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SYMPLECTIC DUALITY FOR HAMILTONIAN REDUCTIONS; AND ORTHODONTIA FOR DOUBLE GROTHENDIECK POLYNOMIALS

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To my parents

"How much soy sauce would we need to add to that coffee in order to notice a difference?"

—Seraphina Lee

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ABSTRACT

The Hamiltonian reduction $\mathcal{N}/\!\!/\!\!/T$ of the nilpotent cone in $\mathfrak{sl}_n$ by the torus of diagonal matrices is a Nakajima quiver variety which admits a symplectic resolution $\widetilde{\mathcal{N}/\!\!/\!\!/T}$, and the corresponding BFN Coulomb branch is the affine closure $\overline{T^*(G/U)}$ of the cotangent bundle of the base affine space. We construct a surjective map $\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] \twoheadrightarrow H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right)$ of graded algebras, which the Hikita conjecture predicts to be an isomorphism. Our map is inherited from a related case of the Hikita conjecture and factors through Kirwan surjectivity for quiver varieties. We conjecture that many other Hikita maps can be inherited from that of a related dual pair.

* * *

We give a new formula for double Grothendieck polynomials based on Magyar's orthodontia algorithm for diagrams. Our formula implies a similar formula for double Schubert polynomials $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$. We also prove a curious positivity result: for vexillary permutations $w \in S_n$, the polynomial $x_1^n \ldots x_n^n \mathfrak{S}_w(x_n^{-1}, \ldots, x_1^{-1}; 1, \ldots, 1)$ is a graded nonnegative sum of Lascoux polynomials. We conjecture that this positivity result holds for all $w \in S_n$. This conjecture would follow from a problem of independent interest regarding Lascoux positivity of certain products of Lascoux polynomials.

CHAPTER 1

INTRODUCTION

This thesis consists of my work in representation theory (Chapter 2) and my work in combinatorics (Chapter 3).

The key phenomenon motivating my work in representation theory is the theorem of Borel [Bor53] expressing the cohomology of the flag variety as an explicit coordinate ring: For a complex semisimple Lie group G with T a maximal torus, $\mathfrak{h} = \mathrm{Lie}(T)$ its Lie algebra, W the Weyl group, and $B \supset T$ a Borel subgroup, there are isomorphisms

$$H^*(G/B) \cong \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^W \rangle, \tag{1.1}$$

$$H_T^*(G/B) \cong \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]. \tag{1.2}$$

For many varieties X of representation theoretic interest, the geometry of $\mathrm{Spec}(H_T^*(X))$ and $\mathrm{Spec}(H^*(X))$ is rich and sheds light on X itself [GKM97, GM10].

Motivated by the theorem of de Concini–Procesi [dP81] expressing the cohomology of Springer fibers as coordinate rings of fixed point subschemes of nilpotent orbits, Hikita [Hik17] expressed the cohomology of cotangent bundles of partial flag varieties, of Hilbert schemes of points in $\mathbb{C}^2$, and of hypertoric varieties as coordinate rings of fixed point schemes.

The emerging theory of *symplectic duality* unifies the previous examples and predicts many more: The Hikita conjecture [Hik17] states that if the conical symplectic singularity X has a symplectic resolution $\widetilde{X} \to X$, and $X^!$ denotes the symplectic dual of X, then there is an isomorphism

$$H^*(\widetilde{X}) \cong \mathbb{C}[(X^!)^{T^!}], \tag{1.3}$$

where $T^!$ is the maximal torus of the group $\mathrm{Aut}_{\mathrm{Pois},\mathbb{C}^\times}(X^!)$ of Poisson automorphisms of $X^!$ commuting with dilations.

In the remainder of this introduction, we work in type A: set G to be $\mathrm{SL}_n(\mathbb{C})$, T the torus of diagonal matrices, B the Borel subgroup of upper triangular matrices, and U the unipotent subgroup of upper triangular matrices with diagonal entries equal to 1. The cotangent bundle $T^*(G/U)$ is quasi-affine and has a left action of T and a right action of the torus $B/U \simeq T$. We write $\overline{T^*(G/U)}$ for its affine closure.

Based on considerations from physics, it is predicted [DHK21] that $\overline{T^*(G/U)}$ is the symplectic dual of the Hamiltonian reduction $\mathcal{N}/\!\!/\!\!/T := \{x \in \mathcal{N} \colon \mathrm{diag}(x) = 0\}/\!\!/T$, i.e. the categorical quotient of the variety of zero-diagonal nilpotent matrices by the adjoint action of T. We show [Set24] that the variety $\mathcal{N}/\!\!/\!\!/T$ has a symplectic resolution $\widetilde{\mathcal{N}/\!\!/\!\!/T}$, and proved a partial version of the Hikita conjecture for this pair:

Theorem 2.1.1. *There is a surjective morphism of graded algebras*

$$\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] \twoheadrightarrow H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right).$$

Our proof of Theorem 2.1.1 makes use of another dual pair: we relate the resolution $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ to the resolution $\widetilde{\mathcal{N}} = T^*(G/B)$, as well as the fixed subscheme $\overline{T^*(G/U)}^{T\times B/U}$ to the fixed subscheme $(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h})^T$. Using this, surjectivity is a consequence of Kirwan surjectivity for quiver varieties proven by McGerty–Nevins [MN18], combined with equivariant Hikita for $X = \mathcal{N}$, which amounts to Borel's presentation (1.2).

To our knowledge, this argument is the first in the literature which relates one case of the Hikita conjecture to another case; it is highly suggestive of a general framework to study symplectic duality for Hamiltonian reductions.

Conjecture 2.1.4. *Let $X, X^!$ be symplectic dual varieties such that $X^! = T^*\mathrm{Rep}(Q^!)/\!\!/\!\!/\mathrm{GL}(V^!)$ is a Nakajima quiver variety. Let T denote the maximal torus of the group of Poisson automorphisms of X commuting with dilation. Assume that $X/\!\!/\!\!/T$ has a symplectic resolution. Then the Hikita conjecture holds for the pair $X/\!\!/\!\!/T$ and $X^{!,\uparrow} \overset{\mathrm{def}}{=} T^*\mathrm{Rep}(Q^!)/\!\!/\!\!/\mathrm{SL}(V^!)$.*

My work in combinatorics (Chapter 3), joint with Avery St. Dizier [SS24], is motivated by the theory of *flagged Weyl modules.* The cohomology of the flag variety has a distinguished basis consisting of fundamental classes $[X_w]$ of *Schubert varieties*, indexed by permutations $w \in S_n$. A central problem in algebraic combinatorics is to find a manifestly nonnegative formula for *Littlewood–Richardson coefficients*, which are structure constants

$$[X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w] \tag{1.4}$$

for the Schubert basis.

Schubert polynomials, introduced by Lascoux and Schützenberger [LS82], are distinguished lifts $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$ of the corresponding classes $[X_w]$ (cf. (1.1)). Schubert polynomials are combinatorially very well studied [BJS93, FK96, KM05, HMMS22].

Examples of Schubert polynomials include Schur polynomials, which classically arise as characters of irreducible representations of GL_n. From the fact that representations of GL_n are determined by their character, it follows that multiplicities of irreducible representations in the tensor product

$$V_\lambda \otimes V_\mu = \bigoplus_\nu (V_\nu)^{\oplus c_{\lambda,\mu}^\nu}$$

compute Littlewood–Richardson coefficients (1.4) for those $[X_w]$ whose corresponding Schubert polynomials are Schur polynomials.

More generally, every Schubert polynomial $\mathfrak{S}_w$ is the dual character $\chi_{D(w)}$ of the *flagged Weyl module* of the Rothe diagram $D(w)$ of w [KP87, KP04]. For a more general class of *%-avoiding* diagrams, flagged Weyl modules are spaces of sections of line bundles on Bott–Samelson varieties [Mag98]. In the special case of Schur polynomials, this story amounts to the Borel–Weil theorem realizing irreducible representations of GL_n as spaces of sections of line bundles on the flag variety.

Using the geometry of Bott–Samelson varieties, Magyar [Mag98] showed that χ_D is given by the formula

$$\chi_D = \omega_1^{k_1} \dots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1} \pi_{i_2}(\omega_{i_2}^{m_2}(\dots \pi_{i_\ell}(\omega_{i_\ell}^{m_\ell}) \dots))).$$

Here, $\omega_i = x_1 \dots x_i$ is a fundamental weight, $\pi_i = \partial_i x_i$ is a Demazure operator, and

$$\mathbf{k}(D) = (k_1, \dots, k_n), \qquad \mathbf{i}(D) = (i_1, \dots, i_\ell), \qquad \mathbf{m}(D) = (m_1, \dots, m_\ell)$$

is combinatorial data associated to the *orthodontic sequence* of D, which builds a %-avoiding diagram from "smaller" %-avoiding diagrams.

With St. Dizier [SS24], and building on earlier joint work with Mészáros and St. Dizier [MSS22], we extend Magyar's formula to *double Grothendieck polynomials*, which are lifts of structure sheaves $[\mathcal{O}_{X_w}]$ in the equivariant K-theory of the flag variety. Schubert polynomials can be obtained from double Grothendieck polynomials by setting $y_i \mapsto 0$ (corresponding to forgetting equivariance) and taking the lowest degree part (corresponding to taking the associated graded in K-theory). The combinatorics of Schubert polynomials $\mathfrak{S}_w$ often extends to (double) Grothendieck polynomials.

There is no known K-theoretic analogue of flagged Weyl modules. Despite this, we use the combinatorics of orthodontia to extend Magyar's formula.

Theorem 3.1.1. *Let D be a %-avoiding diagram with double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$. Define*

$$\mathscr{G}_D(\mathbf{x}, \mathbf{y}) := \overline{\omega}_1^{K_1} \overline{\omega}_2^{K_2} \dots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1, j_1}(\overline{\omega}_{i_1}^{M_1} \overline{\pi}_{i_2, j_2}(\overline{\omega}_{i_2}^{M_2} \dots \overline{\pi}_{i_\ell, j_\ell}(\overline{\omega}_{i_\ell}^{M_\ell}) \dots)). \qquad (3.1)$$

When $D = D(w)$ is the Rothe diagram of a permutation, then $\mathscr{G}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{G}_w(\mathbf{x}, \mathbf{y})$.

If the columns of D can be ordered by inclusion, the polynomial χ_D is a *key polynomial*. Dual characters of %-avoiding diagrams, and in particular Schubert polynomials, can be expressed as a nonnegative sum of key polynomials [RS98]. Key polynomials are the lowest degree part of *Lascoux polynomials*. One inhomogeneous extension of the key positivity [RS98] of Schubert polynomials is the Grothendieck-to-Lascoux expansion [RY21, SY23]. Our formula (3.1) gives another inhomogeneous extension of the key positivity [RS98]:

Theorem 3.1.2. *Let $D \subseteq [n] \times [m]$ be a diagram whose columns are ordered by inclusion. Let $\mathscr{S}_D(\mathbf{x}, \mathbf{y})$ be the lowest degree part of $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$. Then, the polynomial*

$$x_1^m \dots x_n^m \mathscr{S}_D(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \dots, x_n)$.

In joint work in progress with Tianyi Yu, we extend Theorem 3.1.2 to all %-avoiding diagrams; this appears as Conjecture 3.1.4 in [SS24].

CHAPTER 2

HIKITA SURJECTIVITY FOR $\mathcal{N} /\!\!/\!\!/ T$

2.1 Introduction

Certain conical symplectic singularities X are expected to have a *symplectic dual* $X^!$ which satisfies many striking properties. Although there is no formal definition, nor a systematic procedure to find the dual, in many cases there is a consensus on what the dual ought to be. Examples of symplectic dual pairs include nilpotent orbits in $\mathfrak{sl}_n$ and Slodowy slices through conjugate orbits, hypertoric varieties and their Gale duals, and Nakajima quiver varieties and BFN Coulomb branches.

Symplectic duality is expected to interchange seemingly unrelated invariants. For example, if X has a symplectic resolution $\widetilde{X} \to X$, the Hikita conjecture predicts that the cohomology $H^*(\widetilde{X})$ is isomorphic to the coordinate ring $\mathbb{C}[(X^!)^{T^!}]$ of the scheme-theoretic fixed points of $X^!$ with respect to the maximal torus $T^!$ of the group $\mathrm{Aut}_{\mathrm{Pois},\mathbb{C}^\times}(X^!)$ of Poisson automorphisms of $X^!$ commuting with dilations. When $\widetilde{X}$ is a Slodowy variety, the Hikita conjecture amounts to the theorem [dP81] of deConcini–Procesi expressing the cohomology of a Springer fiber as the coordinate ring of the scheme-theoretic intersection of a nilpotent orbit with the Cartan. Hikita [Hik17] observed that a similar phenomenon holds when $\widetilde{X}$ is the cotangent bundle of a partial flag variety, the Hilbert scheme of n points in $\mathbb{C}^2$, and a hypertoric variety. The recent preprint [HKM24] gave a counterexample to the Hikita conjecture (using a conjecturally dual pair from [LMM21]) and proposed a refined version. However, the original version of the Hikita conjecture and various generalizations have been verified for many other dual pairs [KTW$^+$19a, KMP21, KS22, Hoa24, Shl24, CHY23].

Throughout, we set G to be $\mathrm{SL}_n(\mathbb{C})$, T the torus of diagonal matrices, B the Borel subgroup of upper triangular matrices, and U the unipotent subgroup of upper triangular matrices with diagonal entries equal to 1. The Hamiltonian reduction of the nilpotent cone $\mathcal{N} := \mathcal{N}_{\mathfrak{sl}_n}$ by the maximal torus is defined to be the categorical quotient $\mathcal{N}/\!\!/\!\!/T := \{x \in \mathcal{N} \colon \operatorname{diag}(x) = 0\}/\!\!/T$ of variety of zero-diagonal nilpotent matrices by the adjoint action of T. The variety $\mathcal{N}/\!\!/\!\!/T$ has a realization as a Nakajima quiver variety for the so-called *bouquet quiver*. It follows that $\mathcal{N}/\!\!/\!\!/T$ is a conical symplectic singularity ([BS21, Thm 1.2]).

It is believed [DHK21, §8] that the conical symplectic singularities $\mathcal{N}/\!\!/\!\!/T$ and $\overline{T^*(G/U)}$ should be symplectic dual. There is now good evidence for this belief: Gannon and Williams [GW23] showed that the Coulomb branch of the 3-dimensional $\mathcal{N} = 4$ quiver gauge theory associated to the bouquet is $\overline{T^*(G/U)}$. This places conjectured duality between $\mathcal{N}/\!\!/\!\!/T$ and $\overline{T^*(G/U)}$ in the larger context of duality between Nakajima quiver varieties and BFN Coulomb branches.

Bellamy [Bel23] showed that Coulomb branches are conical symplectic singularities: For $\overline{T^*(G/U)}$, this was proven earlier by Jia [Jia21, Thm 1.1]. More generally, it is known ([Gan24, Thm 1.1]) that $\overline{T^*(G/U)}$ has symplectic singularities in all types, verifying a conjecture of Ginzburg and Kazhdan [GK22, Conj 1.3.6].

Bellamy and Schedler [BS21] characterized the Nakajima quiver varieties admitting a symplectic resolution. We show that $\mathcal{N}/\!\!/\!\!/T$ satisfies their criteria and hence has a symplectic resolution $\widetilde{\mathcal{N}/\!\!/\!\!/T}$. We explicitly construct a variety (Corollary 2.3.17) diffeomorphic to $\widetilde{\mathcal{N}/\!\!/\!\!/T}$. The variety $T^*(G/U)$ has commuting actions of T (induced by left multiplication on G/U) and B/U (induced by right multiplication on G/U). From these actions one can form the scheme-theoretic fixed points $\overline{T^*(G/U)}^{T\times B/U}$. This scheme is nonreduced and has one closed point, so its coordinate ring is finite-dimensional as a graded $\mathbb{C}$-algebra.

For the conical symplectic singularity $X = \mathcal{N}/\!\!/\!\!/T$, the Hikita conjecture predicts that

$$H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right) \cong \mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] \tag{2.1}$$

as graded rings. For $n = 2$ and $n = 3$, the variety $\mathcal{N}/\!\!/\!\!/T$ is the point and the type D_4

Kleinian singularity respectively, while $\overline{T^*(G/U)}$ is the affine space $\mathbb{C}^4$ and the minimal nilpotent orbit in $\mathfrak{so}_8$ ([Jia21, Thm 1.3]), respectively. The Hikita conjecture is known to hold in these examples ([Shl24]).

Our main result states:

Theorem 2.1.1. *There is a surjective morphism of graded algebras*

$$\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] \twoheadrightarrow H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right).$$

The coordinate ring can be realized as a quotient of $\mathbb{C}[\mathfrak{h}\times_{\mathfrak{h}/W}\mathfrak{h}]$ in the following way. The scheme $\overline{T^*(G/U)}^{B/U}$ is a subscheme of $\overline{T^*(G/U)}/\!\!/(B/U) = \mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}$ and hence $\overline{T^*(G/U)}^{T\times B/U}$ is a subscheme of $(\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h})^T = \mathfrak{h}\times_{\mathfrak{h}/W}\mathfrak{h}$. We compute the defining ideal of this subscheme: Write $x_1,\dots,x_n,y_1,\dots,y_n$ for coordinates on $\mathfrak{h}\times\mathfrak{h}$, and for subsets $S,T\subset[n]:=\{1,\dots,n\}$, set

$$f_{S,T} := \prod_{\substack{s\in S\\ t\in T}}(x_s-y_t).$$

Theorem 2.1.2. *We have*

$$\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] = \frac{\mathbb{C}[\mathfrak{h}\times_{\mathfrak{h}/W}\mathfrak{h}]}{\left\langle f_{S,T} : \begin{matrix} S,T\subseteq[n] \\ |S|+|T|=n\end{matrix}\right\rangle}.$$

We show that $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ is diffeomorphic to the quotient of a variety $Y_{\lambda,\delta}$ by a free action of the torus T_{PGL}. From the construction of $Y_{\lambda,\delta}$, there is a natural T-equivariant inclusion $Y_{\lambda,\delta}\hookrightarrow\widetilde{\mathfrak{g}}$.

Theorem 2.1.3. *The inclusion $Y_{\lambda,\delta}\hookrightarrow\widetilde{\mathfrak{g}}$ induces a surjection*

$$\mathbb{C}[\mathfrak{h}\times_{\mathfrak{h}/W}\mathfrak{h}]\cong H^*_T(\widetilde{\mathfrak{g}})\twoheadrightarrow H^*_T(Y_{\lambda,\delta})\cong H^*(\widetilde{\mathcal{N}/\!\!/\!\!/T})$$

and the kernel contains the functions $\{f_{S,T}\colon |S|+|T|=n\}$.

Theorems 2.1.2 and 2.1.3 together imply Theorem 2.1.1.

As $H^*(\widetilde{\mathcal{N}/\!\!/\!\!/T})$ is a finite-dimensional $\mathbb{C}$-algebra, the Hikita conjecture (2.1) would follow from the equality

$$\dim_{\mathbb{C}}\left(H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right)\right)=\dim_{\mathbb{C}}\left(\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right]\right).$$

The arguments in this paper suggest that the surjectivity $\mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right]\twoheadrightarrow H^*\left(\widetilde{\mathcal{N}/\!\!/\!\!/T}\right)$ can be inherited from the self-duality $\mathcal{N}^!_{\mathfrak{sl}_n}=\mathcal{N}_{\mathfrak{sl}_n}$. We use in a crucial way (via Kirwan surjectivity for quiver varieties [MN18]) that $\mathcal{N}/\!\!/\!\!/T$ can be expressed as a quotient of a subvariety of the universal deformation $\widetilde{\mathfrak{g}}$ of $\widetilde{\mathcal{N}}$ by a free action of a torus. We also use (via the Gelfand–Graev action [GR15, Wan21]) that $\overline{T^*(G/U)}$ is the Hamiltonian reduction of a quiver representation space by a product of special linear groups $\mathrm{SL}(V_i)$ and that the corresponding Nakajima quiver variety $\overline{T^*(G/U)}/\!\!/\!\!/(\mathrm{GL}(V)/\mathrm{SL}(V))$ is the nilpotent cone $\mathcal{N}$. We conjecture:

Conjecture 2.1.4. *Let* $X, X^!$ *be symplectic dual varieties such that* $X^!=T^*\mathrm{Rep}(Q^!)/\!\!/\!\!/\mathrm{GL}(V^!)$ *is a Nakajima quiver variety. Let* T *denote the maximal torus of the group of Poisson automorphisms of* X *commuting with dilation. Assume that* $X/\!\!/\!\!/T$ *has a symplectic resolution. Then the Hikita conjecture holds for the pair* $X/\!\!/\!\!/T$ *and* $X^{!,\uparrow}\overset{\mathrm{def}}{=}T^*\mathrm{Rep}(Q^!)/\!\!/\!\!/\mathrm{SL}(V^!)$.

For $X=X^!=\mathcal{N}$, Conjecture 2.1.4 amounts to our case (2.1) of the Hikita conjecture. In Appendix 2.5, we give other examples of symplectic dual pairs which arise as $(X/\!\!/\!\!/T, X^{!,\uparrow})$.

Our original motivation for studying Conjecture 2.1.4, and the pair $(\mathcal{N}/\!\!/\!\!/T, \overline{T^*(G/U)})$ in particular, comes from the theory of BFN Coulomb branches. By definition, $X^!$ is the Higgs branch $\mathcal{M}_H(\mathrm{GL}(V),\mathbf{N})$ of a certain 3-dimensional $\mathcal{N}=4$ supersymmetric gauge theory. According to the expected duality between Higgs and Coulomb branches, the variety X is the corresponding Coulomb branch $\mathcal{M}_C(\mathrm{GL}(V),\mathbf{N})$. On the other hand, $X^{!,\uparrow}$ is the Higgs branch $\mathcal{M}_H(\mathrm{SL}(V), \mathbf{N})$. According to [BFN18, Prop 3.18], corresponding to the short exact sequence

$$1 \longrightarrow \mathrm{SL}(V) \longrightarrow \mathrm{GL}(V) \longrightarrow \frac{\mathrm{GL}(V)}{\mathrm{SL}(V)} \longrightarrow 1$$

there is an action of the torus $T_F := \mathrm{GL}(V)/\mathrm{SL}(V)$ on $\mathcal{M}_C(\mathrm{GL}(V), \mathbf{N}) = X$ and an isomorphism

$$\mathcal{M}_C(\mathrm{SL}(V), \mathbf{N}) \cong \mathcal{M}_C(\mathrm{GL}(V), \mathbf{N}) /\!\!/\!\!/ T_F.$$

(For many theories $(\mathrm{GL}(V), \mathbf{N})$ of interest, the T_F action has an explicit description [BFN19a, Rem 3.12].) In conclusion, the expected duality between Higgs and Coulomb branches predicts that $X^{!,\uparrow}$ is symplectic dual to the Hamiltonian reduction $X /\!\!/\!\!/ T_F$. We thank Vasily Krylov for also pointing out this connection.

2.2 Coordinate ring of $\overline{T^*(G/U)}^{T\times B/U}$

The goal of this section is to prove Theorem 2.1.2. To this end, we express the defining ideal of $\overline{T^*(G/U)}^{T\times B/U}$ inside $\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}$ in terms of the Gelfand–Graev action (Proposition 2.2.5) and uses an explicit description of this action from [Wan21] (Lemma 2.2.8).

2.2.1 Scheme-theoretic fixed points

For an algebraic group H acting on an affine variety X, the scheme-theoretic fixed points X^H [DG70, VIII,Ex 6.5(d),(e)] (see also [Fog73, Thm 2.3]) is the (non-reduced, in general) affine scheme defined by

$$X^H \stackrel{\text{def}}{=} \mathrm{Spec}\left(\frac{\mathbb{C}[X]}{\langle f - h(f)\colon f \in \mathbb{C}[X], h \in H\rangle}\right).$$

When a torus T acts on X, the coordinate ring $\mathbb{C}[X]$ decomposes into a direct sum of weight spaces $\oplus_{\alpha\in\Lambda_T}\mathbb{C}[X]_\alpha$, and there is an isomorphism (cf. [KS22, Prop 1.4])

$$\frac{\mathbb{C}[X]}{\langle f - t(f) \colon f \in \mathbb{C}[X], t \in T\rangle} \cong \frac{\mathbb{C}[X]_0}{\sum_{\alpha\neq 0}\mathbb{C}[X]_\alpha\mathbb{C}[X]_{-\alpha}}. \tag{2.2}$$

2.2.2 Geometry of $T^*(G/U)$ and Gelfand-Graev action

Write $T^*(G/U)$ for the cotangent bundle of the base affine space G/U, and write $\widetilde{\mathfrak{g}}$ for the Grothendieck simultaneous resolution of $\mathfrak{g}$. Using the Killing form to identify $\mathfrak{g} \leftrightarrow \mathfrak{g}^*$, there are identifications $T^*(G/U) \cong G \times_U \mathfrak{b} := \frac{G\times\mathfrak{b}}{U}$ and $\widetilde{\mathfrak{g}} \cong G \times_B \mathfrak{b} := \frac{G\times\mathfrak{b}}{B}$. Write $\mathcal{B} := G/B$ for the flag variety.

The quotient $T^*(G/U) \to \widetilde{\mathfrak{g}}$ makes $T^*(G/U)$ into a B/U-torsor. Write $\Lambda_{B/U}$ for the (B/U)-weight lattice. For $\alpha \in \Lambda_{B/U}$, write $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha)$ for the pullback of the line bundle $\mathcal{O}_{\mathcal{B}}(\alpha)$ along $\widetilde{\mathfrak{g}} \to \mathcal{B}$, so that a regular function on $T^*(G/U)$ of B/U-weight α is precisely a section of $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha)$. As the left T-action on $T^*(G/U)$ descends to $\widetilde{\mathfrak{g}}$, we get an isomorphism

$$\mathbb{C}[T^*(G/U)] = \bigoplus_{\alpha\in\Lambda_{B/U}} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))$$

of T-representations.

Lemma 2.2.3 ([Bro93, Prop 2.6], cf. also [GR15, Lem 3.6.2])**.** *If $\lambda, \mu \in \Lambda^+_{B/T}$ are dominant weights, the multiplication map $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \to \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda+\mu))$ is surjective.*

Proof. For any dominant weight $\alpha \in \Lambda_{B/U}$, let $V_\alpha \subset \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))$ denote the space of sections obtained by pulling back sections of $\mathcal{O}_{\mathcal{B}}(\alpha)$ along the vector bundle map $\widetilde{\mathfrak{g}} \to \mathcal{B}$. A result of Broer [Bro93, Prop 2.6] implies that when α is dominant the subspace X_α generates $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))$ as a $\mathbb{C}[\widetilde{\mathfrak{g}}]$-module.

The lemma follows from the fact that the multiplication map $X_\lambda \otimes X_\mu \to X_{\lambda+\mu}$ is surjective ([BK04, Thm 3.1.2(c)].) □

The coordinate ring $\mathbb{C}[T^*(G/U)]$ is also equipped with an action of W, called the *Gelfand-Graev action*, which restricts to isomorphisms

$$w.\colon \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha)) \xrightarrow{\sim} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(w.\alpha))$$

of T-representations. Given a weight $\beta \in \Lambda_T$ of T, let $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta$ denote the β-weight space of $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))$. Note that the Gelfand-Graev action preserves $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0))$ as well as its 0-weight space $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0))_0$.

It is known [GR15, Prop 5.5.1] that the restriction of the Gelfand-Graev action to $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0)) = \mathbb{C}[\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}]$ agrees with the one induced by the W-action on $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ given by acting trivially on $\mathfrak{g}$ and naturally on $\mathfrak{h}$.

2.2.4 Fixed points in $\overline{T^*(G/U)}$

Consider the algebra

$$R := \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0))_0$$

and the ideals

$$\begin{aligned} I &:= \sum_{(\alpha,\beta)\neq(0,0)} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta} \\ J &:= \sum_{\beta\neq 0} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(0))_{-\beta} \end{aligned}$$

of R. Equation (2.2) implies that $R/J = \mathbb{C}[(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h})^T] = \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$ and that

$$\begin{aligned} \mathbb{C}\left[\overline{T^*(G/U)}^{T\times B/U}\right] &= R/I \\ &\cong \frac{\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]}{I/J}. \end{aligned}$$

By construction, the Gelfand-Graev action on R descends to the W-action on $R/J = \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$ given by acting on the second $\mathfrak{h}$ factor.

Proposition 2.2.5. *The ideal I/J of the ring R/J is generated by the image of the W-orbit of*

$$\sum_{\omega_i,\lambda} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))_\lambda \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))_{-\lambda}$$

under the projection $R \to R/J$. (The sum runs over fundamental weights $\omega_i \in \Lambda_{B/U}$ and all weights $\lambda \in \Lambda_T$.)

Proof. Let $\alpha \in \Lambda_{B/U}$ be a nonzero dominant weight of B/U and let $\beta \in \Lambda_T$ be any weight of T. Pick a fundamental weight $\omega_i \in \Lambda_{B/U}$ so that $\alpha - \omega_i$ is dominant. Note that $w_0.(-\alpha)$, $w_0.(-\omega_i)$, and $w_0.(-\alpha + \omega_i)$ are all dominant as well. Lemma 2.2.3 implies that the multiplication maps

$$\sum_{\lambda \in \Lambda_T} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha - \omega_i))_{\beta-\lambda} \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))_\lambda \longrightarrow \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \tag{$\dagger$}$$

$$\sum_{\lambda \in \Lambda_T} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha + \omega_i))_{-\beta-\lambda} \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))_\lambda \longrightarrow \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta} \tag{$\ddagger$}$$

are surjective. Taking the tensor product of the maps in ($\dagger$) and ($\ddagger$) and composing with multiplication $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta} \twoheadrightarrow \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta}$ gives a surjective map

$$\sum_{\lambda,\mu \in \Lambda_T} \underbrace{\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha - \omega_i))_{\beta-\lambda} \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha + \omega_i))_{-\beta-\mu}}_{\subset \mathbb{C}[\widetilde{\mathfrak{g}}]_{-\lambda-\mu}} \otimes \underbrace{\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))_\lambda \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))_\mu}_{\subset \mathbb{C}[\widetilde{\mathfrak{g}}]_{\lambda+\mu}}$$

$$\longrightarrow \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta}.$$

Under the projection map $I \to I/J$, the images of the subspaces on the left hand side vanish unless $\lambda + \mu = 0$. We deduce that the ideal of R/J generated by the image of

$$\sum_{\lambda} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))_\lambda \otimes \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))_{-\lambda}$$

contains the image of $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta}$.

Since I/J is generated by the image of

$$W\left(\sum_{\substack{\alpha \in \Lambda^+_{B/T} \setminus 0 \\ \beta \in \Lambda_T}} \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))_\beta \cdot \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\alpha))_{-\beta}\right),$$

the result follows. □

Recall that $V_\alpha \subset \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\alpha))$ denotes the space of sections obtained by pulling back sections of $\mathcal{O}_{\mathcal{B}}(\alpha)$ along the vector bundle map $\widetilde{\mathfrak{g}} \to \mathcal{B}$. The space V_α is stable under the T-action. Fix a basis $h_{(\omega_i,\lambda),j}$ for each weight space $(V_{\omega_i})_\lambda$.

Proposition 2.2.6. *The W-orbits of*

$$h_{(\omega_i,\mu),j} \cdot (w_0.h_{(\omega_{n-i},-\mu),k})$$

generate I/J as an ideal of R/J.

Proof. Let

$$f_1 \in \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))_\lambda, \qquad f_2 \in \Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))_{-\lambda}.$$

Because V_{ω_i} and $V_{\omega_{n-i}}$ generate $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(\omega_i))$ and $w_0.\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}(-\omega_i))$ respectively as a $\mathbb{C}[\widetilde{\mathfrak{g}}]$-

module, we can write

$$f_1 = \sum_k g_k h_{(\omega_i,\mu_k),k}, \qquad g_k \in \mathbb{C}[\widetilde{\mathfrak{g}}]_{\lambda-\mu_k}$$

$$f_2 = w_0.\left(\sum_k g'_k h_{(\omega_{n-i},\mu'_k),k}\right), \qquad g'_k \in \mathbb{C}[\widetilde{\mathfrak{g}}]_{-\lambda-\mu'_k}$$

and hence

$$f_1 f_2 = \sum_{k,\ell} g_k (w_0.g'_\ell) \cdot h_{(\omega_i,\mu_k),k} \cdot (w_0.h_{(\omega_{n-i},\mu'_\ell),\ell}).$$

Under the projection map $I \to I/J$, the images of the terms on the right hand side vanish unless $\mu_k + \mu'_\ell = 0$. We conclude that the image of $f_1 f_2$ in I/J can be written as a R/J-linear combination of $h_{(\omega_i,\mu),j} \cdot (w_0.h_{(\omega_{n-i},-\mu),k})$.

The claim now follows from Proposition 2.2.5. □

For subsets $S, T \subset [n]$ with $|S| = |T|$, define the function

$$\Delta_{S,T} \colon \mathrm{SL}_n \to \mathbb{C}$$

$$g = (g_{ij})_{i,j\in[n]} \mapsto \det(g_{st})_{s\in S, t\in T}$$

The functions $\Delta_S := \Delta_{S,\{1,\ldots,|S|\}}$ are U-invariant under right translations and descend to sections of certain line bundles on the flag variety $\mathcal{B}$: specifically, for each i, the set of functions $\{\Delta_S \colon |S| = i\}$ forms a weight basis for the representation $\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\omega_i))$. The T-weight of Δ_S is $\sum_{s\in S} \mu_s$, where $\mu_i \colon T \to \mathbb{C}^\times$ is the character $\mu_i \colon (t_1, \ldots, t_n) \mapsto t_i$.

Let p denote the projection $\widetilde{\mathfrak{g}} = \frac{G\times\mathfrak{b}}{B} \to G/B = \mathcal{B}$ onto the first factor; the map p makes $\widetilde{\mathfrak{g}}$ into a vector bundle over $\mathcal{B}$.

Proposition 2.2.7. *Let $p \colon \widetilde{\mathfrak{g}} \to \mathcal{B}$ denote the vector bundle map. The W-orbits of*

$$p^*\Delta_S \cdot (w_0.p^*\Delta_{[n]\setminus S})$$

generate I/J as an ideal of $R/J = \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$.

Proof. The map $p^*\colon \Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\omega_i)) \xrightarrow{\sim} V_{\omega_i}$ is an isomorphism of G-representations, so the set $\{p^*\Delta_S\colon |S| = i\}$ forms a weight basis of V_{ω_i}. Proposition 2.2.6 implies that $p^*\Delta_S \cdot (w_0.p^*\Delta_{[n]\setminus S})$ generates I/J as an ideal of R/J. □

Finally, we will use an explicit description of the Gelfand-Graev action in type A on the regular semisimple locus. Recall the identification $T^*(G/U) \cong \frac{G\times\mathfrak{b}}{U}$. Let $(T^*(G/U))^{\mathrm{rs}}$ denote the image of

$$\varphi\colon G \times \mathfrak{h}^{\mathrm{rs}} \hookrightarrow G \times \mathfrak{b} \to G \times_U \mathfrak{b} = T^*(G/U).$$

Lemma 2.2.8 ([Wan21, Prop 4.5.1]). *Let $y = \operatorname{diag}(y_1, \ldots, y_n) \in \mathfrak{h}^{\mathrm{rs}}$. Let $s_k(y)$ denote the $n \times n$ matrix obtained from the identity matrix by replacing the $\{k, k+1\}$-th submatrix by*

$$\begin{bmatrix} 0 & \frac{1}{y_k - y_{k+1}} \\ y_{k+1} - y_k & 0 \end{bmatrix}.$$

The Gelfand–Graev action of the transposition s_k interchanging $k \longleftrightarrow k+1$ is given by

$$\begin{aligned} \sigma_k\colon (T^*(G/U))^{\mathrm{rs}} &\to (T^*(G/U))^{\mathrm{rs}} \\ [g, y] &\mapsto [g s_k(y), \mathrm{Ad}_{s_k^{-1}}(y)]. \end{aligned}$$

Lemma 2.2.9. *Let $w_0^{(n)}$ denote the longest element in S_n. The action of $w_0^{(n)}$ on $(T^*(G/U))^{\mathrm{rs}}$ is given by*

$$[(g_{ij}), (y_i)] \mapsto \left[\left(g_{i,n+1-j} \frac{\prod_{\ell>j}(y_{n+1-j} - y_{n+1-\ell})}{\prod_{\ell<j}(y_{n+1-j} - y_{n+1-\ell})}\right), w_0^{(n)}.(y_i)\right]$$

Example 2.2.10. For $n = 4$, the matrix $\left(g_{i,n+1-j} \frac{\prod_{\ell>j}(y_{n+1-j}-y_{n+1-\ell})}{\prod_{\ell<j}(y_{n+1-j}-y_{n+1-\ell})} \right)$ is given by

$$\begin{bmatrix} g_{14}(y_4-y_1)(y_4-y_2)(y_4-y_3) & g_{13}\frac{(y_3-y_1)(y_3-y_2)}{y_3-y_4} & g_{12}\frac{y_2-y_1}{(y_2-y_3)(y_2-y_4)} & g_{11}\frac{1}{(y_1-y_2)(y_1-y_3)(y_1-y_4)} \\ g_{24}(y_4-y_1)(y_4-y_2)(y_4-y_3) & g_{23}\frac{(y_3-y_1)(y_3-y_2)}{y_3-y_4} & g_{22}\frac{y_2-y_1}{(y_2-y_3)(y_2-y_4)} & g_{21}\frac{1}{(y_1-y_2)(y_1-y_3)(y_1-y_4)} \\ g_{34}(y_4-y_1)(y_4-y_2)(y_4-y_3) & g_{33}\frac{(y_3-y_1)(y_3-y_2)}{y_3-y_4} & g_{32}\frac{y_2-y_1}{(y_2-y_3)(y_2-y_4)} & g_{31}\frac{1}{(y_1-y_2)(y_1-y_3)(y_1-y_4)} \\ g_{44}(y_4-y_1)(y_4-y_2)(y_4-y_3) & g_{43}\frac{(y_3-y_1)(y_3-y_2)}{y_3-y_4} & g_{42}\frac{y_2-y_1}{(y_2-y_3)(y_2-y_4)} & g_{41}\frac{1}{(y_1-y_2)(y_1-y_3)(y_1-y_4)} \end{bmatrix}$$

△

Proof of Lemma 2.2.9. We argue by induction. When $n = 2$, Lemma 2.2.8 asserts that the action of $w_0^{(2)} = s_1$ is given by

$$\sigma_1\colon \left[\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, (y_1, y_2) \right] \mapsto \left[\begin{bmatrix} g_{21}(y_2 - y_1) & g_{11}\frac{1}{y_1-y_2} \\ g_{22}(y_2 - y_1) & g_{12}\frac{1}{y_1-y_2} \end{bmatrix}, (y_2, y_1) \right],$$

as claimed.

Let M denote the $n \times n$ matrix with

$$m_{ij} = \begin{cases} g_{i,n-j} \frac{\prod_{\ell>j}(y_{n-j}-y_{n-\ell})}{\prod_{\ell<j}(y_{n-j}-y_{n-\ell})} & \text{if } j \leq n-1 \\ g_{i,n} & \text{if } j = n \end{cases}$$

Let $\iota\colon S_{n-1} \hookrightarrow S_n$ denote the standard embedding. By induction, $\iota(w_0^{(n-1)})$ acts on $(T^*(\mathrm{SL}_n/U))^{\mathrm{rs}}$ by

$$[(g_{ij}), (y_i)] \mapsto \left[M, \iota(w_0^{(n-1)}).(y_i)\right].$$

Using the equality $w_0^{(n)} = s_1 \ldots s_{n-1}\iota(w_0^{(n-1)})$, repeated application of Lemma 2.2.8 gives the result. □

Recall that for subsets $S, T \subset [n]$, the function $f_{S,T} \in \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$ is

$$f_{S,T} = \prod_{\substack{s \in S \\ t \in T}} (x_s - y_t),$$

where $x_1, \ldots, x_n, y_1, \ldots y_n$ are the coordinate functions on $\mathfrak{h} \times \mathfrak{h}$.

Theorem 2.1.2. *We have*

$$\mathbb{C}\left[\overline{T^*(G/U)}^{T \times B/U}\right] = \frac{\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]}{\left\langle f_{S,T} \colon \begin{array}{c} S, T \subseteq [n] \\ |S| + |T| = n \end{array} \right\rangle}.$$

Proof of Theorem 2.1.2. Let $S \subset [n]$ with $|S| = i$. By Lemma 2.2.9, the restriction of the function $w_0.p^*\Delta_S$ to $(T^*(G/U))^{\mathrm{rs}}$ is

$$\begin{aligned}
[g, y] &\mapsto \det\left(g_{s,n-t} \frac{\prod_{\ell > t}(y_{n+1-t} - y_{n+1-\ell})}{\prod_{\ell < t}(y_{n+1-t} - y_{n+1-\ell})}\right)_{\substack{s \in S \\ t \in [i]}} \\
&= \det(g_{s,n-t})_{\substack{s \in S \\ t \in [i]}} \frac{\prod_{\substack{a > b \\ a > n-|S|}} (y_a - y_b)}{\prod_{\substack{a < b \\ a > n-|S|}} (y_a - y_b)} \\
&= \varepsilon_S \det(g_{s,n-t})_{\substack{s \in S \\ t \in [i]}} \prod_{\substack{a > n-|S| \\ b \le n-|S|}} (y_a - y_b), \qquad \varepsilon_S \in \{\pm 1\}.
\end{aligned}$$

In particular, the function $p^*\Delta_S \cdot (w_0.p^*\Delta_{[n]\setminus S})$ restricts to $(T^*(G/U))^{\mathrm{rs}}$ as

$$[g, y] \mapsto \varepsilon_S \det(g_{st})_{\substack{s \in S \\ t \in [i]}} \det(g_{st})_{\substack{s \in [n]\setminus S \\ t \in [n]\setminus[i]}} \prod_{\substack{a > i \\ b \le i}} (y_a - y_b).$$

Given a permutation w, pick a matrix $P_w \in \mathrm{SL}_n$ whose entries are nonzero only at positions $(i, w(i))$, and whose nonzero entries are equal to ± 1. The function $p^*\Delta_S \cdot (w_0.p^*\Delta_{[n]\setminus S})$ restricts to a nonzero function only when $w(S) = [i]$ and in this case it is given by the formula

$$[P_w, y] \mapsto \varepsilon'_S \prod_{\substack{a>i \\ b\leq i}} (y_a - y_b), \qquad \varepsilon'_S \in \{\pm 1\}.$$

It follows that the image of $p^*\Delta_S \cdot (w_0.p^*\Delta_{[n]\setminus S}) \in R = \mathbb{C}[T^*(G/U)]_{(0,0)}$ under the projection $R \to R/J = \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$ is given by the formula

$$(w.y, y) \mapsto \begin{cases} \varepsilon'_S \prod_{s\in[n]\setminus S}(y_{w(s)} - y_1)\dots(y_{w(s)} - y_i) & \text{if } w(S) = [i] \\ 0 & \text{else} \end{cases}$$

By construction, this is the restriction of

$$\varepsilon'_S f_{[n]\setminus S,[i]} = \varepsilon'_S \prod_{\substack{s\in[n]\setminus S \\ t\in[i]}} (x_s - y_t)$$

to $\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}$.

The claim follows from the fact that the W-orbit of $f_{[n]\setminus S,[i]}$ is $\{f_{[n]\setminus S,T} \colon |T| = i\}$. $\square$

Remark 2.2.11. The set $\{f_{S,T} \colon |S| + |T| = n\}$ is not a minimal generating set of the ideal $\langle f_{S,T}\rangle$; for example one can compute that

$$f_{S,T} = (-1)^{|S||T|} f_{[n]\setminus S,[n]\setminus T}$$

as functions on $\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}]$. $\triangle$

2.3 Symplectic resolution of $\mathcal{N}/\!\!/\!\!/T$

In this section we prove that $\mathcal{N}/\!\!/\!\!/T$ is a Nakajima quiver variety (Proposition 2.3.8) with a symplectic resolution $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ (Theorem 2.3.10). Using a standard construction for Nakajima quiver varieties in general, we give an explicit description of the diffeomorphism type of

$\widetilde{\mathcal{N}/\!\!/\!\!/T}$ in Corollary 2.3.17: the variety $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ is diffeomorphic to the quotient $Y_{\lambda,\delta}/(\mathbb{C}^\times)^{n-1}$ for sufficiently generic $\lambda, \delta \in \{(x_1, \ldots, x_n) \in \mathbb{C}^n \colon x_1 + \cdots + x_n = 0\}$, where

$$Y_{\lambda,\delta} := \left\{ (x, F_\bullet) \in \widetilde{\mathfrak{g}} \colon \begin{array}{c} x|_{F_k/F_{k-1}} = \lambda_k \mathrm{Id} \\ \mathrm{diag}(x) = \delta \end{array} \right\},$$

and the torus $(\mathbb{C}^\times)^{n-1}$ acts on $Y_{\lambda,\delta}$ by

$$(t_1, \ldots, t_{n-1}) \cdot (x, F_\bullet) = (\mathrm{Ad}_{t'}(x), t' \cdot F_\bullet), \qquad t' := \mathrm{diag}(t_1, \ldots, t_{n-1}, 1).$$

2.3.1 Nakajima quiver varieties

Let $Q = (I, E)$ be a quiver, and fix dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I_{\geq 0}$. Let $s, t \colon E \to I$ denote the source and target maps respectively. Write

$$\begin{aligned} \mathbb{M}(Q, \mathbf{v}, \mathbf{w}) := & \bigoplus_{e \in E} \mathrm{Hom}(\mathbb{C}^{v_{s(e)}}, \mathbb{C}^{v_{t(e)}}) \oplus \bigoplus_{e \in E} \mathrm{Hom}(\mathbb{C}^{v_{t(e)}}, \mathbb{C}^{v_{s(e)}}) \\ & \oplus \bigoplus_{i \in I} \mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) \\ \cong & T^* \left(\bigoplus_{e \in E} \mathrm{Hom}(\mathbb{C}^{v_{s(e)}}, \mathbb{C}^{v_{t(e)}}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i}) \right). \end{aligned} \tag{$\heartsuit$}$$

The vector space $\mathbb{M}$ is equipped with a canonical symplectic form and an action of the group

$$\mathbb{G}_{\mathbf{v}} := \prod_{i \in I} \mathrm{GL}(v_i)$$

induced by the action of $\mathrm{GL}(v_i)$ on $\mathbb{C}^{v_i}$.

Write $\mathfrak{g}_{\mathbf{v}} := \mathrm{Lie}(\mathbb{G}_{\mathbf{v}})$ and let $\mu \colon \mathbb{M} \to \mathfrak{g}_{\mathbf{v}}^*$ denote the moment map of this action.

The Nakajima quiver variety is the GIT quotient

$$\mathfrak{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) := \mu^{-1}(\lambda)/\!\!/_{\theta}\,\mathbb{G}_{\mathbf{v}}$$

where $\lambda \in Z(\mathfrak{g}_{\mathbf{v}}^*)$ and $\theta = (\theta_i)_{i\in I} \in \mathbb{Z}^I$ is a stability condition, encoding a character $\chi_\theta \colon \mathbb{G}_{\mathbf{v}} \to \mathbb{C}^\times$ via $\chi_\theta((g_i)_{i\in I}) := \prod_{i\in I} \det(g_i)^{\theta_i}$.

Given dimension vectors $\mathbf{v} = (v_i)_{i\in I}$ and $\mathbf{v}' = (v_i')_{i\in I}$, write $\mathbf{v}' < \mathbf{v}$ if $v_i' \leq v_i$ for all i and $\mathbf{v}' \neq \mathbf{v}$.

Definition 2.3.2 ([MN18, Defn 3.1])**.** The stability condition θ is *nondegenerate* if $\sum_i \theta_i \cdot v_i' \neq 0$ for all nonzero dimension vectors $\mathbf{v}' < \mathbf{v}$. △

In the special case $\mathbf{w} = \mathbf{e}_j$ (i.e., exactly one vertex is framed and the framing is one-dimensional), and the stability parameter $\theta = (\theta_i)_{i\in I}$ is in $\mathbb{Z}^I_{>0}$ (hence is nondegenerate), the locus of θ-stable points is particularly easy to compute using King stability conditions [Kin94, Prop 3.1].

Lemma 2.3.3 ([Nak98, Lem 3.8, Lem 3.10])**.** *Let* $\mathbf{w} = \mathbf{e}_j$ *and* $\theta \in \mathbb{Z}^I_{>0}$*. Given a point* $p \in \mathbb{M}(Q,\mathbf{v},\mathbf{w})$*, write* $j(p)$ *for the framing map* $\mathbb{C}^{v_i} \to \mathbb{C}$*. The following are equivalent:*

1. *p is θ-stable*
2. *p is θ-semistable*
3. $\ker(j(p))$ *contains no nonzero p-stable I-graded subspace.*

The action of $\mathbb{G}_{\mathbf{v}}$ *is free on the stable locus* $\mathbb{M}(Q,\mathbf{v},\mathbf{w})^{\theta\text{-s}}$ *and consequently*

$$\mathfrak{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}) = (\mu^{-1}(\lambda) \cap \mathbb{M}^{\theta\text{-s}})/\mathbb{G}_{\mathbf{v}}.$$

The inclusion $\mu^{-1}(0)^{\theta\text{-s}} \hookrightarrow \mu^{-1}(0)$ induces a map in equivariant cohomology

$$H^*_{\mathbb{G}_{\mathbf{v}}}(\mathrm{pt}) \cong H^*_{\mathbb{G}_{\mathbf{v}}}(\mu^{-1}(0)) \to H^*_{\mathbb{G}_{\mathbf{v}}}(\mu^{-1}(0)^{\theta\text{-s}}) \cong H^*(\mathfrak{M}_{0,\theta}(\mathbf{v},\mathbf{w}))$$

called the *Kirwan map*. (The last isomorphism follows from Lemma 2.3.3.)

A special case of *Kirwan surjectivity*, due to McGerty–Nevins, reads:

Theorem 2.3.4 ([MN18, Thm 1.2]). *Let* $\mathfrak{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ *be a smooth Nakajima quiver variety. Then the Kirwan map is surjective.*

We say that a Nakajima quiver variety is *unframed* if $\mathbf{w} = 0$. When the dimension vector $\mathbf{v}$ of an unframed Nakajima quiver variety is in a certain combinatorially defined set $\Sigma_0(Q)$, Bellamy and Schedler provide a criterion for $\mathfrak{M}$ to admit a projective symplectic resolution given by deforming the stability parameter.

Following the notation in [BS21], the Ringel form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\alpha \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

Definition 2.3.5 ([BS21, §2.2]). The vector $\hat{\mathbf{v}}$ is *anisotropic* if $\langle \alpha, \alpha \rangle < 0$. △

Proposition 2.3.6 ([BS21, Thm 1.5]). *Let* $\mathbf{v} \in \Sigma_0(Q)$. *Then the Nakajima quiver variety* $\mathfrak{M}_{0,0}(\mathbf{v}, \mathbf{0})$ *admits a projective symplectic resolution if* $\mathbf{v}$ *is indivisible. If* $\mathbf{v}$ *is anisotropic, a resolution is given by moving to a generic stability parameter.*

2.3.7 Presentation of $\mathcal{N} /\!\!/\!\!/ T$ *as a Nakajima quiver variety*

The *(trimmed) bouquet quiver* Q_n is the quiver with vertices $\{s_1, \dots, s_{n-1}\} \sqcup \{b_1, \dots, b_{n-1}\}$ and edges

$$\{(s_i, s_{i+1}) \colon i \in [n-1]\} \sqcup \{(s_{n-1}, b_i) \colon i \in [n-1]\}.$$

See Figure 2.1 for an example.

We write dimension vectors $\mathbf{v}$ for Q_n as tuples $(v_{s_1}, \dots, v_{s_{n-1}}; v_{b_1}, \dots, v_{b_{n-1}})$. The vertices s_i and b_i are called *stem* and *bouquet* vertices. Associated to this partition of the vertices of Q_n, we write

$$\mathbb{G}_{\text{stem}} := \prod_{\substack{i \in I(Q_n) \\ i = s_j}} \mathrm{GL}(v_i), \qquad \mathbb{G}_{\text{bouq}} := \prod_{\substack{i \in I(Q_n) \\ i = b_j}} \mathrm{GL}(v_i),$$

and $\mu_{\text{stem}}, \mu_{\text{bouq}}$ for the corresponding moment maps.

The *abundant bouquet quiver* Q_n^+ has an extra bouquet vertex: it is the quiver with vertices $\{s_1, \dots, s_{n-1}\} \sqcup \{b_1, \dots, b_n\}$ and edges

$$\{(s_i, s_{i+1})\colon i \in [n-1]\} \sqcup \{(s_{n-1}, b_i)\colon i \in [n]\}.$$

See Figure 2.1 for an example.

We write dimension vectors $\hat{\mathbf{v}}$ for Q_n^+ as tuples $(v_{s_1}, \dots, v_{s_{n-1}}; v_{b_1}, \dots, v_{b_n})$.

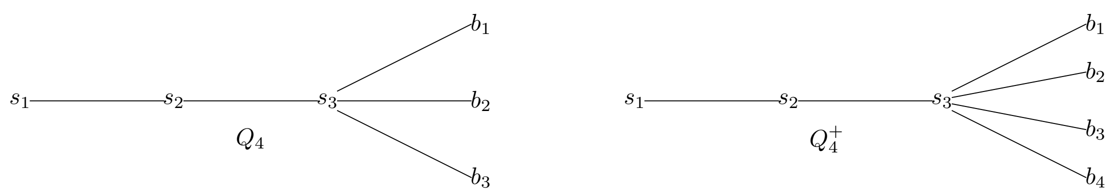


Figure 2.1: Left: The bouquet quiver Q_4, with vertices labelled. Right: The abundant bouquet quiver Q_4^+, with vertices labelled.

A special case of the Crawley-Boevey trick [Cra01, pg. 261] says that the Nakajima quiver variety $\mathfrak{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{e}_{s_{n-1}})$ for the trimmed bouquet Q_n is isomorphic to the unframed Nakajima quiver variety $\mathfrak{M}_{\hat{\lambda},\hat{\theta}}(\hat{\mathbf{v}},\mathbf{0})$ for the abundant bouquet quiver Q_n^+, where:

$$\begin{aligned}
\hat{\mathbf{v}} &:= (\mathbf{v}, 1),\\
\hat{\lambda} &:= \left(\lambda, -\sum_{i\in V(Q_n)} \lambda_i v_i\right),\\
\hat{\theta} &:= \left(\theta, -\sum_{i\in V(Q_n)} \theta_i v_i\right).
\end{aligned}$$

Proposition 2.3.8. *The variety $\mathcal{N}/\!\!/\!\!/T$ is isomorphic to the unframed Nakajima quiver variety $\mathfrak{M}_{0,0}(\hat{\mathbf{v}},\mathbf{0})$ for the quiver Q_n^+ and dimension vector $\hat{\mathbf{v}} = (1,\dots,n-1;1,\dots,1)$.*

Proof. Observe that the restriction of Q_n to the stem vertices is a type A_{n-1} Dynkin quiver, and write $\hat{\mathbf{v}}|_{\text{stem}} := (1,\dots,n-1)$ for the restriction of $\hat{\mathbf{v}}$ to the stem vertices. The Hamiltonian reduction $\mathbb{M}(Q_n^+,\hat{\mathbf{v}},\mathbf{0})/\!\!/\!\!/\mathbb{G}_{\text{stem}} := \mu_{\text{stem}}^{-1}(0)/\!\!/\mathbb{G}_{\text{stem}}$ is the Nakajima quiver variety

$$\mathfrak{M}_{0,0}(\hat{\mathbf{v}}|_{\text{stem}}, n\,\mathbf{e}_{s_{n-1}}),$$

which is known [Nak94, Thm 7.2] to be nilpotent cone $\mathcal{N}$. The map μ_{bouq} descends to $\mathbb{M}(Q_n^+,\hat{\mathbf{v}},\mathbf{0})/\!\!/\!\!/\mathbb{G}_{\text{stem}} \cong \mathcal{N}$ and coincides with the map sending a nilpotent matrix $x = (x_{ij})_{i,j\in[n]}$ to the tuple $(x_{11},\dots,x_{n-1,n-1})$ of diagonal entries of x, and the residual action of $\mathbb{G}_{\text{bouq}}$ coincides with the adjoint action of the torus $T' := \{\mathrm{diag}(t_1,\dots,t_{n-1},1)\}$ on $\mathcal{N}$.

By Hamiltonian reduction in stages ([MMO+07, Thm 5.2.9], see also [Mor14, Thm 3.3.1]),

$$\begin{aligned}
\mathfrak{M}_{0,0}(\hat{\mathbf{v}},\mathbf{0}) &\cong (\mathbb{M}(Q_n^+,\hat{\mathbf{v}},\mathbf{0})/\!\!/\!\!/\mathbb{G}_{\text{stem}})/\!\!/\!\!/\mathbb{G}_{\text{bouq}}\\
&\cong \mathcal{N}/\!\!/\!\!/\mathbb{G}_{\text{bouq}}\\
&= \mu_{\text{bouq}}^{-1}(0)/\!\!/T'.
\end{aligned}$$

As nilpotent matrices are traceless we have $\mu_{\mathrm{bouq}}^{-1}(0) = \{x \in \mathcal{N} \colon \mathrm{diag}(x) = 0\}$. Furthermore, the composite $T' \hookrightarrow T_{\mathrm{GL}_n} \twoheadrightarrow T_{\mathrm{PGL}_n}$ is an isomorphism, and the adjoint action of T factors through $T \to T_{\mathrm{PGL}_n}$; it follows that

$$\mu_{\mathrm{bouq}}^{-1}(0) /\!\!/ T' \cong \{x \in \mathcal{N} \colon \mathrm{diag}(x) = 0\} /\!\!/ T. \qquad \square$$

Remark 2.3.9. By the Crawley-Boevey trick, $\mathcal{N} /\!\!/\!\!/ T$ is also isomorphic to the Nakajima quiver variety $\mathfrak{M}_{0,0}(\mathbf{v}, \mathbf{e}_{s_{n-1}})$ for the quiver Q_n and dimension vector $\mathbf{v} = (1, 2, \ldots, n-1; 1, \ldots, 1)$. △

Theorem 2.3.10. *The variety $\mathcal{N} /\!\!/\!\!/ T$ has a symplectic resolution $\widetilde{\mathcal{N} /\!\!/\!\!/ T}$ given by moving to a generic stability parameter.*

Proof. A routine computation (Lemma 2.6.2) shows that the dimension vector $\hat{\mathbf{v}}$ is in the set $\Sigma_0(Q_n^+)$. Lemma 2.6.3 guarantees that $\hat{\mathbf{v}}$ is anisotropic when $n \geq 4$. The claim follows from Proposition 2.3.6.

For $n = 3$, the quiver Q_n^+ is the affine $\widetilde{D}_4$ Dynkin quiver, and the dimension vector $\hat{\mathbf{v}}$ is the minimal imaginary root; thus $\mathcal{N} /\!\!/\!\!/ T$ is the Kleinian singularity of type D_4. The claim is then a special case of Kronheimer's construction [Kro89] of the minimal resolution via quiver varieties. □

As the diffeomorphism type of a smooth Nakajima variety does not depend on the choice of (generic) moment map and stability parameter ([Nak94, Cor 4.2]), Theorem 2.3.10 implies the following claim.

Proposition 2.3.11 (cf. [Nak94, Cor 4.2])**.** *Let $\theta := (1, \ldots, 1)$. For sufficiently generic $(\nu; \gamma) \in Z(\mathfrak{g}_{\mathbf{v}}^*)$, the variety $\widetilde{\mathcal{N} /\!\!/\!\!/ T}$ is diffeomorphic to $\mathfrak{M}_{(\nu;\gamma),\theta}(\mathbf{v}, \mathbf{e}_{s_{n-1}})$.*

2.3.12 Stable loci conditions

Next, we use King stability conditions to compute stable loci explicitly (Lemma 2.3.13) in order to give an explicit description of the diffeomorphism type of $\widetilde{\mathcal{N}/\!\!/\!\!/T}$.

As in the previous subsection, let $\theta := (1, \ldots, 1)$. We use the notation of Subsection 2.3.1 for the quiver Q_n. Viewing $\mathcal{N}/\!\!/\!\!/T$ and $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ as quiver varieties for Q_n, an element $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ of the corresponding representation space $\mathbb{M}(Q_n, \mathbf{v}, \mathbf{e}_{s_{n-1}})$ consists of maps $x_i \colon \mathbb{C}^i \to \mathbb{C}^{i+1}$, $y_i \colon \mathbb{C}^{i+1} \to \mathbb{C}^i$ between stem vertices, along with n maps $\alpha_i \colon \mathbb{C}^{n-1} \to \mathbb{C}$ and n maps $\beta_i \colon \mathbb{C} \to \mathbb{C}^{n-1}$, where for $i \in [n-1]$ the maps α_i, β_i are incident to s_{n-1} and b_i and α_n, β_n are framing maps on s_{n-1}; see Figure 2.2.

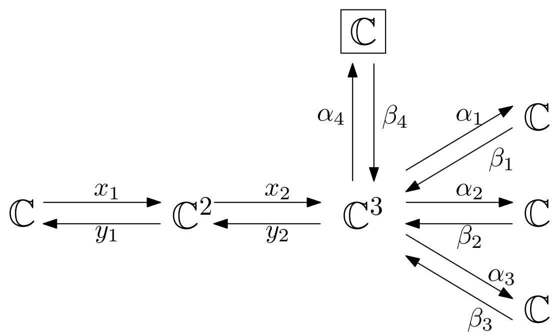


Figure 2.2: An element of the representation space $\mathbb{M}(Q_4, \mathbf{v}, \mathbf{e}_{s_3})$. (The boxed $\mathbb{C}$ is the framing associated with the vertex s_3.)

For $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ in $\mathbb{M}$, it will be useful to combine the α and β into the maps

$$\begin{aligned} \varphi \colon \mathbb{C}^{n-1} &\to \mathbb{C}^n \\ v &\mapsto (\alpha_1(v), \ldots, \alpha_n(v)) \end{aligned}$$

and

$$\begin{aligned} \psi \colon \mathbb{C}^n &\to \mathbb{C}^{n-1} \\ (v_1, \ldots, v_n) &\mapsto \beta_1(v_1) + \cdots + \beta_n(v_n). \end{aligned}$$

The map $\varphi\psi\colon \mathbb{C}^n \to \mathbb{C}^n$ is given by the matrix M whose ij-th entry is $M_{ij} = \alpha_j\beta_i(1)$.

Lemma 2.3.13. *We have*

$$\mathbb{M}^{\theta\text{-s}} = \left\{ (\mathbf{x}, \mathbf{y}, \alpha, \beta) \colon \begin{array}{c} x_i, \varphi \text{ injective} \\ \varphi\psi \text{ satisfies } (*) \end{array} \right\},$$

where the key stability condition is given by:

$$x\colon \mathbb{C}^n \to \mathbb{C}^n \text{ does not preserve any nonzero coordinate subspace of } \mathbb{C}^{n-1} \subset \mathbb{C}^n. \qquad (*)$$

Proof. The condition that $\varphi\psi$ satisfies $(*)$ is equivalent to:

$$\text{For all nonempty } S \subseteq [n-1], \text{ there exists } i \in S \text{ such that } \operatorname{img} \beta_i \not\subset \ker \alpha_j \text{ for some } j \notin S. \qquad (**)$$

(Above, j may be equal to n.) In other words, there may not exist a subset $S \subseteq [n-1]$ where $\operatorname{img} \alpha_i \subset \ker \beta_j$ whenever $i \in S$ and $j \notin S$: such an S exists if and only if $\varphi\psi$ fixes the coordinate subspace $\mathbb{C}^S \subset \mathbb{C}^{n-1}$.

By Lemma 2.3.3, the point $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ is θ-stable if and only if the only $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$-stable subspace of $\ker \alpha_n$ is zero.

We first claim that if any x_k has nonzero kernel, the subspace

$$\ker x_k \oplus \bigoplus_{i=1}^{k-1} y_i y_{i+1} \dots y_{k-1}(\ker x_k)$$

of $\ker \alpha_n$ is $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$-stable, as

$$\begin{aligned}
x_i y_i y_{i+1} \dots y_{k-1}(\ker x_k) &= y_{i+1} x_{i+1} y_{i+1} \dots y_{k-1}(\ker x_k) \\
&= y_{i+1} y_{i+2} x_{i+2} \dots y_{k-1}(\ker x_k) \\
&\vdots \\
&= y_{i+1} y_{i+2} \dots y_{k-1} x_{k-1} y_{k-1}(\ker x_k) \\
&= y_{i+1} y_{i+2} \dots y_{k-1} y_k x_k(\ker x_k) \\
&= 0.
\end{aligned}$$

Similarly, if φ has nonzero kernel, then the subspace

$$\ker \varphi \oplus \bigoplus_{i=1}^{n-2} y_i y_{i+1} \dots y_{n-2} \ker \varphi$$

of $\ker \alpha_n$ is $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$-stable: as above, one iteratively applies the moment map equations $x_j y_j = y_{j+1} x_{j+1}$ when $j \leq n-3$, and then applies the moment map condition $x_{n-2} y_{n-2} = \psi\varphi$.

Next, suppose that there is a nonzero coordinate subspace $\mathbb{C}^S \subset \mathbb{C}^{n-1}$ fixed by $\varphi\psi$. Identifying $\mathbb{C}^S$ with the direct sum of the one-dimensional vector spaces $\mathbb{C}$ at the vertices $\{b_i \colon i \in S\}$, we claim the subspace

$$\mathbb{C}^S \oplus \psi(\mathbb{C}^S) \oplus \bigoplus_{i=1}^{n-2} y_i y_{i+1} \dots y_{n-2} \psi(\mathbb{C}^S)$$

of $\ker \alpha_n$ is $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$-stable: applying moment map equations we compute that

$$\begin{aligned}
x_i(y_i y_{i+1} \dots y_{n-2} \psi(\mathbb{C}^S)) &= y_{i+1} \dots y_{n-2} \psi\varphi\psi(\mathbb{C}^S) \\
&= y_{i+1} \dots y_{n-2} \psi(\mathbb{C}^S)
\end{aligned}$$

because $\varphi\psi$ fixes $\mathbb{C}^S$; furthermore, the equality the equality $\alpha_i = \pi_i\varphi$ (with $\pi_i\colon \mathbb{C}^n \to \mathbb{C}$ the projection to the i-th coordinate) implies

$$\alpha_i\psi(\mathbb{C}^S) \subseteq \begin{cases} \mathbb{C} & \text{if } i \in S \\ 0 & \text{else} \end{cases}$$

because $\varphi\psi$ fixes $\mathbb{C}^S$, while $\beta_i(\mathbb{C}) = \psi(\mathbb{C}^{\{i\}})$ implies

$$\beta_i(\mathbb{C}) \subset \psi(\mathbb{C}^S) \qquad \text{when } i \in S.$$

We deduce that

$$\mathbb{M}^{\theta\text{-s}} \subseteq \left\{ (\mathbf{x}, \mathbf{y}, \alpha, \beta)\colon \begin{array}{c} x_i, \varphi \text{ injective} \\ \varphi\psi \text{ satisfies } (*) \end{array} \right\}.$$

We now show that if x_i, φ are injective and $\varphi\psi$ satisfies $(*)$ then $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$ is θ-stable. Let $V = \bigoplus_{i\in[n-1]} V_{s_i} \oplus \bigoplus_{i\in[n-1]} V_{b_i}$ be an $(\mathbf{x}, \mathbf{y}, \alpha, \beta)$-stable subspace of $\ker\alpha_n$. Let S denote the set of $i \in [n-1]$ for which V_{b_i} is nonzero. Since V is (α, β)-stable, we know $\operatorname{img}\beta_i \subset \ker\alpha_j$ whenever $i \in S$ and $j \notin S$ (cf. $(**)$); thus S must be empty. Since φ is injective and $V \subset \ker\alpha_n$, we deduce $V_{s_{n-1}} = 0$ as well. Finally, since the x_i are injective we conclude that $V_{s_i} = 0$ for $i \in [n-1]$. □

Lemma 2.3.14. *The group $\mathbb{G}_{\text{stem}}$ acts freely on $\{(\mathbf{x}, \mathbf{y}, \alpha, \beta) \in \mu^{-1}(Z(\mathfrak{g}^*_{\mathbf{v}}))\colon x_i, \varphi \text{ injective}\}$, and there is an isomorphism*

$$\{(\mathbf{x}, \mathbf{y}, \alpha, \beta) \in \mu^{-1}(Z(\mathfrak{g}^*_{\mathbf{v}}))\colon x_i, \varphi \text{ injective}\}/\mathbb{G}_{\text{stem}} \xrightarrow{\sim} \widetilde{\mathfrak{g}}$$
$$[(\mathbf{x}, \mathbf{y}, \alpha, \beta)] \longmapsto \left(\varphi\psi - \frac{1}{n}\operatorname{tr}(\varphi\psi)\cdot \operatorname{Id}, F_\bullet\right),$$

where $F_\bullet$ is the flag defined by $F_k = \mathrm{img}(\varphi x_{n-2} \dots x_k)$. Furthermore,

$$(\varphi\psi)|_{F_k/F_{k-1}} = \nu_{n-1} + \cdots + \nu_k, \tag{2.3}$$

where $\nu = (\nu_1, \dots, \nu_{n-1}) := \mu_{\mathrm{stem}}(\mathbf{x}, \mathbf{y}, \alpha, \beta)$. (In particular, $(\varphi\psi)|_{F_n/F_{n-1}} = 0$.)

Proof. Recall that the restriction of Q_n to the stem vertices is the type A_{n-1} Dynkin quiver, and observe that $\mathbb{M}(Q_n, \mathbf{v}, \mathbf{e}_{s_{n-1}})$ and $\mathbb{M}(Q_n|_{\mathrm{stem}}, \mathbf{v}|_{\mathrm{stem}}, n\mathbf{e}_{s_{n-1}})$ are isomorphic as vector spaces via the map $(\mathbf{x}, \mathbf{y}, \alpha, \beta) \mapsto (\mathbf{x}, \mathbf{y}, \varphi, \psi)$.

It is well known that

$$\{(\mathbf{x}, \mathbf{y}, \alpha, \beta) \in \mu^{-1}(Z(\mathfrak{g}_{\mathbf{v}}^*)) \colon x_i, \varphi \text{ injective}\} = \mu_{\mathrm{stem}}^{-1}(Z(\mathfrak{g}_{\mathrm{stem}}^*))^{\theta'\text{-s}},$$

where $\theta' := (1, \dots, 1)$ is a stability condition for $Q_n|_{\mathrm{stem}}$. As μ_{stem} is the moment map for $Q_n|_{\mathrm{stem}}$, Lemma 2.3.3 implies that $\mathbb{G}_{\mathrm{stem}}$ acts freely on

$$\{(\mathbf{x}, \mathbf{y}, \alpha, \beta) \in \mu^{-1}(Z(\mathfrak{g}_{\mathbf{v}}^*)) \colon x_i, \varphi \text{ injective}\}.$$

Furthermore, [DKS13, Thm 7.18] (see also [Wan21, (4.5)]) implies that there is an isomorphism

$$\begin{aligned} (\mu_{\mathrm{stem}})^{-1}(Z(\mathfrak{g}_{\mathrm{stem}}^*))^{\theta'\text{-s}}/\mathbb{G}_{\mathrm{stem}} &\xrightarrow{\sim} \widetilde{\mathfrak{g}} \\ [(\mathbf{x}, \mathbf{y}, \psi, \varphi)] &\longmapsto \left(\varphi\psi - \frac{1}{n}\mathrm{tr}(\varphi\psi) \cdot \mathrm{Id}, F_\bullet\right). \end{aligned}$$

It is left to prove Equation (2.3). To this end, fix any $v \in F_k$, so that $v = \varphi x_{n-2} \dots x_k(w)$

for some w. The moment map equations give

$$\begin{aligned}
\varphi\psi(v) &= \varphi\psi\varphi x_{n-2}\dots x_k(w)\\
&= \varphi \circ [x_{n-2}y_{n-2} + \nu_{n-1}\mathrm{Id}] \circ x_{n-2}\dots x_k(w)\\
&= \varphi x_{n-2}y_{n-2}x_{n-2}\dots x_k(w) + \nu_{n-1}v\\
&= \varphi x_{n-2} \circ [x_{n-3}y_{n-3} + \nu_{n-2}\mathrm{Id}] \circ x_{n-3}\dots x_k(w) + \nu_{n-1}v\\
&= \varphi x_{n-2}x_{n-3}y_{n-3}x_{n-3}\dots x_k(w) + \nu_{n-1}v + \nu_{n-2}v\\
&\vdots\\
&= \underbrace{\varphi x_{n-2}\dots x_k x_{k-1}y_{k-1}(w)}_{\in F_{k-1}} + \nu_{n-1}v + \dots + \nu_k v,
\end{aligned}$$

so that $\varphi\psi$ preserves $F_\bullet$ and $\varphi\psi|_{F_k/F_{k-1}} = \nu_{n-1} + \dots + \nu_k$. □

Proposition 2.3.15. *For any $\nu \in Z(\mathfrak{g}^*_{\mathrm{stem}})$ and $\gamma \in Z(\mathfrak{g}^*_{\mathrm{bouq}})$, we have*

$$\mu^{-1}(\nu,\gamma)^{\theta\text{-s}}/\mathbb{G}_{\mathrm{stem}} = \left\{ (x, F_\bullet) \in \widetilde{\mathfrak{g}}\colon \begin{array}{c} x|_{F_k/F_{k-1}} = \lambda_k \\ \mathrm{diag}(x) = \delta \\ x \text{ satisfies } (*) \end{array} \right\}$$

where the action of $\mathbb{G}_{\mathrm{bouq}} \cong (\mathbb{C}^\times)^{n-1}$ is given by

$$(t_1,\dots,t_{n-1})\cdot(x,F_\bullet) = (\mathrm{Ad}_{t'}(x), t'\cdot F_\bullet), \qquad t' := \mathrm{diag}(t_1,\dots,t_{n-1},1), \tag{2.4}$$

and where λ and δ are given by

$$\lambda_i = \sum_{j=i}^{n-1}\nu_j - \frac{1}{n}(\nu_1 + 2\nu_2 + \dots + (n-1)\nu_{n-1}), \qquad i \in \{1,\dots,n\} \tag{2.5}$$

$$\delta_i = \gamma_i - \frac{1}{n}(\nu_1 + 2\nu_2 + \dots + (n-1)\nu_{n-1}), \qquad i \in \{1,\dots,n\}. \tag{2.6}$$

(Above, we set $\sum_{j=n}^{n-1}\nu_j := 0$ and $\gamma_n := 0$.) In particular,

$$\mathfrak{M}_{(\nu,\gamma),\theta}(\mathbf{v},\mathbf{e}_{s_{n-1}}) = \left\{ (x,F_\bullet)\in\widetilde{\mathfrak{g}}\colon \begin{array}{c} x|_{F_k/F_{k-1}} = \lambda_k \\ \mathrm{diag}(x) = \delta \\ x \text{ satisfies } (*) \end{array} \right\} /\mathbb{G}_{\mathrm{bouq}}.$$

Proof. Lemmas 2.3.13 and 2.3.14 imply that

$$\mu^{-1}(Z(\mathfrak{g}_{\mathbf{v}}^*))^{\theta\text{-s}}/\mathbb{G}_{\mathrm{stem}} \xrightarrow{\sim} \widetilde{\mathfrak{g}}^\theta := \{(x,F_\bullet)\in\widetilde{\mathfrak{g}}\colon x \text{ satisfies } (*)\}.$$

Equation (2.3) implies that μ_{stem} and μ_{bouq} are given by the formulas

$$\begin{aligned} &\mu_{\mathrm{stem}}\colon \widetilde{\mathfrak{g}}^\theta \to Z(\mathfrak{g}^*_{\mathrm{stem}}) \\ &\quad (x,F_\bullet)\mapsto(\nu_1,\dots,\nu_{n-1}), \qquad \nu_i = x|_{F_i/F_{i-1}} - x|_{F_{i+1}/F_i} \\ &\mu_{\mathrm{bouq}}\colon \widetilde{\mathfrak{g}}^\theta \to Z(\mathfrak{g}^*_{\mathrm{bouq}}) \\ &\quad (x,F_\bullet)\mapsto(\gamma_1,\dots,\gamma_{n-1}), \qquad \gamma_i = x_{ii} + \frac{1}{n}(\nu_1+2\nu_2+\dots+(n-1)\nu_{n-1}). \end{aligned}$$

The action of $\mathbb{G}_{\mathrm{bouq}} \cong (\mathbb{C}^\times)^{n-1}$ on $\widetilde{\mathfrak{g}}^\theta$ is given by the formula (2.4). □

Proposition 2.3.16. *For a generic $(\nu;\gamma)$, the condition $(*)$ automatically holds, that is,*

$$\left\{ (x,F_\bullet)\in\widetilde{\mathfrak{g}}\colon \begin{array}{c} x|_{F_\bullet/F_{\bullet-1}} = \lambda \\ \mathrm{diag}(x) = \delta \\ x \text{ satisfies } (*) \end{array} \right\} = \underbrace{\left\{ (x,F_\bullet)\in\widetilde{\mathfrak{g}}\colon \begin{array}{c} x|_{F_\bullet/F_{\bullet-1}} = \lambda \\ \mathrm{diag}(x) = \delta \end{array} \right\}}_{=Y_{\lambda,\delta}},$$

where λ and δ are given by (2.5) and (2.6). In particular,

$$\mathfrak{M}_{(\nu,\gamma),\theta}(\mathbf{v},\mathbf{e}_{s_{n-1}}) = Y_{\lambda,\delta}/\mathbb{G}_{\mathrm{bouq}},$$

where the action of $\mathbb{G}_{\text{bouq}}$ *is given by* (2.4).

Proof. As λ is generic, x is semisimple and there are eigenvectors v_k so that $x(v_k) = \lambda_k v_k$. If x preserves $\mathbb{C}^S \subseteq \mathbb{C}^{n-1}$, then there exist $|S|$ many eigenvectors $v_{i_1}, \ldots, v_{i_{|S|}}$ of x in $\mathbb{C}^S$. It follows that

$$\sum_{s \in S} \delta_s = \operatorname{tr}(x|_{\mathbb{C}^S}) = \sum_{j=1}^{|S|} \lambda_{i_j},$$

contradicting genericity of (ν, γ). □

Corollary 2.3.17. *For sufficiently generic* $(\nu; \gamma)$*, the variety* $\widetilde{\mathcal{N} /\!\!/\!\!/ T}$ *is diffeomorphic to* $Y_{\lambda,\delta}/\mathbb{G}_{\text{bouq}}$.

Proof. Combine Propositions 2.3.11 and 2.3.16. □

2.4 Cohomology of $\widetilde{\mathcal{N} /\!\!/\!\!/ T}$

The goal of this section is to prove Theorem 2.1.3. We construct vector bundles $\mathcal{E}_{S,T}$ on $\widetilde{\mathfrak{g}}$ whose Euler class is $f_{S,T}$ (Lemma 2.4.1), along with a section whose zero locus is djsoint from $Y_{\lambda,\delta}$ (Lemma 2.4.4) and has maximal codimension (Lemma 2.4.5).

For a generic $\lambda = (\lambda_1, \ldots, \lambda_n)$, write $\mathcal{O}_\lambda := \{(x, F_\bullet) \in \widetilde{\mathfrak{g}} \colon x|_{F_k/F_{k-1}} = \lambda_k \mathrm{Id}|_{F_k/F_{k-1}}\}$. By forgetting the flag, the subvariety $\mathcal{O}_\lambda$ of $\widetilde{\mathfrak{g}}$ can be identified with a regular semisimple orbit in $\mathfrak{g}$; in particular the group GL_n acts transitively on $\mathcal{O}_\lambda$ and the stabilizer is a maximal torus.

For $k \in [n]$, let $\mathcal{F}_k$ denote the tautological bundle over the flag variety $\mathcal{B}$ where the fiber over $F_\bullet \in \mathcal{B}$ is the vector space F_k. The bundle $\mathcal{F}_k$ has a natural T-equivariant structure. Also let χ_k denote the trivial bundle $\mathcal{B} \times \mathbb{C}$, endowed with the action of T given by $t \cdot (F_\bullet, z) = (tF_\bullet, \alpha_k(t) z)$, where $\alpha_k \colon T \to \mathbb{C}^\times$ is the character $t = (t_1, \ldots, t_n) \mapsto t_k$.

It is well-known (see e.g. [AF24, Ex 3.1.2, Prop 4.4.1]) that there is an isomorphism

$$\begin{aligned}\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}] &\xrightarrow{\sim} H_T^*(\mathcal{B}) \\ x_k &\mapsto c_1^T((\mathcal{F}_k/\mathcal{F}_{k-1})^\vee) \\ y_k &\mapsto c_1^T(\chi_k).\end{aligned}$$

As $\mathcal{O}_\lambda$ is a retract of $\widetilde{\mathfrak{g}}$, there is an isomorphism

$$\begin{aligned}\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}] &\xrightarrow{\sim} H_T^*(\mathcal{O}_\lambda), \\ x_k &\mapsto c_1^T(i^*p^*((\mathcal{F}_k/\mathcal{F}_{k-1})^\vee)) \\ y_k &\mapsto c_1^T(i^*p^*(\chi_k)),\end{aligned} \qquad (\diamondsuit)$$

where $p\colon \widetilde{\mathfrak{g}} \to \mathcal{B}$ and $i\colon \mathcal{O}_\lambda \to \widetilde{\mathfrak{g}}$ denote the vector bundle map and inclusion respectively.

Lemma 2.4.1. *Under the isomorphism* $(\diamondsuit)$, *the equivariant Euler class of the bundle*

$$\mathcal{E}_{S,T} := i^*p^*\left(\left(\bigoplus_{s\in S}(\mathcal{F}_s/\mathcal{F}_{s-1})^\vee\right) \otimes \left(\bigoplus_{t\in T}\chi_t^\vee\right)\right)$$

corresponds to the polynomial $f_{S,T}(\mathbf{x},\mathbf{y}) \in \mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}] \cong H_T^*(\mathcal{O}_\lambda)$.

Proof. The bundle $\mathcal{E}_{S,T}$ is a direct sum of the line bundles

$$\mathcal{E}_{\{s\},\{t\}} = i^*p^*((\mathcal{F}_s/\mathcal{F}_{s-1})^\vee \otimes \chi_t^\vee), \qquad s \in S, t \in T,$$

and the equivariant Euler class of $\mathcal{E}_{\{s\},\{t\}}$ is identified with $x_s - y_t$ under $(\diamondsuit)$. □

For an integer s and a point $(x, F_\bullet) \in \mathcal{O}_\lambda$, write

$$M_x^{(s)} := (x - \lambda_1 \cdot \mathrm{Id}) \circ \cdots \circ (x - \lambda_{s-1} \cdot \mathrm{Id}).$$

Also write $z_1, \dots, z_n$ for the coordinate functions on $\mathbb{C}^n$.

Lemma 2.4.2. *There is a T-equivariant section of $\mathcal{E}_{\{s\},\{t\}} \to \mathcal{O}_\lambda$ given by*

$$\begin{aligned} \varphi_{\{s\},\{t\}} \colon \mathcal{O}_\lambda &\to \mathcal{E}_{\{s\},\{t\}} \\ (x, F_\bullet) &\mapsto ((x, F_\bullet), (z_t \circ M_x^{(s)})|_{F_s} \otimes 1) \end{aligned}$$

Proof. Observe that the linear map $M_x^{(s)}$ preserves the flag $F_\bullet$ and acts by the zero map on each F_k/F_{k-1} for $1 \le k \le s-1$. In particular $M_x^{(s)}$ acts by the zero matrix on F_{s-1}; thus, the function

$$(z_t \circ M_x^{(s)})|_{F_s} \colon F_s \to \mathbb{C}$$

vanishes on F_{s-1}. Hence φ is a section of $\mathcal{E}_{\{s\},\{t\}} \to \mathcal{O}_\lambda$.

Furthermore, for any $a = \operatorname{diag}(a_1, \dots, a_n) \in T$, we compute that

$$\begin{aligned} (z_t \circ M_{axa^{-1}}^{(s)})|_{aF_s} \otimes 1 &= (a_t z_t \circ M_x^{(s)})|_{aF_s} \otimes 1 \\ &= (z_t \circ M_x^{(s)})|_{aF_s} \otimes a_t; \end{aligned}$$

hence φ is equivariant. □

Lemma 2.4.3. *The zero locus of*

$$\varphi_{S,T} := \bigoplus_{\substack{s \in S \\ t \in T}} \varphi_{\{s\},\{t\}} \colon \mathcal{O}_\lambda \to \mathcal{E}_{S,T}$$

is the set

$$Z_{S,T} := \{(x, F_\bullet) \in \mathcal{O}_\lambda \colon v_s \in \mathbb{C}^{[n]\setminus T} \text{ for all } s \in S\}, \qquad \text{where } x(v_s) = \lambda_s v_s.$$

For subsets $S, T \subset [n]$ with $|S| + |T| = n$, every $(x, F_\bullet) \in Z_{S,T}$ satisfies $x(\mathbb{C}^{[n]\setminus T}) = \mathbb{C}^{[n]\setminus T}$.

Proof. The zero locus of $\varphi_{S,T}$ consists of points $(x, F_\bullet) \in O_\lambda$ such that $(z_t \circ M_x^{(s)})|_{F_s} \colon F_s \to \mathbb{C}$ is the zero function for all $s \in S$ and $t \in T$. This is equivalent to the condition that $v_s \in \mathbb{C}^{[n]\setminus T}$ for all $s \in S$.

When $|S| = n - |T|$, the eigenvectors v_s of x span $\mathbb{C}^{[n]\setminus T}$. It follows that every $(x, F_\bullet) \in Z_{S,T}$ preserves the coordinate subspace $\mathbb{C}^{[n]\setminus T}$. □

Lemma 2.4.4. *For subsets $S, T \subset [n]$ with $|S| + |T| = n$ and generic (λ, δ), the varieties $Z_{S,T}$ and $Y_{\lambda,\delta}$ are disjoint.*

Proof. Recall that $Y_{\lambda,\delta} = \{(x, F_\bullet) \in \mathcal{O}_\lambda \colon \operatorname{diag}(x) = \delta\}$. Lemma 2.4.3 asserts that any $(x, F_\bullet) \in Z_{S,T}$ satisfies $x(\mathbb{C}^{[n]\setminus T}) = \mathbb{C}^{[n]\setminus T}$, so that

$$\sum_{i \in [n]\setminus T} \delta_i = \operatorname{tr}(x|_{\mathbb{C}^{[n]\setminus T}}) = \sum_{s \in S} \lambda_s,$$

contradicting the genericity of (λ, δ). □

Lemma 2.4.5. *For subsets $S, T \subset [n]$ with $|S| + |T| = n$, we have* $\operatorname{codim}_{\mathcal{O}_\lambda}(Z_{S,T}) = \operatorname{rk}(\mathcal{E}_{S,T}) = |S| \cdot |T|$.

Proof. The group GL_n acts transitively on $\mathcal{O}_\lambda$ by conjugation, and the stabilizer of any point is a maximal torus. The subspace $Z_{S,T} \subseteq \mathcal{O}_\lambda$ is an orbit of the subgroup

$$\begin{aligned} G &:= \{g \in \mathrm{GL}_n \colon g(\mathbb{C}^{[n]\setminus T}) = \mathbb{C}^{[n]\setminus T}\} \\ &= \{(g_{ij}) \in \mathrm{GL}_n \colon g_{ab} = 0 \text{ for all } a \in [n] \setminus T \text{ and } b \in T\} \end{aligned}$$

and in particular the stabilizer of any point is an n-dimensional torus. Rearranging the equality $\dim(\mathrm{GL}_n) - \dim(\mathcal{O}_\lambda) = \dim(G) - \dim(Z_{S,T})$ gives

$$\begin{aligned}\operatorname{codim}_{\mathcal{O}_\lambda}(Z_{S,T}) &= \operatorname{codim}_{\mathrm{GL}_n}(G)\\ &= (n - |T|)|T|\\ &= |S| \cdot |T|.\end{aligned}$$

□

Theorem 2.1.3. *The inclusion* $Y \hookrightarrow \widetilde{\mathfrak{g}}$ *induces a surjection*

$$\mathbb{C}[\mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}] \cong H^*_T(\widetilde{\mathfrak{g}}) \twoheadrightarrow H^*_T(Y_{\lambda,\delta}) \cong H^*(\widetilde{\mathcal{N} /\!\!/\!\!/ T})$$

and the kernel contains the functions $\{f_{S,T} \colon |S| + |T| = n\}$.

Proof of Theorem 2.1.3. We first prove surjectivity. There are inclusions

$$\mu^{-1}(\lambda,\delta)^{\theta\text{-s}} \hookrightarrow \{(\mathbf{x},\mathbf{y},\alpha,\beta) \in \mu^{-1}(Z(\mathfrak{g}_{\mathbf{v}})^*) \colon x_i, \varphi \text{ injective}\} \hookrightarrow \mu^{-1}(Z(\mathfrak{g}^*_{\mathbf{v}})).$$

Since $\mu^{-1}(\lambda,\delta)^{\theta\text{-s}}/\mathbb{G}_{\mathbf{v}}$ is a smooth Nakajima quiver variety, the composite map induced on cohomology

$$\begin{aligned}H^*_{\mathbb{G}_{\mathbf{v}}}(\mathrm{pt}) \cong H^*_{\mathbb{G}_{\mathbf{v}}}(\mu^{-1}(Z(\mathfrak{g}^*_{\mathbf{v}}))) \to H^*_{\mathbb{G}_{\mathbf{v}}}&\Big(\{(\mathbf{x},\mathbf{y},\alpha,\beta) \in \mu^{-1}(Z(\mathfrak{g}_{\mathbf{v}})^*) \colon x_i, \varphi \text{ injective}\}\Big)\\ &\xrightarrow{\eta} H^*_{\mathbb{G}_{\mathbf{v}}}(\mu^{-1}(\lambda,\delta)^{\theta\text{-s}})\end{aligned}$$

is surjective by Kirwan surjectivity (Theorem 2.3.4); in particular, the map η is surjective. Lemma 2.3.14 implies that the map η descends to the map

$$\eta \colon H^*_{\mathbb{G}_{\mathrm{bouq}}}(\widetilde{\mathfrak{g}}) \twoheadrightarrow H^*_{\mathbb{G}_{\mathrm{bouq}}}(Y_{\lambda,\delta})$$

induced by the inclusion $Y_{\lambda,\delta} \hookrightarrow \widetilde{\mathfrak{g}}$, where $\mathbb{G}_{\mathrm{bouq}}$ acts as the torus of diagonal matrices whose bottom right entry is 1.

The composite map $\mathbb{G}_{\mathrm{bouq}} \to T_{\mathrm{GL}} \to T_{\mathrm{PGL}}$ is an isomorphism, and it follows that η is a surjection $H^*_{T_{\mathrm{PGL}}}(\widetilde{\mathfrak{g}}) \twoheadrightarrow H^*_{T_{\mathrm{PGL}}}(Y_{\lambda,\delta})$. Corollary 2.3.17 guarantees that $H^*_{T_{\mathrm{PGL}}}(Y_{\lambda,\delta}) \cong H^*(\widetilde{\mathcal{N}/\!\!/\!\!/T})$.

We now show that the kernel contains the functions $f_{S,T}$. As the quotient $T = T_{\mathrm{SL}} \to T_{\mathrm{PGL}}$ induces isomorphisms $H^*_T(\widetilde{\mathfrak{g}}) \cong H^*_{T_{\mathrm{PGL}}}(\widetilde{\mathfrak{g}})$ and $H^*_T(Y_{\lambda,\delta}) \cong H^*_{T_{\mathrm{PGL}}}(Y_{\lambda,\delta})$ it suffices to show that $f_{S,T}$ vanishes under $H^*_T(\widetilde{\mathfrak{g}}) \twoheadrightarrow H^*_T(Y_{\lambda,\delta})$. Lemma 2.4.1 asserts that $\mathrm{eu}^T(\mathcal{E}_{S,T}) = f_{S,T}$. As the zero locus $Z_{S,T}$ of our section $\varphi_{S,T}$ of $\mathcal{E}_{S,T}$ has maximum possible codimension (Lemma 2.4.5), we have $\mathrm{eu}^T(\mathcal{E}_{S,T}) = [Z_{S,T}]^T$ in $H^*_T(\mathcal{O}_\lambda)$ ([AF24, §2.3]). Finally, fundamental classes $[V]^T \in H^*_T(X)$ vanish under the restriction $H^*_T(X) \to H^*_T(X \setminus V)$ ([AF24, pg. 398]), and the varieties Z and $Y_{\lambda,\delta}$ are disjoint (Lemma 2.4.4); in particular $f_{S,T} = [Z_{S,T}]^T$ vanishes under $H^*_T(\mathcal{O}_\lambda) \to H^*_T(Y_{\lambda,\delta})$. Since the inclusion $\mathcal{O}_\lambda \to \widetilde{\mathfrak{g}}$ induces an isomorphism in cohomology, the result follows. □

Remark 2.4.6. Theorem 2.1.3 implies that $\widetilde{\mathcal{N}/\!\!/\!\!/T}$ has vanishing odd cohomology. (The vanishing of odd cohomology is known for all smooth quiver varieties [Nak01, Thm 7.3.5].)

Because $\overline{T^*(G/U)}^{T\times B/U}$ is a zero-dimensional (non-reduced) scheme, its coordinate ring is also finite-dimensional as a $\mathbb{C}$-algebra.

As Theorem 2.1.3 gives a surjective map between two finite-dimensional $\mathbb{C}$-algebras, the Hikita conjecture in fact predicts that the map in Theorem 2.1.3 is an isomorphism. △

2.5 Appendix: Other examples of Conjecture 2.1.4

Let $e \in \mathcal{N}$ be a nilpotent matrix with Jordan type given by a partition λ, and let $e^\vee$ be a nilpotent matrix with Jordan type given by the transpose partition λ'. The closure X of the nilpotent orbit containing e is a symplectic singularity and its symplectic dual $X^!$ is the Slodowy slice in $\mathcal{N}$ through $e^\vee$ ([Hik17]). This behavior is an instance of the "matching of strata" phenomenon in [Kam22, §5.3]. Below, we give examples of Conjecture 2.1.4 arising from the duality between nilpotent orbits and Slodowy slices.

Example 2.5.1. Let $X = \overline{\mathbb{O}_{\min}}$ be the minimal nilpotent orbit closure in type A_n. Then $X^!$ is the Slodowy slice through a subregular nilpotent matrix; in particular $X^! \cong \mathbb{C}^2/A_n$ can be expressed as the Nakajima quiver variety $\mathfrak{M}_{0,0}(\mathbf{1}, \mathbf{0})$ for the affine $\widetilde{A}_n$ Dynkin quiver.

The coordinate ring of the Hamiltonian reduction $X/\!\!/\!\!/T := \{x \in \overline{\mathbb{O}_{\min}} \colon \operatorname{diag}(x) = 0\}/\!\!/T$ is isomorphic to $\mathbb{C}$; thus $X/\!\!/\!\!/T$ is a point. On the other hand, $X^{!,\uparrow} \cong T^*\mathbb{C}^n$ is smooth and the fixed point subscheme $(X^{!,\uparrow})^{T^{!,\uparrow}}$ is one reduced point.

In this case, the Hikita conjecture

$$H^*\left(\widetilde{X/\!\!/\!\!/T}\right) \cong \mathbb{C}\left[(X^{!,\uparrow})^{T^{!,\uparrow}}\right]$$

holds as both sides are isomorphic to $\mathbb{C}$. △

Example 2.5.2. Let X be the Slodowy slice through a nilpotent matrix whose Jordan type has one block of size $n-2$ and another block of size 2, so that $X^!$ is the nilpotent orbit closure

$$X^! = \{x \in \mathcal{N}_{\mathfrak{sl}_n} \colon x^2 = 0, \dim \operatorname{img}(x) = 2\}.$$

The variety $X^!$ has an interpretation as the Nakajima quiver variety $\mathfrak{M}_{0,0}((2),(n))$ associated to the quiver with one vertex and no edges [KP79].

A result of Maffei [Maf05] guarantees that X is the Nakajima quiver variety $\mathfrak{M}_{0,0}(\mathbf{v}, \mathbf{w})$ for the A_{n-1} Dynkin quiver, where $\mathbf{v} = (1, 2, 2, \ldots, 2, 2, 1)$ and $\mathbf{w} = (0, 1, 0, \ldots, 0, 1, 0)$. The torus T acting on X can be identified with the residual action of $\prod_{i \in I} \mathrm{GL}(w_i)$; in particular $X/\!\!/\!\!/T$ is the Nakajima quiver variety $\mathfrak{M}_{0,0}(\mathbf{v}', \mathbf{0})$ for the affine $\widetilde{D}_n$ Dynkin quiver with $\mathbf{v}'$ equal to the minimal imaginary root (see Figure 2.3). In particular $X/\!\!/\!\!/T$ is the Kleinian singularity $\mathbb{C}^2/D_n$.

On the other hand, $X^{!,\uparrow} = T^* \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^n)/\!\!/\!\!/\mathrm{SL}_2$ is the minimal nilpotent orbit closure in $\mathfrak{so}_{2n}$ [Jia21, Prop 3.18]. In this case, the Hikita conjecture for $X/\!\!/\!\!/T$ and $X^{!,\uparrow}$ was verified

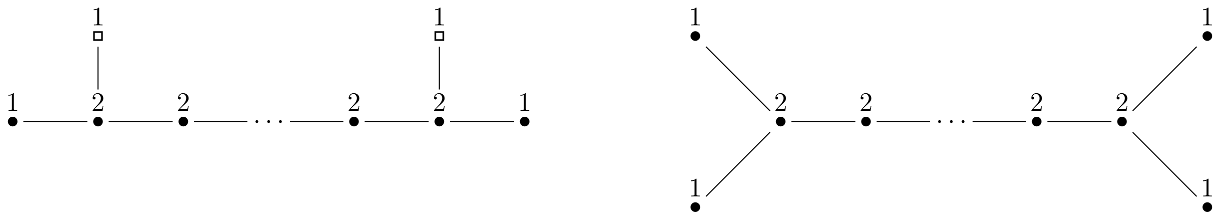


Figure 2.3: Left: The A_{n-1} Dynkin quiver defining the Slodowy slice X. Right: The affine $\widetilde{D}_n$ Dynkin quiver, with dimension vector $\mathbf{v}'$, defining $X/\!\!/\!\!/T = \mathbb{C}^2/D_n$.

in [Shl24]. △

Remark 2.5.3. Now swap the roles of X and $X^!$ in Example 2.5.2: Let X be the nilpotent orbit closure

$$X = \{x \in \mathcal{N}_{\mathfrak{sl}_n} : x^2 = 0, \dim \operatorname{img}(x) = 2\}.$$

Then $X^! = \mathfrak{M}_{0,0}(\mathbf{v}, \mathbf{w})$ is a Nakajima quiver variety for the type A_{n-1} quiver where $\mathbf{v} = (1, 2, 2, \ldots, 2, 2, 1)$ and $\mathbf{w} = (0, 1, 0, \ldots, 0, 1, 0)$ and $X = \mathfrak{M}_{0,0}((2), (n))$ is a Nakajima quiver variety associated to the quiver with one vertex and no edges.

In this case, the Hamiltonian reduction $X/\!\!/\!\!/T$ is the Nakajima quiver variety $\mathfrak{M}_{0,0}(\mathbf{v}, \mathbf{0})$ associated to the star shaped quiver Q_n^{star} with vertices $\{*\} \sqcup \{1, 2, \ldots, n\}$, edges $\{\{*, i\} : i \in [n]\}$, and dimension vector $\mathbf{v}'' = (2; 1, \ldots, 1)$ (see Figure 2.4).

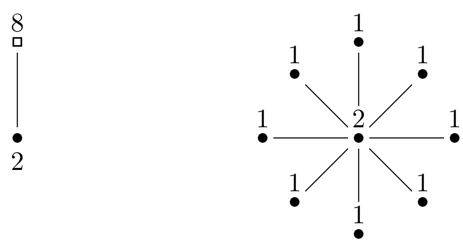


Figure 2.4: Left: The quiver defining the nilpotent orbit X in the case $n = 8$. Right: The star quiver Q_n^{star}, with dimension vector $\mathbf{v}''$, defining $X/\!\!/\!\!/T$, also for $n = 8$.

By [BFN19b, Thm 5.1, Prop 5.20] (see also [DG19, (4.58)]), the corresponding BFN Coulomb branch $\mathcal{M}_C$ associated to Q_n^{star} can be identified with the Hamiltonian reduction $\mathbb{M}(Q, \mathbf{v}, \mathbf{w})/\!\!/\!\!/ \prod_{i \in I} \mathrm{SL}(v_i)$, where Q is the type A_{n-3} Dynkin quiver, $\mathbf{v} = (2, 2, \ldots, 2, 2)$, and $\mathbf{w} = (2, 0, \ldots, 0, 2)$. In particular, $X/\!\!/\!\!/T$ is conjecturally symplectic dual to $\mathcal{M}_C = X^{!,\uparrow}$ (see

Figure 2.5).

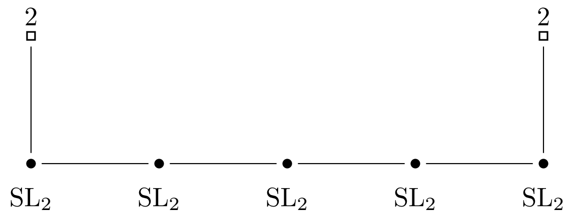


Figure 2.5: The variety $\mathcal{M}_C$, for $n = 8$, expressed as a Hamiltonian reduction of the form $(T^* \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^2))^6 /\!\!/\!\!/ (\mathrm{SL}_2)^5$.

The variety $\widetilde{X /\!\!/\!\!/ T} = \mathfrak{M}_{\theta,0}(\mathbf{v}'', \mathbf{0})$ has appeared in the literature as the *hyperpolygon space* and the cohomology is known [Kon02, Thm 7.1] (see also [HP05, Thm 3.1, Thm 3.2]) to be

$$H^*(\widetilde{X /\!\!/\!\!/ T}) = \mathbb{C}[z_1, \ldots, z_n, p]/I,$$

where I is the ideal generated by all elements $p - z_i^2$ as well as all monomials of degree $2(n-2)$, where $\deg(z_i) := 2$ and $\deg(p) := 4$. The Hikita conjecture predicts that this ring is also the coordinate ring of $(X^{!,\uparrow})^{T^{!,\uparrow}}$. To the best of our knowledge, this case of the Hikita conjecture is still open. △

2.6 Appendix: Bellamy–Schedler criterion

Recall that the Ringel form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\alpha \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

The corresponding Euler form is defined by

$$\begin{aligned}(\alpha, \beta) &:= \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \\ &= 2 \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} (\alpha_{t(a)} \beta_{h(a)} + \beta_{t(a)} \alpha_{h(a)}).\end{aligned} \tag{$\star$}$$

We use a criterion by Crawley-Boevey to determine when $\mathbf{v}$ is in Σ_0:

Proposition 2.6.1 ([Cra01, Cor 5.7])**.** *If* $\mathbf{v} \in \mathbb{N}^I$ *then* $\mathbf{v} \in \Sigma_0$ *if and only if* $\mathbf{v} > 0$ *and* $(\mathbf{w}, \mathbf{v} - \mathbf{w}) \leq -2$ *whenever* $\mathbf{w} \in \mathbb{N}^I$ *and* $0 < \mathbf{w} < \mathbf{v}$*.*

Henceforth, the quiver Q is fixed to be the augmented bouquet quiver Q_n^+.

Lemma 2.6.2. *The dimension vector* $\hat{\mathbf{v}} = (1, 2, \ldots, n-1; 1, 1, \ldots, 1)$ *is in* Σ_0*.*

Proof. We use Proposition 2.6.1. Write $\mathbf{w} = (w_{s_1}, \ldots, w_{s_{n-1}}; w_{b_1}, \ldots, w_{b_n})$, so that

$$\hat{\mathbf{v}} - \mathbf{w} = (1 - w_{s_1}, 2 - w_{s_2}, \ldots, (n-1) - w_{s_{n-1}}; 1 - w_{b_1}, \ldots, 1 - w_{b_n}).$$

Replacing $\mathbf{w}$ with $\hat{\mathbf{v}} - \mathbf{w}$ if necessary, we may assume that $w_{s_{n-1}} \leq (n-1) - w_{s_{n-1}}$. Equation $(\star)$ reads

$$\begin{aligned}(\mathbf{w}, \hat{\mathbf{v}} - \mathbf{w}) = 2\sum_{i=1}^{n-1} w_{s_i}(i - w_{s_i}) + 2\sum_{i=1}^{n} w_{b_i}(1 - w_{b_i}) \\ - \sum_{i=1}^{n-2}\Big(w_{s_i}(i + 1 - w_{s_{i+1}}) + w_{s_{i+1}}(i - w_{s_i})\Big) \\ - \sum_{i=1}^{n}\Big(w_{s_{n-1}}(1 - w_{b_i}) + (n - 1 - w_{s_{n-1}})w_{b_i}\Big)\end{aligned}$$

Since $w_{b_i} \in \{0, 1\}$ we know $w_{b_i}(1 - w_{b_i}) = 0$. Furthermore, since $w_{s_{n-1}} \leq n - 1 - w_{s_{n-1}}$ the quantity $(\mathbf{w}, \hat{\mathbf{v}} - \mathbf{w})$ maximized when $1 - w_{b_i} = 1$ for all i. We conclude

$$\begin{aligned}(\mathbf{w}, \mathbf{v} - \mathbf{w}) &\leq 2\sum_{i=1}^{n-1} w_{s_i}(i - w_{s_i}) - \sum_{i=1}^{n-2}\Big(w_{s_i}(i + 1 - w_{s_{i+1}}) + w_{s_{i+1}}(i - w_{s_i})\Big) - nw_{s_{n-1}} \\ &= 2\sum_{i=2}^{n-1} w_{s_i}(i - w_{s_i}) - \sum_{i=1}^{n-2}\Big(w_{s_i}(i + 1 - w_{s_{i+1}}) + w_{s_{i+1}}(i - w_{s_i})\Big) - nw_{s_{n-1}} \\ &= 2(-w_{s_2}^2 - \cdots - w_{s_{n-1}}^2) + 2(w_{s_1}w_{s_2} + w_{s_2}w_{s_3} + \cdots + w_{s_{n-2}}w_{s_{n-1}}) - 2w_{s_1} \\ &= 2(-w_{s_1}^2 - w_{s_2}^2 - \cdots - w_{s_{n-1}}^2) + 2(w_{s_1}w_{s_2} + w_{s_2}w_{s_3} + \cdots + w_{s_{n-2}}w_{s_{n-1}}),\end{aligned}$$

where the first and last equality follow from the fact that $w_{s_1} \in \{0,1\}$.

Let m and M be the minimum and maximum index so that $w_{s_m}, w_{s_M} \neq 0$. Then

$$2(-w_{s_1}^2 - w_{s_2}^2 - \cdots - w_{s_{n-1}}^2) + 2(w_{s_1}w_{s_2} + w_{s_2}w_{s_3} + \cdots + w_{s_{n-2}}w_{s_{n-1}})$$

is equal to

$$2(-w_{s_m}^2 - w_{s_{m+1}}^2 - \cdots - w_{s_M}^2) + 2(w_{s_m}w_{s_{m+1}} + \cdots + w_{s_{M-1}}w_{s_M}).$$

The AM-GM inequality implies that

$$-w_{s_m}^2 - 2w_{s_{m+1}}^2 - \cdots - 2w_{s_{M-1}}^2 - w_{s_M}^2 + 2w_{s_m}w_{s_{m+1}} + \cdots + 2w_{s_{M-1}}w_{s_M} \leq 0.$$

It follows that

$$2(-w_{s_m}^2 - w_{s_{m+1}}^2 - \cdots - w_{s_M}^2) + 2(w_{s_m}w_{s_{m+1}} + \cdots + w_{s_{M-1}}w_{s_M}) \leq -w_{s_m}^2 - w_{s_M}^2 \leq -2$$

since $w_{s_m}, w_{s_M} \neq 0$. □

Recall that the vector $\hat{\mathbf{v}}$ is *anisotropic* if $\langle \alpha, \alpha \rangle < 0$.

Lemma 2.6.3. *For $n \geq 4$, the dimension vector $\hat{\mathbf{v}} = (1, 2, \ldots, n-1; 1, \ldots, 1)$ is anisotropic.*

Proof. We compute that

$$\begin{aligned}
\langle \hat{\mathbf{v}}, \hat{\mathbf{v}} \rangle &= \sum_{i \in Q_0} v_i^2 - \sum_{e \in Q_1} v_{t(e)}v_{h(e)} \\
&= \left(\sum_{i=1}^{n-1} i^2 \right) + n - \left(\sum_{i=1}^{n-1} i(i+1) \right) \\
&= n - \frac{n(n-1)}{2}
\end{aligned}$$

is negative when $n \geq 4$. $\square$

CHAPTER 3
DOUBLE ORTHODONTIA FORMULAS AND LASCOUX POSITIVITY

This chapter is joint work with Avery St. Dizier.

3.1 Introduction

Orthodontia and flagged Weyl modules

Schubert polynomials $\mathfrak{S}_w(x_1, \ldots, x_n)$, introduced by Lascoux and Schützenberger [LS82], are distinguished representatives of Schubert varieties in the cohomology of the flag variety of $\mathbb{C}^n$. These polynomials have very rich combinatorial structure [BB93, BJS93, FK96, KM05, LLS21, HMMS22] and play a central role in algebraic combinatorics.

Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$ are indexed by permutations $w \in S_n$. The Rothe diagram $D(w)$ of $w \in S_n$ encodes a plethora of information about $\mathfrak{S}_w(\mathbf{x})$. More generally, for any %-avoiding diagram D, there is a representation of the group B of invertible upper triangular matrices called the *flagged Weyl module* $\mathcal{M}_D$. When $D = D(w)$ is a Rothe diagram, the dual character χ_D of the representation $\mathcal{M}_D$ is equal to the Schubert polynomial $\mathfrak{S}_w(\mathbf{x})$ [KP87, KP04]. The study of dual characters χ_D as a whole has shed light on Schubert and related polynomials [FMS18, HMSS24].

Using a geometric interpretation of $\mathcal{M}_D$ as the space of sections of a certain line bundle on a variety, Magyar [Mag98] showed that χ_D is given by the formula

$$\chi_D = \omega_1^{k_1} \ldots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1} \pi_{i_2}(\omega_{i_2}^{m_2}(\ldots \pi_{i_\ell}(\omega_{i_\ell}^{m_\ell}) \ldots))).$$

Here, $\omega_i = x_1 \dots x_i$ is a fundamental weight, $\pi_i = \partial_i x_i$ is a Demazure operator, and

$$\mathbf{k}(D) = (k_1, \dots, k_n), \qquad \mathbf{i}(D) = (i_1, \dots, i_\ell), \qquad \mathbf{m}(D) = (m_1, \dots, m_\ell)$$

is combinatorial data associated to the *orthodontic sequence* of D, which builds a %-avoiding diagram from "smaller" %-avoiding diagrams.

Doubled orthodontia

Double Grothendieck polynomials $\mathfrak{G}_w(x_1, \dots, x_n; y_1, \dots, y_n)$, introduced by Lascoux [Las90], are distinguished representatives of Schubert varieties in the equivariant K-theory of the flag variety. Schubert polynomials can be obtained from double Grothendieck polynomials by setting $y_i \mapsto 0$ (corresponding to forgetting equivariance) and taking the lowest degree part (corresponding to taking the associated graded in K-theory). There is considerable interest in extending our understanding of Schubert polynomials to double Grothendieck polynomials and their various specializations [FK94, KM04, Wei21, LLS23, CCMM23, BFH$^+$23, PSW24, HMSS24].

There is no known K-theoretic or equivariant analogue of $\mathcal{M}_D$. Despite this, we can combinatorially extend Magyar's formula to double Grothendieck polynomials:

Theorem 3.1.1. *Let D be a %-avoiding diagram with double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$. Define*

$$\mathscr{G}_D(\mathbf{x}, \mathbf{y}) := \overline{\omega}_1^{K_1} \overline{\omega}_2^{K_2} \dots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1, j_1} (\overline{\omega}_{i_1}^{M_1} \overline{\pi}_{i_2, j_2} (\overline{\omega}_{i_2}^{M_2} \dots \overline{\pi}_{i_\ell, j_\ell} (\overline{\omega}_{i_\ell}^{M_\ell}) \dots)). \tag{3.1}$$

When $D = D(w)$ is the Rothe diagram of a permutation, then $\mathscr{G}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{G}_w(\mathbf{x}, \mathbf{y})$.

The operators $\overline{\omega}_i^M$ and $\overline{\pi}_{i,j}$ are defined in Equations (3.2) and (3.3).

Double Schubert polynomials $\mathfrak{S}_w(\mathbf{x}, \mathbf{y})$ are equal to the lowest degree part of $\mathfrak{G}_w(\mathbf{x}, -\mathbf{y})$, and Theorem 3.1.1 gives a similar formula for double Schubert polynomials (Corollary 3.3.12). Theorem 3.1.1 also extends our previous joint work with Mészáros [MSS22, Thm 1.1], which gave a similar formula for ordinary Grothendieck polynomials.

As is the case with Magyar's formula for χ_D, and in contrast to the usual degree-*decreasing* recurrence defining Grothendieck polynomials via divided difference operators, the formula (3.1) is degree-*increasing*. This makes the formula (3.1) well suited for some combinatorial applications.

Lascoux positivity

When $D = D(\alpha)$ is the skyline diagram of a composition $\alpha \in \mathbb{N}^n$, the dual character χ_D is the *key polynomial* $\kappa_\alpha(x_1, \ldots, x_n)$ [RS95]. Key polynomials are minimal among the dual characters χ_D of %-avoiding diagrams: every dual character, and in particular every Schubert polynomial, is a nonnegative sum of key polynomials [RS98].

The Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \ldots, x_n)$, indexed by compositions $\alpha \in \mathbb{N}^n$, are inhomogeneous polynomials which often play the same role to the key polynomials as Grothendieck polynomials do to Schubert polynomials. One inhomogeneous extension of the key positivity [RS98] of Schubert polynomials is the Grothendieck-to-Lascoux expansion [RY21,SY23]. One application of our formula (3.1) is another inhomogeneous extension of this key positivity:

Theorem 3.1.2. *Let $D \subseteq [n] \times [m]$ be a diagram whose columns are ordered by inclusion. Let $\mathscr{S}_D(\mathbf{x}, \mathbf{y})$ be the lowest degree part of $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$. Then, the polynomial*

$$x_1^m \ldots x_n^m \mathscr{S}_D(x_n^{-1}, \ldots, x_1^{-1}; -1, \ldots, -1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \ldots, x_n)$.

Theorem 3.1.2 can be modified to incorporate β-Lascoux polynomials: the β-Lascoux expansion of $x_1^m \ldots x_n^m \mathscr{S}_D(x_n^{-1}, \ldots, x_1^{-1}; \beta, \ldots, \beta)$ is $\mathbb{Z}_{\geq 0}[\beta]$-nonnegative.

Theorem 3.1.2 and Corollary 3.3.12 together imply the following corollary.

Corollary 3.1.3. *Let $w \in S_n$ be a vexillary permutation and write $\mathfrak{S}_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$ for the double Schubert polynomial. Then the polynomial*

$$x_1^n \ldots x_n^n \mathfrak{S}_w(x_n^{-1}, \ldots, x_1^{-1}; 1, \ldots, 1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \ldots, x_n)$.

We conjecture that the extra assumption on the diagram D in Theorem 3.1.2 is unnecessary.

Conjecture 3.1.4. *For any %-avoiding diagram $D \subseteq [n] \times [m]$, the polynomial*

$$x_1^m \ldots x_n^m \mathscr{S}_D(x_n^{-1}, \ldots, x_1^{-1}; -1, \ldots, -1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \ldots, x_n)$. In particular, the same holds for

$$x_1^n \ldots x_n^n \mathfrak{S}_w(x_n^{-1}, \ldots, x_1^{-1}; 1, \ldots, 1),$$

for any $w \in S_n$.

In Proposition 3.4.4, we show that Conjecture 3.1.4 would follow from the following conjecture of independent interest:

Conjecture 3.1.5. *For any $\alpha \in \mathbb{N}^n$ and $i \in [n]$, the product $x_1 \ldots x_i(1-x_{i+1}) \ldots (1-x_n)\mathfrak{L}_\alpha$ is a graded nonnegative linear combination of Lascoux polynomials.*

Outline of the paper

Our proof of Theorem 3.1.1 is based on an improvement on the ideas in [MSS22]: we induct on a partial order on S_n called the *orthodontic sort order* (Definition 3.2.16), and a key step is to give a precise description of the first few terms of the orthodontic sequences of *sorted permutations* (Theorem 3.3.9). Although the argument is narratively similar to that in [MSS22], we need different methods.

Our proof of Theorem 3.1.2 examines the orthodontic sequence of diagrams D ordered by inclusion (Corollary 3.4.6) and studying the behavior of the operators ω_i^M and $\pi_{i,j}$ when specializing $y_j \mapsto -1$ and intertwining with the operator

$$f(x_1, \ldots, x_n) \mapsto x_1^m \ldots x_n^m f(x_n^{-1}, \ldots, x_1^{-1}).$$

3.2 Background

Conventions

We write permutations $w \in S_n$ in one-line notation. For $j \in [n-1]$, the notation s_j denotes the adjacent transposition in S_n which swaps j and $j+1$. For $w \in S_n$, let $\ell(w)$ denote the length of w. Permutations act on the right: ws_j is equal to w with $w(j)$ and $w(j+1)$ swapped.

Difference operators and families of polynomials

For $i \in [n-1]$, define the *divided difference operator* $\partial_i \colon \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ by the formula

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}}.$$

The *isobaric divided difference operators* $\overline{\partial}_i$, *Demazure operators* π_i, and *Demazure–Lascoux operators* $\overline{\pi}_i$ are defined on $\mathbb{R}[x_1,\ldots,x_n]$ by the formulas

$$\begin{aligned}\overline{\partial}_i(f) &:= \partial_i((1-x_{i+1})f),\\ \pi_i(f) &:= \partial_i(x_i f),\\ \overline{\pi}_i(f) &:= \overline{\partial}_i(x_i f).\end{aligned}$$

Lemma 3.2.1. *Let φ_i denote any of ∂_i, $\overline{\partial}_i$, π_i, or $\overline{\pi}_i$. Then*

$$\varphi_i\varphi_{i+1}\varphi_i = \varphi_{i+1}\varphi_i\varphi_{i+1} \qquad \text{and} \qquad \varphi_i\varphi_j = \varphi_j\varphi_i \text{ if } |i-j|\geq 2.$$

The *double Grothendieck polynomial* $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ of $w \in S_n$ is defined recursively on the weak Bruhat order, starting from the longest permutation $w_0 \in S_n$. The polynomial $\mathfrak{G}_w(\mathbf{x},\mathbf{y})$ is defined by

$$\mathfrak{G}_w(\mathbf{x},\mathbf{y}) = \begin{cases} \prod_{i+j\leq n}(x_i+y_j-x_iy_j) & \text{if } w=w_0,\\ \overline{\partial}_i\mathfrak{G}_{ws_i}(\mathbf{x},\mathbf{y}) & \text{if } \ell(w)<\ell(ws_i).\end{cases}$$

Lemma 3.2.1 guarantees that double Grothendieck polynomials are well-defined. The *double Schubert polynomial* $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$ is defined to be the lowest degree part of $\mathfrak{G}_w(\mathbf{x},-\mathbf{y})$. In particular,

$$\mathfrak{S}_w(\mathbf{x},\mathbf{y}) = \begin{cases} \prod_{i+j\leq n}(x_i-y_j) & \text{if } w=w_0,\\ \partial_i\mathfrak{S}_{ws_i}(\mathbf{x},\mathbf{y}) & \text{if } \ell(w)<\ell(ws_i).\end{cases}$$

The ordinary *Grothendieck polynomials* $\mathfrak{G}_w(\mathbf{x})$ and ordinary *Schubert polynomials* $\mathfrak{S}_w(\mathbf{x})$ are defined to be the specializations $\mathfrak{G}_w(\mathbf{x},\mathbf{0})$ and $\mathfrak{S}_w(\mathbf{x},\mathbf{0})$ of their doubled counterparts.

In particular,

$$\mathfrak{G}_w(\mathbf{x}) = \begin{cases} x_1^n x_2^{n-1} \dots x_n & \text{if } w = w_0, \\ \overline{\partial}_i \mathfrak{G}_{ws_i}(\mathbf{x}) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^n x_2^{n-1} \dots x_n & \text{if } w = w_0, \\ \partial_i \mathfrak{S}_{ws_i}(\mathbf{x}) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

The symmetric group S_n acts on compositions via permuting coordinates. The *Lascoux polynomial* $\mathfrak{L}_\alpha$ of a composition $\alpha \in \mathbb{N}^n$ is defined recursively by

$$\mathfrak{L}_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \cdots \geq \alpha_n \\ \overline{\pi}_i \mathfrak{L}_{\alpha \cdot s_i}(\mathbf{x}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

Lemma 3.2.1 guarantees that Lascoux polynomials are well-defined. The *key polynomial* $\kappa_\alpha(\mathbf{x})$ is defined to be the lowest degree part of $\mathfrak{L}_\alpha(\mathbf{x})$. In particular,

$$\kappa_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \cdots \geq \alpha_n \\ \pi_i \kappa_{\alpha \cdot s_i}(\mathbf{x}) & \text{if } \alpha_i < \alpha_{i+1}. \end{cases}$$

Pipe dreams

A *pipe dream* is a filling of a triangular grid $\{(i,j) \in [n] \times [n] \colon i + j \leq n\}$ with crossing tiles and bumping tiles. By placing half-bumping tiles at $\{(i,j) \in [n] \times [n] \colon i+j = n+1\}$, a pipe dream forms a network of n pipes running from the top edge of the grid to the left edge (see Figure 3.1).

For any pipe dream P, there is an associated permutation $\partial(P) \in S_n$ given by labeling the pipes 1 through n along the top edge, tracing the pipes and ignoring any crossings between any pair of pipes which have already crossed, i.e. replacing redundant crossing tiles with bump tiles; reading the labels of the pipes along the left edge from top to bottom gives a string of numbers $(\partial(P))(1), (\partial(P))(2), \ldots, (\partial(P))(n)$ that defines $\partial(P) \in S_n$. (The permutation $\partial(P)$ is the *Demazure product* of the transpositions $s_i \in S_n$ corresponding to antidiagonals on which the crosses sit, reading right to left, starting from the top row; cf. [KM04, Ex 5.1], [Wei21, §6.1]. Our convention agrees with original definition [BB93]: the pipe at the i^{th} row is connected to the $(\partial(P))(i)^{\text{th}}$ column.)

Identify P with the set $\{(i,j)\colon P \text{ has } \boxplus \text{ at } (i,j)\}$ of crosses of P, and denote by $P^{(i)}$ the pipe of P entering in the i^{th} column.

Example 3.2.2. All five pipe dreams P such that $\partial(P) = 1423$ are displayed in Figure 3.1. Redundant crossing tiles $\boxplus$ are highlighted in red. △

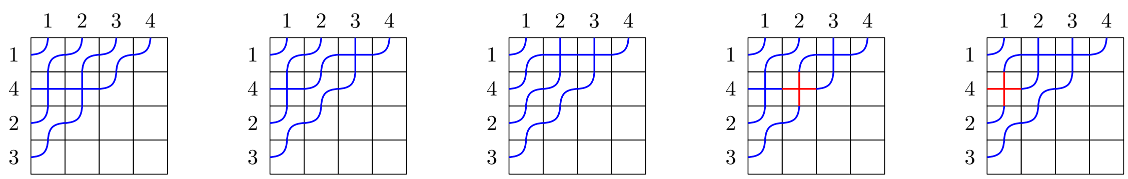


Figure 3.1: The five pipe dreams in PD(1423).

Theorem 3.2.3 ([Wei21, Thm 6.1]; see also [KM04, Cor 5.4], [FK94, Thm 2.3])**.** *The double Grothendieck polynomial of $w \in S_n$ is given by the formula*

$$\mathfrak{G}_w(\mathbf{x}, \mathbf{y}) = \sum_{P \in \mathrm{PD}(w)} \prod_{(i,j) \in P} (x_i + y_j - x_i y_j).$$

Orthodontic sequence

A *diagram* is defined to be a subset $D \subseteq [n] \times [m]$. View $D = (D_1, \ldots, D_m)$ as a subset of an $n \times m$ grid, where $D_j := \{i \in [n]\colon (i,j) \in D\}$ encodes the j^{th} column: an element $i \in D_j$ corresponds to a box in row i and column j.

Definition 3.2.4. Let $w \in S_n$. The Rothe diagram $D(w)$ is defined to be

$$D(w) = \{(i,j) \in [n] \times [n] \colon i < w^{-1}(j) \text{ and } j < w(i)\}. \qquad \triangle$$

Example 3.2.5. The Rothe diagram $D(31542)$ consists of the purple squares in Figure 3.2.

$\triangle$

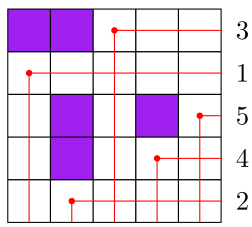


Figure 3.2: The Rothe diagram for $w = 31542$.

Definition 3.2.6 ([RS98])**.** A diagram D is called *%-avoiding* when $i_2 \in D_{j_1}$ and $i_1 \in D_{j_2}$ for $i_1 < i_2$ and $j_1 < j_2$ implies $i_1 \in D_{j_1}$ or $i_2 \in D_{j_2}$. $\triangle$

Equivalently, a diagram is %-avoiding if it does not have a pair of rows and a pair of columns to which its restriction looks like the configuration in Figure 3.3.

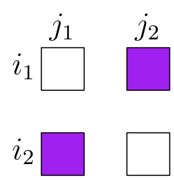


Figure 3.3: A forbidden configuration in a %-avoiding diagram.

Given a %-avoiding diagram D, we describe an algorithm to compute its *double orthodontic sequence*

$$\mathbf{K}(D) = (K_1, \ldots, K_n), \quad \mathbf{i}(D) = (i_1, \ldots, i_\ell), \quad \mathbf{j}(D) = (j_1, \ldots, j_\ell), \quad \mathbf{M}(D) = (M_1, \ldots, M_\ell).$$

Begin by setting $K_i := \{j \colon D_j = [i]\}$, and let D_- denote the diagram obtained by replacing any such columns with the empty column. If D_- is empty, then $\mathbf{i} = \mathbf{j} = \mathbf{M}$ is the empty vector and the algorithm terminates.

An *orthodontic step* applied to D_- outputs a diagram $(D_-)''$ which we now describe. A *missing tooth* of a column $C \subseteq [n]$ is an integer $i \in [n]$ so that $i \notin C$ and $i+1 \in C$; any nonempty column of D_- has a missing tooth. Set i_1 to be the smallest missing tooth of the leftmost nonempty column $(D_-)_j$ of D_-. Set j_1 to be $j - \#\{a \leq i_1 : a \notin (D_-)_j\}$. Now swap rows i_1 and i_1+1 to get a diagram $(D_-)'$. Any column of $(D_-)'$ which is a standard interval is necessarily equal to $[i_1]$; set $M_1 = \{j : (D_-)'_j = [i_1]\}$. Let $(D_-)''$ denote the diagram obtained from $(D_-)'$ by removing all columns equal to $[i_1]$.

Lemma 3.2.7 ([RS98, Prop 12], see also [Mag98, pg. 12])**.** *Any %-avoiding diagram D will be the empty diagram after sufficiently many orthodontic steps.*

Repeatedly apply orthodontic steps to the resulting diagrams $(D_-)''$, keeping track of the data $\mathbf{i}, \mathbf{j}, \mathbf{M}$ at each step, until the diagram is empty. These diagrams form the double orthodontic sequence of D.

Example 3.2.8. The orthodontic sequence for the Rothe diagram $D(31542)$ is shown in Figure 3.4. △

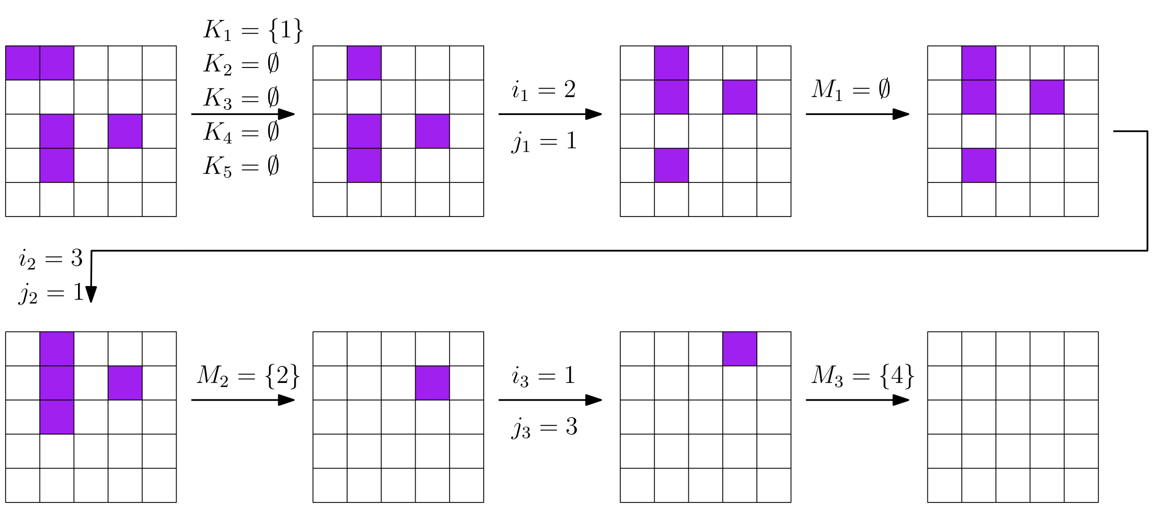


Figure 3.4: The double orthodontic sequence for $w = 31542$.

For a polynomial $f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$, define the operators

$$\pi_{i,j} := \partial_i((x_i + y_j)f), \qquad \overline{\pi}_{i,j} := \overline{\partial}_i((x_i + y_j - x_i y_j)f), \tag{3.2}$$

and for $i \in [n]$ and $M \subseteq [n]$ define the quantities

$$\omega_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m), \qquad \overline{\omega}_i^M := \prod_{\substack{j \in [i] \\ m \in M}} (x_j + y_m - x_j y_m). \tag{3.3}$$

Definition 3.2.9. Let D be a %-avoiding diagram and let $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ denote its double orthodontic sequence. Define

$$\mathscr{G}_D(\mathbf{x}, \mathbf{y}) := \overline{\omega}_1^{K_1} \overline{\omega}_2^{K_2} \dots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1, j_1} (\omega_{i_1}^{M_1} \overline{\pi}_{i_2, j_2} (\omega_{i_2}^{M_2} \dots \overline{\pi}_{i_\ell, j_\ell} (\omega_{i_\ell}^{M_\ell}) \dots)),$$

and write

$$\mathscr{S}_D(\mathbf{x}, \mathbf{y}) := \omega_1^{K_1} \omega_2^{K_2} \dots \omega_n^{K_n} \pi_{i_1, j_1} (\omega_{i_1}^{M_1} \pi_{i_2, j_2} (\omega_{i_2}^{M_2} \dots \pi_{i_\ell, j_\ell} (\omega_{i_\ell}^{M_\ell}) \dots)).$$ △

Remark 3.2.10. It can be shown by inducting down the orthodontic sequence that $\mathbf{x}^D$ appears in $\mathscr{S}_D$ with coefficient 1. As $\mathscr{S}_D$ is nonzero, it follows that $\mathscr{S}_D$ is the lowest degree part of $\mathscr{G}_D$. △

Remark 3.2.11. The polynomial $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$ is not invariant under permuting columns, in contrast to previous work [Mag98, MSS22] on the orthodontia algorithm. This is by design: the Rothe diagram $D(2413)$ can be obtained from the Rothe diagram $D(132)$ by permuting the columns and adding a column equal to $\{1, 2\}$, and although the ordinary Grothendieck polynomials satisfy $\mathfrak{G}_{2413}(\mathbf{x}) = x_1 x_2 \mathfrak{G}_{132}(\mathbf{x})$, there does not exist a polynomial $g(\mathbf{x}, \mathbf{y})$ so that $\mathfrak{G}_{2413}(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}) \cdot \mathfrak{G}_{132}(\mathbf{x}, \mathbf{y})$. △

Following [MSS22], we recall some basic facts about sorted permutations and the sorting projection.

In what follows, a *standard interval* is a set of the form $[j]$ for some $j \geq 0$. Recall that a permutation $w \in S_n$ is called *dominant* if it is 132-avoiding. A permutation w is dominant if and only if its Rothe diagram $D(w) = (D(w)_1, \dots, D(w)_n)$ consists only of standard intervals, which necessarily satisfy $\#D(w)_1 \geq \cdots \geq \#D(w)_n$.

Definition 3.2.12. Fix a permutation $w \in S_n$. The *primary column data* of w, denoted $(h, C, \alpha, i_1, \beta)$, is defined as follows.

If w is not dominant, the diagram $D(w)$ has a column which is not a standard interval. Set h to be the smallest integer such that the column $D(w)_{h+1}$ is not a standard interval. Set C to be the column $D(w)_{h+1} \subseteq [n]$. Set α to be the largest integer such that $[\alpha] \subseteq C$. Set i_1 to be the smallest missing tooth of C. Lastly, set $\beta = i_1 - \alpha$, the size of the "uppermost gap" of C.

If w is dominant, set $h = n$, $C = \emptyset$, $\alpha = 0$, $i_1 = n$, and $\beta = n$. △

Lemma 3.2.13 ([MSS22, Lem 3.4])**.** *Any permutation w restricts to a bijection $[\alpha+1, i_1] \to [h - \beta + 1, h]$. The corresponding permutation $\sigma \in S_\beta$ is dominant.*

See Figure 3.5 for an example.

For $w \in S_n$, write $\sigma(w)$ for the dominant permutation obtained by restricting w to $[\alpha + 1, i_1]$.

Definition 3.2.14. The permutation w is *sorted* if $\sigma(w)$ is the identity. The *sorting* of w, denoted w_{sort}, is the permutation obtained from w by reordering $w(\alpha + 1), \dots, w(i_1)$ to be in increasing order. △

Example 3.2.15. Let $w = 68342751$. The primary column data of w is

$$h = 4,\ C = \{1, 2, 6\},\ \alpha = 2,\ i_1 = 5,\ \beta = 3.$$

and w restricts to a bijection $\{3, 4, 5\} \to \{2, 3, 4\}$; the corresponding permutation $\sigma(w) \in S_3$ is $\sigma(w) = 231$. The sorting of w is $w_{\text{sort}} = 68234751$. The Rothe diagrams of w and w_{sort} are displayed in Figure 3.5. △

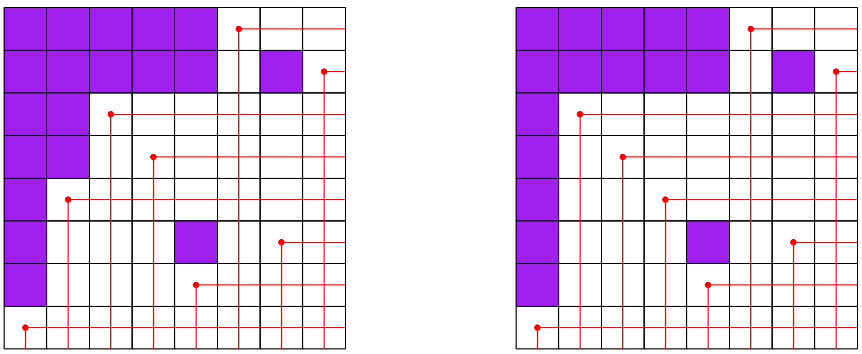


Figure 3.5: The Rothe diagram of $w = 68342751$ on the left, and of its sorting $w_{\text{sort}} = 68234751$ on the right. In this example, $\sigma(w) = 231$.

Definition 3.2.16. The *orthodontic sort order* $\leq_{\text{os}}$ is the reflexive and transitive closure of the relations

$$w_{\text{sort}} \preceq w \tag{†}$$

$$ws_{i_1} \dots s_\alpha \preceq w \text{ whenever } w \text{ is nonidentity and sorted.} \tag{‡}$$

△

Proposition 3.2.17 ([MSS22, Prop 5.6])**.** *The relation $\leq_{\text{os}}$ is a partial order on S_n and the identity permutation is the minimum element.*

3.3 Orthodontia and double Grothendieck polynomials

We establish Theorem 3.1.1 by studying the relationship between $\mathfrak{G}_w$ and $\mathfrak{G}_{w_{\mathrm{sort}}}$ (Proposition 3.3.7), the relationship between $\mathscr{G}_{D(w)}$ and $\mathscr{G}_{D(ws_{i_1}\cdots s_\alpha)}$ (Theorem 3.3.9), and then inducting on the orthodontic sort order (Definition 3.2.16).

The following example motivates Proposition 3.3.2 and Corollary 3.3.3.

Example 3.3.1. Let $w = 68342751$, as in Example 3.2.15, so that $w_{\mathrm{sort}} = 68234751$. Observe that $D(w_{\mathrm{sort}}) = D(w) \setminus \{(2,3),(2,4)\}$ is obtained from $D(w)$ by removing bottom-aligned portions of standard interval columns. Proposition 3.3.2 asserts that this phenomenon holds in general.

Write $\mathbf{K}(w) = (K_1,\ldots,K_n)$ and $\mathbf{K}(w_{\mathrm{sort}}) = (K'_1,\ldots,K'_n)$. Then:

$$K_1 = \emptyset, \quad K_2 = \{3,4\}, \quad K_3 = \emptyset, \quad K_4 = \{2\}, \quad K_5 = \emptyset, \quad K_6 = \emptyset, \quad K_7 = \{1\}$$

and

$$K'_1 = \emptyset, \quad K'_2 = \{2,3,4\}, \quad K'_3 = \emptyset, \quad K'_4 = \emptyset, \quad K'_5 = \emptyset, \quad K'_6 = \emptyset, \quad K'_7 = \{1\}.$$

The fact that $K'_4 = K_4 \setminus \{2\}$ and $K'_2 = K_2 \cup \{2\}$ can be viewed as a consequence of the fact that $D(\sigma(w)) = D(231)$ has Rothe diagram consisting of one standard interval column of size two.

As $D(w_{\mathrm{sort}})$ can be obtained from $D(w)$ by removing bottom-aligned portions of standard interval columns, the two diagrams agree after removing standard interval columns. It follows that the $\mathbf{i},\mathbf{j},\mathbf{M}$ orthodontic sequences of w and w_{sort} agree: $\mathbf{i}(w) = \mathbf{i}(w_{\mathrm{sort}})$, $\mathbf{j}(w) = \mathbf{j}(w_{\mathrm{sort}})$, and $\mathbf{M}(w) = \mathbf{M}(w_{\mathrm{sort}})$. This phenomenon holds more generally, and is stated precisely in Corollary 3.3.3. △

Proposition 3.3.2. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$. Set*

$\lambda_j^\sigma := \#D(\sigma(w))_j$. *The Rothe diagram* $D(w_{\text{sort}})$ *can be obtained from* $D(w)$ *by removing bottom-aligned portions of standard-interval columns, and*

$$D(w) \setminus D(w_{\text{sort}}) = \{(\alpha + a, h - \beta + b) \colon 1 \leq a \leq \lambda_b^\sigma \text{ and } 1 \leq b \leq \beta\}. \tag{3.4}$$

Proof of Proposition 3.3.2. As w_{sort} is obtained from w by reordering numbers $\{w^{-1}(j) \colon j \in [h - \beta + 1, h]\}$, the columns $D(w_{\text{sort}})_j$ and $D(w)_j$ agree whenever $j \notin [h - \beta + 1, h]$. By definition of the primary column data and of the sorting of a permutation, $D(w)_{h-\beta+b} = [\alpha + \lambda_b^\sigma]$ and $D(w_{\text{sort}})_{h-\beta+b} = [\alpha]$. □

Corollary 3.3.3 (cf. [MSS22, Prop 4.4])**.** *Let* $(h, C, \alpha, i_1, \beta)$ *denote the primary column data of* $w \in S_n$*. Set*

$$\mathcal{S}_a := \left\{ h - \beta + j \colon \begin{array}{c} j \in [\beta] \\ \#D(\sigma(w))_j = a \end{array} \right\}.$$

Write $\mathbf{K}(w) = (K_1, \ldots, K_n)$ *and* $\mathbf{K}(w_{\text{sort}}) = (K'_1, \ldots, K'_n)$*. Then*

$$\mathbf{i}(w) = \mathbf{i}(w_{\text{sort}}), \qquad \mathbf{j}(w) = \mathbf{j}(w_{\text{sort}}), \qquad \mathbf{M}(w) = \mathbf{M}(w_{\text{sort}}),$$

and

$$K_j = \begin{cases} K'_j & \text{if } j \leq \alpha - 1, \\ K'_j \setminus \{h - \beta + 1, \ldots, h\} \cup \mathcal{S}_0 & \text{if } j = \alpha, \\ K'_j \cup \mathcal{S}_a & \text{if } j = \alpha + a, \text{ where } a \in [\beta], \\ K'_j & \text{if } j \geq i_1 + 1. \end{cases}$$

In particular,

$$\mathscr{G}_{D(w)} = \left(\prod_{(a,b) \in D(w) \setminus D(w_{\text{sort}})} (x_a + y_b - x_a y_b) \right) \cdot \mathscr{G}_{D(w_{\text{sort}})}.$$

Proof. Proposition 3.3.2 implies that the diagrams $D(w)$ and $D(w_{\mathrm{sort}})$ agree after removing their standard interval columns. It follows that the orthodontic sequences $\mathbf{i}, \mathbf{j}, \mathbf{M}$ of w and w_{sort} agree. The formulas for K_j in terms of K'_j hold by (3.4). The relationship between $\mathscr{G}_{D(w)}$ and $\mathscr{G}_{D(w_{\mathrm{sort}})}$ follows from the construction (3.1) of $\mathscr{G}_D$. □

Lemma 3.3.4. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$. For $j \in [h]$, set*

$$\lambda_j := \#D(w)_j \quad \text{and} \quad \nu_j := \#\{i \le j \colon \#D(w)_i \ge w^{-1}(j)\}.$$

Fix any $P \in \mathrm{PD}(w)$. Then for every $j \in [h]$, the pipe $P^{(j)}$ begins by traversing λ_j crossing tiles vertically, ends by traversing ν_j tiles horizontally, and otherwise traverses only bump tiles. In particular, every tile in

$$C_w := \left\{ (i,j) \colon \begin{array}{c} 1 \le i \le \lambda_j, \\ 1 \le j \le h \end{array} \right\}$$

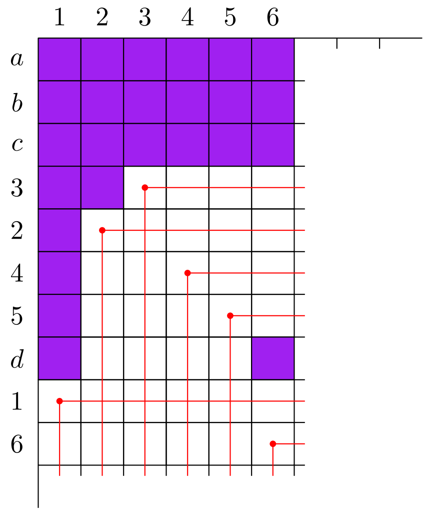
is a cross, and every tile in

$$E_w := \left\{ (i,j) \colon \begin{array}{c} h - \beta + 1 \le i \le h, \\ \alpha + 1 \le j \le i_1, \\ i + j \le h + \alpha + 1 \end{array} \right\} \setminus C_w$$

is an elbow.

Example 3.3.5. Figure 3.6 gives an example of a typical pipe dream, with the structure guaranteed by Lemma 3.3.4 highlighted. △

Proof of Lemma 3.3.4. Induct on j. The pipe $P^{(1)}$ necessarily traveres λ_1 cross tiles vertically before traversing a bump tile to exit at the $w^{-1}(1)^{\mathrm{th}}$ row. Now assume that, for all $k < j$, the pipes $P^{(k)}$ traverse λ_k cross tiles vertically, then traverse bump tiles, before ending by traversing ν_k tiles horizontally.



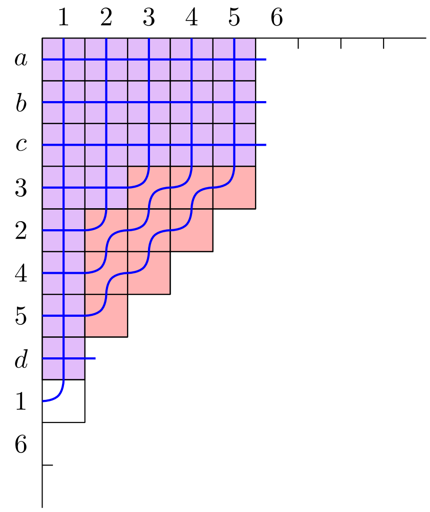


Figure 3.6: Left: The Rothe diagram of a permutation $w = abc3245d16...$, for $a, b, c, d > 6$. Right: A typical pipe dream $P \in \mathrm{PD}(w)$. The boxes in C_w are shaded purple and the boxes in E_w are shaded red.

As $\lambda_1 \geq \cdots \geq \lambda_j$, the inductive hypothesis guarantees that all tiles

$$\left\{ (i,k) \colon \begin{array}{c} 1 \leq i \leq \lambda_j \\ 1 \leq k \leq j-1 \end{array} \right\}$$

are cross tiles. Thus if $P^{(j)}$ traverses a bump tile after traversing $a < \lambda_j$ cross tiles vertically, pipe $P^{(j)}$ is forced to travel horizontally until exiting at the $(a+1)^{\text{th}}$ row. As $a + 1 \leq \lambda_j < w^{-1}(j)$, this is a contradiction.

As λ_i is weakly decreasing, $\#D(w)_i \geq w^{-1}(j)$ if and only if $i \in [\nu_j]$. The inductive hypothesis guarantees that the pipe exiting at row $w^{-1}(j)$ traverses ν_j crossing tiles horizontally before traversing any bump tiles.

It remains to show that $P^{(j)}$ does not traverse any other cross tiles. To this end, observe that any such cross tile (a, b) satisfies $a > \lambda_j$ and $b > \nu_j$. It follows that any such cross tile cannot involve $P^{(w^{-1}(k))}$ for $k \leq \lambda_j$ as the paths of these pipes are contained in $\{(a, b) \colon a \leq k\}$, and cannot involve $P^{(k)}$ for $k \leq \nu_j$ as the paths of these pipes are contained in $\{(a, b) \colon b \leq \nu_j\}$. In other words, any such cross tile involves $P^{(j)}$ and $P^{(k)}$ for $k \notin w^{-1}([\lambda_j])$ and $k \notin [\nu_j]$.

Then, the first part of the result follows from the fact that the inversions of w involving j are

$$\{(k, j)\colon k \in w^{-1}([\lambda_j]) \text{ or } k \in [\nu_j]\}.$$

The rest of the lemma follows from the fact that the set of pipes entering the top edge of the triangular grid

$$\left\{ (i,j)\colon \begin{array}{c} h-\beta+1 \le i \le h, \\ \alpha+1 \le j \le i_1, \\ i+j \le h+\alpha+1 \end{array} \right\},$$

namely the set $\{P^{(j)}\colon h-\beta+1 \le j \le h\}$, is equal to the set of pipes exiting the left edge, hence restricts to a pipe dream for $\sigma(w)$. □

Corollary 3.3.6. *Let $(h, C, \alpha, i_1, \beta)$ denote the primary column data of $w \in S_n$, and let $\mathcal{S} := \{h-\beta+1, \ldots, h\}$. Fix $P \in \mathrm{PD}(w)$.*

- *For $i, j \in \mathcal{S}$ and $k \notin \mathcal{S}$, the pipes $P^{(i)}$ and $P^{(k)}$ cross if and only if $P^{(j)}$ and $P^{(k)}$ cross.*
- *For $i \in \mathcal{S}$ and $j \in [n]$, the pipes $P^{(i)}$ and $P^{(j)}$ cross at most once.*
- *For $i, j \in \mathcal{S}$, the pipes $P^{(i)}$ and $P^{(j)}$ can cross only at $C_w \setminus C_{w_{\mathrm{sort}}}$, and conversely every tile in $C_w \setminus C_{w_{\mathrm{sort}}}$ is a crossing of pipes $P^{(i)}$ and $P^{(j)}$ for $i, j \in \mathcal{S}$.*

Proof. For any $i \in \mathcal{S}$, Lemma 3.3.4 guarantees that pipe $P^{(i)}$ begins by traversing α many cross tiles vertically, crossing pipes $P^{(k)}$ for $k \in w^{-1}([\alpha])$, and ends by traversing $h-\beta$ many cross tiles horizontally, crossing pipes $P^{(k)}$ for $k \in [h-\beta]$. Lemma 3.3.4 also guarantees that the tiles in $C_w \cup E_w \setminus C_{w_{\mathrm{sort}}}$ involve only the pipes $P^{(i)}$ for $i \in \mathcal{S}$. All three parts of the claim follow.

□

Proposition 3.3.7 (cf. [MSS22, Prop 3.9])**.** *Let $w \in S_n$ and set $\lambda_i := \#D(\sigma(w))_i$. Then*

$$\mathfrak{G}_w = \left(\prod_{(a,b) \in D(w) \setminus D(w_{\text{sort}})} (x_a + y_b - x_a y_b) \right) \cdot \mathfrak{G}_{w_{\text{sort}}}.$$

Proof. Fix a pipe dream $P \in \text{PD}(w)$. By Proposition 3.3.2 and Lemma 3.3.4, the tiles in $D(w) \setminus D(w_{\text{sort}})$ are crosses. Let $F(P)$ be the pipe dream obtained from P by replacing the crosses in $D(w) \setminus D(w_{\text{sort}})$ with elbows. By uncrossing all pipes $P^{(i)}$ and $P^{(j)}$ with $i, j \in \mathcal{S}$, Corollary 3.3.6 guarantees that the Demazure word $\partial(F(P))$ of $F(P)$ is w_{sort}.

For any pipe dream $P \in \text{PD}(w_{\text{sort}})$, let $G(P)$ be the pipe dream obtained from P by replacing the elbows in $D(w) \setminus D(w_{\text{sort}}) \subseteq E_{w_{\text{sort}}}$ by crosses. Corollary 3.3.6 guarantees that the Demazure word $\partial(G(P))$ is w. As $FG\colon \text{PD}(w_{\text{sort}}) \to \text{PD}(w_{\text{sort}})$ and $GF\colon \text{PD}(w) \to \text{PD}(w)$ is the identity, the map F is a bijection.

The claim follows from the fact that $F\colon \text{PD}(w) \to \text{PD}(w_{\text{sort}})$ scales the weight of a pipe dream by the monomial

$$\prod_{(a,b) \in D(w) \setminus D(w_{\text{sort}})} (x_a + y_b - x_a y_b). \qquad \square$$

The following example motivates Theorem 3.3.9, which gives a precise description of the first few terms of the orthodontic sequences of sorted permutations.

Example 3.3.8 (cf. [MSS22, Ex 4.1])**.** Consider the sorted permutation $w = 68234751$, with Rothe diagram depicted in the left of Figure 3.7. The Rothe diagram of $w' := ws_5s_4s_3$ "almost" appears in the double orthodontic sequence for $D(w)$, as shown in the top of Figure 3.7. Part (6) of Theorem 3.3.9 makes this connection precise.

The primary column $(h, C, \alpha, i_1, \beta)$ of w are given by $h = 4$, $C = \{1, 2, 6\}$, $\alpha = 2$, $i_1 = 5$, and $\beta = 3$. From that the uppermost gap has size $\beta = 3$ and the first missing tooth $i_1 = 5$ occurs in column $h + 1 = 5$, it follows the next missing teeth are $i_2 = i_1 - 1 = 4$ and

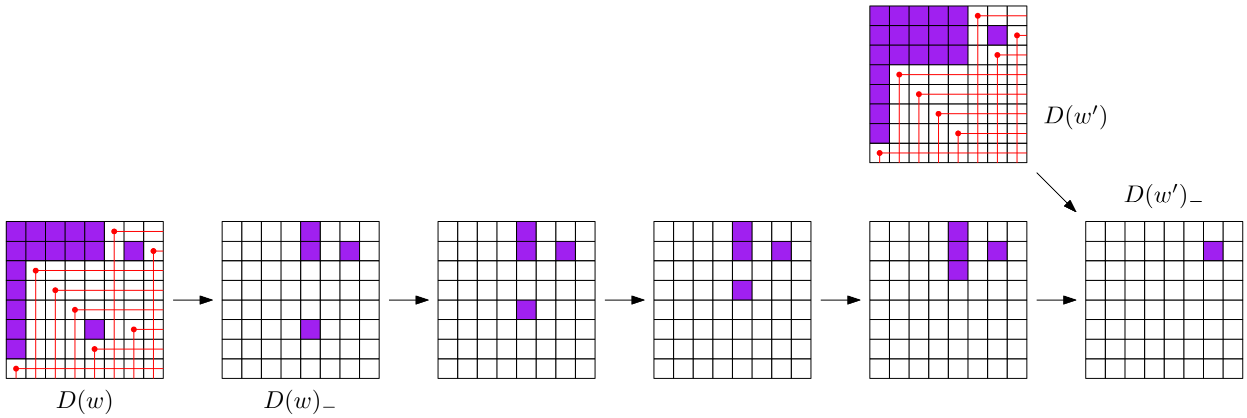


Figure 3.7: Left: Rothe diagram $D(w)$ for $w = 68234751$. Middle: first few orthodontic steps for $D(w)$. Top: Rothe diagram of $D(w')$ for $w' = ws_5s_4s_3 = 68723451$.

$i_3 = i_2 - 1 = 3$ and which occur in the fifth column. This phenomenon is captured as part (1) of Theorem 3.3.9. Direct computation gives $j_1 = 5 - 3 = 2$, $j_2 = j_1 + 1 = 3$, and $j_3 = j_2 + 1 = 4$; this is part (2) of Theorem 3.3.9. None of these row-swap orthodontic moves introduce standard interval columns; this is part (5) of Theorem 3.3.9.

As w is sorted, the three columns immediately to the left of column $h + 1 = 5$ (i.e., columns $2, 3, 4$) are standard intervals equal to $[\alpha] = [2]$. This is part (3) of Theorem 3.3.9. Furthermore, there are no standard interval columns of size k for any $k \in [\alpha+1, i_1] = \{3, 4, 5\}$. This is part (4) of Theorem 3.3.9. △

Theorem 3.3.9 (cf. [MSS22, Thm 4.2])**.** *Let $w \in S_n$ be a nonidentity sorted permutation, and suppose w has orthodontic sequence*

$$\mathbf{i} = (i_1, \ldots, i_\ell), \qquad \mathbf{j} = (j_1, \ldots, j_\ell), \qquad \mathbf{K} = (K_1, \ldots, K_n), \qquad \mathbf{M} = (M_1, \ldots, M_\ell).$$

Let $(h, C, \alpha, i_1, \beta)$ be the primary column data of w. Write $\mathcal{S} := \{h - \beta + 1, \ldots, h\}$. Then:

1. *For $k \in [\beta]$, we have $i_k = i_1 - k + 1$.*

2. *For $k \in [\beta]$, we have $j_k = h - \beta + k$.*

3. If $\alpha > 0$, then $K_\alpha \supseteq \mathcal{S}$.

4. For $k \in [\alpha+1, i_1]$, we have $K_k = \emptyset$.

5. For $k \in [\beta - 1]$, we have $M_k = \emptyset$.

6. The permutation $w' := ws_{i_1} \dots s_{\alpha+1}$ has orthodontic sequence

$$\mathbf{i}(w') = (i_{\beta+1}, \dots, i_\ell), \qquad \mathbf{j}(w') = (j_{\beta+1}, \dots, j_\ell), \qquad \mathbf{M}(w') = (M_{\beta+1}, \dots, M_\ell),$$

and

$$\mathbf{K}(w') = \begin{cases} (K_1, \dots, K_{\alpha-1}, K_\alpha \setminus \mathcal{S}, \mathcal{S} \sqcup M_\beta, K_{\alpha+2}, \dots, K_n) & \text{if } \alpha > 0 \\ (\mathcal{S} \sqcup M_\beta, K_2, \dots, K_n) & \text{if } \alpha = 0. \end{cases}$$

In particular,

$$\mathscr{G}_{D(w)} = \overline{\pi}_{i_1, h+1-\beta} \dots \overline{\pi}_{\alpha+1,h} \left(\prod_{s \in \mathcal{S}} (x_{\alpha+1} + y_s - x_{\alpha+1} y_s)^{-1} \cdot \mathscr{G}_{D(w')} \right). \qquad (\star)$$

Proof. By definition, $C = D(w)_{h+1}$ contains $[\alpha] \cup \{i_1 + 1\}$ and does not contain any of $\alpha+1, \dots, i_1$. Thus, the leftmost missing teeth of the diagrams in the orthodontic sequence of $D(w)$ are equal to $(i_1, i_1 - 1, \dots, \alpha+1)$, and all occur at column $h+1$, with corresponding sequence of uppermost gaps $(\beta, \beta-1, \dots, 1)$. Parts (1) and (2) follow.

As w is sorted, the columns $D(w_{\text{sort}})_{h-\beta+b}$ are equal to $[\alpha]$, and part (3) follows. Parts (4) and (5) are proven as [MSS22, Thm 4.2, (iii), (iv)] respectively.

In order to relate the orthodontic sequences of w and w', consider the diagrams

$$E = (\underbrace{\emptyset, \ldots, \emptyset}_{h \text{ many}}, D(w)_{h+1}, \ldots, D(w)_n),$$
$$E' = (\underbrace{\emptyset, \ldots, \emptyset}_{h \text{ many}}, D(w')_{h+1}, \ldots, D(w')_n).$$

Note that $E' = E \cdot s_{i_1} \ldots s_{\alpha+1}$ and that $M_\beta = \{j \colon E'_j = [\alpha+1]\}$. As the first h columns of $D(w)$ and $D(w')$ are standard intervals, it follows that $D(w')_-$ occurs in the orthodontic sequence of diagrams for $D(w)$ and furthermore that

$$\mathbf{i}(w') = (i_{\beta+1}, \ldots, i_\ell), \qquad \mathbf{j}(w') = (j_{\beta+1}, \ldots, j_\ell), \qquad \mathbf{M}(w') = (M_{\beta+1}, \ldots, M_\ell).$$

From the facts that

- $D(w)_j = D(w')_j$ for $j \leq h - \beta$,
- $D(w)_j \cup \{\alpha+1\} = D(w')_j$ for $h - \beta + 1 \leq j \leq h$, and
- $M_\beta = \{j \colon E'_j = [\alpha+1]\}$,

the formula for $\mathbf{K}(w')$ in terms of $\mathbf{K}(w)$ follows. This gives part (6).

Finally, write $D(w')_-$ for the diagram obtained from $D(w')$ by removing all standard interval columns. The equation $(\star)$ is a consequence of part (4) of the claim via the string of equalities

$$\begin{aligned}
\mathscr{G}_{D(w)} &= \overline{\omega}_1^{K_1} \ldots \overline{\omega}_n^{K_n} \overline{\pi}_{i_1,h+1-\beta} \cdots \overline{\pi}_{\alpha+1,h}(\mathscr{G}_{D(w')_-}) \\
&\overset{(4)}{=} \overline{\pi}_{i_1,h+1-\beta} \cdots \overline{\pi}_{\alpha+1,h} \left(\overline{\omega}_1^{K_1} \ldots \overline{\omega}_n^{K_n} \mathscr{G}_{D(w')_-}\right) \\
&= \overline{\pi}_{i_1,h+1-\beta} \cdots \overline{\pi}_{\alpha+1,h} \left(\prod_{s \in \mathcal{S}} (x_{\alpha+1} + y_s - x_{\alpha+1} y_s)^{-1} \cdot \mathscr{G}_{D(w')}\right).
\end{aligned} \qquad \square$$

In order to relate the formula from Theorem 3.3.9 to Grothendieck polynomials, the following lemmas will be useful.

Lemma 3.3.10. *Let* $\overline{\Delta}_a := \overline{\partial}_{i+a} \circ \cdots \circ \overline{\partial}_i$. *For* $a = -1$, *set* $\overline{\Delta}_a := 1$ *to be the identity operator. Given* $g \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *and* $\ell \geq 1$ *such that* $\partial_{i+\ell}(g) = 0$, *the equality*

$$\overline{\pi}_{i+\ell, j_\ell} \circ \overline{\Delta}_{\ell-1}(g) = \overline{\partial}_{i+\ell} \circ \overline{\pi}_{i+\ell-1, j_\ell} \circ \overline{\Delta}_{\ell-2}(g)$$

holds. In particular, for any $\ell \geq 0$,

$$\overline{\pi}_{i+\ell, j_\ell} \circ \overline{\Delta}_{\ell-1}(g) = \overline{\Delta}_\ell((x_i + y_{j_\ell} - x_i y_{j_\ell})g)$$

whenever $\partial_{i+1}(g) = \cdots = \partial_{i+\ell}(g) = 0$.

Proof. Work in the ring of endomorphisms of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$. For $h \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$, write m_h for the "multiplication by h" operator, given by $f \mapsto fh$. As

$$\partial_i \circ m_{x_i} = (m_{x_{i+1}} \circ \partial_i) + 1 \qquad \text{and} \qquad \partial_i \circ m_{y_j} = m_{y_j} \circ \partial_i \text{ for all } j,$$

observe that

$$\begin{aligned} m_{x_{i+1} - y_j + x_{i+1} y_j} \circ \partial_i &= (m_{x_{i+1}} \circ m_{1+y_j} - m_{y_j}) \circ \partial_i \\ &= (\partial_i \circ m_{x_i} - 1) \circ m_{1+y_j} - \partial_i \circ m_{y_j} \\ &= \partial_i \circ (m_{x_i - y_j + x_i y_j}) - m_{1+y_j}. \end{aligned} \tag{$\diamondsuit$}$$

Hence

$$\begin{aligned}\overline{\pi}_{i+\ell,j_\ell} \circ \overline{\Delta}_{\ell-1} &= \overline{\partial}_{i+\ell} \circ m_{x_{i+\ell}+y_{j_\ell}-x_{i+\ell}y_{j_\ell}} \circ \partial_{i+\ell-1} \circ m_{1-x_{i+\ell}} \circ \overline{\Delta}_{\ell-2} \\ &\overset{(\diamondsuit)}{=} \overline{\partial}_{i+\ell} \circ (\partial_{i+\ell-1} \circ m_{x_{i+\ell-1}+y_{j_\ell}-x_{i+\ell-1}y_{j_\ell}} - m_{1+y_{j_\ell}}) \circ m_{1-x_{i+\ell}} \circ \overline{\Delta}_{\ell-2} \\ &= \overline{\partial}_{i+\ell} \circ \overline{\pi}_{i+\ell-1,j_\ell} \circ \overline{\Delta}_{\ell-2} - \overline{\partial}_{i+\ell} \circ m_{1+y_{j_\ell}} \circ m_{1-x_{i+\ell}} \circ \overline{\Delta}_{\ell-2}. \end{aligned} \tag{$*$}$$

Note that $\overline{\partial}_{i+\ell}$ and $m_{1-x_{i+\ell}}$ commute with $\overline{\Delta}_{\ell-2}$, so that

$$\begin{aligned}\overline{\partial}_{i+\ell} \circ m_{1+y_{j_\ell}} \circ m_{1-x_{i+\ell}} \circ \overline{\Delta}_{\ell-2}(g) &= m_{1+y_{j_\ell}} \circ \overline{\Delta}_{\ell-2} \circ \overline{\partial}_{i+\ell} \circ m_{1-x_{i+\ell}}(g) \\ &= m_{1+y_{j_\ell}} \circ \overline{\Delta}_{\ell-2} \circ \underbrace{\partial_{i+\ell}((1-x_{i+\ell+1})(1-x_{i+\ell})g)}_{=0}. \end{aligned} \tag{$**$}$$

Applying the equality $(*)$ of operators to the function g, and using $(**)$, the claimed equality

$$\overline{\pi}_{i+\ell,j_\ell} \circ \overline{\Delta}_{\ell-1}(g) = \overline{\partial}_{i+\ell} \circ \overline{\pi}_{i+\ell-1,j_\ell} \circ \overline{\Delta}_{\ell-2}(g)$$

follows.

The second part of the lemma, for $\ell = 0$, is nothing but the identity

$$\overline{\pi}_{i,j_0}(g) = \overline{\partial}_i((x_i - y_{j_0})g), \tag{3.5}$$

and in general it follows from repeated application of the first part of the lemma combined with the identity (3.5). □

Lemma 3.3.11 (cf. [MSS22, Lem 5.11])**.** *Let g be a polynomial with*

$$\partial_{i+1}(g) = \cdots = \partial_{i+k}(g) = 0.$$

Then

$$\overline{\pi}_{i+k,j_k}\overline{\pi}_{i+k-1,j_{k-1}}\dots\overline{\pi}_{i,j_0}(g) = \overline{\partial}_{i+k}\dots\overline{\partial}_i\left(\prod_{a=0}^{k}(x_i + y_{j_a} - x_i y_{j_a})\cdot g\right)$$

Proof. Induct on k as follows. The base case $k = 0$ is the identity (3.5). For $k > 0$, the inductive hypothesis guarantees that

$$\pi_{i+k-1,j_{k-1}}\dots\overline{\pi}_{i,j_0}(g) = \overline{\Delta}_{k-1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_i y_{j_a})\cdot g\right).$$

Then, because

$$\partial_{i+1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_i y_{j_a})g\right) = \dots = \partial_{i+k}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_i y_{j_a})g\right) = 0,$$

Lemma 3.3.10 implies that

$$\overline{\pi}_{i+k,j_k}\circ\overline{\Delta}_{k-1}\left(\prod_{a=0}^{k-1}(x_i + y_{j_a} - x_i y_{j_a})\cdot g\right) = \overline{\Delta}_k\left(\prod_{a=0}^{k}(x_i + y_{j_a} - x_i y_{j_a})\cdot g\right).$$ □

Theorem 3.1.1. *Let D be a %-avoiding diagram with double orthodontic sequence $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$. Define*

$$\mathscr{G}_D(\mathbf{x}, \mathbf{y}) := \overline{\omega}_1^{K_1}\overline{\omega}_2^{K_2}\dots\overline{\omega}_n^{K_n}\overline{\pi}_{i_1,j_1}(\overline{\omega}_{i_1}^{M_1}\overline{\pi}_{i_2,j_2}(\overline{\omega}_{i_2}^{M_2}\dots\overline{\pi}_{i_\ell,j_\ell}(\overline{\omega}_{i_\ell}^{M_\ell})\dots)). \tag{3.1}$$

When $D = D(w)$ is the Rothe diagram of a permutation, then $\mathscr{G}_D(\mathbf{x}, \mathbf{y}) = \mathfrak{G}_w(\mathbf{x}, \mathbf{y})$.

Proof of Theorem 3.1.1. Induct on the orthodontic sort order (Definition 3.2.16). In the base case $w = \text{id}$, both $\mathscr{G}_{D(\text{id})} = \mathscr{G}_\emptyset$ and $\mathfrak{G}_{\text{id}}$ are equal to 1.

Assume that w is not sorted and that $\mathscr{G}_{D(w_{\text{sort}})} = \mathfrak{G}_{w_{\text{sort}}}$. Corollary 3.3.3 and Proposition 3.3.7 imply that

$$\begin{aligned}\mathscr{G}_{D(w)} &= \left(\prod_{(a,b)\in D(w)\setminus D(w_{\mathrm{sort}})} (x_a + y_b - x_a y_b)\right) \mathscr{G}_{D(w_{\mathrm{sort}})} \\ &= \left(\prod_{(a,b)\in D(w)\setminus D(w_{\mathrm{sort}})} (x_a + y_b - x_a y_b)\right) \mathfrak{G}_{w_{\mathrm{sort}}} \\ &= \mathfrak{G}_w.\end{aligned}$$

Now assume that w is sorted and that $\mathscr{G}_{D(ws_{i_1}\dots s_\alpha)} = \mathfrak{G}_{ws_{i_1}\dots s_\alpha}$. Writing $w' := ws_{i_1}\dots s_\alpha$ and $\mathcal{S} := \{h-\beta+1,\dots,h\}$, Theorem 3.3.9 and Lemma 3.3.11 imply that

$$\begin{aligned}\mathscr{G}_{D(w)} &= \overline{\pi}_{i_1,h+1-\beta}\dots\overline{\pi}_{\alpha+1,h}\left(\prod_{s\in\mathcal{S}} (x_{\alpha+1} + y_s - x_{\alpha+1} y_s)^{-1} \cdot \mathscr{G}_{D(w')}\right) \\ &= \overline{\partial}_{i_1}\dots\overline{\partial}_\alpha \left(\mathscr{G}_{D(w')}\right) \\ &= \overline{\partial}_{i_1}\dots\overline{\partial}_\alpha \left(\mathfrak{G}_{w'}\right) \\ &= \mathfrak{G}_w.\end{aligned}$$

□

Corollary 3.3.12. *If $D = D(w)$ is a Rothe diagram, then $\mathscr{S}_D(\mathbf{x},\mathbf{y}) = \mathfrak{S}_w(\mathbf{x},-\mathbf{y})$.*

Proof. By definition, $\mathfrak{S}_w(\mathbf{x},\mathbf{y})$ is the lowest degree part of $\mathfrak{G}_w(\mathbf{x},-\mathbf{y})$. Remark 3.2.10 guarantees that $\mathscr{S}_D(\mathbf{x},\mathbf{y})$ is the lowest degree part of $\mathscr{G}_D(\mathbf{x},\mathbf{y})$. The result now follows from Theorem 3.1.1. □

3.4 Lascoux positivity

We prove Theorem 3.1.2 by reducing the problem to the special case where every column is a standard interval (cf. Lemma 3.4.3 and Corollary 3.4.6). This case can be checked explicitly (Proposition 3.4.11).

Lemma 3.4.1. *For $i \in [n]$ and $M \subseteq [n]$, and $f \in \mathbb{C}[\mathbf{x}]$ satisfying $\deg_{x_i}(f) \leq m$, the equalities*

$$\omega_i^M|_{y_j \mapsto -1} = (x_1 - 1)^{|M|} \dots (x_i - 1)^{|M|} \tag{3.6}$$

$$r_{m+1,n}\left((x_1 - 1) \dots (x_i - 1) f\right) = x_1 \dots x_{n-i}(1 - x_{n-i+1}) \dots (1 - x_n) r_{m,n}(f) \tag{3.7}$$

$$\pi_{i,j}(f)|_{y_j \mapsto -1} = \partial_i((x_i - 1)(f|_{y_j \mapsto 1})) \tag{3.8}$$

$$r_{m,n}(\partial_i((x_i - 1)f)) = \overline{\pi}_{n-i}(r_{m,n}(f)) \tag{3.9}$$

hold.

Proof. Equations (3.6) and (3.8) are immediate from the definition of ω_i^M and $\pi_{i,j}$. Equations (3.7) and (3.9) can be proven by a manual check when f is a monomial and applying linearity. (Equation (3.9) also follows from [Yu23, Lem 3.4]). □

Let φ_i denote the operator

$$\varphi_i \colon f \mapsto x_1 \dots x_i (1 - x_{i+1}) \dots (1 - x_n) f.$$

Corollary 3.4.2. *The polynomial $r_{m,n}(\mathscr{S}_D(\mathbf{x}, -\mathbf{1}))$ can be obtained from the polynomial $1 \in \mathbb{C}[\mathbf{x}]$ by repeated application of operators of the form $f \mapsto \overline{\pi}_i(f)$ and φ_i.*

Proof. By the orthodontic formula (3.1), along with Equations (3.6) and (3.8), $\mathscr{S}_D(\mathbf{x}, -\mathbf{1})$ can be obtained from the polynomial $1 \in \mathbb{C}[\mathbf{x}]$ by repeated application of operators of the form

$$f \mapsto \partial_i (x_i - 1) f \qquad \text{and} \qquad f \mapsto (x_1 - 1) \dots (x_i - 1) f.$$

Using Equations (3.7) and (3.9) to commute the operator $r_{m,n}$ past the two operators above, it follows that $r_{m,n}(\mathscr{S}_D(\mathbf{x}, \mathbf{1}))$ can be obtained by repeated application of the operators $\overline{\pi}_i$ and φ_{n-i}. □

Lemma 3.4.3. *The equality*

$$\overline{\pi}_i(\mathfrak{L}_\alpha) = \begin{cases} \mathfrak{L}_\alpha & \text{if } \alpha_i > \alpha_{i+1} \\ \mathfrak{L}_{\alpha\cdot s_i} & \text{if } \alpha_i < \alpha_{i+1} \end{cases}$$

holds; in particular, $\overline{\pi}_i$ preserves Lascoux positivity.

Proof. When $\alpha_i > \alpha_{i+1}$, the definition of Lascoux polynomials guarantees that $\mathfrak{L}_\alpha = \overline{\pi}_i\mathfrak{L}_{\alpha\cdot s_i}$. The claim then follows from the fact that $\overline{\pi}_i$ is idempotent. When $\alpha_i < \alpha_{i+1}$, the claim follows from the definition of $\mathfrak{L}_{\alpha\cdot s_i}$. □

Proposition 3.4.4. *Conjecture 3.1.5 implies Conjecture 3.1.4.*

Proof. By Corollary 3.4.2, it suffices to show that the operators $\overline{\pi}_i$ and φ_i preserve Lascoux positivity. This follows from Lemma 3.4.3 and Conjecture 3.1.5, respectively. □

Lemma 3.4.5. *Assume that the columns of D are ordered by inclusion, and write $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ for the orthodontic sequence. Then:*

1. *Every diagram appearing in the orthodontic sequence of D also has all columns ordered by inclusion.*
2. *If $K_k \neq \emptyset$, then $i_j \neq k$ for all j.*
3. *If $M_k \neq \emptyset$ for some k, then $i_j \neq i_k$ for all $j > k$.*

Proof. Part 1 of the lemma follows from the fact that orthodontic moves either remove a column or swap two rows; both of these moves preserve the inclusion property of the columns.

If $K_k \neq \emptyset$, then D has a column equal to $[k]$. As the columns of D are ordered by inclusion, every other column of D is either contained in $[k]$ or is equal to $[k] \sqcup S$ for some S; in particular, no empty box in the k^{th} row has a square below it in its column. As orthodontic moves remove columns or move boxes up, no empty box in the k^{th} row of any diagram D' appearing in the orthodontic sequence of D has a square below it in its column; in particular, k can never be a missing tooth of D'. Part 2 of the lemma follows.

Similarly, if $M_k \neq \emptyset$ for some k, then there is a diagram D' in the orthodontic sequence of D which has a column equal to $[i_k]$. Arguing as above, it follows that i_k can never be a missing tooth of a diagram in the orthodontic sequence of D'. Part 3 of the lemma follows. $\square$

Corollary 3.4.6. *Let D be a diagram whose columns are ordered by inclusion, and let $\mathbf{K}, \mathbf{i}, \mathbf{j}, \mathbf{M}$ denote the orthodontic sequence of D. Then,*

$$\mathscr{S}_D(\mathbf{x}, \mathbf{y}) = \pi_{i_1,j_1}(\dots \pi_{i_\ell,j_\ell}(\omega_1^{K_1} \dots \omega_n^{K_n} \cdot \omega_{i_1}^{M_1} \dots \omega_{i_\ell}^{M_\ell}) \dots).$$

Proof. Because ω_i^J is symmetric with respect to $x_1, \dots, x_i$ and with respect to $x_{i+1}, \dots, x_n$, it follows that ω_i^J commutes with $\pi_{i',j}$ whenever $i' \neq i$. Lemma 3.4.5 guarantees that the ω_i^J in the orthodontic formula (3.1) can be commuted past the $\pi_{i',j}$ occurring to the right. The claim follows. $\square$

Let $C_{n,k,\ell}$ be the set of $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_1 = \dots = \alpha_k = \ell$ and $\alpha_j \leq \ell$ for all j.

Lemma 3.4.7. *Let $\alpha \in C_{n,k,\ell}$ and $i \leq k$. Then*

$$\mathfrak{L}_\alpha(x_1, \dots, x_n) = x_1^\ell \dots x_i^\ell \mathfrak{L}_{\alpha(i)}(x_{i+1}, \dots, x_n), \qquad \text{where } \alpha(i) := (\alpha_{i+1}, \dots, \alpha_n).$$

Proof. Write $\lambda := \operatorname{sort}(\alpha)$ for the partition obtained by reordering α so that the components are weakly decreasing. Then

$$\mathfrak{L}_\alpha = \overline{\pi}_{i_1} \dots \overline{\pi}_{i_\ell}(\mathbf{x}^\lambda),$$

where $i_j > k$ for all j. As $\lambda_1 = \cdots = \lambda_k = \ell$,

$$\mathfrak{L}_\alpha = x_1^\ell \dots x_i^\ell \overline{\pi}_{i_1} \dots \overline{\pi}_{i_\ell}(\mathbf{x}^{\lambda(i)}), \qquad \lambda(i) := (\lambda_{i+1}, \dots, \lambda_n)$$

for any $i \leq k$. The claim follows from the fact that

$$\overline{\pi}_{i_1} \dots \overline{\pi}_{i_\ell}(\mathbf{x}^{\lambda(i)}) = \mathfrak{L}_{\alpha(i)}(x_{i+1}, \dots, x_n). \qquad \square$$

Lemma 3.4.8. *The set of Lascoux polynomials* $\{\mathfrak{L}_\alpha \colon \alpha \in C_{n,k,\ell}\}$ *forms a basis for the vector space* $V_{k,\ell}$ *of polynomials of the form*

$$x_1^\ell \dots x_k^\ell \cdot f(x_{k+1}, \dots, x_n), \qquad \deg_{x_i}(f) \leq \ell.$$

Proof. The set of polynomials $\mathfrak{L}_\alpha(x_{k+1}, \dots, x_n)$ with $\alpha_i \leq \ell$ forms a basis for the set of polynomials $\mathbb{C}[x_{k+1}, \dots, x_n]$ such that $\deg_{x_i}(f) \leq \ell$ for all ℓ. Lemma 3.4.7 implies that $\{\mathfrak{L}_\alpha \colon \alpha \in C_{n,k,\ell}\}$ spans $V_{k,\ell}$. Since Lascoux polynomials are linearly independent, the claim follows. $\square$

Recall that the stable Grothendieck polynomial $G_w(x_1, \dots, x_n)$ of a permutation $w \in S_m$ is defined to be the limit $\lim_{N \to \infty} \mathfrak{G}_{1^N \times w}(x_1, \dots, x_n, 0, 0, \dots)$.

Example 3.4.9. For $w = 21$, the stable Grothendieck polynomial $G_{21}(x_1, \dots, x_n)$ is equal to

$$G_{21}(x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^{i+1} \mathbf{e}_i(\mathbf{x}),$$

where $\mathbf{e}_i$ is the i^{th} elementary symmetric polynomial. (More generally, for permutations with a unique descent, $G_w(x_1, \dots, x_n)$ can be computed e.g. using [Len00, Thm 2.2].)

In particular,

$$(1 - x_1) \dots (1 - x_n) = 1 - G_{21}(x_1, \dots, x_n). \qquad \triangle$$

The key ingredient for the Lascoux positivity in Theorem 3.1.2 is the following result by Orelowitz and Yu.

Theorem 3.4.10 ([OY23, Thm 1.3])**.** *The product $\mathfrak{L}_\alpha(x_1, \ldots, x_n) \cdot G_w(x_1, \ldots, x_n)$ is a graded nonnegative sum of Lascoux polynomials:*

$$\mathfrak{L}_\alpha(x_1, \ldots, x_n) \cdot G_w(x_1, \ldots, x_n) = \sum_\beta c_\beta (-1)^{|\beta| - \ell(w) - |\alpha|} \mathfrak{L}_\beta(x_1, \ldots, x_n), \qquad c_\beta \geq 0.$$

Orelowitz and Yu prove more in [OY23, Thm 1.3]: they express the product as a sum over certain tableaux, each of which contribute a Lascoux polynomial in the expansion.

Proposition 3.4.11. *Assume that $\varphi_i^{a_i} \varphi_{i+1}^{a_{i+1}} \ldots \varphi_n^{a_n}$ is a graded nonnegative sum of Lascoux polynomials for some $a_i, a_{i+1}, \ldots, a_n \geq 0$. Then*

$$\varphi_i \cdot \varphi_i^{a_i} \varphi_{i+1}^{a_{i+1}} \ldots \varphi_n^{a_n}$$

is again a graded nonnegative sum of Lascoux polynomials.

Proof. For $i = n$, the result is trivial.

Assume that $i \leq n - 1$ and write $f := \varphi_i^{a_i} \varphi_{i+1}^{a_{i+1}} \ldots \varphi_n^{a_n}$. Write $\ell := a_i + \cdots + a_n$ and observe that $f \in V_{i,\ell}$. Lemma 3.4.8 implies that the Lascoux expansion of f, graded nonnegative by assumption, reads

$$f = \sum_{\alpha \in C_{n,i,\ell}} (-1)^{|\alpha| - \ell} c_\alpha \mathfrak{L}_\alpha(x_1, \ldots, x_n), \qquad c_\alpha \geq 0.$$

By Lemma 3.4.7, the polynomial f expands as

$$f = \sum_{\alpha \in C_{n,i,\ell}} (-1)^{|\alpha| - \ell} c_\alpha x_1^\ell \ldots x_i^\ell \cdot \mathfrak{L}_{\alpha(i)}(x_{i+1}, \ldots, x_n), \qquad \text{where } \alpha(i) := (\alpha_{i+1}, \ldots, \alpha_n).$$

As $\varphi_i = x_1 \dots x_i(1 - G_{21}(x_{i+1}, \dots, x_n))$ (cf. Example 3.4.9), the polynomial $\varphi_i f$ expands as

$$\varphi_i f = \sum_{\alpha \in C_{n,i,\ell}} (-1)^{|\alpha|-\ell} c_\alpha x_1^{\ell+1} \dots x_i^{\ell+1} \cdot (1 - G_{21}(x_{i+1}, \dots, x_n)) \cdot \mathfrak{L}_{\alpha(i)}(x_{i+1}, \dots, x_n). \quad (3.10)$$

Theorem 3.4.10 implies that

$$G_{21}(x_{i+1}, \dots, x_n) \mathfrak{L}_{\alpha(i)}(x_{i+1}, \dots, x_n) = \sum_{\beta(i)} c_{\alpha(i),\beta(i)} (-1)^{|\beta(i)|-1-|\alpha(i)|} \mathfrak{L}_{\beta(i)}(x_{i+1}, \dots, x_n).$$

As $\deg_{x_j}(\mathfrak{L}_{\alpha(i)}) \leq \ell$ and $\deg_{x_j}(G_{21}(x_{i+1}, \dots, x_n)) = 1$ for $i + 1 \leq j \leq n$, it follows that $\deg_{x_j}(\mathfrak{L}_{\beta(i)}) \leq \ell + 1$ for $i + 1 \leq j \leq n$. Writing $\beta(i) := (\beta_{i+1}, \dots, n)$, Lemma 3.4.7 implies that

$$x_1^{\ell+1} \dots x_i^{\ell+1} \mathfrak{L}_{\beta(i)}(x_{i+1}, \dots, x_n) = \mathfrak{L}_\beta(x_1, \dots, x_n), \qquad \beta := (\underbrace{\ell, \dots, \ell}_{i \text{ many}}, \beta_{i+1}, \dots, \beta_n).$$

It follows that each summand in Equation (3.10) has a graded nonnegative Lascoux expansion. □

Theorem 3.1.2. *Let $D \subseteq [n] \times [m]$ be a diagram whose columns are ordered by inclusion. Let $\mathscr{S}_D(\mathbf{x}, \mathbf{y})$ be the lowest degree part of $\mathscr{G}_D(\mathbf{x}, \mathbf{y})$. Then, the polynomial*

$$x_1^m \dots x_n^m \mathscr{S}_D(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \dots, x_n)$.

Proof. Corollary 3.4.6 asserts

$$\mathscr{S}_D(\mathbf{x}, \mathbf{y}) = \pi_{i_1,j_1}(\dots \pi_{i_\ell,j_\ell}(\omega_1^{K_1} \dots \omega_n^{K_n} \cdot \omega_{i_1}^{M_1} \dots \omega_{i_\ell}^{M_\ell}) \dots).$$

Equations (3.6)–(3.9), along with the fact that the φ_i commute with each other, imply that

$$r_{m,n}(\mathscr{S}_D(\mathbf{x}, -\mathbf{1})) = \overline{\pi}_{n-i_1} \dots \overline{\pi}_{n-i_\ell}(\varphi_{k_1} \dots \varphi_{k_m}(1))$$

for $k_1 \geq \cdots \geq k_m$. Proposition 3.4.11 implies that $\varphi_{k_1} \dots \varphi_{k_m}(1)$ is Lascoux positive, and the result follows from Lemma 3.4.3. □

Corollary 3.1.3. *Let $w \in S_n$ be a vexillary permutation and write $\mathfrak{S}_w(x_1, \dots, x_n; y_1, \dots, y_n)$ for the double Schubert polynomial. Then the polynomial*

$$x_1^n \dots x_n^n \mathfrak{S}_w(x_n^{-1}, \dots, x_1^{-1}; 1, \dots, 1)$$

is a graded nonnegative sum of Lascoux polynomials $\mathfrak{L}_\alpha(x_1, \dots, x_n)$.

Proof. By Corollary 3.3.12, the double Schubert polynomial $\mathfrak{S}_w(\mathbf{x}, \mathbf{1})$ is equal to the specialization $\mathscr{S}_{D(w)}(\mathbf{x}, -\mathbf{1})$. The result follows from Theorem 3.1.2. □

Example 3.4.12. Replacing $w \in S_n$ with $w(n+1) \in S_{n+1}$ amounts to replacing

$$f := x_1^n \dots x_n^n \mathfrak{S}_w(x_n^{-1}, \dots, x_1^{-1}; \mathbf{1}) \quad \text{with} \quad x_1^{n+1} x_2 \dots x_n x_{n+1} f(x_2, \dots, x_{n+1}).$$

This operation preserves Lascoux positivity by Lemma 3.4.7. For example, if $w = 321$, then

$$\begin{aligned}\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) &= (x_1 - y_1)(x_2 - y_1)(x_1 - y_2) \\ &= x_1^2 x_2 - x_1^2 y_1 - x_1 x_2 y_1 - x_1 x_2 y_2 + x_1 y_1^2 + x_1 y_1 y_2 + x_2 y_1 y_2 - y_1^2 y_2,\end{aligned}$$

which gives

$$\begin{aligned} x_1^3x_2^3x_3^3\mathfrak{S}_w(x_3^{-1},x_2^{-1},x_1^{-1};\mathbf{1}) &= x_1^3x_2^2x_3 - x_1^3x_2^3x_3 - x_1^3x_2^2x_3^2 - x_1^3x_2^2x_3^2 \\ &\quad + x_1^3x_2^3x_3^2 + x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^3 - x_1^3x_2^3x_3^3 \\ &= (\mathfrak{L}_{321}) - (2\mathfrak{L}_{322} + \mathfrak{L}_{331}) + (\mathfrak{L}_{323} + \mathfrak{L}_{332})\,. \end{aligned}$$

Taking $w = 3214$ gives

$$\begin{aligned} x_1^4x_2^4x_3^4x_4^4\mathfrak{S}_w(x_4^{-1},x_3^{-1},x_2^{-1},x_1^{-1};\mathbf{1}) &= x_1^4x_2^4x_3^3x_4^2 - x_1^4x_2^4x_3^4x_4^2 - x_1^4x_2^4x_3^3x_4^3 - x_1^4x_2^4x_3^3x_4^3 \\ &\quad + x_1^4x_2^4x_3^4x_4^3 + x_1^4x_2^4x_3^4x_4^3 + x_1^4x_2^4x_3^3x_4^4 - x_1^4x_2^4x_3^4x_4^4 \\ &= (\mathfrak{L}_{4432}) - (2\mathfrak{L}_{4433} + \mathfrak{L}_{4442}) + (\mathfrak{L}_{4434} + \mathfrak{L}_{4443})\,. \qquad \triangle \end{aligned}$$

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