L}^2 (l - k + 1) \Delta \tilde{K}_{\mathcal{I}} \tilde{\alpha}^{l-k} \left(\left\| \prod_{i=k}^l \Phi_i \right\| + \left\| \prod_{i=k}^l \tilde{\Phi}_i \right\| \right) \leq \tilde{C}_1 n \Delta \tilde{K}_{\mathcal{I}} (l - k + 1) \tilde{\alpha}^{2l-2k+1}.
\end{aligned}$$

where $\tilde{C}_1 = 2L^2 \tilde{L}^3$, the first inequality holds because of the norm triangle inequality and $\|\Phi\|_F \leq \sqrt{n} \|\Phi\|$ for $\Phi \in \mathbb{R}^{n \times n}$, and the third inequality holds because of (3.48). Therefore,

we obtain

$$\begin{aligned}
\left| \|D_{Q_1} S_x\|_F^2 - \|D_{Q_1} \tilde{S}_x\|_F^2 \right| &\leq \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \left| \|Q_1 \Phi_{t-m,t-1}\|_F^2 - \|Q_1 \tilde{\Phi}_{t-m,t-1}\|_F^2 \right| \\
&\leq \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \tilde{C}_1 n \Delta \tilde{K}_{\mathcal{I}} m \tilde{\alpha}^{2m-1} = \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \frac{\tilde{C}_1}{\tilde{\alpha}} n \Delta \tilde{K}_{\mathcal{I}} m \tilde{\alpha}^{2m} \\
&\leq \frac{|\mathcal{I}| n \Delta \tilde{K}_{\mathcal{I}} \tilde{C}_1}{\tilde{\alpha}} \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m} \leq \frac{2\tilde{C}_1}{\tilde{\alpha}(1-\tilde{\alpha}^2)^3} n (\Delta \tilde{K}_{\mathcal{I}}) |\mathcal{I}|.
\end{aligned}$$

where the last inequality holds due to (3.42). For the term $|E[\bar{x}_0^T C_x^T D_Q C_x \bar{x}_0] - E[\bar{x}_0'^T \tilde{C}_x^T D_Q \tilde{C}_x \bar{x}_0']|$ which consists of $\|Q_1 \Phi_{t_1,k}\|$ terms where $t_1 \leq k \leq t_2 - 1$, we express these terms as

$$\|Q_1 \Phi_{t_1,k}\| \leq L^{1/2} \tilde{L} \tilde{\alpha}^{k-t_1+1} := \tilde{C}_{x_0}^{1/2} \tilde{\alpha}^{k-t_1+1},$$

where $\tilde{C}_{x_0} = L\tilde{L}^2$. Then, we give an upper bound for $|E[\bar{x}_0^T C_x^T D_Q C_x \bar{x}_0] - E[\bar{x}_0'^T \tilde{C}_x^T D_Q \tilde{C}_x \bar{x}_0']|$ as follows

$$\begin{aligned}
|E[\bar{x}_0^T C_x^T D_Q C_x \bar{x}_0] - E[\bar{x}_0'^T \tilde{C}_x^T D_Q \tilde{C}_x \bar{x}_0']| &\leq E[\bar{x}_0^2] \cdot \left(\sum_{m=t_1}^{t_2-1} \|Q_1 \Phi_{t_1,m}\| \right)^2 \\
&\leq n \cdot \bar{C}_{\bar{x}_0} \cdot \tilde{C}_{x_0} \left(\sum_{m=0}^{t_2-t_1-1} \tilde{\alpha}^{m+1} \right)^2 \leq \frac{\tilde{C}_{x_0} \bar{C}_{\bar{x}_0}}{(1-\tilde{\alpha})^2} n.
\end{aligned}$$

Then, let us look at $\sum_{t=t_1+1}^{t_2-1} u_t^T R u_t$ and $\sum_{t=t_1+1}^{t_2-1} \tilde{u}_t^T R \tilde{u}_t$, we obtain

$$\begin{aligned}
E \left[\sum_{t=t_1+1}^{t_2-1} u_t^T R u_t \right] &= E \left[\mathbf{w}_u^T S_u^T D_R S_u \mathbf{w}_u \right] + E \left[\bar{x}_0^T C_u^T D_R C_u \bar{x}_0 \right], \\
E \left[\sum_{t=t_1+1}^{t_2-1} \tilde{u}_t^T R \tilde{u}_t \right] &= E \left[\mathbf{w}_u^T \tilde{S}_u^T D_R \tilde{S}_u \mathbf{w}_u \right] + E \left[\bar{x}_0'^T \tilde{C}_u^T D_R \tilde{C}_u \bar{x}_0' \right].
\end{aligned}$$

where $\tilde{S}_u$ and $\tilde{C}_u$ are defined similarly as S_u and C_u . Since R is symmetric and positive definite, there exists a unitary matrix U_R and a diagonal matrix Λ_R such that

$$R = U_R \Lambda_R U_R^T = U_R (\Lambda_R)^{\frac{1}{2}} \left(U_R (\Lambda_R)^{\frac{1}{2}} \right)^T = R_1^T R_1, \quad R_1 = \left(U_R (\Lambda_R)^{\frac{1}{2}} \right)^T. \quad (3.49)$$

Therefore, we can write D_R as $D_R = D_{R_1}^T D_{R_1}$ where D_{R_1} is the block diagonal matrix setting diagonal matrix as R_1 . By Assumption 3.4.2 and $\mathbf{w}_u \sim N(\mathbf{0}, \psi^2 I_{n \cdot (t_2 - t_1 - 1)})$, we obtain

$$E \left[\mathbf{w}_u^T S_u^T D_R S_u \mathbf{w}_u - \mathbf{w}_u^T \tilde{S}_u^T D_R \tilde{S}_u \mathbf{w}_u \right] = \psi^2 \left(\|D_{R_1} S_u\|_F^2 - \|D_{R_1} \tilde{S}_u\|_F^2 \right).$$

We first give the upper bound for $\|\Psi_{k,l} - \tilde{\Psi}_{k,l}\|$ as follows

$$\begin{aligned} \|\Psi_{k,l} - \tilde{\Psi}_{k,l}\| &= \|K_{l+1} \Pi_{i=k}^l \Phi_i - \tilde{K}_{l+1} \Pi_{i=k}^l \tilde{\Phi}_i\| \\ &= \|(K_{l+1} - \tilde{K}_{l+1}) \Pi_{i=k}^l \Phi_i + \tilde{K}_{l+1} \Delta \Phi_{k,l}\| \\ &\leq \Delta \tilde{K}_{\mathcal{I}} \tilde{L} \tilde{\alpha}^{l-k+1} + \tilde{K}_u \Delta \tilde{K}_{\mathcal{I}} L \tilde{L}^2 (l - k + 1) \tilde{\alpha}^{l-k} \\ &:= 2\tilde{K}_u L \tilde{L}^2 (l - k + 1) \Delta \tilde{K}_{\mathcal{I}} \alpha^{l-k}, \end{aligned} \quad (3.50)$$

Since $\|D_{R_1} S_u\|_F^2 - \|D_{R_1} \tilde{S}_u\|_F^2$ consists of $\|R_1 \Psi_{k,l}\|_F^2 - \|R_1 \tilde{\Psi}_{k,l}\|_F^2$ where $k \leq l+1$, $k, l = t_1, t_1 + 1, \dots, t_2 - 2$, for each term, we have

$$\begin{aligned} \left| \|R_1 \Psi_{k,l}\|_F^2 - \|R_1 \tilde{\Psi}_{k,l}\|_F^2 \right| &\leq n \|R_1 \Psi_{k,l} - R_1 \tilde{\Psi}_{k,l}\| (\|R_1 \Psi_{k,l}\| + \|R_1 \tilde{\Psi}_{k,l}\|) \\ &\leq nL \|\Psi_{k,l} - \tilde{\Psi}_{k,l}\| (\|\Psi_{k,l}\| + \|\tilde{\Psi}_{k,l}\|) \leq 2nL \tilde{K}_u \tilde{L} \tilde{\alpha}^{l-k+1} \|\Psi_{k,l} - \tilde{\Psi}_{k,l}\| \\ &\leq 2n \Delta \tilde{K}_{\mathcal{I}} L^2 \tilde{K}_u^2 \tilde{L}^3 (l - k + 1) \tilde{\alpha}^{2l-2k+1} = \tilde{C}_2 n \Delta \tilde{K}_{\mathcal{I}} (l - k + 1) \tilde{\alpha}^{2l-2k+1}. \end{aligned}$$

where $\tilde{C}_2 = 2L^2\tilde{K}_u^2\tilde{L}^3$; and the first inequality holds because of the norm triangle inequality and $\|\Phi\|_F \leq \sqrt{n}\|\Phi\|$ for $\Phi \in \mathbb{R}^{n \times n}$. Therefore, we obtain

$$\begin{aligned} \left| \|D_{R_1} S_u\|_F^2 - \|D_{R_1} \tilde{S}_u\|_F^2 \right| &\leq \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \left| \|R_1 \Psi_{t-m,t-1}\|_F^2 - \|R_1 \tilde{\Psi}_{t-m,t-1}\|_F^2 \right| \\ &\leq \tilde{C}_2 n \Delta \tilde{K}_{\mathcal{I}} \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m-1} \leq \frac{\tilde{C}_2}{\tilde{\alpha}} n |\mathcal{I}| \Delta \tilde{K}_{\mathcal{I}} \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m} \leq \frac{2\tilde{C}_2}{\tilde{\alpha}(1-\tilde{\alpha}^2)^3} n |\mathcal{I}| \Delta \tilde{K}_{\mathcal{I}}. \end{aligned}$$

where the last inequality holds due to (3.42). Since $|E[\bar{x}_0^T C_u^T D_R C_u \bar{x}_0] - E[\bar{x}_0^T \tilde{C}_u^T D_R \tilde{C}_u \bar{x}_0^T]|$ which consists of $\|R_1 \Psi_{t_1,k}\|$ terms where $t_1 \leq k \leq t_2 - 2$. We express these terms as

$$\|R_1 \Psi_{t_1,k}\| \leq L^{1/2} \tilde{L} \tilde{K}_u \tilde{\alpha}^{k-t_1+1} := \hat{C}_{x_0}^{1/2} \tilde{\alpha}^{k-t_1+1},$$

where $\hat{C}_{x_0} = L \tilde{L}^2 \tilde{K}_u^2$. Then, we give an upper bound for $|E[\bar{x}_0^T C_u^T D_R C_u \bar{x}_0] - E[\bar{x}_0^T \tilde{C}_u^T D_R \tilde{C}_u \bar{x}_0^T]|$ as follows

$$\begin{aligned} |E[\bar{x}_0^T C_u^T D_R C_u \bar{x}_0] - E[\bar{x}_0^T \tilde{C}_u^T D_R \tilde{C}_u \bar{x}_0^T]| &\leq E[\bar{x}_0^2] \cdot \left(\sum_{m=t_1}^{t_2-2} \|R_1 \Psi_{t_1,m}\| \right)^2 \\ &\leq \bar{C}_{\bar{x}_0} \cdot n \cdot \hat{C}_{x_0} \cdot \left(\sum_{m=0}^{t_2-t_1-2} \tilde{\alpha}^{m+1} \right)^2 \leq \frac{\bar{C}_{\bar{x}_0} \hat{C}_{x_0}}{(1-\tilde{\alpha})^2} n. \end{aligned}$$

For $|E[u_0^T R u_0 - \tilde{u}_0^T R \tilde{u}_0]|$ and $|E[x_0^T R x_0 - (x'_0)^T R x'_0]|$, we obtain

$$\begin{aligned} \left| E[u_0^T R u_0 - \tilde{u}_0^T R \tilde{u}_0] \right| &= \left| E[x_0^T K_0^T R K_0 x_0 - x_0'^T \tilde{K}_0^T R \tilde{K}_0 x'_0] \right| \leq \tilde{K}_u^2 L \bar{C}_{\bar{x}_0} \cdot n, \\ \left| E[x_0^T R x_0 - (x'_0)^T R x'_0] \right| &\leq L \bar{C}_{\bar{x}_0} \cdot n. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \left| E[J_{\mathcal{I}}(K) - \tilde{J}_{\mathcal{I}}(\tilde{K})] \right| \leq \frac{1}{2} \left| E \left[\sum_{t=t_1+1}^{t_2-1} u_t^T R u_t - \sum_{t=t_1+1}^{t_2-1} \tilde{u}_t^T R \tilde{u}_t \right] \right| \\
& + \frac{1}{2} |E_{\bar{x}_0}[u_0^T R u_0 - \tilde{u}_0^T R \tilde{u}_0]| + \frac{1}{2} |E[x_0^T R x_0 - (x'_0)^T R x'_0]| \\
& + \frac{1}{2} \left| E \left[\sum_{t=t_1+1}^{t_2} x_t^T Q x_t - \sum_{t=t_1+1}^{t_2} \tilde{x}_t^T Q \tilde{x}_t \right] \right| \\
& \leq \tilde{C}_1(\tilde{\alpha}, \tilde{L}, \tilde{K}_u) n + \tilde{C}_2(\tilde{\alpha}, \tilde{L}, \tilde{K}_u) n (\Delta \tilde{K}_{\mathcal{I}}) |\mathcal{I}|,
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\tilde{C}_{x_0} \bar{C}_{\bar{x}_0} + \bar{C}_{\bar{x}_0} \hat{C}_{x_0}}{(1 - \tilde{\alpha})^2} + \tilde{K}_u^2 L \bar{C}_{\bar{x}_0} + L \bar{C}_{\bar{x}_0} \leq \bar{C}_{\bar{x}_0} \frac{L \tilde{L}^2 + L \tilde{L}^2 \tilde{K}_u^2 + \tilde{K}_u^2 L + L}{(1 - \tilde{\alpha})^2} \\
& \leq \tilde{C}_1(\bar{C}_{\bar{x}_0}, \tilde{\alpha}, \tilde{L}, \tilde{K}_u) := \bar{C}_{\bar{x}_0} \frac{4L \tilde{L}^2 \tilde{K}_u^2}{(1 - \tilde{\alpha})^2}, \\
& \frac{2\tilde{C}_1 \psi^2 + 2\tilde{C}_2 \psi^2}{\tilde{\alpha}(1 - \tilde{\alpha}^2)^3} \leq \frac{4L^2 \tilde{L}^3 \psi^2 + 4L^2 \tilde{K}_u^2 \tilde{L}^3 \psi^2}{\tilde{\alpha}(1 - \tilde{\alpha}^2)^3} \leq \tilde{C}_2(\tilde{\alpha}, \tilde{L}, \tilde{K}_u) := \frac{8L^2 \tilde{K}_u^2 \tilde{L}^3 \psi^2}{\tilde{\alpha}(1 - \tilde{\alpha}^2)^3}.
\end{aligned}$$

If $\Delta K_t \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-t}$, we rewrite (3.48) and (3.50) as

$$\begin{aligned}
& \|\Phi_{k,l} - \tilde{\Phi}_{k,l}\| \leq \Delta \tilde{K}_{\mathcal{I}} L \tilde{L}^2 (l - k + 1) \tilde{\alpha}^{l-k} \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-l} L \tilde{L}^2 (l - k + 1) \tilde{\alpha}^{l-k}, \\
& \left| \|Q_1 \Phi_{k,l}\|_F^2 - \|Q_1 \tilde{\Phi}_{k,l}\|_F^2 \right| \leq \tilde{C}_1 n \Delta \tilde{K}_{\mathcal{I}} (l - k + 1) \tilde{\alpha}^{2l-2k+1} \\
& \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-l} \tilde{C}_1 n (l - k + 1) \tilde{\alpha}^{2l-2k+1}, \\
& \|\Psi_{k,l} - \tilde{\Psi}_{k,l}\| \leq 2\tilde{K}_u L \tilde{L}^2 (l - k + 1) \Delta \tilde{K}_{\mathcal{I}} \alpha^{l-k} \\
& \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-l} 2\tilde{K}_u L \tilde{L}^2 (l - k + 1) \alpha^{l-k}, \\
& \left| \|R_1 \Psi_{k,l}\|_F^2 - \|R_1 \tilde{\Psi}_{k,l}\|_F^2 \right| \leq \tilde{C}_2 n \Delta \tilde{K}_{\mathcal{I}} (l - k + 1) \tilde{\alpha}^{2l-2k+1} \\
& \leq C_{\tilde{\rho}} \tilde{\rho}^{t_2-l} \tilde{C}_2 n (l - k + 1) \tilde{\alpha}^{2l-2k+1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \left| \|D_{Q_1} S_x\|_F^2 - \|D_{Q_1} \tilde{S}_x\|_F^2 \right| \leq \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \left| \|Q_1 \Phi_{t-m,t-1}\|_F^2 - \|Q_1 \tilde{\Phi}_{t-m,t-1}\|_F^2 \right| \\
& \leq \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} C_{\tilde{\rho}} \tilde{\rho}^{t_2-t+1} \tilde{C}_1 n m \tilde{\alpha}^{2m-1} = C_{\tilde{\rho}} \sum_{t=t_1+1}^{t_2} \tilde{\rho}^{t_2-t+1} \sum_{m=0}^{t-t_1-1} \frac{\tilde{C}_1}{\tilde{\alpha}} n m \tilde{\alpha}^{2m} \\
& \leq \frac{n \tilde{C}_1 C_{\tilde{\rho}}}{\tilde{\alpha} (1 - \tilde{\rho})} \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m} \leq \frac{2 \tilde{C}_1 C_{\tilde{\rho}}}{\tilde{\alpha} (1 - \tilde{\alpha}^2)^3 (1 - \tilde{\rho})} n, \\
& \left| \|D_{R_1} S_u\|_F^2 - \|D_{R_1} \tilde{S}_u\|_F^2 \right| \leq \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \left| \|R_1 \Psi_{t-m,t-1}\|_F^2 - \|R_1 \tilde{\Psi}_{t-m,t-1}\|_F^2 \right| \\
& \leq \tilde{C}_2 n \sum_{t=t_1+1}^{t_2-1} C_{\tilde{\rho}} \tilde{\rho}^{t_2-t+1} \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m-1} \leq \frac{\tilde{C}_2 C_{\tilde{\rho}}}{\tilde{\alpha} (1 - \tilde{\rho})} n \sum_{m=0}^{t-t_1-1} m \tilde{\alpha}^{2m} \\
& \leq \frac{2 \tilde{C}_2 C_{\tilde{\rho}}}{\tilde{\alpha} (1 - \tilde{\alpha}^2)^3 (1 - \tilde{\rho})} n.
\end{aligned}$$

We complete the proof by

$$\begin{aligned}
& \left| E[J_{\mathcal{I}}(K) - \tilde{J}_{\mathcal{I}}(\tilde{K})] \right| \leq \frac{1}{2} \left| E \left[\sum_{t=t_1+1}^{t_2-1} u_t^T R u_t - \sum_{t=t_1+1}^{t_2-1} \tilde{u}_t^T R \tilde{u}_t \right] \right| \\
& + \frac{1}{2} |E_{\bar{x}_0} [u_0^T R u_0 - \tilde{u}_0^T R \tilde{u}_0]| + \frac{1}{2} |E[x_0^T R x_0 - (x'_0)^T R x'_0]| \\
& + \frac{1}{2} \left| E \left[\sum_{t=t_1+1}^{t_2} x_t^T Q x_t - \sum_{t=t_1+1}^{t_2} \tilde{x}_t^T Q \tilde{x}_t \right] \right| \leq (\bar{C}_{\bar{x}_0} + 1) \frac{12 L^2 \tilde{L}^3 \tilde{K}_u^2 C_{\tilde{\rho}}}{\tilde{\alpha} (1 - \tilde{\alpha}^2)^3 (1 - \tilde{\rho})} n
\end{aligned}$$

Proof of Lemma 3.5.2 By (3.45), $E[w_i] = \mathbf{0}$ for $i \in [T]$, and w_i and w_j are independent if $i \neq j$, we can expand $E[J_{\mathcal{I}}(K)]$ as

$$\begin{aligned} E[J_{\mathcal{I}}(K) | \bar{x}_0] &= \frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Phi_{t-m,t-1}^T Q \Phi_{t-m,t-1} \right) \\ &\quad + \frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Psi_{t-m,t-1}^T R \Psi_{t-m,t-1} \right) \\ &\quad + \frac{1}{2} \sum_{t=t_1}^{t_2} \bar{x}_0^T \Phi_{t_1,t-1}^T Q \Phi_{t_1,t-1} \bar{x}_0 + \frac{1}{2} \sum_{t=t_1}^{t_2-1} \bar{x}_0^T \Psi_{t_1,t-1}^T R \Psi_{t_1,t-1} \bar{x}_0, \end{aligned} \quad (3.51)$$

For $\text{Tr} \left(\Phi_{t-m,t-1}^T Q \Phi_{t-m,t-1} \right)$, we write it as

$$\text{Tr} \left(\Phi_{t-m,t-1}^T Q \Phi_{t-m,t-1} \right) = \sum_{i=1}^n (\Phi_{t-m,t-1})_i^T Q (\Phi_{t-m,t-1})_i,$$

where $(\Phi_{t-m,t-1})_i$ is a column vector of $\Phi_{t-m,t-1}$. Similarly, we can write $\text{Tr} \left(\Psi_{t-m,t-1}^T R \Psi_{t-m,t-1} \right)$ as a quadratic of $(\Psi_{t-m,t-1})_i$ with $i \in [n]$. We know that $\Phi_{t',t+1} = A\Phi_{t',t} + B\Psi_{t',t}$ which implies that $(\Phi_{t',t+1})_i = A(\Phi_{t',t})_i + B(\Psi_{t',t})_i$. Then, take the expectation of (3.51) with respect to $\bar{x}_0$ and get

$$\begin{aligned} E[J_{\mathcal{I}}(K)] &= \underbrace{\frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Phi_{t-m,t-1}^T Q_1^T Q_1 \Phi_{t-m,t-1} \right) + \frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Psi_{t-m,t-1}^T R_1^T R_1 \Psi_{t-m,t-1} \right)}_{E_{J_{\mathcal{I}}(K),1}} \\ &\quad + \underbrace{\frac{1}{2} E \left[\sum_{t=t_1}^{t_2} \bar{x}_0^T \Phi_{t_1,t-1}^T Q \Phi_{t_1,t-1} \bar{x}_0 \right] + \frac{1}{2} E \left[\sum_{t=t_1}^{t_2-1} \bar{x}_0^T \Psi_{t_1,t-1}^T R \Psi_{t_1,t-1} \bar{x}_0 \right]}_{E_{J_{\mathcal{I}}(K),2}}, \end{aligned} \quad (3.52)$$

where Q_1 is defined in (3.47) and R_1 is defined in (3.49). We can write $E_{J_{\mathcal{I}}(K),1}$ in (3.52) as a quadratic function with respect to $\left(\Phi_{t',t''}\right)_i$ and $\left(\Psi_{t',t''}\right)_i$ where $i = 1, \dots, n$ and $t' \leq t'' + 1$. We can write the optimization problem as

$$\min_{\phi, \varphi} \quad \frac{1}{2} \phi^T Q_{\phi, \Sigma} \phi + \frac{1}{2} \varphi^T R_{\varphi, \Sigma} \varphi, \quad (3.53a)$$

$$\text{s.t.} \quad \phi_{(j,i+1)} = A_i \phi_{(j,i)} + B_i \varphi_{(j,i)} \quad t_1 \leq j \leq i+1, \quad t_1 - 1 \leq i \leq t_2 - 2 \quad (3.53b)$$

$$\phi_{(i+1,i)}^T = (e_1^T, e_2^T, \dots, e_n^T) \quad t_1 - 1 \leq i \leq t_2 - 1. \quad (3.53c)$$

$$\varphi_{(i+1,i)}^T = ((K_{i+1})_1^T, (K_{i+1})_2^T, \dots, (K_{i+1})_n^T) \quad t_1 - 1 \leq i \leq t_2 - 1. \quad (3.53d)$$

where $Q_{\phi, \Sigma}$ is a function parameterized by Q_1 and Σ ; $R_{\varphi, \Sigma}$ is a function parameterized by R_1 and Σ ; e_k is a one-hot vector with one in the k -th entry and zero in the other entries; and $\phi^T = (\phi_{(t_1-1)}^T, \phi_{(t_1)}^T, \dots, \phi_{(t_2-1)}^T)$ with $\phi_{(t_1-1)}^T = (\phi_{(t_1, t_1-1)}^T)$ and $\phi_{(i)}^T = (\phi_{(t_1, i)}^T, \phi_{(t_1+1, i)}^T, \dots, \phi_{(i+1, i)}^T)$ where $\phi_{(j,i)}^T = ((\Phi_{j,i})_1^T, (\Phi_{j,i})_2^T, \dots, (\Phi_{j,i})_n^T)$ with $t_1 \leq j \leq i+1$. We apply a similar definition to φ . We can write constraints (3.53b) - (3.53d) as a linear system of ϕ and φ which is $M_{I-A}\phi + M_{-B}\varphi + m_C = \mathbf{0}$. For the optimal ϕ^* , φ^* and non-optimal $\tilde{\phi}$, $\tilde{\varphi}$, we have

$$\begin{aligned} \tilde{\phi} &= \phi^* + \Delta\phi, \quad \tilde{\varphi} = \varphi^* + \Delta\varphi, \\ \mathbf{0} &= Q_{\phi, \Sigma} \phi^* + M_{I-A}^T \lambda, \quad \mathbf{0} = R_{\varphi, \Sigma} \varphi^* + M_{-B}^T \lambda. \end{aligned} \quad (3.54)$$

where the last line holds because of the KKT condition. Since $M_{I-A}\tilde{\phi} + M_{-B}\tilde{\varphi} + m_C = \mathbf{0}$, $M_{I-A}\Delta\phi + M_{-B}\Delta\varphi = \mathbf{0}$. Combined with (3.54), we have $(\Delta\phi)^T Q_{\phi} \phi^* + (\Delta\varphi)^T R_{\varphi} \varphi^* = \mathbf{0}$ and

$$\frac{1}{2} \tilde{\phi}^T Q_{\phi, \Sigma} \tilde{\phi} + \frac{1}{2} \tilde{\varphi}^T R_{\varphi, \Sigma} \tilde{\varphi} - \left(\frac{1}{2} \phi^T Q_{\phi} \phi + \frac{1}{2} \varphi^T R_{\varphi} \varphi \right) = \frac{1}{2} \Delta\phi^T Q_{\phi} \Delta\phi + \frac{1}{2} \Delta\varphi^T R_{\varphi} \Delta\varphi,$$

Therefore, we have

$$\begin{aligned}
& |E[J_{\mathcal{I}}(K) - J_{\mathcal{I}}^*(K^*)]| \\
& \leq \frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Phi_{t-m,t-1}^T Q_1^T Q_1 \Delta \Phi_{t-m,t-1} \right) \\
& \quad + \frac{\psi^2}{2} \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Psi_{t-m,t-1}^T R_1^T R_1 \Delta \Psi_{t-m,t-1} \right) \\
& \quad + \frac{1}{2} \left| E \left[\sum_{t=t_1}^{t_2} \bar{x}_0^T \Phi_{t_1,t-1}^T Q \Phi_{t_1,t-1} \bar{x}_0 \right] - E \left[\sum_{t=t_1}^{t_2} \bar{x}_0^{*T} \Phi_{t_1,t-1}^{*T} Q \Phi_{t_1,t-1}^* \bar{x}_0^* \right] \right| \\
& \quad + \frac{1}{2} \left| E \left[\sum_{t=t_1}^{t_2-1} \bar{x}_0^T \Psi_{t_1,t-1}^T R \Psi_{t_1,t-1} \bar{x}_0 \right] - E \left[\sum_{t=t_1}^{t_2-1} \bar{x}_0^{*T} \Psi_{t_1,t-1}^{*T} R \Psi_{t_1,t-1}^* \bar{x}_0^* \right] \right|,
\end{aligned}$$

First, we want to give the upper bound for $\|\Delta \Phi_{k,l}\|$ and $\|\Delta \Psi_{k,l}\|$. Similar to (3.48) and (3.50), we have

$$\|\Delta \Phi_{k,l}\| \leq C_1(l-k+1)\Delta K_{\mathcal{I}}\alpha_1^{l-k}, \quad \|\Delta \Psi_{k,l}\| \leq C_2(l-k+1)\Delta K_{\mathcal{I}}\alpha_1^{l-k},$$

where $C_1 = LL_1^2$ and $C_2 = 2\tilde{K}_u LL_1^2$. Combined with $\|Q_1\| \leq L^{1/2}$ and (3.42), we have

$$\begin{aligned}
& \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Phi_{t-m,t-1}^T Q_1^T Q_1 \Delta \Phi_{t-m,t-1} \right) = \sum_{m=0}^{t-t_1-1} \|\Delta \Phi_{t-m,t-1}\|_F^2 \\
& \leq n \sum_{m=0}^{t-t_1-1} \|\Delta \Phi_{t-m,t-1}\|^2 \leq nLC_1^2 (\Delta K_{\mathcal{I}})^2 \sum_{m=0}^{t-t_1-1} m^2 \alpha_1^{2(m-1)} \\
& = nLC_1^2 (\Delta K_{\mathcal{I}})^2 \sum_{p=0}^{t-t_1-2} (p+1)^2 \alpha_1^{2p},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{t=t_1+1}^{t_2} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Phi_{t-m,t-1}^T Q_1^T Q_1 \Delta \Phi_{t-m,t-1} \right) \\
& \leq n L C_1^2 (\Delta K_{\mathcal{I}})^2 \sum_{t=t_1+1}^{t_2} \sum_{p=0}^{t-t_2-2} (p+1)^2 \alpha_1^{2p} \\
& \leq n L C_1^2 (\Delta K_{\mathcal{I}})^2 |\mathcal{I}| (U(|\mathcal{I}|) + 2V(|\mathcal{I}|) + W(|\mathcal{I}|)) \\
& \leq n (\Delta K_{\mathcal{I}})^2 |\mathcal{I}| \frac{8 L C_1^2}{(1 - \alpha_1^2)^3}.
\end{aligned}$$

Similarly, since $\|R_1\| \leq L^{1/2}$, we have

$$\begin{aligned}
& \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Psi_{t-m,t-1}^T R_1^T R_1 \Delta \Psi_{t-m,t-1} \right) = \sum_{m=0}^{t-t_1-1} \|R_1 \Delta \Psi_{t-m,t-1}\|_F^2 \\
& \leq n \sum_{m=0}^{t-t_1-1} \|R_1 \Delta \Psi_{t-m,t-1}\|^2 \leq n L C_2^2 (\Delta K_{\mathcal{I}})^2 \sum_{m=0}^{t-t_1-1} m^2 \alpha_1^{2(m-1)} \\
& = n L C_2^2 (\Delta K_{\mathcal{I}})^2 \sum_{p=0}^{t-t_1-2} (p+1)^2 \alpha_1^{2p},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{t=t_1+1}^{t_2-1} \sum_{m=0}^{t-t_1-1} \text{Tr} \left(\Delta \Psi_{t-m,t-1}^T R_1^T R_1 \Delta \Psi_{t-m,t-1} \right) \\
& \leq n L C_2^2 (\Delta K_{\mathcal{I}})^2 \sum_{t=t_1+1}^{t_2-1} \sum_{p=0}^{t-t_1-2} (p+1)^2 \alpha_1^{2p} \\
& \leq n L C_2^2 (\Delta K_{\mathcal{I}})^2 |\mathcal{I}| (U(|\mathcal{I}|) + 2V(|\mathcal{I}|) + W(|\mathcal{I}|)) \\
& \leq n (\Delta K_{\mathcal{I}})^2 |\mathcal{I}| \frac{8 L C_2^2}{(1 - \alpha_1^2)^3}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \frac{1}{2} \left| E \left[\sum_{t=t_1}^{t_2} \bar{x}_0^T \Phi_{t_1,t-1}^T Q \Phi_{t_1,t-1} \bar{x}_0 \right] - E \left[\sum_{t=t_1}^{t_2} \bar{x}_0^{*T} \Phi_{t_1,t-1}^{*T} Q \Phi_{t_1,t-1}^* \bar{x}_0^* \right] \right| \\
& \leq \frac{n \bar{C}_{\bar{x}_0}}{2} \sum_{t=t_1}^{t_2} L L_1^2 \alpha_1^{2(t-t_1)} \leq n \frac{\bar{C}_{\bar{x}_0} L L_1^2}{1 - \alpha_1^2}, \\
& \frac{1}{2} \left| E \left[\sum_{t=t_1}^{t_2-1} \bar{x}_0^T \Psi_{t_1,t-1}^T R \Psi_{t_1,t-1} \bar{x}_0 \right] - E \left[\sum_{t=t_1}^{t_2-1} \bar{x}_0^{*T} \Psi_{t_1,t-1}^{*T} R \Psi_{t_1,t-1}^* \bar{x}_0^* \right] \right| \\
& \leq \frac{n \bar{C}_{\bar{x}_0}}{2} \sum_{t=t_1}^{t_2} L L_1^2 \tilde{K}_u^2 \alpha_1^{2(t-t_1)} \leq n \frac{\bar{C}_{\bar{x}_0} L L_1^2 \tilde{K}_u^2}{1 - \alpha_1^2}.
\end{aligned}$$

Therefore, we complete the proof by

$$\begin{aligned}
& \left| E [J_{\mathcal{I}}(K) - J_{\mathcal{I}}^*(K^*) | \bar{x}_0] \right| \leq n \frac{2 \bar{C}_{\bar{x}_0} L L_1^2 \tilde{K}_u^2}{1 - \alpha_1^2} + \frac{4 \psi^2 L}{(1 - \alpha_1^2)^3} (C_1^2 + C_2^2) |\mathcal{I}| n (\Delta K_{\mathcal{I}})^2 \\
& \leq n \frac{2 \bar{C}_{\bar{x}_0} L L_1^2 \tilde{K}_u^2}{1 - \alpha_1^2} + \frac{4 \psi^2 L}{(1 - \alpha_1^2)^3} \cdot 5 \left(\tilde{K}_u L L_1^2 \right)^2 |\mathcal{I}| n (\Delta K_{\mathcal{I}})^2 \\
& \leq n \frac{2 \bar{C}_{\bar{x}_0} L L_1^2 \tilde{K}_u^2}{1 - \alpha_1^2} + \frac{20 \psi^2 L^3 L_1^4 \tilde{K}_u^2}{(1 - \alpha_1^2)^3} \cdot |\mathcal{I}| n (\Delta K_{\mathcal{I}})^2
\end{aligned} \tag{3.55}$$

Proof of Theorem 3.5.1 Recall from (3.23), we have

$$x_{\mathcal{I}} = S_{\mathcal{I},x} w_{\mathcal{I}} + \nu S_{\mathcal{I},B,x} \eta_{\mathcal{I}} + C_{\mathcal{I},x} \bar{x}_0^e, \quad u_{\mathcal{I}} = S_{\mathcal{I},u} w_{\mathcal{I}} + \nu S_{\mathcal{I},B,u} \eta_{\mathcal{I}} + C_{\mathcal{I},u} \bar{x}_0^e.$$

Let us look at $\sum_{t=t_1+1}^{t_2} x_t^T Q x_t$ first. Since $E[\mathbf{w}_x] = \mathbf{0}$, we obtain

$$\begin{aligned}
& E \left[\sum_{t=t_1}^{t_2} x_t^T Q x_t \middle| \bar{x}_0^e \right] = E \left[x_{\mathcal{I}}^T D_Q x_{\mathcal{I}}^T \middle| \bar{x}_0^e \right] \\
& = E \left[(S_{\mathcal{I},x} w_{\mathcal{I}} + \nu S_{\mathcal{I},B,x} \eta_{\mathcal{I}} + C_{\mathcal{I},x} \bar{x}_0^e)^T D_Q (S_{\mathcal{I},x} w_{\mathcal{I}} + \nu S_{\mathcal{I},B,x} \eta_{\mathcal{I}} + C_{\mathcal{I},x} \bar{x}_0^e) \middle| \bar{x}_0^e \right], \\
& = E \left[w_{\mathcal{I}}^T S_{\mathcal{I},x}^T D_Q S_{\mathcal{I},x} w_{\mathcal{I}} \right] + \nu^2 E \left[\eta_{\mathcal{I}}^T S_{\mathcal{I},B,x}^T D_Q S_{\mathcal{I},B,x} \eta_{\mathcal{I}} \right] + (\bar{x}_0^e)^T C_{\mathcal{I},x}^T D_Q C_{\mathcal{I},x} \bar{x}_0^e,
\end{aligned}$$

and

$$\begin{aligned}
E \left[\sum_{t=t_1}^{t_2} x_t^T Q x_t \right] &= E \left[w_{\mathcal{I}}^T S_{\mathcal{I},x}^T D_Q S_{\mathcal{I},x} w_{\mathcal{I}} \right] + \nu^2 E \left[\eta_{\mathcal{I}}^T S_{\mathcal{I},B,x}^T D_Q S_{\mathcal{I},B,x} \eta_{\mathcal{I}} \right] \\
&+ E \left[(\bar{x}_0^e)^T C_{\mathcal{I},x}^T D_Q C_{\mathcal{I},x} \bar{x}_0^e \right].
\end{aligned}$$

By Assumption 3.4.2 and $\eta_t \stackrel{i.i.d.}{\sim} \mathcal{N}(\mathbf{0}, I_d)$, we obtain

$$\begin{aligned}
E[\eta_{\mathcal{I}}^T S_{\mathcal{I},B,x}^T D_Q S_{\mathcal{I},B,x} \eta_{\mathcal{I}}] &= \text{Tr} \left((D_{Q_1} S_{\mathcal{I},B,x})^T (D_{Q_1} S_{\mathcal{I},B,x}) \right) = \|D_{Q_1} S_{\mathcal{I},B,x}\|_F^2, \\
\|D_{Q_1} S_{\mathcal{I},B,x}\|_F^2 &= \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2} \|Q_1 \tilde{\Phi}_{t,m-1} B_{t-1}\|_F^2 \\
&\leq n \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2} \|Q_1 \tilde{\Phi}_{t,m-1} B_{t-1}\|^2 \\
&\leq n \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2} L_2^2 L^3 \alpha_2^{2(m-t)} \leq n|\mathcal{I}| \frac{L_2^2 L^3}{1 - \alpha_2^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
E \left[\sum_{t=t_1}^{t_2-1} u_t^T R u_t \right] &= E \left[w_{\mathcal{I}}^T S_{\mathcal{I},u}^T D_R S_{\mathcal{I},u} w_{\mathcal{I}} \right] \\
&+ \nu^2 E \left[\eta_{\mathcal{I}}^T S_{\mathcal{I},B,u}^T D_R S_{\mathcal{I},B,u} \eta_{\mathcal{I}} \right] + E \left[(\bar{x}_0^e)^T C_{\mathcal{I},u}^T D_R C_{\mathcal{I},u} \bar{x}_0^e \right], \\
E[\eta_{\mathcal{I}}^T S_{\mathcal{I},B,u}^T D_R S_{\mathcal{I},B,u} \eta_{\mathcal{I}}] &= \text{Tr} \left((D_{R_1} S_{\mathcal{I},B,u})^T (D_{R_1} S_{\mathcal{I},B,u}) \right) = \|D_{R_1} S_{\mathcal{I},B,u}\|_F^2, \\
\|D_{R_1} S_{\mathcal{I},B,u}\|_F^2 &= \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2-1} \|R_1 \tilde{\Psi}_{t,m-1} B_{t-1}\|_F^2 + \sum_{m=t_1+1}^{t_2} \|I\|_F^2 \\
&\leq n \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2} \|R_1 \tilde{\Psi}_{t,m-1} B_{t-1}\|^2 + n|\mathcal{I}| \leq n \sum_{t=t_1+1}^{t_2} \sum_{m=t}^{t_2} L_2^2 L^3 \tilde{K}_u \alpha_2^{2(m-t)} + n|\mathcal{I}| \\
&\leq n|\mathcal{I}| \frac{2L_2^2 L^3 \tilde{K}_u}{1 - \alpha_2^2}.
\end{aligned}$$

Using Lemma 3.5.2, we complete the proof by

$$\begin{aligned}
& \left| E \left[J_{\mathcal{I}}^e(K, \nu) - J_{\mathcal{I}}^*(K^*) \right] \right| \\
& \leq n|\mathcal{I}|\nu^2 \frac{2L_2^2 L^3 \tilde{K}_u}{1 - \alpha_2^2} + n|\mathcal{I}|\nu^2 \frac{L_2^2 L^3}{1 - \alpha_2^2} + n\bar{C}_{\bar{x}_0} \frac{2LL_2^2 \tilde{K}_u^2}{1 - \alpha_2^2} + \frac{20\psi^2 L^3 L_2^4 \tilde{K}_u^2}{(1 - \alpha_2^2)^3} |\mathcal{I}|n\epsilon^2 \\
& \leq \nu^2 \frac{5L^3 L_2^2 \tilde{K}_u^2 (\bar{C}_{\bar{x}_0} + 1)}{1 - \alpha_2^2} |\mathcal{I}|n + \frac{20\psi^2 L^3 L_2^4 \tilde{K}_u^2}{(1 - \alpha_2^2)^3} |\mathcal{I}|n\epsilon^2.
\end{aligned} \tag{3.56}$$

Proof of Theorem 3.5.2 Before the formal proof starts, we first formally define $J_{\mathcal{I}}^*$, $J_{\mathcal{I}}^{\mathcal{I},*}(K_{\mathcal{I}}^{\mathcal{I},*})$, $\hat{J}_{\mathcal{I}}(\hat{K}_{\mathcal{I}})$, and $\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\hat{K}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}))$ as follows. By (3.11) and denoting $\mathcal{I} = [t_1, t_2]$, we can write $J_{\mathcal{I}}^{\mathcal{I},*} = J_{\mathcal{I}}(K_{\mathcal{I}}^{\mathcal{I},*})$ where we obtain $\{K_t^{\mathcal{I},*}\}_{t \in \mathcal{I}}$ by solving the following optimization problem,

$$\begin{aligned}
& \min_{\substack{\{x_{\tau}\}_{\tau \in [t, t_2+1]} \\ \{u_{\tau}\}_{\tau \in [t, t_2]}}} \frac{1}{2} x_{t_2}^T Q x_{t_2} + \sum_{\tau=t}^{t_2-1} \frac{1}{2} x_{\tau}^T Q x_{\tau} + \frac{1}{2} u_{\tau}^T R u_{\tau}, \\
& \text{s.t. } x_t = \text{observed}, \quad x_{\tau+1} = A_{\tau} x_{\tau} + B_{\tau} u_{\tau}, \quad \tau \in [t, t_2].
\end{aligned} \tag{3.57}$$

Similarly, for $\hat{J}_{\mathcal{I}}(\hat{K}_{\mathcal{I}}(\tilde{\mathcal{I}}))$, we obtain $\{\hat{K}_t(\tilde{\mathcal{I}})\}_{t \in [t_1, t_2]}$ by solving the following optimization problem,

$$\begin{aligned}
& \min_{\substack{\{x_{\tau}\}_{\tau \in [t, T+1]} \\ \{u_{\tau}\}_{\tau \in [t, T]}}} \frac{1}{2} x_T^T Q x_T + \sum_{\tau=t}^{T-1} \frac{1}{2} x_{\tau}^T Q x_{\tau} + \frac{1}{2} u_{\tau}^T R u_{\tau}, \\
& \text{s.t. } x_t = \text{observed}, \quad x_{\tau+1} = \hat{A}_{\tilde{\mathcal{I}}} x_{\tau} + \hat{B}_{\tilde{\mathcal{I}}} u_{\tau}, \quad \tau \in [t, T].
\end{aligned} \tag{3.58}$$

where we estimate A and B as $\hat{A}_{\tilde{\mathcal{I}}}$ and $\hat{B}_{\tilde{\mathcal{I}}}$ from the interval $\tilde{\mathcal{I}}$.

Similarly, for $\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\hat{K}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}))$, we obtain $\{\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}})\}_{t \in [t_1, t_2]}$ by solving the following optimiza-

tion problem,

$$\begin{aligned}
& \min_{\substack{\{x_\tau\}_{\tau \in [t, t_2+1]} \\ \{u_\tau\}_{\tau \in [t, t_2]}}} \frac{1}{2} x_{t_2}^T Q x_{t_2} + \sum_{\tau=t}^{t_2-1} \frac{1}{2} x_\tau^T Q x_\tau + \frac{1}{2} u_\tau^T R u_\tau, \\
& \text{s.t. } x_t = \text{observed}, \quad x_{\tau+1} = \hat{A}_{\tilde{\mathcal{I}}} x_\tau + \hat{B}_{\tilde{\mathcal{I}}} u_\tau, \quad \tau \in [t, t_2].
\end{aligned} \tag{3.59}$$

where we estimate A and B as $\hat{A}_{\tilde{\mathcal{I}}}$ and $\hat{B}_{\tilde{\mathcal{I}}}$ from the interval $\tilde{\mathcal{I}}$.

We write $|J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W|$ as

$$\begin{aligned}
& |J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| \\
& = |J_{\mathcal{I}}^* - J_{\mathcal{I}}^{\mathcal{I},*} + J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) + \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) + \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| \\
& \leq |J_{\mathcal{I}}^* - J_{\mathcal{I}}^{\mathcal{I},*}| + |J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})| + |\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}})| + |\hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})|,
\end{aligned}$$

We will analyze the above four terms respectively. First, we analyze $|J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})|$. By Lemma 3.5.1, we only need to obtain an upper bound for $\|K_t^{\mathcal{I},*} - \hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}})\|$. We denote $H_{t,\mathcal{I}}$ as the KKT matrix of eq. (3.57) and $\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})$ as the KKT matrix of eq. (3.59). Since $K_t^{\mathcal{I},*}$ is a submatrix of $H_{t,\mathcal{I}}^{-1}$, $H_{t,\mathcal{I}}$ is a submatrix of H , and $\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}})$ is a submatrix of $\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})^{-1}$, we obtain

$$\begin{aligned}
& \|K_t^{\mathcal{I},*} - \hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}})\| \\
& = \|(H_{t,\mathcal{I}})_{(0,0)}^{-1} - (\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}}))_{(0,0)}^{-1}\| = \|(H_{t,\mathcal{I}}^{-1} - \hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})^{-1})_{(0,0)}\| \\
& \leq \|H_{t,\mathcal{I}}^{-1} - \hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})^{-1}\| = \|H_{t,\mathcal{I}}^{-1} - (H_{t,\mathcal{I}} + \Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}}))^{-1}\| \\
& = \|(H_{t,\mathcal{I}} + \Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}}))^{-1} ((H_{t,\mathcal{I}} + \Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}})) H_{t,\mathcal{I}}^{-1} - I)\| \\
& \leq \|\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})^{-1}\| \|\Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}})\| \|H_{t,\mathcal{I}}^{-1}\|.
\end{aligned}$$

Since $H_{t,\mathcal{I}}$ has a similar structure to $H_{\mathcal{I}}$, and results in Lemma 3.8.10 are independent of the time interval $\mathcal{I}$, we can apply Lemma 3.8.10 to $H_{t,\mathcal{I}}$. Similarly, we can apply Lemma 3.8.11

to $\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})$. Therefore, we have upper bounds for $\|\hat{H}_{t,\mathcal{I}}(\tilde{\mathcal{I}})^{-1}\|$ and $\|H_{t,\mathcal{I}}^{-1}\|$. Then we only need to give an upper bound for $\|\Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}})\|$ where

$$\Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}}) = \begin{bmatrix} \mathbf{0} & \Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}})^T \\ \Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}}) & \mathbf{0} \end{bmatrix},$$

$$\Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}}) = \begin{bmatrix} \mathbf{0} & & & & & & & \\ -\Delta A_t(\tilde{\mathcal{I}}) & -\Delta B_t(\tilde{\mathcal{I}}) & & \mathbf{0} & & & & \\ & & -\Delta A_{t+1}(\tilde{\mathcal{I}}) & -\Delta B_{t+1}(\tilde{\mathcal{I}}) & \mathbf{0} & & & \\ & & & \ddots & & & & \end{bmatrix}.$$

where $\Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}}) \in \mathbb{R}^{((t_2-t+1)n) \times ((n+d)(t_2-t)+n)}$, $\Delta A_\tau(\tilde{\mathcal{I}}) = A_\tau - \hat{A}_{\tilde{\mathcal{I}}}$ and $\Delta B_\tau = B_\tau - \hat{B}_{\tilde{\mathcal{I}}}$ with $\tau \in [t, t_2]$. Since $\|\Delta H_{t,\mathcal{I}}(\tilde{\mathcal{I}})\| \leq \|\Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}})\|$ and $\|\Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}})\| = \max_{\|v_t\|=1} \|\Delta F_{t,\mathcal{I}}(\tilde{\mathcal{I}})v_t\|$ where

$$v_t^T = \begin{bmatrix} v_{t,1}^T & v_{t,2}^T & v_{t+1,1}^T & v_{t+1,2}^T & \cdots & v_{t_2-1,1}^T & v_{t_2-1,2}^T & v_{t_2,1}^T \end{bmatrix},$$

$$\Delta F_{t,|\mathcal{I}|}v_t = - \begin{bmatrix} \mathbf{0} \\ \Delta A_t(\tilde{\mathcal{I}})v_{t,1} + \Delta B_t(\tilde{\mathcal{I}})v_{t,2} \\ \vdots \\ \Delta A_{t_2-1}(\tilde{\mathcal{I}})v_{t_2-1,1} + \Delta B_{t_2-1}(\tilde{\mathcal{I}})v_{t_2-1,2} \end{bmatrix},$$

$$\|\Delta F_t v_t\| = \sqrt{\sum_{i=t}^{t_2-1} \left\| \begin{bmatrix} \Delta A_i(\tilde{\mathcal{I}}) \\ \Delta B_i(\tilde{\mathcal{I}}) \end{bmatrix} \begin{bmatrix} v_{i,1} \\ v_{i,2} \end{bmatrix} \right\|^2}$$

$$\leq \sqrt{\max_{\tau \in \mathcal{I}} \left(\|\Delta A_\tau(\tilde{\mathcal{I}})\|^2, \|\Delta B_\tau(\tilde{\mathcal{I}})\|^2 \right) \left(\sum_{i=t}^{t_2-1} \|v_{i,1}\|^2 + \|v_{i,2}\|^2 \right)}$$

$$\leq \max_{\tau \in \mathcal{I}} \left(\|\Delta A_\tau(\tilde{\mathcal{I}})\|, \|\Delta B_\tau(\tilde{\mathcal{I}})\| \right).$$

Therefore, we have

$$\|K_t^{|\mathcal{I}|,*} - \hat{K}_t^{|\mathcal{I}|}(\tilde{\mathcal{I}})\| \leq \gamma_H^{-1} \cdot \hat{\gamma}_H^{-1} \cdot \max_{\tau \in \mathcal{I}} \left(\|\Delta A_\tau(\tilde{\mathcal{I}})\|, \|\Delta B_\tau(\tilde{\mathcal{I}})\| \right) \leq \gamma_H^{-1} \cdot \hat{\gamma}_H^{-1} \cdot \epsilon. \quad (3.60)$$

Now, let us derive the upper bound for $\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\|$. We know that $\hat{K}_t(\tilde{\mathcal{I}})$ is a submatrix of $\hat{H}_t(\tilde{\mathcal{I}})^{-1}$, and $\hat{K}_t^W(\tilde{\mathcal{I}})$ is a submatrix of $\hat{H}_{t,W}(\tilde{\mathcal{I}})^{-1}$ where $\hat{H}_t(\tilde{\mathcal{I}})$ is the KKT matrix of eq. (3.58) and $\hat{H}_{t,W}(\tilde{\mathcal{I}})$ as the *KKT* matrices of eq. (3.4). We permute $\hat{H}_t(\tilde{\mathcal{I}})$ as $\hat{H}_t^P(\tilde{\mathcal{I}})$ with primal-dual variables ordered as $\begin{bmatrix} x_t^T & u_t^T & \lambda_t^T & \cdots & x_{T-1}^T & u_{T-1}^T & \lambda_{T-1}^T & x_T^T & \lambda_T^T \end{bmatrix}$ and we permute $\hat{H}_{t,W}$ as $\hat{H}_{t,W}^P$ in the similar way. We note that the permutation will not change the exponential decay property of the matrix. Therefore, $\hat{H}_t^P$ will have a similar exponential decay structure as $\hat{H}_t$. Now, we write

$$\hat{H}_t^P(\tilde{\mathcal{I}}) = \begin{bmatrix} \hat{H}_{t,W}^P(\tilde{\mathcal{I}}) & \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \\ \hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) & \hat{H}_{t,-W}^P(\tilde{\mathcal{I}}) \end{bmatrix},$$

$$\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)^{-1} = \begin{bmatrix} \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_W^{-1} & \left(\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1} \right)^T \\ \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1} & \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{-W}^{-1} \end{bmatrix}.$$

Since $\hat{H}_t^P(\tilde{\mathcal{I}}) \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)^{-1} = I$, we obtain

$$\begin{aligned}
I &= \hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_W^{-1} + \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1}, \\
\left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)^{-1} - \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_W^{-1} &= \left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)^{-1} \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1}, \\
\left(\left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)^{-1} - \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_W^{-1} \right)_{[0,0]} \\
&= \left(\left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)^{-1} \right)_{[0,:]} \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1} \right)_{[:,0]}, \\
&\quad \left\| \left(\left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)^{-1} - \left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_W^{-1} \right)_{[0,0]} \right\| \\
&\leq \left\| \left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)_{[0,:]}^{-1} \right\| \left\| \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1} \right)_{[:,0]} \right\|.
\end{aligned}$$

By Lemma 3.8.11, we know $\left\| \left(\hat{H}_{t,W}^P(\tilde{\mathcal{I}}) \right)_{[0,:]}^{-1} \right\| \leq \hat{\gamma}_H^{-1}$. The structure of $\left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T$ is

$$\left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{A}_{\tilde{\mathcal{I}}}^T & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{B}_{\tilde{\mathcal{I}}}^T & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \quad \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)_{[W,0]}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\hat{A}_{\tilde{\mathcal{I}}}^T \\ \mathbf{0} & \mathbf{0} & -\hat{B}_{\tilde{\mathcal{I}}}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

By Lemma 3.8.11, we have

$$\begin{aligned}
&\left\| \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)_{W'}^{-1} \right)_{[:,0]} \right\| = \left\| \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)_{[W,0]}^T \left(\left(\hat{H}_t^P(\tilde{\mathcal{I}}) \right)^{-1} \right)_{[0,W+1]} \right\| \\
&\leq \left\| \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)_{[W,0]} \right\| \hat{c}_1 \hat{\rho}^{W+1} \leq \hat{L}_H \hat{c}_1 \hat{\rho}^{W+1}.
\end{aligned}$$

where the last inequality holds because of $\left\| \left(\hat{H}_{t,W'}^P(\tilde{\mathcal{I}}) \right)_{[W,0]} \right\| \leq \|\hat{H}_t^P(\tilde{\mathcal{I}})\| \leq \hat{L}_H$ by Lemma 3.8.11. Therefore, we have

$$\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\| \leq \left\| \left(\left(\hat{H}_t^P \right)^{-1} - \left(\hat{H}_W^P \right)^{-1} \right)_{[0,0]} \right\| \leq \frac{\hat{c}_1 \hat{\rho} \hat{L}_H}{\hat{\gamma}_H} \hat{\rho}^W. \quad (3.61)$$

For $\|\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{K}_t(\tilde{\mathcal{I}})\|$ and $\|K_t^{\mathcal{I},*} - K_t^*\|$, we can rewrite $\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}}) = \hat{K}_t^{\bar{W}_t}(\tilde{\mathcal{I}})$ and $K_t^{\mathcal{I},*} = K_t^{\bar{W}_t,*}$, consider them as the truncated MPC approach with the truncated horizon length $\bar{W}_t = t_2 - t + 1$, and follow the similar proof of $\|\hat{K}_t(\tilde{\mathcal{I}}) - \hat{K}_t^W(\tilde{\mathcal{I}})\|$. Therefore, we have

$$\|\hat{K}_t^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{K}_t(\tilde{\mathcal{I}})\| \leq \frac{\hat{c}_1 \hat{L}_H}{\hat{\gamma}_H} \hat{\rho}^{t_2-t}, \quad \|K_t^{\mathcal{I},*} - K_t^*\| \leq \frac{c_1 L_H}{\gamma_H} \rho^{t_2-t}. \quad (3.62)$$

Combining (3.60), (3.61), (3.62), Lemmas 3.8.7 and 3.8.9, and Lemmas 3.5.1 and 3.5.2, we have

$$\begin{aligned} c_1^* &= \max(2L, c_1, \hat{c}_1), \quad \rho_1^* = \max(\alpha^{1/2}, \rho, \hat{\rho}^{1/4}), \quad C_{\rho_1}^* = \max\left(\frac{c_1 L_H}{\gamma_H}, \frac{\hat{c}_1 \hat{L}_H}{\hat{\gamma}_H}\right), \\ |J_{\mathcal{I}}^* - J_{\mathcal{I}}^{\mathcal{I},*}| &\leq \frac{(2\bar{x}^2 + 1)12L^2(c_1^*)^3 K_u^2 C_{\rho_1}^*}{\rho_1^*(1 - (\rho_1^*)^2)^3(1 - \rho_1^*)} n, \\ |\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}})| &\leq \frac{(2\bar{x}^2 + 1)12L^2(c_1^*)^3 K_u^2 C_{\rho_1}^*}{\rho_1^*(1 - (\rho_1^*)^2)^3(1 - \rho_1^*)} n, \\ |J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})| &\leq \frac{4\bar{x}^2 L(c_1^*)^2 K_u^2}{1 - (\rho_1^*)^2} n + \frac{20\psi^2 L^3(c_1^*)^4 K_u^2}{(1 - (\rho_1^*)^2)^3 \gamma_H^2 \hat{\gamma}_H^2} |\mathcal{I}| n \epsilon^2, \\ |\hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| &\leq \frac{8L(c_1^*)^2 K_u^2 \bar{x}^2}{(1 - \rho_1^*)^2} n + \frac{8L^2 K_u^2 (c_1^*)^3 \psi^2 C_{\rho_1}^*}{(1 - (\rho_1^*)^2)^3} n |\mathcal{I}| \hat{\rho}^W. \end{aligned}$$

We complete the proof by

$$\begin{aligned}
& |J_{\mathcal{I}}^* - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| \\
& \leq |J_{\mathcal{I}}^* - J_{\mathcal{I}}^{\mathcal{I},*}| + |J_{\mathcal{I}}^{\mathcal{I},*} - \hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}})| + |\hat{J}_{\mathcal{I}}^{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}})| + |\hat{J}_{\mathcal{I}}(\tilde{\mathcal{I}}) - \hat{J}_{\mathcal{I}}^W(\tilde{\mathcal{I}})| \\
& \leq \tilde{a} \cdot n + \left(\hat{a}\epsilon^2 + \hat{b}\hat{\rho}^W \right) \cdot |\mathcal{I}| \cdot n, \\
& \quad \frac{20\psi^2 L^3 (c_1^*)^4 K_u^2}{(1 - (\rho_1^*)^2)^3 \gamma_H^2 \hat{\gamma}_H^2} \leq \hat{a} := \frac{20\psi^2 L^3 (c_1^*)^4 K_u^2}{(1 - \rho_1^*)^3 \gamma_H^2 \hat{\gamma}_H^2}, \\
& \quad \frac{8L^2 K_u^2 (c_1^*)^3 \psi^2 C_{\rho_1}^*}{(1 - (\rho_1^*)^2)^3} \leq \hat{b} := \frac{8L^2 K_u^2 (c_1^*)^3 \psi^2 C_{\rho_1}^*}{(1 - \rho_1^*)^3}, \\
& \quad \frac{2(2\bar{x}^2 + 1)12L^2 (c_1^*)^3 K_u^2 C_{\rho_1}^*}{\rho_1^* (1 - (\rho_1^*)^2)^3 (1 - \rho_1^*)} + \frac{4\bar{x}^2 L (c_1^*)^2 K_u^2}{1 - (\rho_1^*)^2} + \frac{8L (c_1^*)^2 K_u^2 \bar{x}^2}{(1 - \rho_1^*)^2} \\
& \leq (2\bar{x}^2 + 1) \frac{24L^2 (c_1^*)^3 K_u^2 C_{\rho_1}^*}{\rho_1^* (1 - \rho_1^*)^4} + \frac{2\bar{x}^2 \cdot 2L (c_1^*)^2 K_u^2}{1 - \rho_1^*} + \frac{2\bar{x}^2 \cdot 4L (c_1^*)^2 K_u^2}{(1 - \rho_1^*)^2} \\
& \leq \tilde{a} := (2\bar{x}^2 + 1) \frac{30L^2 (c_1^*)^3 K_u^2 C_{\rho_1}^*}{\rho_1^* (1 - \rho_1^*)^4}.
\end{aligned}$$

Proof of Theorem 3.5.3 Before giving the proof of Theorem 3.5.3, we first introduce a direct result obtained from Theorem 3.4.1 and Theorem 3.5.2 and it will help prove Theorem 3.5.3.

Lemma 3.8.16 (Regret of $\mathcal{I}_i^r$). *Under Assumption 3.4.1-3.4.2, $N = V_T^{4/5} T^{1/5}$, and $L_{\mathcal{I}} = T/N = V_T^{-4/5} T^{4/5}$, for $\forall \delta \in (0, 1)$, if*

$$\begin{aligned}
T & \geq \frac{32L^5 (\tilde{C}_e k_T + C_e)^5 V_T K_u^5}{(\Delta^*)^5}, \quad T_2 = \frac{2000(n_\theta)}{9} \left(2 \log \frac{1}{\delta^*} + \log \frac{(n+d)\bar{z}}{\bar{C}_1} \right), \\
T_3 & = \frac{2\hat{c}_1^2 (L^*)^2}{\Delta^*}, \quad \delta^* = \frac{\delta}{8N^2 n(n_\theta)}, \quad L_{\mathcal{I}} \geq \max(T_2^2, T_3), \\
C_e & = \frac{3200C_Q \lambda_{\max, M} \psi(n_\theta) \sqrt{\log \frac{1}{\delta^*}}}{9\bar{C}_1 \lambda_{\min, M}}, \quad \tilde{C}_e = \frac{6400C_Q^2(n_\theta)}{9\bar{C}_1 \lambda_{\min, M}}, \quad \Delta^* = \min(\Delta, \hat{\Delta}, \tilde{\Delta}), \\
L^* & = \max(2L, K_u), \quad W \geq \frac{1}{2} \log_{\hat{\rho}^{-1}} L_{\mathcal{I}}.
\end{aligned}$$

with the probability at least $1-\delta/N$, we obtain

$$|J_{\mathcal{I}_i^r}^* - \hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e)| \leq n \cdot \tilde{C}_J \cdot V_T^{-2/5} T^{2/5}, \quad \tilde{C}_J = \tilde{a} + \hat{a} L^2 \left(\tilde{C}_e k_T + C_e \right)^2 + \hat{b}.$$

where α , K_u , and L are defined in Assumption 3.4.2; $\bar{C}_1$, $\bar{z}$, $\bar{x}^2$, c_1 , ρ , L_H , γ_H , $\hat{c}_1$, $\hat{\rho}$, $\hat{L}_H$, and $\hat{\gamma}_H$ are defined in Theorem 3.4.1; $\hat{\Delta}$, $\tilde{\Delta}$, c_1^* , ρ_1^* , $C_{\rho_1}^*$, $\tilde{a}$, $\hat{a}$, and $\hat{b}$ are defined in Theorem 3.5.2.

Proof. For $t \in \mathcal{I}_i^r$, $\Delta A_t(\mathcal{I}_i^e) = \sum_{k=1}^{n_A} A_k \cdot (\Delta \hat{\theta}_{i,t})_k$ and $\Delta B_t(\mathcal{I}_i^e) = \sum_{l=1}^{n_B} B_l \cdot (\Delta \hat{\theta}_{i,t})_{l+n_A}$ where $(\Delta \hat{\theta}_{i,t})_j$ denotes the j -th element of vector $\Delta \hat{\theta}_{i,t}$. Combined with Assumption 3.4.2, we have $\|(\Delta A_t(\mathcal{I}_i^e), \Delta B_t(\mathcal{I}_i^e))\| \leq L \|\Delta \hat{\theta}_{i,t}\|$ and $\max_{\substack{i \in [N] \\ t \in \mathcal{I}_i^r}} \|(\Delta A_t(\mathcal{I}_i^e), \Delta B_t(\mathcal{I}_i^e))\| \leq L \max_{\substack{i \in [N] \\ t \in \mathcal{I}_i^r}} \|\Delta \hat{\theta}_{i,t}\|$.

By Theorem 3.4.1, we know that if $|\mathcal{I}_i^e| = L_{\mathcal{I}}^{1/2} > T_2$, for $\forall t \in \mathcal{I}_i^r$, we have

$$\|\hat{\theta} - \theta_t^*\| \leq \tilde{C}_e \max_{\tau \in \mathcal{I}_i} \|\theta_\tau^* - \theta_t^*\| + \frac{C_e}{\sqrt{|\mathcal{I}_i^e|}} = \tilde{C}_e \max_{\tau \in \mathcal{I}_i} \|\theta_\tau^* - \theta_t^*\| + \frac{C_e}{L_{\mathcal{I}}^{1/4}},$$

We know $\|\theta_\tau^* - \theta_t^*\| \leq \sum_{k \in \mathcal{I}_i} \|\theta_{k+1}^* - \theta_k^*\| \leq k_T V_T / N = k_T V_T^{1/5} T^{-1/5}$, and we obtain

$$\begin{aligned} \|\hat{\theta} - \theta_t^*\| &\leq \tilde{C}_e k_T V_T^{1/5} T^{-1/5} + C_e V_T^{1/5} T^{-1/5} \leq (\tilde{C}_e k_T + C_e) V_T^{1/5} T^{-1/5}, \\ \|\hat{\theta} - \theta_t^*\|^2 &\leq (\tilde{C}_e k_T + C_e)^2 V_T^{2/5} T^{-2/5}. \end{aligned}$$

When the following holds,

$$T \geq \left(2LK_u \left(\tilde{C}_e k_T + C_e \right) V_T^{1/5} / \Delta^* \right)^5, \quad \max_{\substack{i \in [N] \\ t \in \mathcal{I}_i^r}} \|\Delta \hat{\theta}_{i,t}\| \leq \Delta^* / 2LK_u$$

$$\max_{\substack{i \in [N] \\ t \in \mathcal{I}_i^r}} \|(\Delta A_t(\mathcal{I}_i^e), \Delta B_t(\mathcal{I}_i^e))\| \leq \Delta^* / 2K_u$$

by Theorem 3.5.2, we have

$$\begin{aligned} |J_{\mathcal{I}_i^r}^* - \hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e)| &\leq \tilde{a} \cdot n + \left(\hat{a} L^2 \left(\tilde{C}_e k_T + C_e \right)^2 V_T^{2/5} T^{-2/5} + \hat{b} L_{\mathcal{I}}^{-1/2} \right) \cdot L_{\mathcal{I}} \cdot n \\ &\leq \tilde{a} \cdot n + \left(\hat{a} L^2 \left(\tilde{C}_e k_T + C_e \right)^2 + \hat{b} \right) V_T^{-2/5} T^{2/5} \cdot n. \end{aligned}$$

□

Theorem 3.5.3 is the direct result of Theorem 3.5.1 and Lemma 3.8.16. We know that

$$|J^* - \hat{J}^W| = \sum_{i=0}^{N-1} |J_{\mathcal{I}_i^e}^* - J_{\mathcal{I}_i^e}(K_{stab})| + \sum_{i=0}^{N-1} |J_{\mathcal{I}_i^r}^* - \hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e)|.$$

Since $\|K_t^* - K_{stab}\| \leq 2K_u$ and $|\mathcal{I}_i^e| = L_{\mathcal{I}}^{1/2}$, by Theorem 3.5.1, we know that

$$\begin{aligned} |J_{\mathcal{I}_i^e}^* - J_{\mathcal{I}_i^e}(K_{stab})| &\leq \nu^2 \frac{5L^3 (c_1^*)^2 K_u^2 (2\bar{x}^2 + 1)}{1 - (\rho_1^*)^2} \cdot n L_{\mathcal{I}}^{1/2} + \frac{20\psi^2 L^3 (c_1^*)^4 K_u^2}{(1 - (\rho_1^*)^2)^3} (2K_u)^2 \cdot n L_{\mathcal{I}}^{1/2} \\ &\leq \frac{85L^3 (c_1^*)^4 K_u^4 (2\bar{x}^2 + 1)(\nu^2 + \psi^2)}{(1 - \rho_1^*)^3} \cdot n L_{\mathcal{I}}^{1/2}. \end{aligned}$$

Since $L_{\mathcal{I}}^{1/2} N = V_T^{2/5} \cdot T^{3/5}$, the upper bound of $\sum_{i=0}^{N-1} |J_{\mathcal{I}_i^e}^* - J_{\mathcal{I}_i^e}(K_{stab})|$ is given by

$$\sum_{i=0}^{N-1} |J_{\mathcal{I}_i^e}^* - J_{\mathcal{I}_i^e}(K_{stab})| \leq \frac{85L^3 (c_1^*)^4 K_u^4 (2\bar{x}^2 + 1)(\nu^2 + \psi^2)}{(1 - \rho_1^*)^3} \cdot n \cdot V_T^{2/5} \cdot T^{3/5}. \quad (3.63)$$

By Lemma 3.8.16, we know that

$$\sum_{i=0}^{N-1} |J_{\mathcal{I}_i^r}^* - \hat{J}_{\mathcal{I}_i^r}^W(\mathcal{I}_i^e)| \leq n \cdot \tilde{C}_J \cdot V_T^{-2/5} T^{2/5} N \leq n \cdot \tilde{C}_J \cdot V_T^{2/5} \cdot T^{3/5}. \quad (3.64)$$

By (3.63) and (3.64), we obtain the expected result

$$|J^* - \hat{J}^W| \leq \left(\frac{85L^3 (c_1^*)^4 K_u^4 (2\bar{x}^2 + 1)(\nu^2 + \psi^2)}{(1 - \rho_1^*)^3} + \tilde{C}_J \right) \cdot n \cdot T^{3/5}.$$

CHAPTER 4

SAMPLE-EFFICIENT AND SCALABLE ALGORITHMS FOR THE IDENTIFICATION OF UNKNOWN, STATIONARY, PARAMETRIC PARTIALLY OBSERVED LINEAR SYSTEMS

4.1 Introduction

We investigate the finite-horizon partially observed LQR problem with the stationary dynamics, where the linear system dynamics of the unobserved variables are governed by unknown matrices and the dynamics of the observed variables are determined by known matrices. Under some assumptions, the proposed algorithm is expected to have three contributions: (1) The computational cost per time step of the algorithm is $O(n)$ where n is the dimension of the unobserved variable. It indicates a substantial improvement over the classic algorithm based on the Ho-Kalman algorithm [30], which incurs a cost of $O(n^3)$ per time step. (2) The required sample complexity to learn unknown parameters is $O(\log m)$ where m is the dimension of the observed variable. (3) The estimation error is independent of n . We assume that the linear system dynamics of the unobserved variables is governed by A and B that can be expressed as a linear combination of a set of known basis sparse matrices. Therefore, we estimate these coefficients (vectors) instead of learning matrices as is done in [54, 34]. In the proposed algorithm, given a single input and output trajectory, we first use the ordinary least squares (OLS) to estimate B . Using the estimated B , we apply OLS again to estimate A with the same input and output trajectory.

The OLS imposes a stability constraint on A , which limits its applicability to a narrower class of systems. To address this limitation, we propose an alternative algorithm inspired by the Expectation-Maximization (EM) framework, which retains the computational efficiency of the OLS method while offering the potential to generalize to a broader class of partially observed LQR problems. The theoretical development of this EM-like algorithm is ongoing:

we outline a roadmap for the complete analysis, present partial theoretical results, and defer the remaining analysis and validation to future work.

4.2 Background

While classical studies of the Linear Quadratic Regulator (LQR) focus on systems with fully observable states, many real-world scenarios involve uncertainty in state measurements. In such cases, only noisy observations are available, and the underlying states must be inferred—leading to the formulation of partially observed linear dynamical systems. This partially observed LQR framework has gained prominence in both control theory and reinforcement learning [36], driven by its relevance in domains like robotics with incomplete feedback [75], biological system modeling [10], and power grid control [58]. Because many nonlinear processes can be approximated by time-varying linear models, LQR methods remain widely applicable. However, when direct access to state variables is unavailable, the standard fully observed LQR approach falls short, making its partially observed counterpart essential.

We consider the discrete-time finite-horizon partially observed linear quadratic regulator problem with stationary dynamics:

$$\begin{aligned} \min_{\substack{\{x_t\}_{t \in [T+1]} \\ \{u_t\}_{t \in [T]}}} & \frac{1}{2} x_T^T Q x_T + \sum_{t=0}^{T-1} \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t, \\ \text{s.t. } & x_0 = \bar{x}_0, \\ & x_{t+1} = A x_t + B u_t + w_t, \quad t \in [T], \\ & y_t = C x_t + D u_t + v_t, \quad t \in [T+1]. \end{aligned}$$

where C and D are known sparse matrices, A and B are unknown sparse matrices, T is the length of horizon, $\bar{x}_0 \sim \mathcal{N}(\mathbf{0}, \Sigma_0)$, $x_t \in \mathbb{R}^n$ is the state variable which is not observed,

$u_t \in \mathbb{R}^d$ is the control variable, $y_t \in \mathbb{R}^m$ is the observed variable, and $w_t \sim \mathcal{N}(\mathbf{0}, \sigma_x^2 I_n)$ and $v_t \sim \mathcal{N}(\mathbf{0}, \sigma_y^2 I_m)$ are random noise. We assume that A and B can be expressed as linear combinations of fixed basis matrices:

$$\begin{aligned} A &= \sum_{j=1}^{n_A} \alpha_j A_j, \quad \alpha_j \in \mathbb{R}, \quad A_j \in \mathbb{R}^{n \times n}, \quad \alpha^T = \begin{bmatrix} \alpha_1 & \cdots & \alpha_{n_A} \end{bmatrix} \quad \alpha \in \mathbb{R}^{n_A}, \\ B &= \sum_{k=1}^{n_B} \beta_k B_k, \quad \beta_k \in \mathbb{R}, \quad B_k \in \mathbb{R}^{n \times d}, \quad \beta^T = \begin{bmatrix} \beta_1 & \cdots & \beta_{n_B} \end{bmatrix} \quad \beta \in \mathbb{R}^{n_B}. \end{aligned} \tag{4.1}$$

where coefficients α and β are unknown constant vectors, while the basis matrices $\{A_j\}_{j=1}^{n_A}$ and $\{B_j\}_{j=1}^{n_B}$ are known constant sparse matrices. Therefore, we estimate vectors instead of learning matrices. Learning two matrices A and B is equivalent to learn $2 \cdot n^2$ unknown entries, while estimating α and β only requires to learn $n_A + n_B$ unknown entries. Therefore, this scalable setup is useful for the large-scale system settings and particularly powerful when the system dimension is much larger than the number of fixed matrices ($n \gg n_\theta = n_A + n_B$). This setup is applicable to a variety of settings, including multi-agent systems [19], networked systems [62], and discretized PDEs systems [25], where the system dynamics can be represented as linear combinations of fixed basis matrices. Notably, the number of such basis matrices remains independent of the system's dimensionality.

In the classic algorithm in [54], they first generate an input/output trajectory and apply OLS to learn the system Markov parameters. Then, one obtains the realization of unknown parameters by applying the Ho-Kalman algorithm to Markov parameters [54]. However, this approach suffers from several drawbacks. First, it does not yield a unique approximation of A and B , as the Ho-Kalman algorithm only recovers a system realization up to a similarity transformation. Consequently, the resulting estimates of A and B are not uniquely identifiable, which limits their interpretability and usefulness in applications where parameter uniqueness is critical. Then, the computational cost of the algorithm is $O(n^3)$ per time step, making it computationally intensive for high-dimensional systems. Finally, the sample

complexity required to achieve accurate parameter estimation is $O(n \log(N \cdot n))$ where N is the number of samples, and the estimation error scales as $O(\sqrt{n})$. These limitations make the method prohibitively expensive for large-scale systems, particularly when high accuracy is desired. Therefore, we want to propose an algorithm resolving the drawbacks mentioned above.

We propose a scalable approach to learn A and B by leveraging their parametric and sparsity structure as defined in (4.1). Given a trajectory $\{u_t, y_t\}_{t=2}^{N+2}$, we first apply OLS to learn the estimated β , yielding an estimate of B . Using this estimated B , then perform a second OLS procedure to estimate α , and obtain the estimate of A . The proposed algorithm aims to have four contributions: (1) We demonstrate that the computational cost of the proposed algorithm scales linearly with n , where n is the state dimension. This results in a highly efficient algorithm for large-scale problems. (2) The required sample complexity is $O(\log n)$, making it significantly more efficient for large-scale systems, particularly in regimes where $n \gg n_\theta = n_A + n_B$. (3) The estimation error is $O(\log n)$ independent of n . (4) The proposed algorithm can give a unique approximation of α and β which in turn implies the uniqueness of the estimated system matrices A and B . This property enhances the interpretability and reliability of the learned model, especially in settings where parameter identifiability is essential. However, the OLS-based approaches for estimating the unknown parameters in partially observed LQR systems require the stability of A , thereby excluding a broad class of systems. At present, we do not see a viable way to relax this stability requirement within the OLS framework.

The classic Expectation Maximization (EM) algorithm is proposed to learn unknown parameters of partially observed LQR in [65]. Inspired by the EM algorithm, we propose a generalized EM algorithm to adapt to our setting, which is a candidate to be extended to more general classes of problems-reducing the restriction on A . At the beginning of the algorithm, we generate input sequences $\{u_t\}_{t=1}^N$ and $\{y_t\}_{t=1}^N$ where $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$ and N

is the number of samples. Then, at the iteration τ , we first obtain the estimated unobserved state variable $\{\hat{x}_t^{(\tau)}\}_{t=1}^N$ based on the current estimation $\alpha^{(\tau)}$ and $\beta^{(\tau)}$. Treating $\{\hat{x}_t^{(\tau)}\}_{t=1}^N$ as the observed data, we obtain $\alpha^{(\tau+1)}$ and $\beta^{(\tau+1)}$ by performing the ordinary least square (OLS) on $\{\hat{x}_t^{(\tau)}\}_{t=1}^N$. This iterative process repeats until convergence. In τ -th iteration, we obtain $\{\hat{x}_t^{(\tau)}\}_{t=1}^N$ by

$$\begin{aligned}\hat{x}_t^{(\tau)} &= \hat{x}_{t|t-1}^{(\tau)} + \text{Kalman_L} \left(y_t - C\hat{x}_{t|t-1}^{(\tau)} - Du_t, \hat{A}^{(\tau)}, C, \sigma_x^2, \sigma_y^2, W \right), \\ \hat{x}_{t|t-1}^{(\tau)} &= \hat{A}^{(\tau)}\hat{x}_{t-1}^{(\tau)} + \hat{B}^{(\tau)}u_{t-1}, \quad \hat{A}^{(\tau)} = \sum_{j=1}^{n_A} \alpha_j^{(\tau)} A_j, \quad \hat{B}^{(\tau)} = \sum_{j=1}^{n_B} \beta_j^{(\tau)} B_j.\end{aligned}$$

where Kalman_L algorithm computes two MPC problems with the truncated horizon, which approximates $L(\hat{A}^{(\tau)}) \cdot (y_t - C\hat{x}_{t|t-1}^{(\tau)} - Du_t)$. The proposed Kalman_L algorithm approximates the exact Kalman gain $L(\hat{A}^{(\tau)})$ with $L(\hat{A}^{(\tau)}, W)$ by solving two MPC problems with the truncated horizon W . We show that the proposed method significantly reduces computational cost, and that the approximation error of the Kalman gain decays exponentially with the horizon length W . In contrast, we note that $L(\hat{A}^{(\tau)})$ is the exact Kalman gain of $\hat{A}^{(\tau)}$ requires solving a Riccati equation, which incurs substantially higher computational cost. The work is still in progress. We give the framework the current version of the algorithm, present the proposed generalized EM algorithm, outline the steps of the theoretical analysis, complete the part of the analysis, and give future directions. We postpone the remaining analysis to future work.

In the remaining chapter, the full algorithm of the OLS approach is given in Section 4.3. In Section 4.4, we give the complete sample complexity and estimation error. In Section 4.5, we introduce the EM-like algorithm, outline the steps of the theoretical analysis, complete a portion of it, give future direction on how to relax the assumption, and leave the remaining analysis for future work.

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In Section 4.4, we give the complete sample complexity and estimation error. In Section 4.5, we introduce the EM-like algorithm, outline the steps of the theoretical analysis, complete a portion of it, give future direction on how to relax the assumption, and leave the remaining analysis for future work.

4.3 Algorithm

We present our proposed algorithm *System identification for unknown, parametric, stationary, partially observed systems* in Section 4.3. Generally, we first obtain a trajectory $\{u_{t-1}, y_t\}_{t=2}^{N+2}$ where N is the number of samples and u_t is the input which follows Gaussian distribution $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$. Since we can write y_t as

$$\begin{aligned} y_t &= Cx_t + Du_t + v_t \\ &= C \left(Ax_{t-1} + \left(\sum_{i=1}^{n_B} \beta_i B_i \right) u_{t-1} + w_{t-1} \right) + Du_t + v_t \\ &= Du_t + \left(\sum_{i=1}^{n_B} \beta_i C B_i \right) u_{t-1} + e_{t,B} + Cw_{t-1} + v_t \end{aligned} \tag{4.2}$$

where $e_{t,B} = CAx_{t-1}$ captures the error induced by the effect of the state at time $t-1$, and $e_{t,B}$, Cw_t , and v_t are treated as the additive noise and apply the OLS on $\{u_{t-1}, y_t\}_{t=2}^{N+1}$ to obtain the estimates of $\beta^T = \begin{bmatrix} \beta_1 & \dots & \beta_{n_B} \end{bmatrix}$. With the estimated β , we can obtain the estimated B . Then, we can continue to expand y_t as

$$\begin{aligned} y_t &= Cx_t + Du_t + v_t \\ &= C(Ax_{t-1} + Bu_{t-1} + w_{t-1}) + Du_t + v_t \\ &= C(A(Ax_{t-2} + Bu_{t-2} + w_{t-2}) + Bu_{t-1} + w_{t-1}) + Du_t + v_t \\ &= Du_t + CBu_{t-1} + \sum_{i=1}^{n_A} \alpha_i CA_i Bu_{t-2} + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t \end{aligned} \tag{4.3}$$

where $e_{t,A} = CA^2x_{t-2}$ captures the error induced by the effect of the state at time $t-2$, and $e_{t,A}$, Cw_t , CAw_{t-2} , and v_t are treated as the additive noise. Since we obtain the estimated B , we can apply OLS on $\{u_{t-2}, y_t\}_{t=3}^{N+2}$ with the estimated B to obtain the estimates of $\alpha^T = \begin{bmatrix} \alpha_1 & \cdots & \alpha_{n_B} \end{bmatrix}$. We will show in the next section that the estimation error of α and β is of $O(1/\sqrt{N})$ when $N = O(\log n)$.

Algorithm 3 System identification for unknown, parametric, stationary, partially observed systems

- 1: **Input:** the variance of u_t for the exploration stage σ_u with $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$; the exploration length N ;
- 2: **for** $t = 0, 1, 2, \dots, N+2$ **do**
- 3: Generate $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$ and y_t .
- 4: **end for**
- 5: Obtain $\hat{\beta}$ by

$$\begin{aligned} \hat{\beta} &= \arg \min_{\beta} \sum_{t=2}^{N+1} \|y_t - Du_t - Q_{B,t-1}^T \beta\|^2, \\ Q_{B,t-1}^T &= [CB_1 u_{t-1} \quad \cdots \quad CB_{n_B} u_{t-1}]. \end{aligned} \tag{4.4}$$

- 6: Obtain $\hat{\alpha}$ by

$$\begin{aligned} \hat{\alpha} &= \arg \min_{\alpha} \sum_{t=3}^{N+2} \|y_t - Du_t - C\hat{B}u_{t-1} - \hat{Q}_{A,t-2}^T \alpha\|^2, \\ \hat{Q}_{A,t-2}^T &= [CA_1 \hat{B}u_{t-2} \quad \cdots \quad CA_{n_A} \hat{B}u_{t-2}], \quad \hat{B} = \sum_{i=1}^{n_B} \hat{\beta}_i B_i. \end{aligned} \tag{4.5}$$

4.4 Estimation Analysis

The system identification is crucial for solving the partially observed LQR with unknown parameters. Given the high cost of collecting data, sample complexity and estimation error become central concerns in parameter estimation. However, existing results in the literature [54, 34] show that both sample complexity and estimation error scale unfavorably with

the state (x_t) dimension n , making these approaches impractical for large-scale systems. Therefore, in this section, we analyze the sample complexity and estimation error associated with estimating the unknown parameters by exploiting the structure of A and B given in (4.1).

Before giving analysis, we first give two necessary assumptions.

Assumption 4.4.1. *We assume that when $\|\hat{B} - B\|$ is sufficiently small, there exists constant number $\lambda_{\min,B}$, $\lambda_{\max,B}$, $\lambda_{\min,A}$, and $\lambda_{\max,A}$ such that*

$$\begin{aligned} \lambda_{\min} \left(\sum_{k=1}^m \tilde{M}_k \tilde{M}_k^T \right) &\geq n \cdot \lambda_{\min,B}, & \max_{k=1, \dots, m} \|\tilde{M}_k\| &\leq \lambda_{\max,B}, \\ \lambda_{\min} \left(\sum_{k=1}^m \tilde{G}_k \tilde{G}_k^T \right) &\geq n \cdot \lambda_{\min,A}, & \max_{k=1, \dots, m} \|\tilde{G}_k\| &\leq \lambda_{\max,A}. \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} M_i = CB_i &= \begin{bmatrix} M_{1,i} \\ \vdots \\ M_{m,i} \end{bmatrix} & i = 1, \dots, n_B, & \tilde{M}_j &= \begin{bmatrix} M_{j,1} \\ \vdots \\ M_{j,n_B} \end{bmatrix} & j = 1, \dots, m, \\ G_i = CA_i\hat{B} &= \begin{bmatrix} G_{1,i} \\ \vdots \\ G_{m,i} \end{bmatrix} & i = 1, \dots, n_A, & \tilde{G}_j &= \begin{bmatrix} G_{j,1} \\ \vdots \\ G_{j,n_A} \end{bmatrix} & j = 1, \dots, m. \end{aligned} \quad (4.7)$$

This assumption is extended from Assumption 3.4.1. Then, we give the second assumption widely used in the literature of partially observed LQR [34].

Assumption 4.4.2. *There exists constants $\alpha, \tilde{\rho} \in (0, 1)$, and $\gamma \in (0, 1]$ such that*

1. $\|A\|, \|B\|, \|C\|, \|D\|, \|\Sigma_0\|, \sigma_x^2, \sigma_y^2 \leq L$, $\|A_i\|, \|B_j\| \leq L$ for $i = 1, \dots, n_A$ and $j = 1, \dots, n_B$, $\rho(A) < 1$.

2. For any integer k , $\|A^k\| \leq L \cdot \tilde{\rho}^k$.

3. (A, B) is controllable which means $\begin{bmatrix} B & AB & AB^2 & \dots & A^{n-1}B \end{bmatrix}$ has full row rank.
 (A, C) is observable, which means $\begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \dots & (CA^{n-1})^T \end{bmatrix}^T$ has full column rank.

Since $\rho(A) < 1$, it is clear that we can find such L and $\tilde{\rho}$. Then, we are ready to analyze $\|\beta^* - \hat{\beta}\|$ and $\|\alpha^* - \hat{\alpha}\|$.

The OLS problem of β is defined in (4.4) and its OLS estimator is defined as follows

$$\begin{aligned}
y_t &= Du_t + Q_{B,t-1}^T \beta^* + e_{t,B} + Cw_{t-1} + v_t, \\
\hat{\beta} &= \left(\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T \right)^{-1} \sum_{t=2}^{N+1} Q_{B,t-1} (y_t - Du_t) \\
&= \left(\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T \right)^{-1} \sum_{t=2}^{N+1} Q_{B,t-1} (Q_{B,t-1}^T \beta^* + e_{t,B} + Cw_{t-1} + v_t) \\
&= \beta^* + \left(\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T \right)^{-1} \sum_{t=2}^{N+1} Q_{B,t-1} (e_{t,B} + Cw_{t-1} + v_t).
\end{aligned} \tag{4.8}$$

Then, we express $\|\hat{\beta} - \beta^*\|$ as

$$\begin{aligned}
\|\hat{\beta} - \beta^*\| &= \left\| \left(\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T \right)^{-1} \sum_{t=2}^{N+1} Q_{B,t-1} (e_{t,B} + Cw_{t-1} + v_t) \right\| \\
&\leq \|\Gamma_{B,N}^{-1}\| \cdot \left(\left\| \sum_{t=2}^{N+1} Q_{B,t-1} e_{t,B} \right\| + \left\| \sum_{t=2}^{N+1} Q_{B,t-1} Cw_{t-1} \right\| + \left\| \sum_{t=2}^{N+1} Q_{B,t-1} v_t \right\| \right)
\end{aligned} \tag{4.9}$$

where $\Gamma_{B,N} = \sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T$. The theorem of $\|\hat{\beta} - \beta^*\|$ is given as follows

Theorem 4.4.1. *Under Assumption 4.4.1-4.4.2, if we have*

$$N \geq \frac{16n_B \cdot \sigma_u^2 \cdot \log \frac{11n_B \cdot m}{\delta}}{\sigma_u^2} \cdot \log \frac{11n_B^2 \cdot m}{\delta}, \tag{4.10}$$

for small δ , it holds with the probability at least $1 - \delta$,

$$\|\hat{\beta} - \beta^*\| \leq 4n_B \frac{C_{B,e} + (\sigma_x L + \sigma_y) \lambda_{\max,B}}{\sigma_u \lambda_{\min,B}} \sqrt{\frac{\log\left(\frac{11n_B \cdot m}{\delta}\right)}{N}}, \quad C_{B,e} = \frac{2L^6}{1 - \tilde{\rho}} (\sigma_x + \sigma_u + 1). \quad (4.11)$$

where $\lambda_{\min,B}$ and $\lambda_{\max,B}$ are defined in Assumption 4.4.1; L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. The proof is completed by four steps

1. In Lemma 4.4.1, we obtain the lower bound of $\lambda_{\min}(\Gamma_{B,N})$ to get the upper bound of $\|\Gamma_{B,N}^{-1}\|$.
2. In Lemma 4.4.3, we obtain the upper bound of $\|\sum_{t=2}^{N+1} Q_{B,t-1} e_{t,B}\|$.
3. In Lemma 4.4.4, we obtain the upper bound of $\|\sum_{t=2}^{N+1} Q_{B,t-1} C w_{t-1}\|$.
4. In Lemma 4.4.5, we obtain the upper bound of $\|\sum_{t=2}^{N+1} Q_{B,t-1} v_t\|$.

□

Theorem 4.4.1 shows that the required sample complexity to estimate β is only of $O(n_B \cdot \log m)$. In contrast to the sample complexity in [54, 34] as of $O(m \log(N \cdot m))$, the proposed algorithm significantly reduces the sample complexity and makes it particularly favorable for the large-scale system where $n \gg n_B$. Additionally, the estimation error is only of $O(\log m)$ while the estimation error in [54, 34] is of $O(\sqrt{n})$. Therefore, our algorithm can estimate A and B with much fewer samples while achieving desired accuracy by exploring the structure of A and B .

Then, we want to give similar estimation results of α . The OLS problem of α is defined

in (4.5) and its OLS estimator is defined as follows

$$\begin{aligned}
y_t &= Du_t + CBu_{t-1} + \hat{Q}_{A,t-2}^T \alpha^* + \Delta Q_{A,t-2}^T \alpha^* + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t, \\
Q_{A,t-2}^T &= \begin{bmatrix} CA_1 Bu_{t-2} & \cdots & CA_{n_A} B u_{t-2} \end{bmatrix}, \\
\hat{\alpha} &= \left(\sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \hat{Q}_{A,t-2}^T \right)^{-1} \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} (y_t - Du_t - C\hat{B}u_{t-1}) \\
&= \left(\sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \hat{Q}_{A,t-2}^T \right)^{-1} \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} (\hat{Q}_{A,t-2}^T \alpha^* + \Delta Q_{A,t-2}^T \alpha^* \\
&\quad + C\Delta B u_{t-1} + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t) \\
&= \alpha^* + \Gamma_{A,N}^{-1} \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} (\Delta Q_{A,t-2}^T \alpha^* + C\Delta B u_{t-1} + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t).
\end{aligned}$$

where $\Gamma_{A,N} = \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \hat{Q}_{A,t-2}^T$, $\Delta Q_{A,t-2}^T = Q_{A,t-2}^T - \hat{Q}_{A,t-2}^T$ and $\Delta B = B - \hat{B}$.

Then, we want to analyze $\|\hat{\alpha} - \alpha^*\|$ which is expressed as

$$\begin{aligned}
&\|\hat{\alpha} - \alpha^*\| \\
&= \|\Gamma_{A,N}^{-1}\| \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} (\Delta Q_{A,t-2}^T \alpha^* + C\Delta B u_{t-1} + e_{t,A} + Cw_{t-1} + CAw_{t-2} + v_t) \right\| \\
&\leq \|\Gamma_{A,N}^{-1}\| \sum_{i=1}^6 \mathcal{J}_i, \\
&\mathcal{J}_1 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\|, \quad \mathcal{J}_2 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C\Delta B u_{t-1} \right\|, \\
&\mathcal{J}_3 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} e_{t,A} \right\|, \quad \mathcal{J}_4 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} Cw_{t-1} \right\|, \\
&\mathcal{J}_5 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} CAw_{t-2} \right\|, \quad \mathcal{J}_6 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} v_t \right\|.
\end{aligned}$$

Theorem 4.4.2. *Under Assumption 4.4.1-4.4.2, if we have*

$$N \geq \frac{16n_A \cdot \sigma_u^2 \cdot \log \frac{11n_A \cdot m}{\delta_1}}{\sigma_u^2} \cdot \log \frac{11n_A^2 \cdot m}{\delta_1}, \quad (4.12)$$

for small δ , with $\delta_1 = \delta$, it holds with the probability at least $1 - \delta$,

$$\begin{aligned} & \|\hat{\alpha} - \alpha^*\| \\ & \leq 2n_A \cdot n_B^{3/2} \frac{c\sigma_u C_{A,Q} + C_{B,e} + (2\sigma_x L^2 + \sigma_y)\lambda_{\max,A}}{\lambda_{\min,A}\sigma_u} \sqrt{\frac{\log\left(\frac{11n_B \cdot m}{\delta}\right) \log\left(\frac{11n_A \cdot m}{\delta}\right)}{N}}. \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} c = d/m, \quad C_{B,e} &= \frac{2L^6}{1 - \tilde{\rho}}(\sigma_x + \sigma_u + 1), \\ C_{A,Q} = 4C_B \cdot L^6, \quad C_B &= 4 \frac{C_{B,e} + (\sigma_x L + \sigma_y)\lambda_{\max,B}}{\sigma_u \lambda_{\min,B}}. \end{aligned} \quad (4.14)$$

$\lambda_{\min,B}$, $\lambda_{\max,B}$, $\lambda_{\min,A}$, and $\lambda_{\max,A}$ are defined in Assumption 4.4.1; L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. The proof is completed by four steps

1. In Lemma 4.4.6, we obtain the lower bound of $\lambda_{\min}(\Gamma_{A,N})$ to get the upper bound of $\|\Gamma_{A,N}^{-1}\|$.
2. In Lemma 4.4.7-4.4.8, we obtain the upper bound of $\mathcal{J}_1$ and $\mathcal{J}_2$ respectively where we apply Theorem 4.4.1 to $\Delta Q_{A,t-2}$ and ΔB .
3. In Lemma 4.4.9, we obtain the upper bound of $\mathcal{J}_3$ where we apply Theorem 4.4.1.
4. In Lemma 4.4.10-4.4.11, we obtain the upper bound of $\mathcal{J}_4$ and $\mathcal{J}_5$ respectively.
5. In Lemma 4.4.12, we obtain the upper bound of $\mathcal{J}_6$.

□

Theorem 4.4.2 gives similar results to the results of Theorem 4.4.1. It establishes that the sample complexity required to estimate α is only of $O(n_A \cdot \log m)$. This represents a significant improvement over the sample complexity bounds in [54, 34], which are on the order of $O(m \log(N \cdot m))$. The proposed algorithm is therefore particularly well-suited for large-scale systems where $n \gg n_A$, offering substantial gains in efficiency. Additionally, the estimation error achieved by our algorithm is only $O(\log m)$ while the estimation error in [54, 34] is of $O(\sqrt{n})$. By exploiting the underlying structure of A and B our method is able to estimate these parameters with significantly fewer samples while still attaining the desired level of accuracy.

4.4.1 Lower bound of $\|\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T\|$

Lemma 4.4.1. *Under Assumption 4.4.1-4.4.2, when*

$$N \geq \frac{16n_B \cdot \sigma_u^2 \cdot \log \frac{1}{\delta}}{\sigma_u^2} \cdot \log \frac{n_B}{\delta}, \quad (4.15)$$

with the probability at least $1 - m \cdot \delta$,

$$\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T \succeq N \cdot m \cdot \lambda_{\min,B} \cdot \sigma_u^2 / 2 I_{n_B}. \quad (4.16)$$

where $\lambda_{\min,B}$ is defined in Assumption 4.4.1.

Proof. We first rewrite $Q_{B,t-1}$ as

$$Q_{B,t-1}^T = \begin{bmatrix} M_{1,1}u_{t-1} & \cdots & M_{1,n_B}u_{t-1} \\ \vdots & \cdots & \vdots \\ M_{m,1}u_{t-1} & \cdots & M_{m,n_B}u_{t-1} \end{bmatrix},$$

where M_i is defined in Assumption 4.4.1. If we denote $q_{k,t-1}^T = \begin{bmatrix} M_{k,1}u_{t-1} & \cdots & M_{k,n_B}u_{t-1} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$\begin{aligned} Q_{B,t-1}Q_{B,t-1}^T &= \sum_{k=1}^m q_{k,t-1}q_{k,t-1}^T = \sum_{k=1}^m \tilde{M}_k u_{t-1} u_{t-1}^T \tilde{M}_k^T, \quad q_{k,t-1} = \tilde{M}_k u_{t-1}, \\ \sum_{t=2}^{N+1} Q_{B,t-1}Q_{B,t-1}^T &= \sum_{k=1}^m \tilde{M}_k \left(\sum_{t=2}^{N+1} u_{t-1} u_{t-1}^T \right) \tilde{M}_k^T \\ &= \sum_{k=1}^m U_{B,k} \Sigma_{B,k} \left(\sum_{t=2}^{N+1} V_{B,k}^T u_{t-1} u_{t-1}^T V_{B,k} \right) \Sigma_{B,k} U_{B,k}^T. \end{aligned}$$

where $\tilde{M}_k$ is defined in Assumption 4.4.1 and the last inequality uses the economy singular value decomposition of $\tilde{M}_k$ as $\tilde{M}_k = U_{B,k} \Sigma_{B,k} V_{B,k}^T$. Then, we want to apply the Matrix Chernoff inequality on $\sum_{t=2}^{N+1} V_{B,k}^T u_{t-1} u_{t-1}^T V_{B,k}$ to obtain the lower bound of its eigenvalue for each k where denote $\tilde{u}_{k,t-1} = V_{B,k}^T u_{t-1}$ and $\tilde{u}_{k,t-1} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_{n_B})$. We first recall the Matrix Chernoff inequality as follows

Lemma 4.4.2 (Matrix Chernoff I). *Consider a finite sequence $\{\mathbf{X}_k\}$ of independent, random, self-adjoint matrices with dimension d . Assume that each random matrix satisfies that*

$$\mathbf{X}_k \succeq \mathbf{0}, \quad \lambda_{\max}(\mathbf{X}_k) \leq R, \quad (4.17)$$

almost surely. Define

$$\mu_{\min} = \lambda_{\min} \left(\sum_k E \mathbf{X}_k \right) \quad (4.18)$$

Then

$$\mathbb{P} \left(\lambda_{\min} \left(\sum_k \mathbf{X}_k \right) \leq (1 - \delta) \mu_{\min} \right) \leq d \cdot \left[\frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^{\mu_{\min}/R}. \quad (4.19)$$

We have

$$\begin{aligned}
\|\tilde{u}_{k,t-1}\|^2 &= \sum_{l=1}^{n_B} \tilde{u}_{k,t-1,l}^2, \quad \mathbb{P}(\tilde{u}_{k,t-1,l} > \sqrt{2\sigma_u^2 \log \delta^{-1}}) \leq \delta, \\
\mathbb{P}\left(\|\tilde{u}_{k,t-1,l}\|^2 > n_B \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta}\right) &\leq n_B \cdot \delta, \quad \lambda_{\max}(\tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T) \leq \|\tilde{u}_{k,t-1}\|^2, \\
\mathbb{P}\left(\lambda_{\max}(\tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T) > n_B \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta}\right) &\leq n_B \cdot \delta.
\end{aligned} \tag{4.20}$$

We also know

$$\lambda_{\min} \left(E \left[\sum_{t=2}^{N+1} \tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T \right] \right) = N \cdot \sigma_u^2. \tag{4.21}$$

Then, by Lemma 4.4.2, we have for any $\epsilon \in (0, 1)$,

$$\begin{aligned}
&\mathbb{P} \left(\lambda_{\min} \left(\sum_{t=2}^{N+1} \tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T \right) \leq (1 - \epsilon) N \cdot \sigma_u^2 \right) \\
&\leq n_B \cdot \left[\frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right]^{N \cdot \sigma_u^2 / (n_B \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta})} \\
&\leq n_B \cdot \exp \left(- \frac{N \sigma_u^2}{n_B \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta}} \cdot ((1 - \epsilon) \log(1 - \epsilon) + \epsilon) \right) \\
&\leq n_B \cdot \exp \left(- \frac{N \sigma_u^2}{n_B \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta}} \cdot \frac{\epsilon^2}{2} \right).
\end{aligned} \tag{4.22}$$

Therefore, when $N \geq \frac{4n_B \cdot \sigma_u^2 \cdot \log \frac{1}{\delta}}{\sigma_u^2} \cdot \frac{1}{\epsilon^2} \cdot \log \frac{n_B}{\delta}$, $\mathbb{P} \left(\lambda_{\min} \left(\sum_{t=2}^{N+1} \tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T \right) \leq (1 - \epsilon) N \cdot \sigma_u^2 \right) < \delta$. When we set $\epsilon = \frac{1}{2}$, we have when $N \geq \frac{16n_B \cdot \sigma_u^2 \cdot \log \frac{1}{\delta}}{\sigma_u^2} \cdot \log \frac{n_B}{\delta}$,

$$\mathbb{P} \left(\lambda_{\min} \left(\sum_{t=2}^{N+1} \tilde{u}_{k,t-1} \tilde{u}_{k,t-1}^T \right) \leq N \cdot \sigma_u^2 / 2 \right) \leq \delta. \tag{4.23}$$

Now, we have with the probability at least $1 - m \cdot \delta$

$$\begin{aligned}
\sum_{t=2}^{N+1} Q_{B,t-1} Q_{B,t-1}^T &= \sum_{k=1}^m U_{B,k} \Sigma_{B,k} \left(\sum_{t=2}^{N+1} V_{B,k}^T u_{t-1} u_{t-1}^T V_{B,k} \right) \Sigma_{B,k} U_{B,k}^T \\
&\succeq N \cdot \sigma_u^2 / 2 \sum_{k=1}^m U_{B,k} \Sigma_{B,k} \Sigma_{B,k} U_{B,k}^T \succeq N \cdot \sigma_u^2 / 2 \sum_{k=1}^m \tilde{M}_k \tilde{M}_k^T \succeq N \cdot m \cdot \sigma_u^2 / 2 \cdot \lambda_{\min, B}.
\end{aligned} \tag{4.24}$$

□

4.4.2 Upper bound of $\| \sum_{t=2}^{N+1} Q_{B,t-1} e_{t,B} \|$

Lemma 4.4.3. *Under Assumption 4.4.1-4.4.2, when $N > 16 \log \frac{1}{\delta}$, with the probability at least $1 - 4m \cdot n_B \cdot \delta$,*

$$\left\| \sum_{t=2}^{N+1} Q_{B,t-1} e_{t,B} \right\| \leq m \cdot n_B \cdot 2\sigma_u \cdot C_{B,e} \cdot \sqrt{\log \left(\frac{1}{\delta} \right) N}, \quad C_{B,e} = \frac{2L^6}{1 - \tilde{\rho}} (\sigma_x + \sigma_u + 1). \tag{4.25}$$

where L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. We first write $Q_{B,t-1} \cdot e_{t,B}$ as

$$\begin{aligned}
Q_{B,t-1} \cdot e_{t,B} &= \begin{bmatrix} u_{t-1}^T B_1^T C^T C A x_{t-1} \\ \vdots \\ u_{t-1}^T B_{n_B}^T C^T C A x_{t-1} \end{bmatrix}, \\
\sum_{t=2}^{N+1} Q_{B,t-1} \cdot e_{t,B} &= \begin{bmatrix} \sum_{t=2}^{N+1} u_{t-1}^T B_1^T C^T C A x_{t-1} \\ \vdots \\ \sum_{t=2}^{N+1} u_{t-1}^T B_{n_B}^T C^T C A x_{t-1} \end{bmatrix}, \\
\left\| \sum_{t=2}^{N+1} Q_{B,t-1} \cdot e_{t,B} \right\| &\leq \sum_{l=1}^{n_B} \left| \sum_{t=2}^{N+1} u_{t-1}^T B_l^T C^T C A x_{t-1} \right|.
\end{aligned}$$

For each $l = 1, \dots, n_B$, we have

$$\begin{aligned} \sum_{t=2}^{N+1} u_{t-1}^T B_l^T C^T C A x_{t-1} &= \sum_{k=1}^m \sum_{t=2}^{N+1} u_{t-1,k} (B_l^T C^T C A x_{t-1})_k \\ &= \sum_{k=1}^m \sum_{t=2}^{N+1} u_{t-1,k} \langle e_k, B_l^T C^T C A x_{t-1} \rangle \end{aligned} \quad (4.26)$$

where e_k is an one hot vector with 1 in k -th entry and 0 in other entries. By Lemma 4.7.1, we know with the probability at least $1 - 2\delta$,

$$\left| \sum_{t=2}^{N+1} u_{t-1,k} \langle e_k, B_l^T C^T C A x_{t-1} \rangle \right| \leq \sigma_u \sqrt{2 \log \frac{1}{\delta} \sum_{t=2}^{N+1} \langle e_k, B_l^T C^T C A x_{t-1} \rangle^2}. \quad (4.27)$$

Since $x_t = A^t x_0 + \sum_{k=0}^{t-1} A^k (B u_{t-1-k} + w_{t-1-k})$, we can write

$$\begin{aligned} \mathbf{x}_N &= S_{N,x} \mathbf{w}_N + S_{N,u} \mathbf{u}_N + C_{N,x} x_0, \\ S_{N,x} &= \begin{bmatrix} I_n & & & \\ A & I_n & & \\ \vdots & \ddots & \ddots & \\ A^{N-1} & \dots & A & I_n \end{bmatrix}, \quad S_{N,u} = \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}B & \dots & AB & B \end{bmatrix}, \\ C_{N,x} &= \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{u}_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad \mathbf{w}_N = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}. \end{aligned}$$

We have

$$\mathbf{x}_N = \underbrace{\begin{bmatrix} S_{N,x} & S_{N,u} \end{bmatrix}}_{M_{S,N}} \underbrace{\begin{bmatrix} \sigma_x I_{n \cdot N} & \mathbf{0} \\ \mathbf{0} & \sigma_u I_{d \cdot N} \end{bmatrix}}_{M_{x,N}} M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} + C_{N,x} x_1 \quad (4.28)$$

To have an upper bound of $M_{S,N}$, we want the upper bound of each block. We first look at $S_{N,x}$ which can be written as

$$S_{N,x} = \begin{bmatrix} I_n & & & \\ & I_n & & \\ \vdots & \ddots & \ddots & \\ & \cdots & & I_n \end{bmatrix} + \begin{bmatrix} \mathbf{0} & & & \\ A & & & \\ \vdots & \ddots & \ddots & \\ & \cdots & A & \mathbf{0} \end{bmatrix} + \cdots$$

Therefore, we have $\|S_{N,x}\| \leq \frac{L}{1-\tilde{\rho}}$. Similarly, we have

$$\|S_{N,u}\| \leq \frac{L^2}{1-\tilde{\rho}}, \quad \|C_{N,x}\| \leq \frac{L}{1-\tilde{\rho}}, \quad \|M_{S,N}\| \leq \|S_{N,x}\| + \|S_{N,u}\| \leq \frac{2L^2}{1-\tilde{\rho}}. \quad (4.29)$$

where we use Assumption 4.4.2. Then, we have

$$\begin{aligned} & \sum_{t=2}^{N+1} \langle e_k, B_t^T C^T C A x_{t-1} \rangle^2 \\ &= \left\| \underbrace{\text{diag}(e_k^T B_1^T C^T C A)}_{M_{E,k}} \left(M_{S,N} M_{x,N} \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) + C_{N,x} \Sigma_0 \Sigma_0^{-1} x_0 \right) \right\|^2, \end{aligned} \quad (4.30)$$

where $M_{E,k} \in \mathbb{R}^{N \times (d \cdot N)}$. We apply the singular value decomposition on $M_{E,k} M_{S,N} M_{x,N} = U_{x,k} \Sigma_{x,k} V_{x,k}^T$ and $M_{E,k} C_{N,x} \Sigma_0 = U_{x_0,k} \Sigma_{x_0,k} V_{x_0,k}^T$ where

$$\begin{aligned} \|U_{x,k} \Sigma_{x,k}\| &\leq \|M_{E,k} M_{S,N} M_{x,N}\| \leq \frac{2L^6}{1-\tilde{\rho}} (\sigma_x + \sigma_u), \\ \|U_{x_0,k} \Sigma_{x_0,k}\| &\leq \|M_{E,k} C_{N,x} \Sigma_0\| \leq \frac{L^6}{1-\tilde{\rho}}. \end{aligned} \quad (4.31)$$

. We have

$$\begin{aligned}
& \sum_{t=2}^{N+1} \langle e_k, B_l^T C^T C A x_{t-1} \rangle^2 \\
&= \left\| U_{x,k} \Sigma_{x,k} V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) + U_{x_0,k} \Sigma_{x_0,k} V_{x_0,k}^T x_0 \right\|^2 \\
&\leq \left(\|U_{x,k} \Sigma_{x,k}\| \|V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right)\| + \|U_{x_0,k} \Sigma_{x_0,k}\| \|V_{x_0,k}^T \Sigma_0^{-1} x_0\| \right),
\end{aligned} \tag{4.32}$$

By Laurent-Massart bound [35], when $N \geq 16 \log \frac{1}{\delta}$

$$\begin{aligned}
& \mathbb{P} \left(\left\| V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) \right\|^2 \geq 2N \right) \leq \delta, \quad \mathbb{P} \left(\|V_{x_0,k}^T \Sigma_0^{-1} x_0\|^2 \geq 2N \right) \leq \delta, \\
& \mathbb{P} \left(\sum_{t=2}^{N+1} \langle e_k, B_l^T C^T C A x_{t-1} \rangle^2 \geq 2 \left(\frac{2L^6}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \right)^2 \cdot N \right) \leq 2\delta, \\
& \mathbb{P} \left(\sqrt{\sum_{t=2}^{N+1} \langle e_k, B_l^T C^T C A x_{t-1} \rangle^2} \geq \frac{2L^6}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \cdot \sqrt{2N} \right) \leq 2\delta.
\end{aligned} \tag{4.33}$$

Combined with (4.27), we have with the probability at least $1 - 4\delta$ that,

$$\left| \sum_{t=2}^{N+1} u_{t-1,k} \langle e_k, B_l^T C^T C A x_{t-1} \rangle \right| \leq \frac{4L^6 \sigma_u}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right) N}. \tag{4.34}$$

Therefore, we have with the probability at least $1 - 4m \cdot \delta$,

$$\left| \sum_{t=2}^{N+1} u_{t-1}^T B_l^T C^T C A x_{t-1} \right| < m \cdot \frac{4L^6 \sigma_u}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right) N}, \tag{4.35}$$

With the probability at least $1 - 4m \cdot n_B \cdot \delta$,

$$\left| \sum_{t=2}^{N+1} Q_{B,t-1} \cdot e_{t,B} \right| \leq m \cdot n_B \cdot \frac{4L^6 \sigma_u}{1 - \tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right)} N. \quad (4.36)$$

□

4.4.3 Upper bound of $\left\| \sum_{t=2}^{N+1} Q_{B,t-1} C w_{t-1} \right\|$

Lemma 4.4.4. *Under Assumption 4.4.1-4.4.2, when $N \geq 16 \log(\frac{1}{\delta})$, it holds with the probability at least $1 - 3n_B \cdot m \cdot \delta$ that*

$$\left\| \sum_{t=2}^{N+1} Q_{B,t-1} C w_{t-1} \right\| \leq 2\sigma_u \sigma_x \lambda_{\max,B} L \cdot n_B \cdot m \cdot \sqrt{\log \frac{1}{\delta}} \cdot N. \quad (4.37)$$

where $\lambda_{\max,B}$ is defined in Assumption 4.4.1 and L is defined in Assumption 4.4.2.

Proof. We first rewrite $Q_{B,t-1}$ as

$$Q_{B,t-1}^T = \begin{bmatrix} M_{1,1}u_{t-1} & \cdots & M_{1,n_B}u_{t-1} \\ \vdots & \cdots & \vdots \\ M_{m,1}u_{t-1} & \cdots & M_{m,n_B}u_{t-1} \end{bmatrix},$$

where M_i is defined in Assumption 4.4.1. If we denote $q_{k,t-1}^T = \begin{bmatrix} M_{k,1}u_{t-1} & \cdots & M_{k,n_B}u_{t-1} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$Q_{B,t-1} = \begin{bmatrix} q_{1,t-1} & \cdots & q_{m,t-1} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}
Q_{B,t-1}Cw_{t-1} &= \sum_{k=1}^m q_{k,t-1} \cdot (Cw_{t-1})_k = \sum_{k=1}^m \tilde{M}_k \cdot u_{t-1} \cdot (Cw_{t-1})_k, \\
q_{k,t-1} &= \tilde{M}_k u_{t-1}, \quad \tilde{M}_k u_{t-1} \cdot (Cw_{t-1})_k = \begin{bmatrix} (\tilde{M}_k u_{t-1})_1 \cdot (Cw_{t-1})_k \\ \vdots \\ (\tilde{M}_k u_{t-1})_{n_B} \cdot (Cw_{t-1})_k \end{bmatrix}, \\
\| \sum_{t=2}^{N+1} Q_{B,t-1}Cw_{t-1} \| &= \| \sum_{k=1}^m \sum_{t=2}^{N+1} \tilde{M}_k u_{t-1} \cdot (Cw_{t-1})_k \| \\
&\leq \sum_{k=1}^m \| \sum_{t=2}^{N+1} \tilde{M}_k u_{t-1} \cdot (Cw_{t-1})_k \|_1 \leq \sum_{k=1}^m \sum_{l=1}^{n_B} | \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot (Cw_{t-1})_k |.
\end{aligned}$$

where $\tilde{M}_k$ is defined in Assumption 4.4.1. By Lemma 4.7.1 and Assumption 4.4.1, we know for $l = 1, \dots, d$

$$\mathbb{P} \left(\left| \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot (Cw_{t-1})_k \right| \geq \sigma_u \lambda_{\max, B} \sqrt{2 \log \frac{1}{\delta} \sum_{t=2}^{N+1} (Cw_{t-1})_k^2} \right) \leq 2\delta, \quad (4.38)$$

By Laurent-Massart bound [35] and Assumption 4.4.2, when $N \geq 16 \log(\frac{1}{\delta})$,

$$\begin{aligned}
\mathbb{P} \left(\sum_{t=2}^{N+1} (Cw_{t-1})_k^2 \geq 2L^2 \sigma_x^2 N \right) &\leq \delta, \\
\mathbb{P} \left(\left| \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot (Cw_{t-1})_k \right| \geq 2\sigma_u \sigma_x \lambda_{\max, B} L \sqrt{\log \frac{1}{\delta} \cdot N} \right) &\leq 3\delta.
\end{aligned} \quad (4.39)$$

Therefore, we have

$$\mathbb{P} \left(\left\| \sum_{t=2}^{N+1} Q_{B,t-1}Cw_{t-1} \right\| \geq 2\sigma_u \sigma_x \lambda_{\max, B} L \cdot n_B \cdot m \cdot \sqrt{\log \frac{1}{\delta} \cdot N} \right) \leq 3n_B \cdot m \cdot \delta. \quad (4.40)$$

□

4.4.4 Upper bound of $\|\sum_{t=2}^{N+1} Q_{B,t-1} v_t\|$

Lemma 4.4.5. *Under Assumption 4.4.1-4.4.2, when $N \geq 16 \log(\frac{1}{\delta})$, it holds with the probability at least $1 - 3n_B \cdot m \cdot \delta$ that*

$$\left\| \sum_{t=2}^{N+1} Q_{B,t-1} v_t \right\| \leq 2\sigma_u \sigma_y \lambda_{\max, B} \cdot n_B \cdot m \cdot \sqrt{\log \frac{1}{\delta} \cdot N}. \quad (4.41)$$

where $\lambda_{\max, B}$ is defined in Assumption 4.4.1.

Proof. We first rewrite $Q_{B,t-1}$ as

$$Q_{B,t-1}^T = \begin{bmatrix} M_{1,1}u_{t-1} & \cdots & M_{1,n_B}u_{t-1} \\ \vdots & \cdots & \vdots \\ M_{m,1}u_{t-1} & \cdots & M_{m,n_B}u_{t-1} \end{bmatrix},$$

where M_i is defined in Assumption 4.4.1. If we denote $q_{k,t-1}^T = \begin{bmatrix} M_{k,1}u_{t-1} & \cdots & M_{k,n_B}u_{t-1} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$Q_{B,t-1} = \begin{bmatrix} q_{1,t-1} & \cdots & q_{m,t-1} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}
Q_{B,t-1}v_t &= \sum_{k=1}^m q_{k,t-1} \cdot v_{t,k} = \sum_{k=1}^m \tilde{M}_k \cdot u_{t-1} \cdot v_{t,k}, \quad q_{k,t-1} = \tilde{M}_k u_{t-1}, \\
\sum_{t=2}^{N+1} Q_{B,t-1}v_t &= \sum_{k=1}^m \sum_{t=2}^{N+1} \tilde{M}_k u_{t-1} \cdot v_{t,k}, \quad \tilde{M}_k u_{t-1} \cdot v_{t,k} = \begin{bmatrix} (\tilde{M}_k u_{t-1})_1 \cdot v_{t,k} \\ \vdots \\ (\tilde{M}_k u_{t-1})_{n_B} \cdot v_{t,k} \end{bmatrix}, \\
\left\| \sum_{t=2}^{N+1} Q_{B,t-1}v_t \right\| &\leq \sum_{k=1}^m \left\| \sum_{t=2}^{N+1} \tilde{M}_k u_{t-1} \cdot v_{t,k} \right\|_1 \leq \sum_{k=1}^m \sum_{l=1}^{n_B} \left| \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot v_{t,k} \right|.
\end{aligned}$$

where $\tilde{M}_k$ is defined in Assumption 4.4.1. By Lemma 4.7.1 and Assumption 4.4.1, we know for $l = 1, \dots, d$

$$\mathbb{P} \left(\left| \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot v_{t,k} \right| \geq \sigma_u \lambda_{\max, B} \sqrt{2 \log \frac{1}{\delta} \sum_{t=2}^{N+1} v_{t,k}^2} \right) \leq 2\delta, \quad (4.42)$$

By Laurent-Massart bound [35] and Assumption 4.4.2, when $N \geq 16 \log(\frac{1}{\delta})$,

$$\begin{aligned}
\mathbb{P} \left(\sum_{t=2}^{N+1} v_{t,k}^2 \geq 2\sigma_y^2 N \right) &\leq \delta, \\
\mathbb{P} \left(\left| \sum_{t=2}^{N+1} (\tilde{M}_k u_{t-1})_l \cdot v_{t,k} \right| \geq 2\sigma_u \sigma_y \lambda_{\max, B} \sqrt{\log \frac{1}{\delta} \cdot N} \right) &\leq 3\delta.
\end{aligned} \quad (4.43)$$

Therefore, we have

$$\mathbb{P} \left(\left\| \sum_{t=2}^{N+1} Q_{B,t-1}v_t \right\| \geq 2\sigma_u \sigma_y \lambda_{\max, B} \cdot n_B \cdot m \cdot \sqrt{\log \frac{1}{\delta} \cdot N} \right) \leq 3n_B \cdot m \cdot \delta. \quad (4.44)$$

□

4.4.5 Lower bound of $\sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \hat{Q}_{A,t-2}^T$

Lemma 4.4.6. *Under Assumption 4.4.1-4.4.2, when*

$$N \geq \frac{16n_A \cdot \sigma_u^2 \cdot \log \frac{1}{\delta_1}}{\sigma_u^2} \cdot \log \frac{n_A}{\delta_1}, \quad (4.45)$$

with the probability at least $1 - m \cdot \delta_1$,

$$\sum_{t=3}^{N+2} Q_{A,t-2} Q_{A,t-2}^T \succeq N \cdot m \cdot \lambda_{\min,A} \cdot \sigma_u^2 / 2I_{n_A}. \quad (4.46)$$

where $\lambda_{\min,A}$ is defined in Assumption 4.4.1.

Proof. We first rewrite $Q_{A,t-2}$ as

$$Q_{A,t-2}^T = \begin{bmatrix} G_{1,1}u_{t-2} & \cdots & G_{1,n_A}u_{t-2} \\ \vdots & \cdots & \vdots \\ G_{m,1}u_{t-2} & \cdots & G_{m,n_A}u_{t-2} \end{bmatrix},$$

where G_i is defined in Assumption 4.4.1. If we denote $\tilde{q}_{k,t-2}^T = \begin{bmatrix} G_{k,1}u_{t-2} & \cdots & G_{k,n_A}u_{t-2} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$\begin{aligned} Q_{A,t-2} Q_{A,t-2}^T &= \sum_{k=1}^m \tilde{q}_{k,t-2} \tilde{q}_{k,t-2}^T = \sum_{k=1}^m \tilde{G}_k u_{t-2} u_{t-2}^T \tilde{M}_k^T, \\ \sum_{t=3}^{N+2} Q_{A,t-2} Q_{A,t-2}^T &= \sum_{k=1}^m \tilde{G}_k \left(\sum_{t=3}^{N+2} u_{t-2} u_{t-2}^T \right) \tilde{G}_k^T \\ &= \sum_{k=1}^m U_{A,k} \Sigma_{A,k} \left(\sum_{t=3}^{N+2} V_{A,k}^T u_{t-2} u_{t-2}^T V_{A,k} \right) \Sigma_{A,k} U_{A,k}^T. \end{aligned}$$

where $\tilde{G}_k$ is defined in Assumption 4.4.1 and the last inequality uses the economy singular value decomposition of $\tilde{G}_k$ as $\tilde{G}_k = U_{A,k} \Sigma_{A,k} V_{A,k}^T$. Then, we want to apply the Matrix

Chernoff inequality on $\sum_{t=3}^{N+2} V_{A,k}^T u_{t-2} u_{t-2}^T V_{A,k}$ to obtain the lower bound of its eigenvalue for each k where denote $\bar{u}_{k,t-2} = V_{A,k}^T u_{t-2}$ and $\bar{u}_{k,t-2} \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_{n_A})$. We have

$$\begin{aligned} \|\bar{u}_{k,t-2}\|^2 &= \sum_{l=1}^{n_A} \bar{u}_{k,t-2,l}^2, \quad \mathbb{P}(\bar{u}_{k,t-2,l} > \sqrt{2\sigma_u^2 \log \delta_1^{-1}}) \leq \delta_1, \\ \mathbb{P}\left(\|\bar{u}_{k,t-2}\|^2 > n_A \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta_1}\right) &\leq n_A \cdot \delta_1, \quad \lambda_{\max}(\bar{u}_{k,t-2} \bar{u}_{k,t-2}^T) \leq \|\bar{u}_{k,t-2}\|^2, \\ \mathbb{P}\left(\lambda_{\max}(\bar{u}_{k,t-2} \bar{u}_{k,t-2}^T) > n_A \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta_1}\right) &\leq n_A \cdot \delta_1. \end{aligned} \quad (4.47)$$

We also know

$$\lambda_{\min}\left(E \left[\sum_{t=3}^{N+2} \bar{u}_{k,t-2} \bar{u}_{k,t-2}^T \right]\right) = N \cdot \sigma_u^2. \quad (4.48)$$

Then, by Lemma 4.4.2, we have for any $\epsilon \in (0, 1)$,

$$\begin{aligned} &\mathbb{P}\left(\lambda_{\min}\left(\sum_{t=3}^{N+2} \bar{u}_{k,t-2} \bar{u}_{k,t-2}^T\right) \leq (1-\epsilon)N \cdot \sigma_u^2\right) \\ &\leq n_A \cdot \left[\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}}\right]^{N \cdot \sigma_u^2 / (n_A \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta_1})} \\ &\leq n_A \cdot \exp\left(-\frac{N \sigma_u^2}{n_A \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta_1}} \cdot ((1-\epsilon) \log(1-\epsilon) + \epsilon)\right) \\ &\leq n_A \cdot \exp\left(-\frac{N \sigma_u^2}{n_A \cdot \sigma_u^2 \cdot 2 \log \frac{1}{\delta_1}} \cdot \frac{\epsilon^2}{2}\right). \end{aligned} \quad (4.49)$$

Therefore, when $N \geq \frac{4n_A \cdot \sigma_u^2 \cdot \log \frac{1}{\delta_1}}{\sigma_u^2} \cdot \frac{1}{\epsilon^2} \cdot \log \frac{n_A}{\delta_1}$, $\mathbb{P}\left(\lambda_{\min}\left(\sum_{t=3}^{N+2} \bar{u}_{k,t-2} \bar{u}_{k,t-2}^T\right) \leq (1-\epsilon)N \cdot \sigma_u^2\right) < \delta_1$. When we set $\epsilon = \frac{1}{2}$, we have when $N \geq \frac{16n_A \cdot \sigma_u^2 \cdot \log \frac{1}{\delta_1}}{\sigma_u^2} \cdot \log \frac{n_A}{\delta_1}$,

$$\mathbb{P}\left(\lambda_{\min}\left(\sum_{t=2}^{N+1} \bar{u}_{k,t-2} \bar{u}_{k,t-2}^T\right) \leq N \cdot \sigma_u^2 / 2\right) \leq \delta_1. \quad (4.50)$$

Now, we have with the probability at least $1 - m \cdot \delta_1$

$$\begin{aligned} \sum_{t=3}^{N+2} Q_{A,t-2} Q_{A,t-2}^T &= \sum_{k=1}^m U_{A,k} \Sigma_{A,k} \left(\sum_{t=2}^{N+1} V_{A,k}^T u_{t-2} u_{t-2}^T V_{A,k} \right) \Sigma_{A,k} U_{A,k}^T \\ &\succeq N \cdot \sigma_u^2 / 2 \sum_{k=1}^m U_{A,k} \Sigma_{A,k} \Sigma_{A,k} U_{A,k}^T \succeq N \cdot \sigma_u^2 / 2 \sum_{k=1}^m \tilde{G}_k \tilde{G}_k^T \succeq N \cdot m \cdot \sigma_u^2 / 2 \cdot \lambda_{\min,A}. \end{aligned} \quad (4.51)$$

□

4.4.6 Upper bound of $\mathcal{J}_1$

Lemma 4.4.7. *Under Assumption 4.4.1-4.4.2, with the probability at least $1 - n_A \cdot \delta_1$,*

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\| \leq d \cdot n_A \cdot n_B^{3/2} \cdot \sigma_u^2 \cdot C_{A,Q} \cdot \sqrt{\log \left(\frac{11n_B \cdot m}{\delta} \right)} \cdot \sqrt{N}. \quad (4.52)$$

where

$$C_{A,Q} = 4C_B \cdot L^6, \quad C_B = 4 \frac{C_{B,e} + (\sigma_x L + \sigma_y) \lambda_{\max,B}}{\sigma_u \lambda_{\min,B}}, \quad C_{B,e} = \frac{2L^6}{1 - \tilde{\rho}} (\sigma_x + \sigma_u + 1). \quad (4.53)$$

$\lambda_{\min,B}$ and $\lambda_{\max,B}$ are defined in Assumption 4.4.1; L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. We recall that $\mathcal{J}_1 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\|$, $\Delta Q_{A,t-2}^T = Q_{A,t-2}^T - \hat{Q}_{A,t-2}^T$, and $\Delta B = B - \hat{B}$. Then we have as

$$\begin{aligned} \Delta Q_{A,t-2}^T \alpha^* &= \sum_{i=1}^{n_A} C A_i \Delta B u_{t-2} \alpha_i^* = C A \Delta B u_{t-2}, \\ \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* &= \begin{bmatrix} u_{t-2}^T \hat{B}^T A_1^T C^T C A \Delta B u_{t-2} \\ \vdots \\ u_{t-2}^T \hat{B}^T A_{n_A}^T C^T C A \Delta B u_{t-2} \end{bmatrix}, \\ \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\| &\leq \sum_{l=1}^{n_A} \left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A \Delta B u_{t-2} \right|. \end{aligned} \quad (4.54)$$

Then, we want to obtain the upper bound of $|\sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A \Delta B u_{t-2}|$ as

$$\begin{aligned} \|\hat{B}\| &= \|B - \Delta B\| \leq 2\|B\|, \\ |\sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A \Delta B u_{t-2}| &\leq \|\hat{B}^T\| \|A_l^T\| \|C\|^2 \|A\| \|\Delta B\| \sum_{t=3}^{N+2} \|u_{t-2}\|^2 \end{aligned} \quad (4.55)$$

By Laurent-Massart bound [35], when $N \geq 16 \log \frac{1}{\delta}$,

$$\begin{aligned} \mathbb{P} \left(\sum_{t=3}^{N+2} \|\sigma_u^{-1} u_{t-2}\|^2 \geq |a|_1 + 2|a|_2 \sqrt{\log \frac{1}{\delta_1}} + 2|a|_\infty \log \frac{1}{\delta} \right) &\leq \delta, \\ |a|_1 &:= d \cdot N, \quad |a|_2 = \sqrt{N \cdot d}, \quad |a|_\infty = 1. \end{aligned} \quad (4.56)$$

Since $d \cdot N + 2\sqrt{N \cdot d \frac{1}{\delta_1}} + 2 \log \frac{1}{\delta_1} \leq 2N$ when $N \geq 16 \log \left(\frac{1}{\delta} \right) / d$,

$$\mathbb{P} \left(\sum_{t=3}^{N+2} \|u_{t-2}\|^2 \geq 2\sigma_u^2 N \right) \leq \delta_1, \quad (4.57)$$

By $\Delta B = \sum_{h=1}^{n_B} (\beta_h^* - \hat{\beta}_h) B_h$ and Assumption 4.4.2, we have

$$\|\Delta B\| \leq L \|\beta^* - \hat{\beta}\|_1 \leq L \cdot \sqrt{n_B} \cdot \|\beta^* - \hat{\beta}\| \quad (4.58)$$

Combined with Theorem 4.4.1, with the probability at least $1 - \delta_1$,

$$|\sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A \Delta B u_{t-2}| \leq 4d \cdot \sqrt{n_B} \cdot C_B \cdot L^6 \cdot \sigma_u^2 \cdot \log \frac{1}{\delta_1} \cdot \sqrt{N} \quad (4.59)$$

where $C_B = 4 \frac{C_{B,e} + (\sigma_x L + \sigma_y) \lambda_{\max, B}}{\sigma_u \lambda_{\min, B}}$. Then, with the probability at least $1 - n_A \delta_1$,

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\| \leq 4d \cdot C_B \cdot n_A \cdot n_B^{3/2} \cdot L^6 \cdot \sigma_u^2 \cdot \sqrt{\log \left(\frac{11n_B \cdot m}{\delta} \right)} \cdot \sqrt{N}. \quad (4.60)$$

□

4.4.7 Upper bound of $\mathcal{J}_2$

Lemma 4.4.8. *Under Assumption 4.4.1-4.4.2, with the probability at least $1 - n_A \cdot \delta_1$,*

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C \Delta B u_{t-1} \right\| \leq d \cdot n_A \cdot n_B^{3/2} \cdot \sigma_u^2 \cdot \frac{C_{A,Q}}{L} \cdot \sqrt{\log \left(\frac{11n_B \cdot m}{\delta} \right)} \cdot \sqrt{N}. \quad (4.61)$$

where $C_{A,Q}$ is defined in Lemma 4.4.7; $\lambda_{\min,B}$ and $\lambda_{\max,B}$ are defined in Assumption 4.4.1; L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. We recall that $\mathcal{J}_2 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C \Delta B u_{t-1} \right\|$ and $\Delta B = B - \hat{B}$. Then we have

$$\begin{aligned} \hat{Q}_{A,t-2} C \Delta B u_{t-1} &= \begin{bmatrix} u_{t-2}^T \hat{B}^T A_1^T C^T C \Delta B u_{t-1} \\ \vdots \\ u_{t-2}^T \hat{B}^T A_{n_A}^T C^T C \Delta B u_{t-1} \end{bmatrix}, \\ \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C \Delta B u_{t-1} \right\| &\leq \sum_{l=1}^{n_A} \left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C \Delta B u_{t-1} \right|. \end{aligned} \quad (4.62)$$

Then, we want to obtain the upper bound of $\left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C \Delta B u_{t-1} \right|$ as

$$\begin{aligned} \|\hat{B}\| &= \|B - \Delta B\| \leq 2\|B\|, \\ \left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C \Delta B u_{t-2} \right| &\leq \|\hat{B}^T\| \|A_l^T\| \|C\|^2 \|\Delta B\| \sum_{t=3}^{N+2} \|u_{t-2}\|^2 \end{aligned} \quad (4.63)$$

By Laurent-Massart bound [35], when $N \geq 16 \log \frac{1}{\delta}$,

$$\mathbb{P} \left(\sum_{t=3}^{N+2} \|\sigma_u^{-1} u_{t-2}\|^2 \geq |a|_1 + 2|a|_2 \sqrt{\log \frac{1}{\delta_1}} + 2|a|_\infty \log \frac{1}{\delta} \right) \leq \delta, \quad (4.64)$$

$$|a|_1 := d \cdot N, \quad |a|_2 = \sqrt{N \cdot d}, \quad |a|_\infty = 1.$$

Since $d \cdot N + 2\sqrt{N \cdot d \frac{1}{\delta_1}} + 2 \log \frac{1}{\delta_1} \leq 2N$ when $N \geq 16 \log \left(\frac{1}{\delta} \right) / d$,

$$\mathbb{P} \left(\sum_{t=3}^{N+2} \|u_{t-2}\|^2 \geq 2\sigma_u^2 N \right) \leq \delta_1, \quad (4.65)$$

By $\Delta B = \sum_{h=1}^{n_B} (\beta_h^* - \hat{\beta}_h) B_h$ and Assumption 4.4.2, we have

$$\|\Delta B\| \leq L \|\beta^* - \hat{\beta}\|_1 \leq L \cdot \sqrt{n_B} \cdot \|\beta^* - \hat{\beta}\| \quad (4.66)$$

Combined with Theorem 4.4.1, with the probability at least $1 - \delta_1$,

$$\left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C \Delta B u_{t-2} \right| \leq 4d \cdot \sqrt{n_B} \cdot C_B \cdot L^5 \cdot \sigma_u^2 \cdot \log \frac{1}{\delta_1} \cdot \sqrt{N} \quad (4.67)$$

where $C_B = 4 \frac{C_{B,e} + (\sigma_x L + \sigma_y) \lambda_{\max, B}}{\sigma_u \lambda_{\min, B}}$. Then, with the probability at least $1 - n_A \delta_1$,

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} \Delta Q_{A,t-2}^T \alpha^* \right\| \leq 4d \cdot C_B \cdot n_A \cdot n_B^{3/2} \cdot L^5 \cdot \sigma_u^2 \cdot \sqrt{\log \left(\frac{11n_B \cdot m}{\delta} \right)} \cdot \sqrt{N}. \quad (4.68)$$

□

4.4.8 Upper bound of $\mathcal{J}_3$

Lemma 4.4.9. *Under Assumption 4.4.1-4.4.2, when $N > 16 \log \frac{1}{\delta}$, with the probability at least $1 - 4m \cdot n_A \cdot \delta$,*

$$\left\| \sum_{t=3}^{N+2} Q_{A,t-2} e_{t,A} \right\| \leq m \cdot n_A \cdot 2\sigma_u \cdot C_{A,e} \cdot \sqrt{\log \left(\frac{1}{\delta} \right) N}, \quad C_{A,e} = \frac{4L^8}{1 - \tilde{\rho}} (\sigma_x + \sigma_u + 1). \quad (4.69)$$

where L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. We first write $Q_{B,t-1} \cdot e_{t,B}$ as

$$\begin{aligned}
Q_{A,t-2} \cdot e_{t,A} &= \begin{bmatrix} u_{t-2}^T \hat{B}^T A_1^T C^T C A^2 x_{t-2} \\ \vdots \\ u_{t-2}^T \hat{B}^T A_{n_A}^T C^T C A^2 x_{t-2} \end{bmatrix}, \\
\sum_{t=3}^{N+2} Q_{A,t-2} \cdot e_{t,A} &= \begin{bmatrix} \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_1^T C^T C A^2 x_{t-2} \\ \vdots \\ \sum_{t=2}^{N+1} u_{t-1}^T \hat{B}^T A_{n_A}^T C^T C A^2 x_{t-2} \end{bmatrix}, \\
\left\| \sum_{t=3}^{N+2} Q_{A,t-2} \cdot e_{t,A} \right\| &\leq \sum_{l=1}^{n_A} \left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A^2 x_{t-2} \right|.
\end{aligned}$$

For each $l = 1, \dots, n_A$, we have

$$\begin{aligned}
\sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A^2 x_{t-2} &= \sum_{k=1}^m \sum_{t=3}^{N+2} u_{t-2,k} (\hat{B}^T A_l^T C^T C A^2 x_{t-2})_k \\
&= \sum_{k=1}^m \sum_{t=3}^{N+2} u_{t-2,k} \left\langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \right\rangle,
\end{aligned} \tag{4.70}$$

where e_k is an one hot vector with 1 in k -th entry and 0 in other entries. By Lemma 4.7.1, we know with the probability at least $1 - 2\delta$

$$\left| \sum_{t=3}^{N+2} u_{t-2,k} \left\langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \right\rangle \right| \leq \sigma_u \sqrt{2 \log \frac{1}{\delta} \sum_{t=3}^{N+2} \left\langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \right\rangle^2}. \tag{4.71}$$

Since $x_t = A^t x_0 + \sum_{k=0}^{t-1} A^k (Bu_{t-1-k} + w_{t-1-k})$, we can write

$$\begin{aligned} \mathbf{x}_N &= S_{N,x} \mathbf{w}_N + S_{N,u} \mathbf{u}_N + C_{N,x} x_0, \\ S_{N,x} &= \begin{bmatrix} I_n & & & \\ A & I_n & & \\ \vdots & \ddots & \ddots & \\ A^{N-1} & \dots & A & I_n \end{bmatrix}, \quad S_{N,u} = \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \ddots & \ddots & \\ A^{N-1}B & \dots & AB & B \end{bmatrix}, \\ C_{N,x} &= \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{u}_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}, \quad \mathbf{w}_N = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}. \end{aligned}$$

We have

$$\mathbf{x}_N = \underbrace{\begin{bmatrix} S_{N,x} & S_{N,u} \end{bmatrix}}_{M_{S,N}} \underbrace{\begin{bmatrix} \sigma_x I_{n \cdot N} & \mathbf{0} \\ \mathbf{0} & \sigma_u I_{d \cdot N} \end{bmatrix}}_{M_{x,N}} M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} + C_{N,x} x_0 \quad (4.72)$$

To have an upper bound of $M_{S,N}$, we want the upper bound of each block. We first look at $S_{N,x}$ which can be written as

$$S_{N,x} = \begin{bmatrix} I_n & & & \\ & I_n & & \\ \vdots & \ddots & \ddots & \\ & \dots & & I_n \end{bmatrix} + \begin{bmatrix} \mathbf{0} & & & \\ A & & & \\ \vdots & \ddots & \ddots & \\ \dots & A & \mathbf{0} \end{bmatrix} + \dots$$

Therefore, we have $\|S_{N,x}\| \leq \frac{L}{1-\tilde{\rho}}$. Similarly, we have

$$\|S_{N,u}\| \leq \frac{L^2}{1-\tilde{\rho}}, \quad \|C_{N,x}\| \leq \frac{L}{1-\tilde{\rho}}, \quad \|M_{S,N}\| \leq \|S_{N,x}\| + \|S_{N,u}\| \leq \frac{2L^2}{1-\tilde{\rho}}. \quad (4.73)$$

where we use Assumption 4.4.2. Then, we have

$$\begin{aligned} & \sum_{t=3}^{N+2} \langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \rangle^2 \\ &= \left\| \underbrace{\text{diag}(e_k^T \hat{B}^T A_l^T C^T C A^2)}_{M_{E,k}} \left(M_{S,N} M_{x,N} \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) + C_{N,x} \Sigma_0 \Sigma_0^{-1} x_0 \right) \right\|^2 \end{aligned}$$

where $M_{E,k} \in \mathbb{R}^{N \times (d \cdot N)}$. We apply the singular value decomposition on $M_{E,k} M_{S,N} M_{x,N} = U_{x,k} \Sigma_{x,k} V_{x,k}^T$ and $M_{E,k} C_{N,x} \Sigma_0 = U_{x_0,k} \Sigma_{x_0,k} V_{x_0,k}^T$ where

$$\begin{aligned} \|U_{x,k} \Sigma_{x,k}\| &\leq \|M_{E,k} M_{S,N} M_{x,N}\| \leq \frac{4L^8}{1 - \tilde{\rho}} (\sigma_x + \sigma_u), \\ \|U_{x_0,k} \Sigma_{x_0,k}\| &\leq \|M_{E,k} C_{N,x} \Sigma_0\| \leq \frac{2L^8}{1 - \tilde{\rho}}. \end{aligned} \tag{4.74}$$

. We have

$$\begin{aligned} & \sum_{t=3}^{N+2} \langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \rangle^2 \\ &= \left\| U_{x,k} \Sigma_{x,k} V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) + U_{x_0,k} \Sigma_{x_0,k} V_{x_0,k}^T x_0 \right\|^2 \\ &\leq \left(\|U_{x,k} \Sigma_{x,k}\| \|V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right)\| + \|U_{x_0,k} \Sigma_{x_0,k}\| \|V_{x_0,k}^T \Sigma_0^{-1} x_0\| \right)^2 \end{aligned} \tag{4.75}$$

By Laurent-Massart bound [35], when $N \geq 16 \log \frac{1}{\delta}$

$$\begin{aligned}
& \mathbb{P} \left(\left\| V_{x,k}^T \left(M_{x,N}^{-1} \begin{bmatrix} \mathbf{w}_N \\ \mathbf{u}_N \end{bmatrix} \right) \right\|^2 \geq 2N \right) \leq \delta, \quad \mathbb{P} \left(\|V_{x_0,k}^T \Sigma_0^{-1} x_0\|^2 \geq 2N \right) \leq \delta, \\
& \mathbb{P} \left(\sum_{t=3}^{N+2} \langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \rangle^2 \geq 2 \left(\frac{4L^8}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \right)^2 \cdot N \right) \leq 2\delta, \quad (4.76) \\
& \mathbb{P} \left(\sqrt{\sum_{t=3}^{N+2} \langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \rangle^2} \geq \frac{4L^8}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \cdot \sqrt{2N} \right) \leq 2\delta.
\end{aligned}$$

Combined with (4.71), we have with the probability at least $1 - 4\delta$ that

$$\left| \sum_{t=3}^{N+2} u_{t-2,k} \langle e_k, \hat{B}^T A_l^T C^T C A^2 x_{t-2} \rangle \right| \leq \frac{8L^8 \sigma_u}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right) N}. \quad (4.77)$$

Therefore, we have with the probability at least $1 - 4m \cdot \delta$,

$$\left| \sum_{t=3}^{N+2} u_{t-2}^T \hat{B}^T A_l^T C^T C A^2 x_{t-2} \right| < m \cdot \frac{8L^8 \sigma_u}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right) N}, \quad (4.78)$$

With the probability at least $1 - 4m \cdot n_A \cdot \delta$,

$$\left| \sum_{t=3}^{N+2} Q_{A,t-2} \cdot e_{t,A} \right| \leq m \cdot n_B \cdot \frac{8L^8 \sigma_u}{1-\tilde{\rho}} (\sigma_x + \sigma_u + 1) \sqrt{\log \left(\frac{1}{\delta} \right) N}. \quad (4.79)$$

□

4.4.9 Upper bound of $\mathcal{J}_4$ - $\mathcal{J}_6$

Lemma 4.4.10. *Under Assumption 4.4.1-4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$, it holds with the probability at least $1 - 3n_A \cdot m \cdot \delta_1$ that*

$$\left\| \sum_{t=3}^{N+2} Q_{A,t-2} C w_{t-1} \right\| \leq 2\sigma_u \sigma_x \lambda_{\max,A} L \cdot n_A \cdot m \cdot \sqrt{\log\left(\frac{1}{\delta_1}\right) \cdot N}. \quad (4.80)$$

where $\lambda_{\max,A}$ is defined in Assumption 4.4.1 and L is defined in Assumption 4.4.2.

Proof. We first recall that $\mathcal{J}_4 = \left\| \sum_{t=3}^{N+2} Q_{A,t-2} C w_{t-1} \right\|$ and rewrite $Q_{A,t-1}$ as

$$Q_{A,t-2}^T = \begin{bmatrix} G_{1,1}u_{t-2} & \cdots & G_{1,n_A}u_{t-2} \\ \vdots & \cdots & \vdots \\ G_{m,1}u_{t-2} & \cdots & G_{m,n_A}u_{t-2} \end{bmatrix},$$

where G_i is defined in Assumption 4.4.1. If we denote $\tilde{q}_{k,t-2}^T = \begin{bmatrix} G_{k,1}u_{t-2} & \cdots & G_{k,n_A}u_{t-2} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$Q_{A,t-2} = \begin{bmatrix} \tilde{q}_{1,t-2} & \cdots & \tilde{q}_{m,t-2} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}
Q_{A,t-2}Cw_{t-1} &= \sum_{k=1}^m \tilde{q}_{k,t-2} \cdot (Cw_{t-1})_k = \sum_{k=1}^m \tilde{G}_k \cdot u_{t-2} \cdot (Cw_{t-1})_k, \\
\tilde{q}_{k,t-2} &= \tilde{G}_k u_{t-2}, \quad \tilde{G}_k u_{t-2} \cdot (Cw_{t-1})_k = \begin{bmatrix} (\tilde{G}_k u_{t-2})_1 \cdot (Cw_{t-1})_k \\ \vdots \\ (\tilde{G}_k u_{t-2})_{n_A} \cdot (Cw_{t-1})_k \end{bmatrix}, \\
\| \sum_{t=3}^{N+2} Q_{A,t-2}Cw_{t-1} \| &= \| \sum_{k=1}^m \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot (Cw_{t-1})_k \| \\
&\leq \sum_{k=1}^m \| \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot (Cw_{t-1})_k \|_1 \leq \sum_{k=1}^m \sum_{l=1}^{n_A} | \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (Cw_{t-1})_k |.
\end{aligned}$$

where $\tilde{G}_k$ is defined in Assumption 4.4.1. By Lemma 4.7.1 and Assumption 4.4.1, we know for $l = 1, \dots, d$

$$\mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (Cw_{t-1})_k \right| \geq \sigma_u \lambda_{\max,A} \sqrt{2 \log \frac{1}{\delta_1} \sum_{t=3}^{N+2} (Cw_{t-1})_k^2} \right) \leq 2\delta_1, \quad (4.81)$$

By Laurent-Massart bound[35] and Assumption 4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$,

$$\begin{aligned}
\mathbb{P} \left(\sum_{t=3}^{N+2} (Cw_{t-1})_k^2 \geq 2L^2 \sigma_x^2 N \right) &\leq \delta, \\
\mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (Cw_{t-1})_k \right| \geq 2\sigma_u \sigma_x \lambda_{\max,A} L \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N} \right) &\leq 3\delta.
\end{aligned} \quad (4.82)$$

Therefore, we have

$$\mathbb{P} \left(\left\| \sum_{t=3}^{N+2} Q_{A,t-2}Cw_{t-1} \right\| \geq 2\sigma_u \sigma_x \lambda_{\max,A} L \cdot n_A \cdot m \cdot \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N} \right) \leq 3n_A \cdot m \cdot \delta. \quad (4.83)$$

□

Lemma 4.4.11. *Under Assumption 4.4.1-4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$, it holds with the probability at least $1 - 3n_A \cdot m \cdot \delta_1$ that*

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C A w_{t-2} \right\| \leq 2\sigma_u \sigma_x \lambda_{\max,A} L^2 \cdot n_A \cdot m \cdot \sqrt{\log\left(\frac{1}{\delta_1}\right) \cdot N}. \quad (4.84)$$

where $\lambda_{\max,A}$ is defined in Assumption 4.4.1.

Proof. We first recall that $\mathcal{J}_5 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C A w_{t-2} \right\|$ and rewrite $Q_{A,t-1}$ as

$$Q_{A,t-2}^T = \begin{bmatrix} G_{1,1}u_{t-2} & \cdots & G_{1,n_A}u_{t-2} \\ \vdots & \cdots & \vdots \\ G_{m,1}u_{t-2} & \cdots & G_{m,n_A}u_{t-2} \end{bmatrix},$$

where G_i is defined in Assumption 4.4.1. If we denote $\tilde{q}_{k,t-2}^T = \begin{bmatrix} G_{k,1}u_{t-2} & \cdots & G_{k,n_A}u_{t-2} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$Q_{A,t-2} = \begin{bmatrix} \tilde{q}_{1,t-2} & \cdots & \tilde{q}_{m,t-2} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}
Q_{A,t-2}CAw_{t-2} &= \sum_{k=1}^m \tilde{q}_{k,t-2} \cdot (CAw_{t-2})_k = \sum_{k=1}^m \tilde{G}_k \cdot u_{t-2} \cdot (CAw_{t-2})_k, \\
\tilde{q}_{k,t-2} &= \tilde{G}_k u_{t-2}, \quad \tilde{G}_k u_{t-2} \cdot (CAw_{t-2})_k = \begin{bmatrix} (\tilde{G}_k u_{t-2})_1 \cdot (CAw_{t-2})_k \\ \vdots \\ (\tilde{G}_k u_{t-2})_{n_A} \cdot (CAw_{t-2})_k \end{bmatrix}, \\
\| \sum_{t=3}^{N+2} Q_{A,t-2}CAw_{t-2} \| &= \| \sum_{k=1}^m \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot (CAw_{t-2})_k \| \\
&\leq \sum_{k=1}^m \| \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot (CAw_{t-2})_k \|_1 \leq \sum_{k=1}^m \sum_{l=1}^{n_A} | \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (CAw_{t-2})_k |.
\end{aligned}$$

where $\tilde{G}_k$ is defined in Assumption 4.4.1. By Lemma 4.7.1 and Assumption 4.4.1, we know for $l = 1, \dots, d$

$$\mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (CAw_{t-2})_k \right| \geq \sigma_u \lambda_{\max,A} \sqrt{2 \log \frac{1}{\delta_1} \sum_{t=3}^{N+2} (CAw_{t-2})_k^2} \right) \leq 2\delta_1, \quad (4.85)$$

By Laurent-Massart bound[35] and Assumption 4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$,

$$\begin{aligned}
\mathbb{P} \left(\sum_{t=3}^{N+2} (CAw_{t-2})_k^2 \geq 2L^4 \sigma_x^2 N \right) &\leq \delta_1, \\
\mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot (CAw_{t-2})_k \right| \geq 2\sigma_u \sigma_x \lambda_{\max,A} L^2 \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N} \right) &\leq 3\delta_1.
\end{aligned} \quad (4.86)$$

Therefore, we have with the probability at least $1 - 3n_A \cdot m \cdot \delta_1$

$$\| \sum_{t=3}^{N+2} Q_{A,t-2}CAw_{t-2} \| \leq 2\sigma_u \sigma_x \lambda_{\max,A} L^2 \cdot n_A \cdot m \cdot \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N}. \quad (4.87)$$

□

Lemma 4.4.12. *Under Assumption 4.4.1-4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$, it holds with the probability at least $1 - 3n_A \cdot m \cdot \delta_1$ that*

$$\left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} v_t \right\| \leq 2\sigma_u \sigma_y \lambda_{\max,A} \cdot n_A \cdot m \cdot \sqrt{\log \frac{1}{\delta} \cdot N}. \quad (4.88)$$

where $\lambda_{\max,A}$ is defined in Assumption 4.4.1.

Proof. We first recall that $\mathcal{J}_5 = \left\| \sum_{t=3}^{N+2} \hat{Q}_{A,t-2} C A w_{t-2} \right\|$ and rewrite $Q_{A,t-1}$ as

$$Q_{A,t-2}^T = \begin{bmatrix} G_{1,1} u_{t-2} & \cdots & G_{1,n_A} u_{t-2} \\ \vdots & \cdots & \vdots \\ G_{m,1} u_{t-2} & \cdots & G_{m,n_A} u_{t-2} \end{bmatrix},$$

where G_i is defined in Assumption 4.4.1. If we denote $\tilde{q}_{k,t-2}^T = \begin{bmatrix} G_{k,1} u_{t-2} & \cdots & G_{k,n_A} u_{t-2} \end{bmatrix}$ for $k = 1, \dots, m$, then we have

$$Q_{A,t-2} = \begin{bmatrix} \tilde{q}_{1,t-2} & \cdots & \tilde{q}_{m,t-2} \end{bmatrix}$$

Therefore, we have

$$\begin{aligned} Q_{A,t-2} v_t &= \sum_{k=1}^m \tilde{q}_{k,t-2} \cdot v_{t,k} = \sum_{k=1}^m \tilde{G}_k \cdot u_{t-2} \cdot v_{t,k}, \quad \tilde{q}_{k,t-2} = \tilde{G}_k u_{t-2}, \\ \sum_{t=3}^{N+2} Q_{A,t-2} v_t &= \sum_{k=1}^m \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot v_{t,k}, \quad \tilde{G}_k u_{t-2} \cdot v_{t,k} = \begin{bmatrix} (\tilde{G}_k u_{t-2})_1 \cdot v_{t,k} \\ \vdots \\ (\tilde{G}_k u_{t-2})_{n_A} \cdot v_{t,k} \end{bmatrix}, \\ \left\| \sum_{t=3}^{N+2} Q_{A,t-2} v_t \right\| &\leq \sum_{k=1}^m \left\| \sum_{t=3}^{N+2} \tilde{G}_k u_{t-2} \cdot v_{t,k} \right\|_1 \leq \sum_{k=1}^m \sum_{l=1}^{n_A} \left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot v_{t,k} \right|. \end{aligned}$$

where $\tilde{G}_k$ is defined in Assumption 4.4.1. By Lemma 4.7.1 and Assumption 4.4.1, we know

for $l = 1, \dots, d$

$$\mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l v_{t,k} \right| \geq \sigma_u \lambda_{\max, A} \sqrt{2 \log \frac{1}{\delta_1} \sum_{t=3}^{N+2} v_{t,k}^2} \right) \leq 2\delta_1, \quad (4.89)$$

By Laurent-Massart bound[35] and Assumption 4.4.2, when $N \geq 16 \log(\frac{1}{\delta_1})$,

$$\begin{aligned} \mathbb{P} \left(\sum_{t=3}^{N+2} v_{t,k}^2 \geq 2\sigma_y^2 N \right) &\leq \delta_1, \\ \mathbb{P} \left(\left| \sum_{t=3}^{N+2} (\tilde{G}_k u_{t-2})_l \cdot v_{t,k} \right| \geq 2\sigma_u \sigma_y \lambda_{\max, A} \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N} \right) &\leq 3\delta_1. \end{aligned} \quad (4.90)$$

Therefore, we have

$$\mathbb{P} \left(\left\| \sum_{t=3}^{N+2} Q_{A,t-2} v_t \right\| \geq 2\sigma_u \sigma_y \lambda_{\max, A} \cdot n_A \cdot m \cdot \sqrt{\log \left(\frac{1}{\delta_1} \right) \cdot N} \right) \leq 3n_A \cdot m \cdot \delta_1. \quad (4.91)$$

□

4.5 EM-like algorithm for Partially Observed LQR

A major limitation of the OLS-based approaches [54, 34] lies in their reliance on the stability of the system matrix of A . While this assumption is essential for the theoretical guarantees of the OLS method, we currently see no clear path to relaxing it within that framework to accommodate a broader class of systems. Therefore, we turn our attention to alternative methods. The classical Expectation-Maximization (EM) algorithm has previously been applied to parameter estimation in partially observed LQR systems. Drawing inspiration from this approach, we propose an EM-like algorithm that significantly reduces the computational cost compared to the classical EM algorithm. Although the current version of the proposed algorithm still requires the stability of A , we believe it lays the groundwork for future ex-

tensions that may relax this restriction. Toward this goal, we outline several directions for future research aimed at relaxing the stability assumption.

In this section, we first review the classic EM algorithm for partially observed LQR in Section 4.5.1. In Section 4.5.2, we give the details of the proposed EM-like algorithm. In Section 4.5.3, we give the outline and partial proof of the theoretical proof for the proposed algorithm. Finally, we outline future directions for completing the theoretical analysis of the current version of the proposed algorithm and present preliminary ideas for relaxing the stability assumption on A in Section 4.5.4.

4.5.1 Review of Classic EM Algorithm for Partially Observed LQR

We first want to introduce how the classic Expectation-Maximization (EM) algorithm is designed for the partially observed linear quadratic regulator (LQR) problem with stationary dynamics [65]. Authors consider the following state and observation equations

$$\begin{aligned} x_t &= Ax_{t-1} + w_t, \\ y_t &= Cx_{t-1} + v_t. \end{aligned} \tag{4.92}$$

where x_t is not observed while y_t is observed, $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$, $w_t \sim \mathcal{N}(\mathbf{0}, \Sigma_w)$, and $v_t \sim \mathcal{N}(\mathbf{0}, \Sigma_v)$. They assume $\Theta = (\mu_0, \Sigma_0, \Sigma_w, \Sigma_v, A)$ is unknown. The basic idea is if the state variables, $x_{0:N} = \{x_0, x_1, \dots, x_N\}$, are observable, then $\{x_{0:N}, y_{1:N}\}$ can be considered as the complete data, with joint density

$$p_{\Theta}(x_{0:N}, y_{1:N}) = p_{\mu_0, \Sigma_0}(x_0) \prod_{t=1}^N p_{A, \Sigma_w}(x_t | x_{t-1}) \prod_{t=1}^N p_{\Sigma_v}(y_t | x_t) \tag{4.93}$$

By the Gaussian assumption and ignoring constants, the equation (4.93) can be simplified as

$$\begin{aligned}
-2 \log L_{X,Y}(\Theta) = & \log |\Sigma_0| + (x_0 - \mu_0)^T \Sigma_0 (x_0 - \mu_0) \\
& + N \log |\Sigma_w| + \sum_{t=1}^N (x_t - Ax_{t-1})^T \Sigma_w^{-1} (x_t - Ax_{t-1}) \\
& + N \log |\Sigma_v| + \sum_{t=1}^N (y_t - Cx_t)^T \Sigma_v^{-1} (y_t - Cx_t)
\end{aligned} \tag{4.94}$$

Therefore, if the complete data is available, the maximum likelihood estimation (MLE) of Θ can be obtained by minimizing eq. (4.94). Since the complete data is not accessible in the current setup, the EM algorithm as an iterative approach is used to find the MLEs of Θ by repeatedly maximizing the conditional expectation of the complete data likelihood with respect to the observed data $\{y_t\}_{t=1}^N$. Before giving the complete version of the EM algorithm for learning the unknown parameters of the partially observed LQR, we first recall Property 4.5.1-4.5.3 defined in [65] for the EM algorithm.

Property 4.5.1 (Property 6.1 in [65]). *For the state space defined in (4.92), for $t = 1, \dots, N$, we define the Kalman K_t as follows*

$$K_t = P_t^{t-1} C^T \left[C P_t^{t-1} C^T + \Sigma_v \right]^{-1},$$

where

$$P_t^{t-1} = C P_{t-1}^{t-1} C^T + \Sigma_w, \quad P_t^t = [I - K_t C] P_t^{t-1}.$$

Then, with the initial condition $x_0^0 = \mu_0$ and $P_0^0 = \Sigma_0$, the prediction x_t is obtained by

$$x_t^{t-1} = Ax_{t-1}^{t-1}, \quad x_t^t = x_t^{t-1} + K_t \epsilon_t, \tag{4.95}$$

The innovation error (prediction error) ϵ_t and its corresponding variance-covariance matrix

$\Sigma_t(\Theta)$ are

$$\begin{aligned}\epsilon_t &= y_t - E[y_t|y_{1:t-1}] = y_t - Cx_t^{t-1}, \\ \Sigma_t(\Theta) &= \text{Var}(\epsilon_t) = CP_t^{t-1}C^T + \Sigma_v.\end{aligned}\tag{4.96}$$

Property 4.5.2 (Property 6.2 in [65]). *For the state space defined in (4.92) and $t = N, N-1, \dots, 1$, x_{t-1}^N and P_{t-1}^N are obtained via*

$$\begin{aligned}x_{t-1}^N &= x_{t-1}^{t-1} + J_{t-1}(x_t^N - x_t^{t-1}), \quad P_{t-1}^N = P_{t-1}^{t-1} + J_{t-1} \left(P_t^N - P_t^{t-1} \right) J_{t-1}^T, \\ J_{t-1} &= P_{t-1}^{t-1} A^T [P_{t-1}^{t-1}]^{-1}.\end{aligned}$$

where Property 4.5.1 is used to obtain initial conditions x_N^N and P_N^N .

Property 4.5.3 (Property 6.3 in [65]). *For the state space defined in (4.92), first use Property 4.5.1-4.5.2 to obtain $\{K_t\}_{t=1}^N$, $\{J_t\}_{t=1}^N$, and P_N^N . Then, with the initial condition $P_{N,N-1}^N = (1 - K_N C) A P_{N-1}^{N-1}$,*

$$P_{t-1,t-2}^N = P_{t-1}^{t-1} J_{t-2}^T + J_{t-1} (P_{t,t-1}^N - A P_{t-1}^{t-1}) J_{t-2}^T, \quad t = N, N-1, \dots, 2.$$

Then, we are ready to summarize the complete EM algorithm introduced in [65] as follows
Initialization:

1. Choose the initial guess for the parameter $\Theta^{(0)} = \{\mu_0, \Sigma_0, \Sigma_w, \Sigma_v, A\}$, and use (4.97) to compute the incomplete-data likelihood $-\log L_Y \left(\Theta^{(0)} \right)$.

$$-\log L_Y(\Theta) = \frac{1}{2} \sum_{t=1}^N \log |\Sigma_t(\Theta)| + \frac{1}{2} \sum_{t=1}^N \epsilon_t(\Theta)^T \Sigma_t(\Theta)^{-1} \epsilon_t(\Theta) \tag{4.97}$$

where N is the number of samples; $\epsilon_t(\Theta)$ and $\Sigma_t(\Theta)$ are defined in Property 4.5.1.

For iteration τ , ($\tau = 1, 2, \dots$):

2. E-step: (Smoothing Procedure): Use Property 4.5.1-4.5.3 with $\Theta^{(\tau-1)}$ (the estimator of the last iteration) to obtain the smoothed values $\{x_t^N\}_{t=1}^N$, $\{P_t^N\}_{t=1}^N$ and $\{P_{t,t-1}^N\}_{t=1}^N$.
 (Conditional Expectation): Calculate

$$Q\left(\Theta|\Theta^{(\tau-1)}\right) = E\left\{-2\log L_{X,Y}(\Theta)|y_{1:N}, \Theta^{(\tau-1)}\right\}$$

where $-2\log L_{X,Y}(\Theta)$ is defined in (4.94).

3. M-step: Using the fact that $E[x_t x_t^T | y_{1:N}] = x_t^N (x_t^N)^T + P_{t,t}^N$ and $E[x_t x_{t-1}^T | y_{1:N}] = x_t^N (x_{t-1}^N)^T + P_{t,t-1}^N$ for $t \leq N$, maximize $Q\left(\Theta|\Theta^{(\tau-1)}\right)$ to get $\Theta^{(\tau)}$.
 4. Compute $-\log L_Y(\Theta^{(\tau)})$, the incomplete-data likelihood.
 5. Repeat Steps (2)-(4) until convergence criteria met.

4.5.2 Proposed Generalized EM algorithm

We consider the discrete-time finite-horizon partially observed linear quadratic regulator (LQR) problem with the stationary dynamics:

$$\begin{aligned} \min_{\substack{\{x_t\}_{t \in [T+1]} \\ \{u_t\}_{t \in [T]}}} & \frac{1}{2} x_T^T Q x_T + \sum_{t=0}^{T-1} \frac{1}{2} x_t^T Q x_t + \frac{1}{2} u_t^T R u_t, \\ \text{s.t. } & x_0 = \bar{x}_0, \\ & x_{t+1} = A x_t + B u_t + w_t, \quad t \in [T], \\ & y_t = C x_t + D u_t + v_t, \quad t \in [1, T+1]. \end{aligned} \tag{4.98}$$

where $x_0 \sim (\mathbf{0}, \Sigma_0)$ and

$$\begin{aligned} A &= \sum_{j=1}^{n_A} \alpha_j A_j, \quad \alpha_j \in \mathbb{R}, \quad A_j \in \mathbb{R}^{n \times n} \quad \forall j \in [n_A], \\ B &= \sum_{j=1}^{n_B} \beta_j B_j, \quad \beta_j \in \mathbb{R}, \quad B_j \in \mathbb{R}^{n \times d} \quad \forall j \in [n_B] \end{aligned} \tag{4.99}$$

C and D are sparse matrices, T is the length of horizon, the state variable $x_t \in \mathbb{R}^n$ is not observable, $y_t \in \mathbb{R}^m$ is the observation variable, $u_t \in \mathbb{R}^d$ is the control variable, $w_t \sim \mathcal{N}(\mathbf{0}, \sigma_x^2 I_n)$ is the random noise, and $v_t \sim \mathcal{N}(\mathbf{0}, \sigma_y^2 I_m)$ is the random noise.

We assume we know σ_x^2 , σ_y^2 , Σ_0 , C , and D , but do not know $\alpha = \begin{bmatrix} \alpha_1 & \dots & \alpha_{n_A} \end{bmatrix}$ and $\beta = \begin{bmatrix} \beta_1 & \dots & \beta_{n_B} \end{bmatrix}$. We want to extend the classic EM algorithm in Section 4.3 to our setting to learn unknown parameter α and β . We propose an algorithm in Algorithm 4.

Algorithm 4 Generalized EM Algorithm for Stationary partially observed LQR

- 1: **Input:** initial guess of $\hat{x}_0$, $\alpha^{(0)} = \{\alpha_j^{(0)}\}_{j \in [n_A]}$ and $\beta^{(0)} = \{\beta_j^{(0)}\}_{j \in [n_B]}$; the variance of u_t for the exploration stage σ_u with $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$; MPC horizon W ; the length of total horizon T ; the exploration length N ; the number of outer iterations $L_{\max}$;
- 2: Generate a trajectory $\{u_g\}_{g=0}^N$ and $\{y_g\}_{g=0}^N$.
- 3: **for** $\tau = 0, 1, 2, \dots, L_{\max}$ **do**
- 4: **Expectation Step**
- 5: Compute $\hat{A}^{(\tau)}$ and $\hat{B}^{(\tau)}$ with $\hat{A}^{(\tau)} = \sum_{j=1}^{n_A} \alpha_j^{(\tau)} A_j$ and $\hat{B}^{(\tau)} = \sum_{j=1}^{n_B} \beta_j^{(\tau)} B_j$.
- 6: **for** $t = 1, 2, \dots, N$ **do**
- 7: Step 1: $\hat{x}_{t|t-1}^* = \hat{A}^{(\tau)} \hat{x}_{t-1}^* + \hat{B}^{(\tau)} u_{t-1}$
- 8: Step 2: $\xi_0^{(\tau)} = \text{Kalman_L} \left(y_t - C \hat{x}_{t|t-1}^{(\tau)} - D u_t, \hat{A}^{(\tau)}, C, \sigma_x^2, \sigma_y^2, W \right)$;
- 9: Step 3: $\hat{x}_t^{(\tau)} = \hat{x}_{t|t-1}^{(\tau)} + \xi_0^{(\tau)}$;
- 10: **end for**
- 11: **Maximization Step**
- 12: Obtain $\alpha^{(\tau+1)}$ and $\beta^{(\tau+1)}$ as

$$\theta^{(\tau+1)} = \begin{bmatrix} \alpha^{(\tau+1)} \\ \beta^{(\tau+1)} \end{bmatrix} = \left(\sum_{t=1}^N \hat{Q}_{t-1}^{(\tau)} (\hat{Q}_{t-1}^{(\tau)})^T \right)^{-1} \left(\sum_{t=1}^N \hat{Q}_{t-1}^{(\tau)} \hat{x}_t^{(\tau)} \right), \quad (4.100)$$

$$(\hat{Q}_t^{(\tau)})^T = \begin{bmatrix} A_1 \hat{x}_t^{(\tau)} & \dots & A_{n_A} \hat{x}_t^{(\tau)} & B_1 u_t & \dots & B_{n_B} u_t \end{bmatrix}.$$

by minimizing (4.101)

$$-2 \log \mathcal{L}_{xy}(\alpha, \beta) = \sum_{t=1}^N \left\| \hat{x}_t^{(\tau)} - \sum_{j=1}^{n_A} \alpha_j A_j \hat{x}_{t-1}^{(\tau)} - \sum_{j=1}^{n_B} \beta_j B_j u_{t-1} \right\|^2 \quad (4.101)$$

- 13: Compute $\|\theta^{(\tau+1)} - \theta^{(\tau)}\|$
 - 14: **end for**
-

Algorithm 5 Algorithm to Compute $L \cdot b$. Kalman_L($b, \hat{A}, C, \sigma_x^2, \sigma_y^2, W$)

- 1: **Input:** input vector $b \in \mathbb{R}^m$; output vector $\tilde{b} \in \mathbb{R}^n$, LQR parameters $\hat{A} \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$, $\Sigma_w \in \mathbb{R}^{n \times n}$, and $\Sigma_v \in \mathbb{R}^{m \times m}$; MPC horizon W .
- 2: **Step 0:** Initialize $\tilde{b}$ as the zero vector.
- 3: **Step 1:** Solve $\tilde{\mathcal{L}}_L(b, \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)$.

$$\begin{aligned} \tilde{\mathcal{L}}_L(b, \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W) = & \min_{\substack{\{x_t\}_{\tau \in [W+1]} \\ \{u_t\}_{\tau \in [W]}}} \frac{\sigma_x^2}{2} x_W^T x_W + \sum_{\tau=0}^{W-1} \frac{\sigma_x^2}{2} (x_\tau - r_\tau)^T (x_\tau - r_\tau) + \frac{\sigma_y^2}{2} u_\tau^T u_\tau, \\ \text{s.t. } & r_1 = b \text{ and } r_\tau = \mathbf{0} \text{ for } \tau \neq 1, \\ & x_0 = \mathbf{0}, \quad x_{\tau+1} = \hat{A}^T x_\tau + C^T u_\tau, \tau \in [W]. \end{aligned} \tag{4.102}$$

- Denote $\tilde{H}_L(\hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)$ as the KKT matrix of $\tilde{\mathcal{L}}_L(b, \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)$. Denote $\mathcal{I}_1$ and $\mathcal{I}_2$ as the intervals to represent the location of u_0 and x_1 . Initialize a zero vector g with the dimension the same as the column dimension of $\tilde{H}_L$. Set $g[\mathcal{I}_1] = b$ and solve the linear system to get $\tilde{g} = \tilde{H}_L(\hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)^{-1} g$.
- 4: **Step 2:** Solve $\mathcal{L}_L(-\sigma_x^{-2} \cdot \tilde{g}[\mathcal{I}_2], \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)$ to get λ_0 and $\tilde{b} = \lambda_0$ where $\mathcal{L}_L(-\sigma_x^{-2} \cdot \tilde{g}[\mathcal{I}_2], \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W)$ is defined as

$$\begin{aligned} \mathcal{L}_L(\bar{x}_0, \hat{A}^T, C^T, \sigma_x^2, \sigma_y^2, W) = & \min_{\substack{\{x_t\}_{\tau \in [W+1]} \\ \{u_t\}_{\tau \in [W]}}} \frac{\sigma_x^2}{2} x_W^T x_W + \sum_{\tau=0}^{W-1} \frac{\sigma_x^2}{2} x_\tau^T x_\tau + \frac{\sigma_y^2}{2} u_\tau^T \hat{R} u_\tau, \\ \text{s.t. } & x_0 = \bar{x}_0, (\lambda_0) \\ & x_{\tau+1} = \hat{A}^T x_\tau + C^T u_\tau, \tau \in [W]. \end{aligned} \tag{4.103}$$

- 5: **Output:** $\tilde{b}$
-

First, we want to distinguish our algorithm from the classic EM algorithm in Section 4.3. In the M step of classic EM algorithm, they use the fact that $E[x_s x_t^T | y_{0:N}] = x_s^N (x_t^N)^T + P_{s,t}^N$ to maximize $Q(\Theta | \Theta^{(\tau-1)}) = E\{-2 \log L_{X,Y}(\Theta) | y_{1:N}, \Theta^{(\tau-1)}\}$, the conditional expectation of the complete data likelihood. However, in step 3 (perform M-step) of our algorithm, in τ -th iteration, we obtained $\theta^{(\tau+1)}$ by solving an ordinary least square problem (4.101) where $\hat{x}^{(\tau)}$ is obtained by the current parameter estimator $\theta^{(\tau)}$ and the observed data $\{y_{0:N}\}$. Therefore, what we minimize (4.101) is not equivalent to the M-step of the classic EM algorithm where they maximize the conditional expectation of the complete data likelihood with respect to the observed data. The algorithm we proposed is called the generalized EM algorithm instead of the classic EM algorithm. Now, we want to give some intuition regarding why we want to propose this generalized EM algorithm.

1. The original EM algorithm treats $-\log L_Y(\Theta^{(\tau)})$ as a stopping signal. When the change of $-\log L_Y(\Theta^{(\tau)})$ is small enough, the algorithm will stop the iterations. However, evaluating $-\log L_Y(\Theta^{(\tau)})$ is computationally expensive, requiring $O(n^3)$ per iteration where we recall that n is the dimension of x_t . Therefore, in Algorithm 4, we prefer to perform more iterations or monitor $\|\theta^{(\tau+1)} - \theta^{(\tau)}\|$ instead of evaluating $-\log L_Y(\Theta^{(\tau)})$. Monitoring $\|\theta^{(\tau+1)} - \theta^{(\tau)}\|$ only requires $O(1)$ computational effort.
2. In Step 2 (perform E-step) of the classic EM algorithm, computing the smoothed value $\{x_t^N\}_{t=1:N}$ for every iteration τ is very expensive, which requires $O(n^3 \cdot N)$ per iteration by Property 4.5.2-4.5.3. We tried exploring the sparsity of our setting to see whether we can reduce the computational cost of the smoothed value $\{x_t^N\}_{t=1:N}$. But after careful analysis, the computational cost of computing $\{x_t^N\}_{t=1:N}$ is still $O(n^3 \cdot N)$ under our problem setup. Therefore, in Algorithm 4, instead of computing the smoothed value $\{x_t^N\}_{t=1:N}$, we compute $\{\hat{x}_t^{(\tau)}\}_{t=1:N}$ with the approximated Kalman filter (Algorithm 5) which only requires $O(n \cdot W \cdot N)$ per iteration where W is the MPC horizon length.

3. Similarly, in Step 3 (perform M-step) of the classic EM algorithm in Section 4.3, they use $E[x_s x_t^T | y_{0:N}] = x_s^N (x_t^N)^T + P_{s,t}^N$ which requires to compute the innovation matrices $P_{s,t}^N$ and computing $P_{s,t}^N$ needs $O(n^3)$ for every pair (s, t) as well. However, we do not need $P_{s,t}^N$ in Algorithm 4.
4. We analyze the total computational cost of Algorithm 4. For line 5, its computational cost is $O(1)$ given $n_A, n_B \ll n$ and the sparsity structure in $\{A_i\}_{i=1}^{n_A}$ and $\{B_j\}_{j=1}^{n_B}$. Line 7 and Line 9 requires $O(n)$ computational cost. The computational cost of line 8 is equivalent to the computational cost of Algorithm 5. Algorithm 5 involves solving two MPC problems (4.102)-(4.103) which requires $O(n \cdot W)$ computational cost. Line 12-13 requires $O(1)$ computational cost given $n_A, n_B \ll n$. Therefore, the total computational cost of Algorithm 4 will be $O(n \cdot W \cdot N)$ per outer iteration τ , while the classic EM algorithm in Section 4.3 requires $O(n^3 \cdot N)$ per iteration.

4.5.3 Analysis of the Algorithm

In this section, we want to prove the convergence of our algorithm and analyze the error between the convergence limit point $\theta^{(\infty)}$ and true parameter $\theta^* = \begin{bmatrix} \alpha^* \\ \beta^* \end{bmatrix}$. Here are potential steps on how we want to proceed with the analysis.

1. Given $\hat{A}$, $\hat{B}$, C , σ_x^2 , and $\sigma_y^2 I_n$, the exact $\hat{x}_t$ should be

$$\hat{x}_{t|t-1} = \hat{A}\hat{x}_{t-1} + \hat{B}u_{t-1}, \quad \hat{x}_t = \hat{x}_{t|t-1} + L(\hat{A}) \left(y_t - C\hat{x}_{t|t-1} - Du_t \right),$$

where $L(\hat{A})$ is the exact Kalman gain given $\hat{A}$. However, in our algorithm, we approximate $L(\hat{A}) \left(y_t - C\hat{x}_{t|t-1} - Du_t \right)$ with $\text{Kalman_L}(y_t - C\hat{x}_{t|t-1} - Du_t, \hat{A}, C, \sigma_x^2, \sigma_y^2, W)$. For algorithm $\text{Kalman_L}(y_t - C\hat{x}_{t|t-1} - Du_t, \hat{A}, C, \sigma_x^2 I_n, \sigma_y^2 I_n, W)$, we will show that it gives $L(\hat{A}, W) \cdot \left(y_t - C\hat{x}_{t|t-1} - Du_t \right)$ where $L(\hat{A}, W)$ is the approximation of the

true Kalman gain $L(\hat{A})$ and the approximation quality depends on the truncated MPC horizon W . In addition, we want to prove that $\|L(\hat{A}, W) - L(\hat{A})\|$ decays exponentially with respect to the MPC horizon W . We note that the reason we omit other parameters in $L(\hat{A}, W)$ and $L(\hat{A})$ is because other parameters are fixed during the iterations.

2. We want to show that $\lim_{W \rightarrow \infty} \hat{\theta}(W) = \hat{\theta}$ where $\hat{\theta}(W)$ is defined as follows

$$\begin{aligned}\hat{\theta}(W) &:= \begin{bmatrix} \hat{\alpha}(W) \\ \hat{\beta}(W) \end{bmatrix} = \left(\sum_{t=1}^N \hat{Q}_{t-1} \hat{Q}_{t-1}^T \right)^{-1} \left(\sum_{t=1}^N \hat{Q}_{t-1} \hat{x}_t \right) = \arg \min_{\theta} l(\theta, W), \\ l(\theta, W) &= \sum_{t=1}^N \|\hat{x}_t - A(\theta) \hat{x}_{t-1} - B(\theta) u_{t-1}\|^2, \quad A(\theta) = \sum_{j=1}^{n_A} \alpha_j A_j, \\ \hat{Q}_t^T &= \begin{bmatrix} A_1 \hat{x}_t & \cdots & A_{n_A} \hat{x}_t & B_1 u_t & \cdots & B_{n_B} u_t \end{bmatrix}, \quad B(\theta) = \sum_{j=1}^{n_B} \beta_j B_j, \\ \hat{x}_{t|t-1} &= A \hat{x}_{t-1} + B u_{t-1}, \quad \hat{x}_t = \hat{x}_{t|t-1} + L(A, W) \cdot (y_t - C \hat{x}_{t|t-1} - D u_t).\end{aligned}\tag{4.104}$$

where $L(A, W) \cdot (y_{t+1} - C \hat{x}_{t|t-1} - D u_{t+1})$ is obtained by `Kalman_L` ($y_{t+1} - C \hat{x}_{t|t-1} - D u_{t+1}, A, C, \sigma_x^2 I_n, \sigma_y^2 I_m$). We note that $\hat{x}_t$ is generated with true A and B but with the approximated Kalman gain $L(A, W)$. We defined $\hat{\theta}$ as follows

$$\begin{aligned}\hat{\theta} &:= \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \left(\sum_{t=1}^N \hat{Q}_{t-1}^* (\hat{Q}_{t-1}^*)^T \right)^{-1} \left(\sum_{t=1}^N \hat{Q}_{t-1}^* \hat{x}_t^* \right) = \arg \min_{\theta} l^*(\theta), \\ l^*(\theta) &= \sum_{t=1}^N \|\hat{x}_t^* - A(\theta) \hat{x}_{t-1}^* - B(\theta) u_{t-1}\|^2, \\ (\hat{Q}_t^*)^T &= \begin{bmatrix} A_1 \hat{x}_t^* & \cdots & A_{n_A} \hat{x}_t^* & B_1 u_t & \cdots & B_{n_B} u_t \end{bmatrix}, \\ \hat{x}_{t|t-1}^* &= A \hat{x}_{t-1}^* + B u_{t-1}, \quad \hat{x}_t^* = \hat{x}_{t|t-1}^* + L(A) \cdot (y_t - C \hat{x}_{t|t-1}^* - D u_t).\end{aligned}\tag{4.105}$$

where we note that $\hat{x}_t^*$ is generated with true A and B but with the exact Kalman gain $L(A)$. Therefore, we note that $\hat{\theta}(W)$ and $\hat{\theta}$ are Ordinary Least Square (OLS) estimators based on $\{\hat{x}_t\}$ and $\{\hat{x}_t^*\}$ respectively; $\{\hat{x}_t\}$ and $\{\hat{x}_t^*\}$ are all generated with the true A and B but $\{\hat{x}_t\}$ uses the approximated Kalman gain $L(A, W)$ while $\{\hat{x}_t^*\}$ uses the exact Kalman gain $L(A)$.

Also, we want to analyze $\|\hat{\theta} - \theta^*\|$ in order to analyze the error of our estimator $\hat{\theta}(W)$.

3. Then, we want to prove the convergence of the proposed algorithm. Specifically, We want to show when the initial guess $\theta^{(0)}$ is in the local neighborhood of θ^* , $\lim_{\tau \rightarrow \infty} \theta^{(\tau)} = \hat{\theta}(W)$.
4. Finally, combined with the above three steps, we want to give the error analysis of $\|\theta^{(\tau)} - \theta^*\|$.

Before we start the analysis, we first introduce some necessary assumptions.

Assumptions

Assumption 4.5.1. *We assume there exists a constant number $\lambda_{\min, M}$ such that*

$$\lambda_{\min} \left(\sum_{i=1}^n \tilde{M}_i \tilde{M}_i^T \right) \geq n \cdot \lambda_{\min, M}, \quad (4.106)$$

where

$$A_i = \begin{bmatrix} A_{1,i} \\ \vdots \\ A_{n,i} \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{bmatrix}, \quad \tilde{A}_j = \begin{bmatrix} A_{j,1} \\ \vdots \\ A_{j,n_A} \end{bmatrix}, \quad \tilde{B}_j = \begin{bmatrix} B_{j,1} \\ \vdots \\ B_{j,n_B} \end{bmatrix}, \quad \tilde{M}_j = \begin{bmatrix} \tilde{A}_j & \\ & \tilde{B}_j \end{bmatrix}, \quad (4.107)$$

for $i \in [1, n_A + 1]$, $k \in [1, n_B + 1]$, $j \in [1, n + 1]$, $\tilde{A}_j \in \mathbb{R}^{n_A \times n}$, $\tilde{B}_j \in \mathbb{R}^{n_B \times d}$, and $\tilde{M}_j \in \mathbb{R}^{(n_\theta) \times (n+d)}$.

Assumption 4.5.2. When θ is in the close neighborhood of θ^* , there exists constants $K_A \geq 1$, $\alpha, \rho_A \in (0, 1)$, and $\gamma \in (0, 1]$ such that

1. $\|A(\theta)\|, \|B(\theta)\|, \|C\|, \|D\|, \|\Sigma_0\|, \sigma_x^2, \sigma_y^2 \leq L, \sigma_x, \sigma_y \geq \sqrt{\gamma}, \rho(A(\theta)) < 1$.
2. For any integer k , $\|A(\theta)^k\| \leq L \cdot \rho_A^k$.
3. $(A(\theta)^T, C^T)$ is (L, α) -stabilizable.
4. $(A(\theta)^T, \sigma_x I_n)$ is (L, α) -detectable.

where $A(\theta)$ and $B(\theta)$ are defined in (4.104).

Proof of Step 1 We want to show the following two things in Theorem 4.5.1.

1. We want to show that $\text{Kalman_L}(y_t - C\hat{x}_{t|t-1} - Du_t, \hat{A}, C, \sigma_x^2, \sigma_y^2, W)$ (defined in Algorithm 5) can be written as $L(\hat{A}, W) \cdot (y_t - C\hat{x}_{t|t-1} - Du_t)$ where $L(\hat{A}, W)$ is the approximation of the true Kalman gain $L(\hat{A})$.
2. We want to show that the upper bound of $\|L(\hat{A}, W) - L(\hat{A})\|$ decays exponentially with respect to W . That also indicates that $\lim_{W \rightarrow \infty} L(\hat{A}, W) = L(\hat{A})$.

We first want to show $\text{Kalman_L}(y_t - C\hat{x}_{t|t-1} - Du_t, \hat{A}, C, \sigma_x^2, \sigma_y^2, W)$ (defined in Algorithm 5) can be written as $L(\hat{A}, W) \cdot (y_t - C\hat{x}_{t|t-1} - Du_t)$. For $\text{Kalman_L}(y_t - C\hat{x}_{t|t-1} - Du_t, \hat{A}, C, \sigma_x^2, \sigma_y^2, W)$, the linear dynamic system is given by

$$\begin{aligned} x_{t+1} &= \hat{A}x_t + Bu_t + w_t, \quad w_t \sim \mathcal{N}(\mathbf{0}, \Sigma_w), \\ y_t &= Cx_t + Du_t + v_t, \quad v_t \sim \mathcal{N}(\mathbf{0}, \Sigma_v). \end{aligned} \tag{4.108}$$

where $\Sigma_w = \sigma_x^2 I_n$, $\Sigma_v = \sigma_y^2 I_m$, $x_0 \sim (\mathbf{0}, \Sigma_0)$. The exact Kalman gain $L(\hat{A})$ is given as

$$\begin{aligned} L(\hat{A}) &= P_L(\hat{A})C^T(CP_L(\hat{A})C^T + \Sigma_v)^{-1} \\ P_L(\hat{A}) &= \hat{A}P_L(\hat{A})\hat{A}^T + \Sigma_w - \hat{A}P_L(\hat{A})C^T(CP_L(\hat{A})C^T + \Sigma_v)^{-1}CP_L(\hat{A})\hat{A}^T. \end{aligned} \tag{4.109}$$

where we show that $P_L(\hat{A})$ is the Riccati solution of $\tilde{\mathcal{L}}_0(\cdot, \hat{A}^T, C^T, \Sigma_w, \Sigma_v)$ by Lemma 4.5.1.

Lemma 4.5.1. *For $\tilde{\mathcal{L}}_0(\cdot, \hat{A}^T, C^T, \Sigma_w, \Sigma_v)$ defined in (4.110)*

$$\begin{aligned} \tilde{\mathcal{L}}_0(\bar{x}_0, \hat{A}^T, C^T, \Sigma_w, \Sigma_v) : \min_{\{x_\tau\}, \{u_\tau\}} & \frac{1}{2} \sum_{\tau=0}^{\infty} x_\tau^T \Sigma_w x_\tau + u_\tau^T \Sigma_v u_\tau, \\ \text{s.t. } & x_0 = \bar{x}_0, (\tilde{\lambda}_0) \\ & x_{\tau+1} = \hat{A}^T x_\tau + C^T u_\tau, (\tilde{\lambda}_{\tau+1}) \tau = 0, \dots, \infty. \end{aligned} \quad (4.110)$$

we can prove that the Riccati equation is given by

$$P = \hat{A}P\hat{A}^T + \Sigma_w - \hat{A}PC^T(CPC^T + \Sigma_v)^{-1}CP\hat{A}^T. \quad (4.111)$$

Furthermore, $\tilde{\lambda}_0 = P \cdot \bar{x}_0$.

Proof. The value function $V_k(x_k)$ of (4.110) takes the quadratic form: $2V_k(x_k) = x_k^T P_k x_k$ where P_k is a symmetric matrix. The Bellman equation is:

$$2V_k(x_k) = \min_{u_k} \left(x_k^T \Sigma_w x_k + u_k^T \Sigma_v u_k + 2V_{k+1}(x_{k+1}) \right),$$

Substituting $x_{k+1} = \hat{A}^T x_k + C^T u_k$:

$$2V_k(x_k) = \min_{u_k} \left(x_k^T \Sigma_w x_k + u_k^T \Sigma_v u_k + 2V_{k+1}(\hat{A}^T x_k + C^T u_k) \right),$$

Expanding $2V_{k+1}(x_{k+1})$ as $2V_{k+1}(x_{k+1}) = (\hat{A}^T x_k + C^T u_k)^T P_{k+1} (\hat{A}^T x_k + C^T u_k)$. And expanding all terms:

$$2V_k(x_k) = \min_{u_k} \left(x_k^T (\Sigma_w + \hat{A}P_{k+1}\hat{A}^T) x_k + u_k^T (\Sigma_v + CP_{k+1}C^T) u_k + 2x_k^T (\hat{A}P_{k+1}C^T u_k) \right), \quad (4.112)$$

The optimal control u_k satisfies:

$$\frac{\partial}{\partial u_k} \left[u_k^T (\Sigma_v + CP_{k+1}C^T)u_k + 2x_k^T (\hat{A}P_{k+1}C^T u_k) \right] = 0,$$

which gives $u_k = K_k x_k$ with $K_k = -(\Sigma_v + CP_{k+1}C^T)^{-1}CP_{k+1}\hat{A}^T x_t$. We substitute $u_k = K_k x_k$ into the right hand side of (4.112) and replace the left hand side of (4.112) with $2V_k(x_k) = x_k^T P_k x_k$, and we get

$$P_k = \hat{A}P_{k+1}\hat{A}^T + \Sigma_w - \hat{A}P_{k+1}C^T(CP_{k+1}C^T + \Sigma_v)^{-1}CP_{k+1}\hat{A}^T, \quad (4.113)$$

Since the horizon of (4.110) is infinite, (4.113) will become (4.111) which completes the proof of the first part.

Then, we will prove $\tilde{\lambda}_0 = P \cdot \bar{x}_0$. By The Lagrangian function of (4.110) is

$$\tilde{L}_{0,L} = \tilde{\lambda}_0^T (x_0 - \bar{x}_0) + \sum_{\tau=0}^{\infty} \frac{1}{2} x_{\tau}^T \Sigma_w x_{\tau} + \frac{1}{2} u_{\tau}^T \Sigma_v u_{\tau} + \tilde{\lambda}_{\tau+1}^T (x_{\tau+1} - \hat{A}^T x_{\tau} - C^T u_{\tau}),$$

And its KKT condition is

$$\begin{aligned} \nabla_{x_t} \tilde{L}_{0,L} &= \Sigma_w x_t + \hat{A} \tilde{\lambda}_{t+1} - \tilde{\lambda}_t = \mathbf{0} \Rightarrow \tilde{\lambda}_t = \Sigma_w x_t + \hat{A} \tilde{\lambda}_{t+1}, \\ \nabla_{u_t} \tilde{L}_{0,L} &= \Sigma_v u_t + C \tilde{\lambda}_{t+1} = \mathbf{0} \Rightarrow u_t = -\Sigma_v^{-1} C \tilde{\lambda}_{t+1}, \end{aligned}$$

By induction, we assume $\tilde{\lambda}_{t+1} = P_{t+1} x_{t+1}$ and we want to show $\tilde{\lambda}_t = P_t x_t$.

$$\begin{aligned} \tilde{\lambda}_t &= \Sigma_w x_t + \hat{A} \tilde{\lambda}_{t+1} = \Sigma_w x_t + \hat{A} P_{t+1} x_{t+1} \\ &= \Sigma_w x_t + \hat{A} P_{t+1} (\hat{A}^T x_t + C^T u_t) = \Sigma_w x_t + \hat{A} P_{t+1} (\hat{A}^T x_t + C^T \tilde{K}_t x_t) \\ &= \Sigma_w x_t + \hat{A} P_{t+1} \hat{A}^T x_t - \hat{A} P_{t+1} C^T (\Sigma_v + CP_{t+1}C^T)^{-1} CP_{t+1} \hat{A}^T x_t = P_t x_t. \end{aligned}$$

where the third inequality holds due to $K_t = -(\Sigma_v + CP_{t+1}C^T)^{-1}CP_{t+1}\hat{A}^T$; and the last

inequality holds due to (4.113). So we show that $\tilde{\lambda}_t = P_t x_t$ holds for all t . Since we consider the infinite horizon case, we have $K_t = K = -(\Sigma_v + CPC^T)^{-1}CP\hat{A}^T$ and $P_t = P$. We have $\tilde{\lambda}_t = Px_t$ and we complete the proof. $\square$

By Lemma 4.5.1, for any vector b , we can get approximated $P_L(\hat{A}) \cdot b$ by solving the truncated MPC problem $\mathcal{L}_L(b, \hat{A}^T, C^T, \Sigma_w, \Sigma_v, W)$ in (4.103).

Lemma 4.5.2. *The first optimal control policy (u_0) of the LQR defined in (4.114) is $u_0 = K_u^W \cdot b = (\Sigma_v + CP_1C^T)^{-1}C\Sigma_w \cdot b$.*

$$\begin{aligned} \mathcal{L}_1(b, \hat{A}^T, C^T, \Sigma_w, \Sigma_v, W) : \min_{\{x_\tau\}, \{u_\tau\}} & \frac{1}{2} \sum_{\tau=0}^W (x_\tau - r_\tau)^T \Sigma_w (x_\tau - r_\tau) + u_\tau^T \Sigma_v u_\tau, \\ \text{s.t. } & x_0 = \mathbf{0}, (\lambda_0) \\ & x_{\tau+1} = \hat{A}^T x_\tau + C^T u_\tau (\lambda_{\tau+1}), \tau \in [W], \\ & r_1 = b \text{ and } r_\tau = \mathbf{0} \text{ for } \tau \neq 1. \end{aligned} \tag{4.114}$$

where P_1 fulfills

$$P_1 = \hat{A}P_2\hat{A}^T + \Sigma_w - \hat{A}P_2C^T(CP_2C^T + \Sigma_v)^{-1}CP_2\hat{A}^T$$

Additionally, the first optimal control policy (u_0) of the LQR defined in (4.115) is $u_0 = K_u \cdot b = (\Sigma_v + CPC^T)^{-1}C\Sigma_w \cdot b$.

$$\begin{aligned} \mathcal{L}_0(b, \hat{A}^T, C^T, \Sigma_w, \Sigma_v) := \min_{\{x_\tau\}, \{u_\tau\}} & \frac{1}{2} \sum_{\tau=0}^{\infty} (x_\tau - r_\tau)^T \Sigma_w (x_\tau - r_\tau) + u_\tau^T \Sigma_v u_\tau, \\ \text{s.t. } & x_0 = \mathbf{0}, (\lambda_0) \\ & x_{\tau+1} = \hat{A}^T x_\tau + C^T u_\tau (\lambda_{\tau+1}), \tau = 0, \dots, \infty, \\ & r_1 = b \text{ and } r_\tau = \mathbf{0} \text{ for } \tau \neq 1. \end{aligned} \tag{4.115}$$

where P fulfills

$$P = \hat{A}P\hat{A}^T + \Sigma_w - \hat{A}PC^T(CPC^T + \Sigma_v)^{-1}CP\hat{A}^T$$

Proof. The value function of $\mathcal{L}_1(b, \hat{A}^T, C^T, \Sigma_w, \Sigma_v, W)$ takes the quadratic form: $2V_k(x_k) = x_k^T P_k x_k + 2p_k^T x_k + c_k$ where P_k is a symmetric matrix, p_k is a vector, and c_k is a scalar. The Bellman equation is:

$$2V_k(x_k) = \min_{u_k} \left((x_k - r_k)^T \Sigma_w (x_k - r_k) + u_k^T \Sigma_v u_k + 2V_{k+1}(x_{k+1}) \right),$$

Substituting $x_{k+1} = \hat{A}^T x_k + C^T u_k$:

$$2V_k(x_k) = \min_{u_k} \left((x_k - r_k)^T \Sigma_w (x_k - r_k) + u_k^T \Sigma_v u_k + 2V_{k+1}(\hat{A}^T x_k + C^T u_k) \right),$$

Expanding $(x_k - r_k)^T \Sigma_w (x_k - r_k)$:

$$(x_k - r_k)^T \Sigma_w (x_k - r_k) = x_k^T \Sigma_w x_k - 2x_k^T \Sigma_w r_k + r_k^T \Sigma_w r_k,$$

Expanding $2V_{k+1}(x_{k+1})$:

$$2V_{k+1}(x_{k+1}) = (\hat{A}^T x_k + C^T u_k)^T P_{k+1} (\hat{A}^T x_k + C^T u_k) + 2p_{k+1}^T (\hat{A}^T x_k + C^T u_k) + c_{k+1},$$

Expanding all terms:

$$\begin{aligned} 2V_k(x_k) = \min_{u_k} & \left(x_k^T (\Sigma_w + \hat{A}P_{k+1}\hat{A}^T) x_k + u_k^T (\Sigma_v + CP_{k+1}C^T) u_k \right. \\ & \left. + 2x_k^T (\hat{A}P_{k+1}C^T u_k + \hat{A}p_{k+1} - \Sigma_w r_k) + 2p_{k+1}^T C^T u_k + c_{k+1} + r_k^T \Sigma_w r_k \right), \quad (4.116) \end{aligned}$$

The optimal control u_k satisfies:

$$\begin{aligned} \frac{\partial}{\partial u_k} \left[u_k^T (\Sigma_v + CP_{k+1}C^T)u_k + 2x_k^T \hat{A}P_{k+1}C^T u_k + 2p_{k+1}^T C^T u_k \right] &= \mathbf{0}, \\ \Rightarrow 2(\Sigma_v + CP_{k+1}C^T)u_k + 2CP_{k+1}\hat{A}^T x_k + 2Cp_{k+1} &= \mathbf{0}, \end{aligned}$$

Solving for u_k :

$$\begin{aligned} u_k &= K_k x_k + K_{r,k} p_{k+1}, \\ K_k &= -(\Sigma_v + CP_{k+1}C^T)^{-1}CP_{k+1}\hat{A}^T, \quad K_{r,k} = -(\Sigma_v + CP_{k+1}C^T)^{-1}C, \end{aligned}$$

We substitute $u_k = K_k x_k + K_{r,k} p_{k+1}$ into the right hand side of (4.116), and replace the left hand side of (4.116) with $2V_k(x_k) = x_k^T P_k x_k + 2p_k^T x_k + c_k$. Then, matching the quadratic terms gives Ricatti equation:

$$P_k = \Sigma_w + \hat{A}P_{k+1}\hat{A}^T - \hat{A}P_{k+1}C^T(\Sigma_v + CP_{k+1}C^T)^{-1}CP_{k+1}\hat{A}^T, \quad (4.117)$$

Matching the linear terms gives:

$$p_k = \left(\hat{A} + \hat{A}P_{k+1}C^T K_{r,k} + K_k^T C + K_k^T (\Sigma_v + CP_{k+1}C^T) K_{r,k} \right) p_{k+1} - \Sigma_w r_k. \quad (4.118)$$

Therefore, when $r_1 \neq \mathbf{0}$ and $r_\tau = \mathbf{0}$ for $\tau \neq 1$, $p_W = \mathbf{0}$. Therefore, $p_t = \mathbf{0}$ for $t > 1$ and $p_1 = -\Sigma_v r_1$. Then, we have $u_0 = K_k x_0 - K_{r,k} \Sigma_v r_1$. When $x_0 = \mathbf{0}$, u_0 is reduced to $u_0 = (\Sigma_v + CP_1C^T)^{-1}C\Sigma_w r_1 = (\Sigma_v + CP_1C^T)^{-1}C\Sigma_w \cdot b$ and we complete the proof of the first part. When $W \rightarrow \infty$, we have $P_1 = P$. Then, we have $u_0 = (\Sigma_v + CPC^T)^{-1}C\Sigma_w r_1 = (\Sigma_v + CPC^T)^{-1}C\Sigma_w \cdot b$ for infinite horizon (4.115) and we complete the proof of the second part. $\square$

Theorem 4.5.1. *Under Assumption 4.5.2, when θ is in a close neighborhood of θ^* , for any vector b , we can obtain $L(A(\theta), W) \cdot b$, an approximation of $L(A(\theta)) \cdot b$ by the truncated MPC*

approach with the horizon length as W with the following steps.

1. Step 1: Denote the KKT matrix of $\tilde{\mathcal{L}}_L(b, A(\theta)^T, C^T, \sigma_x^2, \sigma_y^2, W)$ defined in (4.102) as $H_W(\theta)$. Denote $\mathcal{I}_1$ and $\mathcal{I}_2$ as the intervals representing the location of u_0 and x_1 . We solve the linear system to obtain $\tilde{b} = (H_W(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1] \cdot b$.
2. Step 2: Obtain $L(A(\theta), W) \cdot b$ by solving $\mathcal{L}_L(\sigma_x^{-2}\tilde{b}, A(\theta)^T, C^T, \sigma_x^2, \sigma_y^2, W)$ defined in (4.103). Denote $\mathcal{I}_3$ and $\mathcal{I}_4$ as intervals representing the location of x_0 and λ_0 ; observe that the KKT matrix of $L(A(\theta), W) \cdot b$ is also H_W ; and we have $L(A(\theta), W) \cdot b = \sigma_x^{-2} \cdot (H_W(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3] \cdot \tilde{b} = \sigma_x^{-2} \cdot (H_W(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3] \cdot (H_W(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1] \cdot b$.

Furthermore, denote $L(A(\theta), W) = \sigma_x^{-2} \cdot (H_W(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3] \cdot (H_W(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1]$ and we have

$$\|L(A(\theta), W) - L(A(\theta))\| \leq C_L \rho^W, \quad C_L = \frac{2c_1 \rho L_H}{\sigma_x^2 \gamma_H} \left(\|P_L(A(\theta))\| + \frac{\sigma_x^2 L}{\sigma_y^2} \right). \quad (4.119)$$

where $P_L(A(\theta))$ is defined in (4.109); and γ_H , ρ , L_H , and c_1 are defined in Lemma 4.7.3.

Proof. We observe that the KKT matrix of $\tilde{\mathcal{L}}_0(b, A(\theta)^T, C^T, \sigma_x^2 I_n, \sigma_y^2 I_m)$ in (4.110) and $\mathcal{L}_0(b, A(\theta)^T, C^T, \sigma_x^2 I_n, \sigma_y^2 I_m)$ in (4.115) are the same, and we denote it as $H(\theta)$. By Lemma 4.5.2, we know $u_0 = K_u \cdot b = \sigma_x^2 (\sigma_y^2 I_m + C P C^T)^{-1} C \cdot b$ for $\mathcal{L}_0(b, A(\theta)^T, C^T, \sigma_x^2 I_n, \sigma_y^2 I_m)$. Therefore, we have

$$\begin{aligned} K_u(\theta) &= \sigma_x^2 (\sigma_y^2 I_m + C P_L(A(\theta)) C^T)^{-1} C = (H(\theta))^{-1} [\mathcal{I}_1, \mathcal{I}_2], \\ K_u(\theta)^T &= \sigma_x^2 C^T (\sigma_y^2 I_m + C P_L(A(\theta)) C^T)^{-1} = (H(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1]. \end{aligned} \quad (4.120)$$

Similarly, by Lemma 4.5.1, for $\tilde{\mathcal{L}}_0(b, A(\theta)^T, C^T, \sigma_x^2 I_n, \sigma_y^2 I_m)$, we have

$$P_L(A(\theta)) = (H(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3], \quad (4.121)$$

With (4.120)-(4.121), we have

$$L(A(\theta)) = \sigma_x^{-2} \cdot (H(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3] \cdot (H(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1].$$

Therefore, analyzing $\|L(A(\theta), W) - L(A(\theta))\|$ is equivalent to analyzing

$$\begin{aligned} & \|P_L^W(A(\theta)) \cdot K_u^W(\theta)^T - P_L(A(\theta)) \cdot K_u(\theta)^T\| \\ & \leq \|P_L^W(A(\theta)) - P_L(A(\theta))\| \|K_u^W(\theta)^T\| + \|P_L(A(\theta))\| \|K_u^W(\theta)^T - K_u(\theta)^T\| \end{aligned} \quad (4.122)$$

where we denote $K_u^W(\theta)^T = (H_W(\theta))^{-1} [\mathcal{I}_2, \mathcal{I}_1]$ and $P_L^W(A(\theta)) = (H_W(\theta))^{-1} [\mathcal{I}_4, \mathcal{I}_3]$. We want to analyze $\|K_u^W(\theta)^T - K_u(\theta)^T\|$ and $\|P_L^W(A(\theta)) - P_L(A(\theta))\|$ which is equivalent to analyze $H_W^{-1}(\theta)$ and $H^{-1}(\theta)$. To simplify notations, we omit (θ) for H_W and H for the remaining proof. We observe that H_W is the submatrix of H and we permute H as H^P with primal-dual variables ordered as $\begin{bmatrix} x_t^T & u_t^T & \lambda_t^T \end{bmatrix}$ and we permute H_W as H_W^P in the similar way. We note that the permutation will not change the exponential decay property of the matrix. Therefore, H^P will have a similar exponential decay structure as H . Now, we write

$$H^P = \begin{bmatrix} H_W^P & (H_{W'}^P)^T \\ H_{W'}^P & H_{-W}^P \end{bmatrix}, \quad (H^P)^{-1} = \begin{bmatrix} (H^P)_W^{-1} & \left((H^P)_{W'}^{-1}\right)^T \\ \left((H^P)_{W'}^{-1}\right) & (H^P)_{-W}^{-1} \end{bmatrix}.$$

Since $H^P (H^P)^{-1} = I$, we obtain

$$\begin{aligned} I &= H_W^P (H^P)_W^{-1} + (H_{W'}^P)^T (H^P)_{W'}^{-1}, \\ (H_W^P)^{-1} - (H^P)_W^{-1} &= (H_W^P)^{-1} (H_{W'}^P)^T (H^P)_{W'}^{-1}, \\ \left((H_W^P)^{-1} - (H^P)_W^{-1}\right)_{[0,0]} &= \left((H_W^P)^{-1}\right)_{[0,:]} (H_{W'}^P)^T \left((H^P)_{W'}^{-1}\right)_{[:,0]}, \\ \left\| \left((H_W^P)^{-1} - (H^P)_W^{-1}\right)_{[0,0]} \right\| &\leq \left\| (H_W^P)^{-1}_{[0,:]} \right\| \left\| (H_{W'}^P(\tilde{\mathcal{I}}))^T \left((H^P)_{W'}^{-1}\right)_{[:,0]} \right\|. \end{aligned}$$

$$\begin{aligned} \left(\left(H_W^P \right)^{-1} - \left(H^P \right)_W^{-1} \right)_{[0:1,0:1]} &= \left(\left(H_W^P \right)^{-1} \right)_{[0:1,:]} \left(H_{W'}^P \right)^T \left(\left(H^P \right)_{W'}^{-1} \right)_{[:,0:1]}, \\ \left\| \left(\left(H_W^P \right)^{-1} - \left(H^P \right)_W^{-1} \right)_{[0:1,0:1]} \right\| &\leq \left\| \left(H_W^P \right)^{-1}_{[0:1,:]} \right\| \left\| \left(H_{W'}^P(\tilde{\mathcal{I}}) \right)^T \left(\left(H^P \right)_{W'}^{-1} \right)_{[:,0:1]} \right\|. \end{aligned}$$

By Lemma 4.7.3, we know $\left\| \left(H_W^P \right)^{-1}_{[0,:]} \right\|, \left\| \left(H_W^P \right)^{-1}_{[0:1,:]} \right\| \leq \gamma_H^{-1}$. The structure of $\left(H_{W'}^P \right)^T$ is

$$\left(H_{W'}^P \right)^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -C & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \quad \left(\hat{H}_{W'}^P \right)_{[W,0]}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -A \\ \mathbf{0} & \mathbf{0} & -C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

By Lemma 4.7.3, we have

$$\begin{aligned} &\left\| \left(H_{W'}^P \right)^T \left(\left(H^P \right)_{W'}^{-1} \right)_{[:,0]} \right\| \\ &= \left\| \left(H_{W'}^P \right)_{[W,0]}^T \left(\left(H^P \right)^{-1} \right)_{[0,W+1]} \right\| \leq \left\| \left(H_{W'}^P \right)_{[W,0]} \right\| c_1 \rho^{W+1} \leq L_H c_1 \rho^{W+1}. \\ &\left\| \left(H_{W'}^P \right)^T \left(\left(H^P \right)_{W'}^{-1} \right)_{[:,0:1]} \right\| \\ &= \left\| \left(H_{W'}^P \right)_{[W,0]}^T \left(\left(H^P \right)^{-1} \right)_{[0,W+1]} + \left(H_{W'}^P \right)_{[W,0]}^T \left(\left(H^P \right)^{-1} \right)_{[0,W+2]} \right\| \\ &\leq 2 \left\| \left(H_{W'}^P \right)_{[W,0]} \right\| c_1 \rho^{W+1} \leq 2 L_H c_1 \rho^{W+1}. \end{aligned}$$

where the last inequality holds because of $\left\| \left(H_{W'}^P \right)_{[W,0]} \right\| \leq \|H^P\| \leq L_H$ by Lemma 4.7.3.

Therefore, we have

$$\begin{aligned}\|K_u(\theta) - K_u^W(\theta)\| &\leq \left\| \left((H^P)^{-1} - (H_W^P)^{-1} \right)_{[0:1,0:1]} \right\| \leq \frac{2c_1\rho L_H}{\gamma_H} \rho^W, \\ \|P_L(\theta) - P_L^W(\theta)\| &\leq \left\| \left((H^P)^{-1} - (H_W^P)^{-1} \right)_{[0,0]} \right\| \leq \frac{c_1\rho L_H}{\gamma_H} \rho^W.\end{aligned}\tag{4.123}$$

Therefore, by (4.122), we have

$$\begin{aligned}&\|L^{(W)}(A(\theta)) - L(A(\theta))\| \\ &\leq \sigma_x^{-2} \left(\|P_L^W(A(\theta)) - P_L(A(\theta))\| \|K_u^W(\theta)^T\| + \|P_L(A(\theta))\| \|K_u^W(\theta)^T - K_u(\theta)^T\| \right) \\ &\leq \frac{c_1\rho L_H}{\sigma_x^2\gamma_H} \left(\|K_u^T(\theta)\| + 2\|P_L^W(\theta)\| \right) \rho^W \leq \frac{c_1\rho L_H}{\sigma_x^2\gamma_H} \left(\|K_u^T(\theta)\| + 2\|P_L^W(\theta)\| \right) \rho^W.\end{aligned}$$

where the second inequality holds because of (4.123) and the last inequality holds because of $\|K_u(\theta)\| = \|\sigma_x^2(\sigma_y^2 I_m + CP_L(A(\theta))C^T)^{-1}C\| \leq \frac{\sigma_x^2 L}{\sigma_y^2}$ and Assumption 4.5.2. $\square$

Proof of Step 2 There are two parts in Step 2. For the first part, we want to show $\lim_{W \rightarrow \infty} \hat{\theta}(W) = \hat{\theta}$. In Theorem 4.5.1, we prove when θ is in the close neighborhood of θ^* , we have $\|L(A(\theta), W) - L(A(\theta))\| \leq C_L \rho_L^W$. Therefore, when $W \rightarrow \infty$, $L^{(W)}(A) \rightarrow L(A)$ where $A = A(\theta^*)$. Therefore, it is clear that $\lim_{W \rightarrow \infty} \hat{\theta}(W) = \hat{\theta}$. We postpone the analysis of quantifying $\|\hat{\theta}(W) - \hat{\theta}\|$ to future work.

For Step 2, we generate $\hat{x}_t^*$ with the true parameter θ^* , and we perform OLS with $\{\hat{x}_t^*\}_{t=0}^{N-1}$ to get $\hat{\alpha}$ and $\hat{\beta}$. We want to analyze $\|\hat{\theta} - \theta^*\|$ where N is defined in Algorithm 4 and represents the sample complexity. We obtain $\hat{\alpha}$ and $\hat{\beta}$ by solving the following OLS problems

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \arg_{\theta^T = \begin{bmatrix} \alpha^T & \beta^T \end{bmatrix}} \min \sum_{t=0}^{N-1} \|\hat{x}_{t+1}^* - (\hat{Q}_t^*)^T \theta\|^2 \tag{4.124}$$

where we recall that

$$\begin{aligned}
(\hat{Q}_t^*)^T &= \begin{bmatrix} A_1 \hat{x}_t^* & \cdots & A_{n_A} \hat{x}_t^* & B_1 u_t & \cdots & B_{n_B} u_t \end{bmatrix}, \quad x_{t+1} = Ax_t + Bu_t + w_t, \\
A &= A(\theta^*), \quad B = B(\theta^*), \quad \hat{x}_{t+1|t}^* = A\hat{x}_t^* + Bu_t, \quad e_t^* = x_t - \hat{x}_t^*, \quad L^* = L(A), \\
\hat{x}_{t+1}^* &= \hat{x}_{t+1|t}^* + L^*(y_{t+1} - C\hat{x}_{t+1|t}^* - Du_t) \\
&= A\hat{x}_t^* + Bu_t + L^*(C(x_{t+1} - \hat{x}_{t+1|t}^*) + Du_t + v_{t+1} - Du_t) \\
&= A\hat{x}_t^* + Bu_t + L^*(C(Ae_t^* + w_t) + v_{t+1}) \\
&= (\hat{Q}_t^*)^T \theta^* + L^*(CAe_t^* + v_{t+1} + Cw_t).
\end{aligned} \tag{4.125}$$

Therefore, we have

$$\begin{aligned}
\hat{\theta} &= \frac{\partial}{\partial \theta} \sum_{t=0}^{N-1} \|\hat{x}_{t+1}^* - (\hat{Q}_t^*)^T \theta\|^2 \Rightarrow \sum_{t=0}^{N-1} \hat{Q}_t^* (\hat{x}_{t+1}^* - (\hat{Q}_t^*)^T \theta) = 0, \\
&\Rightarrow \sum_{t=0}^{N-1} \hat{Q}_t^* ((\hat{Q}_t^*)^T \theta^* + L^*(CAe_t^* + v_{t+1} + Cw_t) - (\hat{Q}_t^*)^T \theta) = 0, \\
&\Rightarrow \hat{\theta} = \theta^* + (\hat{\Gamma}_N^*)^{-1} \sum_{t=0}^{N-1} \hat{Q}_t^* L^*(CAe_t^* + v_{t+1} + Cw_t). \quad \hat{\Gamma}_N^* = \sum_{t=0}^{N-1} \hat{Q}_t^* (\hat{Q}_t^*)^T.
\end{aligned}$$

We want to analyze $\|\hat{\theta} - \theta^*\|$ by find the upper bound of $\|(\hat{\Gamma}_N^*)^{-1} \sum_{t=0}^{N-1} \hat{Q}_t^* L^*(CAe_t^* + v_{t+1} + Cw_t)\|$ with two steps as follows

1. Obtain the lower bound of the eigenvalues of $\hat{\Gamma}_N^*$ in Lemma 4.5.4.
2. Obtain the upper bound of $\|\sum_{t=0}^{N-1} \hat{Q}_t^* L^*(CAe_t^* + v_{t+1} + Cw_t)\|$. (We postpone the analysis of this step to future work.)

Step 1: Lower Bound of $\lambda_{\min}(\hat{\Gamma}_N^*) = \lambda_{\min}(\sum_{t=0}^N \hat{Q}_t^* (\hat{Q}_t^*)^T)$

We first prove a helper lemma before proving the main lemma.

Lemma 4.5.3. Suppose $\hat{q}_t^* = \tilde{A} \cdot \hat{z}_t^*$ where $\hat{q}_t \in \mathbb{R}^{n_q}$, $\tilde{A} \in \mathbb{R}^{n_q \times (n+d)}$, and $(\hat{z}_t^*)^T = \begin{bmatrix} (\hat{x}_t^*)^T & u_t^T \end{bmatrix}$. We can show $\hat{z}_t^* | \mathcal{F}_{t-1} \sim \mathcal{N}(\bar{z}_t, \hat{\Sigma}_t^*)$ and $\hat{\Sigma}_t^* \succeq \hat{\Sigma}^* := \min(\sigma_u^2, \sigma_h^2) I_{n+d} \succ \mathbf{0}$ where $\sigma_h^2 = \lambda_{\min}(BB^T \sigma_u^2 + L^*C(L^*C)^T \sigma_x^2 + L^*(L^*)^T \sigma_y^2)$, $\{\mathcal{F}_t\}_{t \in [T]}$ is the filtration generated by $\{w_0, u_0, v_1, w_1, u_1, v_2, \dots, w_{T-1}, u_{T-1}, v_T\}$, and $\bar{z}_t, \hat{\Sigma}_t^*$ are $\mathcal{F}_{t-1}$ -measurable. Therefore, we have $q_t^* | \mathcal{F}_{t-1} \sim \mathcal{N}(\tilde{A}\bar{z}_t, \tilde{A}\hat{\Sigma}_t^*\tilde{A}^T)$. Let

$$\mathcal{E} \text{ be any event for which } \bar{\Lambda}_T^* := E \left[\left(\sum_{t=1}^T q_t^* (q_t^*)^T \right) \mathbb{I}_{\mathcal{E}} \right] \text{ satisfies } \text{Tr}(\bar{\Lambda}_T^*) \leq \hat{J}^* T \quad (4.126)$$

for some constant $\hat{J}^* \geq 0$

Then for

$$T \geq \frac{2000}{9} \left(2n_q \log \left(\frac{100}{3} \right) + n_q \log \frac{\hat{J}^*}{\lambda_{\min}(\tilde{A}\hat{\Sigma}^*\tilde{A}^T)} \right), \quad (4.127)$$

it holds that, for $\Lambda_0(\tilde{A}) := \frac{9T}{1600} \tilde{A}\hat{\Sigma}^*\tilde{A}^T$,

$$\mathbb{P} \left[\left\{ \sum_{i=1}^T q_i^* (q_i^*)^T \not\preceq \Lambda_0(\tilde{A}) \right\} \cap \mathcal{E} \right] \leq 2 \exp \left(-\frac{9}{2000(n_q + 1)} T \right). \quad (4.128)$$

Proof. We first denote $e_t^* = x_t - \hat{x}_t^*$ and with (4.125) we have

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t, \quad \hat{x}_{t+1}^* = A\hat{x}_t^* + Bu_t + L^*(C(Ae_t^* + w_t) + v_{t+1}), \\ e_{t+1}^* &= Ae_t^* + w_t - L^*(CAe_t^* + Cw_t + v_{t+1}) = U^*e_t^* + (I - L^*C)w_t - L^*v_{t+1} \\ &= (U^*)^{t+1}e_0^* + \sum_{k=0}^t (U^*)^k ((I - L^*C)w_{t-k} - L^*v_{t+1-k}). \end{aligned} \quad (4.129)$$

where $U^* = (I - L^*C)A$. Since $u_t \sim \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$ and u_t is independent of x_t , we have

$\hat{z}_{t+1}^* | \mathcal{F}_t \sim \mathcal{N}(\bar{z}_t, \hat{\Sigma}_t^*)$ where

$$\begin{aligned}\hat{\Sigma}_t^* &= \text{diag} \left(\hat{\Sigma}_{t,1}^*, \hat{\Sigma}_{t,2}^* \right), \quad \hat{\Sigma}_{t,2}^* = \sigma_u^2 I_d, \\ \hat{\Sigma}_{t,1}^* &= BB^T \sigma_u^2 + L^* C (L^* C)^T \sigma_x^2 + L^* (L^*)^T \sigma_y^2 + 2(L^* C A (U^*)^t) \Sigma_0 (L^* C A (U^*)^t)^T \succeq \sigma_h^2 I_n,\end{aligned}$$

where we recall that $\hat{x}_0, x_0 \sim (\mathbf{0}, \Sigma_0)$ and $e_0^* \sim (\mathbf{0}, 2\Sigma_0)$. Therefore, $\hat{\Sigma}_t^* \succeq \hat{\Sigma}^*$. The remaining proof will be almost the same as the proof of Lemma 3.8.13, and we omit it here. $\square$

Now, we are ready to give our main lemma for step 1.

Lemma 4.5.4. *For the exploration interval $\mathcal{I}^e = [0, N]$, recall $\hat{\Gamma}_N^* := \sum_{t=0}^{N-1} \hat{Q}_t^* (\hat{Q}_t^*)^T$, if we have*

$$|\mathcal{I}^e| \geq \frac{2000}{9} \left(2(n_\theta) \log \left(\frac{1}{\delta} \right) + (n_\theta) \log \left(\frac{n(2\bar{x}_m + 2\bar{e}_m + \sigma_u^2)}{\min(\sigma_u^2, \sigma_h^2) \lambda_{\min, M}} \right) \right), \quad (4.130)$$

then for $\delta \leq 3/100$, it holds with probability at least $1 - 2n\delta$ that

$$\lambda_{\min} \left(\hat{\Gamma}_N^* \right) \geq \frac{9n|\mathcal{I}^e|}{1600} \min(\sigma_u^2, \sigma_h^2) \lambda_{\min, M}. \quad (4.131)$$

where σ_h^2 is defined in Lemma 4.5.3, n_θ , $\lambda_{\min, M}$ are defined in Assumption 4.5.1, and $\bar{x}_m$, $\bar{e}_m$ are defined in Lemma 4.7.6.

Proof. Recall A_k for $1 \leq k \leq n_A$ and B_l for $1 \leq l \leq n_B$ defined in Assumption 4.5.1, and rewrite $(\hat{Q}_t^*)^T$ as follows

$$(\hat{Q}_t^*)^T = \begin{bmatrix} A_{1,1} \hat{x}_t^* & \cdots & A_{1,n_A} \hat{x}_t^* & B_{1,1} u_t & \cdots & B_{1,n_B} u_t \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ A_{n,1} \hat{x}_t^* & \cdots & A_{n,n_A} \hat{x}_t^* & B_{n,1} u_t & \cdots & B_{n,n_B} u_t \end{bmatrix},$$

If we denote $q_{k,t}^T = \begin{bmatrix} A_{k,1}\hat{x}_t^* & \cdots & A_{k,n_A}\hat{x}_t^* & B_{k,1}u_t & \cdots & B_{k,n_B}u_t \end{bmatrix}$ for $k = 1, \dots, n$, then we have

$$\hat{Q}_t^*(\hat{Q}_t^*)^T = \sum_{k=1}^n q_{k,t} q_{k,t}^T, \quad q_{k,t} = \begin{bmatrix} \tilde{A}_k \hat{x}_t^* \\ \tilde{B}_k u_t \end{bmatrix} = \begin{bmatrix} \tilde{A}_k \\ \tilde{B}_k \end{bmatrix} \hat{z}_t^* = \tilde{M}_k \hat{z}_t^*, \quad k = 1, 2, \dots, n.$$

where $\tilde{M}_k$ is defined in Assumption 4.5.1. By singular value decomposition, there exist $U_{M,k}$ and $V_{M,k}$ such that

$$\tilde{M}_k = U_{M,k}^T \Sigma_{M,k} V_{M,k}, \quad k = 1, 2, \dots, n.$$

where $V_{M,k} V_{M,k}^T = I_{n_{M,k}}$. We consider vector $\tilde{q}_{t,k} = V_{M,k} \hat{z}_t^*$ and first want to prove

$$\begin{aligned} & \text{Tr} \left(E \left(\sum_{t \in \mathcal{I}^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T \right) \right) = \text{Tr} \left(V_{M,k} E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) V_{M,k}^T \right) \\ &= \text{Tr} \left(V_{M,k}^T V_{M,k} E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) \leq \text{Tr} \left(E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \text{Tr} \left(\left(I_{n+d} - V_{M,k}^T V_{M,k} \right) E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) \\ &= \text{Tr} \left(U_{M,I,k} \Lambda_{M,I,k}^{\frac{1}{2}} \Lambda_{M,I,k}^{\frac{1}{2}} U_{M,I,k}^T E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) \\ &= \text{Tr} \left(P_{M,k} P_{M,k}^T E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) = \text{Tr} \left(P_{M,k}^T E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) P_{M,k} \right) \\ &= \sum_{l=1}^{n+d} e_l^T P_{M,k}^T E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) P_{M,k} e_l \geq 0 \end{aligned}$$

where $I_{n+d} - V_{M,k}^T V_{M,k} = U_{M,I,k} \Lambda_{M,I,k} U_{M,I,k}^T$ by singular value decomposition with the

unitary matrix $U_{M,I,k}$, the diagonal matrix $\Lambda_{M,I,k}$, and $P = U_{M,I,k}\Lambda_{M,I,k}^{\frac{1}{2}}$. Since

$$\begin{aligned} \text{Tr} \left(\left(I_{n+d} - V_{M,k}^T V_{M,k} \right) E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) &\geq 0, \\ \text{Tr} \left(V_{M,k}^T V_{M,k} E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) &\leq \text{Tr} \left(E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right), \end{aligned}$$

we have

$$\begin{aligned} \text{Tr} \left(E \left(\sum_{t \in \mathcal{I}^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T \right) \right) &\leq \text{Tr} \left(E \left(\sum_{t \in \mathcal{I}^e} \hat{z}_t^* (\hat{z}_t^*)^T \right) \right) \leq E \left[\sum_{t \in \mathcal{I}^e} \|\hat{x}_t^*\|^2 + \|u_t\|^2 \right] \\ &\leq |\mathcal{I}^e| \cdot n \cdot \left(2(\bar{x}_m + \bar{e}_m) + \sigma_u^2 \right), \end{aligned} \quad (4.132)$$

where the third inequality is because of Lemma 4.7.6 and $E[\|u_t\|^2] = \sigma_u^2 \cdot d \leq \sigma_u^2 \cdot n$. Setting $\mathcal{E} = \Omega$ (the probability space) and

$$\Gamma_{0,k} = \frac{9|\mathcal{I}^e|}{1600} \min \left\{ \frac{\nu^2}{2}, \frac{\psi^2 \nu^2}{\nu^2 + 2\psi^2 K_u^2} \right\} I_{n_{M,k}},$$

Since $V_{M,k} V_{M,k}^T = I_{n_{M,k}}$ and $\hat{\Sigma}^* = \min(\sigma_u^2, \sigma_h^2) I_{n+d}$, we have

$$V_{M,k} \hat{\Sigma}^* V_{M,k}^T = \min(\sigma_u^2, \sigma_h^2) I_{n_{M,k}},$$

Let $\delta \leq 3/100$ and let

$$\begin{aligned} |\mathcal{I}^e| &\geq \frac{2000}{9} \left(2n_{M,k} \log \left(\frac{1}{\delta} \right) + n_{M,k} \log \frac{n \cdot (2(\bar{x}_m + \bar{e}_m) + \sigma_u^2)}{\min(\sigma_u^2, \sigma_h^2)} \right) \\ &\geq \frac{2000}{9} \left(2n_{M,k} \log \left(\frac{100}{3} \right) + n_{M,k} \log \frac{n \cdot (2(\bar{x}_m + \bar{e}_m) + \sigma_u^2)}{\min(\sigma_u^2, \sigma_h^2)} \right), \end{aligned}$$

By Lemma 4.5.3, we have

$$\begin{aligned}
\mathbb{P} \left[\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{t,k} \tilde{q}_{t,k}^T \not\preceq \Gamma_{0,k} \right] &\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot |\mathcal{I}^e| \right) \\
&\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot \frac{2000}{9} \left(2n_{M,k} \log \frac{1}{\delta} \right) \right) \\
&\leq 2 \exp \left(-\frac{9}{2000(n_{M,k} + 1)} \cdot \frac{2000}{9} \left((n_{M,k} + 1) \log \frac{1}{\delta} \right) \right) = 2\delta,
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{t \in \mathcal{I}^e} \hat{Q}_t^* (\hat{Q}_t^*)^T &= \sum_{t \in \mathcal{I}^e} \sum_{k=1}^n q_{k,t} q_{k,t}^T = \sum_{t \in \mathcal{I}^e} \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \tilde{q}_{k,t} \tilde{q}_{k,t}^T \Sigma_{M,k} U_{M,k} \\
&= \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \left(\sum_{t \in \mathcal{I}_i^e} \tilde{q}_{k,t} \tilde{q}_{k,t}^T \right) \Sigma_{M,k} U_{M,k} \succeq \frac{9|\mathcal{I}^e|}{1600} \min(\sigma_u^2, \sigma_h^2) \sum_{k=1}^n U_{M,k}^T \Sigma_{M,k} \Sigma_{M,k}^T U_{M,k} \\
&= \frac{9|\mathcal{I}^e|}{1600} \min(\sigma_u^2, \sigma_h^2) \sum_{k=1}^n \tilde{M}_k \tilde{M}_k^T \succeq \frac{9|\mathcal{I}^e|}{1600} \min(\sigma_u^2, \sigma_h^2) n \lambda_{\min, M} I_{n_\theta}.
\end{aligned}$$

where the last inequality holds because of Assumption 4.5.1. Therefore, we have

$$\mathbb{P} \left[\sum_{t \in \mathcal{I}^e} \hat{Q}_t^* (\hat{Q}_t^*)^T \not\preceq \frac{9n|\mathcal{I}^e|}{1600} \min(\sigma_u^2, \sigma_h^2) \lambda_{\min, M} I_{n_\theta} \right] \leq 2\delta.$$

We complete the proof. $\square$

Proof of Step 3 Since Algorithm 4 is not a classic EM algorithm where the M-step of Algorithm 4 is not equivalent to the M-step of the classic EM algorithm, we cannot use the convergence theory of the classic EM algorithm to analyze Algorithm 4. In this section, we propose a potential approach to proving the convergence of the algorithm where we provide the outline of the proof and postpone the full proof to future work. We want to prove the

following theorem

Theorem 4.5.2 (Convergence Analysis). *Under Assumption 4.5.2, when $\theta^{(\tau)}$ is in a close neighborhood of θ^* , there exists a constant $\rho_\theta \in (0, 1)$ such that*

$$\|\theta^{(\tau+1)} - \hat{\theta}(W)\| < \rho_\theta \|\theta^{(\tau)} - \hat{\theta}(W)\| \quad (4.133)$$

where $\hat{\theta}(W)$ is defined in (4.104).

Proof. We note that when $\|\hat{\theta}(W) - \theta^*\|$ is small enough, $\theta^{(\tau)}$ in a close neighborhood of θ^* is equivalent to $\theta^{(\tau)}$ in a close neighborhood of $\hat{\theta}(W)$. We want to prove the theorem with three steps

1. We define $l_\tau(\theta) = \sum_{t=1}^N \|\hat{x}_t^{(\tau)} - A(\theta)\hat{x}_{t-1}^{(\tau)} - B(\theta)u_{t-1}\|^2$ where $A(\theta)$ and $B(\theta)$ are defined in (4.104). Compute $\nabla_\theta(l_\tau(\theta) - l(\theta, W))$ in (4.138)-(4.140).
2. We want to find a relative tight constant $\tilde{C}_{c,1}$ such that

$$\|\nabla_\theta(l_\tau(\theta) - l(\theta, W))\| \leq \tilde{C}_{c,1} \cdot n \cdot N \cdot \|\theta - \hat{\theta}(W)\|, \quad (4.134)$$

3. We compute the Hessian of $l(\theta, W)$ where

$$\nabla_\theta^2 l(\theta, W) = \sum_{t=1}^N \hat{Q}_t(\hat{Q}_t)^T \quad (4.135)$$

where we want to have a lower bound of the minimum eigenvalue of $\sum_{t=1}^N \hat{Q}_t(\hat{Q}_t)^T$ and we expect that there exists a constant $\tilde{C}_{c,2}$ such that

$$\sum_{t=1}^N \hat{Q}_t(\hat{Q}_t)^T \succeq \tilde{C}_{c,2} \cdot n \cdot N \quad (4.136)$$

where the proof of (4.136) will be similar to Lemma 4.5.3. Therefore, we prove that $l(\theta, W)$ is a strongly convex function with respect to θ .

4. We complete the proof and find $\rho_\theta = \tilde{C}_{c,1}/\tilde{C}_{c,2}$.

$$\begin{aligned}
\|\nabla l(\theta^{(\tau+1)}, W)\| &= \|\nabla l(\theta^{(\tau+1)}, W) - \nabla l(\hat{\theta}(W), W)\| \\
&\geq \tilde{C}_{c,2} \cdot n \cdot N \cdot \|\theta^{(\tau+1)} - \hat{\theta}(W)\| \\
\|\nabla l(\theta^{(\tau+1)}, W)\| &= \|\nabla l(\theta^{(\tau+1)}, W) - \nabla l_\tau(\theta^{(\tau+1)})\| \\
&\leq \tilde{C}_{c,1} \cdot n \cdot N \cdot \|\theta^{(\tau)} - \hat{\theta}(W)\| \\
\|\theta^{(\tau+1)} - \hat{\theta}(W)\| &\leq \frac{\tilde{C}_{c,1}}{\tilde{C}_{c,2}} \|\theta^{(\tau)} - \hat{\theta}(W)\|
\end{aligned} \tag{4.137}$$

We define $e_t(\theta) = \hat{x}_t - A(\theta)\hat{x}_{t-1} - B(\theta)u_{t-1}$, and $\epsilon_t^{(\tau)} = \hat{x}_t^{(\tau)} - \hat{x}_t$. Then, we have

$$\begin{aligned}
l_\tau(\theta) &= \sum_{t=1}^N \|\hat{x}_t - A(\theta)\hat{x}_{t-1} - B(\theta)u_{t-1} + \epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)}\|^2 \\
&= \sum_{t=1}^N \|e_t(\theta) + \epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)}\|^2 \\
&= \sum_{t=1}^N \|e_t(\theta)\|^2 + \|\epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)}\|^2 + 2e_t(\theta)^T(\epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)}) \\
&= l(\theta, W) + \sum_{t=1}^N \|\epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)}\|^2 + 2e_t(\theta)^T(\epsilon_t^{(\tau)} - A(\theta)\epsilon_{t-1}^{(\tau)})
\end{aligned} \tag{4.138}$$

Then, we look at $\nabla_{\theta}(l_{\tau}(\theta) - l(\theta, W))$ which is expressed as follows

$$\begin{aligned}
(Q_{A,t}^{(\tau)})^T &= \begin{bmatrix} A_1 \epsilon_t^{(\tau)} & \cdots & A_{n_A} \epsilon_t^{(\tau)} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \\
l_{\tau}(\theta) - l(\theta, W) &= \sum_{t=1}^N \|\epsilon_t^{(\tau)} - A(\theta) \epsilon_{t-1}^{(\tau)}\|^2 + 2e_t(\theta)^T (\epsilon_t^{(\tau)} - A(\theta) \epsilon_{t-1}^{(\tau)}) \\
&= \sum_{t=1}^N \|\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta\|^2 + 2 \left(\hat{x}_t - \hat{Q}_{t-1}^T \theta \right)^T (\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta) \\
&= \sum_{t=1}^N \left(\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta + 2(\hat{x}_t - \hat{Q}_{t-1}^T \theta) \right)^T (\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta)
\end{aligned} \tag{4.139}$$

$$\begin{aligned}
\nabla_{\theta}(l_{\tau}(\theta) - l(\theta, W)) &= \sum_{t=1}^N (-Q_{A,t-1}^{(\tau)} - \hat{Q}_{t-1}) (\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta) \\
&\quad - \sum_{t=1}^N Q_{A,t-1}^{(\tau)} \left(\epsilon_t^{(\tau)} - (Q_{A,t-1}^{(\tau)})^T \theta + 2(\hat{x}_t - \hat{Q}_{t-1}^T \theta) \right),
\end{aligned} \tag{4.140}$$

□

4.5.4 Future Direction

According to Results 3.3.1 and 3.3.2 in [24, Chapter 3], the stability assumption on A can be replaced by a weaker condition: the detectability and stabilizability of the system, (A, B) is stabilizable and (A, C) is detectable. These conditions are sufficient to guarantee the stability of the Kalman filter and are closely related to Points 3 and 4 in Assumption 3.4.2. However, existing EM algorithms for partially observed LQR systems—such as those in [24, 65]—only consider settings without control inputs. As a result, the requirement that the spectral radius of A is less than one is implicitly enforced to ensure stability.

We now discuss potential directions for modifying the EM algorithm to relax the stability assumption on A . In Algorithm 4, the input sequence is generated as $u_t \stackrel{i.i.d.}{\sim} \mathcal{N}(\mathbf{0}, \sigma_u^2 I_d)$, which necessitates the stability of A to prevent the state x_t from growing exponentially. A

potential alternative is to generate inputs according to $u_t = \tilde{K}x_t$ where $\tilde{K}$ is a matrix is a feedback matrix chosen such that $(A + \tilde{K}B)$ is stable and (A, B) is stabilizable. In the partially observed setting, however, the true state x_t is not directly observable, and must be estimated using the current model parameters. To prevent both x_t and its estimate $\hat{x}_t^{(\tau)}$ from diverging, we must ensure that both (A, B) and $(\hat{A}^{(\tau)}, \hat{B}^{(\tau)})$ are both stabilizable. At present, we do not know whether such $\tilde{K}$ exists, and we also do not have a method for finding an appropriate $\tilde{K}$ that satisfies this condition.

To the best of our knowledge, no prior work has provided a theoretical analysis of the sample complexity, estimation error, or convergence rate of the classical EM algorithm in the context of partially observed LQR systems. Therefore, due to the limited literature, we are currently only able to compare the computational efficiency of our algorithm to that of the classical EM approach. However, because sample complexity and estimation error are fundamental concerns in parameter estimation problems, a complete theoretical analysis of the classical EM algorithm is a necessary prerequisite before conducting a comprehensive analysis of the proposed method.

4.6 Conclusion

We presented both an OLS-based approach and an EM-like algorithm to estimate the parameterized dynamics of a stationary, partially observed LQR system. Our results show that the OLS approach offers significantly lower computational cost, estimation error, and sample complexity. In particular, its sample complexity scales only with $\log m$, which is substantially smaller than the bounds in [54]. Unlike [54], where the estimation error depends on n , the estimation error of our method is independent of n . The computational cost per time step is also independent of n , in contrast to the $O(n^3)$ cost in [54]. Moreover, our algorithm identifies a unique estimator, whereas the estimator in [54] is not unique. A key assumption of the OLS method is the stability of A , which may limit its applicability to a broader class

of systems. Therefore, we propose an EM-like algorithm that, while still relying on stability assumptions, is intended as a potential path toward relaxing the stability constraint. Our preliminary results demonstrate that the method substantially reduces computational cost. This work is ongoing. In the current version, we present the algorithmic framework of the generalized EM algorithm, outline the theoretical analysis steps, and complete part of the analysis. We also discuss future research directions. The remaining analysis and extensions are deferred to future work.

4.7 Help Definition and Lemmas

Lemma 4.7.1 (Lemma 4.2 in [67]). *Denote $\{\mathcal{F}_t\}_{t \geq 0}$ as a filtration. If the real valued random process $\{Z_t\}_{t \geq 0}$ is adapted to $\mathcal{F}_{t+1}$, the real valued random process $\{W_t\}_{t \geq 0}$ is adapted to $\mathcal{F}_{t+1}$, and $W_t|\mathcal{F}_t$ is mean zero and σ^2 -sub-Gaussian, then for any $\alpha, \beta \in \mathbb{R}^+$, we have*

$$\mathbb{P} \left[\left\{ \sum_{t=1}^T Z_t W_t \geq \alpha \right\} \cap \left\{ \sum_{t=1}^T Z_t^2 \leq \beta \right\} \right] \leq \exp \left(-\frac{\alpha^2}{2\sigma^2\beta} \right). \quad (4.141)$$

Lemma 4.7.2. *For x_t defined as follows*

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

we have

$$\| \text{Var}(x_{t+1}) \| \leq \gamma_\infty, \quad \gamma_\infty = \frac{(\sigma_u^2 + \sigma_x^2 + 1)L^4}{1 - \tilde{\rho}^2}. \quad (4.142)$$

where L and $\tilde{\rho}$ are defined in Assumption 4.4.2.

Proof. We can write x_{t+1} as

$$x_{t+1} = Ax_t + Bu_t + w_t = A^{t+1}x_0 + \sum_{k=0}^t A^k (Bu_{t-k} + w_{t-k}),$$

Then, we have the variance of x_{t+1} as follow

$$\begin{aligned} \text{Var}(x_{t+1}) &= A^{t+1}\Sigma_0(A^{t+1})^T + \sum_{k=0}^t \sigma_u^2 A^k B B^T (A^k)^T + \sigma_x^2 A^k (A^k)^T, \\ \|\text{Var}(x_{t+1})\| &\leq L^3 \tilde{\rho}^{2t+2} + \frac{(\sigma_u^2 + \sigma_x^2)L^4}{1 - \tilde{\rho}^2} \leq \frac{(\sigma_u^2 + \sigma_x^2 + 1)L^4}{1 - \tilde{\rho}^2}. \end{aligned} \quad (4.143)$$

where the first inequality uses Assumption 4.4.2. $\square$

Lemma 4.7.3. *Under Assumption 4.5.2, for the KKT matrix of $\tilde{\mathcal{L}}_L(b, A(\theta)^T, C^T, \sigma_x^2, \sigma_y^2, W)$*

defined in (4.102) $H = \begin{bmatrix} G & F^T \\ F & 0 \end{bmatrix} \in \mathbb{R}^{n_H \times n_H}$ where $A(\theta)$ is defined in (4.104) and

$$\begin{aligned} G &= \begin{bmatrix} \sigma_x^2 I_n & & & & \\ & \sigma_y^2 I_m & & & \\ & & \ddots & & \\ & & & \sigma_x^2 I_n & \\ & & & & \sigma_y^2 I_m \\ & & & & & \sigma_x^2 I_n \end{bmatrix}, \\ F &= \begin{bmatrix} I & & & & \\ -A(\theta)^T & -C^T & I & & \\ & -A(\theta)^T & -C^T & I & \\ & & \ddots & & \\ & & -A(\theta)^T & -C^T & I \\ & & & -A(\theta)^T & -C^T & I \end{bmatrix}, \end{aligned}$$

the following statements hold with constants,

$$1. \gamma_H \leq \sigma(H) \leq L_H;$$

$$2. FF^T \succeq \gamma_F I;$$

3. $\text{Re}H(G, F) \geq \gamma_G I$;

4. We consider a partition $\{I_i\}_{i=0}^T$ of $\mathbb{I}_{[1, n_H]}$. If $H[I_i, I_j] = \mathbf{0}$ for any $|i - j| > 1$, then

$$\left\| (H)^{-1} [I_i, I_j] \right\| \leq c_1 \rho^{|i-j|}, \quad \forall i, j \in \mathbb{I}_{[0, T]};$$

where $\text{Re}H(G, F) = Z^T G Z$ with Z is a null space of F ; and

$$\begin{aligned} \rho &:= \left(\frac{L_H^2 - \gamma_H^2}{L_H^2 + \gamma_H^2} \right)^{1/2}, \quad \gamma_F := \frac{(1 - \alpha)^2}{(1 + L)^2 L^2}, \quad \gamma_G := \frac{\gamma(1 - \alpha)^2}{2(1 + L)^2 L^4}, \quad c_1 := \frac{L_H}{\gamma_H^2 \rho}, \\ \gamma_H &:= \left(\frac{2}{\gamma_G} + \left(1 + \frac{4L_H}{\gamma_G} + \frac{4L_H^2}{\gamma_G^2} \right) \frac{L_H(1 + \bar{\mu}L_H)}{\gamma_F} + \bar{\mu} \right)^{-1}, \quad L_H := 2L + 1, \\ \bar{\mu} &:= \frac{2L_H^2/\gamma_G + \gamma_G + L_H}{\gamma_F}. \end{aligned} \quad (4.144)$$

Proof. The proof of the lower bound of $\sigma(H)$ is similar to the proof of [64, Theorem A.3] and we omit it here. The remaining part of the lemma is obtained directly from [63, Lemma 2, Theorem 5]. $\square$

Lemma 4.7.4. *Under Assumption 4.5.2, when θ in a close neighborhood of θ^* and W is large enough, there exists a constant $C_P > 1$ and $\rho_u \in (0, 1)$ such that*

$$\| ((I - L(A(\theta), W)C)A(\theta))^k \| \leq C_P \cdot C \rho_u^k, \quad k \in \mathbb{N}^+, \quad (4.145)$$

where $L(A(\theta), W)$ is the approximated Kalman gain of the exact Kalman gain $L(A(\theta))$.

Proof. For the $\hat{x}_t(\theta)$ generated by $A(\theta)$ and $B(\theta)$, we have

$$\begin{aligned}
\hat{x}_{t+1|t}(\theta) &= A(\theta)\hat{x}_t(\theta) + B(\theta)u_t, \quad y_t = Cx_t + Du_t + v_t, \\
\hat{x}_{t+1}(\theta) &= \hat{x}_{t+1|t}(\theta) + L(A(\theta), W)(y_{t+1} - C\hat{x}_{t+1|t}(\theta) - Du_{t+1}), \\
x_{t+1}(\theta) &= A(\theta)x_t(\theta) + B(\theta)u_t + w_t, \quad e_{t+1}(\theta) = \hat{x}_{t+1}(\theta) - x_{t+1}(\theta), \\
e_{t+1}(\theta) &= \hat{x}_{t+1|t}(\theta) + L(A(\theta), W)(y_{t+1} - C\hat{x}_{t+1|t}(\theta) - Du_{t+1}) - x_{t+1} \\
&= \hat{x}_{t+1|t} - x_{t+1} + L(A(\theta), W)(Cx_{t+1} + Du_{t+1} + v_{t+1} - C\hat{x}_{t+1|t}(\theta) - Du_{t+1}) \\
&= (I - L(A(\theta), W)C)(\hat{x}_{t+1|t} - x_{t+1}) + L(A(\theta), W)v_{t+1} \\
&= (I - L(A(\theta), W)C)(A(\theta)e_t(\theta) - w_t) + L(A(\theta), W)v_{t+1} \\
&= U(\theta, W)e_t(\theta) + \underbrace{L(A(\theta), W)v_{t+1} + (L(A(\theta), W)C - I)w_t}_{\eta_t(\theta)}.
\end{aligned}$$

where we omit the dependence of $\hat{x}_t(\theta)$, $e_t(\theta)$, and $\eta_t(\theta)$ on W to simplify notations; $U(\theta, W) = (I - L(A(\theta), W)C)A(\theta)$. We first prove that $E[\eta_t(\theta, W)\eta_t^T(\theta, W)] \succ 0$ where $E[\eta_t(\theta)\eta_t^T(\theta)] = L(A(\theta), W)(L(A(\theta), W))^T\sigma_y^2 + (L(A(\theta), W)C - I)(L(A(\theta), W)C - I)^T\sigma_x^2$. It is also equivalent to there not existing a non-zero vector $v \in \mathbb{R}^n$ such that $(L(A(\theta), W))^T v = \mathbf{0}$ and $(L(A(\theta), W)C - I)^T v = \mathbf{0}$. Assume there exists a non-zero vector v such that

$$(L(A(\theta), W))^T v = \mathbf{0}, \quad (L(A(\theta), W)C - I)^T v = \mathbf{0},$$

But we know $(L(A(\theta), W)C - I)^T v = C^T(L(A(\theta), W))^T v - v = \mathbf{0} \Rightarrow v = \mathbf{0}$. Therefore, such a vector v does not exist, $E[\eta_t(\theta)\eta_t^T(\theta)] \succ \mathbf{0}$. Since w_t and v_{t+1} are independent of $e_t(\theta)$, and $E[\eta_t(\theta)] = \mathbf{0}$, we denote $M(\theta) = E[e_{t+1}(\theta)e_{t+1}^T(\theta)]$ for the steady state where we also

omit the dependence of $M(\theta)$ on W for simplifying notations, and we have

$$\begin{aligned} E[e_{t+1}(\theta)e_{t+1}^T(\theta)] &= E[U(\theta, W)e_t(\theta)e_t^T(\theta)U(\theta, W)^T] + E[\eta_t(\theta)\eta_t^T(\theta)], \\ M(\theta) &= U(\theta, W)M(\theta, W)U(\theta, W)^T + E[\eta_t(\theta, W)\eta_t^T(\theta, W)], \\ M(\theta) &\succ U(\theta, W)M(\theta)U(\theta, W)^T. \end{aligned}$$

Therefore, there exists $\rho_u \in (0, 1)$ such that

$$\|U(\theta, W)^T\|_M = \sup_{x \neq \mathbf{0}} \sqrt{\frac{x^T U(\theta, W)M(\theta)U(\theta, W)^T x}{x^T M(\theta)x}} \leq \rho_u, \quad (4.146)$$

Then, we have

$$\begin{aligned} \|U(\theta, W)^T\| &= \sup_{x \neq \mathbf{0}} \frac{\|U(\theta, W)^T x\|}{\|x\|} = \sup_{x \neq \mathbf{0}} \sqrt{\frac{\lambda_{\max}(M(\theta))}{\lambda_{\min}(M(\theta))}} \sqrt{\frac{\lambda_{\min}(M(\theta))}{\lambda_{\max}(M(\theta))} \frac{x^T U(\theta, W)U(\theta, W)^T x}{x^T x}} \\ &\leq \sqrt{\kappa(M(\theta))} \sup_{x \neq \mathbf{0}} \sqrt{\frac{x^T U(\theta, W)M(\theta)U(\theta, W)^T x}{x^T M(\theta)x}} \leq \sqrt{\kappa(M(\theta))} \|U(\theta, W)^T\|_M \leq \sqrt{\kappa(M(\theta))} \rho_u \end{aligned}$$

where $\kappa(M(\theta)) = \frac{\lambda_{\max}(M(\theta))}{\lambda_{\min}(M(\theta))}$ is the condition number of $M(\theta)$. Since θ is in the close neighborhood of θ^* , we can always find a constant C_P such that $\kappa(M(\theta))^{1/2} \leq C_P$ when W is large enough. Similarly, we can prove

$$\|U(\theta, W)^k\| \leq C_P \cdot \rho_u^k.$$

□

Lemma 4.7.5. *Under Assumption 4.5.2, there exists a constant $\tilde{C}_L \in (0, 1)$ such that for any matrix M*

$$E \left[MC^T L^T(A(\theta))L(A(\theta))CM^T \right] \preceq \tilde{C}_L E \left[MM^T \right] \quad (4.147)$$

where $L(A(\theta))$ is the Kalman gain. It also implies that $\|C^T L^T(A(\theta))L(A(\theta))C\| < 1$ and $\|L(A(\theta))C\| < 1$.

Proof. To simplify the proof, we omit $(A(\theta))$ in the remaining proof for L and P_L .

$$\begin{aligned} E \left[M C^T L^T L C M^T \right] &= E \left[M P_L^{-1/2} P_L^{1/2} C^T L^T P_L^{-1/2} P_L P_L^{-1/2} L C P_L^{1/2} P_L^{-1/2} M^T \right] \\ &= E \left[M P_L^{-1/2} G_1^T P_L G_1 P_L^{-1/2} M^T \right] \\ G_1(\theta) &= P_L^{-1/2} L C P_L^{1/2} \end{aligned} \quad (4.148)$$

By the definition of the Kalman filter, we have

$$G_1(\theta) = P_L^{-1/2} \left(P_L C^T (C P_L C^T + R)^{-1} \right) C P_L^{1/2} = P_L^{1/2} C^T (C P_L C^T + R)^{-1} C P_L^{1/2} = \tilde{G}_1^T \tilde{G}_0 \tilde{G}_1$$

where $\tilde{G}_1 = C P_L^{1/2}$. By the singular value decomposition (SVD), we have $\tilde{G}_1 = U_G \Sigma_G V_G^T$ and $C P_L C^T = \tilde{G}_1 \tilde{G}_1^T = U_G \Sigma_G \Sigma_G^T U_G^T$ where $U_G \in \mathbb{R}^{m \times m}$, $\Sigma_G \in \mathbb{R}^{m \times n}$, and $V_G \in \mathbb{R}^{n \times n}$. In our case $R = \sigma_y^2 I_m$, by singular value decomposition, we have

$$\begin{aligned} C P_L C^T + R &= U_G (\Sigma_G \Sigma_G^T + \sigma_y^2 I_m) U_G^T \\ (C P_L C^T + R)^{-1} &= U_G (\Sigma_G \Sigma_G^T + \sigma_y^2 I_m)^{-1} U_G^T \\ P_L^{1/2} C^T (C P_L C^T + R)^{-1} C P_L^{1/2} &= V_G \Sigma_G^T (\Sigma_G \Sigma_G^T + \sigma_y^2 I_m)^{-1} \Sigma_G V_G^T \\ \Sigma_G \Sigma_G^T &= \text{diag}(\sigma_{G,1}^2, \dots, \sigma_{G, \text{rank}(G)}^2, 0, \dots, 0) \in \mathbb{R}^{m \times m}. \end{aligned}$$

So, it is clear that

$$\Sigma_G^T (\Sigma_G \Sigma_G^T + \sigma_y^2 I_m)^{-1} \Sigma_G \preceq \max_{i \in [1, \text{rank}(G)+1]} \left(\frac{\sigma_{G,i}^2}{\sigma_{G,i}^2 + (\sigma_y / \sigma_{G,i})^2} I_m \right) \preceq \frac{1}{1 + \sigma_y^2 / \lambda_{\max}(\Sigma_G \Sigma_G^T)} I_m$$

Therefore, we get $G_1 \preceq \tilde{\gamma}_L I_n$ where $\tilde{\gamma}_L = \frac{1}{1 + \sigma_y^2 / \lambda_{\max}(\Sigma_G \Sigma_G^T)} \in (0, 1)$. Then (4.148) can be

written as

$$E \left[MC^T L^T L C M^T \right] \preceq \tilde{\gamma}_L^2 E \left[M P_L^{-1/2} P_L P_L^{-1/2} M^T \right] = \tilde{C}_L E \left[M M^T \right] \quad (4.149)$$

where $\tilde{C}_L = \tilde{\gamma}_L^2$. □

Lemma 4.7.6. *For $\hat{x}_t^*$ and x_t defined as follows*

$$\begin{aligned} x_{t+1} &= A x_t + B u_t + w_t, \quad y_{t+1} = C x_{t+1} + D u_{t+1} + v_{t+1}, \quad e_t^* = x_t - \hat{x}_t^*, \\ \hat{x}_{t+1}^* &= A \hat{x}_t^* + B u_t + L^*(y_{t+1} - C(A \hat{x}_t^* + B u_t) - D u_{t+1}). \end{aligned} \quad (4.150)$$

we have

$$\begin{aligned} E[\|\hat{x}_t^*\|^2] &\leq (2\bar{x}_m + 2\bar{e}_m) \cdot n, \\ \bar{x}_m &= \frac{12L^4 \max(1, \sigma_x, \sigma_u)^2}{(1 - \rho_A)^2}, \quad \bar{e}_m = \frac{32C_P^2 L^2 \max(1, \sigma_x^2, \sigma_y^2)}{(1 - \rho_u)^2}. \end{aligned} \quad (4.151)$$

where L and ρ_A are defined in Assumption 4.5.2; C_P and ρ_u are defined in Lemma 4.7.4.

Proof. Since $\hat{x}_t^* = x_t - e_t^*$, we have $\|\hat{x}_t^*\|^2 \leq 2(\|x_t\|^2 + \|e_t^*\|^2)$ and $E[\|\hat{x}_t^*\|^2] \leq 2(E[\|x_t\|^2] + E[\|e_t^*\|^2])$.

Therefore, we want to find the upper bound for $E[\|x_t\|^2]$ and $E[\|e_t^*\|^2]$.

For $E[\|x_t\|^2]$, we have

$$\begin{aligned} x_{t+1} &= A x_t + B u_t + w_t = A^{t+1} x_0 + \sum_{k=0}^t A^k (B u_{t-k} + w_{t-k}), \\ \|x_{t+1}\| &\leq \|A^{t+1} \Sigma_0^{1/2}\| \|\Sigma_0^{-1/2} x_0\| + \sum_{k=0}^t \|A^k B \sigma_u\| \|u_{t-k}/\sigma_u\| + \|A^k \sigma_x\| \|w_{t-k}/\sigma_x\|. \end{aligned}$$

Then, we know

$$\begin{aligned}
E \left[\|A^{t+1} \Sigma_0^{1/2} \| \Sigma_0^{-1/2} x_0 \| \right] &\leq \rho_A^{t+1} L^{3/2} \cdot \sqrt{n}, \quad \text{Var} \left(\|A^{t+1} \Sigma_0^{1/2} \| \Sigma_0^{-1/2} x_0 \| \right) \leq \rho_A^{2(t+1)} L^3 \cdot n, \\
E \left[\sum_{k=0}^t \|A^k B \cdot \sigma_u \| \|u_{t-k}/\sigma_u \| \right] &\leq \frac{L^2 \sigma_u}{1 - \rho_A} \cdot \sqrt{d}, \quad \text{Var} \left(\sum_{k=0}^t \|A^k B \cdot \sigma_u \| \|u_{t-k}/\sigma_u \| \right) \leq \frac{L^4 \sigma_u^2}{1 - \rho_A^2} \cdot d, \\
E \left[\sum_{k=0}^t \|A^k \cdot \sigma_x \| \|w_{t-k}/\sigma_x \| \right] &\leq \frac{L \sigma_x}{1 - \rho_A} \cdot \sqrt{n}, \quad \text{Var} \left(\sum_{k=0}^t \|A^k \cdot \sigma_x \| \|w_{t-k}/\sigma_x \| \right) \leq \frac{\sigma_x^2 L^2}{1 - \rho_A^2} \cdot n, \\
E[\|x_{t+1}\|] &\leq \frac{L^{3/2} + L^2 \sigma_u + L \sigma_x}{1 - \rho_A} \sqrt{n}, \quad \text{Var}[\|x_{t+1}\|] \leq \frac{L^3 + L^4 \sigma_u^2 + L^2 \sigma_x^2}{(1 - \rho_A)^2} \cdot n.
\end{aligned}$$

where we use Assumption 4.5.2; and the fact that when $\eta \sim \mathcal{N}(\mathbf{0}, I_l)$ for $l \in \mathbb{N}^+$, we have $E[\|\eta\|] \leq \sqrt{l}$, and $\text{Var}(\|\eta\|) \leq l$. Therefore, we have

$$E[\|x_{t+1}\|^2] = \text{Var}[\|x_{t+1}\|] + E[\|x_{t+1}\|]^2 \leq n \cdot \bar{x}, \quad \bar{x}_m = \frac{12L^4 \max(1, \sigma_x, \sigma_u)^2}{(1 - \rho_A)^2}.$$

For $E[\|e_t^*\|^2]$, from (4.129), we have

$$\begin{aligned}
\|e_{t+1}^*\| &\leq \|(U^*)^{t+1} (2\Sigma)^{1/2} \| (2\Sigma)^{-1/2} e_0^* \| + \sum_{k=0}^t \|(U^*)^k (I - L^* C) \sigma_x \| \|w_{t-k}/\sigma_x \| \\
&\quad + \|(U^*)^k L^* \sigma_y \| \|v_{t+1-k}/\sigma_y \|.
\end{aligned}$$

Then, by Lemmas 4.7.4 and 4.7.5, we know $\|I - L^*C\| \leq 2$ and

$$\begin{aligned}
E \left[\|(U^*)^{t+1}(2\Sigma)^{1/2} \|(2\Sigma)^{-1/2} e_0^* \| \right] &\leq C_P \rho_u^{t+1} \sqrt{2L} \cdot \sqrt{n}, \\
\text{Var} \left(\|(U^*)^{t+1}(2\Sigma)^{1/2} \|(2\Sigma)^{-1/2} e_0^* \| \right) &\leq 2C_P \rho_u^{2(t+1)} L \cdot n, \\
E \left[\sum_{k=0}^t \|(U^*)^k (I - L^*C) \sigma_x \| \|w_{t-k}/\sigma_x\| \right] &\leq \frac{2C_P \sigma_x}{1 - \rho_u} \cdot \sqrt{n}, \\
\text{Var} \left(\sum_{k=0}^t \|(U^*)^k (I - L^*C) \sigma_x \| \|w_{t-k}/\sigma_x\| \right) &\leq \frac{4C_P^2 \sigma_x^2}{1 - \rho_u^2} \cdot n, \\
E \left[\sum_{k=0}^t \|(U^*)^k L^* \sigma_y \| \|v_{t+1-k}/\sigma_y\| \right] &\leq \frac{C_P L \sigma_y}{1 - \rho_u} \cdot \sqrt{m}, \\
\text{Var} \left(\sum_{k=0}^t \|(U^*)^k L^* \sigma_y \| \|v_{t+1-k}/\sigma_y\| \right) &\leq \frac{C_P^2 L^2 \sigma_y^2}{1 - \rho_u^2} \cdot m, \\
E[\|e_{t+1}^*\|] &\leq \frac{5C_P L \max(1, \sigma_x, \sigma_y)}{1 - \rho_u} \sqrt{n}, \quad \text{Var}[\|e_{t+1}^*\|] \leq \frac{7C_P^2 L^2 \max(1, \sigma_x^2, \sigma_y^2)}{(1 - \rho_u)^2} \cdot n.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
E[\|e_{t+1}\|^2] &= \text{Var}[\|e_{t+1}\|^2] + E[\|e_{t+1}\|]^2 \leq \bar{e}_m \cdot n, \quad \bar{e}_m = \frac{32C_P^2 L^2 \max(1, \sigma_x^2, \sigma_y^2)}{(1 - \rho_u)^2}, \\
E[\|\hat{x}_t^*\|^2] &\leq 2 \left(E[\|x_t\|^2] + E[\|e_t\|^2] \right) \leq (2\bar{x}_m + 2\bar{e}_m) \cdot n.
\end{aligned}$$

□

CHAPTER 5

FUTURE WORK

We outline future work in two main directions: extending the results of Chapter 2 and expanding upon the developments in Chapter 4.

For the extension of Chapter 2, We identify three promising directions for follow-up work: (1) Develop a unified framework that merges the process of modifying Hessian matrices with the solution of linear-quadratic subproblems into a single computational step. (2) Since incorporating overlapping graph decomposition (OGD) into a sequential quadratic programming (SQP) framework has demonstrated superior theoretical and empirical performance, it would be worthwhile to investigate the integration of OGD into the ADMM framework—commonly used as a baseline in decomposition algorithms—to evaluate potential improvements in performance. (3) Embed OGD into other advanced SQP variants and compare their respective performance across a range of benchmark problems.

For the extension of Chapter 4, one clear follow-up direction is to complete the regret analysis of the proposed algorithm and compare it with the results in [34]. Additionally, it is possible to extend the current work to the non-stationary setting. We believe that the proof techniques developed in Chapter 3 can serve as a useful guideline for this extension. We anticipate that similar favorable results on sample complexity, estimation error, and computational efficiency can be achieved in the non-stationary case. Another important direction is to establish a complete theoretical analysis of the classical EM algorithm for partially observed LQR systems, which remains an open problem in the literature. Addressing this gap would provide a solid foundation for evaluating and improving EM-based methods. Finally, it is also worthwhile to investigate approaches for relaxing the stability assumption on A , as discussed in Section 4.5.4.

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