THE UNIVERSITY OF CHICAGO

INDUCED SUBGRAPHS OF HIGHLY SYMMETRIC GRAPHS: SIZE AND MAXIMUM DEGREE

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To my family

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ABSTRACT

We study the maximum degree of large induced subgraphs of some highly symmetric graphs. This type of problem for Boolean hypercubes was formulated as an equivalent formulation of the Sensitivity Conjecture in the early 90s. A breakthrough result of Hao Huang in 2019 showed that for subsets U of vertices of a n-dimensional Boolean hypercube, if $|U| > 2^{n-1}$ then U induces a subgraph with maximum degree at least $\sqrt{n}$. As a corollary, this implies that the degree of a Boolean function is upper bounded by the square of its sensitivity, confirming the Sensitivity Conjecture.

Huang's theorem raised a natural question — does a similar property about the size and maximum degree of an induced subgraph hold for other graphs? In this thesis, we study this question for general Cayley graphs and the Hamming graph $H(n, 3)$.

We show that for abelian Cayley graphs $G = (V, E)$, if $U \subseteq V$ has size $|U| > |V|/2$, then U induces a subgraph of G with maximum degree at least $\sqrt{(d + t)/2}$ where d is the degree and t is the number of generators of order 2. This bound on the maximum degree is tight. On the other hand, for non-abelian Cayley graphs, there are known constructions of infinite families of non-abelian Cayley graphs that contain an induced 1-regular subgraph on more than half of the vertices.

Our result shows that for bipartite abelian Cayley graphs, any induced subgraph of size exceeding the independence number must have a high degree vertex. However, for non-bipartite abelian Cayley graphs, the independence number can be much smaller. In particular, for the Hamming graph $H(n, k)$, the independence number $\alpha(H(n, k))$ is only $1/k$ times the number of vertices. Moreover, $H(n, k)$ contains an induced subgraph with maximum degree 1 which has size $\alpha(H(n, k)) + 1$.

In this thesis, we focus on the case $k = 3$. We show that there are induced subgraphs of $H(n, 3)$ with maximum degree 1 that have size larger than $\alpha(H(n, 3)) + 1$ but under an extra assumption, any subgraphs of $H(n, 3)$ with maximum degree 1 have size $\alpha(H(n, 3)) + O(1)$.

Specifically, if $U \subseteq \mathbb{Z}_3^n$ and U induces a subgraph of $H(n,3)$ with maximum degree at most 1 then

1. If U is disjoint from a maximum size independent set of $H(n,3)$ then $|U| \leq 3^{n-1}+1$. Moreover, all such U with size $3^{n-1}+1$ are isomorphic to each other.

2. For $n \geq 6$, there exists such a U with size $|U| = 3^{n-1}+18$ and this is optimal for $n = 6$.

3. If $U \cap \{x, x+e_1, x+2e_1\} \neq \phi$ for all $x \in \mathbb{Z}_3^n$ then $|U| \leq 3^{n-1}+729$.

CHAPTER 1

INTRODUCTION

In Computer Science, determining the amount of resources needed for computing a Boolean function f in various ways is a fundamental problem. How fast can a Turing machine compute f? What is the smallest Boolean circuit that represents f? What is the shortest program that generates the truth table of f? How much information do we need to know about the input in order to compute f? Each of these problems is tied to a specific model of computation and leads to a different aspect of complexity. Some of these problems capture the most difficult open problems in mathematics.

1.1 Decision trees and complexity measures

Among various models of computation, decision trees are one of the simplest. The associated model complexity, namely query complexity, captures the amount of information that we need to know about the input x in order to compute $f(x)$. Even for query complexity, proving lower bounds can be tricky since there are exponentially many decision trees that compute the same function. Fortunately, due to the simplicity of decision trees, one can observe that functions with low query complexity must have certain combinatorial or algebraic properties. This led researchers to define complexity measures that could be explicitly determined from the function itself without relying on any model of computation.

In the early 90s, a lot of progress was made on understanding query complexity. A number of complexity measures were introduced, including sensitivity [CDR86], block sensitivity [Nis91], certificate complexity, and degree of the real polynomial representation [NS94]. The relationships between them as well as their connections to query complexity and even other computational models were studied extensively. It is not hard to see that these measures serve as a lower bound for the query complexity. Interestingly, it is also possible to construct a decision tree with depth depending only on these measures. Nisan [Nis91] introduced block sensitivity and showed that query complexity is upper bound by some polynomial in block sensitivity. The relationship between query complexity and certificate complexity was discovered in [BI87, HH87, Tar89]. Nisan and Szegedy [NS94] studied the degree of the real polynomial representation and showed that degree is polynomially related to block sensitivity. As a corollary, this placed degree into the family of measures that are polynomially related to query complexity. At the time, it was not known if sensitivity — one of the earliest measures which was introduced — belongs to the same family.

1.2 Sensitivity versus block sensitivity

The *sensitivity versus block sensitivity* problem asks what the correct relationship between sensitivity and block sensitivity is. It was stated as an open problem in [NS94] and the assertion that block sensitivity is upper bound by some polynomial in sensitivity was known as the Sensitivity Conjecture.

The separation between block sensitivity and sensitivity was known to be at least quadratic since the work of Rubinstein [Rub95]. Virza [Vir11] and then Ambainis and Sun [AS11] improved the separation by a constant factor. The largest known separation to this day is still quadratic. The first super-quadratic separation between sensitivity and any other relevant measures was achieved by Tal [Tal16], who gave a 2.1 power separation between sensitivity and query complexity. This separation was then improved by Ben-David, Hatami and Tal [BHT17] to a cubic separation between quantum query complexity and sensitivity. In the same paper they also showed a 2.22 power separation between sensitivity and certificate complexity.

On the other hand, Simon [Sim83] proved that if a Boolean function depends on n variables, then it must have sensitivity at least $\Omega(\log n)$. This gave an exponential upper bound on the other measures in terms of sensitivity. Kenyon and Kutin [KK04] introduced ℓ-block sensitivity and improved the bound by reducing the exponent by a constant factor. Ambainis et al. [ABG+14, APV16, HLS16] improved the upper bound using a combinatorial argument that is similar to Simon's proof. Finally, in 2019, Hao Huang used an elegant spectral argument to solve the sensitivity versus degree version of the problem, which improved the previous bound exponentially and completely resolved the Sensitivity Conjecture. Although Huang's result implies a tight relationship between degree and sensitivity and implies a polynomial relationship between sensitivity and block sensitivity, it is not known if the resulting relationship between them is tight.

Huang's method is dramatically different from previous attempts. Specifically, he showed that any set of $2^{n-1}+1$ vertices of the n-dimensional Boolean hypercube induces a subgraph with maximum degree at least $\sqrt{n}$. This claim is known to imply the Sensitivity Conjecture via a formulation due to Gotsman and Linial [GL92]. Huang's Theorem implies a tight relationship between sensitivity and degree as Chung et. al. [CFGS88] showed that the n-dimensional Boolean hypercube has an induced subgraph on $2^{n-1}+1$ vertices with maximum degree at most $\lceil\sqrt{n}\rceil$.

1.3 Induced subgraphs of highly symmetric graphs

In his paper, Huang [Hua19] asked the following:

Question 1. *Given a "nice" graph G with high symmetry, denote by $\alpha(G)$ its independence number. Let $f(G)$ be the minimum of the maximum degree of an induced subgraph of G on $\alpha(G)+1$ vertices. What can we say about $f(G)$?*

Since then there has been a considerable amount of work trying to extend Huang's result. Huang's argument has been generalized to Cartesian products of directed l-cycles [Tik22] and Cartesian products of paths [ZH23], as well as other products of graphs [HLL20]. Alon and Zheng [AZ20] considered arbitrary Cayley graphs over $\mathbb{Z}_2^n$ and showed that Huang's theorem implies any induced subgraph on more than half of the vertices must have maximum degree $\sqrt{d}$ where d is the degree of the Cayley graph. Potechin and Tsang [PT20] showed that this can be generalized to any abelian Cayley graph. This answers Huang's question in the case of bipartite abelian Cayley graphs.

It was tempting to conjecture that being a Cayley graph is sufficiently symmetric so that a similar conclusion about the maximum degree of an induced subgraph on more than half of the vertices can be drawn. However, it turned out to be not the case as Lehner and Verret [LV20] constructed families of non-abelian Cayley graphs that contain 1-regular induced subgraphs on more than half of the vertices. García-Marco and Knauer [GMK22] constructed further examples of infinite families of non-abelian Cayley graphs that contain 1-regular induced subgraphs on more than half of the vertices.

1.4 Our contributions

Our first contribution is about understanding induced subgraphs of abelian Cayley graphs. Specifically, we show that

Theorem 1.4.1. *For any Cayley graph $G = \Gamma(X, S)$ such that X is abelian and any $U \subseteq X$ of size $|U| > |X|/2$, the induced subgraph $G(U)$ of G on U has maximum degree at least $\sqrt{(|S|+t)/2}$ where t is the number of elements in S of order 2.*

For the case of bipartite abelian Cayley graphs G, it shows that any induced subgraph on $\alpha(G)+1$ vertices must have maximum degree at least $\sqrt{d/2}$. This bound is tight and it answers Huang's question (Question 1) for the class of bipartite abelian Cayley graphs.

However, when a regular graph $G = (V, E)$ is not bipartite, its independence number can be much smaller than $|V|/2$. In particular, the Hamming graph $H(n, k)$ with vertex set equal to the set of strings of length n over k alphabet has independence number $\alpha(H(n, k)) = k^{n-1}$, which is only $1/k$ of the vertices. The Boolean hypercube is precisely $H(n, 2)$ and hence

Huang's Theorem shows that any induced subgraph of $H(n, 2)$ of size $\alpha(H(n, 2)) + 1$ must have maximum degree at least $\sqrt{n}$. The construction of Chung et al [CFGS88] shows that the $\sqrt{n}$ bound on the maximum degree is the best possible.

Dong [Don21] generalized the construction in [CFGS88] to $H(n, k)$ and showed that there is an induced subgraph of $H(n, k)$ of size $\alpha(H(n, k)) + 1$ that has maximum degree at most $\lceil \sqrt{n} \rceil$. For $k \geq 3$, Tanday [Tan22] gave a construction of an induced subgraph on $\alpha(H(n, k)) + 1$ with maximum degree 1. [GMK22] independently observed that $H(n, 3)$ has a 1-regular induced subgraph on $3^{n-1} + 1$ vertices.

We further investigate the Hamming graph $H(n, 3)$. In particular, we investigate the following question:

Question 2. *Let $U \subseteq \mathbb{Z}_3^n$. If the induced subgraph of $H(n, 3)$ on U has maximum degree at most 1, how large can U be?*

We know that $|U|$ can be at least $3^{n-1} + 1$ according to [GMK22, Tan22]. Our first result shows that this is the largest possible under an extra assumption on U.

Theorem 1.4.2. *Let $U \subseteq \mathbb{Z}_3^n$. If U is disjoint from a maximum size independent set of $H(n, 3)$ and the induced subgraph of $H(n, 3)$ on U has maximum degree at most 1 then $|U| \leq 3^{n-1} + 1$.*

However, without the assumption of being disjoint from a maximum size independent set, U can be somewhat larger. We construct examples of size $3^{n-1} + K$ where $K = 2, 6, 18$ for $n = 4, 5, 6$ respectively. Our example for $n = 6$ can be extended to give an example of size $3^{n-1} + 18$ for any $n \geq 6$.

Theorem 1.4.3. *For $n \geq 6$, there exists $U \subseteq \mathbb{Z}_3^n$ such that $|U| = 3^{n-1} + 18$ and U induces a subgraph of $H(n, 3)$ with maximum degree 1.*

It can be shown that our examples for $n \in [6]$ are the largest possible. Interestingly, they all share a common property that there exists $i \in [n]$ such that every line along direction

i intersects with them. We say that the set is i-saturated if it satisfies this intersection property with lines along direction i. Our main result is an upper bound on the size of i-saturated subsets.

Theorem 1.4.4. *Let $U \subseteq \mathbb{Z}_3^n$ and $i \in [n]$. If U is i-saturated and U induces a subgraph of $H(n,3)$ with maximum degree at most 1 then $|U| \leq 3^{n-1} + 729$.*

Although our proof involves analyzing cases in low dimensions, it does not rely on a computer-assisted argument. That being said, by verifying a certain property using a SAT solver, our size upper bound for i-saturated subsets of $\mathbb{Z}_3^n$ with induced degree 1 can be improved to $3^{n-1} + 81$.

CHAPTER 2

COMPLEXITY OF BOOLEAN FUNCTIONS

In this chapter, we define decision trees and the complexity measures related to it. We present selected results from the literature to show the relationships between them that are relevant to our discussions in the later chapters.

2.1 Complexity measures

A *decision tree* is a rooted binary tree $\mathcal{T}$ such that each internal node is labeled by a variable x_i for some $i \in [n]$ and each leaf is labeled by either 0 or 1. Given $x \in \{0,1\}^n$, $\mathcal{T}$ computes $\mathcal{T}(x)$ as follows: start at the root, if it is a leaf then output the value of this leaf, otherwise query the variable labeling the current node. Recurse on the left and right subtree if the outcome of the query is 0 and 1 respectively. We say that $\mathcal{T}$ *computes* f if $\mathcal{T}(x) = f(x)$ for all $x \in \{0,1\}^n$.

Definition 2.1.1 (Query complexity)**.** The *query complexity* $D(f)$ *of* f is the minimum depth of any decision tree that computes f.

Let $x \in \{0,1\}^n$, and $B \subseteq [n]$. We denote by x^B the string obtained from x by flipping all of the bits in B, i.e., $(x^B)_i = x_i$ if $i \notin B$ and $(x^B)_i = 1 - x_i$ otherwise. When $B = \{i\}$ is a singleton, we use x^i to denote $x^{\{i\}}$.

Definition 2.1.2 (Sensitivity)**.** We say that f is *sensitive to the i-th variable (or coordinate) on x* if $f(x) \neq f(x^i)$. The *sensitivity* $s(f,x)$ *on* x is the number of variables which f is sensitive to on x. For $b \in \{0,1\}$, the *b-sensitivity* $s^b(f)$ is defined as $\max_{x \in f^{-1}(b)}\{s(f,x)\}$ and the *sensitivity* $s(f)$ *of* f is defined as $\max\{s^0(f), s^1(f)\}$.

Definition 2.1.3 (Block sensitivity)**.** Let $B \subseteq [n]$. We say that f is *sensitive to the block B on x* if $f(x) \neq f(x^B)$. The *block sensitivity* $bs(f,x)$ *on* x is the maximum number of disjoint blocks which f is sensitive to x, and the *block sensitivity* $bs(f)$ *of* f is defined as $\max_{x \in \{0,1\}^n} bs(f, x)$.

Definition 2.1.4 (Certificate complexity)**.** We say that a subcube $Q \ni x$ is a *certificate of* f *on* x if f is constant on Q, i.e. $f(y) = f(x)$ for all $y \in Q$. The *certificate complexity* $C(f, x)$ *on* x is the minimum codimension of any certificate on x. The *b-certificate complexity* $C^b(f)$ is defined as $\max_{x \in f^{-1}(b)} C(f, x)$, and the *certificate complexity* $C(f)$ *of* f is defined as $\max\{C^0(f), C^1(f)\}$.

Any Boolean function f on $\{0,1\}^n$ admits a unique multilinear polynomial representation over any given field $\mathbb{F}$:

$$f(x) = \sum_{S \subseteq \{0,1\}^n} c_S \prod_{i:S_i=1} x_i$$

where the additions and multiplications are over $\mathbb{F}$.

Definition 2.1.5 ($\mathbb{F}$-degree)**.** Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $\mathbb{F}$ be a field. The $\mathbb{F}$-*degree* $\deg_{\mathbb{F}}(f)$ of f is the degree of the unique multilinear polynomial representation of f. We use deg to denote $\deg_{\mathbb{R}}$.

When $\mathbb{F} = \mathbb{R}$, by identifying 0 with $+1$ and 1 with -1, it is helpful to consider f as a function from $\{+1,-1\}^n$ to $\{+1,-1\}$. f still admits a unique multilinear polynomial representation over $\mathbb{R}$ and such a representation is known as the Fourier representation of f:

$$f(x) = \sum_{S \subseteq \{0,1\}^n} \hat{f}(S) \chi_S(x)$$

where $\chi_S(x) = \prod_{i \in [n]} x_i$ and the *Fourier coefficients* $\hat{f}(S)$ can be computed by

$$\hat{f}(S) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x) \chi_S(x).$$

Note that the maximum Hamming weight of S for which $\hat{f}(S) \neq 0$ is equal to $deg(f)$.

2.2 Relationships among the measures

In this section, we review some of the known relationships between the complexity measures we introduced.

Theorem 2.2.1. *For all boolean functions f, we have*

1. $s(f) \leq bs(f) \leq C(f)$.

2. *[Nis91]* $C(f) \leq s(f)bs(f)$.

3. *[NS94]* $bs(f) \leq 2\deg(f)^2$.

4. $s(f), bs(f), C(f)$ *and* $\deg(f)$ *are at most* $D(f)$.

Item 1 and 4 follow easily from the definitions. For item 2, we observe that fixing all variables in each sensitivity block determines the value of the function. Let $x \in \{0,1\}^n$, We can assume each sensitive block $B \subseteq [n]$ on x has size at most $s(f)$ since either there is a proper sensitive block $B' \subset B$ or $f(x^{B\setminus\{i\}}) \neq f(x^B)$ for all $i \in B$, which implies $|B| \leq s(f, x^B) \leq s(f)$.

To prove item 3, Nisan and Szegedy [NS94] used a result from approximation theory which asserts that a univariate polynomial which is bounded on an interval and whose derivative is large at the boundary of the interval must have large degree. Let $k = bs(f)$. Let x be the input that attains block sensitivity k and let $B_1, \ldots, B_k$ be the sensitive blocks. [NS94] defined a new Boolean function $g(y_1, \ldots, y_k) = f(y_1\mathbf{1}_{B_1} + \cdots + y_k\mathbf{1}_{B_k})$ where $\mathbf{1}_S \in \{0,1\}^n$ is an indicator vector of the set S, i.e., $(\mathbf{1}_S)_i = 1$ if $i \in S$ and 0 otherwise. They observed that the symmetrization [MP68] of g is a univariate polynomial with large derivative on the interval $[0,1]$. Since the degree of g is at most $\deg(f)$, the result follows. We refer to the original paper [NS94] for the detailed proofs. The survey by Buhrman and de Wolf [BdW02] also contains the proofs of these results as well as many other relevant results and references.

On the other hand, by constructing a decision tree for f in a certain way, it can be shown that $D(f)$ is upper bounded by the square of the certificate complexity of f. This result was discovered by several people independently [BI87, HH87, Tar89].

Theorem 2.2.2. *For all boolean functions f, $D(f) \leq C^0(f)C^1(f)$.*

Proof. Consider the following query algorithm for computing f:

1. If no 1-certificate is consistent with the variables we have queried so far, output 0.

2. Similarly, if no 0-certificate is consistent with the variables we have queried so far, output 1.

3. Otherwise, pick an arbitrary consistent 0-certificate, query all the variables associated to the certificate and repeat the whole process.

It is clear that the algorithm correctly computes $f(x)$ since it will only output 0 (resp. 1) when there is no 1-certificate (resp. 0-certificate) that is consistent with x. In each iteration we query at most $C^0(f)$ bits. Moreover, any 0-certificate that we picked in step 3 must have a new variable in common with any 1-certificate that are still consistent since otherwise there would be both a 0-certificate and 1-certificate that are consistent with x and hence $0 = f(x) = 1$, a contradiction. It follows that there are at most $C^1(f)$ many iterations before all 1-certificates are eliminated. $\square$

Together with Theorem 2.2.1, this shows that $bs(f), C(f)$, $\deg(f)$ and $D(f)$ are polynomially related, i.e., any one of them can be upper bounded by some polynomial of the others. Note that the results we selected to present are meant to demonstrate these measures are closely related to each other. There is a large body of literature improving upon these early results such as tighter relationships between them as well as separation results showing a gap between them.

However, despite the extensive research the following question remained:

Question 3. *Is $s(f)$ polynomially related to $bs(f)$?*

This question was raised by Nisan and Szegedy [NS94] in the early 90's when they showed that $bs(f) \leq 2\deg(f)^2$, and the Sensitivity Conjecture asserts that $s(f)$ is indeed polynomially related to $bs(f)$.

For special functions, it was known that the Sensitivity Conjecture is true for symmetric functions, graph properties [Tur84, Sun11], bipartite graph properties [GMSZ13], minterm-trasitive functions [Cha05], monotone functions [Nis91], functions with constant alternating number [LZ16], and various classes of functions defined based on the circuits that compute them [Mor14, BLTV16, KT16]. It was also known to be false for real-valued functions on Boolean hypercubes with range $[0, 1]$ [Tal16].

However, for general Boolean functions, the best known upper bound on block sensitivity in terms of sensitivity was exponential for a very long time. The first such upper bound was proved by Simon [Sim83]. Simon showed that if a Boolean function depends on n variables, then it must have sensitivity at least $\Omega(\log n)$. It implies an exponential upper bound on all $bs(f), C(f), \deg(f), D(f)$ in terms of $s(f)$. Kenyon and Kutin [KK04] introduced ℓ-block sensitivity and obtained a better upper bound on block sensitivity in terms of sensitivity. Ambainis et al. [ABG$^+$14, APV16, HLS16] improved the upper bound on $C(f)$ using a combinatorial argument that is similar to Simon's argument.

There are also a number of equivalent formulations. See the survey [HKP11] by Hatami, Kulkarni and Pankratov for the progress and equivalent formulations in the 25 years after the Sensitivity Conjecture was introduced.

In an attempt to obtain weaker conjectures, the author [Tsa14] compared the sensitivity with a measure that can be much smaller.

Definition 2.2.3. The *minimum certificate complexity* of a Boolean function f is defined as $\min_{x\in\{0,1\}^n} C(f, x)$.

Note that minimum certificate complexity is not a common complexity measure. Unlike other measures that we have introduced, the minimum certificate complexity is not polynomially related to any of them. In fact, it can be arbitrarily smaller than the others. For instance, the AND_n function on n variables has minimum certificate complexity 1 but $s(\mathrm{AND}_n) = bs(\mathrm{AND}_n) = C(\mathrm{AND}_n) = \deg_{\mathbb{F}}(\mathrm{AND}_n) = D(\mathrm{AND}_n) = n$. Similarly, $\mathbb{F}_2$-degree can also be arbitrarily smaller than the others. The parity function $\textsc{Parity}_n$ on n variables has $\mathbb{F}_2$-degree 1 but $s(\textsc{Parity}_n) = bs(\textsc{Parity}_n) = C(\textsc{Parity}_n) = \deg_{\mathbb{R}}(\textsc{Parity}_n) = D(\textsc{Parity}_n) = n$.

Even though both minimum certificate complexity and $\mathbb{F}_2$-degree can be arbitrarily smaller than sensitivity, it turns out that if we can obtain a polynomial upper bound on both measures in terms of sensitivity, the Sensitivity Conjecture would follow.

Definition 2.2.4. We define $C'_{\min}(f)$ to be the largest minimum certificate of the restriction of f on any subcube. More precisely, let $\mathcal{Q}_n$ be the set of all subcubes of the n-dimensional Boolean cube, and let $f|_Q$ be the restriction of f on the subcube Q. Then $C'_{\min}(f) = \max_{Q \in \mathcal{Q}_n} C_{\min}(f|_Q)$.

Theorem 2.2.5. $D(f) \leq C'_{\min}(f) \deg_{\mathbb{F}}(f)$.

Proof. By definition $C_{\min}(f|_Q) \leq C'_{\min}(f)$ for any subcube Q. Let Q be a subcube of codimension $m = C_{\min}(f)$ on which f is a constant. Since restricting f on Q is equivalent to fixing some variables of f, it follows that the set of fixed variables corresponding to Q intersects with all monomials of degree $\deg_{\mathbb{F}}(f)$ in the $\mathbb{F}$-polynomial representation of f (since otherwise the resulting function would not be a constant). Thus, querying all those variables reduces the $\mathbb{F}$-degree by at least 1. By repeating this process at most $\deg_{\mathbb{F}}(f)$ times the function will become a constant and hence we have a decision tree for f. The depth of the tree is at most $\max_{Q \in \mathcal{Q}_n} C_{\min}(f|_Q) \cdot \deg_{\mathbb{F}}(f)$, as desired. $\square$

Corollary 2.2.6. *Let $\mathbb{F}$ be a field and let $a, b > 0$ be constants. If for all boolean functions f we have $C_{\min}(f) \leq s(f)^a$ and $\deg_{\mathbb{F}}(f) \leq s(f)^b$, then $\deg_{\mathbb{R}}(f) \leq s(f)^{a+b}$ for all f.*

However, since then not much progress was made on the Sensitivity Conjecture until the breakthrough result of Huang [Hua19] in 2019. This result resolved the Sensitivity Conjecture completely using a beautiful spectral argument. Huang's proof, which we will present in the next chapter, is short and elegant.

CHAPTER 3

THE SENSITIVITY THEOREM AND INDUCED SUBGRAPHS OF BOOLEAN HYPERCUBES

In 2019, Hao Huang [Hua19] used an elegant spectral argument to solve the sensitivity vs. degree problem, which improved the previous bound exponentially and completely resolved the Sensitivity Conjecture. Huang proved a remarkable property of induced subgraphs of Boolean hypercubes. The connection between induced subgraphs and sensitivity vs. degree was discovered by Gotsman and Linial [GL92], which we will present next.

3.1 Gotsman-Linial formulation of the Sensitivity Conjecture

Let Q_n be the n-dimensional Boolean hypercube. Gotsman and Linial observed that the sensitivity versus degree problem is equivalent to the following combinatorial problem regarding induced subgraphs of the Boolean hypercube:

Question 4. *Is it true that if $U \subseteq \{0,1\}^n$ has size $|U| \neq 2^{n-1}$, then either U or $\{0,1\}^n \setminus U$ induces a subgraph of Q_n with maximum degree at least n^c for some absolute constant $c > 0$?*

To see the connection, recall that any function $f : \{0,1\}^n \to \mathbb{R}$ admits a Fourier representation

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S(x)$$

where $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$ and the Fourier coefficients $\hat{f}(S)$ can be computed by

$$\hat{f}(S) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)\chi_S(x).$$

Note that $\deg(f)$ is equal to the largest S such that $\hat{f}(S) \neq 0$.

Since the sensitivity of a function is non-increasing under restriction, we can assume f has $\deg(f) = n$ in the Sensitivity Conjecture. In this case, the Fourier representation of f has $\hat{f}([n]) \neq 0$. Consider the function $g(x) = f(x) \oplus \texttt{Parity}(x)$ where $\oplus$ is the exclusive-OR operation. $f(x)$ and $g(x)$ satisfy the equation $1 - 2g(x) = (1 - 2f(x))\chi_{[n]}(x)$ and hence

$$g(x) = \frac{1}{2} - \frac{\chi_{[n]}(x)}{2} + f(x)\chi_{[n]}(x),$$

which has Fourier coefficients $\hat{g}(\phi) = 1/2 + \hat{f}([n])$, $\hat{g}([n]) = \hat{f}(\phi) - 1/2$ and $\hat{g}(S) = \hat{f}([n] \setminus S)$ for non-empty proper subsets S of $[n]$. Since f has degree n, $\hat{f}([n]) \neq 0$ and hence $\hat{g}(\phi) \neq 1/2$. Since $\hat{g}(\phi) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} g(x)\chi_\phi(x) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} g(x)$, $\hat{g}(\phi) \neq 1/2$ implies $|g^{-1}(1)| \neq 2^{n-1}$. On the other hand, for each $x \in \{0,1\}^n$, the number of neighbors y of x in Q_n such that $g(x) = g(y)$ is precisely the sensitivity of f on x. Thus, the sensitivity of f is equal to the maximum degree of the subgraph induced by $g^{-1}(0)$ or $g^{-1}(1)$. This proves the following result of Gotsman and Linial.

Theorem 3.1.1 ([GL92]). *The following two statements are equivalent:*

1. *For all Boolean functions f, $\deg(f) \leq s(f)^c$.*
2. *For all $U \subseteq \{0,1\}^n$, if $|U| \neq 2^{n-1}$, then either U or $\{0,1\}^n \setminus U$ induces a subgraph of Q_n with maximum degree at least n^c.*

3.2 Huang's Theorem

The proof of the Sensitivity Conjecture or the Gotsman-Linial formulation remained out of reach until in 2019 Huang found a proof of a slightly stronger statement of the Gotsman-Linial formulation. Specifically, Huang proved the following theorem, which implies the second statement in Theorem 3.1.1.

Theorem 3.2.1. *For all $U \subseteq \{0,1\}^n$, if $|U| > 2^{n-1}$, then U induces a subgraph of the n-dimensonal Boolean hypercube with maximum degree at least $\sqrt{n}$.*

The $\sqrt{n}$ lower bound on the maximum degree is tight [CFGS88]. By the proof of 3.1.1, to obtain an induced subgraph of Q_n that has size greater than 2^{n-1} and maximum degree equals to $\sqrt{n}$, it suffices to construct a Boolean function with degree n and sensitivity $\sqrt{n}$. One such example is the AND-OR tree. The AND-OR tree on n^2 variables is the AND function of n OR functions on disjoint sets of n variables. This function has degree n^2 and sensitivity n.

By Theorem 3.1.1, Huang's Theorem implies $\deg(f) \leq s(f)^2$ for all Boolean functions f, settling the Sensitivity Conjecture.

We present Huang's proof of Theorem 3.2.1 in the rest of the chapter. The proof is remarkably short and elegant. We start with the following lemma, which is well-known for standard adjacency matrices but the same proof can also be applied to signed adjacency matrices.

Lemma 3.2.2. *Let $G = (V, E)$ be an undirected graph and A be a signed adjacency matrix of G, i.e., $A_{u,v} \in \{+1, -1\}$ if $\{u, v\} \in E$ and $A_{u,v} = 0$ otherwise. If λ is the largest eigenvalue of A, then the maximum degree of G is at least λ.*

Proof. Let x be an eigenvector corresponding to λ, i.e., $\lambda x = Ax$. Let u be a vertex such that $|x_u|$ is maximized (i.e., $|x_u| \geq |x_v|$ for all $v \in V$). Observe that

$$|\lambda||x_u| = \Big|\sum_{v \in V} A_{u,v} x_v\Big| \leq \sum_{v \in V} |A_{u,v}||x_u| = \deg(u)|x_u|.$$

Since $\mathrm{Tr}(A) = 0$, $\lambda \geq 0$. It follows that $\deg(u) \geq |\lambda| = \lambda$. $\square$

To complete the proof of Theorem 3.2.1, Huang designed a signed adjacency matrix for the Boolean hypercubes such that all eigenvalues have absolute value $\sqrt{n}$ and then used the

Cauchy interlacing theorem to conclude that any large principal submatrix must also have an eigenvalue at least $\sqrt{n}$.

Definition 3.2.3. Let $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $n \geq 2$, define

$$A_n = \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix}.$$

It is not hard to see that A_n is indeed a signed adjacency matrix of Q_n. A remarkable property of A_n is that all its eigenvalues have absolute value $\sqrt{n}$. More precisely, we have the following lemma.

Lemma 3.2.4. *A_n has eigenvalues $-\sqrt{n}$ and $\sqrt{n}$, each with multiplicity 2^{n-1}.*

Proof. It suffices to show that $A_n^2 = nI$ as this implies all eigenvalues of A_n have absolute value $\sqrt{n}$. The result then follows from the fact that $\mathrm{Tr}(A_n) = 0$.

We prove $A_n^2 = nI$ by induction on n. The case of $n = 1$ is trivial. We assume $A_k^2 = kI$ and proceed to show that $A_{k+1}^2 = (k+1)I$. Indeed, by the recursive definition of A_{k+1} and the inductive hypothesis, we have

$$A_{k+1}^2 = \begin{pmatrix} A_k & I \\ I & -A_k \end{pmatrix} \begin{pmatrix} A_k & I \\ I & -A_k \end{pmatrix} = \begin{pmatrix} A_k^2 + I^2 & A_k - A_k \\ A_k - A_k & (-A_k)^2 + I^2 \end{pmatrix} = \begin{pmatrix} kI + I & 0 \\ 0 & kI + I \end{pmatrix},$$

as desired. □

Let A be a $n \times n$ matrix. A principal submatrix B of A is a submatrix of A obtained by deleting the same set of rows and columns. The Cauchy interlacing theorem asserts that the spectrum of A interlaces with the spectrum of its principal submatrix.

Theorem 3.2.5 (Cauchy interlacing theorem)**.** *Let A be a $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let B be a $m \times m$ principal submatrix of A with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then for all $i \in [m]$*

$$\lambda_i \geq \mu_i \geq \lambda_{i+(n-m)}$$

.

Proof of Theorem 3.2.1. Let $U \subset \{0,1\}^n$ of size $|U| > 2^{n-1}$. Let B be the principal submatrix of A_n obtained by deleting the rows and columns that do not correspond to vertices in U. Let μ_1 be the largest eigenvalue of B. By Lemma 3.2.4, since A_n has eigenvalue $\sqrt{n}$ with multiplicity 2^{n-1} and $|U| > 2^{n-1}$, $1 + (2^n - |U|) \leq 2^{n-1}$ and the Cauchy interlacing theorem implies

$$\mu_1 \geq \lambda_{1+(2^n-|U|)} \geq \lambda_{2^{n-1}} = \sqrt{n}.$$

Since B is a signed adjacency matrix of the subgraph induced by U, Lemma 3.2.2 implies that this induced subgraph has a vertex with degree at least $\mu_1 \geq \sqrt{n}$ and the proof is completed. $\square$

CHAPTER 4

CAYLEY GRAPHS

4.1 Introduction

Huang's Theorem settled the Sensitivity Conjecture affirmatively and optimally, as Chung et al [CFGS88] showed that the Boolean hypercube Q_n has an induced subgraph on $2^{n-1}+1$ vertices such that the maximum degree is at most $\lceil\sqrt{n}\rceil$. Huang asked what we can say about the maximum degree of induced subgraphs of a graph G on more than $\alpha(G)$ vertices if G is highly symmetric, where $\alpha(G)$ denotes the independence number of G. This led to a line of research on studying similar problem for various classes of graphs [Tik22, ZH23, HLL20, AZ20, PT20].

Cayley graphs are highly symmetric which makes them a natural candidate to study regarding Huang's question. Alon and Zheng [AZ20] considered general Cayley graphs over $\mathbb{Z}_2^n$ and generalized Huang's theorem to this class of graphs. They showed that while Huang's method of constructing a special signed adjacency matrix does not work for general Cayley graphs over $\mathbb{Z}_2^n$, Huang's theorem itself can be used to deduce this generalization. Potechin and Tsang [PT20] further generalized Huang's theorem to any abelian Cayley graph and conjectured that being abelian is not necessary. However, this turned out to not be the case as Lehner and Verret [LV20] constructed families of non-abelian Cayley graphs that contain 1-regular induced subgraphs on more than half of the vertices. García-Marco and Knauer [GMK22] constructed further examples of infinite families of non-abelian Cayley graphs that contain 1-regular induced subgraphs on more than half of the vertices.

In this chapter, we will start with some basic examples which show that a certain degree of symmetry is needed in order to guarantee that large induced subgraphs must have large maximum degree. We will then present our result for abelian Cayley graphs, followed by the constructions of Lehner and Verret [LV20], and García-Marco and Knauer [GMK22] which showed that our result cannot be extended to non-abelian Cayley graphs.

4.2 Basic examples of graphs with large induced 1-regular subgraphs

We give a simple example of a regular bipartite graph $G = (L, R, E)$ which has a subset of size $|L|+1$ that induces a subgraph with maximum degree 1. Let A, B, C, D be disjoint sets of size $|A| = |C| = n+1$ and $|B| = |D| = n$. Let $L = A \cup B$ and $R = C \cup D$. Let E be the union of a perfect matching between A and C, the set of all edges between A and D, and the set of all edges between B and C. It is straightforward to check that G is $(n+1)$-regular, but the set $A \cup B$ has size $2(n+1) = |L|+1$ and induces a subgraph with maximum degree 1. A concrete drawing of such a graph for $n = 2$ is shown in Figure 4.1.

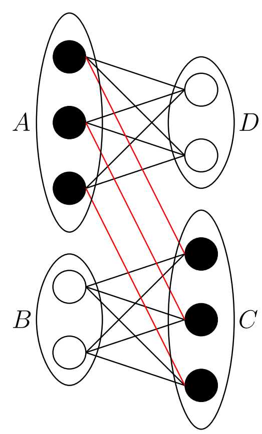


Figure 4.1: An illustration of the graph G for $n = 2$.

The second example is the odd graph [BD20] $G = (V, E)$ with vertices $\{S \subseteq [2n+1] : |S| = n\}$ and edges $\{\{S, T\} : S \cap T = \phi\}$. It is clear that the automorphism group of the odd graph contains the symmetric graph S_n. Moreover it is not hard to see that G is edge-transitive. Let $U = \{S \subseteq V : 1 \notin S\}$. Then $|U| = \binom{2n}{n} = \frac{n+1}{2n+1} \cdot \binom{2n+1}{n} > \frac{1}{2}|V|$ and for each $S \in U$, the only neighbor that is also in U is $T = [2n+1] \setminus (\{1\} \cup S)$. When $n = 2$, the odd graph is precisely the well-known Petersen graph, which is shown in Figure 4.2

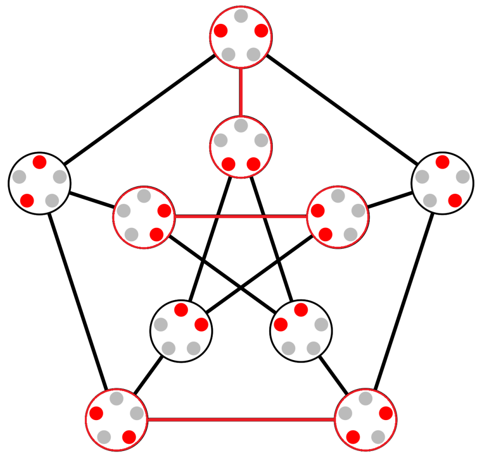


Figure 4.2: An illustration of the odd graph for $n = 2$. The set U consists of the endpoints of the red edges.

4.3 Abelian Cayley graphs

We need the following basic definitions. All the groups we consider in this chapter are finite.

Definition 4.3.1 (Cayley Graphs)**.** Given a group X and a set of non-identity elements S of X, the Cayley graph $G = \Gamma(X, S) = (V, E)$ is the graph with vertices $V = X$ and edges $E = \{(x, sx) : x \in X, s \in S\}$. We consider undirected Cayley graphs and hence we assume that S is symmetric, i.e., if $s \in S$ then $s^{-1} \in S$.

Without loss of generality, we can assume that S generates X as otherwise, letting X' be the subgroup of X which is generated by S, $\Gamma(X, S)$ consists of $\frac{|X|}{|X'|}$ disjoint copies of $\Gamma(X', S)$.

The n-dimensional Boolean hypercube Q_n can be viewed as a Cayley graph over the group $\mathbb{Z}_2^n$.

Definition 4.3.2 (Boolean Hypercube)**.** $Q_n = \Gamma(Z_2^n, \{e_i : i \in [n]\})$ where e_i is the standard basis, i.e., $(e_i)_j = 1$ if $j = i$ and $(e_i)_j = 0$ otherwise.

We can now state our main result.

Theorem 4.3.3. *For any Cayley graph $G = \Gamma(X, S)$ such that X is abelian and any $U \subseteq X$ of size $|U| > |X|/2$, the induced subgraph $G(U)$ of G on U has maximum degree at least $\sqrt{(|S| + t)/2}$ where t is the number of elements in S of order 2.*

We prove this theorem in two steps:

1. We show that Huang's theorem implies that the same property holds for products of cycles.
2. We generalize the argument used by Alon and Zheng [AZ20] for $X = \mathbb{Z}_2^n$ to prove the result for all abelian X.

4.3.1 From the Boolean hypercube to products of cycles

We recall Huang's theorem (Theorem 3.2.1) asserts that for Q_n, any induced subgraph on more than half of the vertices has maximum degree $\sqrt{n}$. We show that it implies the same property for products of cycles.

Corollary 4.3.4. *Let $X = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_d}$, $S = \{\pm e_1, \ldots, \pm e_d\}$, and $G = \Gamma(X, S)$. For any $U \subseteq X$ of size $|U| > |X|/2$, there is an element $u \in U$ and $k \geq \sqrt{d}$ distinct indices $i_1, \ldots, i_k \in [d]$ such that for all $j \in [k]$, either $u + e_{i_j} \in U$ or $u - e_{i_j} \in U$.*

Proof. To prove this, we cover G with copies of d-dimensional Boolean hypercube Q_d.

Definition 4.3.5. Let $U_r = \{r + \sum_{i \in T} e_i : T \subseteq [d]\}$.

Observe that $\mathbb{E}_r|U_r \cap U| > 2^{n-1}$ where the expectation is over uniform random $r \in X$. Thus, there must be some $x \in X$ that satisfies $|U_x \cap U| > 2^{n-1}$. Since $G(U_x)$ is isomorphic to the Boolean cube Q_d of dimension d, by Huang's theorem, the induced subgraph $G(U_x \cap U)$ of G on $U_x \cap U$ has maximum degree at least $\sqrt{d}$. $\square$

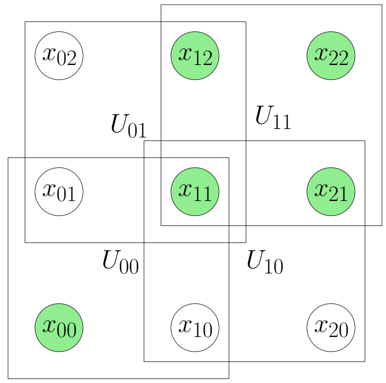


Figure 4.3: An illustration of some of the sets $\{U_r : r \in G\}$ when $d = 2$ and $m_1 = m_2 = 3$. The sets $U_{02}, U_{12}, U_{20}, U_{21}, U_{22}$ wrap around and are not shown.

4.3.2 From products of cycles to abelian Cayley graphs

We now apply an argument of Alon and Zheng [AZ20] to prove Thereom 4.3.3.

Proof of Theorem 4.3.3. By the fundamental theorem of finite abelian groups, we can assume $X = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$. Denote $S = \{s_1, \ldots, s_t, s_{t+1} \ldots, s_d, -s_{t+1}, \ldots, -s_d\}$ and let $m = lcm(m_1, \ldots, m_k)$. We consider the Cayley graph $G' = \Gamma(\mathbb{Z}_m^d, T)$ where $T = \{\pm e_1, \ldots, \pm e_d\}$. Let $A : \mathbb{Z}_m^d \to X$ be a linear map defined by $A(e_i) = s_i$. Note that A is well-defined because $\text{ord}(s_i)|m$ for all $i \in [d]$.

Since S is a generating set of X, the linear map A is onto. Thus, for all $x \in X$, $A^{-1}(x)$ has size $m^d/|X|$. It follows that $A^{-1}(U)$ has size $|A^{-1}(U)| = (m^d/|X|)|U| > (m^d/|X|)(|X|/2) = m^d/2$. By Corollary 4.3.4, there is a vertex $h \in A^{-1}(U)$ and $k \geq \sqrt{d}$ distinct indices $i_1, \ldots, i_k \in [d]$ such that for all $j \in [k]$, either $h + e_{i_j}$ or $h - e_{i_j}$ is in $A^{-1}(U)$. Take $h_j \in \{h + e_{i_j}, h - e_{i_j}\}$ so that $h_j \in U$ (if both elements are in U then this choice is arbitrary) and observe that for all $j' \neq j \in [k]$,

$$A(h_{j'}) - A(h_j) = A(h) \pm A(e_{i_{j'}}) - A(h) \mp A(e_{i_j}) = \pm s_{i_{j'}} \mp s_{i_j} \neq 0.$$

Thus, all $A(h_1), \ldots, A(h_k)$ are distinct, contained in U, and adjacent to $A(h) \in U$ in $G(U)$ where $G = \Gamma(X, S)$. Finally, $|S| = t + 2(d - t)$ and hence $d = (|S| + t)/2$, as desired. $\square$

Since the n-dimensional Boolean hypercube Q_n is an abelian Cayley graph (Definition 4.3.2) and every element in $\mathbb{Z}_2^n$ has order 2, Theorem 4.3.3 reduces to Huang's Theorem when the graph is Q_n. The dependence on the number of elements in S of order 2 is necessary as it was pointed out by Alon and Zheng [AZ20] that for the 6-cycle, $C_6 = \Gamma(\mathbb{Z}_6, \{+1, -1\})$, it has an induced 1-regular subgraph on 4 vertices. Neither 1 and -1 has order 2, hence we have $d = (2+0)/2 = 1$ in Theorem 4.3.3, which also shows that the bound is tight.

4.4 Counterexamples for non-abelian Cayley graphs

It is natural to ask if abelian is necessary in Theorem 4.3.3. Specifically, is it true that for all Cayley graphs $G = \Gamma(X, S)$ where $S = S^{-1}$, any $U \subseteq X$ of size $|U| > |X|/2$ induces a subgraph with maximum degree at least $|S|^c$ for some absolute constant $c > 0$? It turned out that the answer is negative. In fact there exists infinite families of non-abelian Cayley graphs with unbounded degree that contain induced subgraphs with maximum degree 1 on more than half of the vertices. The first such example was constructed by Lehner and Verret [LV20] using iterated wreath products.

Let X be a group. X acts naturally on the set of functions $\mathbb{Z}_2^X$ from X to $\mathbb{Z}_2$ by permuting the domain. More precisely, let $a \in \mathbb{Z}_2^X$ and $x \in X$, the element a^x is defined as the function $a^x(y) = a(x^{-1}y)$ for all $y \in X$. The set $\mathbb{Z}_2^X$ itself is also a group under point-wise addition. The wreath product $\mathbb{Z}_2 \wr X$ is the group consisting of elements of the form (a, x) where $a \in \mathbb{Z}_2^X$ and $x \in X$ with the binary operation defined by

$$(a, x)(b, y) = (a + b^x, xy).$$

Let $a_x \in \mathbb{Z}_2^X$ be the indicator function of the element x, i.e., $a_x(y) = 1$ if $y = x$ and $a_x(y) = 0$ otherwise. Lehner and Verret proved the following:

Lemma 4.4.1 ([LV20])**.** *Let X be a group with identity element e and S be a generating* *set for X. Let $Y = \mathbb{Z}_2 \wr X$ and $T = \{(a_e, e)\} \cup \{(0, s) : s \in S\}$. If $\Gamma(X, S)$ has an induced subgraph of maximum degree 1 on more than half of its vertices, then the same is true for $\Gamma(Y, T)$.*

Starting with $X = \mathbb{Z}_2$ and $S = \{1\}$, by iteratively applying the lemma, Lehner and Verret constructed an infinitely family of non-abelian Cayley graphs with unbounded degree that have an induced subgraph of maximum degree 1 on more than half of the vertices.

They also constructed an infinitely family of 3-regular Cayley graphs over dihedral groups that contain induced subgraph of maximum degree 1 on more than half of the vertices. García-Marco and Knauer [GMK22] showed how to construct Cayley graphs over dihedral groups of unbound degree that have a similar property. More precisely, let D_n denote the dihedral group

$$D_n = \langle a, b | a^n = b^2 = (ab)^2 = 1 \rangle.$$

For a positive integer m, denote by $[m]_3 \in \{1, 2\}$ the right-most nonzero entry in its representation in base 3.

Theorem 4.4.2 ([GMK22])**.** *Let $n = 3^d$ and $S = \{a^{3^i} b : 0 \leq i \leq d\}$. Then the Cayley graph $\Gamma(D_n, S)$ has an induced matching on $n + 1$ vertices. Specifically, the set $M = \{a^i : [i]_3 = 1\} \cup \{a^i b : [i]_3 = 2\} \cup \{1, b\}$ induces a matching with $n + 1$ vertices.*

They also considered Cayley graphs over the symmetric groups and provided further constructions of Cayley graphs that contain an induced subgraph of maximum degree 1 on more than half of the vertices. In particular, they showed that the star graph $SG_n = \Gamma(S_n, \{(12), (13), \ldots, (1n)\})$, which is an edge-transitive Cayley graph, has such a property. Specifically, let $\mathrm{supp}(\pi) = \{i \in [n] : \pi(i) \neq i\}$ be the set of elements that are not fixed by π.

Let $f : S_n \to \{u_1, u_2, u_3, v_1, v_2, v_3\}$ be defined as

$$f(\pi) = \begin{cases} u_1 & \text{if } |\mathrm{supp}(\pi) - \{1\}| = n-1 \text{ and } \pi \in A_n, \\ v_1 & \text{if } |\mathrm{supp}(\pi) - \{1\}| = n-1 \text{ and } \pi \notin A_n, \\ u_2 & \text{if } |\mathrm{supp}(\pi) - \{1\}| = n-2 \text{ and } \pi \notin A_n, \\ v_2 & \text{if } |\mathrm{supp}(\pi) - \{1\}| = n-2 \text{ and } \pi \in A_n, \\ u_3 & \text{if } |\mathrm{supp}(\pi) - \{1\}| < n-2 \text{ and } \pi \in A_n, \\ v_3 & \text{if } |\mathrm{supp}(\pi) - \{1\}| < n-2 \text{ and } \pi \notin A_n. \end{cases}$$

Theorem 4.4.3 ([GMK22])**.** *Let $U = f^{-1}(\{u_1, u_2, v_3\})$ and $V = f^{-1}(\{v_1, v_2, u_3\})$. Then $|U| \neq |V|$ and both U and V induced a subgraph of SG_n with maximum degree 1.*

It is not difficult to show that for d-regular graphs on n vertices, any induced subgraph with maximum degree d has size at most $\frac{dn}{2d-1}$. García-Marco and Knauer constructed Cayley graphs that contains an induced matching achieving this bound, answering a question of Lehner and Verret [LV20].

Theorem 4.4.4 ([GMK22])**.** *Let m be a positive integer. Let $X = S_{2m+1}$ if m is odd and $X = A_{2m+1}$ if m is even. Let $c_k \in S_{2m+1}$ be the order 2 permutation defined by*

$$c_k(i) = \begin{cases} i+m & \textit{if } i < k-m, \\ i+m+1 & \textit{if } k-m \leq i \leq m, \\ i-m & \textit{if } m < i < k, \\ i & \textit{if } i = k, \\ i-m-1 & \textit{if } k < i \leq 2m+1. \end{cases}$$

Let $S = \{c_k \in X : m+1 \leq k \leq 2m+1\}$. Then the Cayley graph $\Gamma(X, S)$ has degree $m+1$ and it contains an induced matching with $\frac{m+1}{2m+1}|X|$ vertices. Specifically, the set

$M = \{\pi \in X : \pi(1) \geq m+1\}$ *has size* $\frac{m+1}{2m+1}|X|$ *and it induces a matching in* $\Gamma(X, S)$.

CHAPTER 5

HAMMING GRAPH $H(n, 3)$

5.1 Introduction

We showed in the last chapter that for abelian Cayley graphs with a generating set of size d, any induced subgraph on more than half of its vertices must have maximum degree at least $\sqrt{d/2}$. When the graph is also bipartite, it shows that abelian Cayley graphs exhibit a similar behavior as Boolean hypercubes that there is a "jump" in the maximum degree when the size of the induced subgraph becomes bigger than its independence number.

However, our result on abelian Cayley does not imply a similar threshold behavior for non-bipartite abelian Cayley graphs. In particular, the Hamming graph $H(n, k)$ with vertex set equal to the set of strings of length n over k alphabet has independence number $\alpha(H(n, k)) = k^{n-1}$, only $1/k$ of the number of vertices. It turned out the Hamming graph does not admit the same threshold behavior despite being a similar to Boolean hypercubes. Dong [Don21] showed that there is an induced subgraph of $H(n, k)$ of size $\alpha(H(n, k)) + 1$ that has maximum degree at most $\lceil \sqrt{n} \rceil$. This was improved by Tanday [Tan22] who showed that $H(n, k)$ has an induced subgraph on $\alpha(H(n, k)) + 1$ vertices with maximum degree 1. [GMK22] independently discovered that $H(n, 3)$ has a 1-regular induced subgraph on $3^{n-1} + 1$ vertices.

In this chapter, we further investigate the maximum size of the induced subgraph of $H(n, 3)$ with maximum degree 1.

5.2 Preliminaries

Definition 5.2.1 (Hamming graph $H(n, 3)$)**.** The Hamming graph $H(n, 3)$ is the graph with vertices $\mathbb{Z}_3^n$ and edges $\{\{x, y\} \in \mathbb{Z}_3^n \times \mathbb{Z}_3^n : d_H(x, y) = 1\}$ where d_H is the Hamming distance.

Definition 5.2.2 (Standard basis for $\mathbb{Z}_3^n$)**.** We use the standard basis $e_1, \ldots, e_n$ for $\mathbb{Z}_3^n$. In other words, we take $e_i \in \mathbb{Z}_3^n$ to be the point which is 1 in coordinate i and 0 in the other coordinates.

Definition 5.2.3 (Induced degree)**.** We say that $U \subseteq \mathbb{Z}_3^n$ has *induced degree* d if U induces a subgraph of $H(n, 3)$ with maximum degree d.

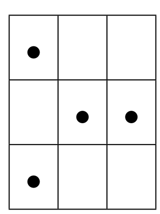
Throughout the paper, we will draw diagrams representing subsets $U \subseteq \mathbb{Z}_3^n$. For these diagrams, we will use the following conventions.

1. Each square represents a subset of $\mathbb{Z}_3^k$ for some $k \geq 0$. When $k = 0$, we represent a point of U by $\bullet$ and we represent the empty set ϕ by an empty space.

2. For each 3×3 block, the first column corresponds to $x_{k+1} = 0$, the second column corresponds to $x_{k+1} = 1$, and the third column corresponds to $x_{k+1} = 2$. Similarly, if there is more than one row, the first row corresponds to $x_{k+2} = 0$, the second column corresponds to $x_{k+2} = 1$, and the third column corresponds to $x_{k+2} = 2$.

3. When there is more than one block, each additional direction represents an additional coordinate.

Example 1. The following diagram shows the set of points $U = \{(0, 0), (0, 2), (1, 1), (2, 1)\}$. Note that $|U| = 4$ and U induces a subgraph of $H(n, 3)$ where every vertex has degree 1.



We now describe some basic facts about $H(n, 3)$ and maximum size independent sets of $H(n, 3)$.

Proposition 5.2.4. *The automorphisms of the Hamming graph $H(n,3)$ are generated by the following operations.*

1. *Permuting the coordinates.*
2. *Multiplying a coordinate by* 2.
3. *Adding $x \in \mathbb{Z}_3^n$ to every point.*

Remark. When we say two subsets of $\mathbb{Z}_3^n$ are isomorphic, we mean that there is an automorphism of the Hamming graph $H(n,3)$ mapping one subset to the other.

Definition 5.2.5. Given a graph G, let $\alpha(G)$ denote the maximum size of an independent set of G.

For the Hamming graph $H(n,3)$, we have that $\alpha(H(n,3)) = 3^{n-1}$ and the maximum size independent sets of $H(n,3)$ are hyperplanes of the vector space $\mathbb{Z}_3^n$.

Definition 5.2.6. Let $n \geq 2$ be a natural number. Given a set of vertices $S \subseteq \mathbb{Z}_3^{n-1}$, for each $c \in \mathbb{Z}_3^n$, we define $(S,c) \subseteq \mathbb{Z}_3^n$ to be the set of points

$$(S,c) = \{(x_1,\dots,x_n) \in \mathbb{Z}_3^n : (x_1,\dots,x_{n-1}) \in S, x_n = c\}$$

Proposition 5.2.7. *I is a maximum size independent set of $H(n,3)$ if and only if there exist $b \in \{1,2\}^n$ and $c \in \mathbb{Z}_3$ such that $b_1 = 1$ and $I = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^n b_i x_i \equiv c \mod 3\}$. Moreover, for each I of this form, the only independent sets of $H(n,3)$ of size 3^{n-1} which are disjoint from I are the independent sets $I' = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^n b_i x_i \equiv c+1 \mod 3\}$ and $I'' = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^n b_i x_i \equiv c+2 \mod 3\}$.*

Proof. We prove this by induction. For the base case $n = 1$, the only independent sets of size 1 are $\{0\}$, $\{1\}$, and $\{2\}$ which are given by $\{x \in \mathbb{Z}_3 : x \equiv 0 \mod 3\}$, $\{x \in \mathbb{Z}_3 : x \equiv 1$

mod 3$\}$, and $\{x \in \mathbb{Z}_3 : x \equiv 2 \mod 3\}$ respectively. All three of these indepdendent sets are disjoint.

For the inductive step, assume that the statement is true for $n \leq k$. By the inductive hypothesis, maximum size independent sets of $H(k,3)$ have size 3^{k-1} so the only way to obtain an independent set I of $H(k+1,3)$ of size 3^k is if $I = (I_0, 0) \cup (I_1, 1) \cup (I_2, 2)$ where I_0, I_1, I_2 are disjoint independent sets of Z_3^k of size 3^{k-1}. By the inductive hypothesis, $I_0 = \{x \in \mathbb{Z}_3^k : \sum_{i=1}^{k} b_i x_i \equiv c \mod 3\}$ for some $b \in \{1,2\}^k$ and $c \in \mathbb{Z}_3$ such that $b_1 = 1$. Moreover, since the only independent sets of $H(k,3)$ of size 3^{k-1} which are disjoint from I_0 are $I' = \{x \in \mathbb{Z}_3^k : \sum_{i=1}^{k} b_i x_i \equiv c+1 \mod 3\}$ and $I'' = \{x \in \mathbb{Z}_3^k : \sum_{i=1}^{k} b_i x_i \equiv c+2 \mod 3\}$, we must either have that $I_1 = I'$ and $I_2 = I''$ or $I_1 = I''$ and $I_2 = I'$. In the first case, we have that $I = \{x \in \mathbb{Z}_3^k : \sum_{i=1}^{k} b_i x_i + 2x_{k+1} \equiv c \mod 3\}$. In the second case, we have $I = \{x \in \mathbb{Z}_3^k : \sum_{i=1}^{k} b_i x_i + x_{k+1} \equiv c \mod 3\}$.

To show the moreover statement, observe that if $b, b' \in \{1,2\}^n$, $b_1 = b'_1 = 1$ and $b' \neq b$ then for any $c, c' \in \mathbb{Z}_3$, the linear equations $\sum_{i=1}^{n} b_i x_i \equiv c \mod 3$ and $\sum_{i=1}^{n} b'_i x_i \equiv c' \mod 3$ have 3^{n-2} common solutions. $\square$

For our analysis, it is useful to fix three disjoint maximum size independent sets of $H(n,3)$.

Definition 5.2.8. For each $n \in \mathbb{N}$, we define $A_n = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^{n} x_i = 0\}$, $B_n = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^{n} x_i = 1\}$ and $C_n = \{x \in \mathbb{Z}_3^n : \sum_{i=1}^{n} x_i = 2\}$.

The following recursive definition of A_n, B_n, and C_n is useful.

Proposition 5.2.9. *$A_1 = \{0\}$, $B_1 = \{1\}$ and $C_1 = \{2\}$ and for all $n \in \mathbb{N}$,*

1. *$A_{n+1} = (A_n, 0) \cup (C_n, 1) \cup (B_n, 2)$. We can represent this visually as* $A_{n+1} = \begin{array}{|c|c|c|}\hline A_n & C_n & B_n \\ \hline \end{array}$.

2. *$B_{n+1} = (B_n, 0) \cup (A_n, 1) \cup (C_n, 2)$. We can represent this visually as* $B_{n+1} = \begin{array}{|c|c|c|}\hline B_n & A_n & C_n \\ \hline \end{array}$.

3. *$C_{n+1} = (C_n, 0) \cup (B_n, 1) \cup (A_n, 2)$. We can represent this visually as* $C_{n+1} = \begin{array}{|c|c|c|}\hline C_n & B_n & A_n \\ \hline \end{array}$.

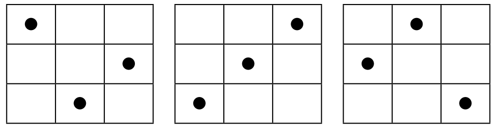


Figure 5.1: This figure shows the independent set A_3. Note that the first block of A_3 is A_2, the second block of A_3 is C_2, and the third block of A_3 is B_2.

For an illustration of the independent set A_3, see Figure 5.1.

5.2.1 Collapsing one dimension of $\mathbb{Z}_3^n$

For our analysis, it is very useful to collapse one dimension of $\mathbb{Z}_3^n$. To do this, we define the following subsets of $\mathbb{Z}_3$.

Definition 5.2.10. We define $A = A_1 = \{0\}$, $B = B_1 = \{1\}$, $C = C_1 = \{2\}$, $X = \mathbb{Z}_3 \setminus A = \{1, 2\}$, $Y = \mathbb{Z}_3 \setminus B = \{0, 2\}$, and $Z = \mathbb{Z}_3 \setminus C = \{0, 1\}$.

For an illustration of A, B, C, X, Y, and Z, see Figure 5.2.

Proposition 5.2.11. *A subset has maximum induced degree at most* 1 *if and only if the following conditions are satisfied:*

1. *Every A is adjacent to at most one other A and is not adjacent to any Y or Z.*
2. *Every B is adjacent to at most one other B and is not adjacent to any X or Z.*
3. *Every C is adjacent to at most one other C and is not adjacent to any X or Y.*
4. *Every X is only adjacent to A or ϕ.*
5. *Every Y is only adjacent to B or ϕ.*
6. *Every Z is only adjacent to C or ϕ.*

It is also useful to represent a subset U by a function $U_f(x)$ such that $U = \cup_{x \in \mathbb{Z}_3^{n-1}} (U_f(x) \times \{x\})$.

$$A = \begin{array}{|c|c|c|}\hline \bullet & & \\ \hline\end{array}, \quad B = \begin{array}{|c|c|c|}\hline & \bullet & \\ \hline\end{array}, \quad C = \begin{array}{|c|c|c|}\hline & & \bullet \\ \hline\end{array},$$
$$X = \begin{array}{|c|c|c|}\hline & \bullet & \bullet \\ \hline\end{array}, \quad Y = \begin{array}{|c|c|c|}\hline \bullet & & \bullet \\ \hline\end{array}, \quad Z = \begin{array}{|c|c|c|}\hline \bullet & \bullet & \\ \hline\end{array}$$

Figure 5.2: This figure shows A, B, C, X, Y, and Z.

Definition 5.2.12. Let U be a subset of $\mathbb{Z}_3^n$ such that for each $x \in \mathbb{Z}_3^{n-1}$, the line $\{x, x+e_1, x+2e_1\}$ contains at most two points of U. We define U_f to be the function $U_f : \mathbb{Z}_3^{n-1} \to \{\phi, A, B, C, X, Y, Z\}$ such that $U = \cup_{x \in \mathbb{Z}_3^{n-1}} (U_f(x) \times \{x\})$.

Definition 5.2.13 (Affine subsets)**.** We define an affine subset of $\mathbb{Z}_3^n$ to be a subset $H \subseteq \mathbb{Z}_3^n$ of the form

$$H = \{x \in \mathbb{Z}_3^n : \text{For all } i \in R_H, x_i = c_i\}$$

for some $R_H \subseteq [n]$ and elements $\{c_i : i \in R_H\}$ where each $c_i \in \mathbb{Z}_3$.

We say that an affine subset H contains direction e_i if $i \notin R_H$ (i.e., the value of x_i is not restricted by H). We say that two distinct affine subsets H and H' are *parallel* if $H' = H \pm e_j$ for some $j \in [n]$.

Definition 5.2.14. Given an affine subset H of $\mathbb{Z}_3^n$ which contains direction e_1, we define H_{red} to be the affine subset of $\mathbb{Z}_3^n$ such that $H = \mathbb{Z}_3 \times H_{red}$.

Definition 5.2.15 (i-saturated)**.** Given $U \subseteq \mathbb{Z}_3^n$ and $i \in [n]$, we say that U is *i-saturated* if $U \cap \{x, x+e_i, x+2e_i\} \neq \phi$ for all $x \in \mathbb{Z}_3^n$. In other words, U is i-saturated if and only if every line in direction e_i contains at least one point of U.

Proposition 5.2.16. *If $U \subseteq \mathbb{Z}_3^n$, U is 1-saturated and U induces a subgraph of maximum degree at most 1 then for all $x \in \mathbb{Z}_3^{n-1}$, $U_f(x) \in \{A, B, C, X, Y, Z\}$.*

5.3 Canonical sets

We first consider subsets of $\mathbb{Z}_3^n$ that have induced degree 1 and are disjoint from a maximum size independent set of $H(n,3)$. It was known that such subsets can have size $3^{n-1} +$

1 [GMK22]. In this section, we show that this is the maximum possible size for such a subset. Specifically, if $U \subseteq \mathbb{Z}_3^n$ has induced degree 1 and U is disjoint from a maximum size independent set of $H(n,3)$, then $|U| \leq 3^{n-1}+1$. Moreover, up to isomorphism, there is only one such subset of size $3^{n-1}+1$.

Definition 5.3.1. We say that $U \subseteq \mathbb{Z}_3^n$ is a *canonical set* if $|U| \geq 3^{n-1}+1$, U is disjoint from a maximum size independent set of $H(n,3)$, and U has induced degree at most 1.

5.3.1 Definition of D_n and facts about D_n

Up to isomorphism, the only canonical set in $\mathbb{Z}_3^n$ is the set D_n which is defined as follows.

Definition 5.3.2. We define D_n recursively.

1. $D_1 = \{1,2\}$

2. For all $n \in \mathbb{N}$, $D_{n+1} = (D_n, 0) \cup (A_n, 1) \cup (A_n, 2)$.

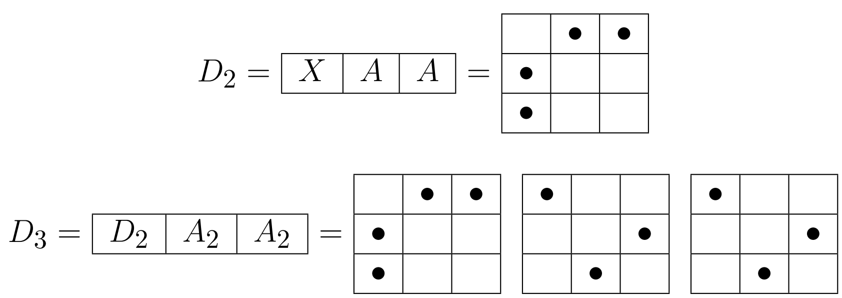


Figure 5.3: This figure shows D_2 and D_3.

Proposition 5.3.3. *For all $n \in \mathbb{N}$, $A_n \cap D_n = \phi$ and all vertices in D_n have degree* 1.

Proof. We prove this by induction. For $n = 1$, $D_1 = \{1,2\}$ so both of the vertices in D_1 have degree 1. Since $A_1 = \{0\}$, $D_1 \cap A_1 = \phi$, as needed.

For the inductive step, assume that all vertices in D_n have degree 1 and $A_n \cap D_n = \phi$. Recall that

1. $A_{n+1} = \boxed{A_n \mid C_n \mid B_n}$

2. $D_{n+1} = \boxed{D_n \mid A_n \mid A_n}$

Since $A_n \cap D_n = \phi$, $A_n \cap B_n = \phi$, and $A_n \cap C_n = \phi$, $D_{n+1} \cap A_{n+1} = \phi$. To see that all vertices in D_{n+1} have degree 1, observe the following:

1. By the inductive hypothesis, $A_n \cap D_n = \phi$ and each vertex in D_n has degree 1.

2. Each vertex in one of the copies of A_n is only adjacent to the same vertex in the other copy of A_n.

$\square$

Proposition 5.3.4. *For all $n \in \mathbb{N}$,*

1. *There is exactly one line of the form $\{x, x + e_1, x + 2e_1\}$ which contains two points of D_n.*

2. *For all $k \in [2, n]$, there are exactly 3^{k-2} lines of the form $\{x, x + e_k, x + 2e_k\}$ which contain two points of D_n.*

Corollary 5.3.5. *For all $n \geq 2$, D_n is 1-saturated and 2-saturated but is not i-saturated for any $i \geq 3$.*

We now observe that for all $n \in \mathbb{N}$, there is exactly one other maximum size independent set of $H(n, 3)$ which is disjoint from D_n.

Definition 5.3.6. Define

$$A'_n = \left\{ x \in \mathbb{Z}_3^n : \left(\sum_{i=1}^{n-1} x_i \right) + 2x_n \equiv 0 \mod 3 \right\} = (A_{n-1}, 0) \cup (B_{n-1}, 1) \cup (C_{n-1}, 2).$$

Proposition 5.3.7. *For all $n \in \mathbb{N}$, if I is a maximum size independent set of $H(n, 3)$ and $I \cap D_n = \phi$ then $I = A_n$ or $I = A'_n$.*

Proof. We prove this for $n+1$ rather than n to make the diagrams nicer.

Recall that $D_{n+1} = \begin{array}{|c|c|c|}\hline D_n & A_n & A_n \\ \hline\end{array}$ and let I be an independent set of $H(n+1,3)$ of maximum size which is disjoint from D_{n+1}. Since the only independent sets of $H(n,3)$ of maximum size which are disjoint from A_n are B_n and C_n, I must be equal to one of the following two possibilities

1. $I = \begin{array}{|c|c|c|}\hline I' & B_n & C_n \\ \hline\end{array}$
2. $I = \begin{array}{|c|c|c|}\hline I' & C_n & B_n \\ \hline\end{array}$

for some independent set I' of $H(n,3)$ of maximum size. Since A_n is the only independent set of $H(n,3)$ of maximum size which is disjoint from B_n and C_n, we must have $I' = A_n$. This implies that $I = A_{n+1}$ or $I = A'_{n+1}$, as needed. $\square$

Before proving our uniqueness theorem, we need one more fact.

Lemma 5.3.8. *If $U \subseteq \mathbb{Z}_3^n$ is isomorphic to D_n and $A_n \cap U = \phi$ then $|U \cap B_n| = |U \cap C_n| = \frac{|U|}{2}$.*

Proof. Recall that every vertex of D_n has degree 1 so there is a matching M of size $\frac{3^{n-1}+1}{2}$ between the vertices of U. For each edge $\{u,v\} \in M$, at most one of u and v are in B_n as B_n is an independent set. Similarly, at most one of u and v are in C_n. Since $u, v \notin A_n$ as $U \cap A_n = \phi$, either u in B_n and $v \in C_n$ or v in B_n and $u \in C_n$. Since this is true for all edges $\{u,v\} \in M$, the result follows. $\square$

5.3.2 Uniqueness of canonical sets

We are now ready to prove our uniqueness theorem.

Theorem 5.3.9. *Let $U \subseteq \mathbb{Z}_3^n$. If $|U| \geq 3^{n-1}+1$, U has induced degree 1 and U is disjoint from a maximum size independent set of $H(n,3)$, then there exists $\sigma \in \mathrm{Aut}(H(n,3))$ such that $\sigma(U) = D_n$.*

Proof. We prove this by induction on n. For $n = 1$ and $n = 2$, it can be checked directly that up to isomorphism, D_n is the unique subset of $\mathbb{Z}_3^n$ of size at least $3^{n-1} + 1$ which has induced degree 1.

For the inductive step, assume the result is true for $\mathbb{Z}_3^{n+1}$ and let U be a subset of $\mathbb{Z}_3^{n+2}$ such that $|U| \geq 3^{n+1} + 1$, U has induced degree 1, and U is disjoint from a maximum size independent set of $H(n + 2, 3)$.

Writing $U = (U_0, 0) \cup (U_1, 1) \cup (U_2, 2)$, at least one of U_0, U_1, and U_2 must have size larger than 3^n. By applying an appropriate translation, we can assume that $|U_0| > 3^n$. Since U_0 is disjoint from an independent set, we can apply an automorphism of $H(n + 1, 3)$ so that $U_0 = D_{n+1}$. After we do this, we have that

$$U = \begin{array}{|c|c|c|} \hline D_n & A_n & A_n \\ \hline U_{01} & U_{11} & U_{21} \\ \hline U_{02} & U_{12} & U_{22} \\ \hline \end{array}$$

for some subsets $U_{01}, U_{11}, U_{21}, U_{02}, U_{12}, U_{22}$ of $\mathbb{Z}_3^n$. Following similar logic as before and swapping the second and third rows and/or the second and third columns if needed, we can assume that U is disjoint from the independent set

$$A_{n+2} = \begin{array}{|c|c|c|} \hline A_n & C_n & B_n \\ \hline C_n & B_n & A_n \\ \hline B_n & A_n & C_n \\ \hline \end{array}$$

We now make the following observations:

1. Since $|U| \geq 3^{n+1} + 1$, $|U_{01}| + |U_{11}| + |U_{21}| + |U_{02}| + |U_{12}| + |U_{22}| \geq 6 \cdot 3^{n-1}$.

2. $|U_{01}| + |U_{02}| \leq 2 \cdot 3^{n-1}$ as otherwise the first column would be disjoint from an independent set and would have more than $3^n + 1$ points, which would contradict the inductive hypothesis.

3. $U_{11} \subseteq C_n$ as U_{11} is disjoint from both A_n and B_n.

4. $U_{22} \subseteq B_n$ as U_{22} is disjoint from both A_n and C_n.

5. $|U_{12}| \leq 3^{n-1} + 1$ and if $|U_{12}| = 3^{n-1} + 1$ then $|U_{11}| \leq 3^{n-1} - \frac{|U_{12}|}{2}$ and $|U_{22}| \leq 3^{n-1} - \frac{|U_{12}|}{2}$. To see this, observe that if $|U_{12}| > 3^{n-1}$ then since U_{12} is disjoint from A_n, U_{12} is a canonical set. By the inductive hypothesis, $|U_{12}| = 3^{n-1} + 1$ and every vertex of U_{12} has degree 1. This implies that $U_{12} \cap U_{11} = \phi$ and $U_{12} \cap U_{22} = \phi$. By Lemma 5.3.8, $|U_{12} \cap C_n| = \frac{|U_{12}|}{2}$ so since $U_{11} \subseteq C_n$ and $U_{11} \cap U_{12} = \phi$, $|U_{11}| \leq 3^{n-1} - \frac{|U_{12}|}{2}$

6. Following similar logic, $|U_{21}| \leq 3^{n-1} + 1$ and if $|U_{21}| = 3^{n-1} + 1$ then $|U_{11}| \leq 3^{n-1} - \frac{|U_{21}|}{2}$ and $|U_{22}| \leq 3^{n-1} - \frac{|U_{21}|}{2}$.

Combining these observations, there are two possibilities for the sizes of $U_{01}, U_{11}, U_{21}, U_{02}, U_{12}, U_{22}$.

1. $|U_{01}| + |U_{02}| = 2 \cdot 3^{n-1}$ and $|U_{11}| = |U_{12}| = |U_{21}| = |U_{22}| = 3^{n-1}$.

2. $n = 1$, $|U_{01}| + |U_{02}| = 2$, $|U_{12}| = |U_{21}| = 2$, and $|U_{11}| = |U_{22}| = 0$.

If $|U_{01}| + |U_{02}| = 2 \cdot 3^{n-1}$ and $|U_{11}| = |U_{12}| = |U_{21}| = |U_{22}| = 3^{n-1}$, we have that

$$U = \begin{array}{|c|c|c|} \hline D_n & A_n & A_n \\ \hline U_{01} & C_n & U_{21} \\ \hline U_{02} & U_{12} & B_n \\ \hline \end{array}$$

We make the following further observations:

1. $U_{12} = B_n$ or $U_{12} = C_n$. To see this, observe that $U_{12} \cap A_n = \phi$ and U_{12} must be an independent set as for every vertex $u \in U_{12}$, either $u \in U_{11} = C_n$ or $u \in U_{22} = B_n$ so if u had degree 1 in U_{12} then U would have a degree 2 vertex.

2. Following similar logic, $U_{21} = B_n$ or $U_{21} = C_n$.

3. $U_{01} = U_{02} = A_n$. To see this, observe that by the first two observations, $U_{12} = C_n$ and $U_{21} = B_n$ or $U_{12} = B_n$ and $U_{21} = C_n$. In either case, U_{01} and U_{02} must be disjoint from both B_n and C_n as U is disjoint from A_{n+2}.

Combining these observations, we have that

$$U = \begin{array}{|c|c|c|}\hline D_n & A_n & A_n \\ \hline A_n & C_n & B_n \\ \hline A_n & C_n & B_n \\ \hline \end{array} \text{ or } \begin{array}{|c|c|c|}\hline D_n & A_n & A_n \\ \hline A_n & C_n & C_n \\ \hline A_n & B_n & B_n \\ \hline \end{array}$$

The first case is D_{n+2} and the second case can be transformed into D_{n+2} by swapping coordinates $n+1$ and $n+2$ (which corresponds to swapping the rows and columns.)

If $n = 1$, $|U_{01}| + |U_{02}| = 2$, $|U_{12}| = |U_{21}| = 2$, and $|U_{11}| = |U_{22}| = 0$ then we must have that

$$U = \begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & & X \\ \hline A & X & \\ \hline \end{array} \simeq \begin{array}{|c|c|c|}\hline & \bullet & \bullet \\ \hline \bullet & & \\ \hline \bullet & & \\ \hline \end{array} \; \begin{array}{|c|c|c|}\hline \bullet & & \\ \hline & & \bullet \\ \hline & \bullet & \\ \hline \end{array} \; \begin{array}{|c|c|c|}\hline \bullet & & \\ \hline & & \bullet \\ \hline & \bullet & \\ \hline \end{array}.$$

Note that the right hand side is obtained from the left hand side by replacing A with $\{0\}$ and $X = \{1, 2\}$ where the replacement is done in the third coordinate rather than the first coordinate. As shown by Figure 5.3, the right hand side is D_3 so we have that $U \simeq D_3$, as needed. $\square$

We now make some useful observations about the structure of canonical sets.

Definition 5.3.10 (Popular direction)**.** Let U be a canonical subset U of $\mathbb{Z}_3^n$. We define the *popular direction* of U to be the unique $i \in [n]$ such that there are exactly 3^{n-2} lines of the form $\{x, x + e_i, x + 2e_i\}$ which contains two points of U.

Definition 5.3.11 (Extra point)**.** Given a 1-saturated canonical subset U of $\mathbb{Z}_3^n$, writing $U = \cup_{x \in \mathbb{Z}_3^{n-1}} (U_f(x) \times \{x\})$, we define the *extra point* x_U of U to be the unique $x_U \in \mathbb{Z}_3^{n-1}$ such that $U_f(x_U)$ is X, Y, or Z.

Corollary 5.3.12. *If U is a 1-saturated canonical subset U of $\mathbb{Z}_3^n$ then for all affine subsets $H \subseteq \mathbb{Z}_3^n$ containing the direction e_1, if H contains the extra point of U then $U \cap H$ is a canonical set for H.*

Proof. We prove this statement by induction. The base case $n = 1$ is trivial. For the inductive step, assume that the result is true when $n = k$ and consider the case when $n = k + 1$.

Since U is a 1-saturated canonical set, there is a popular direction $i \in [k+1]$ for U which is not 1. Let $U_0 = \{x \in U : x_i = 0\}$, $U_1 = \{x \in U : x_i = 1\}$, and $U_2 = \{x \in U : x_i = 2\}$. Similarly, let $H_0 = \{x \in H : x_i = 0\}$, $H_1 = \{x \in H : x_i = 1\}$, and $H_2 = \{x \in H : x_i = 2\}$. We must have that when we ignore coordinate i, two of U_0, U_1, and U_2 are copies of the same independent set I while the third is a canonical set which is disjoint from I. Without loss of generality, we can assume that when we ignore coordinate i, $U_1 = U_2 = I$ and U_0 is a canonical set which is disjoint from I. There are two cases to consider.

1. H contains the direction e_i. In this case, H_0 contains the extra point of U which is also the extra point of U_0. By the inductive hypothesis, $U_0 \cap H_0$ is a canonical set for H_0. When we ignore coordinate i, $U_1 = U_2 = I$ and $U_0 \cap I = \phi$ so we must have that $U_1 \cap H_1$ and $U_2 \cap H_2$ are two copies of the same independent set which are disjoint from $U_0 \cap H_0$. This implies that $U \cap H$ is a canonical set.

2. H does not contain the direction e_i. In this case, since H contains the extra point of U, we must have that $H = H_0$. Since $H = H_0$ contains the extra point of U_0 and U_0 is a canonical set, the result follows from the inductive hypothesis.

□

5.3.3 Canonical paths

For our analysis, it is useful to define canonical paths on U_f. We will show that if U is a 1-saturated subset of $\mathbb{Z}_3^n$ which induces a subgraph of maximum degree 1 then for each $x \in \mathbb{Z}_3^{n-1}$ such that $U_f(x) \in \{X, Y, Z\}$, there must be a large number of $x' \in \mathbb{Z}_3^{n-1}$ such that the canonical path starting at x' ends at x. This implies that there cannot be too many $x \in \mathbb{Z}_3^{n-1}$ such that $U_f(x) \in \{X, Y, Z\}$.

Definition 5.3.13. Let U be a 1-saturated subset of $\mathbb{Z}_3^n$ which induces a subgraph of maximum degree at most 1 and write $U = \cup_{x \in \mathbb{Z}_3^{n-1}}(U_f(x) \times \{x\})$. For each $x \in \mathbb{Z}_3^{n-1}$, we define the canonical path P_x starting at x as follows.

If there is a direction $i \in [n-1]$ such that $U_f(x+e_i) = U_f(x)$ or $U_f(x+2e_i) = U_f(x)$ then we do the following:

1. If $U_f(x+e_i) = U_f(x)$ then we take the point $y = x + 2e_i$ and then take the canonical path P_y starting from y.

2. If $U_f(x+2e_i) = U_f(x)$ then we take the point $y = x + e_i$ and then take the canonical path P_y starting from y.

If there is no direction $i \in [n-1]$ such that $U_f(x+e_i) = U_f(x)$ or $U_f(x+2e_i) = U_f(x)$ then we end P_x at x.

A key observation for our upper bound in section 5.5 is that on 1-saturated canonical sets U, all canonical paths end at the extra point of U.

Lemma 5.3.14. *For all $n \in \mathbb{N}$, writing $D_n = \{D_{n,f}(x) \times x : x \in \mathbb{Z}_3^{n-1}\}$, we have that for all $x \in \mathbb{Z}_3^{n-1}$, the canonical path P_x starting at x ends at the extra point $x_{D_n} = (0, \ldots, 0)$ of D_n.*

Proof. We prove this statement by induction. The base case $n = 1$ is trivial. For the inductive step, assume that the result holds for D_n and consider $D_{n+1} = \boxed{D_n \mid A_n \mid A_n}$.

If $x \in D_n$ then the result follows from the inductive hypothesis. If x is in one of the copies of A_n then the next point y will be in D_n so the result follows from the inductive hypothesis. $\square$

Example 2. Observe that

$$D_4 = \begin{array}{|c|c|c|}\hline D_3 & A_3 & A_3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & C & B \\ \hline A & C & B \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}$$

If we start with the A at $x = (2, 1, 2)$ (the middle right point of the right block), the next points of P_x are the B at $(2, 1, 0)$, the A at $(2, 0, 0)$, and the X at $(0, 0, 0)$, i.e., the middle right, upper right, and upper left points of the left block.

Lemma 5.3.15. *Let U be a 1-saturated subset of $\mathbb{Z}_3^n$ which has induced degree at most 1 and let $H' = \mathbb{Z}_3 \times H'_{red}$ and $H'' = \mathbb{Z}_3 \times H''_{red}$ be two affine subsets of $\mathbb{Z}_3^n$ containing the direction e_1. Taking $U' = U \cap H'$ and $U'' = U \cap H''$ and writing $U' = \cup_{x \in H'_{red}}(U'_f(x) \times \{x\})$ and $U'' = \cup_{x \in H''_{red}}(U''_f(x) \times \{x\})$,*

1. *If U' is a canonical set for H' then for all $x \in H'_{red}$, the canonical path P_x starting at x ends at the extra point $x_{U'}$ of U'.*

2. *If U' is a canonical set for H', U'' is a canonical set for H'', and $H' \cap H'' \neq \phi$ then $x_{U'} = x_{U''}$ and $U' \cap U''$ is a canonical set for $H' \cap H''$.*

Proof. We prove the first statement by induction. For the first statement, the base case $n = 1$ is trivial. For the inductive step, assume that the first statement is true when $n \leq k$ and consider the case when $U \subseteq \mathbb{Z}_3^{k+1}$.

Observe that if U' is a canonical set, since U and thus U' are 1-saturated, there is a popular direction $i \in [k+1]$ for U' which is not 1. Let $U'_0 = \{x \in U' : x_i = 0\}$, $U'_1 = \{x \in U' : x_i = 1\}$, and $U'_2 = \{x \in U' : x_i = 2\}$. Observe that $U'_0 = U_0 \cap H'_0$,

$U_1' = U_1 \cap H_1'$, and $U_2' = U_2 \cap H_2'$ where $U_0 = \{x \in U : x_i = 0\}$, $U_1 = \{x \in U : x_i = 1\}$, $U_2 = \{x \in U : x_i = 2\}$, $H_0' = \{x \in H' : x_i = 0\}$, $H_1' = \{x \in H' : x_i = 1\}$, and $H_2' = \{x \in H' : x_i = 2\}$.

We must have that when we ignore coordinate i, two of U_0', U_1', and U_2' are copies of the same independent set I while the third is a canonical set which is disjoint from I. Without loss of generality, we can assume that when we ignore coordinate i, $U_1' = U_2' = I$ and U_0' is a canonical set which is disjoint from I. By the inductive hypothesis, $(H_0')_{red}$ contains the extra point $x_{U'}$ of U' and all canonical paths starting in $(H_0')_{red}$ end at $x_{U'}$. Since all canonical paths starting in $(H_1')_{red}$ or $(H_2')_{red}$ reach U_0' after their first step, all canonical paths starting in H'_{red} must end at $x_{U'}$.

We now show the second statement. Observe that by the first statement, for all $x \in H'_{red}$, P_x ends at $x_{U'}$. Similarly, for all $x \in H''_{red}$, P_x ends at $x_{U''}$. Thus, we must have that $x_{U'} = x_{U''}$. Since $x_{U'} \in H'_{red}$ and $x_{U''} \in H''_{red}$, $x_{U'} = x_{U''} \in (H' \cap H'')_{red}$. By Corollary 5.3.12, $U' \cap U'' = U' \cap (H' \cap H'') = U'' \cap (H' \cap H'')$ is a canonical set for $H' \cap H''$. $\square$

The fact that $U' \cap U''$ is a canonical set for $H' \cap H''$ depends on the fact that U is 1-saturated and H and H' contain the direction e_1. When we consider general subsets of $\mathbb{Z}_3^n$ rather than 1-saturated subsets, it is possible for two canonical sets to have an intersection which is not a canonical set. We show this by giving a $U \subseteq \mathbb{Z}_3^6$ where there are two canonical subsets of U which are isomorphic to D_4 and intersect in a 2-dimensional affine subset where they only have two points.

The first canonical set is $S = \{x \in U : x_4 = x_5 = 0\}$ (i.e., the points in the first set of blocks) and the second canonical set is $S' = \{x \in U : x_1 = 1, x_3 = 2\}$ (i.e., all of the B shown in gray cells). Note that $S \cap S'$ consists of the two B in the bottom row of the first set of blocks which are marked with a star.

X	A	A
A	B	B
A	C	C

A	C	B
B	A	C
C	B^*	A

A	C	B
B	A	C
C	B^*	A

A	C	C
C	A	A
	B	B

C	B	A
A	C	B
B	A	C

C	B	A
A	C	B
B	A	C

A	C	C
C	A	A
	B	B

C	B	A
A	C	B
B	A	C

C	B	A
A	C	B
B	A	C

C	B	B
Z	C	C
C	A	A

B	A	C
C	B	A
A	C	B

B	A	C
C	B	A
A	C	B

A	B	B
	C	C
B	A	A

B	A	C
C	B	A
A	C	B

B	A	C
C	B	A
A	C	B

	A	A
C	B	B
B	C	C

A	C	B
B	A	C
C	B	A

A	C	B
B	A	C
C	B	A

A	B	B
	C	C
B	A	A

B	A	C
C	B	A
A	C	B

B	A	C
C	B	A
A	C	B

	A	A
C	B	B
B	C	C

A	C	B
B	A	C
C	B	A

A	C	B
B	A	C
C	B	A

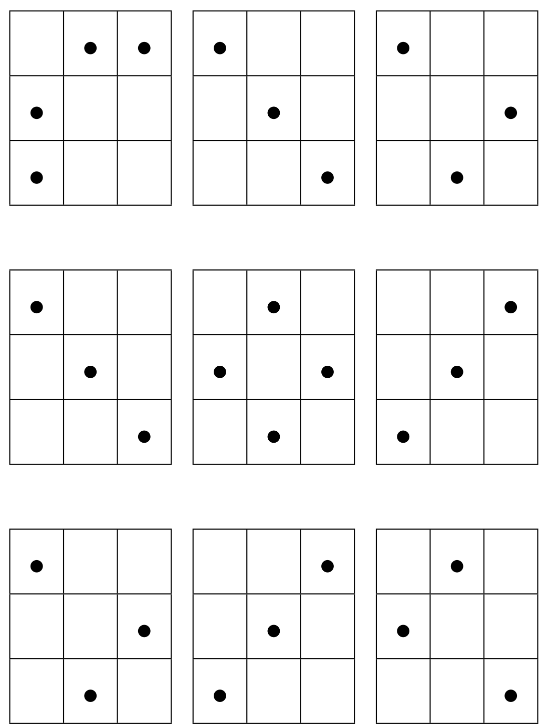
B	C	C
B	A	A
Y	B	B

C	B	A
A	C	B
B	A	C

C	B	A
A	C	B
B	A	C

5.4 Size lower bound

Recall that $\alpha(H(n,3)) = 3^{n-1}$ is the size of the largest independent set of $H(n,3)$. For $n = 1$ and 2, the largest induced degree 1 subset of $\mathbb{Z}_3^n$ has size $\alpha(H(n,3))+1$ and these subsets are unique up to isomorphism. For $n = 3$, it is not hard to show that the largest induced degree 1 subset still has size $\alpha(H(n,3)) + 1$ but there are two non-isomorphic subsets. For $n = 4$, it is possible to have an induced degree 1 subset of size strictly greater than $\alpha(H(n,3)) + 1$. The following set has size $\alpha(H(n,3)) + 2 = 29$.



In fact, it can be shown that not only is this set the largest possible subset of $\mathbb{Z}_3^4$ with induced degree 1, but it is also the only one up to isomorphism.

Theorem 5.4.1. *Up to isomorphism, there is a unique set $X \subseteq \mathbb{Z}_3^4$, X has maximum induced degree 1, and $|X| = 3^3 + 2$.*

Proof.

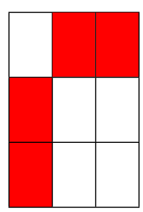
Proposition 5.4.2. *For any $n \geq 3$ and any set $X \subseteq \mathbb{Z}_3^n$, if there are two parallel affine* *subsets of dimension* 2 *(i.e.,* 3×3 *blocks) which each contain at least* 4 *points of* X *then the maximum induced degree of* X *is at least* 2*.*

Proof. Assume that $X \subseteq \mathbb{Z}_3^n$, there are two parallel affine subsets of dimension 2 (i.e., 3×3 blocks) which each contain at least 4 points of X, and the maximum induced degree of X is at most 1.

Up to isomorphism, there is only one subset of $\mathbb{Z}_3^2$ of size 4 with maximum induced degree at most 1 so without loss of generality we can assume that one of the affine subsets is as follows.

	•	•
•		
•		

Since the maximum induced degree of X is 1, in the second parallel affine subset, X cannot contain any of the points shown in red.



Since X can contain at most three of the remaining points, X cannot contain 4 points of this affine subset which gives a contradiction. □

We now consider the possible ways for the points of X to be divided up when we split $\mathbb{Z}_3^4$ into nine 3×3 blocks. By Proposition 5.4.2, if X has maximum induced degree 1 and $|X| = 3^3 + 2 = 29$ then up to isomorphism, the only possibilities are as follows.

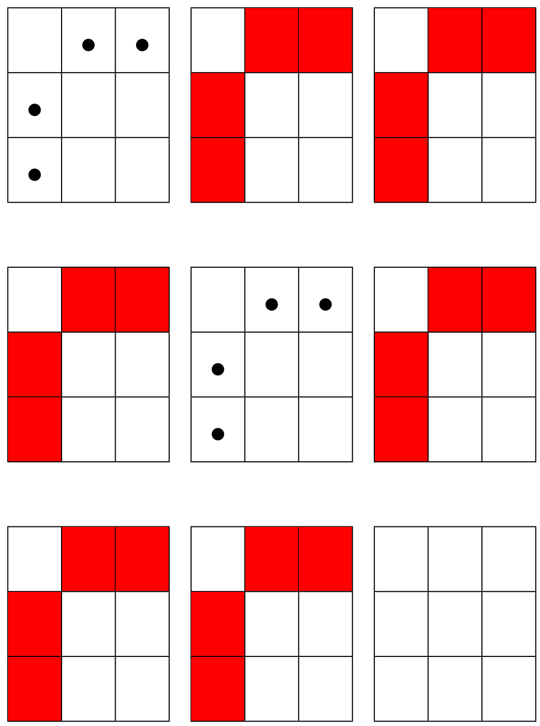
4	3	3
3	4	3
3	3	3

or

4	3	3
3	4	3
2	3	4

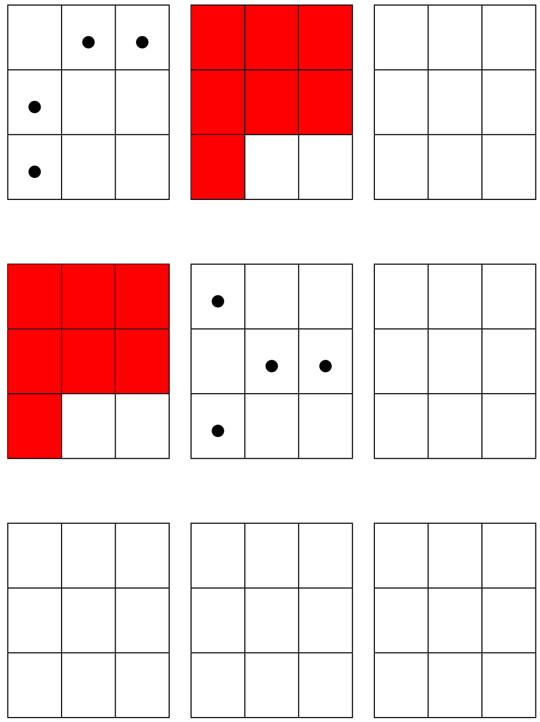
If we consider the possibilities for two of the 3×3 blocks with four points in X then up to isomorphism, there are three possibilities.

First, these 3×3 blocks may be the same. In this case, if we look at the six neighboring blocks, X cannot contain any of the points shown in red.



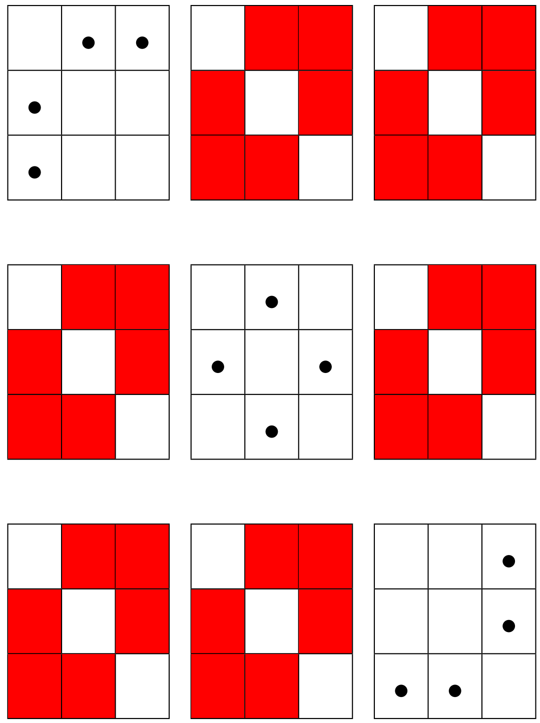
Observe that X can contain at most 4 of the six upper left corners so at least two of these neighboring 3×3 blocks can only contain 2 points of X. This is impossible as all but one of these 3×3 blocks must have at least 3 points of X.

The second possibility is that one of the 3×3 blocks is obtained from the other by either swapping the row containing two points of X with a row containing one point of X or swapping the column containing two points of X with a column containing one point of X (but not both). In this case, if we look at the two blocks which neighbor both of these blocks, the points shown in red cannot be in X so both of these blocks contain at most 2 points of X. Again, this gives a contradiction.



The third possibility is that the one of the 3×3 blocks is obtained from the other by swapping the row containing two points of X with a row containing one point of X and swapping the column containing two points of X with a column containing one point of X.

We now show that there is no way for the points of X to have the division $\begin{array}{|c|c|c|}\hline 4 & 3 & 3 \\ \hline 3 & 4 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}$ and up to isomorphism, there is a unique way to have the division $\begin{array}{|c|c|c|}\hline 4 & 3 & 3 \\ \hline 3 & 4 & 3 \\ \hline 3 & 3 & 3 \\ \hline \end{array}$. To see that there is no way to have the division $\begin{array}{|c|c|c|}\hline 4 & 3 & 3 \\ \hline 3 & 4 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}$, observe that if each pair of 3×3 blocks with 4 points of X satisfies the third possibility then up to ismorphism, these points must be arranged as follows.

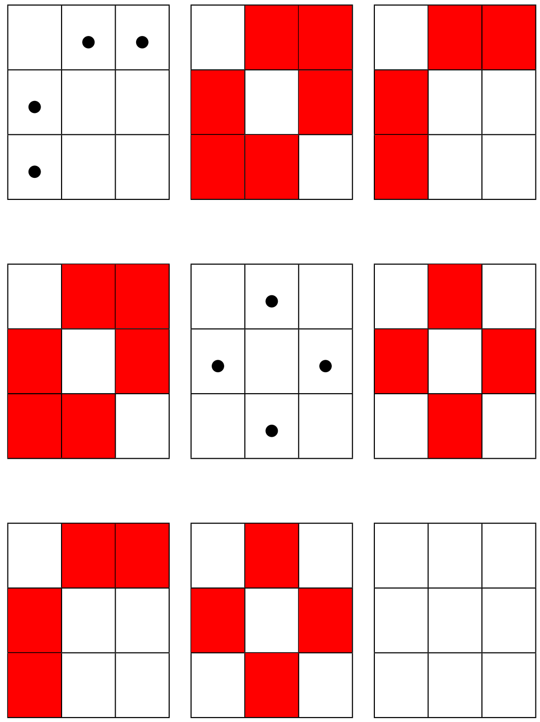


Observe that X can only contain four of the six points in the upper left corners of the six remaining blocks. Similarly, X can only contain 4 of the 6 middle points of these blocks and X can only contain 4 of the 6 bottom right points of these blocks.

Finally, we consider the case where the points of X are divided as

4	3	3
3	4	3
3	3	3

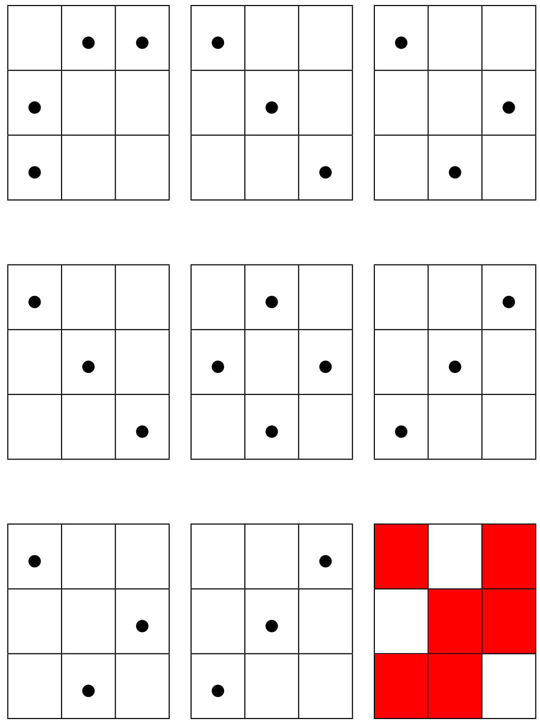
and the pair of blocks with four points of X satisfy the third possibility. In this case, up to isomorphism, these blocks must have the following points. This implies that the points shown in red cannot be in X.



For the bottom left and upper right blocks, X must contain the top left point. Moreover, we must either have that the middle and bottom right points are in X or the middle right and bottom middle points are in X. Observe that the first case is impossible as this would eliminate too many possible points from the middle right or bottom middle block. Thus, for the bottom left and upper right blocks, X must contain the top left, middle right, and bottom middle points. Note that this eliminates the top left point of the middle right and bottom middle blocks.

For the middle right and bottom middle blocks, observe that X cannot contain the bottom right point as this would eliminate both the top right and the bottom left points. Thus, for the the middle right and bottom middle blocks, X must contain the top right, middle, and bottom left points.

Thus, the points for all of the blocks except the bottom right block must be as follows.



For the bottom right block, the points shown in red cannot be in X. There are exactly three points remaining so X must contain these points. $\square$

This set demonstrates some interesting properties. Denote the set as X_4 and let $N(x)$ denote the set of neighbors of x:

1. There exists a subset $O \subseteq \mathbb{Z}_3^4$ such that $\cup_{x \in O} N(x) \subseteq X_4$. Moreover, the only vertices with degree 1 in the induced subgraph are the vertices in $\cup_{x \in O} N(x)$.

2. X_4 is i-saturated for all $i \in [4]$.

Almost all the extremal subsets for $n \leq 4$ satisfy both properties, with D_3 the canonical set of dimension 3 being the only exception. D_3 satisfies neither of the two properties, but it satisfies a weaker form of property 2: it is i-saturated for some $i \in [3]$.

All the extremal subsets for $n \leq 4$ are i-saturated for some i. By permutating the coordinates if needed, we can assume without loss of generality that they are 1-saturated and hence we can represent them by functions $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ such that $U = \cup_{x \in \mathbb{Z}_3^{n-1}} (U_f(x) \times \{x\})$. The extremal subset X_4 for $\mathbb{Z}_3^4$ can be written as follows

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & Y & B \\ \hline C & B & A \\ \hline \end{array}\;
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}\,.
$$

5.4.1 Finding subsets via SAT solvers

By Proposition 5.2.11, subsets U with induced degree at most 1 can be characterized by a set of conditions on the adjacent blocks. If in addition U also 1-saturated, then it can be characterized as a solution to a certain CNF formula. Specifically, Proposition 5.2.11 can be rephrased as the following fact regarding the function representation.

Proposition 5.4.3. *Let $U_f : \mathbb{Z}_3^{n-1} \to \{A, B, C, X, Y, Z\}$ and $U = \cup_{x \in \mathbb{Z}_3^{n-1}} (U_f(x) \times \{x\})$. Then U has induced degree 1 if and only if for all $x \in \mathbb{Z}_3^{n-1}$ and distinct $y, z \in N(x)$, U_f satisfies the following constraints:*

1. *$U_f(x), U_f(y)$ and $U_f(z)$ are not all equal.*

2. *If $U_f(x) = X$, then $U_f(y) = A$.*

3. *If $U_f(x) = Y$, then $U_f(y) = B$.*

4. *If $U_f(x) = Z$, then $U_f(y) = C$.*

To construct our SAT instance, we create a variable $v_{x,E}$ for each $x \in \mathbb{Z}_3^{n-1}$ and $E \in \{A, B, C, X, Y, Z\}$ that indicates if $U_f(x) = E$. We have the following formula for each $x \in \mathbb{Z}_3^{n-1}$.

$$Assign_x(v) = \Big(\bigvee_{E \in \{A,B,C,X,Y,Z\}} v_{x,E} \Big) \wedge \Big(\bigwedge_{E,F \in \{A,B,C,X,Y,Z\}, E \neq F} \neg v_{x,E} \vee \neg v_{x,F} \Big).$$

For each $x \in \mathbb{Z}_3^{n-1}$ and distinct $y, z \in N(x)$, the first constraint in Proposition 5.4.3 can be represented by the CNF

$$NAE_{x,y,z}(v) = \bigwedge_{E \in \{A,B,C,X,Y,Z\}} (\neg v_{x,E} \vee \neg v_{y,E} \vee \neg v_{z,E}).$$

The remaining constraints can be represented by

$$Disj_{x,y}(v) = (\neg v_{x,X} \vee v_{y,A}) \wedge (\neg v_{x,Y} \vee v_{y,B}) \wedge (\neg v_{x,Z} \vee v_{y,C}).$$

Thus U_f corresponds to a satisfiable assignment to

$$\bigwedge_{x \in \mathbb{Z}_3^{n-1}, y,z \in N(x), y \neq z} Assign_x(v) \wedge NAE_{x,y,z}(v) \wedge Disj_{x,y}(v).$$

Note that we do not try to minimize the size of the formula and there are redundant clauses.

5.4.2 Examples of subsets with 6 and 18 extra points

With the help of a SAT solver, we found an induced degree 1 subset in $\mathbb{Z}_3^5$ of size $\alpha(H(5,3))+6$ and an induced degree 1 subset in $\mathbb{Z}_3^6$ of size $\alpha(H(6,3)) + 18$. We illustrate them below in terms of U_f using A, B, C, X, Y, Z blocks. Note that each block with a value in $\{X, Y, Z\}$ (i.e., values in the the gray cells) contributes one additional point to the set. We say that a subset of $\mathbb{Z}_3^n$ has m extra points if $|U| - \alpha(H(n,3)) = m$. For 1-saturated subsets the number of extra points is precisely the number of X, Y, Z blocks.

Example of 6 extra points in $\mathbb{Z}_3^5$.

X	A	A		A	B	C		A	C	B
A	C	B		B	C	A		C	Z	C
A	B	C		C	A	B		B	C	A

A	B	C		B	Y	B		C	B	A
C	A	B		A	B	C		B	C	A
B	C	A		C	B	A		A	A	X

A	C	B		C	B	A		B	A	C
B	B	Y		C	A	B		A	C	B
C	A	B		Z	C	C		C	B	A

Example of 18 extra points in $\mathbb{Z}_3^6$.

X	A	A		A	B	C		A	C	B		A	B	C		B	C	A		C	A	B
A	C	B		B	C	A		C	Z	C		B	A	C		Y	B	B		B	C	A
A	B	C		C	A	B		B	C	A		C	C	Z		B	A	C		A	B	C

A	B	C		B	Y	B		C	B	A		C	A	B		C	B	A		Z	C	C
C	A	B		A	B	C		B	C	A		A	X	A		B	A	C		C	A	B
B	C	A		C	B	A		A	A	X		B	A	C		A	C	B		C	B	A

A	C	B		C	B	A		B	A	C		B	C	A		A	A	X		C	B	A
B	B	Y		C	A	B		A	C	B		C	A	B		B	C	A		A	B	C
C	A	B		Z	C	C		C	B	A		A	B	C		C	B	A		B	Y	B

A	C	B
C	B	A
B	A	C

C	A	B
B	A	C
A	X	A

B	B	Y
A	C	B
C	A	B

B	C	A
B	A	C
Y	B	B

A	B	C
C	C	Z
B	A	C

C	A	B
A	B	C
B	C	A

C	Z	C
A	C	B
B	C	A

B	C	A
A	B	C
C	A	B

A	C	B
X	A	A
A	B	C

Each 1-saturated induced subset U has a natural extension to higher dimensions that preserves the number of extra points. This allows us to generate an example with 18 extra points for all $n \geq 6$.

Lemma 5.4.4. *Let $U \subseteq \mathbb{Z}_3^n$ be a set corresponding to $U_f : \mathbb{Z}_3^{n-1} \to \{A, B, C, X, Y, Z\}$. If U has induced degree 1 then there exists $V \subseteq \mathbb{Z}_3^{n+1}$ such that $|V| - \alpha(H(n+1, 3)) = |U| - \alpha(H(n, 3))$ and V has induced degree 1.*

Proof. Define $V_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ as follows. For each $y \in \mathbb{Z}_3^{n-1}$,

1. If $U_f(y) = X$ then $V_f(0, y) = X$, $V_f(1, y) = A$, and $V_f(2, y) = A$.

2. If $U_f(y) = Y$ then $V_f(0, y) = Y$, $V_f(1, y) = B$, and $V_f(2, y) = B$.

3. If $U_f(y) = Z$ then $V_f(0, y) = Z$, $V_f(1, y) = C$, and $V_f(2, y) = C$.

4. If $U_f(y) = A$ then $V_f(0, y) = A$, $V_f(1, y) = C$, and $V_f(2, y) = B$.

5. If $U_f(y) = B$ then $V_f(0, y) = B$, $V_f(1, y) = A$, and $V_f(2, y) = C$.

6. If $U_f(y) = C$ then $V_f(0, y) = C$, $V_f(1, y) = B$, and $V_f(2, y) = A$.

It is straightforward to verify that V_f satisfies the constraints in Proposition 5.4.3 and hence the corresponding subset $V = \cup_{x \in \mathbb{Z}_3^{n-1}}(V_f(x) \times \{x\})$ has induced degree 1. Since $|V| - \alpha(H(n+1, 3)) = |\{x \in \mathbb{Z}_3^n : V_f(x) \in \{X, Y, Z\}\}|$ and $|\{x \in \mathbb{Z}_3^n : V_f(x) \in \{X, Y, Z\}\}| = |\{x \in \mathbb{Z}_3^{n-1} : U_f(x) \in \{X, Y, Z\}\}|$ by construction, the result follows. □

Corollary 5.4.5. *For all $n \geq 6$, there exists $U \subseteq \mathbb{Z}_3^n$ such that U has induced degree 1 and $|U| = \alpha(H(n, 3)) + 18$.*

Note that our example of 6 extra points contains a subset that is isomorphic to the extremal set X_4 in $\mathbb{Z}_3^4$, i.e., it is an extension to X_4. Similarly, the example of 18 extra points is an extension to the example of 6 extra points. However, the SAT solver determined that there is no 1-saturated subset in $\mathbb{Z}_3^7$ with induced degree 1 and size greater than $\alpha(H(n, 7)) + 18$ which extends our 18 extra points example in $\mathbb{Z}_3^6$.

5.5 Size upper bound

As induced degree is non-increasing under restriction, the extremal subset of $\mathbb{Z}_3^4$ we presented in the previous section implies an upper bound of 6 and 18 extra points for $\mathbb{Z}_3^5$ and $\mathbb{Z}_3^6$. Hence both examples we showed are the largest possible in the corresponding dimensions. Moreover, since the extremal subset of $\mathbb{Z}_3^4$ is unique up to isomorphism and it is i-saturated for all $i \in [4]$, we have

Proposition 5.5.1. *For $n \in [6]$, if $U \subseteq \mathbb{Z}_3^n$ has induced degree 1 and has the maximum size, then U is i-saturated for some $i \in [n]$.*

In this section we prove the following.

Theorem 5.5.2. *Let $U \subseteq \mathbb{Z}_3^n$. If U has induced degree 1 and U is 1-saturated, then $|U| \leq 3^{n-1} + 729$.*

Recall that 1-saturated subsets with induced degree 1 can be identified as functions $U_f : \mathbb{Z}_3^{n-1} \to \{A, B, C, X, Y, Z\}$. We will use the function representation extensively in this section. In order to simplify the notation, we will consider functions with domain $\mathbb{Z}_3^n$ instead of $\mathbb{Z}_3^{n-1}$ and note that these functions correspond to subsets of $\mathbb{Z}_3^{n+1}$.

All subsets of $\mathbb{Z}_3^n$ we consider in this section are 1-saturated and it will be convenient to extend our definitions for subsets to their corresponding functions.

Definition 5.5.3 (Canonical functions)**.** We say that a function $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ is *canonical* if the corresponding subset is canonical.

Definition 5.5.4 (Induced degree)**.** The *induced degree* of a function $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ is defined as the maximum degree of the subgraph induced by U where U is the corresponding subset of U_f.

Definition 5.5.5. We say that that two functions $U_f, V_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ are *isomorphic* if their corresponding subsets are isomorphic, i.e., there exists $\sigma \in \mathrm{Aut}(H(n, 3))$ such that $\sigma(U) = V$ where U, V are the corresponding subsets of U_f and V_f.

5.5.1 The proof strategy

The main observation is that if U_f has induced degree 1 and $U_f(x) \in \{X, Y, Z\}$, then not only does it determine the values of $U_f(y)$ for all $y \in N(x)$, it also imposes a strong restriction on what the values can be on $U_f(z)$ for z where $d_H(x, z) = 2$. Specifically, we will show that for $n \geq 8$, if U_f has induced degree 1 and $U_f(x) \in \{X, Y, Z\}$, then for all $i \in [n]$ there exists $j \in [n] \setminus \{i\}$ such that either

$$U_f(x + e_j) \quad = \quad U_f(x + 2e_j),$$

$$\begin{aligned}
U_f(x + e_i + e_j) &= U_f(x + e_i + 2e_j),\\
U_f(x + 2e_i + e_j) &= U_f(x + 2e_i + 2e_j).
\end{aligned}$$

or

$$\begin{aligned}
U_f(x + e_i) &= U_f(x + 2e_i),\\
U_f(x + e_j + e_i) &= U_f(x + e_j + 2e_i),\\
U_f(x + 2e_j + e_i) &= U_f(x + 2e_j + 2e_i).
\end{aligned}$$

For illustration, suppose $U_f(x) = X$ and let $i \in [n]$, then there must be a different direction $j \in [n] \setminus \{i\}$ such that the function looks like one of the following:

$$\begin{array}{c}\overset{i}{\rightarrow}\\ j\downarrow \begin{array}{|c|c|c|}\hline X & A & A\\ \hline A & E & F\\ \hline A & E & F\\ \hline\end{array} \cdots\\ \vdots\end{array} \quad \text{or} \quad \begin{array}{c}\overset{i}{\rightarrow}\\ j\downarrow \begin{array}{|c|c|c|}\hline X & A & A\\ \hline A & E & E\\ \hline A & F & F\\ \hline\end{array} \cdots\\ \vdots\end{array}$$

where $E, F \in \{B, C\}$ and $E \neq F$. In either case, we find a canonical set containing x. Then we show that it is possible to find a much larger canonical set by choosing different directions i. Specifically, we will show that there exists a d-dimensional canonical set containing x where $d \geq n - 6$. By constructing a large canonical set containing each x for which $U_f(x) \in \{X, Y, Z\}$ and showing that canonical sets containing different x are disjoint, we conclude that $|\{x \in \mathbb{Z}_3^n : U_f(x) \in \{X, Y, Z\}\}| \leq 3^6$. Since the size of U is precisely $3^n + |\{x \in \mathbb{Z}_3^n : U_f(x) \in \{X, Y, Z\}\}|$, the theorem follows.

5.5.2 The line extension lemma

The first step of our proof is the following lemma.

Lemma 5.5.6 (Line extension lemma)**.** *Let $n \geq 8$ and $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$. If U_f has induced degree 1 and $U_f(0) = X$, then for all $i \in [n]$ there exists $j \in [n] \setminus \{i\}$ such that the restriction of U_f on* $\mathrm{Span}(e_i, e_j)$ *is canonical.*

Remark. Using a SAT solver, it can be shown that $n \geq 6$ is sufficient for Lemma 5.5.6.

Definition 5.5.7 (i-skew functions)**.** We say that a function $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ is *i-skew* if U_f satisfies the following:

1. U_f has induced degree 1.

2. $U_f(0) = X$.

3. For all $j \in [n] \setminus \{i\}$, the restriction of U_f on $\mathrm{Span}(e_i, e_j)$ is not canonical.

In other words, Lemma 5.5.6 asserts that if a function is i-skew for some $i \in [n]$, then $n \leq 7$. We will give a complete characterization for 1-skew functions for $n = 2, 3$ and use that to prove a property for them for $n = 4$ which is sufficient for proving Lemma 5.5.6.

$n = 2$

Let $U_f : \mathbb{Z}_3^2 \to \{A, B, C, X, Y, Z\}$ and $R = \{(1,1), (1,2), (2,1), (2,2)\}$. If U_f has induced degree 1 and $U_f(0,0) = X$, then it is clear that $A \notin U_f(R)$ since otherwise U_f will have induced degree at least 2, Similarly, $\{X, Y, Z\} \cap U_f(R) = \phi$ since X, Y and Z intersect with each other and each of them will force the neighbors to be their complements. Since any three elements in R form a path of length 2, we cannot have three B or three C on those locations. It follows there are exactly two B and exactly two C on R. If the B are adjacent then it is disjoint from a maximum size independent set and hence is canonical. Thus, the only 1-skew function is isomorphic to the following:

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array} \tag{5.1}$$

The stabilizing actions for this function are generated by (i) swapping the 2nd and 3rd rows and then swapping B and C; (ii) swapping the 2nd and 3rd columns and then swapping B and C; and (iii) swapping the row and column coordinates.

$n = 3$

Definition 5.5.8. Let $n > m$. We say that a function $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ *extends* a function $V_f : \mathbb{Z}_3^m \to \{A, B, C, X, Y, Z\}$ if there exists a function W_f that is isomorphic to U_f and $W_f(x \times 0^{m-n}) = V_f(x)$ for all $x \in \mathbb{Z}_3^m$.

We now show that up to isomorphism, there are precisely three 1-skew functions for $n = 3$. We list the 14 distinct 1-skew functions for $n = 3$ which extend (5.1) and have the first row of the second block equal to (A, B, C) in Section 5.5.5.

Theorem 5.5.9. *For $n = 3$, there are 28 1-skew functions extending (5.1). Each of them is isomorphic to one of the following:*

(a)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & A & C \\ \hline C & B & A \\ \hline \end{array} \tag{5.2}$$

with stabilizing actions generated by swapping the 2nd and 3rd rows, swapping the 2nd and 3rd blocks, and then swapping B and C.

(b)

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}
\tag{5.3}
$$

with stabilizing actions generated by (i) swapping the row and column coordinates; and (ii) swapping the 2nd and 3rd rows, swapping the 2nd and 3rd columns, and then swapping the 2nd and 3rd blocks.

(c)

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & A & B \\ \hline B & B & E \\ \hline \end{array}
\tag{5.4}
$$

where $E \in \{A, C, Y\}$. *The stabilizing actions are generated by (i) swapping the row and column coordinates; and (ii) swapping the row and block coordinates.*

Proof. Since the restriction of U_f on $\mathrm{Span}(e_1, e_2)$ is not canonical, by swapping B and C if necessary we can assume the restriction of U_f on $\mathrm{Span}(e_1, e_2)$ is identical to (5.1). Furthermore, by swapping the second and third block, we can assume the function is of the following form. Note that it also reduces the number of extensions by half. We consider the following partial function for which the values of the empty entries will be determined later.

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline & & \\ \hline & & \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline & & \\ \hline & & \\ \hline \end{array}
$$

We now consider all the possible values for the entries in the first column of the middle and last blocks which do not immediately create a vertex of degree at least 2. More precisely, let $S = \{(0,1,1), (0,1,2), (0,2,1), (0,2,2)\}$ be the set of coordinates of these entries. Then $U_f(x) \in \{B, C\}$ for all $x \in S$, and exactly two of them are equal to B or otherwise U_f will have induced degree 2.

- **Case 1: adjacent B in S.** If $U_f(0,1,1) = U_f(0,1,2) = B$ (the $(0,1)$ entries of the middle and third blocks), then the entries below them must be equal to C. This allows us to fill the remaining entries as follows. The subscripts in the table below indicate the order in which they are deduced.

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C_7 & A_1 \\ \hline C & A_5 & B_3 \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & A_6 & C_4 \\ \hline C & B_8 & A_2 \\ \hline \end{array}.$$

If the adjacent B are in the first column of the middle block, i.e., $U_f(0,1,1) = U_f(0,2,1) = B$, then the two entries in the first column of the last block must be equal to C. By swapping the row coordinate with the block coordinate, it can be reduced to the case above and hence the remaining entries will be uniquely identified. The result is as follows.

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline B & A & C \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & A & B \\ \hline C & B & A \\ \hline \end{array}.$$

Similarly, we can uniquely deduce the functions for the remaining two cases and they are both isomorphic to one of the functions above.

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline B & C & A \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}.$$

and

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline C & B & A \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & C & A \\ \hline B & A & C \\ \hline \end{array}.$$

By swapping the 2nd and 3rd block, we have 8 1-skew functions in total and each of them is isomorphic to (5.2).

- **Case 2: no adjacent B in S.**

  There are two cases for which there are no adjacent B in the first column of the middle and last block. These two cases are isomorphic by swapping the 2nd and 3rd columns and then swapping B and C. So it suffices to consider the case that the $(0, 1)$ entry in the middle block is B.

  Consider the 6-cycle consisting of the cells in gray.

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & & \\ \hline C & & \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & & \\ \hline B & & \\ \hline \end{array}.$$

  None of these entries can be C. They also cannot be in $\{X, Y, Z\}$ and hence they are either A or B. However, there cannot be a pair of adjacent B in this cycle or otherwise the induced degree will be at least 2. It follows that this 6-cycle must either contain a pair of adjacent As, or it consists of alternating A and B.

  - **Adjacent A in the 6-cycle.**

    The values of all entries will be determined by where we place the adjacent A and there are three possibilities. Suppose the adjacent A are at the $(1, 2)$ entry of the middle and last block. Then the function will be as follows:

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & & A \\ \hline C & A & B \\ \hline \end{array}
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & \\ \hline \end{array},
$$

which further implies the remaining entries must be C:

$$
\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline \end{array}
\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}.
$$

If the adjacent A are in the last column of the middle block, then it is isomorphic to the case above by swapping the row and block coordinates. Moreover, it is also isomorphic to the case that the adjacent A are in the last row of the middle block by swapping the row and column coordinates. Thus, all three cases are isomorphic to each other.

We have another three different 1-skew functions if the $(0, 1)$ entry in the middle block is C by the same logic. By swapping the 2nd and 3rd blocks we have a total of 12 1-skew functions extending (5.1) and all of them are isomorphic to (5.3).

– **No adjacent A in the 6-cycle.** It remains to consider the case for which the A and B in the 6-cycle are alternating. Suppose the $(2, 1)$ entry in the middle block is A, then we have the following partial function:

$$
\begin{array}{|c|c|c|}\hline X & A & A \\\hline A & B & C \\\hline A & C & B \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & B & C \\\hline B & & A \\\hline C & A & B \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & C & B \\\hline C & A & B \\\hline B & B & \\\hline\end{array}.
$$

The empty entry of the middle block must be C but the empty entry of the last block can be A, C or Y. So we must have

$$
\begin{array}{|c|c|c|}\hline X & A & A \\\hline A & B & C \\\hline A & C & B \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & B & C \\\hline B & C & A \\\hline C & A & B \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & C & B \\\hline C & A & B \\\hline B & B & E \\\hline\end{array}
$$

where $E \in \{A, C, Y\}$, which is precisely (5.4).

The other function is the following, which is isomorphic to the function above by swapping both the 2nd and 3rd rows, columns and then blocks.

$$
\begin{array}{|c|c|c|}\hline X & A & A \\\hline A & B & C \\\hline A & C & B \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & B & C \\\hline B & E & B \\\hline C & B & A \\\hline\end{array}
\;
\begin{array}{|c|c|c|}\hline A & C & B \\\hline C & B & A \\\hline B & A & C \\\hline\end{array}.
$$

There are 2 ways to place the A and B alternatively in the 6-cycle. Similar to the case of adjacent A, the rest of the functions can be generated by (i) swapping the 2nd and 3rd columns and then swapping B and C (note that swapping B and C also swaps Y and Z) and (ii) swapping the 2nd and 3rd blocks. In total there are 8 1-skew functions with alternating A in the corresponding 6-cycle which extend (5.1) and all of them are isomorphic to (5.4)

$\square$

$n = 4$

In this case, there are a lot more 1-skew functions and we will not give a complete list of them. Instead, we prove the following lemma, which is sufficient for proving Lemma 5.5.6.

Lemma 5.5.10. *Let $U_f : \mathbb{Z}_3^4 \to \{A, B, C, X, Y, Z\}$. If $U_f(x_1, x_2, x_3, 0)$ is identical to (5.2) or (5.3), and $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.2) or (5.3). Then U_f has induced degree at least 2.*

The proof involves analyzing the 1-skew functions for $n = 3$ and it is presented in a separate section. The following corollary is what we need.

Corollary 5.5.11. *Let $U_f : \mathbb{Z}_3^4 \to \{A, B, C, X, Y, Z\}$ be a 1-skew function. Then U_f is an extension of (5.4).*

5.5.3 Proof of Lemma 5.5.6

We now prove Lemma 5.5.6 which says that if $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ is a 1-skew function then $n \leq 7$.

Suppose for contradiction there exists a 1-skew function U_f for $n \geq 8$. Since there exists a restriction of a 1-skew function that is also 1-skew, we can assume $n = 8$. For $i = 3, 4, \ldots, 8$, let $H_i = \{x \in \mathbb{Z}_3^8 : x_k = 0 \ \forall k \in [8] \setminus \{1, 2, i\}\}$. We claim that there are at least 5 H_is for which the restriction of U_f on H_i is isomorphic to (5.4). Suppose not, then there are $H_i \neq H_j$ such that both the restrictions of U_f on them are not isomorphic to (5.4). However, by Lemma 5.5.10, this implies that U_f has induced degree 2, contradicting the assumption that it has induced degree 1.

Now, let $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \times 0^6$. Since for each i such that the restriction of U_f on H_i is isomorphic to (5.4), there exist a $x \in R$ and a neighbor $y \in H_i$ of x such that $U_f(x) = U_f(y)$, by the pigeonhole principle there is a $x \in R$ and distinct neighbors y, y'

of x such that $U_f(x) = U_f(y) = U_f(y')$, contradicting the assumption that U_f has induced degree 1.

5.5.4 Growing the canonical set

We now use Lemma 5.5.6 to prove that for $n \geq 8$, if $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ has induced degree 1 and $U_f(0) = X$, then there exists $n' \geq n - 6$ such that U_f is an extension of a canonical function on $\mathbb{Z}_3^{n'}$.

By restricting U_f on an affine subset, the following is immediate.

Corollary 5.5.12. *Let $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ with induced degree 1 and $U_f(0) = X$. Let $I = \{i_1, \ldots, i_d\} \subseteq [n]$. If $n - d \geq 7$, then for each $i \in I$, there exists $j \notin I$ such that the restriction of U_f on $\mathrm{Span}(e_i, e_j)$ is canonical.*

Proof. Without loss of generality, let $i = i_1$ and $H = \{x \in \mathbb{Z}_3^n : x_{i_2} = x_{i_3} = \cdots = x_{i_d} = 0\}$. The dimension of H is $n - d + 1 \geq 8$. Since $0 \in H$ and the induced degree is non-increasing under restriction, by Lemma 5.5.6 there exists $j \in [n] \setminus I$ such that the restriction of U_f on $\mathrm{Span}(e_i, e_j)$ is canonical, as desired. □

Theorem 5.5.13. *Let $n \geq 8$ and $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$. If U_f has induced degree 1 and $U_f(0) = X$, then there exists $I \subseteq [n]$ such that $|I| \geq n - 6$ and the restriction of U_f on $H = \{x \in \mathbb{Z}_3^n : x_i = 0\ \forall i \notin I\}$ is canonical.*

The following lemma will be useful.

Lemma 5.5.14. *Let $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$ with induced degree 1. If there exists $i \in [n]$ and an affine subset H such that the restriction of U_f on H is an independent set I and $U_f(x) = U_f(x + e_i)$ for all $x \in H$, then*

1. *If there exists $x \in H$ such that $U_f(x + e_j) = U_f(x + e_j + e_i)$, then the restriction of U_f on $H + e_j$ is an independent set disjoint from I and $U_f(y) = U_f(y + e_i)$ for all $y \in H + e_j$.*

2. *If there exists $x \in H$ such that $U_f(x+e_j) = U_f(x+2e_j)$, then the restriction of U_f on $H+e_j$ is an independent set disjoint from I and $U_f(y) = U_f(y+e_j)$ for all $y \in H+e_j$.*

Proof. We only present the proof of item 1, the proof of item 2 is similar and is omitted.

Suppose H contains directions $i_1, \ldots, i_d$. Let $K_m = \mathrm{Span}(e_{i_1}, \ldots, e_{i_m}) + x + e_j$. We show that $U_f(y) = U_f(y+e_i)$ for all $y \in K_m$ and $m \in [d]$ by induction on m.

The base case $m = 0$ is trivial since $K_0 = \{x + e_j\}$, $y = x + e_j$ and hence $U_f(x + e_j) = U_f(x + e_j + e_i)$ by the assumption. Suppose $U_f(y) = U_f(y + e_i)$ for all $y \in K_k$. Consider $y \in K_{k+1}$. If $y \in K_k$, we are done. Otherwise, either $y + e_{k+1} \in K_k$ or $y + 2e_{k+1} \in K_k$.

Suppose $y + e_{k+1} \in K_k$ (the case of $y + 2e_{k+1} \in K_k$ is similar), then by the inductive hypothesis, we have $U_f(y + e_{k+1}) = U_f(y + e_{k+1} + e_i)$. Since U_f has induced degree 1, $U_f(y), U_f(y + 2e_{k+1}) \neq U_f(y + e_{k+1})$. We have the following two cases:

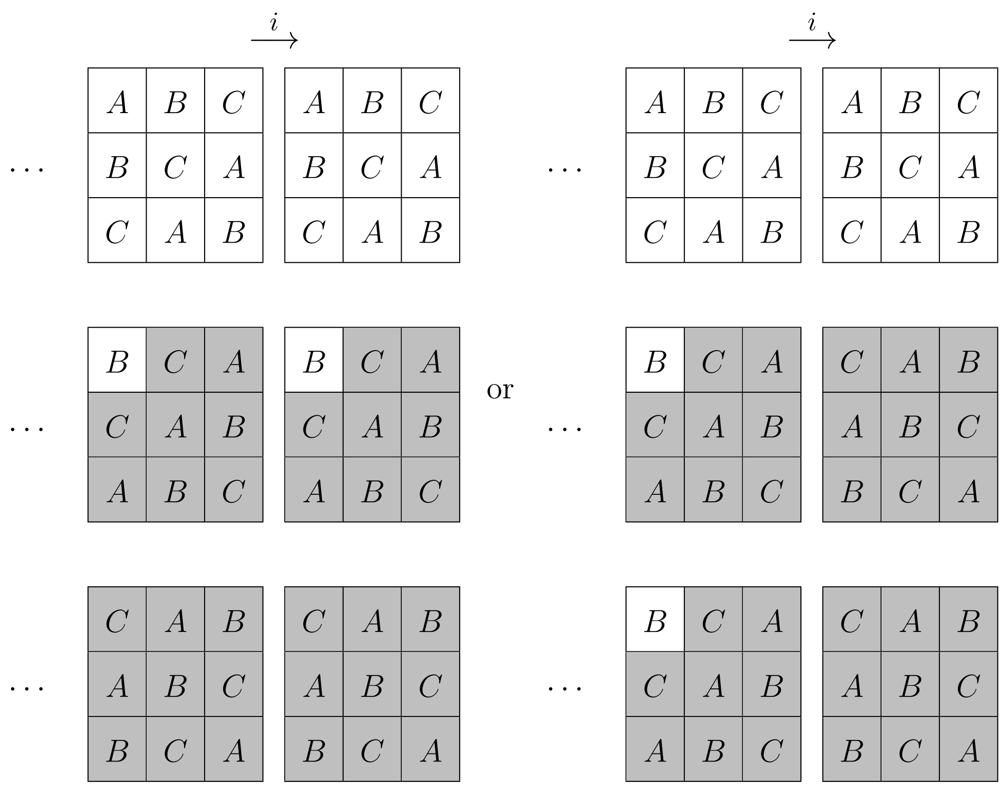
1. If $U_f(y + e_{k+1}) \neq U_f(y - e_j)$, then $U_f(y) \in \{A, B, C\} \setminus \{U_f(y + e_{k+1}), U_f(y - e_j)\}$ since $y - e_j \in H$ and $U_f(y - e_j) = U_f(y - e_j + e_i)$. It follows that $U_f(y + e_i) \in \{A, B, C\} \setminus \{U_f(y+e_{k+1}+e_i), U_f(y-e_j+e_i)\} = \{A, B, C\} \setminus \{U_f(y+e_{k+1}), U_f(y-e_j)\}$. So $U_f(y) = U_f(y + e_i)$, as desired.

2. If $U_f(y+e_{k+1}) = U_f(y-e_j)$, then $U_f(y+e_{k+1}) \neq U_f(y+2e_{k+1}-e_j)$. By the same logic as case 1, $U_f(y+2e_{k+1}) = U_f(y+2e_{k+1}+e_i)$ which further implies $U_f(y) = U_f(y+e_i)$.

□

We illustrate this lemma and its proof below. This lemma asserts that if there are two identical independent sets which are adjacent to each other in direction i then if we consider the neighboring affine subsets in direction j, if there is a pair of identical elements in these neighboring affine subsets which are adjacent in direction i or j then this determines all of the entries of these neighboring affine subsets. Moreover, it forces these affine subsets to be identical independent sets which are adjacent in the same direction as the pair of identical elements.

In the figures below, the top two blocks are the two identical independent sets which are adjacent in direction i. In the figure on the left, the pair of B in the first row of the middle blocks are an identical pair of elements which are adjacent in direction i. This pair determines the values in all of the gray cells and makes these affine subsets into identical independent sets which are adjacent in direction i. In the figure on the right, the pair of B in the first column of the middle left and bottom left blocks are an identical pair of elements which are adjacent in direction j. This pair determines the values in all of the gray cells and makes these affine subsets into identical independent sets which are adjacent in direction j.



Proof of Theorem 5.5.13. We proceed by induction on n. For $n = 8$, Lemma 5.5.6 asserts that there exists $i \neq j$ such that U_f is canonical on $\mathrm{Span}(e_i, e_j)$.

Suppose the claim is true for functions on $\mathbb{Z}_3^k$ for some $k \geq 8$. Let U_f be a function on $\mathbb{Z}_3^{k+1}$ that satisfies the assumption. Since the restriction of U_f on $\{x \in \mathbb{Z}_3^{k+1} : x_{k+1} = 0\}$

is a function on $\mathbb{Z}_3^k$ and it also satisfies the assumption of the theorem. By the inductive hypothesis, there exists $I = \{i_1, \ldots, i_d\} \subseteq [k]$ such that $d \geq k-6$ and the restriction of $U_f(x_1, \ldots, x_k, 0)$ on $H = \{x \in \mathbb{Z}_3^k : x_i = 0\ \forall i \notin I\}$ is canonical. If $d > k-6$, we are done. However, if $d = k-6$, then $k+1-d = 7$. Thus, by Corollary 5.5.12 for each $i \in I$, there exists $j \notin I$ such that the restriction of U_f on $\mathrm{Span}(e_i, e_j)$ is canonical. Let i be the popular direction of the restriction of U_f on H, and j be the direction asserted by Corollary 5.5.12. We claim that U_f is canonical on $H' = \{x \in \mathbb{Z}_3^{k+1} : x_i = 0\ \forall i \notin I'\}$ where $I' = I \cup \{j\}$.

Let $H'_{a,b} = H' \cap \{x \in \mathbb{Z}_3^{k+1} : x_i = a, x_j = b\}$ where $a, b \in \{0,1,2\}$. Since U_f is canonical on $\mathrm{Span}(e_i, e_j)$, we have either

$$\begin{aligned} U_f(e_i + e_j) &= U_f(e_i + 2e_j), \\ U_f(2e_i + e_j) &= U_f(2e_i + 2e_j). \end{aligned}$$

or

$$\begin{aligned} U_f(e_j + e_i) &= U_f(e_j + 2e_i), \\ U_f(2e_j + e_i) &= U_f(2e_j + 2e_i). \end{aligned}$$

Suppose the former case holds. By Lemma 5.5.14, U_f on $H'_{1,1}$ and $H'_{1,2}$ are a pair of identical independent sets. So U is isomorphic to the following on H':

$$\begin{array}{cc} & \overset{i}{\rightarrow} \\ j\downarrow & \begin{array}{|c|c|c|} \hline D_{k-1} & A_{k-1} & A_{k-1} \\ \hline E & B_{k-1} & C_{k-1} \\ \hline F & B_{k-1} & C_{k-1} \\ \hline \end{array} \end{array}$$

where B_{k-1} and C_{k-1} are distinct independent sets of size 3^{k-2}. Since U has induced degree

1 and it is 1-saturated, E and F must be disjoint from both B_{k-1} and C_{k-1} and have size at least 3^{k-2}. It implies that $E = F = A_{k-1}$. Thus U is canonical on H'.

For the latter case, by Lemma 5.5.14, the restriction of U on H' is isomorphic to the following:

$$\begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{3}{c}{\overset{i}{\rightarrow}} \\
\cline{2-4}
 & D_{k-1} & A_{k-1} & A_{k-1} \\
\cline{2-4}
j\downarrow & E & B_{k-1} & B_{k-1} \\
\cline{2-4}
 & F & C_{k-1} & C_{k-1} \\
\cline{2-4}
\end{array}$$

By Proposition 5.3.7, $E, F \in \{A_{k-1}, A'_{k-1}\}$. Since E and F are disjoint from B_{k-1} and C_{k-1} respectively, $E = A_{k-1}$ and $F = A_{k-1}$. Thus U_f is canonical on H' and the proof is completed. $\square$

Lemma 5.5.15. *Let $U_f : \mathbb{Z}_3^n \to \{A, B, C, X, Y, Z\}$. If U_f has induced degree 1, and there exist $x \neq y$ such that $U_f(x), U_f(y) \in \{X, Y, Z\}$. Let H_x and H_y be affine subsets which contain x and y respectively and the restrictions of U_f on them are canonical, then $H_x \cap H_y = \phi$.*

Proof. Suppose for contradiction $H_x \cap H_y \neq \phi$. By Lemma 5.3.15, the restrictions of U_f on H_x and H_y have the same extra point. However, there can be exactly one extra point in a canonical set, hence $x = y$, contradicting our assumption. $\square$

Proof of Theorem 5.5.2. By translating each x such that $U_f(x) \in \{X, Y, Z\}$ to 0^n and Theorem 5.5.13, there exists an affine subset H_x of dimension at least $n - 6$ on which U_f is canonical. The size of U is precisely 3^n plus the number of such affine subsets. By Lemma 5.5.15, these H_xs are disjoint and hence there are at most 3^6 of them. $\square$

Remark. It can be shown by using a SAT solver that it sufficient to have $n \geq 6$ in Lemma 5.5.6. Using this, the result in Theorem 5.5.13 can be improved which can reduce the final bound from $3^{n-1} + 3^6$ to $3^{n-1} + 3^4$.

5.5.5 *1-skew functions for $n = 3$ and a proof of Lemma 5.5.10*

We list all 1-skew functions for $n = 3$ which extend (5.1) and have the first row of the second block equal to (A, B, C).

Isomorphic to (5.2):

(i)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & A & C \\ \hline C & B & A \\ \hline \end{array}\,.$$

(ii)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline B & A & C \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & A & B \\ \hline C & B & A \\ \hline \end{array}\,.$$

(iii)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline B & C & A \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline \end{array}\,.$$

(iv)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline C & B & A \\ \hline \end{array}\;\begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & C & A \\ \hline B & A & C \\ \hline \end{array}\,.$$

Isomorphic to (5.3):

(v)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & A & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & B & A \\ \hline B & A & C \\ \hline\end{array}.$$

(vi)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & B \\ \hline C & A & A \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & A & A \\ \hline B & B & C \\ \hline\end{array}.$$

(vii)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline B & C & A \\ \hline C & B & A \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline C & A & B \\ \hline B & A & C \\ \hline\end{array}.$$

(viii)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline B & C & A \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & A & C \\ \hline C & B & A \\ \hline\end{array}.$$

(ix)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & C & B \\ \hline B & A & A \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & A & A \\ \hline C & B & C \\ \hline\end{array}.$$

(x)

$$\begin{array}{|c|c|c|}\hline X & A & A \\ \hline A & B & C \\ \hline A & C & B \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & B & C \\ \hline C & A & B \\ \hline B & A & C \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline A & C & B \\ \hline B & C & A \\ \hline C & B & A \\ \hline\end{array}.$$

Isomorphic to (5.4):

(xi)

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	A	B
A	C	B	C	A	B	B	B	E

where $E \in \{A, C, Y\}$.

(xii)

X	A	A	A	B	C	A	C	B
A	B	C	B	E	B	C	B	A
A	C	B	C	B	A	B	A	C

where $E \in \{A, C, Y\}$.

(xiii)

X	A	A	A	B	C	A	C	B
A	B	C	C	A	B	B	C	A
A	C	B	B	C	A	C	F	C

where $F \in \{A, B, Z\}$.

(xiv)

X	A	A	A	B	C	A	C	B
A	B	C	C	C	F	B	A	C
A	C	B	B	A	C	C	B	A

where $F \in \{A, B, Z\}$.

We are ready to prove Lemma 5.5.10. Recall that the statement is as follows:

Lemma 5.5.16. *Let $U_f : \mathbb{Z}_3^4 \to \{A, B, C, X, Y, Z\}$. If $U_f(x_1, x_2, x_3, 0)$ is identical to (5.2) or (5.3), and $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.2) or (5.3). Then U_f has induced degree at least 2.*

Proof. We analyze the two major cases separately.

- **$U_f(x_1, x_2, x_3, 0)$ is identical to (5.2).**

  We show that when $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.2) or (5.3), U_f must have induced degree at least 2. We consider the following partial function, where the entries are to be determined.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	B	A	C
A	C	B	C	A	B	C	B	A

A	B	C						

A	C	B						

  Now the first column must be identical to one of the cases among (i) to (x) in Appendix 5.5.5. Each of case (i)-(iv) enjoys the same symmetry as (5.2). Thus we need to consider them one by one.

– Case (i). The B in the gray cells immediately cause U_f to have induced degree 2.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	B	A	C
A	C	B	C	A	B	C	B	A

A	B	C						
B	C	A						
C	A	B						

A	C	B						
B	A	C						
C	B	A						

.

– Case (ii). The subscripts of the entries indicate the order in which they are deduced. The entries in the gray cells certify the induced degree is at least 2.

X	A	A
A	B	C
A	C	B

A	B	C
B	C	A
C	A	B

A	C	B
B	A	C
C	B	A

A	B	C
B	C	A
B	A	C

A	C	B
C	A	B
C	B	A

A_1		

A_2		C_3

– Case (iii). The gray cell at $(0, 2, 1, 1)$ cannot be A, B or C, otherwise U_f will have induced degree 2. But it cannot be X, Y, Z either.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	B	A	C
A	C	B	C	A	B	C	B	A

A	B	C						
C	A	B	A_1			A_2		
B	C	A						

A	C	B						
C	B	A						
B	A	C	A_3			A_4		

– Case (iv). The entries with the same subscript are deduced at the same time given the entries with smaller subscripts. There is no assignment to the gray cell will result in a 1-saturated function with induced degree 1.

X	A	A
A	B	C
A	C	B

A	B	C
B	C	A
C	A	B

A	C	B
B	A	C
C	B	A

A	B	C
C	A	B
C	B	A

C_3	A_4	
A_1	B_6	
B_2	C_5	

C_3		
A_1		
B_2		

A	C	B
B	C	A
B	A	C

B_3	A_4	
C_2	B_5	
A_1		

B_3		
C_2		
A_1		

For case (v) to (x), the stabilizing action for (5.2) partitions them into three orbits $\{(\text{v}), (\text{viii})\}$, $\{(\text{vi}), (\text{ix})\}$ and $\{(\text{vii}), (\text{x})\}$. So it suffices to consider case (v), (vi) and (vii).

– Case (v). The three B in the gray cells immediately certify that the function has induced degree at least 2.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	B	A	C
A	C	B	C	A	B	C	B	A

A	B	C						
B	C	A						
C	A	B						

A	C	B						
C	B	A						
B	A	C						

- Case (vi).

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	B	A	C
A	C	B	C	A	B	C	B	A

A	B	C						
B	C	B						
C	A	A						

A	C	B						
C	A	A						
B	B	C	A_1	C_3	B_5	A_2	C_4	

– Case (vii).

X	A	A		A	B	C		A	C	B
A	B	C		B	C	A		B	A	C
A	C	B		C	A	B		C	B	A

A	B	C				B_6				
B	C	A				B_5				B_7
C	B	A				C_1				

A	C	B				A_4				
C	A	B				C_3				
B	A	C				A_2				

- $U_f(x_1, x_2, x_3, 0)$ **is identical to** (5.3)**.**

  If $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.2), then it is isomorphic to one of the cases we have analyzed above by swapping the row and column indices of the blocks. So it remains to consider the case that $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.3).

  If $U_f(x_1, x_2, 0, x_4)$ is isomorphic to (5.3), then it is identical to one of the functions among (v) to (x) we listed above. We start with the following partial function, where

E are the entries to be determined.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	B	A
A	C	B	C	A	B	B	A	C
A	B	C						
A	C	B						

Recall that the stabilizing actions for (5.3) includes (a) swapping the row and column coordinates, and (b) swapping 2nd and 3rd rows, columns and then blocks. These actions together with the action of swapping the 2nd and 3rd blocks partition cases (v) to (x) into four orbits: $\{(\text{v})\}$, $\{(\text{vi}), (\text{vii})\}$, $\{(\text{viii})\}$, $\{(\text{ix}), (\text{x})\}$. Thus it suffices to analyze case (v), (vi), (viii) and (ix) respectively.

– Case (v). The three B in the gray cells certify the induced degree is at least 2 immediately.

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	B	A
A	C	B	C	A	B	B	A	C

A	B	C						
B	C	A						
C	A	B						

A	C	B						
C	B	A						
B	A	C						

– Case (vi).

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	B	A
A	C	B	C	A	B	B	A	C

A	B	C						
B	C	B						
C	A	A						

A	C	B						
C	A	A					B_3	
B	B	C		C_1			C_2	

– Case (viii).

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	B	A
A	C	B	C	A	B	B	A	C

A	B	C						
C	A	B						
B	C	A		B_2	C_4		B_1	C_3

A	C	B						
B	A	C						
C	B	A						

– Case (ix).

X	A	A	A	B	C	A	C	B
A	B	C	B	C	A	C	B	A
A	C	B	C	A	B	B	A	C

A	B	C						
C	C	B						
B	A	A						

A	C	B						
B	A	A		B_3			C_2	
C	B	C		C_4	A_1	A_6	B_5	A_7

In all cases U_f must have induced degree at least 2 and this completes the proof. $\square$

CHAPTER 6

FUTURE DIRECTIONS

6.1 Further questions about $H(n,3)$

We analyzed the maximum size of a subset $U \subseteq \mathbb{Z}_3^n$ which induces a subgraph with maximum degree 1. We showed that if U is disjoint from a maximum size independent set then $|U| \leq \alpha(\mathbb{Z}_3^n) + 1$ but U can be larger if U is not disjoint from a maximum size independent set. In particular, for $n \geq 6$, there exists such a U with size $\alpha(\mathbb{Z}_3^n) + 18$ and this is optimal when $n = 6$. We also showed that if U is i-saturated for any $i \in [n]$ then $|U| \leq \alpha(\mathbb{Z}_3^n) + 729$.

The assumption of being i-saturated for some i was motivated by the fact that it is a common property shared by all extremal subsets of $\mathbb{Z}_3^n$ for $n \in [6]$ and works well with SAT-solvers. We conjecture that similar results hold if we remove the assumption of being i-saturatedfor some $i \in [n]$ but this remains to be proven.

Conjecture 1. *All induced degree 1 subsets of $\mathbb{Z}_3^n$ have size $\alpha(H(n,3)) + O(1)$.*

We can also ask what happens if we consider subsets of $\mathbb{Z}_3^n$ with larger induced degree.

Question 5. *Given $d, n \in \mathbb{N}$, what is the largest subset of $\mathbb{Z}_3^n$ with induced degree at most d?*

We observe that there is a nice construction which has at least $3^{\lfloor \frac{(d-1)n}{d} \rfloor}$ extra points. That said, it is possible that there are larger constructions.

Lemma 6.1.1. *For all $d, n \in \mathbb{N}$, there is a subset U of $\mathbb{Z}_3^n$ with induced degree at most d such that $|U| \geq 3^{n-1} + 3^{\lfloor \frac{(d-1)n}{d} \rfloor}$ and U is disjoint from a maximum size independent set of $H(n,3)$.*

Proof. We prove this lemma by induction. If $n \leq d$, we can take $U = \mathbb{Z}_3^n \setminus A_n$ and we will have that $|U| = 3^{n-1} + 3^{n-1} \geq 3^{n-1} + 3^{\lfloor \frac{(d-1)n}{d} \rfloor}$ and U is disjoint from A_n.

If $n > d$ then by the inductive hypothesis, there is a subset $U_{n-d} \subseteq \mathbb{Z}_3^{n-d}$ of size at least $3^{n-1} + 3^{\lfloor \frac{(d-1)n}{d} \rfloor - (d-1)}$ which is disjoint from A_{n-d}. We can now do the following:

1. Start with the independent set A_d.

2. Replace each of the 3^{d-1} points in A_d with a copy of U_{n-d} and replace each point which is not in A_d with a copy of A_{n-d}.

It is not hard to verify that this subset has size at least $3^{n-1} + 3^{\lfloor \frac{(d-1)n}{d} \rfloor}$ and is disjoint from the independent set A_n.

We illustrate this construction for $d = 2$ below. When $d = 2$, we have that

$$U_2 = \begin{array}{|c|c|c|} \hline & \bullet & \bullet \\ \hline \bullet & \bullet & \\ \hline \bullet & & \bullet \\ \hline \end{array} \text{ and for } k \geq 1,\ U_{2(k+1)} = \begin{array}{|c|c|c|} \hline U_{2k} & A_{2k} & A_{2k} \\ \hline A_{2k} & A_{2k} & U_{2k} \\ \hline A_{2k} & U_{2k} & A_{2k} \\ \hline \end{array}. \qquad \square$$

6.2 Other classes of graphs

Although the constructions in [LV20, GMK22] showed that for non-abelian Cayley graphs, a large induced subgraph does not necessarily have large maximum degree. It would be an interesting direction to identify a natural class of non-abelian Cayley graphs for which this property holds. García-Marco and Knauer showed a promising direction for Cayley graph over the Coxeter groups.

Another way to interpret Huang's Theorem for Boolean hypercubes is that there is a threshold for which the maximum degree exhibits a "jump" when the size of the subgraph pass that threshold. Frankl and Kupavskii [FK20] and Chau et al. [CEFL23] studied the maximum degree of induced subgraphs of Kneser graphs and showed that there is a "jump" in minimum maximum degree when the size of the induced subgraph increases. This would be an interesting direction to explore further.

APPENDIX A

PARALLEL INDEPENDENT SETS LEMMA

By formulating the problem of finding 1-saturated subsets with induced degree 1 as a CNF formula, we are able to use SAT solver to discover crucial properties about the solution. The following lemma is one of the examples. Although we did not use this for any of our results for $H(n, 3)$, we find it to be interesting.

Lemma A.0.1. *Let $U_f : \mathbb{Z}_3^4 \to \{A, B, C, X, Y, Z\}$. If U_f has induced degree 1, $U_f(0) = U_f(e_2)$, $U_f(e_1) = U_f(e_1 + e_2)$ and $U_f(e_3) = U_f(e_1 + e_3)$, then $U_f(x) = U_f(x + e_2)$ for all $x \in \mathbb{Z}_3^n$ such that $x_3 = x_2 = 0$.*

Roughly speaking, it asserts that a pair of parallel independent sets can be extended by an additional match. To illustrate, if we have the following partial function:

$$
\begin{array}{ccc}
\begin{array}{|c|c|c|}\hline A & B & C \\ \hline A & B & C \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline C & C & \\ \hline & & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\
\\
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \\
\\
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} &
\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}
\end{array} \tag{A.1}
$$

Then this function must be isomorphic to the following partial function, hence having a pair of larger independent sets in the first column:

A	B	C	C	C				
A	B	C						

B	C	A						
B	C	A						

C	A	B						
C	A	B						

Notation and convention. In the proof, we will consider many cases in which we deduce entries a certain order. For presentation purposes, we highlight the deduced entries in different colors. Entries with the same color are deduced at the same time given the entries that are already determined at that point. Entries in red are deduced first (also indicated by the subscript 1), then followed by blue (with subscript 2), green (with subscript 3), orange (with subscript 4) and finally yellow (with subscript 5).

Proof. We start with the partial function (A.1) and we want to show that for each block in the left hyperplane, the first two rows must be equal.

- **Cases where the first and second rows are not independent sets.**

  Up to symmetry, there are two cases where the first and second rows in the blocks of the left hyperplane are not independent sets. The first case is as follows (here C does not appear twice in the first or second row of the blocks of the left hyperplane):

A	B	C	C	C	A			
A	B	C						

C	A	A						
B	C	B						

B	C	B						
C	A	A						

It is straightforward to make the deductions shown in red, blue, green, and orange (in that order) which gives a contradiction at the B in the gray cell:

A	B	C	C	C	A	B_1	A_1	B_2
A	B	C	B_1	A_1	B_2	C_2	C_2	A_3

C	A	A		B_1	B_4			C_3
B	C	B						

B	C	B	A_1		C_2			
C	A	A		B_4	C_3		B	B_4

The second case is as follows:

A	B	C
A	B	C

C	C	

C	C	B
B	A	A

B	A	A
C	C	B

It is straightforward to make the deductions shown in red, blue, green, and orange (in that order):

A	B	C
A	B	C

C	C	
B_1	A_1	

B_1	A_1	
C_2	C_2	

C	C	B
B	A	A

E	E	

E	E	
A_4	B_3	

B	A	A
C	C	B

A_2	B_1	
E	E	

E	E	

where $E \in \{A, B\}$. It implies the gray cells must be C, i.e.,

A	B	C
A	B	C

C	C	
B	A	

B	A	
C	C	

C	C	B
B	A	A

C	C	

A	B	

B	A	A
C	C	B

A	B	

C	C	

Then we can make the deductions shown in red, blue, green, orange, and yellow (in that order):

A	B	C
A	B	C

C	C	B_3
B	A	A_2

B	A	A_2
C	C	B_3

C	C	B
B	A	A

B_5	A_4	A_5
C	C	B_1

A_4	B_3	C_4
A	B	C_4

B	A	A
C	C	B

A	B	C_4
A_4	B_3	C_4

C	C	B_1
B_5	A_4	A_5

where the gray cells must all be B, which is a contradiction.

- **Case where the first and second rows are independent sets.** We now consider the cases when the first and second rows of the blocks in the left hyperplane are all independent sets. One such case is as follows:

A	B	C	C	C				
A	B	C						

B	C	A						
C	A	B						

B	C	A						
C	A	B						

It is straightforward to make the deductions shown in red and blue (in that order) which gives a contradiction at the red and blue B in the gray cells:

A	B	C		C	C			B_1	A_1	
A	B	C		B_1	A_1			C_2	C_2	

B	C	A		A_1	B_2					
C	A	B		B_2						

B	C	A		A_1	B_2					
C	A	B		B_2						

The final case is as follows:

A	B	C		C	C	A				
A	B	C								

B	C	A								
C	A	B								

C	A	B								
B	C	A								

It is straightforward to make the deductions shown in red, blue, and green (in that order).

A	B	C	C	C	A	B_1	A_1	B_2
A	B	C	B_1	A_1	B_2	C_2	C_2	A_3
B	C	A						
C	A	B						
C	A	B						
B	C	A						

The gray cell is either A or C. Suppose it is C, then we have

A	B	C	C	C	A	B	A	B
A	B	C	B	A	B	C	C	A
B	C	A		A_4				
C	A	B	C	B_1	A_1		A_4	
C	A	B		A_4				
B	C	A	A_1	B_3	C_2		A_4	

where the A in the gray cells lead to a contradiction. Thus, this entry must be an A and we have the following partial function, where the gray cell is either A or B:

A	B	C	C	C	A	B	A	B
A	B	C	B	A	B	C	C	A
B	C	A						
C	A	B	A					
C	A	B						
B	C	A						

We claim that the gray cell must be B. Assume for contradiction it is A, then it is straightforward to make the deductions shown in red, blue, and green (in that order) below, which leads to a contradiction at the gray cells.

A	B	C	C	C	A	B	A	B
A	B	C	B	A	B	C	C	A

B	C	A	A	B_1		C_1		
C	A	B	A		C_1	B_1		

C	A	B	B_1	A_2	C_3	A_2		C_3
B	C	A	C_1	B_3	A_2	A_2	B_3	

Thus, we must have the following partial function

A	B	C	C	C	A	B	A	B
A	B	C	B	A	B	C	C	A

B	C	A	B					
C	A	B	A					

C	A	B						
B	C	A						

However, we can now make the deductions shown in red, blue, green, and orange (in that order), leading to a contradiction at the gray cells.

A	B	C
A	B	C

C	C	A
B	A	B

B	A	B
C	C	A

B	C	A
C	A	B

B	A_1	C_1
A		A_4

C	A	B
B	C	A

A_1	B_2	C_3
C_2		A_4

$\square$

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