Bound Entanglement in Thermalized States and Black Hole Radiation

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We study the mixed-state entanglement structure of chaotic quantum many-body systems at late times using the recently developed *equilibrium approximation*. A rich entanglement phase diagram emerges when we generalize this technique to evaluate the logarithmic negativity for various universality classes of macroscopically thermalized states. Unlike in the infinite-temperature case, when we impose energy constraints at finite temperature, the phase diagrams for the logarithmic negativity and the mutual information become distinct. In particular, we identify a regime where the negativity is extensive but the mutual information is subextensive, indicating a large amount of *bound entanglement*. When applied to evaporating black holes, these results imply that there is quantum entanglement within the Hawking radiation long before the Page time, although this entanglement may not be distillable into Einstein-Podolsky-Rosen pairs. We claim that at this earlier time, rather than the Page time, information about diaries thrown into the black hole first starts to leak out.

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Introduction.—The entanglement of a bipartite system, $\mathcal{H}_A \otimes \mathcal{H}_B$, in a pure state can heuristically be captured by some number of Einstein-Podolsky-Rosen (EPR) pairs, as it is always possible to convert the pure state into these EPR pairs and vice versa using quantum operations that are local to $\mathcal{H}_A$ or $\mathcal{H}_B$, respectively, as well as classical communications between observers local to $\mathcal{H}_A$ and $\mathcal{H}_B$ (known as LOCC). This kind of interpretation becomes more complicated for mixed states. The interconversion between mixed states and EPR pairs is, in general, irreversible; the number of EPR pairs needed to prepare the state using LOCC operations (so-called entanglement cost) can be greater than the number that can be extracted from it (distillable entanglement) [1]. In particular, there exist *bound-entangled states* [2], defined as states which have nonzero entanglement cost, but from which no EPR pairs can be distilled.

While mixed-state entanglement carries important physical information, the corresponding operational measures such as entanglement cost and distillable entanglement are extremely hard to calculate even for few-qubit systems. The logarithmic negativity provides a more calculable measure [3–9] but is still very difficult to compute in many-body systems and field theories.

To define the logarithmic negativity, we consider a (generally mixed) density matrix ρ_{AB}. The partial transpose with respect to $\mathcal{H}_B$ is defined as having matrix elements $\langle a_1, b_1|\rho_{AB}^{T_B}|a_2, b_2\rangle = \langle a_1, b_2|\rho_{AB}|a_2, b_1\rangle$, where $|a, b\rangle$ forms a basis. Unentangled mixed states are *separable*, of the form

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \tag{1}$$

and their partial transpositions are positive operators. The logarithmic negativity captures the nonseparability (entanglement) in the density matrix by quantifying how negative the partially transposed operator is,

$$\mathcal{E}(A, B) = \log |\rho_{AB}^{T_B}|_1, \tag{2}$$

where $|\cdot|_1$ is the one-norm. Separable states have trivial negativity, so states with nonzero negativity are necessarily entangled. Crucially, the *mutual information*, defined as $I(A, B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, does not measure entanglement, because it may be nonzero even for separable states. Here, $S(\rho) \coloneqq -\mathrm{Tr}\rho \log \rho$ is the von Neumann entropy.

In this Letter, we generalize a method developed in Ref. [10], called the equilibrium approximation, to obtain the logarithmic negativity of a macroscopically equilibrated mixed state. The approximation applies to general chaotic systems at late times, when the macroscopic properties of the state, such as expectation values of local operators, are

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close to those of a thermal ensemble, although the state is far from the thermal density matrix by measures like trace distance. We find that there can be a rich entanglement structure, captured by an intricate entanglement phase diagram. A particularly surprising result is that, in the thermodynamic limit, there can be a finite region in the parameter space where the logarithmic negativity is extensive but the mutual information is subextensive, implying that there is a large amount of bound entanglement. This phenomenon does not take place in the infinite-temperature case previously studied in Ref. [11] but arises in a variety of universality classes of equilibrated states at finite temperature studied here.

A main physical application we have in mind is an evaporating black hole. Consider a black hole formed from the gravitational collapse of matter in a pure state. The black hole emits Hawking radiation and eventually evaporates completely. The dynamics of black holes are expected to be highly chaotic. Hence, if the evaporation process respects the usual rules of quantum mechanics, general results on the quantum-informational properties of a chaotic quantum many-body system can be used to make predictions about the black hole and its radiation. The verification of such predictions using gravity calculations can then provide highly nontrivial checks of the consistency of black hole physics with quantum-mechanical principles and can also lead to new insights into quantum gravitational dynamics. One good example is the celebrated Page curve, a prediction for the time evolution of the entanglement entropy of the radiation [12,13]. This curve was recently derived in Refs. [14–17] with gravity calculations, which not only confirmed that black holes obey unitarity, but also revealed new geometric features known as "islands" and "replica wormholes."

The evaporation process is very slow in microscopic scales, so that at any time in the process we can treat the remaining black hole as well as the radiation as being in macroscopic equilibrium. Page considered the von Neumann entropies of the black hole and the radiation at infinite temperature. However, in general, it is important to study the entanglement structure at finite temperature and also to probe it with other quantum-informational quantities such as logarithmic negativity. The general results obtained in this Letter serve as predictions for the entanglement structure *within* the radiation at both infinite and finite temperatures. At infinite temperature, before the Page time, the radiation is maximally entangled with the black hole, and there is no entanglement within the radiation itself. In contrast, at finite temperature, we find that there is a new timescale t_b *before* the Page time t_p, when nontrivial entanglement within the radiation starts to emerge (see Fig. 1). For $t \in (t_b, t_p)$, the entanglement is bound entanglement; that is, it cannot be distilled into EPR pairs using LOCC. [It remains a logical possibility that EPR pairs may be distillable using quantum operations that preserve positivity of the partial transpose (PPT operations) or that the distillable entanglement is subextensive but nonzero.] In this Letter, we outline the general ideas and main results, leaving technical details and further elaboration to Ref. [18].

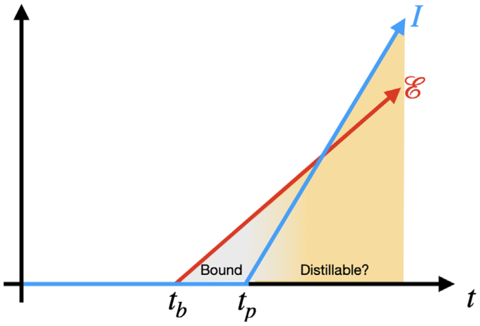

FIG. 1. The general behavior of logarithmic negativity and mutual information is shown for finite-temperature equilibrated pure states, specifically evaporating black holes. While the mutual information within the radiation does not become extensive until the Page time t_p, the negativity becomes large at the earlier time t_b. After the Page time, we expect the entanglement to be distillable.

Setup.—Consider a system A in a mixed state ρ_A, which is in macroscopic equilibrium. To explore the bipartite entanglement structure of A, we would like to evaluate the logarithmic negativity $\mathcal{E}(A_1, A_2)$ and mutual information $I(A_1, A_2)$ between a subsystem A_1 and its complement A_2 in A. The logarithmic negativity $\mathcal{E}$ is nonzero only if ρ_A is not separable and can be used to lower bound the entanglement cost: $\mathcal{E}(A_1, A_2) \leq E_c^{(\mathrm{PPT,exact})}(A_1, A_2)$ [19]. The mutual information I also contains classical information but is, nevertheless, of importance, as it upper bounds the distillable entanglement: $E_d(A_1, A_2) \leq \frac{1}{2} I(A_1, A_2)$ [20]. (See Supplemental Material [21] for various technical definitions, which includes references to Refs. [22–25].)

We can imagine that A is embedded in a larger system $S = A \cup B$, with the total system S in a pure state $|\Psi\rangle$ and $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$. In many situations of interest, such a B naturally exists. For example, for an evaporating black hole, when A is the Hawking radiation, B is the remaining black hole.

In Ref. [10], it was shown that the quantum-informational properties for a system in such an equilibrated pure state $|\Psi\rangle$ can be calculated from properties of an equilibrium density matrix

$$\rho^{(\mathrm{eq})} := \frac{1}{Z_1} \mathcal{I}_\alpha, \qquad Z_1 := \mathrm{Tr} \mathcal{I}_\alpha, \tag{3}$$

which has the same macroscopic thermodynamic behavior as $|\Psi\rangle$. Here, α denotes macroscopic equilibrium parameters such as temperature or chemical potential. Specification of $\mathcal{I}_\alpha$ can be viewed as specifying the universality class of an equilibrated pure state. We assume below that $Z_1 \gg 1$, which corresponds to having a large effective Hilbert space dimension.

From Ref. [10], the Rényi partition function for a subsystem R can be obtained by

$$\mathcal{Z}_{n,R} := \mathrm{Tr}\rho_R^n \approx \frac{1}{Z_1^n}\sum_{\tau\in\mathcal{S}_n}\langle \eta_R \otimes e_{\bar{R}}|\mathcal{I}_\alpha, \tau\rangle, \tag{4}$$

where τ is an element of the permutation group $\mathcal{S}_n$ and $\bar{R}$ is the complement of R in S. When Eq. (4) can be expressed as an analytic function of n, the von Neumann entropy for R can be obtained by analytic continuation of Eq. (4). Alternatively, we can use Eq. (4) to calculate the resolvent

$$\mathcal{R}(\lambda) := \mathrm{Tr}\left(\frac{1}{\lambda\mathbf{1} - \rho_R}\right) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\mathcal{Z}_{n,R}, \tag{5}$$

which can be used to find the spectral density of ρ_R,

$$D(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}\mathcal{R}(\lambda - i\epsilon), \qquad \lambda\in\mathbb{R}, \tag{6}$$

and, hence, the von Neumann entropy

$$S(R) = -\int d\lambda D(\lambda)\lambda\log\lambda. \tag{7}$$

Equations (4)–(7) can be used to calculate the mutual information $I(A_1, A_2)$ by taking R to be the different subsystems.

We generalize the methods of Ref. [10] to calculate the partial transpose partition function

$$\begin{aligned}\mathcal{Z}_n^{(\mathrm{PT})} &:= \mathrm{Tr}_A(\rho_A^{T_2})^n \\ &\approx \frac{1}{Z_1^n}\sum_{\tau}\langle \eta_{A_1}\otimes\eta_{A_2}^{-1}\otimes e_B|\mathcal{I}_\alpha, \tau\rangle,\end{aligned} \tag{8}$$

where T_2 denotes partial transpose of ρ_A with respect to A_2. When Eq. (8) is analytic in *even* n, the logarithmic negativity can be found from analytic continuation as $\mathcal{E}(A_1, A_2) = \lim_{n\to 1/2}\log\mathcal{Z}_{2n}^{(\mathrm{PT})}$. Alternatively, it can be calculated from the resolvent for $\rho_A^{T_2}$ as

$$R_N(\lambda) := \mathrm{Tr}\left(\frac{1}{\lambda\mathbf{1} - \rho_A^{T_2}}\right) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{1}{\lambda^n}\mathcal{Z}_n^{(\mathrm{PT})}, \tag{9}$$

$$D_N(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0}\mathrm{Im}R_N(\lambda - i\epsilon), \qquad \lambda\in\mathbb{R}, \tag{10}$$

$$\mathcal{E}(A_1, A_2) = \log\left(\int d\lambda D_N(\lambda)|\lambda|\right), \tag{11}$$

where $D_N(\lambda)$ is the spectral density of $\rho_A^{T_2}$.

We will consider $\mathcal{E}(A_1, A_2)$ and $I(A_1, A_2)$ at leading order in the thermodynamic limit. In this limit, Eqs. (4) and (8) can both be approximated by terms from a subset of permutations τ, which give the dominant contribution. These sets of permutations can change as we vary two parameters

$$\lambda := \frac{S_{A_1}^{(\mathrm{eq})}}{S_A^{(\mathrm{eq})}}, \qquad c := \frac{S_A^{(\mathrm{eq})}}{S_A^{(\mathrm{eq})} + S_B^{(\mathrm{eq})}}, \tag{12}$$

where $S_{A_1,A,B}^{(\mathrm{eq})}$ are, respectively, the von Neumann entropies for A_1, A, and B in the state $\rho^{(\mathrm{eq})}$. These parameters can be seen as a way of measuring the relative sizes of the subsystems in the general case where the system $S = A\cup B$ is inhomogeneous. The change in the dominant contribution on varying c and λ leads to qualitative changes in the behavior of $\mathcal{Z}_n^{(\mathrm{PT})}$ and $\mathcal{Z}_{n,R}$ and, correspondingly, of $\mathcal{E}(A_1, A_2)$ and $I(A_1, A_2)$. We refer to such changes as entanglement phase transitions. In the black hole setting, c is a monotonic function of evaporation time t with $c(t=0) = 0$, $c = \frac{1}{2}$ corresponding to the Page time and $c = 1$ corresponding to the time at which the black hole has entirely evaporated. While the specific function $c(t)$ is system dependent, c can be used as a system-independent proxy of the evaporation time. Furthermore, λ represents the fraction of the radiation in subsystem A_1.

To justify using the equilibrium approximation for a given quantity Q, we can estimate the size of the fluctuations around the equilibrium value using $\Delta_Q^2 \equiv (Q^2)_{\mathrm{eq\ approx}} - (Q_{\mathrm{eq\ approx}})^2$. As a self-consistent check of whether the approximation is valid, Δ_Q should be small relative to $Q_{\mathrm{eq\ approx}}$. We checked that this criterion is met for the Rényi partition functions in Ref. [10] and for the partial transpose partition function in Appendix B in Ref. [18], when the effective Hilbert space dimension of the system is large. For a recent study of the logarithmic negativity in thermalized chaotic quantum-many body systems, see Ref. [26].

Infinite-temperature phase diagram.—For an equilibrated pure state at infinite temperature, $\mathcal{I}_\alpha$ is given by

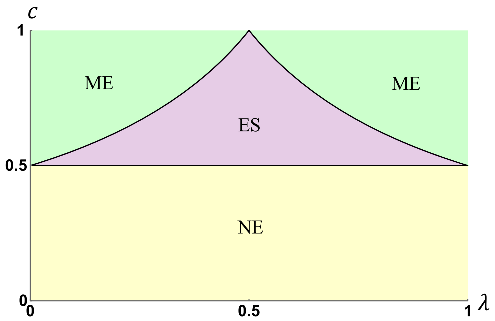

FIG. 2. Entanglement phase diagram for the infinite-temperature equilibrated pure state.

the identity operator $\mathbf{1}_{AB}$. The logarithmic negativity and mutual information have the same phase structure (Fig. 2), coinciding with that obtained from the Haar random states [11] and summarized as follows. (1) *Phase of no entanglement (NE)* ["no entanglement" here should be understood as no "volume-law" entanglement; i.e., there is no contribution at the order of $O(\log d_A)$]: For $c < \frac{1}{2}$, we find

$$\mathcal{E}(A_1, A_2) = I(A_1, A_2) = 0, \tag{13}$$

implying that ρ_A is close to maximally mixed, and all degrees of freedom of A are maximally entangled with those in B. (2) *Maximally entangled phase (ME)*: For $c > \frac{1}{2}$, $\lambda > (1/2c)$, we find

$$\mathcal{E}(A_1, A_2) = \frac{1}{2} I(A_1, A_2) = S_{A_2}^{(\mathrm{eq})}, \tag{14}$$

which implies that A_2 is maximally entangled with A_1. In this regime, the effective number of degrees of freedom in A_2 and B together is smaller than that in A_1. Thus, both A_2 and B should be maximally entangled with A_1. (3) *Entanglement saturation phase (ES)*: For $c > \frac{1}{2}$, $1 - (1/2c) < \lambda < (1/2c)$, we have

$$\mathcal{E}(A_1, A_2) = \frac{1}{2} I(A_1, A_2) = \frac{1}{2}(S_A^{(\mathrm{eq})} - S_B^{(\mathrm{eq})}). \tag{15}$$

Both the negativity and mutual information depend only on the difference $S_A^{(\mathrm{eq})} - S_B^{(\mathrm{eq})}$ and do not change as we vary the size of A_1 (as long as we stay in the aforementioned parameter range). There is a simple intuitive interpretation of Eq. (15) in terms of bipartite entanglement among pairs of subsystems: Since $S_B^{(\mathrm{eq})} < S_A^{(\mathrm{eq})}$, $\log d_B = S_B^{(\mathrm{eq})}$ degrees of freedom in A are entangled with B, and the remaining $\log d_A - \log d_B = S_A^{(\mathrm{eq})} - S_B^{(\mathrm{eq})}$ are entangled between A_1 and A_2. We should emphasize, however, that this "mechanical" way of assigning entanglement likely does not reflect the genuine entanglement structure of the system in this phase, and there are indications of significant multipartite entanglement [18].

The equilibrium approximation.—The infinite-temperature case applies only to a system with a finite-dimensional Hilbert space at sufficiently high energies or without energy conservation. Otherwise, energy constraints must be imposed. Now there are many more possibilities for $\mathcal{I}_\alpha$, which depend on the ensemble chosen.

At finite temperature, each $\mathcal{Z}_n^{(\mathrm{PT})}$, $\mathcal{Z}_{n,R}$ gives rise to a different phase diagram, revealing intricate patterns of entanglement structure. A significant technical complication at finite temperature is that the extraction of the logarithmic negativity and the von Neumann entropies using analytic continuation becomes *a priori* unreliable near phase boundaries, due to the nonuniform dependence on n. $\mathcal{E}$ and I must be calculated using the resolvents, which fortunately may be done for some choices of $\mathcal{I}_\alpha$. (See Supplemental Material [21] for explicit calculations.)

Consider first the mutual information. At finite temperature, to leading order in volume, the equilibrium approximation for $\mathcal{Z}_{n,R}$ leads to the following approximation for the von Neumann entropy:

$$S_R = \min(S_R^{(\mathrm{eq})}, S_{\bar{R}}^{(\mathrm{eq})}). \tag{16}$$

In Ref. [10], this generalization of Page's formula was argued on the basis of analytic continuation. However, this statement remains true in all cases of $\mathcal{I}_\alpha$ that we have studied using the resolvent, including inhomogeneous cases. Given that the von Neumann entropy of a thermal density operator is extensive ($S_{A_1}^{(\mathrm{eq})} + S_{A_2}^{(\mathrm{eq})} = S_A^{(\mathrm{eq})}$), we then find that $I(A_1, A_2)$ has exactly the same behavior as at infinite temperature (13)–(15) and the same phase diagram.

The logarithmic negativity has a much richer structure. The precise phase diagram for $\mathcal{E}$ depends on the choice of $\mathcal{I}_\alpha$, but, in all cases we have studied, there are analogs of the NE, ES, and ME phases, coming from different dominant permutations in Eq. (8). A most surprising feature, which appears to be generic in the examples where the resolvents can be explicitly calculated, is that there is a regime for an $O(1)$ range of the parameter c where $\mathcal{E}$ is extensive while I is subextensive. It is generally believed that, since the mutual information contains both quantum and classical correlations, there cannot be any volumelike quantum entanglement when it is subextensive. Our results indicate that this intuition cannot be correct. Another surprising feature is that there are new phases in $\mathcal{E}$ which cannot be found by analytic continuation of $\mathcal{Z}_n^{(\mathrm{PT})}$. In all cases below, $S_{n,R}^{(\mathrm{eq})}$ refers to the nth Rényi entropy of $\rho_R^{(\mathrm{eq})}$.

Consider first the canonical ensemble

$$\mathcal{I}_\alpha = e^{-\beta_A H_A} \otimes e^{-\beta_B H_B}, \tag{17}$$

where A and B may have different inverse temperatures β_A and β_B. Using Eq. (8), the negativity can be deduced from the phases of $\mathcal{Z}_n^{(\mathrm{PT})}$ by analytic continuation:

$$\mathcal{E}_{\mathrm{NE}} = 0, \tag{18}$$

$$\mathcal{E}_{\mathrm{ME}} = S_{1/2,A_1}^{(\mathrm{eq})}(\beta_A) \quad \text{or} \quad \mathcal{E}_{\mathrm{ME}} = S_{1/2,A_2}^{(\mathrm{eq})}(\beta_A), \tag{19}$$

$$\mathcal{E}_{\mathrm{ES}} = \frac{1}{2}[S_{1/2,A}^{(\mathrm{eq})}(\beta_A) - S_{2,B}^{(\mathrm{eq})}(\beta_B)], \tag{20}$$

generalizing Eqs. (13)–(15). Comparison of $\mathcal{E}_{\mathrm{ES}}$ and $\mathcal{E}_{\mathrm{NE}}$ suggests that the transition between the NE and ES phases is given by the condition

$$S_{1/2,A}^{(\mathrm{eq})} = S_{2,B}^{(\mathrm{eq})}. \tag{21}$$

Because S_n monotonically decreases with n, we have $S_{1/2,A}^{(\mathrm{eq})} > S_A^{(\mathrm{eq})}$ and $S_{2,B}^{(\mathrm{eq})} < S_B^{(\mathrm{eq})}$, so the transition must happen for $S_A^{(\mathrm{eq})} < S_B^{(\mathrm{eq})}$, i.e., at some $c_0 < \frac{1}{2}$, so that there is a region in the phase diagram where the logarithmic negativity is extensive but the mutual information is subextensive. In the case where A is at infinite temperature while B is at finite temperature, we found the exact resolvent in the NE and ES phases and confirmed Eqs. (20) and (21). A special example is the toy model of black hole evaporation in Jackiw-Teitelboim (JT) gravity discussed in Ref. [16], where

$$S_{2,B}^{(\mathrm{eq})} = S_0 + s_2(\beta), \qquad S_B^{(\mathrm{eq})} = S_0 + s_1(\beta),$$
$$s_1(\beta) > s_2(\beta) \sim O(1), \tag{22}$$

for which we have $c_0 = \frac{1}{2}\{1 - [(s_1 - s_2)/2S_0]\}$. In this model, due to the special structure of the density of states for JT gravity, the difference between c_0 and c turns out to be subleading in S_0 unless one takes β to scale with S_0; for higher-dimensional gravity systems, this difference is always $O(1)$. The negativity phase diagram for this model is also discussed in Refs. [27,28].

Suppose A equilibrates to the microcanonical ensemble at fixed energy density, while B equilibrates to infinite temperature

$$\mathcal{I}_\alpha = \sum_{E_{a_1}^{A_1} + E_{a_2}^{A_2} \in I_{E,\Delta}} |a_1\rangle\langle a_1| \otimes |a_2\rangle\langle a_2| \otimes \mathbf{1}_B, \tag{23}$$

where $E_{a_s}^{A_s}$ refer to energies in A_s and $I_{E,\Delta}$ is a narrow energy interval $[E - \Delta, E + \Delta]$. From Eq. (8) and analytic continuation in n, we find

$$\mathcal{E}_{\mathrm{NE}} = 0, \tag{24}$$

$$\mathcal{E}_{\mathrm{ME}} = S_{1/2,A_1}^{(\mathrm{eq})} \quad \text{or} \quad \mathcal{E}_{\mathrm{ME}} = S_{1/2,A_2}^{(\mathrm{eq})}, \tag{25}$$

$$\mathcal{E}_{\mathrm{ES}} = \frac{1}{2}(S_{1/2,A_1}^{(\mathrm{eq})} + S_{1/2,A_2}^{(\mathrm{eq})} - \log d_B), \tag{26}$$

which we confirm using a resolvent calculation. The value $c_0 < \frac{1}{2}$ for the transition between the ES and NE phases is again determined by the condition $\mathcal{E}_{\mathrm{ES}} = 0$.

Next, consider an example where the system A_2 is taken to be at infinite temperature, while $A_1 B$ equilibrates to the microcanonical ensemble as fixed energy density

$$\mathcal{I}_\alpha = \mathbf{1}_{A_2} \otimes \sum_{E_p^{A_1} + E_r^B \in I_{E,\Delta}} (|p\rangle\langle p|)_{A_1} \otimes (|r\rangle\langle r|)_B. \tag{27}$$

We then find that

$$\mathcal{E}_{\mathrm{NE}} = 0, \tag{28}$$

$$\mathcal{E}_{\mathrm{ME}} = S_{1/2,A_1}^{(\mathrm{eq})} \quad \text{or} \quad \mathcal{E}_{\mathrm{ME}} = \log d_{A_2}, \tag{29}$$

$$\mathcal{E}_{\mathrm{ES}} = \frac{1}{2}\log d_{A_2} + S_{1/3,A_1}^{(\mathrm{eq})} - \frac{1}{2}S_{A_1}^{(\mathrm{eq})} - \frac{1}{2}S_B^{(\mathrm{eq})}. \tag{30}$$

The corresponding c_0 is $< \frac{1}{2}$. since $S_{1/3,A_1}^{(\mathrm{eq})} > S_{A_1}^{(\mathrm{eq})}$. In this case, it is possible to use the resolvent to find the phase diagram for all regimes of c and λ, as shown in Fig. 3. In addition to finite-temperature generalizations of the NE, ES, and ME phases, there are also two new phases (ES-ME1 and ES-ME2) that cannot be obtained by analytic continuation of $\mathcal{Z}_n^{(\mathrm{PT})}$. While there is a discontinuity in first derivative $(\partial\mathcal{E}/\partial c)$ in going from the NE phase to the ES or ES-ME2 phases, for the remaining phase transitions, there is a discontinuity only in the second derivative $(\partial^2\mathcal{E}/\partial c^2)$. One feature of the phase diagram is that the logarithmic negativity and the mutual information appear to reflect different physics; in general, at the phase boundaries of the mutual information, there is no qualitative change in the logarithmic negativity and vice versa.

Let us now discuss the operational implications of the above results. Because the distillable entanglement is upper bounded by half of the mutual information [20] while the

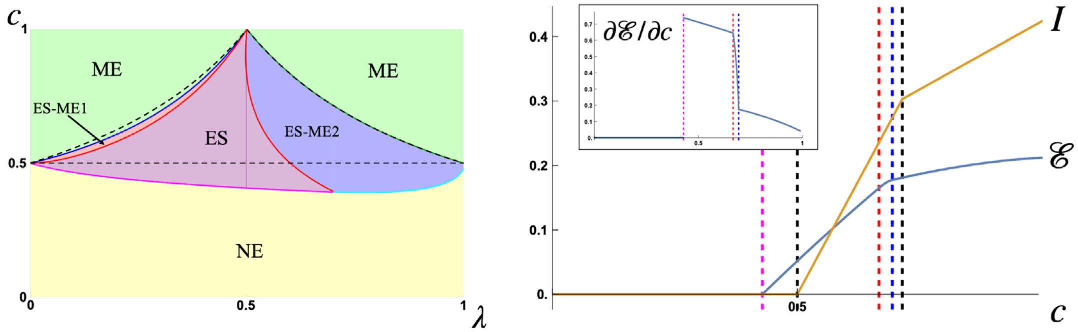

FIG. 3. Left: the phases of the logarithmic negativity for Eq. (27) shown with different colored regions. The phase transition lines for the mutual information are shown with dashed black lines. The magenta line between NE and ES agrees with setting $\mathcal{E}_{\mathrm{ES}}$ in Eq. (30) to zero. Note that the precise shape of the phase transition lines for $\mathcal{E}$ depends on the average energy density ϵ and the form of the entropy density $s(\epsilon)$ for the system; here, we took $s(\epsilon) = \sqrt{\epsilon}$ and $\epsilon = 0.5$. Right: $\mathcal{E}$ and I along a vertical line at $\lambda = 0.3$ in the phase diagram, and the inset shows $(\partial\mathcal{E}/\partial c)$ along the same line, with the phase transitions indicated by vertical grid lines.

exact PPT entanglement cost is lower bounded by the logarithmic negativity [19], we can draw the operational conclusion that the exact entanglement cost is much greater than the distillable entanglement:

$$E_c^{(\mathrm{exact})} \gg E_d^{(\mathrm{exact})}. \tag{31}$$

That is, the system has significant bound entanglement.

Implications for evaporating black holes.—For evaporating black holes, A is the radiation and B is the black hole. In Ref. [13], Page found that the entanglement entropy of the radiation (treated as a random state) undergoes a transition from increasing to decreasing behavior at some timescale $t = t_p$, where $\log d_A = \log d_B$, and is also the time at which entanglement within the radiation starts to become extensive.

The natural finite-temperature generalization of the Page time is when $S_A^{(\mathrm{eq})} = S_B^{(\mathrm{eq})}$ ($c = \frac{1}{2}$), which we confirmed from the equilibrium approximation. However, our results for logarithmic negativity give the surprising prediction that there are significant entanglement correlations within the radiation long before the Page time. This statement is universal at finite temperature and does not depend on the specific ensemble or model chosen. (For example, in the model of black hole evaporation in Ref. [16], the correct ensemble is system B in the canonical ensemble at inverse temperature β with A_1 and A_2 at infinite temperature. In contrast, for black hole evaporation processes that conserve the total energy, it may be more suitable to choose the microcanonical ensemble for the joint $A_1 \cup A_2 \cup B$ system.) This suggests the existence of another timescale t_b when quantum entanglement within the radiation starts becoming extensive. For the canonical ensemble, this is the timescale defined by Eq. (21), while in the microcanonical ensemble, depending on the choice of subsystems with energy conservation, it is given by the time at which Eq. (26) or (30) becomes positive. t_b may then be explicitly written in terms of the black hole and radiation parameters by evaluating the corresponding thermal Rényi entropies using the density of states for the system. For timescales $t \in (t_b, t_p)$, the entanglement correlations within the radiation appear to be bound entanglement. An important outstanding issue in quantum-information theory is whether there exist bound-entangled states with negative partial transpose [29]. Our results suggest a promising avenue to address this problem, though it remains to be seen whether there is a subextensive amount of distillable entanglement in these thermalized states.

It is also instructive to consider the Hayden-Preskill thought experiment at finite temperature. Suppose we throw a diary into the black hole and see when the information of the diary is recoverable from the radiation. Applying the equilibrium approximation to the Petz recovery map, it can be shown that the information of the diary can be recovered from the radiation with $O(1)$ fidelity only after t_p [18]. This can be viewed as giving an operational definition of the Page time. Surprisingly, for diaries with large entropy, the fidelity of recovery first begins to appreciably grow from its minimal value at t_b [18], suggesting an operational definition of this new timescale. Because the correlations within the radiation grow starting at t_b, we expect that this timescale may also be important for other observables.

In Euclidean gravity setups, the calculation of the negativity between parts of the radiation in the ES and ME phases involves replica wormholes which is shown explicitly in Supplemental Material [21]. It would be interesting to understand whether it is possible to obtain a Lorentzian derivation of the nonzero negativity before the Page time and see whether there is any semiclassical or geometric description of bound entanglement. It would also be interesting to understand the new phases which do not correspond to simple analytic continuations in a gravity calculation.

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