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Exact multi-point correlations in the stochastic heat equation for strictly sublinear coordinates

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Abstract

We consider the Stochastic Heat Equation (SHE) in $(1+1)$ dimensions with delta Dirac initial data and spacetime white noise. We prove exact large-time asymptotics for multi-point correlations of the SHE for strictly sublinear space coordinates. The sublinear condition is optimal, in the sense that different asymptotics are known to occur when the space coordinates grow linearly [Lin23, Theorem 1.1]. A notable feature of our result is that the dependence on space coordinates of the SHE's asymptotic multi-point correlations is given by the ground state of the delta-Bose gas.

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1 Introduction

1.1 Background

In this note, we study the Stochastic Heat Equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \xi Z, \qquad Z = Z(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{1.1}$$

starting from Dirac delta initial data $Z(0,\cdot) = \delta(\cdot)$, where $\xi = \xi(t,x)$ is a spacetime white noise. Informally, ξ is defined as the centered Gaussian process with covariance

$$\mathbb{E}[\xi(s,x)\xi(t,y)] = \delta(t-s)\delta(y-x).$$

See, e.g., [Qua12, Sections 2.1–2.6] for a survey of the solution theory of this object. The SHE is an object of fundamental importance in stochastic analysis and mathematical physics due to its connections with various physical models, such as random polymers and the KPZ equation; we refer to [Cor12, Qua12] for a detailed exposition of these (and more) connections. In this note, we are interested in the occurrence of intermittency in $Z(t,\cdot)$ for large t. Informally, intermittency refers to the observation that, as $t \to \infty$, the SHE's solution tends to concentrate most of its mass in tall and narrow peaks separated by deep valleys.

The rigorous study of intermittency in the SHE began with the pair of articles [BC95, BG99], both of which followed the methodology to prove intermittency outlined

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in [Mol91, Page 229]: On the one hand, in [BC95, (2.40)] the authors obtained an explicit expression for the n-moment Lyapunov exponents

$$\mathfrak{L}_n := \lim_{t\to\infty} \frac{\log \mathbb{E}[Z(t,x)^n]}{t}$$

for all $n \in \mathbb{Z}_{\geq 1}$ under the assumption of a constant initial condition $Z(0,\cdot) = c > 0$; namely

$$\mathfrak{L}_n = \frac{n(n^2-1)}{24}. \tag{1.2}$$

Thanks to a simple ergodic theorem/Markov's inequality argument outlined in [BC95, (2.28)–(2.38)], the fact that the moment Lyapunov exponents in (1.2) satisfy $\mathfrak{L}_1 < \mathfrak{L}_2/2 < \mathfrak{L}_3/3 < \cdots$ implies that for any $\alpha > 0$, there exist small islands (which occupy an exponentially small proportion of space that can be quantified using the exponents $\mathfrak{L}_n$) on which $Z(t,\cdot)$ exceeds $e^{\alpha t}$.

On the other hand, in [BG99, Theorem 1.1], the authors show that for every smooth and compactly supported $\phi : \mathbb{R} \to \mathbb{R}$, one has

$$\lim_{t\to\infty} \frac{1}{t} \int_{-\infty}^{\infty} \big(\log Z(t,x)\big)\phi(x)\, dx = -\frac{1}{24}\int_{-\infty}^{\infty} \phi(x)\, dx \qquad \text{in } L^2, \tag{1.3}$$

under the assumption that $Z(0,x) = e^{B(x)}$ where B is a two-sided Brownian motion independent of ξ. Later, (1.3) was proved for more general initial data and $\phi = \delta(\cdot)$, see for example [ACQ11]. When combined with the moment Lyapunov exponents, the additional insight provided by the sample Lyapunov exponent in (1.3) is that the intermittent peaks only persist for a finite amount of time (since the SHE's solution decays exponentially in time). This latter observation is in stark contrast with the intermittency phenomenon observed in the parabolic Anderson model with time-independent noises; see, e.g., the monograph [Kön16].

In recent years, the results in [BC95, BG99] were improved and extended in myriad ways. For instance, [DT21, GL23] generalized (1.2) to all $n > 0$ and an extensive class of initial conditions. For the specific purposes of this paper, one important recent development came from [Che15]. More specifically, as shown in [Che15, (3.2) and (4.1)], the Lyapunov exponents of the SHE for integer powers admit the following variational interpretation:

$$\mathfrak{L}_n = -\inf_{f\in\mathcal{F}_n} \left\{ \frac{1}{2}\int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx - \sum_{1\leq i<j\leq n} \int_{\mathbb{R}^n} \delta(x_i - x_j) f(x)^2\, dx \right\}, \qquad n \in \mathbb{Z}_{\geq 1}, \tag{1.4}$$

where $x = (x_1, \ldots, x_n)$, and $\mathcal{F}_n$ denotes the space of smooth rapidly-decreasing functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_2 = 1$. When combined with (1.2), the equality in (1.4) provided a rigorous proof of the fact that the ground state energy of the Schrödinger operator on $\mathbb{R}^n$ defined as

$$H_n := -\tfrac{1}{2}\Delta - \sum_{1\leq i<j\leq n} \delta(x_i - x_j)$$

is equal to $-\mathfrak{L}_n = -\frac{n(n^2-1)}{24}$. This confirmed physical predictions made in [Kar87, LL63], which relied on the computation of the following "ground state[1]" for H_n using the Bethe ansatz:

$$\Psi_n(x_1, \ldots, x_n) := \exp\left(\sum_{1\leq i<j\leq n} -\frac{|x_i - x_j|}{2} \right). \tag{1.5}$$

[1] Calling Ψ_n a ground state is an abuse of terminology, since although one can convincingly argue that $H_n\Psi_n = \frac{n(n^2-1)}{24}\Psi_n$, the norm $\|\Psi_n\|_2$ is infinite.

1.2 Main result

In this note, we are interested in furthering the insights on the finer details of the geometry of intermittent peaks hinted at by (1.4) and (1.5). More specifically, the proof of (1.4) in [Che15, Section 4] (most notably, its connection with Schrödinger semigroup theory [Che10, Theorem 4.1.6]) strongly suggests the following informal principle: The atypical configurations in $x \mapsto Z(t,x)$ that provide the main contributions to $\mathbb{E}[Z(t,x)^n]$'s size as $t \to \infty$ should be closely related to the ground state Ψ_n.

Further evidence of this connection is provided by the fact that the multi-point correlation functions of the SHE, which we define as

$$u_n(t,x) = \mathbb{E}\left[\prod_{i=1}^n Z(t,x_i)\right], \qquad x = (x_1,\ldots,x_n) \in \mathbb{R}^n,$$

solve the delta-Bose gas PDEs

$$\partial_t u(t,x) = -H_n u(t,x), \qquad u(0,\cdot) = \delta(\cdot);$$

see [BC14a, Proposition 5.4.8] and [Nic21, Corollary 1.7], and also [BC14a, Proposition 6.2.3]. An informal spectral expansion based on this fact suggests that if t is large and $x_1,\ldots,x_n$ are all much smaller than t, then

$$u_n(t,x) \approx e^{t\mathfrak{L}_n}\Psi_n(x). \tag{1.6}$$

Remark 1.1. It is natural to expect that (1.6) should only hold when the x_i's are much smaller than t, since otherwise we see from (1.5) that the product $e^{t\mathfrak{L}_n}\Psi_n(x)$ could be of order $o(e^{t\mathfrak{L}_n})$. In such a case, one expects that the leading order asymptotics of $u_n(t,x)$ are not only explained by H_n's ground state.

In this context, our main result formalizes these heuristics as follows:

Theorem 1.2. *Let Z be as in (1.1) with $Z(0,\cdot) = \delta(\cdot)$. Let $n \in \mathbb{Z}_{\geq 1}$, and for every $1 \leq i \leq n$, let $\big(x_i(t)\big)_{t\geq 0}$ be a strictly sublinear sequence of real numbers, in the sense that $x_i(t) = o(t)$ as $t \to \infty$. It holds that*

$$\mathbb{E}\left[\prod_{i=1}^n Z\big(t,x_i(t)\big)\right] = \frac{(n-1)!\sqrt{2\pi}}{\sqrt{nt}} \exp\left(\mathfrak{L}_n t\right) \Psi_n\big(x_1(t),\ldots,x_n(t)\big)\big(1+o(1)\big) \qquad \text{as } t \to \infty, \tag{1.7}$$

where $\mathfrak{L}_n$ is the Lyapunov exponent in (1.2) and Ψ_n is the ground state in (1.5).

Following-up on Remark 1.1, one can guarantee that the sublinearity assumptions on $x_i(t)$ in Theorem 1.2 are optimal thanks to [Lin23, Theorem 1.1]. Therein, exact asymptotics are derived for multi-point lyapunov exponents of the form

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\prod_{i=1}^n Z(t,tx_i)\right], \qquad x_1,\ldots,x_n \in \mathbb{R}.$$

An examination of this result reveals that the scaling of t in the space coordinates x_i induce a rather different limiting behavior that does not involve Ψ_n, which is explained by an altogether different mechanism from the asymptotic in Theorem 1.2 (i.e., (1.6)).

Remark 1.3. While the presence of $\mathfrak{L}_n$ and Ψ_n in (1.7) can easily be explained conceptually thanks to (1.6), the interpretation of $\frac{(n-1)!\sqrt{2\pi}}{\sqrt{nt}}$ is seemingly less obvious. It would be interesting to investigate whether the presence of this term can be explained in terms of the geometry of intermittent peaks in the SHE; we leave this question open, as our method of proof (i.e., exact asymptotics of contour integral formulas; see Section 2 for the details) does not appear to shed light on this issue.

Finally, we note that the use of asymptotic spatial correlations to investigate finer details of the geometry of intermittency (particularly the optimizers of the variational problems that arise in the moment asymptotics) is not new to this paper. Notably, the work [GdH99] proved a similar result for the discrete parabolic Anderson model with time-independent noises.

2 Outline of proof

In this section, we provide a bird's-eye view of the proof of Theorem 1.2; the proofs of several technical propositions stated here are provided in later sections. For this purpose, going forward, we assume that $n \in \mathbb{Z}_{\geq 1}$ and $x_1(t), \ldots, x_n(t) = o(t)$ are fixed.

The first and main technical ingredient in our proof consists of a contour integral representation for the mixed moment $\mathbb{E}\big[\prod_{i=1}^n Z\big(t, x_i(t)\big)\big]$, which allows to conveniently isolate the leading order contribution of the latter. Before we state this result, we need to introduce some notations:

Definition 2.1. *We say that a vector λ is a partition of n, denoted by $\lambda \vdash n$, if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \geq 1$, $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{\geq 1}$, $\lambda_1 \geq \cdots \geq \lambda_\ell$, and $\sum_{k=1}^{\ell} \lambda_k = n$.*

Given $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, let $\ell(\lambda) := \ell$ denote the length of the partition, and define the combinatorial constant $m(\lambda) := m_1! m_2! \cdots m_n!$, where for every $1 \leq i \leq n$, m_i is the number of times that i appears in λ. Given a complex vector $\vec{w} = (w_1, \ldots, w_{\ell(\lambda)}) \in \mathbb{C}^{\ell(\lambda)}$, we denote

$$\vec{w} \circ \lambda := (w_1, w_1+1, \ldots, w_1+\lambda_1-1, w_2, w_2+1, \ldots, w_2+\lambda_2-1, \ldots, w_{\ell(\lambda)}, \ldots, w_{\ell(\lambda)}+\lambda_{\ell(\lambda)}-1). \tag{2.1}$$

Finally, let S_n denote the set of permutations of $\{1, \ldots, n\}$.

Definition 2.2. *For every $t \geq 0$, let $x_{(1)}(t), \ldots, x_{(n)}(t)$ denote the coordinates $x_i(t)$ ordered in nondecreasing order, i.e., $x_{(1)}(t) \leq x_{(2)}(t) \leq \cdots \leq x_{(n)}(t)$. For every $t > 0$, define the complex function $E_t : \mathbb{C}^n \to \mathbb{C}$ as*

$$E_t(z_1, \ldots, z_n) := \sum_{\sigma \in S_n} \prod_{1 \leq B < A \leq n} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}} \exp\left(\sum_{i=1}^n \left(\frac{t}{2} z_{\sigma(i)}^2 + x_{(i)}(t) z_{\sigma(i)}\right)\right). \tag{2.2}$$

Then, for every $\lambda \vdash n$, define

$$\nu_\lambda(t) := \oint_\gamma^{\otimes \ell(\lambda)} \frac{1}{m(\lambda)} \det\left[\frac{1}{w_i + \lambda_i - w_j}\right]_{i,j=1}^{\ell(\lambda)} E_t(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi \mathbf{i}}, \tag{2.3}$$

where the contour γ is given by $\theta + \mathbf{i}\mathbb{R}$ with arbitrarily fixed $\theta \in \mathbb{R}$, and $\oint_\gamma^{\otimes \ell(\lambda)}$ denotes an $\ell(\lambda)$-fold contour integral on the same contour γ.

Our main technical result is as follows:

Proposition 2.3. *It holds that*

$$\mathbb{E}\left[\prod_{i=1}^n Z\big(t, x_i(t)\big)\right] = \sum_{\lambda \vdash n} \nu_\lambda(t). \tag{2.4}$$

Proposition 2.3 is proved in Section 3.

Remark 2.4. The use of contour integral formulas to study moment asymptotics of the SHE (and other models) is not new to this paper; see. e.g., [Dot10]. In the mathematics literature, [BC14b, CG20] also used the same idea to investigate the one-point moment with a fixed space coordinate (i.e., the setting $x_1(t) = \cdots = x_n(t) = x$). In this scenario, the expression (2.2) in the contour integral formula is significantly simplified due to the symmetry with respect to the components z_k; see, e.g., [BC14a, Proposition 6.2.7].

With this in hand, Theorem 1.2 relies on noting that the asymptotic contribution of the one-element partition $\lambda = (n)$ to the sum (2.4) gives the leading order term in (1.7), and that all the other summands grow at a slower rate:

Proposition 2.5. *As $t \to \infty$, it holds that*

$$\nu_{(n)}(t) = \frac{(n-1)!\sqrt{2\pi}}{\sqrt{nt}} \exp\left(\mathfrak{L}_n t\right) \Psi_n\big(x_1(t), \ldots, x_n(t)\big)\big(1 + o(1)\big).$$

Proposition 2.6. *For every $\lambda \vdash n$ such that $\lambda \neq (n)$, one has*

$$v_\lambda(t) = o\big(v_{(n)}(t)\big) \qquad \text{as } t \to \infty.$$

Remark 2.7. Conceptually, what Proposition 2.3 does is decompose the contour integral in Lemma 3.1 by extracting the various contributions of residues in the integrand. Proposition 2.5 and Proposition 2.6 determine the major contribution of the residues in the asymptotic limit.

Theorem 1.2 then readily follows from (2.4).

3 Contour integral representation - Proof of Proposition 2.3

The starting point of the proof of Proposition 2.3 is the following lemma, which is stated in [BC14a, Proposition 6.2.3] and can be proved by combining [BC14a, Proposition 5.4.8] and [Nic21, Corollary 1.7]:

Lemma 3.1. *It holds that*

$$\mathbb{E}\left[\prod_{i=1}^n Z\big(t, x_i(t)\big)\right] = \oint_{\gamma_1} \cdots \oint_{\gamma_n} \left(\prod_{1\le i<j\le n} \frac{z_i - z_j}{z_i - z_j - 1}\right)\left(\prod_{k=1}^n e^{\frac{t}{2}z_k^2 + x_{(k)}(t) z_k}\right)\prod_{i=1}^n \frac{dz_i}{2\pi\mathbf{i}},$$

where the contour of z_k is given by $\gamma_k = a_k + \mathbf{i}\mathbb{R}$ for some real numbers $a_1, \ldots, a_n$ that can be arbitrary as long as they satisfy $a_j - a_{j+1} > 1$ for $j = 1, \ldots, n-1$.

With this in hand, Proposition 2.3 is an immediate consequence of the following lemma, which is inspired by a similar result that was proved in [Cor18, Theorem 7.7] (see also [BC14a, Proposition 3.2.1] for a full proof of [Cor18, Theorem 7.7] in the special case of a symmetric function):

Lemma 3.2. *Let $N \in \mathbb{Z}_{\ge 1}$ be arbitrary, and let $\gamma_1, \ldots, \gamma_N$ be contours of the form $\gamma_k = a_k + \mathbf{i}\mathbb{R}$ for some $a_k \in \mathbb{R}$ such that $a_j - a_{j+1} > 1$. If $F : \mathbb{C}^N \to \mathbb{C}$ is analytic between the contours γ_k, then*

$$\oint_{\gamma_1} \cdots \oint_{\gamma_N} \left(\prod_{1\le i<j\le N} \frac{z_i - z_j}{z_i - z_j - 1}\right) F(z_1, \ldots, z_N) \prod_{i=1}^N \frac{dz_i}{2\pi\mathbf{i}}$$
$$= \sum_{\lambda \vdash N} \oint_{\gamma}^{\otimes \ell(\lambda)} \frac{1}{m(\lambda)} \det\left[\frac{1}{w_i + \lambda_i - w_j}\right]_{i,j=1}^{\ell(\lambda)} E^F(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi\mathbf{i}},$$

where the contour γ is given by $\theta + \mathbf{i}\mathbb{R}$ with arbitrarily fixed $\theta \in \mathbb{R}$, and we denote

$$E^F(z_1, \ldots, z_N) := \sum_{\sigma \in S_N} \left(\prod_{1\le B<A\le N} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}}\right) F\big(z_{\sigma(1)}, \ldots, z_{\sigma(N)}\big).$$

Proof of Lemma 3.2. Let

$$\mu_N^F := \oint_{\gamma_1} \cdots \oint_{\gamma_N} \left(\prod_{1 \le i < j \le N} \frac{z_i - z_j}{z_i - z_j - 1} \right) F(z_1, \dots, z_N) \frac{dz_1}{2\pi \mathbf{i}} \cdots \frac{dz_N}{2\pi \mathbf{i}}, \tag{3.1}$$

and

$$\nu_\lambda^F := \oint_{\gamma_N}^{\otimes \ell(\lambda)} \frac{1}{m(\lambda)} \det \left[\frac{1}{w_i + \lambda_i - w_j} \right]_{i,j=1}^{\ell(\lambda)} E^F(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi \mathbf{i}}, \tag{3.2}$$

where we can take $\gamma_N = \gamma$. Therefore, it suffices to show that

$$\mu_N^F = \sum_{\lambda \vdash N} \nu_\lambda^F.$$

We apply induction argument. It is clear that the desired equality holds when $N = 1$. Therefore, we only need to prove for every $n \in \mathbb{Z}_{\ge 1}$ that

$$\mu_{n+1}^F = \sum_{\lambda \vdash n+1} \nu_\lambda^F \tag{3.3}$$

knowing that the induction hypothesis holds, i.e.

$$\mu_n^F = \sum_{\lambda \vdash n} \nu_\lambda^F. \tag{3.4}$$

To prove (3.4), we break up our proof into the following steps.

Step 1. We first mention that we can shift the contours as long as we do not cross the poles in the integrand. Such shifting of a contour from a vertical line $a + \mathbf{i}\mathbb{R}$ to $b + \mathbf{i}\mathbb{R}$ is guaranteed by choosing the rectangular contour

$$[a - \mathbf{i}R, a + \mathbf{i}R] \cup [a + \mathbf{i}R, b + \mathbf{i}R] \cup [b + \mathbf{i}R, b - \mathbf{i}R] \cup [b - \mathbf{i}R, a - \mathbf{i}R].$$

The contributions of $[a + \mathbf{i}R, b + \mathbf{i}R]$ and $[b - \mathbf{i}R, a - \mathbf{i}R]$ will vanish as we let $R \to \infty$, thanks to the quadratic term in the exponent of the integrand.

Let $G_{z_1}(z_2, \dots, z_{n+1}) = F(z_1, \dots, z_{n+1})$ and $g_{z_1}(z_2, \dots, z_{n+1}) := \prod_{j=2}^{n+1} \frac{z_1 - z_j}{z_1 - z_j - 1}$. By (3.1), it is straightforward to see that

$$\mu_{n+1}^F = \oint_{\gamma_1} \frac{dz_1}{2\pi \mathbf{i}} \oint_{\gamma_2} \cdots \oint_{\gamma_{n+1}} \prod_{2 \le i < j \le n+1} \frac{z_i - z_j}{z_i - z_j - 1} (g_{z_1} G_{z_1})(z_2, \dots, z_{n+1}) \prod_{i=2}^{n+1} \frac{dz_i}{2\pi \mathbf{i}}.$$

Applying (3.4) to the contour integral $\oint_{\gamma_2} \dots \oint_{\gamma_{n+1}}$, and then using the fact that g_{z_1} is symmetric to factor $E^{g_{z_1} G_{z_1}} = g_{z_1} E^{G_{z_1}}$, we obtain

$$\mu_{n+1}^F = \sum_{\lambda \vdash n} \oint_{\gamma_1} \left(\prod_{i=1}^{\ell(\lambda)} \frac{z_1 - w_i}{z_1 - w_i - \lambda_i} \right) \frac{dz_1}{2\pi \mathbf{i}} \times \oint_{\gamma_{n+1}}^{\otimes \ell(\lambda)} \frac{1}{m(\lambda)} \det \left[\frac{1}{w_i + \lambda_i - w_j} \right]_{i,j=1}^{\ell(\lambda)} E^{G_{z_1}}(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi \mathbf{i}}.$$

Fix λ and $w_1, \dots, w_{\ell(\lambda)}$ and deform the contour of z_1 from γ_1 to γ_{n+1}. Doing so will cross simple poles at $\{w_i + \lambda_i\}_i$. By computing these residues, we get $U_{\lambda,1}, \dots, U_{\lambda,\ell(\lambda)}$ with

$$U_{\lambda,k} := \oint_{\gamma_{n+1}}^{\otimes \ell(\lambda)} \left(\prod_{i=1, i \ne k}^{\ell(\lambda)} \frac{w_k + \lambda_k - w_i}{w_k + \lambda_k - w_i - \lambda_i} \right) \frac{\lambda_k}{m(\lambda)} \times \det \left[\frac{1}{w_i + \lambda_i - w_j} \right]_{i,j=1}^{\ell(\lambda)} E^{G_{w_k + \lambda_k}}(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi \mathbf{i}}. \tag{3.5}$$

When γ_1 arrives at γ_{n+1} we let $z_1 = w_{\ell(\lambda)+1}$ and obtain the contribution

$$U_{\lambda,\ell(\lambda)+1} := \oint_{\gamma_{n+1}}^{\otimes \ell(\lambda)+1} \left(\prod_{i=1}^{\ell(\lambda)} \frac{w_{\ell(\lambda)+1} - w_i}{w_{\ell(\lambda)+1} - w_i - \lambda_i} \right) \times \frac{1}{m(\lambda)} \det \left[\frac{1}{w_i + \lambda_i - w_j} \right]_{i,j=1}^{\ell(\lambda)} E^{G_{w_{\ell(\lambda)+1}}}(\vec{w} \circ \lambda) \prod_{i=1}^{\ell(\lambda)+1} \frac{dw_i}{2\pi \mathbf{i}}. \quad (3.6)$$

The above argument gives the identity

$$\mu_{n+1}^F = \sum_{\lambda \vdash n} \sum_{k=1}^{\ell(\lambda)+1} U_{\lambda,k}. \quad (3.7)$$

Step 2. We seek to rewrite the right hand side of (3.7) into the right-hand side of (3.3). Moreover, when $\lambda_i = \lambda_j$, we can obtain the integrand in $U_{\lambda,i}$ from that in $U_{\lambda,j}$ by swapping the variables w_i and w_j. Since the contours for w_i and w_j are the same, we know that $U_{\lambda,i} = U_{\lambda,j}$. At this point, it is convenient for the purpose of this proof to use an alternate notation for partitions of n and $n+1$, which is as follows:

Notation 3.3. *We write any $\lambda \vdash n$ as $\lambda = a_1^{m_{a_1}} \dots a_s^{m_{a_s}}$ where $a_1 > \dots > a_s \geq 1$ and m_{a_i} denotes the number of times that a_i appears in λ. For any $\bar{\lambda} \vdash n+1$, we instead write $\bar{\lambda} = b_1^{m_{b_1}} \dots b_r^{m_{b_r}}$ where m_{b_i} has a similar meaning. Furthermore, we let $M_{\lambda,k} = \sum_{i=1}^k m_{a_i}$ and $M_{\bar{\lambda},k} = \sum_{j=1}^k m_{b_j}$.*

Using this notation, we get that

$$\sum_{\lambda \vdash n} \sum_{k=1}^{\ell(\lambda)+1} U_{\lambda,k} = \sum_{\lambda \vdash n} \left(\left(\sum_{i=1}^{s} m_{a_i} U_{\lambda, M_{\lambda,i-1}+1} \right) + U_{\lambda,\ell(\lambda)+1} \right). \quad (3.8)$$

Let us define a bijection f from $\{(\lambda, i) : \lambda \vdash n, i \in \{1, \dots, s+1\}\}$ to $\{(\bar{\lambda}, j) : \bar{\lambda} \vdash n+1, j \in \{1, \dots, r\}\}$. Given (λ, i), we define $(\bar{\lambda}, j) = f(\lambda, i)$ by the following rule: $\bar{\lambda}$ is defined via adding the $M_{\lambda,i-1}+1$-th component of λ by 1. j is defined to be the unique number satisfying $M_{\bar{\lambda},j} = M_{\lambda,i-1}+1$. For example, if $\lambda = 4^2 3^2 2^1$ and $i = 2$, then $\bar{\lambda} = 4^3 3^1 2^1$ and $j = 1$.

It is clear that f is invertible and $(\lambda, i) = f^{-1}(\bar{\lambda}, j)$ is given by the following rule: λ is obtained from $\bar{\lambda}$ via subtracting the $M_{\bar{\lambda},j}$-th component by 1 and i is defined to be the unique number satisfying $M_{\lambda,i-1}+1 = M_{\bar{\lambda},j}$.

For $k \in \{1, \dots, r\}$, we let $\bar{\lambda}[k]$ be the vector obtained via subtracting the k-th component of $\bar{\lambda}$ by 1. Note that $\bar{\lambda}[k]$ is a partition if and only if $k = M_{\bar{\lambda},j}$ for some j. From the previous paragraph, it is not hard to verify that if $(\bar{\lambda}, j) = f(\lambda, i)$, then

$$m_{a_i} = m_{b_{j+1}} \mathbf{1}_{\{b_j - 1 = b_{j+1}\}} + 1 \text{ when } i \in \{1, \dots, s\}; \quad 1 = m_{b_{j+1}} \mathbf{1}_{\{b_j - 1 = b_{j+1}\}} + 1 \text{ when } i = s+1. \quad (3.9)$$

Moreover, one can verify that $M_{\bar{\lambda},j} = M_{\lambda,i-1}+1$ and $\bar{\lambda}[M_{\bar{\lambda},j}] = \lambda$, which yields

$$U_{\lambda, M_{\lambda,i-1}+1} = U_{\bar{\lambda}[M_{\bar{\lambda},j}], M_{\bar{\lambda},j}}.$$

The above equality and (3.9) imply that

$$\sum_{\lambda \vdash n} \left(\left(\sum_{i=1}^{s} m_{a_i} U_{\lambda, M_{\lambda,i-1}+1} \right) + U_{\lambda,\ell(\lambda)+1} \right) = \sum_{\bar{\lambda} \vdash n+1} \sum_{j=1}^{r} (m_{b_{j+1}} \mathbf{1}_{\{b_j - 1 = b_{j+1}\}} + 1) U_{\bar{\lambda}[M_{\bar{\lambda},j}], M_{\bar{\lambda},j}}.$$

By the above equality and (3.8) and (3.7), we obtain

$$\mu_{n+1}^F = \sum_{\bar{\lambda} \vdash n+1} \sum_{j=1}^{r} (m_{b_{j+1}} \mathbf{1}_{\{b_j - 1 = b_{j+1}\}} + 1) U_{\bar{\lambda}[M_{\bar{\lambda},j}], M_{\bar{\lambda},j}}. \quad (3.10)$$

Step 3. By (3.10), the proof of (3.3) reduces to showing that for all $\bar\lambda \vdash n+1$,

$$\nu^F_{\bar\lambda} = \sum_{k=1}^{r} (m_{b_{k+1}} \mathbf{1}_{\{b_k - 1 = b_{k+1}\}} + 1) U_{\bar\lambda[M_{\bar\lambda,k}], M_{\bar\lambda,k}}. \tag{3.11}$$

The rest of the proof is devoted to proving (3.11). For $j \in \{1, \dots, n+1\}$, we define $\Lambda(k) := \sum_{i=1}^{k} \bar\lambda_i$ and

$$E^F_j(z_1, \dots, z_{n+1}) := \sum_{\sigma \in S_{n+1}, \sigma(1) = j} \left(\prod_{1 \le B < A \le n+1} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}} \right) F\big(z_{\sigma(1)}, \dots, z_{\sigma(n+1)}\big).$$

Note that $E^F = \sum_{j=1}^{n+1} E^F_j$ and $E^F_j(\vec w \circ \bar\lambda)$ is non-zero only if $j \in \{\Lambda(1), \dots, \Lambda(\ell(\bar\lambda))\}$. By (3.2), we know that

$$\nu^F_{\bar\lambda} := \sum_{k=1}^{\ell(\bar\lambda)} \oint_{\gamma_{n+1}}^{\otimes \ell(\bar\lambda)} \frac{1}{m(\bar\lambda)} \det\left[\frac{1}{w_i + \bar\lambda_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda)} E^F_{\Lambda(k)}(\vec w \circ \bar\lambda) \prod_{i=1}^{\ell(\bar\lambda)} \frac{dw_i}{2\pi \mathbf{i}}.$$

For $k \in \{1, \dots, \ell(\bar\lambda)\}$, define

$$f_{\bar\lambda,k}(\vec w) := \prod_{i=1, i\neq k}^{\ell(\bar\lambda)} \frac{w_k + \bar\lambda_k - w_i}{w_k + \bar\lambda_k - w_i - \bar\lambda_i}.$$

It is straightforward to verify that

$$E^F_{\Lambda(k)}(\vec w \circ \bar\lambda) = \bar\lambda_k f_{\bar\lambda,k}(\vec w) E^{G_{w_k + \bar\lambda_k - 1}}(\vec w \circ \bar\lambda[k]).$$

Therefore, we have

$$\nu^F_{\bar\lambda} = \sum_{k=1}^{\ell(\bar\lambda)} \oint_{\gamma_{n+1}}^{\otimes \ell(\bar\lambda)} \frac{\bar\lambda_k}{m(\bar\lambda)} f_{\bar\lambda,k}(\vec w) \det\left[\frac{1}{w_i + \bar\lambda_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda)} E^{G_{w_k + \bar\lambda_k - 1}}(\vec w \circ \bar\lambda[k]) \prod_{i=1}^{\ell(\bar\lambda)} \frac{dw_i}{2\pi \mathbf{i}}.$$

One can see that if $\bar\lambda_i = \bar\lambda_j$ for some $i, j \in \{1, \dots, \ell(\bar\lambda)\}$, then i-th integrand on the right hand side above can be obtained by swapping the variables w_i and w_j in the j-th integrand. As a consequence, the i-th integral in the summation above is equal to the j-th integral. Merging the terms with the same values in the summation, we have

$$\nu^F_{\bar\lambda} = \sum_{k=1}^{r} \oint_{\gamma_{n+1}}^{\otimes \ell(\bar\lambda)} \frac{m_{b_k}}{m(\bar\lambda)} \bar\lambda_{M_{\bar\lambda,k}} f_{\bar\lambda, M_{\bar\lambda,k}}(\vec w)$$
$$\det\det\left[\frac{1}{w_i + \bar\lambda_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda)} E^{G_{w_{M_{\bar\lambda,k}} + \bar\lambda_{M_{\bar\lambda,k}} - 1}}(\vec w \circ \bar\lambda[M_{\bar\lambda,k}]) \prod_{i=1}^{\ell(\bar\lambda)} \frac{dw_i}{2\pi \mathbf{i}}. \tag{3.12}$$

It is standard to check that

$$\frac{m_{b_k}}{m(\bar\lambda)} = \frac{m_{b_{k+1}} \mathbf{1}_{\{b_k - 1 = b_{k+1}\}} + 1}{m(\bar\lambda[M_{\bar\lambda,k}])} \text{ and } \bar\lambda_{M_{\bar\lambda,k}} - 1 = \bar\lambda[M_{\bar\lambda,k}]_{M_{\bar\lambda,k}}.$$

This, together with (3.12) imply that

$$\nu^F_{\bar\lambda} = \sum_{k=1}^{r} (m_{b_{k+1}} \mathbf{1}_{\{b_k - 1 = b_{k+1}\}} + 1) \oint_{\gamma_{n+1}}^{\ell(\bar\lambda)} \frac{1}{m(\bar\lambda[M_{\bar\lambda,k}])} \bar\lambda_{M_{\bar\lambda,k}} f_{\bar\lambda, M_{\bar\lambda,k}}(\vec w)$$
$$\times \det\left[\frac{1}{w_i + \bar\lambda_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda)} \times E^{G_{w_k + \bar\lambda[M_{\bar\lambda,k}]_{M_{\bar\lambda,k}}}}(\vec w \circ \bar\lambda[M_{\bar\lambda,k}]) \prod_{i=1}^{\ell(\bar\lambda)} \frac{dw_i}{2\pi \mathbf{i}}. \tag{3.13}$$

To prove (3.11), it suffices to show that for $k \in \{1, \dots, r\}$,

$$\text{The } k\text{-th integral in (3.13)} = U_{\bar\lambda[M_{\bar\lambda,k}],M_{\bar\lambda,k}}.$$

It suffices to match the integrands in the integral on both sides. If $\bar\lambda_{M_{\bar\lambda,k}} > 1$, then $U_{\bar\lambda[M_{\bar\lambda,k}],M_{\bar\lambda,k}}$ takes the form of (3.5). By Cauchy determinant formula, one can verify that

$$\bar\lambda_{M_{\bar\lambda,k}} f_{\bar\lambda,M_{\bar\lambda,k}}(\vec w) \det\left[\frac{1}{w_i + \bar\lambda_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda)} = \bar\lambda[M_{\bar\lambda,k}]_{M_{\bar\lambda,k}} \det\left[\frac{1}{w_i + \bar\lambda[M_{\bar\lambda,k}]_i - w_j}\right]_{i,j=1}^{\ell(\bar\lambda[M_{\bar\lambda,k}])} \times \prod_{i=1, i\neq M_{\bar\lambda,k}}^{\ell(\bar\lambda[M_{\bar\lambda,k}])} \frac{w_{M_{\bar\lambda,k}} + \bar\lambda[M_{\bar\lambda,k}]_{M_{\bar\lambda,k}} - w_i}{w_{M_{\bar\lambda,k}} + \bar\lambda[M_{\bar\lambda,k}]_{M_{\bar\lambda,k}} - w_i - \bar\lambda[M_{\bar\lambda,k}]_i}.$$

Applying this equality to the integrand in (3.13), we see that the integrand in (3.13) is equal to that in $U_{\bar\lambda[M_{\bar\lambda,k}],M_{\bar\lambda,k}}$. If instead $\bar\lambda_{M_{\bar\lambda,k}} = 1$, then $U_{\bar\lambda[M_{\bar\lambda,k}],M_{\bar\lambda,k}}$ takes the form of (3.6), a similar argument concludes the matching of the integrands. □

4 Leading order term - Proof of Proposition 2.5

By definition of $\nu_\lambda(t)$ in (2.3), if we take $\lambda = (n)$ and make the choice $\theta = 0$, then we have that

$$\nu_{(n)}(t) = \oint_{\mathbf{i}\mathbb{R}} \frac{1}{1!} \cdot \frac{1}{n} \cdot E_t(w, w+1, \dots, w+n-1) \frac{dw}{2\pi\mathbf{i}}. \tag{4.1}$$

Recall the definition of E_t in (2.2). If a permutation $\sigma \in S_n$ is such that there exists indices $1 \le \beta < \alpha \le n$ with $\sigma(\alpha) = \sigma(\beta) + 1$, then the product

$$\prod_{1\le B<A\le n} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}}$$

contains the factor

$$z_{\sigma(\alpha)} - z_{\sigma(\beta)} - 1 = z_{\sigma(\beta)+1} - (z_{\sigma(\beta)} + 1).$$

Any such term vanishes if we take

$$(z_1, \dots, z_n) = (w, w+1, \dots, w+n-1),$$

as $z_{k+1} = z_k + 1$ for all $1 \le k \le n-1$, including $k = \sigma(\beta)$. Thus, the only permutations that can contribute to

$$E_t(w, w+1, \dots, w+n-1)$$

in the sum (2.2) are those such that $\sigma(\alpha) \neq \sigma(\beta) + 1$ whenever $\beta < \alpha$. The only permutation that satisfies this property is $\sigma(i) = n+1-i$, $1 \le i \le n$. With this particular choice of permutation, we note that

$$\frac{1}{n} \prod_{1\le B<A\le n} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}} = \frac{1}{n} \prod_{1\le B<A\le n} \frac{z_{n+1-A} - (z_{n+1-B} + 1)}{z_{n+1-A} - z_{n+1-B}},$$

and

$$\exp\left(\sum_{i=1}^n \left(\frac{t}{2} z_{\sigma(i)}^2 + x_{(i)}(t) z_{\sigma(i)}\right)\right) = \exp\left(\sum_{i=1}^n \left(\frac{t}{2} z_{n+1-i}^2 + x_{(i)}(t) z_{n+1-i}\right)\right).$$

With the choice of coordinates $z_i = w + i - 1$, this now becomes

$$\frac{1}{n} \prod_{1\le B<A\le n} \frac{z_{\sigma(A)} - z_{\sigma(B)} - 1}{z_{\sigma(A)} - z_{\sigma(B)}} = \frac{1}{n} \prod_{1\le B<A\le n} \frac{1 + (A-B)}{A-B} = (n-1)!$$

and by expanding and completing the square,

$$\exp\left(\sum_{i=1}^{n}\left(\frac{t}{2}z_{\sigma(i)}^2 + x_{(i)}(t)z_{\sigma(i)}\right)\right) = \exp\left(\sum_{i=1}^{n}\left(\frac{t}{2}(w+n-i)^2 + x_{(i)}(t)\big(w+n-i\big)\right)\right)$$
$$= \exp\left(\frac{n(n^2-1)t}{24} + \sum_{i=1}^{n} x_{(i)}(t)\left(\frac{n+1}{2} - i\right)\right.$$
$$\left. - \frac{(\sum_{i=1}^{n} x_{(i)}(t))^2}{2nt} + \frac{nt}{2}\left(w - \left(\frac{1-n}{2} - \frac{\sum_{i=1}^{n} x_{(i)}(t)}{nt}\right)\right)^2\right).$$

At this point, we note that $\exp(\frac{n(n^2-1)t}{24}) = \exp(\mathfrak{L}_n t)$,

$$\exp\left(\sum_{i=1}^{n} x_{(i)}(t)\left(\frac{n+1}{2} - i\right)\right) = \exp\left(-\sum_{1\le i<j\le n}\frac{x_{(j)}(t) - x_{(i)}(t)}{2}\right) = \Psi\big(x_1(t),\ldots,x_n(t)\big),$$

where we recall the convention $x_{(i)}(t) \le x_{(i+1)}(t)$ for all $1 \le i \le n$ from Definition 2.2, and $-\frac{1}{2nt}(\sum_{i=1}^{n} x_{(i)}(t))^2 = o(t)$ because $x_i(t) = o(t)$. Thus, by computing a straightforward Gaussian integral (with variance $\frac{1}{nt}$), the integral in (4.1) yields

$$\nu_{(n)}(t) = \frac{(n-1)!\sqrt{2\pi}}{\sqrt{nt}}\exp\left(\mathfrak{L}_n t\right)\Psi_n\big(x_1(t),\ldots,x_n(t)\big)\big(1+o(1)\big) \qquad \text{as } t\to\infty$$

as desired.

5 Remainder terms - Proof of Proposition 2.6

In the argument that follows, we use $C, c > 0$ to denote positive constants independent of t (but which may depend on some other parameters, such as n) whose values may change from one display to the next. Moreover, we use $\Re$ and $\Im$ to respectively denote the real and imaginary parts of a complex number.

Let $\lambda \vdash n$ be a fixed partition such that $\lambda \ne (n)$. Our aim is to control the moduli of the functions whose products appear inside the integral (2.3) individually (i.e., the determinant and the multiple products in $E_t(\vec{w}\circ\lambda)$), and thus obtain a result that grows at a slower rate than $\nu_{(n)}(t)$ as $t\to\infty$. In this context, looking back at (2.2), we see that the terms that are the most difficult to control are those that appear in $E_t(\vec{w}\circ\lambda)$ due to the divisions by $z_{\sigma(A)} - z_{\sigma(B)}$. In order to get around this, we deform the $\ell(\lambda)$ contours in (2.3) as follows:

$$\nu_\lambda(t) = \oint_{\gamma_1^\varepsilon}\cdots\oint_{\gamma_{\ell(\lambda)}^\varepsilon}\frac{1}{m(\lambda)}\det\left[\frac{1}{w_i+\lambda_i-w_j}\right]_{i,j=1}^{\ell(\lambda)} E_t(\vec{w}\circ\lambda)\prod_{i=1}^{\ell(\lambda)}\frac{dw_i}{2\pi\mathbf{i}}, \tag{5.1}$$

where $\varepsilon \in (0, \frac{1}{n-1})$ is a fixed constant, $\gamma_1^\varepsilon = \gamma = \theta + \mathbf{i}\mathbb{R}$, and for every $1 < k \le \ell(\lambda)$, we define $\gamma_k^\varepsilon = \theta + (k-1)\varepsilon + \mathbf{i}\mathbb{R}$. We recall that $\theta\in\mathbb{R}$ can be chosen arbitrarily (as per Definition 2.2). Moreover, the restriction $\varepsilon < \frac{1}{n-1}$ ensures that no poles in the determinant are crossed when deforming (2.3) into (5.1) because $\theta + (k-1)\varepsilon < \theta + 1 \le \theta + \lambda_i$ for any choice of λ and $1 \le k, i \le \ell(\lambda)$; otherwise ε can be chosen arbitrarily at this time.

With this in hand, we first note that since $|z^{-1}| \le |\Re(z)|^{-1}$, for every $\theta\in\mathbb{R}$ and $\varepsilon\in(0,\frac{1}{n-1})$, there exists some $C > 0$ such that

$$\sup_{w_1\in\gamma_1^\varepsilon,\ldots,w_{\ell(\lambda)}\in\gamma_{\ell(\lambda)}^\varepsilon}\left|\frac{1}{m(\lambda)}\det\left[\frac{1}{w_i+\lambda_i-w_j}\right]_{i,j=1}^{\ell(\lambda)}\right| \le C$$

and

$$\sup_{w_1\in\gamma_1^\varepsilon,\ldots,w_{\ell(\lambda)}\in\gamma_{\ell(\lambda)}^\varepsilon}\left|\prod_{1\leq B<A\leq n}\frac{z_{\sigma(A)}-z_{\sigma(B)}-1}{z_{\sigma(A)}-z_{\sigma(B)}}\right|\leq C\qquad\text{for all }\sigma\in S_n.$$

Secondly, given that $|e^z|=e^{\Re(z)}\leq e^{|\Re(z)|}$ and $\Re(z^2)=\Re(z)^2-\Im(z)^2$, for any $z\in\mathbb{C}$ and $1\leq i\leq n$,

$$\left|\exp\left(\tfrac{t}{2}z^2+x_{(i)}(t)z\right)\right|=\exp\left(\tfrac{t}{2}\Re(z^2)+x_{(i)}(t)\Re(z)\right)\leq\exp\left(\tfrac{t}{2}\Re(z)^2-\tfrac{t}{2}\Im(z)^2+\mathsf{m}(t)|\Re(z)|\right),$$

where we denote

$$\mathsf{m}(t)=\max\{|x_i(t)|:1\leq i\leq n\}=o(t).$$

Therefore, since the real parts of the components of $\vec{w}\circ\lambda$ in (2.1) are bounded for any fixed θ and ε, there exists some $c>0$ such that $|\Re(z)|\leq c$ for all $z\in\vec{w}\circ\lambda$; hence it suffices to prove that there exists a choice of $\theta\in\mathbb{R}$ and $\varepsilon\in(0,\frac{1}{n-1})$ such that

$$e^{c\mathsf{m}(t)}\oint_{\gamma_1^\varepsilon}\cdots\oint_{\gamma_{\ell(\lambda)}^\varepsilon}\exp\left(\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{t}{2}\Re(w_k+i-1)^2-\frac{t}{2}\Im(w_k+i-1)^2\right)\prod_{i=1}^{\ell(\lambda)}\frac{dw_i}{2\pi\mathbf{i}}=o\big(v_{(n)}(t)\big)$$

as $t\to\infty$. Given that

$$\Re(w_k+i-1)^2=\big(\theta+(k-1)\varepsilon+i-1\big)^2\qquad\text{and}\qquad\Im(w_k+i-1)^2=\Im(w_k)^2,$$

by a Gaussian integral

$$\begin{aligned}e^{c\mathsf{m}(t)}\oint_{\gamma_1^\varepsilon}\cdots\oint_{\gamma_{\ell(\lambda)}^\varepsilon}\exp\left(\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{t}{2}\Re(w_k+i-1)^2-\frac{t}{2}\Im(w_k+i-1)^2\right)\prod_{i=1}^{\ell(\lambda)}\frac{dw_i}{2\pi\mathbf{i}}\\ \leq\frac{C}{\sqrt{t}}\exp\left(c\mathsf{m}(t)+t\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{\big(\theta+(k-1)\varepsilon+i-1\big)^2}{2}\right).\end{aligned}\tag{5.2}$$

Thus, we need only prove that the right-hand side of (5.2) is of order $o\big(v_{(n)}(t)\big)$ as $t\to\infty$ for any $C,c>0$.

Toward this end, we note that since $e^{c\mathsf{m}(t)},\Psi_n\big(x_1(t),\ldots,x_n(t)\big)=e^{o(t)}$, it suffices to find a $\theta\in\mathbb{R}$ and $\varepsilon\in(0,\frac{1}{n-1})$ such that

$$\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{\big(\theta+(k-1)\varepsilon+i-1\big)^2}{2}<\mathfrak{L}_n.$$

By expanding the square above, we get that

$$\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{\big(\theta+(k-1)\varepsilon+i-1\big)^2}{2}\leq\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{\big(\theta+i-1\big)^2}{2}+C\varepsilon$$

for some constant $C>0$ that depends on λ and θ, but is independent of ε. Given that ε can be taken arbitrarily small, it suffices to prove that there exists some $\theta\in\mathbb{R}$ such that

$$\sum_{k=1}^{\ell(\lambda)}\sum_{i=1}^{\lambda_k}\frac{\big(\theta+i-1\big)^2}{2}<\mathfrak{L}_n.$$

By the formula $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$ and the fact that $\sum_{k=1}^{\ell(\lambda)} \lambda_k = n$,

$$\sum_{k=1}^{\ell(\lambda)} \sum_{i=1}^{\lambda_k} \frac{\big(\theta + i - 1\big)^2}{2} = \sum_{k=1}^{\ell(\lambda)} \frac{\big(6\lambda_k\theta^2 + 6(\lambda_k^2 - \lambda_k)\theta + 2\lambda_k^3 - 3\lambda_k^2 + \lambda_k\big)}{12}$$
$$= \frac{(6n\theta^2 + 6(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2 - n)\theta + 2\sum_{k=1}^{\ell(\lambda)} \lambda_k^3 - 3\sum_{k=1}^{\ell(\lambda)} \lambda_k^2 + n)}{12}.$$

By elementary calculus, it is easy to see that the above is minimized (with respect to θ) at

$$\theta_\star = \frac{n - \sum_{k=1}^{\ell(\lambda)} \lambda_k^2}{2n}.$$

With this particular choice, the sum simplifies to

$$\sum_{k=1}^{\ell(\lambda)} \sum_{i=1}^{\lambda_k} \frac{\big(\theta_\star + i - 1\big)^2}{2} = \frac{1}{24}\left(4\sum_{k=1}^{\ell(\lambda)} \lambda_k^3 - \frac{3}{n}\left(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2\right)^2 - n\right).$$

Note that the above reduces to $\mathfrak{L}_n$ when $\lambda = (n)$. It now only remains to show that any other choice of permutation yields a quantity that is strictly smaller, that is,

$$\max_{\lambda\vdash n,\ \lambda\neq(n)} \left\{4\sum_{k=1}^{\ell(\lambda)} \lambda_k^3 - \frac{3}{n}\left(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2\right)^2\right\} < n^3 \qquad \text{for every } n \geq 1.$$

For this, we recall that for $p > q \geq 1$, equality of the elementary inequality $\|x\|_{\ell^p} \leq \|x\|_{\ell^q}$ holds if and only if x has at most one nonzero component. Since $\lambda \neq (n)$, the components λ_k cannot all be equal, whence

$$\max_{\lambda\vdash n,\ \lambda\neq(n)} \left\{4\sum_{k=1}^{\ell(\lambda)} \lambda_k^3 - \frac{3}{n}\left(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2\right)^2\right\}$$
$$< \max_{\lambda\vdash n,\ \lambda\neq(n)} \left\{4\left(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2\right)^{3/2} - \frac{3}{n}\left(\sum_{k=1}^{\ell(\lambda)} \lambda_k^2\right)^2\right\} \leq \max_{r\in\mathbb{R}} \left\{4r^3 - \frac{3}{n}r^4\right\}.$$

It is easily seen (using elementary calculus) that the maximum of the function on the right-hand side is n^3, which is achieved at $r = n$. Thus, the proof of Proposition 2.6 (and therefore of Theorem 1.2) is complete.

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