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PROBLEMS AT THE INTERSECTION OF GEOMETRY AND DYNAMICS: LYAPUNOV EXPONENTS, RIGIDITY AND ENTROPY OF GEODESIC FLOWS

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ABSTRACT

This thesis comprises three results relating the geometry of a closed Riemannian manifold of negative sectional curvature to the dynamics of the associated geodesic flow.

The first result proves that generically in the space of $\frac{1}{4}$-pinched negatively curved metrics on a closed manifold the Lyapunov exponents of the geodesic flow all have multiplicity 1, a property known as simple Lyapunov spectrum, with respect to all measures with local product structure.

The second is a rigidity theorem which provides a dynamical counterpart to a theorem of Eberlein showing that the universal cover of a closed manifold of negative sectional curvatures admits a discrete group of isometries, unless the manifold is locally symmetric. In this setting, we define a transformation group of the unit tangent bundle of the universal cover which preserves the Anosov structure of the geodesic flow, and show, under a $\frac{1}{4}$-pinching assumption of sectional curvatures, that it must be discrete unless the underlying manifold is locally symmetric.

Lastly, the third result, which is joint work with K. Butt, A. Erchenko and T. Humbert, shows that for a closed surface of variable negative Gaussian curvature the Liouville entropy of the geodesic flow is strictly increasing along the normalized Ricci flow.

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CHAPTER 1

INTRODUCTION

The modern theory of dynamical systems and its study from a mathematical perspective has its origins in the discovery of the emergence of chaotic, seemingly random behavior in deterministic systems. Hyperbolicity is now the best understood source of chaotic behavior, and is defined by the presence of both expansion and contraction in different directions of the phase space. As such, hyperbolic dynamics is a rich subject from the point of view of ergodic theory, which in recent decades has been shown to be connected to a diverse range of other areas such as differential geometry and number theory. This thesis concerns problems related to the ergodic theory of hyperbolic dynamical systems and its connections to differential geometry in negative curvature.

One of the earliest formalizations of a general hyperbolicity condition is found in Anosov's seminal work showing the ergodicity of the geodesic flow of a manifold of negative curvatures [2]. Given a closed Riemannian manifold (M, g), the geodesic flow $\varphi_t : SM \to SM$ is defined on the unit tangent bundle SM of M by flowing a vector along the unit-speed oriented geodesic tangent to it for time t. If g is of negative sectional curvatures, this flow is uniformly hyperbolic on the entirety of the space, a property known as Anosov:

Definition 1.0.1. *Let N be a smooth manifold. A C^1 flow $\varphi^t : N \to N$ generated by the vector field X is said to be Anosov if there exists a $D\varphi^t$-invariant splitting $TN = E^u \oplus \mathbb{R}X \oplus E^s$ of the tangent bundle and $a > 0$ such that for any $t > 0$, $v^s \in E^s$ and $v^u \in E^u$:*

$$\|D\varphi^t(v^s)\| \leq e^{-at}\|v^s\|, \qquad \|D\varphi^{-t}(v^u)\| \leq e^{-at}\|v^u\|$$

The bundles E^u and E^s, referred to as the unstable and stable bundles of the flow respectively, can be shown to be integrate to foliations W^u and W^s referred to as the unstable and stable foliations.

Note that the geodesic flow is defined on the unit tangent bundle, so the stable and unstable bundles are subbundles of the tangent bundle TSM. In this setting, the leaves of the stable and unstable foliations on SM project to the *horospheres* of M. The stable and unstable foliations $W^{s,u}$ themselves are referred to as the *horospherical foliations*.

Geodesic flows in negative curvature are still the foremost example of Anosov flows, and the study of their properties have in turn led to plethora of results on the study of geometry in negative curvature. We highlight three programs which have driven much of the research within hyperbolic dynamics, and more specifically within this geometric setting:

1. Rigidity: Can a weaker property or invariant of a dynamical system determine a stronger property of the system or even the system itself?

2. Flexibity: Given a class of dynamical systems, are the known constraints on its properties tight? If so, can we construct examples to show this?

3. Genericity: What are the properties of the generic system within a certain class?

This thesis addresses some questions in the above programs for the class of hyperbolic dynamical systems given by geodesic flows of manifolds of negative sectional curvature. We now give a brief overview of these results, whereas full definitions and more background is provided in each respective chapter.

Lyapunov exponents of generic Riemannian metrics

In Chapter 2 we consider the Lyapunov exponents of the geodesic flow associated to a generic negatively curved metric. The Anosov condition just mentioned is an example of uniformly hyperbolic behavior, where expansion and contraction admit a minimal non-zero uniform rate on the entirety of the space. From the point of view of ergodic theory, hyperbolicity is often present *non-uniformly* and is described in terms of *Lyapunov exponents*. For f a C^1 diffeomorphism of a closed Riemannian d-manifold M, given any f-invariant ergodic Borel measure μ on M the limit

$$\lambda_i(f,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(D_x f^n),$$

exists and does not depend on x almost surely, where σ_i represents the i-th largest singular value of a linear map and $f^n = f \circ \cdots \circ f$ n-times. The numbers $\lambda_1(f,\mu) \geq \cdots \geq \lambda_d(f,\mu)$, are known as the Lyapunov exponents of f with respect to μ.

Hyperbolicity of f with respect to μ is then naturally defined by the condition $\lambda_i(f,\mu) \neq 0$ for all i. A refined condition, which we will discuss in the following sections, known as *simple Lyapunov spectrum* is defined by $\lambda_1(f,\mu) > \cdots > \lambda_d(f,\mu)$, that is all exponents have multiplicity one. Simple Lyapunov spectrum provides more detailed information on ergodic averages, and implies in many settings the existence of one dimensional invariant foliations for the system. The above results hold in the more general setting of *linear cocycles*. Given some dynamical system $f : X \to X$, an invertible linear cocycle is a map $A : X \to \mathrm{GL}(d,\mathbb{R})$ from which we can form the products:

$$A^n(x) := A(f^n(x)) \dots A(x), \ \forall x \in X.$$

Both the derivative of a diffeomorphism and a product of i.i.d. sampled matrices can be realized as cocycles. When the base dynamics f is some uniformly hyperbolic map, it can be represented by a transitive subshift of finite type $\sigma : \Sigma \to \Sigma$ through a Markov partition. Given some shift-invariant ergodic Borel measure on Σ and a measurable cocycle $A : \Sigma \to \mathrm{GL}(d,\mathbb{R})$ such that $\log \|A\|^+ \in L^1(\mu)$, the Lyapunov exponents exist as before and are given by the μ-a.e. limit

$$\lambda_i(A,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_i(A^n(x)).$$

The definitions above can be extended to the case where f is instead a continuous time system, such as an Anosov flow. In particular, Lyapunov exponents have been recently studied for geodesic flows in the context of rigidity [20], where it is shown that the exponents over periodic measures are a rigid invariant of locally symmetric metrics, and also in the context of flexibility for surfaces of negative Gaussian curvature [30] where it is shown that the exponent with respect to the Liouville measure, which in this case coincides with metric entropy, is flexible.

For manifolds M of higher dimension the flexibility problem for Lyapunov exponents is open, and in fact little is known overall even for generic metrics. A natural question is to ask if the Lyapunov spectrum is simple in general, that is, all exponents have multiplicity 1, and what can be said in the case where there do exist repeated exponents. The main result of Chapter 2 constructs local perturbations of any negatively curved Riemannian pinched metric which show that generically the spectrum is simple:

Theorem 1.0.2. *[75] For $3 \leq k \leq \infty$ we denote by $\mathcal{G}_k$ the set of C^k Riemannian metrics on a fixed closed manifold M with sectional curvatures $1 \leq -K < 4$. There exists a C^2-open and C^k-dense set in $\mathcal{G}_k$ of metrics such that with respect to the equilibrium state of any Holder potential, such as the Liouville measure or the measure of maximal entropy, the derivative cocycle of the geodesic flow has simple Lyapunov spectrum, i.e., all its Lyapunov exponents have multiplicity one.*

The theoretical underpinning of the proof is the simplicity criterion of Ávila-Viana [3], which holds for a fiber-bunched linear cocycle over a hyperbolic (discrete-time) map. For continuous-time dynamical systems, such as the geodesic flow, the perturbation has to be constructed carefully, specifically considering its effect on the rotation number of the return map of the derivative on the unstable bundle. Crucially, we show how such perturbations can be constructed in the space of Riemannian metrics, making use of the results of [63].

Symmetries of geodesic flows on covers and rigidity

In Chapter 3 we present a rigidity result. A property or invariant of a Riemannian manifold is said to be *rigid* if it is able to distinguish the Riemannian metric itself within a certain class, up to isometry. One of the earliest examples of rigidity is Mostow's rigidity theorem, which shows that in the class of hyperbolic metrics the fundamental group is a rigid invariant. In recent decades, dynamical quantities associated to the geodesic flow have been shown to distinguish locally symmetric metrics of negative sectional curvatures from non-locally symmetric negatively curved metrics. Examples of rigidity theorems that have driven much of the research in the field are the minimal entropy rigidity theorem [9] and marked length spectrum rigidity for surfaces [79] [26]. For a plethora of examples, see [84].

The centralizer is the group of symmetries of a smooth dynamical system: given a diffeomorphism f, it is defined as the group of diffeomorphisms g such that $f \circ g = g \circ f$. Equivalently $f = g \circ f \circ g^{-1}$, and hence g can be seen as a change of parameters under which f is invariant. In recent years, several novel results have shown that a "large" centralizer group for a dynamical system with some hyperbolicity condition, such as Anosov diffeomorphisms and certain classes of partially hyperbolic diffeomorphisms, is a rigid invariant, distinguishing a linear or algebraic system within its class up to smooth conjugacy. This phenomenon is known as centralizer rigidity.

Closed negatively curved Riemannian manifolds display an analogous lack of symmetry in their isometry group: P. Eberlein showed that if the group of isometries of the universal cover $\tilde{M}$ is not virtually the same as the deck transformations $\pi_1(M)$ then g is locally symmetric [29]. Since the isometry group is a Lie group by the Myers-Steenrod theorem, this condition is equivalent to the isometry group of $\tilde{M}$ not being discrete. A natural dynamical equivalent then would be the centralizer group of the geodesic flow on the universal cover:

$$Z = \{\psi \in \mathrm{Diff}^\infty(S\tilde{M}) : \psi \circ \varphi^t = \varphi^t \circ \psi, \text{ for all } t \in \mathbb{R}\}.$$

However, since the notion of hyperbolicity itself is dependent on the choice of a metric in a non-compact setting, the above definition is too loose. For instance, elements of Z can fail to be uniformly continuous with respect to the distance induced by the lifted Riemannian metric, and in fact Z can be shown to be infinite-dimensional in any of the C^k-topologies. A condition which adds this compatibility with the dynamics in a non-compact setting, first introduced in [42], is to consider symmetries which additionally preserve the stable and unstable foliations, or equivalently in our case, the horosopherical foliations:

$$G := \{\psi \in Z : D\psi(E^s) = E^s,\ D\psi(E^u) = E^u\}$$

We endow this group with the compact-open topology, and call it the foliated centralizer. We show that it is rigid in the sense of Eberlein in the setting of pinched negatively curved manifolds:

Theorem 1.0.3. *[76] For $n \geq 3$, let (M^n, g) be a closed Riemmanian manifold with negative strictly $\frac{1}{4}$-pinched sectional curvatures, that is, with sectional curvatures $-4 < K \leq -1$. If the reduced foliated centralizer, i.e. $G/\langle\varphi^t\rangle$, is non-trivial, then (M, g) is homothetic to a real hyperbolic manifold.*

This definition of the foliated centralizer also points to a potential relation to a different type of rigidity theorem which concerns the regularity of the stable and unstable foliations. The most general result [7] shows that for metrics of negative sectional curvature if the horospherical foliation of the unit tangent bundle is sufficiently regular, e.g. C^∞, then the metric is locally symmetric, whereas in general horospherical foliations are no more regular than Hölder continuous. The approach taken in the proof of Theorem 1.0.3 can be seen as taking a different approach to understand this rigidity of metrics with smooth, in the sense of highly symmetric, horospheres.

Monotonicity of Liouville entropy along the Ricci flow

In Chapter 4, which is joint work with Karen Butt, Alena Erchenko and Tristan Humbert, we answer a question first asked by A. Manning in [71].

We begin by recalling that topological entropy h_{top}, first introduced in [1], is a fundamental invariant of continuous systems which roughly captures the exponential rate of growth of the complexity of the system. When the system in question is the geodesic flow of a negatively curved manifold, Manning [69] proves that topological entropy of the geodesic flow agrees with the volume entropy of the manifold, namely, the exponential rate of growth of the volume of balls of radius R as a function of R on the universal cover.

This characterization of topological entropy has proved to be extremely fruitful in establishing connections between the dynamics of the geodesic flow and the geometry of the metric itself, in the setting of negatively curved metrics. A prominent example of this connection is the minimal entropy rigidity theorem of Besson, Courtois and Gallot [9], which implies that, on a closed manifold of dimension at least 3 admitting a hyperbolic metric, topological entropy of the geodesic flow is strictly minimized at metrics of negative sectional curvature.

The proof of this fact for surfaces precedes the minimal entropy rigidity theorem and is due to A. Katok [61]. Katok's theorem relies on yet another view of the topological entropy of the geodesic flow, namely, as the measure-theoretical entropy of the geodesic flow with respect to the measure of maximal entropy, also known in this setting as the Bowen-Margulis measure [72]. Katok's result in fact shows more. Recall that in this setting Liouville entropy h_{Liou} is the measure-theoretical entropy of the geodesic flow with respect to the unique absolutely continuous invariant measure, namely the Liouville measure on the unit tangent bundle. Then Katok's result is that amongst metrics of negative curvature on closed surfaces, topological entropy is minimized at the constant curvature metrics, whereas Liouville entropy is maximized, and the two quantities are equal only at the constant curvature metrics.

In [71], Manning considers the evolution of the topological entropy of the geodesic flow of a negatively curved surface under the normalized Ricci flow defined by Hamilton [51]. For a Riemannian metric on a surface, recall that the normalized Ricci flow $t \mapsto g_t$ is a 1-parameter family of metrics obtained by evolving the metric via the ODE:

$$\frac{d}{dt} g_t = -2(K - \overline{K}) g_t,$$

where K is Gaussian curvature and $\overline{K}$ is its mean with respect to volume. The main result of [71] shows that topological entropy is in fact a decreasing function of the time parameter of the Ricci flow, which is consistent with the fact that topological entropy is minimized at the constant curvature metrics, since under the Ricci flow the metric converges to be of constant Gaussian curvature.

On the other hand, Liouville entropy is maximized at the constant curvature metric, and Manning asks the natural question of whether Liouville entropy is in fact increasing along the normalized Ricci flow. In Chapter 4 we show that this is indeed the case:

Theorem 1.0.4. *Let M be a smooth closed orientable surface of negative Euler characteristic. Let g_0 be a smooth Riemannian metric on M of non-constant negative Gaussian curvature. Let $t \mapsto g_t$ denote the normalized Ricci flow starting from g_0. Then*

$$t \mapsto h_{\text{Liou}}(g_t) \textit{ is strictly increasing on } [0, \infty).$$

CHAPTER 2

SIMPLICITY OF LYAPUNOV SPECTRUM FOR CLASSES OF ANOSOV FLOWS

This chapter presents material first appearing in the paper:

Mitsutani, Daniel, Simplicity of the Lyapunov spectrum for classes of Anosov flows *Ergodic Theory and Dynamical Systems*, 43(7):2437–2463 [75].

2.1 INTRODUCTION

The existence of a positive Lyapunov exponent and more generally the multiplicity of the Lyapunov exponents of a system are of essential interest due to their relation to other dynamical invariants and the geometry of the associated dynamical foliations. In this chapter, we seek to address the question of how often simplicity (i.e. all exponents of multiplicity 1) of Lyapunov spectrum arises for some classes of hyperbolic flows. In the classical settings of random matrix products, criteria for simplicity of spectrum were first established in the seminal papers ([39], [47]) and with a variety of different techniques simplicity has more recently been proved in a large variety of settings ([4], [10], [73], [82]).

In [15], Bonatti and Viana first established a criterion for simplicity of Lyapunov spectrum of a cocycle over a discrete symbolic base which holds in great generality with respect to a large class of measures. Applying a Markov partition construction, the authors also extend the results to cocycles over hyperbolic maps, which naturally leads to the question of whether the criterion generically holds for the derivative cocycle in the space of diffeomorphisms. Indeed, without any further restrictions, the arguments in [15] can be modified without much difficulty to show that such a result would be possible, for appropriate choices of measures.

Here we consider the question of genericity of simple spectrum in the continuous-time setting – in particular in more restrictive classes (geodesic flows, conservative flows, etc.) of Anosov flows, which presents significant differences relative to the discrete-time scenario. We establish a method of constructing appropriate perturbations of the Lyapunov spectrum by perturbing the 1-jet of an appropriate Poincaré map within a given class.

We apply it in different settings to obtain the following results. Let X be a smooth closed manifold; precise definitions of the other terms below are given in Section 2.2:

Theorem 2.1.1 (Geodesic flows). *For $3 \leq k \leq \infty$ we denote by $\mathcal{G}^k$ the set of C^k-Riemannian metrics on X with sectional curvatures $1 \leq -K < 4$.*

There exists a C^2-open and C^k-dense set in $\mathcal{G}^k$ of metrics such that with respect to the equilibrium state of any Hölder potential (e.g. Liouville measure, m.m.e.) the derivative cocycle of the geodesic flow has simple Lyapunov spectrum, i.e., all its Lyapunov exponents have multiplicity one.

Theorem 2.1.2 (Conservative flows). *For a fixed smooth volume m and for $2 \leq k \leq \infty$ let $\mathfrak{X}^k_m(X)$ be the set of divergence-free (with respect to m) C^k vector fields on M which generate (strictly) $\frac{1}{2}$-bunched Anosov flows.*

Then flows in a C^1-open and C^k-dense set of $\mathfrak{X}^k_m(X)$ have simple Lyapunov spectrum with respect to m.

Theorem 2.1.3 (All flows). *For $2 \leq k \leq \infty$ let $\mathfrak{X}^k_A(X)$ be the set of C^k vector fields on M which generate (strictly) $\frac{1}{2}$-bunched Anosov flows.*

Then flows in a C^1-open and C^k-dense set of $\mathfrak{X}^k_A(X)$ have simple Lyapunov spectrum with respect to the equilibrium state of any Hölder potential (e.g. SRB measure, m.m.e.).

As indicated before, the proofs are accomplished by constructing a discrete symbolic system via a Markov partition to apply a simplicity criterion of Avila and Viana [3], which is itself an improvement of the criterion of Bonatti and Viana [15] aforementioned. In each class, we prove or use a previously established perturbational result to obtain density in the theorems above.

One main difficulty particular to the setting of $\mathbb{R}$-cocycles which was already present in [15] arises in attempting to perturb the norms of pairs of complex eigenvalues generically. In [15], through the introduction of rotation numbers which vary continuously with the perturbation for orbits near a periodic point, a small rotation on a periodic orbit is propagated to an arbitrarily large one for a homoclinic point, which can then be made to have real eigenvalues.

While such rotation numbers are well-defined for the particular perturbation of the cocycle introduced in [15], a general construction which allows for perturbations of the base system has only been introduced recently in [40]. However, the constructions in [40] do not apply directly to flows, and so we introduce new ideas to control the eigenvalues of the cocycle in the continuous-time setting.

Since the class of geodesic flows is the substantially more difficult case, we carry out the proof of Theorem 2.1.1 in detail, and in Section 2.5 we prove the analogous results needed for Theorem 2.1.2.

2.1.1 Outline of chapter

In Section 2.2 we give the necessary background for the later sections; we summarize the main results of [63] and [3] and introduce rotation numbers. For a more basic introduction to Lyapunov exponents and cocycles we refer the reader to [87] and for background on geodesic flows [80]. In Sections 2.3 and 2.4 we specialize to the setting of the geodesic flows, giving the main arguments to prove of Theorem 2.1.1. Finally, in Section 2.5 we prove a perturbational result for the volume-preserving class, which by direct adaptation of the arguments of the previous sections proves Theorem 2.1.2 and Theorem 2.1.3.

2.2 PRELIMINARIES

2.2.1 Lyapunov exponents and Simplicity of Spectrum

Here we collect and fix the definitions and background results used in later sections. For a continuous flow $\Phi^t : X \to X$ on a compact metric space X preserving an ergodic measure μ, a continuous linear cocycle over Φ on a linear bundle $\pi : \mathcal{E} \to X$ is a continuous map $\mathcal{A} : \mathbb{R} \times \mathcal{E} \to \mathcal{E}$ such that $\Phi^t \circ \pi = \pi \circ \mathcal{A}^t$, where $\mathcal{A}^t := \mathcal{A}(t, \cdot)$. Moreover, we require that the maps $A^t_{\pi(v)} := \mathcal{A}(t, \cdot)|_{\mathcal{E}_{\pi(v)}}$ are linear isomorphisms $\mathcal{E}_{\pi(v)} \to \mathcal{E}_{\phi^t \pi(v)}$ and satisfy the cocycle property $A^{t+s}(x) = A^s(\Phi^s(x)) \circ A^t(x)$.

Suppose $\log^+ \|A^t(x)\| \in L^1(X, \mu)$ for all $t \in \mathbb{R}$. For some fixed choice of norm $\|\cdot\|$ on the fibers, the fundamental result describing asymptotic growth of vectors under $\mathcal{A}$ is Oseledets' theorem: there exists a set of numbers $\lambda_1, ..., \lambda_n \in \mathbb{R}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$, a measurable splitting $\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^n$ and a set of full measure $Y \subseteq X$ such that for all $x \in Y$ and $t \in \mathbb{R}$ we have $A^t_x \mathcal{E}^i_x = \mathcal{E}^i_{\Phi^t(x)}$ and moreover for $v \in \mathcal{E}^i_x$:

$$\lim_{t \to \pm\infty} \frac{1}{t} \log ||A^t_x v|| = \lambda_i.$$

The numbers λ_i are the Lyapunov exponents of $\mathcal{A}$ with respect to μ.

When all bundles $\mathcal{E}^i$ are 1-dimensional, $\mathcal{A}$ is said to have simple Lyapunov spectrum with respect to μ. When X is a smooth manifold and Φ^t is C^1, the dynamical cocycle on $\mathcal{E} = TX$ is the derivative map $D\Phi^t$ of the flow, we often refer to its Lyapunov exponents as the Lyapunov exponents of Φ with respect to μ. Similarly, we say Φ has simple Lyapunov spectrum when the dynamical cocycle does.

The definitions above hold in the discrete-time setting of [3], with appropriate modifications, where the criterion for simplicity of Lyapunov spectrum we need is proved. Following their notation, we let $\hat{f}$ be the shift map on a subshift of finite type $\hat{\Sigma}$ and $\mathcal{A}$ be a measurable cocycle on $\hat{\Sigma} \times \mathbb{R}^d$ over $\hat{f}$, which alternatively can be equivalently described by some measurable $\hat{A}:\hat{\Sigma}\to GL(d,\mathbb{R})$.

The theorems of Avila and Viana all require the additional bunching assumption:

Definition 2.2.1 (Domination/Holonomies). *$\hat{A}$ is dominated if there exists a distance d in $\hat{\Sigma}$ and constants $\theta<1$ and $\nu\in(0,1]$ such that, up to replacing $\hat{A}$ by some power $\hat{A}^N$:*

(1) $d(\hat{f}(\hat{x}),\hat{f}(\hat{y}))\le\theta d(\hat{x},\hat{y})$ and $d(\hat{f}^{-1}(\hat{x}),\hat{f}^{-1}(\hat{y}))\le\theta d(\hat{x},\hat{y})$ for every $\hat{y}\in W^s_{loc}(\hat{x})$ and $\hat{z}\in W^u_{loc}(\hat{x})$

(2) The map $\hat{x}\mapsto\hat{A}(\hat{x})$ is ν-Hölder continuous and $\|\hat{A}(\hat{x})\|\|\hat{A}^{-1}(\hat{x})\|\theta^\nu<1$ for every $\hat{x}\in\hat{\Sigma}$.

If $\hat{A}$ is either dominated or constant on each cylinder, there exists a family of holonomies $\phi^u_{\hat{x},\hat{y}}$, i.e., linear isomorphisms of $\mathbb{R}^d$ such that for each $\hat{x},\hat{y},\hat{z}\in\hat{\Sigma}$ in the same unstable manifold of $\hat{f}$ there exists $C_1>0$ such that:

(1) $\phi^u_{\hat{x},\hat{x}}=id$ and $\phi^u_{\hat{x},\hat{y}}=\phi^u_{\hat{x},\hat{z}}\circ\phi^u_{\hat{z},\hat{y}}$,

(2) $\hat{A}(\hat{f}^{-1}(\hat{y}))\circ\phi^u_{\hat{f}^{-1}(x),\hat{f}^{-1}(y)}\circ\hat{A}^{-1}(\hat{x})=\phi^u_{\hat{x},\hat{y}}$,

(3) $\|\phi^u_{\hat{x},\hat{y}}-id\|\le C_1 d(\hat{x},\hat{y})^\nu$.

There is a family ϕ^s of holonomies over stable manifolds satisfying analogous properties.

For such cocycles, the holonomies allow to propagate the dynamics over single periodic orbits to obtain data on the Lyapunov spectrum of certain measures. Thus, the adaptation of the original pinching and twisting conditions for a monoid of matrices can be adapted to these cocycles as follows:

Definition 2.2.2 (Simple cocycles). *Suppose $\hat{A}:\hat{\Sigma}\to GL(d,\mathbb{R})$ is either dominated or constant on each cylinder of $\hat{\Sigma}$. We say that $\hat{A}$ is simple if there exists a periodic point $\hat{p}$ and a homoclinic point $\hat{z}$ associated to $\hat{p}$ such that:*

(P) the eigenvalues of $\hat{A}$ on the orbit of $\hat{p}$ have multiplicity 1 and distinct norms – let $\omega_j\in\mathbb{R}P^{d-1}$ represent the eigenspaces, for $1\le j\le d$; and

(T) $\{\psi_{\hat{p},\hat{z}}(\omega_i) : i \in I\} \cup \{\omega_j : j \in J\}$ *is linearly independent, for all subsets* I *and* J *of* $1, ..., d$ *with* $\#I + \#J \leq d$ *where, denoting by* ϕ^u *and* ϕ^s *the holonomies as above,*

$$\psi_{\hat{p},\hat{z}} = \phi^s_{\hat{z},\hat{p}} \circ \phi^u_{\hat{p},\hat{z}}.$$

An invariant probability measure $\hat{\mu}$ has local product structure if for every cylinder $[0 : i]$:

$$\hat{\mu}|[0 : i] = \psi \cdot (\mu^+ \times \mu^-)$$

where $\psi : [0 : i] \to \mathbb{R}$ is continuous and μ^+ and μ^- are the projections of $\hat{\mu}|[0 : i]$ to spaces of one-sided sequences indexed by positive and negative indices respectively. For instance, this property holds for every equilibrium state of $\hat{f}$ associated to a Hölder potential [16].

Theorem 2.2.3. *[3, Theorem A] If* $\hat{A}$ *is a simple cocycle then it has Lyapunov exponents of multiplicity one with respect to any* $\hat{\mu}$ *with local product structure.*

2.2.2 Anosov Flows

The continuous-time hyperbolic systems we study are:

Definition 2.2.4 (C^k-Anosov Flows)**.** *A* C^k *(*$1 \leq k \leq \infty$*) flow* $\Phi^t : X \to X$ *on a smooth manifold* X *is called Anosov if it preserves a splitting* $E^u \oplus E^0 \oplus E^s$ *of* TX *such that* E^0 *is the flow direction and there exist* $\lambda > 0$ *and* $C > 1$ *such that for all* $v \in E^u$ *and* $u \in E^s$*:*

$$||D\Phi^t v|| \geq Ce^{\lambda t}||v||, \quad ||D\Phi^{-t} u|| \geq Ce^{\lambda t}||u||.$$

A significant class of cocycles over Anosov flows related to the theory of partially hyperbolic systems and to the class of dominated cocycles over shift maps is that of fiber bunched cocycles, whose expansion and contraction rates are dominated by the base dynamics:

Definition 2.2.5 (Fiber Bunching)**.** *A β-Hölder continuous cocycle $\mathcal{A} : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ over an Anosov flow $\Phi^t : X \to X$ is said to be α-fiber bunched if $\alpha \leq \beta$ and there exists $T > 0$ such that for all $p \in M$ and $t \geq T$:*

$$\|A_p^t\| \|(A_p^t)^{-1}\| \|D\Phi^t|_{E^s}\|^\alpha < 1, \quad \|A_p^t\| \|A_p^{-t}\| \|D\Phi^{-t}|_{E^u}\|^\alpha < 1.$$

When the cocycles $D\Phi^t|_{E^{i=u,s}}$ themselves satisfy the inequalities above in place of $\mathcal{A}$, the Anosov flow is said to be α-bunched.

Fiber bunching is a partial hyperbolicity condition on the projectivization of the fiber bundle. The strong stable and unstable manifold theorem for this partially hyperbolic system can be interpreted as defining holonomy maps between the fibers:

Theorem 2.2.6. *[59] Suppose $\mathcal{A}$ is β-Hölder and fiber bunched over a base system as in Definition 3.2.5. Then the cocycle admits holonomy maps h^u, that is, a continuous map $h^u : (x, y) \to h^u_{x,y}$, $x \in M$, $y \in W^u(x)$ the strong unstable manifold of x, such that:*

(1) $h^u_{x,y}$ is a linear map $\mathcal{E}_x \to \mathcal{E}_y$,

(2) $h^u_{x,x} = Id$ and $h^u_{y,z} \circ h^u_{x,y} = h^u_{x,z}$,

(3) $h^u_{x,y} = (A_y^t)^{-1} \circ h^u_{\Phi^t(x),\Phi^t(y)} \circ A_x^t$ for every $t \in \mathbb{R}$.

Moreover, the holonomy maps are unique, and, fixing a system of linear identifications $I_{xy} : \mathcal{E}_x \to \mathcal{E}_y$, see [59], they satisfy:

$$\|h^u_{x,y} - I_{x,y}\| \leq C d(x, y)^\beta.$$

Using property (3), for sufficiently small $r > 0$ one may extend these holonomies for all $y \in W_r^{cu}(x)$, the ball of radius r centered at x in the center-unstable manifold sometimes called the local center-unstable manifold, and such holonomies are denoted by h^{cu}. Namely, one lets

$$h^{cu}_{xy} = h^{u}_{\Phi^t(x),y} \circ A^t_x$$

where $t \in \mathbb{R}$ is chosen to minimize $|t|$ among the times such that $\Phi^t(x) \in W^u(y)$. Observe that this is only well-defined locally and that the same construction holds over W^{cs}.

For the case where Φ is itself α-bunched, it is known that [52] the bunching constant is directly related to the regularity of the Anosov splitting: for $\frac{1}{2}$-bunched Anosov flows, the weak stable and unstable bundles $E^{cu,cs} := E^0 \oplus E^{u,s}$ are of class C^1. Thus:

Proposition 2.2.7. *For $\Phi^t : X \to X$ a $\frac{1}{2}$-bunched C^2-Anosov flow, the cocycle $\mathcal{A}^u$ (resp. $\mathcal{A}^s$) on the bundle $Q^u := E^{cu}/E^0$ (resp. $Q^s := E^{cs}/E^0$) given by the derivative $D\Phi^t$ is* 1*-bunched.*

Proof. The cocycle $D\Phi^t|_{E^{cu}/E^0}$ is C^1 by the regularity of the splitting mentioned above, and by hypothesis $D\Phi^t|_{E^{cu}/E^0}$ satisfies the inequalities in the definition of fiber bunching with $\alpha = 1$. Same for E^{cs}. □

Finally, we describe the class of measures with respect to which we prove our results. Fix a topologically mixing C^2-Anosov flow Φ^t. Let $\rho : X \to \mathbb{R}$ be a Hölder-continuous function, which we refer to as a potential. Then an equilibrium state μ_ρ of ρ is an invariant measure satisfying the variational principle:

$$h_{\mu_\rho}(\Phi) + \int \rho \, d\mu_\rho = \sup_{\mu \in \mathcal{M}_\Phi(X)} h_\mu(\Phi) + \int \rho \, d\mu,$$

where $h_\mu(\Phi)$ is the measure-theoretic entropy of Φ with respect to μ and $\mathcal{M}_\Phi(X)$ is the set of invariant measures of Φ. The existence and uniqueness of the equilibrium state μ_ρ, is, in this setting, a foundational result in the theory of the thermodynamical formalism [16].

Important examples of equilibrium states include the case $\rho = 0$, which gives the measure of maximal entropy as the equilibrium state, and $\rho(x) = -\frac{d}{dt} \log J^u(x,t)|_{t=0}$, where $J^u(x,t) = \det D_x\Phi^t|_{E^u}$, which gives the SRB measure. Moreover, the product structure property mentioned in the previous section is also a classical result for equilibrium states proved in [16].

2.2.3 Rotation Numbers

As indicated in the introduction, in order to perturb away complex eigenvalues by a small rotation, one needs the formalism of rotation numbers, which we introduce in complete form here. We roughly follow the discussion in Section 3 of [40].

As a brief introduction, recall that for an orientation preserving homeomorphism of the circle $f : S^1 \to S^1$, the Poincaré rotation number $\rho(f) \in S^1 = \mathbb{R}/2\pi$ of f is defined as:

$$\rho(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n} \pmod{2\pi},$$

for a lift $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f. This limit always exists and is independent of the choice of $x \in \mathbb{R}$ and the lift $\tilde{f}$. For an orientation reversing homeomorphism we define $\rho(f) = 0$.

The Poincaré rotation number measures, on average, how much an element is rotated by an application of f and is a conjugation invariant, i.e., $\rho(g^{-1}fg) = \rho(f)$, for g also a homeomorphism of S^1. In what follows we extend this definition for cocycles on circle bundles.

Throughout this section, let X a compact metric space and Φ^t a continuous flow on X. Let $\mathcal{M}_\Phi(X)$ be the space of probability measures on X invariant under Φ^t with the weak-* topology.

For our purposes, it will suffice to work with trivial bundles $E = X \times S^1$, and a continuous cocycle $\mathcal{A} : \mathbb{R} \times E \to E$ over Φ^t. Then for $(x, \theta) \in E$, the map $t \mapsto A_x^t(\theta)$ is a continuous map from $\mathbb{R} \to S^1$, so it may lifted to some $w_{x,\theta} : \mathbb{R} \to \mathbb{R}$. Let $\tilde{w}_{x,\theta}(t) := w_{x,\theta}(t) - w_{x,\theta}(0)$, so that $\tilde{w}$ does not depend on the lift w.

Definition 2.2.8 (Pointwise Rotation Number)**.** *The average rotation number $\rho : X \to \mathbb{R}$*

is defined by the limit:

$$\rho(x) = \lim_{t\to\infty} \frac{\tilde{w}_{x,\theta}(t)}{t},$$

whenever it exists, and is independent of the choice of θ*.*

Indeed, for any $\theta, \theta' \in S^1$, we have $|\tilde{w}_{x,\theta}(t) - \tilde{w}_{x,\theta'}(t)| < 2\pi$ for any t so the limit does not depend on choice of $\theta \in S^1$.

Now define $\sigma : X \times \mathbb{R} \to \mathbb{R}$ and $\tau : X \times \mathbb{R} \to \mathbb{R}$ by:

$$\sigma^t(x) := \sigma(x,t) = \sup_{\theta\in S^1} \tilde{w}_{x,\theta}(t)$$

$$\tau^t(x) := \tau(x,t) = \inf_{\theta\in S^1} \tilde{w}_{x,\theta}(t),$$

which, by continuity of $\mathcal{A}$ are evidently continuous in t and in x. Moreover, by the cocyle equation for $\mathcal{A}$ it is clear that the functions σ and τ are subadditive and superadditive in the $\mathbb{R}$ coordinate, respectively.

By Kingman's subadditive ergodic theorem for flows, for any $\mu \in \mathcal{M}_\Phi(X)$:

(1) The sequence $\frac{1}{t}\sigma^t$ converges μ-a.e. to a Φ invariant map, which agrees with ρ.

(2) We may compute the integral of ρ by:

$$\rho_\mu := \int \rho \, d\mu = \inf_{t>0} \frac{1}{t} \int \sigma^t \, d\mu. \tag{2.1}$$

The discussion above then implies:

Theorem 2.2.9. *The map* $\mathcal{M}_\Phi(X) \to \mathbb{R}$ *given by* $\mu \mapsto \rho_\mu$ *is continuous.*

Proof. Note that by compactness of X and continuity of $\sigma^t : X \to \mathbb{R}$, the map $\mu \to \int \sigma^t \, d\mu$ is continuous, and hence by:

$$\int \rho \, d\mu = \inf_{t>0} \frac{1}{t} \int \sigma^t \, d\mu$$

and the analogous equation for τ, we obtain upper and lower semicontinuity of $\mu \mapsto \rho_\mu$. ☐

Remark 2.2.10. *When μ is supported on a periodic orbit $\mathcal{O}$, we will often write $\rho_{\mathcal{O}}$ for ρ_μ.*

Next we consider perturbations of cocycles over a fixed base flow. The space of cocycles $\mathcal{C}_\Phi$ over the same Φ has a C^0-topology of uniform convergence defined by the property that $\mathcal{A}_n \to \mathcal{A}$ if for each $x \in X$ and $|t| < 1$ the maps $(A_n)^t_x \to A^t_x$ in $C^0(S^1, S^1)$ uniformly.

Associated to the cocycles $\mathcal{A}$ are rotation numbers $\rho_\mu(\mathcal{A})$ for invariant measures μ defined by Equation (2.1). Then:

Proposition 2.2.11. *For a $\mu \in \mathcal{M}_\Phi(X)$, the map $\mathcal{C}_\Phi \to \mathbb{R}$ given by*

$$\mathcal{A} \mapsto \rho_\mu(\mathcal{A})$$

is continuous.

Proof. The proof is nearly identical to that of Theorem 2.2.9. Namely, one uses continuity of $\mathcal{A} \mapsto \int \sigma^t(\mathcal{A})\, d\mu$ and the subadditive ergodic theorems. □

Now we specialize to the case where $X = \mathcal{O}$ is a hyperbolic periodic orbit of a C^1 flow Φ_0 on a Riemannian manifold N, which will be $N = SM$ with the Sasaki metric in the setting of this chapter. We are interested in how $\rho_{\mathcal{O}}$ varies as the flow Φ varies, for the derivative cocycle on certain circle bundles.

By structural stability of the hyperbolic set $\mathcal{O}$ there exists $\mathcal{U}$ a C^1-neighborhood of Φ_0 and a continuous $h : \mathcal{U} \times \mathcal{O} \to N$ such that the maps $h_\Phi(x) := h(\Phi, x)$ are C^1-diffeomorphisms onto their images, and $\mathcal{O}_\Phi := h_\Phi(\mathcal{O})$ is a closed orbit of Φ. Moreover, since the maps h_Φ are C^1 there exists a continuous $\kappa : \mathcal{U} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ such that $\kappa_\Phi(x, t) := \kappa(\Phi, x, t)$ is C^1 and the flow $\tilde{\Phi}$ (defined on $\mathcal{O}_\Phi$) given by:

$$\tilde{\Phi}^t(x) = \Phi^{\kappa_\Phi(h_\Phi(x),t)}(x),$$

is in fact conjugated to Φ_0 by h_Φ, i.e., $h_\Phi \circ \Phi_0 = \tilde{\Phi} \circ h_\Phi$. .

For any bundle E, we write $\mathcal{P}E$ for its projectivization. Let F_0 be a 2-dimensional trivial subbundle of $TN|_{\mathcal{O}}$ which is part of a dominated splitting $E_0 \oplus_{\leq} F_0 \oplus_{\leq} G_0$ of $TN|_{\mathcal{O}}$. The derivative cocycle $D\Phi$ on $\mathcal{P}F_0$ then is a cocyle on a trivial S^1 bundle, and it has a rotation number $\rho_{\mathcal{O}}$ as before.

Assuming $\mathcal{U}$ is taken sufficiently small, by persistence of dominated splittings for each $\Phi \in \mathcal{U}$ there is a splitting $TN|_{\mathcal{O}_\Phi} = E_\Phi \oplus_{\leq} F_\Phi \oplus_{\leq} G_\Phi$ for Φ and the bundle F_Φ is trivial. Moreover, the splitting is also dominated for the flow $\tilde{\Phi}$, which is simply a time change of Φ. Hence, $D\tilde{\Phi}$ and $D\Phi$ on $\mathcal{P}F_\Phi$ also have well defined rotation numbers $\rho_{\mathcal{O}_{\tilde{\Phi}}}$, $\rho_{\mathcal{O}_\Phi}$, which satisfy the relation:

$$\rho_{\mathcal{O}_{\tilde{\Phi}}} \ell(\mathcal{O}_\Phi) = \rho_{\mathcal{O}_\Phi} \ell(\mathcal{O}),$$

as they differ by a time change, where ℓ denotes the period of a periodic orbit.

With all the objects defined, we now state the continuity with respect to the parameters:

Proposition 2.2.12. *The map* $\mathcal{U} \to \mathbb{R}$ *given by*

$$\Phi \mapsto \rho_{\mathcal{O}_\Phi},$$

is continuous in some open $\mathcal{V} \subseteq \mathcal{U}$ *containing* Φ_0.

Proof. First, we would like to consider all cocycles $D\tilde{\Phi}$ constructed on F_Φ as existing on the same bundle over the same base map.

For $x \in \mathcal{O}$ there exists a unique length-minimizing geodesic segment (from the Riemannian structure on N) from x to $h_\Phi(x)$, as long as h_Φ is close to the identity, which may be ensured by reducing the neighborhood $\mathcal{U}$ to some $\mathcal{V} \subseteq \mathcal{U}$ further if needed. By parallel transport of the bundle F_Φ over $\mathcal{O}_\Phi$ along such segments, one then obtains a 2-dimensional trivial bundle F'_Φ over $\mathcal{O}$. By reducing $\mathcal{V}$ further if needed, the bundle F'_Φ obtained is a given by a graph over F_0 with respect to the fixed Riemannian metric on N, and hence by orthogonal projection they may be identified.

Since all maps above are continuous, the construction above describes a continuous map

$Th : \mathcal{U} \times F_0 \to TN$, so that $Th_\Phi(\cdot) := Th(\Phi, \cdot)$ are bundle isomorphisms $F_0 \to F_\Phi$ fibering over h_Φ. Hence, conjugating by Th_Φ we may regard $D\tilde{\Phi}$ on F_Φ as a cocyle on F_0 over Φ_0. By continuous dependence on Φ, this defines a continuous map $\Phi \to D\tilde{\Phi}$, where $D\tilde{\Phi}$ are now regarded as elements of the space of cocycles over Φ_0 on F_0 with the C^0-topology.

Since all rotation numbers $\rho_{\mathcal{O}_{\tilde{\Phi}}}$ defined previously are preserved by conjugation, it suffices to check continuity of the rotation numbers of the conjugated cocycles, which is given by Proposition 2.2.11. Thus the map $\Phi \mapsto \rho_{\mathcal{O}_{\tilde{\Phi}}}$ is continuous, and finally since

$$\rho_{\mathcal{O}_{\tilde{\Phi}}} \ell(\mathcal{O}_\Phi) = \rho_{\mathcal{O}_\Phi} \ell(\mathcal{O}),$$

and the periods vary continuously, the map $\Phi \mapsto \rho_{\mathcal{O}_\Phi}$ is continuous as well. □

2.2.4 Geodesic Flows

Let M be a smooth closed manifold. Since we consider varying Riemannian metrics, it is useful to work on the sphere bundle over M of oriented directions of the tangent space, which we denote by SM, rather than on the unit tangent bundle. When a metric g is fixed, $T^1_g M$ is canonically diffeomorphic to SM, and one can pullback the Sasaki metric from $T^1_g M$ to SM.

Recall that for $3 \leq k \leq \infty$ we denote by $\mathcal{G}^k$ the set of C^k-Riemannian metrics on M with sectional curvatures $1 \leq -K < 4$. The geodesic flow on the unit tangent bundle of a negatively curved Riemannian manifold is an Anosov flow with the horospherical foliations corresponding to the stable and unstable foliations; moreover, under the pinching condition above it is a $\frac{1}{2}$-bunched (see Anosov flows section) Anosov flow [62, Theorem 3.2.17]. In particular, the bundles $E^{cu,cs}$ are C^1, and, since the flow is contact and the kernel of the C^{k-1} contact form equals $E^u \oplus E^s$, in fact $E^u \oplus E^0 \oplus E^s$ is a (at least) C^1 Anosov spliting.

We describe now the perturbational results of [63] that will be used to perturb the derivative cocycle by perturbing the metric. For a fixed embedded compact interval or closed loop $\gamma \subseteq SM$, the set of metrics for which γ is an orbit segment of the geodesic flow is denoted by $\mathcal{G}^k_\gamma \subseteq \mathcal{G}^k$. For a fixed $g_0 \in \mathcal{G}^k_\gamma$, pick local hypersurfaces Σ_0 and Σ_1 in SM that are transverse to $\dot{\gamma}(t) \in TSM$ at $t = 0$ and $t = 1$, respectively. This allows us to define a Poincaré map

$$P_{g_0} : \Sigma_0 \supseteq U \to \Sigma_1,$$

where U is a neighborhood of $\gamma(0)$, by mapping $\xi \in U$ to $\varphi^{t_1}_{g_0}(\xi)$, where t_1 is the smallest positive time such that $\varphi^{t_1}_{g_0}(\xi) \in \Sigma_1$ and φ_{g_0} is the geodesic flow of the metric g_0. By the Implicit Function Theorem and the fact that $\varphi^t_{g_0}$ is C^{k-1}, the map P is C^{k-1}.

By projecting the tangent spaces of $\Sigma_{i=0,1}$ to $E^u \oplus E^s$ one may give $\Sigma_{i=0,1}$ a symplectic structure ω which is preserved by the Poincaré map, since the symplectic form is invariant by the geodesic flow [63]. With g_0 fixed, we let $\mathcal{G}^k_{g_0,\gamma} \subseteq \mathcal{G}^k_\gamma$ be the set of metrics such that $\pi(\gamma(0)), \pi(\gamma(1)) \notin \operatorname{supp}(g - g_0)$ ($\pi : SM \to M$ is the canonical projection map) that is, metrics unperturbed at the ends of the fixed geodesic segment γ relative to g_0.

We will repeatedly use the main result on generic metrics established by Klingenberg and Takens in [63] to perturb the metric g_0:

Theorem 2.2.13. *[63, Theorem 2] Suppose $g_0 \in \mathcal{G}^\infty_\gamma$, and let Q be some open dense subset of the space of $(k-1)$-jets of symplectic maps $(\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$.*

Then there is arbitrarily C^k-close to g_0 a $g' \in \mathcal{G}^k_{g_0,\gamma}$ such that $P_{g'} \in Q$, where $P_{g'} : (\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$ is the Poincaré map for the geodesic flow of g'.

Remark 2.2.14. *The technical assumption that g_0 is C^∞ needed in [63] is virtually harmless, since by smooth approximation $\mathcal{G}^\infty \subseteq \mathcal{G}^k$ is dense for all k.*

We will need two additional facts about how these perturbations can be made, both of which follow directly from the proof of Theorem 2.2.13 in [63]:

Proposition 2.2.15. *Let $h := g' - g_0$ where g' and g_0 given as in the statement of Theorem 2.2.13. For any tubular neighborhood V of γ, h can be taken to satisfy:*

(1) supp$(h) \subseteq V$;

(2) For a system of coordinates $\{x_0, ..., x_{2n-2}\}$ on V where ∂_{x_0} is parallel to the geodesic flow, the k-jets of h_{00} (where $h = h_{ij}\, dx_i\, dx_j$) vanish identically along $\{x_0 = 0\}$.

In particular, this implies that the parametrization of γ by arc-length in g_0 is the same as that in g', i.e., the geodesic flow for both metrics agree along γ.

Let J_s^{k-1} denote the Lie group of $(k-1)$-jets of C^{k-1} symplectic maps $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ with the standard symplectic form $\sum_i dx^i \wedge dy^i$. If $\mathcal{O}$ is a closed orbit, we may take $v := \gamma(0) = \gamma(1) \in \mathcal{O}$ and fix $\Sigma := \Sigma_0 = \Sigma_1$, so by Darboux's theorem we may choose coordinates that identify the space of $(k-1)$-jets of C^{k-1} symplectic maps $(\Sigma, v) \to (\Sigma, v)$ with J_s^{k-1}.

Corollary 2.2.16. *If $\mathcal{O}$ is a closed geodesic for $g_0 \in \mathcal{G}_{\mathcal{O}}^{\infty}$ and $Q \subseteq J_s^{k-1}$ is an open dense invariant (Q satisfies $\sigma Q \sigma^{-1} = Q$ for any $\sigma \in J_s^{k-1}$) set then there is arbitrarily C^k-close to g_0 a $g' \in \mathcal{G}_{\mathcal{O}}^k$ such that for any $v \in \mathcal{O}$ and any Σ a transverse at v, $P_{g'} \in Q$, where $P_{g'} = P(v, \Sigma)$ is the Poincaré return map for the geodesic flow of g'.*

Proof. The choice of a different section Σ or a different point v of the orbit changes $P_{g'}$ by conjugation, so the property that $P_{g'} \in Q$ needs only to be assured at one fixed point and one fixed section, which is done by Theorem 2.2.13. □

Remark 2.2.17. *Since the map $\pi^{k-1} : J_s^{k-1} \to J_s^1 \cong Sp(2n)$ is a submersion, for Q an open dense invariant subset of $Sp(2n)$, $(\pi^{k-1})^{-1}(Q)$ is an open dense invariant subset of J_s^{k-1}, so in the statement of Corollary 2.2.16 we may take an open dense invariant $Q \subseteq Sp(2n)$ instead, while the approximation is still in $\mathcal{G}^k$.*

In the context of Theorem 2.2.13, the analogous observation holds; that is, one may take Q to be an open dense subset of 1-jets of symplectic maps $(\Sigma_0, \gamma(0)) \to (\Sigma_1, \gamma(1))$, and approximate in $\mathcal{G}^k$.

2.3 PINCHING AND TWISTING FOR FLOWS

In this section, we present the main technical results of the chapter, namely, the construction of perturbations of Anosov flows leading to an appropriate pinching and twisting condition. For the sake of simplicity we specialize to the class of geodessic flows, but the main arguments here adapt to the proofs of the other theorems with adjustments which we describe in the last section. We define pinching and twisting for orbits of the geodesic flow in analogy with Definition 2.2.2, and use the results on generic metrics to show that these are C^1-open and C^k-dense.

We fix the following useful notation. For a metric g such that $\mathcal{O} \subseteq SM$ is a periodic orbit of its geodesic flow with period ℓ, let $v \in \mathcal{O}$ and let $\{\lambda_1, ..., \lambda_{2n}\}$ be the generalized eigenvalues of $D_v\varphi_g^\ell|_{E^u \oplus E^s}$, which do not depend on the choice of v, sorted so that $|\lambda_i| \geq |\lambda_j|$ whenever $i < j$. We write:

$$\vec{\lambda}^u(\mathcal{O}, g) := (\lambda_1, ..., \lambda_n), \quad \vec{\lambda}^s(\mathcal{O}, g) := (\lambda_{n+1}, ..., \lambda_{2n}) \in \mathbb{C}^n,$$

$$\vec{\lambda}(\mathcal{O}, g) := (\lambda_1, ..., \lambda_{2n}) \in \mathbb{C}^{2n}.$$

The i-th coordinates of the vectors above are written as $\vec{\lambda}_i^{u,s,\cdot}(\mathcal{O}, g)$ (where $\cdot$ means no superscript above).

The following continuity lemma about these $\vec{\lambda}$ is the bread and butter of all "openness" arguments which follow:

Lemma 2.3.1. *Fix $k \geq 2$. For a metric $g_0 \in \mathcal{G}^k$ there exists a neighborhood $\mathcal{U} \subseteq \mathcal{G}^k$ of g_0 such that for any $g \in \mathcal{U}$ any orbit $\mathcal{O}$ of the geodesic flow of g_0 has a hyperbolic continuation $\mathcal{O}_g$ for the geodesic flow of g, and the maps $\mathcal{U} \to \mathbb{C}^n$ given by*

$$g \mapsto \vec{\lambda}^{u,s}(\mathcal{O}_g, g)$$

are continuous with respect to the C^2-topology.

Proof. Let Σ be a smooth hypersurface parallel to $E^u \oplus E^s$ at v so that $\mathcal{O} \cap \Sigma =: \{v\}$. The return map for the geodesic flow φ_{g_0} then defines a map $P_{g_0} : U \to \Sigma$, where $U \subseteq \Sigma$ is some neighborhood of v, for which v is a hyperbolic fixed point.

For any g sufficiently close to g_0, we also obtain a map $P_g : U \to \Sigma$ given by the return map of φ_g, and by the standard hyperbolic theory, a fixed point v_g such that $g \mapsto v_g$ is continuous. The geodesic flow φ_g varies in a C^{k-1} fashion as g varies in $\mathcal{G}^k$, and by the implicit functon theorem so does P_g. Then by fixing a coordinate system, since $k \geq 3$ the matrices $D_{v_g} P_g$ vary continuously, so their eigenvalues vary continuously as g varies in $\mathcal{G}^k$.

Finally, the eigenvalues of the matrices $D_{v_g} \varphi_g^{\ell_g}|_{E^u \oplus E^s}$ and $D_{v_g} P_g$ agree, so we obtain the desired result. □

2.3.1 Pinching

Before moving to the definition of pinching, first we verify that generically there exists a periodic orbit with a dominated splitting of $E^u \oplus E^s$ into 1-dimensional subspaces and 2-dimensional subspaces corresponding to conjugate pairs of eigenvalues.

Proposition 2.3.2. *Let*

$$\mathcal{G}_d^k := \{g \in \mathcal{G}^k : \exists \mathcal{O} : |\lambda_i| \neq |\lambda_j|, \text{ unless } \lambda_i = \overline{\lambda}_j, \text{ where } (\lambda_1, ..., \lambda_n) := \vec{\lambda}^u(\mathcal{O}, g)\}.$$

The set $\mathcal{G}_d^k$ is C^2-open and C^k-dense in $\mathcal{G}^k$.

Proof. Openness follows directly from Lemma 2.3.1, since by continuity of $\vec{\lambda}^u$ the continuations of $\mathcal{O}$ will satisfy the same condition defining $\mathcal{G}_d^k$.

For density, we start by assuming that $g_0 \in \mathcal{G}_{\mathcal{O}}^\infty$, for some $\mathcal{O}$, which is possible by density of $\mathcal{G}^\infty$ in $\mathcal{G}^k$. It remains to check that the property defining $\mathcal{G}_d^k$ is indeed an open dense in J_s^{k-1}, so that we may apply Corollary 2.2.16 to finish the proof. By Remark 2.2.17, it suffices to check that having eigenvalues distinct with distinct norms, apart from complex conjugate pairs, is an openand dense $\mathrm{Sp}(2n)$.

Openness is clear, since the eigenvalues depend continuously on the matrix entries. For density, we note that the condition of distinct eigenvalues is given by the complement of the equation $\Delta = 0$, where Δ is the discriminant of the characteristic polynomial, which is a non-empty Zariski open set in $\mathrm{Sp}(2n)$, and thus dense in the analytic topology. In particular, the set of diagonalizable matrices is dense. Since diagonalizable matrices are symplectically diagonalizable, by the lemma following this proof, by a small perturbation on the norm of the diagonal blocks we obtain density of eigenvalues of distinct norms. □

We prove the linear algebra lemma used above, which will also be useful in what follows:

Lemma 2.3.3. *A matrix $A \in Sp(2n)$ with all eigenvalues distinct is symplectically diagonalizable in the sense that there exists $P \in Sp(2n)$ such that $P^{-1}AP$ is in real Jordan form (i.e., given by diagonal blocks which are either trivial or 2×2 conformal).*

Proof. Recall that eigenvalues of $A \in \mathrm{Sp}(2n)$ appear in 4-tuples

$$\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}\}$$

for $\lambda \notin \mathbb{R}$ and in pairs $\{\lambda, \lambda^{-1}\}$ for $\lambda \in \mathbb{R}$. For each λ we let $E_\lambda = E_{\lambda^{-1}}$ be the 2-dimensional subspace spanned by the eigenspaces of λ and λ^{-1}.

Let ω be the canonical symplectic form on $\mathbb{R}^{2n}$ and extend ω and A to $\omega_{\mathbb{C}}$ and $A_{\mathbb{C}}$ in the complexification $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C}$. By definition $A_{\mathbb{C}}$ and $\omega_{\mathbb{C}}$ agree with A and ω on $\mathbb{R}^{2n} \otimes 1$.

The identity for eigenvectors v_λ and v_η:

$$\omega_{\mathbb{C}}(v_\lambda, v_\eta) = \omega_{\mathbb{C}}(A_{\mathbb{C}} v_\lambda, A_{\mathbb{C}} v_\eta) = \lambda\eta\, \omega_{\mathbb{C}}(v_\lambda, v_\eta),$$

implies that, unless $\lambda\eta = 1$, we have $\omega_{\mathbb{C}}(v_\lambda, v_\eta) = 0$. Therefore $E_\lambda \otimes \mathbb{C}$ is symplectically orthogonal to $E_\eta \otimes \mathbb{C}$ for any $\lambda \neq \eta, \eta^{-1}$.

In particular, this implies that the $E_\lambda \otimes 1$ are symplectic subspaces with respect to ω the real form, and symplectically orthogonal to each other. In each E_λ, A can be put in

Jordan real form with respect to a symplectic basis. By orthogonality we may construct a symplectic basis for $\mathbb{R}^{2n}$ by taking the union of symplectic bases for the E_λ. Then let P be the matrix which sends the standard $\mathbb{R}^{2n}$ basis to the constructed symplectic basis. $\square$

The next step is to construct a metric with a periodic orbit with simple real spectrum with an arbitrarily small perturbation of the metric. Following [15], this is accomplished by slightly perturbing a periodic orbit $\mathcal{O}$ rotating a complex eigenspace, and propagating the perturbation to a periodic orbit which shadows a homoclinic orbit of $\mathcal{O}$ that spends a long time near $\mathcal{O}$.

Recall the following definitions: an ε-pseudo-orbit for a flow Φ on a space X is a (possibly discontinuous) function $\gamma : \mathbb{R} \to X$ such that:

$$d(\gamma(t+\tau), \Phi^\tau(\gamma(t))) < \varepsilon \text{ for } t \in \mathbb{R} \text{ and } |\tau| < 1.$$

For γ a ε-pseudo-orbit, γ is said to be δ-shadowed if there exists a point $p \in X$ and a homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t)-t$ has Lipschitz constant δ and $d(\gamma(t), \Phi^{\alpha(t)}(p)) \leq \delta$ for all $t \in \mathbb{R}$.

The classic closing lemma for Anosov flows we need is:

Theorem 2.3.4. *[37] (Anosov Closing Lemma) If Λ is a hyperbolic set for a flow Φ then there are a neighborhood U of Λ and numbers $\varepsilon_0, L > 0$ such that for $\varepsilon \leq \varepsilon_0$ any compact ε-pseudo-orbit in U is $L\varepsilon$-shadowed by a unique compact orbit for Φ.*

We use it to prove the main result of this section:

Proposition 2.3.5. *Let*

$$\mathcal{G}_p^k := \{g \in \mathcal{G}^k : \exists \mathcal{O} : \lambda_i \neq \lambda_j, \lambda_i \in \mathbb{R}, \text{ where } (\lambda_1, ..., \lambda_n) := \vec{\lambda}^u(\mathcal{O}, g).\}.$$

In this situation, we say $\mathcal{O}$ has the pinching property for g. Then $\mathcal{G}_p^k$ is C^2-open and C^k dense in $\mathcal{G}^k$.

Proof. Fix a C^2-open set $\mathcal{U} \subseteq \mathcal{G}^k$. First, since $\mathcal{G}_d^k$ is C^2-open and dense and $\mathcal{G}^\infty$ is C^k-dense in $\mathcal{G}^k$, we may fix some $g_0 \in \mathcal{U} \cap \mathcal{G}_d^\infty$. Let $\mathcal{O}$ be as in Proposition 2.3.2.

Suppose that the vector $\vec{\lambda}^u(\mathcal{O}, g)$ has $2c$ entries in $\mathbb{C} \setminus \mathbb{R}$, for some $c > 0$. It suffices to show that there exists a metric g' in $\mathcal{U}$ which has a periodic orbit $\mathcal{O}'$ such that $\vec{\lambda}^u(\mathcal{O}', g')$ has $2(c-1)$ complex entries and all real entries distinct.

Along $\mathcal{O}$ there is a dominated splitting $E^u = E_1^- \oplus \cdots \oplus E_k^-$ such that each E_i is either 1 or 2-dimensional. Fix the smallest index $i \in \{1, ..., k\}$ such that $E_i^\pm$ is 2-dimensional and let P_{g_0} denote the Poincaré return map of the geodesic flow for a fixed section Σ transverse to the flow small enough so that $\mathcal{O} \cap \Sigma =: \{v\}$. By shrinking $\mathcal{U}$ further if needed we may assume that $\mathcal{U} \subseteq \mathcal{G}_d^k$, i.e., that the dominated splitting for φ_{g_0} along $\mathcal{O}$ persists for the continuation of $\mathcal{O}$ for all $g \in U$; thus, by Lemma 2.3.1 the map $g \mapsto \theta_g := |\arg(\lambda_g)|$ is well defined and continuous, where λ_g is an eigenvalue of $D\varphi_g$ on E_i^- on the continuation of $\mathcal{O}$.

Lemma 2.3.6. *There exists $g_1 \in \mathcal{U} \cap \mathcal{G}_{\mathcal{O}}^k$ such that $\theta_{g_1} \neq \theta_{g_0}$.*

Proof. The derivative of the Poincaré map is conjugate to $D\varphi_g|_{E^u \oplus E^s}$ over the closed orbit $\mathcal{O}$, so θ_{g_0} agrees with the argument of the eigenvalue of DP_{g_0} along the 2-dimensional Jordan block $F^- \subseteq T_v\Sigma$ mapped to E_i^- under the conjugation aforementioned. Moreover, let F^+ be the Jordan block corresponding to E_i^+ in the same manner.

Identifying the space of symplectic maps $T_v\Sigma \to T_v\Sigma$ with $\mathrm{Sp}(2n)$ there exists some neighborhood $\mathcal{V} \subseteq \mathrm{Sp}(2n)$ of the original map DP_{g_0}, such that for $A \in \mathcal{V}$ the Jordan block F has a continuation for A, and we call the norm of the argument of the eigenvalue of A along this continuation θ_A. Let $\mathcal{W} \subseteq \mathcal{V}$ be the set of matrices A such that $\theta_A \neq \theta_{g_0}$. If $\mathcal{W}$ is open and dense in $\mathcal{V}$ then by Remark 2.2.17 we may apply Corollary 2.2.16 to $\mathcal{W} \cup ((\mathrm{Sp}(2n) \setminus \mathrm{Cl}(\mathcal{V}))$, which will be open and dense in $\mathrm{Sp}(2n)$ to find that the set of metrics which has $\theta_{g_1} \neq \theta_{g_0}$ is dense (and open) in $\mathcal{U}$.

It remains to check that $\mathcal{W}$ is open and dense in $\mathcal{V}$. Openness is clear by continuous dependence of eigenvalues on matrix entries. For density, let R_θ be given by rotation of any angle of $\theta > 0$ on the subspaces F^-, F^+ and the identity on the other subspaces, satisfies

$R_\theta \Omega R_\theta^T = \Omega$, where Ω is the standard symplectic form. Then $R_\theta DP_{g_0}$ has $\theta_{R_\theta DP_{g_0}} \neq \theta_{g_0}$; since $\theta > 0$ can be made arbitrarily small, this finishes the proof. □

Let g_1 be given as in the lemma above, and for $0 \leq s \leq 1$ we let $g_s = sg_1 + (1-s)g_0$, which, if g_1 is taken sufficiently close to g_0, also satisfies $\{g_s\} \subseteq \mathcal{U} \cap \mathcal{G}^k_{\mathcal{O}}$. Clearly, the map $[0,1] \to \mathcal{G}^k$ given by $s \mapsto g_s$ is continuous. Also note that, by Proposition 2.2.15 (2), $\mathcal{O}$ is not only a closed orbit of φ_{g_s} for all $s \in [0,1]$, but it in fact has the same arc-length parametrization with respect to all g_s.

For the geodesic flow of g_0, fix w a transverse homoclinic point of v, i.e., $w \in W^u(v) \cap W^{cs}(v)$. Fix some $\varepsilon > 0$ so that the geodesic flow has local product structure at scale 2ε. Then there exists $t_1, t_2 > 0$ such that $\varphi_{g_0}^{-t_2}(w) \in W^u_\varepsilon(v)$, $\varphi_{g_0}^{t_1}(w) \in W^s_\varepsilon(v)$ and also a $C > 0$ such that for all $t > 0$:

$$d(\varphi_{g_0}^{-(t_2+t)}(w), \varphi_{g_0}^{-t}(v)) < C\varepsilon e^{-t},$$

$$d(\varphi_{g_0}^{t_1+t}(w), \varphi_{g_0}^{t}(v)) < C\varepsilon e^{-t}.$$

Hence for $n \in \mathbb{N}$ the $\gamma_n : \mathbb{R} \to SM$ given by

$$\gamma_n(t) = \varphi_{g_0}^{\tilde{t}-(t_2+n\ell)}(w), \text{ where } \tilde{t} = t \bmod (t_2 + t_1 + 2n\ell)$$

are ε_n-pseudo-orbits where $\varepsilon_n < 2C\varepsilon e^{-n\ell}$, by the fact that the minimal expansion of the geodesic flow is $\tau = 1$ by the assumption on curvature.

For n sufficiently large, there exist unique periodic w_n's which $L\varepsilon_n$-shadow γ_n. Let $w_{n,s}$ be continuations of w_n for $0 \leq s \leq 1$ (where $w_{n,0} = w_n$, by definition). Let w_s be the hyperbolic continuations of w. By uniqueness of shadowing, note that the $w_{n,s}$ can also be constructed by shadowing segments of the orbit of w_s. The following proposition shows we can extend the dominated splitting of $\mathcal{O}$ to the new orbits we defined:

Lemma 2.3.7. *There exists N large so that for each $0 < s < 1$ the compact invariant set*

$$K_{N,s} = \bigcup_{n \geq N} \mathcal{O}(w_{n,s}) \cup \mathcal{O}(w_s) \cup \mathcal{O},$$

for the geodesic flow φ_{g_s} of g_s admits a dominated splitting for the bundle $E^u = E^-_{s,1} \oplus \cdots \oplus E^-_{s,k}$ over $K_{m,s}$ coinciding with the dominated splitting of E^u over $\mathcal{O}$, and similarly for $E^s = E^+_{s,1} \oplus \cdots \oplus E^+_{s,k}$.

Proof sketch. See [15], Lemma 9.2. We sketch the proof for $s = 0$ which is almost identical to the result cited. Then since dominated splittings over compact invariant sets persists under C^1-small perturbations by an invariant cone argument, this shows the result for all $s \in [0, 1]$. Consider the case of E^u.

Since $w \in W^u(v) \cap W^{cs}(v)$, one can extend the dominated splitting of $\mathcal{O}$ to $\mathcal{O}(w)$ as follows. Consider the bundles over $\mathcal{O}$ given by $F^i = E^-_1 \oplus \cdots \oplus E^-_{j+1}$, and $G^i = E^-_j \oplus \cdots \oplus E^-_k$ for $i, j = 1, ..., k-1$. Then we define

$$E^j(w) := \phi^{cs}_{v,w} F^j(v) \cap \phi^{cu}_{v,w} G^j(v)$$

and extend the E^j bundles to $\mathcal{O}(w)$ by the derivative of the flow. Proof of continuity and domination of this splitting follows closely that in [15].

For N sufficiently large we observe that $K_{N,0}$ is contained in an arbitrarily small neighborhood of $\mathcal{O} \cup \mathcal{O}(w)$, so the dominated splitting extends by continuity. □

For each n, we let $\theta_n : [0, 1] \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be defined by setting $\theta_n(s)$ to be the argument of the eigenvalue of $D\varphi_{g_s}$ along $E^-_{s,i}$ on the closed orbit $w_{n,s}$. By Lemma 2.3.1, the θ_n are continuous so for each n they may be lifted to some $\tilde{\theta}_n : [0, 1] \to \mathbb{R}$.

The main result about these rotation numbers, whose proof is postponed to the next section due to its length, is:

Lemma 2.3.8. *There exists $n \in \mathbb{N}$ so that $|\tilde{\theta}_n(1) - \tilde{\theta}_n(0)| > 2\pi$.*

By continuity one then finds n, s such that $\tilde{\theta}_n(s)$ is an integer multiple of 2π, i.e., such that the eigenvalues in $\vec{\lambda}^u(\mathcal{O}(w_{n,s}), g_s)$ corresponding to the subspace $E^-_{i,s}$ are real. By another perturbation using Corollary 2.2.16, there exists a metric such that these eigenvalues become distinct. Then by induction on the other eigenspaces with complex eigenvalues, all eigenvalues are real and distinct.

To finish the proof, openness follows again by Lemma 2.3.1, since the requirements on the products of the eigenvalues is an open condition. □

2.3.2 Proof of Lemma 2.3.8

We apply the notions introduced in Section 2.2.3 to give a proof of Lemma 2.3.8.

Proof of Lemma 2.3.8. Fix N large enough so that $K_{N,s}$ satisfies the conclusion of Lemma 2.3.7. We begin with:

Proposition 2.3.9. *There exists $N' > N$, which we denote by N after this proposition, such that the bundles with total spaces E_s defined by the fibers $E_s(x) := E^-_{s,i}(x)$ over $x \in K_{N',s}$ are continuously trivializable, for each $s \in [0,1]$ and where i is some index as set in the proof of Proposition 2.3.5.*

Proof. First, note that it suffices to prove that E_s is trivializable for $s = 0$, since the bundles E_s vary continuously in the ambient space TSM as s varies.

We will construct a non-vanishing section of the frame bundle F associated to E_0 over some $K_{N',0}$ for N' large, which is equivalent to a continuous choice of basis for E_0, proving triviality of the bundle.

For $\delta > 0$ let $B_\delta(\mathcal{O})$ a δ-tubular neighborhood of $\mathcal{O}$. If δ is sufficiently small relative to the scale of local product structure of the Anosov flow, for all $n \geq N'$, and N' sufficiently large, $\mathcal{O}(w_n)_\delta := B_\delta(\mathcal{O}) \cap \mathcal{O}(w_n)$ consists of a connected segment of the embedded circle $\mathcal{O}(w_n)$ and moreover, $\mathcal{O}(w)_\delta := B_\delta(\mathcal{O}) \cap \mathcal{O}(w)$ consists of the complement of a connected closed interval in $\mathcal{O}(w)$, i.e., two immersed connected components.

Note that we may assume that the map $D\varphi_{g_0}^{\ell(w_n)}$ preserves orientation on E_0 over any periodic orbit $\mathcal{O}(w_n)$, since otherwise it would have real eigenvalues (any $A \in \mathrm{GL}(2,\mathbb{R})$ with negative determinant has real eigenvalues) and we would obtain a proof of Lemma 2.3.8. Hence, the bundle E_0 is trivializable over any $\mathcal{O}(w_n)$. It is also clearly so over $\mathcal{O}(w)$, since it is an immersed real line, and we may assume it is too for $\mathcal{O}$, since otherwise, again, we would have real eigenvalues.

By shrinking δ further if necessary, there exists a well-defined closest point projection $p : B_\delta(\mathcal{O}) \to \mathcal{O}$ which is a surjective submersion. Fix a trivialization of E_0 over $\mathcal{O}$, i.e., a non-vanishing section $S : \mathcal{O} \to F$, which is possible by the previous paragraph.

For $x \in B_\delta(\mathcal{O}) \cap K_{N',0} =: K_\delta$, again shrinking δ further if necessary, there exists a unique length-minimizing geodesic segment between x and $p(x)$, and by parallel transporting $E_0(p(x))$ along such segments and then projecting orthogonally onto $E_0(x)$ one obtains a continuous bundle map $E_0|_{K_\delta} \to E_0|_{\mathcal{O}}$ which is an isomorphism on fibers. This map induces a map $F|_{K_\delta} \to F_{\mathcal{O}}$ and so by pulling back the non-vanishing section $S : \mathcal{O} \to F$ we obtain a non-vanishing section, which we now denote by $S : K_\delta \to F$ since its restriction to $\mathcal{O}$ agrees with the previous S, of F over K_δ.

Recall that $\mathcal{O}(w)_\delta$ consists of two connected immersed components homeomorphic to $\mathbb{R}$. Since $\mathcal{O}(w)$ is contractible, it is possible to define a determinant on $F|_{\mathcal{O}(w)}$; up to scalar it is unique, and hence there is a well defined continuous sign function on each fiber. Then we claim that $S|_{\mathcal{O}(w)_\delta}$ has the same determinant sign on both components, so that it may be extended to a continuous section $\mathcal{O}(w) \to F|_{\mathcal{O}(w)}$. Suppose not for a contradiction.

Define the line bundle $L := \bigwedge^2 E^0$ over K, which restricted to individual orbits is trivial since E_0 is. At each point $x \in K$ there is a natural map $F(x) \to L(x)$ given by $(e_1, e_2) \mapsto e_1 \wedge e_2$, which extends to a continuous global map $W : F \to L$. Considering the image of $S|_{\mathcal{O}(w)_\delta}$ under W, we obtain a section $\mathcal{O}(w)_\delta \to L$, which has opposite signs in the two connected components. Let $B : \mathcal{O}(w) \to L$ be any extension of this section to all of $\mathcal{O}(w)$; by the previous remark, B must have an odd number of zeros.

By continuity

$$\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O}) \to \mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$$

as $n \to \infty$, so by continuity of the bundle for n sufficiently large we can parallel transport the section B on $\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$ to $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ to obtain a section B_n on $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ which has the same number of zeros as B on $\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$, i.e., oddly many.

On the other hand as $n \to \infty$,

$$B_n|_{\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})} \to (W \circ S)|_{\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})},$$

and hence for n large enough B_n has constant sign on $\mathcal{O}(w_n)_\delta \setminus B_{\delta/2}(\mathcal{O})$. Thus B_n extends to $\mathcal{O}(w_n)_\delta$ without any zeros. Hence we obtain a global section B_n on $\mathcal{O}(w_n)$ with an odd number of zeros, contradicting the triviality of L over $\mathcal{O}(w_n)$.

Hence we may extend $S|_{\mathcal{O}(w)_\delta}$ continuously to all of $\mathcal{O}(w)$. Since in K_δ the section S is continuous, and again $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O}) \to \mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})$ as $n \to \infty$, we can then continuously extend $S|_{\mathcal{O}(w) \setminus B_{\delta/2}(\mathcal{O})}$ to $\mathcal{O}(w_n) \setminus B_{\delta/2}(\mathcal{O})$ while agreeing with S in K_δ. Since $S|_{\mathcal{O}(w)}$ is non-vanishing, the section obtained in this way is also globally non-vanishing.

□

The projectivization $\mathcal{P}E_s$ of the bundle E_s then defines a trivial circle bundle over $K_{N,s}$, and we fix a trivializing bundle isomorphism $\phi_s : \mathcal{P}E_s \to K_{N,s} \times S^1$. By conjugating with ϕ_s, the derivative of the geodesic flow then defines a continuous cocycle $\mathcal{A}_s$ on $K_{N,s} \times S^1$ over the geodesic flow, so we may apply the results of Section 2.2.3 for $\mathcal{A}_s$.

Then the the rotation numbers have the following characterization over periodic orbits:

Lemma 2.3.10. *For a closed orbit $\mathcal{O}(u)$ of a point $u \in K_{N,s}$, the argument $\theta(u)$ of the eigenvalue of the return map of the geodesic flow on $E^-_{s,i}$ satisfies:*

$$\theta(u) = \ell(u) \cdot \rho_{\mathcal{O}(u)} \, (mod \; 2\pi)$$

where $\ell(u)$ is the period of u, and $\rho_{\mathcal{O}(u)}$ is as in Remark 2.2.10 for the cocycle $\mathcal{A}_s$ defined above.

Proof. On one hand, it follows from the definition of ρ that $\ell(u) \cdot \rho_{\mathcal{O}(u)}$ agrees mod 2π with the Poincaré rotation number for the map

$$(A_s)_u^{\ell(u)} : S^1 \to S^1.$$

On the other, the projectivization of the derivative of the flow also defines on the fiber a homeomorphism $S^1 \to S^1$ with Poincaré rotation number equal to the argument of the eigenvalue of the derivative.

Since the two above differ by a conjugation given by $\pi_2 \circ \phi_s(u, \cdot) : S^1 \to S^1$, where $\pi_2 : K_{N,s} \times S^1$ is the natural projection, by invariance we obtain the result. $\square$

Applying Lemma 2.3.10 to the $\theta_n(s)$, we obtain for $0 \leq s \leq 1$:

$$\theta_n(s) = \ell(w_{n,s})\rho_{\mathcal{O}(w_{n,s})} \,(\mathrm{mod}\, 2\pi),$$

By continuity of the functions θ_n we may lift them to $\tilde{\theta}_n : [0,1] \to \mathbb{R}$ satisfying $\tilde{\theta}_n(0) = \ell(w_{n,0})\rho_{\mathcal{O}(w_{n,0})}$. By the continuity of $\rho_{\mathcal{O}(w_{n,s})}$ in s, given by Proposition 2.2.12, our choice of lift then implies:

$$\tilde{\theta}_n(s) = \ell(w_{n,s})\rho_{\mathcal{O}(w_{n,s})}, \text{ for } 0 \leq s \leq 1.$$

Let $\theta(s)$ be the argument of the eigenvalue of the $D\varphi_{g_s}$ on $E^-_{i,s}$ on the periodic orbit $\mathcal{O}$ (recall $\mathcal{O}$ is a closed geodesic for all g_s with $\ell(\mathcal{O})$ fixed), and repeat the constructions above to obtain $\tilde{\theta}(s)$ as well satisfying

$$\tilde{\theta}(s) = \ell(\mathcal{O})\rho_{\mathcal{O}}(s), \tag{2.2}$$

where $\rho_{\mathcal{O}}(s)$ is $\rho_{\mathcal{O}}$ of the geodesic flow of g_s.

Since $\mu_{\mathcal{O}(w_{n,s})} \to \mu_{\mathcal{O}}$ (where $\mu_{\mathcal{O}}$ is the invariant probability measure supported on the closed orbit $\mathcal{O}$) we have $\rho_{\mathcal{O}(w_{n,s})} \to \rho_{\mathcal{O}}(s)$ as $n \to \infty$ by Theorem 2.2.9. By hypothesis $\theta(1) \neq$

$\theta(0)$, and since $\ell(\mathcal{O})$ is constant as s varies, Equation (2.2) gives that $\rho_{\mathcal{O}}(1) - \rho_{\mathcal{O}}(0) \neq 0$. Hence for n large enough there exists some $\delta > 0$ such that $|\rho_{\mathcal{O}(w_{n,1})} - \rho_{\mathcal{O}(w_{n,0})}| \geq \delta$.

Finally, let $\delta_n = |\ell(w_{n,1}) - \ell(w_{n,0})|$. Again, we defer the proof of the following final proposition we need:

Proposition 2.3.11. *There exists $M_2 > 0$ such that $\delta_n < M_2$ for all $n \in \mathbb{N}$.*

With Lemma 2.3.11, we complete the proof of Lemma 2.3.8:

$$\begin{aligned}
|\tilde{\theta}_n(1) - \tilde{\theta}_n(0)| &= |\ell(w_{n,1})\tilde{\rho}_{\mathcal{O}(w_{n,1})} - \ell(w_{n,0})\tilde{\rho}_{\mathcal{O}(w_{n,0})}| \\
&\geq |\ell(w_{n,1})(\tilde{\rho}_{\mathcal{O}(w_{n,1})} - \tilde{\rho}_{\mathcal{O}(w_{n,0})})| \\
&\quad - |(\ell(w_{n,1}) - \ell(w_{n,0}))\tilde{\rho}_{\mathcal{O}(w_{n,0})}| \\
&\geq \delta\ell(w_{n,1}) - \delta_n|\tilde{\rho}_{\mathcal{O}(w_{n,0})}| \\
&> \delta\ell(w_{n,1}) - M_1 M_2 > 2\pi
\end{aligned}$$

for all n sufficiently large, since $\ell(w_{n,1}) \to \infty$. □

At last, we prove Proposition 2.3.11.

Proof of Proposition 2.3.11. To bound the variations δ_n, we use exponential shadowing and Hölder continuity of the geodesic stretch, defined below. Since the geodesic flow is unperturbed on $\mathcal{O}$ and the orbits $\mathcal{O}(w_{n,s})$ approximate $\mathcal{O}$, the two mentioned properties give us the bound on δ_n.

Recall that the w_n are constructed by shadowing $\gamma_n : \mathbb{R} \to SM$ given by

$$\gamma_n(t) = \varphi_{g_0}^{\tilde{t}-(t_2+n\ell)}(w), \text{ where } \tilde{t} = t \bmod (t_2 + t_1 + 2n\ell),$$

which is a ε_n-pseudo-orbit, where t_1 (resp. t_2) is such that $\varphi_{g_0}^{t_1}(w)$ (resp. $\phi_{g_0}^{-t_2}(w)$) is in $W_\varepsilon^s(v)$ (resp. $W_\varepsilon^u(v)$) and $\varepsilon_n < 2C\varepsilon e^{-n\ell}$.

The following well-known theorem is an adaptation for flows of the usual “exponential" shadowing theorem, which uses the Bowen bracket in its proof. The statement gives a sharper estimate on how well shadowing orbits approximate pseudo-orbits:

Theorem 2.3.12. *[38, Theorem 6.2.4] For a hyperbolic set* Λ *of a flow* Φ *on a closed manifold* $\exists c, \eta > 0$ *such that* $\forall \varepsilon > 0$, $\exists \delta > 0$ *so that: if* $x, y \in \Lambda$, $s : \mathbb{R} \to \mathbb{R}$ *continuous,* $s(0) = 0$ *and* $d(\Phi^t(x), \Phi^{s(t)}(y)) < \delta$ *for all* $|t| \leq T$*, then*

(1) $|t - s(t)| < 3\varepsilon$ *for all* $|t| \leq T$*,*

(2) there exists $t(x, y)$ *with* $|t(x, y)| < \varepsilon$ *so that the* ε*-stable manifold* $\Phi^{t(x,y)}(x)$ *intersects uniquely the* ε*-unstable manifold of* y *and:*

$$d(\Phi^t(y), \Phi^t(\Phi^{t(x,y)}(x))) < ce^{\eta(T-|t|)} \text{ for } |t| < T.$$

In the context of the current proof, we apply the above theorem as follows.

Let $T_n = \ell(w_n)$, $x = \varphi_{g_0}^{T_n/2}(w_n)$ and $y = \varphi_{g_0}^{\tau_n}(w)$ where $\tau_n := T_n/2 - (t_2 + n\ell)$. For n sufficiently large, $d(\varphi_{g_0}^t(x), \varphi_{g_0}^{s(t)}(y)) < \delta$ is satisfied, by the statement of shadowing, for $|t| < T_n/2$ and δ given by the theorem for the $\varepsilon > 0$ fixed before. Then the theorem gives a $t_n \in \mathbb{R}$ such that:

$$d(\varphi_{g_0}^{t_n+t}(w_n), \varphi_{g_0}^{\tau_n+t}(w)) < ce^{\eta(T_n/2-|t|)}, \text{ for } |t| < T_n/2.$$

Now we turn to computing the period of $w_{n,1}$ using the facts established above. By structural stability, there exists $h : SM \to SM$ which conjugates the orbits of φ_{g_0} to those of φ_{g_1}. This conjugacy can be taken to be Hölder continuous and C^1 along the flow direction. Thus, there exists some $a : SM \to \mathbb{R}$ which is Hölder continuous with some exponent $1 \geq \beta > 0$, such that for $u \in SM$:

$$dh(u)X_{g_0}(u) = a(u)X_g(h(u)),$$

where X_g (resp. X_{g_0}) is the vector field generating the geodesic flow for g (resp. g_0). The function a is referred to as the geodesic stretch, and the proof of the facts above can be found, for instance, in [44, p. 12-13]

The period of $w_{n,1}$ is given by the formula:

$$\ell(w_{n,1}) = \int_0^{T_n} a(\varphi_{g_0}^t(w_n))\, dt$$

By Proposition 2.2.15 (2), since $\mathcal{O}$ is a closed geodesic, with same arclength parametrization for g_0 and g_1, it is clear that $a|_{\mathcal{O}} \equiv 1$. Therefore, we may compute the difference $\delta_n = |\ell(w_{n,1}) - \ell(w_{n,0})|$ as follows:

$$\begin{aligned} |\ell(w_{n,1}) - \ell(w_{n,0})| &\leq \int_{-T_n/2}^{T_n/2} |a(\varphi_{g_0}^t(w_n)) - 1|\, dt \\ &\leq M \int_{-T_n/2}^{T_n/2} d(\varphi_{g_0}^{t_n+t}(w_n), \mathcal{O})^\beta\, dt, \end{aligned}$$

since a is β-Hölder continuous and the distance between a point and a compact set is well defined. To estimate the distance, note:

$$\begin{aligned} d(\varphi_{g_0}^{t_n+t}(w_n), \mathcal{O}) &\leq d(\varphi_{g_0}^{t_n+t}(w_n), \varphi_{g_0}^{\tau_n+t}(w)) + d(\varphi_{g_0}^{\tau_n+t}(w), \mathcal{O}) \\ &\leq c(e^{\eta(T_n/2-|t|)} + e^{-|t|}), \text{ for } |t| < T_n/2, \end{aligned},$$

since w is a homoclinic point of $\mathcal{O}$ so $d(\varphi_{g_0}^{\tau_n+t}(w), \mathcal{O}) \leq ce^{-|t|}$ for some $c > 0$ which we assume, by taking the max if necessary, is the same as the previous c. Substituting this inequality into the previous integral, we obtain:

$$|\ell(w_{n,1}) - \ell(w_{n,0})| \leq M \int_{-T_n/2}^{T_n/2} (e^{\eta(T_n/2-|t|)} + e^{-|t|})^\beta\, dt < M_2 < \infty,$$

for M_2 independent of n, as an easy calculus exercise shows. □

2.3.3 Twisting

Following the previous section, we fix a metric $g_0 \in \mathcal{G}_p^k$. Let $\mathcal{O}$ be the orbit with the pinching property, $v \in \mathcal{O}$ and l the period of $\mathcal{O}$. We fix an arbitrary $w \in W^s_{g_0}(v) \cap (W^{cu}_r)_{g_0}(v)$ a transverse homoclinic point of the orbit of v, and consider the following composition of cocycle holonomy maps for the unstable bundle E^u:

$$\psi^{g_0}_{v,w} = h^s_{w,v} \circ h^{cu}_{v,w}$$

given by Theorem 2.2.6 and Proposition 2.2.7. Existence of w satisfying the above properties is given by the existence of homoclinic points and the fact that, by considering $\varphi^t(w)$ if needed we may always choose w to lie in the local center-unstable manifold of v, so that the center-unstable holonomy is well defined (c.f. Theorem 2.2.6 again), and moreover so that $w \in W^s_{g_0}(v)$ simultaneously.

Recall that $\vec{\lambda}^u(\mathcal{O}, g)$ consists of distinct real numbers, so let $\{e_i\}$ be an (non-generalized, real) eigenbasis for E^u. For all $1 \leq j \leq n$ the alternating powers $\Lambda^k E^u(v)$ have a basis obtained as exterior products of the e_i. We write $e_I^j := e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $I = \{i_1, ..., i_j\}$.

Proposition 2.3.13. *For $g_0 \in \mathcal{G}_p^k$ as above we say g_0 has the twisting property for $w \in SM$ with respect to v, and we write $g_0 \in \mathcal{G}^k_{p,t}$, if*

$$\forall e_I^j, e_{I'}^l,\ j + l = n : (\wedge^k \psi^{g_0}_{v,w})(e_I^j) \wedge e^l_{I'} \neq 0,$$

which is to say that the image of any direct sums of eigenspaces intersects any direct sum of eigenspaces of complementary dimension only at the origin.

The set $\mathcal{G}^k_{p,t}$ is C^2-open and C^k-dense in $\mathcal{G}^k$.

Proof. Again, by density of $\mathcal{G}^\infty \subseteq \mathcal{G}^k$ and openness of $\mathcal{G}^k_p$ we may assume that $g_0 \in \mathcal{G}^\infty$ so we can apply Theorem 2.2.13. For some small $\varepsilon > 0$, consider the geodesic segment $\gamma = \varphi^{g_0}_{[0,\varepsilon]}(w)$. Note that since $\mathcal{O}(w)$ accumulates as $|t| \to \infty$ on the compact set $\mathcal{O}$, if we take $\varepsilon > 0$ small enough we may take $\pi(\gamma)$ to be disjoint from $\pi(\mathcal{O}(w) \setminus \gamma) \cup \pi(\mathcal{O})$, where $\pi : SM \to M$ is the projection map.

Then we apply Theorem 2.2.13 to $\gamma' \subseteq \gamma$, where $\gamma' = \varphi^{g_0}_{[\delta, \varepsilon-\delta]}(w)$ for $\delta > 0$ small, to perturb $D_w \varphi^{\varepsilon}_{g_0}$ by perturbing the metric only on a tubular neighborhood $V_{\gamma'}$ of γ' small enough (possible by Proposition 2.2.15 (1)) so that

$$V_{\gamma'} \cap \mathrm{Cl}(\pi(\mathcal{O}) \cup \pi(\mathcal{O}(w) \setminus \gamma)) = \varnothing.$$

where Cl denotes closure.

By equivariance of holonomies the map $\psi^{g_0}_{v,w}$ can be rewritten as:

$$\psi^{g_0}_{v,w} = h^s_{\varphi^{\varepsilon}_{g_0}(w),v} \circ D_w \varphi^{\varepsilon}_{g_0}|_{E_u} \circ h^{cu}_{v,w}.$$

Then observe that perturbations to the metric of the form described in the previous paragraph affect only the $D_w \varphi^{\varepsilon}_{g_0}|_{E_u}$ term in the composition above. Indeed, we recall that $h^{cu}_{w,v}$ depends only on the values of the cocycle on a neighborhood of the $(-\infty, 0]$ part of the orbit $\varphi^t_{g_0}(w)$, and $h^s_{\varphi^{\varepsilon}_{g_0}(w),v}$ on a neighborhood of the $[\varepsilon, \infty)$ part of the orbit $\varphi^t_{g_0}(w)$ and on the cocyle along $\mathcal{O}$. By construction of $V_{\gamma'}$, the cocyle is not perturbed in any of these sets.

It remains to check that for an open and dense set of 1-jets of symplectic maps P from a small transversal to the flow at w to a small transversal section to the flow at $\varphi^{\varepsilon}_{g_0}(w)$ the map $\psi^{g_0}_{v,w}$ has the twisting property (we assume both transversals to be tangent to E^u at w and at $\varphi^{\varepsilon}_{g_0}(w)$, respectively), if we replace $D_w \varphi^{\varepsilon}_{g_0}|_{E_u}$ by $DP|_{E^u}$. This implies by Theorem 2.2.13 that we can construct such a small perturbation in the space of metrics, completing the proof.

Since both holonomy maps in the composition defining $\psi^{g_0}_{v,w}$ as above are symplectic isomorphisms, an open and dense subset of $\mathrm{Sp}(E^u(v) \oplus E^s(v))$ is mapped under composition with the holonomies to an open dense set of the 1-jets of symplectic maps P as above, so it suffices to check that twisting holds when the map $\psi^{g_0}_{v,w}$ takes value in an open and dense subset of $\mathrm{Sp}(E^u(v) \oplus E^s(v))$.

Again, observe that the condition defining twisting is given by a Zariski open subset of the matrices $\mathrm{Sp}(E^u(v) \oplus E^s(v))$. Hence, as long this set is non-empty the twisting set must also be open and dense in the analytic topology. Then by the paragraph above, this translates to an open and dense condition in 1-jets of symplectic maps P, and as there is no condition imposed on higher jets, we obtain the desired result by Remark 2.2.17.

To finish the proof, it thus suffices to check that the Zariski open set defining twisting is non-empty in the symplectic group, which is done below. ☐

Lemma 2.3.14. *There exists a matrix $A \in Sp(2n)$, where $\mathbb{R}^{2n}$ is taken with standard symplectic basis $\{e_i, f_i\}$ such that A preserves $E^u := span\{e_i\}_{i=1}^n$ and*

$$\forall e_I^j, e_{I'}^l,\ j + l = n : (\wedge^k A)(e_I^j) \wedge e_{I'}^l \neq 0.$$

Proof. Note that for fixed $e_I^k, e_{I'}^j$ the property that $(\wedge^j A)(e_I^j) \wedge e_{I'}^l \neq 0$ is open in $\mathrm{Sp}(2n)$. Thus by induction it suffices to show that for any $e_I^j, e_{I'}^l$ one can arrange so that $(\wedge^j A)(e_I^j) \wedge e_{I'}^l \neq 0$ and moreover A still preserves E^u, by an arbitrarily small perturbation of $A \in \mathrm{Sp}(2n)$. Then by induction the proof is completed by performing sucessively small perturbations over all pairs I, I'.

To prove the claim, suppose $(\wedge^j A)(e_I^j) \wedge e_{I'}^l = 0$, and write $(\wedge^j A)(e_I^j) = \sum_J a_J e_J^j$. Since A is invertible, there exists J_0 such that $a_{J_0} \neq 0$ and such that $|J_0 \cap I'|$ is minimal. Since $|J_0| + |I'| = n$, we have $|J_0 \cap I'| = \{1, \ldots, n\} \setminus (J_0 \cup I')$, so we take an arbitrarily chosen bijection $r \mapsto s_r$ from $J_0 \cap I$ to $\{1, \ldots, n\} \setminus (J_0 \cup I')$.

For $\theta > 0$, and $r, s \in \{1, \ldots, n\}$ let $R_\theta^{r,s}$ given by rotating the (oriented) planes $\mathrm{span}(e_r, e_s)$ and $\mathrm{span}(f_r, f_s)$ by θ and preserving the other basis elements. Let A' be obtained by composing A with each of R_θ^{r,s_r} for $r \in J_0 \cap I$ (in any order, since the rotation matrices commute). One checks directly that $R_\theta^{r,s}\Omega(R_\theta^{r,s})^T = \Omega$, where Ω is the standard symplectic form, so $R_\theta^{r,s}$

preserves E^u so A' is symplectic and preserves E^u. Writing

$$\prod_{r\in J_0\cap I'}(\wedge^k R_\theta^{r,s_r})e_J^k=\sum_L b_L e_L^j,$$

by a direct computation one checks that $b_{\{1,\ldots,n\}\setminus I'}\neq 0$ if and only if $J=J_0$, which implies that $(\wedge^j A')(e_I^j)\wedge e_{I'}^l\neq 0$. □

2.4 PROOF OF THEOREM 2.1.1

We finish the proof of the Theorem 2.1.1. In what follows, let $\sigma:\Sigma\to\Sigma$ be the shift map of an invertible subshift of finite type Σ. The suspension of Σ under a continuous $f:\Sigma\to\mathbb{R}^+$ is the compact metric space:

$$\Sigma_f:=(\Sigma\times\mathbb{R})/((x,s)\sim\alpha^n(x,s),\, n\in\mathbb{Z}),$$

where $\alpha(x,s):=(\sigma(x),s-f(x))$. The shift σ lifts to a continuous-time system $\sigma_f^t:\Sigma_f\to\Sigma_f$ given by $\sigma_f^t(x,s)=(x,s+t)$ for $t\in\mathbb{R}$.

First, we need to represent Anosov flows by the suspension of a shift. The following is the standard statement of the construction of a Markov partition for an Anosov flow:

Theorem 2.4.1. *[38, Theorem 6.6.5] Let $\Phi:M\to M$ be a C^1 Anosov flow. There is a semiconjugacy from a hyperbolic symbolic flow to Φ that is finite-to-one and one-to-one on a residual set of points, where the roof function for the subshift of finite type corresponds to the travel times between the local sections for the smooth system.*

At last we prove Theorem 2.1.1.

Proof of Theorem 2.1.1. We prove that the statement holds for all $g\in\mathcal{G}_{p,t}^k$, so the theorem is proved by Proposition 2.3.13. Fix some such $g_0\in\mathcal{G}_{p,t}^k$ and let $v,w\in SM$ be the vectors along whose orbits pinching and twisting hold respectively.

Let Σ_f be a suspension of a subshift of finite type and $P : \Sigma_f \to SM$ be the semi-conjugacy map to the geodesic flow of g_0 given by Theorem 2.4.1. Following the proof of Theorem 2.4.1 in [37], we see that it is possible to construct the Markov partition so that $v \in SM$ has a unique lift (p, t) to the suspension of the shift space Σ_f by enlarging the Markov rectangles by an arbitrarily small amount so that the orbit of v only intersect their interiors, where $f : \Sigma \to \mathbb{R}$ is some roof function. Then by [37, Claim 6.6.9, Corollary 6.6.12] there is also a unique (q, s) which lifts the homoclinic point with twisting w.

We write $\mathcal{E} \to \Sigma_f$ for the pullback of the bundle $E^u \to SM$ to Σ_f under P, and by $A^t : \mathcal{E} \to \mathcal{E}$ the pullback of the derivative cocycle. By using the return map of A^t to the 0 section of Σ_f, the cocycle A^t determines a discrete time cocycle A on $\mathcal{E} \to \Sigma$ identified with $\Sigma \times \{0\} \subseteq \Sigma_f$. Following the propositions in Section 2.1 of [14] there exists a distance on Σ which makes the cocycle A dominated, so that it admits holonomies H^s and H^u.

Recall that the local stable and unstable manifolds $W^s_{loc}(\overline{x})$ (resp. $W^u_{loc}(\overline{x})$) for the shift space Σ are defined as the sequences $\overline{y}$ such that $(\overline{y})_i = (\overline{x})_i$ for all $i \geq 0$ (resp. ≤ 0), where the subscript i denotes the i-th entry of $\overline{x}$ and $\overline{y}$ regarded as a sequence in the shift space Σ. By reducing the size of the rectangles in the original construction if necessary, it is possible to ensure that points in the same local stable/unstable manifold in the shift Σ are mapped to the same local center stable/unstable manifold in SM by the semi-conjugacy P. Then we can prove the following lemma which verifies agreement of holonomies of the geodesic flow and its symbolic discrete representation:

Lemma 2.4.2. *Let $x = \overline{x} \times \{0\} \in \Sigma_f$, and $y = \overline{y} \times \{0\} \in \Sigma_f$, where $\overline{y}$ is in the local stable manifold. Let $v = P(x)$ and $w = P(y)$ which by the previous paragraph lie in the same local center-unstable manifold in SM. Then $h^{cs}_{vw} = H^s_{\overline{x}\overline{y}}$ and the analogous result holds for unstable holonomies.*

Proof. By the proof of existence of holonomies as in [15], one obtains the holonomy map as a limit:

$$H^s_{\overline{x},\overline{y}} = \lim_{n\to\infty}((A^n)_{\overline{x}})^{-1} \circ I_{\sigma^n\overline{x}\sigma^n\overline{y}} \circ (A^n)_{\overline{y}}.$$

As $n \to \infty$, note that $(\sigma^n\overline{x}) \times \{0\}$ and $(\sigma^n\overline{y}) \times \{0\}$ converge to the same stable manifold in Σ_f. Hence, if we let $T_n := \sum_{i=0}^{n-1} f(\sigma^i\overline{x})$ so that $(A^n)_{\overline{x}} = (A^{T_n})_{\overline{x}\times\{0\}}$, then $\sum_{i=0}^{n-1} f(\sigma^i\overline{y}) - (T_n + r) \to 0$, as $n \to \infty$, where $r \in \mathbb{R}$ is such that $\sigma^r_f(y) \in W^s(x)$.

On the other hand, using the formula defining the holonomies and the definition of A^t as a pullback cocycle of $D\varphi^t_{g_0}|_{E^u}$:

$$\begin{aligned} h^{cs}_{vw} &= \lim_{T\to\infty}(D\varphi_{g_0}|^T_{E^u})^{-1}_v \circ I_{\varphi^T_{g_0}(v),\varphi^{T+r}_g(w)} \circ (D\varphi_{g_0}|^T_{E^u})_{\varphi^r_{g_0}(w)} \circ (D\varphi_{g_0}|^r_{E^u})_w, \\ &= \lim_{T\to\infty}(A^T)^{-1}_x \circ I_{\sigma^T_f y,\sigma^{T+r}_f y} \circ (A^T)_{\sigma^r_f y} \circ (A^r)_y, \end{aligned}$$

so letting $T = T_n$ we conclude that $H^s_{\overline{x},\overline{y}} = h^{cs}_{vw}$. □

With the above proposition it is straightforward to verify using the equivariance of holonomies (with respect to A) that the cocycle A over Σ is simple. Let $\rho : SM \to \mathbb{R}$ be a Hölder potential and μ_ρ its associated equilibrium state for the geodesic flow of g_0. Let $\tilde{\rho}$ be the Hölder continuous potential on Σ_f given by $\tilde{\rho} = \rho \circ P$, and $\tilde{\mu}_\rho$ its associated equilibrium state for $\sigma^t_f : \Sigma_f \to \Sigma_f$.

It is a well-known fact (see e.g. [17]) that P is in fact a measurable isomorphism between $(\Sigma_f, \tilde{\mu}_\rho)$ and (SM, μ_ρ). Hence the Lyapunov spectrum of A^t with respect to $\tilde{\mu}_\rho$ agrees with that of $D\varphi^t_{g_0}$ with respect to μ_ρ, and it suffices to show simplicity of the spectrum of the former.

Since $f : \Sigma \to \mathbb{R}$ is Hölder, identifying Σ with $\Sigma \times \{0\} \subseteq \Sigma_f$, the Hölder continuous function:

$$\left(\int_0^{f(x)} \tilde{\rho}(x,t)\,dt\right) - P(\sigma_f, \tilde{\rho}) f_g(x),$$

where $P(\sigma^t_f, \tilde{\rho})$ is the pressure of σ^t_f with respect to $\tilde{\rho}$, defines a potential on $\Sigma = \Sigma \times \{0\} \subseteq$

Σ_{f_g} and has a unique equilibrium state μ which satisfies, for $F \in C^0(\Sigma_f)$:

$$\int_{\Sigma_f} F \, d\tilde{\mu}_\rho = \frac{\int_\Sigma \left(\int_0^{f(x)} F(x,t) \, dt \right) d\mu}{\int_\Sigma f(x) \, d\mu}$$

by [38, Proposition 4.3.17]. In particular, since μ is an equilibrium state it has local product structure.

The product $\mu \times dt$ defines a measure for the suspension flow σ_1^t on Σ_1 (where 1 is the constant function 1) which has the same Lyapunov spectrum as μ. Since $\mu \times dt$ and $\tilde{\mu}_\rho$ are related by a time change, the Lyapunov spectrum of A^t with respect to μ_ρ and the Lyapunov spectrum of A with respect to μ differ by a scalar, see e.g. [20, Proposition 2.15]. Hence applying Theorem 2.2.3 to the simple cocycle A for the measure μ we obtain simplicity of the Lyapunov spectrum for μ_ρ. □

2.5 PROOF OF THEOREMS 2.1.2 AND 2.1.3

In this section we explain the needed modifications to the previous sections to give the proofs of Theorems 2.1.2 and 2.1.3:

Proof of Theorems 2.1.2 and 2.1.3. For $\frac{1}{2}$-bunched Anosov flows, the splitting $E^u \oplus E^0 \oplus E^s$ may not be C^1, so instead we consider the derivative cocycle on the C^1-bundles $Q^u := E^{cu}/E^0$ and $Q^s := E^{cs}/E^0$, which we have shown to be 1-bunched in Proposition 2.2.7. In what follows, we prove simplicity for the spectrum on Q^u and Q^s implies the desired result since $D\Phi$ on $Q^{u,s}$ has the same spectrum as $D\Phi$ on $E^{u,s}$.

For Theorem 2.1.3, recall that topological mixing is C^1-open and C^k-dense in the space of Anosov flows. Then we follow the propositions in Section 2.3 to construct orbits with pinching and twisting for the cocycle on Q^u by a C^k-small perturbation, which in this case is achievable since the analogue of Theorem 2.2.13 is clear in the space of all vector fields and $\mathfrak{X}^k_A(X)$ is open by structural stability in the space of all vector fields and moreover the linear algebra lemmas (Lemma 2.3.3, Lemma 2.3.14) needed for the case of $\mathrm{Sp}(2n)$ are immediate for $\mathrm{GL}(n)$. The C^1-openness of the conditions also is proved similarly. Then by a symmertric argument it is clear that pinching and twisting for both Q^u and Q^s is C^1-open and C^k-dense. The proof then follows the same outline in Section 2.4.

The proof of Theorem 2.1.2 is similar, in that the linear algebra lemmas (Lemma 2.3.3, Lemma 2.3.14) needed for the case of $\mathrm{Sp}(2n)$ are still immediate for $\mathrm{SL}(n)$. Moreover, topological mixing is known for all C^2-volume-preserving Anosov flows. Finally, it remains to prove an analogue of Theorem 2.2.13 for the conservative class, which we do in the next section. With that in hand, the proof also follows the same outline as Theorem 2.1.1. □

2.5.1 Conservative Perturbations

In this section we prove the analogue of Theorem 2.2.13 in the volume-preserving category. To the best of the author's knowledge the result is not found anywhere in the literature so the complete proof is included here. Throughout, we let $X \in \mathfrak{X}^\infty_m(M)$ be a non-vanishing vector field generating the flow φ_X on the smooth manifold M which preserves the smooth volume m. Fix an embedded segment of a flow orbit $l : [0, \varepsilon] \to M$ parametrized by the time-parameter and a small transversal smooth hypersurface $\Sigma(0)$ to X at $l(0)$.

For $t \in [0, \varepsilon]$, set $\Sigma(t) = \varphi^t_X(\Sigma(0))$ so that $\iota_X m$ is a volume form on the hypersurfaces $\Sigma(t)$. The following result, whose proof is elementary except for an application of the conservative pasting lemma, shows that it is possible to perturb the k-jets in the conservative setting generically by C^k-small perturbations.

Theorem 2.5.1. *Let Q be some dense subset of the space of k-jets of volume-preserving maps $(\Sigma(0), \iota_X m, l(0)) \to (\Sigma(\varepsilon), \iota_X m, l(\varepsilon))$.*

Then there is arbitrarily C^k-close to X an m-preserving X' such that:

(a) $Y := X' - X$ is supported in an arbitrarily small tubular neighborhood B of $l([\delta, \varepsilon - \delta])$, for some $0 < \delta < \varepsilon$;

(b) $Y = 0$ on $l([0,1])$ and Y is tangent to the hypersurfaces $\Sigma(t)$;

(c) The flow of X' generates a map $(\Sigma(0), l(0)) \to (\Sigma(\varepsilon), l(\varepsilon))$ with k-jets in Q.

Proof. If B is sufficiently small we may assume that it is foliated by the transversals $\Sigma(t)$ and, moreover, by passing to a further neighborhood we may assume that the transverse sections are mapped diffeomorphically onto each other by the flow X, i.e., we may construct the perturbation in a flow box with transversals given by the $\Sigma(t)$.

In the flowbox, a classic application of Moser's trick allows us to assume that the flow is in normal coordinates $\varphi_X^t(x_1, ..., x_{n-1}, s) \mapsto (x_1, ..., x_{n-1}, s+t)$, where the image of l is contained in $\{x_1 = ... = x_n = 0\}$ and $m = dx^1 \wedge ... \wedge dx^n$. In these coordinates, we may regard $B \cong U \times [0, \varepsilon]$, where $U \subseteq \mathbb{R}^{n-1}$ is a domain and so $Q \subseteq J_m^k(n-1, \mathbb{R})$, where $J_m^k(n-1, \mathbb{R})$ is the Lie group of k-jets of volume preserving maps fixing the origin.

Using the flow to identify the fibers of $U \times \mathbb{R} \to \mathbb{R}$, the problem is thus reduced to the construction, for each $\delta > 0$, of a time dependent vector field $\{Y_t\}_{t \in [0,\varepsilon]}$ on $\mathbb{R}^{n-1}$ with the following properties:

(a) $Y_t(0) = 0$ and $\mathrm{supp}(Y_t) \subseteq U$ for $t \in [0, \varepsilon]$;

(b) $Y_t \equiv 0$ on $[0, \delta]$ and $Y_t \equiv 0$ on $[\varepsilon - \delta, \varepsilon]$;

(c) Y_t is divergence free for all $t \in [0, \varepsilon]$;

(d) The time-ε map $f : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^{n-1}, 0)$ of Y_t has derivative at 0 in Q;

(e) $\|Y_t\|_{C^\infty} < \delta$ for all $t \in [0, \varepsilon]$;

The construction is given by first specifying the time-ε map f and then finding an appropriate isotopy within the volume-preserving category to the identity.

Fix some $\theta \in Q$ sufficiently close to the k-jets of I (the identity map) and a map $F : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^{n-1}, 0)$ whose k-jet at the origin is given by θ. Take some C^∞ bump function $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ which interpolates between the constant function 1 in $B(0, \eta/2)$ to the constant function 0 outside of $B(0,\eta)$ for some η small. Let $F' = \rho F$; if θ is sufficiently close to 0, then $||F' - I||_{C^k}$ is small so in particular $F' \in \mathrm{Diff}(\mathbb{R}^{n-1})$. Applying Moser's trick, we can find an $f \in \mathrm{Diff}_m(\mathbb{R}^{n-1})$, i.e. preserving m, which is C^k-close to the identity and which agrees with F' where it is conservative, namely, everywhere except $B(0,\eta) \setminus B(0,\eta/2)$. In particular, the k-jet of f at the origin equals $\theta \in Q$.

To obtain such an f, we construct a family $s \mapsto h_s \in \mathrm{Diff}(\mathbb{R}^{n-1})$ such that $h_1 = F$ and $h_0 =: f$ is conservative. Let $r : B \to \mathbb{R}$ be the smooth function C^k close to 1 satisfying $F_*\mu = r\mu$. Then for $s \in [0,1]$ we solve $(h_s)_*\mu = r^s\mu$, namely $\mathrm{div} Z_s = r^s \log r^s$, where $Z_s = \partial_s h_s$. Moreover, the proof of the Poincaré Lemma shows that we can take Z_s to be constant equal to 0 outside of of $B(0,\eta)$. By the conservative pasting lemma [85], there exists W_s which agrees with Z_s on a neighborhood of $\mathbb{R}^{n-1} \setminus (B(0,\eta) \setminus B(0,\eta/2))$ and is divergence-free. Then $Z'_s = Z_s - W_s$ also satisfies $\mathrm{div} Z'_s = r^s \log r^s$ and it is identically 0 where $r = 1$, so that $h_s(x) = F(x)$ on $B(0,\eta) \setminus B(0,\eta/2)$, where now $\partial_s h_s = Z'_s$. In particular, $f := h_0$ is the identity outside of $B(0,\eta)$ and its k-jet at the origin is given by θ. This constructs the desired f.

Now let $\alpha : [0,\varepsilon] \to [0,\varepsilon]$ be a C^∞ function such that $\alpha \equiv 0$ on $[0,\delta]$ and $\alpha \equiv \varepsilon$ on $[\varepsilon - \delta, \varepsilon]$. If $\|f - I\|_{C^k}$ is sufficiently small (which is ensured by taking θ closer to the jets of the identity), the maps $g_t := \alpha(t)f + (1 - \alpha(t))I$ are all diffeomorphisms and $t \mapsto \partial_t g_t$ is a time-dependent vector field that satisfies all desired properties except for being divergence-free.

To repair that, again Moser's trick constructs a family $s \mapsto g_{t,s}$ such that $g_{t,0} = g_t$ and $g_{t,1}$ is conservative as follows. Let $r_t : B \to \mathbb{R}$ be the smooth 1-parameter family of smooth functions C^k close to 1 satisfying $(g_t)_*\mu = r_t\mu$. Then for $s \in [0,1]$ we solve $(g_{t,s})_*\mu = r_t^s\mu$, namely $\mathrm{div} Z_{t,s} = r_t^s \log r_t^s$, where $Z_{t,s} = \partial_s g_{t,s}$. It is an easy consequence of the proof of the Poincaré lemma that the family $Z_{t,s}$ may be taken to be smooth in t with small t derivatives, since $t \mapsto r_t$ as a 1-parameter family has the same properties. Moreover, we can take supp $Z_{t,s} \subseteq \mathrm{supp}\,(r_t - 1)$. The C^k norm of the $Z_{t,s}$ is a continuous function of the C^k norm of

$r_t^s \log r_t^s$, so that taking $Y_t = \partial_t g_{t,1}$ finishes the proof. $\square$

CHAPTER 3

SYMMETRIES OF GEODESIC FLOWS ON COVERS AND RIGIDITY

This chapter presents material first appearing in the article:

Mitsutani, Daniel, "Symmetries of geodesic flows on covers and rigidity,"

Preprint, arXiv:2402.09576v1, https://doi.org/10.48550/arXiv.2402.09576 [76].

3.1 INTRODUCTION

It is a widely explored theme in geometry that symmetries occur mostly in special circumstances. For instance, a generic Riemannian metric g on a closed manifold M admits no isometries. For metrics of negative Ricci curvature an old result of Bochner shows that closed manifolds admit only finitely many isometries. While it is easy to perturb a locally symmetric metric of negative sectional curvature on a closed manifold without changing its isometry group, in this setting it has been known that rigidity occurs when such a metric admits "too many" isometries in some sense. The idea, pioneered by Eberlein, is to look for excess symmetries on covers: if the isometry group of the universal cover of an (M, g) of negative sectional curvature is not discrete then g is locally symmetric [22]. This result was later extended with other methods in great generality in [32] to any aspherical Riemannian manifold M with an arbitrary metric.

In this chapter we will assume that M is a closed manifold and g is a smooth Riemannian metric of negative sectional curvature $K \leq -1$. Associated to g is a hyperbolic dynamical system on the unit tangent bundle SM referred to as the geodesic flow φ_t, which flows a unit tangent vector along its tangent unit-speed oriented geodesic for time t. In our setting, the geodesic flow φ_t is an Anosov flow, meaning that there exists a $D\varphi^t$-invariant splitting $TSM = E^u \oplus E^s \oplus \mathbb{R}X_g$ and constants $\tau > 1, C > 1$ so that for $t \geq 0$:

$$C^{-1}e^t\|v_u\| \leq \|D\varphi^t(v_u)\| \leq Ce^{\tau t}\|v_u\|,\, v_u \in E^u,$$
$$C^{-1}e^t\|v_s\| \leq \|D\varphi^{-t}(v_s)\| \leq Ce^{\tau t}\|v_s\|,\, v_s \in E^s,$$

and where X_g generates the flow φ_t. The bundles E^u and E^s are known to integrate to unstable and stable foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ of SM which project to the horospherical foliations of M, that is, the level sets of the Busemann functions on M as a Hadamard space (see Subsection 3.2.3) .

In terms of dynamics, the symmetries of a C^r-flow $\varphi^t : X \to X$, for $r \geq 0$ an integer, are described by the smooth centralizer group: the group $Z^r(\varphi^t)$ of C^r-diffeomorphisms of X commuting with the system, i.e.:

$$Z^r(\varphi^t) = \{f \in \mathrm{Diff}^r(X) : f \circ \varphi^t = \varphi^t \circ f,\, \forall t \in \mathbb{R}\}.$$

The elements of $Z^r(\varphi^t)$ are known to be rare in a generic sense as well [13]. Moreover, when the dynamical system has some hyperbolicity such as the Anosov condition, in many settings it has been shown that a large centralizer group implies rigidity, generally meaning that the dynamical system is conjugate to an algebraic one. Our main result (Theorem 3.1.2) shows that, in the spirit of Eberlein's result [22], this occurs for the geodesic flow of a closed Riemannian manifold of negative sectional curvature, provided one looks at the universal cover to find excess symmetries.

First we have to define a symmetry of the geodesic flow on unit tangent bundle $\widetilde{SM}$ of the universal cover. The centralizer group of the geodesic flow on the universal cover, which would be a natural candidate, turns out to always be too large, i.e. always infinite-dimensional (see Proposition 3.3.1) since it is possible to construct perturbations which are localized around a wandering orbit. In non-compact settings usually some uniformity of the conjugacy map is assumed, since the notion of hyperbolicity itself depends on the metric structure chosen: for instance a uniformly continuous diffeomorphism of $S\widetilde{M}$ must preserve the stable and unstable foliations of the geodesic flow, whereas without uniformity this is not necessarily true.

In fact, this latter condition – that the centralizer element also preserves the stable and unstable foliation – will suffice for our theorem, and turns out to be a natural requirement in the classification up to conjugacy of Anosov systems in non-compact settings. For instance, recently in the study of Anosov diffeomorphisms in non-compact settings Groisman and Nitecki [42] have introduced the notion of a foliated conjugacy – a conjugacy between two diffeomorphisms preserving stable and unstable foliations – and constructed many Anosov diffeomorphisms on $\mathbb{R}^2$ which can be distinguished by foliated conjugacy classes.

This leads us to the following definition for the group of symmetries of the geodesic flow and the stable and unstable foliations simultaneously. Again, recall that (M, g) is a negatively curved closed Riemannian manifold and let $S\widetilde{M}$ be the unit tangent bundle of the universal cover of M and let $\mathrm{Diff}^{\infty}(N)$ be the group of all smooth diffeomorphisms of a smooth manifold N, with possibly unbounded derivatives if N is non-compact.

Definition 3.1.1. *The foliated centralizer of the geodesic flow on $S\widetilde{M}$ is the group $G < \mathrm{Diff}^{\infty}(S\widetilde{M})$ of all $\psi \in \mathrm{Diff}^{\infty}(S\widetilde{M})$ satisfying:*

(a) $d\psi(X_g) = X_g$, where X_g is the geodesic spray,

(b) ψ preserves the horospherical foliations of $S\widetilde{M}$:

$$\psi(W^{u,s}(v)) = W^{u,s}(\psi(v)).$$

For instance, for n-dimensional hyperbolic space the foliated centralizer $G \cong \mathrm{SO}(n, 1) \times \mathbb{R}$, where $\mathrm{SO}(n, 1)$ acts on the left by lifts of isometries to the unit tangent bundle and $\mathbb{R}$ acts on the right by the geodesic flow.

As is the case with isometries, we expect the presence of too many symmetries for the geodesic flow on the universal cover to determine a rigid situation. This is the main result we prove. In what follows, the foliated centralizer G as defined above is endowed with the compact-open topology. Let $G_c = G/\langle \varphi^t \rangle$ be the reduced foliated centralizer, obtained by taking the quotient of G by the geodesic flow on the universal cover φ^t. We will say that G is trivial if G_c is discrete. A priori G_c is only a topological group, but as we will see later (Theorem 3.1.4) under our assumptions G_c is in fact always a Lie group.

Theorem 3.1.2. *For $n \geq 3$, let (M^n, g) be a closed Riemmanian manifold with negative strictly $\frac{1}{4}$-pinched sectional curvatures, that is, with sectional curvatures $-4 < K \leq -1$. If the reduced foliated centralizer G_c is non-trivial, then (M, g) is homothetic to a real hyperbolic manifold.*

It is well-known that the pinching assumption can be taken to be pointwise rather than global for the results we introduce in Section 3.2 to hold. That is, one may instead assume that for every $x \in M$ the ratio of the sectional curvatures at x is $\frac{1}{4}$-pinched. Using this and replacing the use of [7] at the end of the proof of the theorem by the analogous result in [55] it is easy to see that the result also holds for negatively curved surfaces even without the pinching assumption.

The $\frac{1}{4}$-pinching hypothesis is necessary for the theorem to be true – indeed, any closed, complex hyperbolic manifold has non-trivial foliated centralizer. This however opens up the question of whether the theorem can be true without the $\frac{1}{4}$-pinching hypothesis but replacing the conclusion with (M, g) is locally symmetric. A major difficulty in proving this generalization is that the Anosov splitting is not always C^1 if the sectional curvatures are not $\frac{1}{4}$-pinched.

3.1.1 Hidden Symmetries.

Farb and Weinberger [31] proved that on arithmetic manifolds only locally symmetric Riemannian metrics admit infinitely many isometries on finite covers which are not lifts of other isometries. Since these isometries can only be detected on covers, they call them hiddden symmetries. For manifolds of negative sectional curvature, this result is immediately recovered from Eberlein's result, since $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] = \infty$ implies that $\mathrm{Isom}(\widetilde{M})$ is not discrete for M compact.

Theorem 3.1.2 almost allows us to recover an analogous result for the centralizer groups of finite covers of SM. Recall that $\psi \in \mathrm{Diff}^\infty(SM)$ is a centralizer element of the geodesic flow φ_t of a closed Riemannian manifold if $\psi \circ \varphi_t = \varphi_t \circ \psi$ for all $t \in \mathbb{R}$ or equivalently if $d\psi(X_g) = X_g$, where X_g is the geodesic spray. Centralizer elements on closed Riemannian manifolds of negative sectional curvature preserve the horospherical foliations, so their lifts to the universal cover lie in the foliated centralizer. As such, it is natural to ask:

Question 3.1.3. *For $n \geq 3$, let (M^n, g) be a closed Riemmanian manifold with strictly $\frac{1}{4}$-pinched negative sectional curvature. Suppose there exist infinitely many (up to the action of the flow itself) centralizer elements of the geodesic flow of finite Riemannian covers of (M, g) which are not lifts of centralizer elements of other covers of SM. Then is (M, g) homothetic to a real hyperbolic manifold?*

Unlike the isometry case, where $[\mathrm{Isom}(\widetilde{M}) : \pi_1(M)] = \infty$ immediately implies that $\mathrm{Isom}(\widetilde{M})$ is not discrete (since $\mathrm{Isom}(\widetilde{M})/\pi_1(M)$ is compact), for the centralizer it is not obvious that the existence of infinitely many centralizer elements on finite covers implies that the foliated centralizer is not trivial. The differential of an isometry of a Riemannian manifold is a centralizer element of the geodesic flow, so this result would evidently extend the hidden symmetry theorem for negatively curved $\frac{1}{4}$-pinched metrics to "hidden symmetries" of the geodesic flow.

An interesting open question in the negative curvature setting is whether in fact every conjugacy of geodesic flows is the differential of an isometry, up to the action of the flow. In this direction, a positive answer to Question 3.1.3 would also show that there can only be finitely many more centralizer elements, i.e. self-conjugacies, on all finite covers of M than there are isometries.

3.1.2 Rigid Geometric Structures.

Lastly, we point out that the foliated centralizer has in fact been previously studied–with other rigidity applications in mind–as the group of isometries of the geometric structure determined by the geodesic spray X_g and the bundles E^u and E^s tangent to the horospherical foliations. It has long been known that (X_g, E^u, E^s) determines a pseudo-Riemannian metric h_{KC} on SM, referred to as the Cartan-Kanai form (see Subsection 3.2.3.3). Since h_{KC} is determined by (X_g, E^u, E^s), the foliated centralizer G is a subgroup of the (global) isometry group of h_{KC}.

When the horospherical foliations are smooth, h_{KC} is a rigid geometric structure in the sense of Gromov. In its most general formulation, a rigid geometric structure T on a closed Riemannian manifold V in the sense of Gromov [43] is a geometric structure, such as a tensor, whose pseudo-groups of local isometries are finite-dimensional. Typical examples of such T are affine connections and smooth Riemannian or pseudo-Riemannian metrics. In a well-known paper, Benoist, Foulon and Labourie [7] use Gromov's results on rigid geometric structures to show that a closed Riemannian manifold with negative sectional curvature and C^∞ horospherical foliations must be locally symmetric.

When M is a closed manifold of negative curvature with C^r horospherical foliations, the pseudo-Riemannian metric h_{KC} is still well-defined but only has C^r regularity. On one hand, closed Riemannian manifolds of negative sectional curvature admit only Hölder continuous horospherical foliations in general so h_{KC} is also only Hölder continuous. Hence h_{KC} does not define a rigid geometric structure in the sense of Gromov in the general case, since in low regularity pseudo-Riemannian metrics can admit infinite dimensional local isometry pseudo-groups. On the other hand, while requiring C^2 regularity of the horospherical foliations would easily give that G is a Lie group, this proves too restrictive since conjecturally this situation can only happen when M is locally symmetric.

A natural class of intermediary C^1 regularity of horospherical foliations occurs when the sectional curvatures of M are $\frac{1}{4}$-pinched. In this situation h_{KC} is still only C^1, so it is not immediately obvious that the foliated centralizer is finite-dimensional. The second result of the chapter, which is essential to the proof of Theorem 3.1.2, is to show that G is a Lie group under C^1 horospherical regularity:

Theorem 3.1.4. *Let (M, g) be a closed Riemannian manifold with C^1 horospherical foliations. Then the foliated centralizer G is a Lie group.*

We emphasize that Theorem 3.1.4 does not use non-discreteness of G_c and only C^1 horospherical regularity, as opposed to $\frac{1}{4}$-pinching. The proof of Theorem 3.1.4 uses the fact that elements of G define affine transformations with respect to the Kanai connection ∇, an affine connection associated to h_{KC}, to show that G is locally compact (Section 3.3). Then by the classical work of Montgomery and Zippin on Hilbert's fifth problem this implies that G is a finite-dimensional Lie group.

Remark 3.1.5. *The use of the classical theory of Montgomery-Zippin to show certain transformation groups in dynamics are finite dimensional Lie groups has recently yielded many rigidity results for centralizers; for instance, simultaneously to the writing of this work, Damjanovic-Wilkinson-Xu [27] have also announced rigidity results which use this to classify the centralizer of certain partially hyperbolic diffeomorphisms.*

3.1.3 Outline of the Proof.

After we conclude that G is a Lie group (Section 3.3) we assume as in the hypothesis of Theorem 3.1.2 that G_c is not discrete. Then non-discreteness of G_c implies by classic results also of [77] that it must define a smooth 1-parameter action on $\widetilde{V}$. We show that the partition of $\widetilde{V}$ by the orbits of this action must contain an open set of full dimension by an argument due to Hamenstadt using 1/4-pinching, so that the action of G admits an open and dense orbit in which the horospherical foliations must be smooth (Section 3.4). Finally, using the dynamics of the action of $\pi_1(M)$ on $\partial\widetilde{M}$, we show that the horospherical foliations must in fact be smooth everywhere. The result then follows from the main theorem of [7] and the minimal entropy rigidity theorem [9].

3.2 PRELIMINARIES

In this section, we review important constructions and give precise definitions needed for the proofs to follow.

3.2.1 Topological Transformation Groups

We begin by reviewing some concepts related to transformation groups and Montgomery-Zippin's theory that are fundamental to our results.

3.2.1.1 C^r-topologies

Let $0 \leq r \leq \infty$ be an integer. For non-compact manifolds, there are two available choices of topologies on function spaces. For our purposes, we focus on the weak C^r-topology on $C^r(M, N)$ for M and N smooth manifolds, which is defined as follows. For $f \in C^r(M, N)$, (φ, U) and (ψ, V) charts on M and N respectively, $K \subseteq M$ a compact set such that $f(K) \subseteq V$, $\varepsilon > 0$ let $N^r(f; (\varphi, U), (\psi, V), K, \varepsilon)$ be given by

$$\{g \in C^k(M, N) : g(K) \subseteq V, \|D^k(\psi f \varphi^{-1})(x) - D^k(\psi f g \varphi^{-1})(x)\| < \varepsilon\},$$

where D^k represents the k-th partial derivatives of order and k runs over $k = 0, ..., r$ (when $r = \infty$, k runs over $\mathbb{N}$). The weak C^r-topology is defined to have the sets

$$N^r(f; (\varphi, U), (\psi, V), K, \varepsilon)$$

as a basis. From now on we simply refer to the weak C^r-topologies as the C^r-topologies, since we do not make use of the strong topologies. In other words, we use the topology of uniform C^r-convergence on compact sets. We recall that the weak C^r-topologies are second countable [54, p. 35].

3.2.1.2 Montgomery-Zippin's solution to Hilbert's Fifth Problem

We recall some facts about group actions on manifolds. Throughout, let M be a smooth manifold. A topological (resp. Lie) group of C^r-transformations is a Hausdorff topological (resp. Lie) group G with a homomorphism into $\mathrm{Diff}^r(M)$. Partially giving an answer to Hilbert's fifth problem, Gleason, Montgomery-Zippin and Yamabe proved a criterion for when topological groups admit a compatible (finite-dimensional) smooth manifold structure, i.e., are Lie groups, in terms of the no small subgroups (NSS) criterion.

We will use the following consequence of their theorem, together with a result of Bochner-Montgomery [12, Thm. 1] showing that a locally compact subgroup of $\mathrm{Diff}^r(M)$ has the no small subgroups property. Recall that we say the action of G is effective if its kernel into $\mathrm{Diff}^r(M)$ is trivial.

Theorem 3.2.1. *[77, 90] Let G be a topological group of C^r transformations on a smooth manifold M, with $r \geq 1$. If G is effective and locally compact, then G admits a unique smooth structure which makes it a Lie group of C^r transformations.*

Moreover, it follows from [12, Thm. 4] that in the situation above the natural action map $G \times M \to M$ is of class C^r. In particular, as we will need later, the action of 1-parameter subgroups of G corresponds to a C^r action of $\mathbb{R}$ by C^r-diffeomorphisms on M.

3.2.2 C^r-foliations

We lay out standard definitions and results related to smooth foliations, using the same notations as [20]. We let M be a smooth m-dimensional C^s-manifold throughout. For $1 \leq k \leq m-1$ and $0 \leq r \leq s$, a k-dimensional C^r-foliation is a decomposition of M into disjoint subsets $\{\mathcal{W}_i\}_{i\in I}$ such that each $\mathcal{W}_i$ is a connected C^s-submanifold of M and for each $x \in M$ there is an open neighborhood U of x and a C^r-coordinate chart $\psi : U \to$

$B_k \times B_{k-m} \subseteq \mathbb{R}^k \times \mathbb{R}^{k-m}$ such that $\psi(x) = (0,0)$ and for each $i \in I$ and each connected component W of $U \cap \mathcal{W}_i$ there is a unique $p \in B_{k-m}$ such that $\psi(W) = B_k \times \{p\}$.

3.2.2.1 Foliations with uniformly C^r-leaves

A foliation $\mathcal{W}$ of a C^s-manifold M is said to have uniformly C^r leaves if there is an atlas of foliation charts $(\psi_j, U_j)_{j\in J}$ for $\mathcal{W}$ with $\psi_j : U_j \to B_k \times B_{m-k}$ such that for $p \in B_{m-k}$ the compositions:

$$\zeta_{j,p} = \psi_j^{-1} \circ i_p : B_k \to M,$$

are C^r and vary continuously on p in $C^r(B_k, M)$, where $i_p(x) = (x,p)$.

Given another C^r manifold N, a function $f : M \to N$ is said to be uniformly C^r along $\mathcal{W}$ if for each chart (ψ_j, U_j) the compositions $f \circ \zeta_{j,p}$ depend continuously on $p \in B_{m-k}$ inside $C^r(B_k, N)$.

Remark 3.2.2. *Although we say $\mathcal{W}$ has uniformly C^r-leaves, as this is common terminology in the literature, note that in a non-compact setting this has no implication of uniform continuity of the $\zeta_{j,p}$ on the entirety of the non-compact manifold.*

3.2.2.2 Connections along foliations

Let $r \geq 0$ be an integer. Throughout, a C^r connection on a C^{r+1} vector bundle E over M is a linear map

$$\nabla : \Gamma^{r+1}(E) \to \Gamma^r(E \otimes T^*M),$$

where $\Gamma^{r+1}(E)$ is the space of C^{r+1}-sections of E, which satisfies the Leibniz rule:

$$\nabla(fX) = df \otimes X + f\nabla X,$$

for $f \in C^{r+1}(M)$ and $X \in \Gamma^{r+1}(E)$.

Fix M and $\mathcal{W}$ a foliation with uniformly C^{r+1} leaves. Let $E \subseteq TM$ be a vector bundle over M which is uniformly C^{r+1} along leaves. Let $\Gamma^{r+1}_{\mathcal{W}}(E)$ be the space of sections $Z : M \to E$ which are uniformly C^{r+1} along $\mathcal{W}$. Given $X \in \Gamma^{r+1}_{\mathcal{W}}(E)$, we may consider it as a C^{r+1} section $X : M_{\mathcal{W}} \to E$, where $M_{\mathcal{W}} = \sqcup_{i \in I} \mathcal{W}_i$ (i.e., the topologically disjoint union of the leaves). Then a C^r connection on E along $\mathcal{W}$ is a C^r connection on E considered as a C^{r+1} vector bundle over $M_{\mathcal{W}}$. It is said to be uniformly C^r along $\mathcal{W}$ if

$$\nabla(\Gamma^{r+1}_{\mathcal{W}}(E)) \subseteq \Gamma^r_{\mathcal{W}}(T^*\mathcal{W} \otimes E),$$

where we regard ∇ as a map $\Gamma^{r+1}_{\mathcal{W}}(E) \to \Gamma(T^*\mathcal{W} \otimes E)$ by $X \mapsto (Z \mapsto \nabla_X Z)$.

3.2.2.3 Journé's Theorem

Finally, in dealing with maps which are regular along a pair of transverse foliations of complementary dimensions, a basic tool is Journé's Theorem:

Theorem 3.2.3. *[57] Let $\mathcal{F}^s$ and $\mathcal{F}^u$ be two transverse foliations of a manifold M with uniformly C^∞ leaves. If $f \colon M \to N$ is uniformly C^∞ along the leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ then $f \in C^\infty(M, N)$.*

3.2.3 Geodesic Flows in Negative Curvature

From now on, we fix a closed Riemannian manifold (M, g) with negative sectional curvatures $K \leq -1$ and $\widetilde{M}$ its universal cover. Throughout, $V := SM$ denotes the unit tangent bundle, $\widetilde{V}$ denotes the unit tangent bundle of $\widetilde{M}$. The flip map is denoted by ι, i.e. the diffeomorphism of V given by $(x, v) \mapsto (x, -v)$, and it is also defined on $\widetilde{V}$. We write $\Gamma := \pi_1(M)$, and observe that all objects defined in what follows are invariant by the action of Γ by deck transformations on $\widetilde{M}$ and its associated tangent bundle $T\widetilde{M}$.

Let $\mathcal{V} \subseteq TT\widetilde{M}$ be the vertical bundle spanned by vector fields tangent to the fibers of $T\widetilde{M}$ and $\mathcal{H} \subseteq TT\widetilde{M}$ be the horizontal bundle spanned by parallel vector fields on TM with respect to the Riemannian connection defined by g. The Sasaki metric g_{Sas} on $T\widetilde{M}$ is defined by pulling back the metric $g \oplus g$ by the isomorphism $TT\widetilde{M} = \mathcal{V} \oplus \mathcal{H} \cong T\widetilde{M} \oplus T\widetilde{M}$. We denote the Sasaki norm simply by $\|\cdot\|$ and we endow $T\widetilde{M}$ and the unit tangent bundle $S\widetilde{M}$ with the Sasaki metric and with the metric space structure coming from the Riemannian distance d_{Sas} given by the Sasaki metric on $T\widetilde{M}$ and $S\widetilde{M}$ respectively. When g is complete, g_{Sas} is complete and thus so is $S\widetilde{M}$ as a metric space.

The ideal boundary $\partial\widetilde{M}$ is defined as the set of equivalence classes $[\gamma]$ of oriented geodesics γ on $\widetilde{M}$ which are asymptotic to each other as $t \to \infty$, and $\partial\widetilde{M}$ is endowed with the quotient topology. The natural projection map $P : \widetilde{V} \to \partial\widetilde{M}$ is defined by $v \mapsto [\gamma_v]$, where γ_v is the oriented geodesic through v. Throughout, for $v \in SM$, we write $v_+ := P(v)$ and similarly we write $v_- := P(\iota(v))$.

The preimages $\widetilde{W}^{cs}(v) := P^{-1}([\gamma_v])$ form a foliation $\widetilde{\mathcal{W}}^{cs} = \{\widetilde{W}^{cs}(v)\}_{v\in\widetilde{V}}$ of $\widetilde{V}$ known as the center-stable foliation. The center-stable foliation is subfoliated by the stable foliation $\widetilde{\mathcal{W}}^{s} = \{\widetilde{W}^{s}(v)\}_{v\in\widetilde{V}}$ with leaves defined by the condition:

$$\widetilde{W}^s(v) := \{w \in \widetilde{W}^{cs}(v) : d_{\mathrm{Sas}}(\varphi^t(v), \varphi^t(w)) \to 0 \text{ as } t \to \infty\}.$$

The center-unstable $\widetilde{\mathcal{W}}^{cu} = \{\widetilde{W}^{cu}(v)\}_{v\in\widetilde{V}}$ and unstable $\widetilde{\mathcal{W}}^{u} = \{\widetilde{W}^{u}(v)\}_{v\in\widetilde{V}}$ foliations are defined as the images of the center-stable and stable foliations, respectively, under ι. Namely, $\widetilde{W}^{cu}(v) = \iota(\widetilde{W}^{cs}(\iota(v)))$, and similarly for $\widetilde{W}^u$. The leaves of $\widetilde{\mathcal{W}}^{cs}$ and $\widetilde{\mathcal{W}}^{u}$ (resp. $\widetilde{\mathcal{W}}^{cu}$ and $\widetilde{\mathcal{W}}^{s}$) have complementary dimension and intersect transversely, and in fact have a property known as global product structure: for $w \neq \iota(v)$ the leaves $\widetilde{W}^{cs}(v)$ and $\widetilde{W}^{u}(w)$ (resp. $\widetilde{W}^{cu}(v)$ and $\widetilde{W}^{s}(w)$) intersect at a unique point.

Lastly, given $v \in S\widetilde{M}$ with basepoint p, we define the horosphere

$$B_{v_+}(p) := \pi(\widetilde{W}^s(v)) \subseteq \widetilde{M},$$

where $\pi : S\widetilde{M} \to \widetilde{M}$ is the standard projection map to basepoints.

3.2.3.1 Geodesic Flows

For (M, g) as above, let λ^* be the canonical 1-form on the cotangent bundle of the universal cover $T^*\widetilde{M}$ and $\omega^* = d\lambda^*$. Denote by X_g the geodesic spray, i.e., the vector field generating the geodesic flow φ^t on $T\widetilde{M}$. Pulling back these forms by the bundle isomorphism $T^*\widetilde{M} \cong T\widetilde{M}$ induced by the Riemannian metric g we obtain forms λ and ω on $T\widetilde{M}$. Moreover, the isomorphism $TT\widetilde{M} \cong T^*T\widetilde{M}$ induced by the Sasaki metric sends X_g to the 1-form λ, that is $\lambda(Y) = \langle X_g, Y\rangle_{\mathrm{Sas}}$, which provides an alternative description of the geodesic spray X_g. Moreover, from this description it is immediate the 1-form λ is invariant by the geodesic flow, i.e., $\mathcal{L}_{X_g}\lambda = 0$ or equivalently $\varphi^t_*\lambda = \lambda$ for all $t \in \mathbb{R}$.

Recall that for a linear map A between normed linear spaces its co-norm $m(A)$ is given by $m(A) := \inf_{\|v\|=1} \|Av\|$. The geodesic flow generated by X_g is an Anosov flow when restricted to the unit tangent bundle $V := SM$; that is, there exists a $D\varphi^t$ splitting $TV = E^u \oplus E^s \oplus \mathbb{R}X_g$ and constants $\tau > 1, C > 1$ so that for $t \geq 0$:

$$\begin{aligned} C^{-1}e^t \leq m(D\varphi^t|_{E^u}) \leq \|D\varphi^t|_{E^u}\| \leq Ce^{\tau t}, \\ C^{-1}e^t \leq m(D\varphi^{-t}|_{E^s}) \leq \|D\varphi^{-t}|_{E^s}\| \leq Ce^{\tau t}. \end{aligned} \tag{3.1}$$

Remark 3.2.4. *The minimal rate of growth for an Anosov flow does not have to be $C^{-1}e^t$ in general but rather $C^{-1}e^{\beta t}$ for some $\beta > 0$. However in our case since $K \leq -1$ it can be shown that the inequality holds with $C^{-1}e^t$.*

The Anosov property can be used to provide a dynamical description of the foliations defined in the previous section as we now explain. It is known from standard hyperbolic dynamics theory that the unstable and stable bundles E^u, E^s are integrable with tangent foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ with uniformly C^∞ leaves. The leaves W^u and W^s when lifted to the universal cover agree with the leaves of the unstable and stable foliations $\widetilde{\mathcal{W}}^u$ and $\widetilde{\mathcal{W}}^s$ described in the last section, hence their common notation. The center unstable and stable bundles $E^{cu,cs} := E^{u,s} \oplus \mathbb{R}X_g$ have associated foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ with uniformly C^∞

leaves with the same properties.

3.2.3.2 Regularity of the splitting, fiber-bunching, holonomies

For an Anosov flow the bundles $E^{cu,cs}$ are in fact Hölder-continuous with Hölder constant determined by the constant τ in the inequalities (3.1). For geodesic flows, the constant τ is directly related to the pinching constant of sectional curvature. To be precise, it is a standard fact regarding the solutions to the Ricatti equation that when the pointwise sectional curvatures of M are pinched in the interval $[-a,-1]$, then in fact $\tau = \sqrt{a}$. Based on the considerations above, Hasselblatt proved a sharp result on the regularity of the stable and unstable foliations. We denote C^{r-} regularity, for $r > 0$, to be regularity $C^{\lfloor r\rfloor, r-\lfloor r\rfloor-\varepsilon}$ for all $\varepsilon > 0$. Specifically, it is shown in [52] that the splitting $TSM = E^u \oplus E^s \oplus \mathbb{R}X_g$ is of class $C^{2/\sqrt{a}-}$, and that generically this is the maximum regularity. In particular, under quarter pinching the bundles E^u, E^s are known to be $C^{1,\alpha}$ for some $1 > \alpha > 0$.

Besides regularity of the Anosov splitting, the other main reason we restrict ourselves to pinched metrics is the notion of a fiber-bunched system, which we now define. Recall that a cocycle over a flow $\varphi^t : X \to X$ on a smooth manifold X is a map $\mathcal{A} : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ where $\mathcal{E}$ is a fiber bundle over X such that if $\pi : \mathcal{E} \to X$ is the canonical projection we have $\pi \circ A^t = \varphi^t \circ \pi$, where $A^t(\cdot) := \mathcal{A}(\cdot, t)$.

Definition 3.2.5 (Fiber Bunching). *Let X a smooth closed manifold and let $\mathcal{E}$ be a β-Hölder continuous subbundle of TX. A β-Hölder continuous cocycle $\mathcal{A} : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ over an Anosov flow $\varphi^t : X \to X$ is said to be α-fiber bunched if $\alpha \leq \beta$ and there exists $T > 0$ such that for all $p \in M$ and $t \geq T$:*

$$\|A_p^t\|\|A_p^{-t}\|\|D\varphi^t|_{E^s}\|^\alpha < 1, \quad \|A_p^t\|\|A_p^{-t}\|\|D\varphi^{-t}|_{E^u}\|^\alpha < 1.$$

When the inequalities are satisfied by $A^t = D\varphi^t|_{E^u}$ and $A^t = D\varphi^t|_{E^s}$ the Anosov flow itself is said to be α-bunched.

Fiber-bunched systems admit cocycle holonomies, that is, identifications their stable/unstable bundles over different basepoints on the same stable/unstable leaves, which we now define. For a fiber bundle as in Definition 3.2.5 and for $x, y \in X$ sufficiently close let $I_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ be the identification obtained by parallel transporting (with respect to any fixed smooth Riemannian metric on X) $\mathcal{E}_x \subseteq T_xX$ to T_yX along the unique shortest geodesic from x to y and orthogonally projecting onto $\mathcal{E}_y$. Clearly, the dependence of I_{xy} on x, y is as regular as $x \mapsto \mathcal{E}_x$. The following proposition is standard in this setting, see e.g. [20, Proposition 2.3] :

Proposition 3.2.6 (Cocycle holonomies). *Let $\mathcal{A}$ be a cocycle as in Definition 3.2.5. Then for each $x \in X$, $y \in W^u(x)$, there is a linear isomorphism $H^u_{xy} : \mathcal{E}_x \to \mathcal{E}_y$ with the properties:*

(a) For $x \in M$ and $y, z \in W^u(x)$ we have $H_{xx} = I_x$ and $H^u_{yz} \circ H^u_{xy} = H^u_{xz}$;

(b) For $x \in M$ and $y \in W^u(x)$, $t \in \mathbb{R}$:

$$H^u_{xy} = (A^t_y)^{-1} \circ H^u_{\varphi^t(x)\varphi^t(y)} \circ A^t_x.$$

(c) There is an $r > 0$ and a constant $C > 0$ such that for $x \in M$ and $y \in W^u(x)$ with $d(x, y) \leq r$ we have:

$$\|H^u_{xy} - I_{xy}\| \leq Cd(x, y)^\alpha.$$

Moreover, the family of linear maps H satisfying the above properties is unique. The analogous holonomies H^s over W^s exist, are unique, and satisfy properties analogous to (a), (b) and (c).

For M whose sectional curvatures are pinched in $(-4, -1]$ since horospherical foliations are $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, we see that $D\varphi^t|_{E^{u,s}}$ is 1-bunched, so the derivative cocycle restricted to $E^{u,s}$ admits holonomies as above. Since for any Anosov flow the foliations W^u, W^c and W^s locally define a product structure, and by (b) the holonomies are equivariant with respect to the cocycle $D\varphi^t$, the holonomies may be extended equivariantly with respect to

$D\varphi^t$ to the connected component of $W^{cu}_{loc}(p) := W^{cu}(p) \cap B$, where B is a small neighborhood containing p. With the additional global product structure of the foliations for the geodesic flow explained in the previous section, by flowing back to the box B the cocycle holonomies may be extended to entire center-unstable leaves. We write $H^{cu,cs}$ for the cocycle holonomies over the leaves of $\widetilde{\mathcal{W}}^{cu,cs}$.

3.2.3.3 The Kanai connection

Recall that $\widetilde{V} := S\widetilde{M}$. For this section, we assume crucially that the horospherical foliations are C^1, or what is equivalent, the splitting $T\widetilde{V} = E^s \oplus E^u \oplus \mathbb{R}X_g$ is C^1. In the study of such geodesic flows, a key tool for the study of rigidity relating the geometry of the horospherical foliations and the dynamics of the geodesic flow is the the Kanai connection [60].

To motivate its definition, consider the symmetric bilinear tensor on SM known as the Cartan-Kanai form:

$$h_{KC}(v, w) = \omega(v, Iw) + \lambda \otimes \lambda(v, w),$$

where λ is as in Subsection 3.2.3.1, ω is a symplectic form on TM given by $\omega = d\lambda$ and I is a $(1,1)$-tensor defined by $I(v_u) = v_u$ for $v_u \in E^u$, $I(v_s) = -v_s$ for $v_s \in E^s$ and $I(X_g) = 0$. The tensor h_{KC} is in fact a pseudo-Riemannian metric and it is of regularity as high as that of the bundles E^u and E^s, which in our case are assumed to be C^1.

Since h_{KC} is at least C^1, the pseudo-Riemannian metric defines a connection via Koszul's formula with respect to which it is parallel – this is the Kanai connection, whose properties we list below. For the following, observe that the Lie bracket of C^1 vector fields, and hence the torsion of a C^0-connection (c.f. Subsection 3.2.2.2) such as ∇, is well defined in local coordinates.

Proposition 3.2.7. *[74, Theorem 2.6] There exists a continuous connection ∇, called the Kanai connection, on $T\widetilde{V}$ with the following properties:*

(a) h_{KC} as defined in above is parallel with respect to ∇, i.e. $\nabla h_{KC} = 0$ and ∇ has torsion

$\omega \otimes X_g$;

(b) ∇ is Γ-invariant, φ^t-invariant, $\nabla\omega = 0$, $\nabla_{X_g} = \mathcal{L}_{X_g}$ and $\nabla X_g = 0$;

(c) The Anosov splitting is invariant under ∇, that is, if $X_s \in \Gamma(E^s)$, $X_u \in \Gamma(E^u)$ and Y is any vector field on $\widetilde{V}$ then

$$\nabla_Y X_s \in \Gamma(E^s), \qquad \nabla_Y X_u \in \Gamma(E^u);$$

(d) The restriction of ∇ to the leaves of the foliations $\widetilde{\mathcal{W}}^u$ (resp. $\widetilde{\mathcal{W}}^s$) of $S\widetilde{M}$ is uniformly C^∞ along $\widetilde{\mathcal{W}}^u$ (resp. $\widetilde{\mathcal{W}}^s$), in the sense of Subsection 3.2.2.2. It is moreover flat, complete, and the parallel transport determined by it coincides with the holonomy transport determined by the stable and unstable foliations and with H as in Proposition 3.2.6 by the uniqueness part of that proposition.

Proof. The only non-standard claim not found in [74] is the completeness claim in item (d) which we prove now. Since ∇ is Γ-invariant, it descends to the quotient V. By compactness of V, there exists some constant $c > 0$ such that for $Y^s \in E^s$ with $\|Y^s\| < c$ the ∇-geodesic in the direction of Y^s is defined for at least time 1. But in fact for any $Y^s \in E^s$, there exists a $t > 0$ such that $\|D\varphi^t(Y^s)\| < c$. Then the ∇-geodesic issued from $D\varphi^t(Y^s)$ is well-defined for time at least 1, so by invariance of ∇ by the geodesic flow, so is the ∇-geodesic issued from Y^s. These geodesics lift from V to $\widetilde{V}$. ☐

Remark 3.2.8. *(i) Since we will not use the Levi-Civita connection of the Sasaki metric on $\widetilde{V}$, we use ∇ simply to denote the Kanai connection.*

(ii) The Kanai connection ∇ in fact has Hölder-continuous regularity under $\frac{1}{4}$-pinching, but only C^0 regularity will be needed in what follows.

Finally, we use all the constructions above to induce a C^1-structure on $\partial\widetilde{M}$ and then recall some of its properties. Consider two leaves of the stable foliation $\widetilde{W}^s(x)$ and $\widetilde{W}^s(y)$

lifted to the universal cover $\widetilde{V}$ and let π_y^s and π_x^s be defined as in Subsection 3.2.3.1. There is a well-defined map

$$h_{xy}^{cu} : \widetilde{W}^s(x) \setminus \widetilde{W}^{cu}(\iota(y)) \to \widetilde{W}^s(y) \setminus \widetilde{W}^{cu}(\iota(x))$$

given by $z \mapsto \widetilde{W}^{cu}(z) \cap \widetilde{W}^s(y)$, which we call the (center-unstable) global holonomy map. Observe that this agrees with $(\pi_y^s)^{-1} \circ \pi_x^s$ on its domain of definition and since $\widetilde{\mathcal{W}}^{cu}$ is C^1 so is h_{xy}^{cu} and therefore π_x^s and π_y^s define a C^1-structure on $\partial\widetilde{M}$ with transition map $(\pi_y^s)^{-1} \circ \pi_x^s = h_{xy}^{cu}$.

3.3 THEOREM 3.1.4: LOCAL COMPACTNESS OF THE FOLIATED CENTRALIZER

We start with a straightforward proposition mentioned in the introduction proving that the (non-foliated) centralizer of the geodesic flow in the universal cover is not a Lie group, and hence too large to be a rigid invariant of hyperbolic metrics.

Proposition 3.3.1. *Let $(\widetilde{M}, g)$ be a simply connected manifold with sectional curvatures $K < 0$. The set of diffeomorphisms in $\mathrm{Diff}^\infty(S\widetilde{M})$ commuting with the geodesic flow is infinite dimensional in any of the weak C^k-topologies.*

Proof. Fix some $v_0 \in S\widetilde{M}$. Let S_0 be the C^∞ submanifold of codimension 1 in $S\widetilde{M}$ obtained by restricting the bundle $S\widetilde{M}$ to the horosphere $B_{(v_0)_+}(p_0) \subseteq \widetilde{M}$, where p_0 is the basepoint of v_0. Let $S \subseteq S_0$ be a small precompact neighborhood of v_0 in S_0. Observe that S has codimension 1 in $S\widetilde{M}$ and is transverse to X_g, the geodesic spray.

Let $\Phi : S \times \mathbb{R} \to S\widetilde{M}$ be given by $\Phi(v,t) = \varphi^t(v)$. Then as long as S is chosen sufficiently small, Φ is a diffeomorphism onto its image since for $v \in S$ its trajectory $\varphi^t(v)$ will not intersect $S \subseteq B_{(v_0)_+}(p_0)$ again. Moreover, in Φ-coordinates, X_g is given simply by $(0, \partial_t)$.

Hence, any vector field Z on $\Phi(S \times \mathbb{R})$ of the form $Z = (Y, 0)$ (in Φ-coordinates), where

Y is a vector field compactly supported on S, will commute with $X_g = (0, \partial_t)$ and can be extended to all of $S\widetilde{M}$. The algebra of such vector fields is evidently infinite-dimensional, so the centralizer of X_g on $S\widetilde{M}$ is not finite-dimensional in any of the weak C^k-topologies. □

Now we move to the proof of Theorem 3.1.4, regarding the finite dimensionality of the group G under C^1-horospherical regularity.

Proof of Theorem 3.1.4. We start by constructing normal form coordinates on the unstable/stable leaves with respect to which the elements of G have a simple (affine) representation.

Proposition 3.3.2. *There exists a family of C^∞ diffeomorphisms $\{\mathcal{H}^u_x\}_{x\in\widetilde{V}}$, with $\mathcal{H}^u_x : T_x\widetilde{W}^u(x) \to \widetilde{W}^u(x)$ such that:*

(a) $\mathcal{H}^u_x(0) = x$ and $D_x\mathcal{H}^u_x$ is the identity map for each $x \in \widetilde{V}$;

(b) The maps $\mathcal{H}^u_x$ depend uniformly C^∞ on x along $\widetilde{\mathcal{W}}^u$-leaves and smoothly on x restricted to a leaf of $\widetilde{\mathcal{W}}^u$;

(c) For $\phi \in G$ and any $x \in \widetilde{V}$, we have:

$$\phi|_{\widetilde{W}^u(x)} = \mathcal{H}^u_{\phi(x)} \circ (D_x\phi)|_{E^u(x)} \circ (\mathcal{H}^u_x)^{-1}.$$

The analogous propositions hold for the foliations $\widetilde{\mathcal{W}}^{cu}$, $\widetilde{\mathcal{W}}^s$, and we denote the coordinates respectively as $\mathcal{H}^{cu}$ and $\mathcal{H}^s$.

Proof. By definition any element $\phi \in G$ preserves the bundles E^u, E^s and since $\ker\lambda = E^u \oplus E^s$ as well as (by definition) $d\phi(X_g) = X_g$ it follows that $\phi_*\lambda = \lambda$. Thus, since ϕ preserves h_{KC}, ω and X_g, the Kanai connection is invariant by the action of $\phi \in G$, being the unique connection with respect to which h_{KC} is parallel and with torsion $\omega \otimes X_g$.

Recall that by Proposition 3.2.7 (a), (d), the connection ∇^u_x obtained by restricting the Kanai connection to a single unstable leaf $\widetilde{W}^u(x)$ is smooth, flat and torsion-free, so in particular it is the Levi-Civita connection of a smooth flat Riemannian metric on $\widetilde{W}^u(x)$. By Proposition 3.2.7 (d) ∇^u_x is complete and thus so is the Riemannian metric. Then let $\mathcal{H}^u_x : T_x\widetilde{W}^u(x) \to \widetilde{W}^u(x)$ be the exponential map of ∇^u_x, which is a diffeomorphism, since again ∇^u_x is the connection of a flat complete Riemannian metric on a simply connected space. Moreover, $\mathcal{H}^u_x$ is smooth on leaves of $\widetilde{W}^u$ and uniformly C^∞ along leaves of $\widetilde{W}^u$ by Proposition 3.2.7 (d) again. Finally, for (c), we observe again that elements of G preserve the Kanai connection and so $\phi_*\nabla^u_x = \nabla^u_{\phi(x)}$ from which the statement follows.

The proof for the stable foliation follows using the flip map ι, and the proof for the weak unstable foliation by equivariance of all the constructions above with respect to the geodesic flow φ^t. To be precise, one defines $\mathcal{H}^{cu}$ in the unique φ^t-equivariant way: $\mathcal{H}^{cu}_x(y) = \varphi^t \circ \mathcal{H}^u_x \circ \varphi^{-t}(y)$ where $t \in \mathbb{R}$ is such that $\varphi^{-t}(y) \in \widetilde{W}^u(x)$. Then (a) and (b) are immediate, whereas (c) follows since elements $\phi \in G$ by definition commute with φ^t. □

Remark 3.3.3. *The construction above is similar to a well-known result first proved by Guysinsky and Katok in [48] constructing normal forms for uniformly contracted bundles. In the non-compact setting, this result does not apply so we use the geometric structure provided by* h_{KC}.

For $\psi \in G$, pick some arbitrary $p \in \widetilde{V}$, a compact neighborhood K of p and a bounded open set U such that $\psi(K) \subseteq U$. Then $U_p := \{\phi \in G : \phi(K) \subseteq U\}$ is a neighborhood of ψ in the compact-open topology. We will show that the closure of U_p is compact, which proves Theorem 3.1.4 by Theorem 3.2.1. Recall that the compact-open topology is second countable [54, p. 35], so it suffices to check sequential compactness of $\overline{U}_p$.

Let ϕ_n be a sequence in U_p and observe that by passing to a subsequence we may assume that $q_n := \phi_n(p)$ converges to some $q \in \overline{U}$. Moreover, we claim that we may pass to a further subsequence so that $D_p\phi_n$ also converges to some linear isomorphism $A : T_p\widetilde{V} \to T_q\widetilde{V}$.

To prove the claim, it suffices to show that there is some $C > 1$ such that $C^{-1}\|v\| \leq \|D_p\phi_n(v)\| \leq C\|v\|$ for all n and $v \in T_p\widetilde{V}$. This is evidently true for $v = X_g$. For contradiction, suppose there exists a sequence $v_n \in E^u$ with $\|D_p\phi_n(v_n)\|/\|v_n\| \to \infty$. By Proposition

3.3.2, the normal form charts are uniformly C^∞ along $\widetilde{\mathcal{W}}^u$, which contradicts $\phi_n(K) \subseteq U$, since U is a bounded set and $\|D_p\phi_n(v_n)\|/\|v_n\| \to \infty$. The same holds for $v_n \in E^s$. Finally, if $\|D_p\phi_n(v_n)\|/\|v_n\| \to 0$ and $v_n \in E^u$, since the ϕ_n preserve ω and E^s and E^u are transverse Lagrangian subspaces there must exist some $u_n \in E^s$ such that $\|D_p\phi_n(u_n)\|/\|u_n\| \to \infty$, reducing to the previous case. Similarly there cannot exist $\|D_p\phi_n(v_n)\|/\|v_n\| \to 0$ and $v_n \in E^s$, so the claim is proved.

Using A and the normal form coordinates $\mathcal{H}_x$ defined above we now construct a map $\phi_\infty \in G$ which we show is the C^0-limit of $\{\phi_n\}_{n\in\mathbb{N}}$. Let $\psi^s : \widetilde{W}^s(p) \to \widetilde{W}^s(q)$ and $\psi^{cu} : \widetilde{W}^{cu}(p) \to \widetilde{W}^{cu}(q)$ be given by:

$$\psi^\bullet := \mathcal{H}^\bullet_{\phi(x)} \circ A|_{E^\bullet(x)} \circ (\mathcal{H}^\bullet_x)^{-1}, \text{ where } \bullet \in \{cu, s\},$$

so that the maps $\phi_n|_{W^\bullet(p)}$ converge uniformly C^∞ to $\psi^\bullet$. To extend the above construction to an open set of $\widetilde{V}$, we use the product structure of the foliations on the universal cover. For $x \in \widetilde{V}$ there exists a C^1 diffeomorphism $[\cdot,\cdot]_x : \widetilde{W}^{cu}(x) \times \widetilde{W}^s(x) \to \widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(x)) \cup \widetilde{W}^s(\iota(x)))$ called the Bowen bracket given by

$$(y, z) \mapsto [y, z]_x := \widetilde{W}^s(y) \cap \widetilde{W}^{cu}(z).$$

Define the diffeomorphism $\phi_\infty : \widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(p)) \cup \widetilde{W}^s(\iota(p))) \to \widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(q)) \cup \widetilde{W}^s(\iota(q)))$ by:

$$\phi_\infty([y, z]_p) = [\psi^{cu}(y), \psi^s(z)]_q,$$

where in the equation above we use the fact that the Bowen bracket is a diffeomorphism $\widetilde{W}^{cu}(x) \times \widetilde{W}^s(x) \to \widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(x)) \cup \widetilde{W}^s(\iota(x)))$. Furthermore, since the foliations are C^1 the dependence of the bracket $[\cdot,\cdot]_x$ on x is also C^1. Using the facts just mentioned along with the fact that elements of G preserve the bracket (since they preserve the foliations $\widetilde{\mathcal{W}}^{cu,s}$) we see that ϕ_∞ agrees with the limit of ϕ_n, that is: as maps on $(y, z) \in \widetilde{W}^{cu}(p) \times \widetilde{W}^s(p) \cong$

$\widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(p)) \cup \widetilde{W}^s(\iota(p)))$, we have the following C^1-convergence uniform on compact sets

$$[\phi_n(y), \phi_n(z)]_{q_n} \to [\psi^{cu}(y), \psi^s(z)]_q = \phi_\infty([y,z]_p).$$

Now we verify that ϕ_∞ is uniformly C^∞ along $\widetilde{\mathcal{W}}^{cu}$-leaves on the open set $\widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(p)) \cup \widetilde{W}^s(\iota(p)))$. Fix some $x \in \widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(p)) \cup \widetilde{W}^s(\iota(p)))$. Then by the convergence in the previous paragraph, we have that $(\phi_n)|_{W^{cu}(x)} \to (\phi_\infty)|_{W^{cu}(x)}$ uniformly on compact sets. On the other hand, since $\phi_n \in G$, by Lemma 3.3.2 the restrictions $(\phi_n)|_{W^{cu}(x)}$ are given by:

$$(\phi_n)|_{W^{cu}(x)} = \mathcal{H}^{cu}_{\phi_n(x)} \circ (D_x \phi_n)|_{E^u(x)} \circ (\mathcal{H}^{cu}_x)^{-1},$$

and hence their limit is given by

$$\mathcal{H}^{cu}_{\phi_\infty(x)} \circ A' \circ (\mathcal{H}^{cu}_x)^{-1},$$

where A' is the limit of $(D_x \phi_n)|_{E^u(x)}$. Therefore

$$(\phi_\infty)|_{W^{cu}(x)} = \mathcal{H}^{cu}_{\phi_0(x)} \circ A' \circ (\mathcal{H}^{cu}_x)^{-1},$$

which is C^∞, and uniformly so along leaves since the right-hand side expression clearly is. By the same argument, ϕ_∞ is uniformly C^∞ along $\widetilde{\mathcal{W}}^s$-leaves. Hence we conclude by Journé's Lemma (Lemma 3.2.3) that ϕ_∞ is C^∞ on $\widetilde{V} \setminus (\widetilde{W}^{cu}(\iota(p)) \cup \widetilde{W}^s(\iota(p)))$.

Finally, it remains to extend ϕ_∞ to all of $\widetilde{V}$. To do this, simply repeat the same entire argument with $\iota(p)$ in place of p and then passing to a further subsequence finally constructs a ϕ such that $\phi_n \to \phi$ in the compact-open topology. □

3.4 SMOOTHNESS OF FOLIATIONS: PROOF OF THEOREM 3.1.2

We now turn to the proof of Theorem 3.1.2. Throughout this section we let $\widetilde{V}$ be the universal cover of the unit tangent bundle of a Riemannian manifold (M, g) satisfying the hypotheses of Theorem 3.1.2. Namely, we assume that the sectional curvatures of g are $\frac{1}{4}$-pinching and the reduced foliated centralizer G_c is not discrete.

By Theorem 3.1.4 the foliated centralizer group G is a Lie group and we let $\mathfrak{g} \cong T_eG$ be its Lie algebra. We identify $\mathfrak{g}$ with a finite-dimensional vector subspace of $\Gamma^\infty(\widetilde{V}, T\widetilde{V})$, that is, we consider $\mathfrak{g} \to \Gamma^\infty(\widetilde{V}, T\widetilde{V})$ via $Y \mapsto \partial_s \phi_s^Y|_{s=0}$, where $\phi_s^Y \in G$ is a curve tangent to Y at the identity. Let $\sigma_x : \mathfrak{g} \to T_x\widetilde{V}$ be given by evaluating the vector field at x, namely, $\sigma_x(Y) = \partial_s \phi_s^Y(x)|_{s=0}$. For $x \in \widetilde{V}$ we let $F_x := \sigma_x(\mathfrak{g})$ be the image of $\mathfrak{g}$ under σ_x. For each x, F_x is a vector subspace of $T_x\widetilde{V}$, with dimension depending on x. Since

$$\dim F_x = \dim \mathfrak{g} - \dim \ker(\sigma_x),$$

we see that $x \mapsto \dim F_x$ is lower semi-continuous, and so there exists an open non-empty set $\widetilde{\mathcal{U}} \subseteq \widetilde{V}$ where $\dim F_x$ is constant and maximal.

Recall $T_x\widetilde{V} = E_x^s \oplus E_x^u \oplus E_x^c$, where $E^c = \mathbb{R}X_g$, so that we can define maps $\pi_x^{s,u,c} : T_x\widetilde{V} \to E_x^{s,u,c}$ associated to the unique decomposition of a vector $v \in T_xV$ into $v^s + v^u + v^c$. We write

$$F_x^{s,u,c} := \pi^{s,u,c}(F_x) = (\pi^{s,u,c} \circ \sigma_x)(\mathfrak{g}).$$

and as before, note that there exist open non-empty open sets $\widetilde{\mathcal{U}}^{s,u,c} \subseteq \widetilde{V}$ where $\dim F_x^{s,u,c}$ are constant and attain their maxima over $\widetilde{V}$. We denote these maximal dimensions by $d^{s,u,c}$, respectively.

3.4.1 Subfoliations induced by the G-action.

The goal of this section is to construct subfoliations $\mathcal{F}^s$ of $\widetilde{\mathcal{W}}^s$ using the action of G on $\widetilde{V}$, specifically, using the bundle F^s defined above. Roughly speaking, the leaves of this subfoliation correspond to G-orbits of $\widetilde{\mathcal{W}}^{cu}$-leaves intersected with a fixed leaf of $\widetilde{\mathcal{W}}^s$, which we show can be properly defined on an open and dense subset of $\widetilde{V}$.

We start by showing that F^s and F^u are non-trivial:

Proposition 3.4.1. *The dimensions d^s and d^u are positive.*

Proof. Observe that $d^u = 0$ if and only if $d^s = 0$, since the foliations are images of each other under the flip map ι. Hence, we can suppose both are zero for contradiction.

Since $d^s = d^u = 0$, we have that $F_x \subseteq E^c_x = \mathbb{R}X_g$ for all $x \in \widetilde{V}$, so any vector field in $\mathfrak{g}$ must be of the form fX_g, where $f \in C^\infty(\widetilde{V})$, and X_g is the geodesic spray. But notice that the vector field $fX_g \in \mathfrak{g}$ must preserve the contact form λ on $\widetilde{V}$ since it preserves X_g and $E^u \oplus E^s$ (since it is a vector field in the foliated centralizer). On the other hand:

$$\mathcal{L}_{fX_g}\lambda = d \circ i_{fX_g}\lambda + i_{fX_g}d\lambda = df,$$

since $i_{X_g}d\lambda = 0$ and $\lambda(X_g) = 1$. Hence $\mathcal{L}_{fX_g}\lambda$ is zero if and only if f is constant. Therefore, the action of G_0 (the connected component of the identity) reduces to the action of the geodesic flow, which contradicts the hypothesis that the reduced centralizer $G_c = G/\langle \varphi^t \rangle$ is not discrete. □

Now observe that F_x and $E^{s,u}$ are invariant by the geodesic flow φ_t and by the action of $\Gamma = \pi(M)$ on $\widetilde{V}$. This, along with transitivity of the geodesic flow in the quotient, implies that $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{U}}^s$, and $\widetilde{\mathcal{U}}^u$ are in fact open and dense, and hence so is their intersection, which we denote by $\widetilde{\mathcal{V}} := \widetilde{\mathcal{U}} \cap \widetilde{\mathcal{U}}^s \cap \widetilde{\mathcal{U}}^u$. On $\widetilde{\mathcal{V}}$ the subspaces F^s and F^u have constant dimension, and we now show they moreover are integrable.

Proposition 3.4.2. *The vector bundles $F^{s,u} \to \widetilde{\mathcal{V}}$ are integrable.*

Proof. We work with F^s, the proof for F^u being identical. Fix some $x \in \widetilde{\mathcal{V}}$; since the dimension of F is constant on $\tilde{\mathcal{V}}$, there exists some $Y_1, \dots, Y_m \in \mathfrak{g}$ such that $Y_i(x), \dots, Y_m(x)$ form a basis for F_x and $Y_i, \dots, Y_m$ form a local basis for F in a neighborhood U of x.

For a vector field $Y \in \Gamma(\widetilde{\mathcal{V}}, T\widetilde{V})$ we write $Y^{s,u,c} = \pi^{s,u,c}(Y) \in \Gamma(\widetilde{\mathcal{V}}, E^{s,u,c})$. Then by definition of F^s the vector fields $Y_1^s, \dots, Y_m^s$ form a local basis for F^s on the set U. Note that $Y_i \in \Gamma^\infty(\widetilde{\mathcal{V}}, T\widetilde{V})$ but since E^s is C^1 we have $Y_i^s \in \Gamma^1(\widetilde{\mathcal{V}}, E^s)$. By the C^1-Frobenius theorem, it suffices to show that on U the commutators $[Y_i^s, Y_j^s]$ for $i, j \in \{1, \dots, m\}$ are sections of F^s to conclude the proof. We show this in several steps below.

Claim 3.4.3. *If $X \in \mathfrak{g}$ and $Y \in \Gamma^1(U, E^{s,u,c})$ then $[X, Y] \in \Gamma^0(U, E^{s,u,c})$.*

Proof. By definition

$$[X, Y] = \lim_{t \to 0} \frac{(\phi_t^X)_* Y - Y}{t},$$

where ϕ_t^X is the flow generated by X. But since $X \in \mathfrak{g}$, its flow is in the foliated centralizer and hence $(\phi_t^X)_* E^{s,u,c} = E^{s,u,c}$. Then the expression above is also clearly contained in $E^{s,u,c}$. □

We use Claim 3.4.3 to show the following:

Claim 3.4.4. *For $i, j \in \{1, \dots, m\}$, let Y_i and Y_j be as before. Then*

$$[Y_i^s, Y_j^u] \in \Gamma^0(U, E^c).$$

Proof. We show that $[Y_i^s, Y_j^u] \in \Gamma^0(U, E^{cs})$ and $[Y_i^s, Y_j^u] \in \Gamma^0(U, E^{cu})$, which implies that $[Y_i^s, Y_j^u] \in \Gamma^0(U, E^c)$.

Recall that $Y_i = Y_i^s + Y_i^u + Y_i^c$, so that

$$[Y_i^s, Y_j^u] = [Y_i, Y_j^u] - [Y_i^u, Y_j^u] - [Y_i^c, Y_j^u].$$

Note that $[Y_i^u, Y_j^u] \in \Gamma^0(U, E^u)$ and $[Y_i^c, Y_j^u] \in \Gamma^0(U, E^{cu})$, since E^u and E^{cu} are integrable.

Moreover, $[Y_i, Y_j^u] \in \Gamma^0(U, E^u)$ by Claim 3.4.3, since $Y_i \in \mathfrak{g}$. Hence all terms in the right hand side are in $\Gamma^0(U, E^{cu})$ and thus so is $[Y_i^s, Y_j^u]$.

Similarly, writing $Y_j = Y_j^s + Y_j^u + Y_j^c$, we have

$$[Y_i^s, Y_j^u] = [Y_i^s, Y_j] - [Y_i^s, Y_j^s] - [Y_i^s, Y_j^c],$$

and by the same argument the three terms on the right hand side are in $\Gamma^0(U, E^{cs})$. Hence $[Y_i^s, Y_j^u] \in \Gamma^0(U, E^{cs})$, and the proof is complete.

□

Now we can prove the following:

Claim 3.4.5. *Let Y_i and Y_j be as before. Then $\pi^s([Y_i, Y_j]) = [Y_i^s, Y_j^s]$.*

Proof. Writing $Y_i = Y_i^s + Y_i^u + Y_i^c$ again, and same for Y_j, we have:

$$\begin{aligned} &= [Y_i, Y_j^c] + [Y_i^c, Y_j] - [Y_i^c, Y_j^c] \\ &+ [Y_i^u, Y_j^s] + [Y_i^s, Y_j^u] \\ &+ [Y_i^u, Y_j^u] + [Y_i^s, Y_j^s]. \end{aligned}$$

To show $\pi^s([Y_i, Y_j]) = [Y_i^s, Y_j^s]$, it suffices to show that the first six terms in the right-hand side sum are sections of E^{cu}.

The first two terms $[Y_i, Y_j^c]$ and $[Y_i^c, Y_j]$ are sections of E^c by Claim 3.4.3, as is $[Y_i^c, Y_j^c]$ by integrability of E^c. The terms $[Y_i^u, Y_j^s]$ and $[Y_i^s, Y_j^u]$ are sections of E^c by Claim 3.4.4. Lastly, $[Y_i^u, Y_j^u]$ is a section of E^u by its integrability.

□

Now the last claim completes the proof of Proposition 3.4.2, since it shows that for $i, j \in \{1, \ldots, m\}$, we have that $[Y_i^s, Y_j^s] = \pi^s([Y_i, Y_j])$ is a section of F^s and thus F^s is involutive. □

For the rest of the proof, we denote by $\mathcal{F}^s$ the subfoliation of $\widetilde{\mathcal{W}}^s$ obtained from the integrable bundle F^s on the set $\widetilde{\mathcal{V}}$.

3.4.2 Local transitivity of the G-action

Recall that we can identify $\partial\widetilde{M}$ with the space of center-unstable leaves via the projection $P^- : \widetilde{V} \to \partial\widetilde{M}$ given by $v \mapsto v_-$, which has center-unstable leaves as preimages of points in $\partial\widetilde{M}$. Since the action of G_0 preserves $\widetilde{\mathcal{W}}^{cu}$, we obtain an associated action on $\partial\widetilde{M}$. In this section we use the foliation $\mathcal{F}^s$ defined above to prove the following:

Proposition 3.4.6. *There exists an open set $W \subseteq \partial\widetilde{M}$ such that the foliation $\widetilde{\mathcal{W}}^{cu}$ restricted to $(P^-)^{-1}(W) \subseteq \widetilde{V}$ is C^∞.*

We briefly comment on the proof idea, which is to identify $\partial\widetilde{M}$ with the space of center unstable leaves, as explained, and use $\mathcal{F}^s$ to construct a Γ-invariant foliation of $\partial\widetilde{M}$ whose leaves are given by G_0-orbits of center-unstable leaves. Then a classic argument using the north-south dynamics of the action of Γ on $\partial\widetilde{M}$ shows that the foliation cannot be proper, i.e, that $d^s = \dim E^s$. This in turn implies that the action of G_0 on the space of $\widetilde{\mathcal{W}}^{cu}$-leaves is locally transitive. Since the action of G_0 is C^∞ and commutes with the geodesic flow, $\widetilde{\mathcal{W}}^{cu}$ itself is a C^∞ foliation around points in $\widetilde{\mathcal{V}}$. We begin the proof of Proposition 3.4.6 proving the first statement of the outline above:

Proposition 3.4.7. *For d^s as defined in the beginning of the section, $d^s = \dim E^s$.*

Proof. We start by showing that the subfoliation $\mathcal{F}^s$ of $\widetilde{\mathcal{W}}^s$ defined on $\widetilde{\mathcal{V}}$ can be pushed to define a foliation of the boundary $\partial\widetilde{M}$.

Let $x_0, x_1 \in \widetilde{\mathcal{V}}$ be such that their φ^t-orbit in $V = \widetilde{V}/\Gamma$ is periodic and such that $\widetilde{W}^s(x_1) \neq \widetilde{W}^s(x_0)$, which exist since $\widetilde{\mathcal{V}}$ is open and non-empty. Since $\widetilde{W}^s(x_i)$ is contracted under φ^t, this implies that $\widetilde{W}^s(x_i) \subseteq \mathcal{V}$, by Γ and φ^t-invariance of $\widetilde{\mathcal{V}}$ By Propositions 3.4.1 and 3.4.2

the bundle F^s integrates to foliations $\mathcal{F}_i^s$, $i = 0, 1$ of $\widetilde{W}^s(x_i)$, $i = 0, 1$. Let

$$h^{cu} : \widetilde{W}^s(x_0) \setminus \widetilde{W}^{cu}(\iota(x_1)) \to \widetilde{W}^s(x_1) \setminus \widetilde{W}^{cu}(\iota(x_0))$$

be the center-unstable global holonomy map namely, $h^{cu}(y) = \widetilde{W}^{cu}(y) \cap \widetilde{W}^s(x_1)$ for $y \in \widetilde{W}^s(x_0) \setminus \widetilde{W}^{cu}(x_1)$. The following lemma, which uses crucially the pinching hypothesis on the metric, shows that F^s is invariant under the holonomy maps. The idea is that high regularity subbundles of pinched flows must in fact be holonomy invariant, although extra care has to be taken in a non-compact setting.

Lemma 3.4.8. *The bundle F^s over $\widetilde{W}^s(x_0)$ is mapped to the bundle F^s over $\widetilde{W}^s(x_1)$ by the holonomies h^{cu}. Namely, for all $y \in \widetilde{W}^s(x_0) \setminus \widetilde{W}^{cu}(x_1)$:*

$$Dh_y^{cu}(F_y^s) = F_{h^{cu}(y)}^s.$$

Proof. Denote by H^{cu} the cocycle holonomies of the bundle E^s over leaves of $\widetilde{\mathcal{W}}^{cu}$ as given by Proposition 3.2.6. Note that Dh^{cu} defines a family isomorphisms of E^s over leaves of $\widetilde{\mathcal{W}}^{cu}$ which satisfies the conditions of Proposition 3.2.6 so by uniqueness in the proposition we see that $Dh^{cu} = H^{cu}$, in their common domain of definition. We will now show invariance of F^s under H^{cu} between the two leaves in question.

To deal with the lack of uniformity of F^s in our non-compact setting we verify holonomy invariance for points negatively asymptotic to a periodic orbit in the quotient in the following claim:

Claim 3.4.9. *Let $y \in \widetilde{W}^s(x_0)$ be such that its negative endpoint $y_- \in \partial M$ is the endpoint of some φ^t-periodic orbit $w \in \widetilde{\mathcal{V}}$, i.e., such that $w_- = y_-$ and w is periodic in V. Then $H_{y,h^{cu}(y)}^{cu} F_y^s = F_{h^{cu}(y)}^s$.*

For readability we provide the proof of Claim 3.4.9 shortly after the proof of Proposition 3.4.7. With the claim above, we now conclude the proof of Lemma 3.4.8. For this, it suffices to observe that since $\widetilde{\mathcal{V}}$ is open dense and the periodic orbits of an Anosov flow are also dense, there exists a dense set of points $y \in \widetilde{W}^s(x_0)$ satisfying the hypotheses of Claim 3.4.9. Therefore $Dh^{cu}_y F^s_y = F^s_{h^{cu}(y)}$ holds on a dense subset of $\widetilde{W}^s(x_0)$. Further, the maps Dh^{cu}_y depend C^0 on the basepoint y and so does the distribution F^s, so that the identity $Dh^{cu}_y F^s_y = F^s_{h^{cu}(y)}$ which holds on a dense set has to in fact hold for all $y \in \widetilde{W}^s(x_0) \setminus \widetilde{W}^{cu}(x_1)$, as desired. $\square$

We now conclude the proof of Proposition 3.4.7. Fix some x_0, x_1 as in the hypotheses of Lemma 3.4.8. Since $\widetilde{W}^{cs}(x_1) \neq \widetilde{W}^{cs}(x_0)$, the natural projections:

$$P^-|_{\widetilde{W}^s(x_i)} : \widetilde{W}^s(x_i) \to \partial\widetilde{M},\ i = 0, 1,$$

form a system of charts covering $\partial\widetilde{M}$ with transition maps h^{cu}, which by Subsection 3.2.3.2 are C^1, and thus give $\partial\widetilde{M}$ a C^1-structure. Since $h^{cu}(\mathcal{F}^s_0) = \mathcal{F}^s_1$, the foliations $\mathcal{F}^s_i$ induce a well-defined topological foliation $\mathcal{F}$ of $\partial\widetilde{M}$.

Further, for any $\gamma \in \Gamma$ and $i = 0, 1$ the distribution F defines a foliation of $\widetilde{W}^s(\gamma x_i)$, as the γx_i are also periodic orbits contained in $\widetilde{\mathcal{V}}$. We can consider the leaves of $\widetilde{W}^s(\gamma x_i)$ also as charts on $\partial\widetilde{M}$, and by Lemma 3.4.8 the leaves $\{\widetilde{W}^s(\gamma x_i)\}_{\gamma\in\Gamma,\, i=0,1}$ are all compatible as charts. Therefore we obtain a well-defined action of Γ on $\partial\widetilde{M}$ via its action on the union of the leaves $\widetilde{W}^s(\gamma x_i)$. Moreover, since the action of $\gamma \in \Gamma$ sends the foliation of $\widetilde{W}^s(x)$ to the foliation of $\widetilde{W}^s(\gamma x)$, the action of Γ on $\partial\widetilde{M}$ preserves the induced foliation $\mathcal{F}$. By a well-known argument of Hamenstädt (see [49, Sec. 4]), any Γ-invariant topological foliation of $\partial\widetilde{M}$ is trivial, and since it has positive dimension we have $d^s = \dim E^s$. $\square$

We now prove Proposition 3.4.6:

Proof of Proposition 3.4.6. Fix some $x \in \widetilde{\mathcal{V}}$ and let $\varepsilon > 0$ be such that the neighbhorhood $\widetilde{W}^{cu}_\varepsilon(x)$ of radius ε of x in $\widetilde{W}^{cu}(x)$ satisfies $\widetilde{W}^{cu}_\varepsilon(x) \subseteq \widetilde{\mathcal{V}}$. Let $\mathcal{N}$ be a small tubular neighborhood of $\widetilde{W}^{cu}_\varepsilon(x)$ such that $\mathcal{N} \subseteq \widetilde{\mathcal{V}}$. By Proposition 3.4.7, for $y \in \mathcal{N}$ we have $\dim F^s_y = d^s =$

dim E^s, so that there exists a neighborhood B of the identity of G_0 such that the map $B \times \widetilde{W}^{cu}_{\varepsilon/2}(x) \to \mathcal{N}$ given by the action $(g, y) \mapsto g \cdot y$ is a submersion.

Since the action of G_0 preserves entire stable leaves, the orbit $G_0 \cdot \widetilde{W}^{cu}(x)$ is a $\widetilde{\mathcal{W}}^{cu}$-saturated set, meaning it is a union of whole $\widetilde{\mathcal{W}}^{cu}$-leaves. The first paragraph of the proof then shows that $G_0 \cdot \widetilde{W}^{cu}(x)$ must contain an open set around $\widetilde{W}^{cu}(x)$ in $\partial\widetilde{M}$ under the identification of the space of leaves given by P^-, since locally around x it does so and the set is saturated. Lastly, since the action $G_0 \times \widetilde{V} \to \widetilde{V}$ is C^∞, this implies that the foliation $\widetilde{\mathcal{W}}^{cu}$ itself is C^∞ in a small neighborhood of $\widetilde{W}^{cu}(x)$ in the space of $\widetilde{\mathcal{W}}^{cu}$-leaves. $\square$

It remains to give the proof of Claim 3.4.9.

Proof of Claim 3.4.9. For the proof, we recall a standard bound for linear actions on Grassmannians. We regard F^s as a section of the Grassmanian bundle $\mathcal{P} \to \widetilde{\mathcal{V}}$ with fibers $\mathcal{P}_x$ isomorphic to the Grassmanian $\mathrm{Gr}(d^s, d)$ of d^s-dimensional subspaces of E^s, which admits an action induced by $D\varphi^t$. As is standard, we endow the fibers $\mathcal{P}_x$ with the metric:

$$\rho_x(V, W) = \sup_{v \in V, \|v\|=1} \inf_{w \in W} \|v - w\|.$$

For this metric, given a linear isomorphism A acting on the underlying vector space, the induced action of A on the Grassmannian manifold is bi-Lipschitz with norm bounded by $\|A\|\|A^{-1}\|$. Hence, by the pinching hypothesis, we obtain the following bound for $V, W \in \mathcal{P}_x$:

$$\begin{aligned} \rho_x(V, W) &\le \|D_x\varphi^{-t}|_{E^s}\| \|D_{\varphi^{-t}(x)}\varphi^t|_{E^s}\| \, \rho_{\varphi^{-t}x}(D\varphi^{-t}V, D\varphi^{-t}W) \\ &\le Ce^{-t}e^{\tau t} \, \rho_{\varphi^t x}(D\varphi^{-t}V, D\varphi^{-t}W) \\ &\le Ce^{(\tau-1)t}\rho_{\varphi^t x}(D\varphi^{-t}V, D\varphi^{-t}W) \text{ for all } t > 0 \text{ large,} \end{aligned} \tag{3.2}$$

where $C > 0$ is some constant which does not depend on any of the parameters and where $\tau = \sqrt{a}$, where a is such that the sectional curvatures of M are pinched in the interval $[-a, -1]$ with $a < 4$.

For $t \in \mathbb{R}$, we write $y_t = \varphi^t(y)$ and $z_t = \varphi^t(z)$. Let $t_0 \in \mathbb{R}$ be such that $z_{t_0} \in \widetilde{W}^u(y)$. Recall that by Proposition 3.2.6 (c), H^{cu} is given by the limit $H^{cu}_{yz} = \lim_{t\to\infty} D\varphi^{t-t_0} \circ I_{y_{-t}z_{-t+t_0}} \circ D\varphi^{-t}$, Therefore, to conclude the proof it suffices to show that $\rho_z(D\varphi^{t-t_0} \circ I_{y_{-t}z_{-t+t_0}} \circ D\varphi^{-t}(F^s_y), F^s_z) \to 0$.

Using $D\varphi^t$ invariance of F^s, for $t > t_0$:

$$\begin{aligned}\rho_z(D\varphi^{t-t_0} \circ I_{y_{-t}z_{-t+t_0}} \circ D\varphi^{-t}(F^s_y), F^s_z) &\leq Ce^{(\tau-1)t}\rho_z(I_{y_{-t}z_{-t+t_0}} \circ D\varphi^{-t}(F^s_y), D\varphi^{-t+t_0}F^s_z) \\ &= Ce^{(\tau-1)t}\rho_z(I_{y_{-t}z_{-t+t_0}}F^s_{y_{-t}}, F^s_{z_{-t+t_0}}),\end{aligned}$$

where we use the bound (3.2) to obtain the inequality. It remains to bound this last term $\rho_z(I_{y_{-t}z_{-t+t_0}}F^s_{y_{-t}}, F^s_{z_{-t+t_0}})$. From Proposition 3.2.6 (c), the family of maps I has C^1 dependence on the basepoints since the bundle E^s itself is C^1. The distribution F^s is also of class C^1 on $\widetilde{\mathcal{V}}$ since E^s is, but it may fail to be uniformly so since $\widetilde{\mathcal{V}}$ is an open set. However, since both y_{-t} and z_{-t+t_0} are negatively asymptotic to p_-, i.e., they shadow a point $p \in \widetilde{\mathcal{V}}$ periodic in the quotient, as $t \to \infty$, for sufficiently large t they must lie within some compact set $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{V}}$ which is the lift of a closed tubular neighborhood of the orbit of p in the quotient. Therefore for t sufficiently large y_{-t} and z_{-t+t_0} lie in $\widetilde{\mathcal{N}}$ where F^s is thus uniformly Lipschitz. Therefore:

$$\rho_z(I_{y_{-t}z_{-t+t_0}}F^s_{y_{-t}}, F^s_{z_{-t+t_0}}) \leq C_1 d(y_{-t}, z_{-t+t_0}).$$

Combining the above with the bound $d(y_{-t}, z_{-t+t_0}) \leq C_2e^{-t}$, which is immediate since $y \in \widetilde{W}^{cu}(z)$, we now conclude:

$$\begin{aligned}d_z(D\varphi^{t-t_0} \circ I_{y_{-t}z_{-t+t_0}} \circ D\varphi^{-t}(F^s_y), F^s_z) &\leq Ce^{(\tau-1)t}\rho_z(I_{y_{-t}z_{-t+t_0}}F^s_{y_{-t}}, F^s_{z_{-t+t_0}}) \\ &\leq Ce^{(\tau-1)t}C_1 d(y_{-t}, z_{-t+t_0}) \\ &\leq CC_1C_2e^{(\tau-1)t}e^{-t} = C_3e^{(\tau-2)t} \to 0,\end{aligned}$$

where we absorb all the constants in the last line into one.

□

3.4.3 Proof of Theorem 3.1.2

With Proposition 3.4.6, we now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. The map $\widetilde{V} \to \partial\widetilde{M}$ given by $v \mapsto v_-$ induces a homeomorphism of the space $\widetilde{V}/\widetilde{\mathcal{W}}^{cu}$ of $\widetilde{\mathcal{W}}^{cu}$-leaves to $\partial\widetilde{M}$, and we identify these from now on. By Proposition 3.4.6 there exists an open set $W \subseteq \partial\widetilde{M}$ saturated by $\widetilde{\mathcal{W}}^{cu}$-leaves on which $\widetilde{\mathcal{W}}^{cu}$ is smooth. The subset of $\partial\widetilde{M}$ on which $\widetilde{\mathcal{W}}^{cu}$ is smooth is invariant by the action of Γ on $\partial\widetilde{M}$ (since the image of a C^∞ foliation by a C^∞ map is also a C^∞ foliation) and we use this now to conclude the proof. Each $\gamma \in \Gamma$ acts on $\partial\widetilde{M} \approx S^n$ with north-south dynamics. Take $\gamma \in \Gamma$ such that γ_-, the repelling fixed point of γ on $\partial\widetilde{M}$, lies in W. This is possible since the set of $(\gamma_-, \gamma_+) \in \partial^2\widetilde{M}$ which are endpoints of axes of $\gamma \in \Gamma$ is dense, i.e. the closed geodesics are dense in the space of geodesics. Then $\gamma^n(W)$ converges to $\partial\widetilde{M} - \{\gamma_+\}$, where γ_+ is the attracting fixed point of γ on $\partial\widetilde{M}$. Hence we showed that $\widetilde{\mathcal{W}}^{cu}$ is C^∞ except perhaps at the leaf corresponding to γ_+. To conclude now repeat the argument with any γ' such that $\gamma'_+ \neq \gamma_+$ and $\gamma'_- \neq \gamma_+$.

Therefore we find that the foliation $\widetilde{\mathcal{W}}^{cu}$ is a C^∞ foliation everywhere. Since ι maps $\widetilde{\mathcal{W}}^{cu}$-leaves to $\widetilde{\mathcal{W}}^{cs}$-leaves, we similarly conclude that $\widetilde{\mathcal{W}}^{cs}$ is C^∞. The result now follows from the main theorem of [7], which shows that C^∞ smoothness of horospherical foliations implies that the geodesic flow of (M, g) is C^∞ conjugate to that of a locally symmetric manifold, so that by the pinching hypothesis it must be conjugate to that of a manifold with constant negative sectional curvature. Then the minimal entropy rigidity theorem [9] shows that (M, g) itself must have constant negative sectional curvature. □

CHAPTER 4

MONOTONICITY OF THE LIOUVILLE ENTROPY ALONG THE RICCI FLOW ON SURFACES

This chapter presents material which is joint work with Karen Butt, Alena Erchenko and Tristan Humbert.

4.1 INTRODUCTION

Let (M, g) be a closed negatively curved surface, and let $h_{\mathrm{Liou}}(g)$ denote its *Liouville entropy*, i.e., the measure-theoretic entropy of the geodesic flow on the unit tangent bundle $S^g M$ with respect to the Liouville measure. In this chapter, we answer affirmatively a question raised by Manning in [71] about the monotonicity of $h_{\mathrm{Liou}}(g)$ along the *normalized Ricci flow* on the space of negatively curved metrics on M.

Theorem 4.1.1. *Let M be a smooth closed orientable surface of negative Euler characteristic. Let g_0 be a smooth Riemannian metric on M of non-constant negative Gaussian curvature. Let $\varepsilon \mapsto g_\varepsilon$ denote the normalized Ricci flow starting from g_0. Then*

$$\varepsilon \mapsto h_{\mathrm{Liou}}(g_\varepsilon) \text{ is strictly increasing for all } \varepsilon \geq 0.$$

We recall that in dimension 2, the normalized Ricci flow is given by

$$\frac{\partial}{\partial \varepsilon} g_\varepsilon = -2(K_\varepsilon - \overline{K}) g_\varepsilon, \tag{4.1}$$

where $\overline{K} := \int_M K_\varepsilon$, and is independent of ε by Gauss–Bonnet. Hyperbolic metrics, i.e., metrics of constant Gaussian curvature, are fixed by the Ricci flow; for metrics of non-constant curvature, (4.1) defines a conformal family of negatively curved metrics $\varepsilon \mapsto g_\varepsilon$ of fixed area converging to a hyperbolic metric (of constant curvature $\overline{K}$) as $\varepsilon \to \infty$ [51, Theorem 3.3].

In [71], Manning considered the variation of the *topological entropy* $h_{\mathrm{top}}(g)$ along the normalized Ricci flow in the above setting. This quantity coincides with Liouville entropy if and only if the metric g is hyperbolic, as shown by Katok in [61, Corollary 2.5]. Katok also proved that Liouville entropy (resp. topological entropy) is maximized (resp. minimized) at hyperbolic metrics among negatively curved metrics of the same area. Using Katok's above result for $h_{\mathrm{top}}(g)$, Manning proved the topological entropy decreases along the normalized Ricci flow [71, Theorem 1]. He also asks whether the analogous monotonicity result holds for the Liouville entropy [71, Question 3]. In this chapter, we prove this is indeed the case, as stated in Theorem 4.1.1 above.

In contrast to Manning's proof of [71, Theorem 1], our proof of Theorem 4.1.1 does not use the fact that Liouville entropy is minimized at hyperbolic metrics. As such, this chapter gives a new proof of this fact (shown also in [70, Theorem 1] and [83, Corollary 1]).

Moreover, we obtain a new proof of Katok's aforementioned entropy rigidity result [61, Corollary 2.5]:

Corollary 4.1.2 (Katok [61])**.** *Let (M, g) is a negatively curved surface. Then $h_{\mathrm{top}}(g) = h_{\mathrm{Liouv}}(g)$ if and only if the metric g is* hyperbolic*, i.e., of constant negative curvature.*

To see this, one can combine our above monotonicity result (Theorem 4.1.1) with Manning's [71, Theorem 1]. This implies that for g not hyperbolic, the difference $h_{\mathrm{top}}(g) - h_{\mathrm{Liou}}(g)$ is *strictly* decreasing along the Ricci flow. On the other hand, the variational principle states $h_{\mathrm{top}}(g) - h_{\mathrm{Liouv}}(g) \geq 0$, so the inequality must be strict.

4.1.1 Mean root curvature

Our next result concerns a geometric invariant introduced by Manning [70] known as the mean root curvature, which is defined for a negatively curved metric g on a closed surface M by

$$\kappa(g) := \frac{1}{A(g)} \int_M \sqrt{-K_g}\, dA_g, \tag{4.2}$$

where dA_g is the Riemannian area form of g, and $A(g)$ is the area defined by $A(g) = \int_M dA_g$.

The mean root curvature is small for metrics which concentrate curvature in regions of small area, and is maximized strictly at metrics of constant negative curvature, by Jensen's inequality and the Gauss–Bonnet theorem. In addition, it provides a lower bound for the Liouville entropy: $\kappa(g) \leq h_{\text{Liou}}(g)$ [70, Theorem 2] with equality if and only if g is of constant negative Gaussian curvature [78].

Since the mean root curvature is a purely geometric invariant related to the concentration of Gaussian curvature and to the Liouville entropy, it is natural to ask if it is also strictly increasing along the Ricci flow. We prove that this is indeed the case.

Theorem 4.1.3. *Let M be a smooth closed orientable surface of negative Euler characteristic. Let g_0 be a smooth Riemannian metric on M of non-constant negative Gaussian curvature. Let $\varepsilon \mapsto g_\varepsilon$ denote the normalized Ricci flow starting from g_0. Let $\kappa(g_\varepsilon)$ denote the mean root curvature of g_ε as in* (4.2)*. Then*

$$\varepsilon \mapsto \kappa(g_\varepsilon) \textit{ is strictly increasing for all } \varepsilon \geq 0.$$

4.1.2 Strategy of the proofs

In Section 4.3, we prove Theorem 4.1.3 by first finding the derivative of $\kappa(g)$ along an arbitrary conformal and area-preserving perturbation (Proposition 4.3.1). We deduce positivity of this derivative along the Ricci flow using a Jensen-type inequality (Lemma 4.2.6).

The key ingredient in the proof of Theorem 4.1.1 is a new formula for the derivative of the

Liouville entropy along an arbitrary area-preserving conformal perturbation of a negatively curved metric on a surface. As in the proof of Theorem 4.1.3, we then deduce Theorem 4.1.1 from this formula using Lemma 4.2.6.

Theorem 4.1.4 (Derivative of Liouville entropy for a conformal perturbation)**.** *Let* (M, g_0) *be a smooth negatively curved surface. Let* $g_\epsilon = e^{2\rho_\epsilon} g_0$ *be a smooth conformal area-preserving perturbation of* g_0 *and let* $\dot{\rho}_0 = \frac{d}{d\epsilon}|_{\epsilon=0}\, \rho_\epsilon \in C^\infty(M)$. *Let* $h_{\mathrm{Liouv}}(\varepsilon)$ *denote the Liouville entropy of* g_ε. *Then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = -\frac{1}{2}\int_{SM} \dot{\rho}_0 w^s dm,$$

where m *is the Liouville measure for* g_0 *and* $-w^s(v)$ *is the mean curvature of the stable horosphere* (*or, strictly speaking, the geodesic curvature of the stable horocycle*) *determined by* v*; see* (4.18) *below.*

In Section 4.4, we prove Theorem 4.1.4. We begin with the well-known fact that, in negative curvature, the Liouville entropy can be expressed as the average (with respect to the Liouville measure) of the mean curvature of horospheres (see (4.21) below). This was used by Knieper–Weiss to show the Liouville entropy varies smoothly with respect to the metric for negatively curved surfaces in [64].

In this chapter, we use that the the mean curvature of a horosphere is in turn equal to the Laplacian of the corresponding Busemann function, and can hence be expressed as the divergence of a vector field closely related to the geodesic spray. This formulation of the mean curvature was used by Ledrappier–Shu in [65, 66] to study the differentiability of the *linear drift*. In Subsections 4.4.1 and 4.4.2, we differentiate the horospherical mean curvature using their methods. A key tool, in both their work and ours, is a slightly non-standard decomposition of the unit tangent bundle of the universal cover $\tilde{M}$ as the product of $\tilde{M}$ with $\partial\tilde{M}$, the visual boundary at infinity. As a consequence of this perspective, integrals of certain functions along *half-infinite orbits* of the geodesic flow appear naturally in the computations. In Subsection 4.4.3, we use microlocal methods, more specifically, the formalism of *Pollicott–Ruelle resonances*, to express these integrals in terms of *resolvents* of the geodesic flow, as in the work of Faure–Guillarmou [34]. This key insight allows for dramatic simplification of our derivative formula.

Remark 4.1.5. *Without appealing to microlocal methods, we are able to simplify our derivative formula enough to prove Theorem 4.1.1 for metrics with 1/6-pinched sectional curvature. We present this argument in Appendix 4.5.*

4.2 PRELIMINARIES

In Subsection 4.2.1 we record standard facts on the geometry of the unit tangent bundle of a surface, and in Subsection 4.2.2 we describe the stable and unstable distributions of the geodesic flow in negative curvature. In Subsection 4.2.3, we recall that in our setting, the Liouville entropy has a geometric formulation as the average of the mean curvatures of horospheres, which is the starting point of our proof of Theorem 4.1.4; see equation (4.21). In Subsection 4.2.4, we record a Jensen-type integral inequality that is used in the proofs of Theorems 4.1.1 and 4.1.3.

4.2.1 Geometry of surfaces

In this subsection, we recall some basic facts about the geometry of surfaces and establish some notation. For a textbook account of all these notions, we refer to [80, Chapter 1] and [88, Chapter 2].

4.2.1.1 Geodesic flow and Liouville measure

Consider a smooth closed surface M equipped with a smooth Riemannian metric g. Let K_g denote the sectional curvature of g. We will denote by dA_g the Riemannian area form defined by g on M. The area $A(g)$ is the mass of dA_g, i.e., $A(g) = \int_M dA_g$.

Let $SM = \{(x, v) \in TM \mid \|v\|_g = 1\}$ denote the unit tangent budle of g. The pair (M, g)

defines a natural dynamical system on SM called the *geodesic flow*:

$$\varphi_t : (x, v) \mapsto (\gamma_v(t), \dot{\gamma}_v(t)), \tag{4.3}$$

where $t \mapsto \gamma_v(t)$ is the (projection on M) of the unique geodesic passing through x at time $t = 0$ with velocity v. The *geodesic spray* is the vector field generating φ_t, i.e.,

$$X(x, v) := \frac{d}{dt}\Big|_{t=0} \varphi_t(x, v) \in C^\infty(SM, T(SM)). \tag{4.4}$$

The metric g induces a natural probability measure on SM called the *Liouville measure*, which we will denote by $m = m_g$. This measure has a concrete description which is compatible with the sphere-bundle structure of SM: it is locally given (up to a multiplicative constant) by the product of the Riemannian area dA_g on the base M, together with the spherical Lebesgue measure (arclength) on the circular fibers. This measure also turns out to be geodesic-flow–invariant, as we will discuss below. In summary, the metric g defines a measure-preserving dynamical system $(SM, \varphi_t, m)_g$.

4.2.1.2 Horizontal and vertical spaces

In this subsection, we describe the horizontal-vertical decomposition of the tangent bundle of SM, along with some of its specific features in the case $\dim M = 2$. Let $P : TM \to M$ be the footpoint projection $(x, v) \mapsto x$. The metric g induces an identification of $T_v SM$ with a subspace of $T_x M \oplus T_x M$ which we now recall.

We start by discussing $T_v TM$. Given $W \in T_v TM$, let $c(t)$ be a curve in TM with $c(0) = v$ and $c'(0) = W$. Define the *connector map*

$$\mathcal{K} : T_v TM \to T_p M, \quad W \mapsto \frac{D}{dt} c(0), \tag{4.5}$$

where $\frac{D}{dt}$ denotes covariant differentitation (with respect to g) along the footpoint curve

$P(c(t)) \in M$. Then we have an identification

$$T_v TM \longleftrightarrow T_x M \oplus T_x M, \quad W \mapsto (dP(W), \mathcal{K}(W)).$$

A double tangent vector W in the kernel of dP is called *vertical*, and a vector in the kernel of $\mathcal{K}$ is called *horizontal*. We will refer to $dP(W)$ and $\mathcal{K}(W)$ as the horizontal and vertical components of W, respectively. One can check that elements of $T_v SM \subset T_v TM$ are characterized by having vertical component orthogonal to v.

Via the above identification, the metric g induces a metric on TM (by declaring the above direct sum decomposition to be orthogonal). This metric is called the *Sasaki metric* and we denote it by g_{Sas}. The Sasaki metric on TM restricts to a metric on SM which we will still denote by g_{Sas}.

Suppose now that M is a surface. Then one can give a more explicit description of the vertical and horizontal spaces. The vertical space is one-dimensional in this case, and we define a vertical vector field as follows. An oriented Riemannian surface admits a complex structure. This means that there is a section $J \in \mathrm{End}(TM)$ satisfying $J^2 = -\mathrm{Id}$, and such that the area form associated to g is given by $dA_g = g(J\cdot, \cdot)$. One defines a rotation in the fiber by

$$\rho_\theta : SM \to SM, \quad \rho_\theta(v) = e^{J\theta} v,$$

where $e^{J\theta} v$ is the unit vector obtained by rotating v by an angle θ. The *vertical vector field* V is the generator of this rotation:

$$V := \left.\frac{d}{d\theta}\right|_{\theta=0} \rho_\theta \in C^\infty(SM, T(SM)). \tag{4.6}$$

Next, note that the geodesic vector field X is horizontal, since $\mathcal{K}(X) = 0$ is the definition of a geodesic. Define the horizontal vector field $H := [V, X]$. We will use the following important commutation relations; see for instance [67, Lemma 15.2.1],

$$H = [V, X], \quad [H, V] = X, \quad [X, H] = K_g V. \tag{4.7}$$

One can show that (X, V, H) is a global orthonormal frame for the restriction of the Sasaki metric g_{Sas} on $T(SM)$. It defines a (normalized) Riemmannian volume form on SM, and this coincides with the *Liouville measure* m_g defined above, see [46, Lemma 1.30]. One can show (see [80, Exercise 1.33]) that there is a contact structure on SM for which X is the Reeb vector field and m_g is the Liouville form. In particular, we deduce the important property that the Liouville measure is φ_t-invariant. Moreover, the Liouville measure can be shown to be invariant with respect to H and V, see [46, Proposition 1.47]. In other words,

$$X^* = -X, \quad H^* = -H, \quad V^* = -V, \tag{4.8}$$

where Y^* denotes the $L^2(SM, dm)$-adjoint of a differential operator Y.

4.2.2 The Anosov property and stable manifolds

The main hypothesis in this chapter is that the curvature is negative, that is $K_g < 0$. This ensures that the dynamics of the geodesic flow are *chaotic*.

Proposition 4.2.1 (Anosov)**.** *The geodesic flow on a negatively curved manifold (M, g) is Anosov (uniformly hyperbolic). That is, there exist constants $C, \lambda > 0$, together with a flow-invariant and continuous splitting*

$$T(SM) = E^s \oplus \mathbb{R}X \oplus E^u, \tag{4.9}$$

such that

$$\forall v \in SM, \quad \begin{cases} \|d\varphi_t(v)W^s\|_{g_{\mathrm{Sas}}} & \leq Ce^{-\lambda t}\|W^s\|_{g_{\mathrm{Sas}}},\ W^s \in E^s(v),\ t \geq 0, \\ \|d\varphi_t(v)W^u\|_{g_{\mathrm{Sas}}} & \leq Ce^{-\lambda|t|}\|W^u\|_{g_{\mathrm{Sas}}},\ W^u \in E^u(v),\ t \leq 0. \end{cases} \tag{4.10}$$

The bundle E^s (resp. E^u) is called the stable (resp. unstable) bundle of the flow.

We will not prove this proposition, but we will recall in detail the construction of the stable and unstable bundles E^s and E^u, since we will use their geometric characterization throughout the majority of this chapter. See, for instance, [6] for more details.

4.2.2.1 Stable manifolds and horocycles

We start by describing the stable and unstable *manifolds* of the flow. For any $v \in SM$, these are, by definition, immersed submanifolds

$$\begin{aligned} \mathcal{W}^s(v) &:= \{v' \in SM \mid d(\varphi_t(v), \varphi_t(v')) \to_{t\to+\infty} 0\}, \\ \mathcal{W}^u(v) &:= \{v' \in SM \mid d(\varphi_t(v), \varphi_t(v')) \to_{t\to-\infty} 0\}, \end{aligned} \tag{4.11}$$

called the (strong) stable (resp. unstable) manifolds, such that $T_v\mathcal{W}^s = E^s(v)$ and $T_v\mathcal{W}^u = E^u(v)$. We also define the *weak* stable and unstable manifolds

$$\begin{aligned} \mathcal{W}^{cs}(v) &:= \{v' \in SM \mid \limsup_{t\to+\infty} d(\varphi_t(v), \varphi_t(v')) < +\infty\} = \bigcup_{t\in\mathbb{R}} \varphi_t(\mathcal{W}^s(v)) \\ \mathcal{W}^{cu}(v) &:= \{v' \in SM \mid \limsup_{t\to-\infty} d(\varphi_t(v), \varphi_t(v')) < +\infty\} = \bigcup_{t\in\mathbb{R}} \varphi_t(\mathcal{W}^u(v)). \end{aligned} \tag{4.12}$$

Their tangent spaces are given respectively by $\mathbb{R}X \oplus E^s$ and $\mathbb{R}X \oplus E^u$.

Geometrically, we can describe the strong/weak stable/unstable manifolds in terms of *Busemann functions*. To lighten the presentation, we will describe the stable case only; the unstable case is analogous, and is used minimally in this chapter. Let $\tilde{M}$ denote the universal cover of M and let $\partial\tilde{M}$ denote its visual boundary at infinity; see for instance, [18, Chapter

8], [6, Chapter II]). Let $\pi : S\tilde{M} \to \partial\tilde{M}$ denote the natural forward projection along the geodesic flow. We have the identification

$$\Pi : S\tilde{M} \to \tilde{M} \times \partial\tilde{M}, \quad (x, v) \mapsto (x, \pi(v)). \tag{4.13}$$

For $(x, \xi) \in SM$, let $b_{x,\xi} \in C^\infty(\tilde{M})$ denote the associated *Busemann function*:

$$b_{x,\xi}(p) = \lim_{t\to\infty} (d(p, \gamma_v(t)) - t), \tag{4.14}$$

where γ_v is the geodesic such that $\gamma_v(0) = x$ and $\gamma_v(+\infty) = \xi$ (see, for instance, [6, Chapter II]). For any fixed $\xi \in \partial\tilde{M}$, the dependence of $b_{x,\xi}(p)$ on p and also on x is C^∞ (see e.g. [89, Proposition 2.2]), whereas the dependence on ξ is in general only Hölder continuous, even though g is a smooth metric. Nevertheless, when $\dim M = 2$, it follows from the work of Hurder–Katok [55] that the dependence in ξ is $C^{1+\alpha}$ for some $\alpha > 0$. Level sets of Busemann functions are called *horospheres*, or *horocycles* in the case where $\dim M = 2$.

Fix $\xi \in \partial\tilde{M}$ and define the vector field $X^\xi(y) = -\operatorname{grad} b_{x,\xi}(y)$ for $y \in \tilde{M}$. Then the lift of $\mathcal{W}^s(v)$ to $S\tilde{M}$ is given by the inward normal vector field of the horocycle $\{b_v = 0\}$, that is,

$$\widetilde{\mathcal{W}^s(v)} = \{X^\xi(y) \,|\, y \in \{b_v = 0\}\}. \tag{4.15}$$

This is because X^ξ is the unit vector field on $\tilde{M}$ determined by $\pi(X^\xi(y)) = \xi$ for all $y \in \tilde{M}$, which means the expression (4.15) defines a variation of geodesics centered at v which all asymptotic to ξ. Hence (4.11) holds by the definition of $\partial\tilde{M}$. See also [6, p. 72]. Similarly, the lift of $\mathcal{W}^{cs}(v)$ to $S\tilde{M}$ is given by

$$\widetilde{\mathcal{W}^{cs}(v)} = \{X^\xi(y) \,|\, y \in \tilde{M}\}. \tag{4.16}$$

A Jacobi field associated to the geodesic variation in (4.15) is called a *stable Jacobi* *field*. Since such a Jacobi field is everywhere perpendicular to the geodesic γ_v determined by $v \in SM$, and $\dim M = 2$, we can view it as a real-valued function along the geodesic $\gamma_v(t)$. Letting $j^s(t)$ denote this function, we have $j^s(t) \to 0$ as $t \to \infty$. The exponential decay estimates in the Anosov property (4.10) are equivalent to analogous decay estimates for $j^s(t)$ and $Xj^s(t)$. In constant negative curvature, these are readily obtained by explicitly solving the Jacobi equation, and one can generalize these estimates to variable negative curvature using the Rauch comparison theorem; see, for instance [6, Proposition IV.1.13 and Proposition IV.2.15].

4.2.2.2 The stable vector field

We now specify a vector field e^s which spans the stable bundle E^s. Let $s \mapsto c(s)$ be a parametrization of the horocycle $\{b_{x,\xi} = 0\}$ such that $c(0) = x$ and $c'(0) = J(x, \xi)$, where J is the complex structure of M discussed in the previous subsection. We define the stable vector field e^s on $S\tilde{M}$ by $e^s(v) = \frac{d}{ds}|_{s=0}(c(s), \xi)$. By construction, e^s has integral curves given by $\mathcal{W}^s$. Moreover, since, the horizontal component of e^s is Jv, it is of the form

$$e^s = H + w^s V \tag{4.17}$$

for some function $w^s : S\tilde{M} \to \mathbb{R}$.

By the above discussion, the regularity of w^s is $C^{1+\alpha}$ in the setting $\dim M = 2$ [55], which will be very important for our argument. From the definitions of the connector map (4.5) and the second fundamental form of a hypersurface, one can deduce the following two characterizations of w^s, both of which are used crucially in this chapter:

- $-w^s(v)$ is the (trace of the) second fundamental form, i.e., the mean curvature, of the horosphere $\{b_v = 0\}$ (or, since $\dim M = 2$, the geodesic curvature of the horocycle). Since the trace of the second fundamental form of a level hypersurface is given by the

Laplacian of its defining function, we obtain

$$-w^s(v) = \Delta b_{x,\xi}(x) = -\mathrm{Div}(X^\xi)(x). \tag{4.18}$$

- $w^s = \frac{Xj^s}{j^s}$, where j^s is the stable Jacobi field along γ_v defined above. In particular, w^s is everywhere negative. Moreover, since j^s satisfies the Jacobi equation, a direct computation shows that w^s satisfies the *Riccati equation*

$$X(w^s) = -(w^s)^2 - K. \tag{4.19}$$

Remark 4.2.2. *Note that since $w^s = X(j^s)/j^s$, one has*

$$\frac{\|d\varphi_t(v)e^s(v)\|}{\|e^s(v)\|} = \frac{j^s(\varphi_t(v))}{j^s(v)} = \exp\left(\int_0^t X(\ln(j^s))(\varphi_r v)dr\right) = e^{\int_0^t w^s(\varphi_s p)ds}. \tag{4.20}$$

Since w^s is continuous and negative, this shows that e^s is indeed exponentially contracted along the flow which is consistent with the fact that the stable foliation E^s is tangent to $\mathcal{W}^s$.

4.2.3 Liouville entropy

The main object of study of this chapter is the measure-theoretic entropy of the geodesic flow with respect to the Liouville measure m, which we denote by h_{Liou} from now on. This invariant roughly captures the exponential rate of divergence of nearby geodesics for m-a.e. point; see, for instance, [19] or [37, Appendix A] for a textbook account of this notion. We will use the descriptions of the stable bundle of the geodesic flow from the previous subsection to obtain the following geometric expression of the Liouville entropy in our setting:

$$h_{\mathrm{Liou}} = -\int_{SM} \mathrm{Div}(X^\xi)(x)\, dm(x,\xi). \tag{4.21}$$

This formula will be the starting point for our proof of Theorem 4.1.4.

To deduce the above formula, we recall that in our setting

- the geodesic flow φ_t is Anosov, see Proposition 4.2.1;
- the Liouville measure m is *smooth*, meaning, it is absolutely continuous with respect to (and more specifically, identically equal to) the normalized Riemannian volume on SM induced by the Sasaki metric.

We can thus use the theory of *thermodynamic formalism* to write h_{Liou} in terms of the *stable Jacobian.* The stable Jacobian of a general Anosov flow is given by the following formula

$$J^s(v) := -\frac{d}{dt}\det(d\varphi_t(v)_{|E^s(v)})\Big|_{t=0} = -\frac{d}{dt}\ln\det(d\varphi_t(v)_{|E^s(v)})\Big|_{t=0}.$$

It is well known that the thermodynamic equilibirium measure associated to the potential $-J^s$ is precisely the Liouville measure in our setting; see for instance [37, Proposition 4.3.8]. By [37, Corollary 7.4.5], we then have

$$h_{\text{Liou}}(g) = h_{m_g}(\varphi_1) = \int_{SM} J^s(x)dm_g(x).$$

For the case of a negatively curved surface, we have $\det(d\varphi_t(v)|_{E_s(v)}) = j^s(v,t)$, where $j^s(v,t)$ is the stable Jacobi field along γ_v with initial condition $j^s(v,0) = 1$. Since

$$\frac{d}{dt}\Big|_{t=0} \ln(j^s(v,t)) = \frac{Xj^s(v,0)}{j^s(v,0)},$$

we conclude, using (4.18) and (4.19), that

$$J^s(v) = -w^s(v) = -\text{Div}(X^\xi)(x). \tag{4.22}$$

This shows (4.21).

Remark 4.2.3. *It is more standard to define the Liouville measure using the* unstable Jacobian $J^u = -\frac{d}{dt}|_{t=0}\ln\det(d\varphi_t(v)|_{E^u(v)})$. *Similarly to the case of the stable Jacobian, one can* *check that* $J^u = -w^u$ *where* w^u *is the unstable solution of the Riccati equation* (4.19)*. Using the Riccati equation, one can show that*

$$w^u + w^s = -X(\ln(w^u - w^s)). \tag{4.23}$$

In other words, $-J^s$ *and* J^u *are cohomologous and thus define the same equilibrium state; see for instance [37, Theorem 7.3.24]. However, it will be more natural for us to work with* $-w^s$ *than* w^u *because we will use the specific identification of the unit tangent bundle given by* (4.13)*.*

Remark 4.2.4. *Alternatively, one can deduce (4.21) using* Lyapunov exponents, *via* Pesin's entropy formula *[81]. See, for instance, [70, p. 354] and [64, Appendix A] for accounts of this approach.*

Remark 4.2.5. *The mean root curvature is conceptually related to the Liouville entropy as follows: averaging both sides of the Riccati equation* (4.19) *with respect to Liouville measure shows that the average of* $(w^s)^2$ *coincides with that of* $-K$*; thus one might expect the Liouville entropy, which is the average of* $-w^s$ *(see* (4.18) *and* (4.21)*), to be related to the average of* $\sqrt{-K}$*. Indeed, as mentioned in the introduction, Manning proved the former is always larger than the latter [70, Theorem 2].*

4.2.4 A Jensen-type inequality

To show positivity of the derivatives of both the Liouville entropy and mean root curvature, we will need the following lemma.

Lemma 4.2.6. *Let* (Ω, μ) *be a probability space. Let* $F\colon \Omega \to \mathbb{R}$ *be a measurable non-negative function. Then,*

$$\int_\Omega F^2 \left(F - \int_\Omega F \, d\mu \right) d\mu \geq 0,$$

with equality if and only if F *is* μ*-a.e constant.*

Proof. We denote $c = \int\limits_{\Omega} F\, d\mu \geq 0$. Let $\Omega_c = \{x \in \Omega \,|\, F(x) \leq c\}$. Note that if $x \in \Omega_c$ then $F^2(x) \leq c^2$ and $F(x) - c \leq 0$, so $F^2(x)(F(x) - c) \geq c^2(F(x) - c)$. Similarly, if $x \in \Omega \setminus \Omega_c$, then $F^2(x)(F(x) - c) \geq c^2(F(x) - c)$. Thus,

$$\begin{aligned}
\int\limits_{\Omega} F^2 \left(F - \int\limits_{\Omega} F\, d\mu \right) d\mu &= \int\limits_{\Omega_c} F^2 \left(F - c\right)\, d\mu + \int\limits_{\Omega\setminus\Omega_c} F^2 \left(F - c\right)\, d\mu \\
&\geq c^2 \left(\int\limits_{\Omega_c} (F - c)\, d\mu + \int\limits_{\Omega\setminus\Omega_c} (F - c)\, d\mu \right) \\
&= c^2 \int\limits_{\Omega} (F - c)\, d\mu.
\end{aligned}$$

To complete the proof, we note that since μ is normalized, we have $\int_{\Omega} c d\mu = c = \int_{\Omega} F d\mu$, which shows the last line above equals 0. The equality holds if and only if F is is μ-a.e constant. ☐

Remark 4.2.7. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $\phi(x) = x^2(x - c)$. Then the above inequality $\int_{\Omega} F^2(F - c) \geq c^2 \int_{\Omega}(F - c) = 0$ can be reformulated as $\int_{\Omega} \phi(F) \geq \phi(\int_{\Omega} F)$. However, ϕ is not a convex function on $[0, \infty]$, so the usual Jensen inequality does not apply.*

4.3 MONOTONICITY OF MEAN ROOT CURVATURE

In this section, we prove the mean root curvature (defined in (4.2)) is monotonically increasing along the Ricci flow (Theorem 4.1.3). While the mean root curvature is related to the Liouville entropy, as explained in Remark 4.2.5, the proof of Theorem 4.1.3 takes place entirely in M, and we do not use any of the above background on SM. First, we compute the variation of the mean root curvature with respect to a conformal change which preserves the area. Since we are only interested in the sign of the derivative, we can suppose without loss of generality that $A(g_\varepsilon) \equiv 1$. We will use ϵ as a subscript to indicate that the corresponding objects are taken with respect ot hte metric g_ϵ.

Proposition 4.3.1. *Let (M, g_0) be a closed surface of negative curvature and area 1. Let $\varepsilon \mapsto g_\varepsilon = e^{2\rho_\varepsilon} g_0$ be a conformal area-preserving deformation of g_0. Then we have*

$$\dot{\kappa}_0 := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \kappa(g_\epsilon) = -\int_M \frac{\Delta_0 \dot{\rho}_0}{2\sqrt{-K_0}} dA_0 + \int_M \dot{\rho}_0 \sqrt{-K_0}\, dA_0. \tag{4.24}$$

Proof. We have

$$\dot{\kappa}_0 = \int\limits_M \frac{\partial}{\partial \varepsilon} \sqrt{-K_\varepsilon}|_{\varepsilon=0}\, dA_0 + \int\limits_M \sqrt{-K_0} \frac{\partial}{\partial \varepsilon} \left(dA_\varepsilon\right)|_{\varepsilon=0} \tag{4.25}$$

To simplify the first term, we have the following formula [23, Lemma 5.3] relating the Gaussian curvature of conformal metrics:

$$K_\varepsilon = e^{-2\rho_\varepsilon} \left(-\Delta_0 \rho_\varepsilon + K_0\right)$$

Hence, $\dot{K}_0 = -2\dot{\rho}_0 K_0 - \Delta_0 \dot{\rho}_0$, and thus

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \sqrt{-K_\varepsilon} = \frac{-\dot{K}_0}{2\sqrt{-K_0}} = -\dot{\rho}_0 \sqrt{-K_0} + \frac{\Delta_0 \dot{\rho}_0}{2\sqrt{-K_0}}.$$

For the second term, we note that $dA_\varepsilon = e^{2\rho_\varepsilon} dA_0$. This gives $\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} (dA_\varepsilon) = 2\dot{\rho}_0 dA_0$, which completes the proof. □

Now we specialize to the Ricci flow, i.e., we set $\dot{\rho}_0 = -(K_0 - \overline{K})$. To prove our monotonicity result, we use Lemma 4.2.6 to show positivity of the second term in (4.24).

Proof of Theorem 4.1.3. Letting $\dot{\rho}_0 = -(K_0 - \overline{K})$ in Proposition 4.3.1 and setting $F = \sqrt{-K_0} > 0$ gives

$$\begin{aligned} \dot{\kappa}_0 &= \int_M \frac{\Delta_0 K_0}{2\sqrt{-K_0}} dA_0 - \int_M \sqrt{-K_0}(K_0 - \overline{K}) dA_0 \\ &= \int_M \frac{\Delta_0 F^2}{2F} dA_0 - \int_M (F^3 - F \int_M F^2)\, dA_0. \end{aligned}$$

For the first term, using Stokes' theorem yields

$$\int_M \frac{\Delta_0 F^2}{2F} dA_0 = \int_M \frac{1}{2}\Delta_0 F\, dA_0 + \int_M \frac{\|\nabla_0 F\|^2}{F} = \int_M \frac{\|\nabla_0 F\|^2}{F} \geq 0,$$

which is positive whenever F (hence K_0) is nonconstant. For the second term, we use

$$\int_M F \left(\int_M F^2 dA_0\right) dA_0 = \left(\int_M F dA_0\right)\left(\int_M F^2 dA_0\right) = \int_M F^2 \left(\int_M F dA_0\right) dA_0$$

to obtain

$$\int_M \left(F^3 - F\int_M F^2\right) dA_0 = \int_M F^2 \left(F - \int_M F dA_0\right) dA_0.$$

By Lemma 4.2.6, this term is positive for F non-constant. Hence, $\dot{\kappa} > 0$, which completes the proof. □

4.4 MONOTONICITY OF THE LIOUVILLE ENTROPY

In this section, we will compute the derivative of the Liouville entropy with respect to an arbitrary conformal perturbation (Theorem 4.1.4). We then deduce Theorem 4.1.1.

We start by differentiating (4.21) using some results of Ledrappier and Shu [65] (Proposition 4.4.1). Next, we use an integration by parts formula (Lemma 4.4.10) to simplify a divergence term (Proposition 4.4.5). Then, we use the formalism of Ruelle resonances to rewrite the derivative. In particular, using the work of Faure-Guillarmou [34], we are able to dramatically simplify the expression of the derivative (Theorem 4.1.4). Finally, we deduce our main result (Theorem 4.1.1) using the technical Lemma 4.2.6.

We consider a smooth one-parameter family of conformal area-preserving changes of g_0 :

$$g_\varepsilon = e^{2\rho_\varepsilon} g_0, \qquad A_\epsilon(M) = \int_M e^{2\rho_\epsilon(x)} dA_0(x) \equiv A_0(M). \tag{4.26}$$

We let $\dot{\rho}_0 \in C^\infty(M)$ denote $\frac{d}{d\varepsilon}|_{\varepsilon=0}\rho_\varepsilon$, and we note that by the area-preserving condition, $\dot{\rho}_0$

is mean-zero function.

We start by differentiating the Liouville entropy with respect to this general conformal deformation. In Subsection 4.4.4, we will specialize to the case of the normalized Ricci flow, which corresponds to setting $\dot{\rho}_0 = -(K_0 - \overline{K_0})$ by (4.1).

4.4.1 Using the identification $SM \cong M_0 \times \partial\tilde{M}$

Recall $\pi : S\tilde{M} \to \partial\tilde{M}$ denotes the forward projection along the geodesic flow to the boundary at infinity. Recall from (4.13) the identification

$$\Pi : S\tilde{M} \to \tilde{M} \times \partial\tilde{M}, \quad (x, v) \mapsto (x, \pi(v)).$$

For each $\xi \in \partial\tilde{M}$, we have $\Pi^{-1}(\tilde{M} \times \{\xi\}) = \widetilde{\mathcal{W}^{cs}}(x, \xi)$, the weak stable leaf defined in (4.16). For each $x \in \tilde{M}$, we have $\Pi^{-1}(\{x\} \times \partial\tilde{M}) = S_x\tilde{M}$, which is the leaf of the vertical foliation through (x, ξ).

Let $M_0 \subset \tilde{M}$ be a fundamental domain for the action of the fundamental group of M on $\tilde{M}$. From now on, we will identify SM with the restriction of the above identification to $M_0 \times \tilde{\partial}\tilde{M}$.

Since the metrics g_ε are all quasi-isometric to g_0 (via the identity map), and $\partial\tilde{M}$ is a quasi-isometry invariant (see, for instance, [18, Theorem III.H.3.9]), we will from now on identify all the unit tangent bundles $S^{g_\varepsilon}M$ with the product $M_0 \times \partial\tilde{M}$.

Now let m_ε denote the Liouville measure with respect to g_ε. From now on, we will use $h_{\mathrm{Liou}}(\varepsilon)$ to denote the Liouville entropy of g_ε. Our goal in this subsection is to show:

Proposition 4.4.1. *Let $\varepsilon \mapsto e^{2\rho_\varepsilon} g_0$ be as in (4.26). Then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = -\int_{M_0 \times \partial\tilde{M}} \mathrm{Div}(Y^\xi)\, dm - \int_{M_0 \times \partial\tilde{M}} \dot{\rho}_0 \,\mathrm{Div}(X^\xi)\, dm,$$

where Y is a C^1 vector field on M, which is perpendicular to X^ξ in the g_0 metric, defined in

(4.30) *below.*

The fact that the Liouville entropy depends differentiably on the metric is non-trivial; this is due to Knieper-Weiss [64] for negatively curved surfaces, and to Contreras [25] for general negatively curved manifolds. We will use a slightly different approach from [64] to compute the derivative by starting from (4.21) (the difference being that we integrate $\mathrm{Div}(X)$ instead of the Riccati solution w^s). Formally differentiating (4.21) yields

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Big(-\int_{M_0\times\partial\tilde{M}} \mathrm{Div}_\varepsilon(X^\xi)\, dm(x,\xi) - \int_{M_0\times\partial\tilde{M}} \mathrm{Div}(X^\xi)\, dm_\varepsilon(x,\xi) - \int_{M_0\times\partial\tilde{M}} \mathrm{Div}(X^\xi_\varepsilon)\, dm(x,\xi)\Big). \tag{4.27}$$

We justify that the above formula makes sense by treating each term individually:

- The variation of the divergence can be computed using [8], see Lemma 4.4.2 below.
- The variation of the Liouville measure is computed in Lemma 4.4.3.
- The difficult part is to show that the geodesic spray X^ξ is differentiable when the metric varies, and to compute the derivative. This was first achieved by Ledrappier–Shu in [66, Theorem 3.11] (building on the work of Fathi–Flaminio [33], and in turn on [28, Theorem A.1]), where they compute the derivative of the linear drift along a conformal deformation. We will crucially use their work for our computation of the derivative of the Liouville entropy, see Proposition 4.4.4.

To prove Proposition 4.4.1, we start by showing that the first term in (4.27) vanishes.

Lemma 4.4.2. *With the notation introduced above, the derivative below exists and vanishes at $\varepsilon = 0$:*

$$-\int_{M_0\times\partial\tilde{M}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Div}_\varepsilon(X^\xi)\, dm(x,\xi) = 0.$$

Proof. To compute the variation of Div_ε with respect to ε, we will use [8, Theorem 1.174], which computes the variation of the Levi-Civita connection ∇^ε associated to g_ε. More precisely, for any vector fields $X, Y, Z \in C^\infty(M; TM)$, we have

$$g_0(\partial_\varepsilon|_{\varepsilon=0}\nabla^\varepsilon X(Y), Z) = \frac{1}{2}\big(\nabla_X \dot{g}_0(Y, Z) + \nabla_Y \dot{g}_0(X, Z) - \nabla_Z \dot{g}_0(X, Y)\big).$$

In particular, choosing a local orthonormal frame $(e_i)_{i=1,2}$, we obtain

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0}\mathrm{Div}_\varepsilon(X^\xi) &= -\operatorname{tr}(\partial_\varepsilon|_{\varepsilon=0}\nabla^\varepsilon X^\xi) = -\sum_{i=1}^{2} g_0(\partial_\varepsilon|_{\varepsilon=0}\nabla^\varepsilon X^\xi(e_i), e_i) \\ &= -\frac{1}{2}\sum_{i=1}^{2}\big(\nabla_{X^\xi}\dot{g}_0(e_i, e_i) + \nabla_{e_i}\dot{g}_0(X^\xi, e_i) - \nabla_{e_i}\dot{g}_0(X^\xi, e_i)\big) \\ &= -\frac{1}{2}\operatorname{tr}(\nabla_{X^\xi}\dot{g}_0) = \operatorname{tr}(X^\xi(\dot{\rho}_0)g_0 + \dot{\rho}_0\nabla_{X^\xi}g_0) = 2X^\xi(\dot{\rho}_0).\end{aligned}$$

In the last line, we used the Leibniz rule together with $\dot{g}_0 = 2\dot{\rho}_0 g_0$ (see (4.26)), followed by the fact that $\nabla_{X^\xi} g_0 = 0$. In particular, we see that $\partial_\varepsilon|_{\varepsilon=0}\mathrm{Div}_\varepsilon(X^\xi)$ is a co-boundary. Indeed, one has $X^\xi(\dot{\rho}_0)(x) = X(\dot{\rho}_0)(x, \xi)$, for X as in (4.4). Since the Liouville measure is X-invariant, this shows that the integral in the statement of the lemma vanishes. □

For the second term, we compute the variation of the Liouville measure.

Lemma 4.4.3 (Variation of Liouville measure)**.** *The variation of the Liouville measure with respect to the* (*area-preserving*) *conformal pertubation* (4.26) *is given by*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} dm_\varepsilon = 2\dot{\rho}_0 dm_0.$$

Proof. Since the area of the deformation is constant, the total mass of the Sasaki volume form associated to g_ε is independent of ε. In particular, computing the variation of the Liouville measure reduces to computing the variation of the Sasaki volume form. Recall that the unit tangent bundle locally looks like the base manifold M times the fiber S^1, and the Sasaki volume form has the local product structure which is the product of Riemannian area on the base manifold M and Lebesgue measure on the fiber S^1. For conformal deformations, the Riemannian area form of $g_\varepsilon = e^{2\rho_\varepsilon} g_0$ on M changes exactly by the conformal factor $e^{2\rho_\varepsilon}$. Moreover, conformally changing the metric preserves angles, and hence does not affect the measure on S^1. Differentiating and evaluating at $\varepsilon = 0$ completes the proof. $\square$

We now use the work of Ledrappier and Shu to compute the last term in (4.27). The next proposition is essentially [65, Proposition 4.5] specialized to the case of surfaces.

To state this proposition, we introduce the following notation. For any $f \in C^{1+\alpha}(SM)$, we define a new function I_f on SM by the following integral:

$$I_f(v) = \int_0^{+\infty} \frac{j^s(\varphi_t v)}{j^s(v)} e^s(f)(\varphi_t v)\, dt, \tag{4.28}$$

where $j^s(\varphi_t v)$ is a stable Jacobi solution along the geodesic determined by v, and e^s is the stable vector field defined in (4.17). Although j^s is only defined up to a scalar factor, the ratio $j^s(\varphi_t(v))/j^s(v)$ is well defined along the geodesic defined by v, see (4.20). Note also that the above integral converges because $e^s(f)$ is a continuous, and thus bounded, function on SM, and $j^s(\varphi_t v)/j^s(v)$ decreases exponentially fast by the Anosov property (4.10). Throughout we will frequently use the notation I_f for $f \in C^{1+\alpha}(M)$, where we identify without further comment the function f with its lift to the unit tangent bundle $f \circ P : SM \to \mathbb{R}$.

Further properties of the above “half-orbit" integrals are discussed in Subsection 4.4.3, where we will rewrite them using the meromorphic extension of the resolvent of X.

Proposition 4.4.4. *Fix $v \in SM$ and let $(x, \xi) \in M_0 \times \partial\tilde{M}$ such that $P(v) = x$ and $\pi(v) = \xi$. For $\tau \in \mathbb{R}$, let v_τ denote $\varphi_\tau v$. Let $Jv_\tau \in T_{P(v_\tau)}\tilde{M}$ be the unit vector perpendicular to v, where J is the complex structure associated to the conformal class of g_0. Let $j^s(v_\tau)$ be such that $\tau \mapsto j^s(v_\tau)Jv_\tau$ is a stable Jacobi field along the geodesic generated by v.*

Then the geodesic spray $\varepsilon \mapsto X^\xi_\varepsilon$ is differentiable at $\varepsilon = 0$ with derivative given by

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} X^\xi_\varepsilon = -\dot{\rho}_0 X^\xi + Y^\xi, \tag{4.29}$$

where Y^ξ is a vector field on $\tilde{M}$ perpendicular to X^ξ and given by

$$Y^\xi(x) = -I_{\dot{\rho}_0}(v) Jv := \left(-\int_0^\infty \frac{j^s(v_\tau)}{j^s(v)} Jv_\tau(\dot{\rho}_0(v_\tau))\, d\tau \right) Jv, \tag{4.30}$$

where $v = X^\xi(x)$ and $I_{\dot{\rho}_0}$ is defined in (4.28)*.*

Proof. As in the proof of [65, Proposition 4.5], we note that $\dot{X}^\xi$ can be naturally split into two terms as follows:

$$\dot{X}^\xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(X^\xi_\varepsilon(x) - \frac{X^\xi_\varepsilon(x)}{\|X^\xi_\varepsilon(x)\|_{g_0}} \right) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{X^\xi_\varepsilon(x)}{\|X^\xi_\varepsilon(x)\|_{g_0}} - X^\xi_0(x) \right),$$

The first term records the variation of the g_0-length of X^ξ_ε and is equal to $\frac{d}{d\varepsilon}\|X^\xi_\varepsilon\|_{g_0} X^\xi_0$. Differentiating $\|X^\xi_\varepsilon\|_\varepsilon \equiv 1$ using (4.26) yields

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \|X^\xi\|_\varepsilon = -\frac{1}{2}\dot{g}_0(v,v) = -\dot{\rho}_0.$$

The second term records the change of direction of X^ξ_ε and is a multiple of Jv. The precise formula for the multiple follows from combining the left hand side of the last line in the proof of [65, Proposition 4.5], the expression for $b(t)$ in [65, Proposition 4.3], and the expression for $\Upsilon(t)$ in [65, (5.1)]. □

Proof of Proposition 4.4.1. By Proposition 4.4.4, we have $\frac{d}{d\varepsilon}\mathrm{Div}(X^\xi_\varepsilon) = -\mathrm{Div}(\dot{\rho}_0 X^\xi) + \mathrm{Div}(Y^\xi)$. Note that, up to a coboundary, the first term on the right hand side is equal to $-\dot{\rho}_0 \mathrm{Div}(X^\xi)$. Combining this with Lemmas 4.4.2 and 4.4.3 completes the proof. □

4.4.2 Using an integration by parts formula

In this subsection, we further simplify the expression in Proposition 4.4.1 using an *integration by parts formula* (Lemma 4.4.10 below). More precisely, we obtain the following result.

Proposition 4.4.5. *Let* $\varepsilon \mapsto e^{2\rho_\varepsilon} g_0$ *be as in* (4.26)*. Then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = -\int_{SM} V(w^s) I_{\dot\rho_0} dm - \int_{SM} \dot\rho_0 w^s dm,$$

where $I_{\dot\rho_0}$ *is defined in* (4.28)*.*

We first express the Liouville measure with respect to the weak-stable–vertical identification $SM \cong M_0 \times \partial\tilde M$ in (4.13). This follows from work of Hamenstädt [50, Theorem C].

Lemma 4.4.6 (Hamenstädt)**.** *Let* $(x,\xi) \in \tilde M \times \partial\tilde M \cong S\tilde M$ *and let* $dm(x,\xi)$ *denote the Liouville measure. Then disintegrating* dm *along the projection* $S\tilde M \to \tilde M$ *gives*

$$\forall f \in C^\infty(SM), \quad \int_{SM} f(x,\xi) dm(x,\xi) = \int_{M_0} \left(\int_{\partial\tilde M} f(x,\xi) d\mu_x(\xi) \right) dA(x), \tag{4.31}$$

where $dA(x)$ *is the Riemannian area on* (M,g)*, and* $\{d\mu_x(\xi)\}_{x\in\tilde M}$ *is a mutually absolutely continuous and* $\pi_1(M)$*-equivariant family of probability measures on* $\partial\tilde M$ *whose Radon–Nikodym derivatives are given by*

$$l(x,y,\xi) := \frac{d\mu_x}{d\mu_y}(\xi) = \int_0^\infty \big(w^s(\varphi_t(x,\xi)) - w^s(\varphi_t(y,\xi))\big)\, dt \tag{4.32}$$

for (x,ξ) *and* (y,ξ) *in the same strong stable leaf.*

Remark 4.4.7. *The above improper integral converges because* w^s *is Hölder continuous (in fact* $C^{1+\alpha}$ *by [55] since* $\dim M = 2$*), and because* $d(\varphi_t(x,\xi), \varphi_t(y,\xi))$ *decays exponentially in* t *by the Anosov property* (4.10)*.*

Proof of Lemma 4.4.6. We first discuss the decomposition of the Liouville measure with respect to the weak-stable–unstable *local product structure* of SM (see, for instance, [37, Proposition 6.2.2] for details on the local product structure). Let m^u denote the family of measures on the strong unstable foliation induced by the Riemannian metric g. By the definition of the unstable Jacobian (see Subsection 4.2.3), these transform under the geodesic flow via $\frac{d}{dt}|_{t=0}\, dm^u \circ \varphi_t = w^u\, dm^u$.

Now let f be the *stable* Jacobian $-w^s$ (see (4.22)). Define the measure $\eta^u_f = \phi m^u$, where $\phi = w^u - w^s$. By construction, η^u_f is absolutely continuous with respect to m^u, and moreover, one can check using (4.23) that our choice of ϕ implies $\frac{d}{dt}|_{t=0}\, \eta^u_f \circ \varphi_t = f\eta^u_f$ (see also [24, Corollary 3.11]). Since the pressure of $w^s = -f$ is zero (see (4.23) and [37, Corollary 7.4.5]), we see that η^u_f is as in [50, p. 1069] (note that our notation for the strong vs weak stable foliations differs from Hamenstädt's and that her sign convention for the pressure also differs from ours.)

Let m^s be the analogue of m^u for the strong stable foliation; in particular, these measures transform via $\frac{d}{dt}|_{t=0}\, dm^s \circ \varphi_t = w^s dm^s = -f dm^s$. Let $dm^{cs} = dm^s \times dt$ be the associated measure on weak stable leaves, where dt denotes one-dimensional Lebesgue measure in the flow direction. As discussed in [50, Section 3] (see also [24, Theorem 3.10]), we can "paste" the measures dm^{cs} and $d\eta^u_f$ together using the local product structure into a measure $dm^{cs} \wedge d\eta^u_f$ on SM (Hamenstädt denotes this measure by $d\lambda^s \times d\eta^{su}_f$, see [24, (3.31)] for a more precise formulation). By construction, this "product" measure is absolutely continuous with respect to the Liouville measure. Moreover, by our choice of f, this measure is flow-invariant :

$$\left.\frac{d}{dt}\right|_{t=0} (dm^{cs} \wedge d\eta^u_f) \circ \varphi_t = \left.\frac{d}{dt}\right|_{t=0} (dm^{cs} \circ \varphi_t) \wedge d\eta^u_f + dm^{cs} \wedge \left.\frac{d}{dt}\right|_{t=0} (d\eta^u_f \circ \varphi_t)$$
$$= -f(dm^{cs} \wedge d\eta^u_f) + f(dm^{cs} \wedge d\eta^u_f) = 0.$$

Hence, this measure coincides with the Liouville measure dm.

Theorem C in [50] then states that $dm^{cs} \wedge d\eta^u_f$ (and thus dm) in turn coincides with the measure $dA(x)d\eta_f^x(v)$, where $d\eta_f^x(v)$ is a probability measure on $S_x\tilde{M}$ such that the family of pushforward measures $x \mapsto d\mu_x^f(\xi) := d\eta_x^f(\pi_x^{-1}(v))$ on $\partial\tilde{M}$ has Radon–Nikodym derivatives as in (4.32). See also [24, (3.16)]. $\square$

Remark 4.4.8. *In the above proof, if one replaces $f = -w^s$ by a cohomologous potential, i.e., a potential whose equilibrium state is still the Liouville measure m (for instance $f = w^u$), then the "product" measure $dA(x)d\eta_f^x(v)$ constructed from dm^{cs} and $d\eta_f^u$ in general only yields a measure in the same* measure class *as Liouville (denoted by $\overline{\eta}_f$ in [50]).*

Remark 4.4.9. *On a related note, the family $d\mu_x$ in the conclusion of Lemma 4.4.6 is absolutely continuous with respect to, but not equal to, the usual visual measures dm_x given by the pushforward via $\pi : S\tilde{M} \to \partial\tilde{M}$ of Lebesgue measure on the fibers $S_x\tilde{M}$. The Radon-Nikodym derivatives $dm_x/dm_y(\xi)$ are given by replacing w^s with $-w^u$ in the above formula* (4.32). *(To see this, see, for instance, [58, equation (0.5)], which is equivalent to the fact that the Radon-Nikodym derivative dm_x/dm_y is an appropriate limit of a ratio of spherical Jacobi fields. Then one can replace these spherical Jacobi fields with unstable Jacobi fields in the limit by using the C^2 convergence of the limits in the definition of a Busemann function in* (4.14)*; see for instance [53, Lemma 3.3].) Compare with [24, (3.33) and (3.34)].*

Now we use the decomposition of dm in (4.31) to deduce a useful integration by parts formula. A version of this formula appears in [65, 66]; for the harmonic measure, see, for instance, [65, Equation (5.10)].

Lemma 4.4.10. *Let $\xi \mapsto Y^\xi$ be a continuous family of C^1 vector fields on M. Let $dm(x,\xi)$ be the Liouville measure on $SM \cong M_0 \times \partial\tilde{M}$ and let $\{\mu_x\}_{x\in\tilde{M}}$ and $l(x,y,\xi)$ as in the previous lemma. Then, letting Y denote Y^ξ throughout, we have*

$$\int_{M_0\times\partial\tilde{M}} \mathrm{Div}(Y)\, dm(x,\xi) = -\int_{M_0\times\partial\tilde{M}} \left\langle Y, \nabla_y \log l(y,x,\xi)|_{y=x}\right\rangle\, dm(x,\xi).$$

Proof. Using the above decomposition of $dm(x,\xi)$, together with the definition of the Radon-

Nikodym derivative, we have $dm(x,\xi) = l(x,x_0,\xi)\,dA(x)d\mu_{x_0}(\xi)$. We then have

$$\begin{aligned}
\int_{M_0\times\partial\tilde{M}} \mathrm{Div}(Y)\,dm(x,\xi) &= \int_{\partial\tilde{M}}\int_{M_0} \mathrm{Div}(Y) l(x,x_0,\xi)\,dA(x)\,d\mu_{x_0}(\xi) \\
&= -\int_{\partial\tilde{M}}\int_{M_0} \langle Y, \nabla_y l(y,x_0,\xi)|_{y=x}\rangle\; dA(x)\,d\mu_{x_0}(\xi) \\
&= -\int_{\partial\tilde{M}}\int_{M_0} \left\langle Y, \frac{\nabla_y l(y,x_0,\xi)|_{y=x}}{l(x,x_0,\xi)}\right\rangle l(x,x_0,\xi)\,dA(x)\,d\mu_{x_0}(\xi) \\
&= -\int_{M_0\times\partial\tilde{M}} \langle Y, \nabla_y \log l(y,x,\xi)|_{y=x}\rangle\,dm(x,\xi).
\end{aligned}$$

In the last line, we used $\nabla_y \log l(y,x,\xi)|_{y=x} = \nabla_y \log l(y,x_0,\xi)|_{y=x}$, which follows by taking the gradient of the cocycle relation $\log l(y,x,\xi) = \log l(y,x_0,\xi) + \log l(x_0,x,\xi)$. □

Lemma 4.4.11. *Let Y^ξ as in Propositions 4.4.1 and 4.4.4. Then one has*

$$-\int_{M_0\times\partial\tilde{M}} \mathrm{Div}(Y^\xi)dm = -\int_{M_0\times\partial\tilde{M}} I_{w^s} I_{\dot{\rho}_0} dm,$$

where the notation I_f is defined in (4.28).

Proof. Fix $v = (x,\xi)$. Let $c(s)$ be a curve in $\tilde{M}$ such that $c(0) = x$ and $c'(0) = Jv$. This means $(c(s),\xi)$ is tangent to e^s, defined in (4.17). By (4.30), we have

$$\begin{aligned}
\langle Y, \nabla_y \log l(y,x,\xi)|_{y=x}\rangle &= -I_{\dot{\rho}_0}(v)\,\frac{d}{ds}\bigg|_{s=0} \log l(c(s),x,\xi) \\
&= -I_{\dot{\rho}_0}(v)\,e^s(\log l(y,x,\xi)_{y=x}) \\
&= -I_{\dot{\rho}_0}(v) \int_0^{+\infty} e^s(w^s\circ\varphi_\tau v)d\tau \qquad \text{(by Lemma 4.4.6)} \\
&= -I_{\dot{\rho}_0}(v) \int_0^{+\infty} \frac{j^s(\varphi_\tau v)}{j^s(v)}[e^s w^s](\varphi_\tau v)d\tau = -I_{\dot{\rho}_0}(v) I_{w^s}(v).
\end{aligned}$$

Applying Lemma 4.4.10 completes the proof. □

To complete the proof of Proposition 4.4.5, it remains to show the following:

Lemma 4.4.12. *Let V be the vertical vector field in* (4.6)*. Then for all $v \in SM$, we have*

$$I_{w^s}(v) = V(w^s)(v).$$

Proof. Since $VK = 0$, applying V to both sides of the Riccati equation (4.19) gives $VXw^s = -2w^sVw^s$. Next, we use the commutation relation (4.7) between X and V to get $XVw^s + Hw^s = -2w^sVw^s$, which is equivalent to

$$(X + w^s)Vw^s = -(H + w^sV)w^s = -e^s(w^s).$$

Plugging this into the integral defining I_{w^s} and integrating by parts, we obtain

$$\begin{aligned} I_{w^s} &= \int_0^{+\infty} \frac{j^s(v_\tau)}{j^s(v)}[e^s w^s](v_\tau)d\tau = -\int_0^{+\infty} \frac{j^s(v_\tau)}{j^s(v)}[(X + w^s)Vw^s](v_\tau)d\tau \\ &= \frac{1}{j^s(v)} \int_0^{+\infty} \underbrace{(-X + w^s)j^s(v_\tau)}_{=0} Vw^s(v_\tau)d\tau - \frac{1}{j^s(v)}[j^s(v_\tau)V(w^s)(v_\tau)]_0^{+\infty} = V(w^s), \end{aligned}$$

which completes the proof. □

4.4.3 Using Pollicott–Ruelle resonances

In this subsection, we use microlocal analysis, more specifically, the formalism of Pollicott–Ruelle resonances, to simplify the function $I_{\dot{\rho}_0}$ appearing in Proposition 4.4.5. Our goal is to obtain the following formula for the derivative of the Liouville entropy along a conformal change.

Theorem 4.1.4 (Derivative of the metric entropy for a conformal perturbation)**.** *Let (M, g_0) be a negatively curved surface and let $g_\varepsilon = e^{2\rho_\varepsilon} g_0$ be a conformal area-preserving perturbation of g_0. Then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\text{Liou}}(\varepsilon) = -\frac{1}{2}\int_{SM} \dot{\rho}_0 w^s dm.$$

Our computation of the derivative of h_{Liou} from the previous subsection involves certain functions which are obtained as integrals over half orbits (see (4.28) and Lemma 4.4.11 above). We start by noting that these functions satisfy a differential equation.

Proposition 4.4.13. *Fix $f \in C^{1+\alpha}(SM)$. For any $v \in SM$ and $t \in \mathbb{R}$, let $j^s(\varphi_t v)$ be a stable Jacobi solution along the geodesic determined by v. As in* (4.28)*, define the following integral:*

$$I_f(v) = \int_0^{+\infty} \frac{j^s(\varphi_t v)}{j^s(v)} [e^s(f)](\varphi_t v)\, dt.$$

Then I_f satisfies

$$(X + w^s) I_f = -e^s(f). \tag{4.33}$$

Remark 4.4.14. *In order to show $I_{w^s} = V(w^s)$ in Lemma 4.4.12, we showed the above claim for the function $f = w^s$.*

Proof. As explained below equation (4.28), the above expression is well-defined and the improper integral I_f converges.

To check (4.33), we apply the pullback of the flow to I_f. We will write v_τ as a shorthand notation for $\varphi_\tau(v)$. We have

$$(\varphi_\theta)^* I_f(v) = \int_0^{+\infty} \frac{j^s(v_{\tau+\theta})}{j^s(v_\theta)} [e^s f](v_{\tau+\theta}) d\tau = \frac{1}{j^s(v_\theta)} \int_\theta^{+\infty} j^s(v_\tau)[e^s f](v_\tau) d\tau.$$

Differentiating at $\theta = 0$ gives

$$X(I_f) = -\frac{Xj^s}{j^s}(v) I_f(v) - \frac{1}{j^s(v)} j^s(v_\tau)(e^s f)(\tau)\Big|_{\tau=0}.$$

Using that $Xj^s/j^s = w^s$ completes the proof. ∎

We now rewrite the function I_f using the formalism of *Pollicott–Ruelle resonances.* View the geodesic vector field $X \in C^\infty(SM; T(SM))$ as a differential operator on $C^\infty(SM)$. For

$\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > 0$, define the positive and negative *resolvents* of X:

$$R_\pm(\lambda) := (\mp X - \lambda)^{-1} : \ L^2(SM, dm) \to L^2(SM, dm), \ f \mapsto -\int_0^\infty e^{-t\lambda} e^{\mp tX} f(v) dt, \quad (4.34)$$

where $e^{tX} f(v)$ denotes the propagator $f(\varphi_t v)$ (see, for instance, [67, (9.1.3)]). Note the above integral converges for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, and it defines an L^2 function since the propagator is unitary on $L^2(SM, dm)$. Indeed, one has for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > 0$

$$\forall f \in L^2(SM, dm), \quad \left\| \int_0^\infty e^{-t(\pm X + \lambda)} f(v) dt \right\|_{L^2} \leq \|f\|_{L^2} \int_0^{+\infty} e^{-\mathrm{Re}(\lambda) t} dt = \frac{\|f\|_{L^2}}{\mathrm{Re}(\lambda)}.$$

We will show that the operator $f \mapsto I_f$ above is related to $R_\pm(\lambda)$ for $\lambda = 0$ (Proposition 4.4.16 below). To make this precise, we note that it is now well understood, see for instance [11, 21, 5, 41, 35, 36], that one can construct function spaces tailored to the flow (the so-called *anisotropic spaces*) on which the resolvents defined in (4.34) extend meromorphically to the entire complex plane. The poles of the resulting meromorphic extension are called *Pollicott–Ruelle* resonances, and they encode important properties of the flow such as the exponential decay rate of the correlations [68, 86]. As alluded to above, the resonance $\lambda = 0$ will be of particular importance to us.

We will not recall the exact constructions of these anisotropic spaces (see [67, Section 9.1.2] for an introduction), but we recall the following properties which are needed to state precisely the relation between I_f and the resolvent (Proposition 4.4.16). By the work of Faure and Sjöstrand [36], there exists a family of Hilbert spaces $(\mathcal{H}^s_\pm)_{s>0}$ with the following properties.

1. [67, Lemma 9.1.13] The space $C^\infty(SM)$ is densely included in $\mathcal{H}^s_\pm(SM)$.

2. [67, Lemma 9.1.14] One has $H^s \subset \mathcal{H}^s_\pm \subset H^{-s}$, where H^s is the usual L^2-Sobolev space

of order s. Recall as well from [56, Chapter 7.9] that

$$\forall \alpha \notin \mathbb{N},\ \forall s < \alpha, \quad C^\alpha \subset H^s \subset \mathcal{H}^s_\pm. \tag{4.35}$$

In other words, Hölder continuous functions are in $\mathcal{H}^s_\pm$ for $s > 0$ small enough.

3. [67, Theorem 9.1.5] There exists $c > 0$, such that for any $s > 0$ and any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -cs$, the operators

$$\mp X - \lambda : \mathrm{Dom}(X) \cap \mathcal{H}^s_\pm = \{u \in \mathcal{H}^s_\pm \mid Xu \in \mathcal{H}^s_\pm\} \to \mathcal{H}^s_\pm \tag{4.36}$$

act unboundedly. Moreover, the resolvents

$$R_\pm(\lambda) = (\mp X - \lambda)^{-1} : \mathcal{H}^s_\pm \to \mathcal{H}^s_\pm \tag{4.37}$$

are well defined, bounded and holomorphic for $\{\mathrm{Re}(\lambda) > 0\}$, and have a meromorphic extension to $\{\mathrm{Re}(\lambda) > -cs\}$, which is independent of any choices made in the construction. Thus, the resolvents $R_\pm(\lambda)$, viewed as operators from $C^\infty(M)$ to the space of distributions $\mathcal{D}'(M)$, have meromorphic extensions to the whole complex plane. The poles of this extension are called the *Pollicott–Ruelle resonances* of X.

4. [67, Section 9.2.4] Near the pole $\lambda = 0$, one has the Laurent expansion

$$R_\pm(\lambda) = -\frac{\Pi_0}{\lambda} - R^H_\pm(\lambda) + O(\lambda), \tag{4.38}$$

where $R^H_\pm(\lambda)$ is the holomorphic part of the resolvent, and Π_0 is the orthogonal projection onto constant functions.

5. [67, Lemma 9.2.9 i) and Lemma 9.2.4] For any $s > 0$, the operators $R^H_\pm(0) : H^s \to H^{-s}$

are bounded, and one has the adjoint identity

$$(R^H_\pm(0))^* = R^H_\mp(0). \tag{4.39}$$

6. Applying $(\mp X - \lambda)$ on both the left and right of (4.38), using $X\Pi_0 = \Pi_0 X = 0$, and then taking $\lambda \to 0$, one obtains the commutation relations

$$\pm XR^H_\pm(0) = \pm R^H_\pm(0)X = \mathrm{Id} - \Pi_0, \tag{4.40}$$

which, together with the above adjoint identity, will be key for obtaining the simplified formula in Theorem 4.1.4. (Note, however, that (4.39) and (4.40) are not yet required for the statement and proof of Proposition 4.4.16.)

Remark 4.4.15. *One can give a more explicit description of the operators $R^H_\pm(0)$ by appealing to the exponential mixing of the geodesic flow. Let $f_1, f_2 \in C^\alpha(SM)$ for some $\alpha > 0$, and suppose that $\langle f_1, 1\rangle_{L^2} = \int_{SM} f_1 dm = 0$. Then by a result of Liverani [68],*

$$\exists C > 0, \exists \eta > 0, \forall t \in \mathbb{R}, \quad |\langle f_1 \circ \varphi_t, f_2\rangle_{L^2}| \leq Ce^{-\eta|t|}. \tag{4.41}$$

One can then show [45, Equation (2.6)] that for any $f_1 \in C^\alpha$ such that $\langle f_1, 1\rangle_{L^2} = 0$ and for any $f_2 \in C^\infty(SM)$, the distributional pairing $(R^H_\pm(0)f_1, f_2)$ is given by

$$(R^H_\pm(0)f_1, f_2) = \pm \int_0^{\pm\infty} \langle f_1 \circ \varphi_t, f_2\rangle_{L^2}\, dt = \pm \int_0^{\pm\infty} \int_{SM} f_1 \circ \varphi_t(p) f_2(p)\, dm(p)\, dt, \tag{4.42}$$

where the right-hand side is well-defined by (4.41). Formally exchanging the two integrations, one sees that $R^H_\pm(0)f_1 = \pm \int_0^{\pm\infty} f_1 \circ \varphi_t dt$, where the equality is meant distributionally. This explains how the operators $R^H_\pm(0)$ can be used to make sense of "integrating on half orbits".

We note, however, that (4.42) is not needed for our purpose since we will only use the Laurent expansion of $R_\pm(\lambda)$ near 0. In particular, we will use the work of Faure–Guillarmou

[34] to rewrite the function I_f in terms of $R_-^H(0)$.

Proposition 4.4.16. *Let $f \in C^{1+\alpha}(M)$, then*

$$I_f = e^s R_-^H(0) f. \tag{4.43}$$

Proof. Let $R_{X+w^s}(\lambda) := (X + w^s - \lambda)^{-1}$ be the L^2 resolvent of $X + w^s$ defined for $\mathrm{Re}(\lambda)$ large enough. One has the commutation relation

$$\begin{aligned}[X, e^s] = [X, H + w^s V] &= K_g V - w^s H + X(w^s) V \\ &= -w^s H + (K_g + X(w^s))V = -w^s H - (w^s)^2 V = -w^s e^s.\end{aligned}$$

In particular, for any $\lambda \in \mathbb{C}$, one has

$$e^s(X - \lambda) = X e^s + w^s e^s - \lambda e^s = (X + w^s - \lambda) e^s.$$

If $\mathrm{Re}(\lambda) \gg 1$, both $R_-(\lambda) := (X - \lambda)^{-1}$ and $R_{X+w^s}(\lambda) = (X + w^s - \lambda)^{-1}$ exist. Applying $R_-(\lambda)$ on the right and $R_{X+w^s}(\lambda)$ on the left yields

$$R_{X+w^s}(\lambda) e^s = e^s R_-(\lambda). \tag{4.44}$$

By [34, Corollary 3.6], there exists $s_0 > 0$ such that the above relation extends to $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > -cs_0\}$, that is,

$$\forall\, \mathrm{Re}(\lambda) > -cs_0, \quad e^s R_-(\lambda) = R_{X+w^s}(\lambda) e^s, \tag{4.45}$$

where the equality holds as *analytic* operators $C^\infty(SM) \to \mathcal{D}'(SM)$.

Let us be more explicit about the meaning of (4.45) when λ is a pole of $R_-(\lambda)$, i.e., a Pollicott–Ruelle resonance. We will apply (4.45) at $\lambda = 0$. With the notations of (4.38), near $\lambda = 0$, one has

$$e^s R_-(\lambda) = e^s \left(-\frac{\Pi_0}{\lambda} - R_-^H(0) + O(\lambda) \right) = -\frac{e^s \Pi_0}{\lambda} - e^s R_-^H(0) + O(\lambda).$$

The crucial point is that since Π_0 is the projection onto constant functions, one has $e^s \Pi_0 = 0$. This shows that $e^s R_-(\lambda)$ can be extended to $\lambda = 0$ with value equal to $-e^s R_-^H(0)$.

The (far) more general statement of [34, Corollary 3.10] (which we will not need) is that for any Pollicott–Ruelle resonance $\lambda_0 \in \{\lambda \mid \mathrm{Re}(\lambda) > -cs_0\}$, the polar part of $R_-(\lambda)$ near $\lambda = \lambda_0$ is killed by e^s. In other words, the *generalized resonant states* associated to λ_0 are invariant by the horocycle derivative e^s. We refer to the introduction of [34] for a more detailed discussion of the matter.

Note that using (4.33), we can rewrite the half-orbit integral I_f for $f \in C^\infty$ as

$$I_f = -R_{X+w^s}(0) e^s(f). \tag{4.46}$$

By (4.45) and the previous discussion, for any $f \in C^\infty(SM)$, one has

$$I_f = -R_{X+w^s}(0) e^s f = -e^s R_-(0) f = e^s R_-^H(0) f.$$

Now, choose a sequence of smooth functions $(f_n)_{n \in \mathbb{N}}$ such that $f_n \to f$ in $C^{1+\alpha}$. Using (4.35), this means that $f_n \to f$ and $e^s(f_n) \to e^s(f)$ in H^s for $s < \alpha$. In particular, we obtain (4.43) by passing to the limit, using the boundedness property (4.39) recalled above. □

We are now ready to prove Theorem 4.1.4.

Proof of Theorem 4.1.4. Using Proposition 4.4.5, followed by Proposition 4.4.16, one has

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) &= -\int_{SM} V(w^s) I_{\dot\rho_0} dm - \int_{SM} \dot\rho_0 w^s dm \\ &= -(V(w^s), e^s R_-^H(0) \dot\rho_0) - \int_{SM} \dot\rho_0 w^s dm, \end{aligned}$$

where $(\cdot,\cdot)$ denotes the distributional pairing. The main idea of the proof is that the algebraic properties (4.39) and (4.40) of $R_-^H(0)$ stated above (together with the commutation and adjoint relations in (4.44) and (4.8)), allow us to dramatically simplify this distributional pairing.

In the following, we write $f = \dot{\rho}_0 \in C^\infty(SM)$. Note that the area-preserving condition implies $\Pi_0 f = \langle f, 1\rangle_{L^2} = 0$. Using $(e^s)^* = -e^s - V(w^s)$, we obtain

$$(V(w^s), e^s R_-^H(0) f) = -\big((e^s + V(w^s))V(w^s), R_-^H(0) f\big).$$

The last pairing is well defined since $(e^s + V(w^s))V(w^s) \in C^\alpha$ for some $\alpha > 0$, which means that it belongs to some H^{s_0} for $s_0 < \alpha$ by (4.35). On the other hand, $R_-^H(0) f \in \cap_{s>0} \mathcal{H}_-^s \subset H^{-s_0}$ by the above properties (5) and (1) of $\mathcal{H}^s$, since f is smooth. To further simplify this expression, we first check that

$$[e^s, V] = [H + w^s V, V] = X - V(w^s)V,$$

and thus

$$(e^s + V(w^s))V(w^s) = Ve^s(w^s) + X(w^s) - V(w^s)^2 + V(w^s)^2 = Ve^s(w^s) + X(w^s).$$

In particular, plugging this in the last expression and using $(R_+^H(0))^* = R_-^H(0)$, we get

$$-\int_{SM} V(w^s) I_f \, dm = \big(R_+^H(0)(Ve^s(w^s) + X(w^s)), f\big).$$

Using (4.40) and $\Pi_0(w^s) = -h_{\mathrm{Liou}}(0)$, one has

$$-\int_{SM} V(w^s) I_f \, dm = \big(R_+^H(0) V e^s(w^s), f\big) + (w^s, f) + \underbrace{h_{\mathrm{Liou}}(0)\langle f, 1\rangle_{L^2}}_{=0}.$$

This yields

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = \big(R_+^H(0)Ve^s(w^s), f\big), \tag{4.47}$$

where the pairing is again meant distributionally. Next, applying V to the Ricatti equation (4.19) gives $VX(w^s) = -2w^sV(w^s)$. In particular, using (4.7), one has

$$\begin{aligned}
Ve^s(w^s) &= V\big(H(w^s) + w^sV(w^s)\big) = V\big(H(w^s) - \frac{1}{2}VX(w^s)\big) \\
&= V(H(w^s) - \frac{1}{2}XV(w^s) - \frac{1}{2}H(w^s)) \\
&= \frac{1}{2}V(H(w^s) - XV(w^s)) = -\frac{1}{2}X(w^s) + \frac{1}{2}HV(w^s) - \frac{1}{2}VXV(w^s) \\
&= -\frac{1}{2}X(w^s) + \frac{1}{2}(H - VX)V(w^s) = -\frac{1}{2}X(w^s) - \frac{1}{2}XV^2(w^s).
\end{aligned}$$

We note that since $w^s \in C^{1+\alpha}$, the expression $V^2(w^s)$ is a priori not well defined (as a function), but the previous computation shows that $XV^2(w^s) = -2Ve^s(w^s) - X(w^s)$ is. Plugging this last equality into (4.47), using (4.40) and $(R_+^H(0)XV)^* = VXR_-^H(0)$ gives

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) &= -\frac{1}{2}\big(R_+^H(0)(X(w^s) + XV^2(w^s)), f\big) \\
&= -\frac{1}{2}(w^s + h_{\mathrm{Liou}}(g_0), f) - \frac{1}{2}(V(w^s), VXR_-^H(0)f) \\
&= -\frac{1}{2}(w^s, f) + \frac{1}{2}\langle V(w^s), V(f - \Pi_0 f)\rangle = -\frac{1}{2}(w^s, f),
\end{aligned}$$

where we used that $\Pi_0(w^s) = -h_{\mathrm{Liou}}(g_0)$ and that $V(f - \Pi_0 f) = 0$. ◻

4.4.4 Specializing to the normalized Ricci flow

To prove the Liouville entropy is monotonic along the Ricci flow (Theorem 4.1.1), we will set $\dot\rho_0 = -(K_0 - \overline{K})$ in Theorem 4.1.4, where $\overline{K} = \int_{SM} K_0 dm$. (see (4.1)). We first record the following lemma.

Lemma 4.4.17. *For any negatively curved surface* (M, g)*, we have*

$$\int_{SM} (K_g - \overline{K}) w_g^s \, dm_g \geq 0,$$

with equality if and only if g *has constant curvature.*

Proof. Integrating both sides of the Riccati equation (4.19) gives $\overline{K} = -\int_{SM} (w_g^s)^2 dm_g$. Next, multiplying (4.19) by w_g^s, we get

$$\frac{1}{2} X((w_g^s)^2) = -(w_g^s)^3 - K_g w_g^s.$$

This, together with Lemma 4.2.6, gives

$$\int_{SM} (K_g - \overline{K}) w_g^s dm_g = \int (w_g^s)^2 \left(-w_g^s - \int (-w_g^s) dm_g \right) dm_g \geq 0,$$

with strict inequality if g has non-constant curvature. □

Proof of Theorem 4.1.1. We apply Theorem 4.1.4 to the normalized Ricci flow. Setting $\dot{\rho}_0 = -(K_0 - \overline{K})$ in Theorem 4.1.4, we obtain

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_{\text{Liou}}(\varepsilon) = \frac{1}{2} \int_{SM} (K_0 - \overline{K}) w^s dm.$$

By the previous lemma, we see that

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h_{\text{Liou}}(\varepsilon) \geq 0, \tag{4.48}$$

with equality if and only if g_ε has constant curvature. This shows that for any g_0 with non-constant curvature, the above derivative is strictly positive, which proves the theorem. □

4.5 MONOTONICITY FOR 1/6-PINCHED METRICS

In this section, we prove positivity of the derivative of the Liouville entropy (see Proposition 4.5.3) without using the formalism of Ruelle resonances (Proposition 4.4.16), under the additional assumption of 1/6-pinching of Gaussian curvature. Since the normalized Ricci flow preserves the pinching constant (Proposition 4.5.4) this shows that for a 1/6-pinched metric Liouville entropy is strictly increasing along its full normalized Ricci flow future orbit. We start by showing the following Lemma.

Lemma 4.5.1. *We have the following identity*

$$I_{-K} = -e^s(w^s) + I_{(w^s)^2}. \tag{4.49}$$

Proof. First, since $e^s = H + w^s V$ and K is a function on the base M, we see that $H(K) = e^s(K)$. Applying e^s on both sides of (4.19) yields $e^s(K) = -e^s X w^s - 2e^s(w^s)w^s$. Next, we use (4.7) and (4.19) to compute:

$$\begin{aligned}[X, e^s] &= [X, H + w^s V] = K_g V - w^s H + X(w^s)V \\ &= -w^s H + (K_g + X(w^s))V = -w^s H - (w^s)^2 V = -w^s e^s.\end{aligned}$$

Using $[X, e^s] = -w^s e^s$, this gives finally

$$-e^s(K) = (X + 3w^s)(e^s w^s). \tag{4.50}$$

Plugging (4.50) into (4.30) we obtain, writing $v_\tau = \varphi_\tau v$,

$$\begin{aligned}
-\int_0^\infty \frac{j^s(v_\tau)}{j^s(v)}(e^s K)(v_\tau)\,d\tau &= \int_0^\infty \frac{j^s(v_\tau)}{j^s(v)}(X + w^s)(e^s w^s)(v_\tau)\,d\tau \\
&\quad + 2\int_0^\infty \frac{j^s(v_\tau)}{j^s(0)} w^s(v_\tau) e^s(w^s(v_\tau))d\tau \\
&= \frac{1}{j^s(v)}\left[j^s(v_\tau)(e^s w^s)(v_\tau)\right]_0^{+\infty} + \int_0^\infty \frac{j^s(v_\tau)}{j^s(v)} e^s\big(w^s(v_\tau)^2\big)d\tau \\
&\quad - \frac{1}{j^s(v)}\int_0^\infty \underbrace{(-X + w^s)j^s(v_\tau)}_{=0}(e^s w^s)(v_\tau)\,d\tau \\
&= -e^s w^s(v) + \int_0^\infty \frac{j^s(v_\tau)}{j^s(v)} e^s\big(w^s(v_\tau)^2\big)d\tau.
\end{aligned}$$

In the previous computation, we used the fact that $w^s = X(j^s)/j^s$ and the fact that j^s is a stable Jacobi field to compute the bracket. This concludes the proof of the lemma. □

Using this, together with the differential equations satisfied by I_{w^s} and $I_{(w^s)^2}$ (Proposition 4.4.13), we obtain the following.

Proposition 4.5.2.

$$-\int_{SM} I_{-K} I_{w^s} dm = \int_{SM} \frac{K}{2(w^s)^3}(I_{(w^s)^2} - w^s I_{w^s})^2 dm + \int_{SM} -w^s (I_{w^s})^2 (3 + \frac{K}{2(w^s)^2}) dm. \quad (4.51)$$

Proof. Using Proposition 4.4.11 and Lemma 4.5.1,

$$\begin{aligned}
-\int_{SM} I_{-K} I_{w^s} dm &= \int_{SM} e^s(w^s) I_{w^s} dm - \int_{SM} I_{w^s} I_{(w^s)^2} dm \\
&= -\int_{SM} (X I_{w^s} + w^s I_{w^s}) I_{w^s} dm - \int_{SM} I_{w^s} I_{(w^s)^2} dm \quad \text{(Proposition 4.4.13)} \\
&= \int_{SM} -w^s (I_{w^s})^2 dm - \int_{SM} I_{w^s} I_{(w^s)^2} dm.
\end{aligned}$$

To simplify the second term above, we start by using Proposition 4.4.13 with $f = (w^s)^2$.

This gives

$$
\begin{aligned}
-\int_{SM} I_{w^s} I_{(w^s)^2} dm &= \int_{SM} \frac{I_{w^s}}{w^s}(X I_{(w^s)^2} + 2w^s e^s(w^s))dm \\
&= -\int_{SM} X(I_{w^s}/w^s) I_{(w^s)^2} + \int_{SM} 2e^s(w^s) I_{w^s} dm \\
&= -\int_{SM} X(1/w^s) I_{w^s} I_{(w^s)^2} dm - \int_{SM} \frac{1}{w^s} X(I_{w^s}) I_{(w^s)^2} dm \\
&\quad + \int_{SM} -2w^s (I_{w^s})^2 dm,
\end{aligned}
$$

where we integrated by parts. Next, note that by the Ricatti equation, we have,

$$
-\int_{SM} X(\frac{1}{w^s}) I_{w^s} I_{(w^s)^2} dm = -\int_{SM} I_{w^s} I_{(w^s)^2} - \int_{SM} \frac{K}{(w^s)^2} I_{w^s} I_{(w^s)^2} dm.
$$

Next, Proposition 4.4.13 with $f = w^s$ gives

$$
-\int_{SM} \frac{1}{w^s} X(I_{w^s}) I_{(w^s)^2} dm = \int_{SM} I_{w^s} I_{(w^s)^2} dm - \int_{SM} \frac{e^s(w^s)}{w^s} I_{(w^s)^2} dm.
$$

Hence,

$$
\begin{aligned}
\int_{SM} \mathrm{Div}(Y) dm = &- \int_{SM} 3w^s (I_{w^s})^2 dm - \int_{SM} \frac{e^s(w^s)}{w^s} I_{(w^s)^2} dm \\
&- \int_{SM} \frac{K}{(w^s)^2} I_{w^s} I_{(w^s)^2} dm.
\end{aligned}
$$

To simplify the second term, we use Proposition 4.4.13 with $f = (w^s)^2$, which gives

$$
\begin{aligned}
-2\int_{SM} \frac{e^s(w^s)}{w^s} I_{(w^s)^2} dm &= \int_{SM} \frac{(I_{(w^s)^2})^2}{w^s} dm + \int_{SM} \frac{X(I^2_{(w^s)^2})}{2} \frac{1}{(w^s)^2} dm \\
&= \int_{SM} \frac{(I_{(w^s)^2})^2}{w^s} dm - \int_{SM} (I_{(w^s)^2})^2 \frac{X(w^s)}{(w^s)^3} dm \\
&= \int_{SM} \frac{(I_{(w^s)^2})^2}{w^s} \big(1 - \frac{(w^s)^2 + K}{(w^s)^2}\big) dm \\
&= \int_{SM} \frac{K}{(w^s)^3} (I_{(w^s)^2})^2 dm.
\end{aligned}
$$

Hence,

$$\begin{aligned}\int_{SM} \mathrm{Div}(Y)dm &= \int_{SM} -3w^s(I_{w^s})^2dm + \int_{SM} \frac{K}{2(w^s)^3}((I_{(w^s)^2})^2 - 2w^sI_{w^s}I_{(w^s)^2})dm \\ &= \int_{SM} -3w^s(I_{w^s})^2dm + \int_{SM} \frac{K}{2(w^s)^3}((I_{(w^s)^2})^2 - w^sI_{w^s})^2dm \\ &\quad - \int_{SM} \frac{K}{2w^3}(w^s)^2(I_{w^s})^2dm \\ &= \int_{SM} -w^s(I_{w^s})^2(3 + \frac{K}{2(w^s)^2})dm + \int_{SM} \frac{K}{2(w^s)^3}((I_{(w^s)^2})^2 - w^sI_{w^s})^2dm.\end{aligned}$$

This completes the proof. □

Using Proposition 4.4.5, we have and Lemma 4.4.17,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) = -\int_{SM} V(w^s)I_{\dot\rho_0}dm - \int_{SM} \dot\rho_0 w^s dm = \int_{SM} I_{w^s}I_{-K} + \int_{SM} (K - \overline{K})w^s. \quad (4.52)$$

In light of Lemma 4.4.17, to complete the proof of Theorem 4.1.1 for 1/6-pinched metrics, it suffices to show the following.

Proposition 4.5.3. *Suppose that the metric g is 1/6-pinched, i.e., $-K_2 \leq K_g \leq -K_1 < 0$ with $K_2/K_1 \leq 6$. Then*

$$-\int_{SM} I_{w^s}I_{-K}dm \geq 0.$$

As a consequence,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} h_{\mathrm{Liou}}(\varepsilon) \geq 0$$

with equality if and only if g is hyperbolic.

Proof. By [64, Appendix B, Lemma 1], we have $K_1 \leq (w^s)^2 \leq K_2$. In particular, one obtains

$$3 + \frac{1}{2}\frac{K}{(w^s)^2} \geq 3 - \frac{1}{2}\frac{K_2}{K_1} \geq 3 - \frac{1}{2} \times 6 = 0,$$

under the 1/6-pinching condition. This means that the two integrands in (4.51) are non negative and thus we deduce $-\int_{SM} V(w^s)I_{-K}dm \geq 0$. To conclude, we use (4.52) and

Lemma 4.4.17. □

Finally, we remark that if g_0 is 1/6-pinched, then the same is true for any metric along its normalized Ricci flow. In particular, this shows that $\varepsilon \mapsto h_{\mathrm{Liou}}(\varepsilon)$ is increasing along the Ricci flow starting from any 1/6-pinched metric. The proof of the following proposition is left as an exercise to the reader.

Proposition 4.5.4. *Let g_0 be a negatively curved metric on a closed surface such that $-B \leq K \leq -A < 0$ for $A, B > 0$. Let $(g_t)_{t\geq 0}$ the normalized Ricci-flow starting from g_0. Then*

$$\forall t \geq 0, \quad -B \leq K_t \leq -A < 0.$$

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