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Credences for strict conditionals

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Abstract

Less-than-certain conditional judgments pose notorious problems for strict analyses of conditionals: across their various incarnations, these analyses have trouble making sense of how conditionals could have non-trivial probabilities in the first place; minimal constraints on how such probabilities are to be assigned, moreover, lead to results that seem at odds with a strict outlook on the semantics of conditionals, most notably the validity of Conditional Excluded Middle. I demonstrate that a strict analysis can overcome the trouble if couched in a bilateral dynamic setting that properly extends the familiar Ramsey test for accepting conditionals to other iffy attitudes, most importantly the one of rejecting a conditional. The resulting framework accommodates the appeal of Stalnaker's thesis as well as of Conditional Excluded Middle in a strict setting. A discussion of how to handle the probability of epistemically modalized conditionals and of compounds of conditionals is provided.

1 | THE PLOT

Natural language conditionals, according to a long tradition in the literature, are STRICT conditionals in that they universally quantify over possible worlds: they require, very roughly, that all contextually relevant antecedent-verifying worlds are also consequent-verifying worlds.[1] The tradition has a lot going for it,[2] but it also faces the notorious problem of making sense of how we tend to assign probabilities to conditionals. The problem comes in various shapes and forms but consider, for starters, an ordinary conditional such as (1):

(1) If the die came up even, it came up two.

Assuming we are concerned with a fair six-sided die, it is natural to assign to (1) a probability of 1/3. Pre-theoretically, however, the die might very well have come up four or six; hence it seems certain that *not all* contextually relevant antecedent-verifying worlds are also consequent-verifying worlds—so how could (1) be strict and also have non-zero probability?[3]

Strict analyses thus have trouble making sense of the apparent fact that conditionals can have non-trivial probabilities.[4] Minimal constraints on how such probabilities are to be assigned, moreover, lead to results that seem at odds with a strict outlook on the semantics of conditionals. Of central importance here is the principle of CONDITIONAL EXCLUDED MIDDLE (CEM):

$$\text{(CEM)} \qquad (\phi > \psi) \vee (\phi > \neg\psi)$$

It is a familiar fact that, without non-classical maneuvers, CEM fails to be valid in a strict setting, the simple reason being that whenever some of the relevant ϕ-worlds make ψ true while others make ψ false—and there is no reason to think that could not happen[5]—neither $\ulcorner\phi > \psi\urcorner$ nor $\ulcorner\phi > \neg\psi\urcorner$ will turn out to be true. And yet the invalidity of CEM stands in tension with

[1] Prominent proposals in the tradition I have in mind here include *variably strict* analyses of conditionals (see e.g. Lewis, 1973 on counterfactuals) as well as *dynamic strict* accounts (von Fintel, 2001; Gillies, 2007; Willer, 2017), but also *restrictor analyses* à la Kratzer (1986, 2012) (where what is restricted in a plain conditional is an unarticulated necessity operator) and Yalcin (2007)-style *informational* accounts.

[2] It does, as we will see momentarily, flow naturally from the popular Ramsey test for conditionals (Ramsey, 1931). But see also Veltman (1985), van Benthem (1986), and Gillies (2010) for rigorous arguments to the conclusion that *if* is best understood as a universal quantifier (if *if* is an operator at all).

[3] Edgington (1995, 2008) raises this kind of concern about Lewis (1973)-style variably strict analyses of counterfactuals (see also DeRose, 1994; Moss, 2013; Schulz, 2017): if one is certain that some of the closest antecedent-verifying worlds are not consequent-verifying worlds, one can be certain that the counterfactual is false. Yet in many such cases ordinary speakers are inclined to assign to the conditional at play middling or perhaps even high credence.

[4] One reaction to the apparent trouble that graded conditional judgments pose for strict accounts is that we have just conflated judging the probability of a bare conditional with judging the truth of a hedged conditional: to judge that "If the die came up even, it came up two" has a 1 in 3 chance of being true is really to judge true that "If the die comes up even, there is a 1 in 3 chance that it came up two" (see e.g. Bennett, 2003; Égré & Cozic, 2011; Kratzer, 1986, 2012; Rothschild, 2021). So, what needs explaining is not so much how conditionals can have contents that are more or less likely, but how *if*-clauses can play with hedges such as *probably*, which is a home game at least for some strict accounts. My primary goal here is simply to show that it is not the only available path for someone sympathetic to a strict outlook on conditionals, but I will also highlight some distinct advantages of the strategy pursued here in Section 5.

[5] A proposal by von Fintel (1997a) and Cariani and Goldstein (2020) is that conditionals are universal quantifiers alright but come with a *homogeneity* presupposition that all antecedent-verifying worlds be either consequent-verifying or consequent-falsifying worlds (see also Križ, 2015 and Schlenker, 2004 for discussion). CEM follows right away on this picture, given minimal assumptions about how presupposition failures interact with logical consequence. But as Santorio (2017) observes, embellishing a strict analysis with a homogeneity presupposition makes it even more mysterious how we could ever have middling credences in conditionals, and in general there just seems to be no sense of presupposition failure when we are asked to evaluate a conditional whose antecedent fails to settle the question raised by the consequent in one way or another. So, while CEM is not *per se* incompatible with a strict analysis, enforcing it without unwelcome side-effects in a strict setting is a non-trivial task.

Stalnaker's (1970) famous suggestion that the probability of a conditional equals the conditional probability of the consequent, given the antecedent (if defined):

$$\text{(ST)} \qquad Pr(\phi > \psi) = Pr(\psi|\phi) \text{ whenever } Pr(\phi) > 0$$

While STALNAKER'S THESIS faces several challenges in its original formulation,[6] it retains intuitive appeal at least on some restricted version, e.g. for indicative conditionals whose antecedent and consequent do not contain modals or conditionals (for recent discussion see Ciardelli & Ommundsen, 2024, and Goldstein & Santorio, 2021; Khoo, 2022).[7] And yet van Fraassen (1976) shows that if ST holds, CEM has the air of a validity (see also Santorio, 2022). For it follows from the probability calculus that $Pr(\psi|\phi) + Pr(\neg\psi|\phi) = 1$ and hence by (ST) that $Pr(\phi > \psi) + Pr(\phi > \neg\psi) = 1$. Assuming that $\ulcorner\phi > \psi\urcorner$ and $\ulcorner\phi > \neg\psi\urcorner$ are contradictories and so that $Pr((\phi > \psi) \wedge (\phi > \neg\psi)) = 0$, we have $Pr((\phi > \psi) \vee (\phi > \neg\psi)) = 1$. So, every instance of CEM has the same probability as the tautological proposition, as long as the antecedent has non-zero probability. And if probabilistic equivalence entails logical equivalence, that makes such instances plain tautologies—bad news for strict analyses of all stripes (and, of course, good news for selection function analyses in the style of Stalnaker, 1968; Stalnaker & Thomason, 1970).[8]

Central to all the trouble for the strict analysis is the question of what it could mean to, on the one hand, know full well that some antecedent-verifying worlds fail to be consequent-verifying worlds and still, at the same time, refrain from flat-out rejecting the conditional in question. This kind of "middle ground" between failing to accept some conditional and rejecting it is, indeed, hard to find in off-the-shelf strict analyses of conditionals such as (1),[9] and yet that there *must* be such ground is—or so I shall argue—strongly suggested by the classical motivation for adopting a strict outlook on the semantics of *if* in the first place, i.e. the Ramsey test for conditionals. The first good news is that we can come up with a strict analysis of conditionals that does draw this distinction in just the right way.

[6] Lewis (1976) kicks off a long tradition of TRIVIALITY results demonstrating that ST has implausible consequences, given (allegedly) minimal assumptions about probability and conditionals (see e.g. Bradley, 2000, 2007 and Hájek & Hall, 1994). Cases of conditionals whose probability do not seem to align with Stalnaker's thesis have been discussed by e.g. McGee (2000), Kaufmann (2004), and Khoo (2016, 2022); cf. Ciardelli and Ommundsen (2024). Stalnaker's hypothesis is sometimes encountered in the literature as ADAMS' THESIS (Adams, 1965, 1966, 1975) or simply as THE THESIS.

[7] See also van Fraassen (1976), Kaufmann (2009, 2015), and Bacon (2015) for accounts that try to vindicate ST. There is by now an enormous body of empirical evidence supporting the equation between the probability of a conditional and the corresponding conditional probability, see among many others Evans and Over (2004) and Douven and Verbrugge (2010).

[8] As such, while it is common to refer to the "Lewis-Stalnaker semantics" as the poster child for a variably strict analysis of conditionals, the accounts differ in ways that matter for current purposes and specifically it is Lewis's account, but not Stalnaker's, that treats *if* as a universal quantifier over possible worlds (and thus qualifies as the kind of strict analysis of conditionals that is of interest here).

[9] That is, assume, as seems reasonable, that (1) is epistemic so that your generic strict analysis will treat (1) as true at w given some context c just in case the c-relevant information at w entails that the die came up two assuming it came up even. It does not take much to make a case for thinking that the c-relevant information cannot vary across those ways the world might be that are live in context (Gillies, 2009, 2010). But if so, the mere fact that the the die came up (say) four at some contextual possibility suffices to establish (1) as a guaranteed falsity in context. Unsurprisingly, strict accounts that make *if* sensitive to some world-independent parameter by semantic design—such as standard dynamic or informational accounts—make essentially the same unfortunate prediction here (more on how things play out in a dynamic setting momentarily).

The other good news is that once we come up with a strict setting that firmly distinguishes between failing to accept a conditional and rejecting it, we can also make sense of middling credences for strict conditionals. For failing to accept a conditional differs from rejecting it in that the former leaves room for *coming to accept* the conditional in light of some additional information, and we can exploit this feature in a strict explanation of the phenomena surrounding less-than-certain conditional judgments. Very roughly, the guiding assumption here is that assigning credences is a process of evaluating some sentence over a set of *hypothetical strengthenings* of one's state of information. Strict conditionals can then have non-trivial probabilities because even if a state fails to accept some conditional, it may very well do so in light of some additional information—how likely we deem the information leading to acceptance of some conditional to be, then, determines the probability of the *if*-construction in question. Moreover, whenever some ϕ is *guaranteed* to be accepted by each relevant strengthening of an arbitrary state, ϕ has a distinct air of validity and this, I suggest, is sufficient to capture the appeal of CEM (or some restricted version thereof) in a strict setting. Conditionals, on this view, are strict conditionals alright but their domain gets systematically *restricted* in logical and probabilistic reasoning. Filling in the details of this narrative, and exploring some of its ramifications, is the purpose of this paper.

My plan is as follows. In Section 2, I will tell just enough of my story to build up some confidence that it makes sense to talk about graded conditional judgments in a strict setting. Section 3 shows how the framework can make sense of CEM. Section 4 adds a few missing pieces to the picture by addressing how epistemic modals and conjunction fit into the framework. Section 5 concludes the discussion by highlighting the bare bones of the story told here and by offering some additional contextualization of the proposal within the existing literature on conditionals.

2 | BASICS

Strict analyses of conditionals come in various shapes and forms. Here I will choose an articulation of the strict paradigm that puts its trouble with graded conditional judgment into clear view but also points towards a way forward. Start with a familiar dictum from Ramsey (1931): that a conditional is accepted, given some state of information s, just in case its consequent is (hypothetically) accepted in the derived state of information got by strengthening s with the assumption of its antecedent. Putting things a bit more precisely:

$$s \Vdash \phi > \psi \text{ just in case } s[\phi] \Vdash \psi$$

Here "$\Vdash$" denotes the relation of acceptance and $s[\phi]$ is the result of strengthening (or updating) s with the information carried by ϕ. I am interested in the question of how this proposal can be turned into a semantics for conditionals that also has something useful to say about graded conditional judgments. I will first consider an off-the-shelf proposal from the existing literature which (predictably) fails to do the trick (Section 2.1) and then start working toward a better proposal (Section 2.2).

2.1 | Beginnings

How should we take Ramsey's proposal as a guide towards the semantics of conditionals? *One* way to go here—not the only one but especially well-suited to our purposes—is to take a DYNAMIC route and model the meaning of conditionals in terms of their acceptance conditions.[10] A bit of background: in a dynamic setting, semantic values are modeled in terms of update effects on an information carrier, understood as a set of possible worlds. The following are the entries for atomics, negation, and conjunction from Veltman (1996):[11]

$$\text{(i) } s[p] = \{w \in s : w(p) = 1\} \text{ (ii) } s[\neg\phi] = s \setminus s[\phi] \text{ (iii) } s[\phi \wedge \psi] = s[\phi] \cap s[\psi]$$

An update of an information carrier s with an atomic sentence p eliminates all possible worlds from s at which p is false. Negation is set subtraction and in order to update with a conjunction, intersect the results of updating with its conjuncts.[12] It is then straightforward to translate Ramsey's acceptance conditions for conditionals into a dynamic update procedure (see, e.g., Gillies, 2004 and Veltman, 1985). Say that s accepts ϕ, $s \Vdash \phi$, just in case $s[\phi] = s$, that is, just in case updating s with ϕ idles. Define:

$$\text{(iv) } s[\phi > \psi] = \{w \in s : s[\phi] \Vdash \psi\}$$

A conditional tests whether its consequent is accepted once the input state is strengthened with the antecedent. If it is, then the test is passed and returns the original input; if not, the test fails and returns the absurd state (∅).

Acceptance being essentially a matter of entailment, dynamic conditionals are at their core strict conditionals. And it does not take much to see that our dynamic spin on the Ramsey tests runs into trouble with graded conditional judgments, for whenever a state fails to accept a conditional, it fully accepts its negation. Indeed, consider Yalcin's (2012) proposal for how to assign probabilities to sentences in a dynamic setting. Start with a probability function Pr that assigs to each subset of W (the set of possible worlds) a number in $[0, 1]$ so that $Pr(W) = 1$ and $Pr(p \cup q) = Pr(p) + Pr(q)$ if p and q are disjoint. If $Pr_s(p) = Pr(p|s)$ for all $p \subseteq W$ is the result of conditionalizing Pr on s, then the probability of ϕ in light of some information carrier s, $Pr(s, \phi)$, is the probability of $s[\phi]$ conditional on s:

$$Pr(s, \phi) = Pr_s(s[\phi])$$

Determining probabilities, then, is a matter of updating one's current state of information s and considering the probability of the resulting proposition. That makes perfect sense in principle, but it also means that, since every conditional is either accepted or rejected, its probability cannot be anything but trivial (0 or 1).

I have stated the problem for the strict paradigm in its dynamic manifestation, but the problem persists if we replace talk about acceptance with talk about truth. Yalcin (2007) proposes that a conditional is true given some world w and state of information s just in case s accepts its consequent once updated with the antecedent (see also Kolodny & MacFarlane,

[10] The usual list of classical dynamic frameworks: Discourse Representation Theory (Kamp, 1981; Kamp et al., 2011; Kamp & Reyle, 1993), Dynamic Predicate Logic (Groenendijk & Stokhof, 1991), File Change Semantics (Heim, 1982), and Update Semantics (Veltman, 1985, 1996).

[11] Heim's work on presupposition projection in dynamic semantics is also seminal, see e.g. Heim (1983).

[12] Conjunction is commonly modeled as sequential updating in dynamic semantics—see, for instance, von Fintel and Gillies (2008) and Willer (2015)—but here we do not need to worry about internal dynamics, and so a simple intersective approach will do.

2010). The resulting truth-conditions match those of classical analyses (see e.g. Kratzer, 1986, 2012) whenever the relevant modal domain does not vary across possible worlds in context, as seems to be the case for epistemic conditionals such as (1). All of these analyses will treat (1) as false in context at all points of evaluation, making it no less difficult to see how their probability could exceed 0 than on the dynamic proposal that we have decided to put in the spotlight.

The good news is that it is not difficult to identify the key culprit behind the trouble: that acceptance-failures (or, for that matter, truth-failures) entail that the conditional in question is to be rejected (or deemed false). The principle has little to recommend for it on the surface—why should conditional judgments not allow for middling attitudes—but overcoming it is not an entirely trivial affair. I will make a proposal in the following subsection and show how it leaves room for graded conditional judgments.

2.2 | Refinements

One way to see why the previously discussed off-the-shelf proposal is wrong-headed starts with the observation that Ramsey's dictum only speaks to what it takes to accept a conditional; it is silent on what it takes to reject (or, for that matter, doubt) a conditional. A sensible way to generalize Ramsey's dictum is the SUPPOSITIONAL RULE (see e.g. Williamson, 2020):

> Take an attitude unconditionally to $\ulcorner\phi > \psi\urcorner$ just in case you take it conditionally to ψ on the supposition ϕ.

This is to impose constraints on what it takes to reject a conditional that are strictly stronger than the one we saw in the off-the-shelf model: the former requires that one *reject* the consequent under the supposition of the antecedent, while the latter deems it sufficient that one *fails to accept* the consequent under the supposition of the antecedent. And since a state may be agnostic about some conditional consequent—neither accept nor reject that consequent—even if strengthened with some additional bit of information, middling attitudes towards conditionals are just what we expect, given our suppositional generalization of Ramsey's rule.

Ultimately we thus aim for a proposal that distinguishes between the notions of acceptance ($\Vdash^{+}$) and rejection ($\Vdash^{-}$) in accordance with the following constraints:

$$s \Vdash^{+} \phi > \psi \text{ just in case } s[\phi] \Vdash^{+} \psi$$
$$s \Vdash^{-} \phi > \psi \text{ just in case } s[\phi] \Vdash^{-} \psi$$

Furthermore, we shall tie the attitude of rejection to the acceptance of a negation as follows:

$$s \Vdash^{+} \neg\phi \text{ just in case } s \Vdash^{-} \phi$$
$$s \Vdash^{-} \neg\phi \text{ just in case } s \Vdash^{+} \phi$$

In the current dynamic setting we achieve these outcomes by adopting a BILATERAL setting that distinguishes between a positive update rule $[\cdot]^{+}$ and a negative update rule $[\cdot]^{-}$.[13] Setting

[13] Bilateral approaches have been employed in a variety of semantic settings, including data semantics (Veltman, 1981, 1985), inquisitive semantics (Bledin, 2020), and truth-maker semantics (Fine, 2017).

aside the issue of conjunction for the moment, we say:

$$(\mathcal{A}) \quad \begin{aligned} s[p]^+ &= \{w \in s : w(p) = 1\} \\ s[p]^- &= \{w \in s : w(p) = 0\} \end{aligned} \qquad (\neg) \quad \begin{aligned} s[\neg\phi]^+ &= s[\phi]^- \\ s[\neg\phi]^- &= s[\phi]^+ \end{aligned}$$

A positive update with p eliminates from the input state all possible worlds at which p is false, while a negative update with p eliminates all possible worlds at which p is true. (We will sometimes write $\overline{p}$ for the complement of p, i.e. $w(\overline{p}) = 1$ just in case $w(p) = 0$). A positive update with $\ulcorner\neg\phi\urcorner$ is a negative update with ϕ, and a negative update with $\ulcorner\neg\phi\urcorner$ is a positive update with ϕ.

We will then write update rules for the conditional in the obvious way. Say now that s accepts ϕ, $s \Vdash^+ \phi$, just in case $s[\phi]^+ = s$ and that s rejects ϕ, $s \Vdash^- \phi$, just in case $s[\phi]^- = s$. The basic proposal would then be that a conditional of the form $\ulcorner\phi > \psi\urcorner$ is accepted by s just in case $s[\phi]^+ \Vdash^+ \psi$ and rejected just in case $s[\phi]^+ \Vdash^- \psi$. One minor wrinkle: following standard protocol (see, e.g., von Fintel, 2001; Gillies, 2007; Willer, 2017) we shall now also assume that conditionals presuppose that their antecedent be compatible with the relevant modal domain and since—for now anyway—conditionals impose tests on the input context s, this is just to presuppose that the conditional antecedent is a possibility in the input state. Putting all of this together, we say:

$$(>) \quad \begin{aligned} s[\phi > \psi]^+ &= \{w \in s : s[\phi]^+ \Vdash^+ \psi\}, \text{ defined iff } s[\phi]^+ \neq \emptyset \\ s[\phi > \psi]^- &= \{w \in s : s[\phi]^+ \Vdash^- \psi\}, \text{ defined iff } s[\phi]^+ \neq \emptyset \end{aligned}$$

Positive and negative updates with conditionals both require that their antecedent be compatible with the input state: if so, we will say that the update is defined or that s ADMITS an update with the conditional in question. If defined, a positive update with a conditional tests whether its consequent is accepted once the input state is strengthened with the antecedent. If defined, a negative update with a conditional tests whether its consequent is rejected once the input state is strengthened with the antecedent. As before, a passed test returns the original state; a failed test results in the absurd state ($\emptyset$).[14]

We now allow for states that admit an update with a conditional but neither accept nor reject it—that are, as we shall put it, AGNOSTIC about the conditional in question. Going back to the case in which a fair six-sided die was thrown, the claim that the die came up two is neither accepted nor rejected under the assumption that it came up even, and so "If the die came up even, it came up 2" is neither accepted nor rejected.

How can we exploit this innovation to leave room for graded conditional judgments? States that are agnostic about ϕ can be strengthened with some additional information so that they end taking a definitive stance on the issue. The obvious proposal then is that it is the likelihood of the information that would lead one to accept rather than to reject ϕ that determines the probability of ϕ in light of some information carrier. One (preliminary) way to make this idea more precise is to use the semantic story just told to derive truth-conditions for sentences of our target language, taking Stalnaker's (1968) remark that a possible world is the ontological analogue of an information carrier as source of inspiration. Specifically, we may treat ϕ as true (false) at w in case $\{w\}$ accepts (rejects) ϕ; if $\{w\}$ does not admit an update with ϕ in the first place— say, because ϕ is a

[14] Groenendijk and Roelofsen (2015), Ciardelli (2021), and Willer (2022) explore approaches to (negated) conditionals that are similar in spirit to the one pursued here, but not with an eye toward making sense of graded conditional judgments in a strict setting.

conditional whose antecedent is false at w—a third truth-value "$\frac{1}{2}$" is assigned. So, to introduce some handy notation, we shall say that (i) $w(\phi) = 1$ if $\{w\} \Vdash^{+} \phi$, (ii) $w(\phi) = 0$ if $\{w\} \Vdash^{-} \phi$, and (iii) $w(\phi) = \frac{1}{2}$ otherwise (see also Booth, forthcoming and Starr, 2014). Truth, in other words, is a matter of acceptance under complete information. This gives plain conditionals "de Finetti"-style (1936) truth-conditions in that '$p > q$' is true at all pq-worlds, false at all $p\overline{q}$-worlds, and neither true nor false at all $\overline{p}$-worlds.

This setup allows us to think of the probability of ϕ (given some state of information s) as the likelihood of ϕ being true rather than false:

$$Pr_s(\{w : w(\phi) = 1\} | \{w : w(\phi) \neq \frac{1}{2}\})$$

This is only a preliminary proposal, to be upgraded momentarily. But there are two important facts that we can highlight already at this stage.

First, the proposal immediately delivers Stalnaker's thesis for plain conditionals:

Fact 1 $Pr(s, p > q) = Pr_s(q|p)$

The underlying fact here is that $\{w\}$ can only accept or reject '$p > q$' if w is a p-world, and can accept '$p > q$' only if w is also a q-world. Stalnaker's hypothesis will have to break once we consider more complex cases. But the observation should do enough to stir up some optimism that we can make sense of graded judgment in a strict setting.

Second, while the proposed apparatus for assigning probabilities to strict conditionals does not presuppose a bilateral setting for its technical implementation, it makes little sense in the off-the-shelf setting from Section 2.1, given the minimal assumption that acceptance amounts to credence 1: for then we would have cases in which a conditional such as 'If the die came up even, it came up two or four' has a probability of 2/3, all the while its negation is fully certain.[15] In contrast, in our bilateral setting the probabilities of plain conditionals and their negations work out as they should:

Fact 2 If $s[p > q]^{+}$ is defined, then $Pr(s, p > q) + Pr(s, \neg(p > q)) = 1$

The simple underlying fact here is that if defined, $Pr_s(q|p) + Pr_s(\neg q|p) = 1$. We will come back to this result once it comes to the task of making sense of CEM.

I have proposed a simple strict analysis of conditionals that addresses at least the most foundational question that each such analysis must answer: how can one have middling probabilistic attitudes toward conditionals whose acceptance conditions one knows to be violated? The key move is to insist that failing to accept a (strict) conditional is not equivalent to rejecting it and in particular the former leaves room for the possibility of coming to accept the conditional if better informed. The probability of any sentence of our target language is then modeled as the likelihood of acceptance under complete information—that leaves conditionals strict but at the same time *restricts* their domain when it comes to assigning credences. The proposal is in need of some tweaking, but I submit that it has enough initial promise to warrant further investigation.

[15] A problem of this kind arises for the proposal in Starr (2014), which starts with an off-the-shelf dynamic treatment of conditionals and negation and then proposes an analysis of *probably* as testing whether $Pr(s, \phi) > .5$ (where s is the input state and ϕ is the prejacent, and Pr is defined as above). As an unwelcome consequence, a state may now accept the negation of 'If the die came up even, it came up two or four' while also accepting that it is probably true.

3 | CONDITIONAL EXCLUDED MIDDLE

I have shown how STALNAKER'S THESIS may readily be vindicated—for plain conditionals anyway—in a strict setting. That is progress but the thesis, as we have seen, also gives CONDITIONAL EXCLUDED MIDDLE (CEM) the air of a validity, and that remains a bit of mystery. For take a simple CEM-instance such as '$(p > q) \vee (p > \neg q)$': predicting that its probability is 1 (if admitted) is no trouble, given the minimal assumption that updating with a disjunction is to take the union of updating with its disjuncts.[16] How '$(p > q) \vee (p > \neg q)$' could be a validity is a bit harder to see—a standard "update-to-test" account of logical consequence, which asks whether the conclusion is guaranteed to be accepted by every state once it has been updated with the premises (assuming it admits the series of updates), will not do the trick:

Validity$_1$ (Acceptance) A sequence $\phi_1, \ldots \phi_n$ *guarantees acceptance* of ψ, $\phi_1, \ldots \phi_n \vDash_1 \psi$, iff $s[\phi_1]^+ \cdots [\phi_n]^+ \Vdash^+ \psi$ for all states s such that $s[\phi_1]^+ \cdots [\phi_n]^+[\psi]^+$ is defined.

CEM fails to be valid$_1$ across the board, for whenever (say) a state accepts p but is agnostic about q, $s[p > q]^+ = s[p > \neg q]^+ = \emptyset$, hence $s[(p > q) \vee (p > \neg q)]^+ = \emptyset$ and so $s \nVdash^+ (p > q) \vee (p > \neg q)$.

The previous counterexample to CEM exploits the fact that the assumption of p need not settle the question as to whether q is true. But, one may add, every way of resolving that uncertainty does end up accepting '$p > q$' or '$p > \neg q$.' Strengthenings of a state of information already figured prominently in our previous proposal for assigning probabilities to ϕ given s, for there we essentially asked which maximally opinionated strengthenings of s accept rather than reject ϕ. The natural suggestion then is to introduce a notion of validity that essentially supervaluates over such "precisifications."[17] We begin by stating a notion of a precisification that is a bit more general than what we saw before—it allows, but does not require, a precisification to be maximally opinionated—and then explain what acceptance in light of such refinements amount to (setting aside some complications that will come into view once we look a conjunctions of conditionals).

Refinements (v.1). Given some state s and sentence ϕ, the set of *refinements* of s with respect to ϕ is defined as $\rho(s, \phi) = \{s' \subseteq s : s' \neq \emptyset$ and $s'[\phi]^+$ is defined and $\neg\exists s'' \subseteq s'. s'' \neq \emptyset$ and $s''[\phi]^+$ is defined$\}$, i.e. as the non-absurd strengthenings of s that are as opinionated as possible while still admitting an update with ϕ. A state s *super-accepts* ϕ, $s \Vdash^+_\rho \phi$, just in case for all $t \in \rho(s, \phi)$, $t \Vdash^+ \phi$.

A state may fail to accept ϕ simpliciter but be guaranteed to end up accepting ϕ however things turn out to be (or else the question of whether to accept ϕ becomes moot). That appears to be a bona fide sense in which some body of information can support the conclusion of some argument, and so it seems legitimate to define a notion of validity that asks whether updating with the premises guarantees super-acceptance of the conclusion.

[16] Then a state accepts a disjunction as long as it accepts one of its disjuncts, and so $w((p > q) \vee (p > \neg q)) = 1$ if w is a pq-world or a $p\overline{q}$-world, and otherwise $w((p > q) \vee (p > \neg q)) = \frac{1}{2}$. Accordingly $Pr(s, (p > q) \vee (p > \neg q)) = 1$ as long as s is compatible with p.

[17] Applications of supervaluationist techniques to various philosophical topics can be found in discussions by Fine (1975), van Fraassen (1966), Kamp (1975), Stalnaker (1981), and Thomason (1970), among many others.

Validity$_2$ (Super-Acceptance) A sequence $\phi_1, \ldots \phi_n$ *guarantees super-acceptance* of ψ, $\phi_1, \ldots \phi_n \vDash_2 \psi$, iff $s[\phi_1]^+ \cdots [\phi_n]^+ \Vdash^+_\rho \psi$ for all states s such that $s[\phi_1]^+ \cdots [\phi_n]^+[\psi]^+$ is defined.

In the fragment considered so far, validity$_1$ entails validity$_2$ since acceptance is persistent: if s accepts ϕ, all subsets admitting an update with ϕ do so as well.

But there is a critical difference: assuming that a disjunction is accepted whenever one of its disjuncts is, each CEM-instance with atomic p and q turns out to be a validity if validity amounts to guaranteed super-acceptance:

Fact 3 $\vDash_2 (p > q) \vee (p > \neg q)$

This is straightforward: whenever s is a state and w is a p-world in s, $\{w\}$ qualifies as a refinement of s with respect to '$(p > q) \vee (p > \neg q)$'. And of course, since w is either a q-world or a $\neg q$-world, $\{w\}$ either accepts '$p > q$' or '$p > \neg q$'.

This is not just a re-casting of the familiar Stalnaker-style selection function semantics for conditionals. For starters, the resulting analysis validates Import-Export.[18] But more importantly for current purposes, it actually imposes—given minimal assumptions anyway—certain limits on the validity of CEM. Let me explain.

Consider a scenario such as the following:

(2) There are 100 coins in an urn: 90 gold, 9 silver, and 1 plastic. The gold coins are all winners, while the silver and plastic coins are losers.

In this context the disjunctive antecedent conditional in (3) seems to have two readings, one on which it is highly probable, and one on which it is not (Khoo, 2021):[19]

(3) If John draws a gold coin or a silver coin, he will win.

That (3) has a reading on which its probability is high should not be surprising: this is just what we expect given Stalnaker's thesis, since the probability that John will win, given that he draws gold or silver, is high (approximating .91). And indeed it is easy to offer a line of reasoning in support of this prediction: for if John draws a gold or a silver coin, it will most likely be a gold one, and in that case he will win for sure; so it is likely that if John draws a gold coin or a silver coin, he will win.

[18] That is, $\ulcorner(\phi \wedge \psi) > \chi\urcorner$ and $\ulcorner\phi > (\psi > \chi)\urcorner$ are mutually entailing, and this marks an important contrast with a selection function semantics, as discussed e.g. by Gillies (2009). As a consequence, conditionals such as (a) "If the die landed even, then if it did not land on 2, it landed on 6" ($E > (\neg 2 > 6)$) and (b) "If the die landed even and did not land on 2, it landed on 6" ($(E \wedge \neg 2) > 6$) are equivalent, and they both have a fifty-fifty chance (assuming again the six-sided die is fair). Importantly, note here that $Pr_s(\neg 2 > 6) = \frac{1}{5}$ and that $Pr_s(\neg 2 > 6|E) \leq \frac{2}{5}$ (assuming that $Pr_s(\phi \wedge \psi) \leq Pr_s(\phi)$). It follows that $Pr(\neg 2 > 6|E) \neq Pr(s, E > (\neg 2 > 6))$ and since $Pr(s, E > (\neg 2 > 6)) = Pr_s([\![6]\!]|[\![E]\!] \cap [\![\neg 2]\!])$, that means that updating with some bit of information is not the same as conditionalizing on it. The point and its role for avoiding triviality have been discussed recently by Goldstein and Santorio (2021) and Ciardelli and Ommundsen (2024) (see also Fusco, 2023 for relevant discussion).

[19] Khoo (2021) in fact goes a step further and holds that (3) has a reading on which is certainly false (rather than just failing to be very likely). This is what I will end up saying as well but it is not the only possible outlook on what is going on with (3) on its problematic reading—more on this momentarily—and so I choose a more neutral label.

What about the reading of (3) on which it has little to recommend for it? Disjunctive antecedent conditionals, so a familiar observation goes, have a reading on which they entail their simplifications, i.e. for (3) we have:

(4) (a) If John draws a gold coin, he will win.
(b) If John draws a silver coin, he will win.

On this reading, the probability of (3) cannot be high since its entailment (4b) is certainly false: if John draws a silver coin, he will definitely lose.

The two readings and their diverging probabilities are in need of an explanation; let me briefly outline how this can be done in the strict dynamic framework developed so far (leaving some details to the Appendix).[20] Earlier we said that conditionals come with a 'visibility' presupposition in the sense that their antecedent must be compatible with the relevant modal domain. We now add that disjunctive antecedent conditionals allow for two readings because the visibility constraint for disjunctions may play out in two distinct ways: either at least one disjunct must be visible, or both. Construing the visibility presupposition of disjunctive antecedent conditionals in the latter "strong" fashion guarantees that they simplify, assuming they are strict material conditionals over the relevant modal domain (Willer, 2018); failure to simplify is possible if the visibility presupposition is "weak" and just one disjunct must be visible for the update to be defined.[21]

Corresponding to these potential readings of (3), we expect two possible probability verdicts. The non-simplifying reading behaves in accordance with STALNAKER'S THESIS; its probability, accordingly, is high. Assuming that, on its simplifying reading, the probability of (3) is defined and cannot exceed the probability of the conjunction of its simplifications, and that the probability of this conjunction cannot exceed those of its conjuncts—more on conjunction in Section 4—its probability is 0 (recall that the silver coins are all losers).

If all of this is correct, we also have counterinstances to CEM. Take (5):

(5) If John draws a gold coin or a silver coin, he will lose.

On its simplifying reading, (5) implies that John loses if he draws gold—since the latter is certainly false, the framework proposed here will assign to (5) a probability of 0 on its simplifying reading.[22] Accordingly the framework delivers counterexamples to the validity of CEM since we now have some instances—such as the disjunction of (3) and (5)—whose probability does not amount to 1. This, to be clear, is consistent with van Fraassen's (1976) argument for CEM based on ST that we considered in Section 1: for the credence in (3) to be zero, its probability cannot be the probability of its consequent given the antecedent (since, as we have seen, that approximates.91); ST and, by extension, the case for CEM have no purchase if we are interested in the simplifying reading of disjunctive antecedent conditionals.

The remaining question now is how we can capture this wrinkle in the story about the validity of CEM. The key observation is that our alternative notion of validity is lenient enough to render

[20] See von Fintel (1997b) for an early precursor to the story told here.

[21] Attempts at explaining how disjunctive antecedent conditionals simplify abound. Pragmatic approaches include Franke (2011), Klinedinst (2009), and van Rooij (2010); semantic approaches include Alonso-Ovalle (2009), Ciardelli et al. (2019), van Rooij (2006), Santorio (2018), and Willer (2018).

[22] Note that (assuming that not winning is the same as losing) our bilateral setting treats (5) as equivalent with the negation of (3).

simple instances of CEM valid—those cases in which ϕ and ψ are atomic—while also blocking the validity of the disjunction of (3) and (5) on the simplifying reading:

Fact 4 $\not\models_2 [(p \vee q) > r] \vee [(p \vee q) > \neg r]$

The simplifying reading, recall, is tied to the strong manifestation of the visibility presupposition of disjunctive antecedent conditionals: both disjuncts must be compatible with the input state. Take now any state s including (say) a $p\overline{q}r$-world w and a $\overline{p}q\overline{r}$-world v: clearly $\{w, v\}$ fails to accept '$[(p \vee q) > r] \vee [(p \vee q) > \neg r]$'; furthermore, $\{w, v\}$ is a *minimal* '$[(p \vee q) > r] \vee [(p \vee q) > \neg r]$'-admitting state since neither $\{w\}$ nor $\{v\}$ satisfy the strong visibility presupposition of the disjunctive antecedent conditionals involved. Accordingly, s fails to super-accept the CEM-instance under consideration. This highlights another sense in which validity$_2$ is not just a matter of the premises of some argument guaranteeing the truth of its conclusion: in some cases completely opinionated states are too opinionated to admit an update with the conclusion, and so the question of truth does not arise.

That CEM clashes with the thesis that disjunctive antecedent conditionals have a reading on which they simplify has been observed before (see in particular Cariani & Goldstein, 2020) and it is worth stressing that what I have said about how to react to this clash—that, on their simplifying reading, disjunctive antecedent conditionals carry visibility presuppositions that are so strong that they may lead to CEM failures—is not without alternatives. Others have suggested that (3) as well as (5) are defective in the sense that they suffer from some kind of presupposition failure—this, indeed, is the path chosen by Cariani and Goldstein (2020) and by Santorio (2022), who all appeal to homogeneity effects (though in importantly different ways). CEM, on this view, has undefined but no false instances and is thus valid given minimimal assumptions about how presupposition failures play with validity. How much there is to choose between these paths is a complex question that cannot be settled here. What seems uncontroversial is that CEM has distinct appeal but also comes with instances that are problematic in some way. That a strict account can account for both facts is, I submit, genuine progress.[23]

Nonetheless some important matters of detail need to be taken care of. Perhaps most obviously, the explanation of how the disjunctive antecedent conditional (3) can have a reading on which its probability is low suggested that on this reading it entails the conjunction of its simplification—but we have not said yet how to exactly assign probabilities to conjunctions of conditionals. This is (part of) the task of the next section.

[23] But it is fair to still wonder how the story told here can avoid the problematic implications of a simple material analysis. Clearly Material Negation, which requires negated conditionals to entail the falsity of their antecedent, fails simply in virtue of the bilateral treatment of negated *ifs*. Yet commonly deemed unwelcome principles such as Contraposition and Antecedent Strengthening pass as valid, and regardless of whether we put acceptance or super-acceptance at the center stage. The issue has been addressed in various forms (von Fintel, 2001; Gillies, 2009, 2010; Starr, 2014; Veltman, 1985; Warmbrōd, 1983). The basic thought is that the problematic principles lose the air of implausibility if—as we have done here—the premises and the conclusion are evaluated against a single state satisfying all visibility presuppositions involved. But we can say a bit more: suppose that we allow a state that has been updated with the premises to evolve dynamically in case the conclusion of the argument under consideration brings a hitherto ignored possibility into view. Then we can observe, first, that a state accepting, say, (a) "If Alice comes to the party, it will be fun" need not admit (b) "If Alice and Bert come to the party, it will be fun," as entertaining the possibility that Alice comes is not to entertain the possibility that Alice and Bert come. And, second, there is no guarantee that once the latter has been accommodated, the resulting state will actually accept, or super-accept, (b)—it is in this sense, then, that an update with (a) does not guarantee (super)-acceptance of (b); likewise for problematic instances of Contraposition. Indeed, the current framework allows us to articulate a notion of entailment that preserves everything we have said about CEM while avoiding the paradoxes of material implication. I leave the (straightforward) details to the Appendix.

4 | UPGRADES

The purpose here is to motivate and execute two critical upgrades to the basic system to deal with two kinds of phenomena, the first one concerning conjunctions of conditionals. Our basic analysis, recall, is grounded in a broadly de Finettian approach to conditionals: the question of truth (or support) does not meaningfully arise unless the antecedent is true (or supported). This approach faces a problem similar to the one that McGee (1989) observes for approaches that take subjective probabilities to be reflective of de Finettian betting conditions for conditionals. Consider the following statement about a coin about to be flipped:

(6) If the coin comes up heads, it will come up heads, and if the coin does not come up heads, it will come up tails.

The conjunction in (6) is certainly deserving of high credence, but since the conditionals involved have incompatible antecedents, no $\{w\}$ can support both and so we would expect (6) to have zero credence (on any plausible conception of conjunction anyway).[24] Conjunctions such as the following pose related problems:

(7) a. $p \wedge (p > q)$
b. $(p > q) \wedge (q > p)$
c. $p \wedge (p > p)$

Assume, as seems plausible, that an update with a conjunction is defined only in case an update with both conjuncts is defined. Then the probability of (7a), if defined, equals the one of '$p > q$', which does not seem right; (7b) and (7c) are not certainties but their probability, if defined, is unfortunately guaranteed to be 1 right now. (7c) also shows that the early defined notion of a refinement does not fully generalize, as it would wrongly predict the sentence to be a validity.[25]

The interplay between epistemic modals and conditionals poses another challenge to the basic analysis (as well as to other strict accounts). On the standard dynamic analysis of epistemic *must*, $s \Vdash^+ \Box_e \phi$ just in case $s \Vdash^+ \phi$. This gives conditionals such as (8a) and (8b) the air of equivalence:

(8) a. If the die came up even, it came up two.
b. If the die came up even, it must have come up two.

The prediction seems on target, but as Mandelkern (2018) stresses, (8a) and (8b) are not on par when it comes to graded judgments (see also Ciardelli, 2021, 2022). By everyone's agreement, (9) is the negation of (8b)

[24] The proposal targeted by McGee (1989, p. 496) takes a conjunction of bets to be called off in case at least one of its conjuncts is called off. If subjective probabilities are reflective of expectations of betting outcomes, it is then hard to see how one could have any confidence in (6), for it would articulate a bet to be called off no matter what.

[25] To see the point about (7a), simply observe that $w(p \wedge (p > q)) = 1$ just in case $w(p \wedge q) = 1$ and that $w(p \wedge (p > q)) \neq \frac{1}{2}$ just in case $w(p) = 1$; the fact about (7b) follows from the observation that $w((p > q) \wedge (q > p)) \neq \frac{1}{2}$ just in case $w(p \wedge q) = 1$. Finally, the minimal states admitting (7c) are singleton sets consisting of a p-world, and so every state admitting an update with (7c) super-accepts it. These problems are not surprising given everything that has been observed about trivalent approaches to conditionals in the literature (Bradley, 2002, Section 8; von Fintel & Gillies, 2015, Section 6; Khoo, 2022, Chapter 4.2; a.o.).

(9) If the die came up even, it might have come up four or six.

Since (9) seems to be a certainty (if the die is fair), (8b) should be assigned zero credence. The probability of (8a), in contrast, is 1/3. The basic analysis does not draw this distinction.[26] Let me address this issue before tackling the (more intricate) issue with conjunction.

4.1 | Informational parameters

We couple the common slogan that epistemic *must* is sensitive to what is taken for granted in discourse with the less familiar idea that we need some way to keep track of this information in probabilistic reasoning. Begin by cashing out the first bit by analyzing epistemic *must* as sensitive to a separately provided *informational parameter*:

$$(\Box_e)\qquad \begin{aligned} s[\Box_e\phi]^+_u &= \{w \in s : u[\phi]^+_u = u\} \\ s[\Box_e\phi]^-_u &= \{w \in s : u[\phi]^-_u \neq \emptyset\} \end{aligned}$$

Here we now say that s accepts ϕ, $s \Vdash^+ \phi$, just in case $s \Vdash^+_s \phi$, which in turn is (once again) a question as to whether updating idles, i.e. $s \Vdash^+_u \phi$ just in case $s[\phi]^+_u = s$. (Relatedly, s now rejects ϕ, $s \Vdash^- \phi$, just in case $s \Vdash^-_s \phi$; and s admits ϕ just in case $s[\phi]^+_s$ is defined.) We will also write "$s[\phi]^+$" and "$s[\phi]^-$" as short for "$s[\phi]^+_s$" and "$s[\phi]^-_s$", respectively.

Adding some extra informational parameter to our semantics does not affect our semantics for atoms and negation; the entry for conditionals needs some minor tweaking:

$$(>)\qquad \begin{aligned} s[\phi > \psi]^+_u &= \{w \in s : s[\phi]^+_u \Vdash^+_{u'} \psi\}, \text{ defined iff } s[\phi]^+ \neq \emptyset \text{ and with } u' = u[\phi]^+ \\ s[\phi > \psi]^-_u &= \{w \in s : s[\phi]^+_u \Vdash^-_{u'} \psi\}, \text{ defined iff } s[\phi]^+ \neq \emptyset \text{ and with } u' = u[\phi]^+ \end{aligned}$$

In evaluating a conditional, we continue to check whether the consequent is accepted once we have updated with the antecendent. The new twist: since we now talk about acceptance in light of some informational parameter u, we need to say how that parameter evolves in conditional reasoning. The key point: in evaluating a conditional in light of some u, the consequent is evaluated in light of the result of updating u with the antecedent.

To see why these modifications matter, take the following (preliminary but straightforward) rewrite of how to assign probabilities to sentences of our extended target language. Assign truth-values at possible-worlds relative to some informational parameter so that $w(\phi)_u = 1$ if $\{w\} \Vdash^+_u \phi$, (ii) $w(\phi)_u = 0$ if $\{w\} \Vdash^-_u \phi$, and (iii) $w(\phi)_u = \frac{1}{2}$ otherwise. The probability of ϕ in light of s is now simply $Pr_s(\{w : w(\phi)_s = 1\} | \{w : w(\phi)_s \neq \frac{1}{2}\})$. Note that lack of acceptance in light of u is persistent for epistemic modals in the sense that if $s \nVdash^+_u \Box_e\phi$, then $s' \nVdash^+_u \Box_e\phi$ for all $s' \subseteq s$. As a consequence, epistemic *must* does not allow for middling credence: once u

[26] But the challenge is of a general nature: Mandelkern's (2018) observation raises questions for a variety of strict analyses that treat (8a) and (8b) as equivalent, including those developed by Gillies (2004), Kratzer (1986, 2012), and Yalcin (2007).

is fixed, every state either accepts or rejects the *must*-claim.[27] Correspondingly, while (8a) and (8b) are mutually entailing, they have different probabilities. Let $s = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ and $t = s[\text{even}]_s^+ = \{w_2, w_4, w_6\}$, and observe that $\{w : w(\text{two})_s = 1\} = \{w_2\}$ while $\{w : w(\Box_e\text{two})_s = 1\} = \emptyset$. As such $Pr(s, (8a)) = Pr_s(\{w_2\}|\{w_2, w_4, w_6\}) = 1/3$. But note that $s' \Vdash_s^+ \text{even} > \Box_e\text{two}$ just in case $s' \cap \{w_2, w_4, w_6\} \Vdash_t^+ \Box_e\text{two}$; since $t \nVdash^+ \text{two}$, $\{w : w(\text{even} > \Box_e\text{two})_s = 1\} = \emptyset$, and hence $Pr(s, (8b)) = 0$.

It is worth asking how the proposed change affects what we said earlier about validity. Everything we said about update-to-test consequence may remain in place; the alternative we proposed to handle the appeal of CEM, however, talks about a state and its refinements and we need to clarify how they relate. The only thing needed is a slight re-write of what it takes for a state to accept ϕ in light of its refinements: we now say that a state s super-accepts ϕ, $s \Vdash_\rho^+ \phi$, just in case for all $t \in \rho(s, \phi)$, $t \Vdash_s^+ \phi$, i.e. in checking for local acceptance we keep the global information state (after updating with the premises) fixed. This ensures, for instance, that '$\Box_e p \vee \Box_e \neg p$' remains invalid even on the revised notion of validity.

4.2 | Compounds of conditionals

Conjunctions of conditionals call for a more comprehensive method for assigning probabilities; they also demonstrate the need for a refined conception of a refinement. To address the latter, rather straightforward task, we now no longer talk about a single visibility presupposition of a sentence but of the visibility presuppositions brought into play by its components. A precisification is now composed of maximally opinionated ways of satisfying each of the visibility presuppositions at play.

Visibility Presuppositions. If χ is a sentence other than a (negated) conjunction or disjunction of some ϕ and ψ, then p is a visibility presupposition of χ, $p \in \partial(\chi)$, iff (i) for all s, if $s[\chi]^+$ is defined, then $s \cap p \neq \emptyset$, and (ii) there is no $q \subset p$ such that for all s, if $s[\chi]^+$ is defined, then $s \cap q \neq \emptyset$. Otherwise, $\partial(\chi) = \partial(\phi) \cup \partial(\psi)$.

So for instance, the visibility presupposition of a simple conditional of the form '$p > r$' is thus $\{p\}$, while '$(p \wedge q) > r$' as well as '$p > (q > r)$' presuppose $\{p \cap q\}$. The visibility presuppositions of a conjunction of conditionals '$(p > q) \wedge (r > s)$' would then be $\{p, q\}$, and likewise for its negation. Note that an atomic sentence such as p comes with a "trivial" visibility presupposition $\partial(p) = \{W\}$ and so in particular $\partial(p \wedge (p > p)) = \{W, p\}$.

We can then offer the following refined conception of a refinement, which will appeal to the notion of a choice function:

Choice Function A choice function γ on some collection X of nonempty sets assigns to each set x in that collection some element $\gamma(x)$ of x. If X is a set of sets of possible worlds, then γ is

[27] Whether epistemic modality judgments come in degrees is a difficult question (see e.g. Charlow, 2020 and Moss, 2015 for discussion). A more complex framework may allow for uncertainty about what s is and thus leave room for graded judgments about epistemic modality. I say we already have enough on our plate and set the issue aside for the time being.

centered on w iff for all $x \in X$, if $w \in x$, then $\gamma(x) = w$; γ_w is the result of centering γ on w, i.e. γ_w is like γ except that $\gamma(x) = w$ whenever $w \in x$.

The output of a choice function on the visibility presuppositions of some sentence ϕ admits ϕ by design (if ϕ is admissible at all); it is also "minimal" in the sense that it is a union of maximally opinionated ways of accommodating each visibility presupposition brought into play. We thus let such outputs now play the role of refinements:

Refinements (v.2) Given some state s and sentence ϕ, t is a *refinement* of s with respect to ϕ, $t \in \rho(s, \phi)$, just in case there is a choice function γ on $\partial(\phi)$ such that $t = \{w : w \in s \text{ and } \exists x \in \partial(\phi). \gamma(x) = w\}$. A state s *super-accepts* ϕ, $s \Vdash^+_\rho \phi$, just in case for all $t \in \rho(s, \phi)$, $t \Vdash^+_s \phi$.

This already helps with the earlier observed problem with sentences such as '$p \wedge (p > p)$'. Here the important point is that $\partial(p \wedge (p > p) = \{W, p\}$, and so if s is agnostic about p, there is a refinement of s with respect to '$p \wedge (p > p)$' that includes a $\overline{p}$-word, which does not accept '$p \wedge (p > p)$' given the obvious positive and negative entries for conjunction:

$$(\wedge) \qquad \begin{aligned} s[\phi \wedge \psi]^+_u &= s[\phi]^+_u \cap s[\psi]^+_u \\ s[\phi \wedge \psi]^-_u &= s[\phi]^-_u \cup s[\psi]^-_u \end{aligned}$$

We thus avoid the unwelcome result that our supervaluationist approach renders '$p \wedge (p > p)$' a validity.

The next step is to refine our method for assigning probabilities to conditionals. As a first step we need to define where compounds of conditionals are defined. We will talk here about the "domain" of a sentence and think of it as the union of its visibility presuppositions.

Domain The *domain* of ϕ is defined as $d(\phi) = \bigcup \partial\phi$ and if s is a state of information then $d(\phi, s) = d(\phi) \cap s$, i.e. the worlds in s that lie in the domain of ϕ.

As such the domain of e.g. '$(p > q) \wedge (r > s)$' consists of the union of the p- and r-worlds.

We can then effectively assign to conditionals (and their compounds) partial truth-values at possible worlds, in line with a long tradition in the existing literature.[28] The question, essentially, is how likely some world w is to be "matched" with another so that we arrive at a minimal ϕ-accepting state of information, the matchmaker being a choice function on $\partial(\phi)$ that is centered on w. To that end it is useful to introduce the notion of a "centered" refinement:

Centered Refinements. t is a refinement of s with respect to ϕ that is *centered* on v, $t \in \rho_v(s, \phi)$, just in case there is a choice function γ_v on $\partial(\phi)$ such that $t = \{w : w \in s \text{ and } \exists x \in \partial(\phi). \gamma_v(x) = w\}$.

Centered refinements guarantee that w is matched with another possible world only if $\{w\}$ alone is too opinionated to satisfy all visibility presuppositions involved. So for instance, if $w \in u$ happens to be a pr-world, then $\{w\}$ is the only w-centered refinement of s with respect to '$(p > q) \wedge (r > s)$'. Clearly, the sentence should count as true or false at w, depending on

[28] For classical discussions, see van Fraassen (1976), McGee (1989), Jeffrey (1991), Stalnaker and Jeffrey (1994).

whether or not $\{w\}$ accepts or rejects it. More interesting are those cases in which $w \in u$ is (say) a $pq\overline{r}$-world: here t is a w-centered refinement just in case it couples w with some r-world v from u. The key idea then is that '$(p > q) \wedge (r > s)$' receives a partial truth-value at w equaling the probability that v is an rs- rather than an $r\overline{s}$-world. Generalizing a bit, we shall say:

Truth-values We say that $v(\phi, s, p, w)$ is defined iff $v \in p$ and $v \in t$ for some $t \in \rho_w(s, \phi)$; $v(\phi, s, p, w) = 1$ iff $v \in p$ and $v \in t$ for some $t \in \rho_w(s, \phi)$ such that $t \Vdash^+ \phi$. The *truth-value* of ϕ at w given s is given as

$$\mathrm{TV}(w, s, \phi) = \prod_{p \in \partial(\phi)} Pr_s(\{v : v(\phi, s, p, w) = 1\} | \{v : v(\phi, s, p, w) \text{ is defined}\})$$

Say again that $w \in u$ is a $pq\overline{r}$-world and that $\rho_w(u, (p > q) \wedge (r > s))$ is not empty to begin with: then $\{v : v((p > q) \wedge (r > s), u, p, w) \text{ is defined}\} = \{w\}$ due to the centering constraint and $\{v : v((p > q) \wedge (r > s), u, p, w) = 1\} = \{w\}$ as well since w is a pq-world. Furthermore $\{v : v((p > q) \wedge (r > s), u, r, w) \text{ is defined}\}$ is the set of r-worlds in u while $\{v : v((p > q) \wedge (r > s), u, r, w) = 1\}$ is the set of rs-worlds in u. The truth-value of '$(p > q) \wedge (r > s)$' at w given u is thus $Pr_u(\{w\}|\{w\}) \times Pr_u(r \wedge s|r) = 1 \times Pr_u(r \wedge s|r)$, i.e. the conditional probability of s given r in light of u, as desired.

The probability of ϕ in light of s is then defined in the obvious way:

$$Pr(s, \phi) = \sum_{w \in d(\phi, s)} [Pr_s(\{w\}|d(\phi, s)) \times \mathrm{TV}(w, s, \phi)]$$

For plain conditionals this re-write amounts to the original proposal from the previous section.[29] If we let ϕ be a conjunction of conditionals such as '$(p > q) \wedge (r > s)$', then $\mathrm{TV}(w, t, \phi) = 1$ in case w is a $pqrs$-world and $\mathrm{TV}(w, t, \phi) = 0$ in case it is a $p\overline{q}$-world or a $r\overline{t}$-world, as no minimal ϕ-admitting state including w accepts ϕ. (Any $\overline{pr}$-world is outside the range of ϕ.) So, the only interesting cases are those in which w is a $pq\overline{r}$-world or a $\overline{p}rs$-world, where ϕ receives the (potentially partial) truth-value equaling the probability of '$r > s$' and '$p > q$', respectively.

Nothing I have said here about conjunction is genuinely novel: the proposal matches the accounts found in van Fraassen (1976), McGee (1989), Jeffrey (1991), and Stalnaker and Jeffrey (1994). As such it avoids the problems we earlier observed concerning (7a–c): the probability of '$p \wedge (p > q)$' now equals the one of '$p \wedge q$' (rather than of '$p > q$') and '$p \wedge (p > p)$' is no longer a certainty but equals in probability the one of p; finally, '$(p > q) \wedge (q > p)$' is no longer a certainty but instead we have $Pr(s, (p > q) \wedge (q > p)) = Pr_s(p \wedge q | p \vee q)$.[30] Furthermore, everything said here is compatible with the sophisticated tools that Kaufmann (2004) uses to improve the predictive powers of the established frameworks in the tradition of van Fraassen.[31] The point here is that conjunctions of conditionals pose no insurmountable trouble for the framework proposed here.

[29] Simply recall that $\partial(p > q) = \{p\}$ and that if w is a p-world, $\gamma_w(p) = w$. Depending on whether w is also a q-world, the truth-value of '$p > q$' at w in s is either 1 or 0, as in the original setting.

[30] To see the point about (7a), simply note that its range is W and that its truth-value is 1 at each pq-world but 0 elsewhere; (7b) has W as its range as well and has truth-value 1 at each p-world, 0 at each $\overline{p}$-world; finally, the range of (7c) is the union of p and q, with a truth-value of 1 at all pq-worlds and 0 otherwise.

[31] Kaufmann (2004) shows how ideas from the literature on counterfactuals may be employed to capture dependencies between conditionals that seem to matter for how we assign probabilities to compounds of conditionals and that the basic

And we can put a finishing touch of detail on the story about disjunctive antecedent conditionals that was told in the previous section. There we tied the two possible readings of a disjunctive antecedent conditional to the different ways in which the visibility presupposition of the antecedent may play out: either at least one disjunct must be visible, or both. The former lets a conditional such as "If John draws a gold or a silver coin, he will win" be (fully) true at each world w at which John draws gold and wins; the latter, simplifying reading will limit the truth-value at such worlds to the probability of John winning conditional on his drawing silver, as $\{w\}$ must be enriched with a world at which John draws silver to meet the visibility contraints. The simplifying reading will thus, as required, have a the truth-value 0 at all possible worlds if, as in the scenario we discussed in Section 3, all silver coins are losers.

5 | CONCLUSION

Let me conclude the discussion by offering an overview of the key components of the proposal developed here (section 5.1) as well as by offering some additional contextualization of the story within the existing literature on conditionals (section 5.2).

5.1 | Synopsis

Strict accounts have trouble making sense of how conditionals can have middling credences and support Conditional Excluded Middle. The trouble, I have suggested, disappears once we realize that assigning credences as well as logical reasoning involve a distinct process of adopting a hypothetical stance that is as opinionated as possible while properly accommodating the visibility presuppositions of the natural language constructions involved. Conditionals, to put a label on the account presented here, are strict conditionals alright but their domain gets *restricted* in distinct ways when we ask how likely some conditional is or consider what follows from what. A good deal of the ensuing task was to develop a notion of "refining a state of information" that makes plausible predictions when it comes to ordinary judgments of probability and validity. For simple conditionals, maximally opinionated states of information will do; but as examples become more complex, so does the method of accommodating visibility presuppositions when constructing refinements.

Logical consequence is a question of guaranteed super-acceptance after updating with the premises. In assigning credences, we ask how likely some ϕ is true and truth, in turn, is defined in

model misses. Lance (1991), for instance, considers a scenario in which a werewolf has a 50% chance of roaming John's area; if so, everybody who walks out is killed. In this scenario, "If John walks out the front door, he will be killed, and if John walks out the back door, he will be killed" seems to have a fifty-fifty chance, but McGee-style accounts, as well as the proposal made here, assign it a merely 1 in 4 chance (see also Bradley, 2012 and Edgington, 1991). Kaufmann's response is couched in a Stalnaker-Bernoulli setting, but the key idea can be re-cast in the setting pursued here. Consider a world w at which John walks out the front door and is killed; earlier we suggested that we arrive at a minimal state by enriching $\{w\}$ with some world v at which he walks out the back door. The additional suggestion now is that not any such world will do: v must agree with all the facts at w that are independent of the fact that John leaves through the back door. Assuming that the werewolf's roamings belong to this category, and given that werewolf must stalk John's neighborhood at w—otherwise he would not be killed—it must also stalk John's neighborhood at v. It follows that Lance's conjunction has a truth-value of 1 (rather than.5) at every world at which John is killed, and the whole conjunction a probability of.5, as required. The proposal, in brief, then is that ρ is to be contextually restricted (in Kaufmann's sytem, what is restricted is the set of contextually relevant sequences of possible worlds).

terms of the likelihood that a refinement centered on w accepts rather than rejects ϕ. In the simple case—if the refinements at play are singletons—this reduces to the question of whether ϕ is true rather than false at some world of evaluation, with conditionals receiving de Finettian truth-conditions. If a refinement is less than maximally opinionated, ϕ may receive a partial truth-value depending on how likely w is "matched" with other worlds from the input state in a ϕ-accepting refinement that is centered on w. In this setting, dynamic semantic values are basic and (partial) truth-values are derived.

Technically sound as it may be, the above story would make little sense if rejecting a conditional were the same as failing to accept it, for then we would end up with states assigning middling credences to conditionals they flat out reject. The required distinction does not come for free in a strict setting but a bilateral framework, I have suggested, is a well-motivated background setting that does the trick.

The ideas developed here have received a dynamic gloss because they are, I submit, naturally articulated in terms of what it takes for a state and its refinements to accept or reject a conditional. They can, however, be articulated in other strict settings. An alternative starting point, taking some inspiration from Yalcin (2007), assigns to conditionals truth-conditions relative to a world w and state of information s (with $s_\phi = s \cap [\![\phi]\!]^s$):

$$[\![\phi > \psi]\!]^{w,s} = 1 \text{ iff for all } w' \in s_\phi : [\![\psi]\!]^{w',s_\phi} = 1, \text{ defined iff } s_\phi \neq \emptyset$$

$$[\![\phi > \psi]\!]^{w,s} = 0 \text{ iff for all } w' \in s_\phi : [\![\psi]\!]^{w',s_\phi} = 0, \text{ defined iff } s_\phi \neq \emptyset$$

We may then once again introduce the notion of a refinement and then (to give just one example) say that an argument with ψ as its conclusion is valid just in case for all w and s such that $w \in s$ and that make all the premises true, $[\![\psi]\!]^{w,t} = 1$ for all refinements t of s with respect to ψ. This delivers CEM for plain conditionals. Letting $Pr(s,\phi) = Pr_s(\{w \in s : [\![\phi]\!]^{w,\{w\}}\} | \{w : [\![\phi]\!]^{w,\{w\}} \text{ is defined}\})$ leads to results matching those in Section 2.2. The proposal would need to be enriched so that it covers epistemic modals and generalized to make reasonable predictions e.g. for compounds of conditionals. But there is no reason to think that the path pursued in Section 4 cannot be transferred to a truth-conditional setting. I think of this as a welcome result. If others are willing to work on integrating the ideas developed here into their non-dynamic strict analysis conditionals, all the power to them.

5.2 | Context

Let me conclude the discussion by briefly comparing the proposal with some salient alternative approaches in the existing literature. Throughout I have assumed that it makes good sense to ask how to assign probabilities to bare conditionals, and already noted earlier (in footnote) a long tradition of challenging this basic assumption. Judgments of probabilities of conditionals, according to this tradition, are not really judgments of the probability that some conditional is true but rather iffy probability judgments: to judge that it is likely that if John is Chicago, then Carl is in New York is not to judge that some conditional content is likely but simply to judge true that if John is Chicago, then it is likely that Carl in New York. This move, moreover, is not *ad hoc* but flows naturally from the well-established hypothesis that *if*-clauses are restrictors by design: hedged conditionals are really conditional hedges, where the hedge makes a quantificational claim over a domain that is restricted by the *if*-clause.[32]

This is not the place to demonstrate that a restrictor-based approach to probability judgments is untenable—my main point here is that it is not the only avenue to choose for those sympathetic to a strict analysis of conditionals. But I note that the literature already offers some considerations suggesting that we must be able to assign probabilities to conditional contents (see von Fintel & Gillies, 2015 and Rothschild, 2021), and add here that any story about the probability of conditionals worth its salt must also account for our everyday judgments concerning compounds of conditionals such as the following:

(10) The following is certainly true: if the die came up even, it came up two, or if the die came up even, it came up four or six.

It is unclear how restrictor-based accounts can make sense of (10), which for sure is not equivalent with (11):

(11) It's certainly true that if the die came up even, it came up two, or it's certainly true that if the die came up even, it came up four or six.

How else the *if*-clauses could function as restrictors of some probability operator in (10) is not transparent.[33] The most straightforward reading of (10) is that it pertains to a disjunction of conditionals and that the sum of the probability of these disjuncts amounts to 1—this is just to say that each of these conditional statements must be assigned some probability value.

The account developed has no trouble making sense of judgments such as (10). And despite the distinct de Finettian flavor that the proposal exhibits in Section 2, it differs substantially from existing trivalent approaches in the literature (Booth, forthcoming; Égré et al., 2021, 2023; Huitink, 2008; Rothschild, 2014). For starters, the account developed here draws a distinction between valid and invalid instances of CONDITIONAL EXCLUDED MIDDLE, while (to my knowledge) no such distinction is drawn in existing trivalent settings. Another distinguishing feature of the story told here is its handling of conjunctions of conditionals, which pose notorious problems for trivalent approaches. Sentences of the form '$(p > q) \wedge (q > p)$' and '$p \wedge (p > p)$' are, for instance, wrongly predicted to be validities assuming the standard Kleene truth-tables for conjunction (and that validity amounts to guaranteed truth if defined). As we have seen, the basic apparatus presented in Section 2 runs into the same problem. But we also saw that by evaluating compounds of conditionals at worlds that only partially contribute to the satisfaction of their visibility presuppositions—and by allowing for partial truth-values at possible worlds—a more sophisticated incarnation of the dynamic bilateral approach handles compounds of conditionals successfully.

In contrast, the prospects of offering a satisfying treatment of compounds of conditionals in a trivalent setting remain uncertain. I concur with Égré et al. (2023) that it is tempting to replace

[32] See Kratzer (1986, 2012) for seminal discussion but also Lewis (1975).

[33] One may suggest that (10) reads as "It's certainly true that if the die came up even, it came up two or four or six," but it is unclear why that should be so. Given the discussion in Section 3, we cannot say that a disjunction of conditionals $\ulcorner(\phi_1 > \psi) \vee (\phi_2 > \chi)\urcorner$ is in general equivalent with $\ulcorner(\phi_1 \vee \phi_2) > (\psi \vee \chi)\urcorner$. Specifically, the disjunctive antecedent conditional '$(p \vee q) > (r \vee \neg r)$' is a validity even on its simplifying reading, but the disjunction of '$(p \vee q) > r$' and $(p \vee q) > \neg r$' is not.

Kleene-conjunction with QUASI-CONJUNCTION, which evaluates a conjunction to true in case one conjunct is true and the other is not false, false if one of the conjuncts is, and neither true nor false otherwise (Adams, 1966; Cooper, 1968). Conjunctions of conditionals, however, cannot always be interpreted that way. Recall the case from Section 3:

(2) There are 100 coins in an urn: 90 gold, 9 silver, and 1 plastic. The gold coins are all winners, while the silver and plastic coins are losers.

In this scenario, reading *and* as quasi-conjunction makes the truth of (12) highly probable:

(12) If John draws a gold coin, he will win, and if he draws a silver coin, he will win.

I submit there is no such reading: since the second conjunct is certainly false, (12) is certainly false. The reason why quasi-conjunction misses this is that it allows the probability of a conjunction to exceed the one of its conjuncts. It is difficult to see how such an analysis could serve as a basis for an analysis of compounds of conditionals.[34] Compounds of conditionals continue to cause headaches for trivalent approaches.

There is without doubt much more to say here about other issues. I have remained silent on counterfactuals and while we have talked about compounds of conditionals nothing has been said about "left-nested" conditionals of the form $\ulcorner(\phi > \psi) > \chi\urcorner$. Concerning counterfactuals, the starting thought would be that they are strict as well, though over a different modal domain, and are assigned probabilities on that basis. Concerning left-nested conditionals, the prediction would be that $\ulcorner(\phi > \psi) > \chi\urcorner$ and $\ulcorner(\phi \wedge \psi) > \chi\urcorner$ are equivalent, which does not seem terribly off the mark.[35] Finally, it remains to comment on what the bilateral setting has to offer when it comes to the interaction between quantifiers and conditionals.[36] But not here: the point of this exercise was to demonstrate that it does make sense to attempt to address these and other phenomena surrounding graded conditional judgments in a—suitably set up—strict setting.

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[34] McDermott (1996), Santorio and Wellwood (2023), Ciardelli and Ommundsen (2024), and Égré et al. (2023) all discuss cases in which *and* apparently behaves like quasi-conjunction, e.g. 'If the die came up even, it came up two, and if it came up odd, it came up one or three' just seems to say that the die came up one, two, or three. Its probability, accordingly, should be 1/2 (assuming we are concerned with the single throw of a fair six-sided die). These cases, for sure, call for an explanation, but I submit that the most promising approach is to take a 'standard' reading of *and* as basic and then explain how a quasi-reading can arise in some contexts. Indeed, there seems to be a more general phenomenon here, as compounds of conditionals coordinated by *or* sometimes have non-standard (viz. non-disjunctive) readings, consider e.g. Woods's (1997) 'Either he will stay in America if he is offered tenure or he will return to Europe if he isn't' (see also Geurts, 2005). Making sense of all this must be left to another day.

[35] Assuming here for convenience that $s[\phi]^+ \neq \emptyset$ requires $s[\phi]^+$ to be defined. For then the mere fact that $s[(\phi > \psi) > \chi]^+$ is defined guarantees that $s[\phi \wedge \psi]^+ \neq \emptyset$ and that $s \vDash^+ \chi$. The predicted equivalence, to be clear, is not uncontroversial, but see Lassiter and Baratgin (2021) for a discussion of how to handle some controversial cases.

[36] See Ramotowska et al. (2023) for recent empirical arguments to the conclusion that quantified conditionals pose trouble for a variety of strict and homogeneity-based analyses of conditionals.

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APPENDIX A

This section briefly shows how a few bells and whistles can be added to the existing framework to arrive at a system that delivers SIMPLIFICATION OF DISJUNCTIVE ANTECEDENTS and CONDITIONAL EXCLUDED MIDDLE while avoiding the paradoxes of material implication.

A.1 | Relational bilateral semantics

Specifically, to deal with the phenomenon of SIMPLIFICATION, we will move towards a *relational* bilateral proposal, i.e. positive and negative semantic values are relations between an input and an output state.

$$(\mathcal{A}) \quad \begin{array}{l} s[p]^+_u t \text{ iff } t = \{w \in s : w(p) = 1\} \\ s[p]^-_u t \text{ iff } t = \{w \in s : w(p) = 0\} \end{array} \qquad (\neg) \quad \begin{array}{l} s[\neg\phi]^+_u t \text{ iff } s[\phi]^-_u t \\ s[\neg\phi]^-_u t \text{ iff } s[\phi]^+_u t \end{array}$$

The proposal for conjunction is as follows:

$$(\wedge) \quad \begin{array}{l} s[\phi \wedge \psi]^+_u t \text{ iff } \exists s'.\, s[\phi]^+_u s' \text{ and } s'[\psi]^+_u t \\ s[\phi \wedge \psi]^-_u t \text{ iff } s[\phi]^-_u t \text{ or } s[\psi]^-_u t \end{array}$$

If conjunction and disjunction are duals, we arrive at the following entries for the latter connective:

$$(\vee) \quad \begin{array}{l} s[\phi \vee \psi]^+_u t \text{ iff } s[\phi]^+_u t \text{ or } s[\psi]^+_u t \\ s[\phi \vee \psi]^-_u t \text{ iff } \exists s'.\, s[\phi]^-_u u \text{ and } s'[\psi]^-_u t \end{array}$$

We set aside internal dynamic effects: when updating with a conjunction (disjunction), both conjuncts (disjuncts) are processed in light of the same informational parameter.

Updating some state s proceeds by taking the union of its update relata, treating s itself as the relevant informational parameter. We define the notions of acceptance and rejection on this basis:

$$\text{(i)} \quad s \uparrow \phi = \bigcup\{t : s[\phi]^+_s t\} \qquad \text{(ii)} \quad s \downarrow \phi = \bigcup\{t : s[\phi]^-_s t\}$$

$$\text{(iii)} \quad s \Vdash^+ \phi \text{ iff } s \uparrow \phi = s \qquad \text{(iv)} \quad s \Vdash^- \phi \text{ iff } s \downarrow \phi = s$$

We can then define the update rule for conditionals (with $u' = u \uparrow \phi$):

$$(>) \quad \begin{array}{l} s[\phi > \psi]^+_u t \text{ iff } t = \{w \in s : \text{ for all } s' \text{ such that } s[\phi]^+_u s'.\, s' \Vdash^+_{u'} \psi\} \text{ and } \langle s, \emptyset\rangle \notin [\phi]^+ \\ s[\phi > \psi]^-_u t \text{ iff } t = \{w \in s : \text{ for all } s' \text{ such that } s[\phi]^+_u s'.\, s' \Vdash^-_{u'} \psi\} \text{ and } \langle s, \emptyset\rangle \notin [\phi]^+ \end{array}$$

The visibility presupposition stated thus guarantees that every state accepting some disjunctive antecedent conditional accepts its simplifications. To illustrate: if s is incompatible with p, $\langle s, \emptyset\rangle \in [p]^+_s$ and so $\langle s, \emptyset\rangle \in [p \vee q]^+_s$; similarly if s is incompatible with q. So acceptance of '$(p \vee q) > r$' requires the existence of p-worlds and of q-worlds in s in virtue of its visibility presupposition, and that all of these worlds are r-worlds in virtue of its asserted content.

A non-simplifying reading of disjunctive antecedent conditionals can be captured by introducing a "flattening"-operator familiar from the inquisitive semantics literature (Ciardelli et al., 2019):

$$(!)\quad \begin{aligned} s[\phi!]^+_u t \text{ iff } t &= \bigcup\{v : s[\phi]^+_u v\} \\ s[\phi!]^-_u t \text{ iff } t &= \bigcup\{v : s[\phi]^-_u v\} \end{aligned}$$

Disjunctive antecedent conditionals then allow for two distinct logical forms:

(a) $(\phi \vee \psi) > \chi$

(b) $(\phi \vee \psi)! > \chi$

The reading in (a) is simplifying for reasons already discussed; the reading in (b) merely presupposes the visibility of one of the disjuncts in the modal domain and thus does not enforce simplification.

The entries for epistemic modals are as follows (here C is any consistent proposition):

$$(\Box_e)\quad \begin{aligned} s[\Box_e\phi]^+_u &= \{w \in s : \langle u, C\rangle \notin [\phi]^-_u\} \\ s[\Box_e\phi]^-_u &= \{w \in s : \langle u, \emptyset\rangle \notin [\phi]^-_u\} \end{aligned}$$

Assuming that *must* and *might* are duals, the proposal delivers the FREE CHOICE EFFECT for disjunctions scoping under *might* (Willer, 2018).

A.2 | Entailment

Recall the notions of a *visibility presupposition*, a *refinement* and *super-acceptance* (now relativized to an informational parameter):

Visibility Presuppositions. If χ is a sentence other than a (negated) conjunction or disjunction of some ϕ and ψ, then p is a visibility presupposition of χ, $p \in \partial(\chi)$, iff (i) for all s, if $s[\chi]^+$ is defined, then $s \cap p \neq \emptyset$, and (ii) there is no $q \subset p$ such that for all s, if $s[\chi]^+$ is defined, then $s \cap q \neq \emptyset$. Otherwise, $\partial(\chi) = \partial(\phi) \cup \partial(\psi)$.

Refinements (v.2). Given some state s and sentence ϕ, t is a *refinement* of s with respect to ϕ, $t \in \rho(s, \phi)$, just in case there is a choice function γ on $\partial(\phi)$ such that $t = \{w : w \in s \text{ and } \exists x \in \partial(\phi). \gamma(x) = w\}$. A state s *super-accepts* ϕ, $s \Vdash^+_\rho \phi$, just in case for all $t \in \rho(s, \phi)$, $t \Vdash^+_s \phi$.

The key idea is behind the upcoming proposal for entailment is that visibility presuppositions can be accommodated; what gives (plain instances of) CEM some appeal is that they are super-accepted regardless of how the visibility presupposition is accommodated, while this is not so if we look at instances of e.g. ANTECEDENT STRENGTHENING and CONTRAPOSITION. Accommodating a visibility presupposition requires, of course, expanding the modal horizon; the result of this accommodation is determined by a system of spheres (Gillies, 2007).

Systems of Spheres A *system of spheres* π is a set of (non-empty) sets of possible worlds totally ordered by the subset relation. π is *centered* on s just in case s is the minimal element of π. $\mu(s, \pi, \phi)$ is defined only if (i) π is centered on s and (ii) π includes an element admitting ϕ; if defined, $\mu(s, \pi, \phi)$ is the minimal element of π admitting ϕ.

We then define entailment as guaranteed super-acceptance of the conclusion after accommodating its visibility presuppositions (if any):

Entailment A sequence $\phi_1, \ldots \phi_n$ entails ψ, $\phi_1, \ldots \phi_n \vDash \psi$, iff for all states s and π such that $s \uparrow \phi_1 \ldots \uparrow \phi_n$ and $\mu(s \uparrow \phi_1 \ldots \uparrow \phi_n, \pi, \psi)$ are defined, $\mu\,(s \uparrow \phi_1 \ldots \uparrow \phi_n, \pi, \psi) \Vdash^+_\rho \psi$.

Simply recall that any state admitting p super-accepts '$(p > q) \vee (p > \neg q)$'; hence accommodating the visibility presupposition of '$(p > q) \vee (p > \neg q)$' in any state is guaranteed to result in a state super-accepting '$(p > q) \vee (p > \neg q)$'. At the same time, clearly the visibility of '$p > r$' does not satisfy the visibility presuppositions of '$(p \wedge q) > r$' and '$\neg r > \neg p$'; furthermore, accommodating that visibility presupposition does not result in a state super-accepting these conditionals.

The resulting framework integrates smoothly into the proposed framework for assigning probabilities. Specifically, we can still rely on the notion of a world-based refinement and define potentially partial truth-values at possible worlds.