Multivalued Wess-Zumino-Novikov functional and chiral anomaly in hydrodynamics

P. B. Wiegmann

Kadanoff Center for Theoretical Physics, University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637, USA

(Received 8 April 2024; accepted 1 December 2024; published 30 December 2024)

We present a hydrodynamic framework derived from the action of a perfect fluid, modified by the hydrodynamic analog of Novikov's multivalued functional. This modification introduces spin degrees of freedom into the fluid. The structure closely resembles the Abelian version of the Wess-Zumino functional, commonly applied in field theories with chiral anomalies. The deformation incorporates transport properties of Weyl fermions and, in the case of a charged fluid, exhibits the chiral anomaly. It is also consistent with Onsager's semiclassical quantization of circulation. Additionally, we discuss the hydrodynamic analog of instantons and related topological invariants.

DOI: 10.1103/PhysRevD.110.116032

I. INTRODUCTION AND THE BACKGROUND

The equations of motion for a perfect fluid can be regarded as conservation laws associated with the group of spacetime diffeomorphisms. If no external forces act on the fluid the momentum and energy are conserved resulting in four dynamical equations expressed as a divergence-free condition of the canonical momentum-stress-energy tensor

$$\partial_\mu T^\mu{}_\nu = 0. \tag{1}$$

These equations alone are sufficient for describing flows, when the only dynamical variables are components of the particle number 4-current $n^\mu = (n^0, n^0 \boldsymbol{v})$, where n^0 is the particle number density and $\boldsymbol{v}$ is the fluid velocity. In this case the continuity equation

$$\partial_\mu n^\mu = 0 \tag{2}$$

is not an independent condition. It follows from the conservation of momentum and energy described by (1). Barotropic flows and more general homentropic flows represent this situation.

A more general, baroclinic flow involves additional dynamical variables. In this case, more equations are necessary. They stem from symmetries other than spacetime diffeomorphisms, such as gauge symmetry. Then the continuity equation (2) normally arises as the Noether conservation law for a particle number.

The phenomenon known as the chiral current anomaly presents an obstacle to the conservation of particle number. The issue arises when the conserved Noether current generated by the gauge symmetry, denoted as I^μ, is not gauge invariant; however, its divergence is. In this case, the particle number is not identical to the Noether charge, and is not conserved $\partial_\mu n^\mu \neq 0$. Such system is not isolated and in contact with a reservoir capable to supply or remove particles. At the same time the equations of motion

$$\partial_\mu I^\mu = 0 \tag{3}$$

remains local and gauge invariant. The chiral anomaly signifies that the flow entrains a reservoir capable of supplying and swapping particles.

The chiral anomaly was initially identified as a kinematic property of quantum field theories involving chiral (or Weyl) fermions [1]. A defining feature of the chiral anomaly is that the particle production rate, $\partial_\mu n^\mu$, is locally defined by the flow itself and is unaffected by changes in the spacetime metric. Therefore, the anomaly is largely insensitive to interaction and, when carried over to a liquid state it does not introduce additional spacetime scales beyond already accounted gradients of hydrodynamic fields. Being insensitive to a variation of metric, the chiral anomaly only impacts the continuity equation while leaving the form of the stress tensor and its conservation (1) unaffected.

In recent years, there has been growing confidence that the current anomalies are compatible with classical fluid dynamics. An incomplete list of references is [2–17]. More references could be found in the review [18]. A physical argument supporting this perspective is the existence of liquids composed of Weyl fermions. Such liquids are expected to retain kinematic features of Weyl fermions, including their anomalies. Notable examples are the superfluid ^{3}He A, semiconductors with high spin-orbit interaction, and quark-gluon plasma occurring in heavy-ion collisions (see e.g., [2,11,18,19] for review of each topic).

The gauge symmetry generated Noether current is expected to be equal to a particle number current n^μ modified by a pseudovector field h^μ

$$I^\mu = n^\mu + \frac{k}{2} h^\mu. \tag{4}$$

Here k, referred as the level, is a pseudoscalar parameter representing the strength of the deformation. Then the Noether conservation law (3) becomes the equation for the particle production

$$\partial_\mu n^\mu = -\frac{k}{2} \partial_\mu h^\mu. \tag{5}$$

The imposed properties on h^μ are as follows: it is (i) locally expressed through Eulerian fields, (ii) does not possess any scale, and therefore has no reference to the spacetime metric, (iii) its divergent $\partial_\mu h^\mu$ is gauge invariant.

In fluid mechanics there is only one object of this kind, the 4- *fluid helicity* current. The fluid helicity is the dual to the 3-differential form $h = p \wedge dp$ constructed from of fluid 4-momentum 1-form $p = p_\mu dx^\mu$. In tensor notations

$$h^\mu = \varepsilon^{\mu\nu\lambda\sigma} p_\nu \partial_\lambda p_\sigma, \tag{6}$$

where $\varepsilon^{\mu\nu\lambda\sigma}$ is the Levi-Civita symbol. The timelike component of the helicity current $h_0 = \boldsymbol{p} \cdot \nabla \times \boldsymbol{p}$ is the usual helicity density (in Sec. IV we give a formal definition of the kinematic momentum and its relation to the particle number current). Variants of such deformation, albeit in various different settings had been introduced in Refs. [3–5,16]. An incomplete list of early related works is [6–9].

We remark that a room for deformations of fluid dynamics is limited, as the fluid equations of motion must be covariant under the action of the gauge group and the group of spacetime diffeomorphisms

$$\mathcal{G} = U(1) \rtimes \mathrm{Diff}(\mathcal{M}^4). \tag{7}$$

Why would the deformation (5) be consistent with fluid dynamics? A concise criterion is the Hamilton principle of fluid mechanics. The Hamilton principle asserts the existence of a Hamilton functional or an action, whose invariance under the action of the symmetry group $\mathcal{G}$ yields the desired equations of motion. This is the central part of our approach. We construct the Hamiltonian functional which yields Eqs. (1) and (5) keeping the stress tensor of the perfect fluid intact. The latter is given by

$$T^\mu{}_\nu = n^\mu p_\nu + \delta^\mu{}_\nu P. \tag{8}$$

Here P is the fluid pressure. Being complemented by the equation of state which expresses the fluid momentum in terms of the particle current (discussed below) Eqs. (1), (5), (8) give the complete set of the fluid equations of motions.

We will show that the action or the Hamiltonian functional of the perfect fluid could be uniquely extended by the multivalued Wess-Zumino-Novikov (WZN) functional and that this extension yields Eqs. (1), (5), (8). The multivaluedness of the WZN action provides global obstructions which restricts the parameter k to be an integer in units of the Planck constant. This result is consistent with the Onsager quantization and the known kinematic properties of Weyl fermions developed in the early works of Vilenkin [20]. Our results are summarized by (46), (47), (49) in Sec. X.

Our construct holds for any even spacetime dimension d. In this case, the modification of the Noether current (4) is given by the $(d-1)$-form

$$\frac{k}{(d/2)!} p \wedge (dp)^{d/2-1}. \tag{9}$$

In particular, this formula agrees with the known expression for the Noether current of $(1+1)$-chiral bosons

$$I^\mu = n^\mu + k\varepsilon^{\mu\nu} p_\nu. \tag{10}$$

The concept of the multivalued functionals in a general setting was introduced by Novikov in 1981 [21]. Soon after Novikov's paper, it was recognized that a class of these functionals appeared in the early work of Wess and Zumino [22]. Wess and Zumino constructed the functional whose variation replicates effects of the anomaly. In this paper we introduce the hydrodynamic version of the multivalued functional.

The anomaly is a topological phenomenon in the sense that it is metric independent and therefore could be expressed solely in terms of differential forms. The efficient framework that helps incorporate anomalies into fluid mechanics is the spacetime covariant formulation of hydrodynamics of Lichnerowicz [23] and Carter [24]. For recent reviews, see [25,26], and [12–14] for its adaptations to anomalies. In this approach, the hydrodynamics is expressed in terms of the particle number 4-current n^μ, and its conjugate, a covector, the fluid 4-momentum p_μ, without reference to the spacetime metric. Consequently, the form of the fluid equations of motion appear identical for both relativistic and nonrelativistic fluids.

We begin with a discussion of the relation between Eq. (5) and the traditional form of the anomaly as a linear response to an external electromagnetic field (Sec. II), followed by a semiclassical quantization of the level k (Sec. III), assuming that Eq. (5) is given. Next, we outline the hydrodynamic setup (Secs. IV–VI) and provide a brief account of the Carter covariant Hamilton principle in hydrodynamics (Sec. VII).

After these preliminaries, we will be ready to address the central part of the paper: the multivalued action (Secs. VII–IX). Various forms of the full set of equations of motion are collected in Sec. X. The effect of spin introduced by the multivalued action is briefly outlined in Sec. XI and Appendix C. The entropy production, the relation to homentropic flow and comparison with Weyl fermions, is briefly discussed in the Appendixes A, B, and D.

II. CHIRAL ANOMALY AND THE CHIRAL PHASE

Traditionally the chiral anomaly is understood as a linear response of a charged system to the external electromagnetic field. In the canonical formulation where the particle current and the momentum are treated as independent fields the effect of electromagnetic field is accounted by replacing the kinematic momentum by the canonical momentum

$$\pi_\mu = p_\mu + A_\mu, \tag{11}$$

where A_μ is the gauge potential. Then the helicity in the expression for the Noether current (4) reads

$$h^\mu = \epsilon^{\mu\nu\lambda\sigma}\pi_\nu\partial_\lambda\pi_\sigma. \tag{12}$$

Helicity, and therefore the Noether current, is not gauge-invariant. Under the gauge transformation

$$\pi_\mu \to \pi_\mu + \partial_\mu\Theta \tag{13}$$

it changes as

$$I^\mu \to I^\mu + k\partial_\nu\Theta\,{}^\star\Omega^{\mu\nu}, \tag{14}$$

where

$$\Omega_{\mu\nu} = \partial_\mu\pi_\nu - \partial_\nu\pi_\mu = \partial_\mu p_\nu - \partial_\nu p_\mu + F_{\mu\nu} \tag{15}$$

is the 4-vorticity tensor and ${}^\star\Omega^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\Omega_{\lambda\sigma}$ is the dual tensor. Using the relation $\partial_\mu h^\mu = \frac{1}{4}\Omega_{\mu\nu}{}^\star\Omega^{\mu\nu}$ we write the particle production as

$$\partial_\mu n^\mu = -\frac{k}{4}\Omega_{\mu\nu}{}^\star\Omega^{\mu\nu} \tag{16}$$

and transform it in the form of the linear response. Denote the gauge-invariant part of the Noether current as

$$j^\mu = n^\mu + \frac{k}{2}\Sigma^\mu, \tag{17}$$

where Σ^μ is the kinetic helicity [16]

$$\Sigma^\mu = \epsilon^{\mu\nu\lambda\sigma}p_\nu(\partial_\lambda p_\sigma + F_{\lambda\sigma}). \tag{18}$$

Then the conservation law (3) or equivalently the particle production equation (16) reproduces the commonly known expression for the chiral anomaly

$$\partial_\mu j^\mu = -\frac{k}{4}F_{\mu\nu}{}^\star F^{\mu\nu}, \tag{19}$$

where ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ is the dual field tensor. This follows from the identity $2\partial_\mu\Sigma^\mu = \Omega_{\mu\nu}{}^\star\Omega^{\mu\nu} - F_{\mu\nu}{}^\star F^{\mu\nu}$.

Two terms in Σ^μ are intrinsically related as they both followed from (12). In recent literature, they have been referred to as the chiral vortical effect and the chiral magnetic effect, respectively (see, e.g., [11] and references therein). The coefficients in (4), (17), (18), (19) are quantized topological invariants, as discussed in Sec. III. They are aligned with the result of the direct computation for Weyl fermions of Ref. [20] as discussed in Appendix D. See, also [27].

We will omit the external gauge field in intermediate formulas of Secs. IV and V. The external gauge field could be added upon the use of (11) for the canonical momentum. Even at no electromagnetic field the canonical momentum and the kinematic momentum momenta are related by a gradient of a phase

$$\pi_\mu = p_\mu + \partial_\mu\Theta. \tag{20}$$

In a usual fluid, the chiral phase Θ has no physical significance. However, with the chiral anomaly, the scalar Θ takes on a physical meaning. It does not factor into the equation of motion but enters the Hamilton functional (44), and the fluid action, similarly to the *axion* in the theory of CP violation [28]. The chiral phase is an important part of our construct (see, also [17]).

III. SEMICLASSICAL QUANTIZATION OF FLUID HELICITY, VORTEX INSTANTONS, AND PARTICLE PRODUCTION

While our discussion primarily focuses on classical fluid dynamics, a natural normalization of the fluid helicity arises from Onsager-like semiclassical consideration.

We recall that in semiclassical fluids vorticity is localized in vortex lines (loops in the absence of spatial boundaries), with the vortex circulation $\mathcal{C} = \oint \boldsymbol{\pi} d\boldsymbol{x}$ being quantized [29]. Choosing the Planck constant $2\pi\hbar$ as a unit for the momentum Onsager's quantization states

$$\mathcal{C} = \oint d\Theta = \text{integer}. \tag{21}$$

The quantization of circulation renders the gauge group (and thereby the entire fluid phase space) compact. The gauge group becomes $U(1)$ making the field Θ into a phase that winds over a circle.

At the same time, the total helicity

$$\mathcal{H} \coloneqq \int h^0 d^3\boldsymbol{x} = \int (\boldsymbol{\pi}\cdot\nabla\times\boldsymbol{\pi})d^3\boldsymbol{x} \tag{22}$$

is twice the linking number of vortex loops in units of vortex circulation $\mathcal{H} = 2\mathrm{Lk}[\text{vortex loops}]$ [30,31]. In the chosen units it is quantized as an even number [16].

Integrating the particle production equation (5) over a time interval and over the entire space we find that the left-hand side of (5) is a change in the total particle number ΔN over the time interval. On the right-hand side we obtain the change of the fluid helicity $\Delta\mathcal{H}$, i.e. twice the vortex linking number times $k/2$

$$\Delta N = \frac{k}{2}\Delta\mathcal{H} = k\Delta\,\mathrm{Lk}[\text{vortex loops}]. \tag{23}$$

Hence, a change of the linking of vortex loops by 1 alters the particle number by k. Given that the particle number is an integer, k is also an integer

$$k \in \mathbb{Z}. \tag{24}$$

Also, when momentum is measured in units of the Planck constant, the particle number current n^μ and the helicity current h^μ, the terms in the current (4) and (16), should be treated of comparable order in gradients.

It follows from (16) that the change of the particle number is assisted by "vorticity instantons," a flow which gives a nonzero value to the integral $\frac{1}{4}\int\Omega\wedge\Omega$, where $\Omega \coloneqq d\pi = \frac{1}{2}\Omega_{\mu\nu}dx^\mu \wedge dx^\nu$ is the vorticity 2-form. This integral is the Pontryagin class, a topological invariant, of the fluid cotangent bundle. Vorticity instantons, therefore, are the flow which at an instance changes the vortex linking number.

IV. CURRENTS AND MOMENTUM

As the particle number is not conserved (16), the fluid exchanges particles with a reservoir and, therefore, are nonhomentropic as we now discuss.

In relation to chiral fermions, the fluid could be seen as being composed of particles with right-handed chirality ($k > 0$), and the reservoir represents particles with the opposite (left-handed) chirality treated as a spectator medium (that is a fluid of massless particles devoid of spacelike momentum).

We denote the particle density number by n and the density number of the reservoir by $\bar{n}$ and introduce dimensionless density ratio $S = \bar{n}/n$. Subsequently, the fluid energy density $\varepsilon(n, S)$, being a function of n, is also a function of S [32]. Furthermore, we assume that the fluid and the reservoir are oppositely electrically charged.

We will consider that flows with the density ratio S vary across streamlines. Such flow is called nonhomentropic. It is also baroclinic. Nonhomentropic flows are endowed with a nondegenerate vorticity 2-form Ω, and therefore the top form $\Omega\wedge\Omega$, which appears in the particle production equation (16), is nowhere zero (see Appendix B). This is the essential part of the construction of the multivalued action outlined in Secs. VII and VIII.

In the covariant formulation of hydrodynamics [23–26] that we employ here, the equations of motion of the relativistic or nonrelativistic fluid have the same form, although the derivations are technically simpler in a relativistic setting, which we assume.

Taking advantage of the Lorentz metric we express the particle number density n through the particle current as

$$n^\mu n_\mu = -n^2 \tag{25}$$

and treat the energy density ε as a function of n^μ and S. We introduce the fluid momentum p_μ through a differential of energy taken at a fixed S

$$(d\varepsilon)_S = p_\mu dn^\mu. \tag{26}$$

For isotropic fluid, where the energy density depends on n we express the momentum in terms of "enthalpy" per particle $w = \partial_n\varepsilon$ and the 4-velocity $u_\mu \coloneqq n_\mu/n$, a 4-unit vector collinear to the particle current. Then

$$p_\mu = w u_\mu, \qquad n^\mu = n u^\mu, \qquad u^\mu u_\mu = -1. \tag{27}$$

[In the nonrelativistic case, (27) identifies $-p_0$ with the energy per particle $-p_0 = \boldsymbol{p}^2/(2w) + w$.]

Contrary to particles, the constituencies of the reservoir have no momentum and the relation (25) does not hold for the reservoir as their density and their current are independent.

V. TRANSFORMATIONS OF NATURAL VARIABLES

Let us examine how the natural variables π, Θ, S transform under the action of the symmetry group $\mathcal{G}$. The action of the gauge group is just a variation

$$\Theta \to \Theta + \delta\Theta. \tag{28}$$

The action of the spacetime diffeomorphisms

$$x^\mu \to x^\mu + \epsilon^\mu(x) \tag{29}$$

is carried out by the Lie derivative $\mathcal{L}_\epsilon$, the directional derivative along a vector field ϵ. The density ratio $S = \bar{n}/n$ being a scalar transforms as

$$\delta_\epsilon S \coloneqq \mathcal{L}_\epsilon S = \epsilon^\mu \partial_\mu S. \tag{30}$$

The momenta π_μ transform as the covector defined via a form-valued variation

$$\delta_\epsilon \pi \coloneqq \mathcal{L}_\epsilon \pi = \delta_\epsilon(\pi_\nu dx^\nu) = (\delta_\epsilon \pi_\nu) dx^\nu. \tag{31}$$

Explicit form of the transformed momentum is given by the Cartan formula followed from (31)

$$\delta_\epsilon \pi_\nu = \epsilon^\mu \partial_\mu \pi_\nu + \pi_\mu \partial_\nu \epsilon^\mu. \tag{32}$$

VI. HAMILTON PRINCIPLE OF HYDRODYNAMICS

The Hamiltonian principle asserts that on the equation of motions, the Hamilton functional, which we denote by Λ, is invariant under the action of the group $\mathcal{G}$. In this form, the Hamiltonian principle incorporates the fluid kinematics into the conservation laws associated with the symmetry group $\mathcal{G}$ [33].

The Hamilton functional depends of *natural variables*, which we choose by following Carter [24] as the canonical momentum π, the chiral phase Θ and the density ratio S. Then the variation of the Hamilton functional

$$\delta_\epsilon \Lambda = \int [\mathcal{J}^\mu \delta_\epsilon \pi_\mu + \pi_S \delta_\epsilon S - I^\mu \partial_\mu \delta\Theta] \tag{33}$$

defines the conjugate fields: the flow field $\mathcal{J}^\mu \coloneqq \delta\Lambda/\delta\pi_\mu$, a conjugate to the canonical momentum, the Noether current $I^\mu \coloneqq -\delta\Lambda/\delta(\partial_\mu \Theta)$, and the conjugate to the density ratio $\pi_S \coloneqq \delta\Lambda/\delta S$. Using explicit forms of the variations (30), (32) a simple algebra leads to what Carter referred to as the canonical fluid equation [34]

$$\mathcal{J}^\mu \Omega_{\mu\nu} + \pi_\nu(\partial_\mu \mathcal{J}^\mu) = \pi_S \partial_\nu S, \qquad \partial_\mu I^\mu = 0. \tag{34}$$

The first term of the left-hand side is the force acting on a rotating fluid parcel. It is balanced by the force due to the fluid source and the reservoir source. They are the second term sometimes called the "rocket term" and the "heat" source on the right-hand side of (34). A notable feature of the canonical equation is the absence of a reference to a spacetime metric.

The combined result must be gauge invariant. The gauge phase Θ introduced through the canonical momentum (20) by the rocket term should not enter the equations. If the flow field $\mathcal{J}$ is gauge invariant, this is achieved by setting $\mathcal{J}$ divergence free and killing the rocket term. This is the case of the perfect fluid. However, if $\mathcal{J}$ is not gauge invariant, i.e., depends on $\partial_\mu \Theta$, the Θ in the first term must cancel the Θ in the second term. This requirement imposes a nearly prohibiting condition on $\mathcal{J}$.

The perfect fluid could be defined by the condition that the flow field, the Noether current, and the particle number are all equal $\mathcal{J}^\mu = I^\mu = n^\mu$. This condition determines the Hamilton functional equal to (minus) spacetime integral of the fluid pressure [35,36]

$$\Lambda_0 = -\int_{\mathcal{M}^4} P. \tag{35}$$

It is instructive to check it. For this purpose we need the differential of the fluid pressure with respect to the natural variables. The fluid pressure is defined as $P = n\partial_n \varepsilon - \varepsilon$. In view of the relations (25)–(27) we write the pressure in terms of momentum and particle current as $-P = p_\mu n^\mu + \varepsilon$ and compute its differential as

$$-dP - \partial_S \varepsilon dS = n^\mu dp_\mu = n^\mu d\pi_\mu - n^\mu \partial_\mu d\Theta. \tag{36}$$

It follows that the flow field and the Noether are equal to the particle number and $\pi_S = \partial_S \varepsilon$. We obtain the canonical form of the Euler equation (also referred as the Lichnerowicz equation) for the perfect charged fluid

$$n^\mu \Omega_{\mu\nu} + \pi_\nu(\partial_\nu n^\nu) = (\partial_\nu \varepsilon)_n, \tag{37}$$

where $(\partial_\nu \varepsilon)_n = (\partial_S \varepsilon)\partial_\nu S$ is the gradient of energy at a fixed n plus the continuity equation (2). The canonical equation is equivalent to the momentum-stress-energy conservation laws (1), (8). This is simplified by taking into account the continuity equation which kills the rocket term in the left-hand side of (37).

The deformation of the Hamilton functional disrupts the accidental relation between the currents and alters the mechanism that brings (33) to its gauge-invariant form.

VII. FLUID PHASE SPACE AND A GENERALIZED HOPF FIBRATION

The last general point we need to discuss before introducing the multivalued action is the geometry of the fluid phase space.

In addition to the four-dimensional space of momentum, the phase space includes the scalar S. That makes the phase space five-dimensional, matching the dimension of the manifold of the symmetry group $\mathcal{G}$. We illustrate this important feature by invoking the Clebsch realization of the momentum.

The vorticity 2-form $\Omega = \frac{1}{2}\Omega_{\mu\nu}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)$ of the non-homentropic flow, where $dS \neq 0$, is nondegenerate $\det \Omega_{\mu\nu} \neq 0$. It endows a symplectic structure. Under this condition we may invoke the Daurboux theorem. It asserts that there are four local coordinates α, β, η, S among which one could be chosen to be the density ratio S, in which the symplectic structure takes on a canonical form:

$$\Omega = d\alpha \wedge d\beta + d\eta \wedge dS. \tag{38}$$

As a result, the canonical momentum is locally represented by *five* coordinates

$$\pi = d\Theta + \alpha d\beta + \eta dS. \tag{39}$$

We are endowed with a map of the five-dimensional phase space, denoted by N^5 to the four-dimensional spacetime M^4: $\mathrm{N}^5 \to \mathrm{M}^4$, where a point of a spacetime x is mapped out from a distinct circle S^1, represented by the chiral phase Θ. The local coordinates of N^5 are associated with five Clebsch potentials Θ, α, β, η, S [35–37]. Then the canonical momentum $\pi = (\partial_\mu\Theta + \alpha\partial_\mu\beta + \eta\partial_\mu S)dx^\mu$, being the 1-form in M^4 could be seen as a push-forward of the 1-form (39) in N^5. The map describes a fibration of the phase space $\mathrm{S}^1 \hookrightarrow \mathrm{N}^5 \to \mathrm{M}^4$, where the spacetime is the base of the bundle. The total space N^5 consists of fibers, with each fiber being a circle S^1 spanned by the chiral phase one for each point of the spacetime. This setup is analogous to the classical Hopf fibration (albeit not for spheres), given by the Hopf map $\mathrm{S}^1 \hookrightarrow \mathrm{S}^3 \to \mathrm{S}^2$. It was introduced in [38]. The map is characterized by the invariant, which was referred to in [39] as the Hopf-Novikov invariant. Analogous to the realization of the Hopf invariant in terms of differential forms [40], the Hopf-Novikov invariant is also represented by the integral of the top-form in N^5, which is constructed from the pullback of the canonical momentum [41]

$$\mathrm{H} = \int_{\mathrm{N}^5} \pi \wedge (d\pi)^2, \qquad (d\pi)^2 = d\pi \wedge d\pi. \tag{40}$$

In the context of semiclassical hydrodynamics the invariant is the volume of the compact phase space.

VIII. MULTIVALUED FUNCTIONAL

The five-dimensional phase space allows the following interpretation. Consider a closed five-dimensional space M^5 and treat it as a spacetime of an auxiliary five-dimensional fluid. Then the map $\mathrm{M}^5 \to \mathrm{N}^5$ defines the momentum of the auxiliary fluid via (39) and the invariant (40) is the linking number of singular 3-surfaces

$$\mathrm{H} = \int_{\mathrm{M}^5} \varepsilon^{\mu\nu\lambda\sigma\rho}\pi_\mu\partial_\nu\pi_\lambda\partial_\sigma\pi_\rho d^5x. \tag{41}$$

Consider now a bounded five-dimensional space M^5_+, a half-space of M^5, and identify the boundary of M^5_+ with the physical spacetime $\mathrm{M}^4 = \partial\mathrm{M}^5_+$. Then the boundary layer of the five-dimensional fluid could be identified with the physical fluid. The integrand in (41) is a Jacobian of map $\mathrm{M}^5 \to \mathrm{N}^5$. It is a closed form $\pi \wedge (d\pi)^2 = d\Phi$. Therefore, the integral (41) over M^5_+ is a surface term spanned over physical spacetime, which is the integral over pullback of the 4-form Φ

$$\mathrm{H}_+ = \int_{\mathrm{M}^5_+} \varepsilon^{\mu\nu\lambda\sigma\rho}\pi_\mu\partial_\nu\pi_\lambda\partial_\sigma\pi_\rho d^5x = \int_{\mathrm{M}^4} \Phi \quad (\mathrm{mod}\ \mathrm{H}). \tag{42}$$

This is Novikov's functional. It is defined modulo the invariant H reflecting different choice of M^5_+. In this sense the functional is multivalued. Consequently, the 4-form Φ cannot be expressed in a coordinate-free manner, but it could be elementary computed in chosen coordinates. Choosing the chiral phase as a fifth coordinate, the density Φ, modulo an exact 4-form, is

$$\Phi = \frac{1}{2}\Theta(\Omega \wedge \Omega). \tag{43}$$

IX. MULTIVALUED FUNCTIONAL IN FLUID DYNAMICS

Now we deform the Hamilton functional of the perfect fluid by the multivalued functional as

$$\Lambda \coloneqq \Lambda_0 + \frac{k}{4}\mathrm{H}_+ = \int_{M^4}\left[-P + \frac{k}{8}\Theta(\Omega \wedge \Omega)\right]. \tag{44}$$

While the added functional is not uniquely defined, it nonetheless generates a local equation of motion [43]. The ambiguity of the functional does not extend to its variation as the invariant H does not vary.

Unlike Λ_0, the functional H_+ is not gauge invariant. It opens a channel of inflow of the five-dimensional auxiliary fluid into the physical fluid. Nevertheless, the equations of motion maintain the gauge invariance.

We remark that the multivalued term in (44) is a hydrodynamic version of an *axion*, a Θ angle promoted to a dynamical field (see [28] for a review).

In the semiclassical fluid, the multivaluedness of H_+ leads to the quantization of the level k, as we already discussed in Sec. III. It follows from the requirement for $\exp[(i/\hbar)\Lambda]$ to be single valued under a global gauge transformation which changes the circulation by a unit $\mathcal{C} \to \mathcal{C} + 1$. Then the change of the Hamilton functional is $k/2$ times the total helicity $\Lambda \to \Lambda + \frac{k}{2}\mathcal{H}$. Since the latter is an even integer, k is quantized (cf. [44]).

X. EQUATIONS OF MOTION

Now we turn to equations of motion. We calculate the currents defined by (33), and subsequently, substitute them into canonical equation (34).

First, we vary (44) over Θ, while keeping π fixed. This gives the divergence of the Noether current $\partial_\mu I^\mu = \partial_\mu n^\mu + \frac{k}{4}\Omega_{\mu\nu}{}^\star\Omega^{\mu\nu}$ and yields the particle production equation (16).

The next step is to vary (44) with respect to π, which results in a deformation of the flow field $\mathcal{J}^\mu = n^\mu + k^\star\Omega^{\mu\nu}\partial_\nu\Theta$ by a not gauge-invariant, albeit divergence-free term. The relation $\partial_\mu\mathcal{J}^\mu = \partial_\mu n^\mu$ remains unchanged.

Now we have all of the terms in the canonical equation (34) to verify that the chiral phase Θ cancels out. We see it with the help of the identity

$$\epsilon^{\mu\nu\lambda\sigma}X_\rho + \cdots = 0, \tag{45}$$

which holds for an arbitrary X_μ (the ellipsis denotes the cyclic permutation of five indices) and its consequence $4({}^\star\Omega^{\lambda\mu}\Omega_{\mu\nu}) = \delta^\lambda_\nu({}^\star\Omega^{\lambda\mu}\Omega_{\mu\lambda})$. The net result is the familiar canonical form of the Euler equation for a perfect fluid, Eq. (31). The only difference is that the rocket term no longer vanishes

$$n^\mu\Omega_{\mu\nu} + p_\nu(\partial_\mu n^\mu) = (\partial_\nu\varepsilon)_n. \tag{46}$$

In the form of energy-stress-momentum this equation reads

$$\partial_\mu T^\mu{}_\nu = F_{\nu\lambda}n^\lambda, \qquad T^\mu{}_\nu = n^\mu p_\nu + \delta^\mu{}_\nu P. \tag{47}$$

As expected, the Euler equation is unaffected by the WZN term and remains identical to that of a perfect fluid [45]. Since the WZN term is independent of the metric, it does not alter the form of the stress tensor, but only modifies the continuity equation as

$$\partial_\mu n^\mu = -\frac{k}{4}\Omega_{\mu\nu}{}^\star\Omega^{\mu\nu}. \tag{48}$$

These equations are complemented by the equation of state (27) which connects the particle number current and the momentum. This is the full set.

XI. SPIN AND SPIN-ORBIT COUPLING

We conclude by emphasizing a property that the WZN term imparts to the perfect fluid. It gives the fluid a spin equal to $k/2$.

A hint to it provides the Newtonian form of the Euler equation. In this form the rocket term in (46) expressed in terms of the fluid momentum by virtue of the particle production equation is treated as a force exerted on the fluid. Using (46) and the identity $\frac{1}{2}p_\nu(\partial_\mu\Sigma^\mu) = \Sigma^\mu\Omega_{\mu\nu}$ followed from (45) we bring the Euler equation into the form

$$n^\mu\partial_\mu p_\nu + \partial_\nu P = k\Sigma^\mu\Omega_{\mu\nu}. \tag{49}$$

The term on the right-hand side of (49) indicates that our fluid is spinning, with the spin density $\frac{1}{2}\Sigma^\mu$ (see Appendix C). The spin exerts a force on fluid vorticity given by the right-hand side of (49) (more about the spin density and the spin current is given in Appendix D).

In summary, we presented what we believe to be the only deformation of a single-component perfect fluid that does not introduce additional scales into the system. This deformation captures the chiral anomaly. While it modifies the continuity equation, it does not alter the form of the stress-energy tensor. It requires the system to have an open channel for particle production and breaks inversion symmetry in the same manner as the chiral anomaly. The kinematic and geometric properties of our hydrodynamics are consistent with those of systems composed of Weyl (or chiral) fermions.

ACKNOWLEDGMENTS

The author gratefully acknowledges discussions with G. Volovik and L. Friedlander. Special appreciation is extended to A. G. Abanov and A. Cappelli for their collaboration on this subject. This work was supported by the NSF under Grant No. NSF DMR-1949963.

APPENDIX A: VORTICITY INSTANTONS AND "ENTROPY" PRODUCTION

Here we remark on the role of the anomaly in particle exchange with the reservoir.

We recall that the reservoir density $\bar{n} = nS$ could be interpreted as an "entropy" [32]. Consequently, the "entropy current" is $s^\mu \coloneqq \bar{n}u^\mu = Sn^\mu$. By contracting (46) with n^μ we obtain the relation between particles and entropy productions $\partial_\mu s^\mu = -(\partial\bar{n}/\partial n)_\varepsilon(\partial_\mu n^\mu)$, with $\bar{n}$ being treated as a function of ε and n. This is a general relation for an open system. It means that the entropy propagates along isoenergy hypersurfaces $d\varepsilon = 0$. Specifically to our fluid we may express the entropy production in terms of vorticity as

$$\partial_\mu s^\mu = \frac{k}{4}\left(\frac{\partial\bar{n}}{\partial n}\right)_\varepsilon\Omega_{\mu\nu}{}^\star\Omega^{\mu\nu}. \tag{A1}$$

We conclude that the entropy production goes along with particle production assisted by vorticity instantons. A change of linking number of vortex loops changes the entropy [48].

APPENDIX B: HOMENTROPIC FLOWS

A homentropic flow occurs when the density ratio S is uniform and constant. It is also called uniformly canonical

[24]. In this situation, the flow is barotropic and the vorticity tensor is degenerate $\det\Omega_{\mu\nu} = 0$, having rank 2. Consequently, the rate of particle production ${}^\star\Omega^{\mu\nu}\Omega_{\mu\nu} = \sqrt{\det\Omega_{\mu\nu}}$ vanishes. This prevents the construction of the WZN term since the phase space of a homentropic flow is not symplectic. In this case canonical helicity and particle currents are conserved independently $\partial_\mu n^\mu = \partial_\mu h^\mu = 0$. In this case, the equations of motion are no different from that of the perfect fluid but the kinematic helicity Σ^μ obeys the anomaly equation $\partial_\mu \Sigma^\mu = -\frac{1}{2} F_{\mu\nu}{}^\star F^{\mu\nu}$ [12–17].

APPENDIX C: TOTALLY ANISYMMETRIC SPIN CURRENT

A spinning fluid possesses a spin current, which we denote by $\sigma_{\mu\alpha\beta}$. Given a spin current one finds a spin tensor $\sigma_{\alpha\beta} = -u^\nu \sigma_{\nu\alpha\beta}$ by projecting the spin current onto the 4-velocity. It is customary to use the dual spin tensor ${}^\star\sigma^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$. Then the spin density reads $\sigma^\nu = u_\mu {}^\star\sigma^{\mu\nu}$.

We identify the spin density as a neutral part of the current (17). It is given by half of helicity as Eq. (18) suggests,

$$\sigma^\mu = \frac{1}{2}\Sigma^\mu, \qquad \sigma_{\mu\alpha\beta} = -\frac{1}{6} p_{[\mu}(\partial_\alpha p_{\beta]} + F_{\alpha\beta]}), \tag{C1}$$

Here $[\mu, \alpha, \beta]$ denotes the antisymmetrization over all three indices. We observe that the spin current is totally antisymmetric. This is a distinguished property of spinning fermions.

APPENDIX D: COMPARISON WITH THE KINEMATICS OF WEYL FERMIONS

Our results are consistent with the kinematics of chiral fermions. Here we briefly outline the major points.

Chiral fermions carry an electric charge and also $\frac{1}{2}$ spin. The current, therefore, is composed of a charge (the particle number) current, and a neutral part representing the spin current. The formula (17) represents this composition. Therefore, we identify the neutral part of the current $\sigma^\nu = \frac{1}{2}\Sigma^\nu$ withe the spin. The neutral part can be also obtained as a difference between the particle current and the current of antiparticles $\Sigma^\nu = j^\nu|_\mu - j^\nu|_{-\mu}$, where μ is the chemical potential. The (twice of the) spin current of free chiral fermions had been computed by Vilenkin in a series of papers [20]. Despite that Vilenkin computations are valid for free fermions, some of the results are transferred to a liquid state and could be compared with formulas (17), (18).

Vilenkin computed the equilibrium value of the spacelike component of $\boldsymbol{\Sigma}$ for free rotating chiral fermions in a magnetic field and at a low temperature. His result (in units of Planck constant $2\pi\hbar$) is

$$\boldsymbol{\Sigma} = 2\left[\left(\mu^2 + \frac{\pi^2 T^2}{3}\right)\boldsymbol{\omega} + \mu \boldsymbol{B}\right]. \tag{D1}$$

Now we specify our formula (18) $\Sigma^\mu = \epsilon^{\mu\nu\lambda\sigma} p_\nu(\partial_\lambda p_\sigma + F_{\lambda\sigma})$ for a stationary rotating fluid and check it against Vilenkin's direct computation.

The spacelike component of Σ evaluated for a stationary flow in the leading order of velocity $\boldsymbol{u}$ is

$$\boldsymbol{\Sigma} = w^2 \nabla \times \boldsymbol{u} - 2p_0 \boldsymbol{B}. \tag{D2}$$

We can extend this formula to a rotating fluid, by adding twice the angular velocity of rotation $2\boldsymbol{\omega}$ to vorticity $\boldsymbol{\Sigma} = w^2(\nabla \times \boldsymbol{u} + 2\boldsymbol{\omega}) - 2p_0\boldsymbol{B}$. At equilibrium we drop vorticity. Then

$$\boldsymbol{\Sigma} = 2(w^2\boldsymbol{\omega} - p_0\boldsymbol{B}). \tag{D3}$$

This is a general result. To specify it for fermions we need the value of enthalpy for a degenerate Fermi gas. It could be extracted from the textbook [46] (paragraph 61, problem 2). At low temperature the enthalpy of free fermions is $w \approx \mu(1 + \frac{\pi^2}{6}\frac{T^2}{\mu^2})$. For the Fermi gas we have to set $-p_0$ to be the Fermi level equal the chemical potential $-p_0 = \mu$. Combining, we get Vilenkin's result [47].

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