THE UNIVERSITY OF CHICAGO

LOWER BOUNDS FOR SLOPES OF OVERCONVERGENT EIGENFORMS

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BY
YULIA KOTELNIKOVA

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ABSTRACT

Let $p \geq 3$ be a prime and suppose G is a form of a reductive group, that is compact at infinity and splits at p. Under these assumptions, we can estimate the slopes in the corresponding eigenvariety by finding the lower bound of the Newton polygon of the compact Hecke operator in the spirit of the Liu et al. [2017]. In case GSp_4, we obtain a sharper estimate.

CHAPTER 1

INTRODUCTION

1.1 Background

1.1.1 The eigencurves and eigenvarieties

The eigencurve was first introduced in the work of Coleman and Mazur Coleman and Mazur [1998] as an instrument to better understand families of overconvergent modular forms. Some generalizations of the eigencurve were investigated in numerous subsequent works, including papers of Buzzard (Buzzard [2004]) and Emerton (Emerton [2006]).

For higher (and lower) rank reductive groups, the eigenvarieties were constructed in by Chenevier (Chenevier [2004]), by Buzzard (Buzzard [2004]) as well as many other papers. Buzzard suggested a general construction of the eigenvariety for an arbitrary reductive group in Buzzard [2007].

The study of families of overconvergent modular forms and eigencurves has raised various natural and nontrivial questions about their slopes, such as the halo conjecture and the ghost conjecture. The halo conjecture suggests that the set of slopes of overconvergent automorphic forms over the boundary of the weight space is a union of finitely many arithmetic progressions. The halo conjecture for the eigencurve was proved in many special cases, and it was ultimately resolved by Liu, Wan, and Xiao (Liu et al. [2017]), and by Diao and Yao (Diao and Yao [2023]).

For higher rank eigenvarieties, we do not have a very good understanding of slopes of overconvergent eigenforms. Recently, progress was made by Ye (Ye [2024]), who proved that the rate of growth of the Newton polygon of Hecke operators acting on the overconvergent forms for GU_n is $O\left(x^{1+\frac{2}{n(n-1)}}\right)$.

1.2 Main results

In this paper we always assume that G is a reductive group such that $G(\mathbb{R})$ is compact. In this case, following Buzzard, we define the space of overconvergent automorphic forms of level $K^p \operatorname{Iw}_p$ with central character χ to be

$$\mathcal{S}(\rho)_G^{K^p \operatorname{Iw}_p}(R) = \left\{ \varphi\colon D(\mathbb{Q})\backslash D(\mathbb{A}_f)/K^p \to M \;\middle|\; \varphi(xu^{-1}) = u_p\varphi(x) \text{ for } u_p \in \operatorname{Iw}_p \right\},$$

where M is a representation of Iw_p with coefficients in R, that is also a space of functions with some convergence condition. In this case, under the technical assumption that G does not contain the exceptional group G_2 as a subgroup, we show that, for all but finitely many p, there is a lower bound for the slopes of the eigenvariety over the boundary whose asymptotic is $O\left(x^{1+\frac{1}{\#\Phi_-}}\right)$, where $\#\Phi_-$ is the number of all negative roots in the root system of G.

1.2.1 Proof strategy

In the paper of Liu, Wan, and Xiao (Liu et al. [2017]), the Halo conjecture is proved in the case where G is a compact form of GL_2. The first step in their proof is finding a creative way to evaluate the lower bound on the Newton polygon. In Section 4.2, we give an alternative interpretation of their computation in terms of the norm with respect to the lattice

$$L_p = \left\langle p^n \binom{z}{n} \right\rangle.$$

Moreover, for this norm, the Hecke operator is compact. Then it turns out, that the lower bound on the Newton polygon of the Hecke operator, found in Liu et al. [2017], is the Hodge polygon of the Hecke operator for the new unusual norm.

Furthermore, it turns out that the Hecke operator can be decomposed as a sum of operators of the form $(\operatorname{Iw}_p \, d \operatorname{Iw}_p)_*$ where d is diagonal (as an element of GL_2), and that the lattice L_p is invariant under the Iwahori action. As a result, we are able to prove that the

Hodge polygon of the Hecke operator is equal to that of the operator d_*.

We generalize this approach to the higher rank cases. We find the appropriate analogue L_I of the lattice L_p in the higher rank situations, and show that the lattice L_I is Iwahori invariant. The Hecke operator is again a sum of operators of the form $(\mathrm{Iw}_p\, d\, \mathrm{Iw}_p)_*$, where $d \in T(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$. We find some lower bound on the Hodge polygons of each of the operators of the form $(\mathrm{Iw}_p\, d\, \mathrm{Iw}_p)_*$. A more careful examination of these actions allows us to find a lower bound on the Hodge polygon of the Hecke operator on the lattice L_I, and therefore, evaluate the Newton polygon.

1.3 Organization of the paper

In Chapter 2 we introduce group-theoretic notations; in Chapter 3 we remind the definition of the automorphic forms, the Hecke action, as well as the notion of an integral model introduced in Liu et al. [2017], and define the Hecke action on the integral model.

In Section 4.1, we give a reminder about Newton polygons and Hodge polygons; in Section 4.2 we introduce the lattice L_p and review the proof of Liu, Wan, and Xiao (Liu et al. [2017]). In Section 4.3 the main analysis happens. We outline the argument in Section 4.3.2; the lattice L_I is introduced in Section 4.3.4 and in Section 4.3.5 we prove that the L_I is Iwahori invariant. We investigate the Hodge polygon of operators of the form $\mathrm{Iw}_p\, d\, \mathrm{Iw}_p$ in Example 3.2.0.3. We conclude by some additional analysis in Section 4.3.7, and find the lower bound for the slopes of asymptotic $O\left(x^{1+\frac{1}{\#\Phi_-}}\right)$. Note that this aggrees with the lower bound by Ye (Ye [2024]), however, in Section 5.1, we manage to obtain a better constant in the $O\left(x^{1+\frac{1}{\#\Phi_-}}\right)$.

In Chapter 5, we apply the ideas from Chapter 4 to slopes of overconvergent eigenforms. In Section 5.1 we compare the bound we obtain in case where G is a compact form of GL_n with that in Ye [2024]; in Section 5.2 we study the case where G is a compact form of GSp_4.

CHAPTER 2

REMINDERS FROM LIE THEORY.

In this section we introduce the notation for standard subgroups of Lie groups.

2.1 Lie algebra-related notations

This section largely follows Iwahori and Matsumoto [1965]. For a reductive algebraic group G we fix the Borel subgroup and the torus:

$$G \supset B \supset T.$$

The groups of characters and cocharacters, respectively, are denoted by $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$.

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has a decomposition

$$\mathfrak{g} = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha.$$

Let $\Pi \subset \Phi^+ \subset \Phi \subset X^*(T)$ denote the sets of simple roots, positive, and all roots, respectively; let $\Phi^- \subset \Phi$ be the set of negative roots. For each root α, we choose an isomorphism

$$x_\alpha \colon \mathbb{G}_a \to \underline{\exp(\mathfrak{g}_\alpha)}$$

into the corresponding unipotent subgroup of G.

We denote by $N_+ \subset B$ the unipotent subgroup generated by all elements of the form $x_\alpha(\nu)$ where $\alpha \in \Phi_+$. The Iwahori subgroup Iw_p is generated by $\underline{T}(\mathbb{Z}_p)$, $x_\alpha(\mathbb{Z}_p)$ for $\alpha \in \Phi^+$ and $x_\alpha(p\mathbb{Z}_p)$ for $\alpha \in \Phi^-$. In other words, $\mathrm{Iw}_p(\mathbb{Z}_p) \supset \mathrm{B}(\mathbb{Z}_p)$ to be the preimage of $\mathrm{B}(\mathbb{F}_p)$ under the map $G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$.

Example 2.1.0.1 (Conventions for GSp_4.)**.** In this paper, GSp_4 is a group of linear transformations preserving the form

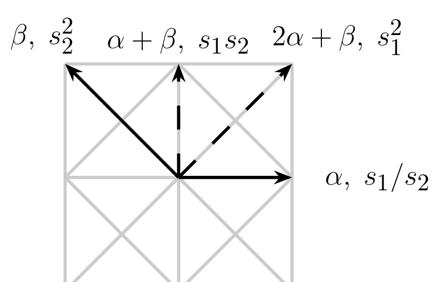
$$J = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

In other words, a matrix M is in GSp_4 if and only if $M^t JMJ = \lambda$ for some constant λ.

If $G = \mathrm{Sp}_4 \subset \mathrm{GSp}_4$, the torus is

$$T = \left\{ \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_2^{-1} & \\ & & & s_1^{-1} \end{pmatrix} \right\}.$$

The root system is known to be B_2; the positive roots are α, β, $\alpha+\beta$, and $2\alpha+\beta$; the simple roots are α and β.



The corresponding unipotent groups are

$$x_\alpha(s) = \left(\begin{array}{cc:cc} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad x_\beta(s) = \left(\begin{array}{c:cc} 1 & 0 & 0 \\ & s & 0 \\ \hdashline 0 & & 1 \end{array}\right)$$

$$x_{\alpha+\beta}(s) = \left(\begin{array}{c:cc} 1 & s & 0 \\ & 0 & s \\ \hdashline 0 & & 1 \end{array}\right) \qquad x_{2\alpha+\beta}(s) = \left(\begin{array}{c:cc} 1 & 0 & s \\ & 0 & 0 \\ \hdashline 0 & & 1 \end{array}\right)$$

The nontrivial commutation relations in N_+ are

$$[x_\alpha(s), x_\beta(t)] = x_{2\alpha+\beta}(s^2 t) x_{\alpha+\beta}(st),$$
$$[x_\alpha(s), x_{\alpha+\beta}(t)] = x_{2\alpha+\beta}(2st).$$

These relations can be checked in various ways, and also follow from Computation 4.3.3.1.

2.2 Coordinates

Recall that $\mathrm{B}(\mathbb{Z}_p)$ was defined to be a group generated by the torus T and the groups $x_\alpha(s)$ for $\alpha > 0$ and $s \in \mathbb{Z}_p$; Iw_p is defined to be generated by $\mathrm{B}(\mathbb{Z}_p)$ and $x_\alpha(ps)$ for $\alpha \in \Phi_-$ and $s \in \mathbb{Z}_p$. Hence, there is an isomorphism

$$\mathbb{Z}_p^{\Phi_-} \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p))$$

$$(\, z_\alpha \,)_{\alpha\in\Phi_-} \longmapsto \sum_{\alpha\in\Phi_-} X_\alpha(p z_\alpha) \overset{\exp}{\longmapsto} \prod_{\alpha\in\Phi_-} x_\alpha(p z_\alpha)$$

Convention 2.2.0.1. We think of the set Φ_- as an ordered set with some fixed natural order. That is, if $\alpha > \beta$ then α goes before β. The notation $\prod_{\alpha \in \Phi_-}$ is used to denote the product in that order.

CHAPTER 3

AUTOMORPHIC FORMS AND HECKE ACTION

In this section we introduce the notation related to automorphic forms and Hecke operators. Exposition here is largely inspired by Buzzard (Buzzard [2004], Buzzard [2007]).

3.1 The space of automorphic forms

Suppose G is a reductive group and that $G(\mathbb{R})$ is compact. Let R be a $\mathbb{Z}_p$-algebra and suppose $\rho\colon B \to R^\times$ is a character of the Borel. We consider the module

$$\operatorname{Ind}(\rho)_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(R) = \left\{ \text{continous functions } f\colon \mathrm{Iw}_p \to R \;\middle|\; f(bg) = \rho(b) f(g) \text{ where } b \in \mathrm{B}(\mathbb{Z}_p) \right\}.$$

with the usual left action of the Iw_p by $u \cdot f(g) = f(gu)$. Then we define the space of R-valued automorphic forms of weight ρ to be

$$\mathcal{S}(\rho)_G^{K^p\,\mathrm{Iw}_p}(R) = \Big\{ \varphi\colon D(\mathbb{Q})\backslash D(\mathbb{A}_f)/K^p \to \operatorname{Ind}(\rho)_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(R) \;\Big|\; \varphi(xu^{-1}) = u_p \varphi(x) \text{ for } u_p \in \mathrm{Iw}_p \Big\}.$$

Remark 3.1.0.1. Following Liu et al. [2017], we use $f(g)$ to denote elements of $\operatorname{Ind}(\rho)_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(R)$ and $\varphi(x)$ for elements of $\mathcal{S}(\rho)_G^{K^p\,\mathrm{Iw}_p}(R)$, and try to be consistent with it.

3.2 Coordinates

Recall that there is an identification between $\mathbb{Z}_p^{\Phi-}$ and $\mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p)$.

Definition 3.2.0.1. We fix the isomorphism

$$\operatorname{Ind}(\rho)_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(R) \xrightarrow{\sim} \mathcal{C}\left(\mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p), R\right) \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R)$$

The second map is defined the way it is discussed in Section 2.2: for a fixed order on Φ_-, $f \in \mathcal{C}\left(\mathrm{Iw}_p \;/\; \mathrm{B}(\mathbb{Z}_p), R\right)$ maps to $F \in \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ if

$$F(\,(\,z_\alpha\,)_{\alpha\in\Phi_-}\,) = f\left(\prod_{\alpha\in\Phi_-} x_\alpha(pz_\alpha)\right)$$

Definition 3.2.0.2. The *left* action of Iw_p on the space $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$, is defined as follows:

$$u_*\, F(\,(\,z_\alpha\,)_{\alpha\in\Phi_-}\,) = \rho(b)\cdot F(\,(\,z_\alpha^{new}\,)_{\alpha\in\Phi_-}\,)$$

where

$$\prod_{\alpha\in\Phi_-} x_\alpha(pz_\alpha)\cdot u = b\cdot \prod_{\alpha\in\Phi_-} x_\alpha(pz_\alpha^{new}), \quad b\in \mathrm{B}(\mathbb{Z}_p).$$

Example 3.2.0.3. For $h \in T$, we have

$$\prod_{\alpha\in\Phi_-} x_\alpha(pz_\alpha)\cdot h = h\cdot \prod_{\alpha\in\Phi_-} x_\alpha(\alpha^{-1}(h)pz_\alpha);$$

$$h_*\, F(\,(\,z_\alpha\,)_{\alpha\in\Phi_-}\,) = \rho(h)\cdot F(\,(\,\alpha^{-1}(h)z_\alpha\,)_{\alpha\in\Phi_-}\,).$$

Remark 3.2.0.4. The right hand side in the above example can be extended to all elements $h \in T$ for which $\rho(h)$ is defined and $\alpha^{-1}(h) \in \mathbb{Z}_p$. This way the action of Iw_p extends to a monoid. See examples below.

Example 3.2.0.5 (Sanity check)**.** In this example we compare the $G = \mathrm{GL}_2$ case to the paper Liu et al. [2017]. For the character

$$\rho\left(\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) = \chi(d),$$

the action defined in Liu et al. [2017] looks like this:

$$F\Big\|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(z) = \chi(cz+d)\cdot F\left(\frac{az+b}{cz+d}\right),$$

where

$$F(z) = f\left(\begin{pmatrix} 1 & 0 \\ pz & 1 \end{pmatrix}\right) \quad \text{for } f \in \mathrm{Ind}_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(\chi).$$

For example, if $F(z) = z^k$, and $\chi(z) = z^k$ then

$$(z \mapsto z^k)\Big\|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = \left(z \mapsto (cz+d)^k \frac{(az+b)^k}{(cz+d)^k} = (az+b)^k\right).$$

This formula extends to the monoid

$$M_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}_p) \text{ where } \ p\,|\,c,\ p \not| d \ \text{ and } ad-bc \neq 0 \right\} \supset \mathrm{Iw}_p\,.$$

In other words, the action of the matrix $\mathrm{diag}(p,1)$ by $z \mapsto pz$ is defined.

With our definition, for the same character, we get the action

$$\begin{aligned}
\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_* F(z) =& f\left(\begin{pmatrix} 1 & 0 \\ pz & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & b \end{pmatrix}\right) = \\
=& f\left(\begin{pmatrix} * & * \\ 0 & pbz+d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ p\frac{az+c/p}{pbz+d} & 1 \end{pmatrix}\right) = \\
=& \chi\left(bpz+d\right)\cdot F\left(\frac{az+c/p}{pbz+d}\right).
\end{aligned}$$

In case $b = c = 0$, the map is $z \mapsto \chi(d)\cdot(a/d)z$, which is defined if $\chi(d)$ is and if $a/d \in \mathbb{Z}_p$ and $d \in \mathbb{Z}_p^\times$. Therefore, this action also extends to a modoid generated by Iw_p and matrices $\mathrm{diag}(a,d)$ where $d \in \mathbb{Z}_p^\times$. In other words, we can also define the action of the matrix $\mathrm{diag}(p,1)$

by $z \mapsto pz$.

Example 3.2.0.6. If $G = \mathrm{GSp}_4$, there are natural coordinates on the space $\mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p)$, that is a map $\mathbb{Z}_p^4 \to \mathrm{Lie}(\mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p)) \to \mathrm{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p)$

$$\begin{aligned} &(\, z_{-\alpha},\, z_{-\beta},\, z_{-\alpha-\beta},\, z_{-2\alpha-\beta}) \mapsto \\ &\mapsto X_{-\alpha}(pz_{-\alpha}) + X_{-\beta}(pz_{-\beta}) + X_{-\alpha-\beta}(pz_{-\alpha-\beta}) + X_{-2\alpha-\beta}(pz_{-2\alpha-\beta}) \\ &\mapsto x_{-\alpha}(pz_{-\alpha})\, x_{-\beta}(pz_{-\beta})\, x_{-\alpha-\beta}(pz_{-\alpha-\beta})\, x_{-2\alpha-\beta}(pz_{-2\alpha-\beta}) \end{aligned}$$

as well as the following isomorphism:

$$\mathrm{Fun}(\mathbb{Z}_p^4) \xrightarrow{\quad\quad\sim\quad\quad} \mathrm{Ind}(\rho)_{\mathrm{B}(\mathbb{Z}_p)}^{\mathrm{Iw}_p}(R)$$

$$F(\, z_{-\alpha}, \dots,\, z_{-2\alpha-\beta}) \longmapsto f\left(x_{-\alpha}(p\, z_{-\alpha}) \cdot \dots x_{-2\alpha-\beta}(p\, z_{-2\alpha-\beta})\right).$$

Here, the fixed order on Φ_- is $(-\alpha, -\beta, -\alpha-\beta, -2\alpha-\beta)$.

Moreover, by Remark 3.2.0.4, the action can be extended to the monoid, generated by Iw_p and matrices of the form $\mathrm{diag}(a, b, c, d)$ where $a/b \in \mathbb{Z}_p$, $b/c \in \mathbb{Z}_p$. For instance, the following maps are defined actions

$$\begin{aligned} \mathrm{diag}(p, p, 1, 1)_* (\, z_{-\alpha},\, z_{-\beta},\, z_{-\alpha-\beta},\, z_{-2\alpha-\beta}) &= (\, z_{-\alpha},\, pz_{-\beta},\, pz_{-\alpha-\beta},\, pz_{-2\alpha-\beta}) \\ \mathrm{diag}(p^2, p, p, 1)_* (\, z_{-\alpha},\, z_{-\beta},\, z_{-\alpha-\beta},\, z_{-2\alpha-\beta}) &= (\, pz_{-\alpha},\, z_{-\beta},\, pz_{-\alpha-\beta},\, p^2 z_{-2\alpha-\beta}) \end{aligned}$$

3.3 Neatness.

A level subgroup K of $G(\mathbb{A}_f)$ is called *neat* if $G(\mathbb{A}_f)$ a disjoint union of the form

$$G(\mathbb{A}_f) = \bigsqcup_{i \in \{0, \dots n-1\}} G(\mathbb{Q}) \gamma_i K.$$

We assume from now on that $K = K^p \operatorname{Iw}_p$ is neat. In that case, there is an isomorphism

$$\mathcal{S}(\rho)_G^K(R) \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))_i} \bigoplus_{i=0}^{n-1} \mathcal{C}(\operatorname{Iw}_p \,/\, \mathrm{B}(\mathbb{Z}_p), R) \xrightarrow{=} \bigoplus_{i=0}^{n-1} \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R) \ .$$

3.4 Integral model; universal character

The notion of the integral model as defined in Liu et al. [2017] has a natural extension in general. Consider the ring of functions on the torus

$$\Lambda = \mathbb{Z}[\![(\mathbb{Z}_p^\times)^\Pi]\!] = \mathbb{Z}\left[\!\!\left[\prod_{\alpha \in \Pi} (\mathbb{Z}_p^\times)_\alpha\right]\!\!\right] .$$

The *universal character* ρ_{univ} is defined naturally: for a simple root α and $z \in 1 + p\mathbb{Z}_p$,

$$\rho_{univ}(h_\alpha(z)) = (1 + T_\alpha)^{\log(z)/p}$$

We will call the space $\mathcal{S}(\rho_{univ})_G^{K^p \operatorname{Iw}_p}(\Lambda)$ the integral model.

3.5 The double coset operator

Suppose $t \in T(\mathbb{Q}_p)$ and suppose that $t^{-1}B(\mathbb{Z}_p)t \subset B(\mathbb{Z}_p)$. If there is a double coset decomposition

$$\operatorname{Iw}_p t \operatorname{Iw}_p = \coprod v_i \operatorname{Iw}_p,$$

Then we define the corresponding *double coset operator*

$$(U_t \varphi)(x) = \sum_i (v_i)_p \varphi(x v_i).$$

Proposition 3.5.0.1 (Generalizes Proposition 3.1 from Liu et al. [2017])**.** Let U_t be a double coset operator corresponding to the double coset $\operatorname{Iw}_p t \operatorname{Iw}_p$. Consider the map $\mathfrak{U}_t$ defined by the following diagram:

$$\begin{array}{ccc}
\mathcal{S}(\rho)_G^K(R) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))_i} & \bigoplus_{i=0}^{n-1} \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R) \\
\downarrow U_t & & \downarrow \mathfrak{U}_t \\
\mathcal{S}(\rho)_G^K(R) & \xrightarrow{\varphi \mapsto (\varphi(\gamma_i))_i} & \bigoplus_{i=0}^{n-1} \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R)
\end{array}$$

Then for $F \in \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R)$, each component of $\mathfrak{U}_t(F)$ lies in the space $(\mathrm{Iw}_p\, t\, \mathrm{Iw}_p)_* \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R)$ (that is, the space generated by $\delta_* G$ for $G \in \mathcal{C}(\mathbb{Z}_p^{\Phi-}, R)$ and $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$).

Proof. The proof is identical to one given in the paper Liu et al. [2017]. The map $\mathfrak{U}_t$ has the form $(\varphi(\gamma_i))_i \mapsto ((U_t\varphi)(\gamma_i))_i$. For each component, we have

$$(U_t\varphi)(\gamma_i) = \sum_\eta (\eta)_p \varphi(\gamma_i \eta)$$

Here, η are in the coset $\mathrm{Iw}_p\, t$. By the assumption in Section 3.3, $\gamma_i \eta = d\gamma_j u$ for some j, some global d, and some $u \in \mathrm{Iw}_p$; by the definition of $\mathcal{S}(\rho)_G^K$,

$$\eta_p \varphi(\gamma_i \eta) = \eta_p \varphi(\gamma_j u) = u_p^{-1} \eta_p \varphi(\gamma_j) = (\eta u^{-1})_* \, \varphi(\gamma_j).$$

□

CHAPTER 4

p-ADIC ANALYSIS.

In this and further sections we consider automorphic forms with values in the ring

$$R = \Lambda = \mathbb{Z}[\![(\mathbb{Z}_p^\times)^\Pi]\!] = \mathbb{Z}\left[\!\!\left[\prod_{\alpha\in\Pi}(\mathbb{Z}_p^\times)_\alpha\right]\!\!\right] = \mathbb{Z}_p[\![(T_\alpha)_{\alpha\in\Pi}]\!] \times \mathbb{Z}_p[\text{ finite set }]$$

and the character is $\rho = \rho_{univ}$. In Proposition 3.5.0.1, we defined the map $\mathfrak{U}_t$

$$\bigoplus_{i=0}^{n-1} \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda) \xrightarrow{\mathfrak{U}_t} \bigoplus_{i=0}^{n-1} \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda)$$

By abuse of notation we will simply write $\mathfrak{U}_t$ for the map $(\mathfrak{U}_t)_{i,j}$ from the i-th summand to the j-th summand.

The goal of this section is to find a lower bound for the Newton polygon of $\mathfrak{U}_t$.

4.1 Newton polygons and infinite-dimensional operators

Definition 4.1.0.1. Let $M = (M_{ij})_{i,j=1}^{\infty}$ be a matrix. We define the characteristic power series $\mathrm{Char}(M)(t)$ to be the limit of characteristic polynomials of $n\times n$ submatrices:

$$\mathrm{Char}(M)(t) = \lim_{n\to\infty} \mathrm{Char}(M_{ij})_{i,j=1}^{n}(t).$$

The above limit in general does not necessarily exist; if it does, the characteristic power series and the Newton polygon in general depend on the choice of basis. If M is a compact operator, the characteristic power series does not depend on the basis.

Moreover, in this situation the Newton polygon lies on or above the Hodge polygon. Indeed, if M is a finite-dimensional operator, this is well-known (see Katz [1979]); the statement for a compact operator follows from the above definition. In other words, one can learn how an operator acts on a vector space by choosing an appropriate lattice and analyzing how the operator acts on the lattice.

Remark 4.1.0.2. We will really only use a special case of the Newton-above-Hodge inequality, specifically, if n-th column of the matrix of the operator is divisible by $p^{f(n)}$, and $f(n)$ is an increasing function, then the Newton polygon is bounded below by the polygon whose vertices are

$$\left(n, \sum_{i=0}^{n} f(i)\right).$$

Lemma 4.1.0.3. Let M be a matrix and d be a diagonal matrix. If $\mathrm{Char}(M)(t)$ exist, then

$$\mathrm{Char}(d^{-1}Md)(t) = \mathrm{Char}(M)(t).$$

Proof. This follows from the observation that

$$((d^{-1}Md)_{ij})_{i,j=1}^{n} = ((d^{-1})_{ij})_{i,j=1}^{n} \cdot (M_{ij})_{i,j=1}^{n} \cdot (d_{ij})_{i,j=1}^{n}.$$

□

4.2 Lower bound on the Newton polygon: the argument of Liu-Wan-Xiao

We give an interpretation of the argument in Section 3 of Liu et al. [2017] in terms of computing the Hodge polygon. In their case, G is a compact form of GL_2 and $t = \mathrm{diag}(p, 1)$. We use a different convention for the action.

By Proposition 3.1 in Liu et al. [2017] or Proposition 3.5.0.1 here, $\mathfrak{U}_t$ is a sum of elements of the form $\delta_* G$ for $G \in \mathcal{C}(\mathbb{Z}_p, R)$ and $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$. We consider the ring

$$R = \Lambda[\![pT^{-1}]\!] = \mathbb{Z}_p[\![T, pT^{-1}]\!] \times \mathbb{Z}_p\{\text{ finite set }\}$$

and the universal character

$$\chi(1+pz) = \chi_{univ}(1+pz) = (1+T)^{\log(1+pz)/p}.$$

Consider the restriction of the operator δ_* to the order L_T generated by elements $T^n\binom{z}{n}$. Up until the very last step one can think of it as the lattice generated by $p^n\binom{z}{n}$. This is rationally a change of basis that does not change the Newton polygon (see Lemma 4.1.0.3). Moreover, the operator δ_* acts compactly on L_p, as we are about to see. This lattice is mentioned in Section 5.4 of Liu et al. [2017]; we want to translate the entire computation into this language.

The notion of tilted degree introduced in Liu et al. [2017](Definition 3.8) translates easily into the norm in this lattice. Namely, $f(z)$ has tilted degree $\leq n$ (by definition in Liu et al. [2017]) if and only if $f(z) = \sum b_i\binom{z}{i}$ where $v_p(b_i) \geq i - n$, i.e. if and only if $\|f\|_{L_p} \leq p^n$.

For convenience, we rewrite two lemmas from Liu et al. [2017] in this language.

Lemma 4.2.0.1 (Liu et al. [2017], Lemmas 3.12, 3.13)**.**

1. Let $f(z) = \sum_{i\geq 0} a_i p^i z^i$ where $a_i \in \mathbb{Z}_p$. Then
$$p^n\binom{f(z)}{n} \in p^{n-\lfloor n/p \rfloor} L_p.$$
2. Let $f(z) = a_0 + \sum_{i\geq 0} a_i p^{i-1} z^i/i$ where $a_i \in \mathbb{Z}_p$. Then the map $z \mapsto f(z)$ that acts by $\binom{z}{n} \mapsto \binom{f(z)}{n}$ preserves the lattice L_p.

It is not hard to see, assuming $p \in (T)$, that Lemma 4.2.0.1 holds if L_p is replaced by L_T.

We can study the image δ_* of the lattice L_T. We can assume that $\delta = u_{ut} \cdot t \cdot u_{lt} \cdot d$ where $d \in T(\mathbb{Z}_p) \subset \mathrm{Iw}_p$, u_{lt} is lower-triangular and u_{ut} is upper-triangular, u_{lt}, $u_{ut} \in \mathrm{Iw}_p$. We can study the action of each of the elements individually.

Step 1. Let $u_{ut} = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$

$$(u_{ut\,*})p^n\binom{z}{n} = p^n\binom{\frac{z}{1+paz}}{n}\cdot\chi(1+paz) = T^n\binom{\frac{z}{1+paz}}{n}\cdot(1+T)^{\log(1+paz)/p}$$

By part 2 of Lemma 4.2.0.1

$$\begin{aligned}
\binom{\frac{z}{1+paz}}{n} &= \sum_{i=0}^{n} b_{in}\binom{z}{i} \qquad \text{where } v_p(b_{in}) \geq i-n;\\
p^n\binom{\frac{z}{1+paz}}{n} &\in L_p;\\
\binom{\frac{\log(1+paz)}{p}}{n} &= \sum_{i=0}^{n} c_{in}\binom{z}{i} \qquad \text{where } v_p(b_{in}) \geq i-n;\\
T^n\binom{\frac{\log(1+paz)}{p}}{n} &\in L_T.
\end{aligned}$$

Step 2. $t_*z = pz$. By part 1 of Lemma 4.2.0.1,

$$\begin{aligned}
t_*\binom{z}{n} = \binom{pz}{n} &= \sum_{i=0}^{n} b_{in}\binom{z}{i} \qquad \text{where } v_p(b_{in}) \geq i - \left\lfloor\frac{n}{p}\right\rfloor;\\
p^n\binom{pz}{n} &= \sum_{i=0}^{n} \widetilde{b_{in}}\cdot p^i\binom{z}{i} \qquad \text{where } v_p(\widetilde{b_{in}}) \geq n - \left\lfloor\frac{n}{p}\right\rfloor;\\
T^n\binom{pz}{n} &= \sum_{i=0}^{n} \widehat{b_{in}}\cdot T^i\binom{z}{i} \qquad \text{where } v_T(\widehat{b_{in}}) \geq n - \left\lfloor\frac{n}{p}\right\rfloor.
\end{aligned}$$

It follows that t_*L_T is the sublattice generated by

$$T^{n-\left\lfloor\frac{n}{p}\right\rfloor}\cdot p^n\binom{z}{n},$$

n-th column of the matrix of t_* is divisible by $T^{n-\left\lfloor\frac{n}{p}\right\rfloor}$.

Step 3. Finally, consider d_* and $u_{lt\,*}$. If $d = \mathrm{diag}(a,b)$ for some $a, b \in \mathbb{Z}_p^\times$, then $d_*z = az\cdot\chi(b)$;

$u_{lt\,*}z = z + c$ for some $c \in \mathbb{Z}_p$. Since $\binom{az+c}{n} \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$, we have

$$
\begin{aligned}
u_{lt\,*}d_* \binom{z}{n} = \chi(b)\binom{az+c}{n} &= \chi(b) \cdot \sum_{i=0}^{n} b_{in}\binom{z}{i} \quad \text{where } b_{in} \in \mathbb{Z}_p; \\
p^n \binom{az+c}{n} &= \sum_{i=0}^{n} (p^{n-i} b_{in}) p^i \binom{z}{i}; \\
T^n \binom{az+c}{n} &= \sum_{i=0}^{n} (T^{n-i} b_{in}) T^i \binom{z}{i}.
\end{aligned}
$$

Hence, $(u_{lt\,*}d_*)L_p = L_p$ and $(u_{lt\,*}d_*)L_T = L_T$ (this is also a case of Remark 3.9 and Lemma 3.13 in Liu et al. [2017]). Moreover the matrix of $(u_{lt\,*}d_*)$ on L_T is upper-triangular with coefficients $\widetilde{b_{in}}$ such that $v_p(\widetilde{b_{in}}) \geq n - i$.

Let us call $(\delta_*)_{ij}$ the coefficient of the corresponding matrix in the chosen basis in L_p. Combining all steps together we get

$$\delta_* = u_{ut\,*} \cdot t_* \cdot u_{lt\,*} \cdot d_*$$

is the product of matrices where

$$
\begin{aligned}
v_T((u_{lt\,*} \cdot d_*)_{ij}) &\geq j - i; \\
v_T((t_*)_{ij}) &\geq j - \left\lfloor \frac{j}{p} \right\rfloor; \\
v_T((t_* \cdot u_{lt\,*} \cdot d_*)_{ij}) &\geq \min_{k \leq j} \left((j-k) + k - \left\lfloor \frac{k}{p} \right\rfloor \right) = j - \left\lfloor \frac{j}{p} \right\rfloor; \\
v_T((u_{ut\,*} \cdot t_* \cdot u_{lt\,*} \cdot d_*)_{ij}) &\geq j - \left\lfloor \frac{j}{p} \right\rfloor.
\end{aligned}
$$

It follows that the Hodge polygon of δ_* on L_T is bounded below by the polygon with slopes $n - \left\lfloor \frac{n}{p} \right\rfloor$. The Hodge polygon of the restriction of the original operator $\mathfrak{U}_t$ to L_T, and therefore the Newton polygon has the same bound, as desired.

Remark 4.2.0.2. We have shown that i-th column of the matrix δ_* on the lattice L_T is divisible by $p^{i-\lfloor j/p \rfloor}$. In other words, the matrix of δ_* in the basis $T^n\binom{z}{n}$ has a decomposition

$$\delta_* = Q \cdot \mathrm{diag}(i - \lfloor j/p \rfloor)_{j \geq 0},$$

where $Q = Q(\delta, t)$ preserves the lattice L_T. Then, by Proposition 3.5.0.1,

$$\mathfrak{U}_t = \sum \delta_* = \left(\sum Q(\delta, t)\right) \mathrm{diag}(i - \lfloor j/p \rfloor)_{j \geq 0}.$$

As a result, the Hodge polygon of $\mathfrak{U}_t$ is bounded below by that of the matrix $\mathrm{diag}(i - \lfloor j/p \rfloor)_{j \geq 0}$. This will be the intuition behind the argument of Corollary 4.3.7.7, outlined in Section 4.3.2.

Remark 4.2.0.3. If we remove the assumption that $p/T \in R$, we can show that n, k,

$$\delta_* T^n p^k \binom{z}{n+k} \in (T, p)^{n+k-\left\lfloor \frac{n+k}{p} \right\rfloor} L_{(T,p)},$$

where $L_{(T,p)}$ is generated by elements of the form $p^i T^j \binom{z}{i+j}$. Assuming that $p \in (T)$ would yield the bound

$$\delta_* T^n \binom{z}{n} \in T^{n-\left\lfloor \frac{n}{p} \right\rfloor} L_T.$$

Remark 4.2.0.4. The lower bound is precise infinitely many times (Liu et al. [2017], 3.23, Step 1). That is, the Newton polygon of U_t is approximated by the parabola

$$y = \frac{1}{2}\left(1 - \frac{1}{p}\right) x^2 \cdot |T|.$$

4.3 Multi-variable p-adic analysis

The goal of this section is to extend the above lower bound to higher rank groups.

4.3.1 Mahler basis

If R is a $\mathbb{Z}_p$-algebra, then the space of functions $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ is generated by the Mahler basis. If $\mathbf{z} = (z_\alpha)_{\alpha\in\Phi_-}$ is a tuple of variables and $\mathbf{n} = (n_\alpha)_{\alpha\in\Phi_-}$ are integers, we use the following notation for the corresponding monomials

$$\mathbf{z}^{\mathbf{n}} = \prod_{\alpha\in\Phi_-} z_\alpha^{n_\alpha},$$
$$\mathbf{z}^{\mathbf{n}\downarrow} = \binom{\mathbf{z}}{\mathbf{n}} = \prod_{\alpha\in\Phi_-} \binom{z_\alpha}{n_\alpha}.$$

Then for every continuous function $f \in \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$, there exists a series

$$f(\mathbf{z}) = \sum_{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^{\Phi_-}} c_{\mathbf{n}}\, \mathbf{z}^{\mathbf{n}\downarrow}.$$

If $m_\alpha < n_\alpha$ for all α , we say that $\mathbf{m} < \mathbf{n}$. We also use the notation $\sum \mathbf{n} = \sum_{\alpha\in\Phi_-} n_\alpha$; we denote $p^{\sum_\alpha n_\alpha}$ by $p^{\mathbf{n}}$ and $p^{\mathbf{n}}/p^{\mathbf{m}}$ by $p^{\mathbf{n}-\mathbf{m}}$.

4.3.2 Outline of the computation in general

We adapt the argument from Section 4.2 for a general reductive group. By Proposition 3.5.0.1, the operator $\mathfrak{U}_t$ on the space $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda)$ has a decomposition as a sum of operators δ_* where $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$.

We would like to study the action on the lattice $L_{(T_\gamma)_{\alpha\in\Pi}+(p)}$ generated by elements of the form

$$\prod_{\gamma\in\Pi} T_\gamma^{n_\gamma} \cdot p^{n_p} \mathbf{z}^{\mathbf{m}\downarrow},$$

where

$$\sum \mathbf{m} = \sum_{\gamma\in\Pi} n_\gamma + n_p.$$

The idea behind the proof is to show that δ_* has the form QD, where $Q = Q(t, \delta)$ and

$D = D(t)$ both are endomorphisms of the lattice $L_{(T_\gamma)_{\alpha\in\Pi}+(p)}$ and $D = D(t)$ does not depend on δ. Then

$$\mathfrak{U}_t = \sum \delta_* = \sum Q(t,\delta)D(t) = \left(\sum Q(t,\delta)\right) D(t).$$

Since $\sum Q(t,\delta)$ is an endomorphism of the lattice $L_{(T_\gamma)_{\alpha\in\Pi}+(p)}$, the Hodge polygon of $\mathfrak{U}_t$ is bounded below by the Hodge polygon of $D(t)$.

Like before, we consider $\delta = d \cdot u_{ut} \cdot t \cdot u_{lt}$ where $d \in T$, u_{ut} is upper-triangular and u_{lt} is lower-triangular. We denote the ideal $(T_\gamma)_{\alpha\in\Pi} + (p)$ by I. We study the action of each factor in isolation. We want to prove the following statements:

$$u_*(L_I) \subset L_I \quad \text{for } u \in \mathrm{Iw}_p; \qquad \text{See Section 4.3.5} \tag{4.3.1}$$

$$t_*\left(I^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}\right) \subset I^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)),n_\gamma)} L_I, \tag{4.3.2}$$

where $\mathit{slope}(m,n)$ is defined in Section 4.3.6. Then we study how $u_{lt\,*}$ affects the estimate in (4.3.2).

For instance, in case $G = \mathrm{GSp}_4$, in Section 5.2, we show that, for each $\mathbf{n}$,

$$\delta_*\left((T_\alpha, T_\beta, p)^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}\right) \subset (T_\alpha, T_\beta, p)^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)),n_\gamma)} L_{(T_\alpha,T_\beta,p)},$$

and therefore,

$$\mathfrak{U}_t\left((T_\alpha, T_\beta, p)^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}\right) \subset (T_\alpha, T_\beta, p)^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)),n_\gamma)} L_{(T_\alpha,T_\beta,p)}. \tag{4.3.3}$$

From this relation we can deduce a lower bound on the Hodge polygon. For instance, if we assume that $T_\alpha|T_\beta|p$, and consider the matrix of $\mathfrak{U}_t$ in the basis $T_\alpha^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}$, then the column of this matrix that corresponds to $T_\alpha^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}$ is divisible by $T_\alpha^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)),n_\gamma)}$. Hence, the corresponding slope of the Hodge polygon is at least $\sum_\gamma \mathit{slope}(v_p(\gamma(t)), n_\gamma)$.

We will refer to statements like Equation (4.3.3) as *lower bounds*.

4.3.3 General computations in Lie algebras.

This section contains technical details that will allow us to analyze the coordinates introduced in Section 2.2. The computations are largely inspired by Schneider and Teitelbaum [2000].

In what follows, we assume that $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{Z}_p$, $\mathfrak{g}$ does not contain the exceptional algebra $\mathfrak{g}_2$ as a subalgebra. Let $G = \exp(\mathfrak{g})$.

Computation 4.3.3.1 (Baker–Campbell–Hausdorff formula in a very nilpotent case)**.** *Suppose $\mathfrak{g}$ is a reductive Lie algebra.*

1. *Suppose $\mathfrak{g} \not\supset \mathfrak{g}_2$ and $\alpha \neq -\beta$. Then*

$$[x_\alpha(\nu), x_\beta(\mu)] = x_{\alpha+\beta}(c_1 \cdot \nu\mu) x_{2\alpha+\beta}(c_2 \cdot \nu^2\mu) x_{\alpha+2\beta}(c_3 \cdot \nu\mu^2)$$

   where $c_i \in 1/2 \cdot \mathbb{Z}_p$ and $x_\gamma(s) = 1$ if γ is not a root.

2.
$$\begin{aligned} x_\alpha(\nu)x_{-\alpha}(\mu) &= h_\alpha(1+\nu\mu)\, x_{-\alpha}(\mu \cdot (1+\nu\mu))\, x_\alpha(\nu/(1+\nu\mu)) \\ &= x_{-\alpha}(\mu/(1+\nu\mu))\, h_\alpha(1+\nu\mu)\, x_\alpha(\nu/(1+\nu\mu)) \end{aligned}$$

Proof. (1) It suffices to check that the logarithms of both sides are the same. Let $\mathfrak{n}$ be the nilpotent algebra generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$. Note that $\mathfrak{n}$ is a nilpotent subalgebra with a rank 2 root system and if $\mathfrak{n} \not\subset \mathfrak{g}_2$ then $[\mathfrak{n}, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]] = 0$. For $X, Y \in \mathfrak{n}$, we can show that

$$\exp(X)\exp(Y)\exp(-X)\exp(-Y) = \exp([X,Y])\exp(1/2\,[X,[X,Y]])\exp(1/2\,[Y,[X,Y]]).$$

If we set $x_\alpha(\nu) = X$, $x_\beta(\mu) = Y$, then

$$\begin{aligned}\exp([X,Y]) =& x_{\alpha+\beta}(c_1 \cdot \nu\mu),\\ \exp([X,[X,Y]]) =& x_{2\alpha+\beta}(c_2 \cdot \nu^2\mu),\\ \exp([Y,[X,Y]]) =& x_{\alpha+2\beta}(c_3 \cdot \nu\mu^2),\end{aligned}$$

where c_i are some integers, and we get the desired formula. First, observe that $[X,Y]$, $[X,[X,Y]]$ and $[Y,[X,Y]]$ commute, and

$$\begin{aligned}&\exp([X,Y])\exp(1/2\,[X,[X,Y]])\exp(1/2\,[Y,[X,Y]]) =\\ &\exp([X,Y] + 1/2\,[X,[X,Y]] + 1/2\,[Y,[X,Y]]).\end{aligned}$$

The left-hand side can be computed by Baker–Campbell–Hausdorff formula:

$$\begin{aligned}&\exp(X)\exp(Y)\exp(-X)\exp(-Y) =\\ &\exp(X + Y + 1/2\,[X,Y] + 1/12[X,[X,Y]] + 1/12[Y,[Y,X]])\\ &\exp(-X - Y + 1/2\,[X,Y] - 1/12[X,[X,Y]] - 1/12[Y,[Y,X]]) =\\ &\exp([X,Y] + 1/2[X + Y + 1/2\,[X,Y], -X - Y + 1/2\,[X,Y]]) =\\ &\exp([X,Y] + 1/2[X + Y,\ [X,Y]]).\end{aligned}$$

(2) can be checked in SL_2. □

Remark 4.3.3.2. The case of the exceptional algebra $\mathfrak{g}_2$ would require a separate computation and, possibly, exclusion of some more primes. However, an analogous result should also be true for $\mathfrak{g}_2$.

Computation 4.3.3.3. *Let $p \neq 2$, and let x be an element of* $\exp(\mathfrak{n}_-)$ *and*

$$x = \prod_{j\in J} x_{\alpha_j}(\nu_j)$$

where $\alpha_j < 0$ may repeat. Then there exists a decomposition

$$x = \prod_{\alpha \in \Phi_-} x_\alpha(z_\alpha),$$

where Φ_- is ordered in some fixed way. Moreover

$$z_\alpha = \sum_{\alpha_i = \alpha} \nu_i + r$$

where r is a sum of monomials of the form $c \cdot \prod_{k \in K} \nu_k$ where $c \in \mathbb{Z}_p$ and $\sum_{k \in K} \alpha_k = \alpha$. To simplify the notation we will refer to this property as $P(\alpha)$.

Proof. Consider the lower central series and the root space decomposition on $\mathfrak{n}$

$$\mathfrak{n}_- = \mathfrak{n}_0 \supset [\mathfrak{n}_0, \mathfrak{n}_0] = \mathfrak{n}_1 \supset [\mathfrak{n}_0, [\mathfrak{n}_0, \mathfrak{n}_0]] = \mathfrak{n}_2 \supset \dots$$
$$\mathfrak{n}_- = \mathfrak{n} = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha.$$

Note that they are compatible, meaning that a subset of $\mathfrak{g}_\alpha$'s generates $\mathfrak{n}_i$ for every i. To see this, note that the center $Z(\mathfrak{n})$ is preserved by the Cartan algebra. Then we can consider the action on the Cartan algebra on $\mathfrak{n}/Z(\mathfrak{n})$. This is also a nilpotent algebra, we can proceed by induction.

Consider the stratification

$$\Phi_- = \Phi_{-,0} \supset \Phi_{-,1} \supset \Phi_{-,2} \supset \dots$$

on the ordered set Φ_-, such that $\Phi_{-,i}$ is an ordered set that contains the set of α's that generates $\mathfrak{n}_i$.

We will argue by induction in i, The induction step is as follows: that there exists a decomposition

$$x = \prod_{\alpha \in \Phi_- \setminus \Phi_{-,i}} \left(x_\alpha(z_\alpha) \cdot \text{remainder}_{\alpha,i} \right),$$

where z_α is of the desired form and $\text{remainder}_{\alpha,i} \in \exp(\mathfrak{n}_{i+1})$ is a product of factors of the form $x_\beta(w)$ where $\beta \in \Phi_{-,i+1}$ and either w has the properties $P(\beta)$ or $w = \nu_i$ and $\alpha_i = \beta$.

To do the induction step, we need to find all x_α for $\alpha \in \Phi_{-,i+1} \setminus \Phi_{-,i+2}$ and put them in their places. This is done by commuting $x_\alpha(w)$ with other elements. The commutators will be elements of $\mathfrak{n}_{i+2}$ and will satisfy the induction hypothesis by Computation 4.3.3.1. □

4.3.4 The lattice L_I.

In the following sections we introduce the lattice L_I and show that the group Iw_p acts on it.

Definition 4.3.4.1. If $I \subset R$ is an ideal, we denote by L_I the lattice generated by elements of $I^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow}$.

We will mainly be concerned with the case $I = (T_\alpha)_{\alpha \in \Pi} + (p)$.

Lemma 4.3.4.2. The lattice L_I is a ring.

Remark 4.3.4.3. In particular, if I is a prime ideal, then there is a well-defined I-adic valuation with respect to L_I. That is, if $a \in I^n L_I$ and $b \in I^m L_I$, then $ab \in I^{n+m} L_I$.

Proof. It suffices to check that if $s \in I^{\mathbf{n}}$, $t \in I^{\mathbf{m}}$ then $s\mathbf{z}^{\mathbf{n}\downarrow} \cdot t\mathbf{z}^{\mathbf{m}\downarrow} \in L_I$. Observe that $\mathbf{z}^{\mathbf{n}\downarrow} \cdot \mathbf{z}^{\mathbf{m}\downarrow}$ is a polynomial of the total degree $\sum \mathbf{n} + \sum \mathbf{m}$. It follows that

$$s\mathbf{z}^{\mathbf{n}\downarrow} \cdot t\mathbf{z}^{\mathbf{m}\downarrow} \in I^{\mathbf{n}+\mathbf{m}} \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)^{\deg \, \leq \, \sum \mathbf{n} + \sum \mathbf{m}}.$$

The R-module $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)^{\deg \, \leq \, \sum \mathbf{n} + \sum \mathbf{m}}$ is generated by Mahler monomials $\mathbf{z}^{\mathbf{j}\downarrow}$ where $\sum \mathbf{j} \leq \sum \mathbf{n}$. Hence,

$$s\mathbf{z}^{\mathbf{n}\downarrow} \cdot t\mathbf{z}^{\mathbf{m}\downarrow} \;\in\; I^{\mathbf{n}+\mathbf{m}} \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)^{\deg \, \leq \, \sum \mathbf{n} + \sum \mathbf{m}} \;\subset\; L_I \cap \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)^{\deg \, \leq \, \sum \mathbf{n} + \sum \mathbf{m}}.$$

$\square$

We denote by $\mathfrak{m} \subset \mathbb{Z}_p[(pz_\alpha)_{\alpha\in\Phi_-}]$ the ideal generated by the elements pz_α and by $\widehat{\mathfrak{m}}$ the ideal generated by the elements pz_α in $\mathbb{Z}_p[\![(pz_\alpha)_{\alpha\in\Phi_-}]\!]$.

If $f(\mathbf{z})\colon \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R) \to \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ is a ring map, then $f(\mathbf{z})$ preserves L_I if and only if $s\binom{(f(\mathbf{z}))_\alpha}{n} \in L_I$ for any $\alpha \in \Phi_-$ and any $s \in I^n$.

Lemma 4.3.4.4. Suppose f, $g\colon \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R) \to \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ are continuous operators that act as identity on all functions that do not depend on z_α, and suppose both maps preserve L_I.

1. The the $h(\mathbf{z})$ acting by

$$z_\varepsilon \mapsto \begin{cases} (f(\mathbf{z}))_\alpha + (g(\mathbf{z}))_\alpha & \text{if } \varepsilon = \alpha; \\ z_\varepsilon & \text{otherwise.} \end{cases}$$

   preserves L_I.

2. For any $c \in \mathbb{Z}_p$, the map

$$z_\varepsilon \mapsto \begin{cases} c(f(\mathbf{z}))_\alpha & \text{if } \varepsilon = \alpha; \\ z_\varepsilon & \text{otherwise.} \end{cases}$$

   preserves L_I.

3. For any $\mathbf{m} \in \mathbb{Z}_{\geq 0}^{\Phi_-}$ and any $t \in I^{\sum \mathbf{m} - 1}$, the map

$$z_\varepsilon \mapsto \begin{cases} t\mathbf{z}^{\mathbf{m}} & \text{if } \varepsilon = \alpha; \\ z_\varepsilon & \text{otherwise.} \end{cases}$$

   preserves L_I.

Proof. For (1), it suffices to check that

$$s\binom{(f(\mathbf{z}))_\alpha + (g(\mathbf{z}))_\alpha}{n} \in L_I$$

where $s = s_1 s_2 \dots s_n$, $s_j \in I$. Observe that

$$s\binom{(f(\mathbf{z}))_\alpha + (g(\mathbf{z}))_\alpha}{n} = \sum_i s_1 \dots s_i \binom{(f(\mathbf{z}))_\alpha}{i} s_{i+1} \dots s_n \binom{(g(\mathbf{z}))_\alpha}{n-i},$$

the desired now follows.

For $c \in \mathbb{N}$, (2) follows from (1). By continuity, (2) is true for all $c \in \mathbb{Z}_p$.

To check (3), we need to show that

$$s\binom{t\ \mathbf{z}^{\mathbf{m}}}{n} \in L_I$$

where $s = s_1 s_2 \dots s_n$, $s_j \in I$. Let $\Delta_\gamma \colon f(z_\gamma) \mapsto f(z_\gamma + 1) - f(z_\gamma)$ be the discrete differentiation with respect to the variable z_γ. Then $f(\mathbf{z}) \in L_I$ if and only if $\Delta_\gamma f(\mathbf{z}) \in IL_I$ for all γ, as can be seen by considering separately each monomial.

We argue by induction in the total degree $\sum \mathbf{m}$ and in n. For the base of induction, we remark that the statement is true if $\sum \mathbf{m} = 1$ and $n \leq 1$. To verify the step of induction, we use the observation about the derivatives:

$$s\Delta_\gamma \binom{t\ \mathbf{z}^{\mathbf{m}}}{n} = s\left(\binom{(z_\gamma + 1)^{m_\gamma} \cdot t\ \frac{\mathbf{z}^{\mathbf{m}}}{z_\gamma^{m_\gamma}}}{n} - \binom{t\ \mathbf{z}^{\mathbf{m}}}{n}\right) =$$

$$= s_1 \sum_{i \geq 1} s_2 \dots s_i \binom{((z_\gamma + 1)^{m_\gamma} - z_\gamma^{m_\gamma}) \cdot t\ \frac{\mathbf{z}^{\mathbf{m}}}{z_\gamma^{m_\gamma}}}{i} \cdot s_{i+1} \dots s_n \binom{t\ \mathbf{z}^{\mathbf{m}}}{n-i}$$

The first factor is a polynomial of lower total degree, and for the second factor, we use the induction in n. □

4.3.5 Iwahori acts on $L_{((T_\alpha)_{\alpha \in \Pi}, p)}$

Computation 4.3.5.1. *Let $I \subset R$ be an ideal such that $p \in I$, and $L_I \subset \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ the corresponding lattice.*

1. *If $d \in T \subset G(\mathbb{Z}_p)$, then $d_* L_I \subset L_I$.*

2. *Consider $\gamma \in \Phi_-$ and $\nu \in \mathbb{Z}_p$. Then $x_\gamma(p\nu)_* L_I \subset L_I$.*

3. *Suppose $f(\mathbf{z}) \in \widehat{\mathfrak{m}} \subset \mathbb{Z}_p[\![(pz_\alpha)_{\alpha\in\Phi_-}]\!]$.*

$$(1+T)^{\log(1+f(\mathbf{z}))/p} \in L_{(T,p)}.$$

4. *Consider $\gamma \in \Pi$, $\nu \in \mathbb{Z}_p$, and let $I = (T_\alpha)_{\alpha\in\Pi} + (p)$. Then $x_\gamma(\nu)_* L_I \subset L_I$.*

Proof. It suffices to check (1) and (2) on the monomials of the form $s\binom{z_\varepsilon}{n_\varepsilon}$ where $s \in I^{n_\alpha}$.

$$d_*\ s\ \binom{z_\varepsilon}{n_\varepsilon} = \chi(d)\cdot\ s\ \binom{\varepsilon^{-1}(d)\cdot z_\varepsilon}{n_\varepsilon} = b\cdot\ s\ \binom{cz_\varepsilon}{n_\varepsilon}.$$

for some constants $c \in \mathbb{Z}_p$, $b \in R$. The result follows from Lemma 4.3.4.4.

To see how $x_\gamma(p\nu)_*$ acts on $\mathbf{z}$, we need to compute

$$\prod_{\varepsilon\in\Phi_-} x_\varepsilon(pz_\varepsilon)\ \cdot\ x_\gamma(p\nu) = \prod_{\varepsilon\in\Phi_-} x_\varepsilon(pz_\varepsilon^{new}).$$

From Computation 4.3.3.3, it follows that for any root ε,

$$z_\varepsilon^{new} \in z_\varepsilon + \delta_{\varepsilon,\gamma}\nu + \mathfrak{m},$$

where $\mathfrak{m} \subset \mathbb{Z}_p[(p\cdot z_\varepsilon)_{\varepsilon\in\Phi_-}]$ be the ideal generated by the elements $p\cdot z_\varepsilon$.

It suffices to check that the maps $z_\varepsilon \mapsto z_\varepsilon + b$ and $z_\varepsilon \mapsto z_\varepsilon + cp^{\mathbf{m}}\mathbf{z}^{\mathbf{m}}$ preserve L_I, and the result follows from Lemma 4.3.4.4.

(3) We want to show that

$$T^n\binom{\log(1+f(\mathbf{z}))/p}{n} \in L_{(T,p)}.$$

It suffices to consider each summand in the series $\log(1+f(z))$. We want to show that

$$T^n \binom{f(\mathbf{z})^m/pm}{n} \in L_{(T,p)}.$$

The map

$$z_\varepsilon \mapsto \begin{cases} f(\mathbf{z})^m/pm & \text{if } \varepsilon = \alpha; \\ z_\varepsilon & \text{otherwise} \end{cases}$$

is the composition of maps $z_\alpha \mapsto f(\mathbf{z})$ and $z_\alpha \mapsto z_\alpha^m/pm$. It follows from Lemma 4.3.4.4 and Lemma 4.2.0.1, that both maps preserve the lattice $L_{(T,p)}$.

(4) To see how $x_\gamma(\nu)_*$ acts on $\mathbf{z}$, we need to compute

$$\prod_{\alpha \in \Phi_-} x_\alpha(pz_\alpha) \; \cdot \; x_\gamma(\nu).$$

By Computation 4.3.3.3, to simplify the notation, we may assume that $-\gamma$ is the first element of the product. Then

$$\prod_{\alpha \in \Phi_-} x_\alpha(pz_\alpha) \cdot x_\gamma(\nu) =$$

$$x_\gamma\left(\frac{\nu}{1+\nu p z_{-\gamma}}\right) h_\gamma(1+\nu p z_{-\gamma})^{-1} \, x_{-\gamma}\left(\frac{pz_{-\gamma}}{1+\nu p z_{-\gamma}}\right) \cdot \prod_{\alpha \in \Phi_- \setminus \{-\gamma\}} x_\alpha(pz_\alpha) \cdot [x_\alpha(pz_\alpha), x_\gamma(\nu)]$$

If γ is simple, then for any $\alpha \in \Phi_-$, if $\alpha + \gamma \neq 0$, then $\alpha + \gamma \in \Phi_-$, and then $\alpha + 2\gamma$ and $2\alpha + \gamma$ cannot be positive by the same argument.

Each of the commutators $(x_\alpha(pz_\alpha), x_\gamma(\nu))$ can be found by Computation 4.3.3.1 and they are in N_-. It follows that

$$x_\gamma(\nu)_* f(\mathbf{z}) = \chi_{univ}(h_\gamma(1+\nu p z_{-\gamma})^{-1}) f(\mathbf{z}^{new}),$$

where

$$pz_\alpha^{new} \in \mathfrak{m} + \left(\frac{\mathfrak{m}}{1 + p\nu z_{-\gamma}}\right) \subset \widehat{\mathfrak{m}} \subset \mathbb{Z}_p[\![(pz_\beta)_{\beta\in\Phi_-}]\!]$$

for all α by Computation 4.3.3.3. The result follows from Lemma 4.3.4.4 and the previous step. □

Corollary 4.3.5.2. Suppose $G = \mathrm{GSp}_4(\mathbb{Z}_p)$; then $\mathrm{Iw}_p \subset G$ acts on the lattice L_I where $I = (T_\alpha)_{\alpha\in\Pi} + (p) = (T_\alpha, T_\beta, p)$.

Proof. Let α and β be the simple roots like in Example 2.1.0.1 and $p \neq 2$. Then we can check that the group $N_+(\mathbb{Z}_p)$ is generated by elements of the form $x_\alpha(\nu)$ and $x_\beta(\nu)$ for $\nu \in \mathbb{Z}_p$. It is enough to check that $x_{\alpha+\beta}(\nu)$ and $x_{2\alpha+\beta}(\nu)$ can be expressed in terms of $x_\alpha(\cdot)$ and $x_\beta(\cdot)$. One checks by direct computation that

$$[x_\alpha(\nu_1), [x_\alpha(\nu_2), x_\beta(\nu_3)]] = x_{2\alpha+\beta}(2\nu_1\nu_2\nu_3);$$
$$[x_\alpha(\nu/2), [x_\alpha(1), x_\beta(1)]] = x_{2\alpha+\beta}(\nu);$$
$$[x_\alpha(1), x_\beta(\nu)]x_{2\alpha+\beta}(-\nu) = x_{\alpha+\beta}(\nu).$$

□

Remark 4.3.5.3. It follows from Computation 4.3.5.1, that the group generated by elements of the form $x_\alpha(\nu)$ for $\alpha \in \Pi$ and $\nu \in \mathbb{Z}_p$ and $x_\alpha(p\nu)$ for $\alpha \in \Phi_-$ and $\nu \in \mathbb{Z}_p$ acts on L_I. Hence, the group Iw_p acts on L_I, if elements of the form $x_\alpha(\nu)$ for $\alpha \in \Pi$ generate $N_+(\mathbb{Z}_p)$. For a fixed G, this is so for all but finitely many primes p.

Indeed, the Lie algebra $\mathfrak{n}_+(\mathbb{Q})$ is generated as a $\mathbb{Q}$-algebra by the elements X_α for $\alpha \in \Pi$. Hence, the elements $\exp(X_\alpha) = x_\alpha(1)$ for $\alpha \in \Pi$ generate a finite index subgroup in $N_+(\mathbb{Z})$, so they generate $N_+(\mathbb{Z}_p)$ for all but finitely many p.

Remark 4.3.5.4. It is also not hard to see that Iw_p act on L_I for $G = \mathrm{GL}_n$ for all n: we can adapt the argument of Computation 4.3.5.1(4) in that case. If $\gamma \in \Phi_+$ is not necessarily simple, then

$$\prod_{\alpha\in\Phi_-} x_\alpha(pz_\alpha) \;\cdot\; x_\gamma(\nu) =$$

$$x_\gamma\left(\frac{\nu}{1+\nu p z_{-\gamma}}\right) h_\gamma(1+\nu p z_{-\gamma})^{-1}\, x_{-\gamma}\left(\frac{pz_{-\gamma}}{1+\nu p z_{-\gamma}}\right)\cdot$$

$$\cdot \prod_{\alpha\in\Phi_-\setminus\{-\gamma\}} x_\alpha(pz_\alpha)\cdot[x_\alpha(pz_\alpha), x_\gamma(\nu)].$$

All commutators $[x_\alpha(\mu), x_\gamma(\nu)]$ are equal to $x_{\alpha+\gamma}(\mu\nu)$. If the root $\alpha+\gamma$ is positive for $\alpha < 0$ and $\beta > 0$, then it is less deep in the lower central series of $\mathfrak{n}_+$ than γ. We can proceed by induction.

4.3.6 The torus action.

Recall that $\mathfrak{U}_t$ is a sum of operators of the form $(\mathrm{Iw}_p\, t\, \mathrm{Iw}_p)_*$, and that Iw_p preserves the lattice L_I. In this section we study the action of t_*.

Computation 4.3.6.1.

$$p^n\binom{p^m z_\alpha}{n} \in p^{slope^{exact}(m,n)} L_p$$

where

$$slope^{exact}(0,n) = 0$$

$$slope^{exact}(1,n) = n - \left\lfloor \frac{n}{p} \right\rfloor$$

$$slope^{exact}(m+1,n) = slope^{exact}(m,n) + \left\lceil \frac{n}{p^m}\right\rceil - \left\lfloor \frac{\left\lceil \frac{n}{p^m}\right\rceil}{p}\right\rfloor.$$

Proof. For $m = 1$, this is the statement of Lemma 4.2.0.1. We proceed by induction in m. Observe that $slope^{exact}(m,n)$ is non-strictly monotonic in n for any fixed m. Indeed, this is true for $m = 1$, and

$$slope^{exact}(m+1,n) = slope^{exact}(m,n) + slope^{exact}\left(1, \left\lceil \frac{n}{p^m}\right\rceil\right).$$

By the induction hypothesis, the right hand side is a sum of two monotonic functions in n.

We also observe that if

$$p^n \binom{p^m z_\alpha}{n} = p^{slope^{exact}(m,n)} \left(\sum_i c_i p^i \binom{z}{i} \right),$$

then the left hand side is equal to 0 if $0 \leq p^m z < n$. Hence $c_i = 0$ for $i < n/p^m$. It follows that

$$\begin{aligned} p^n \binom{p^{m+1} z_\alpha}{n} =& p^{slope^{exact}(m,n)} \left(\sum_i c_i p^i \binom{pz}{i} \right) = \\ =& p^{slope^{exact}(m,n)} \left(\sum_i \sum_j d_{i,j} p^{slope^{exact}(1,i)} p^i \cdot \binom{z}{i} \right). \end{aligned}$$

The result follows. □

Corollary 4.3.6.2. If $p \in I$, then

$$I^n \binom{p^m z_\alpha}{n} \subset I^{slope^{exact}(m,n)} L_I;$$
$$I^{\mathbf{n}} \prod_\alpha \binom{p^{m_\alpha} z_\alpha}{n_\alpha} \subset I^{\sum slope^{exact}(m_\alpha, n_\alpha)} L_I.$$

Remark 4.3.6.3. It follows from Computation 4.3.5.1 and the above corollary, that for an element δ_* of the form $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$, its Hodge polygon has slopes

$$\left(\sum_{\Phi_-} slope^{exact}(v_p(\gamma(t)), n_\gamma) \right)$$

for $(n_\gamma)_{\gamma \in \Phi_-} \in \mathbb{Z}_{\geq 0}^{\Phi_-}$.

In the following sections, we will get estimates of the form

$$\delta_* \left(I^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow} \right) \subset I^{\sum_\gamma \text{something}(\mathbf{n})} L_I.$$

4.3.7 Compact Hecke operators and $u_{lt\,*}$ action

In this and further sections, we assume that $v_p(\gamma(t)) \geq 1$ for all $\gamma \in \Phi_+$. In this case, the corresponding Hecke operator is compace on L_I, as we are about to see. Some results can still be stated if $v_p(\gamma(t)) = 0$ for some γ; however, in this case, the corresponding Hecke operator is not compact on L_I, and, probably, not on any its full-rank sublattice.

Definition 4.3.7.1. We define

$$slope(m,n) = n - \left\lfloor \frac{n}{p^m} \right\rfloor.$$

Computation 4.3.7.2. *$slope^{exact}(m,n) \geq slope(m,n)$.*

Proof. Observe that

$$slope(m,n) = \sum_{i=0}^{m-1} \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^{i+1}} \right\rfloor,$$

and that

$$\left\lceil \frac{n}{p^i} \right\rceil - \left\lfloor \frac{\left\lceil \frac{n}{p^i} \right\rceil}{p} \right\rfloor \geq \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^{i+1}} \right\rfloor.$$

Indeed, if $p^i | n$, then the left hand side and the right hand side coincide; otherwise, observe that

$$\left\lfloor \frac{\left\lceil \frac{n}{p^i} \right\rceil}{p} \right\rfloor \leq \left\lfloor \frac{n}{p^{i+1}} \right\rfloor + 1.$$

It follows that

$$\left\lceil \frac{n}{p^i} \right\rceil - \left\lfloor \frac{\left\lceil \frac{n}{p^i} \right\rceil}{p} \right\rfloor \geq \left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^{i+1}} \right\rfloor.$$

□

Computation 4.3.7.3. *If $p \in I$ and $a \in R$, then*

$$I^n \binom{p^m z_\alpha + a}{n} \subset I^{slope(m,n)} L_I.$$

Proof.

$$\binom{p^m z_\alpha + a}{n} = \sum_j \binom{p^m z_\alpha}{j}\binom{a}{n-j};$$

Combining results of Section 4.3.6 with Computation 4.3.7.2, we get

$$I^n \binom{p^m z_\alpha}{j} \subset I^{n-j} I^{slope^{exact}(m,j)} L_I \subset I^{n-j} I^{slope(m,j)} L_I \subset I^{slope(m,n)} L_I.$$

□

Computation 4.3.7.4. *Suppose $u_{lt} \in \mathrm{Iw}_p \in G(\mathbb{Z}_p)$ is a lower-triangular matrix, suppose that $v_p(\gamma(t)) \geq 1$ for all $\gamma \in \Phi_+$, and let $I \ni p$. Then*

$$t_* u_{lt\,*}\left(I^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow}\right) \subset I^{\mu(\mathbf{n})} L_I,$$

where

$$\mu(\mathbf{n}) = \sum_{\gamma \in Pi_-} slope(1,n) + \sum_{\gamma \in \Phi_- \setminus \Pi_-} slope(2,n).$$

Proof. First, observe that if β is simple, then, by Computation 4.3.3.3,

$$t_* u_{lt\,*}(z_\beta) = t_*(z_\beta + const) = t_* z_\beta + const.$$

It follows from Computation 4.3.7.3, that

$$t_* u_{lt\,*} I^n \binom{z_\beta}{n} =\subset I$$

We need to show that if β is not simple, then

$$t_* u_{lt\,*}(z_\beta) = const + p^2 z_\beta^{new},$$

where $z_\beta^{new} \in L_I$. Then, by Computation 4.3.7.3 and Lemma 4.3.4.4, for every n,

$$I^n \binom{t_* u_{lt\,*}(z_\beta)}{n} \subset I^{slope(2,n)} L_I.$$

Note that $v_p(\gamma(\beta)) \geq 2$. It suffices to give the proof for $u_{lt} = x_{-\gamma}(p\nu)$. If γ and β do not compare or $\beta > \gamma$, then $u_{lt\,*} z_\beta = z_\beta$.

If $\gamma = \beta$, then

$$t_* u_{lt\,*} z_\beta = t_* z_\beta + \nu,$$

as desired.

Finally, if $\beta < \gamma$, then, by Computation 4.3.3.3,

$$u_{lt\,*} z_\beta = z_\beta + \sum \text{ monomials of the form } c_{\mathbf{m}} \cdot (p\nu)^{m_\nu} p^{\mathbf{m}-1} \mathbf{z}^{\mathbf{m}},$$

where $m_\nu \gamma + \sum m_\epsilon \epsilon = \beta$ and $m_\nu \geq 1$ and $\sum \mathbf{m} \geq 1$. Then

$$\begin{aligned} t_* u_{lt\,*} z_\beta =& t_* z_\beta + p \cdot \sum c_{\mathbf{m}} \cdot (p\nu)^{m_\nu - 1}\ t_* p^{\mathbf{m}-1} \mathbf{z}^{\mathbf{m}} = \\ & t_* z_\beta + p^2 \cdot \sum c_{\mathbf{m}} \cdot (p\nu)^{m_\nu - 1} \cdot p^{-1} t_* p^{\mathbf{m}-1} \mathbf{z}^{\mathbf{m}}. \end{aligned}$$

Since $\sum \mathbf{m} \geq 1$, we have $p^{-1} t_* p^{\mathbf{m}-1} \mathbf{z}^{\mathbf{m}} \in L_I$, as desired.

□

Corollary 4.3.7.5. For a given group G, for all but finitely many primes p, the following lower bound holds: suppose that $v_p(\gamma(t)) \geq 1$ for all $\gamma \in \Phi_+$, and let $I = (T_\alpha)_{\alpha \in \Pi} + (p)$. Then

$$\mathfrak{U}_t \left(I^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow} \right) \subset I^{\sum_\gamma n_\gamma - \left\lfloor \frac{n_\gamma}{p} \right\rfloor} L_I.$$

Proof. By Proposition 3.5.0.1, we know that, for any $F \in \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ the element $\mathfrak{U}_t(F)$ is a sum of elements $\delta_* G$, where $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$. As pointed out in Section 4.3.2, δ can be decomposed as $\delta = d u_{ut} t u_{lt}$ where d is diagonal, u_{ut} and u_{lt} are upper-triangular and lower-triangular respectively, and $d, u_{ut}, u_{lt} \in \mathrm{Iw}_p$.

By Computation 4.3.5.1,

$$d_* u_{ut\,*} L_I \subset L_I.$$

By Computation 4.3.7.4,

$$d_* u_{ut\,*} t_* u_{lt\,*} \left(I^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow} \right) \subset d_* u_{ut\,*} I^{\sum_\gamma n_\gamma - \left\lfloor \frac{n_\gamma}{p} \right\rfloor} L_I \ \subset I^{\sum_\gamma n_\gamma - \left\lfloor \frac{n_\gamma}{p} \right\rfloor} L_I.$$

□

Notation 4.3.7.6. We call $mult_{N,t}^{naive,\,\Phi_-}$ number of integer solutions of the system of inequalities:

$$\begin{gathered} n_\gamma \geq 0; \quad \gamma \in \Phi_- \\ \sum_{\gamma \in \Pi_-} slope(1, n_\gamma) + \sum_{\gamma \in \Phi_- \setminus \Pi_-} slope(2, n_\gamma) \leq N. \end{gathered} \tag{4.3.4}$$

Let $mult_{N,t}^{naive,\,\Phi_-} = mult_{\leq N,t}^{naive,\,\Phi_-} - mult_{\leq N-1,t}^{naive,\,\Phi_-}$.

Corollary 4.3.7.7. Suppose that $|T_{\alpha_1}| > |T_{\alpha_2}| > \ldots > 1/p$ for some assignment of α_1, α_2, $\ldots \in \Pi$. Consider the ring

$$\Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p} = \mathbb{Z}_p[\![(T_{\alpha_i})_{\alpha_i \in Pi}]\!]\ [T_{\alpha_1} T_{\alpha_2}^{-1}, T_{\alpha_2} T_{\alpha_3}^{-1}, \ldots p T_{\alpha_n}^{-1}],$$

and the operator $\mathcal{U}_t$ on $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda) \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}$. The Newton polygon of $\mathcal{U}_t$ is bounded below by the polygon that has $mult_{N,t}^{naive,\,\Phi_-}$ edges with slope N for each $N \geq 0$ (we assume the slope of T_{α_1} is 1).

Proof. We assume that $\Lambda = \mathbb{Z}_p[\![(T_{\alpha_i})_{\alpha_i \in \Pi}]\!]$ is a subring of the ring $\Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}$. The elements $T_{\alpha_1}^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow}$ form a basis in the submodule

$$L_{(T_{\alpha_i})_{\alpha_i \in \Pi} + (p)} \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p} = L_{(T_{\alpha_1})} \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}.$$

By Corollary 4.3.7.5,

$$U_t\ T_{\alpha_1}^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow} \in T_{\alpha_1}^{\sum_\gamma n_\gamma - \lfloor n_\gamma/p \rfloor} L_{(T_{\alpha_i})_{\alpha_i\in\Pi}+(p)} \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}.$$

That is, if we consider the matrix $\mathfrak{U}_t \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}$ in the basis $T_{\alpha_1}^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}$, then the column corresponding to the basis element $T_{\alpha_1}^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}$ is divisible by $T_{\alpha_1}^{\sum_\gamma n_\gamma - \lfloor n_\gamma/p \rfloor}$.

For each N, the number of columns divisible by at most T_α^N is at most $mult_{\leq N,t}^{naive,\,\Phi_-}$. Thus, the Hodge polygon of $\mathfrak{U}_t \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}$ on $L_{T_{\alpha_1}}$ is bounded below by the polygon that has $mult_{N,t}^{naive,\,\Phi_-}$ edges of slope N. □

Corollary 4.3.7.8. Suppose

$$G(\mathbb{A}_f) = \bigsqcup_{i\in\{0,\ldots n-1\}} G(\mathbb{Q})\gamma_i K$$

is a union of n double cosets (see Section 3.3). The Newton polygon of the operator $\mathcal{U}_t$ on the space of overconvergent automorphic forms of radius r is bounded below by the polygon that has $n \cdot mult_{N,t}^{naive,\,\Phi_-}$ edges with slope N for each $N \geq 0$.

Proof. By Section 3.3 in Johansson and Newton [2016], the space of overconvergent forms of radius r has an orthonormal basis of the form

$$\mathfrak{n}(r,\mathbf{m})\mathbf{z}^{\mathbf{m}\downarrow},$$

where $\mathfrak{n}(r,\mathbf{m})$ is a coefficient that should be thought of as a power of p. Hence, Lemma 4.1.0.3 applies to the operator $\mathfrak{U}_t$ on

$$L_{(T_{\alpha_1})} \otimes \Lambda_{|T_{\alpha_1}|>|T_{\alpha_2}|>\ldots>1/p}$$

and the corresponding operator on the space of overconvergent forms of radius r. □

CHAPTER 5

APPLICATIONS

In the remainder of this paper, we explore some applications of the ideas from Chapter 4.

5.1 The lower bound in the unitary case

In this section we compare our lower bound to that obtained in Ye [2024]. Consider the case where G is a compact form of GL_n, and $t = \mathrm{diag}(p^{n-1}, p^{n-2}, \ldots, p, 1)$. The main result of Ye [2024] states that the asymptotic of the Newton polygon of the U_t operator is $C \cdot x^{1+\frac{2}{n(n-1)}}$. More precisely, in Theorem 5.1.1 of Ye [2024], the following lower bound on the Newton polygon is established, with the help of some ideas from Johansson and Newton [2016]: the Newton polygon of U_t is bounded by the polygon that, for each N, has

$$\binom{N + n(n-1)/2 - 1}{n(n-1)/2 - 1}$$

edges of slope $N - \lfloor N/p \rfloor$. To rephrase, if the basis in the space on which the Hecke operator acts is indexed by multi-indices $(i_1, \ldots i_{\binom{n}{2}})$, then the slope corresponding to this multi-index is

$$\sum_{k=1}^{\binom{n}{2}} i_k - \left\lfloor \frac{\sum_{k=1}^{\binom{n}{2}} i_k}{p} \right\rfloor .$$

According to Corollary 4.3.7.7, the multiplicity of each slope N for $N < p - 1$ is equal to the number of solutions of the system

$$\begin{cases} i_\gamma \geq 0; \quad \gamma \in \Phi_- \\ \sum_{\gamma \in \Pi_-} \mathit{slope}(1, n_\gamma) + \sum_{\gamma \in \Phi_- \setminus \Pi_-} \mathit{slope}(2, n_\gamma) = N. \end{cases} \tag{4.3.4}$$

Another way of saying this is that for a basis indexed by multi-indices $(i_\gamma)_{\gamma \in \Phi_-}$, the slope corresponding to each multi-index is

$$\sum_{\gamma\in\Pi_-} slope(1,n_\gamma) + \sum_{\gamma\in\Phi_-\setminus\Pi_-} slope(2,n_\gamma) \geq$$
$$\geq \sum_{\gamma} i_\gamma - \sum_{\gamma\in\Pi_-}\left\lfloor\frac{i_\gamma}{p}\right\rfloor - \sum_{\gamma\notin\Pi_-}\left\lfloor\frac{i_\gamma}{p^2}\right\rfloor \geq$$
$$\geq \sum_{\gamma\in\Phi_-} i_\gamma - \sum_{\gamma\in\Phi_-}\left\lfloor\frac{i_\gamma}{p}\right\rfloor \geq \sum_{\gamma\in\Phi_-} i_\gamma - \left\lfloor\frac{\sum_\gamma i_\gamma}{p}\right\rfloor.$$

For instance, for $N < p-1$, for the lower bound in Ye [2024], the multiplicity of slope N is

$$\binom{N+n(n-1)/2-1}{n(n-1)/2-1},$$

and the multiplicity of the slope $N = p-1$ is

$$\binom{p+n(n-1)/2-2}{n(n-1)/2-1} + \binom{p+n(n-1)/2-1}{n(n-1)/2-1}.$$

If we apply corollary 4.3.7.7 to the case $N < p-1$, then for every solution, for all γ, $n_\gamma < p-1$, so, $n_\gamma - \lfloor n_\gamma/p\rfloor = n_\gamma$, and the number of solutions is equal to

$$\binom{N+n(n-1)/2-1}{n(n-1)/2-1}.$$

If $N = p-1$, then there are two types of solutions: ones, where $n_\gamma \leq p-1$ for all γ, and ones where $n_\gamma = p$ for one simple γ, and the remaining variables are 0. Hence, the total number of solutions is equal to

$$\binom{p+n(n-1)/2-2}{n(n-1)/2-1} + n - 1;$$

that is, according to Corollary 4.3.7.7, the total number of slopes $\leq p-1$ is slightly lower than the total number of slopes $\leq p-1$ in Ye [2024].

The asymptotic of the lower bound in Ye [2024] is

$$y = \left(x^{1+\frac{2}{n(n-1)}} \cdot \frac{\binom{n}{2}}{\binom{n}{2}+1} \cdot \left(\binom{n}{2}! \right)^{-\frac{2}{n(n-1)}} \cdot \left(1 - \frac{1}{p}\right) + o\left(x^{1+\frac{2}{n(n-1)}} \right) \right) \cdot \max_{\alpha}(|T_\alpha|),$$

which generalizes Liu et al. [2017] (see Remark 4.2.0.4).

To find the asymptotic of the bound that follows from Corollary 4.3.7.7, we observe that the number of points is approximated by the volume of the simplex

$$\begin{gathered} i_\gamma \geq 0; \quad \gamma \in \Phi_- \\ \sum_{\gamma \in \Pi_-} n_\gamma - \frac{n_\gamma}{p} + \sum_{\gamma \in \Phi_- \setminus \Pi_-} n_\gamma - \frac{n_\gamma}{p^2} = N, \end{gathered}$$

which is approximately equal to

$$N^{\binom{n}{2}-1} \cdot \frac{1}{(\binom{n}{2}-1)!}.$$

Using this estimate, we can show that the upper bound that follows from Corollary 4.3.7.7 is

$$y = \left(x^{1+\frac{2}{n(n-1)}} \cdot \frac{\binom{n}{2}^{\frac{2}{n(n-1)}}}{\binom{n}{2}+1} \cdot \left(\binom{n}{2}! \right)^{-\frac{2}{n(n-1)}} \cdot C^{\frac{2}{n(n-1)}} + o\left(x^{1+\frac{2}{n(n-1)}} \right) \right) \cdot \max_{\alpha}(|T_\alpha|),$$

where

$$C = \left(1 - \frac{1}{p}\right)^{n-1} \left(1 - \frac{1}{p^2}\right)^{\binom{n-1}{2}} > \left(1 - \frac{1}{p}\right)^{\binom{n}{2}}.$$

5.2 The lower bound on the Newton polygon: the GSp_4 case.

In this section we assume $G = \mathrm{GSp}_4$ and improve the estimates obtained in Corollary 4.3.7.7. This yields a better constant for the asymptotic upper bound than one in Corollary 4.3.7.5.

In this section we use the same notations as in the series of examples Example 2.1.0.1, Example 3.2.0.6. We only consider the operator $\mathfrak{U}_t$ for $t = \mathrm{diag}(p^3, p^2, p, 1)$, (that is, $\alpha(t) = \beta(t) = p$). This operator is compact, as follows from Corollary 4.3.7.7.

5.2.1 $u_{lt\,*}$ *action for* GSp_4

Computation 5.2.1.1. *Suppose* $u_{lt} \in \mathrm{Iw}_p \in \mathrm{GSp}_4(\mathbb{Z}_p)$ *is a lower-triangular matrix and let* $I \ni p$. *Then*

$$t_* u_{lt\,*}\left(I^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}\right) \subset I^{\sum_\gamma slope(v_p(\gamma(t)),n_\gamma)} L_I.$$

Proof. We can assume that

$$u_{lt} = \prod_{\varepsilon \in \Phi_-} x_\varepsilon(p\nu_\varepsilon).$$

Then

$$t_* u_{lt\,*}\mathbf{z} = \left(\nu_{-\alpha},\ \nu_{-\beta},\ \nu_{-\alpha-\beta},\ \nu_{-2\alpha-\beta}\right) + \left(p\ z_{-\alpha},\ p\ z_{-\beta},\ p^2\ (\nu_{-\alpha}z_{-\beta} + z_{-\alpha-\beta}),\ p^3\ (\nu_{-\alpha}^2 z_{-\beta} + 2\nu_{-\alpha}z_{-\alpha-\beta} + z_{-2\alpha-\beta})\right).$$

By Lemma 4.3.4.4, the right hand side is t_* composed with a map that preserves the lattice L_I. The result follows from Section 4.3.6 and Computation 4.3.7.3. □

5.2.2 The lower bound for the Newton polygon

The following proposition follows from the above computations:

Proposition 5.2.2.1.

$$\mathfrak{U}_t\left((T_\alpha, T_\beta, p)^{\mathbf{n}}\mathbf{z}^{\mathbf{n}\downarrow}\right) \subset (T_\alpha, T_\beta, p)^{\sum_\gamma slope(v_p(\gamma(t)),n_\gamma)} L_{(T_\alpha,T_\beta,p)}.$$

Proof. By Proposition 3.5.0.1, we know that, for any $F \in \mathcal{C}(\mathbb{Z}_p^{\Phi_-}, R)$ the element $\mathfrak{U}_t(F)$ is a sum of elements $\delta_* G$, where $\delta \in \mathrm{Iw}_p\, t\, \mathrm{Iw}_p$. As pointed out in Section 4.3.2, δ can be decomposed as $\delta = d u_{ut} t u_{lt}$ where d is diagonal, u_{ut} and u_{lt} are upper-triangular and lower-triangular respectively, and $d, u_{ut}, u_{lt} \in \mathrm{Iw}_p$.

By Computation 4.3.5.1,

$$d_* u_{ut\,*} L_{(T_\alpha, T_\beta, p)} \subset L_{(T_\alpha, T_\beta, p)}.$$

By Computation 5.2.1.1,

$$d_* u_{ut\,*} t_* u_{lt\,*} \left((T_\alpha, T_\beta, p)^{\mathbf{n}} \mathbf{z}^{\mathbf{n}\downarrow} \right) \subset d_* u_{ut\,*} (T_\alpha, T_\beta, p)^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)), n_\gamma)} L_{(T_\alpha, T_\beta, p)} \subset$$
$$\subset (T_\alpha, T_\beta, p)^{\sum_\gamma \mathit{slope}(v_p(\gamma(t)), n_\gamma)} L_{(T_\alpha, T_\beta, p)}.$$

□

Notation 5.2.2.2. Let $\mathit{mult}^{\Phi_-}_{\leq N, t}$ denote the number of integer solutions of the system of inequalities:

$$\begin{gathered} n_\gamma \geq 0; \quad \gamma \in \Phi_- \\ \sum_{\gamma \in \Phi_-} \mathit{slope}(v_p(\gamma(t)), n_\gamma) \leq N. \end{gathered} \tag{5.2.1}$$

Let $\mathit{mult}^{\Phi_-}_{N,t} = \mathit{mult}^{\Phi_-}_{\leq N,t} - \mathit{mult}^{\Phi_-}_{\leq N-1,t}$. The functions *slope* are defined in Section 4.3.6.

Remark 5.2.2.3. If $t = \mathrm{diag}(p^3, p^2, p, 1)$, then the number of solutions of Equation (5.2.1) is approximately equal to the number of integer points in the simplex bounded by the hyperplane

$$n_{-\alpha} \cdot \left(1 - \frac{1}{p}\right) + n_{-\beta} \cdot \left(1 - \frac{1}{p}\right) + n_{-\alpha-\beta} \cdot \left(1 - \frac{1}{p^2}\right) + n_{-2\alpha-\beta} \cdot \left(1 - \frac{1}{p^3}\right) \leq N$$

and the hyperplanes $n_\gamma \geq 0$ for $\gamma \in \Phi_-$. The number of lattice points in the simplex is approximately equal to its volume. It follows that

$$mult^{\Phi_-}_{\leq N,t} \sim \frac{1}{24}N^4 \cdot \left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{p^3}\right)^{-1},$$

and

$$mult^{\Phi_-}_{N,t} \sim \frac{1}{6}N^3 \cdot \left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{p^3}\right)^{-1}.$$

Corollary 5.2.2.4. Suppose $t = \mathrm{diag}(p^3, p^2, p, 1)$.

1. Consider the ring $\Lambda_{|T_\alpha|>|T_\beta|>1/p} = \mathbb{Z}_p[\![T_\alpha, T_\beta]\!][T_\alpha T_\beta^{-1}, pT_\alpha^{-1}]$ in which $|T_\alpha| > |T_\beta| > 1/p$ and the operator U_t on $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda) \otimes \Lambda_{|T_\alpha|>|T_\beta|>1/p}$. The Newton polygon of U_t is bounded below by the polygon that has $mult^{\Phi_-}_{N,t}$ edges with slope N for each $N \geq 0$.

2. Consider the ring $\Lambda_{|T_\beta|>|T_\alpha|>1/p} = \mathbb{Z}_p[\![T_\alpha, T_\beta]\!][T_\beta T_\alpha^{-1}, pT_\beta^{-1}]$ in which $|T_\beta| > |T_\alpha| > 1/p$ and the operator U_t on $\mathcal{C}(\mathbb{Z}_p^{\Phi_-}, \Lambda) \otimes \Lambda_{|T_\beta|>|T_\alpha|>1/p}$. The Newton polygon of U_t is bounded below by the polygon that has $mult^{\Phi_-}_{N,t}$ edges with slope N for each $N \geq 0$.

Proof. The proof is identical to that of Corollary 4.3.7.7. □

Remark 5.2.2.5. In the GSp_4 case, Corollary 5.2.2.4 tells us that the Newton polygon of the Hecke operator U_t where $t = \mathrm{diag}(p^3, p^2, p, 1)$ is bounded below by the polygon that has slope N (in the ring $\Lambda_{|T_\alpha|>|T_\beta|>1/p}$ or in the ring $\Lambda_{|T_\beta|>|T_\alpha|>1/p}$)

$$mult^{\Phi_-}_{N,t} \sim \frac{1}{6}N^3 \cdot \left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{p^3}\right)^{-1}$$

times. In other words, the Newton polygon is *almost* bounded below by the curve

$$y = x^{5/4} \cdot \frac{4}{5} \cdot \left(6\left(1-\frac{1}{p}\right)^{2}\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^3}\right)\right)^{1/4} \cdot \max(|T_\alpha|, |T_\beta|).$$

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