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INFINITESIMAL STRUCTURE OF THE MODULI SPACE OF PRINCIPAL G-BUNDLES OVER A CURVE.

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Dedicated to my parents.

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ABSTRACT

Given a semisimple reductive group G and a smooth projective curve X over an algebraically closed field k of arbitrary characteristic, let Bun_G denote the moduli space of principal G-bundles over X. For a bundle $P \in \mathrm{Bun}_G$ without infinitesimal symmetries, we provide a description of all divided-power infinitesimal jet spaces, $J_P^{n,PD}(\mathrm{Bun}_G)$, of Bun_G at P. The description is in terms of differential forms on X^n with logarithmic singularities along the diagonals and with coefficients in $(\mathfrak{g}_P^*)^{\boxtimes n}$. Furthermore, we show the pullback of these differential forms to the Fulton-Macpherson compactification of the configuration space, $\hat{X}^n$, is an isomorphism. Thus, we relate the two constructions of (BD; BG), and as a consequence, give a connection between divided-power infinitesimal jet spaces of Bun_G and the $\mathcal{L}ie$ operad.

CHAPTER 1

INTRODUCTION

Let G be a semisimple algebraic group over an algebraically closed field k of arbitrary characteristic and let X be a smooth projective curve over k of genus at least 2. Let Bun_G denote the moduli stack of principal G-bundles over X, and let $\mathcal{M} \subset \mathrm{Bun}_G$ denote the smooth locus consisting of stable bundles. The main purpose of this thesis is to describe the local geometry of $\mathcal{M}$ and to relate several existing descriptions of it.

Let us elaborate on the main goal. The algebro-geometric analogue of an n^{th}-order Taylor approximation of a regular function on $\mathcal{M}$ around a point $P \in \mathcal{M}$ is given by taking the germ about this point, modulo the relation that it vanishes of order $n+1$ there. This *infinitesimal* n^{th}*-order jet space* is denoted by $J^n_P(\mathcal{M}) := \mathcal{O}_{\mathcal{M},P}/m_P^{n+1}$ for the local ring $(\mathcal{O}_{\mathcal{M},P}, m_P)$. For $n = 1$, this is the cotangent space $T^*_P\mathcal{M}$.

In characteristic p, there is another notion of jet spaces involving *divided powers* attached to any variety, due to Berthelot and Ogus (BO). For example, over the affine line, the symbols $t^{(n)}$ heuristically represent the polynomials $t^n/n!$ (which do not exist when p divides n), and they generate the *free divided power algebra* $k[t^{(i)} : i > 0]/\big(t^{(n)}t^{(m)} = \binom{n+m}{n}t^{(n+m)}\big)$. Also, an n^{th}-order Taylor approximation about $t = 0$ is then an element in this algebra modulo the ideal defined by $t^{(m)} = 0$ for all $m > n$. More generally, the *infinitesimal* n^{th} *order divided power jet space around* $P \in \mathcal{M}$, $J^{n,PD}_P(\mathcal{M})$, is a quotient of the free divided power algebra of rank $\dim(\mathcal{M})$ by the ideal generated by symbols of degree at least $n+1$. If the characteristic of k is zero, then $J^{n,PD}_P(\mathcal{M}) = J^n_P(\mathcal{M})$.

The main theorem, Theorem 1.2.1, provides an explicit description of the divided power jet spaces $J^{n,PD}_P(\mathcal{M})$ for arbitrary characteristic in a canonical, coordinate-free way.

1.1 First-order jet space

The first-order infinitesimal jet space is classically computed using the Kodaira-Spencer isomorphism

$$J^1_P(\mathcal{M}) := T^*_P(\mathcal{M}) = H^0(X, \Omega_X \otimes \mathfrak{g}^*_P), \tag{1.1}$$

where Ω_X is the sheaf of differential 1-forms on X and $\mathfrak{g}^*_P = (\mathfrak{g}^* \times P)/G$ is the associated vector bundle to P given by the adjoint action of G on the dual Lie algebra $\mathfrak{g}^*$. It is illuminating to sketch a proof of this result using the local geometry of Bun_G.

Fix a point $x \in X$ and consider the formal completion $O_x := \hat{\mathcal{O}}_{X,x}$ and its field of fractions $K_x := \mathrm{Frac}(O_x)$. Let $\mathcal{O}_{\mathrm{out}} := \mathcal{O}_X|_{X \setminus x}$ denote the sheaf of regular functions on the punctured curve. Since X is a curve, we may choose a uniformizer and identify $O_x \simeq k[[t_x]]$ and $K_x \simeq k((t_x))$. Also, we may embed $\mathcal{O}_{\mathrm{out}} \hookrightarrow K_x$ using the Taylor expansion about $x \in X$. Any principal G-bundle over the curve may be trivialized on the punctured curve $X \setminus x$ and the formal disk $D_x = \mathrm{Spec}(\mathcal{O}_x)$. Together, $\{X \setminus x, D_x\}$ define an open cover of X and consequently the bundle is determined by the transition map on the overlap $D^\circ_x := \mathrm{Spec}(K_x)$. This sketches a proof of the Beauville – Laszlo uniformization theorem, which states there is an isomorphism of stacks

$$\mathrm{Bun}_G = G(\mathcal{O}_{\mathrm{out}}) \backslash G(K_x)/G(O_x).$$

The uniformization theorem immediately provides an explicit description of the tangent spaces. Let $\mathfrak{g}_{K_x} = \mathfrak{g} \otimes K_x, \mathfrak{g}_{\mathcal{O}_x} = \mathfrak{g} \otimes \mathcal{O}_x$, $\mathfrak{g}_{\mathrm{out}} = \mathfrak{g} \otimes \mathcal{O}_{\mathrm{out}}$, and let $\phi \in G(K)$ be a lift of $P \in \mathcal{M}$ via the projection $\pi : G(K_x) \to \mathcal{M}$. Thus

$$T_P(\mathcal{M}) = \mathfrak{g}_{K_x} \big/ (\mathrm{Ad}_\phi(\mathfrak{g}_{\mathcal{O}_x}) + \mathfrak{g}_{\mathrm{out}}).$$

The right-hand-side is precisely the result of computing $H^1(X, \mathfrak{g}_P)$ using the Cech covering

$\{X \setminus x, D_x\}$. Taking linear duals, and using Serre duality, we deduce Equation 1.1

1.2 Higher-order jet spaces

The above description of $J^1_P(\mathcal{M})$ naturally generalizes to a similar description for higher-order jet space. In Section 3.2, we explain how $J^n_P(\mathcal{M})$ pairs perfectly with $\mathcal{D}^{\text{crys}}_P(\mathcal{M})_n$ the fiber at P of the sheaf of (crystalline) differential operators on $\mathcal{M}$ of order at most n. Then, (BMR) shows the latter space is just given by the n^{th} filtered component of the PBW filtration of the enveloping algebra $U(T_P(\mathcal{M}))$. Thus we may identify (see Corollary 3.3.4)

$$J^\infty_P(\mathcal{M}) \simeq U(T_P(\mathcal{M}))^* \tag{1.2}$$

The latter space is called the space of conformal blocks. In fact, conformal blocks arise naturally as the output of a *localization* functor in the geometric Langlands program. Let us explain the localization of $\mathcal{D}$-modules construction in general, following the original ideas of Beilinson and Bernstein (BB).

Let Y be a variety equipped with an action of a reductive group G, and let $\mathfrak{g} = \text{Lie}(G)$ be its Lie algebra. Let $\mathcal{D}(Y)$ be the ring of differential operators on Y. Differentiating the action map yields an algebra map $\mu : U(\mathfrak{g}) \to \mathcal{D}(Y)$ called the quantum moment map. Then, given a $\mathfrak{g}$-module V, the induced $\mathcal{D}(Y)$-module is $\mathcal{D}(Y) \otimes_{U(\mathfrak{g})} V$. More generally, if $K \subset G$ is a subgroup acting compatibly with the $\mathfrak{g}$-action on V, then the map $V \mapsto (\mathcal{D}(Y) \otimes_{U(\mathfrak{g})} V)^K$ into K-invariants becomes a $\mathcal{D}(Y/K)$-module. This procedure may be applied to produce $\mathcal{D}$-modules on Bun_G by using the uniformization theorem. Thus the pair $(\mathfrak{g}, K)$ is the loop algebra $\mathfrak{g} \otimes k((t))$ and the group $G(k[[t]])$, respectively. Elements of the loop algebra define the gluing data, and the group $G(k[[t]])$ acts simply transitively on the space Y parameterizing G-bundles together with a trivialization over the formal disk. This is precisely the coset space $Y = G(\mathcal{O}_{\text{out}}) \backslash G(K_x)$. Consequently, $\text{Bun}_G = Y/K$. The output of localization in the case $V = \mathrm{Ind}_{\mathfrak{g}_{O_x}}^{\mathfrak{g}_{K_x}} k$ is called the vector bundle of coinvariants, and linear functionals on it are called the space of conformal blocks. Conformal blocks originated in two-dimensional conformal field theory, where the $\mathfrak{g} \otimes k((t))$-representation is known as a vacuum module. Its elements are termed observables, and linear functionals on the space of observables are called correlators, which yield expected values.

It turns out jet spaces of $\mathcal{M}$, and consequently conformal blocks, have a relation to the compactification of the configuration space (FM94),(BG), (Loo99). This compactification is a resolution $p : \hat{X}^n \to X^n$ of the diagonal $D := \bigcup_{i \neq j}\{x_i = x_j\} \subset X^n$ such that the new diagonal divisor $\hat{D} := p^{-1}(D)$ has irreducible components $\hat{D}_I$, indexed by subsets I of $\{1, \ldots, n\}$ of cardinality at least two, and such that the codimension of the intersection of k many components is k (i.e normal-crossing). For $I = \{1, \ldots, n\}$, let $\mathcal{L}ie(I)$ denote the vector space spanned by all nested Lie bracket expressions where each x_i (for $i \in I$) appears once. There is a natural linear map $\psi_I : \mathfrak{g}^{\otimes I} \otimes \mathcal{L}ie(I) \to \mathfrak{g}$ given by insertion into the Lie bracket expression. Dualizing this map and moving point by point induces an embedding Ψ_I^* of vector bundles whose fibers are the maps $\mathrm{Id} \otimes \psi_I^*$.

Let $(\hat{\mathfrak{g}}_P^*)^{\boxtimes n}$ be the pullback of $(\mathfrak{g}_P^*)^{\boxtimes n}$ along $p : \hat{X}^n \to X^n$. Define *the BG sheaf* $\hat{\mathcal{G}}_n$ consisting of sections ω that are $(\hat{\mathfrak{g}}_P^*)^{\boxtimes n}$-valued top-degree differential forms on $\hat{X}^n$ and: (1) regular on $\hat{X}^n \setminus \hat{D} = X^n \setminus D$, (2) with simple poles on the diagonal divisor $\hat{D}$, and (3) with residue $\mathrm{Res}_{\hat{D}_I}(\omega) \in \mathrm{Im}(\Psi_I^*)$. The symmetric group acts by permuting points of X^n and this induces an action on $\hat{X}^n$ and $\hat{\mathcal{G}}_n$. Our main theorem reads:

Theorem 1.2.1. *(Gra24) Let k be an algebraically closed field of arbitrary characteristic. Then for any smooth projective curve X of genus $g \geq 2$ and semisimple group G over k, there is a canonical isomorphism*

$$J_P^{n,PD}(\mathcal{M}) \simeq \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{sgn}, \tag{1.3}$$

where "sgn" denotes the sign-invariants under the S_n-action.

One advantage of this realization of jet spaces is it leads to an explicit formula for the flat projective connection for the space of conformal blocks using only the local geometry of Bun_G. The existence of such a connection has a long history: it was first conjectured by Witten, then proven by Hitchin in (Hi), and later reaffirmed by various authors using a range of diverse approaches, e.g (BK91), (Fa), (Gin95), (TUY).

1.3 Deformation theory perspective

In this section we specialize to characteristic 0 and recall some known descriptions of the infinitesimal jet spaces. From the deformation-theoretic perspective, it has long been known that the local geometry of moduli spaces in characteristic zero is governed by differential graded Lie algebras (dglas). In the case of a principal G-bundle P, the relevant dgla is the derived global sections $R\Gamma(X, \mathfrak{g}_P)$ of the associated vector bundle $\mathfrak{g}_P$. Thus, the infinite-order jet spaces are given by the Lie algebra homology

$$J_P^\infty(\mathcal{M}) \simeq H_*(R\Gamma(X, \mathfrak{g}_P)). \tag{1.4}$$

A "dual" description of the local geometry of Bun_G using deformation theory may be found in (EV94). There, the authors construct a higher-order analogue of the Kodaira-Spencer isomorphism $T_P(\mathcal{M}) \to H^1(X, \mathfrak{g}_P)$ by finding a complex of sheaves $\mathcal{A}^\bullet(n)$ on X^n whose i^{th} term consists of sheaves supported on the various diagonal embeddings of $X^i \hookrightarrow X^n$. Let $\mathcal{D}^n_{\mathcal{M}}$ be the sheaf of differential operators on $\mathcal{M}$ of order at most n, and let $\mathcal{D}^n_{\mathcal{M},P}$ be its (geometric) fiber at $P \in \mathcal{M}$. Then the "higher Kodaira-Spencer" morphism is a map

$$\mathcal{D}^n_{\mathcal{M},P} \to \mathbf{H}^n(X^n, \mathcal{A}^\bullet(n))$$

from differential operators on $\mathcal{M}$ to the hypercohomology of the complex. Furthermore, it is obtained by composing various coboundary maps on the (hyper)cohomology of the complex.

On the smooth locus $\mathcal{M} \subset \mathrm{Bun}_G$, the bundles $P \in \mathcal{M}$ have no infinitesimal automorphisms, i.e. $H^0(X, \mathfrak{g}_P) = 0$. Then, (EV94) show the higher Kodaira-Spencer morphism is injective and that its image consists of the invariant sections under the symmetric group action.

Another philosophy in deformation theory is that the n^{th} order deformations of objects related to X should be controlled by some sheaf on the space parameterizing n-tuples of points of X. Using this, Z. Ran initiated a series of papers studying the local geometry of moduli spaces (Ran93; Ran00; Ran06) by working with the Ran space $\mathrm{Ran}(X)$ instead of X. The Ran space $\mathrm{Ran}(X)$ is a prestack whose k-points parameterize finite subsets of X. Although it is not a scheme (or even an ind-scheme), it still makes sense to consider complexes of sheaves due to the foundational work of (BD04). For example, we declare a sheaf $\mathcal{F}$ on $\mathrm{Ran}(X)$ to be a collection of sheaves $\mathcal{F}_I$ for each finite subset $I \subset X$ which satisfy various compatibility relations with respect to diagrams of arrows between finite subsets.

This allows one to define "dglas" on $\mathrm{Ran}(X)$ and consequently, one can consider the Chevalley complex of the dgla. Z. Ran showed (in the topological setting) the Chevalley complex associated to the tangent sheaf on X gives rise to the universal deformation ring, i.e. the space of infinite-order jets on $\mathcal{M}$. Furthermore, using the Chevalley complex, Ran constructs both a higher Kodaira-Spencer morphism and a flat connection on moduli spaces in general. This recovers Hitchin's flat connection on the space of conformal blocks discussed previously. For a nice exposition of Z. Ran's work on this subject (re-written in the algebraic setting of (BD04), we refer the reader to (Yan16)

Finally, we may reformulate Ran's results in terms of factorization and chiral algebras following Rozenblyum's work (Roz; Roz21). A factorization algebra $\mathcal{A}$ on $\mathrm{Ran}(X)$, in the sense of Beilinson-Drinfeld, is a $\mathcal{D}$-module on $\mathrm{Ran}(X)$ satisfying an additional factorization condition that for disjoint subsets $S = S_1 \sqcup S_2$, there are compatible isomorphisms

$$\mathcal{A}_S \simeq \mathcal{A}_{S_1} \otimes \mathcal{A}_{S_2}.$$

For a factorization algebra we may define the chiral homology using the Lie algebra homology via the Chevalley-Eilenberg complex:

$$H_{ch}(X, \mathcal{A}) := H_{dR}(\mathrm{Ran}(X), \mathcal{A}).$$

A factorization algebra $\mathcal{A}$ on $\mathrm{Ran}(X)$ is equivalent to the data of a $\mathcal{D}$-module $\mathcal{B} := \mathcal{A}[1]$ on X together with a chiral Lie bracket on $\mathcal{B}$. This data defines a chiral algebra. Given a chiral algebra L, the corresponding factorization algebra $\mathcal{A}(L)$ is called the chiral envelope. Then, we may compute the chiral homology of $\mathcal{A}(L)$ using the Lie algebra homology $H_{ch}(X, \mathcal{A}(L)) = C_*(H_{dR}(X, L))$

For Bun_G, the chiral algebra on X associated to $\mathfrak{g}_P$ is

$$L_{\mathfrak{g}_P} := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathfrak{g}_P$$

equipped with a chiral bracket coming from the pointwise Lie bracket on $\mathfrak{g}$. In (Roz, Theorem 17), the chiral envelope for $L_{\mathfrak{g}_P}$ is explicitly constructed. Over the fiber at $x \in X$, it matches with the vacuum module $\mathbf{M}_\phi := U(\mathfrak{g}_K)/\mathrm{Ad}_\phi \mathfrak{g}_O \cdot U(\mathfrak{g}_K)$, where $\phi \in G(K)$ is a representative of $P \in \mathcal{M}$ using the uniformization theorem.

Thus we obtain the (derived) isomorphisms

$$H_{ch}(X, \mathcal{A}(L_{\mathfrak{g}_P})) \simeq C_*(H_{dR}(X, L_{\mathfrak{g}_P})) \simeq C_*(R\Gamma(X, \mathfrak{g}_P)) \tag{1.5}$$

which at homological degree 0, recovers Equation 1.4. The equivalence between the Lie homology description (1.4) and the chiral homology (1.5) is an instance of chiral Koszul duality, which relates Lie algebras in the category of $\mathcal{D}$-modules to their corresponding factorization coalgebras – see (Roz11; FG11).

1.4 Organization

The paper is organized as follows. In Section 2, we recall some general properties on differential forms with logarithmic singularities along a normal-crossing divisor. In Section 3, we recall the definition of a divided-power algebra and show divided-power infinitesimal jet spaces pair perfectly with the "space of coinvariants" of Bun_G. In Section 4, we prove the main result that divided-power infinitesimal n^{th}-order jet space is isomorphic to a space of logarithmic differential forms on X^n with special residue along the diagonals. This is summarized in diagram 4.8. Finally, in Section 5, we review the combinatorics of the Fulton-Macpherson resolution of the diagonal and prove that the pull-back of global sections of (BD) sheaf give global sections of (BG) sheaf. We conclude by discussing how the latter is related to the $\mathcal{L}ie$ operad.

CHAPTER 2

RESIDUES ON X^N

In this chapter, we recall some results on residues which will be used throughout this paper. As before, X is a smooth projective curve over an arbitrary algebraically closed field k.

2.1 Local Residue

By a local residue, we mean a residue defined using formal neighborhoods around closed points. Fix a closed point $x \in X$ on the curve and let t be a uniformizer. There is an identification $k[[t]] \cong O_x := \hat{\mathcal{O}}_{X,x}$, the ring of regular functions on the completion of the stalk at x. And, on fraction fields we have $k((t)) \cong K_x := \mathrm{Frac}(\hat{\mathcal{O}}_{X,x})$. Let $D_x = \mathrm{Spec}(O_x), D_x^\times = \mathrm{Spec}(K_x)$ be the formal disk, formal punctured disk, respectively.

The notion of a residue along a formal disk inside a smooth projective curve X was first developed by (Tate). Given $\omega \in \Omega^1(X)$, express $\omega = f(t)dt$, for $f \in k((t))$, and write

$$\mathrm{Res}_x \omega := \mathrm{Res}_{t=0} f(t)dt. \tag{2.1}$$

Tate showed the residue is independent of choice of uniformizer. A fundamental result on residues is that for projective curves X, the sum of residues is 0: $\sum_{x \in X} \mathrm{Res}_x \omega = 0$. The following result, dubbed the "strong residue theorem", provides a converse:

Theorem 2.1.1. *(BZF, Sect. 9.2.9)(Strong Residue Theorem) Let $\mathcal{E}$ be a locally free sheaf over X. Then a section $s \in \mathcal{E}(D_x^\times)$ extends to $\tilde{s} \in \mathcal{E}(X \setminus x)$ if and only if*

$$Res_x(\langle s, \omega \rangle) = 0 \quad \textit{for all } \omega \in (\mathcal{E}^* \otimes \Omega_X)(X \setminus x).$$

As a corollary, there is a perfect pairing, given by the residue at x, between

$$\mathcal{E}(D_x^\times)/\mathcal{E}(X \setminus x) \times (\mathcal{E}^* \otimes \Omega_X)(X \setminus x) \xrightarrow{Res_x} k$$

Proof. The following proof essentially follows the proof of (Ueo08, Theorem 1.22). If a section $s \in \mathcal{E}(D_x^\times)$ extends to $\tilde{s} \in \mathcal{E}(X \setminus x)$, then $\langle s, \omega \rangle \in \Omega_X(X \setminus x)$ is a 1-form which is regular everywhere except possibly at x. But the sum of residues equals zero, hence $\mathrm{Res}_x(s, \omega) = 0$.

Conversely, suppose $\mathrm{Res}_x(\langle s, \omega \rangle) = 0$ for all 1-forms $\omega \in (\mathcal{E}^* \otimes \Omega_X)(X \setminus x)$. Let $j : X \setminus x \to X$ and $i : \{x\} \to X$ be the natural inclusions. Consider the short exact sequence of sheaves on X

$$0 \to \mathcal{E} \to j_* j^* \mathcal{E} \to i_*(\Gamma(D_x^\times, \mathcal{E})/\Gamma(D_x, \mathcal{E})) \to 0$$

where the projection map sends a meromorphic section $s \in j_* j^* \mathcal{E}$ to $[s]$, its Taylor series expansion at x modulo the regular part $\Gamma(D_x, \mathcal{E})$. Note, the quotient is the skyscraper sheaf on $x \in X$ encoding the polar part (portion consisting of purely negative powers of t) of the section. Taking global sections, we obtain a long exact sequence

$$0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X \setminus x, \mathcal{E}) \to \mathcal{E}_x \otimes \big(k((t))/k[[t]]\big) \xrightarrow{\delta} H^1(X, \mathcal{E}) \to 0$$

Since $X \setminus x$ is affine, $H^1(X \setminus x, \mathcal{E}) = 0$. By Serre duality, we may identify $H^1(X, \mathcal{E})$ with $H^0(X, \mathcal{E}^* \otimes \Omega_X)^*$. Under this identification, the residue pairing $\mathrm{Res}_x(\langle s, - \rangle)$ coincides with the standard pairing $\langle \delta([s]), - \rangle$. The assumption on the residue vanishing for all test $\omega \in \Gamma(X \setminus x, \mathcal{E}^* \otimes \Omega_X)$ implies in particular it vanishes on $\Gamma(X, \mathcal{E}^* \otimes \Omega_X)$, hence $[s] \in \ker \delta$. Namely, s comes from a section in $\Gamma(X \setminus x, \mathcal{E})$ as desired. □

Next, we introduce a notion of residues along formal disks of higher dimensions which generalizes Tate's definition in Equation 2.1 in the case of curves. Our notion is used implicitly in (BD), but we make this computation explicit. Let

$$O_n := k[[z_1, \ldots, z_n]], \quad A_n := k[[z_1, \ldots, z_n]][z_j^{-1}, (z_i - z_j)^{-1}]_{1 \le i < j \le n}.$$

There is a natural inclusion $O_n \to A_{n-1}((z_{n-1} - z_n))$ given by $z_n \mapsto z_{n-1} - (z_{n-1} - z_n)$. By Taylor expanding in the $z_{n-1} - z_n$ direction, we see this inclusion extends uniquely to produce an algebra embedding $T_{z_{n-1}-z_n} : A_n \to A_{n-1}((z_{n-1} - z_n))$. Indeed, we only need to check the Taylor expansions of z_n^{-1} and $(z_i - z_n)^{-1}, i < n$ about $z_{n-1} - z_n$ lands in $A_{n-1}((z_{n-1} - z_n))$:

$$\frac{1}{z_{n-1} - z_n} \in A_{n-1}((z_{n-1} - z_n))$$

$$\frac{1}{z_i - z_n} = \frac{1}{z_i - z_{n-1} + (z_{n-1} - z_n)} = \sum_{k \ge 0} (-1)^k \frac{(z_{n-1} - z_n)^k}{(z_i - z_{n-1})^{k+1}} \in A_{n-1}((z_{n-1} - z_n))$$

$$\frac{1}{z_n} = \frac{1}{z_{n-1} - (z_{n-1} - z_n)} = \sum_{k \ge 0} \frac{(z_{n-1} - z_n)^k}{z_{n-1}^{k+1}} \in A_{n-1}((z_{n-1} - z_n))$$

Similarly, the natural inclusion $O_n \to A_{n-1}((z_n))$ given by $z_n \mapsto z_n$ extends to a unique algebra embedding $T_{z_n} : A_n \to A_{n-1}((z_n))$ by Taylor expansion about z_n. These simple observations allow us to define the generalized formal local residue:

Definition 2.1.2. *Suppose $f \in A_n$. We say the expansion of f in the $z_{n-1} - z_n$ direction is*

$$T_{z_{n-1}-z_n}(f) = \sum_{i \ge d} P_i \cdot (z_{n-1} - z_n)^i \in A_{n-1}((z_{n-1} - z_n))$$

for some integer d. We define the formal local residue map by

$$Res_{z_{n-1}=z_n}(f) := \text{ The } (z_{n-1}-z_n)^{-1} \text{ coefficient of } T_{z_{n-1}-z_n}(f).$$

Similarly, the expansion of f in the z_n direction is $T_{z_n}(f)$ and then we declare

$$Res_{z_n=0}(f) := \text{ The } z_n^{-1} \text{ coefficient of } T_{z_n}(f).$$

In both cases, the residue map is an O_n-linear map $A_n \to A_{n-1}$. Finally, we recall a fundamental result on residues:

Lemma 2.1.3. *(Parshin (Par76)) Suppose $f \in A_3$. Then*

$$Res_{z_1=0}Res_{z_2=0}(f) - Res_{z_2=0}Res_{z_1=0}(f) = -Res_{z_1=0}Res_{z_1=z_2}(f)$$

Proof. This is readily checked by brute force on $f = \frac{1}{z_1^a z_2^b (z_1-z_2)^c}$ for $a, b, c \in \mathbf{Z}_{\geq 0}$. □

2.2 Global Residue

Next, we recall the definition of logarithmic differential forms along a reduced, normal-crossing divisor D of a smooth algebraic variety S over k. We also construct the residue map and recall several of its basic properties. As k is allowed to be an arbitrary field, we work over the algebraic (étale) topology. A nice survey of these classical facts and constructions may be found in the original source, Deligne Hodge II, or e.g., (EV92; S79).

Since D is normal-crossing in S, we may cover S via open affine U such that

- U is étale over $\mathbf{A}_k^n$, via coordinates $x_1, \ldots, x_n$.
- $D|_U$ is defined by an equation $x_1 \ldots x_s = 0$ (i.e D is the inverse image of the union of

the first $s \leq n$ coordinate hyperplanes of $\mathbf{A}_k^n$).

Let $j : S \setminus D \hookrightarrow S$ be the inclusion and suppose locally $D|_U = (h = 0)$. Define $\Omega_S^\bullet(\log D)$, the locally-free $\mathcal{O}_S$-module of logarithmic differential forms on S along D, as the subsheaf of $j_*(\Omega_{S \setminus D}^\bullet)$ consisting of forms ω such that $h\omega$ and $dh \wedge \omega \in \Omega_S^\bullet(U)$. Equivalently, we may specify that over each open affine U as above, $\Omega_S^1(\log D)(U)$ has basis $\frac{dx_1}{x_1}, \dots, \frac{dx_s}{x_s}, dx_{s+1}, \dots, dx_n$. Then $\Omega_S^i(\log D) = \wedge_{\mathcal{O}_S}^i(\Omega_S^1(\log D))$.

Now, fix $p \in D$ and choose local coordinates $x_1, \dots, x_n$ of $D|_U = (h_p = 0)$ around p. Since D is normal-crossing, we may assume $h_p = x_1 \cdots x_{s'}$ for $s' \leq s \leq n$, and that $(D, p) = (D_1, p) \cup \cdots \cup (D_{s'}, p)$ where $(D_i, p) = (x_i = 0)$ are the irreducible components of the divisor D locally around p. Write $\mathcal{O}_{S,p}$ and $\Omega_{S,p}^1(\log D)$ for the stalks at p. Suppose $\omega_p \in \Omega_{S,p}^1(\log D)$ and consider a local expansion

$$\omega_p = \sum_{i=1}^{s'} f_i \frac{dx_i}{x_i} + \sum_{i=s'+1}^{n} g_i dx_i$$

where $f_i, g_i \in \mathcal{O}_{S,p}$. Define $\mathrm{Res}_{D,p}$ to be the $\mathcal{O}_{S,p}$-linear map

$$\mathrm{Res}_{D,p} : \omega_p \mapsto (f_1|_{D_1,p}, \dots, f_{s'}|_{D_{s'},p}) \in \bigoplus_{i=1}^{s'} \mathcal{O}_{D_i,p}. \tag{2.2}$$

That the residue is independent of choice of presentation follows from the "generalized de Rham lemma", which, as formulated in (S76), is characteristic independent. Further details may be found in e.g., (S79).

Let $i_j : D_j \hookrightarrow S$ be the inclusion. There is an exact sequence of coherent $\mathcal{O}_S$-modules

$$0 \to \Omega_S^1 \to \Omega_S^1(\log D) \xrightarrow{\mathrm{Res}_D} \bigoplus_{j=1}^{s} (i_j)_* \mathcal{O}_{D_j} \to 0 \tag{2.3}$$

which on stalks recovers the residue map in Equation 2.2. Let us now explain how to upgrade the residue construction to higher-order differential forms to give a map

$$\Omega^p_S(\log D) \xrightarrow{\operatorname{Res}_D} \bigoplus_{j=1}^{s} (i_j)_* \Omega^{p-1}_{D_j}(\log(D - D_j)).$$

The map Res_D is again defined as the sum of residues Res_{D_i} along each irreducible component D_i of D. Consider $i = 1$ and assume the component D_1 is locally $D_1 = (x_1 = 0)$. Thus, we may write $\omega \in \Omega^p_S(\log D)$ as $\omega = \frac{dx_1}{x_1} \wedge \xi + \eta$ where $\xi \in \Omega^{p-1}_S(\log(D - D_1))$ and $\eta \in \Omega^p_S(\log(D - D_1))$. Then define

$$\operatorname{Res}_{D_1}(\frac{dx_1}{x_1} \wedge \xi + \eta) := \xi|_{D_1} \in \Omega^{p-1}_{D_1}(\log(D - D_1)|_{D_1}).$$

This fits into an exact sequence

$$0 \to \Omega^p_S(\log(D - D_1)) \to \Omega^p_S(\log D) \xrightarrow{\operatorname{Res}_{D_1}} (i_1)_* \Omega^{p-1}_{D_1}(\log(D - D_1)|_{D_1}) \to 0. \tag{2.4}$$

Finally, we cite a useful lemma whose proof may be found in (V82, Lemma 1.6).

Lemma 2.2.1. *Let $f : X \to Y$ be a smooth map of algebraic varieties and suppose D is a normal-crossing divisor in Y. Suppose $\hat{D} := (f^{-1}(D))_{red}$ is a normal-crossing divisor of X. Then $f^*(\Omega_Y(\log D)) \subset \Omega_X(\log \hat{D})$*

Proof. Let $x \in X$ and $y = f(x)$. Since D and $\hat{D}$ are normal-crossing divisors, we may work étale locally and assume $\mathcal{O}_{Y,y} = k[y_1, \ldots, y_n], D = \{y_1 \ldots y_r = 0\}$ and $\Omega_Y(\log D)$ is generated by $\frac{dy_1}{y_1}, \ldots, \frac{dy_r}{y_r}, dy_{r+1}, \ldots, dy_n$. Also, we may assume $\mathcal{O}_{X,x} = k[x_1, \ldots, x_m]$, $\hat{D} = \{x_1 \ldots x_s = 0\}$, and $\Omega_X(\log \hat{D})$ is generated by $\frac{dx_1}{x_1}, \ldots, \frac{dx_s}{x_s}, dx_{s+1}, \ldots, dx_m$.

Since $\hat{D}$ is normal-crossing, we may write

$$f^* y_i = u_i \prod_{j=1}^{s} x_j^{n_{ij}}, \quad \text{for unit } u \text{ and } n_{ij} \geq 0$$

Thus,

$$f^*(\frac{dy_i}{y_i}) = \frac{df^*y_i}{f^*y_i} = \sum_{j=1}^{s} n_{ij}\frac{dx_j}{x_j} + \frac{du_i}{u_i} \in \Omega_X(\log \hat{D})$$

□

CHAPTER 3

JETS ON BUN$_G$

3.1 Divided power algebras

We recall the notion of divided power structure, following the exposition in (BO, Section 3). Let A be a commutative ring and I an ideal. Then a *divided power structure on* I means a collection of maps $\gamma_i : I \to A$, for all integers $i \geq 0$, satisfying:

- For all $x \in I$, $\gamma_0(x) = 1, \gamma_1(x) = x, \gamma_i(x) \in I$ if $i \geq 1$.
- For $x, y \in I$, $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.
- For $\lambda \in A, x \in I, \gamma_k(\lambda x) = \lambda^k \gamma_k(x)$.
- For $x \in I, \gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$.
- $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$.

We will also say (A, I, γ) *is a P.D ring.* The motivation for divided power structure is to emulate elements $\frac{x^n}{n!}$, which do not make sense in characteristic p, with elements $\gamma_n(x)$. The 5 axioms then capture all the properties we would like for $\gamma_n(x)$ to satisfy, were it to be $\frac{x^n}{n!}$. It is a theorem ((BO, Theorem 3.9)) that given any commutative k-algebra A and an A-module I, there exists a P.D. algebra, denoted $(\Gamma_A(I), \Gamma_A^+(I), \gamma)$, satisfying a natural universal property. We call $\Gamma_A(I)$ the divided power envelope of I in A. There is a grading on $\Gamma_A(I)$ with $\Gamma_0(I) = A, \Gamma_1(I) = I$, and $\Gamma_A^+(I) = \oplus_{i\geq 1}\Gamma_i(I)$. In particular, there is an A-linear map $I \to \Gamma_A(I)^+$.

Let us describe the *P.D polynomial algebra.* Suppose $\{x_1, \cdots, x_n\}$ is an A-module basis for I. Denote $x^{[n]} = \gamma_n(x) \in \Gamma_n(I)$. Then

$$\{x^{[q]} := x_1^{[q_1]} \dots x_n^{[q_n]} : \sum q_i = n, x_i \in I\} \quad \text{forms an A-module basis for } \Gamma_n(I)$$

and we denote $\Gamma_A(I) := A\langle x_1, \dots x_n\rangle$ and call it the P.D polynomial A-algebra. For example, let $V = k^n$ be a vector space and consider $A := \mathrm{Sym}(V) := TV/\langle a\otimes b - b\otimes a\rangle$, and $I = \mathrm{Sym}(V)^+$. Then

$$\Gamma_A(I) = k\langle x_1, \dots x_n\rangle \simeq \mathrm{Sym}^{PD}(V) := \bigoplus_{i\geq 0} \left(V^{\otimes i}\right)^{S_i},$$

where the isomorphism is given by $x^{[q]} \mapsto \sum_{\omega\in O_q}\omega$ and O_q is the orbit $S_{q_1+\dots+q_n}.x_1^{\otimes q_1}\otimes \dots \otimes x_n^{\otimes q_n}$. One readily checks the natural pairing of V, V^* induces the perfect pairing

$$\langle,\rangle:\ \mathrm{Sym}^{PD}(V)\times \mathrm{Sym}(V^*)\to k.$$

3.2 Divided power jet spaces

In this section, we recall the definition of divided power jet spaces, as done in (BO). Suppose $\mathcal{M}$ is a smooth scheme over k, and denote $\mathcal{O} := \mathcal{O}_{\mathcal{M}}$, $\mathcal{O}^e := \mathcal{O}_{\mathcal{M}}\otimes\mathcal{O}_{\mathcal{M}}$. Let $\Delta : \mathcal{M}\hookrightarrow \mathcal{M}\times\mathcal{M}$ be the diagonal embedding and I the ideal sheaf of Δ in $\mathcal{M}\times\mathcal{M}$. Then I is locally generated by $\langle a\otimes 1 - 1\otimes a\rangle$ — the kernel of the multiplication map $\mathcal{O}^e\to\mathcal{O}$. The usual *sheaf of n-jets of functions on $\mathcal{M}$* (without divided-powers) is defined as

$$J^n(\mathcal{M}) := \mathcal{O}^e/I^{n+1}. \tag{3.1}$$

This is naturally a $\mathcal{O}^e$-module. The fiber over $P\in\mathcal{M}$ will be denoted $J^n_P(\mathcal{M})$ and is called the vector space of *n^{th}-order infinitesimal jet-spaces of $P\in\mathcal{M}$*. For example, $J^1(\mathcal{M}) = \mathcal{O}_{\mathcal{M}}\oplus T^*_{\mathcal{M}}$ and $J^1_P(\mathcal{M})\simeq k\oplus T^*_P(\mathcal{M})$ is (scalars plus) the Zariski cotangent space. Considering $J^n(\mathcal{M})$ as a left $\mathcal{O}$-module, we define the sheaf of *Grothendieck differential operators of order at most n on $\mathcal{M}$* as

$$\mathcal{D}^{Gr}(\mathcal{M})_n := \mathcal{H}om_{\mathcal{O}}(J^n(\mathcal{M}),\mathcal{O}). \tag{3.2}$$

Next, we explain the divided-power analogues. The divided power envelope of I inside $\mathcal{O}^e$ is (étale) locally isomorphic to the P.D polynomial algebra:

$$\Gamma_{\mathcal{O}^e}(I) \simeq \mathcal{O}\langle \xi_1, \ldots, \xi_n \rangle, \tag{3.3}$$

where $\xi_i = 1 \otimes x_i - x_i \otimes 1$ are the $\mathcal{O}$-basis of I, and x_i are the local coordinates of $\mathcal{O}$. Let $(\Gamma_{\mathcal{O}^e}(I), \bar{I}, \gamma)$ be the associated P.D. ring. Define the n^{th}*-order divided power neighborhood*, also called the sheaf of n-PD jets, by

$$J^{n,PD}(\mathcal{M}) := \Gamma_{\mathcal{O}^e}(I)/\bar{I}^{n+1}, \tag{3.4}$$

$$J^{\mathrm{PD}}(\mathcal{M}) := \varprojlim_n J^{n,\mathrm{PD}}(\mathcal{M}). \tag{3.5}$$

The fiber over $P \in \mathcal{M}$ is denoted $J_P^{n,PD}(\mathcal{M})$ and called the n^{th}*-order infinitesimal divided-power jet-spaces of* $P \in \mathcal{M}$. Viewing $J_P^{n,PD}(\mathcal{M})$ as a left $\mathcal{O}$-module, we define the sheaf of P.D. differential operators of order at most n, also called the sheaf of *crystalline differential operators*, by

$$\mathcal{D}^{crys}(\mathcal{M})_n := \mathcal{H}om_{\mathcal{O}}(J^{n,PD}(\mathcal{M}), \mathcal{O}).$$

By definition,

$$\langle , \rangle : \mathcal{D}_P^{crys}(\mathcal{M})_n \times J_P^{n,PD}(\mathcal{M}) \to k \quad \text{is a perfect pairing.}$$

We caution that the pairing is *not* given by evaluating a crystalline differential operator on a function with divided powers!

Recall in Equation 3.3 the $\mathcal{O}$- basis $\{\xi^{[q]} := \xi_1^{[q_1]} \ldots \xi_n^{[q_n]} : |q| \leq m\}$ of $J^{m,PD}(\mathcal{M})$. Let $D_{[q]}$ denote the corresponding dual basis of $\mathcal{D}^{crys}(\mathcal{M})$. The composition in $\mathcal{D}^{crys}(\mathcal{M})$ is defined as follows. Let

$$\delta : \mathcal{O}^e \to \mathcal{O}^e \otimes_{\mathcal{O}} \mathcal{O}^e, \;\; \xi \mapsto \xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi \;\text{ if } \xi = 1 \otimes x - x \otimes 1, \mathbf{1} = 1 \otimes 1.$$

By universal property, this induces a map $\delta : \Gamma_{\mathcal{O}^e}(I) \to \Gamma_{\mathcal{O}^e}(I) \otimes_{\mathcal{O}} \Gamma_{\mathcal{O}^e}(I)$ which satisfies

$$\delta^{n,m} : J^{n+m,PD}(\mathcal{M}) \to J^{n,PD}(\mathcal{M}) \otimes_{\mathcal{O}} J^{m,PD}(\mathcal{M}), \;\; \xi^{[q]} \mapsto \sum_{i+j=q} \xi^{[i]} \otimes \xi^{[j]}.$$

Then define composition of $f \in \mathcal{D}^{crys}(\mathcal{M})_n, g \in \mathcal{D}^{crys}(\mathcal{M})_m$ is defined by the formula

$$f \circ g : J^{n+m,PD}(\mathcal{M}) \xrightarrow{\delta^{n,m}} J^{n,PD}(\mathcal{M}) \otimes_{\mathcal{O}} J^{m,PD}(\mathcal{M}) \xrightarrow{1 \otimes f} J^{n,PD}(\mathcal{M}) \xrightarrow{g} \mathcal{O}$$

It is immediate from this formula that we have the following relations inside $\mathcal{D}^{crys}(\mathcal{M})$:

$$D_{[q]} \circ D_{[q']} = D_{[q+q']} \quad \text{and } fD_{[q]} = \sum_{|i|+|j|=|q|} \binom{|i|+|j|}{|i|} D_{[i]}(f) D_{[j]} \text{ for } f \in J^{PD}(\mathcal{M}) \tag{3.6}$$

The *universal enveloping algebra of the tangent Lie algebroid* $\mathcal{T}_{\mathcal{M}}$, denoted $U_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}})$, is defined as the $\mathcal{O}_{\mathcal{M}}$ algebra generated by $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{T}_{\mathcal{M}}$, subject to the relations $f \cdot \partial = f\partial, \partial \cdot f - f \cdot \partial = \partial(f), \partial \cdot \partial' - \partial' \cdot \partial = [\partial, \partial'], \quad \text{for } f \in \mathcal{O}_{\mathcal{M}}, \partial, \partial' \in \mathcal{T}_{\mathcal{M}}$. By Equation 3.6, it is clear

$$\mathcal{D}^{crys}(\mathcal{M}) \xrightarrow{\sim} U_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}), \;\; D_{[q]} \mapsto D_1^{q_1} \cdot D_2^{q_2} \cdots D_n^{q_n} \tag{3.7}$$

is an isomorphism of $\mathcal{O}_{\mathcal{M}}$-algebras (see also (BMR)).

3.3 Pairing fibers of jets with the coinvariants

As a starting point, we cite the uniformization theorem for Bun_G. Recall $O_x \simeq k[[t_x]], K_x \simeq k((t_x))$, and denote $\mathcal{O}_{out} := \mathcal{O}_X|_{X \backslash x}$.

Theorem 3.3.1. *(Uniformization Theorem) Let G be a semisimple reductive group over k, X a smooth projective curve over k, and $x \in X$ a closed point. Then there is an isomorphism of stacks*

$$Bun_G(X) = G(\mathcal{O}_{out})\backslash G(K_x)/G(O_x).$$

A proof of the uniformization theorem, as formulated, may be found in (Sor99, Theorem 5.1.1). We will use the uniformization theorem to produce an explicit description of the infinitesimal jet spaces on the underlying smooth locus $\mathcal{M}$ of Bun_G. In particular, we only consider $P \in \mathrm{Bun}_G$ whose associated adjoint bundle $\mathfrak{g}_P$ has no global sections: $H^0(X, \mathfrak{g}_P) = 0$.

Let

$$\mathfrak{g}_K := \mathfrak{g} \otimes K, \mathfrak{g}_O := \mathfrak{g} \otimes O.$$

Under the uniformization map, we may associate to P the triple $(\tau_{X\backslash x}, \tau_{D_x}, \phi)$, where

$$\tau_{D_x} : P|_{D_x} \xrightarrow{\sim} G \times D_x, \quad \tau_{X\backslash x} : P|_{X\backslash x} \xrightarrow{\sim} G \times (X \setminus x)$$

are trivializations and $\phi \in G(K)$ is the transition function on the overlap $D_x^\times$. This means the composition of maps

$$\mathfrak{g}_{\text{out}} \xrightarrow{\tau_{X\backslash x}^{-1}} \Gamma(X \setminus x, \mathfrak{g}_P) \xrightarrow{\text{Restrict}} \Gamma(D_x^\times, \mathfrak{g}_P) \xrightarrow{\tau_{D_x}} \mathfrak{g}_K \tag{3.8}$$

has image landing in $\mathrm{Ad}_\phi \mathfrak{g}_O \subset \mathfrak{g}_K$.

Definition 3.3.2. *Let $\phi \in G(K)$. The "vacuum module with central charge 0" is*

$$\mathbf{M}_\phi := U(\mathfrak{g}_K)/Ad_\phi \mathfrak{g}_O \cdot U(\mathfrak{g}_K).$$

The "space of coinvariants" is

$$\mathbf{M}_{\phi}^{out} := \mathbf{M}/\mathbf{M}\mathfrak{g}_{out},$$

where $\mathbf{M}_{\phi}$ is viewed as a $\mathfrak{g}_{out}$-module under the embedding $\mathfrak{g}_{out} \hookrightarrow \mathfrak{g}_K$ in Equation 3.8.

The PBW filtration on $U(\mathfrak{g}_K)$ induces a PBW filtration on $\mathbf{M}_{\phi}^{out}$, where the nth order filtration is denoted $\mathbf{M}_{\phi,n}^{out}$. Explicitly, we find

$$\mathbf{M}_{\phi,n} \;:=\; \frac{\operatorname{span}\left\langle\, \xi_1 \cdots \xi_r \;\middle|\; r \le n,\ \xi_i \in \mathfrak{g}_K \right\rangle}{\operatorname{span}\left\langle\, \xi_1 \cdots \xi_r \;\middle|\; r \le n,\ \xi_i \in \mathfrak{g}_K, \xi_1 \in \operatorname{Ad}_\phi \mathfrak{g}_O \right\rangle}.$$

$$\mathbf{M}_{\phi,n}^{\text{out}} \;:=\; \frac{\operatorname{span}\left\langle\, \xi_1 \cdots \xi_r \;\middle|\; r \le n, \xi_i \in \mathfrak{g}_K \right\rangle}{\operatorname{span}\left\langle\, \xi_1 \cdots \xi_r \;\middle|\; r \le n,\ \xi_i \in \mathfrak{g}_K, \xi_1 \in \operatorname{Ad}_\phi \mathfrak{g}_O \text{ or } \ \xi_r \in \mathfrak{g}_{\text{out}} \right\rangle}.$$

Lemma 3.3.3. *There is an isomorphism of vector spaces between the fiber of crystalline differential operators on $\mathcal{M}$ and the space of coinvariants:*

$$\mathcal{D}_P^{crys}(\mathcal{M}) \simeq \mathbf{M}_{\phi}^{out}$$

Proof. By 3.7, $\mathcal{D}^{crys}(\mathcal{M}) \simeq U_{\mathcal{O}_M}(\mathcal{T}_{\mathcal{M}})$. Thus the fiber over $P \in \mathcal{M}$ is just $U(T_P(\mathcal{M}))$, the enveloping algebra of the tangent space $T_P(\mathcal{M})$. It is well known, e.g by using the uniformization theorem for Bun_G, that $T_P\mathcal{M}$ is isomorphic to $\mathfrak{g}_K/(\operatorname{Ad}_\phi \mathfrak{g}_O + \mathfrak{g}_{\text{out}})$. The claim $U(T_P(\mathcal{M})) \simeq \mathbf{M}_{\phi}^{out}$ follows by comparing the associated graded of both sides. $\square$

Corollary 3.3.4. *The natural pairing of crystalline differential operators with divided-power jets induces a perfect pairing*

$$\phi : \mathbf{M}_{\phi,n}^{out} \times J_P^{n,PD}(\mathcal{M}) \to k.$$

CHAPTER 4

THE "UNIVERSAL SHEAF" OF LOG-DIFFERENTIAL FORMS

Fix a closed point $x \in X$. Following the notation of subsection 2.1, we consider the n-fold formal, resp. punctured disk on X^n:

$$(D_x)^n := \mathrm{Spec}((\widehat{\mathcal{O}}_{X,x})^{\hat{\otimes} n}), \qquad (D_x^\times)^n := \mathrm{Spec}(\mathrm{Frac}(\widehat{\mathcal{O}}_{X,x}^{\hat{\otimes} n})).$$

So, after choosing uniformizers, we may identify

$$\Gamma((D_x)^n, \mathcal{O}_{X^n}) = k[[z_1, \dots, z_n]], \quad \Gamma((D_x^\times)^n, \mathcal{O}_{X^n}) = \mathrm{Frac}(k[[z_1, \dots, z_n]]).$$

In higher dimension, the trivialization (3.8) reads

$$\tau_{D_x} : \Gamma(D_x^n, \mathfrak{g}_P^{\boxtimes n}) \xrightarrow{\sim} \mathfrak{g}^{\otimes n} \otimes k[[z_1, \dots, z_n]] = (\mathfrak{g}_O)^{\hat{\otimes} n}.$$

Let S_n denote the symmetric group of n letters. For $\sigma \in S_n$, let

$$D_\sigma = \bigcup_{i=1}^{n-1} \{x_{\sigma(i)} = x_{\sigma(i+1)}\} \subset X^n.$$

Observe each D_σ is a normal-crossing divisor, but the full diagonal divisor $D := \bigcup_{\sigma \in S_n} D_\sigma$ is not as soon as $n > 2$. Let $\Omega_{X^n}(\log(D_\sigma))$ denote the sheaf of top degree log-differential forms on X^n with simple poles along D_σ, as defined in Section 2. From here on, $\Omega_S := \Omega_S^{\dim S}$ will always mean top-degree forms on a smooth scheme S.

Definition 4.0.1. *Let $j : X^n \setminus D \hookrightarrow X^n$ be the inclusion. Define* [1]

$$\tilde{\Omega}_{X^n}(\log D) := \sum_{\sigma \in S_n} \Omega_{X^n}(\log D_\sigma) \subset j_*(\Omega_{X^n \setminus D})$$

Thus, restriction to $(D_x^\times)^n$ and then trivializing about τ_{D_x} produces a map

$$\Gamma(X^n, (\mathfrak{g}_P^*)^{\boxtimes n} \otimes \tilde{\Omega}_{X^n}(\log D)) \hookrightarrow (\mathfrak{g}^*)^{\otimes n} \otimes k[[z_1, \ldots, z_n]][(z_j - z_k)^{-1}] dz_1 \cdots dz_n, \quad \omega_n \mapsto \omega_n|_{(D_x^\times)^n}.$$

By Equation 3.8, the image lands in

$$\omega_n|_{(D_x^\times)^n} \in (\mathrm{Ad}_\phi \mathfrak{g}_O^*)^{\hat{\otimes} n}[(z_i - z_j)^{-1}, i, j \leq n] dz_1 \cdots dz_n. \tag{4.1}$$

Similarly, for $\omega_n \in \Gamma(D_x^n, (\mathfrak{g}_P^*)^{\boxtimes n} \otimes \tilde{\Omega}_{X^n}(\log D))$, we denote by $\omega_n|_{(D_x^\times)^n}$ to be the restriction to the formal punctured disk, followed by the τ_{D_x}-trivialization. This still has image as in Equation 4.1.

In what follows, we study properties of log-differential forms on X^n by analyzing their power-series expansion. We are now ready to define the main object:

4.1 The universal space controlling jets

Definition 4.1.1. *Define the "universal space" Ω_n over X as follows. Over an open $U \subset X$, define the sections of $\Omega_n(U)$ to consist of $(n+1)-$ tuples*

$$(\omega_0, \ldots, \omega_n)$$

1. To avoid overcount, we can index the sum over S_n/τ, where $\tau : i \mapsto n - i - 1$ is the Type A_n Dynkin automorphism.

where $\omega_0 \in k$ *and* $\omega_i \in \Gamma(U^i, (\mathfrak{g}_P^*)^{\boxtimes i} \otimes \tilde{\Omega}_{X^i}(\log D))^{-S_i}, i > 0$ *are such that, after restricting to* $(D_x)^i$*,*

$$\omega_i|_{(D_x)^i} \in \left((\mathfrak{g}^*)^{\otimes i} \otimes k[[z_1, \dots, z_i]][(z_j - z_k)^{-1}]\right)^{-S_i} dz_1 \dots dz_i$$

and $\omega_i|_{(D_x)^i}$ *has the expansion in the* $z_{i-1} - z_i$ *direction, as defined in Definition 2.1.2:*

$$T_{z_{i-1}-z_i}(\omega_i|_{(D_x)^i}) = \frac{\Phi_i^*(\omega_{i-1}|_{(D_x)^{i-1}})}{z_{i-1} - z_i} dz_i + reg. \tag{4.2}$$

Here, $(-)^{-S_i}$ *denotes the* S_i*-sign-invariants and* Φ_i^* *is the map induced by the Lie bracket:*

$$\Phi_i : \mathfrak{g}^{\otimes i} \to \mathfrak{g}^{\otimes i-1}, \quad \xi_1 \otimes \dots \otimes \xi_i \mapsto \xi_1 \otimes \dots \otimes [\xi_{i-1}, \xi_i].$$

And, "reg" stands for an expression whose expansion in the $z_{i-1} - z_i$ *direction has no* $(z_{i-1} - z_i)^d$ *terms when* $d < 0$*.*

Remark 1. *In defining the "universal space"* Ω_n*, we chose to work with sections* $\omega_i \in \tilde{\Omega}_{X^i}(\log D)$ *instead of sections in the larger space* $\Omega_{X^i}(\log D)$ *in order to avoid discussing log differential forms on non-normal-crossing divisors and to better illuminate what sorts of poles are allowed. It turns out the residue constraint in Equation 4.2 forces these two options to be equivalent. We explain this below.*

Suppose Ω_n *is instead defined as in Definition 4.1.1, except* $\tilde{\Omega}_{X^i}(\log D)$ *is replaced with* $\Omega_{X^i}(\log D)$*. We use (S79) to define logarithmic differential forms and their residue along the (non-normal-crossing) diagonal divisor* $D \subset X^i$ *for each* i*. The key difference is the residue in Equation 2.3 a priori now lands in rational sections of the (normalization of the) divisor instead of regular sections. However, the residue constraint in Equation 4.2 ensures this does not happen:*

Indeed, let $\omega = (\omega_i)_{0 \leq i \leq n} \in \Omega_n(U)$*. Suppose for contradiction there was a term of* $\omega_n|_{((D_x)^\times)^n}$ *with a "loop" in the denominator:* $(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \dots (z_{i_m} - z_{i_1})$*. Then the* *iterated residue*

$$\tilde{\omega}_{n-m+2} := Res_{z_{i_1}=z_{i_2}} \circ Res_{z_{i_2}=z_{i_3}} \circ \cdots \circ Res_{z_{i_{m-2}}=z_{i_{m-1}}}(\omega_n)$$

creates an order 2-pole for $\tilde{\omega}_{n-m+2}$ *at* $z_{i_1} - z_{i_m}$. *For example, if* $m = 3$,

$$Res_{z_1=z_2}\frac{1}{(z_1-z_2)(z_2-z_3)(z_3-z_1)} = \frac{-1}{(z_1-z_3)^2}.$$

But the composition of Lie bracket maps, Φ_T^* *(see 5.11), is* $k((z_1,\dots,z_n))$*-linear. Hence according to definition of* $\Omega_n(U)$, $(\Phi_T^*)^{-1}(\tilde{\omega}_{n-m+2}) = \omega_{n-m+2}$ *should still have log poles on the diagonals. Contradiction. Thus, there exist no "loops" in* ω_n*, and we conclude if* $(z_{i_1}-z_{i_2})(z_{i_2}-z_{i_3})\dots(z_{i_{m-1}}-z_{i_m})$ *is a denominator of* $\omega_n|_{(D_x^\times)^n}$*, then* $m \leq n$ *and* i_j *are all distinct. But this is precisely how Definition 4.0.1 is setup, and we conclude* $\omega_n \in \tilde{\Omega}_{X^n}(\log D)$. *So we could have equivalently worked with* $\Omega_{X^n}(\log D)$ *instead of* $\tilde{\Omega}_{X^n}(\log D)$

We will only consider sections of the "universal space" Ω_n on $U = D_x^\times, D_x$, and X. In these cases, we prove $\Omega_n(U)$ is naturally isomorphic to the infinitesimal jet spaces of $G(K), \mathrm{Gr}_G$ and Bun_G, respectively. As defined, the sheaf $\Omega_n(X)$ depends on $x \in X$ since we check the residue condition (Equation 4.2) locally around the formal disk about x. As a consequence of Theorem 4.1.4 or Subsection 4.2, this definition is in fact independent of the choice of point on the curve.

Let us now formulate the main results of (BD) as Proposition 4.1.2 and Proposition 4.1.3.

Proposition 4.1.2. *(BD) Let* k *be an arbitrary field. Then there is a perfect pairing*

$$\Phi_K : U^{\leq n}(\mathfrak{g}_K) \times \Omega_n(D_x^\times) \to k$$

induced by the pairing: given $\xi_1, \dots, \xi_k \in \mathfrak{g}_K$ *and* $\omega := (\omega_0, \dots, \omega_n)$*,*

$$\Phi_K(\xi_1 \dots \xi_k, \omega) := Res_{z_1=0} \dots Res_{z_k=0} \langle \xi_1(z_1) \otimes \cdots \otimes \xi_k(z_k), \omega_k \rangle \tag{4.3}$$

where $\xi_1(z_1) \otimes \cdots \otimes \xi_k(z_k) \in (\mathfrak{g}_K)^{\hat{\otimes} k}$ *and* ω_k *is a differential form with values in* $(\mathfrak{g}^*)^{\otimes k}$*. So,* $\langle , \rangle$ *is a scalar-valued differential form. And,* $Res_{z_1=0} \dots Res_{z_k=0}$ *means we first compute the residue* $z_k = 0$ *and treat* $z_1, \dots, z_{k-1}$ *as scalars, etc, as explained in Definition 2.1.2.*

Remark 2. *A heuristic explanation for why to expect (BD) to hold in characteristic p is that under the perfect pairing,* $\Omega(D_x^\times)$ *plays the role of the algebra with divided powers and* $U(\mathfrak{g}_K)$ *plays the role of the algebra without divided powers. A more convincing, general construction of a "divided power module" will be considered in Subsection 5.4.*

Proof. For the reader's convenience, we present the proof found in (BD). First, we must check the residue pairing on $\mathfrak{g}_K^{\hat{\otimes} n} \times \Omega_n(D_x^\times) \to k$ defined by Equation 4.3 descends to $U^{\leq n}(\mathfrak{g}_K)$. This amounts to checking

$$\mathrm{Res}_{z_1=0} \dots \mathrm{Res}_{z_k=0} \langle \xi_1(z_1) \otimes \dots \big(\xi_i(z_i) \otimes \xi_{i+1}(z_{i+1}) - \xi_{i+1}(z_i) \otimes \xi_i(z_{i+1})\big) \cdots \otimes \xi_k(z_k), \omega_k \rangle \tag{4.4}$$

$$= \mathrm{Res}_{z_1=0} \dots \mathrm{Res}_{z_k=0} \langle \xi_1(z_1) \otimes \dots [\xi_i(z_i), \xi_{i+1}(z_i)] \otimes \xi_{i+2}(z_{i+1}) \cdots \otimes \xi_k(z_{k-1}), \omega_k \rangle. \tag{4.5}$$

Now, the S_2 anti-invariance of ω_2, together with a relabelling of variables $z_1 \leftrightarrow z_2$, implies:

$$\begin{aligned} \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} \langle \xi_2(z_1) \otimes \xi_1(z_2), \omega(z_1, z_2) \rangle &= -\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} \langle \xi_1(z_2) \otimes \xi_2(z_1), \omega(z_2, z_1) \rangle \\ &= -\mathrm{Res}_{z_2=0}\mathrm{Res}_{z_1=0} \langle \xi_1(z_1) \otimes \xi_2(z_2), \omega(z_1, z_2) \rangle. \end{aligned}$$

And in general, S_k anti-invariance of ω_k implies Equation 4.4 equals

$$\mathrm{Res}_{z_1=0}\dots\mathrm{Res}_{z_{i+1}=0}\eta(z_1,\dots,z_{i+1}) - \mathrm{Res}_{z_1=0}\dots\mathrm{Res}_{z_{i+1}=0}\mathrm{Res}_{z_i=0}\eta(z_1,\dots,z_{i+1})$$

where $\eta(z_1,\dots,z_{i+1}) = -\mathrm{Res}_{z_{i+2}=0}\dots\mathrm{Res}_{z_k=0}\langle\xi_1(z_1)\otimes\cdots\otimes\xi_k(z_k),\omega_k\rangle$. But, η only has poles along the hyperplanes $z_m = 0$ and $z_m = z_l$. Thus we may apply Parshin's residue formula, Lemma 2.1.3, to deduce

$$\mathrm{Res}_{z_i=0}\mathrm{Res}_{z_{i+1}=0}\eta - \mathrm{Res}_{z_{i+1}=0}\mathrm{Res}_{z_i=0}\eta = -\mathrm{Res}_{z_i=0}\mathrm{Res}_{z_i=z_{i+1}}\eta.$$

We remark that the right-hand side of the above formula is again interpretted using Definition 2.1.2. But now, we are done because the expansion of η in the $z_i - z_{i+1}$ direction has the form as in Equation 4.2; consequently its residue at $z_i - z_{i+1}$ induces the $[\xi_i(z_i),\xi_{i+1}(z_i)]$ expression in Equation 4.5. This proves the claim that the pairing is well-defined.

Now, we will show the pairing is perfect by constructing an isomorphism between $(U^{\leq n}\mathfrak{g}_K)^*$ and $\Omega_n(D_x^\times)$. Let $e_i \in \mathfrak{g}$ be a basis and e^i the dual basis of $\mathfrak{g}^*$. Let $e_i^{(l)} = e_i z^l \in \mathfrak{g}_K \subset U(\mathfrak{g}_K)$. We claim there exists a map

$$\psi_K : (U^{\leq n}\mathfrak{g}_K)^* \to \Omega_n(D_x^\times), \quad \lambda \mapsto (w_r), \quad \text{where}$$

$$w_r = \sum_{l_1,\dots,l_n\in\mathbf{Z}}\sum_{i_1,\dots,i_r}\lambda(e_{i_1}^{(l_1)}\dots e_{i_r}^{(l_n)})e^{i_1}\otimes\cdots\otimes e^{i_r} z_1^{-l_1-1}\dots z_r^{-l_r-1}dz_1\dots dz_r \tag{4.6}$$

As written, we have

$$\omega_r \in (\mathfrak{g}^*)^{\otimes r}\otimes k((z_1))\cdots((z_r))dz_1\dots dz_r.$$

To check that (ω_r) lands in $\Omega_n(D_x^\times)$, we must first show ω_r is the expansion of an element $\hat{\omega}_r \in \left((\mathfrak{g}^*)^{\otimes r}\otimes k[[z_1,\dots,z_r]][(z_i-z_j)^{-1}z_k^{-1}]_{i,j,k}dz_1\dots dz_r\right)^{-S_r}$, where by expansion we mean as introduced in Definition 2.1.2. We make the subtle remark that S_r does not act on the "expansion" because $k((z_1))((z_2)) \neq k((z_2))((z_1))$. So once we show ω_r is the expansion of some (unique) $\hat{\omega}_r$, we must actually define the map ψ_K to be $\lambda \mapsto (\hat{\omega}_r)$.

The existence of such an expansion follows by the locality property of the affine Kac-Moody vertex algebra (see (BZF, Theorem 4.5.2) for the general proof for vertex algebras). Indeed, introduce the "fields" $A_i(z) := \sum_{l\in \mathbf{Z}} e_i^{(l)} z^{-l-1}$. These fields satisfy the locality property $(z-w)^2[A_i(z), A_j(w)] = 0$ and the operator product expansion (OPE)

$$A_i(z)A_j(w) = \frac{c_{ij}}{(z-w)^2} + \sum_q f_{ij}^q \frac{A_q(w)}{z-w} + :A_i(z)A_j(w):$$

Here, c_{ij} are scalars corresponding to central charge c (so $c_{ij} = 0$ for us), f_{ij}^q are structure constants for $\mathfrak{g}$, and $:A_i(z)A_j(w):$ is the normally ordered product, which lives in $\mathfrak{g}^{\otimes 2} \otimes k[[z,w]][z^{-1}, w^{-1}]$.

Now, rewrite Equation 4.6 as

$$\omega_r = \sum_{i_1,\dots,i_r} \lambda(A_{i_1}(z_1)\cdots A_{i_r}(z_r)) e^{i_1} \otimes \cdots \otimes e^{i_r} dz_1 \dots dz_r.$$

Consequently, this expression of ω_r, together with the locality property, implies

$$\omega_r \prod_{i<j} (z_i - z_j)^2$$

is the expansion of an element of $\left((\mathfrak{g}^*)^{\otimes r} \otimes k[[z_1,\dots,z_r]][(z_i-z_j)^{-1} z_k^{-1}]_{i,j,k} dz_1 \dots dz_r\right)^{-S_r}$. Thus the same holds for ω_r. Next, the OPE directly implies the residue constraint 4.2. Thus, $(\omega_r) \in \Omega_n(D_x^\times)$. Finally, it is clear ψ_K is injective: if ω_r vanish for all r and z_i, then $\phi = 0$.

Next, it is also clear the map $\psi'_K : \Omega_n(D_x^\times) \to (U^{\leq n}(\mathfrak{g}_K))^*$ induced by the residue pairing Φ_K is the inverse to ψ_K. Indeed, this boils down to formula 4.6. Thus ψ_K being injective and $\psi'_K \circ \psi_K = 1$ implies ψ_K is isomorphism. □

Proposition 4.1.3. *(BD) Let k be an arbitrary field. Let $\phi \in G(K)$ and let $\mathbf{M}_{\phi,n}$ denote the n^{th}-filtered component of the vacuum module (see Definition 3.3.2). There is a perfect* *pairing induced by restricting Equation 4.3 to* $\Omega_n(D_x) \subset \Omega_n(D_x^\times)$

$$\Phi_{K/O} : \mathbf{M}_{\phi,n} \times \Omega_n(D_x) \to k.$$

Proof. First, we check the restriction of Φ_K to $\Omega_n(D_x) \subset \Omega_n(D_x^\times)$ factors through $\mathbf{M}_{\phi,n}$. This amounts to the claim

If $\omega = (\omega_k|_{(D_x^\times)^k}) \in \Omega_n(D_x)$ and $\xi_1 \in \mathrm{Ad}_\phi(\mathfrak{g}_O), \xi_2, \ldots, \xi_n \in \mathfrak{g}_K$, then $\Phi_K(\xi_1 \cdots \xi_k, \omega) = 0$.

Recall $\phi \in G(K)$ is the transition function associated to the G-bundle P, and that Equation 4.1 says $\omega_k|_{(D_x^\times)^k} \in (\mathrm{Ad}_\phi \mathfrak{g}_O^*)^{\hat{\otimes} n}[(z_i - z_j)^{-1}, i, j \leq n] dz_1 \cdots dz_n$. The standard pairing $\langle , \rangle : \mathfrak{g}_K \times \mathfrak{g}^* \otimes K dz \to 0$ is clearly Ad_ϕ-invariant, in the sense that

$$\mathrm{Res}_{z=0} \langle \mathrm{Ad}_\phi(\xi), \mathrm{Ad}_\phi \omega \rangle = \mathrm{Res}_{z=0} \langle \xi, \omega \rangle.$$

Let $\tilde{\omega}_1 := \mathrm{Res}_{z_2=0} \ldots \mathrm{Res}_{z_k=0} \langle \xi_1(z_1) \otimes \cdots \otimes \xi_k(z_k), \omega_k|_{(D_x^\times)^k} \rangle \in \mathrm{Ad}_\phi(\mathfrak{g}_O^*) dz_1$. Then $\langle \xi_1, \tilde{\omega_1} \rangle \in k[[z_1]] dz_1$, which implies its residue at z_1 is zero. This proves existence of the pairing $\Phi_{K/O}$ as claimed in the Proposition. Thus restricting ψ'_K, from the proof of Proposition 4.1.2, to $\Omega_n(D_x) \subset \Omega_n(D_x^\times)$ produces a map $\psi'_{K/O} : \Omega_n(D_x) \to \mathbf{M}_{\phi,n}$.

To show $\Phi_{K/O}$ is perfect, it just remains to show restricting $\psi_K : (U^{\leq n} \mathfrak{g}_K)^* \to \Omega_n(D_x^\times)$ to $\mathbf{M}_{\phi,n}$ has image landing in $\Omega_n(D_x)$. More precisely, define

$$\psi_{K/O} := \mathrm{Ad}_\phi \circ \psi_K|_{\mathbf{M}^*_{\phi,n}} : \mathbf{M}^*_{\phi,n} \to \Omega_n(D_x^\times).$$

where

$$\mathrm{Ad}_\phi(\omega_r(z_1, \ldots, z_r)) := (\mathrm{Ad}_{\phi(z_1)} \otimes \cdots \otimes \mathrm{Ad}_{\phi(z_r)})(\omega_r(z_1, \ldots, z_r)).$$

Then as we saw before, Equation 4.6 implies the composite $\Omega_n(D_x) \xrightarrow{\psi'_{K/O}} \mathbf{M}_{\phi,n} \xrightarrow{\psi_{K/O}}$

$\Omega_n(D_x^\times)$ is the identity map.

Suppose $\lambda \in \mathbf{M}_{\phi,n}^*$. Then

$$\mathrm{Ad}_\phi.\lambda(e_{i_1}^{(l_1)} \dots e_{i_r}^{(l_r)}) = \lambda(\mathrm{Ad}_\phi(e_{i_1}^{(l_1)}) \dots \mathrm{Ad}_\phi(e_{i_r}^{(l_r)})) = 0$$

for all $l_1 \geq 0$ because $e_{i_1}^{(l_1)} \in \mathfrak{g}_O$ in that case. Thus, no negative powers of z_1 appear in ω_r. Then S_r-invariance of ω_r implies no negative powers of z_i appear for any i. Hence $\psi_{K/O}(\lambda) \in \Omega_n(D_x)$. □

Finally, we prove the global analogue:

Theorem 4.1.4. *Let k be an arbitrary field. Let $\mathbf{M}_{\phi,n}^{out}$ denote n^{th} filtered component of the space of coinvariants (see Definition 3.3.2). Then there is a perfect pairing*

$$\Phi_X : \mathbf{M}_{\phi,n}^{out} \times \Omega_n(X) \to k$$

induced by the pairing: given $\xi_1, \dots, \xi_k \in \mathfrak{g}_K$ and $\omega := (\omega_0, \dots, \omega_n) \in \Omega_n(X)$,

$$\Phi_X(\xi_1 \dots \xi_k, \omega) := Res_{z_1=0} \dots Res_{z_k=0} \langle \xi_1(z_1) \otimes \dots \otimes \xi_k(z_k), \omega_k|_{(D_x^\times)^k} \rangle$$

As a corollary of Corollary 3.3.4,

$$J_P^{n,PD}(\mathcal{M}) \simeq \Omega_n(X).$$

Proof. Using Proposition 4.1.3, we just need to show restricting $\psi'_{K/O}$ to $\Omega_n(X)$ has image landing in $(\mathbf{M}_{\phi,n}^{out})^*$ and restricting $\psi_{K/O} : \mathbf{M}_{\phi,n}^* \to \Omega_n(D_x)$ to $(\mathbf{M}_{\phi,n}^{out})^*$ has image landing in $\Omega_n(X)$.

Let us first discuss $\psi'_{K/O}$. Suppose $\omega = (\omega_k) \in \Omega_n(X)$ and suppose $\xi_1, \dots, \xi_{k-1} \in \mathfrak{g}_K, \xi_k \in \mathfrak{g}_{\mathrm{out}}$. Then $\langle \xi_1 \dots \xi_k, \omega \rangle$ is regular on $(D_x^\times)^{\times(k-1)} \times (X \setminus x)$. So on the final component, it may only possibly have a residue at x. As X is projective, this forces

$$\mathrm{Res}_{z_k=0}\langle \xi_1 \dots \xi_k, \omega|_{(D_x^\times)^k}\rangle = 0.$$

Next, if we assume $\xi_1 \in \mathrm{Ad}_\phi(\mathfrak{g}_O), \xi_2, \dots, \xi_k \in \mathfrak{g}_K$, then we already saw in Proposition 4.1.3 that $\Phi_X(\xi_1 \dots \xi_k, \omega) = 0$. Thus, $\psi'_X := \psi'_{K/O}|_{\Omega_n(X)} : \Omega_n(X) \to (\mathbf{M}_{\phi,n}^{out})^*$.

Now, suppose $\lambda \in (\mathbf{M}_{\phi,n}^{out})^*$, and define $\psi_{K/O}(\lambda) := (\omega_k)$ as in Proposition 4.1.3. There, we showed ω_k is regular on $(D_x)^{\times k}$. Now, using λ kills $\phi \mathfrak{g}_O \phi^{-1}$ on the left-most factor and $\mathfrak{g}_{out}$ on the right-most factor, we conclude ω_k may be extended to $(D_x)^{\times(k-1)} \times X$ by the Strong Residue Theorem 2.1.1. Since ω_k is S_k-invariant, it is thus regular on $(D_x)^{\times i} \times X \times (D_x)^{\times(k-i-2)}$ for all i. We will conclude it is then automatically regular on X^k by the following remarkable fact from algebraic geometry, which is a generalization of the strong residue theorem:

Lemma 4.1.5. [2] *Suppose X is a smooth projective curve over a field k, and fix a point $x_0 \in X$. Suppose $\mathcal{E}$ is a locally free sheaf on X^m, $m \geq 2$, and suppose s is a section of $\mathcal{E}$ defined locally on*

$$s \in \Gamma\Big(\bigcup_{1 \leq i \leq m} (D_{x_0})^{\times i} \times X \times (D_{x_0})^{\times(m-i-2)}, \mathcal{E}\Big).$$

Then s extends uniquely to a global section of $\mathcal{E}$.

Proof. (The following proof was communicated by Sasha Beilinson). The proof will proceed by induction. Let us prove the $m = 2$ case first.

Since $\mathcal{E}$ is locally free over X^2, we may find a very ample line bundle $\mathcal{L}$ over X such that $\mathcal{E}^* \otimes (\mathcal{L} \boxtimes \mathcal{L})$ is generated by global sections. This means the map

$$V \otimes \mathcal{O}_{X\times X} \to \mathcal{E}^* \otimes (\mathcal{L} \boxtimes \mathcal{L}), \;\; V := \Gamma(X^2, \mathcal{E}) \otimes (\mathcal{L} \boxtimes \mathcal{L})$$

2. This lemma for $m = 2$ appears as Theorem 10.3.3 of (BZF). Several misprints occur in the published proof, so we provide a self-contained argument here.

is surjective. Taking the dual, we find $\mathcal{E} \hookrightarrow V^* \otimes (\mathcal{L} \boxtimes \mathcal{L})$ is injective. Thus, we see the statement for $\mathcal{L} \boxtimes \mathcal{L}$ implies the statement for $\mathcal{E}$. Furthermore, we may assume $\mathcal{L}$ has large enough degree so that $H^1(X, \mathcal{L}) = 0$.

Pick n sufficiently large so that $H^0(X, \mathcal{L}(-nx_0)) = 0$. Let $D_n := \operatorname{Spec}(\mathcal{O}_X/m_{x_0}^{n+1})$ be the n^{th} infinitesimal neighborhood of x_0. Consider the short exact sequence of sheaves on X:

$$0 \to \mathcal{L}(-nx_0) \to \mathcal{L} \to \mathcal{L}|_{D_n} \to 0. \tag{4.7}$$

Now, consider the external tensor product on the left with $(X, \mathcal{L})$ and then with $(D_n, \mathcal{L}|_{D_n})$. The restriction map $D_n \times X \to X \times X$ induces a map on cohomologies:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X^2, \mathcal{L} \boxtimes \mathcal{L}) & \longrightarrow & H^0(X \times D_n, \mathcal{L} \boxtimes \mathcal{L}|_{D_n}) & \xrightarrow{\delta} & H^1(X \times X, \mathcal{L} \boxtimes \mathcal{L}(-nx_0)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(D_n \times X, \mathcal{L}|_{D_n} \boxtimes \mathcal{L}) & \longrightarrow & H^0(D_n^2, \mathcal{L}|_{D_n} \boxtimes \mathcal{L}|_{D_n}) & \xrightarrow{\delta} & H^1(D_n \times X, \mathcal{L}|_{D_n} \boxtimes \mathcal{L}(-nx_0)) & \longrightarrow & 0
\end{array}$$

Thus, the obstruction of lifting $s \in H^0(X \times D_n, \mathcal{L} \boxtimes \mathcal{L})$ to $H^0(X^2, \mathcal{L}^{\boxtimes 2})$ is $\delta(s) \in H^1(X \times X, \mathcal{L} \boxtimes \mathcal{L}(-nx_0))$. By Kunneth, we compute this H^1 equals $H^0(X, \mathcal{L}) \otimes H^1(X, \mathcal{L}(-nx_0))$. Next, we know s restricted to $D_n \times D_n$ extends to $D_n \times X$, so the restriction of $\delta(s)$ to $H^1(D_n \times X, \mathcal{L}|_{D_n} \boxtimes \mathcal{L}(-nx_0))$ vanishes. By Kunneth, this H^1 equals $H^0(D_n, \mathcal{L}|_{D_n}) \otimes H^1(X, \mathcal{L}|_{D_n})$. Finally, the restriction map

$$H^0(X, \mathcal{L}) \otimes H^1(X, \mathcal{L}(-nx_0)) \to H^0(D_n, \mathcal{L}|_{D_n}) \otimes H^1(X, \mathcal{L}(-nx_0))$$

is injective because $H^0(X, \mathcal{L}(-nx_0)) = 0$. Thus, $\delta(s) = 0$ and s extends to a global section of $\mathcal{E}$. This concludes the $m = 2$ case.

Now suppose the lemma holds for X^{m-1}. We may again reduce to the case $\mathcal{E} = \mathcal{L}^{\boxtimes m}$ for a very ample line bundle $\mathcal{L}$. Suppose a section s of $\mathcal{L}^{\boxtimes m}$ is defined on $X \times D_n^{\times(m-1)}$ and all m permutations. By the induction hypothesis, it may be extended to $X^{m-1} \times D_n$. Then consider the exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(X^m, \mathcal{L}^{\boxtimes m}) & \longrightarrow & H^0(X^{m-1} \times D_n, \mathcal{L}^{\boxtimes m-1} \boxtimes \mathcal{L}|_{D_n}) & \xrightarrow{\delta} & H^1(X^m, \mathcal{L}^{\boxtimes m-1} \boxtimes \mathcal{L}(-nx_0)) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(D_n^{m-1} \times X, \mathcal{L}|_{D_n}^{\boxtimes m-1} \boxtimes \mathcal{L}) & \longrightarrow & H^0(D_n^m, \mathcal{L}|_{D_n}^{\boxtimes m}) & \xrightarrow{\delta} & H^1(D_n^{m-1} \times X, \mathcal{L}|_{D_n}^{\boxtimes m-1} \boxtimes \mathcal{L}(-nx_0)) & \to 0 \end{array}$$

where the top row is induced by the cohomology of the external tensor product of the exact sequence 4.7 with $(X^{m-1}, \mathcal{L}^{\boxtimes m-1})$. The vertical arrows are restricting the first $m-1$ factors to D_n^{m-1}. Repeating the $m = 2$ argument, we find the restriction of

$$\delta(s) \in H^1(X^m, \mathcal{L}^{\boxtimes m-1} \boxtimes \mathcal{L}(-nx_0)) = H^0(X, \mathcal{L})^{\otimes m-1} \otimes H^1(X, \mathcal{L}(-nx_0))$$

to $H^1(D_n^{m-1} \times X, \mathcal{L}|_{D_n}^{\boxtimes m-1} \boxtimes \mathcal{L}(-nx_0)) = H^0(D_n, \mathcal{L}|_{D_n})^{\otimes m-1} \otimes H^1(X, \mathcal{L}(-nx_0))$ is 0 because of the assumption that the restriction of s to $D_n^{\times m}$ extends to $D_n^{m-1} \times X$. Moreover, we know the restriction map on H^1 is injective because $\mathcal{L}$ is very ample. Thus, the section s lifts to a global section $s \in \Gamma(X^m, \mathcal{L}^{\boxtimes m})$. □

To complete Theorem 4.1.4, we may apply Lemma 4.1.5 to the locally free sheaf $\mathcal{E} := (\mathfrak{g}_P^*)^{\boxtimes k} \otimes \tilde{\Omega}_{X^k}(\log D)$ on X^k, and the section $s = \omega_k$. This shows

$$\psi_X := \psi_{K/O}|_{(\mathbf{M}_{\phi,n}^{out})^*} : (\mathbf{M}_{\phi,n}^{\text{out}})^* \to \Omega_n(X)$$

as desired, and we are finished. □

We conclude with a concise summary of the main results proven in this section. There is a commutative diagram where each horizontal line is a vector space isomorphism. Moreover, the top row is compatible with the algebra structures.

$$\begin{array}{ccccc} J_\phi^{n,PD}(G(K)) & \xrightarrow{\sim} & ((U\mathfrak{g}_K)_{\leq n})^* & \xrightarrow{\sim} & \Omega_n(D_x^\times) \\ \uparrow & & \uparrow & & \uparrow \\ J_{\phi G(O)}^{n,PD}(\mathrm{Gr}_G) & \xrightarrow{\sim} & (\mathbf{M}_{\phi,n})^* & \xrightarrow{\sim} & \Omega_n(D_x) \\ \uparrow & & \uparrow & & \uparrow \\ J_P^{n,PD}(\mathcal{M}) & \xrightarrow{\sim} & (\mathbf{M}_{\phi,n}^{\mathrm{out}})^* & \xrightarrow{\sim} & \Omega_n(X) \end{array} \tag{4.8}$$

4.2 Changing the point $x \in X$

There is a classical fact that "1-point coinvariants are isomorphic to the 2-point coinvariants." In the case of the Heisenberg Lie algebra, and in characteristic 0, this is proved in Appendix 9.6 of (BZF). For Bun_G, this follows from the fact that 1-point uniformization is isomorphic to the 2-point uniformization. Let us formulate this precisely. Let $x, y \in X$ and consider the vacuum module

$$\mathbf{M}_{x,y} := \big(U(\mathfrak{g}_{K_x}) \otimes_{U(\mathfrak{g}_{O_x})} 1_x\big) \otimes_k \big(U(\mathfrak{g}_{K_y}) \otimes_{U(\mathfrak{g}_{O_y})} 1_y\big).$$

Let $\mathcal{O}_{out(x,y)} := \mathcal{O}(X \setminus \{x, y\})$. Then we have injective (Lie algebra) morphism

$$\mathfrak{g}_{out(x,y)} := \mathfrak{g} \otimes \mathcal{O}_{out(x,y)} \hookrightarrow \mathfrak{g}_{K_x} \oplus \mathfrak{g}_{K_y}.$$

Then define

$$\mathbf{M}_{x,y}^{out(x,y)} := \mathbf{M}_{x,y} / \mathfrak{g}_{out(x,y)} \cdot \mathbf{M}_{x,y}.$$

The PBW filtration on $\mathbf{M}_x, \mathbf{M}_y$ naturally induces a tensor-product filtration on $\mathbf{M}_{x,y}$, which in turn induces a filtration on $\mathbf{M}_{x,y}^{out(x,y)}$.

Theorem 4.2.1. *There is an isomorphism of graded vector spaces*

$$\mathbf{M}_x^{out(x)} \xrightarrow{\sim} \mathbf{M}_{x,y}^{out(x,y)}$$

As a consequence, $\mathbf{M}_x^{out(x)} \simeq \mathbf{M}_y^{out(y)}$ for all $x, y \in X$. This justifies our notation $\mathbf{M}_n^{out} := (\mathbf{M}_x^{out(x)})_n$ used in Chapter 4, and moreover shows the independence of choosing $x \in X$ in defining Ω_n.

CHAPTER 5

PULLBACK TO THE FULTON-MACPHERSON COMPACTIFICATION

In this section we recall the geometry of the Fulton-Macpherson compactification of the configuration space of n-points on a curve, $\hat{X}^n$. The main result is Theorem 5.3.4, which relates the global sections of the two sheaves considered by (BD; BG). Finally, in subsection 5.4, we provide a relationship between the (BG) sheaves and the Lie operad.

5.1 Resolution of the diagonal

Let us briefly recall the geometry of the Fulton-Macpherson compactification of the configuration space which will be used. Let X be a smooth projective curve and $n > 1$ a positive integer. Let $\mathring{X}^n \subset X^n$ be the open set of all n-tuples of pairwise distinct points of X, and let $D := X^n \setminus \mathring{X}^n$ denote the big diagonal divisor. It is not a normal-crossing divisor for any $n > 2$. We will now construct a smooth projective variety $\hat{X}^n$ and projective morphism $p : \hat{X}^n \to X^n$ which is an isomorphism over $\mathring{X}^n$ and such that $\hat{D} := (p^{-1}(D))_{red}$ is a normal-crossing divisor. Our construction is taken from (BG) and is done through a sequence of blowups along diagonals. A similar construction, done through a different sequence of blowups, may be found in (FM94).

For each subset $I \subset [n] := \{1, 2, \dots, n\}$, define the diagonal divisor

$$D_I := \{(x_i) \in X^n : x_i = x_j \text{ for all } i, j \in I\}.$$

For two or three-element subsets, we adopt the notation $D_{ij} := D_{\{i,j\}}$ and $D_{ijk} := D_{\{i,j,k\}}$. Let $\pi_n : X^n_n \to X^n$ be the blowup of X^n along $D_{[n]} \simeq X$. Now, we will construct by downward induction a sequence of varieties

$$\hat{X}^n = X_2^n \to X_3^n \to \cdots \to X_{n-1}^n \to X_n^n \to X^n.$$

Suppose $\pi_k : X_k^n \to \cdots \to X_n^n \to X^n$ has been constructed and let $\tilde{D}_I = \pi_k^{-1}(D_I)$. Then define X_{k-1}^n to be the result of blowing up all subvarieties $\tilde{D}_I \subset X_k^n$ with $|I| = k-1$. The order may be taken arbitrarily because the intersection of any number of subvarieties $\tilde{D}_I$ with $|I| = k-1$ is either empty or transverse. Let $\hat{X}^n = X_2^n$ be the final step of this inductive construction, and let $\pi := \pi_2 : \hat{X}^n \to X^n$. Let $\hat{D} := \pi^{-1}(D)$. The irreducible components of $\hat{D}$ are $\{\hat{D}_I : |I| \geq 2\}$, and each of them is smooth.

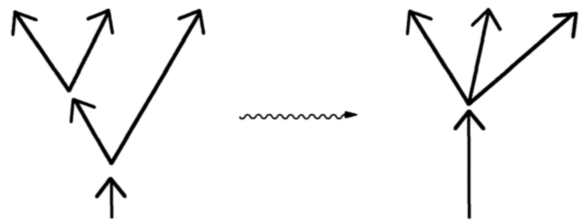
By a *tree*, we mean a graph without loops such that there is exactly one ingoing edge and at least 2 outgoing edges at each vertex of the graph. A connected component of a tree has a unique ingoing external edge. Such a component may consist of a single line, in which case it has no vertices and that line is viewed as both the outgoing and ingoing external edge. A connected tree with a single vertex is called a *star*. An $[n]$-tree consists of a tree together with a bijection between the set $[n]$ and the set of outgoing external edges of the tree. The symmetric group S_n acts naturally on the set of outgoing edges; we consider two labelings of an $[n]$-tree the same if they are in the same coset under the isotropy group $S_n(T) \subset S_n$ action.

There is a natural ordering among $[n]$-trees, where we say $T \leq T'$ if T' is obtained from T via a sequence of operations consisting of either contraction of an internal edge, or deletion of a star containing an ingoing external edge.

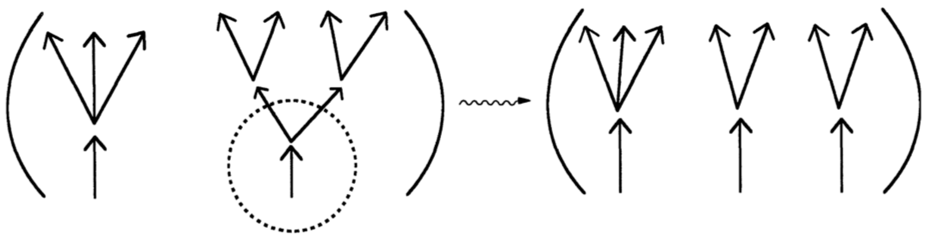
There is a natural stratification of $\hat{X}^n$ – we summarize its main properties in the following proposition.

Proposition 5.1.1. *There is a stratification $\hat{X}^n = \sqcup_T S_T$ by smooth locally-closed algebraic subvarieties S_T such that*

(E) **contraction of an internal edge, e.g.,**



(V) **deleting a star containing an ingoing external edge, e.g.,**



1. *the strata* $\{S_T\}$ *are indexed by all* $[n]$*-trees,*

2. *The codimension of* S_T *equals the number of vertices of* T*,*

3. $S_T \subset \bar{S}_{T'} \Leftrightarrow T \leq T'$

4. *If* T *consists of* n *connected components, then* $S_T = \mathring{X}^n$*, the unique open stratum,*

5. *If* T *consists of* $[n] \setminus I$ *connected components, where one consists of an* I*-star and the rest are just lines, then* $\bar{S}_T = \hat{D}_I$*, an irreducible divisor.*

6. *Given a vertex* $v \in T$*, let* $E_v \subset [n]$ *denote the subset of all the labels attached to all outgoing external edges of* T *that come out of vertex* v*. Then*

$$\bar{S}_T = \bigcap_{\textit{vertices } v \textit{ in } T} \hat{D}_{E_v}.$$

We may interpret Proposition 5.1.1 heuristically as follows. There is a natural bijection between the indexing set of strata of $\hat{X}^n$, namely $[n]$-trees, and the set of ways for n particles in the curve X to collide, while recording the order in which collisions occur. Indeed, at each internal vertex, the set of outgoing edges specifies which particles collide and become a cluster at that step. And the ordering of vertices determines the order in which collisions occur. Thus, the number of internal vertices equals the number of collisions, and the number of connected components equals the number of clusters that survive.

We may apply the construction of $\hat{X}^n$ to $X = \mathbf{A}_k$, the affine line over k. The group of affine transformations Aff of the line induces an action on $\hat{X}^n$, which acts freely on $\hat{X}^n \setminus \hat{D}_{[n]} \simeq X^n \setminus D_{[n]}$. We find $\mathbf{P}_k^{n-2} \simeq (\mathbf{A}_k^n \setminus D_{[n]})/\mathrm{Aff}$ and define

$$\hat{\mathbf{P}}_k^{n-2} := (\hat{\mathbf{A}}_k^n \setminus \hat{D}_{[n]})/\mathrm{Aff}.$$

More generally, given a finite set I and curve X, define X^I to be the set of X-valued functions on I. Then $X^I \simeq X^{\#I}$ and $\hat{X}^I = \hat{X}^{\#I}$. Define $\mathbf{P}_k^I := (\mathbf{A}_k^I \setminus D_I)/\mathrm{Aff}$ and $\hat{\mathbf{P}}_k^I := (\hat{\mathbf{A}}_k^I \setminus \hat{D}_I)/\mathrm{Aff}$. We caution that $\mathbf{P}_k^I \simeq \mathbf{P}_k^{\#I-2}$. In what follows, we will omit the index k unless we wish to specify a phenomenon specific to characteristic p.

The variety $\hat{\mathbf{P}}^I$ also has a stratification $\hat{\mathbf{P}}^I = \sqcup_T \mathbf{P}_T^\circ$, parameterized by connected I-trees with at least one vertex, where each $\mathbf{P}_T^\circ$ is locally closed with smooth closure $\mathbf{P}_T$.

Proposition 5.1.2. *For each connected tree T, there are canonical isomorphisms*

$$\mathring{\mathbf{P}}_T = \prod_{\text{vertices } v \in T} \mathring{\mathbf{P}}^{I(v)}, \quad \mathbf{P}_T = \prod_{\text{vertices } v \in T} \hat{\mathbf{P}}^{I(v)}$$

where $I(v)$ denotes the set of all outgoing edges at the vertex v.

If T consists of just a line, let $\mathbf{P}_T = \mathring{\mathbf{P}}_T = \{pt\}$. For a tree T with connected components $T_1, \ldots, T_r$, let $\mathring{\mathbf{P}}_T = \mathring{\mathbf{P}}_{T_1} \times \cdots \times \mathring{\mathbf{P}}_{T_r}$. For any stratum $S_T \subset \hat{X}^n$, there are canonical isomorphisms

$$\bar{S}_T \simeq \mathbf{P}_T \times \hat{X}^J, \quad S_T \simeq \mathring{\mathbf{P}}_T \times \mathring{X}^J \tag{5.1}$$

where J denotes the set of connected components of T. As a corollary, we find

$$\hat{D}_I \simeq \hat{\mathbf{P}}^I \times \hat{X}^{[n]/I}.$$

5.2 Construction of the [BG] Sheaf

The main goal in this section is to define a space of log differential forms on the compactification of the configuration space whose global sections produce the infinitesimal jet spaces of $\mathcal{M} \subset \mathrm{Bun}_G$. This is accomplished in Definition 5.2.2.

Let $I \subset [n]$ be any subset of size at least 2. Let $[n]/I := [n] \backslash I \cup \{I\}$ be a set of cardinality $n - |I| + 1$. Let $\mathrm{Lie}(I)$ denote the Lie operad on I. Let $\Omega_{\hat{X}^n, \mathring{X}^n}$, respectively $\Omega(\hat{X}^n, \mathring{X}^n)$, denote sheaf, resp. global sections of the sheaf, of top degree logarithmic differential forms on $\hat{X}^n$ with log poles on $\hat{D}^n$.[1]. The following proposition is well-known – see e.g. (Loo99, Prop 3.5) for a proof. We sketch the proof for the reader's convenience.

Proposition 5.2.1. *(BG, Prop 4.3) Let $k = \mathbf{Z}$. There exists a perfect pairing*

$$Res : Lie_k(I) \times \Omega(\hat{\mathbf{P}}^I_k, \mathring{\mathbf{P}}^I_k) \to k.$$

Furthermore, both above k-modules are free.

Proof. Let $I = [d]$. Let us first define the pairing. It is well known $\mathrm{Lie}(I)$ has dimension $(d-1)!$ and a basis is given by $\mathbf{x}_\sigma := [x_{\sigma(1)}, [\ldots, [x_{\sigma(d-1)}, x_d] \ldots]]$ for $\sigma \in S_d$ such that $\sigma(d) = d$. To each $\mathbf{x}_\sigma$, we associate the linear map

$$\mathrm{Res}_\sigma := \mathrm{Res}_{\hat{D}_{\sigma(1)} \cap \cdots \cap \hat{D}_{\sigma(d)}} \circ \cdots \circ \mathrm{Res}_{\hat{D}_{\sigma(r-1)} \cap \hat{D}_{\sigma(r)}} \circ \mathrm{Res}_{\hat{D}_{\sigma(r)}} : \Omega(\hat{\mathbf{P}}^I_k, \mathring{\mathbf{P}}^I_k) \to k$$

1. We slightly change notation here to match notation with (BG)

Then the bilinear pairing is the linear extension of

$$(x_\sigma, \omega) \mapsto \mathrm{Res}_\sigma(\omega).$$

To show that this pairing is well defined, namely that it satisfies the Jacobi relations, it suffices to consider the $I = [3]$ case. Then, we may identify $\hat{\mathbf{P}}^{[3]} = \mathbf{P}^1$, $\mathring{\mathbf{P}}^{[3]} = \mathbf{P}^1 \setminus \{[0:1], [1:0], [1:1]\}$, and

$$\Omega(\hat{\mathbf{P}}^{[3]}, \mathring{\mathbf{P}}^{[3]}) = \{\lambda_{23}\frac{d(z_2 - z_3)}{z_2 - z_3} + \lambda_{13}\frac{d(z_1 - z_3)}{z_1 - z_3} + \lambda_{12}\frac{d(z_1 - z_2)}{z_1 - z_2} : \lambda_{ij} \in k, \lambda_{23} + \lambda_{13} + \lambda_{12} = 0\}.$$

The Jacobi identity on Lie(3) then corresponds to sum of residues/coefficients of $\omega \in \Omega(\hat{\mathbf{P}}^{[3]}, \mathring{\mathbf{P}}^{[3]})$ equalling 0.

Since $\hat{D}$ is normal-crossing, the pullback of the forms $\wedge_{p=1}^{d} d\log(z_{i_p} - z_{j_p})$ generate the space of log differential forms (ESV). This shows surjectivity of the induced map $\Omega(\hat{\mathbf{P}}^I_k, \mathring{\mathbf{P}}^I_k) \to \mathrm{Lie}(I)^*$. Injectivity then follows by a dimension count: it is well known (Br73, Lemma 5),(SV) $\Omega(\hat{\mathbf{P}}^I_k, \mathring{\mathbf{P}}^I_k)$ is isomorphic to the top degree cohomology of the hyperplane complement $H^n(\mathring{\mathbf{P}}^I_k, k)$, and the latter is free of rank $(d-1)!$. [2] □

This pairing of Proposition 5.2.1 provides a canonical distinguished linear map

$$\phi_I : \mathfrak{g}^{\otimes I} \to \mathfrak{g} \otimes \Omega(\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I)$$

characterized by the property that, for all I-binary trees T, $\mathrm{Res}_T(\Phi_I(\xi_1 \otimes \cdots \otimes \xi_I))$ consists of a Lie bracket expression involving insertion $[\xi_i, \xi_j]$ whenever edges i, j share a vertex in T. Here,

$$\mathrm{Res}_T := \mathrm{Res}_{\hat{D}_1 \cap \cdots \cap \hat{D}_r} \circ \cdots \circ \mathrm{Res}_{\hat{D}_{r-1} \cap \hat{D}_r} \circ \mathrm{Res}_{\hat{D}_r} \tag{5.2}$$

where $\hat{D}_1, \ldots, \hat{D}_r$ are the irreducible diagonal divisors corresponding to each of the r vertices

2. In fact, this algebra has a generators and relations presentation due to Orlik and Solomon (OS)

of the binary tree T. The order of the $\hat{D}_i$ taken does not affect Res_T because of normal-crossing. Denote the linear adjoint of ϕ_I by

$$\phi_I^* : \mathfrak{g}^* \to (\mathfrak{g}^*)^{\otimes I} \otimes \Omega(\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I). \tag{5.3}$$

We now explain ϕ_I^* is Ad_g-invariant for any $g \in G$. Using Proposition 5.2.1, we may replace $\Omega(\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I)$ with $\mathrm{Lie}(I)^*$ since the Residue is Ad_g-equivariant. For simplicity, write $I = [d]$. Recall the basis $\mathbf{x}_\sigma$ of Lie introduced in the proof of Proposition 5.2.1. Then explicitly,

$$\phi_I^* : A \mapsto (x_1 \otimes \cdots \otimes x_d \mapsto A(\mathbf{x}_\sigma))_{\sigma \in S_{d-1}}.$$

This expression is clearly Ad_g-invariant, as desired.

Thus, we can construct a relative version of ϕ_I^*, denoted by Φ_I^*. By relative, we mean an $\mathcal{O}_{\hat{D}_I}$-module morphism whose fibers are ϕ_I^*. Let P be a principal G-bundle over X. Let $\mathfrak{g}_P := \mathfrak{g} \otimes \mathcal{O}_P/G$ be the associated adjoint bundle on X. Let $p : \hat{X}^n \to X^n$ be the Fulton-Macpherson compactification, and denote $\hat{\mathfrak{g}}_P^{\boxtimes n} := p^*(\mathfrak{g}_P^{\boxtimes n})$ the associated bundle on $\hat{X}^n$. Similarly, denote $(\hat{\mathfrak{g}}_P)^{\boxtimes [n]/I} := p^*(\mathfrak{g}_P^{\boxtimes [n]/I})$ for $p : \hat{X}^{[n]/I} \to X^{[n]/I}$.

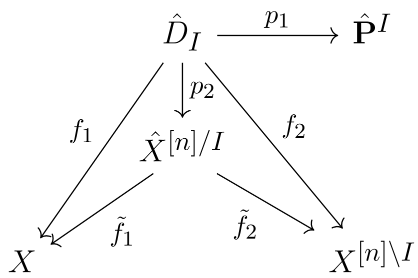
First, we tensor ϕ_I^* on the left by $\mathcal{O}_P$ and (external) on the right by $\mathcal{O}_{\hat{\mathbf{P}}^I}$ to obtain the following morphism of $\mathcal{O}_{X \times \hat{\mathbf{P}}^I}$-modules

$$\Phi_I^* : \mathcal{O}_P \otimes \mathfrak{g}^* \boxtimes \mathcal{O}_{\hat{\mathbf{P}}^I} \to \mathcal{O}_P \otimes (\mathfrak{g}^*)^{\otimes I} \boxtimes \Omega_{\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I}.$$

The Ad_G equivariance allows us to descend to a map of $\mathcal{O}_{X \times \hat{\mathbf{P}}^I}$-modules

$$\Phi_I^* : \mathfrak{g}_P^* \boxtimes \mathcal{O}_{\hat{\mathbf{P}}^I} \to (\mathfrak{g}_P^*)^{\otimes I} \boxtimes \Omega_{\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I}. \tag{5.4}$$

Now, we have $\hat{D}_I \simeq \hat{X}^{[n]/I} \times \hat{\mathbf{P}}^I$ and $D_I \simeq X^{[n]\backslash I} \times X$. Consider the following commutative diagram



(5.5)

Then we have

$$\Omega_{\hat{D}_I,\mathring{D}_I} \simeq p_1^*(\Omega_{\hat{\mathbf{P}}^I,\mathring{\mathbf{P}}^I}) \otimes p_2^*(\Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}}) \tag{5.6}$$

$$(\hat{\mathfrak{g}}_P^*)^{\boxtimes n}|_{\hat{D}_I} \simeq f_1^*(\mathfrak{g}_P^*)^{\otimes I} \otimes f_2^*((\mathfrak{g}_P^*)^{\boxtimes [n]\setminus I}) \tag{5.7}$$

$$p_2^*((\hat{\mathfrak{g}}_P^*)^{\boxtimes [n]/I}) \simeq f_1^*(\mathfrak{g}_P^*) \otimes f_2^*((\mathfrak{g}_P^*)^{\boxtimes [n]\setminus I}) \tag{5.8}$$

Next, if we take Equation 5.4 and pullback along the map $(f_1, p_1) : \hat{D}_I \to X \times \hat{\mathbf{P}}^I$, and using pullback commutes with tensor product, and that pullback along a projection corresponds to taking external tensor product with structure sheaf, we obtain the following morphism of $\mathcal{O}_{\hat{D}_I}$-modules:

$$\Phi_I^* : f_1^*(\mathfrak{g}_P^*) \to f_1^*(\mathfrak{g}_P^*)^{\otimes I} \otimes p_1^*(\Omega_{\hat{\mathbf{P}}^I,\mathring{\mathbf{P}}^I}). \tag{5.9}$$

Next, we tensor the above map by $- \otimes f_2^*((\mathfrak{g}_P^*)^{\boxtimes [n]\setminus I}) \otimes p_2^*(\Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}})$ and use the three identifications in 5.6 to obtain the map of $\mathcal{O}_{\hat{D}_I}$-modules [3]

$$\Psi_I^* : p_2^*\big((\hat{\mathfrak{g}}_P^*)^{\boxtimes [n]/I} \otimes \Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}}\big) \to (\hat{\mathfrak{g}}_P^*)^{\boxtimes n}|_{\hat{D}_I} \otimes \Omega_{\hat{D}_I,\mathring{D}_I} \tag{5.10}$$

Next, observe that since $\mathfrak{g}$ is semi-simple, the Lie bracket map is surjective. From this it directly follows ϕ_I^* is injective. Then at each step of the construction leading to Ψ_I^*, we preserved left-exactness (since we first pulled-back, then took tensor product with a locally-

3. This corrects a few typos present in Equation 7.3 [BG]

free module). Thus Ψ_I^* is an embedding and its image $\mathrm{Im}(\Psi_I^*)$ is a locally-free sheaf on $\hat{D}_I$ with fiber $(\mathfrak{g}^*)^{[n]/I}$.

We are ready to state the main definition:

Definition 5.2.2. *Define the "BG sheaf" on $\hat{X}^n$ of logarithmic differential forms with prescribed residue along the diagonal*

$$\hat{\mathcal{G}}_n := \{\omega \in (\hat{\mathfrak{g}}_P^*)^{\boxtimes n} \otimes \Omega_{\hat{X}^n, \mathring{X}^n} : \ Res_{\hat{D}_I}(\omega) \in Im(\Psi_I^*)\}$$

Note, S_n acts naturally on X^n, and this induces, by functoriality, an action on $\hat{X}^n$ making $(\hat{\mathfrak{g}}_P^*)^{\boxtimes n}, \Omega_{\hat{X}^n, \mathring{X}^n}, \hat{\mathcal{G}}_n$ all into S_n-equivariant sheaves. Next, S_n acts on all $[n]$-trees by permuting the labelings. Given an $[n]$-tree T and $\sigma \in S_n$, we have $\sigma(S_T) = S_{\sigma(T)}$. Thus, $S_n(T)$, the isotropy group of T, acts on S_T, and we denote:

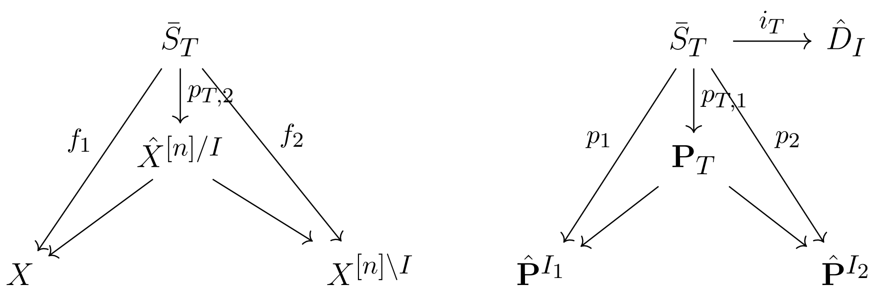
Definition 5.2.3. *Let*

$$H^0(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n} := \left\{ s \in H^0(\hat{X}^n, \hat{\mathcal{G}}_n) : \begin{array}{l} \textit{For each strata } S_T \textit{ and } \sigma \in S_T, \\ \textit{we have } \sigma Res_T(s) = sign(\sigma) \cdot Res_T(s) \end{array} \right\}$$

Now, we explain how to upgrade the construction of Ψ_I^* to Ψ_T^*, where T is a tree. Let $I_1, I_2 \subset [n]$ and suppose $i \in I_1$. Let $I := I_1 \circ_i I_2 := (I_1 \setminus i) \sqcup I_2$. Given a connected I_1-grove T_1 and connected I_2-grove T_2, each with a single vertex (i.e "stars"), we may define $T := T_1 \circ_i T_2$ by inserting at the outgoing edge i of T_1, the unique ingoing external edge of T_2. We caution that $T_2 \circ T_1 \neq T_1 \circ T_2$ in general. Let $\hat{D}_{I_1}, \hat{D}_{I_2}$ be the corresponding irreducible divisors. We know the corresponding stratum $\bar{S}_T = \hat{D}_{I_1} \cap \hat{D}_I$. By Proposition 5.1.2 and Equation 5.1, we know

$$\Omega_{\mathbf{P}_T, \mathring{\mathbf{P}}_T} \simeq \Omega_{\hat{\mathbf{P}}^{I_1}, \mathring{\mathbf{P}}^{I_1}} \boxtimes \Omega_{\hat{\mathbf{P}}^{I_2}, \mathring{\mathbf{P}}^{I_2}}, \quad \text{and } \Omega_{\bar{S}_T, S_T} = \Omega_{\mathbf{P}_T, \mathring{\mathbf{P}}_T} \boxtimes \Omega_{\hat{X}^{[n]/I}, \mathring{X}^{[n]/I}}.$$

Consider the following commutative diagrams involving S_T:



Thus, taking Equation 5.9 for I_1, restricting to $S_T \hookrightarrow \hat{D}_{I_1}$, and then applying $\circ_i \Phi^*_{I_2}$ (i.e apply $\Phi^*_{I_2}$ at the ith copy of $f_1^*(\mathfrak{g}_P^*)$ in $f_1^*(\mathfrak{g}_P^*)^{\otimes I_1}$), we obtain a morphism of sheaves on $\bar{S}_T$

$$\Phi^*_T : f_1(\mathfrak{g}_P^*) \xrightarrow{\Phi^*_{I_1}} f_1^*(\mathfrak{g}_P^*)^{\otimes I_1} \otimes p_1^*(\Omega_{\hat{\mathbf{P}}^{I_1}, \mathring{\mathbf{P}}^{I_1}}) \xrightarrow{i \circ \Phi^*_{I_2}} f_1^*(\mathfrak{g}_P^*)^{\otimes I} \otimes p_{T,1}^*(\Omega_{\mathbf{P}_T, \mathring{\mathbf{P}}_T}) \qquad (5.11)$$

(Recall $I = I_1 \circ_i I_2$) Finally, tensor both sides by $f_2^*((\mathfrak{g}_P^*)^{\boxtimes [n]\setminus I}) \otimes p_{T,2}^*(\Omega_{\hat{X}^{[n]/I}, \mathring{X}^{[n]/I}})$ to obtain the desired morphism of $\bar{S}_T$-modules

$$\Psi^*_T : p_{T,2}^*\big((\hat{\mathfrak{g}}_P^*)^{\boxtimes [n]/I} \otimes \Omega_{\hat{X}^{[n]/I}, \mathring{X}^{[n]/I}}\big) \to (\hat{\mathfrak{g}}_P^*)^{\boxtimes n}|_{\bar{S}_T} \otimes \Omega_{\bar{S}_T, S_T} \qquad (5.12)$$

Again, since $\mathfrak{g}$ is semisimple, this morphism is a locally-split embedding, and its image is a locally free sheaf on $\bar{S}_T$ with fibers $(\mathfrak{g}^*)^{\otimes I}$. We make the remark that Ψ^*_T may only be defined on $\bar{S}_T$, for the composition of $\Phi^*_{I_1}$ and $\Phi^*_{I_2}$ is only well-defined on the intersection of their supports, i.e $\bar{S}_T$.

Next, observe the key defining property of Ψ^*_I is the commutative diagram of sheaves on $\bar{S}_T$ for all I trees T (still keeping the $I = I_1 \circ_i I_2 \circ \cdots, T = T_1 \circ_i T_2 \circ \cdots$ notation, and where Res_T is as defined in Equation 5.2):

$$\begin{array}{ccc} p_{T,2}^*((\hat{\mathfrak{g}}_P^*)^{\boxtimes[n]/I} \otimes \Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}}) & \xrightarrow{\Psi_I^*} & (\hat{\mathfrak{g}}_P^*)^{\boxtimes n}|_{\bar{S}_T} \otimes \Omega_{\hat{D}_I,\mathring{D}_I}|_{\bar{S}_T} \\ & \searrow^{\Psi_T^*} & \downarrow \mathrm{Res}_T \\ & & (\hat{\mathfrak{g}}_P^*)^{\boxtimes n}|_{\bar{S}_T} \otimes \Omega_{\bar{S}_T,S_T} \end{array}$$

In summary, we have

$$\mathrm{Res}_T \circ (\Psi_I^*|_{S_T}) = \Psi_T^* \quad \text{for all I-trees T.} \tag{5.13}$$

Next, let I be a d element subset of $[n]$, which for simplicity of notation we suppose $I = [d]$. Let T_0 denote that binary I-tree $T_0 = [1,[2,\ldots,[d-1,d]\ldots]]$, and let $\bar{S}_{T_0} := \hat{D}_{[d]} \cap \hat{D}_{\{2,3,\ldots,d\}} \cdots \cap \hat{D}_{\{d-1,d\}}$ denote the corresponding strata in $\hat{D}_{[d]}$, and denote by $p_0 = p_{T_0,1}(S_{T_0})$ be the corresponding point in $\hat{\mathbf{P}}^I$ so that $\bar{S}_{T_0} \simeq \hat{X}^{[n]/I} \times \{p_0\}$. Next, for $\sigma \in S_{d-1} = \mathrm{Stab}_{S_d}(d)$, denote the binary I-tree $T_\sigma := \sigma.T_0 = [\sigma(1),[\sigma(2),\ldots,[\sigma(d-1),d]\ldots]]$. Similarly, let $p_\sigma := \sigma(p_0) \in \hat{\mathbf{P}}^I$ be the corresponding point. Let $i_\sigma : \{p_\sigma\} \hookrightarrow \hat{\mathbf{P}}^I$ denote the inclusion. Finally, let $\mathrm{adj}_T : \mathcal{F} \to (i_T)_* i_T^*(\mathcal{F})$ be the unit map for $\mathcal{F}$ a sheaf on $\hat{\mathbf{P}}^I$, and denote the locally free $\hat{X}^{[n]/I}$-modules by

$$\mathcal{F}_{[n]/I} := (\hat{\mathfrak{g}}_P^*)^{\boxtimes[n]/I} \otimes \Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}}.$$

$$\mathcal{E}^{[n]/I} := \tilde{f}_1(\mathfrak{g}_P^*)^{\otimes I} \otimes \tilde{f}_2((\mathfrak{g}_P^*)^{\boxtimes[n]\setminus I}) \otimes \Omega_{\hat{X}^{[n]/I},\mathring{X}^{[n]/I}}$$

Thus, we may rewrite the compatibility equation 5.13 as a commutative diagram of $\hat{D}_I = \hat{X}^{[n]/I} \times \hat{\mathbf{P}}^I$-modules:

$$
\begin{array}{ccc}
\mathcal{F}_{[n]/I} \boxtimes \mathcal{O}_{\hat{\mathbf{P}}^I} & \xrightarrow{\Psi_I^*} & \mathcal{E}_{[n]/I} \boxtimes \Omega_{\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I} \\
\downarrow (\mathrm{adj}_{T_\sigma}) & & \downarrow (\mathrm{Res}_{T_\sigma}) \\
\bigoplus_{\sigma \in S_{d-1}} \mathcal{F}_{[n]/I} \boxtimes (i_\sigma)_*(\{p_\sigma\}) & \xrightarrow[(\Psi_{T_\sigma}^*)]{} & \bigoplus_{\sigma \in S_{d-1}} \mathcal{E}_{[n]/I} \boxtimes (i_\sigma)_*(\{p_\sigma\})
\end{array} \tag{5.14}
$$

On global sections, the left vertical map is the diagonal embedding, the right vertical is an isomorphism, by Proposition 5.2.1, and the horizontal maps are embeddings.

5.3 The pullback

Definition 5.3.1. *Define the "BD sheaf" $\mathcal{G}_n$ over X^n whose local sections are those forms ω such that*

$$\mathcal{G}_n := \{\omega \in (\mathfrak{g}_P^*)^{\boxtimes n} \otimes \tilde{\Omega}_{X^n}(\log(D)) : \textit{For all } d \leq n \textit{ and } \sigma \in S_{d-1},\ Res_{T_\sigma}(\omega) \in Im(\Psi_{T_\sigma}^*)\}$$

where for a binary d-tree $T_\sigma = [\sigma(1), [\dots, [\sigma(d-1), \sigma(d)] \dots]]$, we denote

$$Res_{T_\sigma} := \sigma(Res_{D_{[d]}} \circ Res_{D_{\{2,\dots,d\}}} \circ \dots \circ Res_{D_{\{d-1,d\}}}),$$

and $\Psi_{T_\sigma}^*$ is the corresponding Lie cobracket expression as defined in Equation 5.12.

Note, we denote Ψ_T^* for both the map of sheaves on X^n and on $\hat{X}^n$ because the latter is induced by the former under pullback, and they are both induced by the same corresponding cobracket expression on fibers. Also, analogous to Definition 5.2.3, we may define $H^0(X^n, \mathcal{G}_n)^{-S_n}$. Since $p : \hat{X}^n \to X^n$ is a proper map, the global sections of $\mathcal{G}_n$ coincide with those of $p^*\mathcal{G}_n$. The sheaf $\mathcal{G}_n$ over X^n is related to the universal sheaf Ω_n over X (recall Definition 4.1.1) in the following way:

$$\Gamma(X, \Omega_n) \xrightarrow{\sim} \Gamma(X^n, \mathcal{G}_n)^{-S_n} \xleftarrow{\sim} \Gamma(\hat{X}^n, p^*\mathcal{G}_n)^{-S_n}$$

where first map is $\omega = (\omega_i)_{0 \leq i \leq n} \mapsto \omega_n$.

Next, we wish to relate $p^* \mathcal{G}_n$ with $\hat{\mathcal{G}}_n$, where $p : \hat{X}^n \to X^n$ is the Fulton-Macpherson compactification. A simple yet important observation is that on the smooth locus of D,

$$\mathrm{Res}_{\hat{D}}(p^* \omega)|_{\hat{D}_{ij}} = p^*(\mathrm{Res}_D(\omega)|_{D_{ij}})|_{\hat{D}_{ij}} \tag{5.15}$$

This follows because the singular locus of D is $D^{\mathrm{sing}} = \bigcup_{ijk} D_{ijk}$ and p is an isomorphism on $p^{-1}(X^n \setminus D^{\mathrm{sing}}) = \hat{X}^n \setminus \bigcup_{I \subset [n] : |I| \geq 3} \hat{D}_I$.

Following the same reasoning,[4] we find in general that for any tree of the form T_σ,

$$\mathrm{Res}_{T_\sigma}(p^* \omega) = \lambda_\sigma p^*(\mathrm{Res}_{T_\sigma}(\omega))|_{S_{T_\sigma}}. \tag{5.16}$$

where Res_{T_σ} is computed using Definition 5.2 on the left and Definition 5.3.1 on the right. The scalar $\lambda_\sigma \in \mathbf{Z}_{>0}$ appears because $\hat{D}_I$, for $|I| \geq 3$, appears with some multiplicity inside the non-reduced locus $p^{-1}(D)$, and there is the remarkable property that $\frac{dh^\alpha}{h^\alpha} = \alpha \frac{dh}{h}$. Note that in our case, local sections $\omega \in \mathcal{G}_n$ may be written as $\omega = \sum_{\sigma \in S_n} a_\sigma \omega_\sigma$ for $\omega_\sigma \in (\mathfrak{g}_P^*)^{\boxtimes n} \otimes \Omega_{X^n}(D_\sigma)$. And, $\mathrm{Res}_{T_\sigma}(\omega) = a_\sigma \mathrm{Res}_{T_\sigma}(\omega_\sigma)$. So, Equation 5.16 is really a statement about forms with poles along a normal-crossing divisor, and consequently there is no ambiguity in the order in which we take $\mathrm{Res}_{T_\sigma}(\omega)$ downstairs. We suppress this integer λ_σ from notation as it does not affect the results.

Lemma 5.3.2. *Suppose $\omega \in \Gamma(X^n, \mathcal{G}_n)^{-S_n}$. Then $p^* \omega \in \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n}$.*

Proof. First, Lemma 2.2.1 (and see Remark 1) shows

$$p^* \omega \in \Gamma(\hat{X}^n, (\hat{\mathfrak{g}}_P^*)^{\boxtimes n} \otimes \Omega_{\hat{X}^n}(\log(\hat{D})))^{-S_n}.$$

4. For instance, the regular locus D_{123}^{reg} inside D_{12} is $D_{123} \setminus \bigcup_{i \geq 4} D_{1234}$, and this equals the regular locus $(\hat{D}_{12} \cap \hat{D}_{123})^{\mathrm{reg}} = \hat{D}_{123} \setminus \bigcup_{[3] \subset I, |I| \geq 4} \hat{D}_I$ inside $\hat{D}_{12}$.

Thus, it remains to check $\mathrm{Res}_{\hat{D}_I}(p^*\omega) \in \mathrm{Im}(\Psi_I^*)$ for all $I \subset [n]$. By symmetry, it suffices to check for $I = [d]$ for each $2 \leq d \leq n$. A summary of the following proof is that there are no global sections of $\Omega_{\hat{\mathbf{P}}^I}$, so logarithmic differential forms $\Omega(\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I)$ are completely determined by all possible residues $\mathrm{Res}_T(\omega)$, for T an I-binary tree, in the sense that there exists a unique log form with those prescribed residues.

We know by 5.16 that for any binary d-tree of the form T_σ,

$$\mathrm{Res}_{T_\sigma}(p^*\omega) = p^*\mathrm{Res}_{T_\sigma}(\omega).$$

This means for each binary d-tree of the form $T_\sigma, \sigma \in S_{d-1}$, we have

$$\mathrm{Res}_{T_\sigma}(p^*\omega) \in \mathrm{Im}(\Psi_{T_\sigma}^*).$$

Because of transverse intersection, the order in which we compute the iterated residue in $\mathrm{Res}_{T_\sigma}(p^*\omega)$ does not matter, up to sign. So if we start with $\mathrm{Res}_{\hat{D}_I}(p^*\omega) \in \mathcal{E}_{[n]/I} \boxtimes \Omega_{\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I}$, then from diagram 5.14, we can conclude its image under right vertical arrow comes from the image of the left horizontal arrow. Namely, there exist $A_\sigma \in \mathcal{F}_{[n]/I}$ such that

$$\mathrm{Res}_{T_\sigma}(p^*\omega) = \Psi_{T_\sigma}^*(A_\sigma). \tag{5.17}$$

Next, we show S_n-sign-invariance of $p^*\omega$ implies $A_\sigma = \mathrm{sign}(\sigma)A_0$ for all σ. Indeed:

$$\Psi_{T_\sigma}^*(A_\sigma) = \mathrm{Res}_{T_\sigma}(p^*\omega) = \mathrm{sign}(\sigma).\sigma\mathrm{Res}_{T_0}(p^*\omega) = \mathrm{sign}(\sigma).\sigma\Psi_{T_0}^*(A_0). \tag{5.18}$$

Next, recall $\Psi_{T_\sigma}^*$ was defined so that its fibers are the maps

$$\psi_{T_\sigma}^* : \mathfrak{g}^* \to (\mathfrak{g}^*)^{\otimes I}, \;\; A_\sigma \mapsto (x_1 \cdots x_d \mapsto A_\sigma([x_{\sigma(1)}, \ldots, [x_{\sigma(d-1)}, x_d] \ldots]))$$

So, $\Psi^*_{T_\sigma}(A_\sigma) = \sigma\Psi^*_{T_0}(A_\sigma)$ and consequently equation 5.18 implies

$$\Psi^*_{T_0}(A_\sigma) = \Psi^*_{T_0}(\text{sign}(\sigma)A_0).$$

Since $\mathfrak{g}$ is semisimple, $\psi^*_{T_0}$, and consequently $\Psi^*_{T_0}$, are injective. Thus $A_\sigma = \text{sign}(\sigma)A_0$ for all σ. The left vertical arrow in 5.14 is just the diagonal inclusion (up to signs), thus we conclude $A_0 \in \mathcal{F}_{[n]/I} \boxtimes \mathcal{O}_{\hat{\mathbf{P}}^I}$ is such that $A_0 \to (A_\sigma)_{\sigma\in S_{d-1}}$. Finally, commutativity of the diagram and injectivity of both horizontal arrows forces $\text{Res}_{\hat{D}_I}(p^*\omega) = \Psi^*_I(A_0)$, as desired! □

Lemma 5.3.3. *We have a morphism of $\mathcal{O}_{\hat{X}^n}$-modules*

$$\hat{R}_n := (\Psi^*_{\{n-1,n\}})^{-1} \circ \mathit{Res}_{\hat{D}_{\{n-1,n\}}} : \hat{\mathcal{G}}_n \to (\Delta_{n-1,n})_*(\hat{\mathcal{G}}_{n-1})$$

where $\Delta_{n-1,n} : \hat{D}_{n-1,n} \hookrightarrow \hat{X}^n$ is the inclusion.

Proof. The inverse of Ψ^*_I is well-defined on $\text{Res}_{\hat{D}_I}\hat{\mathcal{G}}_n$ since it is an isomorphism on it. Next, observe

$$\{I \subset [n] : \hat{D}_I \cap \hat{D}_{n-1,n} \neq , |I| \geq 2\} = \{I \circ_i \{n-1,n\} : i \in I \text{ and } I \subset [n-1], |I| \geq 2\}.$$

Thus, we must check $\text{Res}_{\bar{S}_T}\text{Res}_{\hat{D}_I}(\omega) \in \text{Im}(\Psi^*_T)$ for all sections $\omega \in \hat{\mathcal{G}}_n$. Here, $\tilde{I} = I \circ_i \{n-1,n\}, I \subset [n-1]$, and T is the $\tilde{I}$-tree obtained by inserting $\{n-1,n\}$ at vertex i of the connected I-star.

Since $\omega \in \hat{\mathcal{G}}_n$, there exists $\omega_{\tilde{I}}$ such that $\Psi^*_{\tilde{I}}(\omega_{\tilde{I}}) = \text{Res}_{\hat{D}_{\tilde{I}}}\omega$ and $\omega_{n-1,n}$ such that $\Psi^*_{n-1,n}(\omega_{n-1,n}) = \text{Res}_{\hat{D}_{n-1,n}}(\omega)$. Transverse intersection and Equation 5.13, imply

$$\text{Res}_{\bar{S}_T}(\Psi^*_{n-1,n}(\omega_{n-1,n})) = \text{Res}_{\bar{S}_T}\text{Res}_{\hat{D}_{n-1,n}}(\omega) = \text{Res}_{\bar{S}_T}\text{Res}_{\hat{D}_I}(\omega) = \text{Res}_{\bar{S}_T}\Psi^*_{\tilde{I}}(\omega_{\tilde{I}}) = \Psi^*_T(\omega_{\tilde{I}}) \in \text{Im}(\Psi^*_T),$$

as desired. □

Theorem 5.3.4. *Let $p : \hat{X}^n \to X^n$ be the Fulton-Macpherson compactification over any algebraically closed field. Then* $\Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n} \simeq \Gamma(\hat{X}^n, p^*(\mathcal{G}_n))^{-S_n}$.

Proof. Consider the map $R_n := (\Psi^*_{n-1,n})^{-1} \circ \mathrm{Res}_{D_{n-1,n}} : \mathcal{G}_n \to (\Delta_{n-1,n})_*(\mathcal{G}_{n-1})$ of $\mathcal{O}_{X^n}$-modules. On global sections, it maps sign-invariants to sign-invariants by definition 5.2.3. The map

$$R_n : \Gamma(X^n, \mathcal{G}_n)^{-S_n} \to \Gamma(X^{n-1}, \mathcal{G}_{n-1})^{-S_{n-1}}$$

has kernel

$$\mathrm{Ker}(R_n) = \Gamma(X^n, (\mathfrak{g}^*_P)^{\otimes n} \otimes \Omega_{X^n})^{-S_n} \simeq (\Gamma(X, \mathfrak{g}^*_P \otimes \Omega_X)^{\otimes n})^{S_n} = S^n(J^1_P(\mathcal{M}))$$

where $S^n(J^1_P(\mathcal{M}))$ denotes the nth symmetric power of the 1st infinitesimal jet space of Bun_G. Indeed, suppose $\omega \in \Gamma(X^n, \mathcal{G}_n)^{-S_n}$ is such that $\mathrm{Res}_{D_{n-1,n}}(\omega) = 0$. Then S_n-anti-invariance of ω forces $\mathrm{Res}_{D_{ij}}(\omega) = 0$ for all i, j, and consequently ω is regular on X^n, thus proving the claim.

Next, we consider $\mathcal{O}_{\hat{X}^n}$-modules. Sections of $p^*(S^n(\mathcal{G}_1))$ are regular on $\hat{X}^n$, hence automatically meet the residue constraint. Thus, $p^*(S^n(\mathcal{G}_1)) \hookrightarrow \hat{\mathcal{G}}_n$. We claim $\Gamma(\hat{X}^n, p^*(S^n(\mathcal{G}_1)))$ is the kernel of

$$\hat{R}_n := (\Psi^*_{n-1,n})^{-1} \circ \mathrm{Res}_{\hat{D}_{n-1,n}} : \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n} \to \Gamma(\hat{X}^{n-1}, \hat{\mathcal{G}}_{n-1})^{-S_{n-1}}.$$

Again, S_n-sign-invariance forces a section $\omega \in \mathrm{Ker}(\hat{R}_n)$ to vanish on all $\hat{D}_{ij}$. But then we follow the argument used in Lemma 5.3.2 to deduce $\mathrm{Res}_{\hat{D}_I}(\omega) = 0$ for all $I \subset [n]$. Indeed, $\omega_I := \mathrm{Res}_{\hat{D}_I}(\omega)$ is a form on $\hat{D}_I \simeq \hat{X}^{[n]/I} \times \hat{\mathbf{P}}^I$ which satisfies $\mathrm{Res}_{T_\sigma}(\omega_I) = 0$ for all binary trees $T_\sigma, \sigma \in S_I$, because of the commutativity of residue along normal-crossing divisor property. Then Diagram 5.14, together with injectivity of $\Psi^*_{T_\sigma}, \Psi^*_I$, forces $\omega_I = 0$.

We summarize the preceding discussions with the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\hat{X}^n, p^*(S^n(\mathcal{G}_1))) & \longrightarrow & \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n)^{-S_n} & \xrightarrow{\hat{R}_n} & \Gamma(\hat{X}^{n-1}, \hat{\mathcal{G}}_{n-1})^{-S_{n-1}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Gamma(\hat{X}^n, p^*(S^n(\mathcal{G}_1))) & \longrightarrow & \Gamma(\hat{X}^n, p^*(\mathcal{G}_n))^{-S_n} & \xrightarrow{R_n} & \Gamma(\hat{X}^{n-1}, p^*\mathcal{G}_{n-1})^{-S_n} & \longrightarrow & 0 \end{array}$$

The theorem tautologically holds for $n = 1, 2$. Thus we are finished by induction and the 5 lemma. Note, Theorem 4.1.4 implies both above exact sequences, on global sections, are

$$0 \to S^n(J_P^{1,PD}(\mathcal{M})) \to J_P^{n,PD}(\mathcal{M}) \to J_P^{n-1,PD}(\mathcal{M}) \to 0. \qquad \square$$

5.4 Jet spaces and the Lie cooperad

There is a close relationship between the sheaves $\hat{\mathcal{G}}_n$ and the Lie operad $\mathcal{L}ie$ which we now explain. We will only provide a brief exposition to the theory of operads – for a careful treatment, we refer the reader to e.g (LV).

5.4.1 Operads

Let k be an algebraically closed field, and let $\mathcal{P}$ be a k-linear operad. Let V be a k-vector space. The *free $\mathcal{P}$-algebra generated by V*, resp. *free $\mathcal{P}$-algebra with divided powers generated by V*, defined by (Fre), is:

$$\mathcal{P}\langle V\rangle := \bigoplus_{n\geq 1} (\mathcal{P}(n) \otimes_k V^{\otimes n})_{S_n}, \quad \text{resp.} \tag{5.19}$$

$$\mathcal{P}^{PD}\langle V\rangle := \bigoplus_{n\geq 1} (\mathcal{P}(n) \otimes_k V^{\otimes n})^{S_n} \tag{5.20}$$

The difference between these algebras is the former involves S_n-coinvariants while the latter involves S_n-invariants.

We in fact work with cooperads, so let us recall definitions in this setting: A *comodule over a $\mathcal{P}$-coalgebra A* is a k-vector space M together with a collection of linear maps

$$q_n : M \to M \otimes A^{\otimes(n-1)} \otimes \mathcal{P}(n)$$

satisfying coassociativity, equivariance, and counit axioms. A *comodule $\mathcal{M}$ over an cooperad $\mathcal{P}$* ((KM)), is a collection of vector spaces $\mathcal{M}(n)$ with S_n-actions together with linear maps

$$p_{\alpha_1,\ldots,\alpha_k} : \mathcal{M}(\alpha_1 + \cdots + \alpha_k) \to \mathcal{M}(k) \otimes \mathcal{P}(\alpha_1) \otimes \cdots \otimes \mathcal{P}(\alpha_k)$$

satisfying coassociativity, equivariance, and counit axioms.

5.4.2 Jet spaces and Lie operad

Let us recall the $\mathcal{L}ie$ operad. Let $x_1, \ldots, x_n$ be variables, with the natural S_n-action. Consider the $(n-1)!$-dimensional vector space

$$\mathcal{L}ie(n) := \mathrm{Span}\langle [x_{s(1)}, [\ldots, [x_{s(n-1)}, x_n] \ldots]] : s \in S_{n-1} \rangle.$$

Insertion of the bracket compositions defines the operad structure. Next, define

$$\Omega_{\hat{\mathbf{P}}} := \{\Omega(\hat{\mathbf{P}}^n, \mathring{\mathbf{P}}^n) \otimes \bigwedge^n k^n\}_{n\geq 1}.$$

Under the perfect pairing in Proposition 5.2.1, this forms a cooperad, which we will call *the Lie cooperad*. It is instructive to explain the composition maps explicitly.

Recall the stratification on $\hat{\mathbf{P}}^n$: The codimension 1 strata are normal-crossing, labeled by $\tilde{D}_I$, for $I \subset [n]$ a proper subset of size at least 2. And the closure of the codimension r

strata are

$$\tilde{S}_{I_1,\ldots,I_r} := \tilde{D}_{I_1} \cap \tilde{D}_{I_1 \circ I_2} \cap \cdots \cap \tilde{D}_{I_1 \circ \cdots \circ I_r} \simeq \hat{\mathbf{P}}^{I_1} \times \hat{\mathbf{P}}^{I_2} \times \cdots \times \hat{\mathbf{P}}^{I_r} \times \hat{\mathbf{P}}^{n/(I_1 \circ \cdots \circ I_r)} \tag{5.21}$$

This shows $\hat{\mathbf{P}} := \{\hat{\mathbf{P}}^n\}_{n \geq 1}$ is a topological operad. Thus the composition maps of $\Omega_{\hat{\mathbf{P}}}$ are

$$\mathrm{Res}_{T_{I_1,\ldots,I_r}} : \Omega_{\hat{\mathbf{P}}}(\alpha_1 + \cdots + \alpha_k) \to \Omega_{\hat{\mathbf{P}}}(k) \otimes \bigotimes_{i=1}^{k} \Omega_{\hat{\mathbf{P}}}(\alpha_i).$$

The coassociativity axiom is satisfied because the divisors are normal-crossing, so the order in which the residue is computed is the same (up to a sign, which is accounted for by the determinant).

Next, for X a smooth projective curve over k, the collection of the resolution of the diagonals, $\hat{X} := \{\hat{X}^n\}_{n \geq 1}$, forms a module over the topological operad $\hat{\mathbf{P}}$ because the strata S_T of $\hat{X}^n$ satisfy 5.1. Moreover,

$$\Omega_{\hat{X}} := \{\Omega(\hat{X}^n, \mathring{X}^n) \otimes \bigwedge^{n} k^n\}_{n \geq 1}$$

forms a comodule over the cooperad $\Omega_{\hat{\mathbf{P}}}$ via the composition maps

$$\mathrm{Res}_{T_{I_1,\ldots,I_r}} : \Omega_{\hat{\mathbf{X}}}(\alpha_1 + \cdots + \alpha_k) \to \Omega_{\hat{\mathbf{X}}}(k) \otimes \bigotimes_{i=1}^{k} \Omega_{\hat{\mathbf{P}}}(\alpha_i).$$

Next, we sheafify. Taking global sections of the $\hat{\mathbf{P}}^I$-contribution in Equation 5.9 produces a morphism of $\mathcal{O}_X$-modules:

$$\Phi_I^* : \mathfrak{g}_P^* \to (\mathfrak{g}_P^*)^{\otimes I} \otimes \Omega(\hat{\mathbf{P}}^I, \mathring{\mathbf{P}}^I).$$

This gives $\mathfrak{g}_P^*$ the structure of a sheaf of coalgebras over the Lie-cooperad. Now, consider:

$$\Omega_{\hat{X}}^{PD}\langle \mathfrak{g}_P^* \rangle := \bigoplus_{n\geq 1} \left(\Gamma(\hat{X}^n, (\hat{\mathfrak{g}}_P^*)^{\boxtimes n} \otimes \Omega_{\hat{X}^n, \mathring{\hat{X}}^n} \otimes \omega_{\hat{X}^n}) \right)^{S_n}$$

where $\omega_{\hat{X}^n}$ is the determinant line bundle on $\hat{X}^n$, and $(\hat{\mathfrak{g}}_P^*)^{\boxtimes n}$ is the pullback of $(\mathfrak{g}_P^*)^{\boxtimes n}$ under $p : \hat{X}^n \to X^n$, respectively. The comodule morphisms (Recall diagram 5.5 for the notation):

$$\mathrm{Res}_{\hat{D}_\alpha} : \left((\hat{\mathfrak{g}}_P^*)^{\boxtimes m} \otimes \Omega_{\hat{X}^m, \mathring{\hat{X}}^m} \right)|_{\hat{D}_\alpha} \to \left((\hat{\mathfrak{g}}_P^*)^{\boxtimes [m]/\alpha} \otimes \Omega_{\hat{X}^{[m]/\alpha}, \mathring{\hat{X}}^{[m]/\alpha}} \otimes \tilde{f}_1 (\mathfrak{g}_P^*)^{\otimes(\alpha-1)} \right) \boxtimes \Omega_{\hat{\mathbf{P}}^\alpha, \mathring{\hat{\mathbf{P}}}^\alpha}. \tag{5.22}$$

endow $\Omega_{\hat{X}^n}^{PD}\langle \mathfrak{g}_P^* \rangle$ with the structure of a comodule over $\mathfrak{g}_P^*$ over the Lie cooperad. If we take global sections over the $\hat{\mathbf{P}}^\alpha$ contribution and then take a fiber over $\hat{X}^{[n]/\alpha}$, this morphism just says the free divided power space generated by $\mathfrak{g}^*$,

$$\Omega_{\hat{\mathbf{P}}}^{PD}\langle \mathfrak{g}^* \rangle := \bigoplus_{n\geq 1} \left((\mathfrak{g}^*)^{\otimes n} \otimes \Omega(\hat{\mathbf{P}}^{[n]}, \mathring{\hat{\mathbf{P}}}^{[n]}) \right)^{S_n},$$

is a comodule over $\mathfrak{g}^*$ over the Lie cooperad. These observations are what allow us to make the final claim, and in particular suggest why $\Gamma(\hat{X}^n, \mathcal{G}_n)^{-S_n}$ has divided powers.

Corollary 5.4.1. *The collection of BG-sheaves $\{\hat{\mathcal{G}}_n\}_{n\geq 1}$ form a sheaf over $\hat{X} := \{\hat{X}^n\}_{n\geq 1}$, in the sense of (GK, Section 1.5). Here, $\hat{X}$ is viewed as a topological module over the operad $\hat{\mathbf{P}}$. Moreover,*

$$\bigoplus_{n\geq 1} \Gamma(\hat{X}^n, \hat{\mathcal{G}}_n \otimes \omega_{\hat{X}^n})^{S_n}$$

is a sub $\mathfrak{g}_P^*$-comodule of $\Omega_{\hat{X}}^{PD}\langle \mathfrak{g}_P^* \rangle$. Here, $\mathfrak{g}_P^*$ is viewed as a sheaf of coalgebras over the Lie cooperad.

Proof. Both of the claims follow by observing that restricting the residue map in Equation

5.22 to $\hat{\mathcal{G}}_n$ induces the map of $\mathcal{O}_{\hat{D}_\alpha}$-modules (recall diagram 5.5 for the notation):

$$\mathrm{Res}_{\hat{D}_\alpha} : \hat{\mathcal{G}}_m|_{\hat{D}_\alpha} \to \hat{\mathcal{G}}_{[m]/\alpha} \otimes \tilde{f}_1^*(\mathfrak{g}_P^*)^{\otimes(\alpha-1)} \boxtimes \Omega_{\hat{\mathbf{P}}^\alpha, \mathring{\mathbf{P}}^\alpha}. \tag{5.23}$$

Indeed, checking this boils down to a similar proof as given in Lemma 5.3.3. The main point is there is a compatibility 5.13

$$\mathrm{Res}_T \circ (\Psi_I^*|_{S_T}) = \Psi_T^* \quad \text{for all } I\text{-trees } T. \qquad \square$$

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