Local conjugacy and primary-type decompositions in nonabelian cohomology

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Aim

- We will develop & connect three related notions from finite group theory
  1. **Local conjugacy** for complements & supplements over normal nilpotent subgroups, from Losey & Stonehewer (Warwick)
  2. **Primary-type decompositions** in group cohomology
  3. **Fixed point theorems** for non-coprime actions, from Glauberman (Chicago)

Two subgroups H and H' are *locally conjugate* if for each prime p, a Sylow p-subgroup of H is conjugate to a Sylow p-subgroup of H'

Motivation

- In 1964, Glauberman proved:

> *Suppose J acts on N via automorphisms and the induced semi-direct product $N \rtimes J$ acts on a non-empty set Ω where the action of N is transitive. If*
>
> *(Z) each complement of N in G is conjugate to J, and*
>
> *(S) each supplement S of N in G splits over $S \cap N$,*
>
> *then J fixes some element of Ω.*

- By the Schur-Zassenhaus theorem (1937), if $|J|$ and $|N|$ are coprime (and J or N is soluble, until Feit-Thompson in 1962) then (Z) & (S) hold

Motivation II

- Conditions (Z) and (S) are sufficient, but not necessary
- Consider the proof:
  - Let G_α denote the stabiliser subgroup of G fixing some $\alpha \in \Omega$
  - As N acts transitively, G_α supplements N in G
  - By (S), there exists a complement J' to $G_\alpha \cap N$ in G_α
    - J' also complements N in G
  - By (Z), $J' = J^g \, (= g^{-1}Jg)$ for some $g \in G$
  - $J^g \leq G_\alpha$ so that J fixes $g \cdot \alpha$
- All we really need is $J^g \leq G_\alpha$ for some $g \in G$ (an inclusion result)
  - So G_α must split over $G_\alpha \cap N$ & a complement of $G_\alpha \cap N$ in G_α must be G-conjugate to J

Previous results (non-coprime)

- 1952 (Gaschütz): when N is abelian and each Sylow p-subgroup of H splits over $N \cap H$, then H splits over $N \cap H$
- 1954 (D. G. Higman): If in $N \rtimes J$ (N abelian), for each prime p there is a Sylow p-subgroup S of G such that any two complements of $N \cap S$ are conjugate within S, then any two complements of N in G are conjugate
- 1979 (Losey & Stonehewer): if J and J' are locally conjugate supplements to nilpotent N in soluble G and one of the following holds: (A) G/N is nilpotent, (B) N is abelian, or (C) the Sylow p-subgroups of G have class at most two, then J and J' are conjugate
- 1988 (Evans & Shin): two locally conjugate supplements to a normal abelian subgroup N are conjugate (G need not be soluble)

Main theorem

- We will show:

Suppose J acts on nilpotent N via automorphisms and the induced semi-direct product $N \rtimes J$ acts on a non-empty set Ω, where the action of N is transitive. If for each prime p, a Sylow p-subgroup of J fixes an element of Ω, and one of the following holds:

- *N is abelian,*
- *$N \rtimes J$ is supersoluble, or*
- *J is nilpotent,*

then J fixes some element of Ω.

Agenda

- Notation & background
- Lemma 1: a primary-type decomposition of the first cohomology set under some conditions on J and N
- Corollaries on when locally conjugate complements are conjugate
- Lemma 2: an analogue of Loosey and Stonehewer's results for subgroup inclusion
- Proof of theorem

Notation

- Suppose a group J acts on a group N via automorphisms (N is a J-group)
  - We write $n^j = j^{-1}nj$ and have an induced semidirect product $N \rtimes J$
- A (1-)cocycle is map $\phi : J \to N$ such that, for all j, j':

$$\phi(jj') = \phi(j)\phi(j')^{j^{-1}}$$

- Let $Z^1(J, N)$ denote the set of all such maps
  - For nonabelian N, this is a 'pointed set' with distinguished point $\phi \equiv 1_N$
- Two cocycles ϕ and ϕ' are cohomologous if:

$$\phi'(j) = n^{-1}\phi(j)n^{j^{-1}}$$

  - This gives an equivalence relation; we write $\phi \sim \phi'$
- The first cohomology set is then $H^1(J, N) = Z^1(J, N)/\sim$

Correspondence | complements & cocycles

given K;
for each $j \in J$, write $j = n^{-1}k$ for unique $n \in N, k \in K$;
let $\phi_K(j) = n = kj^{-1}$; then ϕ_K is a cocycle

$\{K\colon K \text{ complements } N \text{ in } N \rtimes J\}$

$Z^1(J, N)$

given ϕ;
$K_\phi = \{\phi(j)\,j\}_{j \in J}$ complements N

Correspondence | conj. classes & cohomology

given K, K^ν for some $\nu \in N$;
for each $j \in J$, we have $j = n^{-1}k = (\nu^{-1} n \nu^{j^{-1}})^{-1} k^\nu$
for unique $n \in N, k \in K$;
then $\phi_{K^\nu}(j) = \nu^{-1}\phi_K(j)\nu^{j^{-1}}$ so that $\phi_{K^\nu} \sim \phi_K$

$\{[K]\colon K \text{ complements } N \text{ in } N \rtimes J\}$ $\qquad$ $H^1(J, N)$

given $\phi \sim \phi'$ so
that $\phi'(j) = \nu^{-1}\phi(j)\nu^{j^{-1}}$ for some $\nu \in N$;
then $K_{\phi'} = \{\phi'(j)j\}_{j\in J} = \{\nu^{-1}\phi(j)\nu^{j^{-1}} j\}_{j\in J} = \{\nu^{-1}\phi(j)j\nu\}_{j\in J} = K_\phi^\nu$

Notation II

- For a subgroup $K \leq J$ and $\phi \in Z^1(J,N)$, we can consider the restriction $\phi|_K$
  - This induces a map in cohomology $\mathrm{res}^J_K : H^1(J,N) \to H^1(K,N)$
- For $\phi \in Z^1(K,N)$ and $j \in J$, define $\phi^j(x) = \phi(x^{j^{-1}})^j$. Call ϕ J-invariant if for all $j \in J$:

$$\phi|_{K\cap K^j} \sim \phi^j|_{K\cap K^j}$$

  - For normal K, this simplifies to $\phi \sim \phi^j$
- Let $\mathrm{inv}_J H^1(K,N)$ denote the set of J-invariant elements of $H^1(K,N)$
  - Note that restrictions are always invariant, i.e. $\mathrm{res}^J_K H^1(J,N) \subseteq \mathrm{inv}_J H^1(K,N)$, as

$$\phi^j(x) = \phi(x^{j^{-1}})^j = \phi(jxj^{-1})^j = \phi(j)^j\phi(xj^{-1}) = \phi(j)^j\phi(x)\phi(j^{-1})^{x^{-1}} = n^{-1}\phi(x)n^{x^{-1}}$$

where $n = \phi(j^{-1})$ as $1_N = \phi(1_J) = \phi(j^{-1}j) = \phi(j^{-1})\phi(j)^j$

Primary decomposition: abelian case

- For abelian N we have the standard decomposition

Let J act on abelian N via automorphisms. Then:

$$H^1(J,N) \cong \bigoplus_{p\in\mathcal{D}} \mathrm{inv}_J H^1(J_p,N)$$

where $\mathcal{D}$ denotes the prime divisors of $|J|$ and $J_p \in \mathrm{Syl}_p(J)$ for each $p \in \mathcal{D}$. Explicitly, $H^1(J,N) \cong \bigoplus_{p\in\mathcal{D}} H^1(J,N)_{(p)}$ where $H^1(J,N)_{(p)}$ denotes the p-primary component and for each p, we have the isomorphism:

$$\mathrm{res}^J_{J_p} : H^1(J,N)_{(p)} \xrightarrow{\cong} \mathrm{inv}_J H^1(J_p,N)$$

Brown's text, ch. 3, §10

Does N need to be abelian?

- Mark Grant (Aberdeen) posed:

mathoverflow

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primary decomposition for nonabelian cohomology of finite groups

Asked 6 years, 6 months ago Modified 6 years, 5 months ago Viewed 594 times

18

Let G be a finite group, and let M be a group on which G acts (via a homomorphism $G \to \mathrm{Aut}(M)$).

If M is abelian, hence a $\mathbb{Z}G$-module, there is a primary decomposition

$$H^k(G;M) = \bigoplus_p H^k(G;M)_{(p)}$$

for each $k > 0$, where p ranges over the primes dividing $|G|$ and $H^k(G;M)_{(p)}$ denotes the p-primary component of $H^k(G;M)$. It follows that if $H^1(G;M)$ is nonzero, then $H^1(P;M)$ is nonzero for some p-Sylow subgroup P of G. See Section III.10 of K.S. Brown's "Cohomology of Groups".

If M is nonabelian, then $H^1(G;M)$ is still defined, but is not a group in general, so that asking for a decomposition such as above would not make any sense. But one could still ask the following weaker question:

> With G and M as above, suppose that $H^1(G;M)$ is non-trivial. Then must $H^1(P;M)$ be non-trivial for some p-Sylow subgroup P of G?

https://mathoverflow.net/questions/304554

Neat, if true

- the most popular response argues that things still work
- this would imply that if all complements to N in $N \rtimes J$ were locally conjugate to J, then all such complements would be conjugate, with no further restrictions on J or N

12

Yes, this is true.

Let F be the free product of all the Sylow subgroups of G:

$$F = \mathop{*}_{P \text{ Sylow}} P$$

The inclusions $P \to G$ together give a group homomorphism $F \to G$. On cohomology, this induces the map

$$H^1(G; M) \to H^1(F; M) = \prod_P H^1(P; M)$$

and we are done if we can verify that this map is injective.

However, the map $F \to G$ is surjective: given any element $g \in G$ of order n, generating a copy of $\mathbb{Z}/n$, we can use the Chinese remainder theorem to write $\mathbb{Z}/n$ as a product $\prod \mathbb{Z}/p^r$ of primary groups. This allows us to conclude that $g = g^{e_1} g^{e_2} \ldots g^{e_k}$ where each g^{e_i} has order a prime power, hence is in a Sylow subgroup, and hence is in the image of P. Thus g is in the image of the amalgamated product F.

This now reduces us to showing: If $F \to G$ is a surjective map of groups and M is a nonabelian group with G-action, then the map $H^1(G; M) \to H^1(F; M)$ is injective. This is straightforward from the cocycle definition: if we have two cocycles $f, h : G \to M$ which become equal in $H^1(F; M)$, then by definition there is an element $m \in M$ such that $h(x) = m^{-1} \cdot f(x) \cdot {}^x m$ for all $x \in F$, but both sides of this identity only depend on the image of x in G.

Not true in general

- Losey and Stonehewer provide a counter-example
  - Let $J = S_3$ act on $N = Q_8$ as in $GL(2,3)$
  - Then there is a second complement J' to N that is locally conjugate but not conjugate to J
  - Thus $H^1(J, N)$ has order 2, but $H^1(P, N)$ is trivial for each Sylow p-subgroup P of J
  - Also, this is right out:

```
gap> G := SmallGroup(48,29);; # GL(2,3)
gap> N := First(NormalSubgroups(G), s -> Order(s) = 8);; # Q8
gap> Js := ComplementClassesRepresentatives(G,N); # there are 2
[ <pc group with 2 generators>, <pc group with 2 generators> ]
gap> # check local conjugacy
gap> SJ1 := SylowSystem(Js[1]);;
gap> SJ2 := SylowSystem(Js[2]);;
gap> ForAll([1..Length(SJ1)], i -> IsConjugate(G,SJ1[i],SJ2[i]));
true
```

COROLLARY 3.5. *Suppose that G is a soluble group and Q is a nilpotent normal subgroup of G. If U and V are supplements to Q in G such that U and V are locally G-conjugate, then U and V are G-conjugate.*

Lemma 1

- With some restrictions on J and N, we do have a decomposition:

Suppose J acts on nilpotent N via automorphisms. If

- $N \rtimes J$ *is supersoluble, or*
- J *is nilpotent,*

then $\phi \mapsto \times_p \phi|_{J_p}$ induces an isomorphism of pointed sets:

$$H^1(J, N) \cong \times_{p \in \mathcal{D}} \operatorname{inv}_J H^1(J_p, N)$$

What remains true in the nonabelian case

‘The extent of knowledge required is nothing like so great as is sometimes supposed’. —Littlewood (1944)

- Serre’s *Galois Cohomology* (ch. I, §5) dedicates 15 pages to this topic. We have 2 exact sequences:
  - Inflation-restriction, for $Q \trianglelefteq J$:

$$1 \to H^1(J/Q, N^Q) \to H^1(J, N) \xrightarrow{\mathrm{res}^J_Q} H^1(Q, N)^{J/Q}$$

  - By inclusion, for $M \trianglelefteq N$:

$$1 \to H^1(J, M) \xrightarrow{\mathrm{inc}^N_M} H^1(J, N) \to H^1(J, N/M)$$

Proof of Lemma 1

- The projection maps $N \to N_p$ induce the decomposition:

$$\times_p \pi_p : H^1(J, N) \xrightarrow{\cong} \times_p H^1(J, N_p)$$

- For each prime p, we have the restriction map:

$$\mathrm{res}^J_{J_p} : H^1(J, N_p) \to \mathrm{inv}_J H^1(J_p, N_p)$$

  and the inclusion map:

$$\mathrm{inc}^N_{N_p} : \mathrm{inv}_J H^1(J_p, N_p) \to \mathrm{inv}_J H^1(J_p, N)$$

- The composition of these maps gives us $\phi \mapsto \times_p \phi|_{J_p}$ so it suffices to show that each is an isomorphism, i.e.:

$$H^1(J, N) \cong \times_p H^1(J, N_p) \cong \times_p \mathrm{inv}_J H^1(J_p, N_p) \cong \times_p \mathrm{inv}_J H^1(J_p, N)$$

Proof of Lemma 1 | $N \rtimes J$ is supersoluble

- Claim: $\mathrm{res}^J_{J_p} : H^1(J, N_p) \to \mathrm{inv}_J H^1(J_p, N_p)$ is an isomorphism
- Inducting on $|J|$
- Let $Q \lhd J$ be a Sylow q-subgroup where q is the largest prime divisor of $|J|$; then $J \cong Q \rtimes M$ for some Hall q'-subgroup M
- Consider the inflation-restriction exact sequence:

$$1 \to H^1(J/Q, N_p^Q) \to H^1(J, N_p) \xrightarrow{\mathrm{res}^J_Q} H^1(Q, N_p)^{J/Q}$$

- Two sub-cases:
  - $q \neq p$
  - $q = p$

Proof of Lemma 1 | $N \rtimes J$ is supersoluble | $q \neq p$

- In this case, $H^1(Q, N_p)$ is trivial so that
$$1 \to H^1(J/Q, N_p^Q) \to H^1(J, N_p) \xrightarrow{\mathrm{res}_Q^J} H^1(Q, N_p)^{J/Q}$$
implies $H^1(J, N_p) \cong H^1(M, N_p^Q)$
- Now $Q \lhd N_pQ$ so that $N_pQ \cong N_p \times Q$ and $N_p^Q = N_p$
- Thus, $H^1(J, N_p) \cong H^1(M, N_p)$ and we claim that res_M^J affords this isomorphism
- Suffices to show that res_M^J is surjective

Proof of Lemma 1 | $N \rtimes J$ is supersoluble | $q \neq p$

- Claim $\mathrm{res}^J_M : H^1(J, N_p) \to H^1(M, N_p)$ is surjective
- Let $\phi \in Z^1(M, N_p)$
- Define $\tilde{\phi} : J \to N_p$ by $\tilde{\phi}(qm) = \phi(m)$ for $q \in Q, m \in M$
  - well-defined as $J \cong Q \rtimes M$
- Then $\tilde{\phi} \in Z^1(J, N_p)$ and $\tilde{\phi}|_M \sim \phi$; thus res^J_M is an isomorphism
- As $\mathrm{res}^M_{J_p}$ is injective by induction it follows that $\mathrm{res}^J_{J_p} = \mathrm{res}^M_{J_p} \circ \mathrm{res}^J_M$ is also injective
- Also, $\mathrm{inv}_J H^1(J_p, N) \subseteq \mathrm{inv}_M H^1(J_p, N) = \mathrm{res}^M_{J_p} H^1(M, N) \subseteq \mathrm{res}^J_{J_p} H^1(J, N)$ so that $\mathrm{res}^J_{J_p} : H^1(J, N_p) \to \mathrm{inv}_J H^1(J_p, N_p)$ is surjective and thus an isomorphism

Proof of Lemma 1 | $N \rtimes J$ is supersoluble | $q{=}p$

- In this case, $Q = J_p$ so $H^1(J/Q, N_p^Q)$ is trivial
- Then $1 \to H^1(J, N_p) \xrightarrow{\mathrm{res}^J_{J_p}} H^1(J_p, N_p)^{J/J_p}$ is exact, where $H^1(J_p, N_p)^{J/J_p} = \mathrm{inv}_J H^1(J_p, N_p)$, so is $\mathrm{res}^J_{J_p}$ injective
- To show $\mathrm{res}^J_{J_p}$ is surjective, suppose $\phi \in Z^1(J_p, N_p)$ is M-invariant
- In this case, define $\tilde{\phi} : J \to N_p$ by $\tilde{\phi}(qm) = \phi(q)$ for $q \in J_p, m \in M$
    - again, well-defined as $J \cong J_p \rtimes M$
- We find that $\tilde{\phi} \in Z^1(J, N_p)$ and clearly $\tilde{\phi}|_{J_p} \sim \phi$ so that again $\mathrm{res}^J_{J_p} : H^1(J, N_p) \to \mathrm{inv}_J H^1(J_p, N_p)$ is surjective and thus an isomorphism

Proof of Lemma 1 | J is nilpotent

- We're (still) inducting on $|J|$
- We have $J \cong J_p \times J_p'$ where $J_p \in \mathrm{Syl}_p(J)$ and J_p' is the Hall p'-subgroup
  - In this case, $H^1(J_p, N_p)^{J_p'} = \mathrm{inv}_J H^1(J_p, N_p)$
- Consider the inflation-restriction exact sequence:

$$1 \to H^1(J_p', N_p^{J_p}) \to H^1(J, N_p) \xrightarrow{\mathrm{res}^J_{J_p}} H^1(J_p, N_p)^{J_p'}$$

- As $H^1(J_p', N_p^{J_p})$ is trivial, $\mathrm{res}^J_{J_p}$ is injective
- For J-invariant $\phi \in Z^1(J_p, N_p)$, define $\tilde{\phi} : J \to N_p$ by $\tilde{\phi}(j_p j_p') = \phi(j_p)$ for $j_p \in J_p, j_p' \in J_p'$. Then $\tilde{\phi} \in Z^1(J, N_p)$ and $\tilde{\phi}|_{J_p} \sim \phi$ so $\mathrm{res}^J_{J_p}$ is surjective

Proof of Lemma 1

- Recap: we now have $H^1(J,N) \cong \times_p H^1(J,N_p) \cong \times_p \operatorname{inv}_J H^1(J_p,N_p)$
- Claim: $\operatorname{inc}_{N_p}^N : \operatorname{inv}_J H^1(J_p,N_p) \to \operatorname{inv}_J H^1(J_p,N)$ is an isomorphism
- We have the exact sequence:

$$1 \to H^1(J_p,N_p) \xrightarrow{\operatorname{inc}_{N_p}^N} H^1(J_p,N) \to H^1(J_p,N/N_p)$$

  where $H^1(J_p,N/N_p)$ is trivial
- As $\operatorname{inv}_J H^1(J_p,N_p) \subseteq \operatorname{inv}_J H^1(J_p,N)$, we can conclude
- In particular,

$$H^1(J,N) \cong \times_p H^1(J,N_p) \cong \times_p \operatorname{inv}_J H^1(J_p,N_p) \cong \times_p \operatorname{inv}_J H^1(J_p,N)$$

Corollaries

- It follows that:

> *In a finite supersoluble group, two complements of a normal nilpotent subgroup are conjugate if and only if they are locally conjugate.*

and:

> *[Losey & Stonehewer] Two nilpotent complements of a normal nilpotent subgroup in a finite group are conjugate if and only if they are locally conjugate.*

Proof of corollaries

- Suppose J and J' are locally conjugate complements as described
- Let $\phi \in Z^1(J, N)$ correspond to J'
- By hypothesis, $\phi|_{J_p} \sim 1|_{J_p}$ for each prime p and some $J_p \in \mathrm{Syl}_p(J)$
- As $\phi \mapsto \times_p \phi|_{J_p}$ induces an isomorphism of pointed sets, $\phi \sim 1$

Remark

- For abelian N, we get a similar decomposition:

$$H^1(J,N) \cong \oplus_p H^1(J,N_p) \cong \oplus_p \operatorname{inv}_J H^1(J_p,N_p)$$

  where N_p denotes the p-primary component of N and $J_p \in \mathrm{Syl}_p(J)$
- Gaschütz showed that $H^1(J_p,N_p) = 1$ implies $H^1(K,N_p) = 1$ for all $K \leq J_p$
- In particular, under D. G. Higman's hypotheses—when N is abelian and for each prime p there is a Sylow p-subgroup S of G such that any two complements of $N \cap S$ are conjugate within S—then N is cohomologically trivial
  - *see Brown's text, ch. VI, prop 8.8*

Lemma 2

- We will now use Lemma 1 to give an inclusion-based result:

> *Given a subgroup H of $N \rtimes J$ where N is nilpotent, suppose H contains a conjugate of some Sylow p-subgroup of J for each prime p. If*
>
> - *N is abelian,*
> - *$N \rtimes J$ is supersoluble, or*
> - *J is nilpotent,*
>
> *then H contains a conjugate of J.*

- Remark: H will supplement N in $N \rtimes J$

Proof of Lemma 2

- We're inducting on $|G|$
- Claim: without loss, N is a p-group
- Suppose multiple primes divide N
  - Then $N \cong N_p \times N_p'$ nontrivially for $N_p \in \mathrm{Syl}_p(N)$ and N_p' the Hall p'-subgroup
- Induction in G/N_p implies $J^{n_0} \leq N_p H$ for some $n_0 \in N_p'$
  - Induction in G/N_p' implies $J^{n_1} \leq N_p' H$ for some $n_1 \in N_p$
- Then $J^{n_0 n_1} \leq N_p H \cap N_p' H = H$
  - If $g \in N_p H \cap N_p' H$, then $g = n_1 h_1 = n_0 h_0$ for some $n_1 \in N_p, h_0, h_1 \in H, n_0 \in N_p'$. So $h_1 h_0^{-1} = n_1^{-1} n_0 \in H$, where n_0, n_1 commute and have co-prime orders so that $n_0, n_1 \in H$ and in particular $g \in H$

Proof of Lemma 2

- Now, we have $N = N_p$ for some prime p
- Claim: $N_p \cap H$ cannot contain a nontrivial normal subgroup $A \lhd G$
- Otherwise, in $G/(N_p \cap H)$, induction would allow us to conclude
- If N_p were abelian, then $N_p \cap H$ would itself be normal, as $N_p \cap H \lhd N_p$
  - This concludes the abelian case

Proof of Lemma 2

- Now, we have $N{=}N_p$ for some prime p
- Switching to a conjugate of H if necessary, $J_p \leq H$ for some $J_p \in \mathrm{Syl}_p(J)$
- We want a proper subgroup $1 \lneq Z \lneq N_p$ with $Z \lhd G$
  - If G is supersoluble, then N_p contains $Z \lhd G$ of order p
  - If J is nilpotent, then N_p contains a nontrivial centre $Z = Z(N_p)$
  - In both cases, $Z \cap H$ is trivial, and Z must be proper for N to be non-abelian
- In G/Z, induction implies $J^g Z/Z \leq ZH/Z$ for some $g \in G$
  - Let $\psi : H \xrightarrow{\cong} ZH/Z$ denote the isomorphism
  - Then $K = \psi^{-1}(ZJ^g/Z) \leq H$ complements $N_p \cap H$ in H (and N_p in G)

Proof of Lemma 2

- Let $\phi \in Z^1(K, N_p)$ correspond to J
  - As $J_p \leq H$, $J_p \leq (N_p \cap H)K_p \in \mathrm{Syl}_p(H)$ for some $K_p \in \mathrm{Syl}_p(K)$
  - Then $\phi|_{K_p} \in \mathrm{inv}_K Z^1(K_p, N_p)$ corresponds to J_p
- Under the correspondence, $\mathrm{im}(\phi|_{K_p}) \leq \langle K_p, J_p \rangle \leq H$, so that:

$$\phi|_{K_p} \in \mathrm{inv}_K Z^1(K_p, N_p \cap H)$$

- As $\mathrm{res}^K_{K_p} : H^1(K, N_p \cap H) \to H^1(K_p, N_p \cap H)$ is an isomorphism, there exists some $\tilde{\phi} \in H^1(K, N_p \cap H)$ such that $\tilde{\phi}|_{K_p} \sim \phi|_{K_p}$
- Thus there exists some complement $L \leq H$ to $N_p \cap H$ with $J_p \leq L$
- Then L and J are locally conjugate as Sylow q-subgroups of L & J are Sylow q-subgroups of G for $q \neq p$, and so are conjugate by Lemma 1

Remark

- For abelian N, the extension over N need not split in order for this to work. I.e.:

> *Suppose H and J each supplement an abelian group N. If H contains a conjugate of some Sylow p-subgroup of J for each prime p, then H contains a conjugate of J.*

- Proof:
  - Let $G = NH$ and induct on $|G|$. Clearly, $|H| \geq |J|$.
  - If $N \cap H \lhd G$ were trivial, then J and H would be locally conjugate complements
  - Otherwise, induction in $G/(N \cap H)$ allows us to conclude

Proof of Theorem

- Hypotheses:
  (1) J acts on nilpotent N via automorphisms and the induced semi-direct product $N \rtimes J$ acts on a non-empty set Ω, where the action of N is transitive

  (2) For each prime p, a Sylow p-subgroup of J fixes an element of Ω

  (3) One of the following: N is abelian, $N \rtimes J$ is supersoluble, or J is nilpotent
- For arbitrary α, let $G_\alpha \leq G$ denote the stabliser subgroup.
- By (2) and the transitivity assumption from (1), G_α contains a conjugate of some Sylow p-subgroup of J for each prime p.
- By (3) and Lemma 2, we have $J^g \leq G_\alpha$ for some $g \in G$.
- Conclusion: J fixes $g \cdot \alpha \in \Omega$

Remark

- Losey and Stonehewer's original paper includes a sequence of arguments (some of which we've now seen) to improve our corollary from Lemma 1:

> *In a supersoluble group, two locally conjugate supplements of a normal nilpotent subgroup are conjugate.*

- Proof: suppose H & J supplement N
  - H & J must be core-free; i.e. there cannot exist $A \lhd G$ with $A \leq H$ or $A \leq J$
  - without loss, N is p-group and we may let $P \in \mathrm{Syl}_p(H) \cap \mathrm{Syl}_p(J)$
  - without loss, $G = \langle H, J \rangle$, otherwise we could induct in $\langle H, J \rangle$
  - $N \cap H \leq P$ so $N \cap P = N \cap H \lhd H$; analogously, $N \cap P = N \cap J \lhd J$ so that $N \cap P \lhd G$; but H & J are core free so we must have $N \cap P = N \cap H = N \cap J = 1$

Potential extensions

- In 1995, Shin extended Losey and Stonehewer's results to profinite groups

In a profinite group G, suppose U and V are closed, locally conjugate supplements of a normal nilpotent closed subgroup N. If:

- *N is abelian,*
- *G/N, or is nilpotent, or*
- *J is the Sylow p-subgroups of G have class at most 2 for every prime p,*

then U and V are conjugate.

Thanks for joining today!

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