THE UNIVERSITY OF CHICAGO

NON-INVERTIBLE SYMMETRIES IN FINITE-GROUP GAUGE THEORY

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Dedicated to Aruê, Lis, Helena, Rogério, and Valéria.

“Eu ouvi o canto da sereia. Ilumina lua cheia, que eu respondo o seu chamado."

Ana Maria Carvalho

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ABSTRACT

Global symmetries are fundamental to our understanding of nature: they organize the spectrum of physical theories, constrain their dynamics, and reveal emergent effective degrees of freedom at low energies. Crucially, global symmetries are non-perturbative tools that shed light on strongly coupled regimes inaccessible to perturbative methods. Recently, it was realized that symmetries have an intrinsic formulation in quantum field theory through topological operators, also known as symmetry defects. This perspective has led to important generalizations, including the discovery of non-invertible symmetries, characterized by categorical fusion rules rather than conventional group multiplication. In this thesis, we investigate these non-invertible symmetries in the context of finite-group gauge theories, which are exactly solvable models and serve as prototypical examples of topological quantum field theories.

In the first part of this work (Chapter 2) we investigate the invertible and non-invertible symmetries of topological finite-group gauge theories in general spacetime dimensions, where the gauge group can be abelian or non-abelian. We focus in particular on the 0-form symmetry. The gapped domain walls that generate these symmetries are specified by boundary conditions for the gauge fields on either side of the wall. We investigate the fusion rules of these symmetries and their action on other topological defects, including the Wilson lines, magnetic fluxes, and gapped boundaries. We illustrate these constructions with various novel examples, including non-invertible electric-magnetic duality symmetry in 3+1d $\mathbb{Z}_2$ gauge theory and non-invertible analogs of electric-magnetic duality symmetry in non-abelian finite-group gauge theories. In particular, we discover topological domain walls that obey Fibonacci fusion rules in 2+1d gauge theory with dihedral gauge group of order 8. We also generalize the Cheshire string defect to analogous defects of general codimensions and gauge groups and show that they form a closed fusion algebra.

In the second part (Chapter 3) we demonstrate how to realize these symmetries as

condensation defects, i.e., as suitable insertions of lower dimensional topological operators. We then compute these symmetries' fusion rules and action using their condensation expression and the algebraic properties of the lower-dimensional objects that make them. We illustrate the discussion in $\mathbb{Z}_N$ gauge theory, where we derive the correspondence between domain walls, labeled by subgroups and topological actions for the doubled gauge group, and higher gauging condensation defects, labeled by subalgebras of the global symmetry. As a primary application, we obtain the condensation expression for the invertible symmetries of abelian gauge theories defined by outer automorphisms of the gauge group. We also show how to use these ideas to derive the action for certain non-abelian groups. For instance, one can obtain the action for the Dihedral group $\mathbb{D}_4$ by gauging a swap symmetry of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory.

CHAPTER 1

INTRODUCTION

1.1 Emergence: from micro to macro

In this section, we present what can be viewed as a high-level motivation behind this thesis: understanding emergence.

1.1.1 Emergence in science

A strategy for chasing reality is *reductionionism*. When confronted with a complex system of some kind, we begin by breaking it into parts and studying each piece separately. From the iteration of this research strategy, we have made remarkable progress, uncovering a hierarchical structure of matter's constituents (see Fig. 1.1). As we probe deeper into the

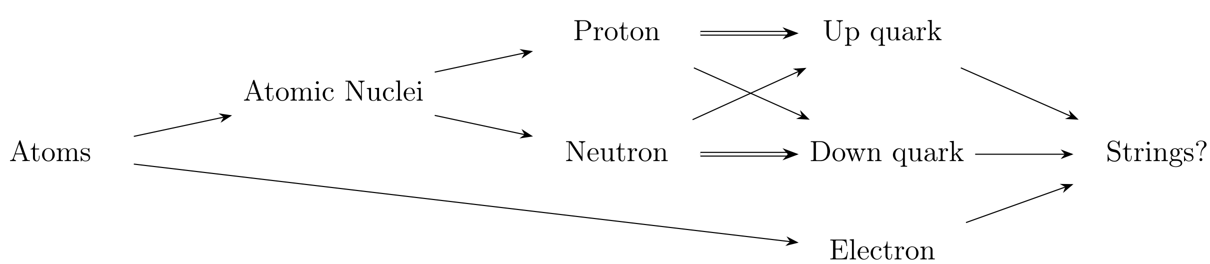


Figure 1.1: Reductionism: from systems to constituents.

fabric of matter, we find that the binding energies holding its components together become increasingly large (see Table 1.1). Consequently, ever-higher energies are needed to break apart ever-smaller building blocks. This is why exploring the smallest structures requires the most massive instruments, such as the Large Hadron Collider (LHC).[1] However, it is striking to observe that this sophisticated scientific endeavor is ultimately driven by the same simple

1. The LHC is a 27-kilometer ring of superconducting magnets equipped with accelerating structures that boost particle energies along the way [CERN(n.d.)]

strategy that perhaps even babies follow when they begin exploring the world: breaking things apart.

System	Constituents	Force	Binding Energy
Hydrogen	Electron, proton	Electromagnetic	1.3×10 eV
Deuterium	Proton, neutron	Strong force	2.2×10^6 eV
Proton	Up, Up, Down	Strong force	9.3×10^8 eV

Table 1.1: The smaller the scales, the higher the energies.

A different strategy to chase reality is *emergentism*, which is an attempt to address the problem of *emergence*. The general problem of emergence lies in deriving the properties of a system from those of its constituents and their interactions, in other words, going from parts to wholes. Developing techniques to attack the problem of emergence is important as we often possess a good description of the parts but lack a good description of the whole. In such cases, a common goal is to use what we know about the parts to derive a better description of the whole (see Fig. 1.2).

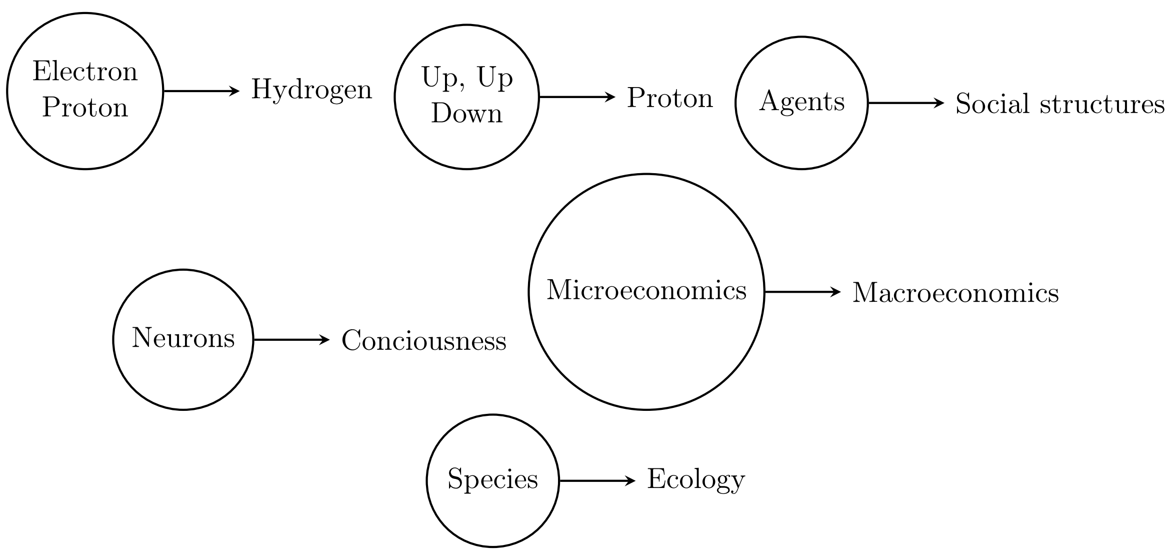


Figure 1.2: Emergentism: from constituents to systems.

The problem of emergence is what most captivates me in science because of its ubiquity across scientific domains. From social structures arising from individual agents, macroeconomics emerging from microeconomic interactions to intelligence and consciousness arising from neuronal activity, emergence appears again and again. Its widespread presence suggests that uncovering general principles of emergence could yield insights across disciplines. Beyond its intrinsic appeal and potential technological relevance, understanding emergence in the theoretically controlled setting of high-energy physics may offer lessons for tackling emergent phenomena more broadly.

The simple yet profoundly rich example of the hydrogen atom (see Fig. 1.2) already offers key lessons for understanding emergence more broadly. In this system, we have two constituents—the proton and the electron—that in the non-relativistic limit interact by the electromagnetic Coulomb potential. When considered separately, each part is described by a free-particle wavefunction, and the non-interacting combined system is given by their tensor product. Once they interact, however, they form a new entity: the hydrogen atom. This bound system exhibits emergent properties that are absent in the non-interacting pair. Specifically, the quantum state of the hydrogen atom, $|n, \ell, m\rangle$, depends on three additional quantum numbers known as the principal n, orbital angular momentum ℓ, and magnetic quantum numbers m. Mathematically, these emergent properties arise as eigenvalue parameters (separation constants) in the differential equation that governs the hydrogen atom. Abstracting away from the particularity of this example reveals a general pattern of emergence: when a differential equation describes interacting components, the properties of the resulting whole can emerge as separation constants.

The second example shown in Fig. 1.2 remains a work in progress, and one of the hopes guiding this thesis is that the tools explored here may help us tackle that more difficult problem. As I have tried to emphasize throughout this introduction, beneath the jargon and specificity of high-energy theoretical physics, this work is ultimately an exploration of a tool for addressing the problem of emergence. That tool is symmetry, a concept that has long played a central role in physics. What is new, is the recent realization that symmetry can be meaningfully generalized. The community is now actively exploring the scope of this generalization and its potential application.

To close this section, let me illustrate at a cartoon level why symmetries might be helpful in addressing the problem of emergence. If the constituents of a system obey a certain symmetry, then any interaction between them must also respect that symmetry. As a consequence, the system they compose will also be invariant under the same symmetry transformations. This inherited invariance often reveals structural or dynamical features of the whole that might otherwise go unnoticed.

1.1.2 Emergence in physics beyond perturbation

The general concept of emergence takes on a precise form in the framework of quantum field theory, where it appears as the renormalization group (RG) flow. This is possible because quantum field theory provides well-defined notions of spacetime and scaling, allowing us to move systematically from short-distance (component-level) descriptions to long-distance (system-level) behavior. Central to this framework are conformal field theories (CFTs), theories that remain invariant under changes of scale.[2] In this context, a quantum field theory can be viewed as a trajectory: an RG flow that connects a short-distance (ultraviolet, or UV) fixed point to a long-distance (infrared, or IR) CFT (see Fig. 1.3).

Quantum field theories display a remarkable feature in this setting: they require both reductionist and emergentist strategies. As one flows along the RG trajectory, the strength of interactions between constituents changes: it can either increase or decrease. In theories where the interaction becomes weaker at large distances, the IR limit resembles a collection of non-interacting particles. A prime example is quantum electrodynamics (QED), where the IR theory is free Maxwell theory, the theory of photons. In such weakly coupled cases, emergence is trivial, as the constituents remain effectively unchanged. However, understanding the short-distance (UV) behavior can be more challenging. For instance, QED is believed to

2. A compelling visual analogy for such scale invariance is provided by fractal geometries, where geometrical patterns repeat across different length scales.

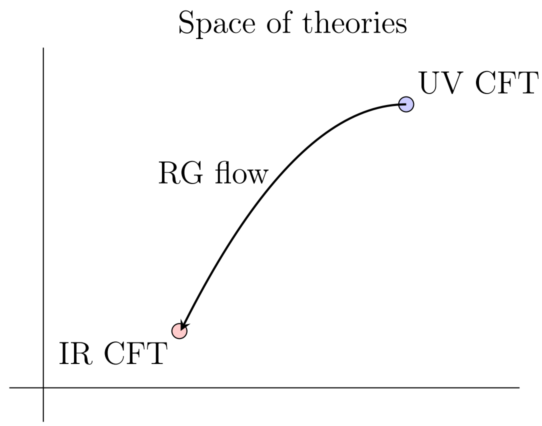


Figure 1.3: A quantum field theory is an RG trajectory between a UV and IR CFT.

suffer from a Landau pole, suggesting the need for a UV completion.

Surprisingly, there are theories with the opposite behavior: instead of weakening, interactions become stronger at long distances. This is the case for quantum chromodynamics (QCD), where the UV behavior is simple—we have weakly coupled quarks and gluons—but the IR behavior is strongly coupled. In these cases, reductionism is trivial, but emergence is not. Strong coupling at large distances renders standard perturbative tools ineffective, and new methods are needed to understand the emergent structure. One powerful non-perturbative tool that has proven useful in this regime is symmetry.

Understanding how symmetries and their generalizations can shed light on strongly coupled systems is a central theme of contemporary theoretical physics, and it provides a background motivation for the investigations pursued in this thesis.

1.2 Global symmetries: a theorist inverted microscope

Microscopes allow us to see the microscopic world from a macroscopic distance, symmetries operate in a kind of inverted way: they allow us to theoretically see the macroscopic world from a microscopic distance. In this section, we elaborate on this idea, concluding that symmetries are powerful tools for understanding emergence. Along these lines, we introduce the main motivation for this work, the recent developments in our understanding of symmetry in quantum field theory.

1.2.1 The power of symmetry

The power of symmetries extends beyond the framework of RG flow and appears more generally in the study of emergence. A simple but important illustration is the hydrogen atom discussed in Section 1.1.1. Even without solving the equations of motion, rotational symmetry alone allows us to predict that the energy spectrum will be organized into representations of the rotation group. Specifically, the hydrogen atom's wavefunction is labeled by the quantum numbers ℓ and m, corresponding to orbital angular momentum and its projection along a fixed axis. These quantum numbers can be seen to emerge purely from symmetry considerations.

Symmetries also yield surprising insights in far more complex settings. One such example arises in the RG flow of QCD-like theories that exhibit chiral symmetry breaking. A highly non-trivial result in this context is the *Nambu–Goldstone theorem*, which states that if a global symmetry is spontaneously broken, then the low-energy, long-distance theory will contain massless excitations, known as Nambu–Goldstone bosons. Pions are a concrete example of such particles.[3] This result is remarkable because we cannot solve QCD analytically. The best we can do is simulate it numerically. At short distances, QCD is weakly coupled and described by quarks and gluons. As we move to larger distances along the RG flow, the theory becomes strongly coupled, exhibiting confinement and hadronization. Despite the complexity of this regime, the spontaneous breaking of chiral symmetry guarantees the presence of low-energy, massless excitations, which again are described by a weakly coupled theory. Symmetry, in this case, reveals the existence of an emergent, low-energy description, even though we cannot directly solve the underlying dynamics. This emergent theory is expressed in terms of new

3. In the real world, chiral symmetry is only approximate, as it is explicitly broken by Yukawa interactions. As a result, pions are not exactly massless. However, their small masses are directly controlled by the strength of the symmetry-breaking terms.

degrees of freedom (the pions), which arise from the interactions of their constituent quarks.

In both examples, we see how robust and powerful symmetries are. They act as non-perturbative tools that reveal features of the emergent system without requiring solving for the microscopic dynamics. If microscopes allow us to see the microscopic world from a macroscopic distance, symmetries operate in a kind of inverted way: they allow us to theoretically see the macroscopic world from a microscopic distance. As such, they are uniquely well-suited for addressing the problem of emergence.

1.2.2 Generalized global symmetries

One of the main drivers behind this work was the realization that the notion of symmetry in quantum field theory admits a powerful generalization. In this section, we review this insight and trace a conceptual path that leads to it.

In formal approaches, a quantum field theory is intrinsically defined by its collection of operators and their correlation functions. Along these lines, a natural question arises: what does it mean for a theory to have a symmetry? The key insight of [Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett] is that a symmetry can also be defined intrinsically as a collection of topological operators. The group structure associated with the symmetry arises from the fusion rules obeyed by these topological operators. Moreover, the action of the symmetry on other operators is realized through linking, that is, by surrounding operator insertions with topological defects. Once symmetries are phrased in this intrinsic and geometric language, a path to generalization naturally opens up.

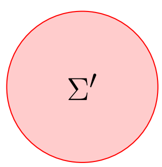
Before turning to the generalization, let us trace a conceptual path that leads to this key insight. Consider a continuous global symmetry group G. By *Noether's theorem*, such a symmetry implies the existence of a conserved current J^μ_a for each generator of the Lie algebra of G. Integrating this current over a codimension-one surface yields a conserved charge

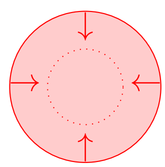
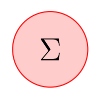
$$Q_a(\Sigma) = \int_\Sigma J^\mu_a dS_\mu, \tag{1.2.1}$$

where Σ is a spatial slice in Lorentzian spacetime, or more generally a codimension-one surface in Euclidean signature. Crucially, current conservation implies that the charge is topological. Specifically, if Σ and Σ', are cobordant (i.e., they bound a region V, so $\partial V = \Sigma' - \Sigma$), then

$$Q_a(\Sigma') - Q_a(\Sigma) = \int_{\Sigma'} J^\mu_a S'_\mu - \int_\Sigma J^\mu_a S_\mu = \int_V \partial_\mu J^\mu_a dv = 0. \tag{1.2.2}$$

This shows that $Q(\Sigma)$ can be smoothly deformed in correlation functions, so long as it does not pass through any operator insertions:

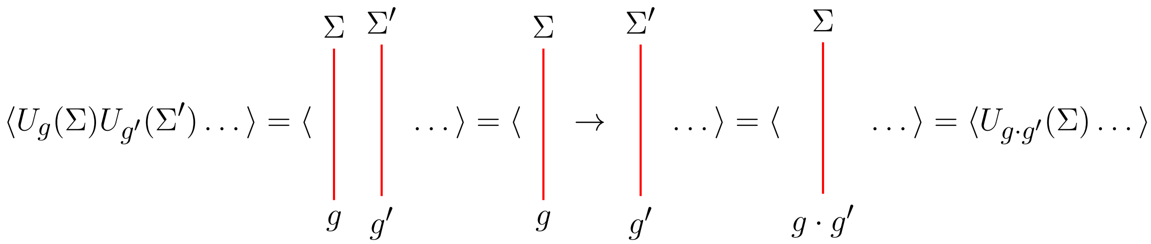
$\langle Q_a(\Sigma') \dots \rangle = \langle$ 

$\dots \rangle = \langle$  $\dots \rangle = \langle$ 

$\dots \rangle = \langle Q_a(\Sigma) \dots \rangle.$ (1.2.3)

The exponential of the charge $U_g(\Sigma) = e^{ig_a Q_a(\Sigma)}$ gives the symmetry defect associated with the group element $g = e^{ig_a T_a} \in G$.

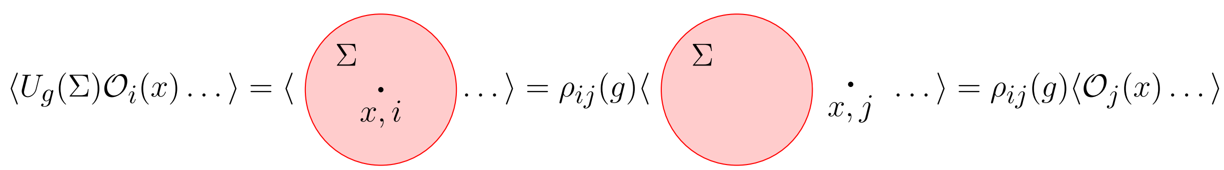
We can now examine the properties of these topological symmetry operators in more detail. First, the group structure associated with the symmetry group G is reflected in the fusion rules of its corresponding topological operators. Specifically, bringing together two operators labeled by elements $g, g' \in G$ yields a third operator labeled by their product $g \cdot g' \in G$, provided the surfaces they are supported on do not intersect other operators. This is illustrated in correlation functions as:



(1.2.4)

where we assumed that Σ' can be smoothly deformed to Σ.

Second, these topological operators act on local operators via linking. Consider a multiplet of local operators $\mathcal{O}_i$ that transform in a representation $\rho : G \to \mathrm{Aut}(V)$ of G. From the *Ward-Takahashi identities*, $\partial_\mu J_a^\mu(x)\mathcal{O}_i(y) = \delta^{(d)}(x-y)\rho_{ij}(T_a)\mathcal{O}_i(y)$, one finds



(1.2.5)

Above, the redish circle represents the codimenion-one topological operator U_g inserted on Σ, and the dot represents the local operator $\mathcal{O}_i$ insertion at x.

The takeaway from this construction is that continuous global symmetries can be intrinsically characterized by operators that:

- are **topological**, meaning they can be freely deformed away from operator insertions;,
- have **codimension-one**, allowing them to link with local operators;
- obey a **group algebra** via fusion.

This perspective can be reversed to define what it means to have a symmetry in a quantum field theory. Crucially, this definition also encompasses discrete symmetries, not just continuous ones. Moreover, by relaxing any of the properties above, we are naturally led to generalized notions of symmetry:

- **Subsystem symmetries** (as in fracton models) relax topological invariance and become partially topological.
- **Higher-form symmetries** involve operators of higher codimension. For instance, 1-form symmetries act on line operators and are supported on codimension-two surfaces (e.g., in Maxwell theory).
- **Non-invertible symmetries** relax the group structure and are described by more general categorical fusion rules, as the Verlinde lines in 2d CFTs.

These directions of generalization are independent, and throughout this work, we will encounter multiple examples that explore these different extensions of symmetry.

1.3 Finite-group gauge theory: a topological quantum field theory

In this final section, we present the setting in which we will explore generalized notions of symmetry: a particular class of topological quantum field theories known as finite-group gauge theories.

1.3.1 Motivations

A quantum field theory is said to be *gapless* if there is no energy gap between the ground state and the first excited state. In this case, the theory contains massless excitations. In contrast, a theory with a nonzero energy gap is called *gapped.* One might expect that in the infrared (IR), a gapped theory flows to a trivial theory with a unique ground state. Such theories are said to be *trivially gapped.* However, this expectation does not always hold. There exist gapped theories that flow in the IR to nontrivial topological quantum field theories (TQFT). These are referred to as *non-trivially gapped.*

A TQFT is a special type of quantum field theory in which observables and correlators do not depend on the metric of spacetime, only on its topology. The class of TQFTs we will focus on in this work is the family of finite-group gauge theories, also known as Dijkgraaf–Witten theories [Dijkgraaf and Witten(1990a)]. As the name suggests, these are gauge theories built from finite groups rather than Lie groups.

There are several motivations for studying these theories. First, it has been shown that they describe the IR behavior of gapped QFTs in $3+1$ dimensions (see Section 3.1 for references). In this context, it is worth noting that while the Standard Model flows to free Maxwell theory in the IR, it is possible that some UV completion, such as a grand unified theory, flows to a combination of Maxwell theory and a finite-group gauge theory. Second, finite-group gauge theories describe certain topological phases of matter, such as the toric code, which plays a key role in proposals for fault-tolerant quantum computation [Kitaev(2003)]. Finally, these theories serve as tractable but general examples of TQFTs. They are defined in all spacetime dimensions and for arbitrary finite groups, making them an ideal playground for developing and testing quantum field theoretic ideas.

Moreover, finite-group gauge theories naturally realize various generalized symmetries, including higher-form and non-invertible symmetries, within a tractable and often exactly solvable framework. Despite their mathematical simplicity, they capture a rich landscape of physical phenomena, ranging from anyonic excitations in 2+1 dimensions to non-trivial topological responses in higher dimensions. These features make them an ideal laboratory for investigating the structure and implications of generalized symmetries. In the chapters that follow, we will investigate these symmetries.

1.3.2 Review

In this section, we will review the basic properties of finite-group gauge theories on the lattice. These theories can be defined in general spacetime dimension D and are classified by tuples $(G, [\alpha_D])$ with G a finite-group and $[\alpha_D] \in H^D(G, U(1))$ a D-cohomology class. Such theories can describe liquid gapped phases, i.e., gapped phases with fully mobile excitations. While gapped phases in $D = 3$ can be described by modular tensor category, gapped phases in $D = 4$ and higher spacetime dimension are more constrained, and topological finite-group gauge theories provide an important class of representative examples [Lan et al.(2018b)Lan, Kong, and Wen, Lan and Wen(2019b), Johnson-Freyd(2022)]. Furthermore, these theories are examples of topological gauge theories and provide an elementary illustration of the categorical approach to quantum field theory [Atiyah(1988), Wakui(1992)]. Now we summarize a few of its properties.

- **Gauge field configurations and gauge transformations.** Let us denote the gauge group by G, and the spacetime manifold by $\mathcal{M}$ of general spacetime dimension $D \geq 2$. We assume $\mathcal{M}$ is orientable and connected. We triangulate the spacetime manifold, enumerate its vertices $\{v_i : 0 \leq i \leq n\}$, and define a *gauge field configuration* as a map $\vec{g}$ that assigns to each edge $[v_i, v_j]$ such that $i < j$ a group element $g_{ij} \equiv \vec{g}([v_i, v_j]) \in G$. A *path* in the triangulation of $\mathcal{M}$ is a sequence of vertices connected by edges $\gamma = (v_{i_1}, \dots, v_{i_n})$, and the *holonomy* along a closed path (with $i_1 = i_n$) is defined by

$$g_\gamma = g_{i_1 i_2} \dots g_{i_{n-1} i_n}, \tag{1.3.1}$$

where $g_{ij} \equiv g_{ji}^{-1}$ whenever $i > j$. A gauge field configuration is said to be *flat* if the holonomy (flux) along the boundary of every 2-simplex $[v_i, v_j, v_k]$ of the triangulation of $\mathcal{M}$ is trivial:

$$g_{(v_i, v_j, v_k)} = g_{ij} \cdot g_{jk} \cdot g_{ki} = 1. \tag{1.3.2}$$

This local flatness condition implies that the holonomy along a closed loop depends only on the homotopy class of the path $\gamma \in \pi_1(\mathcal{M})$. Therefore, a *flat gauge field configuration* can be described globally by a *flat connection* $a \in \mathrm{Hom}(\pi_1(\mathcal{M}), G)$ where $g_\gamma = a(\gamma)$, and Hom indicates that a defines a group homomorphism under concatenation of loops in $\mathcal{M}$.

The gauge field configurations $\vec{g}$ and $\vec{g}'$ are *gauge equivalent* if

$$g'_{ij} = h_i \cdot g_{ij} \cdot h_j^{-1} \tag{1.3.3}$$

for some map $\vec{h}$ that assigns to each vertex v_i a group element $h_i \equiv \vec{h}(v_i) \in G$. We call the map $\vec{h}$ a *gauge transformation* and we say that it changes the gauge field configuration from $\vec{g}$ to $\vec{g}'$. Conversely, two flat connections $a, a' \in \mathrm{Hom}(\pi_1(\mathcal{M}), G)$ are *gauge equivalent* if there exists $h \in G$ such that $a'(\gamma) = h \cdot a(\gamma) \cdot h^{-1}$ for every $\gamma \in \pi_1(\mathcal{M})$. We denote this set by $\mathrm{Hom}(\pi_1(\mathcal{M}), G)/G$.

- **Topological action and group cohomology.** The total action is a product of local terms, one for each D-simplex of the triangulation of $\mathcal{M}$ (which we also denote by $\mathcal{M}$), and is given by

$$\prod_{[v_{i_1},\ldots,v_{i_{D+1}}]\in\mathcal{M}} \alpha_D(g_{i_1 i_2}\ldots, g_{i_D i_{D+1}})^{\epsilon_i} \tag{1.3.4}$$

with $\epsilon_i = \pm 1$[4] depending on whether the orientation of the D-simplex agrees with that of $\mathcal{M}$ and with $[\alpha_D] \in H^D(G, U(1))$. The n-th *group cohomology* $H^n(G, U(1))$ is a finite abelian group defined as the quotient of n-cocycles by n-coboundaries. Specifically, the set of n-cochains C^n is the set of functions $\alpha_n : G^n \to U(1)$ and the coboundary

4. The positive orientation of the D-simplex is obtained by having $i_1 < \cdots < i_{D+1}$.

operator $\delta^{(n)} : C^n \to C^{n+1}$ is

$$\begin{aligned}\delta^{(n)}\alpha_n(g_1,\ldots,g_{n+1}) =& \alpha_n(g_1,\ldots,g_n)^{(-1)^{n+1}}\alpha_n(g_2,\ldots,g_{n+1}) \\ & \times \prod_{i=1}^{n} \alpha_n(g_1,\ldots,g_i \cdot g_{i+1},\ldots,g_{n+1})^{(-1)^i}.\end{aligned} \tag{1.3.5}$$

The set of n-cocycles is defined by $Z^n(G,U(1)) = \{\alpha_n \in C^n : \delta^{(n)}\alpha_n = 1\}$ and the set of n-coboundaries by $B^n(G,U(1)) = \{\alpha_n \in C^n : \alpha_n = \delta^{n-1}\alpha_{n-1},\ \text{with}\ \alpha_{n-1} \in C^{n-1}\}$. It follows from the definition of the coboundary operator that $\delta^{(n)} \cdot \delta^{(n-1)} = 1$ so that the set of n-coboundaries is a subgroup of the set of n-cocycles. The n-th group cohomology of algebraic cocycles of G with $U(1)$ coefficients is defined by:

$$H^n(G,U(1)) = Z^n(G,U(1))/B^n(G,U(1)) = \text{Ker}\ \delta^{(n)}/\text{Im}\ \delta^{(n-1)}. \tag{1.3.6}$$

The fact that the topological action does not depend on the choice of triangulation of $\mathcal{M}$ follows from the cocycle condition $\delta^{(D)}\alpha_D = 1$. When no confusion is possible we will drop the subscript n in α_n and for convenience, we are going to denote by α_n the n-cohomology class and the cocycle used to represent it.

- **Partition function.** We denote by $\mathcal{Z}(G,\mathcal{M},\alpha_D)$ the gauge theory partition function associated with the finite group G and local action $\alpha_D \in H^D(G,U(1))$ on $\mathcal{M}$. We say the theory is *untwisted* if α_D is trivial and *twisted* otherwise. In the first case, we suppress the symbol for the local action. The partition function $\mathcal{Z}(G,\mathcal{M},\alpha_D)$ on the lattice is given by a summation over gauge equivalence classes (1.3.3) of flat gauge field configurations (1.3.2) weighted by the topological action (1.3.4) and normalized by $1/|G|$. This local lattice definition can be recast in a global and manifestly topological invariant way as

$$\mathcal{Z}(G, \mathcal{M}, \alpha_D) = \frac{1}{|G|} \sum_{a \in \mathrm{Hom}(\pi_1(\mathcal{M}), G)/G} \langle a^* \alpha_D, [\mathcal{M}] \rangle, \tag{1.3.7}$$

with $[\mathcal{M}]$ the fundamental class of $\mathcal{M}$ and $\alpha_D \in H^D(BG, U(1))$. Above we used the fact that there is a isomorphism between group cohomology $H^D(G, U(1))$ and topological cohomology $H^D(BG, U(1))$ where BG is a classifying space for G (a space with $\pi_1(BG) = G$ and $\pi_n(BG) = 1$ for $n > 1$). In this setup, the summation is over principal G bundles over $\mathcal{M}$ and the flat connection a defines a homotopy class of maps $a : \mathcal{M} \to BG$ which we use to pull back α_D to spacetime. We see that the theory can be viewed as a sigma model with target space the classifying space BG [Bullivant and Delcamp(2019)].

The normalization factor $1/|G|$ is such that the partition function for untwisted G gauge theory on $S^1 \times S^{D-1}$ equals

$$\mathcal{Z}(G, S^1 \times S^{D-1}) = \begin{cases} 1 & D \geq 3, \\ |G| & D = 2, \end{cases} \tag{1.3.8}$$

which is the dimension of the Hilbert space on S^{D-1}. For $D \geq 3$, the dimension is always one since S^{D-1} is simply connected. For $D = 2$, the space is a circle, and the dimension of Hilbert space is $|G|$. (Recall that we suppress the symbol for the topological action when it is trivial.)

When G is a finite abelian group the theory can be generalized to higher-form G gauge theory. In a p-form G gauge theory, the gauge field configurations are maps that assign group elements to p-simplices, the flatness condition involves the boundary of $(p+1)$-simplices, gauge transformations come from $(p-1)$-simplices and the topological action is classified by $H^D(B^pG, U(1))$ (B^pG is a space with $\pi_p(B^pG) = G$ and $\pi_n(B^pG) = 0$

otherwise). The partition function for untwisted p-form G gauge theory is proportional to $|H^p(\mathcal{M},G)|$ which equals (1.3.7) for $p=1$. This generalization does not work for non-abelian G because of the flatness condition except the $p=0$ case. When $p=0$, a gauge field configuration is a map $\vec{g}$ that assigns a gauge group element to every vertex of $\mathcal{M}$, $\vec{g}(v_i)\equiv g_i$. By the flatness condition $g_{[v_i,v_j]}=g_i g_j^{-1}=1$ for all edges of $\mathcal{M}$. One finds that the map $\vec{g}$ assigns the same group element to every connected component of $\mathcal{M}$ and therefore

$$\mathcal{Z}^0(G,\mathcal{M})=|G|^{|\pi_0(\mathcal{M})|}. \tag{1.3.9}$$

Below we often assume that the spacetime manifold is connected in which case the above is simply $\mathcal{Z}^0(G,\mathcal{M})=|G|$.

- **Hilbert space.** Consider canonical quantization on $\mathcal{M}=\mathbb{R}_{\text{time}}\times M_{\text{space}}$. The partition function on $S^1\times M_{\text{space}}$ gives the dimension of the Hilbert space on M_{space} and can be computed explicitly using the lattice definition. If we view the partition function as a summation over flat connections as in equation (1.3.7) then, for a gauge field with value g in the time direction, the field configurations on M_{space} that label the physical Hilbert space on M_{space} correspond to the flat connections such that the compactification of the topological action $\alpha_D\in H^D(BG,U(1))$ on S^1 is trivial:

$$a^\star i_g\alpha_D=0 \bmod 2\pi\mathbb{Z}\quad \forall g\in G\ , \tag{1.3.10}$$

where $i_g\alpha$ is the slant product (see e.g., Appendix A of [Wang and Wen(2015)]).[5] The

5. Explicitly,

$$\begin{aligned} i_g\alpha_D(g_1,\cdots,g_{D-1}) =&\alpha_D(g,g_1,\cdots,g_{D-1})^{(-1)^{D-1}}\alpha_D(g_1,\ldots,g_{D-1})\\ &\times\prod_{j=1}^{D-2}\alpha_D(g_1,\cdots,g_j,(g_1\cdot g_2,\cdots g_j)^{-1}\cdot g\cdot(g_1\cdot g_2,\cdots g_j),\cdots,g_{D-1})^{(-1)^{D-1+j}}\ . \end{aligned}$$

condition (1.3.10) can also be viewed as the “equation of motion" for the field variation in the temporal direction by the amount g.

In the case of vanishing α_D this Hilbert space is spanned by basis vectors in one-to-one correspondence with elements of $\mathrm{Hom}(\pi_1(M_{\text{space}}), G)/G$ where the quotient is the action by G conjugation, see (1.3.3).

- **Wilson lines.** Wilson lines are one-dimensional extended operators labeled by representations of the gauge group. The Wilson line associated with the representation ρ inserted on a loop γ is given by

$$W_\rho(\gamma) = \chi_\rho(g_\gamma) = \mathrm{Tr}_\rho(g_\gamma) \tag{1.3.11}$$

where $\chi_\rho : G \to \mathbb{C}$ is the character (trace) of the representation ρ and $g_\gamma \in G$ is the holonomy around γ. The operator $W_\rho(\gamma)$ depends on the homotopy class of the cycle γ. We recall that the fundamental group depends on a choice of basepoint. Assuming that the spacetime manifold is connected nothing depends on this choice. However, the presence of a basepoint implies that a loop γ homotopic to $\gamma_1 \cdot \gamma_2$ cannot be viewed as the disjoint union of γ_1 and γ_2. Therefore, in general $W_\rho(\gamma) \neq W_\rho(\gamma_1)W_\rho(\gamma_2)$, even if $\gamma_1 \cdot \gamma_2$ is homotopic to γ. An important exception is when ρ is one-dimensional, which is the case for all irreducible representations of abelian groups. We say that a Wilson line W_ρ has electric charge ρ.

If two Wilson lines are placed along the same loop they fuse according to the tensor product of representations. This follows from the fact that $\chi_\rho(g)\chi_{\rho'}(g) = \chi_{\rho\otimes\rho'}(g)$. Furthermore, representations of G are spanned by irreducible representations. Therefore, given the Wilson lines in representation ρ and ρ' we have:

$$W_\rho(\gamma)W_{\rho'}(\gamma) = W_{\rho\otimes\rho'}(\gamma) = \sum_{\rho_i\in\text{irreps}} c_i W_{\rho_i}(\gamma) \tag{1.3.12}$$

with $c_i \in \mathbb{N}$ the coefficient of ρ_i in the expansion of $\rho\otimes\rho'$ in irreducible representations.

- **General invertible electric defects.** General invertible electric defects are n-dimensional operators labeled by elements of $H^n(G, U(1))$, the n-th group cohomology of G with $U(1)$ coefficients (1.3.6). They are obtained by attaching a topological action along the n-dimensional manifold they are defined on. The general invertible electric operator associated with $\alpha_n \in H^n(G, U(1))$ inserted on the n-dimensional closed manifold Σ_n is given by [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi]:

$$W_{\alpha_n}(\Sigma_n) = \prod_{[v_{i_1},\ldots,v_{i_{n+1}}]\in\Sigma_n} \alpha_n(g_{i_1 i_2}\ldots, g_{i_n i_{n+1}})^{\epsilon_i}. \tag{1.3.13}$$

For $n = 1$, we have $H^1(BG, U(1)) \cong \text{Hom}(G, U(1))$ and these operators reduce to a Wilson line in a one-dimensional representation. For general n, they are submanifolds decorated with topological action for the G gauge fields. Examples of these defects are studied in [Else and Nayak(2017),Hsin and Turzillo(2020),Barkeshli et al.(2023)Barkeshli, Chen, Huang, Kobayashi, Tantivasadakarn, and Zhu, Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi,Hsin(2022), Ji et al.(2022)Ji, Tantivasadakarn, and Xu].

If two general invertible electric defects are placed along the same n-dimensional closed submanifold Σ_n they fuse according to the abelian group structure of $H^n(G, U(1))$. More precisely, given $\alpha_n, \alpha'_n \in H^n(G, U(1))$ we have:

$$W_{\alpha_n}(\Sigma_n)W_{\alpha'_n}(\Sigma_n) = W_{\alpha_n\cdot\alpha'_n}(\Sigma_n). \tag{1.3.14}$$

This is consistent with the property that fusing such domain walls is the same as first stacking the SPT phases with G symmetry labeled by α_n, α'_n on the wall and then gauging the G symmetry [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi].

- **Magnetic defects**. Magnetic defects are codimension-two operators labeled by conjugacy classes of G [Abrikosov(1957), Nielsen and Olesen(1973)]. The insertion of a magnetic defect associated to the conjugacy class of some element $g \in G$ on a closed connected $(D-2)$-submanifold Γ modifies the flatness condition (1.3.2) for the allowed gauge field configurations in the partition function. Specifically, for every 2-simplex $[v_i, v_j, v_k]$ such that $\gamma = (v_i, v_j, v_k)$ links with Γ the insertion of $M_g(\Gamma)$ restricts the holonomy g_γ to be g instead of 1. Here, we view Γ as being spanned by $(D-2)$-simplices in the dual triangulation of $\mathcal{M}$. Note that this implies that the Wilson line W_ρ has nontrivial linking with magnetic defects M_g given by $\frac{\chi_\rho(g)}{\chi_\rho(1)}$. In general, the operator $M_g(\Gamma)$ depends on the isotopy class of Γ. See Fig. 1.4 for illustration. We say that a magnetic defect M_g has magnetic charge g.

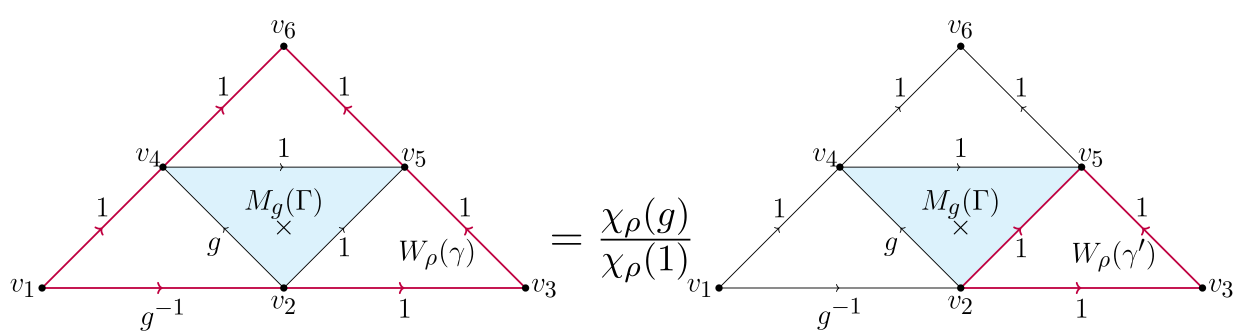


Figure 1.4: Example of valid gauge field configuration on the vicinity of the magnetic defect M_g insertion (indicated by an $\times$), and its linking action on the Wilson line W_ρ (shown in red). The magnetic insertion has linking number one with $\gamma = (v_2, v_4, v_5)$ associated with the 2-simplex $[v_2, v_4, v_5]$ so a valid gauge field configuration should satisfy $g_{(v_2,v_5,v_6)} = g$. Note, however, that the holonomy around the other 2-simplices is trivial. Furthermore, the expectation value for the two insertions with the Wilson line W_ρ along $\gamma = (v_1, v_4, v_6, v_5, v_3, v_2, v_1)$ which is linked with Γ, and along $\gamma' = (v_2, v_5, v_3, v_2)$ which is unlinked with Γ are related by $\frac{\chi_\rho(g)}{\chi_\rho(1)}$.

- **Dyons.** In $D = 3$ magnetic defects are also one-dimensional, therefore one can consider more general one-dimensional operators with electric and magnetic charge. We will focus on the case $\alpha = 0$. This operators are called dyons and they are labeled by the tuple $([g], \rho)$ with $[g]$ a non-trivial conjugacy class of G and ρ a non-trivial representation of the group $C(g) = \{k \in G : kgk^{-1} = g\}$, the centralizer of a fixed element $g \in [g]$ (see e.g., [Beigi et al.(2011)Beigi, Shor, and Whalen]). Wilson lines are dyons with label $([1], \rho)$ since $C(1) \cong G$, and magnetic defects are dyons with label $([g], 1)$ with 1 the trivial representation of $C(g)$.

In $D > 3$ with $\alpha = 0$, the magnetic defects have dimension $(D-2) > 1$. Since on the magnetic defect of conjugacy class $[g]$ the gauge group is reduced to the centralizer $C(g)$, an analog of a "dyon" defect can be defined by decorating the magnetic defect with an invertible electric defect for the unbroken gauge group $C(g)$, labelled by a $(D-2)$-cocycle $\beta \in H^{D-2}(C(g), U(1))$.

When $\alpha \neq 0$, a general dyonic defect is given by decorating the magnetic defect with (higher) projective representation. See e.g., [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi].

- **Fusion of magnetic defects in $D = 3$.** When the G gauge theory has trivial topological action the magnetic defects obey the following fusion rules. Since the magnetic defect M_g reduces the gauge group G to the centralizer subgroup $C(g)$, the Wilson line in irreducible representation ρ in the presence of the magnetic defect decomposes into $\sum_i W_{\rho_i}$ for $\rho = \bigoplus_i \rho_i$ under the stabilizer subgroup $C(g)$. Thus

$$W_\rho \times M_g = \sum_i W_{\rho_i} M_g \ , \tag{1.3.15}$$

where the right-hand side above should be viewed as sum of dyons, and the sum over i is as in the decomposition of ρ above.

Consider fusing magnetic defects $M_m, M_{\bar{m}}$. From the above discussion, this should give the condensation defect of Wilson lines that can terminate on the magnetic defect. Denote $\mathcal{R}_m$ to be the set of irreducible representations of G whose decomposition under the stabilizer subgroup $C(m)$ contains the trivial representation. Then

$$M_m \times M_{\bar{m}} = \frac{1}{|G|} \sum_{g \in G} M_{mg\bar{m}g^{-1}} \sum_{r \in \mathcal{R}_m} d_r W_{\rho_r} \ , \tag{1.3.16}$$

where d_i is the dimension of the representation ρ_i, and the sum is over the representations in $\mathcal{R}_m$. On the right-hand side, we use the property that multiplying two conjugacy classes in general get multiple fusion outcomes. The coefficients on the right-hand side of (1.3.16) can be computed using the method in [Witten(1989), Kaidi et al.(2022a)Kaidi, Komargodski, Ohmori, Seifnashri, and Shao]. Let us denote the right-hand side by $\mathcal{A}_m$, we want to find the coefficient of W_{ρ_r} in $\mathcal{A}_m$. Consider fusing $\mathcal{A}_m$ with $\bar{W}_{\rho_r}$ on $S^{D-1} \times S^1$ with the lines wrapping S^1, the partition function computes $\text{Hom}(\mathcal{A}_m \times \bar{W}_{\rho_r}, 1)$ which is the desired coefficient. On the other hand, view $\mathcal{A}_m$ as an empty cylindrical tube, the configuration is topologically equivalent to $B^{D-1} \times S^1$ with punctured ball B^{D-1} by Wilson line W_{ρ_r} wrapping S^1, and thus the coefficient is dim $\mathcal{H}(B^{D-1}, \rho_r)$, which equals to the dimension of the space of operators living at the intersection of the Wilson line and the boundary, i.e. the dimension of the representation.

For example, if m is in the center of G, the stabilizer $C(m) = G$ is the entire group, then $\mathcal{R}_m$ only contains the trivial representation. The fusion of the magnetic defects reduces to $M_m \times M_{m'} = M_{mm'}$.

When $\alpha \neq 0$, magnetic defects carry additional projective representations, which modify the fusion rules as discussed in [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi].

- **Hamiltonian formalism (quantum double model)**. We can also consider a Hamiltonian formalism with continuous time and discrete space. One possible Hamiltonian model is the quantum double (or its twisted version when the theory has a topological action) discussed in [Kitaev(2003), Hu et al.(2013a)Hu, Wan, and Wu, Moradi and Wen(2015)]. This theory should be viewed as an ultra-violet extension of topological finite-group gauge theory discussed above. Specifically, this Hamiltonian model has excitations with nonzero energy, and its Hilbert space is the tensor product of local Hilbert spaces $\mathbb{C}[G]$ on each edge with basis $\{|g\rangle : g \in G\}$. At low energy, with particular couplings, the ground states realize the Hilbert space of the topological finite-group gauge theory.

More concretely, the topological G gauge theory is realized in the low energy ground states by imposing an energy cost for the configuration that violates the Gauss law $\nabla \cdot E = 0$ for electric field E. To realize the flatness condition on the gauge fields, we also need to impose an energy cost for the fluxes. Thus, the Hamiltonian has the form

$$H = -\sum A_v - \sum_f B_f \ , \tag{1.3.17}$$

where A_v is the energy cost for violation of Gauss law at vertex v, and B_f is the energy cost for fluxes on face f. The explicit form of A_v, B_f are given in [Kitaev(2003), Hu et al.(2013a)Hu, Wan, and Wu, Moradi and Wen(2015)].

CHAPTER 2

NON-INVERTIBLE SYMMETRIES AS DOMAIN WALLS ON THE LATTICE

2.1 Introduction

Symmetries are powerful tools for understanding quantum systems. For instance, symmetries can provide hints about the long-distance behavior of physical systems even when they become strongly coupled. A famous example is the Lieb-Schultz-Mattis (LSM) theorem that constrains the dynamics of lattice models and the connection of these ideas to 't Hooft anomalies of generalized symmetries [Lieb et al.(1961)Lieb, Schultz, and Mattis, Oshikawa(2000), Hastings(2004), Cho et al.(2017)Cho, Ryu, and Hsieh, Cheng and Seiberg(2023), Seifnashri(2024)].

The notion of symmetry was given an intrinsic definition in terms of topological operators and their correlation functions in [Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett]. A frontier area of exploration is symmetries in field theories in general spacetime dimensions, where the topological operators or the corresponding topological quantum field theories are described by higher fusion categories.[1] Topological quantum field theories in general spacetime dimensions also play important role in exploring the dynamical consequences of symmetry in general gapped or gapless quantum systems such as constraining whether the symmetry can be realized on the boundary by symmetric gapped phases or trivially gapped phases [Apte et al.(2023)Apte, Cordova, and Lam, Kaidi et al.(2023a)Kaidi, Nardoni, Zafrir, and Zheng, Zhang and Córdova(2023), Córdova et al.(2023)Córdova, Hsin, and Zhang, Antinucci et al.(2023)Antinucci, Benini, Copetti, Galati, and Rizi] (see also [Ji and Wen(2020)] for related constructions). Thus, understanding the properties of topological operators or topological

1. In this work we will only consider fully topological operators, leaving cases with general subsystem symmetries to future work. See e.g., [Bulmash and Iadecola(2019), Luo et al.(2022)Luo, Spieler, Sun, and Karch, Hsin et al.(2023)Hsin, Luo, and Malladi] and the references therein for examples of gapped domain walls and interfaces in fracton models.

quantum field theories in general spacetime dimensions is important for learning dynamics of general quantum systems from symmetry.

Finite-group topological gauge theories [Dijkgraaf and Witten(1990a)] provide a fruitful playground to explore these notions. They can arise in various settings such as gapped phases of lattice models and quantum field theories and topological codes. Finite-group gauge theories are also relevant in experiments, most prominently $\mathbb{Z}_2$ gauge theory in s-wave superconductors where $U(1)$ electromagnetism is broken to $\mathbb{Z}_2$ by Cooper pairing [Hansson et al.(2004)Hansson, Oganesyan, and Sondhi]. In recent years, there are also experimental realizations of ground state wavefunctions for gauge theories with $\mathbb{Z}_2$ gauge group (e.g., [Iqbal et al.(2023b)]) and dihedral gauge group of order 8 [Iqbal et al.(2024)] in 2+1d by quantum processors.

The invertible symmetries, i.e., symmetries that have an inverse transformation, in finite-group gauge theories have been discussed extensively in the literature of symmetry-enriched topological orders (SET) [Etingof et al.(2010)Etingof, Nikshych, Ostrik, and Meir, Mesaros and Ran(2013), Barkeshli et al.(2019)Barkeshli, Bonderson, Cheng, and Wang, Teo et al.(2015)Teo, Hughes, and Fradkin, Tarantino et al.(2016)Tarantino, Lindner, and Fidkowski, Chen(2017)]. In particular, recently invertible symmetries in finite-group gauge theories have important applications to new fault-tolerant logical gates in topological quantum codes [Barkeshli et al.(2023)Barkeshli, Chen, Huang, Kobayashi, Tantivasadakarn, and Zhu, Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi, Kobayashi(2023), Barkeshli et al.(2024b)Barkeshli, Hsin, and Kobayashi, Zhu et al.(2023)Zhu, Sikander, Portnoy, Cross, and Brown, Fidkowski and Hastings(2023), Kobayashi and Zhu(2023), Hsin et al.(2024b)Hsin, Kobayashi, and Zhu]. Invertible symmetries in finite-group gauge theories also provide construction for new automorphism codes, e.g., [Hastings and Haah(2021), Davydova et al.(2023)Davydova, Tantivasadakarn, and Balasubramanian, Aasen et al.(2022)Aasen, Wang, and Hastings, Hsin and Wang(2023), Kesselring et al.(2024)Kesselring, Magdalena de la

Fuente, Thomsen, Eisert, Bartlett, and Brown].

In addition to invertible symmetries, finite-group gauge theories can have non-invertible symmetry, where the generating topological operators do not obey group-law fusion, and in particular do not have an inverse. Such non-invertible topological defects can be present in various gapless or gapped quantum systems, see [Kapustin and Saulina(2010),Fuchs et al.(2013a)Fuchs, Schweigert, and Valentino, Bhardwaj and Tachikawa(2018), Chang et al.(2019)Chang, Lin, Shao, Wang, and Yin,Choi et al.(2022a)Choi, Córdova, Hsin, Lam, and Shao,Kaidi et al.(2022b)Kaidi, Ohmori, and Zheng, Choi et al.(2022b)Choi, Córdova, Hsin, Lam, and Shao, Bhardwaj et al.(2023c)Bhardwaj, Bottini, Schafer-Nameki, and Tiwari, Choi et al.(2022d)Choi, Lam, and Shao, Córdova and Ohmori(2023)] for early work on this subject. Meanwhile, examples of non-invertible topological domain wall defects in finite-group gauge theory are discussed in various literature [Kitaev(2003),Kapustin and Saulina(2011),Wen(2013),Barkeshli et al.(2019)Barkeshli, Bonderson, Cheng, and Wang,Hsin and Turzillo(2020),Koide et al.(2022)Koide, Nagoya, and Yamaguchi, Hayashi and Tanizaki(2022), Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi, Bhardwaj et al.(2023a)Bhardwaj, Bottini, Pajer, and Schäfer-Nameki, Bhardwaj et al.(2024d)Bhardwaj, Bottini, Schafer-Nameki, and Tiwari,Bhardwaj et al.(2024c)Bhardwaj, Bottini, Schafer-Nameki, and Tiwari, Bhardwaj et al.(2024f)Bhardwaj, Inamura, and Tiwari]. In 2+1d, such topological domain walls or boundaries correspond to certain condensation of bulk topological excitations called Lagrangian algebras [Kong(2014a)]. On the other hand, the gapped domain walls and boundaries in higher dimensions are less understood (see [Zhao et al.(2023)Zhao, Lou, Zhang, Hung, Kong, and Tian, Ji et al.(2022)Ji, Tantivasadakarn, and Xu, Luo(2023)] for recent studies for the gapped boundaries of $\mathbb{Z}_2$ gauge theory in 3+1d).

In this work, we will investigate general symmetries, both invertible and non-invertible, in finite-group topological gauge theories. We will focus on the topological domain walls in general spacetime dimension. Since topological finite-group gauge theories are naturally defined on the lattice (see e.g., [Dijkgraaf and Witten(1990a)]), we will investigate the symmetries by placing the theories on the lattice. A companion paper [Cordova et al.(2024)Cordova, Costa, and Hsin] will explore the relationship of these symmetries to condensations.

2.1.1 Summary of results

Gauge theories with a finite gauge group G can be defined by a path integral on the lattice [Dijkgraaf and Witten(1990a)]. A flat gauge field configuration is a map that assigns to each oriented edge a group element $g_{ij} \in G$ and satisfies a flatness condition for every 2-simplex of the triangulated manifold. Gauge transformations are maps that assign to each vertex a group element $h_i \in G$ and transform a flat gauge field configuration g_{ij} to $h_i g_{ij} h_j^{-1}$. The total action is a product of local terms classified by group cohomology $H^D(G, U(1))$ whose elements are functions that assign a well-defined phase depending on the values of the gauge field configuration in each D-simplex. The partition function is then given by a summation over gauge equivalence classes of flat gauge field configurations and is weighted by the topological action (see Section 1.3 for a review).

Domain walls and gapped boundaries on the lattice Gapped boundaries of untwisted gauge theory with a finite gauge group G in general dimension can be constructed from subgroups $K \leq G$ and a choice of topological action $\alpha \in H^{D-1}(K, U(1))$. Given this data, we construct a gapped boundary $\mathcal{B}_{K,\alpha}$ by restricting the gauge field configurations to be in the subgroup K and by attaching the topological action $\alpha \in H^{D-1}(K, U(1))$ along the boundary $\partial\mathcal{M}$. Motivated by the folding trick, we construct a domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$ by having gauge fields on the subgroup $H \leq G \times G$ and by attaching the topological action $\alpha \in H^{D-1}(H, U(1))$ along the codimension-one submanifold Σ.

Fusion of domain walls and action on gapped boundaries Despite being simple, this definition is generic because it applies to any group G and dimension D. Furthermore, some of the fusion rules for the codimension-one topological operators can be derived in a very simple way from this description. One of the main contributions of this paper is to derive the fusion ring structure of the domain walls $\mathcal{D}_{H,\alpha}$ with subgroup $H \leq G \times G$ and topological action $\alpha \in H^{D-1}(H, U(1))$ as elements of one of the following two families:

- $H = K^{(\phi)} \equiv \{(\phi \cdot k, k) : k \in K\}$ with $K \triangleleft G$ and $\phi \in \mathrm{Aut}(G)$ and a topological action $\alpha \in H^{D-1}(K, U(1))$ evaluated on the right entry of $K^{(\phi)}$.;

- $H = K_L \times K_R$, with $K_L, K_R \triangleleft G$ and $\alpha = \alpha_L \times \alpha_R$ with $\alpha_L \in H^{D-1}(K_L, U(1))$ and $\alpha_R \in H^{D-1}(K_R, U(1))$.

We show that these two families are generated by the domain walls:

- Automorphism domain walls: $\mathcal{D}_{G^{(\phi)}}$, with $\phi \in \mathrm{Aut}(G)$;

- Diagonal domain walls: $\mathcal{D}_{K^{(\mathrm{id})},\alpha}$, with $K \triangleleft G$ and $\alpha \in H^{D-1}(K, U(1))$;

- Magnetic domain wall: $\mathcal{D}_{G\times G}$;

which obey the following fusion rules:

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G^{(\phi')}} = \mathcal{D}_{G^{(\phi\circ\phi')}}, \tag{2.1.1}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{K'^{(\mathrm{id})},\alpha'} = \frac{|G|}{|K \cdot K'|}\mathcal{D}_{(K\cap K')^{(\mathrm{id})},\alpha\cdot\alpha'}, \tag{2.1.2}$$

$$\mathcal{D}_{K_L^{(\mathrm{id})},\alpha_L} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R^{(\mathrm{id})},\alpha_R} = \mathcal{D}_{K_L\times K_R,\alpha_L\times\alpha_R}, \tag{2.1.3}$$

$$\mathcal{D}_{G\times G} \times \mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G\times G} = \mathcal{Z}(K,\alpha)\mathcal{D}_{G\times G}, \tag{2.1.4}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{K^{(\mathrm{id})},\alpha} = \mathcal{D}_{K^{(\phi)},\alpha}, \tag{2.1.5}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{\phi^{-1}(K),\phi^*\alpha} = \mathcal{D}_{(\phi^{-1}(K))^{(\phi)},\phi^*\alpha}, \tag{2.1.6}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G\times G} = \mathcal{D}_{G\times G} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G\times G}, \tag{2.1.7}$$

with $\phi \circ \phi'$ the automorphism composition of $\phi, \phi' \in \mathrm{Aut}(G)$; $\alpha \cdot \alpha'|_{K\cap K'} \in H^{D-1}(K \cap K', U(1))$; $|G|/|K \cdot K'|$ the 0-form partition function of $G/K \cdot K'$ gauge theory on Σ; $\mathcal{Z}(K, \alpha)$

the partition function of K gauge theory twisted by α on Σ; and $\phi^*\alpha$ the pullback of α by $\phi : \phi^{-1}(K) \to K$.

In addition, we show that the domain walls that generate this fusion ring have the following action on the gapped boundaries:

$$\mathcal{D}_{G(\phi)} \times \mathcal{B}_{K,\alpha} = \mathcal{B}_{\phi(K),\phi^{-1*}\alpha}, \tag{2.1.8}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{B}_{K',\alpha'} = \frac{|G|}{|K \cdot K'|}\mathcal{B}_{K\cap K',\alpha\cdot\alpha'}, \tag{2.1.9}$$

$$\mathcal{D}_{G\times G} \times \mathcal{B}_{K,\alpha} = \mathcal{Z}(K,\alpha)\mathcal{B}_{G}. \tag{2.1.10}$$

Transformation on other operators Group elements and gauge transformations along Σ in the presence of $\mathcal{D}_{H,\alpha}(\Sigma)$ are restricted to the subgroup H. From this feature, we can derive the transformation of other operators on the domain walls. As an example, it is easy to show that:

$$\mathcal{D}_1 \cdot W_{\rho_i} = \sum_{\rho_k \in \mathrm{irreps}} d_i d_k W_{\rho_k}, \qquad \mathcal{D}_1 \cdot M_g = 0, \tag{2.1.11}$$

$$\mathcal{D}_{G\times G} \cdot W_{\rho_i} = 0, \qquad \mathcal{D}_{G\times G} \cdot M_g = \sum_{[k]\in \mathrm{Cl}(G)} M_k, \tag{2.1.12}$$

$$\mathcal{D}_{G(\phi)} \cdot W_{\rho_i} = W_{\rho_i \cdot \phi^{-1}}, \qquad \mathcal{D}_{G(\phi)} \cdot M_g = M_{\phi(g)}. \tag{2.1.13}$$

for all simple Wilson lines W_{ρ_i} and magnetic defects M_g where d_i is the dimension of the irreducible representation ρ_i.

Higher codimensional topological operators: Cheshire strings In the definition of the domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$, a crucial ingredient is the orientation of the normal bundle $N\Sigma$. It allows us to consistently define the global meaning of left and right associated with the left and right components of the subgroup $H < G \times G$. Diagonal domain walls, however, are orientation reversal invariant and can be generalized as higher codimensional operators. The dimension-n generalization of the diagonal domain walls are classified by subgroups $K < G$ and a topological action $\alpha \in H^n(K, U(1))$ and obey the fusion rule:

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha}(\Sigma_n) \times \mathcal{D}_{K'^{(\mathrm{id})},\alpha'}(\Sigma_n) = \frac{|G|}{|K \cdot K'|} \mathcal{D}_{(K\cap K')^{(\mathrm{id})},\alpha\cdot\alpha'}(\Sigma_n) \tag{2.1.14}$$

with Σ_n a n-dimensional submanifold of $\mathcal{M}$. This fusion rule generalizes the fusion rule of Cheshire strings [Else and Nayak(2017), Tantivasadakarn and Chen(2024b)].

Non-invertible electric-magnetic duality domain wall Note that in the data that specifies a domain wall, dimension dependence comes from the topological action $\alpha \in H^{D-1}(H, U(1))$. By working out the particular case of $G = \mathbb{Z}_2$ gauge theory in $D = 3$, we compute the fusion, action on boundaries and transformation of other operators for the domain wall associated with the subgroup $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \lhd \mathbb{Z}_2 \times \mathbb{Z}_2$ with the non-trivial topological action $\alpha_2 \in H^2(H, U(1)) = \mathbb{Z}_2$. We find

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} = 1, \tag{2.1.15}$$

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{B}_1 = \mathcal{B}_{\mathbb{Z}_2}, \qquad \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{B}_{\mathbb{Z}_2} = \mathcal{B}_1, \tag{2.1.16}$$

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \cdot W = M, \qquad \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \cdot M = W, \tag{2.1.17}$$

showing that $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ is the electric-magnetic duality symmetry defect. The procedure we follow for the computation is more general and shows that $\mathcal{D}_{G\times G,\alpha}$ generalizes the electric-magnetic duality to higher dimensions, generic gauge groups G and topological action $\alpha \in H^{D-1}(G, U(1))$. This class of domain walls mixes invertible electric and magnetic operators and obeys a non-invertible fusion in general. For instance, in the theory with $G = \mathbb{D}_4$ (the dihedral group of order 8), and $D = 3$, the domain wall associated with the subgroup $H = \mathbb{D}_4 \times \mathbb{D}_4$ and the non-factorized element $\alpha_2 \in H^2(\mathbb{D}_4 \times \mathbb{D}_4, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, obeys the Fibonacci fusion rule:

$$\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} \times \mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} = 1 + \mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)}, \tag{2.1.18}$$

and mixes magnetic and electric operators.

2.2 Codimension-one topological operators on the lattice

In this section, we will first review and define on the lattice the gapped boundaries of finite-group gauge theories, which are related to domain walls via the folding trick. Then we will present a lattice construction of the domain walls using this classification.

2.2.1 Gapped boundaries on the lattice

Gapped boundaries in untwisted finite-group G gauge theory can be constructed from:

- Subgroup $K \leq G$;
- Topological action $\alpha \in H^{D-1}(K, U(1))$.

Given the data (K, α), one constructs the gapped boundary $\mathcal{B}_{K,\alpha}$ by restricting the gauge fields and gauge transformations on the boundary to be elements of K and one decorates the boundary with the corresponding topological action $\alpha \in H^{D-1}(K, U(1))$ as in (1.3.13). The above construction is compatible and generalizes the Beigi-Shor-Whalen classification [Beigi et al.(2011)Beigi, Shor, and Whalen] of gapped boundaries in the quantum double model in $D = 3$.

2.2.2 Domain walls on the lattice from the folding trick

Gapped domain walls can be obtained from gapped boundaries by the folding trick. In particular, we should be able to give a constructive definition of a codimension-one domain wall of G gauge theory from the data that specifies a gapped boundary of $G \times G$ gauge theory, i.e., a subgroup $H \leq G \times G$ and a topological action $\alpha \in H^{D-1}(H, U(1))$. In this section, we outline this construction.

Given a subgroup $H \leq G \times G$ and a codimension-one connected, closed and orientable submanifold Σ, we define the domain wall, $\mathcal{D}_H(\Sigma)$ by restricting the gauge group elements of the connection along Σ to lie in the subgroup H and by properly gluing the H gauge group elements of Σ with the G gauge group elements of the rest of spacetime. We can further decorate the domain wall with a topological action $\alpha \in H^{D-1}(H, U(1))$ which gives the domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$. In more detail, $\mathcal{D}_{H,\alpha}(\Sigma)$ is defined as:

- **Gauge field configurations:** Each edge on Σ has group elements $(h_L, h_R) \in H \leq G \times G$ instead of $g \in G$ (where L and R is defined globally with respect to the orientation of the normal bundle $N\Sigma$). Because of this modification, one needs to specify the appropriate holonomy for 2-simplices that have edges both in and outside Σ, i.e., we need to define the flatness condition of (1.3.2) for such 2-simplices. The holonomy picks a h_L (or h_R) contribution if the edges comes from the left (or right) of Σ. See Fig. 2.1 for an example of a valid flat gauge field configuration.

- **Gauge transformations:** Gauge transformation on the vertices of Σ by $(k_L, k_R) \in H$ transforms the group elements on the edges that meet the vertex:

  - If the edge is on Σ and pointing towards the vertex, the group element (h_L, h_R) on the edge transforms into $(h_L \cdot k_L^{-1}, h_R \cdot k_R^{-1})$. If the edge is pointing away from the vertex, the group element transforms to $(k_L \cdot h_L, k_R \cdot h_R)$.

  - If the edge is outside Σ with group element $g_L \in G$ and joins Σ from the left, the group element transforms into $g_L \cdot k_L^{-1}$. If it leaves Σ to the left the group element transforms into $k_L \cdot g_L$.

  - If the edge is outside Σ with group element $g_R \in G$ and joins Σ from the right, the group element transforms into $g_R \cdot k_R^{-1}$. If it leaves Σ to the right the group element transforms into $k_R \cdot g_R$.

See Fig. 2.2 for illustration.

- **Topological action:** The topological action $\alpha \in H^{D-1}(H, U(1))$ is evaluated for all $(D-1)$-simplices of Σ. Whenever α is trivial we suppress it from our notation for the domain wall. A domain wall with trivial topological action is said to be *untwisted* and *twisted* otherwise.

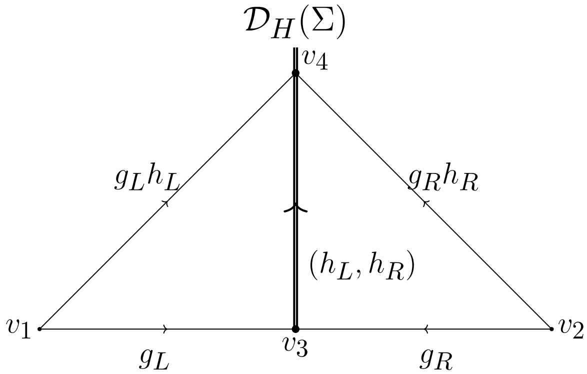


Figure 2.1: Example of valid flat gauge field configuration in a local region of $\mathcal{D}_H(\Sigma)$. Note that the holonomies of (v_1, v_3, v_4, v_1) and (v_2, v_3, v_4, v_2) are trivial, but the holonomy of $(v_1, v_3, v_2, v_4, v_1)$ is not trivial in general.

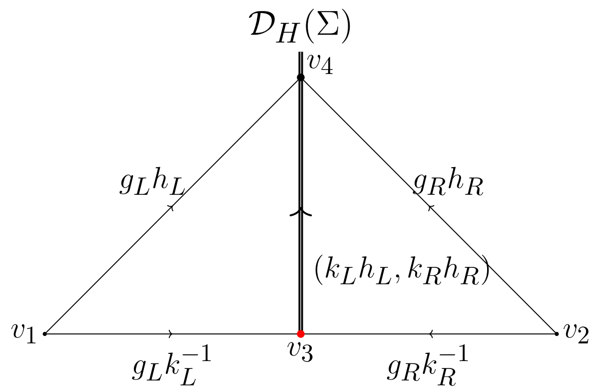


Figure 2.2: Example of equivalent gauge field configuration for the same local region. They are related by a gauge transformation with parameter given by $(k_L, k_R) \in H$ on v_3 and $1 \in H$ on v_1, v_2, v_4.

Notice that depending on the subgroup H, the domain wall can source holonomy for loops that pierce the wall. For example, in Fig. 2.1 one can check that the holonomy along the paths (v_1, v_3, v_4) and (v_2, v_3, v_4) are trivial. However, the holonomy for a contractible path that crosses Σ is not trivial in general, for example, the path $\gamma = (v_1, v_3, v_2, v_4, v_1)$ has holonomy: $g_\gamma = g_L \cdot h_R \cdot h_L^{-1} \cdot g_L^{-1}$ which is not 1 in general. This feature is crucial for constructing the above domain walls as condensations where the non-trivial holonomy is generated by magnetic defect insertions [Cordova et al.(2024)Cordova, Costa, and Hsin].

Because of the modification of gauge transformations along Σ, the holonomy for large loops that pierce the wall do not in general change by conjugation under gauge transformations. This means that a Wilson line inserted in such a loop is not, in general, gauge invariant. Conversely, to have a magnetic defect ending or crossing the domain wall one would need to fix the holonomy for a simplex in Σ to be the conjugacy class of the magnetic operator, but this might not be possible depending on H. We are going to see that these two features can be used to define the action of untwisted domain walls in both operators.

It is straightforward to see how the above definition can be recast in the Hamiltonian formalism of the quantum double model described around equation (1.3.17). For instance, to define a domain wall extended along time in $\mathcal{M} = \mathbb{R}_{\text{time}} \times M_{\text{space}}$, one should change the total Hilbert space on M_{space} by having a local Hilbert space $\mathbb{C}[H]$ with $H \leq G \times G$ for edges on Σ_{space} (a codimension-one submanifold of M_{space}). One should then change accordingly the definition of the A_v and B_f terms of the Hamiltonian for vertices along Σ_{space} and faces with edges contained within Σ_{space}.

We will focus on the domain walls corresponding to the subgroups in Table 2.1. Our

Symbol	Subgroup of $G \times G$	Local action
$\mathcal{D}_{K^{(\phi)},\alpha}$	$K^{(\phi)} \equiv \{(\phi \cdot k, k) : k \in K\}$	$\alpha \in H^{D-1}(K, U(1))$
$\mathcal{D}_{K_L \times K_R, \alpha_L \times \alpha_R}$	$K_L \times K_R$	$\alpha_L \times \alpha_R \in H^{D-1}(K_L \times K_R, U(1))$

Table 2.1: Above K_L, K_R, K are normal subgroups of G and $\phi \in \text{Aut}(G)$. In the first row, the topological action is evaluated on the right entry of $K^{(\phi)}$. More formally, as a topological action of $H^{D-1}(K^{(\phi)}, U(1))$ it is $R^*\alpha$, i.e., the pullback of $\alpha \in H^{D-1}(K, U(1))$ by $R : K^{(\phi)} \to K$ defined by $R(\phi \cdot k, k) = k$.

choice for this particular subset is that they make a closed algebra under fusion. In Section 2.3.4 we are also going to work out examples of domain walls associated with the subgroup $H = G \times G$, with a non-factorized local action, i.e., a local action $\alpha \in H^{D-1}(G \times G, U(1))$ which is not of the form $\alpha_L \times \alpha_R$ with $\alpha_L, \alpha_R \in H^{D-1}(G, U(1))$. In particular, the domain wall that implements electric-magnetic duality in $\mathbb{Z}_2$ gauge theory in $D = 3$ is precisely the domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha}$ with the non-factorized local action $\alpha \in H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. More generally, in Section 2.4.3 we are going to show that defects of this form can mix electric and magnetic operators and, in this sense, generalize the electric-magnetic duality of abelian gauge theories.

2.2.3 Orientation-reversal of domain walls

The definition of the domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$ depends on the orientation of the manifold Σ. In an orientable ambient spacetime (which we assume) an orientation of Σ is equivalent to an orientation of the normal bundle $N\Sigma$. Orientation-reversal of Σ flips the normal vector and exchanges the left and right of the domain wall. Thus the domain wall associated to the subgroup H becomes the image of H under the automorphism of $G\times G$ defined by $T(g_L, g_R) = (g_R, g_L)$. More precisely, let $\overline{\mathcal{D}}_{H,\alpha}$ be the orientation-reversal of $\mathcal{D}_{H,\alpha}$. Then $\overline{\mathcal{D}}_{H,\alpha} = \mathcal{D}_{T(H),T^*\alpha}$ where $T(H)$ denotes the image of $H \leq G\times G$ under T and $T^*\alpha$ is the pullback of $\alpha \in H^{D-1}(H, U(1))$ by $T : T(H) \to H$ (here we used that $T = T^{-1}$). In particular, for the two families of subgroups of Table 2.1 we have:

$$\overline{\mathcal{D}}_{K^{(\phi)},\alpha} = \mathcal{D}_{(\phi(K))^{(\phi^{-1})},\phi^{-1*}\alpha}, \qquad \overline{\mathcal{D}}_{K_L\times K_R,\alpha_L\times\alpha_R} = \mathcal{D}_{K_R\times K_L,\alpha_R\times\alpha_R}. \tag{2.2.1}$$

Note that reversing the orientation of Σ (barred defect above) is the same as taking the CPT conjugate. For invertible operators, this barred operator is thus identified with the inverse and:

$$\mathcal{D}\times\overline{\mathcal{D}} = 1, \qquad \text{(invertible symmetries)} \tag{2.2.2}$$

where the right-hand side denotes the identity operator. Meanwhile, for the more general non-invertible symmetries discussed here, the fusion of $\mathcal{D}$ with its CPT conjugate $\overline{\mathcal{D}}$ is not in general the identity, but rather is a condensation defect [Gaiotto and Johnson-Freyd(2019)] and contains the identity as well as a coherent sum of other operators.[2]

2.3 Fusion rules of domain walls and action on gapped boundaries

In this section, we use our lattice constructions to compute the fusion of domain walls and the action of domain walls on gapped boundaries.

Fusion of domain walls Given two domain walls, $\mathcal{D}_{H,\alpha}, \mathcal{D}_{H',\alpha'}$ associated with the subgroups $H, H' \leq G \times G$ and topological actions $\alpha \in H^{D-1}(H, U(1)), \alpha' \in H^{D-1}(H', U(1))$ defined on Σ, their fusion is defined by placing them "close" together and noticing that one can rewrite the insertion as a sum of other domain walls. The coefficients of the summation are partition functions of topological quantum field theories. The geometry of the two domain walls close together is that of $\Sigma \times [0,1]$ with $\mathcal{D}_{H,\alpha}$ defined on $\Sigma \times 0$ and $\mathcal{D}_{H',\alpha'}$ on $\Sigma \times 1$. In the following computations, we are going to use a cellular decomposition of $\Sigma \times [0,1]$ obtained from two copies of a given triangulation of Σ by joining equivalent vertices of the two copies. See Fig. 2.3 for illustration.

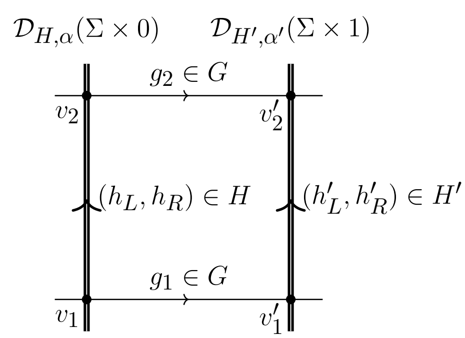


Figure 2.3: Local region of Σ in the presence of $\mathcal{D}_{H,\alpha}(\Sigma) \times \mathcal{D}_{H',\alpha'}(\Sigma)$. We use a cellular decomposition of $\Sigma \times [0,1]$ obtained from two copies of a given triangulation of Σ by joining equivalent vertices of the two copies.

2. We note that the condensation defect can consist of operators of the same dimension as the condensation defect. In such a case, the condensation defect can act on local operators because the operators that constitute the "mesh" in the condensation defect can act on local operators. For instance, the Kramers-Wannier duality σ in 1+1d obeys the fusion rule $\sigma \times \sigma = 1 + \psi$ in the continuum where $\bar{\sigma} = \sigma$, and $1 + \psi$ is a condensation of ψ, which is the $\mathbb{Z}_2$ 0-form symmetry that acts on local operators.

The fusion algebra of the two classes of the domain walls presented in Table 2.1 is generated by the domain walls presented in Table 2.2.

Name	Notation	Subgroup of $G \times G$	Local action
Diagonal	$\mathcal{D}_{K^{(\mathrm{id})},\alpha}$	$K^{(\mathrm{id})} = \{(k,k) : k \in K\}$	$\alpha \in H^{D-1}(K, U(1))$
Automorphism	$\mathcal{D}_{G^{(\phi)}}$	$G^{(\phi)} = \{(\phi \cdot g, g) : g \in G\}$	Trivial
Magnetic	$\mathcal{D}_{G\times G}$	$G \times G$	Trivial

Table 2.2: Generators of the domain walls presented in Table 2.1. Above, $K \lhd G$ and $\phi \in \mathrm{Aut}(G)$.

The fusion of any set of domain walls within the two classes of Table 2.1 can be computed using the fusion rules (derived below):

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G^{(\phi')}} = \mathcal{D}_{G^{(\phi\circ\phi')}}, \tag{2.3.1}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G\times G} = \mathcal{D}_{G\times G} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G\times G}, \tag{2.3.2}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{K^{(\mathrm{id})},\alpha} = \mathcal{D}_{K^{(\phi)},\alpha}, \tag{2.3.3}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{\phi^{-1}(K),\phi^*\alpha} = \mathcal{D}_{(\phi^{-1}(K))^{(\phi)},\phi^*\alpha}, \tag{2.3.4}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{K'^{(\mathrm{id})},\alpha'} = \frac{|G|}{|K \cdot K'|}\mathcal{D}_{(K\cap K')^{(\mathrm{id})},\alpha\cdot\alpha'}, \tag{2.3.5}$$

$$\mathcal{D}_{K_L^{(\mathrm{id})},\alpha_L} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R^{(\mathrm{id})},\alpha_R} = \mathcal{D}_{K_L\times K_R,\alpha_L\times\alpha_R}, \tag{2.3.6}$$

$$\mathcal{D}_{G\times G} \times \mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G\times G} = \mathcal{Z}(K,\alpha)\mathcal{D}_{G\times G}, \tag{2.3.7}$$

with $\phi \circ \phi'$ the automorphism composition of $\phi, \phi' \in \mathrm{Aut}(G)$; $\phi^{-1}(K)$ the image of K under ϕ^{-1}; $\phi^*\alpha$ the pullback of $\alpha : K^{D-1} \to U$ by $\phi : \phi^{-1}(K) \to K$; $\alpha \cdot \alpha'|_{K\cap K'} \in H^{D-1}(K \cap K', U(1))$; $|G|/|K \cdot K'|$ the 0-form partition function of $G/K \cdot K'$ gauge theory on Σ, (see the discussion around (1.3.9)); and $\mathcal{Z}(K,\alpha)$ the partition function of K gauge theory twisted by α on Σ.

As an example, the second class of domain walls presented in Table 2.1 (the factorized domain walls) is generated by the diagonal and the magnetic domain walls. The fusion of factorized domain walls can be derived from (2.3.5), (2.3.6) and (2.3.7) using associativity and is:

$$\mathcal{D}_{K_L \times K_R, \alpha_L \times \alpha_R} \times \mathcal{D}_{K'_L \times K'_R, \alpha'_L \times \alpha'_R} = \frac{|G|}{|K_R \cdot K'_L|} \mathcal{Z}(K_R \cap K'_L, \alpha_R \cdot \alpha'_L) \mathcal{D}_{K_L \times K'_R, \alpha_L \times \alpha'_R}. \tag{2.3.8}$$

Note that the coefficient is again a partition function on Σ: that of a $K_R \cap K'_L$ gauge theory twisted by $\alpha_R \cdot \alpha'_L$ decoupled from an untwisted $G/K_R \cdot K'_L$ zero-form gauge theory. If instead the domain walls were decorated with non-factorized topological actions α and α', the fusion coefficient would depend on the topological actions in a non-trivial way. In particular, the result would not generally be uniform in the spacetime dimension. We will give examples of this in Section 2.3.4.

Action of domain walls on gapped boundaries Similarly, given a domain wall $\mathcal{D}_{H,\alpha}$ and a gapped boundary $\mathcal{B}_{K,\beta}$ associated with the subgroups $H \leq G \times G$, $K \leq G$ and topological actions $\alpha \in H^{D-1}(H, U(1))$, $\beta \in H^{D-1}(K, U(1))$, one can take the domain wall to the boundary which will act on the gapped boundary generating a sum of gapped boundaries. The coefficients of the summation are partition functions of topological quantum field theories. The geometry of the domain wall action on the gapped boundary is that of $\partial\mathcal{M} \times [0,1]$ with $\mathcal{D}_{H,\alpha}$ defined along $\Sigma = \Sigma \times 0$ and $\mathcal{B}_{K,\beta}$ along $\Sigma \times 1$. Similarly to the fusion of domain walls we will use a cellular decomposition of $\partial\mathcal{M} \times [0,1]$ obtained from two copies of a given triangulation of $\partial\mathcal{M}$ by joining equivalent vertices of the two copies. See Fig. 2.4 for illustration.

The above definition gives the action of domain walls on gapped boundaries “from left to right". The action “from right to left" is the same as the action (“from left to right") of the orientation-reversal of the domain wall in consideration.

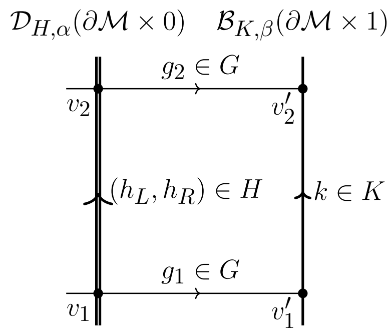


Figure 2.4: Local region of the boundary $\partial\mathcal{M}$ in the presence of $\mathcal{D}_{H,\alpha}(\partial\mathcal{M})\times\mathcal{B}_{K,\beta}(\partial\mathcal{M})$. We use a cellular decomposition of $\partial\mathcal{M}\times[0,1]$ obtained from two copies of a given triangulation of $\partial\mathcal{M}$ by joining equivalent vertices of the two copies.

The domain walls of Table 2.2 have the following action on the gapped boundaries:

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{B}_{K,\alpha} = \mathcal{B}_{\phi(K),\phi^{-1*}\alpha}, \tag{2.3.9}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{B}_{K',\alpha'} = \frac{|G|}{|K\cdot K'|}\mathcal{B}_{K\cap K',\alpha\cdot\alpha'}, \tag{2.3.10}$$

$$\mathcal{D}_{G\times G} \times \mathcal{B}_{K,\alpha} = \mathcal{Z}(K,\alpha)\mathcal{B}_{G}, \tag{2.3.11}$$

with $\phi(K)$ the image of $K \leq G$ under $\phi \in \mathrm{Aut}(G)$; $\phi^{-1*}\alpha$ the pullback of α by $\phi^{-1} : \phi(K) \to K$; $\alpha\cdot\alpha'|_{K\cap K'} \in H^{D-1}(K\cap K', U(1))$; and $\mathcal{Z}(K,\alpha)$ the partition function of K gauge theory twisted by α on $\partial\mathcal{M}$.

2.3.1 Automorphism domain walls

In this section we derive the fusion rules (2.3.1) and (2.3.2) and the action on boundary (2.3.9):

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G^{(\phi')}} = \mathcal{D}_{G^{(\phi\circ\phi')}}, \tag{2.3.12}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{G\times G} = \mathcal{D}_{G\times G} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G\times G}, \tag{2.3.13}$$

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{B}_{K,\alpha} = \mathcal{B}_{\phi(K),\phi^{-1*}\alpha}. \tag{2.3.14}$$

involving the domain walls with automorphism subgroups $G^{(\phi)} = \{(\phi \cdot g, g) : g \in G\} \leq G \times G$ with $\phi \in \mathrm{Aut}(G)$.

We start with the first which we call the *automorphism fusion rule*. Consider the cellular decomposition of $\Sigma \times [0,1]$ illustrated in Fig. 2.3. From the gauge transformation with image $\vec{h}(v_i) = (\phi \cdot g_i^{-1}, g_i^{-1}) \in G^{(\phi)}$ for all v_i in $\mathcal{D}_{G^{(\phi)}}$ we go from a generic gauge field configuration to one with perpendicular edges equal to the identity. We call this gauging fixing condition *temporal gauge*. By the flatness condition explained and illustrated in Fig. 2.1, the holonomy of the path $\gamma = (v_1, v_2, v_2', v_1', v_1)$ should be trivial, which shows that the group elements on the right and left of each domain wall are equal. Gauge transformations with $\vec{h}(v_i) = \vec{h}(v_i')$ preserve the temporal gauge and correspond to the gauge transformations of $\mathcal{D}_{G^{(\phi\circ\phi')}}$. This shows that performing the path integral with the domain walls $\mathcal{D}_{G^{(\phi)}}$ and $\mathcal{D}_{G^{(\phi')}}$ close together is equivalent to performing the path integral with the domain wall $\mathcal{D}_{G^{(\phi\circ\phi')}}$ instead. See Fig. 2.5 for a summary and illustration.

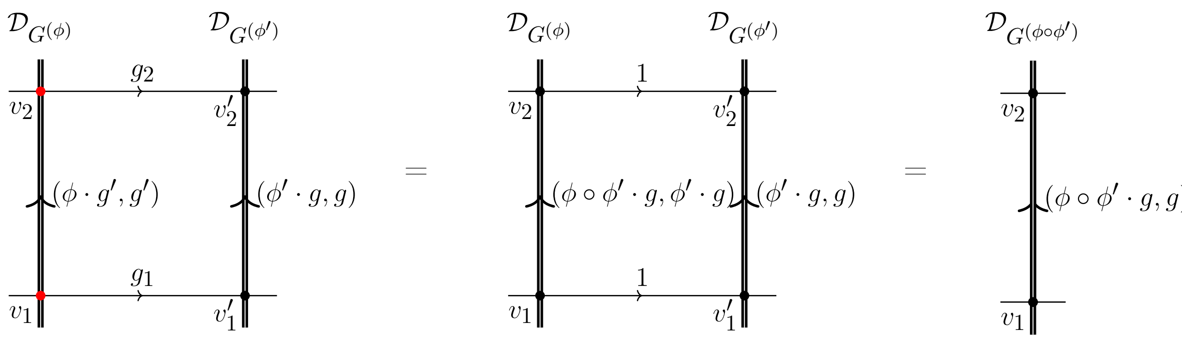


Figure 2.5: Derivation of the automorphism fusion rule (2.3.1). From the gauge transformation with image $\vec{h}(v_i) = (\phi \cdot g_i^{-1}, g_i^{-1}) \in G^{(\phi)}$ for all v_i in $\mathcal{D}_{G^{(\phi)}}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition, the group elements on the right and left of each domain wall are equal. Gauge transformations with $\vec{h}(v_i) = (\phi \cdot \phi' \cdot h, \phi' \cdot h) \in G^{(\phi)}$ and $\vec{h}(v_i') = (\phi' \cdot h, h) \in G^{(\phi')}$ preserve the temporal gauge and make the gauge transformations of $\mathcal{D}_{G^{(\phi\circ\phi')}}$.

The $\mathcal{D}_{G^{(\phi)}}$ defect is associated with an invertible symmetry of discrete gauge theories and fuses according to the automorphism composition. In particular:

$$\mathcal{D}_{G(\phi)} \times \overline{\mathcal{D}}_{G(\phi)} = \mathcal{D}_{G(\phi)} \times \mathcal{D}_{G(\phi^{-1})} = \mathcal{D}_{G(\phi \circ \phi^{-1})} = \mathcal{D}_{G(\mathrm{id})} = 1, \tag{2.3.15}$$

where we used (2.2.1) and (2.3.1). As we are going to see in Section 2.4, the action of the domain wall $\mathcal{D}_{G(\phi)}$ on other operators is insensitive to the action of inner automorphisms (which implement global gauge transformations). If we denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G and by $\mathrm{Inn}(G)$ the subgroup of inner automorphisms, the physical data is the projection of $\phi \in \mathrm{Aut}(G)$ to the quotient $\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ of outer automorphisms. Therefore, generically, the ordinary symmetry of discrete gauge theories contains the subgroup $\mathrm{Out}(G)$.

The fusion rule (2.3.2) and the action on gapped boundary (2.3.9) can be derived following the same method. The derivations are summarized in Fig. 2.6 and Fig. 2.7. The fact that one gets the pullback of $\alpha : K^{D-1} \to G$ by $\phi^{-1} : \phi(K) \to K$ follows from the fact that the topological boundary is evaluated on K group elements, as shown in Fig. 2.7.

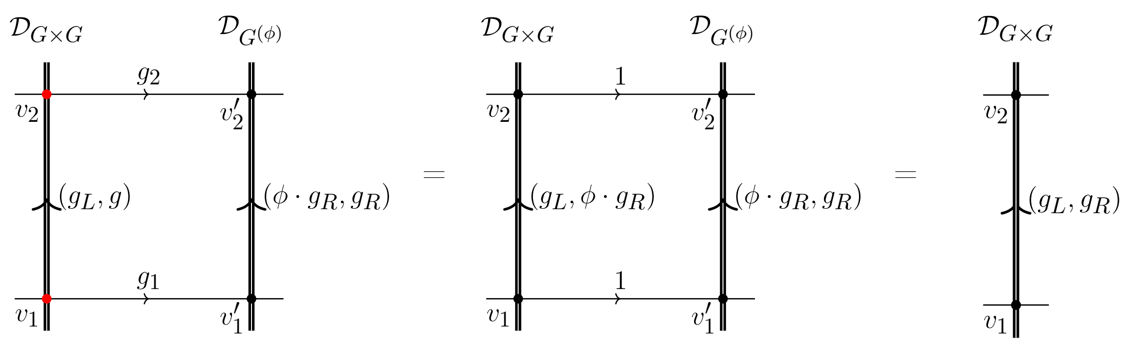


Figure 2.6: Derivation of the fusion rule (2.3.2). From the gauge transformation with image $\vec{h}(v_i) = (1, g_i^{-1}) \in G \times G$ for all v_i in $\mathcal{D}_{G\times G}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition, the group elements on the right and left of each domain wall are equal. Gauge transformations with $\vec{h}(v_i) = (h_L, \phi \cdot h_R) \in G \times G$ and $\vec{h}(v_i') = (\phi \cdot h_R, h_R) \in G^{(\phi)}$ preserve the temporal gauge and make the gauge transformations of the resulting $\mathcal{D}_{G\times G}$.

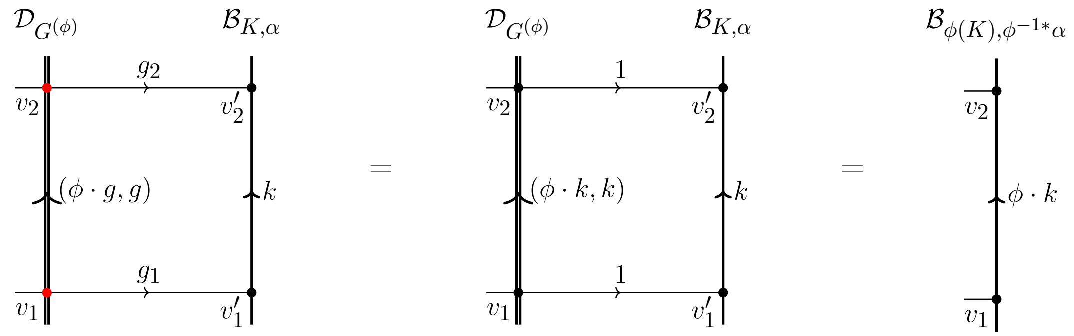


Figure 2.7: Derivation of the action on boundary (2.3.9). From the gauge transformation with image $\vec{h}(v_i) = (\phi \cdot g_i^{-1}, g_i^{-1}) \in G^{(\phi)}$ for all v_i in $\mathcal{D}_{G^{(\phi)}}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition, the group elements on the right of the domain wall and on the boundary are equal. Gauge transformations with $\vec{h}(v_i) = (\phi \cdot h, h) \in G^{(\phi)}$ and $\vec{h}(v'_i) = h \in K$ preserve the temporal gauge and make the gauge transformations of the resulting boundary $\mathcal{B}_{\phi(K),\phi^{-1*}\alpha}$. Note that in the intermediate step the topological action is still evaluated on k group elements, so the fusion outcome has the pullback of α by ϕ^{-1}.

2.3.2 Diagonal domain walls

In this section we derive the fusion rules (2.3.3), (2.3.4) and (2.3.5) and the action on boundaries (2.3.10):

$$\mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{K^{(\mathrm{id})},\alpha} = \mathcal{D}_{K^{(\phi)},\alpha}, \tag{2.3.16}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G^{(\phi)}} = \mathcal{D}_{G^{(\phi)}} \times \mathcal{D}_{\phi^{-1}(K),\phi^*\alpha} = \mathcal{D}_{(\phi^{-1}(K))^{(\phi)},\phi^*\alpha} \tag{2.3.17}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{D}_{K'^{(\mathrm{id})},\alpha'} = \frac{|G|}{|K \cdot K'|}\mathcal{D}_{(K\cap K')^{(\mathrm{id})},\alpha\cdot\alpha'}, \tag{2.3.18}$$

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha} \times \mathcal{B}_{K',\alpha'} = \frac{|G|}{|K \cdot K'|}\mathcal{B}_{K\cap K',\alpha\cdot\alpha'}, \tag{2.3.19}$$

associated to the automorphism subgroup defined in the previous section and the diagonal subgroups $K^{(\mathrm{id})} = \{(k,k) : k \in K\}$ with $K \lhd G$. The topological actions are elements of $\alpha \in H^{D-1}(K, U(1))$, $\alpha' \in H^{D-1}(K', U(1))$ and $\alpha \cdot \alpha'|_{K\cap K'} \in H^{D-1}(K \cap K', U(1))$.

The derivation of the first two fusion rules (2.3.3) and (2.3.4) is similar to what we did in the previous section and is summarized and illustrated in Fig. 2.8 and Fig. 2.9. We remind that we defined the domain wall $\mathcal{D}_{K^{(\phi)},\alpha}$ by having the topological action $\alpha \in H^{D-1}(K, U(1))$ evaluated on the right entry, see Table (2.1). This convention explains the pullback when we invert the order of multiplication.

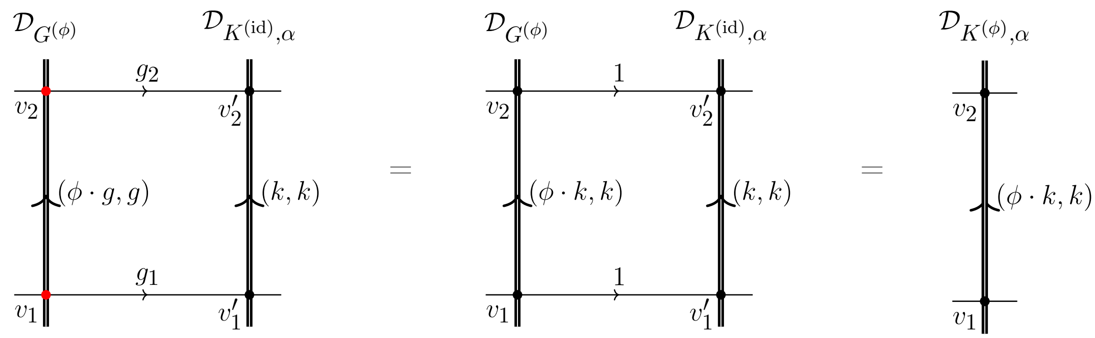


Figure 2.8: Derivation of the fusion rule (2.3.3). From the gauge transformation with parameter $\vec{h}(v_i) = (\phi \cdot g_i^{-1}, g_i^{-1}) \in G^{(\phi)}$ for all v_i in $\mathcal{D}_{G^{(\phi)}}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition, the group elements on the right and left of each domain wall are equal. Gauge transformations with $\vec{h}(v_i) = (\phi \cdot h, h) \in G^{(\phi)}$ and $\vec{h}(v_i') = (h, h) \in K^{(\text{id})}$ preserve the temporal gauge and make the gauge transformations of the resulting $\mathcal{D}_{K^{(\phi)},\alpha}$. The topological action is evaluated on K group elements in all steps so we do not have the pullback of α by ϕ.

The derivation of (2.3.5), which we call the *diagonal fusion rule*, has a novelty. Similarly from what we had in the automorphism derivation we find that the gauge field configuration in $\Sigma \times 0$ will give non-zero result only if the same gauge field configuration appears on $\Sigma \times 1$. This can happen just if the group elements belongs to $\in K \cap K'$. In this case, however, one needs to add the fusion coefficient $|G|/|K \cdot K'|$ that comes from summing the degrees of freedom that cannot be removed from gauge transformations. As a simple example, in the setup we are working in, we can derive the prefactor for the particular case $\mathcal{D}_1 \times \mathcal{D}_1 = |G|\mathcal{D}_1$. By using the same cellular decomposition illustrated in Fig. 2.3, we see that we cannot go to the temporal gauge because the gauge transformations of $\mathcal{D}_1$ are also restricted to the identity subgroup. But by the flatness condition, the group elements on the perpendicular edges should be equal. By summing over all these configurations we get the factor of $|G|$. See Fig. 2.10 for illustration.

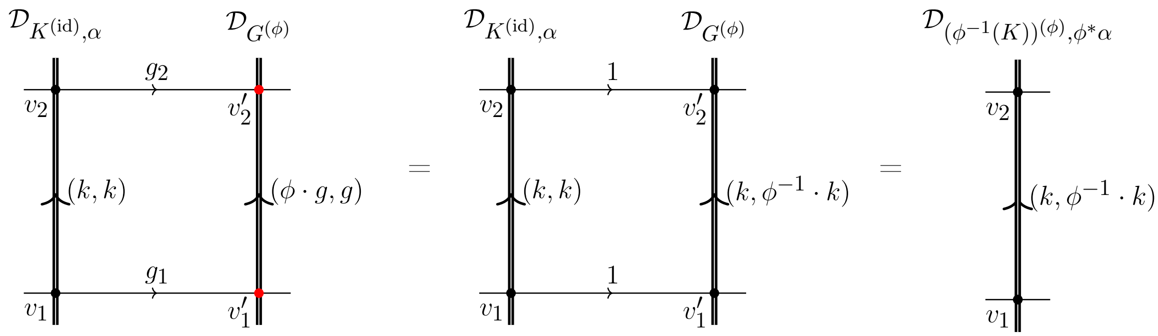


Figure 2.9: Derivation of the fusion rule (2.3.4). From the gauge transformation with parameter $\vec{h}(v_i') = (g_i, \phi^{-1}\cdot g_i) \in G^{(\phi)}$ for all v_i in $\mathcal{D}_{G^{(\phi)}}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition, the group elements on the right and left of each domain wall are equal. Gauge transformations with $\vec{h}(v_i') = (h, \phi^{-1}\cdot h) \in G^{(\phi)}$ and $\vec{h}(v_i) = (h,h) \in K^{(\mathrm{id})}$ preserve the temporal gauge and make the gauge transformations of the resulting $\mathcal{D}_{(\phi^{-1}(K))^{(\phi)},\phi^*\alpha}$. The topological action is evaluated on K group elements in the intermediate step so we need the pullback of α by $\phi : \phi^{-1}(K) \to K$.

The general case can be argued similarly, the difference is that any element g in the middle that can be written in the form kk' with $k \in K$ and $k' \in K'$ can be trivialized by gauge transformations that are not part of the $K \cap K'$ gauge transformations. This means, that the pre-factor is $|G|/|K\cdot K'|$ instead of $|G|$. Note that the fusion rule is consistent with the fact that $\mathcal{D}_{G^{(\mathrm{id})}} = 1$. Note that the prefactor is an integer because for normal subgroups $K\cdot K'$ is also a subgroup and by Lagrange's theorem it divides the order of G.

If the two domain walls $\mathcal{D}_{K^{(\mathrm{id})}}, \mathcal{D}_{K'^{(\mathrm{id})}}$ are decorated with additional topological term $\alpha \in H^{D-1}(K, U(1)), \alpha' \in H^{D-1}(K', U(1))$, their fusion result has decoration given by attaching $\alpha\cdot\alpha'|_{K\cap K'} \in H^{D-1}(K\cap K', U(1))$:

$$\mathcal{D}_{K^{(\mathrm{id})},\alpha}(\Sigma) \times \mathcal{D}_{K'^{(\mathrm{id})},\alpha'}(\Sigma) = \frac{|G|}{|K\cdot K'|}\mathcal{D}_{(K\cap K')^{(\mathrm{id})},\alpha\cdot\alpha'}(\Sigma). \tag{2.3.20}$$

This is more easily seen by thinking of $\mathcal{D}_{K^{(\mathrm{id})},\alpha}(\Sigma)$ as the product of $\mathcal{D}_{K^{(\mathrm{id})}}(\Sigma)$ and the invertible electric defect $W_\alpha(\Sigma)$ defined in (1.3.13). The action of $\mathcal{D}_{K^{(\mathrm{id})},\alpha}$ on gapped

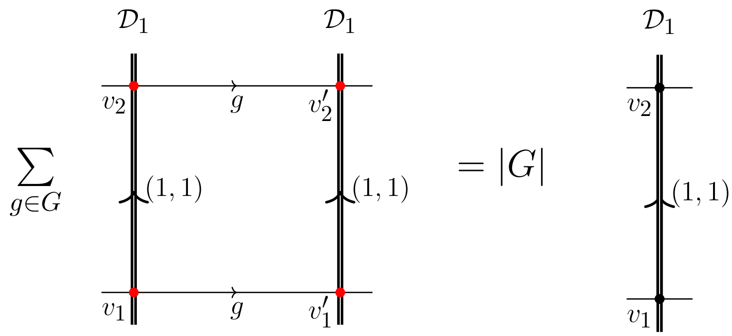


Figure 2.10: Derivation of the fusion rule $\mathcal{D}_1 \times \mathcal{D}_1 = |G|\mathcal{D}_1$. The red dots highlights the source of gauge transformations that are elements of $1 \leq G \times G$ and cannot change the holonomy along the perpendicular edge. By the flatness condition, the group elements on the perpendicular edges are equal. Performing the summation of the middle horizontal edge gives rise to the $|G|$ factor.

boundaries (2.3.10) can be derived in a similar way.

Let us make a few remarks:

- By the lattice definition $\mathcal{D}_{G(\mathrm{id})} = 1$ is the identity defect and $\mathcal{D}_{G(\mathrm{id}),\alpha}(\Sigma)$ is obtained by decorating Σ with the topological action $\alpha \in H^{D-1}(G, U(1))$, which is the codimension-one invertible electric defect defined in (1.3.13). One can see that the fusion of these defects is the group multiplication in $H^{D-1}(G, U(1))$. This is consistent with the property that fusing such domain walls is the same as first stacking the SPT phases with G symmetry labelled by α, α' on the wall and then gauging the G symmetry [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi]. Generically, the full ordinary symmetry of discrete gauge theory is generated by $\mathcal{D}_{G(\phi)}$ and $\mathcal{D}_{G(\mathrm{id}),\alpha}$ which form the group:

$$\mathrm{Out}(G) \ltimes H^{D-1}(G, U(1)). \tag{2.3.21}$$

  In $D = 3$, Wilson lines and magnetic defects are both lines, so there exist other ordinary symmetries that consistently permute the Wilson lines and magnetic defects. Electric-magnetic duality symmetry in $\mathbb{Z}_2$ is such an example. It corresponds to the domain wall associated with the subgroup $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \leq \mathbb{Z}_2 \times \mathbb{Z}_2$ and the non-factorized topological action $\alpha_2 \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$.

- The fusion of diagonal domain walls is commutative. In Section 2.5, we will generalize them to higher-codimension topological defects where the fusion rules must be commutative [Gaiotto et al.(2015a)Gaiotto, Kapustin, Seiberg, and Willett]. This is to be contrasted with the factorized and automorphism domain walls, which obey a non-commutative algebra, and therefore cannot be generalized to higher-codimension.

- As we are going to see in Section 2.6, the fusion coefficient can be interpreted as the partition function of Stueckelberg scalars, which is equal to the 0-form partition function of untwisted $G/K \cdot K'$ gauge theory, (1.3.9). In particular, the coefficient is sensitive to the topology of Σ, more precisely, to $\pi_0(\Sigma)$, which here we have assumed to be 1.

2.3.3 Factorized domain walls

In this section we first derive the elementary fusion rules (2.3.6) and (2.3.7), and the action on boundaries (2.3.11):

$$\mathcal{D}_{K_L^{(\mathrm{id})},\alpha_L} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R^{(\mathrm{id})},\alpha_R} = \mathcal{D}_{K_L\times K_R,\alpha_L\times\alpha_R}, \tag{2.3.22}$$

$$\mathcal{D}_{G\times G} \times \mathcal{D}_{K_R^{(\mathrm{id})},\alpha} \times \mathcal{D}_{G\times G} = \mathcal{Z}(K,\alpha)\mathcal{D}_{G\times G}, \tag{2.3.23}$$

$$\mathcal{D}_{G\times G} \times \mathcal{B}_{K,\alpha} = \mathcal{Z}(K,\alpha)\mathcal{B}_G, \tag{2.3.24}$$

with $\mathcal{Z}(K,\alpha)$ the partition function of K gauge theory twisted by $\alpha \in H^{D-1}(K, U(1))$ on Σ. Then, we use these results to derive (2.3.8).

The factorized domain walls are generated by the diagonal domain walls and $\mathcal{D}_{G\times G}$ from the first equation, (2.3.6). To derive this fusion we proceed similarly to what we did before by using the cellular decomposition of Fig. 2.3. First, we perform gauge transformations on the vertices of $\mathcal{D}_{G\times G}$ to get to temporal gauge; then we use the flatness condition to see that the elements on the middle layer are $(k_L, k_R) \in K_L \times K_R$; and finally we note that the remaining gauge transformations are elements of $K_L \times K_R$. The topological actions are carried along the way. See Fig. 2.11 for an illustration.

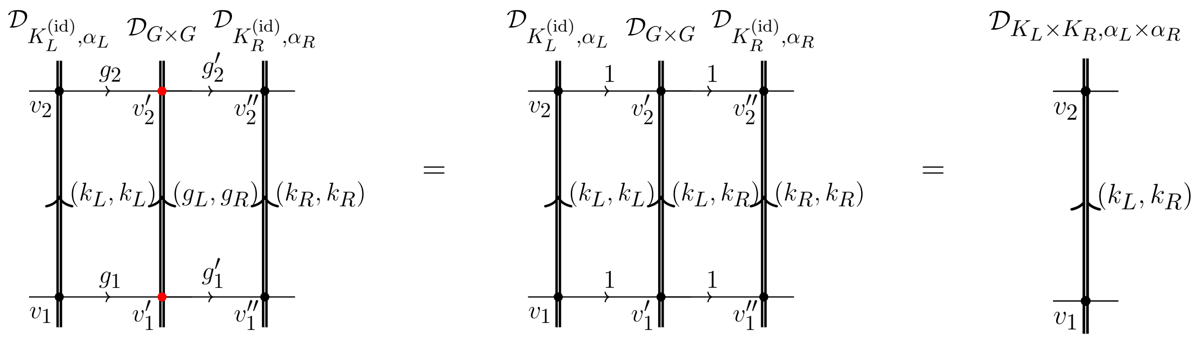


Figure 2.11: Derivation of the factorized fusion rule (2.3.6). The first two figures are related by a gauge transformation with $\vec{h}(v_i') = (g_i, g_i'^{-1}) \in G\times G$ for all v_i' on $\mathcal{D}_{G\times G}$. By the flatness condition the vertical edge on the middle has group elements (k_L, k_R). Gauge transformations with $\vec{h}(v_i) = (h_L, h_L) \in K_L^{(\mathrm{id})}$, $\vec{h}(v_i') = (h_L, h_R) \in G \times G$ and $\vec{h}(v_i'') = (h_R, h_R) \in K_R^{(\mathrm{id})}$ preserve the temporal gauge and make the gauge transformations of $\mathcal{D}_{K_L\times K_R}$. The topological actions are carried along the way.

Now, we derive the second fusion rule, (2.3.7). The result follows from noticing that the $\mathcal{D}_{G\times G}$ domain wall cuts the manifold into two disconnected and independent components. Therefore, one can perform the partition summation for the disconnected part in the middle. This gives the $\mathcal{Z}(K, \Sigma, \alpha)$ partition function because the middle layer is topologically $\Sigma \times [0, 1]$ and retracts to Σ with gauge group elements in K and a topological action $\alpha \in H^{D-1}(K, U(1))$. Conversely, we could follow the same steps as before which are summarized in Fig. 2.12.

The action of $\mathcal{D}_{G\times G}$ on gapped boundaries (2.3.11) can be derived following the same method. The derivation is summarized in Fig. 2.13.

The fusion of factorized domain walls (2.3.8) can be obtained from (2.3.5), (2.3.6) and

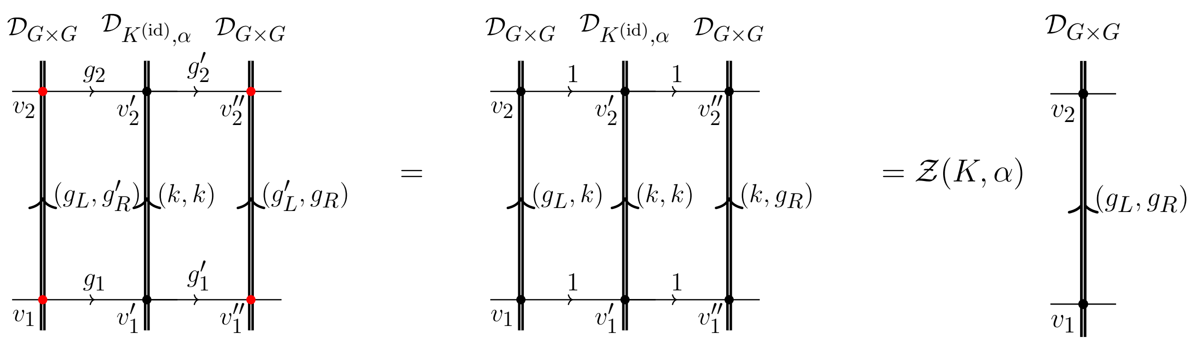


Figure 2.12: Derivation of the factorized fusion rule (2.3.7). The first two figures are related by a gauge transformation with $\vec{h}(v_i) = (1, g_i^{-1}) \in G \times G$ and $\vec{h}(v_i'') = (g_2', 1) \in G \times G$ for all vertices v_i and v_i'' in the two $\mathcal{D}_{G\times G}$. Gauge transformations with $\vec{h}(v_i) = (1, h) \in G \times G$, $\vec{h}(v_i') = (h, h) \in K^{(\mathrm{id})}$ and $\vec{h}(v_i'') = (h, 1) \in G \times G$ make the gauge transformations of the decoupled partition function obtained by summing over k. Gauge transformations with $\vec{h}(v_i) = (h_L, 1) \in G \times G$, $\vec{h}(v_i'') = (1, h_R) \in G \times G$ preserve the temporal gauge and make the gauge transformations of the resulting $\mathcal{D}_{G\times G}$.

(2.3.7) using associativity. For the case with trivial topological action, we have:

$$
\begin{aligned}
\mathcal{D}_{K_L\times K_R} \times \mathcal{D}_{K_L'\times K_R'} &= \mathcal{D}_{K_L^{(\mathrm{id})}} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R^{(\mathrm{id})}} \times \mathcal{D}_{K_L'^{(\mathrm{id})}} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R'^{(\mathrm{id})}}, \\
&= \frac{|G|}{|K_L \cdot K_R|} \mathcal{D}_{K_L^{(\mathrm{id})}} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{(K_R\cap K_L')^{(\mathrm{id})}} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R'^{(\mathrm{id})}}, \\
&= \frac{|G|}{|K_R \cdot K_L'|} \mathcal{Z}(K_R \cap K_L') \mathcal{D}_{K_L^{(\mathrm{id})}} \times \mathcal{D}_{G\times G} \times \mathcal{D}_{K_R'^{(\mathrm{id})}}, \\
&= \frac{|G|}{|K_R \cdot K_L'|} \mathcal{Z}(K_R \cap K_L') \mathcal{D}_{K_L\times K_R'}.
\end{aligned}
\tag{2.3.25}
$$

The derivation for the case with factorized topological action is analogous. We remark that the trivial subgroup domain wall is an example of both factorized and diagonal domain walls. One can easily check that (2.3.8) is equal to (2.3.5) when $K_R = K_L = K_R' = K_L' = 1$.

2.3.4 Twisted domain wall: electric-magnetic duality

To close this section we will show how to compute the fusion and action on gapped boundaries of twisted domain walls $\mathcal{D}_{G\times G,\alpha}$ with $\alpha \in H^{D-1}(G\times G, U(1))$ a non-trivial and non-factorized

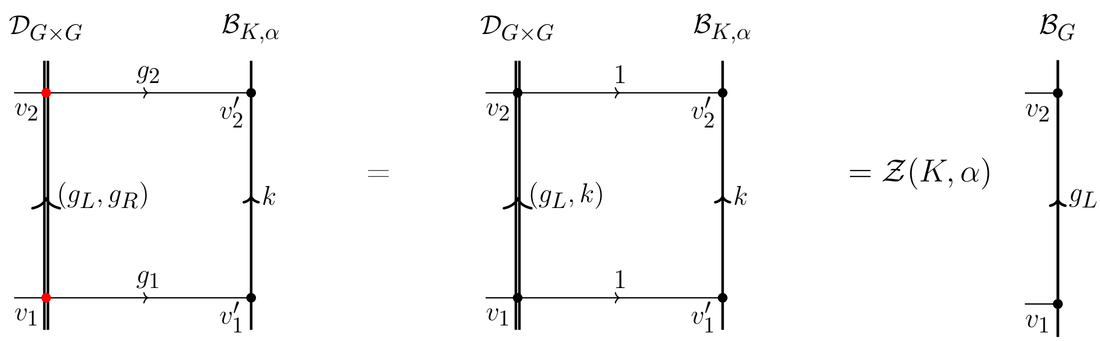


Figure 2.13: Derivation of the action on boundary (2.3.9). From the gauge transformation with image $\vec{h}(v_i) = (1, g_i^{-1}) \in G \times G$ for all v_i in $\mathcal{D}_{G\times G}$ we go from a generic gauge field configuration to temporal gauge in the second figure. By the flatness condition the group elements on the right of the domain wall and on the boundary are equal. Gauge transformations with $\vec{h}(v_i) = (1, h) \in G \times G$ and $\vec{h}(v_i') = h \in K$ preserve the temporal gauge and make the gauge transformations of the decoupled partition function $\mathcal{Z}(K,\alpha)$. Gauge transformations with $\vec{h}(v_i) = (h, 1) \in G \times G$ make the gauge transformations of the resulting boundary $\mathcal{B}_G$.

local action. The result depends on the dimension and gauge group.

The derivation of the fusion $\mathcal{D}_{G\times G,\alpha} \times \mathcal{D}_{G\times G,\alpha'}$ and the action $\mathcal{D}_{G\times G,\alpha} \times \mathcal{B}_{K,\beta}$ is very similar to the derivation of $\mathcal{D}_{G\times G} \times \mathcal{D}_{G\times G} = \mathcal{Z}(G)\mathcal{D}_{G\times G}$ and $\mathcal{D}_{G\times G} \times \mathcal{B}_{K,\beta} = \mathcal{Z}(K,\beta)\mathcal{B}_G$ summarized in Fig. 2.12 and Fig. 2.13 respectively. We start with the cellular decomposition of Fig. 2.3 and Fig. 2.4, we perform a gauge transformation to get to temporal gauge and by the flatness condition, the elements on the right and left are equal. Now, however, the summation over the intermediate gauge group elements does not give a partition function because the topological action is partially evaluated on them. The answer depends on the explicit expression for the cocycles which depends on the dimension and the gauge group.

Let us carry this procedure in detail for $G = \mathbb{Z}_2$ in $D = 3$. Because $H^2(\mathbb{Z}_2, U(1)) = 1$, this theory has just two gapped boundaries $\mathcal{B}_{\mathbb{Z}_2}$ (Neumann) and $\mathcal{B}_1$ (Dirichlet). However, $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ so we can consider a domain wall twisted by a non-factorized action.

We are going to show that

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\times\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}=1, \tag{2.3.26}$$

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\times\mathcal{B}_{\mathbb{Z}_2}=\mathcal{B}_1, \tag{2.3.27}$$

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\times\mathcal{B}_1=\mathcal{B}_{\mathbb{Z}_2}, \tag{2.3.28}$$

where α_2 the non-trivial 2-cocycle of $H^2(\mathbb{Z}_2\times\mathbb{Z}_2,U(1))=\mathbb{Z}_2$.

Let us start with the derivation of the fusion (2.3.26), illustrated in Fig. 2.14. We need

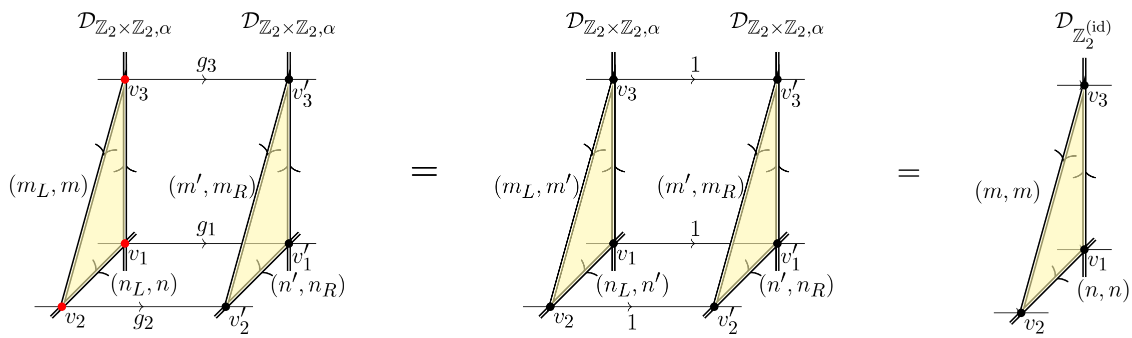


Figure 2.14: Derivation of the fusion rule (2.3.26). The first two figures are related by a gauge transformation with $\vec{h}(v_i)=(1,g_i^{-1})\in\mathbb{Z}_2\times\mathbb{Z}_2$ for all vertices v_i in $\Sigma\times 0$. By the flatness condition the holonomy of (v_1,v_2,v_2',v_1',v_1) and (v_2,v_3,v_3',v_2',v_2) should be trivial, which shows that the outer vertical edges have group elements (m_L,m'), (m',m_R), (n_L,n') and (n',n_R). Summing over n' and m' with the two topological actions forces $m_L=m_R=m$ and $n_L=n_R=n$, see (2.3.30), which corresponds to the $\mathcal{D}_{\mathbb{Z}_2^{(\mathrm{id})}}=1$ domain wall.

to show that the summation over the middle group elements (that would lead to $\mathcal{Z}(\mathbb{Z}_2)$ in the trivial α case) gives a delta function $\delta(m_L-m_R)\delta(n_L-n_R)$. To show that we need the explicit form for the algebraic cocycle $\alpha\in H^2(\mathbb{Z}_2\times\mathbb{Z}_2,U(1))$ which has representative:

$$\alpha(n_L,n_R,m_L,m_R)=e^{\pi i n_L m_R}. \tag{2.3.29}$$

For simplicity, let the surface in consideration be a torus. The product of local actions over 2-simplices for a torus with $\alpha\in H^2(G,U(1))$ is equal to $\frac{\alpha(g,h)}{\alpha(h,g)}$ for $g,h\in G$. Therefore, in the configuration of Fig. 2.14 the total action is

$$\frac{\alpha(n_L, n', m_L, m')\alpha(n', n_R, m', m_R)}{\alpha(m_L, m', n_L, n')\alpha(m', m_R, n', n_R)} = e^{\pi i(n_L m' - m_L n' + n' m_R - m' n_R)}. \tag{2.3.30}$$

where we used (2.3.29). If we sum over n' and m', then we will get $\delta(m_L - m_R)\delta(n_L - n_R)$ that forces $n_L = n_R = n$ and $m_L = m_R = m$. We see that the fusion results in $\mathcal{D}_{\mathbb{Z}_2^{(\mathrm{id})}} = 1$.

The derivation of the action on the gapped boundaries is very similar and is illustrated in Fig. 2.15 for the $\mathcal{B}_{\mathbb{Z}_2}$ case. We need to show that the summation over the middle group

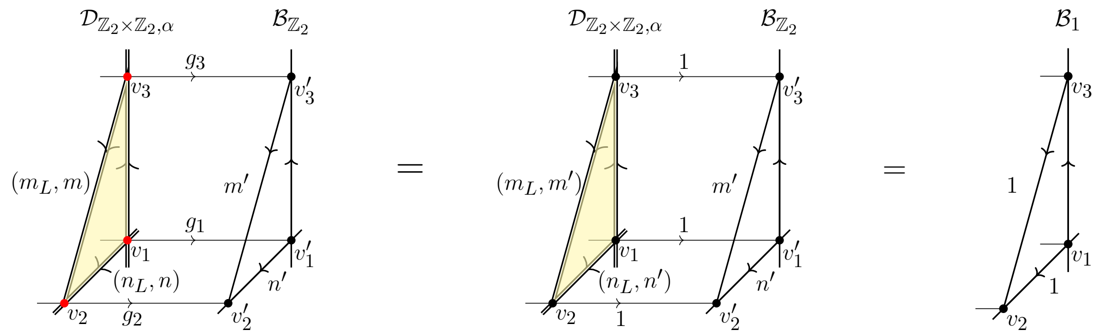


Figure 2.15: Derivation of the action on boundary (2.3.27). The first two figures are related by a gauge transformation with $\vec{h}(v_i) = (1, g_i^{-1}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ for all vertices v_i in $\Sigma \times 0$. By the flatness condition the holonomy of $(v_1, v_2, v_2', v_1', v_1)$ and $(v_2, v_3, v_3', v_2', v_2)$ should be trivial, which shows that the outer vertical edges have group elements (m_L, m), m, (n_L, n) and n. Summing over n' and m' with the topological actions forces $m_L = n_L = 1$, see (2.3.31), which corresponds to the $\mathcal{B}_1$ gapped boundary.

elements gives the delta function $\delta(m_L)\delta(n_L)$. Using again the total action for the torus we have in this case:

$$\frac{\alpha(n_L, n', m_L, m')}{\alpha(m_L, m', n_L, n')} = e^{\pi i(n_L m' - m_L n')}, \tag{2.3.31}$$

which gives $\delta(m_L)\delta(n_L)$ by summing over n' and m'. We see the action results in $\mathcal{B}_1$. The action of the domain wall on $\mathcal{B}_1$ can be derived in the same way. In this case however, the gauge group elements on the boundary are trivial which trivializes the topological action (2.3.29) so the result is the same as the action $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2} \times \mathcal{B}_1 = \mathcal{B}_{\mathbb{Z}_2}$. Consistently, we could compute the action on $\mathcal{B}_1$ using the fusion (2.3.26), the action (2.3.27) and associativity which gives:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{B}_1 = \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \times \mathcal{B}_{\mathbb{Z}_2} = \mathcal{B}_{\mathbb{Z}_2}. \tag{2.3.32}$$

The only invertible symmetry of $\mathbb{Z}_2$ gauge theory is electric-magnetic duality, therefore $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha}$ is the electric-magnetic duality symmetry defect. As we are going to show in Section 2.4.3, operators with the form $\mathcal{D}_{G\times G,\alpha}$ with α non-factorized, generically mix invertible electric operators (1.3.13) and magnetic defects. Note that the same procedure could be carried out in higher dimensions and for generic gauge groups.

2.4 Transformation of Wilson lines and magnetic defects

When we move a topological defect across another operator, the operator can be transformed into another one. In this section, we study the transformation of other operators when they cross the gapped domain walls.

In the following, we will investigate the transformation of Wilson lines and magnetic defects. We start discussing which operators can end on untwisted gapped boundaries $\mathcal{B}_K$ of G gauge theory. Then, we investigate the action of domain walls on other operators separating the discussion into: untwisted domain walls, $\mathcal{D}_H$, and twisted domain walls $\mathcal{D}_{H,\alpha}$ with $\alpha \in H^{D-1}(H, U(1))$ a non-factorized action. In the first class of domain walls, the electric and magnetic operators transform separately, while the second class of domain walls can mix them, and in this sense generalizes electric-magnetic duality. Similar distinctions are discussed in [Hsin et al.(2023)Hsin, Luo, and Malladi].

In our setup, the transformation is derived from the fact that pairs of Wilson lines and magnetic defects can be made gauge invariant if they end on the same locus of the domain wall. We exhibit the collection of all such allowed junctions via the action of the domain wall on simple Wilson lines and magnetic defects:

$$\mathcal{D}_{K,\alpha} \cdot W_{\rho_R} = \sum_{i_L \in \text{irrep}} c_{i_L} W_{\rho_{i_L}}, \qquad \mathcal{D}_{K,\alpha} \cdot M_{g_R} = \sum_{[g_L] \in \text{Cl}(G)} c_{g_L} M_{g_L}, \tag{2.4.1}$$

with c_{i_L} and c_{g_L} larger than zero if there exists a gauge invariant configuration.

2.4.1 Untwisted gapped boundaries

As shown in Section 2.2.1, gapped boundaries in untwisted G gauge theory can be constructed by (K, α) with $K \leq G$ and topological action $\alpha \in H^{D-1}(K, U(1))$. Here we restrict to the case with trivial topological action. The gapped boundary $\mathcal{B}_K$ associated with the subgroup $K \leq G$ is obtained by restricting the gauge group elements on the boundary to be elements of K. From this definition, it follows that:

- **Wilson lines** the Wilson line W_ρ can end on the gapped boundary $\mathcal{B}_K$ if the decomposition of ρ in representations of K contains the singlet. The number of possible junctions is equal to the multiplicity of the trivial representation in the restriction.

- **Magnetic defects** the magnetic defect M_g can end on the domain wall if there exists a group element $k \in [g]$ such that $k \in K$. The number of possible junctions is equal to the number of different conjugacy classes of K of such group elements.

  For illustration, consider $G = S_3$ (the symmetric group on 3 elements), the conjugacy class $[(123)]_{S_3} = \{(123), (132)\}$, and the gapped boundary $\mathcal{B}_{A_3}$ with $A_3 = \{1, (123), (132)\} \leq S_3$ (the alternating group on 3 elements). The magnetic defect $M_{(123)}$ can end on $\mathcal{B}_{A_3}$ because $(123), (132) \in A_3$ and the multiplicity in this case is 2 because $[(123)]_{A_3} = \{(123)\}$ and $[(132)]_{A_3} = \{(132)\}$ are different conjugacy classes of A_3.

In particular, all Wilson lines can end on $\mathcal{B}_1$ with multiplicity equal to the dimension of the corresponding irreducible representation, and all magnetic defects can end on $\mathcal{B}_G$ with multiplicity equal to one.

2.4.2 Untwisted domain walls

Let us begin with the untwisted domain walls $\mathcal{D}_H$. For such domain walls, the Wilson lines and the magnetic defects transform separately:

- **Wilson lines** the Wilson line W_{ρ_L} can be transformed into W_{ρ_R} if the decomposition of $\rho_L \otimes \overline{\rho}_R$ into representations of H contains the trivial representation. If the decomposition of $\rho_L \otimes \overline{\rho}_R$ into representations of H contains the trivial representation, then a configuration with W_{ρ_L} and W_{ρ_R} joining on $\mathcal{D}_H$ can be made gauge invariant. Again, the coefficient of the trivial representation gives the number of possible junctions.

- **Magnetic defects** the magnetic defect M_{g_L} can be transformed into M_{g_R} if there exists a group element $(k_L, k_R) \in [g_L] \times [g_R]$ such that $(k_L, k_R) \in H$. The number of possible junctions equals the number of different conjugacy classes of H that such group elements form.

Let us give some examples to illustrate these transformations.

Example: automorphism domain wall For instance, consider the domain wall that generates automorphism ϕ, it corresponds to the subgroup $G^{(\phi)} = \{(\phi \cdot g, g) : g \in G\} \leq G \times G$. Let us consider the transformation of the magnetic defect M_{g_L} into M_{g_R} and the transformation of Wilson lines W_{ρ_L} into W_{ρ_R}.

- The pair of magnetic defects M_{g_L}, M_{g_R} can have a junction on the wall if and only if $\phi(g_L) = g_R$, where we extend the action of automorphism on the conjugacy class. In other words, the magnetic defect associated with the conjugacy class of g transforms into $\phi^{-1}(g)$ after it passes through the domain wall from left to right. Conversely, it transforms into $\phi(g)$ after it passes through the domain wall from right to left. The multiplicity in this case is one.

- The decomposition of the representation $\rho_L \otimes \overline{\rho}_R$ of $G \times G$ into representations of the subgroup $G^{(\phi)}$ contains the trivial representation when $\rho_L = \rho_R \circ \phi^{-1}$, where $\rho_R \circ \phi^{-1}$ is the representation satisfies $\rho_R \circ \phi^{-1}(g) = \rho_R(\phi^{-1} \cdot g)$ for $g \in G$. In other words, the Wilson line associated with the representation ρ transforms into $\rho \circ \phi$ after it passes through the domain wall from left to right. Conversely, it transforms into $\rho \circ \phi^{-1}$ after it passes through the domain wall from right to left.

Using the notation defined in (2.4.1) the above can be summarized as

$$\mathcal{D}_{G^{(\phi)}} \cdot W_{\rho_i} = W_{\rho_i \circ \phi^{-1}}, \tag{2.4.2}$$

$$\mathcal{D}_{G^{(\phi)}} \cdot M_g = M_{\phi(g)}, \tag{2.4.3}$$

for every irreducible representation ρ_i and conjugacy class $[g]$. These transformation properties are consistent with the fusion of automorphism domain walls (2.3.1). We remark that the transformation preserves the Aharonov-Bohm braiding between the Wilson lines and the magnetic defect, which is given by evaluating the character of the representation of the Wilson line on the conjugacy class of the magnetic defect. See Fig. 2.16 for an illustration.

Example: diagonal domain wall with $H = 1$ Let us consider the domain wall with $H = 1$. In this case, every Wilson line can end on the domain wall. In particular, the trivial Wilson line transforms into the direct sum of all other Wilson lines. An example is $G = \mathbb{Z}_2$ in $D = 3$, where the domain wall is called the Cheshire string domain wall [Else and Nayak(2017)]. No configuration of nontrivial magnetic defects can end on the domain wall with $H = 1$. This happens because it is impossible to fix the holonomy on the simplices of Σ to be an element that is not the identity. Using the notation defined in (2.4.1) the above can

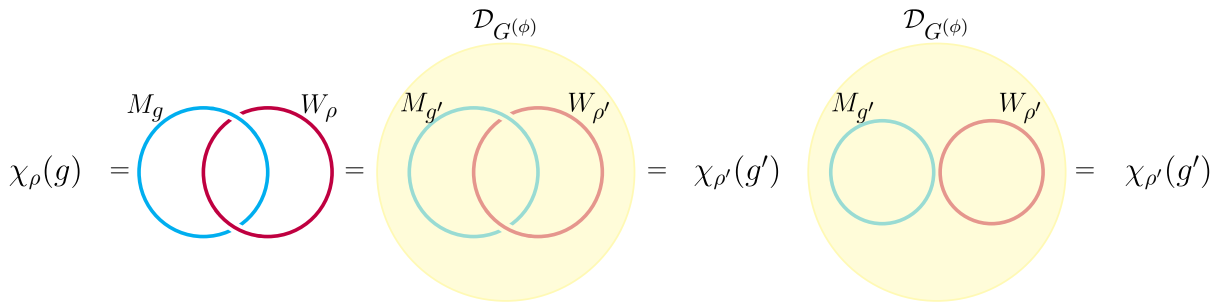


Figure 2.16: A linked configuration of Wilson line, W_ρ and magnetic defect M_g, can be unlinked and contracted giving the associated character $\chi_\rho(g)$. If one nucleates an ordinary symmetry defect $\mathcal{D}_{G(\phi)}$ (oriented outwards) and embed the linked pair inside it, they will be permuted to $W_{\rho'}$ and $M_{g'}$ (where we assumed that the symmetry is invertible). Unlinking the permuted pair inside the symmetry defect and contracting all operators gives the permuted character $\chi_{\rho'}(g')$. These topological manipulations shows that if $W_\rho \mapsto W_{\rho\cdot\phi^{-1}}$, then $M_g \mapsto M_{\phi(g)}$.

be summarized as

$$\mathcal{D}_1 \cdot W_{\rho_i} = \sum_{k\in\text{irreps}} d_i d_k W_{\rho_k}, \tag{2.4.4}$$

$$\mathcal{D}_1 \cdot M_g = 0, \tag{2.4.5}$$

for every irreducible representation ρ_i and conjugacy class $[g]$. Above, d_i is the dimension of the irreducible representation ρ_i. These transformation properties are consistent with the fusion of diagonal domain walls (2.3.5).

Example: factorized domain wall with $H = G \times G$ This case is complementary to the previous example. No Wilson line can end and all magnetic defects can. Using the notation defined in (2.4.1) the above can be summarized as

$$\mathcal{D}_{G\times G} \cdot W_{\rho_i} = 0, \tag{2.4.6}$$

$$\mathcal{D}_{G\times G} \cdot M_g = \sum_{[k]\in\text{Cl}(G)} M_k. \tag{2.4.7}$$

for every irreducible representations ρ_i and conjugacy class $[g]$. These transformation properties are consistent with the fusion of factorized domain walls (2.3.7).

2.4.3 Twisted domain wall: electric-magnetic duality

In this section we show how to understand the action of $\mathcal{D}_{G\times G,\alpha}$ on other operators with $\alpha \in H^{D-1}(G \times G, U(1))$ a non-factorized topological action, i.e., $\alpha \neq \alpha_L \times \alpha_R$ with $\alpha_L, \alpha_R \in H^{D-1}(G, U(1))$. The underlying mechanism determining the action is reminiscent of anomaly inflow [Callan and Harvey(1985)].

As shown in (2.4.7), all magnetic defects can terminate on $\mathcal{D}_{G\times G}$. However, if the domain wall is decorated with a non-factorized topological action $\alpha \in H^{D-1}(G \times G, U(1))$, such junctions are no longer gauge invariant. Conversely, an invertible electric operator (1.3.13) cannot end on $\mathcal{D}_{G\times G}$ because it is not invariant under gauge transformations. Interestingly, the two effects can cancel and a configuration with an invertible electric operator ending on the same locus as a magnetic defect can be made gauge invariant.

Let us illustrate this generic feature using again the simplest example of $G = \mathbb{Z}_2$ gauge theory in $D = 3$. With no other insertions, the Wilson line W coming from the left of $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ cannot end on a vertex v_0 of the domain wall because under a gauge transformation with parameter $\vec{h}(v_0) = (1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ the Wilson line would change to

$$W(\gamma) \to e^{i\pi}\, W(\gamma). \tag{2.4.8}$$

Conversely, consider a local region around v_0 and suppose the magnetic defect M comes from the right and ends on the dual vertex associated with the 2-simplex $[v_0, v_1, v_2]$. That means that we should sum over gauge field configurations with $g_{(v_0,v_1,v_2)} = (0, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. However, with no other insertions, in the presence of the topological action this configuration is not gauge invariant. Under the gauge transformation with $\vec{h}(v_0) = (1, 0)$ considered before the total action in this local region will changes to:

$$\alpha(0,0,0,1)\alpha(0,0,0,0)^2 = 1 \rightarrow \alpha(1,0,0,1)\alpha(1,0,0,0)^2 = e^{i\pi}. \tag{2.4.9}$$

where we used the explicit form for the algebraic cocycle (2.3.29). See Fig. 2.17 for illustration.

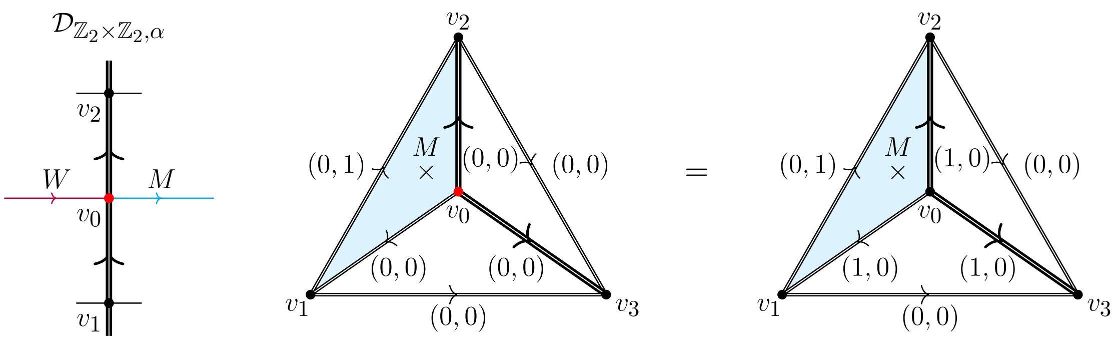


Figure 2.17: The first figure shows a configuration with a magnetic defect M and a Wilson line W ending on the same locus of the domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$. The other two figures shows a local slice of $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ around this locus. Because the magnetic defect M ends on the dual vertex associated with the 2-simplex $[v_0,v_1,v_2]$, we have $g_{(v_0,v_1,v_2)} = (0,1)$. The presence of the magnetic defect makes the total action to changes under gauge transformations with $\vec{h}(v_0) = (1,0) \in \mathbb{Z}_2\times\mathbb{Z}_2$. This change, however, is compensated by the change of the open Wilson line ending on v_0 coming from the left. We see that the mechanism determining the action is reminiscent of anomaly inflow [Callan and Harvey(1985)].

We see that a Wilson line and a magnetic defect cannot end on the domain wall in isolation. However, if the Wilson line ends at a vertex of a simplex dual to the locus of a magnetic defect (with the framing specified by Figure 2.17), their variation under gauge transformations cancels, and the configuration becomes gauge invariant. Using the notation of (2.4.1) we thus derive:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\cdot W = M, \qquad \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\cdot M = W, \tag{2.4.10}$$

confirming that $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ is the electric-magnetic duality symmetry defect.

As already mentioned in Section 2.3.4, this feature, illustrated for $G=\mathbb{Z}_2$ in $D=3$ is a generic feature of domain walls with the form $\mathcal{D}_{G\times G,\alpha}$ and $\alpha \in H^{D-1}(G\times G, U(1))$

non-factorized. Therefore, they generalize electric-magnetic duality to non-abelian groups and to higher dimensions. In higher dimensions, there are gauge invariant configurations with the codimension-two magnetic defect and the codimension-two invertible electric operators defined in (1.3.13) ending on the same codimension-three locus of the domain wall $\mathcal{D}_{G\times G,\alpha}$. For $D=3$ the invertible electric operators of codimension-two are the invertible Wilson lines.

2.5 Higher codimensional topological operators: Cheshire strings

In the definition of the domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$ with $H < G\times G$ and $\alpha \in H^{D-1}(H,U(1))$, a crucial ingredient is the orientation of the normal bundle $N\Sigma$, which gives a consistent global meaning to the "left" and "right" of the domain wall. Note, however, that diagonal domain walls can be constructed without this structure, i.e., diagonal domain walls are orientation-reversal invariant (see Section 2.2.1). Therefore, it is possible to generalize them to higher codimensional operators. From an alternative perspective, note that the holonomy of a contractible path crossing Σ is trivial for a diagonal domain wall (see Fig. 2.1). This feature suggests that these domain walls can be constructed as condensations of Wilson lines [Cordova et al.(2024)Cordova, Costa, and Hsin]. Consequently, it is expected that such domain walls can be generalized to higher codimensional operators through this construction.

The n-dimensional generalization of diagonal domain walls is classified by:

- Subgroup $K \leq G$;
- Topological action $\alpha_n \in H^n(K,U(1))$.

Given the data (K,α_n) we define the n-dimensional operator $\mathcal{D}_{K,\alpha_n}(\Sigma_n)$ by restricting the gauge group elements and gauge transformations along the n-dimensional submanifold Σ_n to be elements of K and by decorating Σ_n with the topological action $\alpha_n \in H^n(K,U(1))$ as in (1.3.13). In $D=3+1$ and $n=2$ these surface defects are discussed in [Else and Nayak(2017)].

The fusion rule of higher codimensional diagonal defects can be computed similarly to (2.3.5) and gives:

$$\mathcal{D}_{K,\alpha_n}(\Sigma_n) \times \mathcal{D}_{K',\alpha'_n}(\Sigma_n) = \frac{|G|}{|K \cdot K'|}\mathcal{D}_{K\cap K',\alpha_n\cdot\alpha'_n}(\Sigma_n). \tag{2.5.1}$$

For $G = \mathbb{Z}_2$, and trivial α_n, the fusion (2.5.1) gives $\mathcal{D}_1 \times \mathcal{D}_1 = 2\mathcal{D}_1$ which is the fusion rule of Cheshire strings [Else and Nayak(2017), Tantivasadakarn and Chen(2024b)]. Note that this fusion rule is universal with respect to spacetime and operator dimension.

We remark that to generalize the domain walls that are not diagonal one would need to introduce a foliation structure, which is ubiquitous in fracton models (see e.g., [Shirley et al.(2019b)Shirley, Slagle, and Chen, Shirley et al.(2019a)Shirley, Slagle, and Chen, Slagle(2021), Hsin and Slagle(2021), Hsin et al.(2024c)Hsin, Stephen, Dua, and Williamson]).

2.6 Examples

In this section, we investigate two examples that have a Lagrangian description: $\mathbb{Z}_N$ gauge theory for N prime in arbitrary spacetime dimensions and $\mathbb{D}_4$ gauge theory in $D = 3$. Using this description, we provide alternative derivations for the lattice results in these specific cases.

2.6.1 $\mathbb{Z}_N$ gauge theory

Consider $\mathbb{Z}_N$ gauge theory with N prime in arbitrary spacetime dimensions. The domain walls that are universal with respect to dimension are: $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$, $\mathcal{D}_{1\times\mathbb{Z}_N}$, $\mathcal{D}_{\mathbb{Z}_N\times 1}$, $\mathcal{D}_1$ and $\mathcal{D}_{\mathbb{Z}_N^{(m)}}$. The last one is the automorphism domain wall associated with the map that sends n to mn with $m \in \mathbb{Z}_N^\times$. Using the fusion rules derived in Section 2.3 we can write the “fusion table" for these domain walls which we present in 2.3.

The $\mathbb{Z}_N$ gauge theory has $N-1$ non-trivial Wilson lines that can be generated by the

Domain walls	$\mathcal{D}_{\mathbb{Z}_N^{(m')}}$	$\mathcal{D}_1$	$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times 1}$
$\mathcal{D}_{\mathbb{Z}_N^{(m)}}$	$\mathcal{D}_{\mathbb{Z}_N^{(mm')}}$	$\mathcal{D}_1$	$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times 1}$
$\mathcal{D}_1$	$\mathcal{D}_1$	$\lvert\mathbb{Z}_N\rvert\mathcal{D}_1$	$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\lvert\mathbb{Z}_N\rvert\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_1$
$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\mathcal{Z}(\mathbb{Z}_N)\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{Z}(\mathbb{Z}_N)\mathcal{D}_{\mathbb{Z}_N\times 1}$
$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_1$	$\mathcal{Z}(\mathbb{Z}_N)\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{Z}(\mathbb{Z}_N)\mathcal{D}_1$
$\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\lvert\mathbb{Z}_N\rvert\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\lvert\mathbb{Z}_N\rvert\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{D}_{\mathbb{Z}_N\times 1}$

Table 2.3: Fusion table for domain walls of $G = \mathbb{Z}_N$ gauge theory with N prime. Above, $\mathbb{Z}_N^{(m)} = \{(mn, n) : n \in \mathbb{Z}_N\} \lhd \mathbb{Z}_N \times \mathbb{Z}_N$ and $m \in \mathbb{Z}_N^\times \cong \mathrm{Aut}(\mathbb{Z}_N)$. Note that the fusion coefficients are decoupled topological quantum field theories on Σ, in particular, $|\mathbb{Z}_N| = \mathcal{Z}^0(\mathbb{Z}_N, \Sigma)$ (see (1.3.9) and recall that we define domain walls on connected submanifolds).

representation that maps $1 \in \mathbb{Z}_N$ to $e^{\frac{2\pi i}{N}}$. We denote the generating Wilson line associated with this representation by W. Similarly, the theory has $N-1$ non-trivial magnetic defects generated by M, which fixes the holonomy to be $1 \in \mathbb{Z}_N$. The transformation properties of these operators can be computed using the methods presented in Section 2.4 and are:

$$\mathcal{D}_1 \cdot M^k = 0 \qquad \mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N} \cdot W^k = 0, \tag{2.6.1}$$

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N} \cdot M^k = \sum_{n=0}^{N-1} M^n, \qquad \mathcal{D}_1 \cdot W^k = \sum_{n=0}^{N-1} W^n, \tag{2.6.2}$$

$$\mathcal{D}_{\mathbb{Z}_N^{(m)}} \cdot M^k = M^{mk}, \qquad \mathcal{D}_{\mathbb{Z}_N^{(m)}} \cdot W^k = W^{\frac{k}{m}}, \tag{2.6.3}$$

where we used the notation of (2.4.1) and $k \in \mathbb{Z}_N$ except in (2.6.1) where $k \in \mathbb{Z}_N^\times$, and the fact that m is invertible. For the other factorized domain walls, the Wilson lines can end on the identity side, and magnetic defects on the $\mathbb{Z}_N$ side.

In addition, there are domain walls specific to each dimension obtained by attaching a topological action for the corresponding subgroup. For example, in $D = 3$ and $N = 2$, we have $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ with α_2 the non-trivial element of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. This domain wall squares to the identity as shown in (2.3.26) and corresponds to the electric-magnetic duality symmetry operator as shown in (2.4.10).

The $\mathbb{Z}_N$ gauge theory in arbitrary dimensions can be described by the BF action using two $U(1)$ gauge fields $a^{(1)}, \tilde{a}^{(D-2)}$ (the superscript denotes the form-degree):

$$S_{\text{BF}} = \frac{iN}{2\pi} \int_{\mathcal{M}} \tilde{a}^{(D-2)} \wedge da^{(1)}, \tag{2.6.4}$$

with gauge transformations $a^{(1)} \to a^{(1)} + d\alpha^{(0)}$ and $\tilde{a}^{(D-2)} \to \tilde{a}^{(D-2)} + d\tilde{\alpha}^{(D-3)}$. In this formulation, the generating Wilson line and magnetic defects are described by:

$$W(\gamma) = \exp\left(i \oint_{\gamma} a^{(1)}\right), \qquad M(\Gamma) = \exp\left(i \oint_{\Gamma} \tilde{a}^{(D-2)}\right), \tag{2.6.5}$$

with γ a closed loop and Γ a closed $(D-2)$-dimensional submanifold. We want to use this Lagrangian description to give an alternative derivation of the results presented above.

Gapped boundaries and boundary conditions

Consider the action (2.6.4) in a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. To define the theory it is necessary to choose boundary conditions. A boundary condition in this setup is a condition for the fields a and $\tilde{a}$ such that there is no surface term in the variation of the action. The bulk equations of motion are standard and imply that all gauge fields are closed. Meanwhile, the boundary term in the variation of the action (2.6.4) is:

$$\delta S_{BF} = \frac{iN}{2\pi} \int_{\partial\mathcal{M}} \tilde{a}^{(D-2)} \wedge \delta a^{(1)}. \tag{2.6.6}$$

There are two boundary conditions for the fields a and $\tilde{a}$ such that $\delta S_{BF} = 0$ which corresponds to the two subgroups of $\mathbb{Z}_N$ with N prime (the trivial and the whole group). The dictionary is summarized in Table 2.4.

Gapped boundary	Boundary condition
$\mathcal{B}_1$	$a = 0$
$\mathcal{B}_{\mathbb{Z}_N}$	$\tilde{a} = 0$

Table 2.4: Dictionary between gapped boundaries and boundary conditions in $\mathbb{Z}_N$ gauge theory.

Domain walls and boundary conditions

To make contact with our lattice construction of domain walls we first divide spacetime into left and right regions $\mathcal{M} = \mathcal{M}_L \cup \mathcal{M}_R$ with a common boundary $\partial\mathcal{M}_L = -\partial\mathcal{M}_R = \Sigma$. The total action in this setup is

$$S_{\rm BF} = \frac{iN}{2\pi}\int_{\mathcal{M}_L} \tilde{a}_L^{(D-2)} \wedge da_L^{(1)} + \frac{iN}{2\pi}\int_{\mathcal{M}_R} \tilde{a}_R^{(D-2)} \wedge da_R^{(1)}, \tag{2.6.7}$$

and one needs to specify boundary conditions on Σ. The bulk equations of motion are standard and imply that all gauge fields are closed. Meanwhile, the boundary term in the variation of the action (2.6.7) is:

$$\delta S_{\rm BF} = \frac{iN}{2\pi}\int_{\Sigma} (\tilde{a}_L^{(D-2)} \wedge \delta a_L^{(1)} - \tilde{a}_R^{(D-2)} \wedge \delta a_R^{(1)}). \tag{2.6.8}$$

There is a correspondence between boundary conditions and domain walls that we summarize in Table 2.5. The boundary conditions in Table 2.5 give the exhaustive set of conditions for

Domain wall	Boundary condition
$\mathcal{D}_{\mathbb{Z}_N^{(m)}}$	$a_L = m a_R$ and $m\tilde{a}_L = \tilde{a}_R$
$\mathcal{D}_1$	$a_L = a_R = 0$
$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\tilde{a}_L = \tilde{a}_R = 0$
$\mathcal{D}_{1\times\mathbb{Z}_N}$	$a_L = \tilde{a}_R = 0$
$\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\tilde{a}_L = a_R = 0$

Table 2.5: Correspondence between domain walls and boundary conditions for $\mathbb{Z}_N$ gauge theory.

the fields such that $\delta S_{\rm BF} = 0$, and correspond to the domain walls that are universal with respect to dimension.

There are other classes of boundary conditions that arise by adding a boundary action term. Because these terms depend on the dimension, these boundary conditions are specific to each dimension and correspond to the twisted domain walls. For simplicity, consider $N=2$, $D=3$ and the boundary action:

$$S_{\alpha_2} = \frac{i}{\pi}\int_\Sigma a_L^{(1)} \wedge a_R^{(1)}. \tag{2.6.9}$$

With this additional boundary interaction term for $a_L^{(1)}$ and $a_R^{(1)}$, the variation of the total action becomes:

$$\delta(S_{\text{BF}} + S_{\alpha_2}) = \frac{i}{\pi}\int_\Sigma [(\tilde{a}_L^{(1)} - a_R^{(1)}) \wedge \delta a_L^{(1)} + (a_L^{(1)} - \tilde{a}_R^{(1)}) \wedge \delta a_R^{(1)}], \tag{2.6.10}$$

enabling the additional boundary condition with $\tilde{a}_L^{(1)} = a_R^{(1)}$ and $a_L^{(1)} = \tilde{a}_R^{(1)}$ on Σ. This boundary condition corresponds to the electric-magnetic duality domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ where α_2 is the non-trivial element of $H^2(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$.

More generally, the domain wall $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N,\alpha_{D-1}}$ associated with an element of $\alpha_{D-1} \in H^{D-1}(\mathbb{Z}_N\times\mathbb{Z}_N, U(1))$ corresponds to a boundary condition that is enabled by adding the corresponding boundary action term constructed from a_L and a_R. The non-factorized terms, such as (2.6.9), give boundary conditions that relate the $a^{(1)}$ and $b^{(D-2)}$ fields on the two sides which implies that they transform electric into magnetic operators and generalize electric-magnetic duality. For example, in $D=4$ we have $H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. The boundary condition that corresponds to the domain wall associated with the non-factorized element of this cohomology group is $b_L^{(2)} = da_R^{(1)}$ and $da_L^{(1)} = b_R^{(2)}$, which is enabled by the boundary action:

$$S_{\alpha_3} = \frac{i}{2\pi}\int_\Sigma a_L^{(1)} \wedge da_R^{(1)}. \tag{2.6.11}$$

As we are going to show, this domain wall is non-invertible. The other two generators of $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ correspond to the boundary conditions $b_L^{(2)} = da_L^{(1)}$ and $da_R^{(1)} = b_R^{(2)}$, which are enabled by the boundary actions with $a_L \wedge da_L$ and $a_R \wedge da_R$ respectively. These other choices do not permute electric and magnetic objects, instead, the magnetic object decorated with the invertible electric operator 1.3.13 can end on it.

Fusion rules

Here we provide an alternative derivation of the fusion rules (2.3.1), (2.3.5) and (2.3.7) for the $G = \mathbb{Z}_N$ case. Then, we rederive (2.3.26) and its generalization to $D = 4$.

Automorphism domain wall The domain wall $\mathcal{D}_{\mathbb{Z}_N^{(m)}}(\Sigma)$ corresponds to the boundary condition:

$$\mathcal{D}_{\mathbb{Z}_N^{(m)}}(\Sigma): \qquad a_L^{(1)}\Big|_\Sigma = m a_R^{(1)}\Big|_\Sigma, \quad m\tilde{a}_L^{(D-2)}\Big|_\Sigma = \tilde{a}_R^{(D-2)}\Big|_\Sigma. \tag{2.6.12}$$

To compute the fusion rule $\mathcal{D}_{\mathbb{Z}_N^{(m)}} \times \mathcal{D}_{\mathbb{Z}_N^{(m')}}$ we can bring two such parallel surfaces on top of each other. We denote the gauge fields on the left, middle, and right layer by $a_L, \tilde{a}_L, a_M, \tilde{a}_M$, and $a_R, \tilde{a}_R$ respectively. Somewhat trivially, the boundary conditions combine: $a_L = m a_M$ & $a_M = m' a_R \Rightarrow a_L = mm' a_R$ and $m\tilde{a}_L = \tilde{a}_M$ & $m'\tilde{a}_M = \tilde{a}_R \Rightarrow mm'\tilde{a}_L = \tilde{a}_R$ so that

$$\mathcal{D}_{\mathbb{Z}_N^{(m)}}(\Sigma) \times \mathcal{D}_{\mathbb{Z}_N^{(m')}}(\Sigma): \qquad a_L^{(1)}\Big|_\Sigma = mm' a_R^{(1)}\Big|_\Sigma, \quad mm'\tilde{a}_L^{(D-2)}\Big|_\Sigma = \tilde{a}_R^{(D-2)}\Big|_\Sigma, \tag{2.6.13}$$

which is the boundary condition for $\mathcal{D}_{\mathbb{Z}_N^{(mm')}}$. Note that there is no remnant degrees of freedom to sum over, this is in contrast with the next two cases. We conclude that

$$\mathcal{D}_{\mathbb{Z}_N^{(m)}} \times \mathcal{D}_{\mathbb{Z}_N^{(m')}} = \mathcal{D}_{\mathbb{Z}_N^{(mm')}}, \tag{2.6.14}$$

which is consistent with (2.3.1).

Diagonal domain wall The domain wall $\mathcal{D}_1(\Sigma)$ corresponds to having Dirichlet boundary condition for a_L and a_R along Σ:

$$\mathcal{D}_1(\Sigma): \qquad a_L^{(1)}\Big|_\Sigma = a_R^{(1)}\Big|_\Sigma = 0. \tag{2.6.15}$$

This boundary condition results in trivial holonomy for loops on Σ. This is also the case for $\mathcal{D}_1(\Sigma)$ as illustrated in Fig. 2.1. From the method presented for the automorphism domain wall, it is clear that $\mathcal{D}_1 \times \mathcal{D}_1$ should be proportional to itself because $a_L = a_M = a_R$. The difference in this case, is that $\tilde{a}_M^{(D-2)}$ is a remant degree of freedom along Σ that we should sum over. This gives the fusion coefficient $\mathcal{Z}^0(\mathbb{Z}_N, \Sigma) = |\mathbb{Z}_N|$.

To see this more explicitly, instead of defining the domain walls by imposing the boundary conditions, we are going to add scalar fields with suitable gauge transformations to cancel the variation of the total action. These fields are sometimes referred to as Stueckelberg scalars, see e.g., [Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett, Kapustin and Seiberg(2014a)] and the mechanism is somewhat analogous to anomaly inflow. Here, we will generalize in dimension the presentation of section 6.3.4 of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] which we refer to for further details.[3] The Dirichlet boundary condition for a_L and a_R can be implemented by the boundary action:

$$\mathcal{D}_1(\Sigma): \qquad -\frac{iN}{2\pi}\int_\Sigma \phi^{(0)} d(\tilde{a}_L^{(D-2)} - \tilde{a}_R^{(D-2)}). \tag{2.6.16}$$

with $\phi^{(0)}$ the Stueckelberg scalar field with gauge transformation $\phi^{(0)} \to \phi^{(0)} + \alpha^{(0)}$ with $\alpha^{(0)}$ the gauge transformation of the a fields. One can check that the total action is gauge invariant and, by integrating out $\tilde{a}$, that a is pure gauge on Σ.

To compute the fusion rule of $\mathcal{D}_1$ with itself we can bring two such parallel surfaces

3. In their analysis, the bulk action has the form $a_L^{(1)} \wedge d\tilde{a}_L^{(D-2)}$ which is related to (2.6.7) by the boundary term $a_L^{(1)} \wedge \tilde{a}_L^{(D-2)}$ under integration by parts. So we assume implicitly here that we added this boundary terms to get to their convention.

on top of each other. We denote the gauge fields on the left, middle, and right layer by $a_L, \tilde{a}_L, a_M, \tilde{a}_M$, and $a_R, \tilde{a}_R$ respectively, and we denote the two Stueckelberg scalar fields by ϕ_1 and ϕ_2. The worldsheet action for the fusion is

$$\mathcal{D}_1(\Sigma) \times \mathcal{D}_1(\Sigma): \qquad -\frac{iN}{2\pi}\int_\Sigma \big[-\phi^{(0)} d\tilde{a}^{(D-2)} + \phi_2^{(0)}(d\tilde{a}_L^{(D-2)} - d\tilde{a}_R^{(d-2)})\big], \tag{2.6.17}$$

with $\phi^{(0)} = \phi_1^{(0)} - \phi_2^{(0)}$ and $\tilde{a}^{(D-2)} = \tilde{a}_M^{(D-2)} - \tilde{a}_L^{(D-2)}$. The first term is the 0-form partition function for a decoupled scalar, i.e., $\mathcal{Z}^0(\mathbb{Z}_N, \Sigma) = |\mathbb{Z}_N|$, and the second term is another copy of the surface defect $\mathcal{D}_1(\Sigma)$. We conclude that

$$\mathcal{D}_1 \times \mathcal{D}_1 = |\mathbb{Z}_N| \mathcal{D}_1, \tag{2.6.18}$$

which is consistent with expression (2.3.5). We see that the prefactor, which we derived before by summing over the intermediate holonomies, can be interpreted, in the abelian case, as the partition function for a decoupled scalar.

Factorized domain wall The domain wall $\mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma)$ corresponds to having Dirichlet boundary condition for $\tilde{a}_L$ and $\tilde{a}_R$:

$$\mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma): \qquad \tilde{a}_L^{(D-2)}\Big|_\Sigma = \tilde{a}_R^{(D-2)}\Big|_\Sigma = 0. \tag{2.6.19}$$

The result of this boundary condition is that the components of da_L and da_R that are perpendicular to Σ are not set to zero after integrating out $\tilde{a}_L$ and $\tilde{a}_R$, therefore, the holonomy for contractible paths that cross Σ is not trivial. This is also the case for $\mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma)$ as illustrated in Fig. 2.1. In this case, the combination of boundary conditions gives $\tilde{a}_L = \tilde{a}_M = \tilde{a}_R$ and it remains to sum over $a_M^{(1)}$ along Σ, which gives the fusion coefficient $\mathcal{Z}(\mathbb{Z}_N, \Sigma)$.

We can see this more explicitly by following the same derivation as outlined before. In this case, we don't add the boundary term $a_L^{(1)} \wedge \tilde{a}_L^{(D-2)}$ and the Dirichlet boundary condition for $\tilde{a}_L$ and $\tilde{a}_R$ is equivalent to the boundary action:

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}(\Sigma): \qquad -\frac{iN}{2\pi}\int_\Sigma \phi^{(D-3)} d(a_L^{(1)} - a_R^{(1)}), \tag{2.6.20}$$

with $\phi^{(D-3)}$ the Stueckelberg scalar field with gauge transformation $\phi^{(D-3)} \to \phi^{(D-3)} - \tilde{\alpha}^{(D-3)}$ with $\tilde{\alpha}^{(D-3)}$ the gauge transformation of the field $\tilde{a}^{(D-2)}$. One can check that the total action is gauge invariant and, by integrating out a, that $\tilde{a}$ is pure gauge on Σ. The fusion in this case leads to:

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}(\Sigma)\times\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}(\Sigma): \quad -\frac{iN}{2\pi}\int_\Sigma \big[-\phi^{(D-3)}\wedge da^{(1)} + \phi_2^{(D-3)}\wedge(da_L^{(1)} - da_R^{(1)})\big], \tag{2.6.21}$$

with $\phi^{(D-3)} = \phi_1^{(D-3)} - \phi_2^{(D-3)}$ and $a^{(1)} = a_M^{(1)} - a_L^{(1)}$. The first term is now the partition function $\mathcal{Z}(\mathbb{Z}_N,\Sigma)$, and the second term is another copy of the surface defect $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$. We conclude that

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}\times\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N} = \mathcal{Z}(\mathbb{Z}_N)\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}, \tag{2.6.22}$$

which is consistent with (2.3.7).

Electric-magnetic duality domain wall in $D=3$

The electromagnetic duality that exchanges the electric and magnetic particles (see e.g. [Bombin(2010), Kitaev and Kong(2012)]) corresponds to the domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}(\Sigma)$ correspond to the boundary conditions:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}(\Sigma): \qquad a_L^{(1)}\Big|_\Sigma = \tilde{a}_R^{(1)}\Big|_\Sigma, \quad \tilde{a}_L^{(1)}\Big|_\Sigma = a_R^{(1)}\Big|_\Sigma. \tag{2.6.23}$$

that are enabled by the boundary action (2.6.9). Similarly to the automorphism fusion we have:

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N,\alpha_2}(\Sigma)\times\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N,\alpha_2}(\Sigma): \qquad a_L^{(1)}\Big|_\Sigma = a_R^{(1)}\Big|_\Sigma, \quad \tilde{a}_L^{(1)}\Big|_\Sigma = \tilde{a}_R^{(1)}\Big|_\Sigma. \tag{2.6.24}$$

which is the boundary condition for $\mathcal{D}_{\mathbb{Z}_2^{\mathrm{id}}} = 1$. Importantly combining the boundary actions leads to:

$$S_{\alpha_2,L} + S_{\alpha_2,R} = \frac{i}{\pi}\int_\Sigma a_L^{(1)}\wedge a_M^{(1)} + \frac{i}{\pi}\int_\Sigma a_M^{(1)}\wedge a_R^{(1)}. \tag{2.6.25}$$

Integrating out a_M simply reproduces the relation $a_L = a_R$. We conclude that:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}\times\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} = 1, \tag{2.6.26}$$

which is (2.3.26).

Non-invertible electric-magnetic duality domain wall in $D = 4$ Consider a generalization of the previous defect to $D = 4$ corresponding to the boundary condition:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}(\Sigma): \qquad da_L^{(1)}\Big|_\Sigma = \tilde{a}_R^{(2)}\Big|_\Sigma, \qquad \tilde{a}_L^{(2)}\Big|_\Sigma = da_R^{(1)}\Big|_\Sigma, \tag{2.6.27}$$

that is enabled by adding the boundary action (2.6.11). Now, in the fusion we get $da_L = da_R$ and $\tilde{a}_L^{(2)} = \tilde{a}_R^{(2)}$ which is not the identity boundary condition. The a_L and a_R gauge fields on the two sides after the fusion can differ by a $\mathbb{Z}_4$ gauge field, instead of being identically equal. The domain wall is non-invertible.

Transformation of Wilson lines and magnetic defects

One way to derive the correspondence between the boundary conditions and the domain walls of Table 2.5 is by the transformations of other operators. Conversely, given the correspondence, we can check the transformation of other operators using the boundary conditions.

Untwisted domain walls Here, we derive the transformation property (2.4.3) for $G = \mathbb{Z}_N$ from the boundary condition $a_L = m a_R$ and $m\tilde{a}_L = \tilde{a}_R$. Consider a path $\gamma = \gamma_L + \gamma_R$ on $\mathcal{M}$ that cross Σ with γ_L in $\mathcal{M}_L$ and γ_R on $\mathcal{M}_R$ such that $\partial\gamma_L = -\partial\gamma_R = v_1 - v_0$ with v_1 and v_0 vertices on Σ. Then, $W_L(\gamma_L)W_R^m(\gamma_R)$ is gauge invariant because

$$W_L(\gamma_L)W_R^m(\gamma_R) \to \exp\Big(\frac{i}{\pi}\int_{\gamma_L} a_L^{(1)} + d\alpha_L^{(0)} + \frac{im}{\pi}\int_{\gamma_R} \tilde{a}_R^{(1)} + d\alpha_R^{(0)}\Big) = W_L(\gamma_L)W_R^m(\gamma_R) \tag{2.6.28}$$

where we used

$$\int_{\gamma_L} d\alpha_L^{(0)} + m\int_{\gamma_R} d\alpha_R^{(0)} = \alpha_L^{(0)}(v_1) - m\alpha_R^{(0)}(v_0) + m\alpha_R^{(0)}(v_1) - \alpha_L^{(0)}(v_1) = 0, \tag{2.6.29}$$

which follows from the boundary condition $\alpha_L^{(0)}|_\Sigma = m\alpha_R^{(0)}|_\Sigma$. We conclude that

$$\mathcal{D}_{\mathbb{Z}_N^{(m)}} \cdot W = W^{\frac{1}{m}}, \tag{2.6.30}$$

which is consistent with (2.4.3). The transformation of the other Wilson lines and magnetic defects can be obtained similarly. Complementarily, they can also be obtained by fusion, see Fig. 2.18 for illustration.

The fact that Wilson lines and magnetic defects can end on the side with Dirichlet boundary conditions for a and $\tilde{a}$ respectively is even easier to derive. For example, suppose we have $a_L^{(1)}|_\Sigma = 0$ and consider a path γ_L such that $\partial\gamma_L = v_1 - v_0$ with v_1 and v_0 vertices

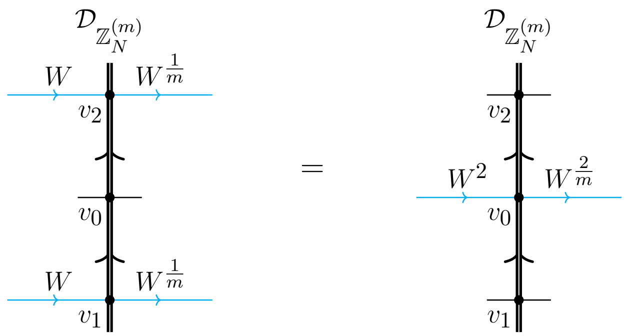


Figure 2.18: Derivation of the transformation of W^2 on $\mathcal{D}_{\mathbb{Z}_N^{(m)}}$ from the transformation of W. By moving the two independent configurations we can fuse them independently on each side.

on Σ. Then, under a gauge transformation we will have

$$W(\gamma_L) \to \exp\left(\frac{i}{\pi}\int_{\gamma_L} d\alpha_L^{(0)}\right) W(\gamma_L) = \exp\left(\alpha_L^{(0)}(v_1) - \alpha_L^{(0)}(v_0)\right) W(\gamma_L) = W(\gamma_L). \quad (2.6.31)$$

where we used the boundary condition $\alpha_L^{(0)}|_\Sigma = 0$. The same argument works for $a_R^{(1)}|_\Sigma = 0$ and for magnetic defects on the domain walls with Dirichlet boundary conditions for $\tilde{a}^{(D-2)}$. These results are consistent with the transformation properties of Wilson lines and magnetic defects for the other untwisted domain walls.

Twisted domain walls Similarly to the automorphism domain wall derivation, the boundary conditions $\tilde{a}_L^{(1)} = a_R^{(1)}$ and $a_L^{(1)} = \tilde{a}_R^{(1)}$ implies:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \cdot M = W, \qquad \mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} \cdot W = M. \quad (2.6.32)$$

There is, however, a complementary point of view to understand this result which uses the analysis of twisted $\mathbb{Z}_2\times\mathbb{Z}_2$ gauge theory in $D = 2$ from [Kapustin and Seiberg(2014b),Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett]. This theory can be described by the action:

$$S_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} = \frac{i}{\pi}\int_\Sigma (\tilde{a}_L^{(0)} \wedge da_L^{(1)} + \tilde{a}_R^{(0)} \wedge da_R^{(1)} + a_L^{(1)} \wedge a_R^{(1)}), \tag{2.6.33}$$

where $\tilde{a}_L$ and $\tilde{a}_R$ are 2π-periodic scalars and the system has gauge symmetry:

$$a_L^{(1)} \to a_L^{(1)} + d\alpha_L^{(0)}, \quad a_R^{(1)} \to a_R^{(1)} + d\alpha_R^{(0)}, \quad b_L^{(0)} \to b_L^{(0)} - \alpha_R^{(0)}, \quad b_R^{(0)} \to b_R^{(0)} - \alpha_L^{(0)}. \tag{2.6.34}$$

In this theory the local magnetic defects $M_L(v_i) = e^{i\tilde{a}_L(v_i)}$ and $M_R(v_i) = e^{i\tilde{a}_R(v_i)}$ are not gauge invariant. Instead, the gauge-invariant operators are $M_L(v_i)W_R(\gamma)M_L(v_j)$, and $M_R(v_i)W_L(\gamma)M_R(v_j)$ with γ an open path with endpoints v_i and v_j. If we extend γ to the bulk and view $M_L(v_i)$ and $M_R(v_j)$ as the endpoints of a bulk magnetic operator, we again confirm the transformation (2.6.32).

2.6.2 $\mathbb{D}_4$ gauge theory

Consider the gauge theory for the Dihedral group of order 8 in $D = 3$. This theory is equivalent to twisted $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory and can be described by the action:

$$S_{\mathbb{D}_4} = \frac{i}{\pi}\int_{\mathcal{M}} \big(a \wedge d\tilde{a} + b \wedge d\tilde{b} + c \wedge d\tilde{c} + \frac{1}{\pi} a \wedge \tilde{b} \wedge c\big). \tag{2.6.35}$$

with $a, \tilde{a}, b, \tilde{b}, c, \tilde{c}$ one-form $U(1)$ gauge fields with correlated gauge transformations such that the action is gauge invariant on closed manifolds. Integrating out $\tilde{b}$ forces $\frac{1}{\pi} a \wedge c = db$, which describes the extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ with the 2-cocycle given by $a \wedge b$. The equivalence with $\mathbb{D}_4$ gauge theory is discussed in [de Wild Propitius(1997), Coste et al.(2000)Coste, Gannon, and Ruelle].

As in the other examples, we divide spacetime into left and right regions $\mathcal{M} = \mathcal{M}_L \cup \mathcal{M}_R$

with a common boundary $\partial\mathcal{M}_L = -\partial\mathcal{M}_R = \Sigma$ with opposite orientation. The total action is (2.6.35) with fields defined on a left and right part, which we denote with the subscripts L and R as we have done in (2.6.7). In general, different boundary conditions correspond to different domain walls.

Diagonal fusion rule

Here we provide an alternative derivation of the fusion rule (2.3.5) for the $G = \mathbb{D}_4$ case. The domain wall $\mathcal{D}_1(\Sigma)$ corresponds to having Dirichlet boundary condition for a, b, c along Σ:

$$\mathcal{D}_1(\Sigma): \qquad a_L\Big|_\Sigma = a_R\Big|_\Sigma = b_L\Big|_\Sigma = b_R\Big|_\Sigma = c_L\Big|_\Sigma = c_R\Big|_\Sigma = 0. \tag{2.6.36}$$

The result of this boundary condition is that the holonomy for loops on Σ are always trivial which is also the case for $\mathcal{D}_1(\Sigma)$ as illustrated in Fig. 2.1. In complete analogy with the $\mathbb{Z}_N$ example, this boundary condition can be implemented by having Stueckeelberg scalar fields ϕ_a, ϕ_b and ϕ_c along Σ. Similar manipulations, then yield three decoupled scalars that make a prefactor equal to $2^3 = |\mathbb{D}_4|$. We see that in this non-abelian example, the pre-factor can also be interpreted as the partition function for decoupled scalars.

Non-invertible electric-magnetic duality

The possible topological actions for the subgroup $H = \mathbb{D}_4 \times \mathbb{D}_4 \leq \mathbb{D}_4 \times \mathbb{D}_4$ are classified by $H^2(\mathbb{D}_4 \times \mathbb{D}_4, U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ (see [Handel(1993)] for the group cohomology of Dihedral groups). In the theory with $\tilde{b}$ integrated out, the domain wall $\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(n,m)}(\Sigma)$ (here $(n,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$) corresponds to the boundary condition:

$$\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(n,m)}(\Sigma): \qquad \tilde{a}_L\Big|_\Sigma = nc_R\Big|_\Sigma,\ na_L\Big|_\Sigma = \tilde{c}_R\Big|_\Sigma,\ \tilde{a}_R\Big|_\Sigma = mc_L\Big|_\Sigma,\ ma_R\Big|_\Sigma = \tilde{c}_L\Big|_\Sigma. \tag{2.6.37}$$

which is enabled by adding the boundary action:

$$S_{(n,m)} = \frac{in}{\pi}\int_\Sigma a_L \wedge c_R + \frac{im}{\pi}\int_\Sigma c_R \wedge b_L. \tag{2.6.38}$$

This correspondence means that the boundary condition in (2.6.37) solves $\delta(S_{\mathbb{D}_4}+S_{(n,m)}) = 0$, where $S_{\mathbb{D}_4}$ is (2.6.35) with spacetime divided into left and right regions as in (2.6.7). Note that in the theory with $\tilde{b}_L$ and $\tilde{b}_R$ integrated out $a_L \wedge c_L$ and $a_R \wedge c_R$ are exact, but not $a_L \wedge c_R$ and $a_R \wedge c_L$, which is what appears in (2.6.38).

Fusion rules To compute the fusion rule of $\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(n,m)}$, instead of imposing the boundary conditions, we will add Stueckelberg fields to make the total action gauge invariant. Let us compute the fusion rule of two such domain walls by dividing the spacetime into the left, middle, and right. The boundary action (2.6.38) combines into $S_{(n_1,m_1),L} + S_{(n_1,m_1),R}$:

$$\frac{n_1 i}{\pi}\int a_L \wedge c_M + \frac{m_1 i}{\pi}\int a_M \wedge c_L + \frac{n_2 i}{\pi}\int a_M \wedge c_R + \frac{m_2 i}{\pi}\int a_R \wedge c_M \tag{2.6.39}$$

Additionally, integrating out the Lagrange multiplier field associated with the condition that $a_M \wedge c_M$ is exact on Σ generates:

$$1 + \exp\Big(\frac{i}{\pi}\int_\Sigma a_M \wedge c_M\Big). \tag{2.6.40}$$

The fusion outcome is different for the two terms in (2.6.40). For the first term, integrating out a_M and c_M leads to $n_1 a_L = m_2 a_R$ and $m_1 c_L = n_2 c_R$. For the second term, integrating out a_M and c_M leads to

$$\frac{i}{\pi}\int (n_1 a_L + m_2 a_R)\wedge(m_1 c_L + n_2 c_R) = \frac{n_1 n_2 i}{\pi}\int a_L \wedge c_R + \frac{m_1 m_2 i}{\pi}\int a_R \wedge c_L, \tag{2.6.41}$$

where the equality used $a_L \wedge c_L$ and $a_R \wedge c_R$ being exact. The above corresponds to the domain wall $\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(n_1n_2,m_1m_2)}(\Sigma)$.

We conclude that the domain wall is non-invertible and we have, for example:

$$\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} \times \mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} = 1 + \mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)}. \tag{2.6.42}$$

This fusion rule is strikingly similar to that of the Fibonacci anyons in $D = 3$ TQFTs [Slingerland and Bais(2001)]. The key difference is that our domain wall is higher dimensional.

Transformation of other operators The theory has four non-trivial Wilson lines, four non-trivial magnetic defects, and thirteen non-trivial Dyons. With the trivial line, this makes up a total of 22 operators [de Wild Propitius and Bais(1996)]. Among them, we have:

$$W_a(\gamma) = e^{i\int_\gamma a}, \qquad W_c(\gamma) = e^{i\int_\gamma c}, \qquad M_a(\gamma) = e^{i\int_\gamma \tilde{a}}, \qquad M_c(\gamma) = e^{i\int_\gamma \tilde{c}}. \tag{2.6.43}$$

where we used a local polarization to write the magnetic defects without a bounding surface [Hsin et al.(2024b)Hsin, Kobayashi, and Zhu]. From the boundary conditions (2.6.37) associated with the domain wall $\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)}(\Sigma)$, we can follow the procedure used in the derivation of (2.6.30) to find:

$$\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} \cdot M_a = W_c + \ldots \qquad \mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} \cdot W_c = M_a + \ldots, \tag{2.6.44}$$

confirming that $\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)}$ is the symmetry defect that implements an electric-magnetic duality. From the two configurations displayed above, and provided the fusion rule $M_a \times W_c = M_a$ we can derive:

$$\mathcal{D}_{\mathbb{D}_4\times\mathbb{D}_4,(1,1)} \cdot M_a = M_a + W_c, \tag{2.6.45}$$

which is consistent with the Fibonacci fusion rule 2.6.42. See Fig. 2.19 for an illustration.

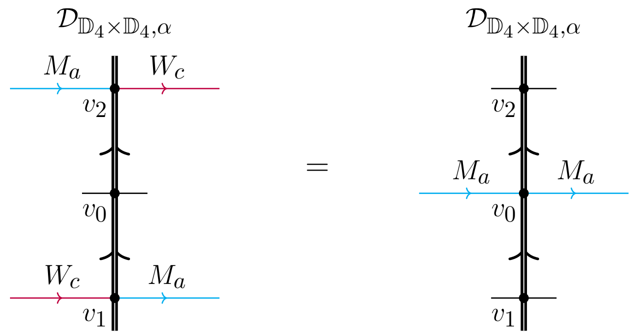


Figure 2.19: Derivation of (2.6.45) from the gauge invariant configurations (2.6.44) and the fusion rule $M_a \times W_c = M_a$. In the figure, we fused M_a and W_c on both sides. The fact that there exists a gauge invariant configuration with M_a makes the transformation consistent with the Fibonacci fusion rule (2.6.42).

CHAPTER 3

NON-INVERTIBLE SYMMETRIES AS CONDENSATION DEFECTS

3.1 Introduction

Symmetries are often the first line of attack for constraining the dynamics of physical systems. As such, new symmetries can give powerful new insights into strongly-coupled theories. Motivated by these considerations, in this work we explore non-invertible symmetries in simple topological field theories: gauge theories with finite gauge group [Dijkgraaf and Witten(1990a)] (see [Córdova et al.(2024)Córdova, Costa, and Hsin] for a review).

The fundamental connection between symmetry and topology was made precise in [Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett] (see also [Córdova et al.(2022a)Córdova, Dumitrescu, Intriligator, and Shao, Shao(2023), Luo et al.(2024)Luo, Wang, and Wang, Bhardwaj et al.(2024a)Bhardwaj, Bottini, Fraser-Taliente, Gladden, Gould, Platschorre, and Tillim, Schafer-Nameki(2024), Iqbal(2024), Brennan and Hong(2023), McGreevy(2023), Gomes(2023), Costa et al.(2024)] for lecture notes and reviews). Every symmetry is defined intrinsically by an extended topological operator, and the selection rules can be deduced through the behavior of these operators in correlation functions. Subsequent developments have indicated an even more precise classification of finite symmetry: topological operators in a quantum field theory in spacetime dimension D can be described by a topological quantum field theory (TQFT) in dimension $D+1$. [Witten(1998), Gaiotto et al.(2015b)Gaiotto, Kapustin, Seiberg, and Willett, Kong et al.(2015)Kong, Wen, and Zheng, Gaiotto and Kulp(2021), Freed et al.(2022)Freed, Moore, and Teleman, Kaidi et al.(2023b)Kaidi, Ohmori, and Zheng, Kong et al.(2020b)Kong, Lan, Wen, Zhang, and Zheng]. There are therefore at least two reasons to be interested in the topological operators in finite-group gauge theories. First, these theories are an illuminating playground where we can make precise many of the abstract ideas that permeate more general models. And second, when $D = 4$ and all particles are bosons, it is expected that every unitary topological field theory is in fact a (possibly twisted) finite-group gauge theory [Lan et al.(2018b)Lan, Kong, and Wen, Lan and Wen(2019b), Johnson-Freyd(2022)].[1]

In the present work, we build on our previous study of defects in finite-group gauge theory [Córdova et al.(2024)Córdova, Costa, and Hsin], focusing on the notion of condensation. The idea of condensation refers to the process of appropriately summing over insertions of topological defects. If the sum is carried out in all of spacetime, this condensation is a form of gauging the symmetry described by the summed defects. In this case, the result of the condensation process is a new quantum field theory, and such constructions have been explored in detail in [Kong(2014b), Eliëns et al.(2014)Eliëns, Romers, and Bais, Lan et al.(2015)Lan, Wang, and Wen, Hung and Wan(2015), Neupert et al.(2016)Neupert, He, von Keyserlingk, Sierra, and Bernevig, Burnell(2018), Tachikawa(2020), Bhardwaj and Tachikawa(2018), Cong et al.(2017)Cong, Cheng, and Wang, Hsin et al.(2019)Hsin, Lam, and Seiberg, Kaidi et al.(2022a)Kaidi, Komargodski, Ohmori, Seifnashri, and Shao, Yu(2021), Décoppet and Yu(2023a), Diatlyk et al.(2024)Diatlyk, Luo, Wang, and Weller, Córdova and García-Sepúlveda(2023), Zhang et al.(2024)Zhang, Vishwanath, and Wen, Córdova and García-Sepúlveda(2024), Kong et al.(2024)Kong, Zhang, Zhao, and Zheng, Perez-Lona et al.(2024b)Perez-Lona, Robbins, Sharpe, Vandermeulen, and Yu].

In contrast to these sums in all of spacetime, in this work, we discuss condensations where the sum is carried out over a submanifold of spacetime. In this case, the result of the condensation procedure is an operator in the initial theory [Carqueville et al.(2018)Carqueville, Runkel, and Schaumann, Gaiotto and Johnson-Freyd(2019), Kong et al.(2020c)Kong, Tian, and Zhang, Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao, Cui et al.(2024)Cui,

1. A closely related statement holds in $D = 5$. Specifically, when all particles are bosons, any unitary TQFT is a finite-group gauge theory coupled to a theory of abelian two-form gauge fields [Johnson-Freyd and Yu(2021), Johnson-Freyd and Yu(2022), Córdova et al.(2023)Córdova, Hsin, and Zhang].

Haghighat, and Ruggeri, Cuiper and Garre-Rubio(2024), Ebisu and Han(2024), Vandermeulen(2023), Lin et al.(2022)Lin, Robbins, and Sharpe, Lyons et al.(2024)Lyons, Lo, Tantivasadakarn, Vishwanath, and Verresen, Carqueville et al.(2020)Carqueville, Runkel, and Schaumann, Mulevičius and Runkel(2023), Koppen et al.(2022)Koppen, Mulevicius, Runkel, and Schweigert, Carqueville et al.(2021)Carqueville, Mulevicius, Runkel, Schaumann, and Scherl, Carqueville et al.(2024)Carqueville, Mulevicius, Runkel, Schaumann, and Scherl, Choi et al.(2023b)Choi, Rayhaun, Sanghavi, and Shao, Choi et al.(2024)Choi, Lu, and Sun, Buican and Radhakrishnan(2024)]. At an intuitive level, one may view the condensation defect as a fine mesh of lower-dimensional objects as illustrated in Fig. 3.1. This picture also accurately captures a key feature of such condensation operators, namely that they have trivial action via linking on other operators – a hole may always be opened in the mesh, which allows the other operator to pass through and unlink.[2] In certain situations, a partial converse to this is also known; for instance, in unitary $D = 3$ TQFTs with a unique identity operator, all surface operators are condensations [Fuchs et al.(2013b)Fuchs, Schweigert, and Valentino].

The concept of condensation operators has been explored in a variety of contexts. In particular, they typically do not obey a group multiplication law and thus constitute an example of non-invertible symmetry [Kapustin and Saulina(2010), Fuchs et al.(2013a)Fuchs, Schweigert, and Valentino, Bhardwaj and Tachikawa(2018), Chang et al.(2019)Chang, Lin, Shao, Wang, and Yin, Komargodski et al.(2021)Komargodski, Ohmori, Roumpedakis, and Seifnashri, Choi et al.(2022a)Choi, Córdova, Hsin, Lam, and Shao, Kaidi et al.(2022b)Kaidi, Ohmori, and Zheng, Chang et al.(2023)Chang, Chen, and Xu]. In $D = 3$ TQFTs there is a detailed dictionary between abstract algebra objects, more physically collections of anyons obeying certain algebraic properties, and surface operators [Eliëns et al.(2014)Eliëns, Romers, and Bais, Kong(2014b), Lan et al.(2015)Lan, Wang, and Wen, Hung and Wan(2015), Neu-

2. More precisely, this is the case for the action by Hopf-linking, where a topological operator of dimension $(D - p)$ acts on operators of dimension $(p - 1)$. Condensation defects can act non-trivially by multilinking. See [Hsin and Turzillo(2020), Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi] for examples.

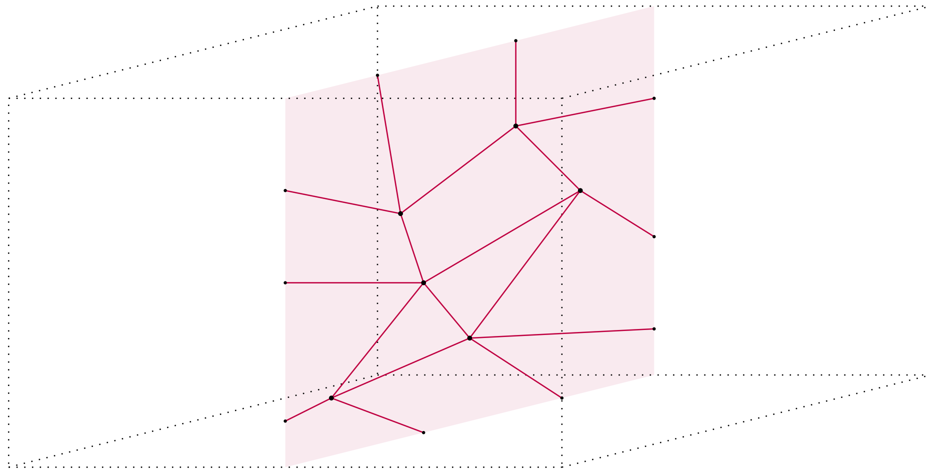


Figure 3.1: Illustration of a condensation defect slice, depicted as the red-shaded plane, formed by the insertion of lower-dimensional topological operators, represented by the lines in the illustration. The condensation defect is embedded within a region of spacetime, shown as the dotted cuboid.

pert et al.(2016)Neupert, He, von Keyserlingk, Sierra, and Bernevig, Burnell(2018), Cong et al.(2017)Cong, Cheng, and Wang, Kaidi et al.(2022a)Kaidi, Komargodski, Ohmori, Seifnashri, and Shao, Yu(2021), Córdova and García-Sepúlveda(2023), Zhang et al.(2024)Zhang, Vishwanath, and Wen]. This dictionary can also be extended to $D = 2$ rational conformal field theories by viewing these as the boundaries of TQFTs (see e.g. [Fuchs et al.(2002)Fuchs, Runkel, and Schweigert, Fuchs et al.(2004a)Fuchs, Runkel, and Schweigert, Fuchs et al.(2004b)Fuchs, Runkel, and Schweigert, Fuchs et al.(2005)Fuchs, Runkel, and Schweigert, Frohlich et al.(2004)Frohlich, Fuchs, Runkel, and Schweigert, Frohlich et al.(2007)Frohlich, Fuchs, Runkel, and Schweigert, Diatlyk et al.(2024)Diatlyk, Luo, Wang, and Weller]). Most closely connected to our approach below is the notion of higher gauging introduced in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. In this context, the authors studied how sums over defects with abelian fusion rules give rise to a wide variety of defects. This also allows one to exhibit the correspondence between the worldvolume action of the defect, itself often a (twisted) finite-group gauge theory, and the resulting defect operator presented as a bulk condensation. Our work will directly build on the analysis of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao], constructing more intricate examples of higher gauging.

More specifically, in this paper, we will provide more examples of higher gauging and related condensation constructions by considering finite-group gauge theories in general spacetime dimension D, where the gauge groups can be abelian or non-abelian. For simplicity of illustration, we will focus on the theories without additional topological terms, i.e. untwisted topological gauge theories. In a recent analysis, [Córdova et al.(2024)Córdova, Costa, and Hsin], we developed a method to construct invertible and non-invertible symmetries of finite-group gauge theories as domain walls on the lattice. In the present work, we demonstrate how to realize these symmetries as condensation defects, i.e., as suitable insertions of lower-dimensional topological operators. This realization allows us to derive properties of the non-invertible symmetries, including fusion rules and the action on other operators, in terms of the constituents that form the condensation defects. (For further examples of topological defects in finite-group gauge theories see e.g. [Kitaev(2003), Kapustin and Saulina(2011), Wen(2013), Barkeshli et al.(2019)Barkeshli, Bonderson, Cheng, and Wang, Hsin and Turzillo(2020), Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi].)

Our results can also be applied to describe invertible and non-invertible symmetry as defects on the lattice with continuous time with the Gauss law imposed energetically by Hamiltonian terms. For symmetry that corresponds to the condensation of particular electric and/or magnetic excitations that have energy cost from suitable Gauss law and flux terms in the Hamiltonian [Kitaev(2003)], the symmetry defect can be constructed as a modification on the Hamiltonian with the Hamiltonian terms along the defect corresponding to the condensed excitations removed. Such a description has been used in the construction of Cheshire string defects where the electric charge condenses [Else and Nayak(2017), Johnson-Freyd(2020)], and the fusion rules of Cheshire strings in $\mathbb{Z}_2$ gauge theory in $D = 3+1$ spacetime dimension can be obtained in this way [Kong et al.(2020c)Kong, Tian, and Zhang].[3] Our results can be

3. Cheshire strings have also been discussed in [Chen et al.(2024)Chen, Dua, Hermele, Stephen, Tantivasadakarn, Vanhove, and Zhao, Tantivasadakarn and Chen(2024a)], but in terms of operators of linear depth

extended to such description to general invertible or non-invertible symmetries in general untwisted gauge theories, which quantum double lattice models can describe [Kitaev(2003)].

3.1.1 Summary of results

Let us now summarize our results in detail. Throughout the following, G denotes the finite gauge group, which in general may be abelian or non-abelian. In this work we aim to establish the correspondence between two descriptions of non-invertible symmetries in finite-group gauge theories: (1) symmetries as domain walls on the lattice [Córdova et al.(2024)Córdova, Costa, and Hsin], and (2) symmetries as condensation defects [Gaiotto and Johnson-Freyd(2019)], i.e., as suitable insertions of lower dimensional topological operators. Let us briefly summarize the two perspectives:

(1) The first construction uses the folding trick to unfold the domain wall into a gapped boundary, which we review in Section 1.3. The domain wall is then labeled by a subgroup $H \leq G \times G$ and a choice of topological action for the given subgroup $\alpha \in H^{D-1}(H, U(1))$. Given this data, we construct a codimension-one domain wall $\mathcal{D}_{H,\alpha}(\Sigma)$ supported on Σ by having the gauge group elements on Σ to be elements of $H \leq G \times G$ and by properly gluing them with the exterior of the domain wall. We further decorate Σ with the topological action α. Here, the left and right components of $H \leq G \times G$ correspond to the left and right of the domain wall respectively, which have a global meaning in terms of the orientation of the normal bundle $N\Sigma$. We remark that this construction applies to all finite gauge groups, including non-abelian ones, and in any spacetime dimension.

(2) When the gauge group G is abelian, condensation defects can be interpreted as the gauging of higher-form symmetries on the support of the 0-form symmetry (i.e., higher

called sequential circuits.

gauging defects), which we review in Section 3.3.1. In $D = 2+1$ spacetime dimensions the higher-form symmetry generated by the electric Wilson lines and magnetic fluxes is a 1-form $G \times G$ symmetry. In this case, the condensation defects are classified by subgroups of the 1-form symmetry $H \lhd G \times G$ with a topological term ("discrete torsion") $f \in H^2(H, U(1))$ [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. Given this data, one can construct a codimension-one topological operator $\mathcal{S}_{H,f}(\Sigma)$ by summing over all insertions of the global symmetry operators in the corresponding subgroup along Σ with torsion equal to f[4]. Here, the left and right components of $H \lhd G \times G$ correspond to the 1-form symmetry generated by the Wilson lines and magnetic defects respectively.

The data for the two classifications is the same in $D = 2+1$ spacetime dimensions, but the meanings are remarkably different. This work aims to relate these two perspectives and extend the relation to the case where G is non-abelian and the spacetime dimension is arbitrary.

Diagonal domain walls: electric condensations Consider first the domain wall associated with the trivial subgroup and let us start the discussion with the abelian case. One can derive the following relation:

$$\mathcal{S}_{G\times 1} = \mathcal{D}_1. \tag{3.1.1}$$

In words, the higher gauging of the 1-form symmetry G generated by the Wilson lines is equivalent to the domain wall that restricts the gauge group elements to be in the subgroup $1 \lhd G \times G$. The equivalence between the two descriptions comes from two facts:

- The higher gauging of the 1-form symmetry G generated by the Wilson lines can be

4. We set as our convention that the Wilson lines are always on the left of the magnetic defects in $\mathcal{S}_{H,f}(\Sigma)$ which corresponds to having them inserted on the left of the magnetic defect insertion.

reorganized as the product of insertions of the Wilson line in the regular representation,

$$W_{\text{reg}} = \sum_{\rho \in \text{irreps}} d_\rho W_\rho, \tag{3.1.2}$$

along the generators of $\pi_1(\Sigma)$;

- The character of the regular representation has the following projection property:

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & g = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Therefore, the operator insertion in the condensation defect $\mathcal{S}_{G\times 1}$ will project the gauge group elements to the trivial subgroup making the domain wall $\mathcal{D}_1$. To be concrete consider $G = \mathbb{Z}_N$, in this case we have:

$$\mathcal{S}_{\mathbb{Z}_N\times 1}(\Sigma) = \frac{N}{|H_1(\Sigma, \mathbb{Z}_N)|} \sum_{\gamma \in H_1(\Sigma, \mathbb{Z}_N)} W(\gamma) = N \prod_{a=1}^{2g} \frac{1}{N} W_{\text{reg}}(\gamma_a) = \mathcal{D}_1(\Sigma), \tag{3.1.4}$$

where $W_{\text{reg}} = 1 + W + W^2 + \ldots W^{N-1}$ is the Wilson line in the regular representation of $\mathbb{Z}_N$ and a run over the generators of $\pi_1(\Sigma)$.

Note that the presentation of $\mathcal{D}_1$ in terms of insertions of W_{reg} along the generators of $\pi_1(\Sigma)$ readily generalizes to non-abelian gauge groups and spacetime dimensions. Therefore, the extension of equation (3.1.1) to the general setting is the leftmost equality in (3.1.4), which states that the domain wall $\mathcal{D}_1$ is equivalent to the condensation of the Wilson line in the regular representation.

These ideas also apply to the other diagonal domain walls, i.e., domain walls associated with diagonal subgroups $K^{(\text{id})} = \{(k,k) : k \in K\} \lhd G \times G$ with $K \lhd G$. The character of the trivial representation of K induced to G, that is $\text{Ind}_K^G 1$, generalizes the property of the regular representation in the sense that

$$\chi_{\mathrm{Ind}_K^G 1}(g) = \begin{cases} \frac{|G|}{|K|} & g \in K, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1.5}$$

and $\rho_{\mathrm{reg}} = \mathrm{Ind}_1^G 1$. Therefore, the insertion of the Wilson line in the representation $\mathrm{Ind}_K^G 1$ along the generators of $\pi_1(\Sigma)$ will project the gauge group elements to K, generating the domain wall $\mathcal{D}_{K^{(\mathrm{id})}}(\Sigma)$. In the $G = \mathbb{Z}_N$ example, this discussion specializes to $\mathcal{S}_{\mathbb{Z}_{N/M}\times 1} = \mathcal{D}_{\mathbb{Z}_M^{(\mathrm{id})}}$ for $M|N$.

Diagonal domain walls are orientation reversal invariant, which allows their generalization to higher-codimensional defects $\mathcal{D}_K(\Sigma_n)$ with $K \lhd G$ and Σ_n a n-dimensional submanifold of spacetime (see Section 1.3 for details). From the perspective of their condensation expression, the fact that they can be generalized is particularly transparent. Wilson lines are one-dimensional objects, and correspondingly can be condensed along submanifolds of dimension greater than or equal to one. Specifically, the domain wall $\mathcal{D}_1(\Sigma_n)$, correspond to the insertion of $W_{\mathrm{Ind}_K^G 1}$ along the generators of $\pi_1(\Sigma_n)$.

At last, the condensation expression for the domain wall $\mathcal{D}_{K^{(\mathrm{id})},\alpha}$ with $\alpha \in H^{D-1}(K, U(1))$ can be obtained by fusing $\mathcal{D}_{K^{(\mathrm{id})}}$ with the general invertible electric defect W_α, see (1.3.13) for its definition.

Magnetic domain wall: magnetic condensation To move to a more general case it is instructive to consider first the domain wall $\mathcal{D}_{G\times G}$. Again, let us start the discussion with the abelian G case in $D = 2+1$ spacetime dimensions. One can derive the following relation:

$$\mathcal{S}_{1\times G} = \mathcal{D}_{G\times G}. \tag{3.1.6}$$

In words, the condensation of the 1-form symmetry G generated by the magnetic defects is equivalent to the domain wall associated to the subgroup $G \times G \lhd G \times G$. Similarly to the diagonal case, the equivalence between the two descriptions comes from two facts:

- That the condensation of the 1-form symmetry G generated by the magnetic defects can be reorganized as the product of insertions of the magnetic operator:

$$M_{\text{reg}} = \sum_{[g]\in \text{Cl}(G)} M_g, \tag{3.1.7}$$

  along the generators of $\pi_1(\Sigma)$.

- That inserting M_{reg} along a generator $\gamma \in \pi_1(\Sigma)$ will source flux for the dual generator $\tilde{\gamma}$ that has linking number one with γ making an effective $G \times G$ boundary. See Fig. 3.2 for an illustration.

Therefore, the operator insertion in the condensation $\mathcal{S}_{1\times G}$ will source the extra holonomies to make the domain wall $\mathcal{D}_{G\times G}$. To be concrete consider again $G = \mathbb{Z}_N$, in this case we have:

$$\mathcal{S}_{1\times \mathbb{Z}_N}(\Sigma) = \frac{1}{N} \sum_{\gamma \in H_1(\Sigma, \mathbb{Z}_N)} M(\gamma) = \frac{1}{N} \prod_{a=1}^{2g} M_{\text{reg}}(\gamma_a) = \mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma), \tag{3.1.8}$$

where $M_{\text{reg}} = 1 + M + M^2 + \cdots + M^{p-1}$ is the magnetic operator analogous to the Wilson line in the regular representation.

Just like the previous case, the presentation of $\mathcal{D}_{G\times G}$ in terms of insertions of M_{reg} along the generators of $\pi_1(\Sigma)$ readily generalizes to non-abelian gauge groups and spacetime dimensions. Therefore, the extension of equation (3.1.6) to the general setting is the leftmost equality in (3.1.8), which states that the domain wall $\mathcal{D}_{G\times G}$ is equivalent to the condensation of the magnetic object M_{reg}, which is the sum of all pure magnetic objects with unit coefficient.

In higher spacetime dimensions, the insertions are over codimension-two closed submanifolds of Σ.

The domain wall $\mathcal{D}_{G\times G}$ cannot be generalized to a higher-codimensional defect. This is evident from its condensation construction. Magnetic defects are already codimension-two objects, and they cannot condense on submanifolds of higher codimension.

Untwisted normal domain walls: electric-magnetic sandwich Here we invert the order of discussion. We start with general gauge group and spacetime dimensions and then consider particular cases. Specifically, the description as higher gauging symmetry defects when G is abelian.

The condensation expression for a domain wall associated with a general subgroup $H \lhd G \times G$ is obtained by sandwiching the magnetic domain wall $\mathcal{D}_{G\times G}$ with the Wilson lines in the representation $\mathrm{Ind}_H^{G\times G}$. By doing this, one projects the $G \times G$ gauge group elements of $\mathcal{D}_{G\times G}$ to its H subgroup. The expression one gets by following this procedure is:

$$\mathcal{D}_H(\Sigma) = \frac{|H|^{n-1}}{|G|^{2n-2}} \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} c^H_{i_1 j_1} W_{i_1}(\gamma_1)\ldots c^H_{i_n j_n} W_{i_n}(\gamma_n) \mathcal{D}_{G\times G}(\Sigma) W_{j_1}(\gamma_1)\ldots W_{j_n}(\gamma_n), \tag{3.1.9}$$

where i_k, j_k run over irreducible representations of G and $\gamma_1, \ldots, \gamma_n$ are n generators of $\pi_1(\Sigma)$. The coefficients are determined by the subgroup $H \lhd G \times G$ as:

$$c^H_{ij} = \langle \chi_{\mathrm{Ind}_H^{G\times G} 1}, \chi_{\rho_i}\chi_{\rho_j} \rangle = \frac{1}{|H|} \sum_{(g_L, g_R)\in H} \chi_{\rho_i}(g_L)\chi_{\rho_j}(g_R), \tag{3.1.10}$$

with ρ_i the irreducible representations of G. We remark that in (3.1.9), writing the Wilson lines on the left and right of $\mathcal{D}_{G\times G}$ corresponds to having them inserted on the left and right of the magnetic insertion, respectively. Moving them from one side to the other costs a braiding phase.

Domain walls as higher gauging condensation defects in $\mathbb{Z}_p$ gauge theory The previous results allow us to derive the dictionary between the domain walls and the condensation definition of codimension-one topological operators. In $D = 2+1$ spacetime dimensions, given a particular group $H \lhd G \times G$ one can rewrite (3.1.9) as the higher gauging of a subgroup of the 1-form global symmetry. By doing this for all subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ when p is prime we find all but the last two rows of the dictionary in Table 3.1, which makes explicit the relation between domain walls and condensation defects in $\mathbb{Z}_p$ gauge theory. In Section 3.3 we derive these results in the general context of non-prime p and arbitrary spacetime dimension (see Table 3.3 for a summary).

Domain wall	Condensation
$\mathcal{D}_{\mathbb{Z}_p^{(\mathrm{id})}}$	$\mathcal{S}_1$
$\mathcal{D}_{\mathbb{Z}_p^{(m)}}$	$\mathcal{S}_{\mathbb{Z}_p \times \mathbb{Z}_p, \frac{m}{1-m}}$
$\mathcal{D}_1$	$\mathcal{S}_{\mathbb{Z}_p \times 1}$
$\mathcal{D}_{1 \times \mathbb{Z}_p}$	$\mathcal{S}_{\mathbb{Z}_p \times \mathbb{Z}_p}$
$\mathcal{D}_{\mathbb{Z}_p \times 1}$	$\mathcal{S}_{\mathbb{Z}_p \times \mathbb{Z}_p, -1}$
$\mathcal{D}_{\mathbb{Z}_p \times \mathbb{Z}_p}$	$\mathcal{S}_{1 \times \mathbb{Z}_p}$
$\mathcal{D}_{\mathbb{Z}_p \times \mathbb{Z}_p, 1}$	$\mathcal{S}_{\mathbb{Z}_p^{(\mathrm{id})}}$
$\mathcal{D}_{\mathbb{Z}_p \times \mathbb{Z}_p, m}$	$\mathcal{S}_{\mathbb{Z}_p^{(m)}}$

Table 3.1: Dictionary between higher gauging condensation defects and domain walls in $\mathbb{Z}_p$ gauge theory. Domain walls are classified by subgroups of the folded theory gauge group $H \lhd \mathbb{Z}_p \times \mathbb{Z}_p$ (with the left and right factors corresponding respectively to the left and right of the domain wall) with a choice of topological action $\alpha \in H^{D-1}(H, U(1))$. In $D = 2+1$ spacetime dimensions, higher gauging condensation defects are classified by subgroups of the 1-form global symmetry $H \lhd \mathbb{Z}_p \times \mathbb{Z}_p$ (with the left and right factors generated respectively by Wilson lines and magnetic defects) with a choice of torsion term $f \in H^2(H, U(1))$. In the table, $\mathbb{Z}_p^{(m)} = \{(mn, n) : n \in \mathbb{Z}_p\} \lhd \mathbb{Z}_p \times \mathbb{Z}_p$ with $1 < m < p$, and $\mathbb{Z}_p^{(1)} = \mathbb{Z}_p^{(\mathrm{id})}$. Note that, because p is prime, $1 - m$ is invertible and $f = \frac{m}{1-m} \in \mathbb{Z}_p$. See Table 3.3 for the non-prime p and higher dimensional generalization.

It remains, to understand the attachment of a topological action in the domain wall formalism and its relation to higher gauging. For $\mathbb{Z}_p$ there are $p-1$ such domain walls which are labeled by $H^2(\mathbb{Z}_p \times \mathbb{Z}_p, U(1)) = \mathbb{Z}_p$, likewise there are $p-1$ different Dyons associated to the automorphism subgroups of the 1-form global symmetry. By viewing the attachment of the topological action as decorating the domain wall $\mathcal{D}_{\mathbb{Z}_p\times\mathbb{Z}_p} = \mathcal{S}_{1\times\mathbb{Z}_p}$ with suitable electric charges it becomes clear the correspondence between the two as displayed in Table 3.1.

In spacetime dimension $D \neq 2+1$, the higher-form symmetries generated by Wilson lines and magnetic defects have a different form degree, resulting in fewer subgroups compared to when they share the same form degree. For example, in this case, there are no diagonal subgroups corresponding to the condensation of Dyons. This feature leads to a mismatch in the number of domain walls and higher gauging condensation defects. This apparent problem is resolved by introducing the concept of “sequential higher gauging". Sequential higher gauging involves the higher gauging of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form global symmetry generated by magnetic defects and the dual symmetry that emerges after gauging the $(D-2)$-form symmetry generated by Wilson lines. Since both factors have the same form degree, all subgroups can be considered and there is no number mismatch between the two constructions. In particular, one can generate the general invertible electric defects (1.3.13) by this procedure. See Section 3.3.3 for details.

Automorphism 0-form symmetry as higher gauging defects in abelian theories As an application of our framework, we can derive an expression for the invertible automorphism domain walls in abelian gauge theories as higher gauging condensation defects. In Section 3.4 we derive these expressions for the domain walls associated with a set of automorphism generators. Here, we present the dictionary for all automorphisms of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ that can be derived using these expressions.

The Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 generators of order 2, which we denote by $A = (1,0)$, $B = (0,1)$ and $C = (1,1)$. The automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to ordered lists of the three generators $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \{\sigma \cdot [A,B,C] : \sigma \in S_3\} \cong S_3$ with group product given by composing the S_3 elements. The bijection is obtained by identifying the first and second elements in the list with the image of $A = (1,0)$ and $B = (0,1)$ under the corresponding automorphism. For instance, the automorphism associated with the list $(23) \cdot [A, B, C] = [A, C, B]$ maps $(1,0) = A \mapsto A = (1,0)$ and $(0,1) = B \mapsto C = (1,1)$. We use this isomorphism to denote the elements of $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ by elements of $S_3 = (\mathrm{id}, (13), (23), (12), (123), (132)\}$. Likewise, denote the automorphism subgroup associated with $\sigma \in S_3$ by $(\mathbb{Z}_2 \times \mathbb{Z}_2)^{(\sigma)} = \{(\sigma \cdot X, X) : X \in \mathbb{Z}_2 \times \mathbb{Z}_2\} \leq (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then, we can derive the dictionary presented in Table 3.2. See Section 3.4.3 for the convention used for the torsion term.

Domain wall	Condensation
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(\mathrm{id})}}$	$\mathcal{S}_1$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(13)}}$	$\mathcal{S}_{(1\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times 1),1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times 1)\times(1\times\mathbb{Z}_2),1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(12)}}$	$\mathcal{S}_{\mathbb{Z}_2^{(\mathrm{id})}\times\mathbb{Z}_2^{(\mathrm{id})},1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(123)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2),(0,1,1,1)}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(132)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2),(1,1,1,0)}$

Table 3.2: Dictionary between higher gauging condensation defects and $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ automorphism domain walls.

Using the higher gauging condensation expression for the domain walls, we can compute the fusion rules and the transformations of other operators by leveraging the algebraic properties of Wilson lines and magnetic defects. Notably, applying this higher gauging expression reveals that the computation of general fusion rules depends in subtle ways on specific number-theoretic properties. However, in all cases where we can compute these rules easily, they consistently agree with the more general fusion rules and transformations derived from the domain wall definition in [Córdova et al.(2024)Córdova, Costa, and Hsin].

Gauging 0-form symmetry: $\mathbb{D}_4$ gauge theory from gauging swap symmetry The group $\mathbb{D}_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ can be constructed as the group extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ with the $\mathbb{Z}_2$ acting by swapping generators. It follows that $\mathbb{D}_4$ gauge theory can be constructed by gauging symmetry generated by the automorphism domain wall $\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}}$. By using its explicit condensation expression reviewed in Table 3.2, i.e., $\mathcal{S}_{(\mathbb{Z}_2\times 1)\times(1\times\mathbb{Z}_2),1}$, one can explicitly derive the action:

$$S_{\mathbb{D}_4} = i\pi \int_{\mathcal{M}} (\tilde{a}^{(D-2)} \cup da^{(1)} + \tilde{b}^{(D-2)} \cup db^{(1)} + \tilde{c}^{(D-2)} \cup dc^{(1)} + a^{(1)} \cup \tilde{b}^{(D-2)} \cup c^{(1)}), \tag{3.1.11}$$

which is the action for $\mathbb{D}_4$. In Section 3.5 we show in detail this derivation and explain how to understand the operators of $\mathbb{D}_4$ gauge theory as combinations of the $\mathbb{Z}_2$ factors associated with the corresponding group extension.

3.2 Domain walls as condensation defects

In Section 1.3 we reviewed the construction of non-invertible symmetries of finite-group gauge theories as domain walls on the lattice [Córdova et al.(2024)Córdova, Costa, and Hsin]. Here, we show how to realize these symmetries as condensation defects, i.e., as suitable insertions of lower dimensional operators along a codimension-one submanifold [Gaiotto and Johnson-Freyd(2019)]. Specifically, we show how to construct the domain walls $\mathcal{D}_H$ for $H \triangleleft G \times G$ as a condensation of Wilson lines and magnetic defects. In summary, we will show that the defect $\mathcal{D}_{G\times G}$ corresponds to the condensation of magnetic objects, and that $\mathcal{D}_H$ can be obtained by further sandwiching the magnetic insertion with suitable Wilson lines, thereby projecting the $G \times G$ gauge fields to its $H \triangleleft G \times G$ subgroup. Provided these condensation expressions, in Section 3.2.5, we compute the fusion of the symmetry defects using the algebraic properties of the lower dimensional objects that make them. This allows us to verify the fusion rules derived using the domain wall lattice definition in [Córdova et al.(2024)Córdova, Costa, and Hsin].

3.2.1 Diagonal domain walls: electric condensations

To understand the general idea of using Wilson lines to project the gauge group elements to a particular subgroup it is useful to start with the diagonal domain walls. They are made out of Wilson lines that project the G gauge group elements along Σ to a particular subgroup $K \triangleleft G$.

Identity diagonal domain wall Let us consider what is arguably the simplest case, the domain wall associated with the trivial subgroup of $G \times G$, i.e., $\mathcal{D}_1(\Sigma)$. The domain wall $\mathcal{D}_1(\Sigma)$ is defined by restricting the gauge field configurations and its gauge transformations along Σ to be elements of $1 \leq G \times G$. This can be achieved in the following way:

- The gauge fields on the wall are enforced to be trivial by inserting the Wilson line in the regular representation divided by $|G|$ in all closed loops of Σ because:

$$\chi_{\rho_{\mathrm{reg}}}(g) = \mathrm{Tr}[\rho_{\mathrm{reg}}(g)] = \begin{cases} |G| & g = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2.1}$$

  for the regular representation ρ_{reg} of G and above $g \in G$.

- In addition to the projection of the gauge field, we need to correct for gauge transformations. In the definition of $\mathcal{D}_1$, gauge transformations are restricted to the subgroup $1 \leq G$. Because we are not changing the gauge transformations when we insert the projector, we need to change the correction due to the volume of gauge transformations which gives $1/|G|$ for each disconnected component. To change $1/|G|$ to 1 we need, therefore, to multiply by $|G|$.

Thus the domain wall can be described in the following equivalent ways:

$$\mathcal{D}_1(\Sigma) = |G| \prod_{\text{closed loops } \gamma} \frac{1}{|G|} W_{\text{reg}}(\gamma) = |G| \prod_{\gamma \in \pi_1(\Sigma)} \frac{1}{|G|} W_{\text{reg}}(\gamma) = |G| \prod_{\gamma \in S} \frac{1}{|G|} W_{\text{reg}}(\gamma) \ , \tag{3.2.2}$$

where the first product is over all closed loops on Σ and S is a generating set of $\pi_1(\Sigma)$. The equivalence between the different expressions can be obtained in the following way. First, one deforms all insertions over homotopy equivalent loops to some representative cycle. Then, one uses the fusion rule $W_{\text{reg}}(\gamma)W_{\text{reg}}(\gamma) = |G|W_{\text{reg}}(\gamma)$ to replace the multiple insertions to a single insertion. At last, consider $\gamma = \gamma_1 \ldots \gamma_k$ with $\gamma_i \in S$, which gives $g_\gamma = g_{\gamma_1} \ldots g_{\gamma_k}$. If $g_{\gamma_i} = e$ for all $\gamma_i \in S$ then $g_\gamma = e$. Hence, $\frac{1}{|G|}W_{\text{reg}}(\gamma)$ equals 1 if $W_{\text{reg}}(\gamma_i)$ are inserted for all $\gamma_i \in S$.

General diagonal domain wall The case of general diagonal domain wall, $\mathcal{D}_{K^{(\text{id})}}$ with $H = K^{(\text{id})} = \{(k,k) : k \in K\} \leq G \times G$ and $K \lhd G$, is similar. Consider the induced representation to G from the trivial representation of $K \lhd G$, i.e., $\text{Ind}_K^G 1$. In the previous case $K = 1$ and $\rho_{\text{reg}} = \text{Ind}_1^G 1$. Then, the character of this induced representation is the induced character to G from the trivial character of $K \lhd G$ and is the permutation character of G acting on the right cosets of K by multiplication from the right. The induced character has the property

$$\chi_{\text{Ind}_K^G 1}(g) = \begin{cases} \frac{|G|}{|K|} & g \in K, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.3}$$

Therefore, the diagonal domain wall can be described by the operator

$$\mathcal{D}_{K^{(\text{id})}}(\Sigma) = \frac{|G|}{|K|} \prod_{\text{closed loops } \gamma} \frac{|K|}{|G|} W_{\text{Ind}_K^G 1}(\gamma), \tag{3.2.4}$$

where the pre-factor corrects for the volume of gauge transformations as explained in the previous case. Similarly to the $K = 1$ case, the insertion defining $\mathcal{D}_{K(\mathrm{id})}$ can be simplified to an insertion over $\pi_1(\Sigma)$ or over a generating set of $\pi_1(\Sigma)$

In addition, we can also modify the operator by stacking with additional invertible electric operators 1.3.13 associated to $\alpha \in H^{D-1}(K, U(1))$ [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi]. This gives $\mathcal{D}_{K(\mathrm{id}),\alpha}$, the twisted diagonal operators.

Higher codimensional operators We remark that the above definition is suitable for the generalization to higher dimensional topological operators discussed in (2.5.1). Instead of inserting along all closed loops of the codimension-one surface Σ, one inserts along all closed loops of the n-dimensional surface Σ_n. Also, to attach a topological action we fuse our defect with the corresponding n-dimensional invertible electric defect (1.3.13).

In [Tantivasadakarn and Chen(2024b)] it was asked "Why did we not include Cheshire strings in the description of $D = 2 + 1$ topological order?". From (3.2.2) we see that in $D = 2 + 1$ the codimension-one operator corresponding to the "Cheshire strings" is the same as the Wilson line in the regular representation, i.e., $\mathcal{D}_1(\gamma) = W_{\mathrm{reg}}(\gamma)$. In $D = 3 + 1$, on the other hand, the equivalent codimension-one operator $\mathcal{D}_1(\Sigma_2)$ is a condensation of W_{reg}.

Frobenius algebra interpretation The composite line $\mathrm{Ind}_K^G 1$ is associated with a symmetric separable Frobenius algebra of $\mathrm{Rep}(G)$ and (3.2.4) should correspond to its generalized gauging along the codimension-one surface Σ [Ostrik(2001), Diatlyk et al.(2024)Diatlyk, Luo, Wang, and Weller]. The character property (3.2.3) connects the idea of generalized gauging and the domain wall lattice construction reviewed in Section 1.3. Further decorating Σ with the invertible electric defect defined in (1.3.13) which is classified by $\alpha \in H^{D-1}(K, U(1))$ should be equivalent to different choices of $(D-1)$-topological junctions for $\mathrm{Ind}_K^G 1$ on Σ.

3.2.2 Factorized domain walls: magnetic condensation

Consider factorized domain walls corresponding to the subgroups $H = K_L \times K_R \leq G \times G$ for $K_L, K_R \lhd G$. The difference between having two diagonal domain walls $\mathcal{D}_{K_L^{(\mathrm{id})}}$ and $\mathcal{D}_{K_R^{(\mathrm{id})}}$ close together (which would fuse according to (2.3.5), the diagonal fusion) and having the domain wall $\mathcal{D}_{K_L \times K_R}$ is the presence of magnetic defects on the middle. The insertion of magnetic defects comes from $\mathcal{D}_{G\times G}$ as in (2.3.7).

Factorized domain wall with $H = G \times G$ The holonomy for a contractible path that crosses the domain wall $\mathcal{D}_{G\times G}$ with group elements $(g_L, g_R) \in G \times G$ can be any element $g_R g_L^{-1} \in G$ as one sees in the example of Fig. 2.1. This suggests that we should have insertions of all magnetic defects along Σ to produce the $\mathcal{D}_{G\times G}$ domain wall. In addition, we showed that all pure magnetic defects can end on (2.4.7), which again shows that they condense on the wall. The operator that condenses is:

$$M_{\mathrm{reg}} = \sum_{[g]\in \mathrm{Cl}(G)} M_g, \tag{3.2.5}$$

which is analogous to the Wilson line in the regular representation. By inserting M_{reg} on codimension-two submanifolds of Σ we can effectively double the holonomies for the paths in Σ. See Fig. 3.2 for an illustration.

General factorized domain wall The general factorized domain wall with $H = K_L \times K_R \leq G \times G$ can be obtained from $\mathcal{D}_{G\times G}$ by projecting the group elements on the left and right sides of the wall to the K_L and K_R subgroups respectively using $\mathcal{D}_{K_L^{(\mathrm{id})}}$ and $\mathcal{D}_{K_R^{(\mathrm{id})}}$, the explicit expression is given in (2.3.6).

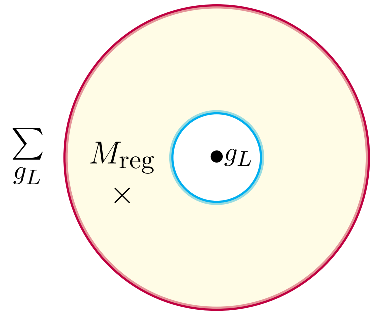


Figure 3.2: The holonomy in the blue path is g_L. Now consider the magnetic operator $M_{g_R g_L^{-1}}$. For this insertion, the holonomy around the red path is g_R. By summing over g_L in the path integral and over conjugacy classes, in the insertion of M_{reg}, we will be summing over all $(g_L, g_R) \in G \times G$ for this non-trivial cycle which produces the $G \times G$ domain wall.

3.2.3 Domain walls of a normal subgroup: electric-magnetic sandwich

The discussion above can be extended to the general normal domain wall $\mathcal{D}_H$ associated with a general normal subgroup $H \lhd G \times G$. In this case, one can consider the projector $1_H^{G\times G}$ of elements of $(g_L, g_R) \in G \times G$ to the subgroup $H \lhd G \times G$. This projector decomposes in irreducible characters of $G \times G$ which are themselves products of irreducible characters of G, and has the property that:

$$\chi_{\mathrm{Ind}_H^{G\times G} 1}(g_L, g_R) = \sum_{i,j \in \text{irreps of } G} c_{ij}^H \chi_{\rho_i}(g_L)\chi_{\rho_j}(g_R) = \begin{cases} \frac{|G|^2}{|H|} & (g_L, g_R) \in H, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.6}$$

with coefficients

$$c_{ij}^H = \langle \chi_{\mathrm{Ind}_H^{G\times G} 1}, \chi_{\rho_i}\chi_{\rho_j} \rangle = \frac{1}{|H|} \sum_{(g_L, g_R)\in H} \chi_{\rho_i}(g_L)\chi_{\rho_j}(g_R), \tag{3.2.7}$$

that one can find by using the orthogonality relations of characters and the defining property of $\chi_{\mathrm{Ind}_H^{G\times G} 1}$. Similar to the diagonal case, the above is the character of the trivial representation in H induced to $G \times G$, *i.e*, $\mathrm{Ind}_H^{G\times G} 1$.

Sandwiching the domain wall $\mathcal{D}_{G\times G}$ with these characters for all loops of Σ will perform the desired projection. Simplifying the insertion to be on the generators of $\pi_1(\Sigma)$ as done in (3.2.2), the general expression can be written as

$$\mathcal{D}_H(\Sigma) = \frac{|H|^{n-1}}{|G|^{2n-2}} \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} c^H_{i_1 j_1} W_{i_1}(\gamma_1)\ldots c^H_{i_n j_n} W_{i_n}(\gamma_n)\mathcal{D}_{G\times G}(\Sigma) W_{j_1}(\gamma_1)\ldots W_{j_n}(\gamma_n), \tag{3.2.8}$$

where i_k, j_k run over irreducible characters of G and $\gamma_1,\ldots,\gamma_n$ are n generators of $\pi_1(\Sigma)$. Here, we also abbreviated $W_{\rho_{i_k}}$ to W_{i_k} as a short hand notation.

It is important to clarify the meaning of this sandwich construction (3.2.8). On the lattice, to define magnetic defects along a codimension-one surface Σ, one needs to consider a thickened surface $\Sigma \times [0,1]$. If not, there are no 2-simplices perpendicular to Σ in which one can fix the holonomies. Along this thickened surface with the inserted magnetic defect, there are two inequivalent ways to insert a Wilson line. It can be positioned either to the right, $\Sigma \times 0$, or to the left, $\Sigma \times 1$, of the magnetic operator. We adopt the convention that $W_\rho(\gamma)M_g(\Gamma)$ represents the Wilson line on the left of the magnetic defect, while $M_g(\Gamma)W_\rho(\gamma)$ places it on the right. Thus, $W_{i_n}(\gamma_n)\mathcal{D}_{G\times G}(\Sigma)Wj_n(\gamma_n)$ indicates that the Wilson line W_{i_n} is inserted along a loop γ_n to the left of the magnetic insertions on $\Sigma \times 0$, while W_{j_n} is inserted along the same loop γ_n, but to the right of the magnetic insertions on $\Sigma \times 1$. Moving them from one side to the other costs a braiding phase.

As a particular example, for factorized domain wall the coefficient (3.2.7) factorizes, i.e., $c^{K_L\times K_R}_{ij} = c^{K_L}_i c^{K_R}_j$ where c^K_i is the coefficient of the irreducible representation ρ_i in the expansion of $\mathrm{Ind}^G_{K_L} 1$, and we get (2.3.6).

3.2.4 Domain walls in abelian theories as condensations

Every subgroup of $G \times G$ is normal when G is abelian. Therefore, (3.2.8) can be applied to generate all untwisted domain walls of abelian theories. In this section, we will simplify (3.2.8) for the abelian case. The simplified expression will then be used to perform consistency checks and compute some fusion rules.

For gauge theories with abelian finite group G, Wilson lines in irreducible representations and simple magnetic defects obey the following commutation relation in a codimension-one submanifold Σ:

$$W_\rho(\gamma)M_g(\Gamma) = \chi_\rho(g)^{\langle\gamma,\Gamma\rangle} M_g(\Gamma)W_\rho(\gamma), \tag{3.2.9}$$

where $\chi_\rho(g)$ is the character of the irreducible representation ρ evaluated in the conjugacy class associated with the element $g \in G$ and $\langle\gamma,\Gamma\rangle$ is the *intersection number*[5] of $\gamma \in H_1(\Sigma,\mathbb{Z})$ and $\Gamma \in H_{D-2}(\Sigma,\mathbb{Z})$ on Σ. The commutation relation (3.2.9) can be understood from the discussion after (3.2.8) and it follows from the linking action of the two operators. Alternatively, one can think of Σ as a time slice and (3.2.9) follows from the canonical quantization of the BF-type action with the $(D-2)$ and 1-form fields being canonically conjugate variables [Witten(1998)].

To further simplify (3.2.8) we assume $\pi_1(\Sigma)$ is torsion-free. With this assumption one can derive the following relations:

$$\mathcal{Z}(G,\Sigma) = \frac{|H_{D-2}(\Sigma,G)|}{|G|} = \frac{|H_1(\Sigma,G)|}{|G|} = |G|^{b_1-1}, \tag{3.2.10}$$

where b_1 is the first betti number of Σ. For the first term we used that $|G|\mathcal{Z}(G,\Sigma) = |\operatorname{Hom}(H_1(\Sigma,\mathbb{Z}),G)| = |G|^{b_1}$ where in the last equality we used that $H_1(\Sigma,\mathbb{Z}) = \mathbb{Z}^{b_1}$ is

5. The intersection number $\langle\gamma,\Gamma\rangle$ is defined explicitly as the integral over Σ of the cup product of the Poincaré dual of $\gamma \in H_1(\Sigma,\mathbb{Z})$ and $\Gamma \in H_{D-2}(\Sigma,\mathbb{Z})$.

torsion-free. For the second term we used Poincaré duality $H_{D-2}(\Sigma, G) \cong H^1(\Sigma, G)$ and the universal coefficient theorem for cohomology $H^1(\Sigma, G) \cong \mathrm{Hom}(H_1(\Sigma, \mathbb{Z}), G) = G^{b_1}$ where the assumption that Σ is connected leads to $\mathrm{Ext}^1(H_0(\Sigma, \mathbb{Z}), G) = \mathrm{Ext}^1(\mathbb{Z}, G) = 0$. Finally, using the universal coefficient theorem for homology we have $H_1(\Sigma, \mathbb{Z}) \otimes G = H_1(\Sigma, G) = G^{b_1}$, where we used that $\mathrm{Tor}_1(H_0(\Sigma, \mathbb{Z}), G) = \mathrm{Tor}_1(\mathbb{Z}, G) = 0$. In this context it is possible to choose a basis $\{\gamma_a : 1 \le a \le b_1\}$ and $\{\Gamma_a : 1 \le a \le b_1\}$ for $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z}^{b_1}$ and $H_{D-2}(\Sigma, \mathbb{Z}) = \mathbb{Z}^{b_1}$ such that $\langle \gamma_a, \Gamma_b \rangle = \delta_{ab}$ where $\langle \cdot, \cdot \rangle : H_1(\Sigma, \mathbb{Z}) \times H_{D-2}(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ is the intersection number defined by integrating over Σ the cup product of the Poincaré dual of the homology classes. We are going to use relation (3.2.10) and this fact to simplify some of the expressions in abelian theories in the constructions below.

When G is abelian, the operator M_{reg} obeys the same fusion rule as W_{reg}, i.e., $M_{\mathrm{reg}} \times M_{\mathrm{reg}} = \sum_g M_g \times \sum_{g'} M_{g'} = \sum_{g,g'} M_{gg'} = |G| \sum_g M_g = |G| M_{\mathrm{reg}}$, where we used the fact that each element of G appears once, and only once in each row and column of the Cayley table of G. Similarly to the derivation of the diagonal domain wall condensation (3.2.2), we can write the operator $\mathcal{D}_{G\times G}$ explicitly in the following equivalent ways:

$$\mathcal{D}_{G\times G}(\Sigma) = \mathcal{Z}(G) \prod_{\Gamma \in H_{D-2}(\Sigma,\mathbb{Z})} \frac{M_{\mathrm{reg}}(\Gamma)}{|G|} = \mathcal{Z}(G) \prod_{\Gamma \in S} \frac{M_{\mathrm{reg}}(\Gamma)}{|G|} = \frac{1}{|G|} \prod_{\Gamma \in S} M_{\mathrm{reg}}(\Gamma) \ , \tag{3.2.11}$$

where S is a set of generators of $H_{D-2}(\Sigma, \mathbb{Z})$ and we used that $|G|\mathcal{Z}(G) = |H^1(\Sigma, G)| = |H_{D-2}(\Sigma, G)|$ by Poincaré duality. Choosing a othonormal basis for the insertion of magnetic and electric operators, (3.2.8) can be written more compactly as

$$\mathcal{D}_H(\Sigma) = \frac{|G|}{|H|} \prod_{a=1}^{b_1} \frac{|H|}{|G \times G|} \sum_{i,j \in \mathrm{irreps}} c^H_{ij} W_{\rho_i}(\gamma_a) M_{\mathrm{reg}}(\Gamma_a) W_{\rho_j}(\gamma_a). \tag{3.2.12}$$

where we used (3.2.11) and we commuted the operators with zero intersection. Let us now illustrate how this definition reproduces expression (3.2.2) for $\mathcal{D}_1$ and the fact that $\mathcal{D}_{G(\mathrm{id})} = 1$.

Example: trivial group domain wall Note that the trivial subgroup domain wall is a nice checking example because it is at the same time a factorized and diagonal domain wall. To show that (3.2.2) follows from (3.2.12) we need to use the commutation relation (3.2.9). Provided the fact that $c^1_{ij} = d_i d_j = 1$, we have:

$$\begin{aligned} W_{\text{reg}}(\gamma)M_{\text{reg}}(\Gamma)W_{\text{reg}}(\gamma) &= \sum_{i,j,g} W_{\rho_i}(\gamma)M_g(\Gamma)W_{\rho_j}(\gamma), \\ &= \sum_{g,i,j} \chi_{\rho_i}(g)M_g(\Gamma)W_{\rho_i\times\rho_j}(\gamma), \\ &= \sum_{g} \chi_{\text{reg}}(g)M_g(\Gamma)W_{\text{reg}}(\gamma), \\ &= |G|W_{\text{reg}}(\gamma), \end{aligned} \tag{3.2.13}$$

where in the first equality we used (3.2.9) assuming for simplicity that $\langle\gamma,\Gamma\rangle = 1$, in the second equality we redefined the dummy variable over irreps and used the definition of the regular character and its property (3.2.1). From this equation, it is easy to derive (3.2.2) from (3.2.12).

Example: automorphism domain walls Let us consider the domain wall that generates group automorphism symmetry for abelian gauge group G. The domain wall for the automorphism $\phi \in \text{Aut}(G)$ corresponds to the subgroup $G^{(\phi)} = \{(\phi\cdot g, g) : g \in G\} \leq G\times G$. The condensation expression for the automorphism domain wall is given by (3.2.8) with

$$c^{G^{(\phi)}}_{ij} = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_i}(\phi\cdot g)\chi_{\rho_j}(g) = \langle\chi_{\rho_i\circ\phi},\chi_{\overline{\rho}_j}\rangle = \begin{cases} 1 & \overline{\rho}_j = \rho_i\circ\phi, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.14}$$

where $\overline{\rho}_i$ is the complex conjugate representation of ρ_i[6] and $\rho_j \cdot \phi$ is the representation obtained by the composition of ϕ and ρ_j.

In particular, the automorphism associated with the identity, i.e., $\phi = \text{id}$, should give 1, the trivial domain wall. Indeed, using (3.2.12) for the identity automorphism we have:

$$
\begin{aligned}
\mathcal{D}_{G(\text{id})}(\Sigma) &= \prod_{a=1}^{b_1} \frac{1}{|G|} \sum_{i \in \text{irreps}} W_{\overline{\rho}_i}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_i}(\gamma_a), \\
&= \prod_{a=1}^{b_1} \frac{1}{|G|} \sum_{i \in \text{irreps}} \sum_{g \in G} \chi_{\overline{\rho}_i}(g) M_g(\Gamma_a) W_{\overline{\rho}_i}(\gamma_a) W_{\rho_i}(\gamma_a), \\
&= \prod_{a=1}^{b_1} \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_{\text{reg}}}(g) M_g(\Gamma_a), \\
&= \prod_{a=1}^{b_1} \sum_{g \in G} \delta_{ge} M_g(\Gamma_a), \\
&= 1,
\end{aligned}
\tag{3.2.15}
$$

where above we used (3.2.9) to permute the Wilson line and magnetic defects, we used that $W_\rho(\gamma) W_{\overline{\rho}}(\gamma) = 1$, and we used the regular representation character identity shown in (3.2.1).

3.2.5 Fusion rules of domain walls from condensation definition

Here, we derive the fusion rules of domain walls reviewed in Section 1.3 by employing the condensation expressions obtained in Section 3.2 and the algebraic properties of Wilson lines and magnetic defects. This approach provides a self-consistency check on the correctness of the condensation expressions and on the fusion rules of domain walls derived by other methods in [Córdova et al.(2024)Córdova, Costa, and Hsin].

6. The complex conjugate representation $\overline{\rho}$ of a representation ρ is the representation such that $\overline{\rho}_i(g) = \overline{\rho_i(g)}$ for all $g \in G$

Diagonal domain walls Let us compute the fusion (2.3.5) for the diagonal domain walls with subgroups $K^{(\mathrm{id})}, K'^{(\mathrm{id})}$ using (3.2.4) and the tensor product of representations. We have:

$$\begin{aligned}\mathcal{D}_{K^{(\mathrm{id})}} \times \mathcal{D}_{K'^{(\mathrm{id})}} &= \frac{|G|}{|K|}\frac{|G|}{|K'|} \prod_{\text{closed loops } \gamma} \frac{|K||K'|}{|G||G|} W_{\mathrm{Ind}_K^G 1}(\gamma) W_{\mathrm{Ind}_{K'}^G 1}(\gamma), \\ &= \frac{|G|}{|K \cdot K'|}\frac{|G|}{|K \cap K'|} \prod_{\text{closed loops } \gamma} \frac{|K \cap K'|}{|G|} W_{\mathrm{Ind}_{K\cap K'}^G 1}(\gamma), \\ &= \frac{|G|}{|K \cdot K'|} \mathcal{D}_{(K\cap K')^{(\mathrm{id})}},\end{aligned} \tag{3.2.16}$$

where we used that

$$\mathrm{Ind}_K^G 1 \otimes \mathrm{Ind}_{K'}^G 1 = \frac{|G|}{|K \cdot K'|} \mathrm{Ind}_{K\cap K'}^G 1, \tag{3.2.17}$$

and $|K||K'| = |K \cdot K'||K \cap K'|$. The same derivation applies to the fusion (2.5.1) associated with the higher codimensional generalization of the diagonal domain wall.

Automorphism domain walls in abelian theories In this section, we derive the fusion rules (2.3.1) and (2.3.2) involving the domain walls with automorphism subgroups $G^{(\phi)} = \{(\phi \cdot g, g) : g \in G\} \leq G \times G$ with $\phi \in \mathrm{Aut}(G)$ using the condensation definition (3.2.12) with c_{ij}^H given by (3.2.14).

For the automorphism fusion rule (2.3.1) we have:

$$
\begin{aligned}
&\mathcal{D}_{G(\phi)} \times \mathcal{D}_{G(\phi')} \\
&= \prod_{a=1}^{b_1} \frac{1}{|G|^2} \sum_{i,j \in \text{irreps}} W_{\overline{\rho}_i}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_i \circ \phi}(\gamma_a) W_{\overline{\rho}_j}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_j \circ \phi'}(\gamma_a), \\
&= \prod_{a=1}^{b_1} \frac{1}{|G|^2} \sum_{i,j \in \text{irreps}} \sum_{g,g' \in G} \chi_{\rho_i \circ \phi}(g) \chi_{\overline{\rho}_j}(g) W_{\overline{\rho}_i}(\gamma_a) M_{gg'}(\Gamma_a) W_{\rho_i \circ \phi}(\gamma_a) W_{\overline{\rho}_j}(\gamma_a) W_{\rho_j \circ \phi'}(\gamma_a), \\
&= \prod_{a=1}^{b_1} \frac{1}{|G|} \sum_{i \in \text{irreps}} W_{\overline{\rho}_i}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_i \circ \phi \circ \phi'}(\gamma_a), \\
&= \mathcal{D}_{G(\phi \circ \phi')},
\end{aligned}
\tag{3.2.18}
$$

where from the second to the third line we redefined $g' = g^{-1}\tilde{g}$ and performed the independent summation on g using the character ortogonality condition, namely:

$$
\frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i \circ \phi}(g) \chi_{\overline{\rho}_j}(g) = \langle \chi_{\rho_i \circ \phi}, \chi_{\rho_j} \rangle = \begin{cases} 1 & \rho_j = \rho_i \circ \phi, \\ 0 & \text{otherwise}, \end{cases}
\tag{3.2.19}
$$

and the fact that $W_\rho(\gamma) W_{\overline{\rho}}(\gamma) = 1$.

Similarly, for (2.3.2) we have:

$$
\begin{aligned}
&\mathcal{D}_{G\times G} \times \mathcal{D}_{G(\phi)},\\
&= \prod_{a=1}^{b_1} \frac{1}{|G|^2} \sum_{i\in \text{irreps}} M_{\text{reg}}(\Gamma_a) W_{\overline{\rho}_i}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_i\circ\phi}(\gamma_a),\\
&= \prod_{a=1}^{b_1} \frac{1}{|G|^2} \sum_{i\in \text{irreps}} \sum_{g,g'\in G} \chi_{\overline{\rho}_i}(g) W_{\overline{\rho}_i}(\gamma_a) M_g(\Gamma_a) M_{g'}(\Gamma_a) W_{\rho_i\circ\phi}(\gamma_a),\\
&= \prod_{a=1}^{b_1} \frac{1}{|G|^2} \sum_{i\in \text{irreps}} \sum_{g,\tilde{g}\in G} \chi_{\overline{\rho}_i}(g) W_{\overline{\rho}_i}(\gamma_a) M_{\tilde{g}}(\Gamma_a) W_{\rho_i\circ\phi}(\gamma_a),\\
&= \prod_{a=1}^{b_1} \frac{1}{|G|} \sum_{i\in \text{irreps}} \delta_{i0} W_{\overline{\rho}_i}(\gamma_a) M_{\text{reg}}(\Gamma_a) W_{\rho_i\circ\phi}(\gamma_a),\\
&= \prod_{a=1}^{b_1} \frac{1}{|G|} M_{\text{reg}}(\Gamma_a),\\
&= \mathcal{D}_{G\times G},
\end{aligned}
\tag{3.2.20}
$$

where we defined $\tilde{g} = g \cdot g'$ and we used the character ortogonality condition $\langle \chi_{\overline{\rho}_i}, \chi_{\rho_0} \rangle = \delta_{i0}$ with $\rho_0 = 1$ the trivial representation. The fusion $\mathcal{D}_{G(\phi)} \times \mathcal{D}_{G\times G} = \mathcal{D}_{G\times G}$ can be derived in a similar way.

Factorized domain walls in abelian theories Let us derive the fusion rule (2.3.7) using the condensation expressions (3.2.4) and (3.2.11). Using the basis for the electric and magnetic operator insertions such that $\langle \gamma_a, \Gamma_b \rangle = \delta_{ab}$, we have:

$$\begin{aligned}\mathcal{D}_{G\times G} \times \mathcal{D}_{K(\mathrm{id})} \times \mathcal{D}_{G\times G} &= \frac{1}{|K||G|} \prod_{a=1}^{b_1} \frac{|K|}{|G|} M_{\mathrm{reg}}(\Gamma_a) W_{\mathrm{Ind}_K^G 1}(\gamma_a) M_{\mathrm{reg}}(\Gamma_a), \\ &= \frac{1}{|K||G|} \prod_{a=1}^{b_1} \frac{|K|}{|G|} \sum_{i,g,g'} \chi_{\rho_i}(g) M_{gg'}(\Gamma_a) c_i W_{\rho_i}(\gamma_a), \\ &= \frac{|K|^{b_1}}{|K||G|} \prod_{a=1}^{b_1} M_{\mathrm{reg}}(\Gamma_a), \\ &= \mathcal{Z}(K)\mathcal{D}_{G\times G}.\end{aligned} \tag{3.2.21}$$

In the first line, we reorganized the product by commuting all operators and leaving just the ones with non-zero intersection number uncommuted. In the second, we used (3.2.9) to permute the simple Wilson lines and magnetic defects and we fused the two simple magnetic defects. In the third line we used the character orthogonality condition:

$$\frac{1}{|G|} \sum_g \chi_{\rho_i}(g) = \langle \chi_{\rho_i}, \chi_1 \rangle = \begin{cases} 1 & \rho_i = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2.22}$$

and our assumption about Σ summarized before (3.2.12) which implies that $\mathcal{Z}(K) = |K|^{b_1 - 1}$.

3.3 Domain walls as higher gauging condensation defects in $\mathbb{Z}_N$ gauge theory

In this section, we investigate $\mathbb{Z}_N$ gauge theory in general spacetime dimensions. First, we consider general untwisted domain walls and show that their condensation expression derived in Section 3.2 can be recast using the concept of higher gauging [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. Consequently, our condensation expression provides a dictionary between the domain wall [Córdova et al.(2024)Córdova, Costa, and Hsin] and the higher gauging [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] constructions of non-invertible symmetries in $\mathbb{Z}_N$ gauge theory. Next, we analyze twisted domain walls and demonstrate that they can be recast using a new concept that generalizes higher gauging, which we term *sequential higher gauging.*

3.3.1 Review of higher gauging condensation defects

Higher gauging is a specific procedure used to construct condensation defects, broadly defined as defects obtained through suitable insertions of lower-dimensional topological operators. Gauging a non-anomalous abelian global symmetry involves summing over insertions of the symmetry generators across spacetime, usually resulting in a different theory. Higher gauging, however, involves gauging a non-anomalous global symmetry on a submanifold rather than the entire spacetime. This procedure creates a defect on the submanifold, known as a condensation defect [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao].

In this section, we will focus on $\mathbb{Z}_N$ gauge theory in general spacetime dimensions D without a Dijkgraaf-Witten topological action. The theory has $\mathbb{Z}_N$ 1-form symmetry generated by the unit charge magnetic operator M and $\mathbb{Z}_N$ $(D-2)$-form symmetry generated by the unit charge Wilson lines W. Higher gauging condensation defects, which we denote by $\mathcal{S}_{\mathbb{Z}_q\times\mathbb{Z}_{q'},f}$, are classified by a choice of subgroup of the higher-form symmetry $\mathbb{Z}_q \times \mathbb{Z}_{q'} \lhd \mathbb{Z}_N \times \mathbb{Z}_N$ generated respectively by Wilson lines and magnetic defects with $q|N$, $q'|N$ and a choice of torsion term, $f \in H^{D-1}(K(\mathbb{Z}_q, D-2) \times K(\mathbb{Z}_{q'}, 1), U(1)) \cong \mathbb{Z}_{\gcd(N/q,N/q')}$ [7]. Explicitly we

7. Given a group G and an integer n, the Eilenberg-Maclane space $K(G, n)$ is defined as a topological space that has n-th homotopy group $\pi_n(G)$ isomorphic to G and all other homotopy groups trivial. In $D = 2 + 1$, the cohomology group that classifies discrete torsion is the same as the group cohomology of $\mathbb{Z}_q \times \mathbb{Z}_{q'}$, because an Eilenberg–MacLane space of type $K(G, 1)$ is isomorphic to BG, the classifying space of G. We emphasize that the form-degree of the symmetries generated by Wilson lines and magnetic defects is such that $H^{D-1}(K(\mathbb{Z}_q, D-2) \times K(\mathbb{Z}_{q'}, 1), U(1))$ is isomorphic to $\mathbb{Z}_{\gcd(N/q,N/q')}$ and does not depend on the spacetime dimension D.

have:

$$\mathcal{S}_{\mathbb{Z}_q\times\mathbb{Z}_{q'},f}(\Sigma) = C(\Sigma) \sum_{\substack{\gamma\in H_1(\Sigma,\mathbb{Z}_q)\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}'_q)}} e^{\frac{2\pi i}{\gcd(N/q,N/q')} f\langle\gamma,\Gamma\rangle} W^{N/q}(\gamma) M^{N/q'}(\Gamma), \tag{3.3.1}$$

with $C(\Sigma)$ a coefficient that depends on the topology of Σ. The conventions we are using is that the first subgroup appearing in the label of $\mathcal{S}_{\mathbb{Z}_q\times\mathbb{Z}_{q'}}$ corresponds to the subgroup of the $(D-2)$-form symmetry generated by Wilson lines and the second to the subgroup of the 1-form symmetry generated by magnetic defect. We also set as a convention that the algebraic expression for the higher gauging condensation defects should have the Wilson lines on the left, which corresponds to having the Wilson lines in the global left of the magnetic defects as reviewed in Section 1.3. Note that in $D = 2 + 1$, one can fuse the Wilson line and magnetic operator into a Dyon with electric and magnetic charge. Therefore, in $D = 2 + 1$ one can gauge “diagonal" 1-form subgroups generated by the Dyons. As we show in Section 3.3.3 this generates the electric-magnetic symmetry discussed in Sections 2.4.3 and 2.3.4.

One can use (3.3.1) and the algebraic properties of Wilson lines and magnetic defects of (3.2.9) to compute the fusion rules of higher gauging condensation defects. In this section, we will check again the fusion rules we derived in the previous sections by following this procedure. We deter to the next section a derivation of some general fusion rules, but we refer the reader to Appendix A of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] for more details.

Similarly, one can use (3.3.1) to compute the transformation of other operators. Here, we reinterpret Eqs. (5.16) and (5.43) of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] to higher dimensions. We can think of the moves performed in their paper as done in a

3d slice, with the 1-form magnetic defect filling all other dimensions. One can then derive:

$$\mathcal{S}_{\mathbb{Z}_q\times\mathbb{Z}_{q'},f}\cdot L=\sum_{\substack{a,a'=0\\ Q_1+\frac{(\gcd(N/q,N/q')+f)}{\gcd(q,q')}a'q=0 \mod q\\ Q_2-faq'=0 \mod q'}}^{q-1,q'-1} W^{Na/q}M^{Na'/q'}L. \tag{3.3.2}$$

with (Q_1,Q_2) the $\mathbb{Z}_q\times\mathbb{Z}_{q'}$ higher-form symmetry charges of the operator L. This expression computes the transformation of operators "from right to left". This comes from the convention used in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] of having the surface defect oriented inwards. The transformation "from left to right" is the same as the action of the orientation reversal (2.2.1).

3.3.2 Untwisted domain walls as higher gauging defects

In this section, we derive the non-trivial correspondence presented in Table 3.3 between the untwisted domain walls and higher gauging condensation defects. We further use these higher gauging expressions to confirm the fusion rules and the transformation of other operators derived in [Córdova et al.(2024)Córdova, Costa, and Hsin] and reviewed in Section 1.3.

Diagonal domain walls

Let us start with the diagonal domain wall, $\mathcal{D}_{\mathbb{Z}_M^{(\mathrm{id})}}$ associated with the subgroup $\mathbb{Z}_N^{(\mathrm{id})}=\{(m,m):m\in\mathbb{Z}_M\}\lhd\mathbb{Z}_N\times\mathbb{Z}_N$ with $M|N$. The representation associated with the projector of (3.2.3) can be written in terms of the unit charge Wilson line as

$$\mathrm{Ind}_{\mathbb{Z}_M}^{\mathbb{Z}_N}1=\sum_{q=0}^{N/M-1}W^{qM}. \tag{3.3.3}$$

Domain wall	Condensation
$\mathcal{D}_{\mathbb{Z}_M^{(\mathrm{id})}}$	$\mathcal{S}_{\mathbb{Z}_{N/M}\times 1}$
$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$	$\mathcal{S}_{1\times\mathbb{Z}_N}$
$\mathcal{D}_{1\times\mathbb{Z}_N}$	$\mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N}$
$\mathcal{D}_{\mathbb{Z}_N\times 1}$	$\mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N,-1}$
$\mathcal{D}_{\mathbb{Z}_N^{(\mathrm{id})}}$	$\mathcal{S}_1$
$\mathcal{D}_{\mathbb{Z}_N^{(m)}}$	$\mathcal{S}_{\mathbb{Z}_{N/\ell}\times\mathbb{Z}_{N/\ell},\ell\frac{m}{1-m}}$

Table 3.3: Dictionary between higher gauging condensation defects and domain walls in $\mathbb{Z}_N$ gauge theory. Domain walls are classified by subgroups of the folded theory $K \triangleleft \mathbb{Z}_N \times \mathbb{Z}_N$ (with the left and right factors corresponding respectively to the left and right of the domain wall) with a choice of topological action $\alpha \in H^{D-1}(K, U(1))$. Higher gauging condensation defects are classified by subgroups of the higher-form global symmetry $\mathbb{Z}_q \times \mathbb{Z}_{q'} \triangleleft \mathbb{Z}_N \times \mathbb{Z}_N$ (with left and right factors generated respectively by Wilson line and magnetic defects) with a choice of torsion term $f \in H^{D-1}(K(\mathbb{Z}_q, D-2) \times K(\mathbb{Z}_{q'}, 1), U(1)) = \mathbb{Z}_{\gcd(N/q,N/q')}$. In the last row, $\mathbb{Z}_N^{(m)} = \{(mn, n) : n \in \mathbb{Z}_N\}$ is the automrphism subgroup associated with the automorphism $1 \mapsto m \in \mathbb{Z}_N^\times$ and $\ell = \gcd(N, m-1)$.

Therefore, using (3.2.4) we have

$$\begin{aligned}
\mathcal{D}_{\mathbb{Z}_M^{(\mathrm{id})}}(\Sigma) &= \frac{N}{M}\prod_{a=1}^{b_1}\frac{1}{|N/M|}\sum_{q_a=0}^{N/M-1} W^{q_a M}(\gamma_a), \\
&= \frac{N/M}{|H_1(\Sigma,\mathbb{Z}_{N/M})|}\sum_{q_1}\sum_{q_2}\cdots\sum_{q_{b_1}} W^{q_1 M}(\gamma_1)\cdots W^{q_b M}(\gamma_{b_1}), \\
&= \frac{N/M}{|H_1(\Sigma,\mathbb{Z}_{N/M})|}\sum_{q_1}\sum_{q_2}\cdots\sum_{q_{b_1}} W^{M}(q_1\gamma_1+\cdots q_{b_1}\gamma_{b_1}), \\
&= \frac{N/M}{|H_1(\Sigma,\mathbb{Z}_{N/M})|}\sum_{\gamma\in H_1(\Sigma,\mathbb{Z}_{N/M})} W^M(\gamma)\ , \\
&= \mathcal{S}_{\mathbb{Z}_{N/M}\times 1}(\Sigma),
\end{aligned} \tag{3.3.4}$$

The Wilson line W^M generate a $\mathbb{Z}_{N/M}$ $(D-2)$-form symmetry in D spacetime dimension, and the domain wall is equivalent to gauging this symmetry on the support of the domain wall instead of the entire spacetime [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao, Choi et al.(2022b)Choi, Córdova, Hsin, Lam, and Shao]. The higher gauging expression in (3.3.4) differs from the result of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] by an Euler counterterm on the wall. Indeed, in $D = 2+1$, Σ is a surface and for a genus-g surface the coefficient in front equals $|N/M|^{1-2g}$. Rescaling by an Euler counter term $\lambda^{\chi(\Sigma)}$ with $\lambda = |N/M|^{-1/2}$ and $\chi(\mathcal{M}) = 2-2g$ the Euler number of Σ, the prefacto becomes $|N/M|^{-g} = 1/\sqrt{|H_1(\Sigma, \mathbb{Z}_{N/M})|}$ which is the same normalization used in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. If instead we used $\lambda = |N/M|^{-1}$, the normalization would become $|N/M|^{-1} = 1/|H^0(\mathcal{M}, \mathbb{Z}_{N/M})|$ which is the same normalization used in [Kaidi et al.(2023b)Kaidi, Ohmori, and Zheng]. In $D = 2+1$ because the Euler counterterm is $2-2g$ it is particularly hard to reason if there exists a more natural normalization. We believe our formalism gives an affirmative answer to this question and here we provide what we believe is the answer.

Consistently, we could also use the general expression (3.2.8) to derive the higher gauging expression (3.3.4). We postpone this to the next section after deriving the higher gauging expression for $\mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}$.

Fusion rule One can use the right side of (3.3.4) and the algebraic properties of Wilson lines and magnetic defects to compute its fusion rule. The detailed calculation takes about two pages and was presented in Appendix A of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. Adapting their derivation to our normalization leads one to

$$\mathcal{S}_{\mathbb{Z}_{N/M}\times 1} \times \mathcal{S}_{\mathbb{Z}_{N/M'}\times 1} = N\frac{\gcd(M,M')}{MM'}\mathcal{S}_{\mathbb{Z}_{N/\gcd(M,M')}\times 1} \tag{3.3.5}$$

where we used that $\gcd(N/M, N/M') = N/(MM'/\gcd(M,M'))$. This agrees with (2.3.5) because $\mathbb{Z}_{N/n}\cap\mathbb{Z}_{N/n'} = \mathbb{Z}_{\gcd(N/n,N/n')} = \mathbb{Z}_{N/(nn'/\gcd(n,n'))}$ and $\mathbb{Z}_M\cdot\mathbb{Z}_{M'} = \mathbb{Z}_{MM'/\gcd(M,M')}$. We see that (2.3.5) specializes to finite-group gauge theory but generalizes in dimensions and gauge group the expression Eq. (5.24) of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao].

Transformation of other operators We can use the condensation expression of (3.3.4) to compute the transformation of other operators using the algebraic properties of Wilson lines and magnetic defects. Using (3.3.2) one can derive

$$\mathcal{S}_{\mathbb{Z}_{N/M}\times 1}\cdot W^{kN/M}=\sum_{n=0}^{M-1}W^{nN/M},\qquad \mathcal{S}_{\mathbb{Z}_{N/M}\times 1}\cdot M^{\ell M}=\sum_{n=0}^{N/M-1}M^{nM},\tag{3.3.6}$$

for all $0\leq k\leq M-1$ and $0\leq \ell\leq N/M-1$ and $\mathcal{S}_{\mathbb{Z}_{N/M}\times 1}\cdot W^{k}=\mathcal{S}_{\mathbb{Z}_{N/M}\times 1}\cdot M^{\ell}=0$ for the other lines. This is consistent with the transformation properties reviewed in Section 1.3. Indeed, the lines $W^{kN/M}$ for $0\leq k,\leq M-1$ are gauge invariant when restricted to $\mathbb{Z}_M$. In particular, all Wilson lines can end on $\mathcal{D}_1$.

Factorized domain walls

Let us consider the factorized domain wall with $K=\mathbb{Z}_N\times\mathbb{Z}_N$. The domain wall operator is (3.2.11): denote a basis for $(D-2)$-cycles on the support Σ of the domain wall operator by $\{\Gamma_a\}$, and the basic magnetic operator operator by M, then

$$\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}(\Sigma)=\frac{1}{N}\prod_{a=1}^{b_1}\sum_{q_a=0}^{N-1}M^{q_a}(\Gamma_a)=\frac{1}{N}\sum_{\Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_N)}M(\Gamma)=\mathcal{S}_{1\times\mathbb{Z}_N}(\Sigma),\tag{3.3.7}$$

As before, we see that $\mathcal{D}_{G\times G}$ is the higher gauging of the $\mathbb{Z}_N$ 1-form magnetic symmetry. Equation (3.3.7) differs from the expression in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] by an Euler conterterm on the wall. Indeed, for a genus-g surface, the coefficient in front equals $|N|^{-1}$. Rescaling by an Euler counterterm $\lambda^{\chi(\Sigma)}$ with $\lambda=|N|^{1/2}$ and $\chi(\Sigma)=2-2g$ the Euler number of Σ, the prefactor becomes $|N|^{-g}=1/\sqrt{H_1(\Sigma,\mathbb{Z}_N)}$ which is the same normalization used in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao].

From the fusion rule (2.3.6) it is now easy to construct the $\mathcal{D}_{\mathbb{Z}_N\times 1},\mathcal{D}_{1\times\mathbb{Z}_N}$. The difference between the two comes from the convention used for the ordering of the Wilson line and magnetic operator in the definition of higher gauging, see (3.3.1). With the convention we are using it follows that:

$$\mathcal{D}_{1\times\mathbb{Z}_N}(\Sigma) = \frac{1}{|H_1(\Sigma,\mathbb{Z}_N)|} \sum_{\substack{\gamma\in H_1(\Sigma,\mathbb{Z}_N)\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_N)}} W(\gamma)M(\Gamma) = \mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N}(\Sigma), \tag{3.3.8}$$

$$\mathcal{D}_{\mathbb{Z}_N\times 1}(\Sigma) = \frac{1}{|H_1(\Sigma,\mathbb{Z}_N)|} \sum_{\substack{\gamma\in H_1(\Sigma,\mathbb{Z}_N)\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_N)}} M(\Gamma)W(\gamma) = \mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N,-1}(\Sigma). \tag{3.3.9}$$

where in the second line we used (3.2.9) to permute $M(\Gamma)W(\gamma)$ giving the torsion term $f = -1 \in H^{D-1}(K(\mathbb{Z}_N, D-2)\times K(\mathbb{Z}_N,1), U(1)) = \mathbb{Z}_N$.

Finally, consistently with the previous section, we can use the general expression (3.2.8) to derive the higher gauging expression for $\mathcal{D}_1$ similarly to what we did in (3.2.13). Using (3.3.4) and (3.3.7) we can write

$$\begin{aligned}
\mathcal{D}_1(\Sigma) &= \frac{N}{|H_1(\Sigma,\mathbb{Z}_N)|^2} \sum_{\substack{\gamma,\gamma'\in H_1(\Sigma,\mathbb{Z}_N)\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_N)}} W(\gamma)M(\Gamma)W(\gamma'),\\
&= \frac{N}{|H_1(\Sigma,\mathbb{Z}_N)|^2} \sum_{\substack{\gamma,\gamma'\in H_1(\Sigma,\mathbb{Z}_N),\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_N)}} e^{\frac{2\pi i}{N}\langle\gamma,\Gamma\rangle} M(\Gamma)W(\gamma+\gamma'),\\
&= \frac{N}{|H_1(\Sigma,\mathbb{Z}_N)|} \sum_{\gamma\in H_1(\Sigma,\mathbb{Z}_N)} W(\gamma),\\
&= \mathcal{S}_{\mathbb{Z}_N\times 1}(\Sigma),
\end{aligned} \tag{3.3.10}$$

where we used (3.2.9) to permute the Wilson line and magnetic operator, we joined the Wilson lines, redefined $\gamma = \gamma - \gamma'$ and summed over γ' forcing $\Gamma = 0$ and giving the $|H_1(\Sigma,\mathbb{Z}_N)|$ factor.

Fusion rules We can use the higher gauging expression for $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$ derived in (3.3.7) and the algebraic properties of Wilson lines and magnetic defects to compute its fusion rule. For example:

$$\begin{aligned}
\mathcal{S}_{1\times\mathbb{Z}_N}\times\mathcal{S}_{1\times\mathbb{Z}_N} &= \frac{1}{N^2}\sum_{\Gamma,\Gamma'} M(\Gamma)M(\Gamma') = \mathcal{Z}(\mathbb{Z}_N,\Sigma)\mathcal{S}_{1\times\mathbb{Z}_N},\\
\mathcal{S}_{\mathbb{Z}_N\times 1}\times\mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N} &= \frac{N}{|H_1(\Sigma,\mathbb{Z}_N)|^2}\sum_{\gamma,\gamma',\Gamma} W(\gamma'+\gamma)M(\Gamma) = N\mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N},\\
\mathcal{S}_{\mathbb{Z}_N\times 1}\times\mathcal{S}_{\mathbb{Z}_N\times\mathbb{Z}_N,-1} &= \frac{N}{|H_1(\Sigma,\mathbb{Z}_N)|^2}\sum_{\gamma,\gamma',\Gamma} e^{-\frac{2\pi i}{N}\langle\gamma,\Gamma\rangle}W(\gamma'+\gamma)M(\Gamma) = \mathcal{S}_{\mathbb{Z}_N\times 1}.
\end{aligned} \tag{3.3.11}$$

For the first derivation we used $M(\Gamma)M(\Gamma')=M(\Gamma+\Gamma')$ and then we redefined the $(D-2)$-cycles, summed over the remaining one giving a factor of $|H_{D-2}(\Sigma,\mathbb{Z}_N)| = |H^1(\Sigma,\mathbb{Z}_N)| = N\mathcal{Z}(\mathbb{Z}_N,\Sigma)$. For the second, we redefined the 1-cyles and summed over the remaining one giving a factor of $|H_1(\Sigma,\mathbb{Z}_N)|$. For the third, we did the same, but the summation over the remaining cycle forced $\Gamma=0$ in addition to giving the factor of $|H_1(\Sigma,\mathbb{Z}_N)|$. This, and the other 13 fusion rules, agree with the elegant expression (2.3.8). For the derivation, one needs to use (3.2.10).

Transformation of other operators Similarly, we can use the higher gauging expression for the domain walls to compute the transformation of other operators using the algebraic properties of Wilson lines and magnetic defects. Using (3.3.2) one can derive

$$\mathcal{S}_{1\times\mathbb{Z}_{N/M}}\cdot M^{kN/M} = \sum_{n=0}^{M-1} M^{nN/M}, \qquad \mathcal{S}_{1\times\mathbb{Z}_{N/M}}\cdot W^{\ell M} = \sum_{n=0}^{N/M-1} W^{nM}, \tag{3.3.12}$$

for all $0\le k\le M-1$ and $0\le \ell\le N/M-1$ and $\mathcal{S}_{1\times\mathbb{Z}_{N/M}}\cdot M^k = \mathcal{S}_{1\times\mathbb{Z}_{N/M}}\cdot W^\ell = 0$ for the other lines. This is consistent with (2.4.7) reviewed in Section 1.3. In particular, for $M=N$ all magnetic defects can end on $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$ and no electric lines can.

Automorphism domain walls

Consider the automorphism $m \in \mathbb{Z}_N^\times \cong \mathrm{Aut}(\mathbb{Z}_N)$ that sends $1 \mapsto m$ and its associated subgroup $\mathbb{Z}_N^{(m)} = \{(mn, n) : n \in \mathbb{Z}_N\}$. Let's first assume that $\gcd(1-m, N) = 1$. Then

$$
\begin{aligned}
\mathcal{D}_{\mathbb{Z}_N^{(m)}}(\Sigma) &= \frac{1}{N^{b_1}} \sum_{i_1, i_2, \ldots, i_{b_1}} W_{\overline{\rho}_{i_1}}(A_1) \ldots W_{\overline{\rho}_{j_{b_1}}}(A_{b_1}) \mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma) W_{\rho_{i_1} \cdot \phi}(A_1) \ldots W_{\rho_{i_{b_1}} \cdot \phi}(A_{b_1}) \\
&= \frac{1}{N^{b_1}} \sum_{i_1=0}^{N-1} \cdots \sum_{i_{b_1}=0}^{N-1} W^{-1}(i_1 A_1 + \cdots + i_{b_1} A_{b_1}) \mathcal{D}_{\mathbb{Z}_N \times \mathbb{Z}_N}(\Sigma) W^m(i_1 A_1 + \cdots + i_{b_1} A_{b_1}) \\
&= \frac{1}{N^{b_1}} \sum_{\substack{\gamma \in H_1(\Sigma, \mathbb{Z}_N) \\ \Gamma \in H_{D-2}(\Sigma, \mathbb{Z}_N)}} W^{-1}(\gamma) M(\Gamma) W^m(\gamma) \\
&= \frac{1}{N^{b_1}} \sum_{\substack{\gamma \in H_1(\Sigma, \mathbb{Z}_N) \\ \Gamma \in H_{D-2}(\Sigma, \mathbb{Z}_N)}} e^{-\frac{2\pi i}{N} m \langle \gamma, \Gamma \rangle} W^{m-1}(\gamma) M(\Gamma) \\
&= \frac{1}{N^{b_1}} \sum_{\substack{\gamma \in H_1(\Sigma, \mathbb{Z}_N) \\ \Gamma \in H_{D-2}(\Sigma, \mathbb{Z}_N)}} e^{\frac{2\pi i}{N} \frac{m}{1-m} \langle \gamma, \Gamma \rangle} W(\gamma) M(\Gamma), \\
&= \mathcal{S}_{\mathbb{Z}_N \times \mathbb{Z}_N, \frac{m}{1-m}}(\Sigma)
\end{aligned}
\tag{3.3.13}
$$

In the first equality, we combined the paths. Then, we passed the Wilson lines through the magnetic operator getting a phase proportional to their intersection number, then we redefined $\gamma \to (1-m)\gamma$ (which is possible under the assumption that $\gcd(1-m, N) = 1$). Notice that if $m = 1$ the Wilson lines would be annihilated and the summation over γ would enforce $\Gamma = 0$ with an extra factor of N^{b_1} resulting in $D_1(\Sigma) = 1$, confirming the previous result. We derived that $\mathcal{D}_{\mathbb{Z}_N^{(m)}}$ is the higher gauging of the global higher-form symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$ generated by the Wilson line and magnetic operator with torsion term given by

$\frac{m}{1-m}$ [8].

Now, consider the more general case $\ell = \gcd(1-m, N) \neq 1$. We would instead have:

$$\begin{aligned}
\mathcal{D}_{\mathbb{Z}_N^{(m)}}(\Sigma) &= \sum_{j_a,k_a=0}^{N-1} e^{-\frac{2\pi i}{N} m j_a k_a} W^{m-1}(k_a\gamma_a) M(j_a\Gamma_a) \\
&= \sum_{l_a,s_a,j_a=0}^{\ell, N/\ell-1, N-1} e^{-\frac{2\pi i}{N} m j_a (l_a N/\ell + s_a)} W^{m-1}(s_a\gamma_a) M(j_a\Gamma_a), \qquad k_a = l_a N/\ell + s_a \\
&= \ell^{b_1} \sum_{s_a,s'_a=0}^{N/\ell-1} e^{-\frac{2\pi i}{N/\ell} m s'_a s_a} W^{m-1}(s_a\gamma_a) M(\ell s'_a \Gamma_a), \\
&= \frac{\ell^{b_1}}{N^{b_1}} \sum_{\substack{\gamma\in H_1(\Sigma,\mathbb{Z}_{N/\ell}) \\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_{N/\ell})}}^{N/\ell-1} e^{\frac{2\pi i}{N/\ell}\ell\frac{m}{1-m}\langle\gamma,\Gamma\rangle} W^{\ell}(\gamma) M^{\ell}(\Gamma), \\
&= S_{\mathbb{Z}_{N/\ell}\times\mathbb{Z}_{N/\ell}, \ell\frac{m}{1-m}}(\Sigma),
\end{aligned} \tag{3.3.14}$$

where from the second to the third line we performed the summation over l_a giving the factor of ℓ^{b_1} and enforcing $j_a N/\ell \equiv 0 \mod N$, that is $j_a = \ell s'_a$ with $0 \leq s' \leq N/\ell - 1$. Finally we redefined $s_a \to \frac{\ell}{1-m} s_a$ which is possible because $(1-m)/\ell$ and N/ℓ are coprime.

Fusion rule One can use the condensation expression of (3.3.14) and the algebraic properties of Wilson lines and magnetic defects to compute its fusion rule. As we are going to see in detail in a more general context in Section 3.4.2, the derivation depends in subtle ways on the number theoretic properties of the particular numbers involved, but in all cases we investigated we checked that the defect fuses according to the automorphism composition

8. If we repeated the above derivation for the defect associated with the inverse automorphism $1 \mapsto m^{-1}$, we would get the higher gauging of the electric and magnetic symmetries with a torsion term equal to $\frac{1}{m-1}$. This gives the deeper underlying reasons behind the "crucial" change of variables $f = \frac{1}{m-1}$ mentioned in [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao].

law as in (2.3.1).

Transformation of other operators Similarly, one can use the condensation expression of (3.3.14) to compute the transformation of other operators of the automorphism domain wall using the algebraic properties of Wilson lines and magnetic defects. Using (3.3.2) one can derive

$$\mathcal{S}_{\mathbb{Z}_{N/\ell}\times\mathbb{Z}_{N/\ell},\ell\frac{m}{1-m}}\cdot M = M^m, \qquad \mathcal{S}_{\mathbb{Z}_{N/\ell}\times\mathbb{Z}_{N/\ell},\ell\frac{m}{1-m}}\cdot W = W^{1/m}. \tag{3.3.15}$$

This is consistent with (2.4.3) reviewed in Section 1.3. Indeed, let ρ and $\rho^{1/m}$ be the representations associated to W and $W^{1/m}$ respectively. Then $\rho^{1/m}\otimes\overline{\rho}$ is the trivial representation of $\mathbb{Z}_N\times\mathbb{Z}_N$ when restricted to the subgroup $\mathbb{Z}_N^{(m)}\lhd\mathbb{Z}_N\times\mathbb{Z}_N$. This means that the configuration with the Wilson line W coming from the right and being permuted to $W^{1/m}$ after passing thru $\mathcal{D}_{\mathbb{Z}_N^{(m)}}$ is gauge invariant. Likewise, the configuration with magnetic operator M coming from the right and being permute to M^m is also gauge invariant. The fact that Eq. (5.43) of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao] also computes the transformation of other operators "from right to left" corresponds to their convention of having the condensation defect oriented inwards. The transformation "from left to right" would corresponds to have the condensation defect oriented outwards which is the same as the transformation of the orientation reversal (2.2.1).

Consistency with electric-magnetic duality The condensation expression (3.3.14) is consistent with electromagnetic duality in $D=2+1$. Under electric-magnetic duality (i.e., $W\leftrightarrow M$) we should have $\mathcal{D}_{\mathbb{Z}_N^{(m)}}\mapsto\mathcal{D}_{\mathbb{Z}_N^{(1/m)}}$. Indeed, an electric-magnetic duality transformation will change the order of M and W in the above condensation. Then, switching the order back to the order conventionally used to define the higher gauging will change the phase to $\frac{m}{1-m}\mapsto\frac{m}{1-m}+1$. Finally, by swapping the paths $\langle\Gamma,\gamma\rangle\mapsto-\langle\gamma,\Gamma\rangle$ the overall transformation will exactly map $\mathcal{D}_{\mathbb{Z}(m)} \mapsto \mathcal{D}_{\mathbb{Z}(1/m)}$. Note that everything is consistent because $\ell \mapsto \gcd(N, 1-\frac{1}{m}) = \gcd(N, 1-m) = \ell$ because m is coprime with N.

3.3.3 Twisted domain walls as sequential higher gauging defects

In the previous section, we derived the correspondence between untwisted domain walls and higher gauging condensation defects. Here, we extend our investigation to twisted domain walls, i.e., domain walls decorated with a non-trivial topological action. To organize the investigation, we introduce the concept of sequential higher gauging. This notion involves the higher gauging of the $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form global symmetry generated by magnetic defects and the dual symmetry that emerges after gauging the $(D-2)$-form symmetry generated by Wilson lines.

Electric-magnetic duality domain wall in $D = 2 + 1$

To begin the discussion, let us show that in $D = 2 + 1$, the $\mathbb{Z}_2$ electric-magnetic duality domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ with non-trivial topological action $\alpha_2 \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ (see Section 1.3), is equivalent to the higher gauging of the $\mathbb{Z}_2$ 1-form global symmetry generated by the Dyon, i.e.,

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2} = \mathcal{S}_{\mathbb{Z}_2^{(\mathrm{id})}} \tag{3.3.16}$$

with $\mathbb{Z}_2^{(\mathrm{id})} \lhd \mathbb{Z}_2 \times \mathbb{Z}_2$ the diagonal $\mathbb{Z}_2$ subgroup of the 1-form global symmetry.

One way to see this is by noticing that one can obtain $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N,\alpha_2}$ by attaching the non-trivial topological action α_2 to $\mathcal{D}_{\mathbb{Z}_N\times\mathbb{Z}_N}$. We saw in Section 3.3.2 that the later is the higher gauging of the 1-form symmetry generated by magnetic defects. If we attach the topological action we should dress the magnetic defects with suitable electric charges [Barkeshli et al.(2022)Barkeshli, Chen, Hsin, and Kobayashi]. Therefore, $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_2}$ should be the condensation of the Dyon.

One can also derive this correspondence explicitly from the BF action (see Section 4.1 of [Córdova et al.(2024)Córdova, Costa, and Hsin] for notation and details). The action for the condensation of the Dyon on Σ can be written as

$$\mathcal{S}_{\mathbb{Z}_2^{(\mathrm{id})}}(\Sigma): \qquad \frac{i}{\pi}\int_{\mathcal{M}} \tilde{a}^{(1)} \wedge da^{(1)} + \frac{i}{\pi}\int_{\Sigma} (a^{(1)} + \tilde{a}^{(1)}) \wedge A^{(1)}. \tag{3.3.17}$$

with $A^{(1)}$ the gauged background field. Having $\mathcal{M} = \mathcal{M}_L \cup \mathcal{M}_R$ such that $\partial\mathcal{M}_L = -\partial\mathcal{M}_R = \Sigma$, we define $a = a_L + a_R$ with a_L defined on $\mathcal{M}_L$ and a_R on $\mathcal{M}_R$ and the same for $\tilde{a}$. Then, one notes that the cross-terms that are integrated on $\mathcal{M}$ are zero except on Σ. However, integrating out A forces $a|_\Sigma = \tilde{a}|_\Sigma$, therefore such terms cancel if we integrate one by part. Integration by parts generates the term $\tilde{a}_R \wedge a_L$ on Σ, and using the condition from integrating out $A^{(1)}$ we derive:

$$\mathcal{S}_{\mathbb{Z}_2^{(\mathrm{id})}}(\Sigma): \qquad \frac{i}{\pi}\int_{\mathcal{M}} (\tilde{a}_L^{(1)} \wedge da_L^{(1)} + \tilde{a}_R^{(1)} \wedge da_R^{(1)}) + \frac{i}{\pi}\int_{\Sigma} a_R^{(1)} \wedge a_L^{(1)}, \tag{3.3.18}$$

which is precisely the defect associated to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup with the non-trivial topological action $\alpha_2 \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$ as showed in [Córdova et al.(2024)Córdova, Costa, and Hsin]. We conclude (3.3.16).

This result generalizes to $\mathbb{Z}_N$ where the $\mathbb{Z}_N^\times$ different Dyons corresponds to the different non-trivial topological actions $H^2(\mathbb{Z}_N, U(1)) = \mathbb{Z}_N$.

Diagonal twisted domain walls and sequential higher gauging

There is a counting incompatibility between domain walls and higher gauging condensation defects that appears in spacetime dimensions different from $D = 2+1$. The global symmetry $\mathbb{Z}_N \times \mathbb{Z}_N$ generated by Wilson lines and magnetic defects have different form degrees if $D \neq 2+1$, which has fewer subgroups than the case when they are of the same form degree.

For example, they do not have diagonal subgroups corresponding to Dyons when $D \neq 2+1$. However, there are still domain walls $\mathcal{D}_H$ associated to every subgroup of $H \lhd \mathbb{Z}_N \times \mathbb{Z}_N$, including diagonal ones. This incompatibility appears because our domain wall classification encompasses general invertible electric operators (1.3.13) obtained by stacking a topological action, and this objects are not generated by the higher gauging of Wilson lines and magnetic defects. However, there is a generalization of the notion of higher gauging that naturally produces them, and also solves the counting incompatibility mentioned before.

Sequential higher gauging If one has a theory with an 1-gaugable abelian q-form global symmetry $G^{(q)}$, gauging $G^{(q)}$ in a codimension-one submanifold Σ generates a higher gauging condensation defect. Similarly to what happens when gauging on the whole spacetime [Bhardwaj and Tachikawa(2018)], there emerges a dual symmetry $\overline{G}^{(D-1-q)}$ after gauging $G^{(q)}$ on Σ. Gauging $\overline{G}^{(D-1-q)}$ cancels the original gauging procedure leading one back to the trivial defect. However, there might be different ways of gauging $\overline{G}^{(D-1-q)}$, labeled by a choice of discrete torsion. Gauging the dual symmetry with a non-trivial choice of discrete torsion produces a non-trivial defect.

Provided this sequential higher gauging notion, the counting problem we started with is resolved because we can consider any subgroup $H \lhd \mathbb{Z}_N \times \mathbb{Z}_N$ with a choice of discrete torsion $H^{D-1}(H, U(1))$. Here, the left factor of H is the 1-form symmetry generated by the magnetic defects, and the right factor is the 1-form dual symmetry that emerges when gauging the $(D-2)$-form symmetry generated by Wilson lines on Σ. In this setup, it is possible to generate the general electric operators of (1.3.13). Let us show that with a particular example.

Consider the generalization to higher codimension of the diagonal defects (see (2.5.1) and the discussion before that). For every $1 \leq n \leq D$ and given $K = \mathbb{Z}_2$ they are classified by

$$H^n(\mathbb{Z}_2, U(1)) \cong \begin{cases} \mathbb{Z}_2 & n \text{ odd}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.3.19}$$

so we just need to consider the case in which n is odd. The domain wall $\mathcal{D}_{\mathbb{Z}_2^{(\mathrm{id})},\alpha_n}(\Sigma_n)$ is obtained by staking the non-trivial topological action $\alpha_n \in H^n(\mathbb{Z}_2, U(1))$ along some n-dimensional submanifold Σ_n. In terms of 1-cocyles, the topological action can be written as $\alpha_n = \frac{i}{\pi}\int_{\Sigma_n} a^{(1)} \wedge (da^{(1)})^{\frac{n-1}{2}}$. One can obtain this defect by gauging the electric $(D-2)$-form $\mathbb{Z}_N$ symmetry along Σ_n, and then one gauges the dual symmetry with the non-trivial discrete torsion. For example, for $n = 3$ we have:

$$\mathcal{D}_{\mathbb{Z}_2^{(\mathrm{id})},\alpha_3}(\Sigma_3) = e^{\frac{i}{\pi}\int_{\Sigma_3} a^{(1)}\wedge da^{(1)}} = \sum_{\substack{A^{(2)}\in H^2(\Sigma_3,\mathbb{Z}_N) \\ \tilde{A}^{(1)}\in H^1(\Sigma_3,\mathbb{Z}_2)}} e^{\frac{i}{\pi}\int_{\Sigma_3}\left(a^{(1)}\wedge A^{(2)} + A^{(2)}\wedge\tilde{A}^{(1)} + \tilde{A}^{(1)}\wedge d\tilde{A}^{(1)}\right)}. \tag{3.3.20}$$

Above, $a^{(1)}$ is the 1-form gauge field of a $\mathbb{Z}_2$ gauge theory, $A^{(2)}$ is the gauged background field for the electric $(D-2)$-form $\mathbb{Z}_2$ symmetry supported on Σ_3, and $\tilde{A}^{(1)}$ is the gauged background field for the dual 1-form global symmetry associated to $A^{(2)}$ that emerged along Σ_3. We see that gauging the electric $(D-2)$-form $\mathbb{Z}_2$ on Σ_3 and then gauging the dual symmetry with the non-trivial torsion term is the same as adding a topological action for a along Σ_3. The equality above generalizes in n and to $\mathbb{Z}_N$.

To conclude, the notion of sequential higher gauging solves the apparent counting problem involving our lattice domain wall formalism and higher gauging.

Non-invertible electric-magnetic duality domain wall in $D = 3+1$

Consider the domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}$ with α_3 the non-factorized topological action of $H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1)) = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. This domain wall is equivalent to the higher gauging of the generalization of a Dyon, that is, the magnetic operator dressed with the invertible electric operator (1.3.13):

$$M_\beta(\Gamma) \equiv M(\Gamma)W_\beta(\Gamma) = e^{i\int_\Gamma(\tilde{a}^{(2)}+da^{(1)})}, \tag{3.3.21}$$

with β the non-trivial element of $H^3(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. From the action for the condensation of M_β, namely:

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}: \qquad \frac{i}{\pi}\int_{\mathcal{M}} a^{(1)}\wedge d\tilde{a}^{(1)} + \frac{i}{\pi}\int_\Sigma (da^{(1)}+\tilde{a}^{(2)})\wedge A^{(1)}, \tag{3.3.22}$$

we can repeat the same manipulations of Section 3.3.3 to derive

$$\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}: \qquad \frac{i}{\pi}\int_{\mathcal{M}} (a_L^{(1)}\wedge d\tilde{a}_L^{(1)} + a_R^{(1)}\wedge d\tilde{a}_R^{(1)}) + \frac{i}{\pi}\int_\Sigma a_R^{(1)}\wedge da_L^{(1)}, \tag{3.3.23}$$

which corresponds to the domain wall associated to the $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup with the non-factorized topological action $\alpha_3 \in H^3(\mathbb{Z}_2\times\mathbb{Z}_2, U(1))$ [Córdova et al.(2024)Córdova, Costa, and Hsin].

Complementarily, we can also use the idea of sequential higher gauging to describe the above domain wall. In this language, the domain wall $\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}$ corresponds to the higher gauging of the full 1-form symmetry $\mathbb{Z}_2\times\mathbb{Z}_2$ (where the right factor is the dual 1-form symmetry that emerges when gauging the electric symmetry) with the non-factorized element of $H^3(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. More precisely,

$$\begin{aligned}\mathcal{D}_{\mathbb{Z}_2\times\mathbb{Z}_2,\alpha_3}(\Sigma) &= \sum_{\Gamma\in H_2(\Sigma,\mathbb{Z}_2)} M_\beta(\Gamma), \\ &= \sum_{\substack{A^{(2)}\in H^2(\Sigma,\mathbb{Z}_2)\\ \bar{A}^{(1)},\tilde{A}^{(1)}\in H^1(\Sigma,\mathbb{Z}_2)}} \exp\Big(\frac{i}{\pi}\int a^{(1)}\wedge A^{(2)} + A^{(2)}\wedge \bar{A}^{(1)} + \tilde{a}^{(2)}\wedge\tilde{A}^{(1)} + \tilde{A}^{(1)}\wedge d\bar{A}^{(1)}\Big),\end{aligned} \tag{3.3.24}$$

where M_β is defined in (3.3.21), $A^{(2)}$ is a background field for the electric 2-form symmetry, $\tilde{A}^{(1)}$ is a background field for the 1-form magnetic symmetry, and $\bar{A}^{(1)}$ is the background field for the 1-form dual symmetry that emerged after gauging the electric symmetry. To go from the second line to the first, we simply integrate out $A^{(2)}$, forcing $a^{(1)} = \bar{A}^{(1)}$ and then we rewrite the summation over $\tilde{A}^{(1)}$ as a summation over its Poncaré dual $\Gamma = \mathrm{PD}(\tilde{A}^{(1)})$.

3.4 Automorphism symmetry as higher gauging defects in abelian theories

We generalize the discussion from the previous section to encompass general finite abelian gauge theories without topological terms. We focus on the invertible 0-form symmetries corresponding to the automorphisms of the gauge group. Our main result is the derivation of the higher gauging expression for a set of automorphism generators. As an illustration, we present the higher gauging condensation defects for the automorphism symmetries of $\mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e., $\mathrm{Aut}(\mathbb{Z}_9) \cong \mathbb{Z}_6$ and $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.

3.4.1 Generators for automorphism symmetry of finite abelian groups

By the fundamental theorem of finite abelian groups, a finite abelian group is isomorphic to a product of $H_p = \mathbb{Z}_{p^{e_1}} \times \cdots \times \mathbb{Z}_{p^{e_L}}$ for different prime p and $1 \le e_1 \le \cdots \le e_L$ are positive integers. From here on, we will denote $p^{e_i} = p_i$. Furthermore, $\mathrm{Aut}(H \times K) = \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ for two finite groups H, K with relatively prime orders, therefore, one just needs to find the automorphism of H_p (see [Hillar and Rhea(2006)] for references).

The automorphisms of H_p are determined by where the L generators are mapped to. Let M_i be the generator of the $\mathbb{Z}_{p_i}$ factor. The automorphisms of H_p are generated by the family $\phi_{(i,j,m)} : H_p \to H_p$ with $1 \leq i, j \leq L$ and $m \in \mathbb{Z}_{p_i}$, or $\mathbb{Z}_{p_j}$, or $\mathbb{Z}_{p_i}^{\times}$ if $j > i, j < i$ or $j = i$ respectively. The automorphism $\phi_{(i,j,m)}$ is defined as $\phi_{(i,j,m)} \cdot M_k = M_k$ for $k \neq i$ and as

$$\phi_{(i,j,m)} \cdot M_i = \begin{cases} M_i M_j^{m p^{\epsilon_j - \epsilon_i}} & i < j, \\ M_i^m & i = j, \\ M_i M_j^m & i > j. \end{cases} \tag{3.4.1}$$

For $i = j$ it is simply an automorphism of the $\mathbb{Z}_{p_i}$ factor. For $i \neq j$ it can be thought of as "dressing" the M_i generator with a subgroup generated by M_j. Furthermore, the power of $\frac{p_j}{p_i} = p^{e_j - e_i}$ for $j > i$ is necessary so that the image is a generator of order p_i.

If $p_j \neq p_i$ for all $j \neq i$, our claim is trivial to check seems compositions of (3.4.1) can be used to map M_i to the most generic generator of order p_i. More explicitly, a generic generator of order p_i has the form

$$M_1^{m_1} \dots M_{i-1}^{m_{i-1}} M_i^{m_i} M_{i+1}^{m_{i+1} p^{e_{i+1} - e_i}} \dots M_L^{m_L p^{e_L - e_i}}, \tag{3.4.2}$$

with $m_i \in \mathbb{Z}_{p_i}^{\times}$, $m_j \in \mathbb{Z}_{p_i}$ for $j > i$ and $m_j \in \mathbb{Z}_{p_j}$ for $j < i$. And an automorphism that sends M_i to the generic generator above is easily seem to be a composition of (3.4.1). Now, suppose exist j such that $p_i = p_j$. In this case, it is necessary to check that the automorphisms above can permute generators with the same order. But the symmetric group is generated by transpositions [Conrad()], so we just need to show that (3.4.1) can generate the transposition of two such generators. Let M_i and M_j be generators of order q, then their transposition is equal to the composition

$$\phi_{(j,j,q-1)} \circ \phi_{(i,j,1)} \circ \phi_{(j,i,1)} \circ \phi_{(i,i,q-1)} \circ \phi_{(i,j,1)}. \tag{3.4.3}$$

Above, we used the fact that $q-1$ and q are always coprimes, i.e., $\gcd(q, q-1) = 1$, so that, $q-1 \in \mathbb{Z}_q^\times$. We conclude that (3.4.1) can generate all automorphisms of H_p.

3.4.2 Domain walls of automorphism generators as higher gauging defects

Now, consider a gauge theory with gauge group $H_p = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_L}$. The theory has a H_p $(D-2)$-form and 1-form symmetries with W_i and M_i the Wilson line and magnetic operator associated to the $\mathbb{Z}_{p_i}$ factors, respectively. From the discussion in the previous section, we can construct all $\mathrm{Aut}(H_p)$ condensation defects by composition if we have the ones corresponding to the family of generators (3.4.1). Given $p_j|q$ define the subgroup $\mathbb{Z}_{q,j} = \{(z_1, \ldots, z_L) : z_j \in \mathbb{Z}_q, z_i = 1 \; \forall \; i \neq j\} \lhd H_p$ so that, for example:

$$\mathcal{S}_{\mathbb{Z}_{q,i} \times \mathbb{Z}_{q,j}, f}(\Sigma) \equiv \frac{1}{q^{b_1}} \sum_{\substack{\gamma \in H_1(\Sigma, \mathbb{Z}_q) \\ \Gamma \in H_{D-2}(\Sigma, \mathbb{Z}_q)}} e^{\frac{2\pi i}{q} f \langle \gamma, \Gamma \rangle} W_i^{p_i/q}(\gamma) M_j^{p_j/q}(\Gamma) \tag{3.4.4}$$

In words, the above is the higher gauging of the subgroup of $(D-2)$-form symmetry with all factors trivial except the $\mathbb{Z}_{p_i}$ factor that has subgroup $\mathbb{Z}_q$ gauged, and the subgroup of the 1-form symmetry with all factors trivial except the $\mathbb{Z}_{p_j}$ factor that has subgroup $\mathbb{Z}_q$ gauged. This gauging is performed with a choice of discrete torsion [Vafa(1986), Vafa and Witten(1995)] equal to $f \in \mathbb{Z}_q$.

Now denote the automormophism subgroup associated with (3.4.1) by $H_p^{(i,j,m)} = \{(\phi_{(i,j,k)} \cdot$

$h, h) : h \in H_p\} \triangleleft H_p \times H_p$. Similarly to the derivation of (3.3.14) we have

$$\mathcal{D}_{H_p^{(i,j,m)}}(\Sigma) = \begin{cases} \mathcal{S}_{\mathbb{Z}_{p_i/\gcd(m,p_i)},i \times \mathbb{Z}_{p_i/\gcd(m,p_i)},j, -\frac{\gcd(m,p_i)}{m}}(\Sigma), & i < j, \\ \mathcal{S}_{\mathbb{Z}_{p_i/\gcd(1-m,p_i)},i \times \mathbb{Z}_{p_i/\gcd(1-m,p_i)},j, \frac{\gcd(1-m,p_i)m}{1-m}}(\Sigma), & i = j, \\ \mathcal{S}_{\mathbb{Z}_{p_j/\gcd(m,p_j)},i \times \mathbb{Z}_{p_j/\gcd(m,p_j)},j, -\frac{\gcd(m,p_j)}{m}}(\Sigma), & i > j. \end{cases} \tag{3.4.5}$$

with a well defined torsion term because $(1-m)/\gcd(1-m,p_i)$ is coprime with $p_i/\gcd(1-m,p_i)$ and has an inverse and likewise for $m/\gcd(m,p_j)$ and $m/\gcd(m,p_i)$. It is striking how the simplicity of the automorphism domain wall translates into a complex higher gauging expression that depends in subtle ways on particular number theoretic properties.

Fusion rules

We can use the condensation expression of (3.4.5) to compute the fusion rule of the automorphism domain wall using the algebraic properties of Wilson lines and magnetic defects. The derivation depends on particular number theoretic properties of the numbers in consideration. By making number theoretic assumptions we can find expressions that agree with the automorphism composition law reviewed in (2.3.1).

Let us illustrate our claim by following similar steps to (A.21) from [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao]. We have for example:

$$\begin{aligned}
&\mathcal{S}_{\mathbb{Z}_{q,i}\times\mathbb{Z}_{q,j},f}\times\mathcal{S}_{\mathbb{Z}_{q,i}\times\mathbb{Z}_{q,j},f'}\\
&=\frac{1}{q^{2b_1}}\sum_{\gamma,\gamma',\Gamma,\Gamma'}e^{\frac{2\pi i}{q}(f\langle\gamma,\Gamma\rangle+f'\langle\gamma',\Gamma'\rangle)}W_i^{\frac{p}{q}}(\gamma)M_j^{\frac{p}{q}}(\Gamma)W_i^{\frac{p}{q}}(\gamma')M_j^{\frac{p}{q}}(\Gamma'),\\
&=\frac{1}{q^{2b_1}}\sum_{\gamma,\gamma',\Gamma,\Gamma'}e^{\frac{2\pi i}{q}(f\langle\gamma,\Gamma\rangle+f'\langle\gamma',\Gamma'\rangle-\frac{p}{q}\delta_{ij}\langle\gamma',\Gamma\rangle)}W_i^{\frac{p}{q}}(\gamma)W_i^{\frac{p}{q}}(\gamma')H_i^{\frac{p}{q}}(\Gamma)M_j^{\frac{p}{q}}(\Gamma),\\
&=\frac{1}{q^{2b_1}}\sum_{\gamma,\gamma',\Gamma,\Gamma'}e^{\frac{2\pi i}{q}(f\langle\gamma,\Gamma\rangle+f'\langle\gamma',\Gamma'\rangle-\frac{p}{q}\delta_{ij}\langle\gamma',\Gamma\rangle)}W_i^{\frac{p}{q}}(\gamma+\gamma')M_j^{\frac{p}{q}}(\Gamma+\Gamma'),\\
&=\frac{1}{q^{2b_1}}\sum_{\gamma,\gamma',\Gamma,\Gamma'}e^{\frac{2\pi i}{q}(f\langle\gamma,\Gamma\rangle+(f+f'+\frac{p}{q}\delta_{ij})\langle\gamma',\Gamma'\rangle-(f+\frac{p}{q}\delta_{ij})\langle\gamma',\Gamma\rangle-f\langle\gamma,\Gamma'\rangle)}W_i^{\frac{p}{q}}(\gamma)M_j^{\frac{p}{q}}(\Gamma),\\
&=\frac{1}{q^{b_1}}\sum_{\gamma,\Gamma}e^{\frac{2\pi i}{q}\frac{ff'}{f+f'+\frac{p}{q}\delta_{ij}}\langle\gamma,\Gamma\rangle}W_i^{\frac{p}{q}}(\gamma)M_j^{\frac{p}{q}}(\Gamma),\\
&=\mathcal{S}_{\mathbb{Z}_{q,j}\times\mathbb{Z}_{q,i},\frac{ff'}{f+f'+\frac{p}{q}\delta_{ij}}},
\end{aligned}\tag{3.4.6}$$

where we assumed that $\gcd(f+f'+\frac{p}{q}\delta_{ij},q)=1$ and summed over Γ' resulting in a Kronecker delta that forced $\gamma'=\frac{f}{f+f'+\frac{p}{q}\delta_{ij}}\gamma$, giving a factor of q^{b_1}.

If we assume that p_i,p_j,m and m' are such that the conditions of the derivation above are satisfied, by plugging the values for f and f' from the relation to the automorphism domain wall derived in (3.4.5) this fusion rule translates to:

$$\mathcal{D}_{H_p^{(i,j,m)}}\times\mathcal{D}_{H_p^{(i,j,m')}}=\begin{cases}\mathcal{D}_{H_p^{(i,j,mm')}} & i=j,\\[2ex] \mathcal{D}_{H_p^{(i,j,m+m')}} & i\neq j,\end{cases}\tag{3.4.7}$$

These, of course, agrees with the automorphisms composition law derived in (2.3.1). But note that this particular derivation is correct just for numbers satisfying the assumptions.

Following the approach outlined in Appendix A of [Roumpedakis et al.(2022)Roumpedakis, Seifnashri, and Shao], a more general expression for the fusion of condensation defects can be derived in roughly two pages of calculation, namely:

$$\mathcal{S}_{\mathbb{Z}_{q,i}\times\mathbb{Z}_{q,j},f}\times\mathcal{S}_{\mathbb{Z}_{q',i}\times\mathbb{Z}_{q',j},f'}=\mathcal{S}_{\mathbb{Z}_{\mathrm{lcm}(q,q'),i}\times\mathbb{Z}_{\mathrm{lcm}(q,q'),j},\frac{ff'}{\left(\frac{fq+f'q'+p\delta_{ij}}{\gcd(q,q')}\right)}} \tag{3.4.8}$$

which holds as long as

$$\gcd\left(\frac{fq+f'q'+p\delta_{ij}}{\gcd(q,q')},\mathrm{lcm}(q,q')\right)=1. \tag{3.4.9}$$

In Section 3.4.3 we will use this expression to show that the automorphism domain walls of $\mathbb{Z}_9$ fuse according to the automorphism composition as expected.

Transformations of other operators

We can use the condensation expression of (3.4.5) to compute the transformation of other operators on the automorphism domain wall using the algebraic properties of Wilson lines and magnetic defects. Using (3.3.2) and the value for q, q' and f associated to the higher gauging expression of $\mathcal{D}_{H_p^{(i,j,m)}}$ derived in (3.4.5) and the fact that $(Q_1, Q_2) = (0,1)$ for the unit charge Wilson line and $(1,0)$ for the unit charge magnetic operator we derive:

$$\mathcal{D}_{H_p^{(i,j,m)}}(\Sigma)\cdot M_i=\begin{cases}M_iM_j^{m\frac{p_j}{p_i}}, \\ M_i^m, \\ M_iM_j^m,\end{cases}\qquad \mathcal{D}_{H_p^{(i,j,m)}}(\Sigma)\cdot W_j=\begin{cases}W_jW_i^{-m}, & i<j, \\ W_j^{\frac{1}{m}}, & i=j, \\ W_jW_i^{-m\frac{p_i}{p_j}}, & i>j,\end{cases} \tag{3.4.10}$$

with the transformation of other operators being trivial.

The derivation above shows that the defect (3.4.5) permutes the magnetic defects according to the automorphism (3.4.1) and that the representation of Wilson lines are permuted by composition with the inverse of (3.4.1). This is consistent with (2.4.3) in particular with our remark that the transformation preserves the Aharonov-Bohm braiding between the Wilson lines and the magnetic operator.

Now consider the $i \neq j$ and for concreteness the automorphism $\phi^{(1,2,m)} : \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \to \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$ for some $m \in \mathbb{Z}_{p_1}$. This automorphism acts on generators according to (3.4.1) which is the same as the action from right to left on magnetic defects, i.e., $\mathcal{D}_{H_p^{(1,2,m)}} \cdot M_1 = M_1 M_2^{mp_2/p_1}$ and $M_2 \mapsto M_2$. In terms of its action on group elements, the same automorphism is defined as $\phi^{(1,2,m)}(n_1, n_2) = (n_1, n_2 + mn_1p_2/p_1)$. Now, consider the irreducible representations $\rho_i(n_1, n_2) = e^{\frac{2\pi i}{p_i} n_i}$ with $i = 1, 2$. It follows that $\rho_1 \cdot \phi^{-1} = \rho_1$ and $\rho_2 \cdot \phi^{-1}(n_1, n_2) = e^{\frac{2\pi i}{p_2}(n_2 - mn_1p_2/p_1)} = \rho_2 \otimes \rho_1^{-m}(n_1, n_2)$. Therefore, $\mathcal{D}_{H_p^{(1,2,m)}} \cdot W_2 = W_2 W_1^{-m}$ and $W_1 \mapsto W_1$. This is but a particular case of the general expression (3.4.10).

3.4.3 Examples

Let us investigate in more detail some particular examples.

$\mathbb{Z}_9$: the $\mathbb{Z}_6$ higher gauging condensation defects

Here we will construct the symmetry defects associated to the automorphisms of $\mathbb{Z}_9$ gauge theory and we check their fusion rules. An automorphism of $\mathbb{Z}_9$ is a map that sends a generator M to M^m with $m \in \mathbb{Z}_9^\times = \{1, 2, 4, 8, 7, 5\} \cong \mathbb{Z}_6$. All this automorphisms are elements of the family defined in (3.4.5). Denote the automorphism subgroup associated to m by $\mathbb{Z}_9^{(m)} = \{(mn, n) : n \in \mathbb{Z}_9\}$. Then we have the correspondence presented in Table 3.4.

Fusion rules Using the fusion rule (3.4.8), we can check that the $5^2 = 25$ fusion rules agree with the composition law of automorphisms. For example,

$$\mathcal{D}_{\mathbb{Z}_9^{(7)}} \times \mathcal{D}_{\mathbb{Z}_9^{(8)}} = \mathcal{S}_{\mathbb{Z}_3 \times \mathbb{Z}_3, 2} \times \mathcal{S}_{\mathbb{Z}_9 \times \mathbb{Z}_9, 4} = \mathcal{S}_{\mathbb{Z}_9 \times \mathbb{Z}_9, 1} = \mathcal{D}_{\mathbb{Z}_9^{(2)}}. \tag{3.4.11}$$

Domain wall	Condensation
$\mathcal{D}_{\mathbb{Z}_9^{(2)}}$	$\mathcal{S}_{\mathbb{Z}_9\times\mathbb{Z}_9,1}$
$\mathcal{D}_{\mathbb{Z}_9^{(4)}}$	$\mathcal{S}_{\mathbb{Z}_3\times\mathbb{Z}_3,1}$
$\mathcal{D}_{\mathbb{Z}_9^{(8)}}$	$\mathcal{S}_{\mathbb{Z}_9\times\mathbb{Z}_9,4}$
$\mathcal{D}_{\mathbb{Z}_9^{(7)}}$	$\mathcal{S}_{\mathbb{Z}_3\times\mathbb{Z}_3,2}$
$\mathcal{D}_{\mathbb{Z}_9^{(5)}}$	$\mathcal{S}_{\mathbb{Z}_9\times\mathbb{Z}_9,7}$

Table 3.4: Dictonary between higher gauging condensation defects and $\mathrm{Aut}(\mathbb{Z}_9) \cong \mathbb{Z}_6$ automorphism domain walls.

which agrees with the automorphism composition rule, seems $7 \times 8 = 56 \equiv 2 \mod 9$.

$\mathbb{Z}_2 \times \mathbb{Z}_2$: the S^3 higher gauging condensation defects

Here we construct the automorphism condensation defects associated with the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and check that their fusion rules obey the $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ group law as in (2.3.1).

The Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 generators of order 2, which we denote by $A = (1,0)$, $B = (0,1)$ and $C = (1,1)$. The automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to ordered lists of the three generators $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \{\sigma \cdot [A,B,C] : \sigma \in S_3\} \cong S_3$ with group product given by the group product of the S_3 elements. The bijection is obtained by identifying the first and second elements in the corresponding list with the image of $A = (1,0)$ and $B = (0,1)$ under the corresponding automorphism. For instance, the automorphism associated with the list $(23) \cdot [A,B,C] = [A,C,B]$ maps $(1,0) = A \mapsto A = (1,0)$ and $(0,1) = B \mapsto C = (1,1)$. We use this isomorphism to denote the elements of $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ by elements of $S_3 = \{\mathrm{id}, (13), (23), (12), (123), (132)\}$. Likewise, denote the automorphism subgroup associated to $\sigma \in S_3$ by $(\mathbb{Z}_2\times\mathbb{Z}_2)^{(\sigma)} = \{(\sigma\cdot n, n) : n \in \mathbb{Z}_2\times\mathbb{Z}_2\} \leq \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

As shown in Section 3.4, the automorphisms of abelian groups can be generated by the family (3.4.1). The two automorphisms in this family for the Klein four group are $(13), (23)$. Using (3.4.5) to write them and the fact that $(12) = (13)(23)(13)$, $(123) = (23)(13)$ and

(132) = (13)(23), we find the correspondence presented in Table 3.5.

Domain wall	Condensation
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(\mathrm{id})}}$	$\mathcal{S}_1$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(13)}}$	$\mathcal{S}_{(1\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times 1),1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times 1)\times(1\times\mathbb{Z}_2),1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(12)}}$	$\mathcal{S}_{\mathbb{Z}_2^{(\mathrm{id})}\times\mathbb{Z}_2^{(\mathrm{id})},1}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(123)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2),(0,1,1,1)}$
$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(132)}}$	$\mathcal{S}_{(\mathbb{Z}_2\times\mathbb{Z}_2)\times(\mathbb{Z}_2\times\mathbb{Z}_2),(1,1,1,0)}$

Table 3.5: Dictionary between higher gauging condensation defects and $\mathrm{Aut}(\mathbb{Z}_2\times\mathbb{Z}_2)\cong S_3$ automorphism domain walls.

The torsion in the last two rows have elements associated with the pairing of $\langle\gamma_1,\Gamma_1\rangle$, $\langle\gamma_1,\Gamma_2\rangle$, $\langle\gamma_2,\Gamma_1\rangle$ and $\langle\gamma_2,\Gamma_2\rangle$. For example:

$$\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(132)}} = \frac{1}{|\mathbb{Z}_2|^{2b_1}} \sum_{\gamma_1,\gamma_2,\Gamma_1,\Gamma_2} e^{\frac{2\pi i}{2}(\langle\gamma_1,\Gamma_1\rangle+\langle\gamma_1,\Gamma_2\rangle+\langle\gamma_2,\Gamma_1\rangle)} W_1(\gamma_1)W_2(\gamma_2)M_1(\Gamma_1)M_2(\Gamma_2). \tag{3.4.12}$$

which we denoted by $\mathcal{S}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,(1,1,1,0)}$ with torsion equal to $(1,1,1,0)$.

Fusion rules Because we constructed the symmetry defects from the generators $\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(13)}}$ and $\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}}$, we just need to check their fusion. Transpositions are their own inverse,

$(13)^2 = (23)^2 = 1$, and we do have

$$
\begin{aligned}
\mathcal{S}_{\mathbb{Z}_{2,2}\times\mathbb{Z}_{2,1},1} \times \mathcal{S}_{\mathbb{Z}_{2,2}\times\mathbb{Z}_{2,1},1} &= \frac{1}{|\mathbb{Z}_2|^{2b_1}} \sum_{\gamma_i,\Gamma'_i} e^{\frac{2\pi i}{2}\langle\gamma,\Gamma\rangle} e^{\frac{2\pi i}{2}\langle\gamma',\Gamma'\rangle} W_2(\gamma) M_1(\Gamma) W_2(\gamma') M_1(\Gamma') \\
&= \frac{1}{|\mathbb{Z}_2|^{2b_1}} \sum_{\gamma,\gamma',\Gamma,\Gamma'} e^{\frac{2\pi i}{2}\langle\gamma,\Gamma\rangle} e^{\frac{2\pi i}{2}\langle\gamma',\Gamma'\rangle} W_2(\gamma+\gamma') M_1(\Gamma+\Gamma') \\
&= \frac{1}{|\mathbb{Z}_2|^{2b_1}} \sum_{\gamma,\gamma',\Gamma,\Gamma'} e^{\frac{2\pi i}{2}(\langle\gamma,\Gamma\rangle - \langle\gamma',\Gamma\rangle + \langle\gamma',\Gamma'\rangle - \langle\gamma',\Gamma\rangle)} W_2(\gamma) M_1(\Gamma') \\
&= \frac{1}{|\mathbb{Z}_2|^{2b_1}} \sum_{\gamma,\gamma',\Gamma,\Gamma'} e^{\frac{2\pi i}{2}(\langle\gamma,\Gamma\rangle + \langle\gamma',\Gamma'\rangle)} W_2(\gamma) M_1(\Gamma') \\
&= 1,
\end{aligned}
$$

where we integrate out Γ and γ' forcing $\gamma = \Gamma' = 0$. The same derivation applies to $\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}}$. Consistently, it also applies to $\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(13)}}$, because

$$
W_\psi M_\psi = W_1 W_2 M_1 M_2 = (-1) M_1 W_1 W_2 M_2 = (-1)^2 M_1 M_2 W_1 W_2 = M_\psi W_\psi. \tag{3.4.13}
$$

At last, because $(23)(13) = (123) \neq (132) = (13)(23)$, we should have $\mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(23)}} \times \mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(13)}} = \mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(123)}} \neq \mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(132)}} = \mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(13)}} \times \mathcal{D}_{(\mathbb{Z}_\times\mathbb{Z}_2)^{(23)}}$, which is also the case for their higher gauging expression. This non-commutativity of the condensation defects comes from the fact that Wilson lines and magnetic defects associated to the same cyclic subgroup braids non-trivially.

3.5 Gauging 0-form symmetry: $\mathbb{D}_4$ gauge theory from gauging swap symmetry

Given a homomorphism $\rho : H \to \mathrm{Aut}(N)$, where H is a group and $\mathrm{Aut}(N)$ is the automorphism group of N, we can construct a group G as a semidirect product of N and H, denoted

$G = N \rtimes_\rho H$. This semidirect product uses the homomorphism ρ to describe how elements of H act on elements of N. In the context of finite-group gauge theories, one can obtain G gauge theory by gauging the H 0-form symmetry of N gauge theory, realized through automorphism domain walls $\mathcal{D}_{N(\phi)}$ with $\phi \in \mathrm{Aut}(H)$.

As a concrete example, the dihedral group of order 8,

$$\mathbb{D}_4 = \langle a, b, c | a^2 = b^2 = c^2 = (ac)^4 = 1, ab = ba = cac, bc = cb = aca \rangle, \tag{3.5.1}$$

is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ with the $\mathbb{Z}_2 = \{1, c\}$ acting under conjugation by swapping two generators, i.e., $cac = ab$ and $cbc = b$. Therefore, $\mathbb{D}_4$ gauge theory can be obtained by gauging the $\mathbb{Z}_2$ 0-form swap symmetry of $\mathbb{Z}_2 \times \mathbb{Z}_2$ associated with the automorphism $\varphi \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ that maps $\varphi : (0,1) \mapsto (1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. This example has been discussed in many places, e.g. [Barkeshli et al.(2019)Barkeshli, Bonderson, Cheng, and Wang, Tantivasadakarn et al.(2023)Tantivasadakarn, Verresen, and Vishwanath], and is also related with fracton topological orders [Bulmash and Barkeshli(2019), Prem and Williamson(2019)]. The group $H^2_\varphi(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$ is trivial, which means that $\mathbb{D}_4$ is the only extension in this setting.

The presentation (3.5.1) can be related to the more standard one $\mathbb{D}_4 = \langle r, s | r^4 = s^2 = 1, srs = r^{-1} \rangle$ with the identification $a = s$, $b = r^2$ and $c = sr$. This more standard presentation make it explicit that $\mathbb{D}_4$ is also isomorphic to $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ with the $\mathbb{Z}_2$ acting under conjugation by the inversion map, i.e., $srs = r^{-1}$. As a consequence, $\mathbb{D}_4$ gauge theory can also be obtained by gauging the $\mathbb{Z}_2$ 0-form symmetry of $\mathbb{Z}_4$ gauge theory associated with the automorphism $\phi \in \mathrm{Aut}(\mathbb{Z}_4) \cong \mathbb{Z}_2$ that maps $\phi : 1 \mapsto 3 \in \mathbb{Z}_4$. The group $H^2_\phi(\mathbb{Z}_4, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$, which means that in addition to $\mathbb{D}_4$, there is a non-split extension in this setting which is $\mathbb{Q}_8$, the quaternion group. Yet another presentation of $\mathbb{D}_4$ is the non-split extension of $\mathbb{Z}_2 \cong Z(\mathbb{D}_4)$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting trivially with a non-trivial extension class $H^2(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The other non-trivial extensions classes produce

$\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Q}_8$. These other presentations are associated with other dual descriptions of $\mathbb{D}_4$ gauge theories. We leave the study of these dual settings to future works.

3.5.1 The symmetry defect for $\mathbb{Z}_2 \times \mathbb{Z}_2$ swap symmetry

From Table 3.5, the domain wall associated with the automorphism that maps $(0,1) \to (1,1)$ can be written as

$$\begin{aligned}
\mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(23)}} &= \mathcal{S}_{\mathbb{Z}_{2,1}\times\mathbb{Z}_{2,2}}(\Sigma),\\
&= \sum_{\substack{\gamma\in H_1(\Sigma,\mathbb{Z}_2)\\ \Gamma\in H_{D-2}(\Sigma,\mathbb{Z}_2)}} (-1)^{\langle\gamma,\Gamma\rangle} W_1(\gamma) M_2(\Gamma),\\
&= \sum_{\substack{A\in H^{(D-2)}(\Sigma,\mathbb{Z}_2),\\ \tilde{A}\in H^1(\Sigma,\mathbb{Z}_2)}} e^{\pi i\int_\Sigma a^{(1)}\cup A^{(D-2)}+\tilde{b}^{(D-2)}\cup\tilde{A}^{(1)}+A^{(D-2)}\cup\tilde{A}^{(1)}},\\
&= e^{\pi i\int_\Sigma a^{(1)}\cup\tilde{b}^{(D-2)}},
\end{aligned} \tag{3.5.2}$$

where we integrated out $\tilde{A}$ forcing $A = \tilde{b}$. In the second line, we are treating $W(\gamma)$ ad $M(\Gamma)$ as commuting field insertions in a path integral and used the Poincaré duals A and $\tilde{A}$ of γ and Γ respectively.

3.5.2 The action from gauging swap symmetry

Gauging the $\mathbb{Z}_2$ 0-form symmetry associated with the swap automorphism (3.5.2) corresponds to the following operator insertion:

$$\sum_{\Sigma\in H_{D-1}(\mathcal{M},\mathbb{Z}_2)} \mathcal{D}_{(\mathbb{Z}_2\times\mathbb{Z}_2)^{(12)}}(\Sigma) = \sum_{c^{(1)}\in H^1(\mathcal{M},\mathbb{Z}_2)} e^{\pi i\int_\mathcal{M} a^{(1)}\cup\tilde{b}^{(D-2)}\cup c^{(1)}}. \tag{3.5.3}$$

where c is the Poincaré dual of Σ. Introducing an auxiliary gauge field $\tilde{c} \in C^{D-2}(\mathcal{M}, \mathbb{Z}_2)$ to make c flat dynamically leads to the action:

$$S_{\mathbb{D}_4} = i\pi \int_{\mathcal{M}} (\tilde{a}^{(D-2)} \cup da^{(1)} + \tilde{b}^{(D-2)} \cup db^{(1)} + \tilde{c}^{(D-2)} \cup dc^{(1)} + a^{(1)} \cup \tilde{b}^{(D-2)} \cup c^{(1)}). \tag{3.5.4}$$

For $D = 2{+}1$, this is the action for $\mathbb{D}_4$ gauge theory we studied in [Córdova et al.(2024)Córdova, Costa, and Hsin]. In the literature it was realized that the modular data of twisted $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory is the same as that of $\mathbb{D}_4$ [de Wild Propitius(1997), Coste et al.(2000)Coste, Gannon, and Ruelle]. Here we derived a more general relation that holds in all dimensions by explicitly gauging the swap symmetry of $\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory. In the next section, we will explain the mapping between the operators of the two theories.

One can explicitly check that the action is invariant under the following gauge transformations:

$$a \to a + d\alpha, \qquad \tilde{a} \to \tilde{a} + d\tilde{\alpha} + \epsilon\tilde{b} + \tilde{\beta}c + \epsilon d\tilde{\beta}, \tag{3.5.5}$$

$$b \to b + d\beta + \epsilon a + \alpha c + \alpha d\epsilon, \qquad \tilde{b} \to \tilde{b} + d\tilde{\beta}, \tag{3.5.6}$$

$$c \to c + d\epsilon, \qquad \tilde{c} \to \tilde{c} + d\tilde{\epsilon} + \alpha\tilde{b} + \tilde{\beta}a + \alpha d\tilde{\beta}, \tag{3.5.7}$$

with $\alpha, \beta, \epsilon \in C^0(\mathcal{M}, \mathbb{Z}_2)$ and $\tilde{\alpha}, \tilde{\beta}, \tilde{\epsilon} \in C^{D-3}(\mathcal{M}, \mathbb{Z}_2)$ (and $\tilde{\alpha} = \tilde{\beta} = \tilde{\epsilon} = 0$ if $D = 2$). The fact that the gauge transformations of c shift b by a descends from the $\mathbb{Z}_2$ action that maps $(0, 1) \to (1, 1)$ and leaves $(1, 0)$ unchanged.

3.5.3 Operators and fusion rules

The set of Wilson lines and pure magnetic defects of $\mathbb{D}_4$ gauge theory, as well as their linking action, are summarized in Table 3.6. Here we are going to show how to construct these operators in terms of gauge invariant combinations of the Wilson lines and magnetic defects of the $\mathbb{Z}_2$ factors that enter in the group extensions $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ summarized in the presentation (3.5.1) (the identification $b = r^2$, $a = s$ and $c = sr$, relates this presentation with the more standard one). Provided these expressions, one can deduce the entries of the character table from the linking action of the $\mathbb{Z}_2$ constituents.

Class	$\{e\}$	$\{b\} = \{r^2\}$	$\{a, ab\} = \{s, sr^2\}$	$\{c, cb\} = \{sr, sr^3\}$	$\{ac, abc\} = \{r, r^3\}$
1	1	1	1	1	1
$\rho_a = \rho_s$	1	1	-1	1	-1
$\rho_c = \rho_{sr}$	1	1	1	-1	-1
$\rho_{a+c} = \rho_r$	1	1	-1	-1	1
$\rho_b = \rho_{r^2}$	2	-2	0	0	0

Table 3.6: Character table of $\mathbb{D}_4$.

Let us start with the Wilson lines. The gauge transformations yield the following gauge invariant, one-dimensional Wilson lines:

$$\mathbb{W}_a(\gamma) = W_a(\gamma) = e^{\pi i \oint_\gamma a}, \quad \mathbb{W}_c(\gamma) = W_c(\gamma) = e^{\pi i \oint_\gamma c}, \quad \mathbb{W}_{a+c}(\gamma) = \mathbb{W}_a(\gamma)\mathbb{W}_c(\gamma), \tag{3.5.8}$$

we use boldface letters to denote the operators of $\mathbb{D}_4$ and no-boldface for their $\mathbb{Z}_2$ constituents. These Wilson lines are associated with the three one-dimensional irreducible representations of $\mathbb{D}_4$ gauge theory, as shown in Table 3.6, with the obvious identification. The integral of b on closed loops is not invariant under gauge transformations because b shifts by a and c. To make it gauge invariant, we impose Dirichlet boundary conditions for a and c on γ, which can be achieved by dressing the operator with the condensation of a and c. Specifically, we have:

$$\mathbb{W}_b = \frac{1}{2} W_b(1 + W_a)(1 + W_c) = \frac{1}{2}(W_b + W_{a+b})(1 + W_c), \tag{3.5.9}$$

where W_b and W_{a+b} are defined in the same way as (3.5.8). In Section 1.3, we defined the diagonal domain walls and their higher codimensional generalizations through a normal subgroup $K \lhd G$. The condensation of Wilson lines projects onto this subgroup as described in Section 3.2.1. Here, the subgroup that is preserved by the condensation of $\mathbb{W}_a$ and $\mathbb{W}_c$ is the center $Z(\mathbb{D}_4) = \mathbb{Z}_2 = \{1, b\} = \{1, r^2\} \lhd \mathbb{D}_4$, i.e.,

$$\mathcal{D}_{\mathbb{Z}_2}(\gamma) = 1 + W_a(\gamma) + W_c(\gamma) + W_{a+c}(\gamma). \tag{3.5.10}$$

We remark that the condensation of W_a and W_c in a loop equals the insertion of $1 + W_a$ and $1 + W_c$ on that loop (3.2.4). Note that (3.5.9) can also be interpreted as the orbit of W_b under the swap symmetry, dressed with the condensation of W_c. This two-dimensional Wilson line is associated with the two-dimensional irreducible representation of $\mathbb{D}_4$ denoted by ρ_b in Table 3.6.

Provided the above expressions, one can directly compute the fusion rules of the Wilson lines of $\mathbb{D}_4$ gauge theory. The one-dimensional irreducible representations have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion structure and the two-dimensional representation obeys:

$$\mathbb{W}_b \times \mathbb{W}_a = \mathbb{W}_a \times \mathbb{W}_b = \mathbb{W}_b \times \mathbb{W}_c = \mathbb{W}_c \times \mathbb{W}_b = \mathbb{W}_b, \tag{3.5.11}$$

$$\mathbb{W}_b \times \mathbb{W}_b = (1 + W_c)(1 + W_a) = 1 + \mathbb{W}_a + \mathbb{W}_c + \mathbb{W}_{a+c}, \tag{3.5.12}$$

which forms a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami category [Tambara and Yamagami(1998)].

Now, let us consider the magnetic defects. In this case, the gauge transformations yield the following gauge invariant one-dimensional magnetic defect associated with the conjugacy class $\{b\} = \{r^2\}$:

$$\mathbb{M}_b(\Gamma) = M_b(\Gamma) = e^{i\pi \oint_\Gamma \tilde{b}}. \tag{3.5.13}$$

Similarly to the two-dimensional Wilson line, the magnetic defects associated with a, c, and $a + c$ need to be dressed with the condensation of M_b and suitable condensations of Wilson lines to be made gauge invariant. These magnetic defects correspond to the three two-dimensional conjugacy classes of $\mathbb{D}_4$ displayed in Table 3.6. Specifically, we have[9]

$$\mathbb{M}_a = \frac{1}{2}M_a(1+M_b)\mathcal{S}_c, \quad \mathbb{M}_c = \frac{1}{2}M_c(1+M_b)\mathcal{S}_a, \quad \mathbb{M}_{a+c} = \frac{1}{2}M_{a+c}(1+M_b)\mathcal{S}_{a+c}, \tag{3.5.14}$$

where M_a, M_c and M_{a+c} are defined in the same way as (3.5.13) and

$$\mathcal{S}_{a+c}(\Gamma) = \frac{2}{|H_1(\Gamma,\mathbb{Z}_2)|}\sum_{\gamma\in H_1(\Gamma,\mathbb{Z}_2)} \mathbb{W}_{a+c}(\gamma) = \mathcal{D}_{\mathbb{Z}_4}(\Gamma), \tag{3.5.15}$$

is the condensation of $\mathbb{W}_{a+c}$ on the codimension-two submanifold Γ and similarly for $\mathcal{S}_a$ and $\mathcal{S}_c$. In terms of the higher codimensional generalization of the diagonal domain walls of $\mathbb{D}_4$ gauge theory reviewed in section 1.3, the condensation $\mathcal{S}_{a+c}$ corresponds to the normal subgroup $\mathbb{Z}_4 = \{1, ac, (ac)^2, (ac)^3\} = \{1, r, r^2, r^3\}$ of $\mathbb{D}_4$. Likewise, the $\mathcal{S}_a$ and $\mathcal{S}_c$ condensations correspond to the two normal Klein four subgroups, i.e., $V_4 = \{1, a, b, ba\} = \{1, s, r^2, r^2 s\}$ and $V_4' = \{1, c, b, cb\} = \{1, sr, r^2, sr^3\}$ respectively. Note that $\mathcal{S}_{a+c} \neq \mathcal{S}_a\mathcal{S}_c$, from (2.5.1) we have

$$\mathcal{D}_{\mathbb{Z}_2}(\Gamma) = \mathcal{D}_{V_4\cap V_4'} = \mathcal{D}_{\mathbb{Z}_4} \times \mathcal{D}_{V_4} = \mathcal{S}_a \times S_c = \frac{4}{|H_1(\Gamma,\mathbb{Z}_2)|^2}\sum_{\gamma,\gamma'\in H_1(\Gamma,\mathbb{Z}_2)} W_a(\gamma)W_c(\gamma'), \tag{3.5.16}$$

with $\mathbb{Z}_2 = \{1, b\} = \{1, r^2\}$, the center of $\mathbb{D}_4$, which is equal to (3.5.10) when we consider the condensation on a one-dimensional submanifold. Notably, the expressions in (3.5.14) are consistent even in $D = 2$. In this degenerate case, the magnetic defects are local operators and $\mathcal{S}_a = \mathcal{S}_a = \mathcal{S}_{a+c} = 2$. As a result, $\mathbb{M}_a$, $\mathbb{M}_c$ and $\mathbb{M}_{a+c}$ are dressed just with the condensation of M_b. Consistently, $\tilde{a}$ and $\tilde{c}$ are not shifted by c and a because $\tilde{\beta} = 0$.

9. In more detail: $\tilde{a}$ shifts by c so it is dressed with $\mathcal{S}_c$, $\tilde{c}$ shifts by a so it is dressed with $\mathcal{S}_a$, and $\tilde{a}+\tilde{c}$ shifts by $a+c$ so it is dressed with $\mathcal{S}_{a+c}$.

Provided the above expressions, one can directly compute the fusion rules of the magnetic defects of $\mathbb{D}_4$ gauge theory. We have, for example:

$$\mathbb{M}_a(\Gamma) \times \mathbb{M}_a(\Gamma) = \mathcal{S}_c(\Gamma) + \mathbb{M}_b(\Gamma) \times \mathcal{S}_c(\Gamma), \tag{3.5.17}$$

i.e., we obtain the condensation of $\mathbb{W}_c$ in the codimension-two surface Γ, plus the magnetic defect $\mathbb{M}_b$ fused with this condensation. These description helps to clarify some of the results in [Heidenreich et al.(2021)Heidenreich, McNamara, Montero, Reece, Rudelius, and Valenzuela, Arias-Tamargo and Rodriguez-Gomez(2023)]. We see that the appearance of the condensation of Wilson lines in the fusion channel of pure fluxes follows from gauge invariance. In summary, the "would-be" pure fluxes are made gauge invariant by dressing them with suitable condensations. Whenever the "would-be" magnetic fluxes fuse to the identity, what remains is the dressing by the electric condensation. Note that in $D = 2 + 1$ spacetime dimensions we have $\mathcal{S}_a = 1 + W_a$, so (3.5.17) is equal to:

$$\mathbb{M}_a \times \mathbb{M}_a = 1 + \mathbb{W}_c + \mathbb{M}_b + \mathbb{M}_b \times \mathbb{W}_c. \tag{3.5.18}$$

However, for $D \neq 2 + 1$ the fusion rule (3.5.18) is not correct because, in this case, Wilson lines and magnetic defects are operators of different dimension. On the other hand, the fusion rule (3.5.17) is correct in arbitrary spacetime dimensions because all $\mathcal{S}_c$, $\mathbb{M}_a$ and $\mathbb{M}_b$ have codimension-two.

By expressing the Wilson lines and magnetic defects of $\mathbb{D}_4$ in terms of gauge-invariant combinations of Wilson lines and magnetic defects from the $\mathbb{Z}_2$ factors, one can directly deduce the elements of the $\mathbb{D}_4$ character table (see Table 3.6) through the linking actions of these components. For instance, $\mathbb{W}_b$ has a linking of zero with $\mathbb{M}_a$, $\mathbb{M}_c$ and $\mathbb{M}_{a+c}$ because M_a, M_c and M_{a+c} have linking zero with $(1 + W_a)(1 + W_b)$. However, $\mathbb{W}_b$ has a linking of -2 with $\mathbb{M}_b$ because M_b has a linking number of -1 with W_b and 4 with $(1 + W_a)(1 + W_c)$.

The other entries can be determined similarly.

In $D = 2+1$, by relabeling $b \leftrightarrow \tilde{b}$, one can view the action (3.5.4) as describing $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory twisted by the cocycle associated with the diagonal $\mathbb{Z}_2$. Then, from the gauge transformations, one can check that the theory has 8 one-dimensional line operators:

$$\mathbb{W}_a(\gamma) = W_a(\gamma) = e^{\pi i \oint_\gamma a}, \quad \mathbb{W}_b(\gamma) = W_b(\gamma) = e^{\pi i \oint_\gamma b}, \quad \mathbb{W}_c(\gamma) = W_c(\gamma) = e^{\pi i \oint_\gamma c}, \tag{3.5.19}$$

and their products, forming a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ fusion ring. Additionally, there are 14 two-dimensional line operators such as:

$$\mathbb{M}_a(\gamma) = \frac{1}{2} M_a(\gamma)\mathcal{S}_b(\gamma)\mathcal{S}_c(\gamma), \qquad \mathbb{M}_a(\gamma)\mathbb{W}_a(\gamma). \tag{3.5.20}$$

which obey, e.g.:

$$\mathbb{M}_a \times \mathbb{W}_b = \mathbb{M}_a \times \mathbb{W}_c = \mathbb{M}_a, \tag{3.5.21}$$

$$\mathbb{M}_a \times \mathbb{M}_a = \mathcal{S}_b\mathcal{S}_c = 1 + \mathbb{W}_b + \mathbb{W}_c + \mathbb{W}_{b+c}, \tag{3.5.22}$$

$$\mathbb{M}_a \times \mathbb{M}_b = \frac{1}{2} M_a M_b \mathcal{S}_a \mathcal{S}_b \mathcal{S}_c = \mathbb{M}_{a+b} + \mathbb{M}_{a+b} \times \mathbb{W}_a. \tag{3.5.23}$$

The relation to $\mathbb{D}_4$ arises by switching back $b \leftrightarrow \tilde{b}$, which amounts to identifying $\mathbb{M}_b$ with the two-dimensional Wilson line and $\mathbb{W}_b$ with the magnetic defect associated with the one-dimensional conjugacy class of $\mathbb{D}_4$.

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