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Asymptotic Properties of Special Function Solutions of the Painlevé III Equation for Fixed Parameters

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ABSTRACT

In this paper, we compute the small and large x asymptotics of the special function solutions of the Painlevé-III equation in the complex plane. We use the representation in terms of Toeplitz determinants of Bessel functions obtained by Masuda. Toeplitz determinants are rewritten as multiple contour integrals using Andréief's identity. The small and large x asymptotics are obtained using elementary asymptotic methods applied to the multiple contour integral. The asymptotics is extended to the whole complex plane using analytic continuation formulas for Bessel functions. The claimed result has not appeared in the literature before. We note that the Toeplitz determinant representation is useful for numerical computations of corresponding solutions of the Painlevé-III equation.

1 | Introduction

Painlevé equations are six nonlinear second-order ordinary differential equations. They are written in the form of $u'' = R(u', u, t)$ with R a rational function. Their solutions have the so-called Painlevé property. This means that the locations of singularities of branching type in the complex plane do not depend on the initial conditions, but the locations of isolated singularities might depend on the initial conditions. They were discovered at the beginning of the 20th century in the works [2, 3], see also [4]. The solutions of the Painlevé equations are classified into three groups: rational and algebraic solutions, solutions expressed in terms of classical special functions, and the rest, see [5]. For generic values of parameters, the solutions are not rational or algebraic and cannot be reduced to classical special functions, so they belong to the third class and are called Painlevé transcendents. We are interested in solutions of the Painlevé III equation expressed in terms of Bessel functions (see [6, 7]). For the

applications of such special function solutions of the Painlevé III equation in random matrix theory, we refer the reader to [8–10].

We start with presenting the Painlevé III equation

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \beta}{x} + u^3 - \frac{1}{u}, \quad \alpha, \beta \in \mathbb{C}. \quad (1)$$

Consider the Toeplitz determinant of cylinder functions:

$$\Delta_n(x, \alpha) = \det \left(\left\{ C_{\frac{\alpha}{2} - j + k}(x) \right\}_{j,k=0}^{n-1} \right), \quad n \in \mathbb{N} \quad (2)$$

with

$$C_\nu(x) = d_1 J_\nu(x) + d_2 Y_\nu(x), \quad d_1, d_2 \in \mathbb{C}, \quad (3)$$

and $J_\nu(x)$, $Y_\nu(x)$ are Bessel functions of first and second kinds. In addition, denote $\Delta_0(x, \alpha) = 1$.

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Proposition 1.1. *The expression*

$$u_n(x, \alpha) = -\frac{\Delta_{n+1}(x, \alpha - 2)\Delta_n(x, \alpha)}{\Delta_{n+1}(x, \alpha)\Delta_n(x, \alpha - 2)}, \quad n \in \mathbb{N} \cup \{0\}, \quad \alpha \in \mathbb{C} \quad (4)$$

solves the Painlevé III equation with shifted parameters

$$u_n'' = \frac{(u_n')^2}{u_n} - \frac{u_n'}{x} + \frac{(\alpha + 2n)u_n^2 + (-\alpha + 2 + 2n)}{x} + u_n^3 - \frac{1}{u_n}. \quad (5)$$

The fact that the Toeplitz determinants of Bessel functions are related to the solutions of the Painlevé III equation is well known; see, for example, [11, 12, (3.5)]. The formula [1, (4.20)] is very similar to (4), but it involves the Wronskian matrix instead of the Toeplitz matrix. It is not difficult to reduce one to another; see [8] or Proposition 3.5. The main advantage of (4) compared to the classical formula [12, (3.5)] is the absence of a derivative operation applied to the corresponding determinants. A similar formula for the case of rational solutions can be found in [13]. Moreover, [1] contains Wronskian formulas for special function solutions of Painlevé-IV, V, and VI without derivatives. For convenience of the reader, we present the proof of Proposition 1.1 in Section 3.

Now we are ready to present the first result of our asymptotic analysis.

Theorem 1.1. *The Toeplitz determinant (2) admits the following $x \rightarrow 0$, $-\pi < \arg(x) < \pi$ asymptotics for fixed $d_1, d_2 \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$*

1. *If $d_2 \neq 0$ and $\operatorname{Re}(\alpha) > 2n - 2$ or $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) = 0$, then*

$$\Delta_n(x, \alpha) \sim (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G\left(\frac{\alpha}{2} + 1\right)}{G\left(\frac{\alpha}{2} - n + 1\right)} \left(\frac{x}{2}\right)^{-\frac{n\alpha}{2}}, \quad \text{as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi.$$

2. *If $d_2 \neq 0$, $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j + 2$ for some $j = 1, 2, \dots, n - 1$, then*

$$\begin{aligned} \Delta_n(x, \alpha) &\sim (-1)^{\frac{n(n-1)}{2} + nj + n - j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)} \left(\frac{x}{2}\right)^{(\alpha - 2n + 2j)j - \frac{n\alpha}{2}}, \\ &\text{as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi. \end{aligned}$$

3. *If $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $\operatorname{Re}(\alpha) < -2n + 2$ or $d_2 = 0$, then*

$$\begin{aligned} \Delta_n(x, \alpha) &\sim \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^n} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} - n + 1\right)} \left(\frac{x}{2}\right)^{\frac{n\alpha}{2}}, \\ &\text{as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi. \end{aligned}$$

where $G(x)$ refers to the Barnes G -function.

For discussion of the necessity of condition $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$, see Remarks 4.2, A.1.

The asymptotic formulas are obtained after rewriting the Toeplitz determinant (2) as multiple contour integral using Andréief's identity and performing an elementary asymptotic analysis. The multiple contour integral formula holds only for $\operatorname{Re}(x) > 0$. However, we know that the power series for cylindrical functions is valid for $-\pi < \arg(x) < \pi$. Since our computation can be interpreted as a calculation of the leading term of the product of many power series, our asymptotic result also holds for $-\pi < \arg(x) < \pi$. Notice that plugging naively the asymptotic of Bessel functions into (4) and trying to derive Theorem 1.1 is a difficult task. One would have to reproduce different leading behaviors for different α and the mechanism for it is unclear to us.

We should mention that the same strategy was applied to special function solutions of the Painlevé-II equation in [14] and to Hankel determinant solutions of Painlevé-VI in [15].

Combining Theorem 1.1 with Proposition 1.1, we derive the asymptotics of $u_n(x, \alpha)$ as $x \rightarrow 0$.

Theorem 1.2. *Solution (4) of the Painlevé-III equation (5) admits the following $x \rightarrow 0$, $-\pi < \arg(x) < \pi$ asymptotics for fixed $d_1, d_2 \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$.*

1. If $d_2 \neq 0$ and $\operatorname{Re}(\alpha) > 2n + 2$ or $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) = 0$, then

$$u_n(x, \alpha) \sim \left(\frac{2}{2n+2-\alpha}\right) \frac{x}{2}, \text{ as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi.$$

2. If $d_2 \neq 0$, $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $2n - 4j < \operatorname{Re}(\alpha) < 2n - 4j + 2$ for some $j = 0, 1, \dots, n$, then

$$u_n(x, \alpha) \sim (-1)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right) \right) \left(\frac{\Gamma\left(-\frac{\alpha}{2} + n - 2j + 1\right)}{\Gamma\left(\frac{\alpha}{2} - n + 2j\right)} \right)^2 \\ \times \frac{\Gamma\left(j + \frac{\alpha}{2}\right) \Gamma(j+1)}{\Gamma\left(-\frac{\alpha}{2} + n - j + 1\right) \Gamma(n-j+1)} \left(\frac{x}{2}\right)^{\alpha-2n+4j-1}, \text{ as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi.$$

3. If $d_2 \neq 0$, $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j$ for some $j = 0, 1, \dots, n-1$, then

$$u_n(x, \alpha) \sim (-1)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right) \right)^{-1} \left(\frac{\Gamma\left(\frac{\alpha}{2} - n + 2j + 1\right)}{\Gamma\left(-\frac{\alpha}{2} + n - 2j\right)} \right)^2 \\ \times \frac{\Gamma\left(-\frac{\alpha}{2} + n - j + 1\right) \Gamma(n-j)}{\Gamma\left(j + \frac{\alpha}{2} + 1\right) \Gamma(j+1)} \left(\frac{x}{2}\right)^{-\alpha+2n-4j-1}, \text{ as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi.$$

4. If $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $\operatorname{Re}(\alpha) < -2n$ or $d_2 = 0$, then

$$u_n(x, \alpha) \sim \left(-\frac{\alpha}{2} - n\right) \left(\frac{x}{2}\right)^{-1}, \text{ as } x \rightarrow 0, \quad -\pi < \arg(x) < \pi,$$

where $\Gamma(x)$ refers to the Gamma function.

We can observe that the leading power of asymptotics in Theorems 1.1, 1.2 is continuous as a function of α , see the illustration for $n = 5$ in Figure 1. That indirectly confirms the validity of our computation. The plot also suggests that the solutions have a qualitatively different behavior for $-2n < \operatorname{Re}(\alpha) < 2n + 2$ and for $\operatorname{Re}(\alpha) > 2n + 2$ or for $\operatorname{Re}(\alpha) < -2n$.

We can compare Theorem 1.2 with the small x asymptotics computed based on the monodromy data in [16, Proposition 1.5], see also [17, 18]. More precisely, consider $w_n(x, \alpha) = -iu_n(-2ix, \alpha)$. It solves the equation

$$w_n'' = \frac{(w_n')^2}{w_n} - \frac{w_n'}{x} + \frac{(2\alpha + 4n)w_n^2 + (2\alpha - 4 - 4n)}{x} + 4w_n^3 - \frac{4}{w_n}.$$

According to [19], solution $w_n(x, \alpha)$ has Riemann–Hilbert representation given by [16, RHP 4.1] with

$$\mathbf{C}_{0\infty}^+ = \mathbf{C}_{0\infty}^- = \begin{pmatrix} 1 & 0 \\ 2b_1 & 1 \end{pmatrix}, \quad \mathbf{S}_1^\infty = \mathbf{S}_1^0 = \mathbb{I}, \quad \Theta_0 = \frac{\alpha}{2} + n, \quad \Theta_\infty = n + 2 - \frac{\alpha}{2}, \\ \mathbf{S}_2^\infty = \begin{pmatrix} 1 & 0 \\ 2(b_1 - b_2 e^{i\pi\alpha}) & 1 \end{pmatrix}, \quad \mathbf{S}_2^0 = \begin{pmatrix} 1 & 0 \\ 2e^{i\pi\alpha}(b_1 - b_2) & 1 \end{pmatrix},$$

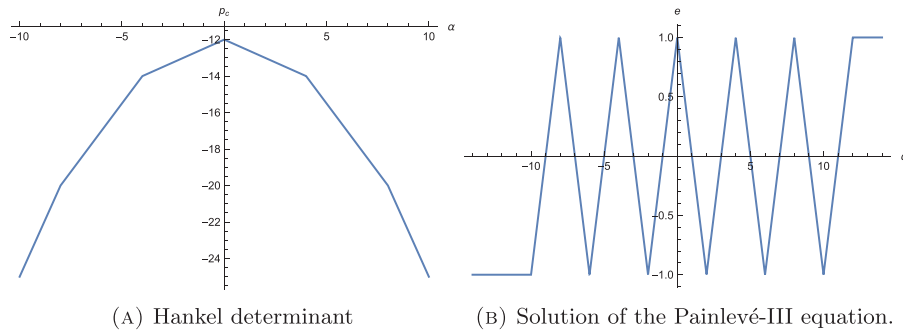


FIGURE 1 | Leading power in the asymptotics for $n = 5$ as a function of α . The piecewise expressions for $p_c(\alpha, n)$ and $e(\alpha, n)$ can be found in (56) and (58), respectively.

where b_1 and b_2 are given by (A10). The monodromy data corresponding to this solution are given by $e_1^2 = e_\infty^{-2} = e_0^2 = (-1)^n e^{\frac{i\pi\alpha}{2}}$, $e_2 = 1$, see [16, Section 4]. We see that it does not satisfy the conditions [16, Definition 1.3]. That means that we are filling the gap in the literature regarding the asymptotics of solutions of the Painlevé-III equation.

Next, we present our results for large x asymptotics.

Theorem 1.3. *The Toeplitz determinant (2) admits the following $x \rightarrow \infty$, asymptotics for fixed $d_1, d_2 \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$.*

1. If $d_1 \pm id_2 \neq 0$, n is even, and $x > 0$, then

$$\Delta_n(x, \alpha) \sim \left(\frac{d_1^2 + d_2^2}{2\pi} \right)^{\frac{n}{2}} \left(G\left(\frac{n}{2} + 1\right) \right)^2 \left(\frac{x}{4} \right)^{-\frac{n^2}{4}}, \text{ as } x \rightarrow \infty, \quad x > 0.$$

2. If $d_1 \pm id_2 \neq 0$, n is odd, and $x > 0$, then

$$\Delta_n(x, \alpha) \sim (-1)^{\frac{n-1}{2}} \left(\frac{d_1^2 + d_2^2}{2\pi} \right)^{\frac{n}{2}} G\left(\frac{n+1}{2}\right) G\left(\frac{n+3}{2}\right) \sin\left(x - \phi + \frac{\pi}{4}(n - \alpha)\right) \left(\frac{x}{4} \right)^{-\frac{n^2+1}{4}},$$

as $x \rightarrow \infty$, $x > 0$.

3. If $d_1 \pm id_2 \neq 0$, and $-\pi < \arg(x) < 0$, then

$$\Delta_n(x, \alpha) \sim (d_1 - id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{-\frac{i\pi n^2}{4}} G(n+1) e^{-\frac{i\pi n\alpha}{4}} e^{inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

4. If $d_1 \pm id_2 \neq 0$, and $0 < \arg(x) < \pi$, then

$$\Delta_n(x, \alpha) \sim (d_1 + id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{\frac{i\pi n^2}{4}} G(n+1) e^{\frac{i\pi n\alpha}{4}} e^{-inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

5. If $d_1 + id_2 = 0$, and $-\pi < \arg(x) < \pi$, then

$$\Delta_n(x, \alpha) \sim (d_1 - id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{-\frac{i\pi n^2}{4}} G(n+1) e^{-\frac{i\pi n\alpha}{4}} e^{inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

6. If $d_1 - id_2 = 0$, and $-\pi < \arg(x) < \pi$, then

$$\Delta_n(x, \alpha) \sim (d_1 + id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{\frac{i\pi n^2}{4}} G(n+1) e^{\frac{i\pi n\alpha}{4}} e^{-inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty,$$

where $G(x)$ refers to the Barnes G -function and $\phi = \frac{1}{2i} \ln(d_1 + id_2) - \frac{1}{2i} \ln(d_1 - id_2)$.

Theorem 1.4. *Solution (4) of the Painlevé-III equation (5) admits the following $x \rightarrow \infty$, $x > 0$ asymptotics for fixed $d_1, d_2 \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C}$,*

1. *If $d_1 \pm id_2 \neq 0$ and n is even, then for some $M > 0$*

$$u_n(x, \alpha) \sim -\cot\left(x - \phi + \frac{\pi}{4}(n+1-\alpha)\right), \text{ as } x \rightarrow \infty, \quad x > 0, \text{ and } |\cot\left(x - \phi + \frac{\pi}{4}(n+1-\alpha)\right)| < M.$$

2. *If $d_1 \pm id_2 \neq 0$ and n is odd, then for some $M > 0$*

$$u_n(x, \alpha) \sim -\tan\left(x - \phi + \frac{\pi}{4}(n-\alpha)\right), \text{ as } x \rightarrow \infty, \quad x > 0, \text{ and } |\tan\left(x - \phi + \frac{\pi}{4}(n-\alpha)\right)| < M.$$

3. *If $d_1 \pm id_2 \neq 0$ and $\text{Im}(x) \neq 0$, then we have*

$$u_n(x, \alpha) - i \sim \left(\frac{d_1 - id_2}{d_1 + id_2}\right) \frac{2^{2n-1}}{(n-1)!} e^{-\frac{i\pi}{2}(n+1+\alpha)} x^{n-1} e^{2ix}, \quad x \rightarrow \infty, \quad 0 < \arg(x) < \pi,$$

$$u_n(x, \alpha) + i \sim \left(\frac{d_1 + id_2}{d_1 - id_2}\right) \frac{2^{2n-1}}{(n-1)!} e^{\frac{i\pi}{2}(n-1+\alpha)} x^{n-1} e^{-2ix}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < 0.$$

4. *If $d_1 + id_2 = 0$ and then the asymptotics holds in larger domain*

$$u_n(x, \alpha) + i \sim \frac{1-\alpha}{2x}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi$$

Similarly for $d_1 - id_2 = 0$, we get

$$u_n(x, \alpha) - i \sim \frac{1-\alpha}{2x}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi.$$

Again, we refer to the fact that the asymptotics of Hankel functions is valid for $-\pi < \arg(x) < \pi$ to extend our result from domain $\text{Re}(x) > 0$ with available contour integral representation to the plane with a cut. To get the asymptotic for $\arg(x) = \pm\pi$, one can use the analytic continuation formulas for cylindrical functions, see Appendix A.3. More precisely, one could replace d_1, d_2 with d_1^\pm, d_2^\pm given by (A22), (A23) and x with $e^{\pm i\pi}x$ and directly use the result for $x > 0$. We can notice that d_1^\pm and d_2^\pm start depending on α in that case, but that change does not affect our computation. This strategy also provides an alternative justification for our result in the domain $\text{Re}(x) < 0$, where contour integral representation does not hold.

We should also mention that for $\alpha - 1 \in 2\mathbb{Z}$ and $d_1 \pm id_2 = 0$, the special function solutions reduce to the rational solutions considered in [20]. It can be seen from the asymptotic formulas (A6) and (A7) that truncate for described choice of α .

Proposition 1.1 is useful for the numerical calculation of the solution $u_n(x, \alpha)$ through direct evaluation of the determinants. We also present a color plot for the argument of $u_n(x, \alpha)$ in the complex plane for various choices of α and n in Figure 2. We present the result for fixed α and large n in Figure 2C. We observe that the pole structure is similar to the pole structure of rational solutions of the Painlevé III equation observed in [20], but the poles now also lie in the regions extending to infinity. The special case with $d_2 = 0$ can be found in Figure 2D, and the case with $d_1 + id_2 = 0$ can be found in Figure 2B. The other case of large α and large n can be seen in Figure 2E. We see that the pole structure looks different in this case. Finally, in Figure 2F, we can see what happens when we take α much larger than n . The analysis of such pictures would require tools like the nonlinear steepest descent method for Riemann–Hilbert problems. The Jupyter notebook with presented plots can be found at <https://github.com/andrei-prokhorov/special-function-solutions-of-PIII.git>.

1.1 | Overview of the Paper

We start with the reminder of the construction of a special function solution using the Ricatti equation in Section 2.1 and the generation of the family of special function solutions using the Bäcklund transformation in Section 2.2.

We prove representation for the special function solutions from Proposition 1.1 in Section 3. We start by introducing the tau function and demonstrating the classical fact that it satisfies the Toda equation in Section 3.2. We use the Deshanot–Jacobi identity and the Toda equation to show the classical Wronskian formula for the tau function associated with the family of special function solutions in Section 3.3. The crucial next step is the identification of the Toeplitz determinant (2) with the Painlevé tau function (19) in the Proposition 3.5, following [8, 12]. The main tools are differential identities (A12), (A13). After a long and tedious computation, we finish the proof of Proposition 1.1.

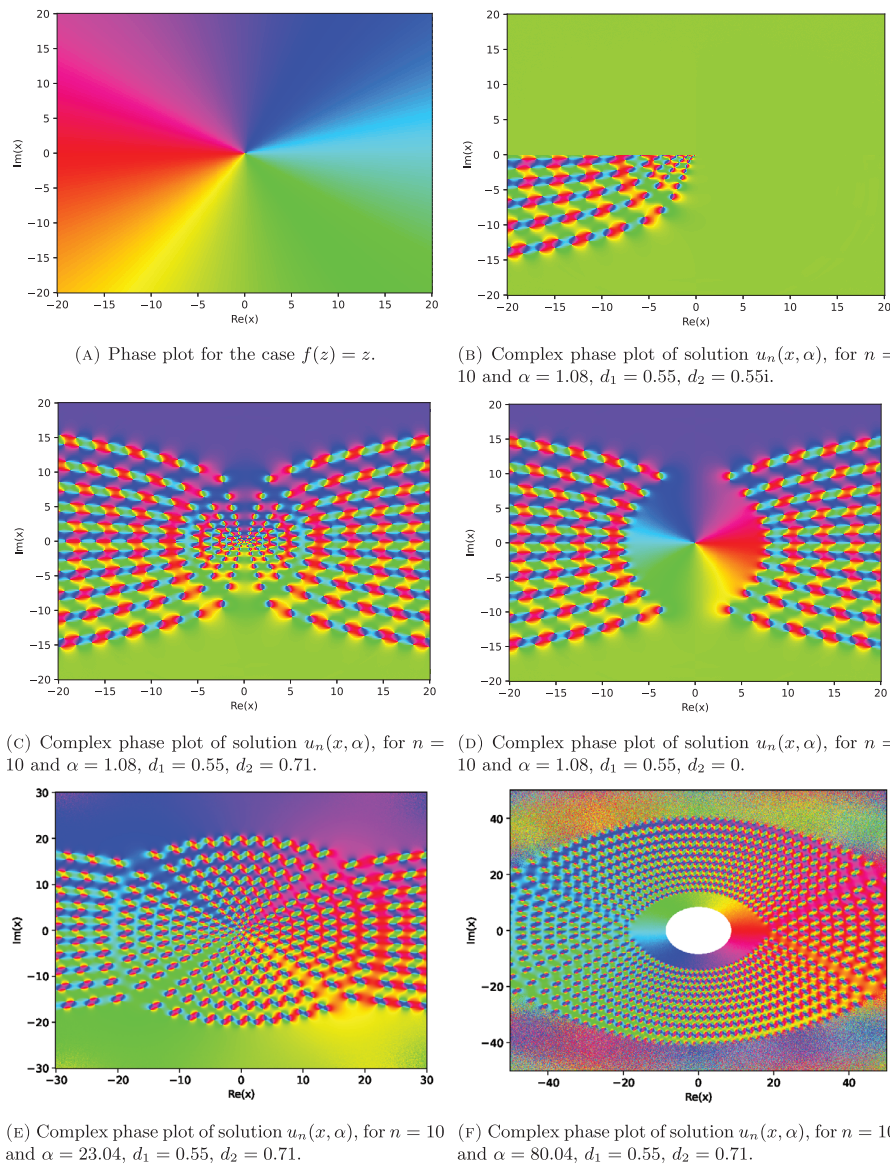


FIGURE 2 | Complex argument plots of solutions for various values of parameters. The color for each value of the argument can be found in Figure 2A.

We prove Theorem 1.1 in Section 4. We start by rewriting the Toeplitz determinant as a multiple contour integral using the Andréief formula in Section 4.1. We start with getting asymptotics for $x > 0$. The key next step in the proof is splitting the multiple contour integral in the sum of other multiple contour integrals so that it is easy to compute the leading term of the asymptotics for the latter integrals. The result can be found in Lemma 4.1. We determine which integral has the largest leading term in Lemma 4.2. The next step in the proof of Theorem 1.1 is the evaluation of the multiple contour integral corresponding to the leading term using formulas from [21]. Finally, we extend our asymptotic formulas to the complex plane with $-\pi < \arg(x) < \pi$ by using the validity of asymptotic series of Bessel functions (A2) in that range. The alternative way of this extension using analytic continuation formulas is presented in the Appendix B.1.

We prove Theorem 1.2 in Section 5. It consists of plugging in of result of Theorem 1.1 in Proposition 1.1 and tedious manipulation with piecewise formulas.

We prove Theorem 1.3 in Section 6.1. We rewrite the cylinder functions in terms of Hankel functions, which are more convenient for large x asymptotics computation. We again start with the case $x > 0$. We apply Andréief identity and use the steepest descent method to determine the final result. The extension to complex plane is provided by the range of validity of asymptotic series of Hankel functions (A6), (A7). Alternative method using analytic continuation formulas is presented in Appendix B.2. Theorem 1.4 is obtained in Section 6.2 by combining Theorem 1.3 and Proposition 1.1. For some cases, we compute additional error terms in Theorem 1.3 to get more meaningful expressions in Theorem 1.4.

In Appendix A, we derive the convenient contour integral representations (A4), (A11) for the cylinder function (3), present differential identities (A12), (A13), and analytic continuation formulas.

2 | Construction of Bessel Function Solutions of the Painlevé III Equation

2.1 | The Simultaneous Solutions of Ricatti and Painlevé III Equations

The standard way to construct the special function solutions of Painlevé equations is to use a Ricatti equation, see [11, Theorem 3.5] and [21, §32.10(iii)]. More precisely, we look for the simultaneous solutions of Painlevé III equation (1) and the Ricatti equation

$$u'(x) = a(x)u^2(x) + b(x)u(x) + c(x), \quad a(x) \neq 0. \quad (6)$$

Taking the first derivative of (6) and plugging in the $u'(x)$, we get

$$u''(x) = 2a^2(x)u^3(x) + (a'(x) + 3a(x)b(x))u^2(x) + (2a(x)c(x) + b^2(x) + b'(x))u(x) + (b(x)c(x) + c'(x)).$$

Meanwhile, plugging (6) into (1), we get

$$u''(x) = (a^2(x) + 1)u^3(x) + \left((2a(x)b(x) - \frac{a(x) - \alpha}{x})u^2(x) + \left(b^2(x) + 2a(x)c(x) - \frac{b(x)}{x} \right)u(x) + \frac{c(x) - 1}{u(x)} + 2b(x)c(x) - \frac{c(x) - \beta}{x} \right).$$

By matching and solving for the coefficients, we have four cases in total. We list them below:

$$a(x) = 1, \quad b(x) = \frac{\alpha - 1}{x}, \quad c(x) = 1, \quad \beta = 2 - \alpha, \quad (7)$$

$$a(x) = -1, \quad b(x) = \frac{-1 - \alpha}{x}, \quad c(x) = -1, \quad \beta = -2 - \alpha, \quad (8)$$

$$a(x) = 1, \quad b(x) = \frac{\alpha - 1}{x}, \quad c(x) = -1, \quad \beta = \alpha - 2, \quad (9)$$

$$a(x) = -1, \quad b(x) = \frac{-1 - \alpha}{x}, \quad c(x) = 1, \quad \beta = \alpha + 2. \quad (10)$$

Notice that if $u(x)$ solves the Ricatti equation, then $w(x) = \exp(-\int a(x)u(x)dx)$ solves the following linear ordinary differential equation:

$$a(x)w''(x) - (a'(x) + a(x)b(x))w'(x) + c(x)a^2(x)w(x) = 0, \quad a(x) \neq 0. \quad (11)$$

From now on, we will only focus on the case (7). Equation (11) becomes

$$w''(x) + \frac{(1 - \alpha)}{x}w'(x) + w(x) = 0. \quad (12)$$

We can notice that $x^{-\frac{\alpha}{2}}w(x)$ solves the Bessel equation (A1) with $\nu = \frac{\alpha}{2}$. For $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$, we denote the solution of (12)

$$w(x, \alpha) = x^{\frac{\alpha}{2}}C_{\frac{\alpha}{2}}(x), \quad (13)$$

where cylinder function $C_\nu(x)$ is given by (3). Here, we assume that $\frac{\alpha}{2}$ is not an integer for convenience in our future computations. As a result, we get the following.

Proposition 2.1 [21, §32.10(iii)]. *Painlevé III equation (1) with $\beta = 2 - \alpha$ and $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$ admits the special function solution*

$$u_0(x, \alpha) = -\frac{d}{dx} \ln(w(x, \alpha)). \quad (14)$$

with $w(x, \alpha)$ given by (13).

Remark 2.1. In case (8), the relevant solution is also given in terms of Bessel functions, while in the cases (9), (10), it is given in terms of modified Bessel functions.

2.2 | Bäcklund Transformation

To construct more solutions for the PIII equation with more general parameters, we need to introduce a powerful tool. Bäcklund transformations for the Painlevé-III equation are given by (see [21, §32.7(iii)])

$$B_1 : (u(x), \alpha, \beta) \rightarrow \left(\frac{xu'(x) + xu^2(x) - \beta u(x) - u(x) + x}{u(x)(xu'(x) + xu^2(x) + \alpha u(x) + u(x) + x)}, \alpha + 2, \beta + 2 \right), \quad (15)$$

$$B_2 : (u(x), \alpha, \beta) \rightarrow \left(-\frac{xu'(x) - xu^2(x) - \beta u(x) - u(x) + x}{u(x)(xu'(x) - xu^2(x) - \alpha u(x) + u(x) + x)}, \alpha - 2, \beta + 2 \right), \quad (16)$$

They are used as follows. Assume that $u(x)$ solves the Painlevé-III equation (5) and denote

$B_1(u(x), \alpha, \beta) = (W(x), \alpha + 2, \beta + 2)$. Then, $W(x)$ solves Painlevé-III equation

$$W''(x) = \frac{(W'(x))^2}{W(x)} - \frac{W'(x)}{x} + \frac{(\alpha + 2)W^2(x) + (\beta + 2)}{x} + W^3(x) - \frac{1}{W(x)}.$$

Similarly, if we denote $B_2(u(x), \alpha, \beta) = (W(x), \alpha - 2, \beta + 2)$, then $W(x)$ solves Painlevé-III equation

$$W''(x) = \frac{(W'(x))^2}{W(x)} - \frac{W'(x)}{x} + \frac{(\alpha - 2)W^2(x) + (\beta + 2)}{x} + W^3(x) - \frac{1}{W(x)}.$$

Proposition 2.2 [21, §32.7(iii)]. Denote $B_1^n(u_0(x, \alpha), \alpha, 2 - \alpha) = (u_n(x, \alpha), \alpha + 2n, -\alpha + 2 + 2n)$ with $u_0(x, \alpha)$ given by (14). Then, $u_n(x, \alpha)$ is the special function solution of Painlev -III equation (5).

We can observe that the parameters of the Painlev -III equation (5) satisfy $\alpha + \beta \in 2 + 4\mathbb{N}$. We will use B cklund transformation B_2 in our future considerations.

Remark 2.2. Using the transformation $u(x) \rightarrow -u(x)$, we can get solutions with $\alpha + \beta \in -2 - 4\mathbb{N}$. Using the transformation $u(x) \rightarrow -iu(-ix)$, we can get solutions with $\alpha - \beta \in 2 + 4\mathbb{N}$.

3 | Toeplitz Determinants of Cylinder Functions

3.1 | Hamiltonian System

We use the formulas presented in [11].

Definition 3.1. We define the momentum associated to the solution of Painlev -III equation using formula

$$v(x) = \frac{1}{2u^2(x)} (xu'(x) + xu^2(x) - x + u(x)(\beta - 1)).$$

Definition 3.2. We define the Hamiltonian associated with the solution of the Painlev -III equation using formula

$$H(x) = v^2(x)u^2(x) - v(x)(xu^2(x) - x + u(x)(\beta - 1)) + 2xu(x)\left(\frac{\beta - (2 + \alpha)}{4}\right).$$

One can show that Painlev -III equation is equivalent to the Hamiltonian system:

$$\begin{aligned} x \frac{du}{dx} &= \frac{\partial H}{\partial v}, \\ x \frac{dv}{dx} &= -\frac{\partial H}{\partial u}. \end{aligned}$$

3.2 | Tau Function and Toda Equation

For details of this section, see [12] and [8].

Definition 3.3. We define the auxiliary Hamiltonian associated with the solution of the Painlev -III equation using formula

$$h(x) = \frac{1}{2} \left(H(x) + u(x)v(x) - x^2 + \frac{1}{4}(\beta - 4)(\beta + (\alpha - 2)) \right).$$

In this section, we will deal with a generic solution $u(x)$ of (1). Since momentum, Hamiltonian, and auxiliary Hamiltonian are expressed in terms of $u(x)$, the action of the B cklund transformation B_1 can be extended to them by formulas $(u(x), \alpha, \beta)$ to $v(x)$, $H(x)$, and $h(x)$. We denote

$$(u_n(x), \alpha + 2n, \beta + 2n) = B_1^n(u(x), \alpha, \beta), \quad (17)$$

$$v_n(x) = v(x)|_{u(x) \rightarrow u_n(x), \beta \rightarrow \beta + 2n},$$

$$H_n(x) = H(x)|_{u(x) \rightarrow u_n(x), v(x) \rightarrow v_n(x), \alpha \rightarrow \alpha + 2n, \beta \rightarrow \beta + 2n},$$

$$h_n(x) = h(x)|_{H(x) \rightarrow H_n(x), u(x) \rightarrow u_n(x), v(x) \rightarrow v_n(x), \alpha \rightarrow \alpha + 2n, \beta \rightarrow \beta + 2n}. \quad (18)$$

On the path to derive the representation of Proposition 1.1, we introduce the tau function associated with the solution $u(x)$.

Definition 3.4. The tau function associated to the solution of Painlev -III equation is defined using the formula

$$x \frac{d}{dx} \ln(\tau_n(x)) = h_n(x). \quad (19)$$

It is defined up to a multiplicative constant.

Proposition 3.1 [8, Proposition 4.2]. Tau function for the Painlev -III equation given by (19) satisfies Toda equation

$$x \frac{d}{dx} x \frac{d}{dx} \ln(\tau_n(x)) = c_n \frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n^2(x)}$$

for some constants c_n . Moreover, multiplicative constants can be chosen in the definition (19) so that $c_n = 1$.

Proof. Using the B cklund transformation B_1 , we can check the identity

$$h_{n+1}(x) = h_n(x) - v_n(x)u_n(x) - \frac{3}{2} + \frac{\alpha}{4} - \frac{3\beta}{4} + 2n. \quad (20)$$

Denote $A_n(x) = x \frac{d}{dx} x \frac{d}{dx} \ln(\tau_n(x))$ and $B_n(x) = \frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n^2(x)}$. We want to show that $A_n(x) = c_n B_n(x)$. Taking a natural log on both sides, we get $\ln A_n(x) = \ln B_n(x) + \ln c_n$. Therefore, it is sufficient to show

$$\frac{d}{dx} (\ln A_n(x) - \ln B_n(x)) = 0. \quad (21)$$

Well, using Definition 3.4 and identity (20), we have

$$\frac{d}{dx} \ln(B_n(x)) = \frac{v_{n-1}(x)u_{n-1}(x) - v_n(x)u_n(x) + 2}{x}, \quad (22)$$

$$\frac{d}{dx} \ln(A_n(x)) = \frac{h'_n(x) + xh''_n(x)}{xh'_n(x)}. \quad (23)$$

Using Definitions 3.3 and (17)–(18), we rewrite (23), (22) in terms of $u_n(x)$. After a long computation, we obtain (21).

Let us show that using transformation $\tau_n(x) \rightarrow a_n \tau_n(x)$, one can make sure that the constant c_n in the Toda equation is 1. We notice that for that to happen a_n has to satisfy a difference equation

$$c_n^{-1} a_n^2 = a_{n+1} a_{n-1}.$$

Its general solution is given by

$$a_n = \frac{a_1^n}{a_0^{n-1}} \prod_{j=1}^{n-1} \prod_{i=1}^j c_i^{-1}, \quad n \in \mathbb{N}.$$

We can pick the initial conditions $a_0 = a_1 = 1$ and choose

$$a_n = \prod_{j=1}^{n-1} \prod_{i=1}^j c_i^{-1}, \quad n \in \mathbb{N}. \quad \square$$

3.3 | Wronskian Solutions of Toda Equation

Toda equation

$$x \frac{d}{dx} x \frac{d}{dx} \ln(\tau_n(x)) = \frac{\tau_{n+1}(x) \tau_{n-1}(x)}{\tau_n^2(x)}$$

determines the tau function recursively given initial conditions. If we want to derive some nice formula for it, we need some properties of the determinants. The Leibniz formula for the determinant of $n \times n$ matrix $A = \{a_{ij}\}_{i,j=1}^n$ is given by

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{k=1}^n a_{k,\sigma(k)},$$

where S_n is the set of permutations of n elements and $\text{sgn}(\sigma)$ is a sign of permutation σ . Directly using the Leibniz formula above, we can show the following formulas for the derivative of a determinant

$$\frac{d}{dx} \det(A) = \sum_{j=1}^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\frac{d}{dx} a_{j,\sigma(j)} \right) \prod_{\substack{k=1 \\ k \neq j}}^n a_{k,\sigma(k)}. \quad (24)$$

Remembering that $\det(A^T) = \det(A)$, we can write the alternative formula

$$\frac{d}{dx} \det(A) = \sum_{j=1}^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\frac{d}{dx} a_{\sigma(j),j} \right) \prod_{\substack{k=1 \\ k \neq j}}^n a_{\sigma(k),k}. \quad (25)$$

Denote by $A_{i|j}$ the matrix obtained from A by deleting its i th row and j th column. The Laplace expansion for the determinant along the j th row is given by

$$\det(A) = \sum_{k=1}^n (-1)^{k+j} a_{jk} \det(A_{j|k}).$$

Proposition 3.2 See [22]. Denote $A_{ij|kl}$ the matrix obtained from A by deleting the i th and j th rows and k th and l th columns. Determinants of these matrices satisfy Deshanot–Jacobi identity

$$\det(A) \det(A_{ij|ij}) = \det(A_{i|i}) \det(A_{j|j}) - \det(A_{i|j}) \det(A_{j|i}), \quad 1 \leq i, j \leq n. \quad (26)$$

Proposition 3.3 [8, (2.43)]. The sequence of functions

$$f_n(x) = \det \left(\left\{ \left(x \frac{d}{dx} \right)^{i+j} f_0(x) \right\}_{i,j=0}^n \right) \quad (27)$$

with infinitely differentiable $f_0(x)$ solves Toda equation corresponding to Painlev -III equation

$$\left(x \frac{d}{dx} \right)^2 \ln(f_n(x)) = \frac{f_{n+1}(x) f_{n-1}(x)}{f_n^2(x)}, \quad n \geq 1. \quad (28)$$

Proof. Specifically, to match the expression in Proposition 3.2, we rewrite (28) as

$$f_{n-1}(x) f_{n+1}(x) = f_n(x) \left(x \frac{d}{dx} \right)^2 f_n(x) - \left(x \frac{d}{dx} f_n(x) \right)^2, \quad n \geq 1$$

Put $f_{n+1}(x) = \det(A)$. It follows that $f_n(x) = \det(A_{n+2|n+2})$ and $f_{n-1}(x) = \det(A_{n+1,n+2|n+1,n+2})$. We take the first derivative of the determinant in (27) by multilinearity with respect to rows using (24). Since a determinant with two identical rows is zero, we end up with $x \frac{d}{dx} f_n(x) = \det(A_{n+1|n+2})$. Since $A_{n+1|n+2} = (A_{n+2|n+1})^T$, this implies that $x \frac{d}{dx} f_n(x) = \det(A_{n+1|n+2}) = \det(A_{n+2|n+1})$. Then, we take the second derivative of the determinant in (27) successively by multilinearity with respect to columns using (25). Similarly, since a determinant with two identical columns is zero, we end up with $\left(x \frac{d}{dx} \right)^2 f_n(x) = \det(A_{n+1|n+1})$. Using (26), we obtain (28). \square

Now, let us return to the special function solutions $u_n(x, \alpha)$. We compute the corresponding auxiliary Hamiltonians $h_0(x, \alpha)$, $h_1(x, \alpha)$, and $h_2(x, \alpha)$. It turns out that corresponding tau functions can be chosen as

$$\tau_0(x, \alpha) = 1, \quad (29)$$

$$\tau_1(x, \alpha) = C_{\frac{\alpha}{2}}(x), \quad (30)$$

$$\tau_2(x, \alpha) = \det \begin{pmatrix} \tau_1(x, \alpha) & x \frac{d}{dx} \tau_1(x, \alpha) \\ x \frac{d}{dx} \tau_1(x, \alpha) & \left(x \frac{d}{dx} \right)^2 \tau_1(x, \alpha) \end{pmatrix}, \quad (31)$$

where $C_\nu(x)$ is given by (3). It indicates that special function solutions can be represented using determinants as in Proposition 3.3 that is not true for arbitrary family of B cklund iterates. Moreover, it is the only family of solutions with this property. More specifically, condition (29) puts the restriction on parameters α and β and imposes the Bessel differential equation for $\tau_1(x, \alpha)$, see [8, Proposition 4.3]. After that, the Toda equation determines the tau function uniquely given the initial condition, which produces determinantal formulas (31), (32), see [8, (2.43)]. To summarize the solution with initial conditions (30), (31) is given by (27) in Proposition 3.3. As a result, we get

Proposition 3.4. Tau functions associated with the special function solutions $u_n(x, \alpha)$ can be chosen as

$$\tau_n(x, \alpha) = \det \left(\left\{ \left(x \frac{d}{dx} \right)^{i+j-2} \tau_1(x, \alpha) \right\}_{i,j=1}^n \right) \quad (32)$$

with $\tau_1(x, \alpha) = C_{\frac{\alpha}{2}}(x)$, $\tau_0(x, \alpha) = 1$.

Proposition 3.5. Formula (32) can be alternatively written as

$$\tau_n(x, \alpha) = x^{n(n-1)} (-1)^{\frac{n(n-1)}{2}} \Delta_n(x, \alpha),$$

where $\Delta_n(x, \alpha)$ is given by (2).

Proof. Using mathematical induction and identity (A12), one can compute the structure of $\left(x \frac{d}{dx}\right)^j C_{\frac{\alpha}{2}}(x)$:

$$\left(x \frac{d}{dx}\right)^j C_{\frac{\alpha}{2}}(x) = x^j C_{\frac{\alpha}{2}-j}(x) + \sum_{k=0}^{j-1} c_{kj} x^k C_{\frac{\alpha}{2}-k}(x), \quad (33)$$

where c_{kj} are constant coefficients. Well, we furtherly simplify the determinant using (33):

$$\det \left(\left\{ \left(x \frac{d}{dx}\right)^{k+j} C_{\frac{\alpha}{2}}(x) \right\}_{k,j=0}^{n-1} \right) = \begin{vmatrix} C_{\frac{\alpha}{2}}(x) & x \frac{d}{dx} C_{\frac{\alpha}{2}}(x) & \cdots \\ x C_{\frac{\alpha}{2}-1}(x) + c_{01} C_{\frac{\alpha}{2}}(x) & x \frac{d}{dx} (x C_{\frac{\alpha}{2}-1}(x) + c_{01} C_{\frac{\alpha}{2}}(x)) & \cdots \\ x^2 C_{\frac{\alpha}{2}-2}(x) + c_{12} x C_{\frac{\alpha}{2}-1}(x) + c_{02} C_{\frac{\alpha}{2}}(x) & x \frac{d}{dx} (x^2 C_{\frac{\alpha}{2}-2}(x) + c_{12} x C_{\frac{\alpha}{2}-1}(x) + c_{02} C_{\frac{\alpha}{2}}(x)) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}.$$

Observe that by elementary row operations, we can always use the previous rows to eliminate the $\sum_{k=0}^{j-1} c_{kj} x^k C_{\frac{\alpha}{2}-k}(x)$ part in a fixed row and the value of the determinant does not change. Doing that, we end up with:

$$\det \left(\left\{ \left(x \frac{d}{dx}\right)^{k+j} C_{\frac{\alpha}{2}}(x) \right\}_{k,j=0}^{n-1} \right) = \begin{vmatrix} C_{\frac{\alpha}{2}}(x) & x \frac{d}{dx} C_{\frac{\alpha}{2}}(x) & \cdots \\ x C_{\frac{\alpha}{2}-1}(x) & x \frac{d}{dx} (x C_{\frac{\alpha}{2}-1}(x)) & \cdots \\ x^2 C_{\frac{\alpha}{2}-2}(x) & x \frac{d}{dx} (x^2 C_{\frac{\alpha}{2}-2}(x)) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} = \det \left(\left\{ \left(x \frac{d}{dx}\right)^k x^j C_{\frac{\alpha}{2}-j}(x) \right\}_{k,j=0}^{n-1} \right).$$

By relation (A13), by induction, we can show

$$\left(x \frac{d}{dx}\right)^k x^j C_{\frac{\alpha}{2}-j}(x) = (-1)^k x^{j+k} C_{\frac{\alpha}{2}-j+k}(x) + \sum_{n=0}^{k-1} d_{nkj} x^{j+n} C_{\frac{\alpha}{2}-j+n}(x). \quad (34)$$

To prove (34) by (A13), we first fix $k = 1$ and induct on j . We have

$$\begin{aligned} x \frac{d}{dx} x^j C_{\frac{\alpha}{2}-j}(x) &= -x^{j+1} C_{\frac{\alpha}{2}-j+1}(x) + x^j \left(\frac{\alpha}{2} - j \right) C_{\frac{\alpha}{2}-j}(x) + j x^j C_{\frac{\alpha}{2}-j}(x) \\ &= -x^{j+1} C_{\frac{\alpha}{2}-j+1}(x) + x^j \frac{\alpha}{2} C_{\frac{\alpha}{2}-j}(x). \end{aligned} \quad (35)$$

After showing (35), we induct on k . We also can notice that coefficient depending in j cancels in the right-hand side of (35). It implies that d_{nkj} actually does not depend on j . So, (34) can be written as

$$\left(x \frac{d}{dx}\right)^k x^j C_{\frac{\alpha}{2}-j}(x) = (-1)^k x^{j+k} C_{\frac{\alpha}{2}-j+k}(x) + \sum_{n=0}^{k-1} d_{nk} x^{j+n} C_{\frac{\alpha}{2}-j+n}(x). \quad (36)$$

Again, we furtherly simplify the determinant by (36):

$$\det \left(\left\{ \left(x \frac{d}{dx}\right)^{k+j} C_{\frac{\alpha}{2}}(x) \right\}_{k,j=0}^{n-1} \right) = \begin{vmatrix} C_{\frac{\alpha}{2}}(x) & -x C_{\frac{\alpha}{2}+1}(x) + d_{01} C_{\frac{\alpha}{2}}(x) & \cdots \\ x C_{\frac{\alpha}{2}-1}(x) & -x^2 C_{\frac{\alpha}{2}}(x) + d_{01} x C_{\frac{\alpha}{2}-1}(x) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}.$$

Similarly, by applying elementary column operations, we can always use the previous columns to eliminate the $\sum_{n=0}^{k-1} d_{nk} x^{j+n} C_{\frac{\alpha}{2}-j+n}^{\alpha}(x)$ part in a fixed column and the value of the determinant does not change. Finally, we will end up with:

$$\det \left(\left\{ \left(x \frac{d}{dx} \right)^{k+j} C_{\frac{\alpha}{2}}^{\alpha}(x) \right\}_{k,j=0}^{n-1} \right) = \det \left((-1)^k x^{j+k} C_{\frac{\alpha}{2}-j+k}^{\alpha}(x) \right).$$

By multilinearity of determinant, we can factor out $(-1)^k x^{j+k}$ and reach the conclusion:

$$\tau_n(x, \alpha) = \det \left((-1)^k x^{j+k} C_{\frac{\alpha}{2}-j+k}^{\alpha}(x) \right) = x^{n(n-1)} (-1)^{\frac{n(n-1)}{2}} \Delta_n(x, \alpha),$$

where $\Delta_n(x, \alpha)$ is given by (2). That completes the proof. \square

3.4 | Proof of Proposition 1.1

Before starting the proof, we need to prove the following lemma.

Lemma 3.1. *The special function solution $u_{n+1}(x, \alpha - 2)$ admits the following formula in terms of $u_n(x, \alpha)$.*

$$(u_{n+1}(x, \alpha - 2), \alpha + 2n, -\alpha + 6 + 2n) = B_1 B_2(u_n(x, \alpha), \alpha + 2n, -\alpha + 2 + 2n). \quad (37)$$

Proof. We start by considering the Bäcklund transformations B_1 and B_2 . Using the explicit formulas (15), (16) and Equation (1), we can show that these transformations commute: $B_1 B_2 = B_2 B_1$.

Let us now consider B_2 applied to the special function solution (14). After using differential equation (12) for $w(x)$, we get

$$B_2(u_0(x, \alpha), \alpha, 2 - \alpha)[1] = \frac{2 - \alpha}{x} + \frac{w(x)}{w'(x)} = \frac{2 - \alpha}{x} + \frac{1}{\frac{\alpha}{2x} + \frac{C'_{\frac{\alpha}{2}}(x)}{C_{\frac{\alpha}{2}}^{\alpha}(x)}}.$$

We use the notation [1] above for the first component of the output of Bäcklund transformation. We use (A13) to get

$$B_2(u_0(x, \alpha), \alpha, 2 - \alpha)[1] = \frac{2 - \alpha}{x} + \frac{C_{\frac{\alpha}{2}}^{\alpha}(x)}{C_{\frac{\alpha}{2}-1}^{\alpha}(x)}.$$

Using identity (A12), we can rewrite it as

$$B_2(u_0(x, \alpha), \alpha, 2 - \alpha)[1] = \frac{2 - \alpha}{2x} - \frac{C_{\frac{\alpha}{2}-1}^{\alpha}(x)}{C_{\frac{\alpha}{2}-1}^{\alpha}(x)} = -\frac{d}{dx} \ln(w(x, \alpha - 2)) = u_0(x, \alpha - 2).$$

Now using commutativity of B_1 and B_2 , we get

$$\begin{aligned} B_2(u_n(x, \alpha), \alpha + 2n, -\alpha + 2 + 2n) &= B_2 B_1^n(u_0(x, \alpha), \alpha, -\alpha + 2) \\ &= B_1^n(u_0(x, \alpha - 2), \alpha - 2, -\alpha + 4) = (u_n(x, \alpha - 2), \alpha - 2 + 2n, -\alpha + 4 + 2n). \end{aligned}$$

Using similar logic, we arrive at (37) and finish the proof. \square

Proof of Proposition 1.1. We start our proof by introducing the following sequence of functions

$$\tilde{u}_n(x, \alpha) = -\frac{\Delta_{n+1}(x, \alpha - 2) \Delta_n(x, \alpha)}{\Delta_{n+1}(x, \alpha) \Delta_n(x, \alpha - 2)}.$$

Using Proposition 3.5, we can rewrite it in terms of tau functions

$$\tilde{u}_n(x, \alpha) = -\frac{\tau_{n+1}(x, \alpha - 2) \tau_n(x, \alpha)}{\tau_{n+1}(x, \alpha) \tau_n(x, \alpha - 2)}. \quad (38)$$

Using the Toda equation (28) and the definition of tau function (19), we can see that

$$\tilde{u}_{n+1}(x, \alpha) = \tilde{u}_n(x, \alpha) \frac{\left(x \frac{d}{dx}\right)^2 \ln(\tau_{n+1}(x, \alpha - 2))}{\left(x \frac{d}{dx}\right)^2 \ln(\tau_{n+1}(x, \alpha))} = \tilde{u}_n(x, \alpha) \frac{\left(x \frac{d}{dx}\right) h_{n+1}(x, \alpha - 2)}{\left(x \frac{d}{dx}\right) h_{n+1}(x, \alpha)}. \quad (39)$$

Using Lemma 3.1 and the definition of $h_{n+1}(x, \alpha)$, we can express the right-hand side of (39) in terms of $u_n(x, \alpha)$. We also provide intermediate formulas

$$\begin{aligned} h_{n+1}(x, \alpha) &= -\frac{1}{8u_n(x, \alpha)^2} \left[u_n(x, \alpha)^2 (\alpha^2 - 4n^2 + 2x^2 + 3) + 2xu_n(x, \alpha)(\alpha - 2n + u'_n(x, \alpha) - 2) \right. \\ &\quad \left. + 2x(\alpha + 2n + 2)u_n(x, \alpha)^3 - x^2 \left((u'_n(x, \alpha))^2 - 1 \right) + x^2 u_n(x, \alpha)^4 \right], \\ h_{n+1}(x, \alpha - 2) &= -\frac{1}{8u_n(x, \alpha)^2} \left[u_n(x, \alpha)^2 (\alpha^2 - 4\alpha - 4n^2 + 2x^2 + 7) + x^2 u_n(x, \alpha)^4 \right. \\ &\quad \left. - 2xu_n(x, \alpha)(-\alpha + 2n + u'_n(x, \alpha) + 4) + 2x(\alpha + 2n)u_n(x, \alpha)^3 - x^2 (u'_n(x, \alpha)^2 - 1) \right]. \end{aligned}$$

As the result, we get

$$\tilde{u}_{n+1}(x, \alpha) = \tilde{u}_n(x, \alpha) \frac{(-3 + \alpha + 2n)u_n(x, \alpha) + xu_n^2(x, \alpha) + x + xu'_n(x, \alpha)}{u_n^2(x, \alpha)((1 + \alpha + 2n)u_n(x, \alpha) + xu_n^2(x, \alpha) + x + xu'_n(x, \alpha))}. \quad (40)$$

Using identity (A12), initial conditions (29), (30), and definition (38), we can observe that

$$\tilde{u}_0(x, \alpha) = u_0(x, \alpha).$$

Using explicit formula (15) for Bäcklund transformation B_1 , relation (40), and mathematical induction, we get the desired result

$$\tilde{u}_n(x, \alpha) = u_n(x, \alpha).$$

□

4 | Asymptotics of Toeplitz Determinant at Zero

4.1 | Andr  ief Identity

To prove our result, we rewrite Toeplitz determinant (2) as a multiple contour integral.

Proposition 4.1 (See [23]). *Andr  ief identity is given by the following formula*

$$\int_{\Gamma} \dots \int_{\Gamma} \det \left(\{f_j(x_k)\}_{j,k=1}^n \right) \det \left(\{g_j(x_k)\}_{j,k=1}^n \right) \prod_{k=1}^n h(x_k) dx_k = n! \det \left(\left\{ \int_{\Gamma} f_j(x) g_k(x) h(x) dx \right\}_{j,k=1}^n \right),$$

where Γ is some contour in the complex plane, such that the corresponding integral is finite.

We apply the Andr  ief identity and get the following result.

Theorem 4.1. *The Toeplitz determinant $\Delta_n(x, \alpha)$ given by (2) can be rewritten as*

$$\Delta_n(x, \alpha) = \frac{1}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \prod_{1 \leq j < k \leq n} (t_k - t_j) \left(\frac{1}{t_k} - \frac{1}{t_j} \right) \prod_{k=1}^n h_1(t_k) dt_k, \quad (41)$$

where

$$h_1(t) = \frac{e^{\frac{x}{2}(t - \frac{1}{t})}}{2\pi i t^{1 + \frac{\alpha}{2}}} \left(\left(d_1 + d_2 \cot \left(\frac{\pi \alpha}{2} \right) \right) \chi_{\Gamma_1}(t) + d_2 \csc \left(\frac{\pi \alpha}{2} \right) e^{\frac{i\pi \alpha}{2}} \chi_{\Gamma_2}(t) \right),$$

and contours of integration Γ_1, Γ_2 are shown in Figure A2 and they do not intersect. We use the notation $\chi_{\Gamma_j}(t)$ for the characteristic function of the contour Γ_j . We assume $-\pi < \arg(t) < \pi$ on the contour Γ_1 and $0 < \arg(t) < 2\pi$ on the contour Γ_2 .

Proof. Using contour integral representation (A4), we get

$$C_{\frac{\alpha}{2}-j+k}(x) = \int_{\Gamma_1 \cup \Gamma_2} t^{-k} t^j h_1(t) dt.$$

Put $g_k(t) = t^{-k}$ and $f_j(t) = t^j$. By Proposition 4.1, we get

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{1}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \det \left(\{f_j(t_k)\}_{j,k=0}^{n-1} \right) \det \left(\{g_j(t_k)\}_{j,k=0}^{n-1} \right) \prod_{k=0}^{n-1} h_1(t_k) dt_k \\ &= \frac{1}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \det \left(\{t_k^j\}_{j,k=0}^{n-1} \right) \det \left(\{t_k^{-j}\}_{j,k=0}^{n-1} \right) \prod_{k=0}^{n-1} h_1(t_k) dt_k. \end{aligned}$$

By the formula for the Vandermonde determinant, we can simplify the integrand and get

$$\begin{aligned} \det \left(\{t_k^j\}_{j,k=0}^{n-1} \right) &= \prod_{0 \leq j < k \leq n-1} (t_k - t_j), \\ \det \left(\{t_k^{-j}\}_{j,k=0}^{n-1} \right) &= \prod_{0 \leq j < k \leq n-1} \left(\frac{1}{t_k} - \frac{1}{t_j} \right). \end{aligned}$$

Thus, the explicit formula for $\Delta_n(x, \alpha)$ is given by

$$\Delta_n(x, \alpha) = \frac{1}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \prod_{0 \leq j < k \leq n-1} (t_k - t_j) \left(\frac{1}{t_k} - \frac{1}{t_j} \right) \prod_{k=0}^{n-1} h_1(t_k) dt_k.$$

For convenience, we shift the index of variables t_k . □

4.2 | Basic Strategies

Up to this point, we have enough preparation to compute the asymptotics at zero. Our goal is to get asymptotics $\Delta_n(x, \alpha) \sim b(n)x^{\alpha(n)}$ when $x \rightarrow 0, x > 0$. This is a reasonable expectation, since the Bessel function $J_\nu(x)$ admits series representation (A2). We summarize several key ideas to achieve this goal.

- The contours Γ_1 and Γ_2 spread to zero and infinity in formula (41). We cannot put $x = 0$ here without losing convergence of the integral.
- Expanding the product $\prod_{k=0}^{n-1} h(t_k)$ in the integrand of (41), we get the sum of expressions, each of them has some of the variables t_k belonging to the contour Γ_1 and others belonging to Γ_2 .
- We apply the change of variables $t = \frac{2}{x}s$ to variables on contours Γ_1 . The integrand will maintain an exponential decay at infinity when we put $x = 0$. On the other hand, we can apply the change of variable $t = \frac{x}{2}s$ to the variables on contours Γ_2 . In this case, the integrand will preserve exponential decay at zero when we put $x = 0$.
- From first glance, it seems that it would be more convenient to use the expression $d_1 J_\nu(x) + d_2 J_{-\nu}(x)$ instead of (3). But unfortunately, Proposition 1.1 would fail if we replace $Y_\nu(x)$ with $J_{-\nu}(x)$. It follows from the fact that the differential identities (A12), (A13) would fail for this alternative choice. And we need them to hold, since they were used extensively in the proof of Proposition 3.5.

4.3 | Expanded Formula for $\Delta_n(x, \alpha)$

We start with the following observation. Let I denote a subset of the set of indices $\{1, \dots, n\}$, r denote its cardinality $|I|$, and I^c denote its complement. The following identity holds:

$$\prod_{k=1}^n (c_1 \chi_{\Gamma_1}(t_k) + c_2 \chi_{\Gamma_2}(t_k)) = \sum_{r=0}^n \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=r}} c_1^r c_2^{n-r} \prod_{k \in I} \chi_{\Gamma_1}(t_k) \prod_{j \in I^c} \chi_{\Gamma_2}(t_j). \quad (42)$$

Now we apply (42) to the expression in Theorem 4.1 to convert the formula into a summation form and decouple the contours. Denote

$$c_1 = d_1 + d_2 \cot\left(\frac{\pi\alpha}{2}\right), \quad c_2 = d_2 \csc\left(\frac{\pi\alpha}{2}\right) e^{\frac{i\pi\alpha}{2}}. \quad (43)$$

We have

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \prod_{m=1}^n (c_1 \chi_{\Gamma_1}(t_m) + c_2 \chi_{\Gamma_2}(t_m)) \prod_{1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{2\pi i t_l^{\frac{\alpha}{2}+1}} dt_l \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \sum_{r=0}^n \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=r}} c_1^r c_2^{n-r} \prod_{i \in I} \chi_{\Gamma_1}(t_i) \prod_{j \in I^c} \chi_{\Gamma_2}(t_j) \prod_{1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{2\pi i t_l^{\frac{\alpha}{2}+1}} dt_l \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n n!} \sum_{r=0}^n \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=r}} c_1^r c_2^{n-r} \int_{\Gamma_1 \cup \Gamma_2} \dots \int_{\Gamma_1 \cup \Gamma_2} \prod_{\substack{j < k \\ j, k \in I}} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{\substack{j < k \\ j, k \in I^c}} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{\substack{j \in I \\ k \in I^c}} \frac{(t_j - t_k)^2}{t_j t_k} \\ &\quad \prod_{l \in I} \chi_{\Gamma_1}(t_l) \prod_{l \in I^c} \chi_{\Gamma_2}(t_l) \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}} dt_l. \end{aligned}$$

We remind that we assume $-\pi < \arg(t) < \pi$ on contour Γ_1 and $0 < \arg(t) < 2\pi$ on contour Γ_2 . We can see that by renaming variables on the right-hand side, we can guarantee that $I = \{1, 2, \dots, r\}$, and $I^c = \{r+1, r+2, \dots, n\}$ for each integral in the sum. Combining the same integrals together, we get

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(-1)^{\frac{n(n-1)}{2}}}{(\pi i)^n} \sum_{r=0}^n \frac{c_1^r c_2^{n-r}}{r!(n-r)!} \int_{\Gamma_1} \dots \int_{\Gamma_1} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq r} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{r+1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{t_j t_k} \\ &\quad \times \prod_{j=1}^r \prod_{k=r+1}^n \frac{(t_j - t_k)^2}{t_j t_k} \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}} dt_l, \end{aligned}$$

where we have r integrals over Γ_1 and $n-r$ integrals over Γ_2 .

For $t_k \in I$, we use change of variable $t_k = \frac{2}{x} s_k$. On the other hand, for $t_j \in I^c$, we use change of variable $t_j = \frac{x}{2} s_j$. Since the only singularities of the integrand are at zero and infinity, we can deform contours of integration back to Γ_1 and Γ_2 . The formula above becomes

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_{r=0}^n \frac{c_1^r c_2^{n-r}}{r!(n-r)!} \int_{\Gamma_1} \dots \int_{\Gamma_1} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq r} \frac{(s_j - s_k)^2}{s_j s_k} \prod_{r+1 \leq j < k \leq n} \frac{(s_j - s_k)^2}{s_j s_k} \\ &\quad \prod_{j=1}^r \prod_{k=r+1}^n \left(\frac{2}{x}\right)^2 \frac{(s_j(1 + \mathcal{O}(x^2)))^2}{s_j s_k} \prod_{l=1}^r \frac{e^{s_l(1 + \mathcal{O}(x^2))}}{\left(\frac{2}{x}\right)^{\frac{\alpha}{2}+1} s_l^{\frac{\alpha}{2}+1}} \frac{2}{x} ds_l \prod_{l=r+1}^n \frac{e^{-\frac{1}{s_l}(1 + \mathcal{O}(x^2))}}{s_l^{\frac{\alpha}{2}+1} \left(\frac{x}{2}\right)^{\frac{\alpha}{2}+1}} \frac{x}{2} ds_l. \end{aligned}$$

Grouping all the $\frac{2}{x}$ factors together and pulling them out of the summation,

$$\Delta_n(x, \alpha) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_{r=0}^n \frac{c_1^r c_2^{n-r}}{r!(n-r)!} \left(\frac{2}{x}\right)^{-\alpha r + 2r(n-r) + \frac{\alpha n}{2}} \int_{\Gamma_1} \dots \int_{\Gamma_1} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq r} \frac{(s_j - s_k)^2}{s_j s_k}$$

$$\prod_{r+1 \leq j < k \leq n} \frac{(s_j - s_k)^2}{s_j s_k} \prod_{j=1}^r \prod_{k=r+1}^n \frac{(s_j(1 + \mathcal{O}(x^2)))^2}{s_j s_k} \prod_{l=1}^r \frac{e^{s_l(1 + \mathcal{O}(x^2))}}{s_l^{\frac{\alpha}{2}+1}} ds_l \prod_{l=r+1}^n \frac{e^{-\frac{1}{s_l}(1 + \mathcal{O}(x^2))}}{s_l^{\frac{\alpha}{2}+1}} ds_l.$$

We also want to group all the products of variables together and separate the integrals based on different contours. We rewrite the following three parts:

$$\prod_{1 \leq j < k \leq r} \frac{(s_j - s_k)^2}{s_j s_k} = \prod_{1 \leq j < k \leq r} (s_j - s_k)^2 \prod_{\substack{j \neq k \\ 1 \leq j, k \leq r}} \frac{1}{\sqrt{s_j s_k}} = \prod_{1 \leq j < k \leq r} (s_j - s_k)^2 \prod_{l=1}^r s_l^{\gamma_1}, \quad (44)$$

$$\prod_{r+1 \leq j < k \leq n} \frac{(s_j - s_k)^2}{s_j s_k} = \prod_{r+1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{\substack{j \neq k \\ r+1 \leq j, k \leq n}} \frac{1}{\sqrt{s_j s_k}} = \prod_{r+1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=r+1}^n s_l^{\gamma_2}, \quad (45)$$

$$\prod_{j=1}^r \prod_{k=r+1}^n \frac{s_j}{s_k} = \prod_{l=1}^r s_l^{\gamma_3} \prod_{l=r+1}^n s_l^{\gamma_4}.$$

To find γ_1 , we can interpret the product in (44) as a product over all elements of $r \times r$ matrix except for the diagonal. The terms with s_j appear along the j th row and j th column, so there are $2(r-1)$ of them. A similar argument can be used for the computation of γ_2 , but the size of the matrix would be $(n-r) \times (n-r)$. Keeping in mind that we still have square root and introducing the power $-\frac{1}{2}$, then we get

$$\gamma_1 = -\frac{2(r-1)}{2} = 1-r,$$

$$\gamma_2 = -\frac{2(n-r-1)}{2} = -n+r+1.$$

To compute γ_3 and γ_4 , we visualize the number of the s -factors using the following matrix:

$$\left\{ \begin{matrix} s_j \\ s_k \end{matrix} \right\}_{\substack{1 \leq j \leq r \\ r+1 \leq k \leq n}} = \begin{pmatrix} \frac{s_1}{s_{r+1}} & \frac{s_1}{s_{r+2}} & \frac{s_1}{s_{r+3}} & \dots \\ \frac{s_2}{s_{r+1}} & \frac{s_2}{s_{r+2}} & \frac{s_2}{s_{r+3}} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can observe that horizontally, for each s_j , there are $(n-r)$ factors. Vertically, for each s_k , there are r factors. We get

$$\gamma_3 = n-r,$$

$$\gamma_4 = -r.$$

As a result, we get the following preliminary asymptotic formula for $\Delta_n(x, \alpha)$.

Lemma 4.1. *The Toeplitz determinant (2) admits the following $x \rightarrow 0$, $x > 0$ asymptotics for fixed $d_1, d_2 \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$*

$$\Delta_n(x, \alpha) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_{r=0}^n \frac{c_1^r c_2^{n-r}}{r!(n-r)!} \left(\frac{2}{x}\right)^{-\alpha r + 2r(n-r) + \frac{\alpha n}{2}} (1 + \mathcal{O}(x^2))$$

$$\times \int_{\Gamma_1} \dots \int_{\Gamma_1} \prod_{1 \leq j < k \leq r} (s_j - s_k)^2 \prod_{l=1}^r e^{s_l} s_l^{-\frac{\alpha}{2} - 2r + n} ds_l \int_{\Gamma_2} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq n-r} (s_j - s_k)^2 \prod_{l=1}^{n-r} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2} - n} ds_l. \quad (46)$$

The coefficients c_1, c_2 are given by (43). The contours of integration Γ_1, Γ_2 are shown in Figure A2. We assume $-\pi < \arg(t) < \pi$ on the contour Γ_1 and $0 < \arg(t) < 2\pi$ on the contour Γ_2 .

4.4 | Asymptotics of $\Delta_n(x, \alpha)$ for $x \rightarrow 0, x > 0$

The asymptotics of $\Delta_n(x, \alpha)$ is the leading term of the asymptotic formula (46). Denote the power of x appearing in (46) as $p(r, \alpha, n)$:

$$p(r, \alpha, n) = \alpha r - \frac{n\alpha}{2} - 2r(n-r). \quad (47)$$

We need to find the minimum of $\text{Re}(p(r, \alpha, n))$ with respect to r . Introduce notation for the index that realizes this minimum

$$\min_{\substack{0 \leq r \leq n \\ r \in \mathbb{N} \cup \{0\}}} \text{Re}(p(r, \alpha, n)) = p(r_c(\alpha, n), \alpha, n). \quad (48)$$

We have the following formula for it.

Lemma 4.2. The critical index $r_c(\alpha, n)$ defined by (48) admits the following piecewise formula:

$$r_c(\alpha, n) = \begin{cases} 0 & \text{if } \text{Re}(\alpha) > 2n - 2 \\ j & \text{if } 2n - 4j - 2 < \text{Re}(\alpha) < 2n - 4j + 2 \\ & \text{and } j = 1, 2, \dots, n-1 \\ n & \text{if } \text{Re}(\alpha) < -2n + 2. \end{cases} \quad (49)$$

Proof. Since $0 \leq r \leq n$ and $r \in \mathbb{Z}$, $p(r, \alpha, n)$ only takes values on that discrete set. It is clear that $\text{Re}(p(r, \alpha, n))$ is an upward parabola in variable r and has a minimum value at $r_{\min}(n, \alpha) = \frac{2n - \text{Re}(\alpha)}{4}$. We will discuss different cases of relative positions between $r_{\min}(\alpha, n)$ and $r_c(\alpha, n)$. If $r_{\min}(\alpha, n) \leq 0$, then $r_c(\alpha, n) = 0$. If $r_{\min}(\alpha, n) \geq n$, then $r_c(\alpha, n) = n$. Let $0 \leq j \leq n$ and $j \leq r_{\min}(\alpha, n) \leq j+1$. If $j \leq r_{\min}(\alpha, n) < j + \frac{1}{2}$, then $r_c(\alpha, n) = j$. If $j + \frac{1}{2} < r_{\min}(\alpha, n) \leq j+1$, then $r_c(\alpha, n) = j+1$. In other words,

- $r_c(\alpha, n) = 0$ when $\frac{2n - \text{Re}(\alpha)}{4} < \frac{1}{2}$,
- $r_c(\alpha, n) = j$ when $j - \frac{1}{2} < \frac{2n - \text{Re}(\alpha)}{4} < j + \frac{1}{2}$ and $j = 1, 2, \dots, n-1$,
- $r_c(\alpha, n) = n$ when $\frac{2n - \text{Re}(\alpha)}{4} > n - \frac{1}{2}$.

These conditions can be rewritten as (49). \square

Remark 4.1. The floor function gives a more compact form for $r_c(\alpha, n)$. Indeed, $\forall 0 \leq j \leq n$, we know $r_c(\alpha, n) = j$ if and only if $j - \frac{1}{2} \leq \frac{n}{2} - \frac{\text{Re}(\alpha)}{4} \leq j + \frac{1}{2}$ or $j \leq \frac{n}{2} - \frac{\text{Re}(\alpha)}{4} + \frac{1}{2} \leq j+1$. Therefore,

$$r_c(\alpha, n) = \begin{cases} n & \text{if } \text{Re}(\alpha) < 2 - 2n, \\ \lfloor \frac{n}{2} - \frac{\text{Re}(\alpha)}{4} + \frac{1}{2} \rfloor & \text{if } 2 - 2n < \text{Re}(\alpha) < 2n - 2, \\ 0 & \text{if } \text{Re}(\alpha) > 2n - 2. \end{cases}$$

Remark 4.2. We consider case $\alpha \in \mathbb{C} \setminus (2\mathbb{Z} + i\mathbb{R})$ to make sure that leading contribution in the asymptotics comes from the one value of r_c . Otherwise we need to include two contributions in the leading term of asymptotics of $\Delta_n(x, \alpha)$, which produces qualitatively oscillating solutions.

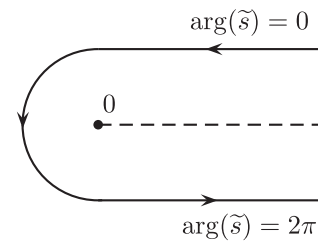


FIGURE 3 | Contour $\widetilde{\Gamma}_1$.

As the result, we have

$$\Delta_n(x, \alpha) \sim \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \frac{c_1^{r_c} c_2^{n-r_c}}{r_c! (n-r_c)!} \left(\frac{x}{2}\right)^{p(r_c, \alpha, n)} \int_{\Gamma_1} \dots \int_{\Gamma_1} \prod_{1 \leq j < k \leq r_c} (s_j - s_k)^2 \prod_{l=1}^{r_c} e^{s_l} s_l^{-\frac{\alpha}{2} - 2r_c + n} ds_l \times \int_{\Gamma_2} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq n-r_c} (s_j - s_k)^2 \prod_{l=1}^{n-r_c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2} - n} ds_l, \quad \text{as } x \rightarrow 0, \quad x > 0.$$

Denote

$$H_1(\alpha, n) = \int_{\Gamma_1} \dots \int_{\Gamma_1} \prod_{1 \leq j < k \leq r_c} (s_j - s_k)^2 \prod_{l=1}^{r_c} e^{s_l} s_l^{-\frac{\alpha}{2} - 2r_c + n} ds_l \quad (50)$$

and

$$H_2(\alpha, n) = \int_{\Gamma_2} \dots \int_{\Gamma_2} \prod_{1 \leq j < k \leq n-r_c} (s_j - s_k)^2 \prod_{l=1}^{n-r_c} e^{-\frac{1}{s_l}} s_l^{-\frac{\alpha}{2} - n} ds_l. \quad (51)$$

To evaluate (50), (51), we reduce them to multiple integrals with Laguerre weight $w(x) = e^{-x} x^\alpha$, $\text{Re}(\alpha) > -1$ on the contour $\Gamma = [0, \infty)$. In $H_1(\alpha, n)$, we make the change of variable $s = \widetilde{s} e^{-i\pi}$. More specifically, the modulus and argument of the variable transform as

$$|s| = |\widetilde{s}|$$

$$\arg(s) = \arg(\widetilde{s}) - \pi, \quad 0 < \arg(\widetilde{s}) < 2\pi.$$

The contour Γ_1 becomes $\widetilde{\Gamma}_1$ as shown in Figure 3.

Also notice that

$$\begin{aligned} s_l^{-\frac{\alpha}{2} - 2r_c + n} &= e^{(-\frac{\alpha}{2} - 2r_c + n) \ln |s_l| + (-\frac{\alpha}{2} - 2r_c + n) i \arg(s_l)} \\ &= e^{(-\frac{\alpha}{2} - 2r_c + n) \ln |\widetilde{s}_l| + (-\frac{\alpha}{2} - 2r_c + n) i (\arg(\widetilde{s}_l) - \pi)} \\ &= \widetilde{s}_l^{-\frac{\alpha}{2} - 2r_c + n} e^{-i\pi(-\frac{\alpha}{2} - 2r_c + n)}. \end{aligned}$$

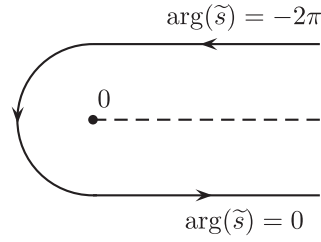


FIGURE 4 | Contour $\widetilde{\Gamma}_2$.

As a result, we have

$$\begin{aligned} H_1(\alpha, n) &= \int_{\widetilde{\Gamma}_1} \dots \int_{\widetilde{\Gamma}_1} \prod_{1 \leq j < k \leq r_c} (\widetilde{s}_j - \widetilde{s}_k)^2 \prod_{l=1}^{r_c} e^{-\widetilde{s}_l} \widetilde{s}_l^{-\frac{\alpha}{2} - 2r_c + n} e^{-i\pi(-\frac{\alpha}{2} - 2r_c + n)} (-1) d\widetilde{s}_l \\ &= (-1)^{r_c} (e^{-i\pi(-\frac{\alpha}{2} - 2r_c + n)})^{r_c} \int_{\widetilde{\Gamma}_1} \dots \int_{\widetilde{\Gamma}_1} \prod_{1 \leq j < k \leq r_c} (\widetilde{s}_j - \widetilde{s}_k)^2 \prod_{l=1}^{r_c} e^{-\widetilde{s}_l} \widetilde{s}_l^{-\frac{\alpha}{2} - 2r_c + n} d\widetilde{s}_l. \end{aligned}$$

To continue our evaluation, we need the following lemma.

Lemma 4.3. *Using the properties of power function, we can show the following identity:*

$$\int_{\widetilde{\Gamma}_1} s^j e^{-s} s^\gamma ds = (e^{2\pi i \gamma} - 1) \int_0^\infty s^j e^{-s} s^\gamma ds, \quad j \in \mathbb{Z}, \quad j + \operatorname{Re}(\gamma) > -1. \quad (52)$$

With the aid of (52), we can rewrite our contour integrals in terms of real line integrals under the convergence condition $\operatorname{Re}(\alpha) < 2n + 2 - 4r_c$.

$$\begin{aligned} H_1(\alpha, n) &= (-1)^{r_c} (e^{-i\pi(-\frac{\alpha}{2} - 2r_c + n)})^{r_c} (e^{2\pi i(-\frac{\alpha}{2} - 2r_c + n)} - 1)^{r_c} \\ &\quad \times \int_0^\infty \dots \int_0^\infty \prod_{1 \leq j < k \leq r_c} (\widetilde{s}_j - \widetilde{s}_k)^2 \prod_{l=1}^{r_c} e^{-\widetilde{s}_l} \widetilde{s}_l^{-\frac{\alpha}{2} - 2r_c + n} d\widetilde{s}_l. \end{aligned}$$

By using [21, (5.14.5)], we get for $\operatorname{Re}(\alpha) < 2n + 2 - 4r_c$

$$H_1(\alpha, n) = (-1)^{nr_c} (2i)^{r_c} \sin^{r_c} \left(\frac{\pi \alpha}{2} \right) \frac{G\left(1 + n - r_c - \frac{\alpha}{2}\right) G(r_c + 2)}{G\left(1 - \frac{\alpha}{2} - 2r_c + n\right)}. \quad (53)$$

We can notice that left- and right-hand sides of (53) are entire functions of α . Therefore, by the uniqueness of analytic continuation, we can say that (53) holds for all $\alpha \in \mathbb{C}$.

Similarly, in $H_2(\alpha, n)$, we put $s = \frac{1}{\widetilde{s}}$. Then, the modulus and argument of the variable respectively transform as

$$\begin{aligned} |s| &= \frac{1}{|\widetilde{s}|}, \\ \arg(\widetilde{s}) &= -\arg(s), \quad -2\pi < \arg(\widetilde{s}) < 0. \end{aligned}$$

The contour Γ_2 becomes $\widetilde{\Gamma}_2$ as shown in Figure 4.

The second multi-integral becomes

$$H_2(\alpha, n) = \int_{\widetilde{\Gamma}_2} \dots \int_{\widetilde{\Gamma}_2} \prod_{1 \leq j < k \leq n - r_c} \left(\frac{1}{\widetilde{s}_j} - \frac{1}{\widetilde{s}_k} \right)^2 \prod_{l=1}^{n - r_c} e^{-\widetilde{s}_l} \widetilde{s}_l^{\frac{\alpha}{2} + n} \left(-\frac{1}{\widetilde{s}_l^2} \right) d\widetilde{s}_l$$

$$\begin{aligned}
 &= (-1)^{n-r_c} \int_{\tilde{\Gamma}_2} \dots \int_{\tilde{\Gamma}_2} \prod_{1 \leq j < k \leq n-r_c} \left(\frac{\tilde{s}_j - \tilde{s}_k}{\tilde{s}_j \tilde{s}_k} \right)^2 \prod_{l=1}^{n-r_c} e^{-\tilde{s}_l} \tilde{s}_l^{\frac{\alpha}{2} + n - 2} d\tilde{s}_l \\
 &= (-1)^{n-r_c} \int_{\tilde{\Gamma}_2} \dots \int_{\tilde{\Gamma}_2} \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{l=1}^{n-r_c} \tilde{s}_l^{2r_c + 2 - 2n} e^{-\tilde{s}_l} \tilde{s}_l^{\frac{\alpha}{2} + n - 2} d\tilde{s}_l \\
 &= (-1)^{n-r_c} \int_{\tilde{\Gamma}_2} \dots \int_{\tilde{\Gamma}_2} \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{l=1}^{n-r_c} e^{-\tilde{s}_l} \tilde{s}_l^{\frac{\alpha}{2} + 2r_c - n} d\tilde{s}_l.
 \end{aligned}$$

Here, we used the square of (45)

$$\prod_{1 \leq j < k \leq n-r_c} \left(\frac{\tilde{s}_j - \tilde{s}_k}{\tilde{s}_j \tilde{s}_k} \right)^2 = \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{\substack{j \neq k \\ 1 \leq j, k \leq n-r_c}} \frac{1}{\tilde{s}_j \tilde{s}_k} = \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{l=1}^{n-r_c} \tilde{s}_l^{2r_c + 2 - 2n}.$$

To continue our evaluation, we need the following analog of Lemma 4.3.

Lemma 4.4. *Using the properties of power function, we can show the following identity:*

$$\int_{\tilde{\Gamma}_2} s^j e^{-s} s^\gamma ds = (1 - e^{-2\pi i \gamma}) \int_0^\infty s^j e^{-s} s^\gamma ds, \quad j \in \mathbb{Z}, \quad j + \operatorname{Re}(\gamma) > -1. \quad (54)$$

With the aid of (54), we can rewrite our contour integrals in terms of real line integrals under the convergence condition $\operatorname{Re}(\alpha) > 2n - 2 - 4r_c$

$$\begin{aligned}
 H_2(\alpha, n) &= (-1)^{n-r_c} \int_{\tilde{\Gamma}_2} \dots \int_{\tilde{\Gamma}_2} \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{l=1}^{n-r_c} e^{-\tilde{s}_l} \tilde{s}_l^{\frac{\alpha}{2} + 2r_c - n} d\tilde{s}_l \\
 &= (-1)^{n-r_c} (1 - e^{-2\pi i (\frac{\alpha}{2} + 2r_c - n)})^{n-r_c} \int_0^\infty \dots \int_0^\infty \prod_{1 \leq j < k \leq n-r_c} (\tilde{s}_j - \tilde{s}_k)^2 \prod_{l=1}^{n-r_c} e^{-\tilde{s}_l} \tilde{s}_l^{\frac{\alpha}{2} + 2r_c - n} d\tilde{s}_l.
 \end{aligned}$$

Using [21, (5.14.5)], we arrive to the formula

$$H_2(\alpha, n) = (-1)^{n-r_c} e^{-i\pi(n-r_c)\frac{\alpha}{2}} (2i)^{n-r_c} \sin^{n-r_c} \left(\frac{\pi\alpha}{2} \right) \frac{G(\frac{\alpha}{2} + r_c + 1) G(n + 2 - r_c)}{G(\frac{\alpha}{2} + 2r_c - n + 1)}. \quad (55)$$

We repeat the uniqueness of analytic continuation argument above to claim that formula (55) holds for all $\alpha \in \mathbb{C}$. Combining (53), (55), (46), and the definitions (50), (51), we get

$$\begin{aligned}
 \Delta_n(x, \alpha) &\sim \frac{(-1)^{\frac{n(n-1)}{2}}}{\pi^n} c_1^{r_c} c_2^{n-r_c} \left(\frac{x}{2} \right)^{p(r_c, \alpha, n)} (-1)^{n-r_c + nr_c} e^{-i\pi(n-r_c)\frac{\alpha}{2}} \sin^n \left(\frac{\pi\alpha}{2} \right) \\
 &\frac{G\left(\frac{\alpha}{2} + r_c + 1\right) G(n + 1 - r_c) G\left(1 + n - r_c - \frac{\alpha}{2}\right) G(r_c + 1)}{G\left(\frac{\alpha}{2} + 2r_c - n + 1\right) G\left(1 - \frac{\alpha}{2} - 2r_c + n\right)}, \quad \text{as } x \rightarrow 0, \quad x > 0.
 \end{aligned}$$

Using definitions (47), (43) and formula (49), we finish the proof of Theorem 1.1 for $x > 0$.

4.5 | Proof of Theorem 1.1

We observe that the asymptotics $x \rightarrow 0, x > 0$ of the Bessel function given by (A2) holds for $-\pi < \arg(x) < \pi$. The same is true for the cylinder function $C_\nu(x)$. Since the Toeplitz determinant $\Delta_n(x, \alpha)$ is the linear combination of products of cylinder functions, its asymptotic formula is obtained by inserting (A2) and computing the leading term. As a result, our formula obtained initially for $x > 0$ is also valid for the entire sector $-\pi < \arg(x) < \pi$.

If we try to use the multiple contour integral representation (41) to get asymptotics for complex values of x , we would notice that it is valid only for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. To follow this path further, we would need to use the analytic continuation formulas presented in the Appendix A.3. We checked that this computation confirms Theorem 1.1 and we present it in Appendix B.1.

5 | Asymptotics of Special Function Solutions at Zero

To deduce the asymptotics of $u_n(x, \alpha)$, we need to use our main result Theorem 1.1 and Proposition 1.1. We have to shift the index of the Toeplitz determinant n and the parameter α .

5.1 | Piecewise Function for the Power of x in the Asymptotic of $u_n(x, \alpha)$

Let us introduce notation for the power of x in the asymptotic of $\Delta_n(x, \alpha)$ in Theorem 1.1

$$p_c(\alpha, n) = p(r_c(\alpha, n), \alpha, n) = \begin{cases} -\frac{\alpha n}{2}, & \text{if } \operatorname{Re}(\alpha) > 2n - 2, \\ (\alpha - 2n + 2j)j - \frac{\alpha n}{2}, & \text{if } 2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j + 2, \\ & \text{for } j = 1, \dots, n - 1 \\ \frac{\alpha n}{2}, & \text{if } \operatorname{Re}(\alpha) < -2n + 2. \end{cases} \quad (56)$$

Then, the power of x in the asymptotic of $u_n(x, \alpha)$ based on Proposition 1.1 is given by

$$e(\alpha, n) = p_c(\alpha - 2, n + 1) - p_c(\alpha - 2, n) + p_c(\alpha, n) - p_c(\alpha, n + 1). \quad (57)$$

Lemma 5.1. *The piecewise function for the power of x in the asymptotic of $u_n(x, \alpha)$ is given by*

$$e(\alpha, n) = \begin{cases} 1 & \text{if } \operatorname{Re}(\alpha) > 2 + 2n \\ \alpha - 2n + 4j - 1 & \text{if } 2n - 4j < \operatorname{Re}(\alpha) < 2n - 4j + 2 \text{ and } j = 0, 1, \dots, n \\ -\alpha + 2n - 4j - 1 & \text{if } 2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j \text{ and } j = 0, 1, \dots, n - 1 \\ -1 & \text{if } \operatorname{Re}(\alpha) < -2n. \end{cases} \quad (58)$$

Proof. We plug in formula (56) in the expression (57).

If $\operatorname{Re}(\alpha) > 2 + 2n$, then we have

$$\begin{aligned} p_c(\alpha, n) &= -\frac{\alpha n}{2}, & p_c(\alpha, n + 1) &= -\frac{\alpha n}{2} - \frac{\alpha}{2}, \\ p_c(\alpha - 2, n) &= -\frac{\alpha n}{2} + n, & p_c(\alpha - 2, n + 1) &= -\frac{\alpha n}{2} + n + 1 - \frac{\alpha}{2}. \end{aligned}$$

That confirms the first case. Similarly, if $2n - 4j < \operatorname{Re}(\alpha) < 2n - 4j + 2$ and $j = 0, 1, \dots, n$, then we have

$$\begin{aligned} p_c(\alpha, n) &= (\alpha - 2n + 2j)j - \frac{\alpha n}{2}, \\ p_c(\alpha, n + 1) &= (\alpha - 2n + 2j)j - 2j - \frac{\alpha n}{2} - \frac{\alpha}{2}, \\ p_c(\alpha - 2, n) &= (\alpha - 2n + 2j)j - 2j - \frac{\alpha n}{2} + n, \\ p_c(\alpha - 2, n + 1) &= (\alpha - 2n + 2j)j - \frac{\alpha n}{2} - n - 1 + \frac{\alpha}{2}. \end{aligned}$$

Notice that for $p_c(\alpha - 2, n + 1)$, we had to shift the index j to $j + 1$ in (56) to get the correct formula. We confirmed the second case. If $2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j$ and $j = 0, 1, \dots, n - 1$, then we have

$$\begin{aligned} p_c(\alpha, n) &= (\alpha - 2n + 2j)j - \frac{\alpha n}{2}, \\ p_c(\alpha, n + 1) &= (\alpha - 2n + 2j)j - 2n + 2j - \frac{\alpha n}{2} + \frac{\alpha}{2}, \\ p_c(\alpha - 2, n) &= (\alpha - 2n + 2j)j + \alpha - n + 2j - \frac{\alpha n}{2}, \end{aligned}$$

$$p_c(\alpha - 2, n + 1) = (\alpha - 2n + 2j)j - \frac{\alpha n}{2} - n - 1 + \frac{\alpha}{2}.$$

Notice that now we had to change the index j to $j + 1$ in (56) for $p_c(\alpha - 2, n + 1)$, $p_c(\alpha, n + 1)$, and $p_c(\alpha - 2, n)$ to obtain the correct formula. We confirmed the third case. Finally, if $\text{Re}(\alpha) < -2n$, then all terms in (57) change sign compared to the case $\text{Re}(\alpha) > 2n - 2$. That confirms the last case and finishes the proof of Lemma 5.1. \square

5.2 | Proof of Theorem 1.2

In this part, we will compute the coefficients in the asymptotics of Theorem 1.2. We introduce the notation for the coefficient in the asymptotics of $\Delta_n(x, \alpha)$ in Theorem 1.1. We included factor of 2 inside of power of x for convenience

$$c(\alpha, n) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)}, & \text{if } \text{Re}(\alpha) > 2n - 2, \\ (-1)^{\frac{n(n-1)}{2} + nj + n - j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j & \text{if } 2n - 4j - 2 < \text{Re}(\alpha) < 2n - 4j + 2, \\ \times \frac{G(n-j+1)G(-\frac{\alpha}{2}+n-j+1)G(j+1)G(j+\frac{\alpha}{2}+1)}{G(-\frac{\alpha}{2}+n-2j+1)G(\frac{\alpha}{2}-n+2j+1)}, & \text{for } j = 1, \dots, n - 1 \\ (-1)^{\frac{n(n-1)}{2} + n^2} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n & \text{if } \text{Re}(\alpha) < -2n + 2 \\ \times \frac{G(n+1)G(-\frac{\alpha}{2}+1)}{G(-\frac{\alpha}{2}-n+1)}. \end{cases} \quad (59)$$

Let us denote the constant coefficient in the asymptotic of $u_n(x, \alpha)$ as $q(\alpha, n)$. We have

$$q(\alpha, n) = -\frac{c(\alpha - 2, n + 1)c(\alpha, n)}{c(\alpha - 2, n)c(\alpha, n + 1)}.$$

Lemma 5.2. The piecewise formula for $q(\alpha, n)$ is given by

$$q(\alpha, n) = \begin{cases} \frac{2}{2n+2-\alpha}, & \text{if } \text{Re}(\alpha) > 2n + 2, \\ (-1)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right) \left(\frac{\Gamma(-\frac{\alpha}{2}+n-2j+1)}{\Gamma(\frac{\alpha}{2}-n+2j)}\right)^2 & \text{if } 2n - 4j < \text{Re}(\alpha) < 2n - 4j + 2, \\ \times \frac{\Gamma(j+\frac{\alpha}{2})\Gamma(j+1)}{\Gamma(-\frac{\alpha}{2}+n-j+1)\Gamma(n-j+1)}, & \text{for } j = 0, \dots, n \\ (-1)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^{-1} \left(\frac{\Gamma(\frac{\alpha}{2}-n+2j+1)}{\Gamma(-\frac{\alpha}{2}+n-2j)}\right)^2 & \text{if } 2n - 4j - 2 < \text{Re}(\alpha) < 2n - 4j, \\ \times \frac{\Gamma(-\frac{\alpha}{2}+n-j+1)\Gamma(n-j)}{\Gamma(j+\frac{\alpha}{2}+1)\Gamma(j+1)}, & \text{for } j = 0, \dots, n - 1 \\ -\frac{\alpha}{2} - n & \text{if } \text{Re}(\alpha) < -2n. \end{cases}$$

Proof. For $\text{Re}(\alpha) > 2n + 2$, we have

$$\begin{aligned} c(\alpha, n) &= (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)}, \\ c(\alpha - 2, n) &= (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)} \frac{\Gamma(\frac{\alpha}{2}-n)}{\Gamma(\frac{\alpha}{2})}, \\ c(\alpha, n + 1) &= (-1)^{\frac{(n+2)(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^{n+1} \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)} \Gamma(n+1) \Gamma\left(\frac{\alpha}{2}-n\right), \\ c(\alpha - 2, n + 1) &= (-1)^{\frac{(n+2)(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^{n+1} \frac{G(n+1)G(\frac{\alpha}{2}+1)}{G(\frac{\alpha}{2}-n+1)} \frac{\Gamma(n+1) \Gamma\left(\frac{\alpha}{2}-n\right) \Gamma\left(\frac{\alpha}{2}-n-1\right)}{\Gamma\left(\frac{\alpha}{2}\right)}, \end{aligned}$$

That implies

$$q(\alpha, n) = -\frac{\Gamma\left(\frac{\alpha}{2} - n - 1\right)}{\Gamma\left(\frac{\alpha}{2} - n\right)} = \frac{2}{2n + 2 - \alpha}.$$

That confirms the first case.

For $2n - 4j < \operatorname{Re}(\alpha) < 2n - 4j + 2$, we have

$$\begin{aligned} c(\alpha, n) &= (-1)^{\frac{n(n-1)}{2} + nj + n - j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)}, \\ c(\alpha - 2, n) &= (-1)^{\frac{n(n-1)}{2} + nj + n} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)\Gamma\left(-\frac{\alpha}{2} + n - j + 1\right)\Gamma\left(\frac{\alpha}{2} - n + 2j\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)\Gamma\left(j + \frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2} + n - 2j + 1\right)}, \\ c(\alpha, n + 1) &= (-1)^{\frac{n(n+1)}{2} + nj + n + 1} \left(\frac{d_2}{\pi}\right)^{n+1} \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)\Gamma\left(-\frac{\alpha}{2} + n - j + 1\right)\Gamma(n-j+1)\Gamma\left(\frac{\alpha}{2} - n + 2j\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)\Gamma\left(-\frac{\alpha}{2} + n - 2j + 1\right)}, \\ c(\alpha - 2, n + 1) &= (-1)^{\frac{n(n+1)}{2} + nj + j + 2n + 2} \left(\frac{d_2}{\pi}\right)^{n+1} \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^{j+1} \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)}\Gamma(j+1)\Gamma\left(-\frac{\alpha}{2} + n - j + 1\right). \end{aligned}$$

Combining these formulas, we confirm the second case. Notice that for $c(\alpha - 2, n + 1)$, we had to shift index j to $j + 1$ in (59) to get correct formula.

For $2n - 4j - 2 < \operatorname{Re}(\alpha) < 2n - 4j$, we have

$$\begin{aligned} c(\alpha, n) &= (-1)^{\frac{n(n-1)}{2} + nj + n - j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)}, \\ c(\alpha - 2, n) &= (-1)^{\frac{n(n-1)}{2} + nj + 2n} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^{j+1} \\ &\quad \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)\Gamma(j+1)\Gamma\left(-\frac{\alpha}{2} + n - 2j\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)\Gamma(n-j)\Gamma\left(\frac{\alpha}{2} - n + 2j + 1\right)}, \\ c(\alpha, n + 1) &= (-1)^{\frac{n(n+1)}{2} + nj + 2n + 1} \left(\frac{d_2}{\pi}\right)^{n+1} \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^{j+1} \end{aligned}$$

$$\begin{aligned} & \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2}+n-j+1\right)G(j+1)G\left(j+\frac{\alpha}{2}+1\right)\Gamma(j+1)\Gamma\left(j+\frac{\alpha}{2}+1\right)\Gamma\left(-\frac{\alpha}{2}+n-2j\right)}{G\left(-\frac{\alpha}{2}+n-2j+1\right)G\left(\frac{\alpha}{2}-n+2j+1\right)\Gamma\left(\frac{\alpha}{2}-n+2j+1\right)}, \\ c(\alpha-2, n+1) &= (-1)^{\frac{n(n+1)}{2}+nj+2n+j+2} \left(\frac{d_2}{\pi}\right)^{n+1} \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^{j+1} \\ & \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2}+n-j+1\right)G(j+1)G\left(j+\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}+n-2j+1\right)G\left(\frac{\alpha}{2}-n+2j+1\right)} \Gamma(j+1)\Gamma\left(-\frac{\alpha}{2}+n-j+1\right). \end{aligned}$$

That confirms the third case. Notice that now we had to shift index j to $j+1$ in (59) for $c(\alpha-2, n+1)$, $c(\alpha, n+1)$, and $c(\alpha-2, n)$ to get correct formula.

Finally, for $\text{Re}(\alpha) < -2n$, we have

$$\begin{aligned} c(\alpha, n) &= \frac{(-1)^{\frac{n(n-1)}{2}+n^2}}{\pi^n} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)}, \\ c(\alpha-2, n) &= \frac{(-1)^{\frac{n(n-1)}{2}+n^2+n}}{\pi^n} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \frac{\Gamma\left(-\frac{\alpha}{2}+1\right)}{\Gamma\left(-\frac{\alpha}{2}-n+1\right)}, \\ c(\alpha, n+1) &= \frac{(-1)^{\frac{(n+1)n}{2}+(n+1)^2}}{\pi^{n+1}} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^{n+1} \\ & \times \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \Gamma(n+1)\Gamma\left(-\frac{\alpha}{2}-n\right), \\ c(\alpha-2, n+1) &= \frac{(-1)^{\frac{(n+1)n}{2}+n+1+(n+1)^2}}{\pi^{n+1}} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^{n+1} \\ & \times \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \Gamma(n+1)\Gamma\left(-\frac{\alpha}{2}+1\right). \end{aligned}$$

That implies

$$q(\alpha, n) = \frac{\Gamma\left(-\frac{\alpha}{2}-n+1\right)}{\Gamma\left(-\frac{\alpha}{2}-n\right)} = -\frac{\alpha}{2} - n.$$

That confirms the last case. □

6 | Asymptotics at Infinity

6.1 | Asymptotics of Toeplitz Determinant

To compute the asymptotics at infinity, it is convenient to use the Hankel functions instead of the Bessel functions. We rewrite the formula (3) as (A9). Following the same argument as in Theorem 4.1, we get the following multiple integral representation using the Andréief identity.

Theorem 6.1. *The Toeplitz determinant $\Delta_n(x, \alpha)$ given by (2) can be rewritten as*

$$\Delta_n(x, \alpha) = \frac{1}{n!} \int_{\Gamma_3 \cup \Gamma_4} \dots \int_{\Gamma_3 \cup \Gamma_4} \prod_{1 \leq j < k \leq n} (t_k - t_j) \left(\frac{1}{t_k} - \frac{1}{t_j}\right) \prod_{k=1}^n h_2(t_k) dt_k, \quad (60)$$

where

$$h_2(t) = \frac{e^{\frac{x}{2}(t - \frac{1}{t})}}{2\pi i t^{1+\frac{\alpha}{2}}} ((d_1 - id_2)\chi_{\Gamma_3}(t) - (d_1 + id_2)\chi_{\Gamma_4}(t)),$$

and contours of integration Γ_3, Γ_4 are shown in Figure A4. We use the notation $\chi_{\Gamma_j}(t)$ for the characteristic function of contour Γ_j . We assume $-\pi < \arg(t) < \pi$ on the contours Γ_3 and Γ_4 .

We would like to compute the large x asymptotics of $\Delta_n(x, \alpha)$. We start by using the expansion (42) and implementing notation (A10)

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \int_{\Gamma_3 \cup \Gamma_4} \dots \int_{\Gamma_3 \cup \Gamma_4} \prod_{m=1}^n (b_1 \chi_{\Gamma_3}(t_m) - b_2 \chi_{\Gamma_4}(t_m)) \prod_{1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{\pi i t_l^{\frac{\alpha}{2}+1}} dt_l \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{(\pi i)^n n!} \sum_{r=0}^n \sum_{\substack{I \subset \{1, \dots, n\} \\ |I|=r}} b_1^r (-b_2)^{n-r} \int_{\Gamma_3 \cup \Gamma_4} \dots \int_{\Gamma_3 \cup \Gamma_4} \prod_{j < k} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{j < k} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{\substack{j \in I \\ k \in I^c}} \frac{(t_j - t_k)^2}{t_j t_k} \\ &\quad \prod_{l \in I} \chi_{\Gamma_3}(t_l) \prod_{l \in I^c} \chi_{\Gamma_4}(t_l) \prod_{l \in I} \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}} dt_l \prod_{l \in I^c} \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}} dt_l. \end{aligned}$$

We can see that by renaming variables on the right-hand side, we can guarantee that $I = \{1, 2, \dots, r\}$, and $I^c = \{r+1, r+2, \dots, n\}$ for each integral in the sum. Combining the same integrals together, we get

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(-1)^{\frac{n(n-1)}{2}}}{(\pi i)^n} \sum_{r=0}^n \frac{b_1^r (-b_2)^{n-r}}{r!(n-r)!} \int_{\Gamma_3} \dots \int_{\Gamma_3} \int_{\Gamma_4} \dots \int_{\Gamma_4} \prod_{1 \leq j < k \leq r} \frac{(t_j - t_k)^2}{t_j t_k} \prod_{r+1 \leq j < k \leq n} \frac{(t_j - t_k)^2}{t_j t_k} \\ &\quad \times \prod_{j=1}^r \prod_{k=r+1}^n \frac{(t_j - t_k)^2}{t_j t_k} \prod_{l=1}^n \frac{e^{\frac{x}{2}(t_l - \frac{1}{t_l})}}{t_l^{\frac{\alpha}{2}+1}} dt_l, \end{aligned} \quad (61)$$

where we have r integrals over Γ_3 and $n-r$ integrals over Γ_4 .

We would like to compute asymptotics of multiple integrals in (61) using the steepest descent method for $x \rightarrow \infty, x > 0$. The critical points of the exponent $\Xi(t) = t - \frac{1}{t}$ are $t = \pm i$. We chose the contours Γ_3 and Γ_4 as the contours of the steepest descent $\text{Im}(\Xi(t)) = \text{Im}(\Xi(\pm i))$. They can be described using cubic equations $(\text{Re } t)^2 (\text{Im } t \pm 2) + \text{Im } t (\text{Im } t \pm 1)^2 = 0$. The main contribution to the $x \rightarrow \infty$ asymptotics comes from the neighborhoods of the critical points. We make a change of variable in the local integrals: $(t_l - i) = s_l e^{\frac{3\pi i}{4}}$ for $l \in I$ and $(t_l + i) = s_l e^{-\frac{3\pi i}{4}}$ for $l \in I^c$. After that, we replace the local integrals with real line integrals.

The result is the following:

$$\begin{aligned} \Delta_n(x, \alpha) &\sim \sum_{r=0}^n \frac{b_1^r b_2^{n-r}}{r!(n-r)!} \frac{4^{r(n-r)}}{\pi^n} e^{\frac{i\pi}{4}(n^2-4r+6nr-4r^2)} e^{\frac{i\pi\alpha}{4}(n-2r)} e^{ix(2r-n)} \\ &\quad \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq r} (s_j - s_k)^2 \prod_{l=1}^r e^{-\frac{x}{2}s_l^2} ds_l \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq n-r} (s_j - s_k)^2 \prod_{l=1}^{n-r} e^{-\frac{x}{2}s_l^2} ds_l, \quad x \rightarrow \infty, \quad x > 0. \end{aligned}$$

We make the change of variables in the integrals $s \rightarrow \frac{s}{\sqrt{x}}$ and evaluate the resulting integrals using standard formula [21, (5.14.6)] rewritten in terms of Barnes G -function.

$$\begin{aligned} \Delta_n(x, \alpha) &\sim \sum_{r=0}^n b_1^r b_2^{n-r} \left(\frac{2}{\pi} \right)^{\frac{n}{2}} 4^{r(n-r)} e^{\frac{i\pi}{4}(n^2-4r+6nr-4r^2)} e^{\frac{i\pi\alpha}{4}(n-2r)} \\ &\quad \times G(r+1)G(n-r+1)e^{i(2r-n)x} x^{-r^2+nr-\frac{n^2}{2}}, \quad x \rightarrow \infty, \quad x > 0. \end{aligned} \quad (62)$$

We notice that this result can be obtained by plugging in asymptotic series of Hankel functions (A6), (A7) in the Toeplitz determinant $\Delta_n(x, \alpha)$. As a result, we claim that (62) holds for $-\pi < \arg(x) < \pi$.

We observe that the multiple contour integral representation (60) holds only for $-\frac{\pi}{2} < \arg x < \frac{\pi}{2}$. We could use the analytic continuation formulas presented in Appendix A.3 to extend the asymptotics to other values of $\arg(x)$. We confirmed that this computation produces the same result and present it in Appendix B.2.

To find the leading term for $x > 0$, we observe that

$$\max_{0 \leq r \leq n} \left(-r^2 + nr - \frac{n^2}{2} \right) = \begin{cases} -\frac{n^2}{4} & \text{when } n\text{-even using } r = \frac{n}{2}, \\ -\frac{n^2}{4} - \frac{1}{4} & \text{when } n\text{-odd using } r = \frac{n \pm 1}{2}. \end{cases}$$

For even n , we plug in $r = \frac{n}{2}$. We combine factorials with Barnes G -function. Since for even n , the number $\frac{3n^2-2n}{4}$ is even, we get

$$\Delta_n(x, \alpha) \sim \left(\frac{d_1^2 + d_2^2}{2\pi} \right)^{\frac{n}{2}} \left(G\left(\frac{n}{2} + 1\right) \right)^2 \left(\frac{x}{4} \right)^{-\frac{n^2}{4}}, \quad x \rightarrow \infty, x > 0.$$

For odd n , we need to combine contributions from $r = \frac{n-1}{2}$ and $r = \frac{n+1}{2}$.

$$\begin{aligned} \Delta_n(x, \alpha) &\sim \left(\frac{d_1^2 + d_2^2}{2\pi} \right)^{\frac{n}{2}} G\left(\frac{n+1}{2}\right) G\left(\frac{n+3}{2}\right) \left(\frac{x}{4} \right)^{-\frac{n^2+1}{4}} e^{\frac{i\pi}{4}(3n^2-2n-1)} \\ &\quad \times \frac{1}{2} \left(e^{\frac{i\pi}{4}(\alpha+2-n)-ix+i\phi} + e^{\frac{i\pi}{4}(-\alpha-2+n)+ix-i\phi} \right) \\ &= \left(\frac{d_1^2 + d_2^2}{2\pi} \right)^{\frac{n}{2}} G\left(\frac{n+1}{2}\right) G\left(\frac{n+3}{2}\right) \left(\frac{x}{4} \right)^{-\frac{n^2+1}{4}} (-1)^{\frac{n-1}{2}} \sin\left(x - \phi + \frac{\pi}{4}(n - \alpha)\right), \quad x \rightarrow \infty, x > 0, \end{aligned}$$

where $\phi = \frac{1}{2i} \ln(d_1 + id_2) - \frac{1}{2i} \ln(d_1 - id_2)$ and $(d_1^2 + d_2^2)^{\frac{n}{2}} = e^{\frac{n}{2} \ln(d_1 + id_2) + \frac{n}{2} \ln(d_1 - id_2)}$. We see that the expression for asymptotics does not depend on the choice of the branch of the logarithm. For real values of d_1 and d_2 , we have $\phi = \arg(d_1 + id_2)$.

For $-\pi < \arg(x) < 0$, the leading term is given by $r = n$:

$$\Delta_n(x, \alpha) \sim (d_1 - id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{-\frac{i\pi n^2}{4}} G(n+1) e^{-\frac{i\pi n\alpha}{4}} e^{inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty. \quad (63)$$

We notice that if $d_1 + id_2 = 0$, then (63) holds for $-\pi < \arg(x) < \pi$.

For $0 < \arg(x) < \pi$, the leading term is given by $r = 0$:

$$\Delta_n(x, \alpha) \sim (d_1 + id_2)^n \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} e^{\frac{i\pi n^2}{4}} G(n+1) e^{\frac{i\pi n\alpha}{4}} e^{-inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty. \quad (64)$$

We notice that if $d_1 - id_2 = 0$, then (64) holds for $-\pi < \arg(x) < \pi$.

As the result, we get Theorem 1.3.

6.2 | Asymptotics of Special Function Solutions

In this part, we still need to use Proposition 1.1 and Theorem 1.3 to get the large x asymptotics of $u_n(x, \alpha)$. We start with the case $x > 0$. Fortunately, life becomes much easier in this scenario. Notice that when n is even, the Toeplitz determinant does not depend on α at all. Hence, when shifting the indices, we have for even n :

$$u_n(x, \alpha) = -\frac{\Delta_{n+1}(x, \alpha - 2)}{\Delta_{n+1}(x, \alpha)} \sim -\frac{\sin\left(x - \phi + \frac{\pi}{4}(n+1-\alpha) + \frac{\pi}{2}\right)}{\sin\left(x - \phi + \frac{\pi}{4}(n+1-\alpha)\right)} = -\cot\left(x - \phi + \frac{\pi}{4}(n+1-\alpha)\right).$$

Similarly, when n is odd and $n + 1$ is even, we get

$$u_n(x, \alpha) = -\frac{\Delta_n(x, \alpha)}{\Delta_n(x, \alpha - 2)} \sim -\frac{\sin\left(x - \phi + \frac{\pi}{4}(n - \alpha)\right)}{\sin\left(x - \phi + \frac{\pi}{4}(n - \alpha) + \frac{\pi}{2}\right)} = -\tan\left(x - \phi + \frac{\pi}{4}(n - \alpha)\right).$$

Regarding the asymptotics in the complex plane to get the expression for the exponential error in the case $b_1, b_2 \neq 0$, it is sufficient to include extra terms from (62). More specifically, we can get the following asymptotics in the upper half plane using only terms with $r = 0$ and $r = 1$ of (62)

$$u_n(x, \alpha) - i \sim \left(\frac{d_1 - id_2}{d_1 + id_2}\right) \frac{2^{2n-1}}{(n-1)!} e^{-\frac{i\pi}{2}(n+1+\alpha)} x^{n-1} e^{2ix}, \quad x \rightarrow \infty, \quad 0 < \arg(x) < \pi.$$

Similarly, in the lower half plane, we use $r = n$ combined with $r = n - 1$ of (62) and get

$$u_n(x, \alpha) + i \sim \left(\frac{d_1 + id_2}{d_1 - id_2}\right) \frac{2^{2n-1}}{(n-1)!} e^{\frac{i\pi}{2}(n-1+\alpha)} x^{n-1} e^{-2ix}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < 0.$$

If $2b_1 = d_1 - id_2 = 0$, then the expression (60) for $\Delta_n(x, \alpha)$ contains only integrals over Γ_4 . To get the error term of asymptotics of $\Delta_n(x, \alpha)$, we need to look at the steepest descent procedure in more details. We start with the change of variables

$$t_l + i = -\frac{s_l}{2} \left(\sqrt{4i + s_l^2} + s_l \right). \quad (65)$$

As the result of this transformation, we get

$$e^{\frac{x}{2} \left(t_l - \frac{1}{t_l} \right)} = e^{-ix - \frac{xs_l^2}{2}}.$$

In the computation of asymptotics of $\Delta_n(x, \alpha)$, we will need the expansion for t_l as $s_l \rightarrow 0$

$$t_l = -i \left(1 + e^{-\frac{i\pi}{4}} s_l - \frac{is_l^2}{2} + \frac{e^{-\frac{3\pi i}{4}}}{8} s_l^3 + \mathcal{O}(s_l^4) \right).$$

Using it and (44), we see that

$$\begin{aligned} \prod_{1 \leq j < k \leq n} \frac{1}{t_j t_k} \prod_{l=1}^n \frac{1}{t_l^{\frac{\alpha}{2}+1}} &= \prod_{l=1}^n t_l^{-n-\frac{\alpha}{2}} = i^{n^2} e^{\frac{i\pi\alpha n}{4}} \left(1 + e^{\frac{3\pi i}{4}} \left(n + \frac{\alpha}{2} \right) \sum_{l=1}^n s_l - i \left(n + \frac{\alpha}{2} \right)^2 \sum_{1 \leq j < k \leq n} s_j s_k \right. \\ &\quad \left. - \frac{i}{2} \left(n + \frac{\alpha}{2} \right)^2 \sum_{l=1}^n s_l^2 + \mathcal{O} \left(\sum_{l=1}^n |s_l|^3 \right) \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \prod_{l=1}^n dt_l &= e^{-\frac{3\pi i n}{4}} \left(1 + e^{-\frac{i\pi}{4}} \sum_{l=1}^n s_l - i \sum_{1 \leq j < k \leq n} s_j s_k - \frac{3i}{8} \sum_{l=1}^n s_l^2 + \mathcal{O} \left(\sum_{l=1}^n |s_l|^3 \right) \right) \prod_{l=1}^n ds_l \\ \prod_{1 \leq j < k \leq n} (t_j - t_k)^2 &= e^{-\frac{3\pi i}{4}(n^2-n)} \prod_{1 \leq j < k \leq n} \left(1 + e^{-\frac{i\pi}{4}} (s_j + s_k) - \frac{i}{2} (s_j^2 + s_k^2) - \frac{3i}{4} s_j s_k + \mathcal{O}(|s_j|^3 + |s_k|^3) \right) \\ &\times \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 = e^{-\frac{3\pi i}{4}(n^2-n)} \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \left(1 + e^{-\frac{i\pi}{4}} \sum_{1 \leq j < k \leq n} (s_j + s_k) - \frac{i}{2} \sum_{1 \leq j < k \leq n} (s_j^2 + s_k^2) \right. \\ &\quad \left. - \frac{3i}{4} \sum_{1 \leq j < k \leq n} s_j s_k - i \sum_{\substack{1 \leq j < k \leq n \\ 1 \leq l < m \leq n \\ (j,k) < (l,m)}} (s_j + s_k)(s_l + s_m) + \mathcal{O} \left(\sum_{l=1}^n |s_l|^3 \right) \right) = e^{-\frac{3\pi i}{4}(n^2-n)} \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \\ &\times \left(1 + e^{-\frac{i\pi}{4}} (n-1) \sum_{l=1}^n s_l - \frac{i}{2} (n-1)^2 \sum_{l=1}^n s_l^2 - \frac{i}{4} (2n-3)(2n-1) \sum_{1 \leq j < k \leq n} s_j s_k + \mathcal{O} \left(\sum_{l=1}^n |s_l|^3 \right) \right), \end{aligned}$$

where symbol $<$ means lexicographic ordering. In the computation above, we used the following observation:

$$\sum_{\substack{1 \leq j < k \leq n \\ 1 \leq l < m \leq n \\ (j,k) < (l,m)}} (s_j + s_k)(s_l + s_m) = \frac{(n-1)^2 - (n-1)}{2} \sum_{l=1}^n s_l^2 + ((n-1)^2 - 1) \sum_{1 \leq j < k \leq n} s_j s_k.$$

This identity is the result of the following computation. First, we notice that the left-hand side is a symmetric polynomial of degree 2 in variables s_j , $j = 1 \dots n$. Therefore, to determine the coefficients near the sums $\sum_{l=1}^n s_l^2$ and $\sum_{1 \leq j < k \leq n} s_j s_k$, it is sufficient to determine the coefficients near s_1^2 and $s_1 s_n$.

The term s_1^2 can only be obtained for $j = l = 1$ in the sum on the left-hand side. Condition $(1, k) < (1, m)$ implies that there are $\frac{(n-1)^2 - (n-1)}{2}$ values for k and m . That determines the first coefficient.

Condition $(j, k) < (l, m)$ implies that the term $s_1 s_n$ can only be obtained for $j = 1$ and $m = n$ on the left-hand side. Indices k and l can take any value except for one choice of $k = n$ and $l = 1$ simultaneously. Therefore, there are $(n-1)^2 - 1$ of them, which determines the second coefficient.

We plug these formulas in (60). We notice that some terms evaluate to zero. We end up with the following result:

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{b_2^n}{\pi^n n!} e^{\frac{i\pi n^2}{4}} e^{\frac{i\pi n\alpha}{4}} e^{-inx} \left(\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right. \\ &\quad \left. + \frac{\alpha}{2} e^{\frac{3\pi i}{4}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \sum_{l=1}^n s_l \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right. \\ &\quad \left. + \left(in^2 - in + \frac{in\alpha}{2} + i \right) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(\sum_{l=1}^n s_l \right)^2 \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right. \\ &\quad \left. + \left(-in^2 + in - \frac{in\alpha}{2} - \frac{i\alpha^2}{8} - \frac{7i}{8} \right) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \sum_{l=1}^n s_l^2 \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right. \\ &\quad \left. + 2 \left(-in^2 + in - \frac{in\alpha}{2} - \frac{i\alpha^2}{8} - \frac{7i}{8} \right) \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \sum_{1 \leq j < k \leq n} s_j s_k \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right) (1 + \mathcal{O}(x^{-1})) \\ &= \frac{b_2^n}{\pi^n n!} e^{\frac{i\pi n^2}{4}} e^{\frac{i\pi n\alpha}{4}} e^{-inx} \left(\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right. \\ &\quad \left. + \frac{i(1 - \alpha^2)}{8} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \left(\sum_{l=1}^n s_l \right)^2 \prod_{1 \leq j < k \leq n} (s_j - s_k)^2 \prod_{l=1}^n e^{-\frac{x}{2} s_l^2} ds_l \right) (1 + \mathcal{O}(x^{-1})). \end{aligned}$$

To evaluate integrals above, we make the change of variables $t_l = \sqrt{2a}y_l - \frac{b}{\sqrt{2a}}$ in the formula [21, (5.14.6)] to get

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq n} (y_j - y_k)^2 \prod_{l=1}^n e^{-ay_l^2 + by_l} dy_l = (2\pi)^{\frac{n}{2}} G(n+2) e^{\frac{nb^2}{4a}} (2a)^{-\frac{n^2}{2}}. \quad (66)$$

Taking the second derivative with respect to b and evaluating $b \rightarrow 0$, $a \rightarrow \frac{x}{2}$, we arrive at

$$\Delta_n(x, \alpha) = \frac{(d_1 + id_2)^n}{(2\pi)^{\frac{n}{2}}} e^{\frac{i\pi n^2}{4}} e^{\frac{i\pi n\alpha}{4}} G(n+1) e^{-inx} x^{-\frac{n^2}{2}} \left(1 + i \frac{(1 - \alpha^2)n}{8x} + \mathcal{O}(x^{-2}) \right).$$

In conclusion,

$$u_n(x, \alpha) - i \sim \frac{1 - \alpha}{2x}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi.$$

For $2b_2 = d_1 + id_2 = 0$, the computation is similar. The expression (60) for $\Delta_n(x, \alpha)$ contains only integrals over Γ_3 . We start with the change of variables

$$t_l - i = -\frac{s_l}{2} \left(\sqrt{-4i + s_l^2} + s_l \right).$$

We notice that it is just a complex conjugation of (65). As a result, all the computations can be complex conjugated and we get

$$\begin{aligned} \Delta_n(x, \alpha) &= \frac{(d_1 - id_2)^n}{(2\pi)^{\frac{n}{2}}} e^{-\frac{in^2}{4}} e^{-\frac{in\alpha}{4}} G(n+1) \\ &\quad e^{inx} x^{-\frac{n^2}{2}} \left(1 - i \frac{(1 - \alpha^2)n}{8x} + \mathcal{O}(x^{-2}) \right) \\ u_n(x, \alpha) + i &\sim \frac{1 - \alpha}{2x}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi. \end{aligned}$$

That proves Theorem 1.4.

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Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Appendix A: Bessel Equation and Its Solutions

Bessel equation is given by

$$u''(x) + \frac{u'(x)}{x} + \left(1 - \frac{\nu^2}{x^2}\right)u(x) = 0. \quad (\text{A1})$$

One of the standard solutions in the form of series representation is given by (see [21, (10.2.2)])

$$J_\nu(x) = \frac{x^\nu}{2^\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)}, \quad -\pi < \arg(x) < \pi, \quad (\text{A2})$$

where $\Gamma(x)$ is the Gamma function.

A.1 | Contour Integral Representations

The contour integral representation for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$ is given by (see [21, (10.9.17)])

$$J_\nu(x) = \frac{1}{2\pi i} \int_{\infty-i\pi}^{\infty+i\pi} e^{x \sinh(z) - \nu z} dz. \quad (\text{A3})$$

The contour of integration in (A3) is shown in Figure A1.

Bessel function of second kind can be written as (see [21, (10.2.3)])

$$Y_\nu(x) = \cot(\pi\nu)J_\nu(x) - \csc(\pi\nu)J_{-\nu}(x).$$

By formula (A3) of $J_\nu(x)$, it follows that cylinder function $C_\nu(x)$ that we introduced in (3) can be written as

$$C_\nu(x) = (d_1 + d_2 \cot(\pi\nu)) \frac{1}{2\pi i} \int_{\infty-i\pi}^{\infty+i\pi} e^{x \sinh(z) - \nu z} dz - d_2 \csc(\pi\nu) \frac{1}{2\pi i} \int_{\infty-i\pi}^{\infty+i\pi} e^{x \sinh(z) + \nu z} dz.$$

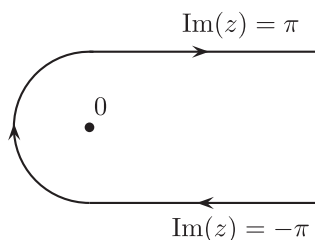


FIGURE A1 | Contour for $J_\nu(x)$.

Making the change of variable $z \rightarrow i\pi - z$ in the second integral, we get

$$C_\nu(x) = (d_1 + d_2 \cot(\pi\nu)) \frac{1}{2\pi i} \int_{\infty-i\pi}^{\infty+i\pi} e^{x \sinh(z) - \nu z} dz + d_2 \csc(\pi\nu) \frac{e^{i\pi\nu}}{2\pi i} \int_{-\infty+2i\pi}^{-\infty} e^{x \sinh(z) - \nu z} dz.$$

Making another change of variable $e^z = t$, we get

$$C_\nu(x) = \int_{\Gamma_1 \cup \Gamma_2} \frac{e^{\frac{x}{2}(t - \frac{1}{t})}}{2\pi i t^{1+\nu}} ((d_1 + d_2 \cot(\pi\nu)) \chi_{\Gamma_1}(t) + d_2 \csc(\pi\nu) e^{i\pi\nu} \chi_{\Gamma_2}(t)) dt = \int_{\Gamma_1 \cup \Gamma_2} h_1(t) dt. \quad (\text{A4})$$

Here, Γ_1 is contour of integration for $J_\nu(x)$ and Γ_2 is contour of integration for $J_{-\nu}(x)$ after change of variable $e^z = t$ shown in Figure A2.

We should make a remark about power function $\frac{1}{t^{1+\nu}}$. When we use it, we assume $-\pi < \arg(t) < \pi$ on contour Γ_1 and $0 < \arg(t) < 2\pi$ on contour Γ_2 .

Remark A.1. For the case $\alpha \in 2\mathbb{Z}$, we need to use formula $Y_n(x) = \frac{1}{\pi} \frac{\partial J_\nu}{\partial \nu} \Big|_{\nu=n} + \frac{(-1)^n}{\pi} \frac{\partial J_\nu}{\partial \nu} \Big|_{\nu=-n}$. It would mean that we need to consider orthogonal polynomials with weight $e^{-t} t^\alpha \ln(t)$. The explicit formula for the corresponding multiple integral can be obtained by using derivatives with respect to parameters, as in (66). The asymptotics of $\Delta_n(x, \alpha)$ would include logarithmic term in that case.

Alternatively, cylinder function $C_\nu(x)$ can be written in terms Hankel functions, which defined as (see [21, (10.4.3)])

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x). \quad (\text{A5})$$

They admit convenient asymptotic series expansions at infinity.

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{ix - \frac{i\pi\nu}{2} - \frac{i\pi}{4}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} - \nu + k\right) \Gamma\left(\frac{1}{2} + \nu + k\right)}{\Gamma\left(\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right) (2ix)^k k!}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi, \quad (\text{A6})$$

$$H_\nu^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-ix + \frac{i\pi\nu}{2} + \frac{i\pi}{4}} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} - \nu + k\right) \Gamma\left(\frac{1}{2} + \nu + k\right)}{\Gamma\left(\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2} + \nu\right) (-2ix)^k k!}, \quad x \rightarrow \infty, \quad -\pi < \arg(x) < \pi. \quad (\text{A7})$$

The contour integral representations for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$ are given by (see [21, (10.9.18)])

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty+i\pi} e^{x \sinh(z) - \nu z} dz, \quad H_\nu^{(2)}(x) = -\frac{1}{\pi i} \int_{-\infty}^{\infty-i\pi} e^{x \sinh(z) - \nu z} dz. \quad (\text{A8})$$

The contours of integration in (A8) are shown in Figure A3.

We would like to also get the alternative integral representation for $C_\nu(x)$. Using (A5), we have

$$C_\nu(x) = b_1 H_\nu^{(1)}(x) + b_2 H_\nu^{(2)}(x), \quad (\text{A9})$$

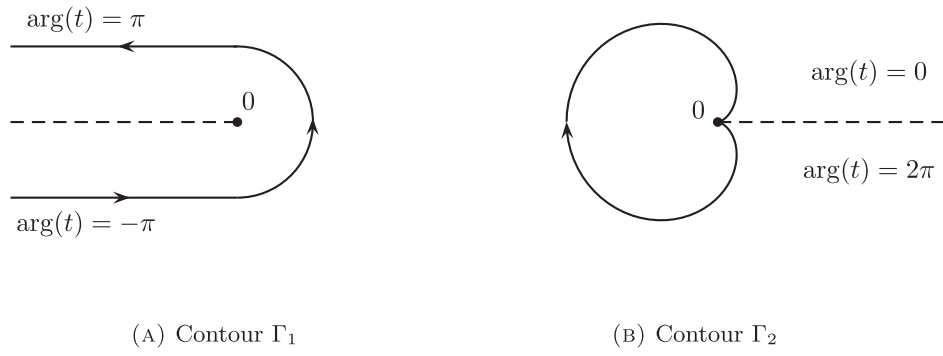


FIGURE A2 | Contours for $C_\nu(x)$.

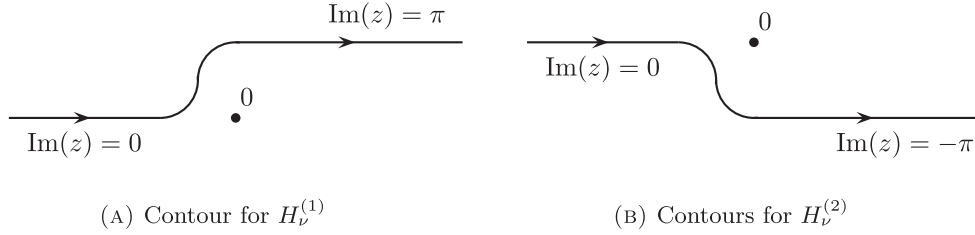


FIGURE A3 | Contours of integration for Hankel functions.

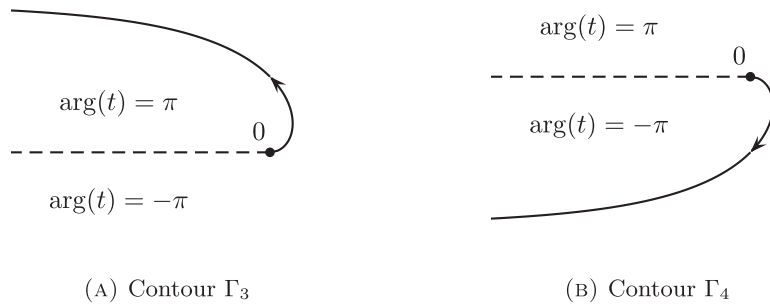


FIGURE A4 | Alternative contours of integration for $C_\nu(x)$.

where

$$b_1 = \frac{d_1 - id_2}{2}, \quad b_2 = \frac{d_1 + id_2}{2}. \quad (\text{A10})$$

Making a change of variable $e^z = t$, we get

$$\begin{aligned} C_\nu(x) &= \int_{\Gamma_3 \cup \Gamma_4} \frac{e^{\frac{x}{2}(t - \frac{1}{t})}}{2\pi i t^{1+\nu}} ((d_1 - id_2)\chi_{\Gamma_3}(t) - (d_1 + id_2)\chi_{\Gamma_4}(t)) dt \\ &= \int_{\Gamma_3 \cup \Gamma_4} h_2(t) dt, \end{aligned} \quad (\text{A11})$$

where contours Γ_3 and Γ_4 are shown in Figure A4.

A.2 | Differential Identities for Cylinder Functions

Cylinder functions $C_\nu(x)$ satisfy the following differential identities (see [21, (10.6.2)]):

$$C'_\nu(x) = \frac{\nu}{x} C_\nu(x) - C_{\nu+1}(x), \quad (\text{A12})$$

$$C'_\nu(x) = C_{\nu-1}(x) - \frac{\nu}{x} C_\nu(x). \quad (\text{A13})$$

A.3 | Analytic Continuation

Cylinder functions satisfy analytic continuation formulas [21, §10.11]. We rewrite them in a more convenient form.

$$J_\nu(x) = e^{-i\pi\nu} J_\nu(e^{i\pi}x), \quad Y_\nu(x) = e^{i\pi\nu} Y_\nu(e^{i\pi}x) - 2i \cos(\pi\nu) J_\nu(e^{i\pi}x), \quad (\text{A14})$$

$$J_\nu(x) = e^{i\pi\nu} J_\nu(e^{-i\pi}x), \quad Y_\nu(x) = e^{-i\pi\nu} Y_\nu(e^{-i\pi}x) + 2i \cos(\pi\nu) J_\nu(e^{-i\pi}x), \quad (\text{A15})$$

$$H_\nu^{(1)}(x) = \csc(\pi\nu) \sin(2\pi\nu) H_\nu^{(1)}(e^{i\pi}x) + e^{-i\pi\nu} H_\nu^{(2)}(e^{i\pi}x),$$

$$H_\nu^{(2)}(x) = -e^{i\pi\nu} H_\nu^{(1)}(e^{i\pi}x), \quad (\text{A16})$$

$$H_\nu^{(1)}(x) = -e^{-i\pi\nu} H_\nu^{(2)}(e^{-i\pi}x),$$

$$H_\nu^{(2)}(x) = \csc(\pi\nu) \sin(2\pi\nu) H_\nu^{(2)}(e^{-i\pi}x) + e^{i\pi\nu} H_\nu^{(1)}(e^{-i\pi}x). \quad (\text{A17})$$

It implies that

$$C_\nu(x) = d_1^+ J_\nu(e^{i\pi}x) + d_2^+ Y_\nu(e^{i\pi}x) \quad (\text{A18})$$

$$= d_1^- J_\nu(e^{-i\pi}x) + d_2^- Y_\nu(e^{-i\pi}x) \quad (\text{A19})$$

$$= b_1^+ H_\nu^{(1)}(e^{i\pi}x) + b_2^+ H_\nu^{(2)}(e^{i\pi}x) \quad (\text{A20})$$

$$= b_1^- H_\nu^{(1)}(e^{-i\pi}x) + b_2^- H_\nu^{(2)}(e^{-i\pi}x), \quad (\text{A21})$$

where

$$d_1^+ = e^{-i\pi\nu} d_1 - 2i \cos(\pi\nu) d_2, \quad d_2^+ = e^{i\pi\nu} d_2, \quad (\text{A22})$$

$$d_1^- = e^{i\pi\nu} d_1 + 2i \cos(\pi\nu) d_2, \quad d_2^- = e^{-i\pi\nu} d_2, \quad (\text{A23})$$

$$b_1^+ = \csc(\pi\nu) \sin(2\pi\nu) b_1 - e^{i\pi\nu} b_2, \quad b_2^+ = e^{-i\pi\nu} b_1, \quad (\text{A24})$$

$$b_1^- = e^{i\pi\nu} b_2, \quad b_2^- = \csc(\pi\nu) \sin(2\pi\nu) b_2 - e^{-i\pi\nu} b_1. \quad (\text{A25})$$

Appendix B: Alternative Proofs of Main Theorems Using Analytic Continuation Formulas

As the general idea stated in Sections 4.5 and 6.1, we can first notice that our method of multiple contour integral representation actually works for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. On the next step, we apply the analytic continuation formulas from Appendix A.3 to the contour integral representations of cylinder functions and extend our results from $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$ to $-\pi < \arg(x) < \pi$. Since the detailed proof along this path involves fairly prolix computations, we presented it in this appendix as a reference.

B.1 | Alternative Proof of Theorem 1.1

In the first step, we need to extend the asymptotic formula from $x > 0$ to $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. We can notice that multiple contour integral representation (41) holds for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. Looking in more detail in the proof of Theorem 1.1, we can notice that the resulting contours after the changes of variables $t_l = \frac{2}{x} s_l$ or $t_l = \frac{x}{2} s_l$ can be deformed back to Γ_1 and Γ_2 . This is possible due to the exponential decay of the integrand in the half plane $\text{Re}(s) < 0$ for the contour Γ_1 and in the half plane $\text{Re}(s) > 0$ for the contour Γ_2 . We perform the deformation in several steps: first, we deform one side of the contour, then move the branch cut, and then move the second part of the contour.

Now we are ready to extend our result to $-\pi < \arg(x) < \pi$. In the first step, we assume $-\pi < \arg(x) < -\frac{\pi}{2}$, which implies $0 < \arg(e^{i\pi}x) < \frac{\pi}{2}$. To obtain the asymptotic formula of the Toeplitz determinant $\Delta_n(x, \alpha)$ at zero, we can apply the analytic continuation formulas (A14). Using them, we express the cylinder function $C_\nu(x)$ as (A18) with d_1^+ and d_2^+ given by (A22).

Since $0 < \arg(e^{i\pi}x) < \frac{\pi}{2}$, we can substitute d_1^+ , d_2^+ , and $e^{i\pi}x$ with $\nu = \frac{\alpha}{2}$ into the asymptotic formulas derived earlier. If $d_2 \neq 0$ and $\text{Re}(\alpha) > 2n - 2$ or $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) = 0$, as $x \rightarrow 0$, $-\pi < \arg(x) < -\frac{\pi}{2}$, we will get

$$\begin{aligned} \Delta_n(x, \alpha) &\sim (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2^+}{\pi}\right)^n \frac{G(n+1)G\left(\frac{\alpha}{2}+1\right)}{G\left(\frac{\alpha}{2}-n+1\right)} \left(\frac{e^{i\pi}x}{2}\right)^{-\frac{n\alpha}{2}} \\ &\sim (-1)^{\frac{n(n+1)}{2}} e^{\frac{i\pi n\alpha}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G\left(\frac{\alpha}{2}+1\right)}{G\left(\frac{\alpha}{2}-n+1\right)} e^{-\frac{i\pi n\alpha}{2}} \left(\frac{x}{2}\right)^{-\frac{n\alpha}{2}} \\ &\sim (-1)^{\frac{n(n+1)}{2}} \left(\frac{d_2}{\pi}\right)^n \frac{G(n+1)G\left(\frac{\alpha}{2}+1\right)}{G\left(\frac{\alpha}{2}-n+1\right)} \left(\frac{x}{2}\right)^{-\frac{n\alpha}{2}}. \end{aligned}$$

If $d_2 \neq 0$, $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $2n - 4j < \text{Re}(\alpha) < 2n - 4j + 2$ for some $j = 0, 1, \dots, n$, as $x \rightarrow 0$, $-\pi < \arg(x) < -\frac{\pi}{2}$, we will get

$$\begin{aligned} \Delta_n(x, \alpha) &\sim (-1)^{\frac{n(n-1)}{2} + nj + n - j} \left(\frac{d_2^+}{\pi}\right)^n \left(\frac{d_1^+}{d_2^+} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\ &\times \frac{G(n-j+1)G\left(-\frac{\alpha}{2} + n - j + 1\right)G(j+1)G\left(j + \frac{\alpha}{2} + 1\right)}{G\left(-\frac{\alpha}{2} + n - 2j + 1\right)G\left(\frac{\alpha}{2} - n + 2j + 1\right)} \left(\frac{e^{i\pi}x}{2}\right)^{(\alpha - 2n + 2j)j - \frac{n\alpha}{2}} \end{aligned}$$

$$\begin{aligned}
& \sim (-1)^{\frac{n(n-1)}{2}+nj+n-j} e^{\frac{i\pi n\alpha}{2}} \left(\frac{d_2}{\pi}\right)^n \left(e^{-i\pi\alpha} \frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) - 2i \sin\left(\frac{\pi\alpha}{2}\right) e^{-\frac{i\pi\alpha}{2}} \cos\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\
& \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2}+n-j+1\right)G(j+1)G\left(j+\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}+n-2j+1\right)G\left(\frac{\alpha}{2}-n+2j+1\right)} \left(\frac{x}{2}\right)^{(\alpha-2n+2j)j-\frac{n\alpha}{2}} e^{i\pi\left((\alpha-2n+2j)j-\frac{n\alpha}{2}\right)} \\
& \sim (-1)^{\frac{n(n-1)}{2}+nj+n-j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j e^{\frac{i\pi n\alpha}{2}-i\pi\alpha j+i\pi\left((\alpha-2n+2j)j-\frac{n\alpha}{2}\right)} \\
& \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2}+n-j+1\right)G(j+1)G\left(j+\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}+n-2j+1\right)G\left(\frac{\alpha}{2}-n+2j+1\right)} \left(\frac{x}{2}\right)^{(\alpha-2n+2j)j-\frac{n\alpha}{2}} \\
& \sim (-1)^{\frac{n(n-1)}{2}+nj+n-j} \left(\frac{d_2}{\pi}\right)^n \left(\frac{d_1}{d_2} \sin\left(\frac{\pi\alpha}{2}\right) + \cos\left(\frac{\pi\alpha}{2}\right)\right)^j \\
& \times \frac{G(n-j+1)G\left(-\frac{\alpha}{2}+n-j+1\right)G(j+1)G\left(j+\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}+n-2j+1\right)G\left(\frac{\alpha}{2}-n+2j+1\right)} \left(\frac{x}{2}\right)^{(\alpha-2n+2j)j-\frac{n\alpha}{2}}.
\end{aligned}$$

If $d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right) \neq 0$, and $\text{Re}(\alpha) < -2n$ or $d_2 = 0$, as $x \rightarrow 0$, $-\pi < \arg(x) < -\frac{\pi}{2}$, we will get

$$\begin{aligned}
\Delta_n(x, \alpha) & \sim \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^n} \left(d_1^+ \sin\left(\frac{\pi\alpha}{2}\right) + d_2^+ \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \left(\frac{x}{2}\right)^{\frac{n\alpha}{2}} \\
& \sim \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^n} \left(e^{-\frac{i\pi\alpha}{2}} d_1 \sin\left(\frac{\pi\alpha}{2}\right) - 2i \sin\left(\frac{\pi\alpha}{2}\right) d_2 \cos\left(\frac{\pi\alpha}{2}\right) + e^{\frac{i\pi\alpha}{2}} d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \\
& \times \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \left(\frac{x}{2}\right)^{\frac{n\alpha}{2}} e^{\frac{i\pi n\alpha}{2}} \\
& \sim \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^n} e^{-\frac{i\pi n\alpha}{2}} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \left(\frac{x}{2}\right)^{\frac{n\alpha}{2}} e^{\frac{i\pi n\alpha}{2}} \\
& \sim \frac{(-1)^{\frac{n(n+1)}{2}}}{\pi^n} \left(d_1 \sin\left(\frac{\pi\alpha}{2}\right) + d_2 \cos\left(\frac{\pi\alpha}{2}\right)\right)^n \frac{G(n+1)G\left(-\frac{\alpha}{2}+1\right)}{G\left(-\frac{\alpha}{2}-n+1\right)} \left(\frac{x}{2}\right)^{\frac{n\alpha}{2}}.
\end{aligned}$$

Therefore, the piecewise formula for sector $-\pi < \arg(x) < -\frac{\pi}{2}$ is consistent with sector $-\frac{\pi}{2} < \arg(x) < 0$ and we can certainly extend the validity of our result to sector $-\pi < \arg(x) < 0$.

Similarly, for $\frac{\pi}{2} < \arg(x) < \pi$, we get $-\frac{\pi}{2} < \arg(e^{-i\pi}x) < 0$. We can apply the analytic continuation formulas (A15). Consequently, the cylinder function $C_\nu(x)$ can be expressed as (A19) with d_1^-, d_2^- given by (A23). Since $-\frac{\pi}{2} < \arg(e^{-i\pi}x) < 0$, we can substitute d_1^-, d_2^- , and $e^{-i\pi}x$ into the asymptotic formulas obtained earlier. After almost the same simplifications, we will arrive at the conclusion that the piecewise formula on the sector $\frac{\pi}{2} < \arg(x) < \pi$ is consistent with the sector $0 < \arg(x) < \frac{\pi}{2}$ and we can also extend the validity of our result to the sector $0 < \arg(x) < \pi$. It follows that our result holds for the entire sector $-\pi < \arg(x) < \pi$. That proves Theorem 1.1.

B.2 | Alternative Proof of Parts (3)–(6) of Theorem 1.3

Again, we start with extending the asymptotics from $x > 0$ to $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. The multiple contour integral representation (60) holds for $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$. We proceed with a steepest descent analysis of (61). The steepest descent contours are given by $\text{Im}(e^{i\arg(x)}\Xi(t)) = \text{Im}(e^{i\arg(x)}\Xi(\pm i))$. Since $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$, the integration over these new contours keeps the integral finite. The main contribution to the asymptotics is provided by critical points. The local changes of the variables are given by $(t_l - i) = s_l e^{-\frac{i\arg(x)}{2}} e^{\frac{3\pi i}{4}}$ for $l \in I$ and $(t_l + i) = s_l e^{-\frac{i\arg(x)}{2}} e^{-\frac{3\pi i}{4}}$ for $l \in I^c$. As a result, we get

$$\Delta_n(x, \alpha) \sim \sum_{r=0}^n \frac{b_1^r b_2^{n-r}}{r!(n-r)!} \frac{4^{r(n-r)}}{\pi^n} (-1)^{\frac{(n-r)(n+r+1)+n(n-1)}{2}} i^{\frac{r(r-1)+(n-r)(n-r-1)-2n}{2}} e^{\frac{\pi i}{4}(2r-n)(1-\alpha)} e^{ix(2r-n)}$$

$$\times e^{i \arg(x)(-r^2 + nr - \frac{n^2}{2})} \int \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq r} (s_j - s_k)^2 \prod_{l=1}^r e^{-\frac{|x|}{2} s_l^2} ds_l \int \dots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq n-r} (s_j - s_k)^2 \prod_{l=1}^{n-r} e^{-\frac{|x|}{2} s_l^2} ds_l,$$

$$x \rightarrow \infty, -\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}.$$

After the change of variable $s \rightarrow \frac{s}{\sqrt{|x|}}$, we recover (62) in the half plane $-\frac{\pi}{2} < \arg(x) < \frac{\pi}{2}$, which implies the validity of parts (3)–(6) of Theorem 1.3 in the same domain.

Now we are ready to the second step of our program. We take $-\pi < \arg(x) < -\frac{\pi}{2}$ which implies $0 < \arg(e^{i\pi}x) < \frac{\pi}{2}$. To get the formula of the Toeplitz determinant $\Delta_n(x, \alpha)$ at infinity, we can apply the analytic continuation formulas (A16). Correspondingly, the cylinder function $C_\nu(x)$ can be expressed as (A20) with b_1^+ and b_2^+ given by (A24). We substitute b_1^+ , b_2^+ , and $e^{i\pi}x$ into (62), then the asymptotic formula of the Toeplitz determinant on the sector $-\pi < \arg(x) < -\frac{\pi}{2}$ is obtained as follows:

$$\Delta_n(x, \alpha) \sim \sum_{r=0}^n (b_1^+)^r (b_2^+)^{n-r} \left(\frac{2}{\pi}\right)^{\frac{n}{2}} 4^{r(n-r)} (-1)^{\frac{(n-r)(n+r+1)+n(n-1)}{2}} i^{\frac{r(r-1)+(n-r)(n-r-1)-2n}{2}} \\ \times G(r+1)G(n-r+1) e^{\frac{\pi i}{4}(2r-n)(1-\alpha)} e^{-i(2r-n)x} (e^{i\pi}x)^{-r^2+nr-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

The leading term is exponentially growing and is given by $r = 0$:

$$\Delta_n(x, \alpha) \sim (b_2^+)^n \left(\frac{2}{\pi}\right)^{\frac{n}{2}} (-1)^{\frac{n(n+1)+n(n-1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{-\frac{\pi i}{4}n(1-\alpha)} e^{inx} (e^{i\pi}x)^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

$$\sim (b_2^+)^n e^{-\frac{in\pi\alpha}{2}} \left(\frac{2}{\pi}\right)^{\frac{n}{2}} (-1)^{\frac{n(n-1)}{2}} e^{\frac{i\pi n(n+1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{-\frac{\pi i}{4}n(1-\alpha)} e^{inx} x^{-\frac{n^2}{2}} e^{-\frac{i\pi n^2}{2}}, \quad x \rightarrow \infty.$$

$$\sim (b_2^+)^n \left(\frac{2}{\pi}\right)^{\frac{n}{2}} (-1)^{\frac{n(n-1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{inx} x^{-\frac{n^2}{2}} e^{\frac{\pi i}{4}n(1-\alpha)}, \quad x \rightarrow \infty.$$

$$\sim (d_1 - id_2)^n \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} (-1)^{\frac{n(n-1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{\frac{\pi i}{4}n(1-\alpha)} e^{inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

Therefore, the formula on the sector $-\pi < \arg(x) < -\frac{\pi}{2}$ is consistent with the sector $-\frac{\pi}{2} < \arg(x) < 0$ and it implies that the validity of our result can be extended to the sector $-\pi < \arg(x) < 0$. That proves formula (63).

On the other hand, let $\frac{\pi}{2} < \arg(x) < \pi$, then $-\frac{\pi}{2} < \arg(e^{-i\pi}x) < 0$. We can apply the analytic continuation formulas (A17). Consequently, the cylinder function $C_\nu(x)$ can be expressed as (A21) with b_1^- and b_2^- given by (A25). We substitute b_1^- , b_2^- , and $e^{-i\pi}x$ into (62), and then the asymptotic formula of the Toeplitz determinant on the sector $\frac{\pi}{2} < \arg(x) < \pi$ is obtained as follows:

$$\Delta_n(x, \alpha) \sim \sum_{r=0}^n (b_1^-)^r (b_2^-)^{n-r} \left(\frac{2}{\pi}\right)^{\frac{n}{2}} 4^{r(n-r)} (-1)^{\frac{(n-r)(n+r+1)+n(n-1)}{2}} i^{\frac{r(r-1)+(n-r)(n-r-1)-2n}{2}} \\ \times G(r+1)G(n-r+1) e^{-\frac{\pi i}{4}(2r-n)(1-\alpha)} e^{-i(2r-n)x} (e^{-i\pi}x)^{-r^2+nr-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

The leading term is exponentially growing and is given by $r = n$:

$$\Delta_n(x, \alpha) \sim (b_1^-)^n \left(\frac{2}{\pi}\right)^{\frac{n}{2}} (-1)^{\frac{n(n-1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{-\frac{\pi i}{4}n(1-\alpha)} e^{-inx} (e^{-i\pi}x)^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

$$\sim (b_1^-)^n e^{\frac{in\pi\alpha}{2}} \left(\frac{2}{\pi}\right)^{\frac{n}{2}} e^{\frac{i\pi n(n-1)}{2}} i^{\frac{n^2-3n}{2}} G(n+1) e^{-\frac{\pi i}{4}n(1-\alpha)} e^{-inx} x^{-\frac{n^2}{2}} e^{\frac{i\pi n^2}{2}}, \quad x \rightarrow \infty.$$

$$\sim (b_1^-)^n \left(\frac{2}{\pi}\right)^{\frac{n}{2}} (-1)^{n^2} i^{\frac{n^2-3n}{2}} G(n+1) e^{inx} x^{-\frac{n^2}{2}} e^{-\frac{\pi i}{4}n(1-\alpha)}, \quad x \rightarrow \infty.$$

$$\sim (d_1 + id_2)^n \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} (-1)^{n^2} i^{\frac{n^2-3n}{2}} G(n+1) e^{-\frac{\pi i}{4}n(1-\alpha)} e^{-inx} x^{-\frac{n^2}{2}}, \quad x \rightarrow \infty.$$

Therefore, the formula for sector $\frac{\pi}{2} < \arg(x) < \pi$ is consistent with sector $0 < \arg(x) < \frac{\pi}{2}$ and again implies that the validity of our result can be extended to sector $0 < \arg(x) < \pi$. That proves formula (64).