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ON THE MONODROMY OF RATIONAL SINGULARITIES IN MIXED
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To my teachers, Georgios Vryzas and Andreas Morikis.

In memory of my fellow number theorist and friend, Michael Zhao.

TABLE OF CONTENTS

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ABSTRACT

We study the ramification on the cohomology of a smooth proper surface X in mixed characteristic, when X degenerates to a surface over $\overline{\mathbb{F}}_p$ with rational singularities, with a focus on the case of rational double points. We find that the associated monodromy action of inertia depends on a formal affine neighborhood of the singularity, and under sufficient restrictions on characteristic p, it is tamely ramified and generated by a conjugacy class representative of an appropriate Weyl group related to the singularity. This naturally extends to a similar monodromy characterization of general rational singularities. Along the way we extend to mixed characteristic some results of Brieskorn and Slodowy concerning simultaneous resolutions of surface singularities. We also compare our Weyl group actions to certain Springer representations constructed by Borho and MacPherson, via the notion of relative perversity as developed by Hansen and Scholze.

CHAPTER 1

INTRODUCTION

Let K be a complete discrete valuation field with valuation ring $\mathcal{O}_K$ of mixed characteristic $(0, p)$. The celebrated Néron–Ogg–Shafarevich theorem, first proven for elliptic curves and then generalized to abelian varieties by Serre–Tate ([54]), states that an abelian variety X over K has *good reduction*, meaning that there exists a smooth proper model $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ with generic fiber X, if and only if the natural action of $G_K = \mathrm{Gal}(\overline{K}/K)$ on the étale cohomology group $\mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for $\ell \neq p$ is *unramified*, i.e. it restricts to a trivial action of inertia I_K. Here I_K is defined as the kernel of the surjection $G_K \to G_k$, where G_k is the absolute Galois group of the residue field k.

1.1   Good reduction beyond abelian varieties.

For a general smooth proper variety X over K there is a subtler relationship between good reduction and *ramification*, i.e. the nontriviality of the monodromy action of I_K on $\mathrm{H}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$. A necessary condition for good reduction is that $\mathrm{H}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all n and $\ell \neq p$, however the converse often fails. For example, there are smooth curves X of genus $g \geq 2$ with unramified $\mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which nevertheless do not have good reduction over K or any finite extension of K ([43], §2.4). Instead, Oda has established that, if X is a smooth proper curve of genus $g \geq 2$, it admits good reduction if and only if a certain G_K-action on the pro-ℓ completion of the geometric étale fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}})_\ell$ is unramified ([48]). More recently, Liedtke–Matsumoto have shown that a K3 surface X with unramified $\mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ admits good reduction after a finite unramified base extension, under the stricter assumption that X admits a potentially semistable model ([43], Thm. 1.3).

1.2 Bad reduction of surfaces and singularities.

In this article, we will consider degenerations (i.e. reductions) of a smooth proper surface X/K. More specifically, we focus on Galois representation $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ and relate the types of singularities that may appear on reductions of X to the ramification of $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Any smooth surface X admits an integral proper flat model $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$: by Nagata's compactification theorem ([62, Tag0F3T]) there exists a proper scheme $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ with an open immersion $X \hookrightarrow \mathcal{X}$, and up to normalizing $\mathcal{X}$ and taking the closure of X in $\mathcal{X}$ we get that $\mathcal{X}$ is reduced and dominates $\mathrm{Spec}(\mathcal{O}_K)$, so $\mathcal{X}$ is integral, proper and flat over $\mathrm{Spec}(\mathcal{O}_K)$. The special fiber $\mathcal{X}_k$ is called the reduction (mod p) of X.

One can ask in general how 'badly' singular the reduction $\mathcal{X}_k$ is. When $\mathcal{X}$ is strictly semistable, i.e. $\mathcal{X}_k$ is a simple normal-crossings (snc) divisor in $\mathcal{X}$, Rapoport–Zink have shown ([51]) that the nearby cycles spectral sequence abutting to $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ gives rise to a weight filtration on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$. Since the weight-monodromy conjecture holds for surfaces in mixed characteristic (*loc. cit.*, Satz 2.13), the weight filtration coincides up to shift with the monodromy filtration induced from the I_K-action. This allows one to determine the full monodromy action on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ by computing the I_K-action on the graded pieces of the weight filtration. These graded pieces are related to the cohomology of special fiber $\mathcal{X}_k$, which admits a nice combinatorial description by the snc property. For general semistable schemes, a cornerstone theorem concerning their ramification is the unipotency of the I_K-action ([3]).

In a somewhat orthogonal direction, one can suppose instead that the proper smooth surface X admits an integral model $\mathcal{X}$ with $\mathcal{X}_k$ having isolated singularities of a certain kind. Typically $\mathcal{X}$ will not be semistable, and so the monodromy action of I_K on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is at worst quasi-unipotent ([3], Exposé I). One can ask whether X admits a potentially semistable or even smooth model, and how the monodromy depends on the type of singularities of $\mathcal{X}_k$. The interest in surface singularities over $\overline{k} \simeq \overline{\mathbb{F}}_p$ here is twofold. Firstly, there has been recent progress on our understanding of positive and mixed characteristic singularities; see

2

[45] for a ring-theoretic approach using perfectoid techniques. Secondly, the relationship with monodromy actions has not been explored a lot beyond varieties acquiring ordinary double points, which have been classically studied in [4].

A recent result of D. Kim ([40]) investigates the monodromy action on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ related to an integral model $\mathcal{X}$ acquiring ordinary double points on the special fiber. Via an explicit calculation of a suitable semistable model of X and the Rapoport–Zink spectral sequence of [51], it is shown that the monodromy action factors through $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/2$, the Galois group of the unique ramified quadratic extension L of K. Thus I_K acts trivially on $\mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ or through a nontrivial quadratic character, dependent on a formal affine neighborhood of the singularity in $\mathcal{X}$.

1.3   Main theorem.

In this paper, we establish a generalization of [40] to a natural class of surface singularities over $\overline{k} = \overline{\mathbb{F}}_p$, that of *rational double points.* These singularities are also known in the literature as simple (surface) singularities, ADE singularities, Kleinian or du Val singularities. Rational double points have the benefit of being amenable to mixed-characteristic extensions, while still being 'mildly' singular with easily computable minimal resolutions. Moreover, there is a McKay correspondence-type relation between rational double points and Lie algebras: each class of a rational double point has a minimal resolution whose exceptional divisor possesses a dual graph isomorphic to a Dynkin diagram of ADE type, and therefore such a class corresponds to a simple simply-laced Lie algebra (over $\mathbb{C}$ or $\overline{\mathbb{F}}_p$ for sufficiently large $p > 0$; see Section 2.1 for details).

We aim to characterize the associated monodromy I_K-actions for models $\mathcal{X}$ of X which degenerate into surfaces $\mathcal{X}_k$ having rational double point singularities. Instead of finding a semistable model of X, we investigate the possible formal affine neighborhoods of the singularities of $\mathcal{X}_k$ in $\mathcal{X}$ via their explicit miniversal deformations. These miniversal deformation

3

equations are determined by $\mathcal{X}_{\breve{K}}$, so throughout this paper (unless stated otherwise) we assume $K = \breve{K}$ is the completion of its maximal unramified extension. In particular the special fiber $\mathcal{X}_k$ and its singularity live over $\overline{k} = k$.

We adapt classical results of Tjurina, Brieskorn and Slodowy ([16], [57], [64]) regarding so-called *simultaneous* resolutions of singularities to the mixed characteristic setting, yielding that X admits a smooth model $\widetilde{\mathcal{X}}$ after a finite base-change L/K. Results of Artin ([7]) show that $\widetilde{\mathcal{X}}$ exists at worst in the category of algebraic spaces, and its fibers are (algebraic) surfaces. Dependent on a restriction on the characteristic p and on a formal affine neighborhood of each singularity, we can make the monodromy I_K-action precise:

Theorem 1.3.1. *Suppose $(K, \mathcal{O}_K, k)$ is the data of a complete DVR of mixed characteristic $(0, p)$ with p sufficiently good, and let X/K be a smooth proper surface with an integral model $\mathcal{X}$ over $\mathcal{O}_K$ so that $\mathcal{X}_k$ has a unique rational double point. Let W be the Weyl group associated to the Dynkin diagram corresponding to the rational double point.*

(i) *The monodromy I_K-action on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ factors through a cyclic subgroup $\langle w \rangle$ of W, dependent up to conjugacy on a formal affine neighborhood of the singularity.*

(ii) *The Weyl element w acts on $\mathrm{H}^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ via a Springer W-representation, and X achieves good reduction after a ramified base-change of degree $\mathrm{ord}(w)$.*

(iii) *In the case of A_n-singularities, for every Weyl conjugacy class there exists an element w in the class and a model $\mathcal{X}$ degenerating to A_n-singular surface $\mathcal{X}_k$ so that w acts as the monodromy operator on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$.*

In particular we recover the results of [40], which in this terminology deal with A_1-singularities (ordinary double points). The main novelty here is to bridge the gap between the characteristic zero and characteristic p cases of monodromy actions on cohomology, since for big enough p the classification of rational double points over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$ is the same. In

4

particular we find that, if the action of monodromy is *tame*, then the monodromy operator acts in the same way as the complex monodromy operator for such singularities. As far as we know, there has not been a thorough investigation of the relationship between conjugacy classes of Weyl groups and degenerations to (rational double point or otherwise) singularities, even over $\mathbb{C}$. We hope to further explore this relationship in the future.

We remark here that Theorem 1.3.1 naturally generalizes to $\mathcal{X}_k$ having any finite number of rational double points, as we may choose disjoint formal affine neighborhoods at each singularity. In this case, the monodromy acts via a product of Weyl group elements, one for each singularity of a fixed Dynkin type. There is also a natural generalization to arbitrary (surface) rational singularities, by using the methods of [67] to reduce the situation to rational double points, and the monodromy acts factors through a product of Weyl groups. There are also possible applications of Theorem 1.3.1 to questions regarding monodromy characterizations of reductions of K3 surfaces in mixed characteristic, as part of the topic of derived equivalences of K3 surfaces (see [32]); we hope to explore this direction in the future as well.

1.4   Outline of the proof.

The main tools for the proof of Theorem 1.3.1 are a theorem of Berkovich ([10]) and the Grothendieck–Springer resolution. Via Berkovich's argument we may relate the nearby cycles of special fiber $\mathcal{X}_k$ to the "formal" nearby cycles of the completion of $\mathcal{X}_k$ at the singularity, and show that the monodromy action on $\mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ depends on a formal neighborhood of the singularity. We may then 'embed' the local picture into the miniversal deformation of the singularity and use a mixed-characteristic incarnation of the Grothendieck–Springer resolution.

Over $\mathbb{C}$, the Grothendieck–Springer resolution (or Grothendieck *alteration*) $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ furnishes a connection between simple Lie algebras $\mathfrak{g}$ and simple surface singularities, which

are exactly the (complex) RDPs. This connection was studied by Brieskorn in the '70s, following a conjecture of Grothendieck ([16]), and full details were written up in ([57]). The Grothendieck alteration may be thought of as an enhancement of simultaneous resolutions of surface singularities on the algebro-geometric side, and as a generalization of the Springer resolution $\widetilde{\mathcal{N}} \to \mathcal{N}$ on the representation-theoretic side. Concretely, one may realize RDPs as generic points in the subregular nilpotent orbit of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. In turn, $\mathfrak{g}$ is connected to the singularity via its Dynkin diagram, which is isomorphic to the dual graph of the exceptional divisor in the minimal resolution of the singularity. These considerations still make sense for Chevalley algebras over $\mathrm{Spec}(\mathcal{O}_K)$.

After a ramified base-change via the Weyl cover, we may simultaneously resolve all singularities appearing on the nilpotent cone, and in particular we obtain a resolution of the singularity in our model $\mathcal{X}$ by pulling back along an appropriate base-change on $\mathrm{Spec}(\mathcal{O}_K)$. Using a recent notion of *relative perversity* from Hansen–Scholze ([29]), we may describe the associated monodromy W-action of the Weyl cover as an action on the relatively perverse sheaf $\mathrm{R}\pi_*\mathbb{Q}_\ell[\dim\mathfrak{g}]$, which ends up being the Springer W-action as constructed by Borho–MacPherson ([13]). Along the way we also derive an ℓ-adic instance of the Springer correspondence and relate Springer theory to the study of nearby cycles over a larger (i.e. > 1-dimensional) mixed-characteristic base. This gives a p-adic picture analogous to the interaction between Springer theory and nearby cycles in the complex setting. The upshot is that, in sufficiently large characteristic p, we may describe explicitly the resulting ramification in the cohomology of X by factoring the I_K-action through a restriction of the Springer W-representation associated with the singularity of $\mathcal{X}_k$. In the case of A_n-singularities we can furthermore check by hand that the monodromy I_K-action can lie in any Weyl conjugacy class.

1.5   Connections with other work.

Our results are parallel to results of Shepherd-Barron ([55], [56]), who extended the Grothendieck–Springer resolution and some related results of Brieskorn to "good" characteristic, via a different method and in the context of groups instead of Lie algebras. Shepherd-Barron also extended arguments of Artin ([7]) regarding simultaneous resolutions to show that one also gets so-called Weyl covers for Brieskorn's resolutions in *all* characteristics. In small characteristic, however, he notes that "one does not have a formula for the action of any reflection of W" on the relevant cohomology groups ([56], Introduction). Our approach is instead a natural extension of the methods described in ([57], [58]), and at the cost of restricting the characteristic we may describe what the corresponding W-action must be.

1.6   Organization of this thesis.

Section 2 contains some background on rational double points, simultaneous resolutions and (miniversal) deformations of isolated singularities in the mixed-characteristic setting. Section 3 covers the necessary Lie-theoretic notions including the description of the nilpotent cone as a fiber of the adjoint quotient and the Grothendieck–Springer resolution, in the context of Chevalley algebras over a mixed-characteristic DVR. Section 4 extends results of Slodowy ([57]) regarding the construction of suitable transverse slices to nilpotent orbits in Lie algebras. Finally Section 5 presents the main argument, involving tools from the study of perverse sheaves and nearby cycles to determine our desired monodromy action in terms of certain Springer representations of the Weyl group.

1.7   Notations and conventions.

All rings are commutative with unity. $\mathcal{O}_K$ denotes a complete mixed-characteristic discrete valuation ring of type $(0, p)$, meaning fraction field K has characteristic zero and residue

field k is *algebraically closed* of characteristic $p > 0$. Unless otherwise stated, we will assume $K = \check{K} = \widehat{K^{\mathrm{unr}}}$ is the completed maximal unramified extension of K in a fixed separable closure $\overline{K}$, so that $\mathcal{O}_K$ may be identified with the Witt vectors $W(k)$ of k. The maximal tamely ramified extension of K is denoted by K^{tr}. The inertia subgroup of Gal_K is $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{unr}})$, the pro-$p$ wild inertia subgroup of I_K is $P = \mathrm{Gal}(\overline{K}/K^{\mathrm{tr}})$ and the tame inertia is defined as I/P, which is topologically generated by one element.

On the geometric side we define $(\mathrm{Spec}(\mathcal{O}_K), \mathrm{Spec}(K), \mathrm{Spec}(k)) = (S, \eta, s)$ to be the data of a (complete) *trait*, with generic point η and closed point $s = \overline{s}$. Separable closures are denoted with a bar, e.g. $\overline{\eta} = \mathrm{Spec}(\overline{K})$. Residue fields of points $x \to X$ of a scheme X are denoted by $k(x)$, e.g. $\overline{K} = k(\overline{\eta})$.

On the Lie-theoretic side, $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ will denote respectively a semisimple Lie algebra along with a choice of Borel and Cartan subalgebra, and $W = W(\mathfrak{g})$ will denote the Weyl group associated to (the Dynkin diagram of) $\mathfrak{g}$. The Coxeter number of $\mathfrak{g}$ is denoted by $\mathrm{Cox}(\mathfrak{g})$. For an affine S-scheme X with an action of an S-group scheme G, $X /\!\!/ G$ denotes the affine GIT quotient with coordinate ring $\mathcal{O}_S[X]^G$.

Unless otherwise stated, H^i denotes étale cohomology $\mathrm{H}^i_{\mathrm{\acute{e}t}}$.

CHAPTER 2

RATIONAL DOUBLE POINTS AND THEIR MINIVERSAL DEFORMATIONS

This section gives some relevant background on rational double points and simultaneous resolutions. Sections 2.3-2.5 describe explicitly the miniversal deformations of these singularities. While the results are known to the experts, the associated deformation problems in this setting have a mixed-characteristic base $\mathcal{O}_K = W(k)$ instead of k, so for completeness we develop the mixed characteristic case here.

2.1   Rational double points.

Let X be an algebraic surface over an algebraically closed field k. A *rational* singularity $x \in X(k)$ is a normal singularity (i.e. $\mathcal{O}_{X,x}$ normal) for which there exists a resolution $f : \widetilde{X} \to X$ satisfying $\mathrm{R}^i f_* \mathcal{O}_{\widetilde{X}} = 0$ for $i \geq 1$.

By Zariski's Main Theorem, the reduced exceptional divisor $E = f^{-1}(x)_{\mathrm{red}}$ is a union of smooth rational curves E_i, elucidating the term 'rational singularity' (see [17], Lemma 3.8). Rational double points are a particular class of rational singularities pinning down the self-intersections of the exceptional divisors E_i:

Definition 2.1.1. A normal surface singularity (X, x) is a *rational double point* (henceforth RDP) if its minimal resolution $f : \widetilde{X} \longrightarrow X$ has reduced exceptional divisor $E = \cup E_i$ so that all E_i are smooth rational curves with self-intersection $E_i^2 = -2$.

There are various equivalent characterizations of RDPs; for example, they are surface singularities (X, x) whose Zariski tangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ has dimension 3 and $\mathcal{O}_{X,x}$ has multiplicity 2 (hence the term 'double points'). The tangent space dimension here implies RDPs are regularly embedded in codimension 1 and are hence hypersurface singularities. Since

smooth points have local rings of multiplicity 1 and tangent spaces of dimension 2, this characterization makes apparent that RDPs are the 'mildest' surface singularities one can ask for.

Example 2.1.2. The following examples may give a rough indication on where RDPs stand within the zoo of surface singularities.

(i) Suppose $E \subset \mathbb{P}^2_k$ is an elliptic curve and $C_E \subset \mathbb{A}^3_k$ is the cone over E. The exceptional divisor of the resolution $\widetilde{C}_E \to C_E$ of the singularity at the origin is E again, hence this singularity is not rational (it is instead an "elliptic singularity").

(ii) The image of the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is a rational normal curve C_d of degree d. The cone $C \subset \mathbb{A}^{d+1}$ has a rational singularity at the origin since the exceptional divisor $E \simeq C_d$. However $E^2 = -d$, so for $d \geq 3$ it is *not* a rational double point.

(iii) The simplest rational double point is the origin in the affine cone $\{x^2 + y^2 + z^2 = 0\}$, which is also known as an ordinary double point..

One property RDPs enjoy is that they are *absolutely isolated*, meaning they can be resolved after a finite number of blowups at points; in fact, each blowup yields an RDP of different type until they are all resolved (see [62, Tag 0BGB]).

A crucial fact about RDPs is that in appropriate characteristic (see Definition 2.1.4 below) they are *taut*, i.e. completely determined up to isomorphism by the dual graph of their minimal resolution. This is classical over $\mathbb{C}$, and has been extended to positive characteristic by [8]:

Theorem 2.1.3. *Let X be a projective surface over k.*

(i) ([17], Thm. 3.32, [6]) *If E is a connected curve on X with smooth rational curve components E_i so that $E_i^2 = -2$, then the only possible dual graphs for E are the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8.*

10

(ii) ([8]) *If $x \in X(k)$ is an RDP and* char(k) *is very good (see Definition 2.1.4) then up to analytic isomorphism $\widehat{\mathcal{O}}_{X,x} \simeq k[\![x,y,z]\!]/(f(x,y,z))$, where $f(x,y,z)$ and the dual graph of the minimal resolution of x are given by the table below:*

$f(x,y,z)$	Dual graph
$z^2 + x^2 + y^{n+1}$	A_n
$z^2 + x^2 y + y^{n-1}$	D_n
$z^2 + x^3 + y^4$	E_6
$z^2 + x^3 + xy^3$	E_7
$z^2 + x^3 + y^5$	E_8

A polynomial $f(x,y,z)$ defining an RDP is usually called a 'normal form'. Since the notion of 'good characteristic' will be ubiquitous in this paper, we record the definition here.

Definition 2.1.4. Given a semisimple simply-laced Lie algebra $\mathfrak{g}$ over k of characteristic $p > 0$, so that its Dynkin diagram has components A_n, D_n or E_n, we say p is

(a) *good* with respect to $\mathfrak{g}$ if $p \neq 2$ if $\mathfrak{g}$ has any D_n components, $p \neq 2, 3$ if $\mathfrak{g}$ has any E_6, E_7 components and $p \neq 2, 3, 5$ if $\mathfrak{g}$ has any E_8 components (note there are no p restrictions for A_n components).

(b) *very good* with respect to $\mathfrak{g}$ if it is good and for any A_n component we have $p \nmid n + 1$

(c) *sufficiently good* with respect to $\mathfrak{g}$ if $p \nmid |W(\mathfrak{g})|$ i.e. it does not divide the order of the associated Weyl group.

For simple Lie algebras it is automatic that sufficiently good $\Rightarrow$ very good $\Rightarrow$ good. We will revisit this definition in Remark 3.2.3 for a more intuitive explanation of these restrictions.

Theorem 2.1.3 identifies RDPs with the corresponding simple simply-laced Lie algebras of the specified Dynkin type, so we can refer to p being (very) good for the singularity, the

11

Lie algebra or the Dynkin diagram interchangeably. Theorem 2.1.3 is no longer true when p is *not* good, and there is more than one equation describing an RDP with the same dual graph; see ([8]) for details.

Remark 2.1.5. We have the following byproduct of the proof of Theorem 2.1.3. We may resolve RDP $x \in X(k)$ by iterated blowups along points, and at each step we get an RDP of different Dynkin type. All the Dynkin diagrams corresponding to each RDP appearing in the resolution process are subdiagrams of the Dynkin diagram of x. Conversely, all subdiagrams of the Dynkin diagram of x correspond to these 'intermediate' RDPs during the iterated blowup process.

2.2   Simultaneous resolutions.

We next describe a particularly rare notion of resolving singularities of schemes in families. In order to be consistent with [7], we enlarge the category of schemes to include separated algebraic spaces, though in practice we will only resolve schemes.

Definition 2.2.1. Let $f : \mathcal{X} \longrightarrow S$ be a finite-type morphism of separated algebraic spaces. A *simultaneous resolution* is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{X}} & \xrightarrow{\pi} & \mathcal{X} \\
\downarrow{\widetilde{f}} & & \downarrow{f} \\
\widetilde{S} & \xrightarrow{\psi} & S
\end{array}
$$

where $\widetilde{f}$ is smooth, π is proper, ψ is a finite surjection and for all geometric points $\widetilde{s} \to \widetilde{S}$ with image $s = \psi(\widetilde{s}) \to S$, the induced morphism on the fibers

$$
\pi_s : \widetilde{\mathcal{X}}_{\widetilde{s}} \longrightarrow \mathcal{X}_s \times_S \widetilde{S}
$$

is a resolution of the singularities of $\mathcal{X}_{s,\text{red}}$. If $S = \text{Spec}(k)$ is a point then we recover the usual notion of a resolution $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ of algebraic spaces over k ([62, Tag 0BHV]).

12

Remark 2.2.2. One generally needs to impose some assumptions on the fibers, e.g. geometrically reduced and excellent, in order for resolutions of singularities to exist in the first place. There are also explicit examples of projective maps $\mathcal{X} \to S$ whose simultaneous resolution $\widetilde{\mathcal{X}}$ is *not* a scheme (see Remark 2.2.8), so in general one needs to consider algebraic spaces. In practice, however, f will be a morphism of henselianized (localized) schemes and the schematic fibers will be equipped with the reduced-induced subscheme structure, so we will not need to mention the above assumptions.

Example 2.2.3. We expound on why simultaneous resolutions rarely exist with the following examples (cf. [41], Ex. 4.27). Assume that $f : \mathcal{X} \longrightarrow S$ is flat, where S is a smooth curve with a fixed closed point s, and that the generic fiber of f is smooth..

(i) If f is a family of curves so that $\mathcal{X}_s$ is a reduced singular curve then a simultaneous resolution does not exist; any resolution will introduce an exceptional locus E in $\widetilde{\mathcal{X}}$ and $E \cap \widetilde{\mathcal{X}}_s$ is a singular set in $\widetilde{\mathcal{X}}_s$.

(ii) If f is a general family of varieties so that $\mathcal{X}_s$ has dimension ≥ 3 and is a complete intersection with (at worst) isolated singularities, then $\mathcal{X}$ is $\mathbb{Q}$-factorial ([28], XI.3.13.(ii)) and hence it is well-known that $\mathcal{X}$ admits no small resolutions. Therefore the exceptional locus will be a divisor and $\widetilde{\mathcal{X}}_s$ cannot be smooth as in the previous example.

There are more examples of obstructions to constructing simultaneous resolutions and it is difficult to come up with sufficiency conditions for their existence. In light of this, it is a surprising theorem that, after a ramified base-change, simultaneous resolutions *do* exist for families of surfaces acquiring RDP singularities. The following was independently discovered by Brieskorn and Tjurina, then generalized by Brieskorn in the complex setting and by Artin in the algebraic setting:

Theorem 2.2.4 ([15],[64], [7])**.** *Let $f : \mathcal{X} \longrightarrow S$ be a flat morphism of schemes and s a closed point of S so that $\mathcal{X}_s$ is a surface with a unique RDP $x \in \mathcal{X}_s$. Let $\widehat{f} : \mathcal{X}_{(x)} \longrightarrow S_{(s)}$*

13

be the induced map of the associated henselianized schemes $\mathcal{X}_{(x)}$ and $S_{(s)}$. Then there exists a finite surjection $\psi : \widetilde{S} \longrightarrow S_{(s)}$ of henselian schemes, branched over a Cartier divisor $\Delta \subset S_{(s)}$, so that $\widehat{f}$ admits a simultaneous resolution $\widetilde{f} : \widetilde{\mathcal{X}} \longrightarrow \widetilde{S}$ fitting into the following diagram:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{X}} & \xrightarrow{\pi_1} & \mathcal{X}_{(x)} \times_S \widetilde{S} & \xrightarrow{\pi_2} & X \\
& \widetilde{f} \searrow & \downarrow & \square & \downarrow \widehat{f} \\
& & \widetilde{S} & \xrightarrow{\psi} & S_{(s)}
\end{array}
$$

Here $\widetilde{f}$ is smooth and π_1, π_2 form the Stein factorization of proper map $\pi = \pi_2 \circ \pi_1$. Cartier divisor Δ is called the discriminant divisor (or the ramification locus) of map $\widehat{f}$.

Remark 2.2.5. In the complex-analytic category one can replace the henselizations with $\widehat{f}$ being a map of *germs* of singularities $(\mathcal{X}, x) \to (S, s)$.

Remark 2.2.6. Suppose $\mathcal{X}, S$ are complex schemes and S is affine. Brieskorn's theorems show that the Galois group of the finite cover $\widetilde{S} \to S$ is the Weyl group W of the Dynkin diagram associated to the RDP in $\mathcal{X}_s$ (cf. Theorem 2.1.3). Moreover, the pullback $\psi^* \Delta$ of the discriminant divisor is a hyperplane arrangement in affine space $\widetilde{S}$, determined up to sign by the root system of the Dynkin diagram. These results do *not* follow from the algebraic methods of Artin ([7]), but later results of Shepherd-Barron ([56]) showed that a "suitable polarization" of Artin's simultaneous resolution functor $\mathrm{Res}_{\mathcal{X}/S}$ yields that $\widetilde{S} \to S$ is a so-called Weyl cover in the algebraic setting too.

Example 2.2.7. Let $S = \mathrm{Spec}(\mathcal{O}_K)$ be a strictly henselian trait with residue characteristic $p \neq 2$, uniformizer π, closed point $s = \mathrm{Spec}(k)$ and generic point $\eta = \mathrm{Spec}(K)$. Let $\mathcal{X} = V(x^2 + y^2 + z^2 - \pi^N) \subseteq \mathbb{A}^3_S$ ($N \geq 1$) be a threefold, flat over S, with singular fiber $\mathcal{X}_s$ over k; note the special fiber $\mathcal{X}_s = V(x^2 + y^2 + z^2) \subseteq \mathbb{A}^3_k$ has an ordinary double point at the origin.

Suppose $N = 2n$ is even. Blowing up $\mathcal{X}$ along ideals $(x, z \pm \pi^n)$ gives two different *small* resolutions $\mathcal{X}_\pm \longrightarrow \mathcal{X}$; smallness here is a consequence of both ideals defining non-Cartier

14

Weil divisors on $\mathcal{X}$. Both $\mathcal{X}_-, \mathcal{X}_+$ are smooth over S with generic fibers $(\mathcal{X}_\pm)_\eta \simeq \mathcal{X}_\eta$ over K. Viewing $\mathcal{X}_s$ as a hyperplane section $V(\pi^N) \cap X$ and setting $u = z - \pi^n$, the universal property of blowing up gives

$$(\mathcal{X}_+)_s \simeq B\ell_{(x,u)}(\mathcal{X}) \times_\mathcal{X} V(x^2 + y^2 + u^2) = B\ell_{(x,y,u)}V(x^2 + y^2 + u^2)$$

i.e. $(\mathcal{X}_+)_s \longrightarrow \mathcal{X}_s$ is the minimal resolution of $\mathcal{X}_{s,\mathrm{red}}$, since a single blowup resolves the ordinary double point. In this situation there exists a birational map $f : \mathcal{X}_- \dashrightarrow \mathcal{X}_+$ induced by $(x, z - \pi^n) \to (x, z + \pi^n)$; it is known as the *Atiyah flop*.

Suppose now $N = 2n+1$ is odd. In this case $\mathcal{X}$ does not admit a simultaneous resolution over S (e.g. when $N = 1$, the obstruction is the smoothness of total space $\mathcal{X}$). To repeat the arguments of the previous paragraph we pass to the unique ramified quadratic extension $L = K(\sqrt{\pi})$ of K so that $\mathcal{X}_L = V(x^2 + y^2 + z^2 + \pi_L^{2N})$ now admits small resolutions by blowing up along ideals $(x, z \pm \pi^N)$. We thus have a simultaneous resolution after base-change $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$, ramified over the closed point s.

Remark 2.2.8. Simultaneous resolutions need not exist in the category of schemes when one considers more 'global' contexts of simultaneously resolving projective families. As Artin's example in ([7], p. 330) shows, if $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ is a projective family so that the generic fiber $\mathcal{X}_\eta$ is a quartic K3 surface of geometric Picard rank 1 (such K3 surfaces exist; see [65], Thm. 3.1), and the special fiber X_s a nodal quartic, then localizing at the node we obtain a situation like that of Example 2.2.7 and neither $\mathcal{X}_-$ nor $\mathcal{X}_+$ are schemes. By ([7], Thm. 1) however, simultaneous resolutions of surfaces will be at worst algebraic spaces whose fibers are schemes, since this is true for any smooth 2-dimensional algebraic space.

A second important point is that simultaneous resolutions are generally non-unique, e.g. Example 2.2.7 yields two non-isomorphic resolutions X_-, X_+ related by a flop $X_- \dashrightarrow X_+$.

2.3 Formal deformations of singularities.

In what follows, let $S = \mathrm{Spec}(\mathcal{O}_K)$ be a complete DVR with algebraically closed residue field k and uniformizer π. Let $\mathcal{X} \to S$ be a flat proper surface with special fiber $\mathcal{X}_k$ containing a unique RDP x. We may choose affine coordinates so that the local ring of the singularity has the form

$$R_0 = \widehat{\mathcal{O}}_{\mathcal{X}_k, x} \simeq \frac{k[\![x, y, z]\!]}{f(x, y, z)}$$

with $f(x, y, z) = 0$ the normal form of the rational double point (2.1.3). We aim to describe the possibilities of what $\widehat{\mathcal{O}}_{\mathcal{X}, x}$, the completed local ring of the *model* $\mathcal{X}$ at point x (viewed as a point in $\mathcal{X}$) can look like, by interpreting $\widehat{\mathcal{O}}_{\mathcal{X}, x}$ as an appropriate deformation of the singularity. We next clarify the notion of deformations we will use.

Notation 2.3.1. Write $V_0 = \mathrm{Spec}(R_0)$ for the affine scheme of the singularity, and Art_k resp. $\widehat{\mathrm{Art}}_k$ for the category of artinian local, resp. complete noetherian local rings with residue field k. Both types of rings become canonically $W(k)$-algebras via the unique lift of the natural surjection $W(k) \twoheadrightarrow k$. For R in $\widehat{\mathrm{Art}}_k$ we set

$$R_n = R/\mathfrak{m}_R^{n+1}, \quad \overline{R} = R/\pi R, \quad \overline{R}_n = R/(\pi, \mathfrak{m}_R^{n+1}) \simeq \overline{R}/\mathfrak{m}_{\overline{R}}^{n+1}$$

so that R_n is in Art_k and $\overline{R}$ resp. $\overline{R}_n$ are complete noetherian local, resp. artinian local k-algebras; note $R \simeq \varprojlim R_n$ and $\overline{R} \simeq \varprojlim \overline{R}_n$. We will also call

$$\mathrm{T}_{\mathfrak{m}}(R) = \mathfrak{m}_R/\mathfrak{m}_R^2, \quad \mathrm{T}_{\mathfrak{m}}(R)_{\mathrm{red}} = \mathfrak{m}_R/(\pi, \mathfrak{m}_R^2) \simeq \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2$$

the cotangent, resp. reduced cotangent spaces of $R \in \{\mathrm{Art}_k, \widehat{\mathrm{Art}}_k\}$; they are both naturally

k-vector spaces. To V_0 we associate two deformation functors

$$\mathrm{Def}_{V_0} : \mathrm{Art}_k \to \mathbf{Set}, \quad \widehat{\mathrm{Def}}_{V_0} : \widehat{\mathrm{Art}}_k \to \mathbf{Set}$$

$$R \in \{\mathrm{Art}_k, \widehat{\mathrm{Art}}_k\} \longmapsto \{\text{isoclasses of deformations } V \to \mathrm{Spec}(R)\}$$

The *tangent space* $\mathrm{Def}_{V_0}(k[\varepsilon])$ parametrizes so-called 1st-order deformations of V_0 and has a canonical k-vector space structure. For R in $\widehat{\mathrm{Art}}_k$ we will call a deformation $V \to \mathrm{Spf}(R)$ of V_0 *formal*, with the understanding that it arises as an inverse limit of deformations of V_0 over artinian local rings. We give a preliminary lemma on what such deformations can look like.

Lemma 2.3.2 ([31], Thm. 9.2). *Let V_0/k be an affine hypersurface singularity defined by polynomial $f(x, y, z)$.*

(i) *For any small extension $\phi : R_{i+1} \to R_i$ in Art_k and deformation $V_i = \mathrm{Spec}\big(\frac{R_i[x,y,z]}{I}\big)$ of V_0 we have $I = (F_i)$ principal and any flat lift of V_i to a deformation $V_{i+1} \to \mathrm{Spec}(R_{i+1})$ is of the form*

$$\mathrm{Spec}\Big(\frac{R_{i+1}[x, y, z]}{(F_{i+1})}\Big), \quad F_{i+1} \equiv F_i \mod (\ker \phi)$$

(ii) *If $R \in \widehat{\mathrm{Art}}_k$ then any formal deformation $V \to \mathrm{Spf}(R)$ of V_0 is of the form*

$$\mathrm{Spf}\Big(\frac{R[\![x, y, z]\!]}{(F)}\Big), \quad F \equiv f \mod \mathfrak{m}_R$$

Remark 2.3.3. In plainer language, Lemma 2.3.2 says that deformations of affine hypersurfaces are still hypersurfaces; more generally, it is true that deformations of local complete intersections are also local complete intersections.

To the deformation theory of V_0 we may associate the Lichtenbaum-Schlessinger functors

$T^i(V_0) = \mathrm{Ext}^i(\Omega_{R_0}, R_0)$, where $T^0(V_0)$ is just the tangent module of R_0. It is known ([66], Prop. 6.4) that $T^1(V_0)$ parametrizes first-order deformations of V_0 and $T^2(V_0)$ contains an obstruction space for their liftings. By the explicit description of V_0 we can be more precise:

Lemma 2.3.4. *We have* $\mathrm{Def}_{V_0}(k[\varepsilon]) \simeq \mathrm{Ext}^1(\Omega_{R_0}, R_0) \simeq R_0/J$ *where* $J = (f_x, f_y, f_z)$ *is the Jacobian ideal, and* $\mathrm{Ext}^2(\Omega_{R_0}, R_0) = 0$. *In particular, deformations of* V_0 *are unobstructed.*

Proof. The first isomorphism is ([53], Thm. 2.4.1 (iv)). The conormal sequence associated to $V_0 \hookrightarrow \mathbb{A}^3_k$ is

$$0 \longrightarrow (f)/(f)^2 \longrightarrow \Omega_{\mathbb{A}^3_k} \otimes R_0 \longrightarrow \Omega_{R_0} \longrightarrow 0$$

which is exact on the left by ([66], Lemma 4.7) and so dualizing we get exact sequence

$$0 \longrightarrow \Omega^\vee_{R_0} \longrightarrow (\Omega_{\mathbb{A}^3_k} \otimes R_0)^\vee \xrightarrow{d^*} ((f)/(f)^2)^\vee \xrightarrow{\partial} \mathrm{Ext}^1(\Omega_{R_0}, R_0) \longrightarrow 0 \qquad (2.3.1)$$

with ∂ surjective as $\Omega_{\mathbb{A}^3_k} \otimes R_0$ is free over R_0. The map $(f)/(f)^2 \to \Omega_{\mathbb{A}^3_k} \otimes R_0$ is given by the Jacobian matrix $f \mapsto df$ and $(f)/(f)^2 \simeq R_0$ is free of rank 1, hence the adjoint map d^* has image $J = (f_x, f_y, f_z)$. So via the boundary map ∂ we get $\mathrm{Ext}^1(\Omega_{R_0}, R_0) \simeq R_0/J$ and, as a byproduct of exact sequence (2.3.1), $\mathrm{Ext}^2(\Omega_{R_0}, R_0) = 0$. $\qquad\square$

Definition 2.3.5. The k-vector space

$$T^1(V_0) = R_0/J \simeq \frac{k[\![x, y, z]\!]}{(f, f_x, f_y, f_z)}$$

is called the *Tjurina algebra* of the singularity in V_0. It has *finite* dimension since $V(f, f_x, f_y, f_z)$ is supported on the unique singular point - in general, for isolated singularities, the Tjurina algebra is finite-dimensional.

2.4 The Kodaira–Spencer map.

This section follows ([66], §6). Given a formal deformation $\phi : V \to \mathrm{Spf}(R)$ of V_0, we aim to attach to it a k-linear map $\mathrm{T}_{\mathfrak{m}}(R)^{\vee}_{\mathrm{red}} \to \mathrm{T}^1(V_0)$ defined in the following steps.

An element of $\mathrm{T}_{\mathfrak{m}}(R)^{\vee}_{\mathrm{red}}$ corresponds to a map $R \to k[\varepsilon]$ which, by $W(k)$-linearity and $\varepsilon^2 = 0$, factors through a map $R \to \overline{R}_1$; write $\overline{\phi}_1$ for the base-change of ϕ along this map. Set

$$\overline{\phi}_0 : V_0 \times_k \mathrm{Spec}(\overline{R}_1) \longrightarrow \mathrm{Spec}(\overline{R}_1)$$

to be the trivial deformation of V_0 over $\overline{R}_1$ so that $\overline{\phi}_0, \overline{\phi}_1 \in \mathrm{Def}_{V_0}(\overline{R}_1)$. By Lemma 2.3.2, both $\overline{\phi}_0, \overline{\phi}_1$ embed as hypersurfaces in $\mathbb{A}^3_{\overline{R}_1}$, defined by ideals $I_0, I_1 \subset \overline{R}_1[\![x,y,z]\!]$ respectively. If $I = (f)$ is the ideal of V_0 in $k[\![x,y,z]\!]$, we may lift a section $F \in I$ to sections $F_0 \in I_0, F_1 \in I_1$. By virtue of the square-zero extension

$$0 \longrightarrow \mathfrak{m}_{\overline{R}_1} \longrightarrow \overline{R}_1 \longrightarrow k \longrightarrow 0$$

the difference $F_0 - F_1$ lies in $\mathfrak{m}_{\overline{R}_1} \otimes_k k[\![x,y,z]\!]$, and its image $[F_0 - F_1]$ under $k[\![x,y,z]\!] \twoheadrightarrow R_0$ depends only on the choice of F. Hence $F \mapsto [F_0 - F_1]$ yields an element

$$\nu \in \mathrm{Hom}_{k[\![x,y,z]\!]}(I, \mathfrak{m}_{\overline{R}_1} \otimes_k R_0) \simeq \mathfrak{m}_{\overline{R}_1} \otimes_k \mathrm{Hom}_{R_0}(I/I^2, R_0) = \mathfrak{m}_{\overline{R}_1} \otimes_k ((f)/(f)^2)^{\vee}$$

Homomorphism ν is well-defined in general for any two deformations of V_0 over an artinian R; see ([66], Prop. 2.8) for more details. We will only be using it in the following definition.

Definition 2.4.1. The *Kodaira–Spencer* map is a k-linear map $\mathrm{KS}_\phi : \mathrm{T}_{\mathfrak{m}}(R)^{\vee}_{\mathrm{red}} \longrightarrow \mathrm{T}^1(V_0)$ defined as the image of homomorphism ν described above under map

$$\mathrm{id} \otimes \partial : \mathfrak{m}_{\overline{R}_1} \otimes_k ((f)/(f)^2)^{\vee} \longrightarrow \mathfrak{m}_{\overline{R}_1} \otimes_k \mathrm{Ext}^1(\Omega_{R_0}, R_0) \simeq \mathrm{Hom}_k(\mathrm{T}_{\mathfrak{m}}(R)^{\vee}_{\mathrm{red}}, \mathrm{T}^1(V_0))$$

where ∂ is the boundary map induced from the conormal exact sequence (2.3.1) of $V_0 \hookrightarrow \mathbb{A}_k^3$.

By construction, map ν depends only on ϕ, hence the Kodaira–Spencer morphism depends only on ϕ - it is well-defined by ([66], Prop. 4.11) and satisfies various functorial properties (*loc. cit.*, Prop. 6.10). We note the following:

Proposition 2.4.2 ([66], Prop. 6.11). *Let $\phi : V \to \mathrm{Spf}(R)$ be a formal deformation of V_0 and for $\alpha \in (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee \simeq \mathrm{Hom}(R, k[\varepsilon])$ let $f_\alpha : R \to k[\varepsilon]$ be the corresponding map. Then the induced 1st-order deformation $f_\alpha^* \phi : V_1 = V \times_R \mathrm{Spec}(k[\varepsilon]) \to \mathrm{Spec}(k[\varepsilon])$ defines a conormal exact sequence via $V_0 \hookrightarrow V_1$, and $\mathrm{KS}_\phi(\alpha) \in \mathrm{T}^1(V_0)$ is this extension class.*

2.5 Miniversal deformations in mixed characteristic.

We can now describe a class of deformations of V_0 from which all others are induced in a certain 'minimal' way. Recall that if $\phi : V \to \mathrm{Spf}(R)$ is a formal deformation of V_0 and $h_R = \mathrm{Hom}(R, -) : \widehat{\mathrm{Art}}_k \to \mathbf{Set}$ is the Hom functor of R, then ϕ induces a natural transformation

$$h_R \to \widehat{\mathrm{Def}}_{V_0}$$

Definition 2.5.1.

(i) ϕ is *versal* when this natural transformation is formally smooth, i.e. if $B \twoheadrightarrow A$ in $\widehat{\mathrm{Art}}_k$ then the map

$$h_R(B) \to h_R(A) \times_{\widehat{\mathrm{Def}}_{V_0}(A)} \widehat{\mathrm{Def}}_{V_0}(B)$$

induced from the obvious commutative diagram is a surjection.

(ii) ϕ is *miniversal* if it is versal and the Kodaira–Spencer map KS_ϕ is an isomorphism.

By Lemma 2.3.4 and the definitions in Notation 2.3.1, it is clear that this definition of miniversality coincides with the more common one (see e.g. [31], §15). We also note that

versality is usually stated in terms of artinian local rings, but one can deduce versality in the above (more general) case from the artinian one ([66], Lemma 7.3).

Proposition 2.5.2 (Schlessinger criteria, [31] Thms. 16.2, 18.1)**.** Def_{V_0} *admits a miniversal deformation if and only if the tangent space* $\mathrm{Def}_{V_0}(k[\varepsilon])$ *is finite-dimensional. In particular there exists a miniversal deformation when* V_0 *has isolated singularities.*

Proposition 2.5.3. *If* $\phi : V \to \mathrm{Spf}(R)$ *is a formal deformation of* V_0 *such that* R *is a power series algebra over* $W(k)$ *and the Kodaira-Spencer map* KS_ϕ *is an isomorphism, then* ϕ *is miniversal.*

Proof. Suppose $R = W(k)[\![t_1, \cdots, t_n]\!]$ and KS_ϕ is an isomorphism. By Proposition 2.5.2 there exists a miniversal formal deformation $\psi : W \to \mathrm{Spf}(S)$, yielding by versality a homomorphism $f : S \to R$ and hence a map $df : \mathrm{T}_{\mathfrak{m}}(S)_{\mathrm{red}} \to \mathrm{T}_{\mathfrak{m}}(R)_{\mathrm{red}}$, dual to the differential map of reduced tangent spaces. Since $\mathrm{KS}_\phi, \mathrm{KS}_\psi$ are isomorphisms and $\mathrm{KS}_\psi = \mathrm{KS}_\phi \circ df$ by the functoriality of the Kodaira–Spencer construction ([66], Prop. 6.10 (b)), df is an isomorphism.

It remains to show f is an isomorphism. We will use df to construct a map $g : R \to S$ so that the dual differentials of (non–reduced) cotangent spaces

$$D(f \circ g) : \mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2, \quad D(g \circ f) : \mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2 \qquad (2.5.1)$$

are *surjective*, then by functoriality of differentials ([66], Lemma 7.5) $f \circ g$ and $g \circ f$ will be isomorphisms so that in particular f is an isomorphism.

To construct g: Observe $\mathrm{T}_{\mathfrak{m}}(R)_{\mathrm{red}} \simeq k[t_1, \cdots, t_n]/(t_1, \cdots, t_n)^2$ has generators $\bar{t}_1, \cdots, \bar{t}_n$. As df is an isomorphism, we can pick basis $\{\bar{g}_i, \cdots, \bar{g}_n\}$ of $\mathfrak{m}_S/(\pi, \mathfrak{m}_S^2)$ so that $df(\bar{g}_i) = \bar{t}_i$ and lift them to a set of representatives $g_i \in \mathfrak{m}_S$. On artinian quotients we define

$$g_m : R_m = \frac{W(k)[t_1, \cdots, t_n]}{(\pi, t_1, \cdots, t_n)^m} \longrightarrow S_m, \quad \bar{t}_i^{(m)} \longmapsto \bar{g}_i^{(m)}$$

21

where $\bar{t}_i^{(m)}, \bar{g}_i^{(m)}$ denote the images of t_i, g_i under $R \to R_m, S \to S_m$ respectively. Since $R \simeq \varprojlim R_m$, $S \simeq \varprojlim S_m$ this defines a homomorphism $g = \varprojlim g_m : R \to S$ mapping $t_i \mapsto g_i$, which by construction induces a surjection $dg : \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \twoheadrightarrow \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2$. In turn dg induces surjections $\mathfrak{m}_{\overline{R}}^m/\mathfrak{m}_{\overline{R}}^{m+1} \twoheadrightarrow \mathfrak{m}_{\overline{S}}^m/\mathfrak{m}_{\overline{S}}^{m+1}$ and as the source and target of this map are respectively the kernels of $\overline{R}_{m+1} \to \overline{R}_m$ and $\overline{S}_{m+1} \to \overline{S}_m$, we inductively get surjections $\overline{R}_m \twoheadrightarrow \overline{S}_m$; the base case is $\overline{R}_1 \simeq k \oplus \mathfrak{m}_{\overline{R}}/\mathfrak{m}_{\overline{R}}^2 \twoheadrightarrow \overline{S}_1 \simeq k \oplus \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2$. Since $\overline{R}_m = R_m/\pi R_m$, by Nakayama's lemma we get that the lifts $g_m : R_m \to S_m$ are surjective, and since R_m, S_m are artinian local rings we get $g = \varprojlim g_m$ surjective, hence the dual differential $Dg : \mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective. Repeating the exact same argument for $f : S \to R$ yields a surjection $Df : \mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_R/\mathfrak{m}_R^2$. Thus the maps in (2.5.1) are surjective and we are done. $\qquad\square$

We now come to the crux of this section: describing the miniversal deformations of V_0.

Proposition 2.5.4 (see [66], Example 7.17). *Let* $r = \dim_k \mathrm{T}^1(V_0)$, $R = W(k)[\![t_1, \cdots, t_r]\!]$ *and choose elements* $g_1, \cdots, g_r$ *in* $W(k)[\![x, y, z]\!]$ *so that* $\{\bar{g}_1, \cdots, \bar{g}_r\}$ *forms a basis of* $\mathrm{T}^1(V_0)$. *Consider the hypersurface* $V = V(F) \subset \mathbb{A}_R^3$ *defined by the vanishing of power series*

$$F(x, y, z, t_1, \cdots, t_r) = f(x, y, z) + \sum_{i=1}^r t_i g_i(x, y, z) \in W(k)[\![x, y, z, t_1, \cdots, t_r]\!] \qquad (2.5.2)$$

Then $V \to \mathrm{Spec}(R)$ *induces a miniversal deformation of* V_0.

Proof. In view of Proposition 2.5.3, it suffices to show the Kodaira–Spencer map KS_ϕ is an isomorphism; we may set $n = r$ in the proof of Proposition 2.5.3 since n there was defined to be the dimension of $\mathrm{T}_\mathfrak{m}(R)_{\mathrm{red}} \simeq \mathrm{T}^1(V_0)$. Let $\mathrm{T}_\mathfrak{m}(R)_{\mathrm{red}} = \langle \bar{t}_1, \cdots, \bar{t}_r \rangle \simeq \mathfrak{m}_{\overline{R}_1}$ for $\bar{t}_i$ the images of indeterminates $t_i \mod (\pi, \mathfrak{m}_R^2)$ and by abuse of notation denote the (dual) basis of $\mathrm{T}_\mathfrak{m}(R)_{\mathrm{red}}$ also by $\bar{t}_i$, then we claim $\mathrm{KS}_\phi(\bar{t}_i) = \bar{g}_i$ so that KS_ϕ maps a basis to a basis and hence is a k-linear isomorphism.

Now Proposition 2.4.2 says that $\bar{t}_i$ corresponds to a map $f : \overline{R}_1 \to k[\varepsilon]$ mapping $\bar{t}_i \mapsto \varepsilon$, $\bar{t}_{j \neq i} \mapsto 0$ via duality on k-vector space $\mathfrak{m}_{\overline{R}_1}$ and $\mathrm{KS}_\phi(\bar{t}_i)$ is the extension class in $\mathrm{T}^1(V_0)$

22

of the pullback of deformation ϕ via $R \to \overline{R}_1 \to k[\varepsilon]$ to the 1st-order deformation $f^*\phi$, i.e.

$$\mathrm{Spec}\Big(\frac{W(k)[\![x,y,z,t_1,\cdots,t_r]\!]}{f(x,y,z)+\sum_{i=1}^{r}t_i g_i(x,y,z)}\otimes_f k[\varepsilon]\Big) \simeq \mathrm{Spec}\Big(\frac{k[\varepsilon,x,y,z]}{f(x,y,z)+\varepsilon\widetilde{g}_i(x,y,z)}\Big) \xrightarrow{f^*\phi} \mathrm{Spec}(k[\varepsilon])$$

where $\widetilde{g}_i$ is the image of g_i in $k[\varepsilon,x,y,z]$. To compute this extension class we trace through the definition of the Kodaira-Spencer map: we have two first-order deformations, $\overline{\phi}_1 = f^*\phi$ and the trivial deformation $\overline{\phi}_0 : V_0 \times_k \mathrm{Spec}(k[\varepsilon]) \to \mathrm{Spec}(k[\varepsilon])$, with corresponding lifts of $f \in (f) \subset k[x,y,z]$ being $f + \varepsilon\widetilde{g}_i$ and f, viewed as elements of $k[\varepsilon,x,y,z]$. Then map ν in the definition of the Kodaira–Spencer map sends $f \mapsto [f + \varepsilon\widetilde{g}_i - f] = [\varepsilon\widetilde{g}_i] \in (\varepsilon) \otimes_k R_0$, hence in the quotient $(\varepsilon) \otimes_k R_0 \to R_0 \to R_0/J = \mathrm{T}^1(V_0)$ (where J is the Jacobian ideal) we get exactly the class $\overline{g}_i$. Thus $\mathrm{KS}_\phi(\overline{t}_i) = \overline{g}_i$. $\qquad\square$

Example 2.5.5. Let R_0 be the local ring of an A_{n-1} singularity $f(x,y,z) = x^2 + z^2 + y^n$. A basis of the Tjurina algebra in good characteristic (see Definition 2.1.4) is given by

$$\frac{k[\![x,y,z]\!]}{(f,f_x,f_y,f_z)} \simeq \frac{k[\![y]\!]}{y^{n-1}} \simeq \bigoplus_{i=0}^{n-2} k \cdot y^i$$

so that $\mathrm{T}^1(\mathrm{Spec}(R_0))$ has dimension $r = n - 1$ and we may choose $g_i(x,y,z) = y^{i-1}$. Then

$$F(x,y,z) = x^2 + z^2 + y^n + t_{n-1}y^{n-2} + \cdots + t_2 y + t_1$$

is a miniversal deformation of $\mathrm{Spec}(R_0)$ with base $W(k)[\![t_1,\cdots,t_{n-1}]\!]$.

We return to the setting in the beginning of Section 2.3, namely $\mathcal{X}/S$ is a flat proper surface with special fiber $\mathcal{X}_k$ containing an RDP $x \in \mathcal{X}_k(k)$.

Corollary 2.5.6. *For $\widehat{\mathcal{O}}_{\mathcal{X},x}$ the completed local ring of $\mathcal{X}$ at $x \to \mathcal{X}_k \subset \mathcal{X}$ we have*

$$\widehat{\mathcal{O}}_{\mathcal{X},x} \simeq \frac{W(k)[\![x,y,z]\!]}{F(x,y,z)} \tag{2.5.3}$$

for some polynomial $F(x, y, z)$ that is the pullback of (2.5.2) *under $W(k)[\![t_1, \cdots, t_r]\!] \to W(k)$ induced by versality. Here $f(x, y, z)$ is the normal form of singularity x, r is the dimension of the Tjurina algebra of the singularity and the t_i are specialized to elements in the maximal ideal $\mathfrak{m}_{W(k)} = (\pi)$ of $W(k)$.*

Proof. This is immediate from Proposition 2.5.4 once we establish (2.5.3) for some power series $F(x, y, z) \in W(k)[\![x, y, z]\!]$, which is the content of Lemma 2.3.2. $\qquad\square$

Remark 2.5.7. While a priori we speak of miniversal deformations as maps of formal schemes $V \to \mathrm{Spf}(R)$, we can regard the miniversal deformation of Proposition 2.5.4 as an algebraic deformation over henselian scheme $\mathrm{Spec}(W(k)[\![t_1, \cdots, t_r]\!])$. This is more generally due to a theorem of Elkik which states that formal deformations of affine schemes with isolated singularities are algebraic ([24]). Thus, by replacing X with an affine étale neighborhood of the singularity in Corollary 2.5.6, we may and do consider the (miniversal or otherwise) deformations of RDPs as usual scheme morphisms over a henselian base.

CHAPTER 3

THE GEOMETRY OF THE GROTHENDIECK ALTERATION

We now collect the necessary Lie-theoretic prerequisites in order to define miniversal deformations of RDP singularities in terms the adjoint quotient of Lie algebras. The main goal is to define and study the classical Grothendieck–Springer resolution over base $\mathrm{Spec}(\mathcal{O}_K)$ instead of $\mathbb{C}$ (Section 3.5). To this end, Sections 3-3.3 extend the relevant notions of nilpotent, semisimple and (sub)regular elements to the setting of Chevalley algebras over $\mathcal{O}_K$.

3.1 Chevalley bases and Chevalley algebras.

Throughout this section we work over base $S = \mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is a mixed-characteristic complete DVR with algebraically closed residue field k and fraction field K; the corresponding closed and generic points of S are respectively s and η.

We recall the existence of Chevalley bases. Let $\mathfrak{g}_{\overline{\eta}}$ be a semisimple Lie algebra over algebraically closed field $\overline{K}$ of characteristic zero, $\mathfrak{h}$ a fixed Cartan subalgebra and Φ the corresponding root system with a basis Δ of simple roots. Via the Cartan decomposition

$$\mathfrak{g}_{\overline{\eta}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha,\overline{\eta}}$$

one may choose $\{e_\alpha \in \mathfrak{g}_{\alpha,\overline{\eta}} \mid \alpha \in \Phi\}$ forming a $\mathbb{Z}$-basis for each 1-dimensional space $\mathfrak{g}_{\alpha,\overline{\eta}}$ and $\{h_\beta \in \mathfrak{h} \mid \beta \in \Delta\}$ fundamental coroots subject to certain compatibility relations ([34], §25.2). Set $\{e_\alpha, h_\beta\}$ forms a *Chevalley basis* with corresponding Chevalley $\mathbb{Z}$-algebra

$$\mathfrak{g}_{\mathbb{Z}} = \bigoplus_{\beta \in \Delta} \mathbb{Z}h_\beta \oplus \bigoplus_{\alpha \in \Phi} \mathbb{Z}e_\alpha$$

One can also construct an associated group scheme G over $\mathbb{Z}$, the *Chevalley group*, playing the role of the Lie group of $\mathfrak{g}_{\mathbb{Z}}$; ([34], §25.4) discusses the adjoint case but we may always

take G to be the simply-connected cover of the adjoint group. We may further base-change to $\mathcal{O}_K$ so that $\mathfrak{g}$ becomes a free $\mathcal{O}_K$-module and G is a group scheme over S. We call $\mathfrak{g}$ a Chevalley algebra of ADE type if $\mathfrak{g}_{\overline{\eta}}$ is simple of ADE type.

From now on we fix a simple, simply-connected, split Chevalley group scheme G/S and a torus and Borel $T \hookrightarrow B$. We have $B = T \rtimes \mathcal{R}_u(B)$ where $\mathcal{R}_u(B)$ is the unipotent radical, a smooth normal subgroup S-scheme in B, and so $T \simeq B/\mathcal{R}_u(B)$. We write $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}$ and $\mathfrak{n}_\mathfrak{b}$ respectively for the Chevalley algebras of G, T, B and $\mathcal{R}_u(B)$, so that $\mathfrak{h}, \mathfrak{b}, \mathfrak{n}_\mathfrak{b}$ form respectively a Cartan, a Borel and the nilradical of the Borel.

One still has a notion of the adjoint action of G on $\mathfrak{g}$ ([19], §5.1) and hence the adjoint action of T decomposes $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \tag{3.1.1}$$

where each $\mathfrak{g}_\alpha$ is a rank 1 free $\mathcal{O}_K$-module and Φ consists of characters $\alpha : T \to \mathbb{G}_m$.

3.2 Root data and adjoint Weyl actions.

Retaining the assumptions and notations of Section 3.1, let T/S be a maximal torus of split group scheme G and let $r = \dim_S(T) = \mathrm{rk}_S(\mathfrak{g})$ be the rank of the associated Chevalley algebra $\mathfrak{g}$. By the split hypothesis there exists a free $\mathbb{Z}$-module $X^*(T)$ of rank r so that

$$T \simeq \underline{\mathrm{Hom}}(X^*(T), \mathbb{G}_m) \otimes_\mathbb{Z} \mathcal{O}_K, \quad \text{i.e.} \quad \mathfrak{h} \simeq \mathrm{Lie}(T) \simeq X^*(T)^\vee \otimes_\mathbb{Z} \mathcal{O}_K$$

Set $X_*(T) = X^*(T)^\vee$ for the $\mathbb{Z}$-dual. The Cartan decomposition of $\mathfrak{g}$ (see Equation (3.1.1)) yields a set $\Phi \subset X^*(T) \setminus \{0\}$ of roots $\alpha : T \to \mathbb{G}_{m, \mathcal{O}_K}$ and a corresponding set of coroots $\alpha^\vee \in \Phi^\vee \subset X_*(T) \setminus \{0\}$.

Definition 3.2.1. The 4-tuple $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a *root datum* for (G, T) and the

quotient $W = W_G(T) = N_G(T)/T$ is the *Weyl group* associated to (G, T).

See ([19], Prop. 5.1.6) for a proof that the 4-tuple in Definition 3.2.1 satisfies the conditions of being a root datum; the Weyl group W is identified with the usual Weyl group associated to Φ and is in particular generated by the set of reflections

$$\{s_\alpha = \mathrm{id} - \alpha^\vee \otimes_\mathbb{Z} \alpha \mid \alpha \in \Phi\}$$

From this description it follows that W is a finite and constant group scheme over S.

We record here two natural actions of the Weyl group on affine spaces, which will be useful for us later on.

Definition 3.2.2. Let W be the Weyl group associated to (G, T) and $\mathfrak{h} = \mathrm{Lie}(T)$.

(i) The adjoint action $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ restricts to an action $N_G(T) \times \mathfrak{h} \to \mathfrak{h}$, since $\mathrm{Ad}_g(h) \in \mathfrak{h}$ for $g \in N_G(T)(\mathcal{O}_K)$ and $h \in \mathfrak{h}(\mathcal{O}_K)$. In particular the adjoint T-action on $\mathfrak{h}$ is trivial since T is abelian, and so the adjoint action descends to the *adjoint Weyl action* $W \to \mathrm{End}(\mathfrak{h})$, $w \longmapsto \mathrm{Ad}_{n_w}$ for a lift $n_w \in N_G(T)$ of $w \in W$. As G is simple, we may identify $\mathfrak{h} \simeq \mathfrak{h}^\vee \simeq X^*(T) \otimes_\mathbb{Z} \mathcal{O}_K$ so that the adjoint Weyl action is identified with the natural reflection action of W on $X^*(T) \otimes_\mathbb{Z} \mathcal{O}_K$.

(ii) There natural action of W on G/T via $(gT) \cdot w = gn_w T$ for a lift $n_w \in N_G(T)$ is the *right multiplication action* of W. This also yields a natural action on $G/T \times_S \mathfrak{h}$ via $(gT, h) \cdot w = (gn_w T, \mathrm{Ad}_{n_w^{-1}}(h))$, which we will make use of later (see Proposition 3.5.3).

Remark 3.2.3 (Good primes and torsion primes). Given a reduced root system $\Phi \subset X^*(T)$, a prime p is said to be *torsion* for a simply-connected Lie group G if there exists a $\mathbb{Z}$-closed root subsystem $\Phi' \subseteq \Phi$ so that the quotient of $\mathbb{Z}$-lattices $\mathbb{Z}\Phi^\vee/\mathbb{Z}(\Phi')^\vee$ has p-torsion. An equivalent definition of a good prime p is that there is no $\mathbb{Z}$-closed root subsystem $\Phi' \subseteq \Phi$ so that $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ has p-torsion, i.e. the good primes are exactly the non-torsion primes ([57],

27

§3.13). Similarly, a prime p is very good if it is good and p does not divide $|(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi|$. See (*loc. cit.*, §3.6) for a table of the values of $|(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi|$ in the case of $\mathfrak{g}$ simple simply-laced; it turns out that the only good but not very good case of prime p occurs when $p \mid n+1$ and $\mathfrak{g}$ is of Type A_n.

3.3 Nilpotent and semisimple elements of Chevalley algebras.

We now come to the definition of the nilpotent scheme and nilpotent sections. Recall that, when $\mathfrak{g}$ is a Lie algebra over k, the nilpotent variety $\mathcal{N}_\mathfrak{g}$ is a reduced closed subscheme of $\mathfrak{g}$ that is the Zariski closure of the nilpotent elements, and by Galois descent $\mathcal{N}_\mathfrak{g}$ is also well-defined over non-algebraically closed fields.

Proposition 3.3.1 ([21], Thm. 4.12)**.** *For a Chevalley $\mathcal{O}_K$–algebra $\mathfrak{g}$, there exists a unique closed S-subscheme $\mathcal{N}_\mathfrak{g}^{\mathrm{sch}} \subset \mathfrak{g}$ that is reduced and $(\mathcal{N}_\mathfrak{g}^{\mathrm{sch}})_{\bar{s}} \simeq \mathcal{N}_{\mathfrak{g}_{\bar{s}}}$ for geometric points $\bar{s} \to S$. Here the right-hand side denotes the usual nilpotent variety over a field.*

Proof. Existence is shown more generally in ([19], Thms. 4.6 and 4.12) so we only discuss uniqueness in our particular case. Suppose X, Y are closed reduced S–subschemes in $\mathbb{A}_S$ so that on geometric fibers $Y_s \simeq X_s \hookrightarrow \mathbb{A}_k^n$ and both X_s, Y_s are reduced, then we claim $X \simeq Y$. The Zariski closure $\overline{X}_\eta$ of X_η is reduced as X_η is an open reduced subscheme of X, hence so is $X' = \overline{X}_\eta \cup X_s$. Since both X', X are reduced closed subschemes of $\mathbb{A}_S^n$ with the same points, $X' = X$. A similar argument for Y yields

$$Y = \overline{Y}_\eta \cup Y_s \simeq \overline{X}_\eta \cup X_s = X$$

$\square$

Since nilpotent schemes behave well under base–change, we will refer to the nilpotent scheme $\mathcal{N}_\mathfrak{g}^{\mathrm{sch}}$ as $\mathcal{N}_\mathfrak{g}$ for any Chevalley $\mathcal{O}_K$-algebra. We can also define:

Definition 3.3.2. An $\mathcal{O}_K$–valued section $x \in \mathfrak{g}(\mathcal{O}_K)$ is *fiberwise nilpotent* if $x_s \in \mathcal{N}_\mathfrak{g}(k)$ and $x_{\overline{\eta}} \in \mathcal{N}_\mathfrak{g}(\overline{K})$. Equivalently x is an $\mathcal{O}_K$–valued section of $\mathcal{N}_\mathfrak{g}$.

For $x \in \mathfrak{g}(\mathcal{O}_K)$, we can define the centralizer $C_G(x)$ as follows. Through the adjoint action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ we obtain a functor $\mathcal{C}_G(x)$ so that on $\mathcal{O}_K$–algebras R,

$$\mathcal{C}_G(x)(R) = \{g \in G(R) \mid \mathrm{Ad}_g(x) = x\}$$

By ([21], Lemma 2.1), $\mathcal{C}_G(x)$ is represented by a closed S–subgroup scheme of G, which we denote by $C_G(x)$; over algebraically closed fields, the reduced scheme underlying $C_G(x)$ is the usual centralizer.

Definition 3.3.3. A section $x \in \mathfrak{g}(\mathcal{O}_K)$ is *regular* if its centralizer subscheme $C_G(x)$ satisfies $\dim(C_G(x)_s) = \dim(C_G(x)_{\overline{\eta}}) = r$ where $r = \mathrm{rank}(G)$. By upper-semicontinuity of fiber dimension for group schemes, it suffices to have $\dim(C_G(x)_s) = r$ since $\dim(C_G(x)) \geq r$ for semisimple G (see [35], §1.6).

Definition 3.3.4. For a Lie algebra $\mathfrak{g}$ over an algebraically closed field we call $x \in \mathfrak{g}$ *subregular* if $\dim(C_G(x)) = r + 2$. For non-regular elements x one has $\dim(C_G(x)) \geq r + 2$ so that subregular elements are are the next "closest" to being regular; see ([35], §4.11) for more details.

If $\mathfrak{g}$ is now a Chevalley $\mathcal{O}_K$–algebra, a nilpotent section $x \in \mathcal{N}_g(\mathcal{O}_K)$ is called *fiberwise subregular* if both $x_s \in \mathcal{N}_\mathfrak{g}(k)$, $x_{\overline{\eta}} \in \mathcal{N}_\mathfrak{g}(\overline{K})$ are subregular nilpotent elements; if we do not require x to be (fiberwise) nilpotent then subregularity in this setting may not make sense (cf. Example 3.3.6).

Definition 3.3.5 ([14], §4.1.5, [1] Exposé XIV.2)**.** An element $x \in \mathfrak{g}(\mathcal{O}_K)$ is called *fiberwise semisimple* if $x_{\overline{s}} \in \mathfrak{g}(k(\overline{s}))$ is semisimple for all geometric points $\overline{s} \to S$. An element $x \in \mathfrak{g}(\mathcal{O}_K)$ is called *regular semisimple* if it lies in some Cartan $\mathfrak{h} \subset \mathfrak{g}$ and for all geometric

points $\bar{s}$ we have

$$\mathfrak{h}_{\bar{s}} = \bigcup_{n \geq 0} (\ker(\mathrm{ad}^n_{x_{\bar{s}}}))$$

This union may be thought of as the centralizer of x so that it has fiberwise dimension equal to $\mathrm{rank}(\mathfrak{g}_{\bar{s}})$. Regular semisimple elements form an open set $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}$.

By ([1], Exposé XIV.2), set $\mathfrak{g}^{\mathrm{rs}}$ is S–fiberwise dense in $\mathfrak{g}$ and its construction commutes with base-change, whence it gives the usual notion of regular semisimple elements of Lie algebras over a field, i.e. $x \in \mathfrak{g}_k$ so that $C_G(x)^{\circ}$ is a maximal torus in G_k. This also makes the density of the semisimple locus in $\mathfrak{g}$ apparent.

Similar methods as in ([21], Lemma 2.1) yield that the functor of regular sections in $\mathfrak{g}$ is represented by an open subscheme $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$ over S. In the case of good characteristic, ([14], Lemma 4.1.6) gives $\mathfrak{g}^{\mathrm{rs}} \subset \mathfrak{g}^{\mathrm{reg}}$ so that both are S–fiberwise dense in $\mathfrak{g}$.

Example 3.3.6. We discuss a couple of pathologies that may occur in Chevalley algebras, which justify the various "fiberwise" conditions in the previous definitions.

(i) The subregular notion need not behave well in fibers. Let

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{sl}_3(\mathcal{O}_K), \quad p = \mathrm{char}(k) > 3$$

then $x_{\bar{\eta}} \in \mathfrak{sl}_3(k(\bar{\eta}))$ is regular nilpotent but special fiber $x_s \in \mathfrak{sl}_3(k)$ is subregular. In general we get that x stays in the same (adjoint) nilpotent orbit if it satisfies a purity assumption in the sense of [20], i.e. constant centralizer dimension on the fibers — see (*loc. cit.*, Prop. 5.10).

(ii) Regular semisimple elements (see Definition 3.3.5 below) can become nilpotent regular

or non-regular elements. Let $p > 2$ and

$$x = \begin{pmatrix} p & 1 \\ 0 & -p \end{pmatrix}, \quad y = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \quad (\text{in } \mathfrak{sl}_2(\mathcal{O}_K))$$

then $x_{\bar{\eta}}, y_{\bar{\eta}}$ are regular semisimple but x_s is regular nilpotent and y_s is subregular nilpotent. For x, even though $\dim_{\mathcal{O}_K}(C_G(x))$ is locally constant, centralizer $C_G(x)$ is not flat.

3.4   The adjoint quotient.

This section and the next are based on some Lie-theoretic observations in [52] and ([14], §4), which hold in greater generality than base scheme $S = \mathrm{Spec}(\mathcal{O}_K)$. In our situation however we may provide simplified proofs and constructions suitable to our purposes.

We retain the notations and assumptions of Section 3.1 for Chevalley group G and Chevalley algebra $\mathfrak{g}$. As an integral scheme, $\mathfrak{g}$ has coordinate ring $\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee)$. The adjoint action $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ yields the categorical quotient $\mathfrak{g}/\!\!/G = \mathrm{Spec}((\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee))^G)$, whose coordinate ring consists of the adjoint G–invariants. Since the restriction of the adjoint action of G to $\mathfrak{h}$ factors through the Weyl group $W = N_G(T)/T$, we get a map $\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee)^G \to \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{h}^\vee)^W$; its schematic version is the *Chevalley map* $\mathfrak{h}/\!\!/W \to \mathfrak{g}/\!\!/G$.

The natural inclusions $\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee)^G \hookrightarrow \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee)$ and $\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{h}^\vee)^W \hookrightarrow \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{h}^\vee)$ yield categorical quotient maps

$$\psi : \mathfrak{h} \longrightarrow \mathfrak{h}/\!\!/W, \quad \chi : \mathfrak{g} \longrightarrow \mathfrak{g}/\!\!/G$$

Map ψ is a finite branched cover, and χ is known as the *adjoint quotient*. For the properties of the counterparts of these morphisms over algebraically closed fields, we refer to [57], §3.10, §3.12 and §3.14. We note here that, by virtue of the Jordan decomposition, two

31

elements $x_1, x_2 \in \mathfrak{g}(k)$ have $\chi(x_1) = \chi(x_2)$ if and only if $x_1^{\mathrm{ss}} \in \overline{\mathrm{Ad}_G(x_2^{\mathrm{ss}})}$ for their semisimple parts, so that $\mathfrak{g}/\!\!/ G$ may be thought of as the space of semisimple conjugacy classes of G in $\mathfrak{g}$ and χ maps x to the class of its semisimple part $[x^{\mathrm{ss}}]$.

Proposition 3.4.1 ([14], Thms. 4.1.10 and 4.1.14). *Suppose that $G, \mathfrak{g}, \mathfrak{h}, W$ are as above, $r = \mathrm{rk}(G)$ and that $\mathrm{char}(k) = p$ is very good for G. Then the Chevalley map is an isomorphism $\mathfrak{h}/\!\!/ W \simeq \mathfrak{g}/\!\!/ G \simeq \mathbb{A}^r_{\mathcal{O}_K}$ and the formation of categorical quotient $\mathfrak{g}/\!\!/ G$ commutes with base-change.*

Note the conditions of both Theorems 4.1.10 and 4.1.14 of [14] are satisfied; p being (very) good implies G root-smooth in the terminology of (*loc. cit.*) and the étale-local assumption on S is trivial since $\mathcal{O}_K$ equals its strict henselization.

Before we investigate the adjoint quotient χ we need a preliminary lemma.

Lemma 3.4.2 ([52], 4.1.3). *Let $f : X \to Y$ be a morphism of finite-type S–schemes with X flat over S. If the base-changed morphism $f_s : X_s \to Y_s$ is smooth over the closed point $s \to S$ then f is smooth.*

Proof. From the fibral criterion of flatness ([62, Tag 00MP]) we get f flat. The smooth locus $U \subseteq X$ is open and dense, so we are done if it contains all closed points $x \to X$. For such an x, the residue field $k(f(x))$ of $f(x) \to Y$ has characteristic p. Since f_s is smooth, x is a smooth point of $X \times_S \mathrm{Spec}(k(f(x)) \to s$. Hence x lies in U. $\square$

Proposition 3.4.3. *Let $\chi : \mathfrak{g} \longrightarrow \mathfrak{g}/\!\!/ G$ be the adjoint quotient morphism as above.*

(i) *χ is flat and its restriction $\chi^{\mathrm{reg}} : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!\!/ G$ is a smooth surjection.*

(ii) *The geometric fibers of χ are normal of codimension r, and the nilpotent scheme is*
$$\mathcal{N}_\mathfrak{g} \simeq \chi^{-1}(0).$$

Proof. Most of the proof is in ([14], 4.1.18, 4.2.6) but in our case of $S = \mathrm{Spec}(\mathcal{O}_K)$ we may be more specific. For (i), note that $\mathfrak{g}/\!\!/ G$ commutes with base-change (Proposition 3.4.1) so

we may pass to geometric fibers $s, \overline{\eta}$, whence the respective adjoint quotients $\chi_{\overline{\eta}}, \chi_s$ have (geometric) irreducible fibers of codimension r ([57], §3.10(iv) and §3.14). Since $\mathfrak{g}/\!\!/G$ is smooth and the fibers have the same dimension, we get χ flat by miracle flatness ([62, Tag 00R4]). Now note that $\mathfrak{g}^{\mathrm{reg}}$ is nonempty and S–fiberwise dense in $\mathfrak{g}$, hence it is reduced and therefore flat over S. To show that $\chi^{\mathrm{reg}} : \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!\!/G$ is smooth we can reduce via Lemma 3.4.2 to showing $\chi_s^{\mathrm{reg}} : \mathfrak{g}_s^{\mathrm{reg}} \to \mathfrak{g}_s/\!\!/G_s$ is smooth, which follows from ([57], §3.10 Thm. (vi)). For the surjectivity of χ^{reg} we refer to ([52], Thm. 4.3.3), where a Kostant section[1] $\mathcal{S} \subset \mathfrak{g}^{\mathrm{reg}}$ is constructed so that $\mathcal{S} \simeq \mathfrak{g}/\!\!/G$ via χ.

We next consider (ii). Let $h \in \mathfrak{g}/\!\!/G \simeq \mathfrak{h}/\!\!/W$ be an $\mathcal{O}_K$–section with geometric generic and special fibers $h_{\overline{\eta}}, h_s$. By ([57], §3.10 Thm. (ii)) we have that $\mathfrak{g}_s^{\mathrm{reg}} \cap \chi_s^{-1}(h_s)$ is open and dense in $\chi_s^{-1}(h_s)$, and a similar statement holds for $\mathfrak{g}_{\overline{\eta}}^{\mathrm{reg}} \cap \chi_{\overline{\eta}}^{-1}(h_{\overline{\eta}})$, thus $\mathfrak{g}^{\mathrm{reg}}$ intersects fiber $\chi^{-1}(h)$ in an open, S–fiberwise dense set. In particular, fibers $\chi^{-1}(h)$ are generically smooth, hence flat over S. Since the geometric fibers $\chi_s^{-1}(h_s), \chi_{\overline{\eta}}^{-1}(h_{\overline{\eta}})$ are normal ([57], §3.10 Thm. (v)), we get $\chi^{-1}(h)$ normal over S by ([46], Thm. 23.9). In particular, for the nilpotent scheme $\mathcal{N}_{\mathfrak{g}}$ we have established that $\mathcal{N}_{\mathfrak{g}}(\overline{K}) = \chi_{\overline{\eta}}^{-1}(0)$ and $\mathcal{N}_{\mathfrak{g}} = \chi_s^{-1}(0)$, and furthermore both $\mathcal{N}_{\mathfrak{g}}$ and $\chi^{-1}(0)$ are reduced. We conclude $\mathcal{N}_{\mathfrak{g}} \simeq \chi^{-1}(0)$ as schemes. $\qquad\square$

Proposition 3.4.4. *Quotient map $\psi : \mathfrak{h} \to \mathfrak{h}/\!\!/W$ is finite flat, the natural W–action on $\mathfrak{h}^{\mathrm{rs}} = \mathfrak{g}^{\mathrm{rs}} \cap \mathfrak{h}$ is free and $C_G(h) \simeq T$ for any $h \in \mathfrak{h}^{rs}(\mathcal{O}_K)$.*

Proof. Finiteness of ψ is automatic as W is finite, and we have ψ flat by the fibral criterion of flatness and miracle flatness; note that the criteria apply because both $\mathfrak{h}$ and $\mathfrak{h}/\!\!/W$ are smooth ([57], §3.15 Remark (ii)). To show that the W–action on $\mathfrak{h}^{\mathrm{rs}}$ is free, it suffices to show that the natural map $W \times_S \mathfrak{h}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}} \times_S \mathfrak{h}^{\mathrm{rs}}, (w, h) \mapsto (w(h), h)$ is a scheme monomorphism, and by ([27], 17.2.6) it suffices to check so on closed points of S. So we can reduce to the algebraically closed field case, where by ([52], Lemma 2.3.3) we have that W acts freely on

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1. One can think of the Kostant section as a Slodowy slice transverse to the unique regular nilpotent orbit at a regular representative $x \in \mathfrak{g}(\mathcal{O}_K)$; see Sections 4.1 and 4.3 for the theory of slices over $\mathrm{Spec}(\mathcal{O}_K)$.

$\mathfrak{h}_s^{\mathrm{rs}}$ and moreover $C_{G_s}(h_s) \simeq T_s$ for $h_s \in \mathfrak{h}_s^{\mathrm{rs}}(k)$. For an alternative (equivalent) viewpoint see ([39], §VI.7). $\qquad\square$

From the above proposition we obtain a finite étale cover $\psi^{\mathrm{rs}} : \mathfrak{h}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}}/\!/W$, and W acts freely-transitively on its fibers. Therefore:

Corollary 3.4.5. *Morphism $\psi^{\mathrm{rs}} : \mathfrak{h}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}}/\!/W$ is a Galois cover with Galois group W.*

3.5 Relative Grothendieck–Springer resolutions.

We retain the notations and assumptions of Section 3.1 for Chevalley group G/S and Chevalley algebra $\mathfrak{g}$. The free $\mathcal{O}_K$-module $\mathfrak{b}$ obtains a B-module structure via the adjoint action of B on $\mathfrak{b}$ and we can therefore form the following associated bundle (also known as *adjoint bundle*)

$$\widetilde{\mathfrak{g}} := G \times^B \mathfrak{b} = G \times_S \mathfrak{b}/\!/B, \quad b \cdot (g, x) = (gb^{-1}, \mathrm{Ad}_b(x)) \text{ for } b \in B, g \in G, x \in \mathfrak{b} \quad (3.5.1)$$

with the induced B-action on $G \times_S \mathfrak{b}$ indicated on the right.

Lemma 3.5.1. $\widetilde{\mathfrak{g}}$ *is a smooth Zariski-locally trivial G–torsor over G/B with fiber $\mathfrak{b}$.*

Proof. Note that any two Borels B_1, B_2 in G are conjugate étale-locally on S ([19], 5.2.11) so G/B exists as a smooth projective S–scheme ([19], 2.3.6) and its generic and special fibers are correspondingly the flag varieties of $G_{\overline{\eta}}$ and G_s. In this case $\pi : G \to G/B$ makes G into a B–torsor on G/B, inducing $\widetilde{\pi} : \widetilde{\mathfrak{g}} \to G/B$, $[g, x] \mapsto \pi(g)$ so that $\widetilde{\mathfrak{g}}$ is a fiber bundle over G/B with fiber $\mathfrak{b}$. Adjoint bundle $\widetilde{\mathfrak{g}}$ is furthermore equipped with a G-action induced from the (left) G–action on G/B, so that $\widetilde{\pi}$ is G–equivariant.

We get that $\widetilde{\mathfrak{g}}$ is a scheme by pulling an open affine cover $\{U_i\}$ of G/B to an open affine cover $\widetilde{\pi}^{-1}(U_i)$ of $\widetilde{\mathfrak{g}}$, the affineness of $\widetilde{\pi}$ being a consequence of ([37], §5.14). Now we can construct Zariski-local sections $G/B \supset U_i \to G$ for the (a priori étale-locally trivial)

34

B–torsor π. It suffices to construct a composition series of S–subschemes for B so that successive quotients are $\mathbb{G}_m$ or $\mathbb{G}_a$, since $\mathbb{G}_m$ and $\mathbb{G}_a$–torsors are Zariski-locally trivial if and only if they're étale-locally trivial ([26], XI.5.1). Since $B = T \ltimes \mathcal{R}_u(B)$, and $T \simeq \mathbb{G}_{m,\mathcal{O}_K}^r$ while $\mathcal{R}_u(B) \simeq \prod_{\alpha>0} U_\alpha$ decomposes into "root groups" $U_\alpha \simeq \mathbb{G}_a$ (i.e. so that $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$) we can build our composition series as in ([19], 5.1.16).

Local triviality implies $G \times_{G/B} U_i \simeq B \times U_i$ and we can lift to Zariski-local sections for $\widetilde{\mathfrak{g}}$ since $U_i \times_{G/B} \widetilde{\mathfrak{g}} \simeq (U_i \times_{G/B} G) \times^B \mathfrak{b} \simeq U_i \times \mathfrak{b}$. Now $\widetilde{\mathfrak{g}}$ is smooth since $\mathfrak{b}$ is (see [37] §5.16), so $\widetilde{\mathfrak{g}}$ is indeed a Zariski-locally trivial smooth G–torsor. $\qquad\square$

We have a closed immersion $\widetilde{\mathfrak{g}} \hookrightarrow G \times^B \mathfrak{g} \simeq G/B \times_S \mathfrak{g}$ into a trivial G–torsor over G/B (which can be checked on the fibers of S), yielding an equivalent description of $\widetilde{\mathfrak{g}}$ as

$$\widetilde{\mathfrak{g}} = \{(B', x) \in G/B \times_S \mathfrak{g} \mid x \in \mathrm{Lie}(B')\}$$

From this description, we can define a dominant morphism

$$\pi : \widetilde{\mathfrak{g}} \longrightarrow G/B \times_S \mathfrak{g} \longrightarrow \mathfrak{g}, \quad [g, x] \longmapsto (gB, \mathrm{Ad}_g(x)) \longmapsto \mathrm{Ad}_g(x) \qquad (3.5.2)$$

It follows that π is proper since G/B is projective over S. The formation of $\widetilde{\mathfrak{g}}$ commutes with base-change on S and on geometric points $s, \overline{\eta}$ we obtain maps $\pi_s : \widetilde{\mathfrak{g}}_s \to \mathfrak{g}_s$ and $\pi_{\overline{\eta}} : \widetilde{\mathfrak{g}}_{\overline{\eta}} \to \mathfrak{g}_{\overline{\eta}}$ as base-changes of (3.5.2). The maps $\pi_s, \pi_{\overline{\eta}}$ are commonly known as *Grothendieck–Springer resolutions* as defined e.g. in ([58], §3.3). They are particular types of simultaneous resolutions (see Section 2.2).

Lemma 3.5.2. $\pi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ *is finite over the open dense locus* $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$.

Proof. As the formations of $\widetilde{\mathfrak{g}}, \pi$ commute with base-change we may check this over the closed point s, where it suffices to show $x \in \mathfrak{g}^{\mathrm{reg}}(k)$ if and only if x is contained in finitely many Lie algebras of Borel subgroups $B_s \subset G_s$; this is found in ([57], §3.8, §3.14). We thus

35

get quasi-finiteness, hence finiteness of $\pi|_{\mathfrak{g}^{\mathrm{reg}}}$ since it comes from base-changing the proper map π. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Since π is proper, dominant and generically finite, we may call it the *Grothendieck alter-ation.* Another map of interest is $\widetilde{\chi} : \widetilde{\mathfrak{g}} \to \mathfrak{h}$, constructed as follows: given our choice of Borel B, $\mathfrak{b}/\mathfrak{n}_{\mathfrak{b}} = \mathrm{Lie}(B/\mathcal{R}_u(B))$ is the "universal Cartan" and $\widetilde{\chi}$ is $(B', x) \mapsto x \mod \mathfrak{n}_{\mathfrak{b}}$. More precisely, there exists a short exact sequence of torsors

$$0 \longrightarrow G \times^B \mathfrak{n}_{\mathfrak{b}} \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow G \times^B \mathrm{Lie}(B/\mathcal{R}_u(B)) \longrightarrow 0$$

induced from $\mathfrak{n}_{\mathfrak{b}} \hookrightarrow \mathfrak{b}$, since all terms are locally trivial over S. Now B acts trivially on $\mathrm{Lie}(B/\mathcal{R}_u(B))$ as the latter is an abelian subalgebra, so that

$$G \times^B \mathrm{Lie}(B/\mathcal{R}_u(B)) \simeq G/B \times \mathrm{Lie}(B/\mathcal{R}_u(B)) \simeq G/B \times \mathfrak{h}$$

and $\widetilde{\chi}$ is the projection of $\widetilde{\mathfrak{g}} \to G/B \times \mathfrak{h}$ onto the second factor. Note $\widetilde{\chi}$ is flat by miracle flatness and the fibral criterion of flatness ([62, Tag 00MP]), hence smooth by Lemma 3.4.2 as $\widetilde{\chi}_s$ is identified with the smooth morphism $\widetilde{\mathfrak{g}}_s \to \mathfrak{h}_s$ defined in ([58], §3.3).

Proposition 3.5.3. *Keeping notations as above, the morphisms π and $\widetilde{\chi}$ fit into a commu-tative diagram*

$$\begin{array}{ccc} \widetilde{\mathfrak{g}} & \overset{\pi}{\longrightarrow} & \mathfrak{g} \\ \downarrow{\scriptstyle\widetilde{\chi}} & & \downarrow{\scriptstyle\chi} \\ \mathfrak{h} & \overset{\psi}{\longrightarrow} & \mathfrak{h}/\!\!/W \end{array} \qquad\qquad (3.5.3)$$

so that, S–fiberwise, π_s and $\pi_{\overline{\eta}}$ induce the Grothendieck–Springer resolutions on $\mathfrak{g}_s$ resp. $\mathfrak{g}_{\overline{\eta}}$. In particular, restriction $\widetilde{\mathfrak{g}}^{\mathrm{rs}} := \pi^{-1}(\mathfrak{g}^{\mathrm{rs}}) \to \mathfrak{g}^{\mathrm{rs}}$ is a W–torsor, $\widetilde{\mathfrak{g}}^{\mathrm{reg}} := \pi^{-1}(\mathfrak{g}^{\mathrm{reg}}) \simeq \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{h}/\!\!/W} \mathfrak{h}$ and $\widetilde{\chi} : \widetilde{\mathfrak{g}}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}}$ is W–equivariant.

Proof. The maps $\pi_s, \pi_{\overline{\eta}}$ are the respective Grothendieck–Springer resolutions for $\mathfrak{g}_s, \mathfrak{g}_{\overline{\eta}}$ by

the discussion preceding Lemma 3.5.2. For the commutativity of (3.5.3) it suffices to have the map $\mathfrak{b} \to \mathfrak{h}/\!\!/W$ (induced by inclusion and χ) factoring through

$$\mathfrak{b} \xrightarrow{\mathrm{pr}} \mathfrak{b}/\mathfrak{n}_\mathfrak{b} \simeq \mathfrak{h} \xrightarrow{\psi} \mathfrak{h}/\!\!/W$$

since $\widetilde{\chi}$ is induced by map pr. Thus, it suffices to check this on global sections, whence

$$
\begin{array}{ccc}
\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{h}^\vee)^W & \lhook\joinrel\longrightarrow & \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{h}^\vee) \\
\wr \big\uparrow & & \\
\mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee)^G & \lhook\joinrel\longrightarrow & \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{g}^\vee) \longrightarrow \mathrm{Sym}_{\mathcal{O}_K}(\mathfrak{b}^\vee)
\end{array}
$$

Over field-valued points we note $\widetilde{\chi}([g, h + n]) = h$ for $h \in \mathfrak{h}, n \in \mathfrak{n}_\mathfrak{b}$, and so the commutativity of the diagram amounts to the fact that the semisimple part of $\mathrm{Ad}_g(h + n)$ is conjugate to h ([58] §3.3).

We next prove the statements involving $\widetilde{\mathfrak{g}}^{\mathrm{rs}}, \widetilde{\mathfrak{g}}^{\mathrm{reg}}$ as follows. Define $X = G/T \times_S \mathfrak{h}^{\mathrm{rs}}$, equipped with the W–action of Definition 3.2.2 (ii). We aim to show $\widetilde{\mathfrak{g}}^{\mathrm{rs}} \simeq X$, which also defines a W–action on $\widetilde{\mathfrak{g}}^{\mathrm{rs}}$. By ([27], 17.9.5) it suffices to show the corresponding map over k and K is an isomorphism, whence it holds by the proof of ([39], Thm. 9.1)[2]. Moreover $X/\!\!/W \simeq \mathfrak{g}^{\mathrm{rs}}$ over k and K, so this isomorphism extends over S again by ([27], 17.9.5).

To show W acts transitively on the S–fibers of $\widetilde{\mathfrak{g}}^{\mathrm{rs}} \to \mathfrak{g}^{\mathrm{rs}}$, it suffices as before to pass to geometric fibers and use $\widetilde{\mathfrak{g}}^{\mathrm{rs}} \simeq X$. Say $(g_1 T, h_1), (g_2 T, h_2) \in X$ mapping to $\mathrm{Ad}_{g_1}(h_1) = \mathrm{Ad}_{g_2}(h_2) \in \mathfrak{g}^{\mathrm{rs}}$. Then $\mathrm{Ad}_{g_2^{-1} g_1}(h_1) = h_2$ so $n_w = g_2^{-1} g_1 \in N_G(T)$ since n_w conjugates the two centralizers $C_G(h_1), C_G(h_2)$, which are both T as $h_1, h_2 \in \mathfrak{h}^{\mathrm{rs}}$. Letting $w = [n_w^{-1}] \in W$ we obtain $(g_1 T, h_1) \cdot w = (g_2 T, h_2)$, establishing that W acts freely transitively on the fibers. Thus $\widetilde{\mathfrak{g}}^{\mathrm{rs}} \to \mathfrak{g}^{\mathrm{rs}}$ is a W–torsor.

Now $\mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{h}/\!\!/W} \mathfrak{h}$ is smooth over S since $\mathfrak{g}^{\mathrm{reg}} \to \mathfrak{h}/\!\!/W$ is (by Proposition 3.4.3), and its restriction $\mathfrak{g}^{\mathrm{reg}} \times_S \mathfrak{h}^{\mathrm{rs}}$ is furthermore a W–torsor over $\mathfrak{g}^{\mathrm{rs}}$ by base-changing along W–torsor

---

2. Note the terminology of ([39]) differs from the standard one - what they call "regular" is regular semisimple in our terminology.

$\mathfrak{h}^{\mathrm{rs}} \to \mathfrak{h}^{\mathrm{rs}} /\!\!/ W$. Then morphism $\widetilde{\mathfrak{g}}^{\mathrm{rs}} = \widetilde{\mathfrak{g}}^{\mathrm{reg}}|_{\mathfrak{g}^{\mathrm{rs}}} \to \mathfrak{g}^{\mathrm{reg}} \times_S \mathfrak{h}^{\mathrm{rs}}$ is W–equivariant as both source and target map to $\mathfrak{h}$ via projection, so we have an isomorphism of W–torsors over $\mathfrak{g}^{\mathrm{rs}}$, which extends to all of $\mathfrak{g}^{\mathrm{reg}}$ by uniqueness of normalizations; see ([14], 4.2.12). In light of the specified W–action on $\widetilde{\mathfrak{g}}^{\mathrm{rs}}$, we likewise obtain that $\widetilde{\chi}_{\widetilde{\mathfrak{g}}^{\mathrm{rs}}}$ is W–equivariant. $\qquad\square$

CHAPTER 4

INTEGRAL SLODOWY SLICES

We now extend Slodowy's construction of transverse slices ([57], §5) to our setting of Cheval-ley $\mathcal{O}_K$–algebras, with the intention of proving a Grothendieck simultaneous resolution state-ment for slices (Section 4.6). Throughout, G will be an affine group scheme, in particular either a Lie group over an algebraically closed field or a split, simple, simply-connected Chevalley group over $S = \mathrm{Spec}(\mathcal{O}_K)$. $T \subset B$ denote a fixed choice of torus and Borel, and the corresponding Lie algebras over k or S are $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$.

4.1 Remarks on Slodowy slices.

We review the theory of transverse slices to G–orbits from ([57], §5) in a slightly more general setting.

Definition 4.1.1. Let G act on an integral scheme X over S. A *transverse Slodowy slice* to the orbit $G \cdot x$ of $x \in X$ is a locally closed subvariety $\mathcal{S} \subseteq X$ so that $x \in \mathcal{S}(\mathcal{O}_K)$, the action morphism $\alpha : G \times \mathcal{S} \to X$, $\alpha(g,s) = g \cdot s$ is smooth and the dimension of $\mathcal{S}$ is minimal with respect to these two conditions.

Based on their definition, we can deduce some useful properties of Slodowy slices.

Lemma 4.1.2 ([57], §5.1 Lemma 3)**.** *Suppose X is smooth and affine with the adjoint action of Chevalley group G over S, Y is another S-scheme with trivial G-action and $f : X \to Y$ is a G-invariant morphism. Let $x, y \in X(\mathcal{O}_K)$ lie in the same G-orbit in X and assume that centralizers $C_G(x), C_G(y)$ are smooth. Suppose $\mathcal{S}_1, \mathcal{S}_2$ are Slodowy slices transverse to the orbit at x resp. y. Then $(\mathcal{S}_1, x), (\mathcal{S}_2, y)$ are ètale-locally isomorphic over Y. In particular the henselizations of $\mathcal{S}_1$ at x and $\mathcal{S}_2$ at y are isomorphic.*

Proof. We may assume $x = y$ (since $y = g \cdot x$ for some $g \in G$). As $C_G(x)$ is smooth, $\mathrm{Lie}(C_G(x)) \subset \mathfrak{g}$ is identified with the Chevalley subalgebra of ad_x-invariants ([19], 2.2.4),

inducing an S-splitting $\mathfrak{g} = \mathrm{Lie}(C_G(x)) \oplus \mathfrak{g}_1$ for some complementary $\mathcal{O}_K$-module $\mathfrak{g}_1$. Then the rest of the proof of ([57], §5.1, Lemma 3) goes through: choose a projection $p : G \to \mathfrak{g}$ étale at the identity (cf. *loc. cit.*, §5.1 Lemma 1's proof) and set $G_1 = p^{-1}(V), G_2 = \{g^{-1} \mid g \in G_1\}$ inside G. By construction, the induced action maps $\mu_i : G_i \times \mathcal{S}_i \to X$ are étale at $(1, x)$ and hence so are the base–changed maps arising from cartesian diagrams

$$
\begin{array}{ccc}
(G_1 \times \mathcal{S}_1) \times_X \mathcal{S}_2 & \longrightarrow & \mathcal{S}_2 \\
\downarrow & & \downarrow \\
G_1 \times \mathcal{S}_1 & \xrightarrow{\ \mu_1\ } & X
\end{array}
\qquad
\begin{array}{ccc}
(G_2 \times \mathcal{S}_2) \times_X \mathcal{S}_1 & \longrightarrow & \mathcal{S}_1 \\
\downarrow & & \downarrow \\
G_2 \times \mathcal{S}_2 & \xrightarrow{\ \mu_2\ } & X
\end{array}
$$

We furthermore have $(G_1 \times \mathcal{S}_1) \times_X \mathcal{S}_2 \simeq (G_2 \times \mathcal{S}_2) \times_x \mathcal{S}_1$ via $(g, s) \mapsto (g^{-1}, g \cdot s)$. Hence we can choose a neighborhood of $(1, x)$ inside this space, with étale maps to $\mathcal{S}_1, \mathcal{S}_2$. $\qquad \square$

Remark 4.1.3. The condition of centralizer $C_G(x)$ being smooth over S is satisfied in the case $X = \mathfrak{g}$ and $x \in \mathfrak{g}(\mathcal{O}_K)$ a fiberwise subregular nilpotent section (cf. Definition 3.3.3) - note that by ([20], Thm. 1.1), the centralizer satisfies all relevant 'purity' assumptions.

Lemma 4.1.4 ([57] §5.1, Lemma 2). *If G acts on schemes X, Y with respective action maps α_X, α_Y, $\mathcal{S} \subset X$ is a locally closed integral subscheme and $f : X \to Y$ is a G-equivariant morphism, then the following diagram is cartesian and the top arrow is smooth if the bottom arrow is smooth:*

$$
\begin{array}{ccc}
G \times (\mathcal{S} \times_Y X) & \xrightarrow{\ \alpha_X\ } & X \\
{\scriptstyle \mathrm{id} \times f} \downarrow & & \downarrow {\scriptstyle f} \\
G \times \mathcal{S} & \xrightarrow{\ \alpha_Y\ } & Y
\end{array}
$$

4.2 Jacobson–Morozov in characteristic p.

The aim of the next few sections is to construct a suitable Slodowy slice $\mathcal{S}$ over S, transverse to a chosen S–fiberwise nilpotent element $x \in \mathfrak{g}$. The goal is to use slice $\mathcal{S}$ to describe a miniversal deformation of RDP $x_s \in \mathcal{N} \cap \mathcal{S}$ and a simultaneous resolution for slices (see Section 4.6). We first make some remarks on $\mathfrak{sl}_2$–representation theory in characteristic $p > 0$, following ([57], §7.1).

Definition 4.2.1. Let k be algebraically closed of characteristic p and fix a standard basis $\{h, x, y\}$ for $\mathfrak{sl}_2(k)$. A representation $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}_n$ is called *good* (in the sense of ([57], §7.1)) when $\rho(x)^{p-1} = \rho(y)^{p-1} = 0$; for $p = 0$ we posit that all $\mathfrak{sl}_2$–representations are good.

If V is an n-dimensional $(n < p)$ k-vector space with basis $\{v_1, \cdots, v_n\}$, we can define an *irreducible* $\mathfrak{sl}_2$–representation ρ_n on V as follows:

$$\rho_n(x)v_1 = 0, \quad \rho_n(y)v_n = 0$$

$$\rho_n(x)v_{i+1} = i(n-i)v_i \mod p, \quad \rho_n(y)v_i = v_{i+1}, \quad \rho_n(h)v_i = (n-2i+1)v_i \mod p \text{ for } i \le n-1$$

Theorem 4.2.2 ([57], §7.1)**.** *Any good $\mathfrak{sl}_2$-representation ρ is completely reducible and decomposes into a sum of good irreducible $\mathfrak{sl}_2$–representations of the above form. For each $n < p$ there is a unique n–dimensional good irreducible representation.*

Denote by V_n the unique n–dimensional good irreducible $\mathfrak{sl}_2$–representation. V_n decomposes further into *weight spaces* $V_n(k)$, which are 1-dimensional eigenspaces for the action of $\rho_n(h)$ as multiplication by k and $-n+1 \le k \le n-1$. Then the nilpotent endomorphisms $\rho_n(x), \rho_n(y)$ respectively induce isomorphisms $V_n(k) \overset{\sim}{\to} V_n(k+2)$ and $V_n(k) \overset{\sim}{\to} V_n(k-2)$; they also respectively annihilate the highest weight space $V_n(n-1)$ and the lowest weight space $V_n(-n+1)$.

Theorem 4.2.3 (Jacobson–Morozov, [63] Thm. 1.1)**.** *Let G be a simple Lie group over k with Lie algebra $\mathfrak{g}$. Suppose $p = 0$ or $p > \mathrm{Cox}(\mathfrak{g})$. For each nilpotent $x \in \mathfrak{g}(k)$ there is a completion to an $\mathfrak{sl}_2$–triple $\{h, x, y\}$ coming from a faithful representation $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ with $x = \rho(x_0)$, $y = \rho(y_0)$, $h = \rho(h_0)$ for $\{h_0, x_0, y_0\}$ the standard basis of $\mathfrak{sl}_2(k)$. The triple $\{h, x, y\}$ is unique up to $C_G(x)$–conjugation and the composite representation $\mathrm{ad} \circ \rho : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{g})$ is a good representation.*

Remark 4.2.4. According to [63], restriction $p > \mathrm{Cox}(\mathfrak{g})$ is optimal for the uniqueness of

$\mathfrak{sl}_2$-triple $\{h, x, y\}$ up to conjugation, and it improves previous bounds such as $4\mathrm{Cox}(\mathfrak{g}) - 2$ appearing in [57].

4.3   Slodowy slices via Jacobson–Morozov.

Let G denote a split, simple, simply-connected Chevalley group of Type A_n over S with Chevalley algebra $\mathfrak{g}$. Consider the setting of Section 4.1, with G acting on $X = \mathfrak{g}$ via the adjoint action and $x \in \mathfrak{g}(\mathcal{O}_K)$ an S-fiberwise subregular nilpotent element; this condition ensures both generic and special fibers $x_{\overline{\eta}}, x_s$ remain in the respective (unique) subregular nilpotent orbit of $\mathfrak{g}_{\overline{\eta}}, \mathfrak{g}_s$. In view of Lemma 4.1.2 we may construct a suitable Slodowy slice at x, transverse to the (adjoint) subregular nilpotent orbits of $\mathfrak{g}_{\overline{\eta}}, \mathfrak{g}_s$ and any other Slodowy slice at x will be étale-locally isomorphic to it. Moreover this allows us to replace x with another representative in its nilpotent orbit, so we may assume x has a 'standard' form.

Proposition 4.3.1. *Assume $p > \mathrm{Cox}(\mathfrak{g})$. There is a choice of subregular nilpotent representative $x \in \mathfrak{g}(\mathcal{O}_K)$ extending to a section of $\mathfrak{sl}_2$-triples $\{h, x, y\} \subset \mathfrak{g}(\mathcal{O}_K)$ i.e. $\{h_{\overline{\eta}}, x_{\overline{\eta}}, y_{\overline{\eta}}\}$ and $\{h_s, x_s, y_s\}$ are $\mathfrak{sl}_2$-triples respectively in $\mathfrak{g}_{\overline{\eta}}$ and $\mathfrak{g}_s$.*

Proof. It is clear by the restrictions (see Theorem 4.2.3) that fiberwise subregular x induces unique $\mathfrak{sl}_2$-triples in Lie algebras $\mathfrak{g}_\eta, \mathfrak{g}_s$ up to the action of $\mathrm{Lie}(C_G(x)) \subset \mathfrak{g}(\mathcal{O}_K)$. Subregular nilpotent elements in Type A_n have standard Levi form in characteristic zero, i.e. they are regular nilpotent in a Levi subalgebra $\mathfrak{l}$ corresponding to a parabolic $\mathfrak{p} \subseteq \mathfrak{g}$ determined by a subset of simple roots $I \subset \Delta$. In this case, if we denote the simple roots as $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, the corresponding Levi and subregular representative are

$$\mathfrak{l} = \mathfrak{h}_{\overline{\eta}} \oplus \bigoplus_{i=1}^{n-1} \mathfrak{g}_{\alpha_i, \overline{\eta}}, \quad x_{\mathrm{subreg}} = \sum_{i=1}^{n-1} e_{\alpha_i} \tag{4.3.1}$$

where Chevalley basis elements e_{α_i} are viewed as root vectors in $\mathfrak{g}_{\overline{\eta}}$ (see Section 3.1). By the discussing preceding this proposition we may replace our subregular x with the "standard"

42

representative $x = x_{\text{subreg}}$ in Equation (4.3.1), which has this form because it is a regular representative in $\mathfrak{l}$ (see e.g. [52], Lemma 3.1.1); as vectors e_{α_i} form a $\mathbb{Z}$–basis for $\mathfrak{g}$, clearly $x \in \mathfrak{g}(\mathcal{O}_K)$ still.

We now complete x to an $\mathfrak{sl}_2$–triple $\{x, y, h\}$. By ([18], §3.6) an element $h = \sum a_i h_{\alpha_i} \in \mathfrak{h}_{\overline{\eta}}$ satisfies $[h, e_{\alpha_i}] = d_i e_{\alpha_i}$ if and only if the weights of the Dynkin diagram corresponding to x are $(d_1, \cdots, d_n)$. Up to relabeling we have $d_i = 2$ for $i \neq n$, $d_n = 0$ (see *loc. cit.*, 3.6.4). Therefore condition $[h, x] = 2x$ yields a system of equations

$$\sum_{j=1}^{n} a_j \frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} = \sum_{j=1}^{n} a_j C_{j,i} = d_i, \quad 1 \leq i \leq n \qquad (4.3.2)$$

where C is the Cartan matrix of $\mathfrak{g}$. This gives a unique h with $\mathcal{O}_K$–coefficients if and only if $\det(C) = n + 1$ is invertible in $\mathcal{O}_K$, which is granted by the restriction $p > \text{Cox}(\mathfrak{g}) = n + 1$. For $y = \sum b_i e_{-\alpha_i}$, a simple calculation using the Chevalley relations yields

$$\sum_{i=1}^{n-1} a_i h_{\alpha_i} = h = [x, y] = \Big[\sum_{i=1}^{n-1} e_{\alpha_i}, \sum_{i=1}^{n-1} b_i e_{-\alpha_i} \Big] = \sum_{i=1}^{n-1} b_i h_{\alpha_i}$$

so coefficients $b_i = a_i$ are still in $\mathcal{O}_K$. Since $[h, y] = -2y$ follows from $[h, x] = 2x$, we have constructed an $\mathfrak{sl}_2$-triple in $\mathfrak{g}(\mathcal{O}_K)$; one can check that $a_i \mod p \neq 0$ so that the mod p reductions $\{\overline{h}, \overline{x}, \overline{y}\}$ still form a subregular $\mathfrak{sl}_2$-triple in $\mathfrak{g}_s$. $\qquad \square$

Lemma 4.3.2. *Fix an integral $\mathfrak{sl}_2$-triple $\{h, x, y\}$ as in* Proposition 4.3.1. *Then $\mathcal{O}_K$-module $\mathfrak{g}/\text{ad}_x(\mathfrak{g})$ is free of rank $r + 2$.*

Proof. By assumption $x \in \mathfrak{g}(\mathcal{O}_K)$ so $\text{ad}_x(\mathfrak{g}) \subset \mathfrak{g}$ and quotient $\mathfrak{g}/\text{ad}_x(\mathfrak{g})$ is an $\mathcal{O}_K$-module. Consider $\{h_\eta, x_\eta, y_\eta\}$ as an $\mathfrak{sl}_2$-triple in the $\overline{K}$-Lie algebra $\mathfrak{g}_{\overline{\eta}}$, then the dimension of $\overline{K}$-vector space $\mathfrak{g}_\eta/\text{ad}_{x_\eta}(\mathfrak{g}_\eta)$ is the dimension of the centralizer $C_{\mathfrak{g}_\eta}(x_\eta)$, which is $r+2$ by subregularity. So $\mathfrak{g}/\text{ad}_x(\mathfrak{g})$ is a rank $r + 2$ module, possibly with torsion.

Now consider the mod p $\mathfrak{sl}_2$-triple $\{h_s, x_s, y_s\}$ inside $\mathfrak{g}_s$. Since $p > \text{Cox}(\mathfrak{g})$, $\mathfrak{g}_s$ is a good

$\mathfrak{sl}_2$–representation decomposing into irreducible good $\mathfrak{sl}_2$–representations $V_{d_1}, \cdots, V_{d_r}$. Each irreducible representation V_{d_i} decomposes further into d_i 1–dimensional eigenspaces

$$V_{d_i}(-d_i + 1), \cdots, V_{d_i}(d_i - 1)$$

for the action of ad_{h_s} and by $\mathfrak{sl}_2$–theory ad_{x_s} maps $V_{d_i}(k)$ isomorphically to $V_{d_i}(k + 2)$ (cf. Section 4.2). So

$$\mathfrak{g}_s/\mathrm{ad}_{x_s}(\mathfrak{g}_s) \simeq \bigoplus_{i=1}^{r} V_{d_i}(-d_i + 1) \qquad (4.3.3)$$

is the direct sum of all lowest weight eigenspaces. Furthermore as ad_x annihilates all *highest* weight eigenspaces $V_{d_i}(d_i - 1)$, the number of irreducible components of $\mathfrak{sl}_2$-representation $\mathfrak{g}$ (and hence the dimension of $\mathfrak{g}_s/\mathrm{ad}_{x_s}(\mathfrak{g}_s)$ equals $r + 2$, the dimension of the centralizer of subregular x_s.

Since forming module quotients commutes with base–change, identification $(\mathfrak{g}/\mathrm{ad}_x(\mathfrak{g})) \otimes k \simeq \mathfrak{g}_s/\mathrm{ad}_{x_s}(\mathfrak{g}_s)$ implies equidimensionality of the fibers of $\mathfrak{g}/\mathrm{ad}_x(\mathfrak{g})$, hence it is flat over $\mathcal{O}_K$ and therefore it is a free rank $r + 2$ module. $\qquad \square$

Corollary 4.3.3. *Retaining the notation of* Lemma 4.3.2, *if $\mathfrak{a}$ denotes a free $\mathcal{O}_K$-submodule of $\mathfrak{g}$ complementary to $\mathrm{ad}_x(\mathfrak{g})$ then $\mathcal{S} = x + \mathfrak{a}$ is a Slodowy slice transverse at x fiberwise to the nilpotent orbits of x_s and $x_{\overline{\eta}}$.*

Proof. The choice of such an $\mathfrak{a}$ is possible via Lemma 4.3.2 since $\mathfrak{g}/\mathrm{ad}_x(\mathfrak{g})$ is free. We observe in the special fiber $\overline{\mathcal{S}} := \mathcal{S} \otimes k = x_s + \mathfrak{a}_s$ that $\mathfrak{a}_s$ is a complement to $\mathrm{ad}_{x_s}(\mathfrak{g}_s)$ in $\mathfrak{g}_s$ and hence isomorphic to $\bigoplus_i V_{d_i}(-d_i + 1)$ (in the notation of the proof of Lemma 4.3.2). It forms a Slodowy slice in the traditional sense by ([58], §2.4) and is transverse at x_s to the subregular orbit. The action map $\mu : G \times \mathcal{S} \to \mathfrak{g}$ is smooth, since by Lemma 3.4.2 it suffices to check smoothness on the special fiber, where $\overline{\mu} : G_s \times \overline{\mathcal{S}} \to \mathfrak{g}_s$ is precisely the smooth action map of Slodowy slice $\overline{\mathcal{S}}$. Since $\dim(\overline{\mathcal{S}}) = \dim(\mathcal{S})$ by flatness, we have checked that $\mathcal{S}$ satisfies all conditions of a transverse Slodowy slice. $\qquad \square$

We may identify $\mathfrak{a} \simeq \ker(\mathrm{ad}_y)$ so that $\mathcal{S} = x + \ker(\mathrm{ad}_y)$ by Corollary 4.3.3; this is the standard formula for Slodowy slices.

4.4   Spaltenstein slices.

We now consider Chevalley algebras of type D_n or E_n[1]. In this case we may construct a transverse Slodowy slice which works for all *good* characteristics, thus improving on the restriction $p > \mathrm{Cox}(\mathfrak{g})$. Following [59], let G be a simple simply-connected split group scheme over S with corresponding Chevalley algebra $\mathfrak{g}$ as before, and fix a maximal torus, Borel $T \subset B$ in G and a fiberwise subregular nilpotent $x \in \mathfrak{g}(\mathcal{O}_K)$.

It is known (cf. [18], §8.2 and §8.4) in the simply-laced cases D_n, E_n that x is a distinguished nilpotent and so by the classification of nilpotent orbits in good characteristic there exists a unique distinguished parabolic $P \subset G$ which contains B and corresponds to the orbit of x; for the notions of distinguished elements and subgroups we refer to [59] and ([18], §8). We may therefore define a 1–parameter subgroup $\lambda : \mathbb{G}_{m,S} \longrightarrow T$ so that, with respect to a root basis $\Delta \subset \Phi$ of G and root system $\Phi_P \subset \Phi$ of P we have

$$\langle \alpha, \lambda(t) \rangle = \begin{cases} 0, & \alpha \in \Delta, \ -\alpha \in \Phi_P \\ 2, & \alpha \in \Delta, \ -\alpha \notin \Phi_P \end{cases}$$

where $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ is the pairing induced from perfect duality on the (co)character lattices.

Theorem 4.4.1 ([59])**.** *Let $\lambda : \mathbb{G}_{m,S} \longrightarrow T$ be constructed as above and let $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ be the $\mathbb{Z}$–graded decomposition of $\mathfrak{g}$ into eigenspaces for the induced $\mathbb{G}_m$–action*

$$\mathfrak{g}(i) = \{x \in \mathfrak{g}(\mathcal{O}_K) \mid \mathrm{Ad}_{\lambda(t)}(x) = t^i x \ \ \forall \, t \in \mathbb{G}_m(\mathcal{O}_K)\}$$

---

1. The construction in this section should work for A_n types as well, thereby eliminating the need for explicit $\mathfrak{sl}_2$–triples; see ([12], 7.1.4).

Then $x \in \mathfrak{g}(2)$ and there exists an affine $\mathbb{G}_m$–stable subspace $\mathfrak{a} \subset \mathfrak{g}$ complementary to $[x, \mathfrak{g}]$, so that $\mathcal{S} = x + \mathfrak{a}$ is a Slodowy slice, transverse at x fiberwise to the nilpotent orbits of $x_s, x_{\bar{\eta}}$.

We will call such slices $\mathcal{S}$ Spaltenstein to differentiate them from the slices constructed in Corollary 4.3.3 via Jacobson–Morozov. The proof in (*loc. cit.*) carries over to $\mathcal{O}_K$ since $\text{rk}[x, \mathfrak{g}] = \dim_k[x_s, \mathfrak{g}_s]$ as a consequence of $\dim_k C_{G_s}(x_s) = \dim_{\mathcal{O}_K} C_G(x)$, and one concludes that $[x, \mathfrak{g}]$ is a direct factor of $\mathfrak{g}$, hence it is free as in Lemma 4.3.2. Furthermore $\mathfrak{g}(i) \subset [x, \mathfrak{g}]$ for $i > 0$ so there exists an affine $\mathbb{G}_m$–stable complement $\mathfrak{a} \subset \bigoplus_{i \leq 0} \mathfrak{g}(i)$ i.e. $\mathfrak{a} = \mathfrak{g}/[x, \mathfrak{g}]$. The action map $\mu : G \times \mathcal{S} \to \mathfrak{g}$, $(g, s) \mapsto \text{Ad}_g(s)$ for $\mathcal{S} = x + \mathfrak{a}$ is smooth in a neighborhood of $(1, 0)$, so by virtue of the homogeneous $\mathbb{G}_m$–action it is smooth (see e.g. [57], §7.4 Cor. 1).

Remark 4.4.2. We can also extend Proposition 4.3.1 to the D_n and E_n cases by writing down a suitable subregular representative $x \in \mathfrak{g}(\mathcal{O}_K)$, (e.g. [18], §5. produces such an x for $\mathfrak{g}$ of type D_n), and then finding h, y is a computational exercise in the vein of the proof of Proposition 4.3.1. We chose to consider Spaltenstein slices here so that we can relax the assumption $p > \text{Cox}(\mathfrak{g})$ to just p being a good prime.

4.5 $\mathbb{G}_m$–actions and $\mathbb{G}_m$–deformations.

Suppose we have constructed a suitable transverse slice $\mathcal{S}$ at a fixed fiberwise subregular nilpotent $x \in \mathfrak{g}(\mathcal{O}_K)$; if $\mathfrak{g}$ is of type A_n, the slice is $\mathcal{S} = x + \ker(\text{ad}_y)$ for $\{x, y, h = [x, y]\}$ forming an integral $\mathfrak{sl}_2$–triple, while for the other simply–laced types $\mathcal{S}$ is a Spaltenstein slice constructed as in Section 4.4. The goal of this section is to discuss some $\mathbb{G}_m$–actions on slices $\mathcal{S}$ and relate them to the notion of $\mathbb{G}_m$-equivariant deformations.

We first define a 1–parameter subgroup $\lambda : \mathbb{G}_{m,S} \to T$ when $\mathfrak{g}$ is of type A_n, similar to Section 4.4. Let x be a fiberwise subregular nilpotent section of $\mathfrak{g}$ and complete it to an $\mathfrak{sl}_2$–triple $\{x, y, h\}$ over S (Section 4.3). Then $\{x_s, y_s, h_s\}$ defines a good $\mathfrak{sl}_2$-representation

on $\mathfrak{g}_s$ (Theorem 4.2.3). Let V_{d_i} be an irreducible summand of this representation, the *unique* irreducible good $\mathfrak{sl}_2$-representation of dimension d_i, and let $\{v_1, \cdots, v_{d_i}\}$ be a basis. There exits a $\mathbb{G}_m$-action on V_{d_i} by linearly extending

$$t \cdot v_k = t^{d_i - 2k + 1} v_k \quad t \in \mathbb{G}_m, \ k \leq d_i \tag{4.5.1}$$

Definition 4.5.1. If $\mathfrak{g} \simeq \bigoplus_i V_{d_i}$ is the decomposition of $\mathfrak{g}$ into irreducible good $\mathfrak{sl}_2$-representations, there is a uniquely defined $\mathbb{G}_m$-action $\lambda : \mathbb{G}_m \to \mathrm{Aut}(\mathfrak{g})$ which operates on each summand V_{d_i} by the rule (4.5.1). It commutes with the Lie bracket of $\mathfrak{g}$ and hence factors through a 1–parameter subgroup, still denoted λ. We call λ a 1–parameter subgroup adapted to x_s.

This decomposition holds over $\mathcal{O}_K$ since $\mathfrak{g}_{\overline{\eta}} \simeq \mathfrak{g}_s$. Moreover, by abuse of notation we denote by V_{d_i} the $\mathcal{O}_K$–module generated by the $\mathbb{Z}$–basis $\{v_1, \cdots, v_{d_i}\}$ for the k–vector space V_{d_i} in (4.5.1). Note that $\lambda(\mathbb{G}_m) \subseteq T$ and the action of ad_{h_s} decomposes $\mathfrak{g}$ into eigenspaces

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad \mathfrak{g}(i) = \{v \in \mathfrak{g} \mid \mathrm{Ad}_{\lambda(t)}(v) = t^i v \ \forall \, t \in \mathbb{G}_m\}$$

which also function as eigenspaces for the λ-action ([57] §7.1, §7.3). In particular $x \in \mathfrak{g}(2)$ and λ coincides with the 1–parameter subgroup constructed for Spaltenstein slices $\mathcal{S}$ (Theorem 4.4.1).

Definition 4.5.2. Let λ be as above and $m : \mathbb{G}_m \to \mathrm{Aut}(\mathfrak{g})$ be the usual left–multiplication action.

(i) The action

$$\mu : t \cdot v = m(t^2) \mathrm{Ad}_{\lambda(t^{-1})}(v), \quad v \in \mathfrak{g}$$

fixes the element x and preserves the Slodowy slice $\mathcal{S}$. Moreover it is a *contracting action* on $\mathcal{S}$, i.e. it extends to an action $\mathbb{A}_{\mathcal{S}}^1 \to \mathrm{Aut}(\mathcal{S})$ with $0 \cdot s = x$ for all $s \in \mathcal{S}$.

47

(ii) Let $\widetilde{\mathcal{S}}$ be the preimage of $\mathcal{S}$ under the Grothendieck alteration π. Define an action on $\widetilde{\mathfrak{g}}$ via

$$\widetilde{\mu} : t \cdot [g, v] = [\lambda(t^{-1})g, t^2 v], \quad t \in \mathbb{G}_m(\mathcal{O}_K), \ g \in G(\mathcal{O}_K), \ v \in \mathfrak{g}(\mathcal{O}_K)$$

This action preserves $\widetilde{\mathcal{S}}$.

By the definition of $\mu, \widetilde{\mu}$ we have that the Grothendieck alteration $\pi : \widetilde{\mathcal{S}} \longrightarrow \mathcal{S}$ is $\mathbb{G}_{m,S}-$equivariant, and a simple calculation yields that $\widetilde{\chi} : \widetilde{\mathcal{S}} \to \mathfrak{h}$ is also $\mathbb{G}_{m,S}-$equivariant when $\mathfrak{h} \simeq \mathbb{A}_S^r$ is equipped with *contracting* $\mathbb{G}_{m,S}-$action $t \cdot h = t^2 h$; in this case, the action contracts $\mathfrak{h}$ to the origin.

We next equip $\mathfrak{h} /\!\!/ W$ with a $\mathbb{G}_{m,S}-$action so that the adjoint quotient $\chi : \mathcal{S} \to \mathfrak{h} /\!\!/ W$ becomes $\mathbb{G}_{m,S}-$equivariant. By ([23], §6 Théoréme 3 and Corollaire), if p is a good prime[2] then $\mathrm{Sym}(X^*(T))^W$ is a graded polynomial $\mathcal{O}_K$-algebra with homogeneous generators $\chi_1, \cdots, \chi_r$ of (homogeneous) degrees $d_1, \cdots, d_r$, and furthermore

$$\mathrm{Sym}(X^*(T) \otimes_{\mathbb{Z}} k)^W \simeq \mathrm{Sym}(X^*(T))^W \otimes_{\mathbb{Z}} k$$

Thus the degrees d_i of the generators χ_i are the same as their mod p versions, which are recorded in ([57], Table in p. 112). As T is split we also have

$$\mathfrak{h} \simeq \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_K}(X^*(T))), \quad \mathfrak{h} /\!\!/ W \simeq \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_K}(X^*(T))^W) \simeq \mathrm{Spec}(\mathcal{O}_K[\chi_1, \cdots, \chi_r])$$

Define a $\mathbb{G}_{m,S}-$action on $\mathfrak{h} /\!\!/ W$ by linearly extending $t \cdot \chi_i = t^{2d_i} \chi_i$. The following proposition carries over to the relative setting without change.

Proposition 4.5.3 ([57], §7.4 Prop. 1). *The adjoint quotient $\chi : \mathcal{S} \longrightarrow \mathfrak{h} /\!\!/ W$ is $\mathbb{G}_{m,S}-$ equivariant with respect to action μ on $\mathcal{S}$ and the above action on $\mathfrak{h} /\!\!/ W$.*

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2. Actually, p non-torsion suffices; see Remark 3.2.3.

Remark 4.5.4. Note that the $\mathbb{G}_{m,S}$–action μ defined on $\mathcal{S}$ has the opposite weights of the $\mathbb{G}_{m,S}$–actions defined in ([52], §4.3) and ([14], §4.2.4) for the Kostant slice $\mathcal{S}$. Both constructions are essentially equivalent; we chose this formulation so as to get a contraction on $\mathfrak{h}$ and stay consistent with the weight conventions discussed in ([57], §7.4) and ([61], §2).

We now explain the notion of $\mathbb{G}_m$-deformations. Suppose X_0 is a singular hypersurface with a $\mathbb{G}_m$-action so that it is $\mathbb{G}_m$-equivariantly isomorphic to $V(f)$ for some weighted–homogeneous polynomial $f(x_1, \cdots, x_n)$. Here weighted homogeneity means that there exists a tuple $(d, k_1, \cdots, k_n)$ so that for any monomial

$$a_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

appearing in f we have $\sum_j i_j k_j = d$. In this case we can study the $\mathbb{G}_m$–equivariant deformation theory of $X_0 \simeq V(f)$ by replacing the objects and maps of the associated deformation functors in Section 2.3 with (formal) schemes equipped with a $\mathbb{G}_{m,S}$–action and $\mathbb{G}_{m,S}$–equivariant morphisms. In particular we may speak of $\mathbb{G}_{m,S}$–miniversal deformations.

It is not obvious that $\mathbb{G}_{m,S}$–miniversal deformations exist. The following theorem is based on the existence of general miniversal deformations and is proven by checking the definition of miniversality after equipping all objects with a $\mathbb{G}_m$–action.

Theorem 4.5.5 ([57], §2.5 Thm). *Suppose $f \in k[x, y, z]$ is weighted–homogeneous and defines a hypersurface $V(f)$ with isolated singularities in $\mathbb{A}_k^3$. Then there exists a $\mathbb{G}_{m,S}$–miniversal deformation of $V(f)$ over $S = \operatorname{Spec}(W(k))$.*

Example 4.5.6. Let us compare the $\mathbb{G}_m$–actions for a specific RDP surface. Polynomial $f(x, y, z) = z^2 + x^2 + y^4$ defines an A_3–singularity (Theorem 2.1.3) and is weighted–homogeneous with weights $(2, 1, 2)$. Its miniversal deformation is given by

$$V(F) \longrightarrow \operatorname{Spec}(\mathcal{O}_K[\![t_1, t_2, t_3]\!]), \quad F(x, y, z, t_1, t_2, t_3) = z^2 + x^2 + y^4 + t_1 y^2 + t_2 y + t_3$$

(see Example 2.5.5). Polynomial F is weighted–homogeneous with weights $(2, 1, 2, 2, 3, 4)$ and as a result base $\mathrm{Spec}(\mathcal{O}_K[\![t_1, t_2, t_3]\!])$ admits a $\mathbb{G}_{m,S}$–action with weights $(2, 3, 4)$, while total space $V(F)$ admits a $\mathbb{G}_{m,S}$–action with weights $(2, 1, 2, 2, 3)$.

On the other hand, for the subregular orbit in $\mathfrak{sl}_4$ (the unique Lie algebra of type A_3), one may compute its decomposition into good irreducible $\mathfrak{sl}_2$–representations (see [57], §7.4 Example) as

$$\mathfrak{sl}_4 \simeq V_4 \oplus V_2 \oplus V_2' \oplus V_2'' \oplus V_0$$

where $\mathfrak{sl}_2$–modules V_i have highest weight i. By inspecting the weights of the μ-action of Definition 4.5.2 we get weights $(6, 4, 4, 4, 2)$; note that in both Sections 4.3 and 4.4, a basis for $\mathcal{S}$ (up to translation with x) is given by a choice of *lowest* weight vectors for each irreducible $\mathfrak{sl}_2$–summand, and there is an identification $\mathcal{S} \simeq \mathrm{Spec}(\mathcal{O}_K[x_1, \cdots, x_5])$ by choosing x_i to be dual to the lowest–weight vectors. This is why we get the aforementioned weights (cf. [57], p. 110). For $\mathfrak{h}/\!\!/W$ we have that the homogeneous degrees of the fundamental generators are $(2, 3, 4)$ (see [58], Table in p. 112); this also follows directly from reading off the degrees of symmetric polynomials $\sigma_2, \sigma_3, \sigma_4$.

Note that if we double the weights of the $\mathbb{G}_m$–action on $V(F)$, then up to reordering variables the miniversal deformation $\phi : V(F) \to \mathrm{Spec}(\mathcal{O}_K[\![t_1, t_2, t_3]\!])$ and $\chi : \mathcal{S} \to \mathfrak{h}/\!\!/W$ are both $\mathbb{G}_{m,S}$–equivariant with the *same* $\mathbb{G}_m$–weights on source and target.

The previous example illustrates a more general principle, which we will clarify in Theorem 4.6.2 in the next section.

4.6   Grothendieck alterations for transverse slices.

We retain the assumptions of Section 4.5 and fix a Slodowy slice $\mathcal{S}$ at a fiberwise subregular nilpotent section x of $\mathfrak{g}$ (either by the Jacobson–Morozov method or the Spaltenstein method). Let $\pi : \widetilde{\mathcal{S}} \longrightarrow \mathcal{S}$ be the restriction of the Grothendieck alteration on $\mathcal{S}$ and define

$\mathbb{G}_{m,S}$–actions $\mu, \widetilde{\mu}$ on $\mathcal{S}, \widetilde{\mathcal{S}}$ as in Section 4.5.

Proposition 4.6.1. *Consider the following commutative diagram induced by restricting diagram* (3.5.3)

$$\begin{array}{ccc} \widetilde{\mathcal{S}} & \xrightarrow{\pi} & \mathcal{S} \\ \downarrow{\scriptstyle\widetilde{\chi}_{\mathrm{res}}} & & \downarrow{\scriptstyle\chi_{\mathrm{res}}} \\ \mathfrak{h} & \xrightarrow{\psi} & \mathfrak{h}/\!\!/W \end{array} \qquad (4.6.1)$$

to $\mathcal{S}$ and its preimage $\widetilde{\mathcal{S}}$ under the Grothendieck alteration. Then this diagram induces S-fiberwise a simultaneous resolution of the singularities of $\chi : \mathcal{S} \to \mathfrak{h}/\!\!/W$.

Proof. Let $\alpha : G \times \mathcal{S} \to \mathfrak{g}$, $\widetilde{\alpha} : G \times \widetilde{\mathcal{S}} \to \widetilde{\mathfrak{g}}$ denote the respective restrictions of the adjoint action of G on $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$. By G–invariance of the adjoint quotient χ and its resolution $\widetilde{\chi}$, their restrictions to $\mathcal{S}, \widetilde{\mathcal{S}}$ yield commutative diagrams:

$$\begin{array}{ccc} G \times \mathcal{S} & \xrightarrow{\alpha} & \mathfrak{g} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle\chi} \\ \mathcal{S} & \xrightarrow{\chi_{\mathrm{res}}} & \mathfrak{h}/\!\!/W \end{array} \qquad \begin{array}{ccc} G \times \widetilde{\mathcal{S}} & \xrightarrow{\widetilde{\alpha}} & \widetilde{\mathfrak{g}} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle\widetilde{\chi}} \\ \widetilde{\mathcal{S}} & \xrightarrow{\widetilde{\chi}_{\mathrm{res}}} & \mathfrak{h} \end{array} \qquad (4.6.2)$$

As $\chi \circ \alpha$ is flat and p_2 is flat and surjective, χ_{res} is flat ([62, Tag 02JZ]). Since $\pi : \widetilde{\mathcal{S}} \to \mathcal{S}$ comes from base-changing proper morphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}$, it is also proper. Applying Lemma 4.1.4 to $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ we get that $\widetilde{\alpha} : G \times \widetilde{\mathcal{S}} \to \widetilde{\mathfrak{g}}$ smooth, so as $p_2, \widetilde{\chi}$ are smooth, it follows that $\widetilde{\chi}_{\mathrm{res}}$ is smooth as well. We may therefore base–change to the special fiber by Lemma 3.4.2 (since formation of diagram (4.6.1 commutes with base–change) and check that we have a simultaneous resolution for $\overline{\mathcal{S}} \to \mathfrak{h}_s/\!\!/W$, where $\overline{\mathcal{S}} = \mathcal{S} \otimes k$ is a Slodowy slice over k. In this setting, the statement is true by ([57], §5.3 Corollary). $\qquad\square$

For ease of notation we will henceforth refer to $\chi_{\mathrm{res}}, \widetilde{\chi}_{\mathrm{res}}$ as χ and $\widetilde{\chi}$ respectively, when no confusion can arise. We now relate Proposition 4.6.1 (i.e. diagram (4.6.1)) to the miniversal

deformation of an RDP singularity *and* its minimal resolution. What follows is essentially the main theorem of [57]; the study of miniversal deformations of minimal resolutions in this context is due to [50].

Theorem 4.6.2. *Let $\chi : \mathcal{S} \longrightarrow \mathfrak{h}/\!\!/W$ denote the localization of map χ obtained by henselianizing $\mathcal{S}$ and $\mathfrak{h}/\!\!/W$ at x and 0 respectively. Let $\widetilde{\mathcal{S}}$ be the preimage of $\mathcal{S}$ under the Grothendieck alteration and $\widetilde{\chi} : \widetilde{\mathcal{S}} \longrightarrow \mathfrak{h}$ the associated map as in diagram (4.6.1). Assume $p = \mathrm{char}(k) > n + 1$ if $\mathfrak{g}$ is of Type A_n, otherwise assume p is good for $\mathfrak{g}$. Then χ is a $\mathbb{G}_{m,S}$–miniversal deformation of the RDP $x_s \in (\mathcal{N} \cap \mathcal{S})(k)$ and $\widetilde{\chi}$ is a miniversal deformation of the minimal resolution of x_s.*

Proof. Let r be the rank of RDP x_s and consider morphism χ first. By Proposition 2.5.4 there exists a miniversal (algebraic) deformation of x_s, which we denote as $\phi : V(F) \to \mathrm{Spec}(R)$ for some $F \in \mathcal{O}_K[\![x, y, z, t_1, \cdots, t_r]\!]$ and $R = \mathcal{O}_K[\![t_1, \cdots, t_r]\!]$. By Theorem 4.5.5 there exists a $\mathbb{G}_{m,S}$–action on $V(F)$ and $\mathrm{Spec}(R)$ making ϕ a $\mathbb{G}_{m,S}$–miniversal deformation.

We know that $\chi^{-1}(0) = \mathcal{N} \cap \mathcal{S}$ has an RDP singularity at x_s and χ is $\mathbb{G}_{m,S}$–equivariant with respect to action μ on $\mathcal{S}$ and the "Weyl exponents" action on $\mathfrak{h}/\!\!/W$ (Proposition 4.5.3), hence by miniversality we get a $\mathbb{G}_{m,S}$–equivariant morphism $f : \mathfrak{h}/\!\!/W \longrightarrow \mathrm{Spec}(R)$. It suffices to show f is an isomorphism, and by the fibral isomorphism criterion ([27], Cor. 17.9.5) it suffices to show $f_s : \mathfrak{h}_s/\!\!/W \longrightarrow \mathrm{Spec}(R \otimes k)$ is an isomorphism over geometric point $s = \mathrm{Spec}(k)$. So we reduce to showing the statement over k of sufficiently good characteristic, whence it follows from ([57], §8.7 Thm) by comparing the $\mathbb{G}_m$–weights on each of $\mathfrak{h}_s/\!\!/W$ and $\mathrm{Spec}(R \otimes k)$. For $\widetilde{\chi}$, the statement holds by ([55], Thm. 3.4). $\qquad\square$

Remark 4.6.3. Shepherd-Barron in ([55], Thm. 4.6) has also shown the above theorem via different methods and in *good* characteristic; the proof similarly contains a passage from characteristic zero to positive characteristic. In particular we may assume that p is good for Theorem 4.6.2 to be true.

Remark 4.6.4. The statement a priori concerns formal miniversal deformations, but since we are dealing with affine (isolated) singularities these deformations are algebraic (see Remark 2.5.7). We may therefore consider the henselianized versions of $\chi, \widetilde{\chi}$ when we refer to the simultaneous resolution diagram (4.6.1) and without loss of generality write $\mathfrak{h} /\!\!/ W \simeq \mathrm{Spec}(\mathcal{O}_K[\![t_1, \cdots, t_r]\!])$.

CHAPTER 5

THE MONODROMY WEYL ACTION

We now come to the central part of the article, the description of monodromy actions in terms of Weyl groups. We discuss the relevant W-actions in Section 5.6, along with the proof of the main theorem. In order to construct said W-actions, multiple tools from the theory of nearby cycles and (relative) perverse sheaves need to be combined, so we explain these concepts next.

5.1 Classical nearby cycles.

Throughout this section we work in the *small* étale topos setting. Our base scheme is a *strictly* henselian trait $S = \mathrm{Spec}(\mathcal{O}_K)$ with closed point $s = \mathrm{Spec}(k)$ and generic point $\eta = \mathrm{Spec}(K)$. We denote geometric points with a bar, e.g. $\bar{\eta}$, with the understanding that the underlying residue field is *separably* closed (e.g. $\bar{\eta} = \mathrm{Spec}(K^{\mathrm{sep}})$). Whenever appropriate we assume $\mathcal{O}_K$ is complete.

Given a finite-type S-scheme X, we denote by $\mathrm{D}^b(X)$ the bounded derived category of $\mathbb{Q}_\ell$-sheaves on X, where $\ell \neq p = \mathrm{char}(k)$. Most of the formalism below is usually developed first for finite coefficient rings $\mathbb{Z}/\ell^n$, but standard reductions via inverse limits and taking $(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ yield the same statements for $\mathbb{Q}_\ell$-coefficients, so we choose not to belabor this point.

Definition 5.1.1. Let $X \to S$ be a finite-type S-scheme with generic fiber X_η, geometric generic fiber $X_{\bar{\eta}}$ and special fiber X_s. Denote the respective inclusions by

$$X_{\bar{\eta}} \overset{\overset{\bar{j}}{\overbrace{}}}{\longrightarrow} X_\eta \overset{j}{\longrightarrow} X \overset{i}{\longleftarrow} X_s$$

(i) The *nearby cycles* functor is $\mathrm{R}\Psi_X : \mathrm{D}^b(X_\eta) \to \mathrm{D}^b(X_s)$, $\mathcal{F} \longmapsto i^* \mathrm{R}\bar{j}_*(\mathcal{F}_{\bar{\eta}})$, where $\mathcal{F}_{\bar{\eta}}$ is the pullback of $\mathcal{F}$ to $X_{\bar{\eta}}$. Complex $\mathrm{R}\Psi_X(\mathcal{F})$ is naturally equipped with an action of inertia $I = \mathrm{Gal}(\bar{\eta}/\eta) = \mathrm{Gal}_K$ ([4], Exposé XIII §1.3).

(ii) For $\mathcal{F} \in \mathrm{D}^b(X)$, adjunction map $\mathcal{F} \to \mathrm{R}\overline{j}_*(\mathcal{F}_{\overline{\eta}})$ gives an exact triangle

$$i^*\mathcal{F} \xrightarrow{\phi} \mathrm{R}\Psi_X(\mathcal{F}) \longrightarrow \mathrm{R}\Phi_X(\mathcal{F}) \xrightarrow{[1]} \tag{5.1.1}$$

defining the *vanishing cycles functor* $\mathrm{R}\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(X_s)$ as the cone of map ϕ.

Nearby and vanishing cycles have various functorial properties. For example, if $f : X \to Y$ is a morphism of S-schemes inducing maps $f_\eta : X_{\overline{\eta}} \to Y_{\overline{\eta}}$ and $f_s : X_s \to Y_s$ between the geometric generic and special fibers, there are natural maps

$$\mathrm{R}\Psi_Y(\mathrm{R}f_{\eta*}\mathcal{F}_{\overline{\eta}}) \longrightarrow \mathrm{R}f_{s*}\mathrm{R}\Psi_X(\mathcal{F}), \quad \mathcal{F} \in \mathrm{D}^b(X) \tag{5.1.2}$$

$$f_s^*\mathrm{R}\Psi_Y(\mathcal{F}) \longrightarrow \mathrm{R}\Psi_X(f_\eta^*\mathcal{F}_{\overline{\eta}}), \quad \mathcal{F} \in \mathrm{D}^b(Y) \tag{5.1.3}$$

Map (5.1.2) is an isomorphism when f is proper, and map (5.1.3) is an isomorphism when f is smooth. Moreover the natural I-action extends to $\mathrm{R}\Phi_X(\mathcal{F})$, making exact triangle (5.1.1) I-equivariant ([4], Exposé XIII, 2.1.7.1, 2.1.7.2, 2.1.2.4).

Given an S-scheme X with $\mathcal{F} \in \mathrm{D}^b(X)$ and a point $x \in X_s$, let $X_{(\overline{x})}$ denote the strict henselization of X at $\overline{x} \to X$. Then the stalks of the nearby cycles are computed as

$$(\mathrm{R}\Psi_X\mathcal{F})_{\overline{x}} \simeq \mathrm{R}\Gamma(X_{(\overline{x})} \times_S \overline{\eta}, \mathcal{F}_\eta)$$

We adopt the terminology of Illusie (cf. [36], §1.3) in saying that $X_{(\overline{x})}$ represents an ℓ-adic Milnor ball and generic fiber $X_{(\overline{x})} \times_S \overline{\eta}$ represents an ℓ-adic Milnor fiber, consistent with the classical fact that stalks of (complex) nearby cycles compute the cohomology of Milnor fibers.

Example 5.1.2. A more direct relationship with the classical Milnor fiber can be seen as follows. Assume X is a flat relative surface over S, smooth outside an isolated *rational*

singularity on the special fiber $\bar{x} \to X_s$, then $\mathrm{R}\Phi_X\mathbb{Q}_\ell$ is supported on the physical point x and we obtain $(\mathrm{R}^0\Phi_X\mathbb{Q}_\ell)_{\bar{x}} \simeq (\mathrm{R}^0\Psi_X\mathbb{Q}_\ell)_{\bar{x}}/\mathbb{Q}_\ell \simeq 0$ and $(\mathrm{R}^i\Phi_X\mathbb{Q}_\ell)_{\bar{x}} \simeq (\mathrm{R}^i\Psi_X\mathbb{Q}_\ell)_{\bar{x}}$ for $i > 0$. In this case we have ([4], Exposé XVI)

$$(\mathrm{R}^i\Phi_X\mathbb{Q}_\ell)_{\bar{x}} \simeq \begin{cases} \Lambda^r, & i = 2 \\ \\ 0, & i \neq 2 \end{cases}$$

where $r \geq 1$ is the dimension of the stalk as a $\mathbb{Q}_\ell$-vector space. This can be thought of as the ℓ-adic analogue of the topological Milnor fiber being a bouquet of n-spheres, hence having only top cohomology.

Since we will only deal with nearby cycles of families $X \to S$ acquiring isolated rational singularities, it is worth re-emphasizing that $\mathrm{R}^i\Psi_X\mathbb{Q}_\ell \simeq \mathrm{R}^i\Phi_X\mathbb{Q}_\ell$ for $i > 0$. Hence any statements regarding stalks of nearby cycles should be compatible with analogous statements in the complex setting, where people usually consider so-called "vanishing homology".

Remark 5.1.3. In the case that x is a hypersurface singularity locally defined by the vanishing of a weighted–homogeneous polynomial $f(x, y, z)$ (such as the normal form of an RDP as defined in Theorem 2.1.3), we have that the dimension of the Tjurina algebra $\dim(\mathrm{T}^1)$ (see Definition 2.3.5) equals the Milnor number $\mu = \mathrm{length}(\mathcal{E}\mathrm{xt}^1(\Omega^1_{X/S}, \mathcal{O}_X))$. If the I-action on $\mathrm{R}\Psi_X\mathbb{Q}_\ell$ is *tamely* ramified, meaning the action factors through tame quotient $I \twoheadrightarrow I_t \simeq I/P$, we have $r = \mu = \dim(\mathrm{T}^1)$. More generally the Deligne–Milnor conjecture states that $\dim(\mathrm{T}^1) = r + \mathrm{Sw}(\mathrm{R}^2\Psi_X\mathbb{Q}_\ell)$, where the Swan conductor term measures the wild ramification of the nearby cycles ([4], Exposé XVI, Conj. 1.9).

We will see that under our restrictions on the characteristic p, for a surface family X/S acquiring RDP singularities we have $\mathrm{R}\Psi_X\mathbb{Q}_\ell \simeq (\mathrm{R}\Psi_X\mathbb{Q}_\ell)^P$ (the wild inertia invariants) i.e. $\mathrm{R}\Psi_X\mathbb{Q}_\ell$ are identified with the *tame* inertial nearby cycles $\mathrm{R}\Psi_X^{\mathrm{tr}}\mathbb{Q}_\ell = \bar{i}^*\mathrm{R}j_*^{\mathrm{tr}}\mathbb{Q}_\ell$, where $j^{\mathrm{tr}} : X_{\eta^{\mathrm{tr}}} \hookrightarrow X$ is induced from inclusion $\eta^{\mathrm{tr}} = \mathrm{Spec}(K^{\mathrm{tr}}) \to S$.

5.2 Nearby cycles on formal schemes.

In this section we assume $S = \mathrm{Spec}(\mathcal{O}_K)$ is a *complete* local trait. In [10], Berkovich constructs a nearby cycles functor variant for a class of formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$, which includes finite-type schemes $\mathcal{X}/S$ completed along a closed subscheme of the special fiber $Y \subseteq \mathcal{X}_s$ ([10], §1). Denote by $\mathfrak{X}$ the formal completion $\widehat{\mathcal{X}}_Y$. Its special fiber $\mathfrak{X}_s$ is identified with finite-type scheme Y, and the generic fiber $\mathfrak{X}_\eta$ is a *rigid-analytic* space over K.

There exists an equivalence between formal schemes étale over $\mathfrak{X}$ and formal schemes étale over $\mathfrak{X}_s$ ([10], Prop. 2.1.(i)) and so composing the associated functor $\mathfrak{X}_s \to \mathfrak{X}$ with the generic fiber functor $\mathfrak{X} \to \mathfrak{X}_\eta$ induces a map of sites $\nu : (\mathfrak{X}_\eta)_{\mathrm{q\acute{e}t}} \longrightarrow (\mathfrak{X}_s)_{\mathrm{\acute{e}t}}$. The source endows $\mathfrak{X}_\eta$ with its *quasi*-etale site, where the quasi-etale covers of analytic spaces are defined in the sense of ([9], §3). There exists also a natural morphism of sites $\mu : (\mathfrak{X}_\eta)_{\mathrm{q\acute{e}t}} \longrightarrow (\mathfrak{X}_{\overline{\eta}})_{\mathrm{\acute{e}t}}$.

Definition 5.2.1. For an étale sheaf $\mathcal{F} \in \mathrm{D}^b(\mathfrak{X}_{\overline{\eta}})$, $\mathrm{R}\Psi_{\mathfrak{X}}^{\mathrm{Ber}}(\mathcal{F}) = \mathrm{R}\nu_*\mu^*(\mathcal{F})$ defines the *Berkovich nearby cycles* functor $\mathrm{R}\Psi_{\mathfrak{X}}^{\mathrm{Ber}} : \mathrm{D}^b(\mathfrak{X}_{\overline{\eta}}) \longrightarrow \mathrm{D}^b(\mathfrak{X}_s)$. It is a sheaf naturally equipped with a $\mathrm{Gal}(\overline{\eta}/\eta) = \mathrm{Gal}_K$-action ([10], Rem. 2.6).

The crux of this construction is a comparison theorem with the algebraic nearby cycles defined on $\mathcal{X}/S$: the rough idea is that $\mathrm{R}\Psi_{\mathcal{X}}\mathbb{Q}_\ell|_Y$ depends only on the formal completion $\mathfrak{X}$ along Y, hence on a formal neighborhood of Y. The comparison theorem below is stated for étale sheaves of torsion prime to p, though as we have remarked before the statement works for $\mathbb{Q}_\ell$ as well.

Theorem 5.2.2 ([10], Thm. 3.1, Cor. 3.5)**.** *Let* $\mathcal{X}, Y, \mathfrak{X}$ *be as above. Let* $\mathcal{F}$ *be an étale constructible sheaf on* $\mathcal{X}_\eta$ *with torsion prime to* p, *and denote by* $\widehat{\mathcal{F}}$ *its pullback to* $\mathfrak{X}_{\overline{\eta}}$. *Suppose* Y/S *is proper (e.g. finite). Then there exist canonical isomorphisms*

$$\mathrm{R}^n\Psi_{\mathcal{X}}\mathcal{F}|_Y \simeq \mathrm{R}^n\Psi_{\mathfrak{X}}^{\mathrm{Ber}}(\widehat{\mathcal{F}}), \quad \mathrm{R}\Gamma(Y, \mathrm{R}\Psi_{\mathcal{X}}\mathcal{F}) \simeq \mathrm{R}\Gamma(\mathfrak{X}_{\overline{\eta}}, \mathcal{F})$$

compatible with the action of $\mathrm{Gal}(\overline{\eta}/\eta)$ *on either side.*

Corollary 5.2.3. *Suppose $\mathcal{X}$ is a proper flat surface over S with smooth generic fiber $\mathcal{X}_\eta$ and special fiber $\mathcal{X}_s$ having exactly one RDP $x \in \mathcal{X}_s(k)$. Then the $\mathrm{Gal}(\overline{\eta}/\eta)$-action on $\mathrm{H}^2_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ depends only on a formal affine neighborhood of $\overline{x}$ in $\mathcal{X}$.*

Proof. Let

$$\widehat{\mathcal{O}}_{\mathcal{X}_s, \overline{x}} \simeq \frac{k[\![x, y, z]\!]}{f(x, y, z)}$$

be the completed local ring at the RDP singularity, f the normal form describing the RDP. By Corollary 2.5.6

$$\widehat{\mathcal{O}}_{\mathcal{X}, \overline{x}} \simeq \frac{\mathcal{O}_K[\![x, y, z]\!]}{F(x, y, z)}, \quad F(x, y, z) \equiv f(x, y, z) \mod p$$

for some polynomial $F(x, y, z)$ that is the pullback of a miniversal equation of the RDP as in Corollary 2.5.6. Now by the comparison of Berkovich (Theorem 5.2.2), we obtain a canonical Galois-equivariant isomorphism

$$(\mathrm{R}^2 \Psi_{\mathcal{X}} \mathbb{Q}_\ell)_{\overline{x}} \simeq \mathrm{H}^2((\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{X}, \overline{x}}))_{\overline{\eta}}, \mathbb{Q}_\ell) \tag{5.2.1}$$

where the generic fiber of the formal completion of $\mathcal{X}$ along $\overline{x}$ is $\mathfrak{X}_{\overline{\eta}} = \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{X}, \overline{x}}))_{\overline{\eta}}$, a rigid-analytic variety over $\overline{\eta}$, and the right-hand side denotes Berkovich's ℓ-adic cohomology for analytic spaces ([10], §3). Now let $\mathcal{Y} \to S$ denote the affine relative surface

$$\mathcal{Y} = \mathrm{Spec}\Big(\frac{\mathcal{O}_K[x, y, z]}{F(x, y, z)}\Big), \quad \mathcal{Y}_s = \mathrm{Spec}\Big(\frac{k[x, y, z]}{f(x, y, z)}\Big)$$

having the same RDP at $\overline{y} \to \mathcal{Y}_s$ as $\overline{x} \to \mathcal{X}_s$. We have $\widehat{\mathcal{O}}_{\mathcal{Y}, \overline{y}} \simeq \widehat{\mathcal{O}}_{\mathcal{X}, \overline{x}}$ and so (5.2.1) gives $(\mathrm{R}^2 \Psi_{\mathcal{X}} \mathbb{Q}_\ell)_{\overline{x}} \simeq (\mathrm{R}^2 \Psi_{\mathcal{Y}} \mathbb{Q}_\ell)_{\overline{y}}$ Galois-equivariantly. To compare this with the Galois action $\mathrm{H}^2_{\mathrm{ét}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$, we use the nearby cycles spectral sequence:

$$E_2^{ij} = \mathrm{H}^i(\mathcal{X}_s, \mathrm{R}^j \Psi_{\mathcal{X}} \mathbb{Q}_\ell) \implies \mathbb{H}^{i+j}(\mathcal{X}_s, \mathrm{R}\Psi_{\mathcal{X}} \mathbb{Q}_\ell) \simeq \mathrm{H}^{i+j}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$$

Since $\mathrm{R}^2\Psi_{\mathcal{X}}\mathbb{Q}_\ell$ is a skyscraper sheaf supported on $\overline{x}$, the E_2-page is

$$\mathrm{H}^0(\mathcal{X}_s, \mathrm{R}^2\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \qquad\qquad 0 \qquad\qquad\qquad 0$$

$$\overset{d_2}{}$$

$$0 \qquad\qquad\qquad 0 \qquad\qquad\qquad\longrightarrow 0$$

$$\mathrm{H}^0(\mathcal{X}_s, \mathbb{Q}_\ell) \qquad\qquad \mathrm{H}^1(\mathcal{X}_s, \mathbb{Q}_\ell) \qquad\qquad \mathrm{H}^2(\mathcal{X}_s, \mathbb{Q}_\ell)$$

hence the spectral sequence degenerates at E_2 and $E_2^{p,q} = \mathrm{gr}_p^F \mathrm{H}^{p+q}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ for the abutt-ment filtration $F^\bullet$. Looking at the nontrivial graded pieces yields $E_2^{0,2} = \mathrm{H}^2(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)/E_2^{2,0}$ i.e. we obtain $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell) \simeq (\mathrm{R}^2\Psi_{\mathcal{X}}\mathbb{Q}_\ell)_x \oplus \mathrm{H}^2(\mathcal{X}_s, \mathbb{Q}_\ell)$ with trivial inertia action on $\mathrm{H}^2(\mathcal{X}_s, \mathbb{Q}_\ell)$ and the induced monodromy $\mathrm{Gal}(\overline{\eta}/\eta)$-action on the stalks of $\mathrm{R}\Psi_{\mathcal{X}}\mathbb{Q}_\ell$; the latter claim follows from the Galois-equivariance of the E_2-page of the spectral sequence. $\qquad\square$

5.3   Nearby cycles on a Grothendieck topos.

Since we will need to describe nearby cycles of ℓ-adic sheaves over bases of dimension > 1, we collect here their general formalism and properties in amenable situations. In order to do this, we will need the language of oriented toposes; for a modern English reference, see ([36], §1).

We retain the conventions of Section 5.1 and consider S-schemes $f : X \to S$, $g : Y \to S$. Bounded derived category of étale sheaves $\mathrm{D}^b(X)$ is assumed to have coefficient ring $\mathbb{Z}/\ell^n$, but the statements below will also work for $\mathbb{Q}_\ell$-coefficients.

Definition 5.3.1. The left oriented 2-product topos $X \overset{\leftarrow}{\times}_S Y$ is the Grothendieck topos defined in a universal way by the data of 2-commutative diagram

$$
\begin{array}{ccc}
X \overset{\leftarrow}{\times}_S Y & \overset{p_2}{\longrightarrow} & Y \\
\downarrow{\scriptstyle p_1} & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle g} \\
X & \overset{f}{\longrightarrow} & S
\end{array}
\qquad\qquad (5.3.1)
$$

where X, Y and S denote the étale toposes associated to the schemes and $\tau : g \circ p_2 \to f \circ p_1$ is a 2-morphism. From these data we get a defining site for $X \overset{\leftarrow}{\times}_S Y$ with the following covering families: for maps $U \to V \leftarrow W$ étale over $X \to S \leftarrow Y$ put $\{U_i \to V \leftarrow W\}$ a covering of $U \to V \leftarrow W$ where $\{U_i\} \to U$ is a covering, and $\{U \to V \leftarrow W_i\}$ a covering of $U \to V \leftarrow W$ where $\{W_i\} \to W$ is a covering. The third type of families is given by coverings $\{U \to V' \leftarrow W\}$ of $U \to V \leftarrow W$ for which the induced square

$$
\begin{array}{ccc}
V' & \longleftarrow & W' \\
\nearrow \quad \downarrow & \square & \downarrow \\
U \longrightarrow V & \longleftarrow & W
\end{array}
$$

is cartesian; note the individual maps $V' \to V$, $W' \to W$ need not be coverings in this case ([36], 1.1.1).

Maps between such oriented toposes $X' \overset{\leftarrow}{\times}_{S'} Y' \longrightarrow X \overset{\leftarrow}{\times}_S Y$ are defined by the data of maps $\{X' \to X, S' \to S, Y' \to Y\}$ and appropriate 2-morphisms between these maps ([36], 1.1.2).

A special case is is $(Y, g) = (S, \mathrm{id})$, where the oriented product $X \overset{\leftarrow}{\times}_S S$ is called the *vanishing topos* of X/S. The points of the vanishing topos consist of triples $(\overline{x}, \overline{\eta}, \mathrm{sp})$ where $\overline{x} \to X, \overline{\eta} \to S$ are geometric points together with a specialization morphism $\mathrm{sp} : \overline{\eta} \to S_{(f(x))}$.

Definition 5.3.2 (Nearby cycles over general bases)**.**

(i) There exists a unique morphism $\Psi : X \longrightarrow X \overset{\leftarrow}{\times}_S S$ compatible with diagram (5.3.1), by the universal property of products. The derived pushforward

$$
\mathrm{R}\Psi_f = \mathrm{R}\Psi_* : \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(X \overset{\leftarrow}{\times}_S S)
$$

is the *nearby cycles functor* relative to f.

(ii) Let $(\overline{x}, \overline{\eta}, \mathrm{sp})$ be a point of $X \overset{\leftarrow}{\times}_S S$ with $\overline{x}$ over a geometric point $\overline{s} \to S$, $\eta \in S$ and $\mathrm{sp} : \overline{\eta} \to S_{(\overline{s})}$ a fixed specialization. There exists a unique map $S_{(\overline{\eta})} \to S_{(\overline{s})}$ compatible

60

with the specialization and map $\overline{\eta} \to S_{(\overline{\eta})}$ ([62, Tag 08HR]). We therefore have natural inclusions

$$i_s : X_{\overline{s}} \hookrightarrow X_{(\overline{s})} = X \times_S S_{(\overline{s})}, \ \ j_\eta^s : X_{(\overline{\eta})} = X \times_S S_{(\overline{\eta})} \to X_{(\overline{s})}$$

and $\mathrm{R}\Psi_\eta^s = i_s^* \mathrm{R} j_{\eta*}^s : \mathrm{D}^b(X_{(\overline{\eta})}) \longrightarrow \mathrm{D}^b(X_{\overline{s}})$ is the *sliced nearby cycles functor*.

The two nearby cycles are related as follows: one identifies topos $X_{\overline{s}} \overset{\leftarrow}{\times}_S \overline{\eta}$ as sheaves on $X_{\overline{s}}$ together with a $\mathrm{Gal}(\overline{\eta}/\eta)$-action, and by functoriality we have a morphism of toposes

$$\overset{\leftarrow}{i}_{(s,\eta)} : X_{\overline{s}} = X_{\overline{s}} \overset{\leftarrow}{\times}_S \overline{\eta} \longrightarrow X_{\overline{s}} \overset{\leftarrow}{\times}_S S \overset{\overset{\leftarrow}{i_s}}{\longrightarrow} X \overset{\leftarrow}{\times}_S S$$

Then $\mathrm{R}\Psi_\eta^s = (\overset{\leftarrow}{i}_{(s,\eta)})^* \mathrm{R}\Psi_f$. See ([36], §1.3 and §1.4) for details. We mention in passing that one can also define a vanishing cycles functor $\mathrm{R}\Phi_f$ in the topos setting, but we will not use it; the construction is given in (*loc. cit.*, 1.2.4).

Nearby cycles in this generality still satisfy functorial properties. An important example is the case of S–schemes $f : X \to S$ and $g : Y \to S$ and a map $h : X \to Y$ of S–schemes. Then the induced commutative diagram

$$\begin{array}{ccccc}
X & \overset{\Psi_f}{\longrightarrow} & X \overset{\leftarrow}{\times}_S S & \overset{\overset{\leftarrow}{i}_{X,(s,\eta)}}{\longleftarrow} & X_{\overline{s}} \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overset{\leftarrow}{h}} & & \downarrow{\scriptstyle h_{\overline{s}}} \\
Y & \overset{\Psi_g}{\longrightarrow} & Y \overset{\leftarrow}{\times}_S S & \underset{\overset{\leftarrow}{i}_{Y,(s,\eta)}}{\longleftarrow} & Y_{\overline{s}}
\end{array} \qquad (5.3.2)$$

yields $\mathrm{R}\Psi_g(\mathrm{R}h_* \mathcal{F}) \simeq \mathrm{R}\overset{\leftarrow}{h}_* \mathrm{R}\Psi_f \mathcal{F}$ for $\mathcal{F} \in \mathrm{D}^b(X)$; furthermore, if h is *proper*, then formation of $\mathrm{R}\overset{\leftarrow}{h}_*$ commutes with base–change on X and S, so that in particular diagram (5.3.2)

induces an isomorphism (see [49], Lemme 8.1.1)

$$\mathrm{R}\Psi_{g,\eta}^s(\mathrm{R}h_*\mathcal{F}) = (\overleftarrow{i}_{Y,(s,\eta)})^*\mathrm{R}\Psi_g(\mathrm{R}h_*\mathcal{F}) \simeq \mathrm{R}(h_{\overline{s}})_*(\overleftarrow{i}_{X,(s,\eta)})^*\mathrm{R}\Psi_f\mathcal{F} = \mathrm{R}(h_{\overline{s}})_*\mathrm{R}\Psi_{f,\eta}^s\mathcal{F}$$

(5.3.3)

There are natural generalizations of ℓ–adic Milnor fibers to this setting (cf. Section 5.1). Given a point $(\overline{x}, \overline{\eta}, \mathrm{sp})$ of $X \overleftarrow{\times}_S S$ with $\overline{x}$ over $\overline{s}$ and a sheaf $\mathcal{F} \in \mathrm{D}^b(X)$, the stalks of nearby cycles may be computed via ([2], Exposé VII, §5.8) as

$$\mathrm{R}\Psi_\eta^s(\mathcal{F})_{\overline{x}} \simeq \mathrm{R}\Psi_f(\mathcal{F})_{(\overline{x},\overline{\eta},\mathrm{sp})} \simeq \mathrm{R}\Gamma(X_{(\overline{x})} \times_{S_{(\overline{s})}} S_{(\overline{\eta})}, \mathcal{F}) \qquad (5.3.4)$$

The scheme $X_{(\overline{x})} \times_{S_{(\overline{s})}} S_{(\overline{\eta})}$ is called the *Milnor tube* at $(\overline{x}, \overline{\eta}, \mathrm{sp})$; it contains Milnor fiber $X_{(\overline{x})} \times_{S_{(\overline{s})}} \overline{\eta}$ as a closed subscheme.

Example 5.3.3. We relate these constructions to the classical nearby cycles. Let S be a strictly henselian trait with closed point $\overline{s}$, generic point η and geometric generic point $\overline{\eta}$. Note $S_{(\overline{s})} = S$ and $S_{(\overline{\eta})} = \overline{\eta}$. For $f : X \to S$ of finite type we have

$$X \overleftarrow{\times}_S S = (X_\eta \overleftarrow{\times}_S S) \cup (X_s \overleftarrow{\times}_S S) = X_\eta \cup X_s \cup (X_s \overleftarrow{\times}_S \overline{\eta})$$

where the last (nontrivial) topos on the right is identified with sheaves on X_s together with a $\mathrm{Gal}(\overline{\eta}/\eta)$-action. The classical nearby cycles $\mathrm{R}\Psi_X(\mathcal{F})$ for $\mathcal{F} \in \mathrm{D}^b(X_\eta, \Lambda)$ are

$$\mathrm{R}\Psi_X(\mathcal{F}_{\overline{\eta}}) = \mathrm{R}\Psi_f(\mathcal{F})|_{X_s \overleftarrow{\times}_S \overline{\eta}}$$

and are therefore identified with the sliced nearby cycles $\mathrm{R}\Psi_\eta^s(\mathcal{F})$. The restriction map is induced from $X_s \to X$ and $\overline{\eta} \to S$. Moreover, for a geometric point $\overline{x} \to X_s$, the Milnor tube $X_{(\overline{x})} \times_S S_{(\overline{\eta})} \simeq X_{(\overline{x})} \times_S \overline{\eta}$ is identified with the Milnor fiber.

In general, nearby cycles $\mathrm{R}\Psi_f\mathcal{F}$ need not be well-behaved; for example, it may not be

constructible. Furthermore, base-changing via $S' \to S$ yields cartesian squares

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g\ } & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
S' & \longrightarrow & S
\end{array}
\qquad\qquad
\begin{array}{ccc}
X' & \xrightarrow{\ \ g\ \ } & X \\
\downarrow{\scriptstyle \Psi_{f'}} & & \downarrow{\scriptstyle \Psi_f} \\
X' \overset{\leftarrow}{\times}_{S'} S' & \xrightarrow{\ \overleftarrow{g}\ } & X \overset{\leftarrow}{\times}_S S
\end{array}
$$

and the associated base-change map

$$
(\overleftarrow{g})^* \mathrm{R}\Psi_f \mathcal{F} \longrightarrow \mathrm{R}\Psi_{f'}(g^* \mathcal{F})
$$

is *not* always an isomorphism: see ([36], 1.7(d)) for a classical example of Deligne which shows that for the origin blowup $f : \widetilde{\mathbb{A}}_k^2 \to \mathbb{A}_k^2$, $\mathrm{R}\Psi_f \mathbb{Q}_\ell$ is not constructible and does not commute with base-change on $\mathbb{A}_k^2$. Moreover, for any point in the exceptional divisor and a nonzero point on $\mathbb{A}_k^2$, the associated Milnor tube is not of finite type.

Nevertheless, a special case where all the above assertions are true is the following.

Theorem 5.3.4 (Deligne, [36] 1.7.(c)). *Let $f : X \to S$ be separated and finite-type, and $\mathcal{F} \in \mathrm{D}_c^b(X)$. Let Z be the complement of the largest open set $U \subseteq X$ so that $f|_U$ is universally locally acyclic over S. If $Z \to S$ is quasifinite, then $\mathrm{R}\Psi_f \mathcal{F}$ is constructible and its formation commutes with any base-change $S' \to S$.*

Remark 5.3.5. We call $(f, \mathcal{F})$ Ψ-*good* in this case. It follows from the theorem that the sliced nearby cycles $\mathrm{R}\Psi_\eta^s(\mathcal{F})$ are also constructible and commute with base-change; in particular the cohomology of Milnor tubes as computed in (5.3.4) restricts isomorphically to the cohomology of Milnor fibers:

$$
(\mathrm{R}\Psi_f \mathcal{F})_{(\overline{x}, \overline{\eta}, \mathrm{sp})} \xrightarrow{\ \sim\ } \mathrm{R}\Gamma(X_{(\overline{x})} \times_{S_{(\overline{s})}} \overline{\eta}, \mathcal{F}) \tag{5.3.5}
$$

5.4 Relative perverse sheaves and the Grothendieck alteration.

We retain the conventions of Section 5.1. In this section we will discuss a recent notion of *relative* perverse t-structures on $\mathrm{D}_c^b(X)$ for S–schemes X, which is compatible (in a sense)

63

with the absolute perverse t-structures on the geometric fibers i.e. $\mathrm{D}^b_c(X_{\overline{\eta}})$ and $\mathrm{D}^b_c(X_{\overline{s}})$. As before we may assume the underlying coefficient field to be $\mathbb{Q}_\ell$.

Following [29], we define a full subcategory of $\mathrm{D}^b(X)$

$$^{p/S}\mathrm{D}(X)^{\leq 0} := \{\mathcal{F} \in \mathrm{D}^b(X) \mid \mathcal{F}|_{X_{\overline{s}}} \in {}^p\mathrm{D}(X_{\overline{s}})^{\leq 0} \text{ for all geometric points } \overline{s} \to S\} \quad (5.4.1)$$

where $(^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ denotes the absolute perverse t-structure for schemes over fields. In the particular case of $S = \mathrm{Spec}(\mathcal{O}_K)$, one just restricts $\mathcal{F}$ to $X_{\overline{\eta}}$ and X_s and checks perversity in the usual absolute sense.

It can be shown that (5.4.1) forms the connective part of a t-structure on $\mathrm{D}^b(X)$ and the goal is then to show that the coconnective part $^{p/S}\mathrm{D}(X)^{\geq 0}$ has the required description in analogy with (5.4.1) and further induces a t-structure on the bounded derived category of constructible complexes $\mathrm{D}^b_c(X)$.

Theorem 5.4.1 ([29], Thm. 6.1). *Let $f : X \to S$ be a finite-type S–scheme. The full subcategories $(^{p/S}\mathrm{D}^{\leq 0}, {}^{p/S}\mathrm{D}^{\geq 0})$ of $\mathrm{D}^b(X)$ define a unique t-structure on $\mathrm{D}^b(X)$, the* relative perverse t-structure, *so that*

(i) *A sheaf $\mathcal{F} \in \mathrm{D}^b(X)$ is in $^{p/S}\mathrm{D}(X)^{\leq 0}$, resp. $^{p/S}\mathrm{D}(X)^{\geq 0}$, if and only if $\mathcal{F}|_{X_{\overline{s}}} \in {}^p\mathrm{D}(X_{\overline{s}})^{\leq 0}$ and $\mathcal{F}|_{X_{\overline{\eta}}} \in {}^p\mathrm{D}(X_{\overline{\eta}})^{\leq 0}$, resp. $\mathcal{F}|_{X_{\overline{s}}} \in {}^p\mathrm{D}(X_{\overline{s}})^{\geq 0}$ and $\mathcal{F}|_{X_{\overline{\eta}}} \in {}^p\mathrm{D}(X_{\overline{\eta}})^{\geq 0}$.*

(ii) *For any morphism $g : \widetilde{S} \to S$ with induced base-change $\widetilde{g} : X_{\widetilde{S}} \to X$, pullback functor $\widetilde{g}^* : \mathrm{D}^b(X) \to \mathrm{D}^b(X_{\widetilde{S}})$ is t-exact with respect to the relative perverse t-structure, hence commutes with the associated truncations $(\tau^{\leq 0}, \tau^{\geq 0})$.*

(iii) *For any open and closed decomposition $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ into S-schemes, the relative perverse t-structure on $\mathrm{D}^b(X)$ is obtained by gluing (recollement) from the relative perverse t-structures on $\mathrm{D}^b(U)$ and $\mathrm{D}^b(Z)$.*

We note that (ii) and (iii) are formal consequences of (i), once the perverse t-structure properties have been established. We also note that if $S = \mathrm{Spec}(k)$ is a field and X is a

finite-type k-scheme, the relative perverse t-structure is identified with the usual (middle) perverse t-structure on $\mathrm{D}^b(X)$, while if $X = S$ then the relative perverse t-structure coincides with the standard t-structure on $\mathrm{D}^b(S)$. In view of (ii), relative perverse sheaves on X pull back to absolute perverse sheaves on $X_{\bar{\eta}}$ and X_s.

We apply the notion of relative perversity to the setting of Chevalley algebras $\mathfrak{g}$ over $S = \mathrm{Spec}(\mathcal{O}_K)$ and their associated bundles $\widetilde{\mathfrak{g}}$. Recall the setting of Sections 3.4 and 3.5, so that in particular $\mathfrak{g}$ is a simple simply-laced Chevalley $\mathcal{O}_K$-algebra and $p = \mathrm{char}(k)$ is very good for $\mathfrak{g}$. The Grothendieck alteration $\pi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ restricts to a Galois W-torsor $\pi^{\mathrm{rs}} : \widetilde{\mathfrak{g}}^{\mathrm{rs}} \longrightarrow \mathfrak{g}^{\mathrm{rs}}$ over the regular semisimple locus (Proposition 3.5.3).

For Lie algebras $\mathfrak{g}$ over algebraically closed field k, the above fact is classically known and furthermore it implies that the (absolute) perverse sheaf

$$\mathcal{F} = \mathrm{R}\pi_* \mathbb{Q}_\ell [\dim \mathfrak{g}]$$

is equipped with a W-action, constructed by Borho–MacPherson (who attribute it originally to Lusztig) in the following way: π is a *small* morphism so in particular $\mathcal{F}$ is an IC sheaf, i.e. the intermediate extension of local system $\mathcal{G} = \pi_*^{\mathrm{rs}} \mathbb{Q}_\ell$ ([13], §1.8). Sheaf $\mathcal{G}$ is W-equivariant and W acts by "deck transformations", so by the functoriality of the intermediate extension functor, this W-action *uniquely* extends to $\mathcal{F}$ (*loc. cit.*, §2.6 Proposition[1]).

We next show that this result of Borho–MacPherson extends to the relative Grothendieck alteration π in the setting of Chevalley $\mathcal{O}_K$-algebras, where now $\mathcal{F}$ is *relatively* perverse and in particular an "intermediate extension" object in $\mathrm{D}^b(\mathfrak{g})$. By an IC sheaf in this setting we mean that, if $j : \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ is the inclusion of the open dense subscheme of regular semisimple elements, then $\mathcal{F} \simeq j_{!*}\mathcal{G}$.

Theorem 5.4.2. *Let $\pi : \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ denote the Grothendieck alteration of Chevalley $\mathcal{O}_K$-algebra*

---

1. Note that Borho–MacPherson state a more general version here, in terms of a parabolic subgroup $P \subset G$; we need only take P to be a Borel for our purposes.

$\mathfrak{g}$. *Then:*

(i) *Complex* $\mathcal{F} = \mathrm{R}\pi_*\mathbb{Q}_\ell[\dim \mathfrak{g}] \in \mathrm{D}_c^b(\mathfrak{g})$ *is relatively perverse.*

(ii) *Complex* $\mathcal{F}$ *is an* IC *sheaf.*

Proof. Denote by $j : \mathfrak{g}_\eta \hookrightarrow \mathfrak{g}$ and $i : \mathfrak{g}_s \hookrightarrow \mathfrak{g}$ the respective open and closed immersions of the geometric generic and special fibers, with the understanding that $\mathfrak{g}_\eta$ is Lie algebra $\mathfrak{g}_{\overline{\eta}} = \mathfrak{g}_{\overline{K}}$ so that the constructions of Sections 3.4 and 3.5 make sense for it too. Part (i) is immediate from Theorem 5.4.1 since proper base-change gives

$$\mathcal{F}|_{\mathfrak{g}_\eta} = j^*\mathrm{R}\pi_*\mathbb{Q}_\ell[\dim \mathfrak{g}] \simeq \mathrm{R}(\pi_\eta)_*\mathbb{Q}_\ell[\dim \mathfrak{g}]$$

and $\mathcal{F}|_{\mathfrak{g}_s} \simeq \mathrm{R}(\pi_s)_*\mathbb{Q}_\ell[\dim \mathfrak{g}]$ as the formation of $\widetilde{\mathfrak{g}}$ commutes with base-change on S in good characteristic. Thus, both restrictions are perverse sheaves on $\mathfrak{g}_\eta$ and $\mathfrak{g}_s$, respectively; in fact they are IC sheaves by the reasoning in ([13], §1.8).

For (ii), let $j' : \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ and $i' : \mathfrak{g}' = \mathfrak{g} \setminus \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ denote respectively the open and closed immersions of the regular semisimple elements and their complement. Denote by $^p\mathcal{H}^0$ the composition $\tau^{\leq 0} \circ \tau^{\geq 0}$ of the relative perverse truncation functors, which is itself a cohomological functor, and set $\mathcal{G} = j'^*\mathcal{F}$. We wish to show

$$\mathcal{F} \simeq j'_{!*}(\mathcal{G}) = \mathrm{im}(^p\mathcal{H}^0(j'_!\mathcal{G}) \to {}^p\mathcal{H}^0(j'_*\mathcal{G}))$$

Note that $\mathcal{G}$ is a lisse sheaf on $\mathfrak{g}^{\mathrm{rs}}$ via proper base-change along cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}^{\mathrm{rs}} & \xrightarrow{\widetilde{j}'} & \widetilde{\mathfrak{g}} \\ \downarrow{\scriptstyle \pi^{\mathrm{rs}}} & & \downarrow{\scriptstyle \pi} \\ \mathfrak{g}^{\mathrm{rs}} & \xrightarrow{j'} & \mathfrak{g} \end{array}$$

That is, $j'^*\mathcal{F} \simeq \pi_*^{\mathrm{rs}}\mathbb{Q}_\ell[\dim \mathfrak{g}]$ and π^{rs} is a Galois W-torsor (Proposition 3.5.3), so that $\pi_*^{\mathrm{rs}}\mathbb{Q}_\ell[\dim \mathfrak{g}]$ is lisse by algebraic Ehresmann ([47], Thm. 20.2). Moreover $\mathcal{F} = {}^p\mathcal{H}^0(\mathcal{F})$ (via

66

the perversity established in (i)) sits in long exact sequences

$$\cdots \longrightarrow {}^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G}) \longrightarrow \mathcal{F} \longrightarrow {}^{p}\mathcal{H}^{0}(i'_{*}i'^{*}\mathcal{F}) \xrightarrow{[1]} \cdots \tag{5.4.2}$$

$$\cdots \longrightarrow {}^{p}\mathcal{H}^{0}(i'_{*}i'^{!}\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow {}^{p}\mathcal{H}^{0}(j'_{*}\mathcal{G}) \xrightarrow{[1]} \cdots \tag{5.4.3}$$

which are respectively coming from the standard triangles (in the *derived* setting)

$$j'_{!}j'^{*}\mathcal{F} \to \mathcal{F} \to i'_{*}i'^{*}\mathcal{F} \xrightarrow{[1]}, \quad i'_{*}i'^{!}\mathcal{F} \to \mathcal{F} \to j'_{*}j'^{*}\mathcal{F} \xrightarrow{[1]} \tag{5.4.4}$$

Set $\mathcal{C}_1 = i'_{*}i'^{*}\mathcal{F} = \mathrm{Cone}(j'_{!}\mathcal{G} \to \mathcal{F})$ and $\mathcal{C}_2 = i'_{*}i'^{!}\mathcal{F}[1] = \mathrm{Cone}(\mathcal{F} \to j'_{*}\mathcal{G})$. In order to show $\mathcal{F} \simeq j'_{!*}\mathcal{G}$ it suffices to show $\mathcal{F} \hookrightarrow {}^{p}\mathcal{H}^{0}(j'_{*}\mathcal{G})$ and ${}^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G}) \twoheadrightarrow \mathcal{F}$ i.e. the respective kernel and cokernel vanish. So it suffices to have ${}^{p}\mathcal{H}^{0}(\mathcal{C}_1) = 0$ and ${}^{p}\mathcal{H}^{-1}(\mathcal{C}_2) = 0$.

A brief note on why checking the above two conditions is enough: suppose ${}^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G}) \twoheadrightarrow \mathcal{F}$, then via right-exactness of i'^{*} we have ${}^{p}\mathcal{H} \circ (i')^{*}$ right-exact ([5], Lemma A.7.14) so that ${}^{p}\mathcal{H}^{0}(i'^{*} \circ {}^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G})) \twoheadrightarrow {}^{p}\mathcal{H}^{0}(i'^{*}\mathcal{F})$. Now ${}^{p}\mathcal{H}^{0} \circ i'^{*} \circ {}^{p}\mathcal{H}^{0} \circ j'_{!}$ is left-adjoint to

$$ {}^{p}\mathcal{H}^{0} \circ j'^{*} \circ {}^{p}\mathcal{H}^{0} \circ i'_{*} \simeq {}^{p}\mathcal{H}^{0} \circ j'^{*} \circ i'_{*} = 0 $$

where the penultimate equivalence is due to the perverse t-exactness of j'^{*} and i'_{*}. Hence ${}^{p}\mathcal{H}^{0}(i'^{*}\mathcal{F}) = \tau^{\geq 0}(i'^{*}\mathcal{F}) = 0$, meaning $i'^{*}\mathcal{F} \in {}^{p/S}\mathrm{D}(\mathfrak{g}')^{\leq -1}$. The dual argument for $\mathcal{F} \hookrightarrow {}^{p}\mathcal{H}^{0}(j'_{*}\mathcal{G})$ gives $i'^{!}\mathcal{F} \in {}^{p/S}\mathrm{D}(\mathfrak{g}')^{\geq 1}$, altogether giving the familiar IC sheaf conditions for $\mathcal{F}$ as a result of *recollement* ([11], 1.4.24).

Consider ${}^{p}\mathcal{H}^{0}(\mathcal{C}_1)$ first; it is enough to show $j^{*}\,{}^{p}\mathcal{H}^{0}(\mathcal{C}_1) = i^{*}\,{}^{p}\mathcal{H}^{0}(\mathcal{C}_1) = 0$ since if ${}^{p}\mathcal{H}^{0}(\mathcal{C}_1)$ had nonempty support, the support would intersect the supports of either of these complexes. Now j^{*} is perverse t-exact by Theorem 5.4.1 (ii), so applying j^{*} to the first triangle in (5.4.4)

and then taking $^p\mathcal{H}^0$, which commutes with j^*, we obtain

$$\cdots \longrightarrow {}^p\mathcal{H}^0(j^* j_!' \mathcal{G}) \longrightarrow j^*\mathcal{F} \longrightarrow {}^p\mathcal{H}^0(j^*\mathcal{C}_1) \xrightarrow{[1]} \cdots \tag{5.4.5}$$

Since $\mathfrak{g}^{\mathrm{rs}}$ represents the open subfunctor in $\mathfrak{g}$ of regular semisimple elements, the following diagram is cartesian

$$\begin{array}{ccc} \mathfrak{g}_\eta^{\mathrm{rs}} & \xrightarrow{\ j^{\mathrm{rs}}\ } & \mathfrak{g}^{\mathrm{rs}} \\ {\scriptstyle j_\eta'}\big\downarrow & & \big\downarrow{\scriptstyle j'} \\ \mathfrak{g}_\eta & \xrightarrow{\ j\ } & \mathfrak{g} \end{array} \tag{5.4.6}$$

where $j^{\mathrm{rs}} : \mathfrak{g}_\eta^{\mathrm{rs}} \to \mathfrak{g}^{\mathrm{rs}}$ is induced from $\bar\eta \to S$ and $j_\eta' : \mathfrak{g}_\eta^{\mathrm{rs}} \to \mathfrak{g}_\eta$ is the open immersion induced from j'. Hence base-change yields $j^* j_!' \simeq j_{\eta!}' j^{\mathrm{rs}*}$ and

$$j^{\mathrm{rs}*}\mathcal{G} = j^{\mathrm{rs}*} j'^* \mathcal{F} \simeq j_\eta'^*(j^*\mathcal{F}) = j_\eta'^*(\mathrm{R}(\pi_\eta)_* \mathbb{Q}_\ell[\dim \mathfrak{g}])$$

so that the long exact sequence (5.4.5) is

$$\cdots \longrightarrow {}^p\mathcal{H}^0(j_{\eta!}'(\mathrm{R}(\pi_\eta)_* \mathbb{Q}_\ell[\dim \mathfrak{g}])|_{\mathfrak{g}_\eta^{\mathrm{rs}}}) \longrightarrow \mathrm{R}(\pi_\eta)_* \mathbb{Q}_\ell[\dim \mathfrak{g}] \longrightarrow {}^p\mathcal{H}^0(j^*\mathcal{C}_1) \xrightarrow{[1]} \cdots \tag{5.4.7}$$

Now the second-left arrow in (5.4.7) is surjective since $\mathrm{R}(\pi_\eta)_* \mathbb{Q}_\ell[\dim \mathfrak{g}]$ is an IC sheaf, and $j^* {}^p\mathcal{H}^1(j_!' j'^* \mathcal{F}) = 0$ since $j_!' j'^*$ is right t-exact, again by recollement ([11], Prop. 1.4.12). We therefore get $j^* {}^p\mathcal{H}^0(C_1) \simeq {}^p\mathcal{H}^0(j^*\mathcal{C}_1) = 0$. A similar argument yields $i^* {}^p\mathcal{H}^0(\mathcal{C}_1) = 0$, via long exact sequence

$$\cdots \longrightarrow {}^p\mathcal{H}^0(i^* j_!' \mathcal{G}) \longrightarrow i^*\mathcal{F} \longrightarrow {}^p\mathcal{H}^0(i^*\mathcal{C}_1) \xrightarrow{[1]} \cdots \tag{5.4.8}$$

and $i^* j_!' \mathcal{G} \simeq j_{s!}' i^{\mathrm{rs}*}\mathcal{G} \simeq j_{s!}' \mathrm{R}(\pi_s)_* \mathbb{Q}_\ell[\dim \mathfrak{g}]|_{\mathfrak{g}_s^{\mathrm{rs}}}$ coming from base-change along diagram

$$\begin{array}{ccc}
\mathfrak{g}_s^{\mathrm{rs}} & \xrightarrow{\ i^{\mathrm{rs}}\ } & \mathfrak{g}^{\mathrm{rs}} \\
{\scriptstyle j_s'}\downarrow & & \downarrow{\scriptstyle j'} \\
\mathfrak{g}_s & \xrightarrow{\ i\ } & \mathfrak{g}
\end{array} \qquad (5.4.9)$$

where i^{rs} is induced from $s \to S$ and j_s' is the base-change of j'. We therefore yield ${}^p\mathcal{H}^0(\mathcal{C}_1) = 0$. For ${}^p\mathcal{H}^{-1}(\mathcal{C}_2)$ we argue in an analogous manner, namely we apply t-exact functors j^*, i^* to the right triangle in (5.4.4) and show $j^{*\,p}\mathcal{H}^{-1}(\mathcal{C}_2) = i^{*\,p}\mathcal{H}^{-1}(\mathcal{C}_2) = 0$. For $j^{*\,p}\mathcal{H}^{-1}(\mathcal{C}_2) = 0$ the reasoning is parallel to showing $j^{*\,p}\mathcal{H}^0(\mathcal{C}_1) = 0$, since $j^* j_*' \simeq j_{\eta *}' j^{\mathrm{rs}*}$ via flat base-change along diagram (5.4.6); once again we reduce to $j^*\mathcal{F}$ being an IC sheaf and ${}^p\mathcal{H}^{-1}(j_*' j'^* \mathcal{F}) = 0$ as $j_*' j'^*$ is left t-exact ([11], Prop 1.4.12). Now t-exactness of i^* yields exact sequence

$$\cdots \longrightarrow i^{*\,p}\mathcal{H}^{-1}(\mathcal{C}_2) \longrightarrow i^*\mathcal{F} \longrightarrow {}^p\mathcal{H}^0(i^* j_*' \mathcal{G}) \xrightarrow{[1]} \cdots \qquad (5.4.10)$$

induced from the right triangle in (5.4.4), but we cannot immediately conclude by a standard base-change theorem. Consider instead the open/closed decompositions

$$\widetilde{\mathfrak{g}}^{\mathrm{rs}} \xrightarrow{\widetilde{j}'} \widetilde{\mathfrak{g}} \xleftarrow{\widetilde{i}'} \widetilde{\mathfrak{g}}' := \widetilde{\mathfrak{g}} \setminus \widetilde{\mathfrak{g}}^{\mathrm{rs}}$$

$$\widetilde{\mathfrak{g}}_s^{\mathrm{rs}} \xrightarrow{\widetilde{j}_s'} \widetilde{\mathfrak{g}}_s \xleftarrow{\widetilde{i}_s'} \widetilde{\mathfrak{g}}_s' = \widetilde{\mathfrak{g}}_s \setminus \widetilde{\mathfrak{g}}_s^{\mathrm{rs}}$$

induced from the respective decompositions on $\mathfrak{g}, \mathfrak{g}_s$ and their Grothendieck alterations π, π_s. Together with $\widetilde{i} : \widetilde{\mathfrak{g}}_s \to \widetilde{\mathfrak{g}}, \widetilde{i}^{\mathrm{rs}} : \widetilde{\mathfrak{g}}_s^{\mathrm{rs}} \to \widetilde{\mathfrak{g}}^{\mathrm{rs}}$, these maps fit into commutative diagrams

$$\begin{array}{ccc}
\widetilde{\mathfrak{g}}_s^{\mathrm{rs}} & \xrightarrow{\widetilde{i}^{\mathrm{rs}}} & \widetilde{\mathfrak{g}}^{\mathrm{rs}} \\
{\scriptstyle \widetilde{j}_s'}\downarrow & & \downarrow{\scriptstyle \widetilde{j}'} \\
\widetilde{\mathfrak{g}}_s & \xrightarrow{\widetilde{i}} & \widetilde{\mathfrak{g}}
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\mathfrak{g}}_s' & \xrightarrow{\widetilde{i}|_{\widetilde{\mathfrak{g}}_s'}} & \widetilde{\mathfrak{g}}' \\
{\scriptstyle \widetilde{i}_s'}\downarrow & & \downarrow{\scriptstyle \widetilde{i}'} \\
\widetilde{\mathfrak{g}}_s & \xrightarrow{\widetilde{i}} & \widetilde{\mathfrak{g}}
\end{array}
\qquad (5.4.11)$$

by the same token as before. We aim to show $i^* j'_* \mathcal{G} \simeq j'_{s*} i^{\mathrm{rs}*} \mathcal{G}$. Via the Grothendieck alteration π, diagram (5.4.9) and the left diagram in (5.4.11) fit into the following commutative cube diagram (5.4.12), all of whose faces are cartesian:

$$
\begin{array}{c}
(5.4.12)
\end{array}
$$

Here $\pi^{\mathrm{rs}}, \pi^{\mathrm{rs}}_s$ denote the obvious restrictions of π. Now, suppressing that π_* is derived for notation purposes, $i^* j'_* \mathcal{G}$ becomes

$$ i^* \circ j'_* \circ j'^* \circ \pi_* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq i^* \circ j'_* \circ \pi^{\mathrm{rs}}_* \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] $$

$$ \simeq i^* \circ \pi_* \circ \widetilde{j}'_* \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq \pi_{s*} \circ \widetilde{i}^* \circ \widetilde{j}'_* \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq \pi_{s*} \circ \widetilde{i}^* \circ \widetilde{j}'_* \mathbb{Q}_\ell [\dim \mathfrak{g}] $$

via proper base-change along the right-face and front-face diagrams, and similarly $j'_{s*} i^{\mathrm{rs}*} \mathcal{G}$ becomes

$$ j'_{s*} \circ i^{\mathrm{rs}*} \circ j'^* \circ \pi_* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq j'_{s*} \circ i^{\mathrm{rs}*} \circ \pi^{\mathrm{rs}}_* \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] $$

$$ \simeq j'_{s*} \circ \pi^{\mathrm{rs}}_{s*} \circ \widetilde{i}^{\mathrm{rs}*} \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq \pi_{s*} \circ \widetilde{j}'_{s*} \circ \widetilde{i}^{\mathrm{rs}*} \circ \widetilde{j}'^* \mathbb{Q}_\ell [\dim \mathfrak{g}] \simeq \pi_{s*} \circ \widetilde{j}'_{s*} \mathbb{Q}_\ell [\dim \mathfrak{g}] $$

via proper base-change along the right-face and back-face diagrams. Then $i^* j'_* \mathcal{G} \simeq j'_{s*} i^{\mathrm{rs}*} \mathcal{G}$ precisely when $\widetilde{i}^* \widetilde{j}'_* \mathbb{Q}_\ell \simeq \widetilde{j}'_{s*} \mathbb{Q}_\ell$, ignoring the dimension shifts. To show this, take the standard exact triangle (on $\widetilde{\mathfrak{g}}$)

$$ \widetilde{i}'_* \widetilde{i}'^! \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow \widetilde{j}'_* \mathbb{Q}_\ell \xrightarrow{[1]} \tag{5.4.13} $$

We have $\widetilde{i}'^!\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell[-2](-1)$ by Lemma 5.4.3, which we prove right after this theorem. Applying the t-exact functor $\widetilde{i}^* : D^b_c(\widetilde{\mathfrak{g}}) \to D^b_c(\widetilde{\mathfrak{g}}_s)$ induces the following diagram from triangle (5.4.13) and the two diagrams in (5.4.11), where the vertical arrows are base-change morphisms:

$$
\begin{array}{ccccc}
\widetilde{i}^*\widetilde{i}'_*\mathbb{Q}_\ell[-2](-1) & \longrightarrow & \mathbb{Q}_\ell & \longrightarrow & \widetilde{i}^*\widetilde{j}'_*\mathbb{Q}_\ell \xrightarrow{+1} \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{h} \\
\widetilde{i}'_{s*}\widetilde{i}|^*_{\widetilde{\mathfrak{g}}'_s}\mathbb{Q}_\ell[-2](-1) & \longrightarrow & \mathbb{Q}_\ell & \longrightarrow & \widetilde{j}'_{s*}\mathbb{Q}_\ell \xrightarrow{+1}
\end{array}
$$

The left vertical arrow is an isomorphism by proper base-change, and so is the middle vertical arrow, hence h is also an isomorphism (eg. by the five lemma). We conclude that $i^*j'_*\mathcal{G} \simeq j'_{s*}i^{\mathrm{rs}*}\mathcal{G}$ and so the long exact sequence (5.4.10) is

$$
\cdots \longrightarrow i^{*p}\mathcal{H}^{-1}(\mathcal{C}_2) \longrightarrow \mathrm{R}(\pi_s)_*\mathbb{Q}_\ell[\dim\mathfrak{g}] \longrightarrow {}^p\mathcal{H}^0(j^*_s\mathrm{R}(\pi_s)_*\mathbb{Q}_\ell[\dim\mathfrak{g}]|_{\mathfrak{g}^{\mathrm{rs}}_s}) \xrightarrow{[1]} \cdots \quad (5.4.14)
$$

so that $i^{*p}\mathcal{H}^{-1}(\mathcal{C}_2) = 0$ as $\mathrm{R}(\pi_s)_*\mathbb{Q}_\ell[\dim\mathfrak{g}]$ is an IC sheaf. This yields $\mathcal{F} \simeq j'_{!*}\mathcal{G} = j'_{!*}j'^*\mathcal{F}$ so $\mathcal{F}$ is indeed an IC sheaf on $\mathfrak{g}$. $\square$

Lemma 5.4.3. *Let $\mathfrak{g}$ be a Chevalley $\mathcal{O}_K$–algebra so that $\mathfrak{g}_{\overline{\eta}}$ and $\mathfrak{g}_s$ are simple Lie algebras of the same Dynkin type, and fix a torus $\mathfrak{h}$ and Borel $\mathfrak{b}$. Let $\widetilde{\chi} : \widetilde{\mathfrak{g}} \to \mathfrak{h}$ be the adjoint bundle associated to $(\mathfrak{b}, \mathfrak{h})$ and write $\widetilde{i}' : \widetilde{\mathfrak{g}}\backslash\widetilde{\mathfrak{g}}^{rs} \to \widetilde{\mathfrak{g}}$ for the inclusion of the complement of the preimage of $\mathfrak{g}^{rs}$ under the Grothendieck alteration. Then $\widetilde{i}'^!\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell[-2](-1)$.*

Proof. Call $\widetilde{\mathfrak{g}}' = \widetilde{\mathfrak{g}}\backslash\widetilde{\mathfrak{g}}^{rs}$, then we aim to show that $\widetilde{\mathfrak{g}}'$ is smooth over S. We first show it is flat. As a consequence of Proposition 3.5.3, restricting $\widetilde{\chi}$ to $\widetilde{\chi}'_s : \widetilde{\mathfrak{g}}'_s \to \mathfrak{h}_s \simeq \mathbb{A}^r_k$ we have that every fiber is smooth over k, so we get flatness of $\widetilde{\chi}_s$ as follows: if $y = \widetilde{\chi}'_s(x)$, the associated local homomorphism $\mathcal{O}_{\mathbb{A}^r,y} \to \mathcal{O}_{\widetilde{\mathfrak{g}}'_s,x}$ has regular source and a regular fiber ring $F = \mathcal{O}_{\widetilde{\mathfrak{g}}'_s,x}/\mathfrak{m}_y\mathcal{O}_{\widetilde{\mathfrak{g}}'_s,x}$, so we may pick a regular system of parameters $(x_1,\cdots,x_r)$ for $\mathcal{O}_{\mathbb{A}^r,y}$ and $(y_1,\cdots,y_s)$ in $\mathcal{O}_{\widetilde{\mathfrak{g}}'_s,x}$ so that their images in F form a regular system of parameters. We

have

$$\dim(\mathcal{O}_{\widetilde{\mathfrak{g}}'_s, x}) = \dim(\mathcal{O}_{\mathbb{A}^r, y}) + \dim(F)$$

so $(x_1, \cdots, x_r, y_1, \cdots, y_s)$ generates the maximal ideal of $\mathcal{O}_{\widetilde{\mathfrak{g}}'_s, x}$ and so $\mathcal{O}_{\widetilde{\mathfrak{g}}'_s, x}$ is regular. Then by miracle flatness ([62, Tag 00R4]) we get $\widetilde{\chi}'_s$ flat. Since every fiber of $\widetilde{\chi}'_s$ is smooth and reduced, we get in fact that $\widetilde{\chi}'_s$ is smooth, hence $\widetilde{\mathfrak{g}}'_s$ is smooth over k and a similar analysis yields $\widetilde{\mathfrak{g}}'_{\overline{\eta}}$ smooth over $\overline{K}$.

The same local algebra argument in the previous paragraph yields $\widetilde{\mathfrak{g}}'$ flat over S, with a minor modification on the local rings: if $\mathrm{Spec}(\mathcal{O}_K) \to \widetilde{\mathfrak{g}}'$ is an $\mathcal{O}_K$-valued section mapping the closed point to $x \in \widetilde{\mathfrak{g}}'$, then it suffices to show the induced local homomorphism $\mathcal{O}_K \to \mathcal{O}_{\widetilde{\mathfrak{g}}', x}$ is flat, which follows from ([46], Thm. 23.7 (ii)[2]) as $\mathcal{O}_{\widetilde{\mathfrak{g}}', x} \otimes k \simeq \mathcal{O}_{\widetilde{\mathfrak{g}}'_s, x_s}$. Now as the generic and special fiber are smooth we get that $\widetilde{\mathfrak{g}}'$ is also smooth over S. Then the proof of ([5], Thm. 2.2.13) shows pair $(\widetilde{\mathfrak{g}}', \widetilde{\mathfrak{g}})$ is *smooth* of relative codimension 1, so that $\widetilde{i}'^! \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell[-2](-1)$, completing the claim. $\qquad\square$

Corollary 5.4.4. *Let $\mathcal{S} \subset \mathfrak{g}$ be a relative Slodowy slice at a fiberwise subregular nilpotent $\mathcal{O}_K$-section $x \in \mathcal{N}_\mathfrak{g}(\mathcal{O}_K)$. Let $\pi_\mathcal{S} : \widetilde{\mathcal{S}} := \pi^{-1}(\mathcal{S}) \longrightarrow \mathcal{S}$ be the restriction of the Grothendieck alteration to $\widetilde{\mathcal{S}}$. Then $\mathcal{F}_\mathcal{S} = \mathrm{R}\pi_{\mathcal{S}*}\mathbb{Q}_\ell[\dim \mathcal{S}]$ is a relative IC sheaf.*

Proof. The arguments of Theorem 5.4.2 carry over to this setting as soon as we have that the version of $\pi_\mathcal{S}$ over a geometric point of $\mathrm{Spec}(\mathcal{O}_K)$ is a small morphism. So we reduce to the case of k an algebraically closed field, $x \in \mathcal{N}_\mathfrak{g}(k)$ a subregular nilpotent element in Lie algebra $\mathfrak{g}$ over k, and $\mathcal{S}$ the canonical Slodowy slice at x, transverse to its orbit ('canonical' here means that, in light of Lemma 4.1.2, any such Slodowy slice has locally the form of the slice described in ([57], §7.4)).

In this setting we know $\mathcal{S}$ is transverse to every adjoint G-orbit ([57], §7.4 Corollary).

---

2. Note that the assumption that B is flat in *loc. cit.* is not needed; see Theorem 51 on the modernized edition at https://aareyanmanzoor.github.io/assets/matsumura-CA.pdf.

Take a stratification of $\mathfrak{g}$ into locally closed subsets

$$\mathfrak{g} = X_{-1} \amalg (X_0 \setminus X_{-1}) \amalg \coprod_{n \geq 1} X_n, \quad X_{-1} = \mathfrak{g}^{\mathrm{rs}}, \quad X_n = \{x \mid \dim(\pi^{-1}(x)) = n\} \ (n \geq 0)$$

which, after a possible refinement, induces a stratification on $\mathcal{S}$. By transversality, $\mathcal{S} \cap X_n = \emptyset$ since $\mathcal{S}$ meets only the regular and subregular orbit in $\mathcal{N}_{\mathfrak{g}}$, so $\mathcal{S} = \mathcal{S}_1 \amalg \mathcal{S}_2 \amalg \mathcal{S}_3$ where $\mathcal{S}_i = \mathcal{S} \cap X_{n-2}$, and for any $y \in \mathcal{S}_i$ we have

$$\dim(\pi_{\mathcal{S}}^{-1}(y)) \leq \frac{1}{2}(\dim(\mathcal{S}) - \dim(\mathcal{S}_i))$$

with equality if and only if $i = 0$ (where we get the dense open stratum $\mathcal{S} \cap \mathfrak{g}^{\mathrm{rs}}$. The only nontrivial case is $y \in \mathcal{S}_3$ as $\dim(\pi_{\mathcal{S}}^{-1}(y)) = 1$; since by Theorem 4.6.2 we have that $\mathcal{S} \longrightarrow \mathfrak{h}/\!\!/W$ realizes a miniversal deformation of surface singularity $x \in \mathcal{N}_{\mathfrak{g}} \cap \mathcal{S}$, locus $\mathcal{S}_3$ consists of the nearby singularities lying over discriminant divisor $\Delta \subset \mathfrak{h}/\!\!/W$, hence by flatness $\dim(\mathcal{S}_3) = r - 1$. The above inequality therefore is just $1 < \frac{3}{2}$. $\qquad\square$

5.5 Weyl–Springer actions.

We retain the definitions and assumptions of Section 3.1. In the setting of Proposition 3.5.3 and Theorem 5.4.2, we have seen that the Grothendieck alteration π restricted to $\pi^{\mathrm{rs}} : \widetilde{\mathfrak{g}}^{\mathrm{rs}} \longrightarrow \mathfrak{g}^{\mathrm{rs}}$ is a finite étale W-torsor so that $\mathcal{G} = \mathrm{R}\pi_* \mathbb{Q}_\ell|_{\mathfrak{g}^{\mathrm{rs}}} \simeq \pi_*^{\mathrm{rs}} \mathbb{Q}_\ell$ is a lisse sheaf on $\mathfrak{g}^{\mathrm{rs}}$. To describe the W-action on it, we define an auxiliary geometric vector bundle related to $\widetilde{\mathfrak{g}}$ as follows. Let

$$\widetilde{\mathfrak{g}}_T := \{(gT, x) \in G/T \times_S \mathfrak{g} \mid x \in \mathrm{Ad}_g(\mathfrak{h})\} \hookrightarrow G/T \times_S \mathfrak{g} \qquad (5.5.1)$$

Here quotient G/T is represented by a smooth quasi-affine S-scheme via ([19], Thm. 2.3.1). Projection to the second factor in (5.5.1) yields a morphism $\rho : \widetilde{\mathfrak{g}}_T \longrightarrow \mathfrak{g}$; note ρ is not

73

necessarily proper like π, since G/T is in general only fiberwise quasi-affine. Projection to the first factor yields a morphism $\widetilde{\mathfrak{g}}_T \longrightarrow G/T$, and the arguments of Lemma 3.5.1 (except Zariski-local triviality) carry over to show:

Lemma 5.5.1. $\widetilde{\mathfrak{g}}_T$ *is a smooth étale-locally trivial G-torsor over G/T with fiber $\mathfrak{h}$.*

Define a *right* W-action on $\widetilde{\mathfrak{g}}_T$ via $(gT, x) \cdot w = (gn_w T, x)$ where $n_w \in N_G(T)$ is a lift of w. The map

$$G/T \times_S \mathfrak{h} \longrightarrow \widetilde{\mathfrak{g}}_T, \quad (gT, h) \mapsto (gT, \mathrm{Ad}_g(h)) \tag{5.5.2}$$

is a W-equivariant isomorphism, where the left-hand side is equipped with the W-action described in Definition 3.2.2 (ii). We also define a morphism $\phi : G/T \times_S \mathfrak{g}^{\mathrm{rs}} \longrightarrow G/B \times_S \mathfrak{g}^{\mathrm{rs}}$ via $(gT, x) \mapsto (gB, x)$ and set $\widetilde{\mathfrak{g}}_T^{\mathrm{rs}} = \{(gT, x) \in \widetilde{\mathfrak{g}}_T \mid x \in \mathfrak{g}^{\mathrm{rs}}\}$.

Lemma 5.5.2. *Restricting ϕ to $G/T \times_S \mathfrak{h}^{\mathrm{rs}}$ induces an isomorphism $\widetilde{\mathfrak{g}}_T^{\mathrm{rs}} \simeq \widetilde{\mathfrak{g}}^{\mathrm{rs}}$.*

Proof. It is clear that $G/T \times_S \mathfrak{h}^{\mathrm{rs}} \simeq \widetilde{\mathfrak{g}}_T^{\mathrm{rs}}$ via the isomorphism in (5.5.2) since $\mathfrak{g}^{\mathrm{rs}} = \mathrm{Ad}_G(\mathfrak{h}^{\mathrm{rs}})$. As before, by the fibral isomorphism criterion ([27], 17.9.5) it suffices to pass to geometric points $\overline{\eta}$ and s of base S and show we have an isomorphism over $\overline{K}$ and k. This is the content of ([38], §13.4 Lemma); we only explain the bijection input over field k.

If $(gT, h) \in \widetilde{\mathfrak{g}}_T^{\mathrm{rs}}(k)$ then $h \in g\mathrm{Lie}(T)g^{-1}$ i.e. $gTg^{-1} \subseteq C_G(h) = T$, so $g \in N_G(T)$ and we may write $\rho^{-1}(h) = \{(n_w T, h) \mid w \in W\}$. Similarly if $(gB, h) \in \widetilde{\mathfrak{g}}^{\mathrm{rs}}(k)$ we have $h \in \mathrm{Lie}(gBg^{-1})$ so $h \in \mathrm{Lie}(T')$ for some maximal torus $T' \subseteq gBg^{-1}$. At the same time $T' \subseteq C_G(h) = T$ so $T' = T$ and $T, g^{-1}Tg$ are conjugate in B, say via $b \in B(k)$. It follows that $gb = n_w \in N_G(T)$ so $gB = n_w B$ and $\pi^{-1}(h) = \{(n_w B, h) \mid w \in W\}$. Hence ϕ maps $\rho^{-1}(h)$ bijectively to $\pi^{-1}(h)$. $\square$

Corollary 5.5.3. $\rho^{rs} : \widetilde{\mathfrak{g}}_T^{rs} \to \mathfrak{g}^{rs}$ *is a finite étale W-torsor and $\mathcal{G} = \pi_*^{rs}\mathbb{Q}_\ell \simeq \rho_*^{rs}\mathbb{Q}_\ell$.*

Proof. This is a direct consequence of Proposition 3.5.3 since $\rho^{\mathrm{rs}} = \pi^{\mathrm{rs}} \circ \phi$ and π^{rs} is a finite étale W-torsor. By Lemma 5.5.2 we have a W-equivariant isomorphism $\widetilde{\mathfrak{g}}_T^{\mathrm{rs}} \simeq \widetilde{\mathfrak{g}}^{\mathrm{rs}}$ so $\mathcal{G} \simeq \pi_*^{\mathrm{rs}} \circ \phi_* \mathbb{Q}_\ell = \rho_*^{\mathrm{rs}}\mathbb{Q}_\ell$. $\square$

The upshot of this construction is that we may define the W-action on $\mathcal{G}$ explicitly via its description as $\rho_*^{\mathrm{rs}}\mathbb{Q}_\ell$. The stalk of constant sheaf $\mathbb{Q}_\ell$ at any geometric point $x \to \widetilde{\mathfrak{g}}_T^{\mathrm{rs}}$ is generated by a distinguished basis element v_x, and for $w \in W$ we have an isomorphism $i_w : w_*\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ so that the map on stalks is $v_{x \cdot w^{-1}} \longmapsto v_x$ via the right W-action on $\widetilde{\mathfrak{g}}_T^{\mathrm{rs}}$. Applying finite map ρ_*^{rs} to all i_w and using $\rho^{\mathrm{rs}} \circ w = \rho^{\mathrm{rs}}$ we have

$$\rho_*^{\mathrm{rs}}\mathbb{Q}_\ell = (\rho^{\mathrm{rs}} \circ w)_*\mathbb{Q}_\ell \simeq \rho_*^{\mathrm{rs}}w_*\mathbb{Q}_\ell \xrightarrow{i_w} \rho_*^{\mathrm{rs}}\mathbb{Q}_\ell \tag{5.5.3}$$

where the rightmost map is denoted i_w again by abuse of notation. By functoriality this is an automorphism of $\rho_*^{\mathrm{rs}}\mathbb{Q}_\ell$ and on each stalk at $\rho^{\mathrm{rs}}(x)$ we have

$$(\rho_*^{\mathrm{rs}}\mathbb{Q}_\ell)_{\rho^{\mathrm{rs}}(x)} = \langle v_{x \cdot u^{-1}} \mid u \in W \rangle$$

so that (5.5.3) maps basis vectors $v_{x \cdot u^{-1}} \xrightarrow{i_w} v_{x \cdot (u^{-1}w)}$. One checks $i_{w_1 w_2} = i_{w_1} \circ i_{w_2}$ and so we obtain an algebra homomorphism

$$\mathbb{Q}_\ell[W] \longrightarrow \mathrm{End}(\rho_*^{\mathrm{rs}}\mathbb{Q}_\ell) \simeq \mathrm{End}(\pi_*^{\mathrm{rs}}\mathbb{Q}_\ell)$$

Via the intermediate extension functor we get an induced action on $\mathcal{F} = \mathrm{R}\pi_*\mathbb{Q}_\ell$ as follows; recall here that $j' : \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$, $i' : \mathfrak{g} \setminus \mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ are the open/closed decompositions of $\mathfrak{g}$ coming from the gluing construction.

Lemma 5.5.4. *Sheaf $\mathcal{F} \simeq j'_{!*}\mathcal{G}$ is W-equivariant via canonical isomorphism*

$$\mathrm{End}_{\mathrm{Perv}(\mathfrak{g}^{rs})}(\mathcal{G}) \simeq \mathrm{End}_{\mathrm{Perv}(\mathfrak{g})}(j'_{!*}\mathcal{G})$$

Proof. Both perverse sheaf categories here are assumed to come from the relative perverse

t-structures defined in Section 5.4. Via adjunction

$$\mathrm{Hom}(\mathcal{G},\mathcal{G}) = \mathrm{Hom}(j'^{*}j'_{!}\mathcal{G},\mathcal{G}) \simeq \mathrm{Hom}(j'_{!}\mathcal{G},j'_{*}\mathcal{G}) \simeq \mathrm{Hom}(^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G}),{}^{p}\mathcal{H}^{0}(j'_{*}\mathcal{G})) \qquad (5.5.4)$$

where the last (canonical) isomorphism follows from $j'_{!}\mathcal{G} \in {}^{p/S}\mathrm{D}^{\leq 0}$, $j'_{*}\mathcal{G} \in {}^{p/S}\mathrm{D}^{\geq 0}$. Since i'_{*} is t-exact, for any $\mathcal{E}$ in $\mathbf{Perv}(\mathfrak{g} \setminus \mathfrak{g}^{\mathrm{rs}})$ we have

$$\mathrm{Hom}(^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G}),i'_{*}\mathcal{E}) = \mathrm{Hom}(j'_{!}\mathcal{G},i'_{*}\mathcal{E}) = \mathrm{Hom}(i'^{*} \circ j'_{!}\mathcal{G},\mathcal{E}) = 0$$

so $^{p}\mathcal{H}^{0}(j'_{!}\mathcal{G})$ has no nontrivial quotient objects from $i'_{*}\mathbf{Perv}(\mathfrak{g} \setminus \mathfrak{g}^{\mathrm{rs}})$. Similarly, by the gluing construction we have $i'^{!} \circ j'_{*} = 0$ hence $^{p}\mathcal{H}^{0}(j'_{*}\mathcal{G})$ has no nontrivial subobjects from $i'_{*}\mathbf{Perv}(\mathfrak{g} \setminus \mathfrak{g}^{\mathrm{rs}})$. So by the definition of $j'_{!*}\mathcal{G}$, the right-hand side in (5.5.4) is precisely $\mathrm{End}(j'_{!*}\mathcal{G})$ and the W-equivariance follows formally (see e.g. [39], §III.15.3). $\qquad \square$

Definition 5.5.5.

(i) For a general $\mathcal{O}_{K}$-section x of the nilpotent scheme $\mathcal{N}_{\mathfrak{g}}$, we denote by

$$\mathcal{B}_{x} := \pi^{-1}(x) = \widetilde{g} \times_{\mathfrak{g}} \overline{\mathrm{im}x}$$

its projective *Springer fiber* with the reduced subscheme structure coming from $\widetilde{\mathfrak{g}}$. Here $\overline{\mathrm{im}x}$ is the closure of the image of section $x : \mathrm{Spec}(\mathcal{O}_{K}) \to \mathcal{N}_{\mathfrak{g}}$.

(ii) The *Springer representations* are $\mathrm{H}^{i}(\mathcal{B}_{x},\mathbb{Q}_{\ell})$ equipped with a W-action inherited from the W-action on $\mathcal{F} = \mathrm{R}\pi_{*}\mathbb{Q}_{\ell}$ constructed above, by taking stalks and proper base-change.

If we consider the special fiber $x_{s} \in \mathcal{N}_{\mathfrak{g}}(k)$ of x, then by Proposition 3.5.3 we get that the special fiber of $\mathcal{B}_{x}$ is $\mathcal{B}_{x_{s}}$, the usual Springer fiber corresponding to a nilpotent element of $\mathfrak{g}_{s}$. Since $\mathcal{B}_{x}$ is projective over $\mathcal{O}$, and the proper base-change isomorphism is canonical

76

([25], Thm. 7.3.1), we obtain an isomorphism of W-modules

$$\mathrm{H}^i(\mathcal{B}_x, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$$

where the right-hand side has the classical Springer action of W defined via the method of Borho–MacPherson ([13], §2.6). In fact, in very good characteristic more is true:

Lemma 5.5.6. *We have* $\mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell)$ *canonically as Galois modules.*

Proof. Let $f : \mathcal{B}_x \to \mathrm{Spec}(\mathcal{O}_K)$ be the projective structure map and let $\nu : \widetilde{\mathcal{B}}_x \to \mathcal{B}_x$ be the normalization; concretely we have

$$\coprod_{i=1}^{r}(\mathbb{P}_S^1)_i = \widetilde{\mathcal{B}}_x \xrightarrow{\ \nu\ } \mathcal{B}_x = \bigcup_{i=1}^{r}(\mathbb{P}_S^1)_i$$
$$\widetilde{f} \searrow \qquad \downarrow f$$
$$S = \mathrm{Spec}(\mathcal{O}_K)$$

where structure map $\widetilde{f}$ is proper and smooth. In particular $\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell$ is a local system on S and so on stalks we have

$$(\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell)_s \xrightarrow{\ \sim\ } (\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell)_{\overline{\eta}}^I \simeq (\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell)_{\overline{\eta}} \qquad (5.5.5)$$

Here the first isomorphism is the cospecialization map given by the data of an étale sheaf on S. Now by proper base–change (as $\widetilde{f}$ and the special and generic fiber counterparts $\widetilde{f}_s, f_s, \widetilde{f}_{\overline{\eta}}, f_{\overline{\eta}}$ are all proper) we have

$$\mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \simeq \mathrm{R}^i(f_s)_* \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{R}^i(\widetilde{f}_s)_* \mathbb{Q}_\ell \simeq (\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell)_s$$

$$\mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell) \simeq \mathrm{R}^i(f_{\overline{\eta}})_* \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{R}^i(\widetilde{f}_{\overline{\eta}})_* \mathbb{Q}_\ell \simeq (\mathrm{R}^i \widetilde{f}_* \mathbb{Q}_\ell)_{\overline{\eta}}$$

Hence, together with isomorphism (5.5.5) we have $\mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell)$ canonically.

The case of interest here is $i = 2$, whence

$$\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \simeq \bigoplus_{i=1}^{r} \mathbb{Q}_\ell(-1) \simeq \mathrm{H}^2(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell)$$

where the I–action factors through the mod ℓ^n cyclotomic characters on the Tate twists, hence is trivial as $K = K^{\mathrm{unr}}$. The essential ingredient here is that x is fiberwise subregular *and* characteristic p is very good, so that both the generic and special subregular Springer fibers are the same arrangement of r projective lines ([68], §1.3.8). □

The above lemma may be interpreted as a 'shadow' of the more general principle that, even though a priori our base scheme is $\mathrm{Spec}(\mathcal{O}_K)$, the adjoint quotient χ and its "resolved" version $\widetilde{\chi}$ have the same properties over K and over k of very good characteristic.

We now make a few remarks on the nature of Springer representations. The surprising fact about them is that the W–action on $\mathrm{H}^i(\mathcal{B}_x, \mathbb{Q}_\ell)$ is not induced by a "physical" W–action on the Springer fiber. Nevertheless, Springer ([60], Thm. 6.10) defined a natural correspondence between $\mathrm{H}^i(\mathcal{B}_x, \mathbb{Q}_\ell)$ and representations of Weyl group (by a manner different from Borho–MacPherson).

Theorem 5.5.7 (Springer correspondence)**.** *Assume G is a simple, simply–connected, simply–laced Lie group G over algebraically closed field k of good characteristic, and let $x \in \mathcal{N}(k)$ be a nilpotent element and $C = C_G(x)/C_G(x)^\circ$ be the connected component group of its centralizer. Set $n = \dim(\mathcal{B}_x)$ and let χ be the character of an irreducible representation of C. There is a natural graded W-action on $\mathrm{H}^\bullet(\mathcal{B}_x, \mathbb{Q}_\ell)$ commuting with the C-action, and in particular*

$$\mathrm{H}^{2n}(\mathcal{B}_x, \mathbb{Q}_\ell) \simeq \bigoplus_{\chi \in \widehat{C}} \chi \otimes V_{x,\chi}$$

where each non-zero χ-isotypic component $V_{x,\chi}$ is an irreducible W-representation. Furthermore each irreducible representation of W appears as a $V_{x,\chi}$ for a unique (up to conjugacy)

pair (x, χ).

The convention we use here is that the trivial W-representation corresponds to x regular nilpotent and trivial χ, and the sign W-representation corresponds to $x = 0$, where one investigates the cohomology of the full flag variety $\mathcal{B}$. This is the opposite convention of [60] but coincides with [13]; in fact the Springer representations we consider differ from [60] by a twist of the sign character.

In our setting, x is subregular (so $n = 1$) and C is trivial for the ADE-type Lie algebras since the centralizer is connected ([57], §7.5 Lemma 4). In this case it is known (see e.g. [68], §1.5.17) that the corresponding Springer representation on $\mathrm{H}^2(\mathcal{B}_x, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^r$ is isomorphic to the irreducible *reflection representation* of W on $\mathfrak{h}^\vee$, which in the A_n-case (where $W = S_{n+1}$) is just the standard S_{n+1}-representation.

5.6 Monodromy Weyl actions and the proof of the main theorem.

We now go back to the setting of Sections 5.1 and 5.2, assuming in particular $S = \mathrm{Spec}(\mathcal{O}_K)$ is a complete trait and $\mathrm{char}(k)$ is *very good*. Since we require notation from previous sections as well, we gather here the relevant objects.

Notation 5.6.1. Let $\mathcal{X}/S$ be an integral proper flat surface with smooth generic fiber $\mathcal{X}_\eta$ and singular special fiber $\mathcal{X}_s$ containing a unique RDP $x_s \in \mathcal{X}_s(k)$. Let r be the rank of the Dynkin diagram associated to the RDP (Theorem 2.1.3 (ii)).

If $\mathfrak{g}$ is the rank–r simple Chevalley $\mathcal{O}_K$-algebra associated to RDP x_s, we may identify x_s with a generic point of the unique subregular orbit inside the nilpotent cone of the Lie k-algebra $\mathfrak{g}_s$ ([57], §6.4 Thm.). Extend x_s to a fiberwise subregular element $x \in \mathfrak{g}(\mathcal{O}_K)$ (Definition 3.3.4) and let $\mathcal{S}$ be the affine relative Slodowy slice which is transverse at x in the sense of Section 4.1.

Unless noted otherwise, $\mathcal{S}_{\overline{h}}$, resp. $\widetilde{\mathcal{S}}_h$ denote the geometric fibers of χ, resp. $\widetilde{\chi}$ over geometric points (i.e. field-valued points) $\overline{h} \to \mathfrak{h}/\!\!/W$ and $h \to \mathfrak{h}$. The respective generic and special

fibers will be denoted $(\mathcal{S})_{\overline{\eta}}$ and $(\mathcal{S})_s$, and similarly for $\widetilde{\mathcal{S}}$ so that we can distinguish them from the geometric fibers of $\chi, \widetilde{\chi}$.

By Corollary 5.2.3 the monodromy action of $I = \mathrm{Gal}(\overline{\eta}/\eta)$ on $\mathrm{H}^2(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell)$ depends only on $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{X}, x_s})$, which by Remark 2.5.7 we can view as an algebraic deformation of the affine singularity $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{X}_s, x_s})$ over base S. In other words, by Theorem 4.6.2 we have two cartesian diagrams

$$
\begin{array}{ccc}
\mathcal{X}_{(x_s)} = \mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{X}, x_s}) & \longrightarrow & \mathcal{S} \\
\downarrow & & \downarrow{\scriptstyle \chi} \\
S & \xrightarrow{\ \phi\ } & \mathfrak{h}/\!\!/ W
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{Z} = \mathcal{S} \times_{\mathfrak{h}/\!\!/ W} \mathfrak{h} & \longrightarrow & \mathfrak{h} \\
\downarrow & & \downarrow{\scriptstyle \psi} \\
S & \xrightarrow{\ \phi\ } & \mathfrak{h}/\!\!/ W
\end{array}
\tag{5.6.1}
$$

where ϕ is induced by miniversality, and the right diagram is the pullback of ϕ along finite cover ψ. Here χ is a map of *henselianized* schemes, $\mathfrak{h}$ and W are part of the Lie algebraic data associated to RDP x_s (see Section 4.6) and by Remark 4.6.4 we may identify the miniversal base as

$$
\mathfrak{h}/\!\!/ W \simeq \mathrm{Spec}(\mathcal{O}_K[\![t_1, \cdots, t_r]\!])
$$

Lemma 5.6.2. *Let $\mathcal{X}/S$, $x_s \in \mathcal{X}_s(k)$ and $\psi : \mathfrak{h} \to \mathfrak{h}/\!\!/ W$ be as above.*

(i) *There exists a finite ramified extension L/K and its associated trait $S_L = \{\eta_L, \overline{s}\}$, which is minimal with respect to the following property: the base-change $\mathcal{X}_L/S_L$ admits a local affine model of the RDP singularity $x_s \in \mathcal{X}_s(k)$ which has a simultaneous resolution.*

(ii) *If $\mathcal{B}_{x_s}$ is the exceptional divisor of the minimal resolution of $\mathcal{X}_s$ induced by the simultaneous resolution of $\mathcal{X}_L$, then the stalks of the nearby cycles are*

$$
(\mathrm{R}\Psi_\mathcal{X}\mathbb{Q}_\ell)_{x_s} \xrightarrow{\ \sim\ } \mathrm{R}\Gamma(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)
\tag{5.6.2}
$$

80

and the isomorphism is $\mathrm{Gal}(\overline{\eta}/\eta)$*-equivariant.*

Proof. For part (i), note that $S \to \mathfrak{h} /\!\!/ W$ maps $s \mapsto 0 \in \mathfrak{h} /\!\!/ W(k)$ and $\eta \mapsto \mathfrak{h}^{\mathrm{rs}} /\!\!/ W(K)$ since the generic fiber $\mathcal{X}_\eta$ is smooth. Thus, $\mathcal{Z}$ is a finite flat, generically étale S-scheme and so in particular it is S-affine, say $\mathcal{Z} = \mathrm{Spec}(R)$ for some $\mathcal{O}_K$-algebra R. By assumption $R \otimes_{\mathcal{O}_K} K$ is a finite étale K-algebra, whence

$$R \otimes_{\mathcal{O}_K} K \simeq \prod_i L_i$$

is a finite product of finite separable extensions L_i/K, totally ramified since $K = \check{K}$. From the right diagram in (5.6.1) we obtain a cartesian diagram

$$
\begin{array}{ccc}
\coprod_i \mathrm{Spec}(L_i) & \longrightarrow & \mathfrak{h}^{\mathrm{rs}} \\
\downarrow & & \downarrow{\scriptstyle \psi^{\mathrm{rs}}} \\
\eta & \longrightarrow & \mathfrak{h}^{\mathrm{rs}} /\!\!/ W
\end{array}
$$

where ψ^{rs} is a Galois W-cover, hence we get a transitive W-action on $\coprod_i \mathrm{Spec}(L_i)$. So all extensions L_i/K are isomorphic and we may fix one $L = L_i$ with Galois group $\mathrm{Gal}(L/K) = W_1 \subseteq W$, the stabilizer of $\mathrm{Spec}(L)$ inside $\coprod \mathrm{Spec}(L_i)$.

Since R is excellent, we can replace it with its normalization $R \longrightarrow \widetilde{R} \simeq \prod_i \mathcal{O}_{L_i}$ which is finite over R. In particular we get a map

$$S_L = \mathrm{Spec}(\mathcal{O}_L) \longrightarrow \widetilde{\mathcal{Z}} = \mathrm{Spec}(\widetilde{R}) \longrightarrow \mathfrak{h}$$

and via Theorem 4.6.2 we may produce a fiber product from this map and the 'versal' simultaneous resolution $\widetilde{\chi} : \widetilde{\mathcal{S}} \longrightarrow \mathfrak{h}$. In other words, define $\mathcal{Y}$ via the cartesian square

$$
\begin{array}{ccc}
\mathcal{Y} = S_L \times_{\mathfrak{h}} \widetilde{\mathcal{S}} & \longrightarrow & \widetilde{\mathcal{S}} \\
\downarrow & & \downarrow{\scriptstyle \widetilde{\chi}} \\
S_L & \longrightarrow & \mathfrak{h}
\end{array}
\qquad (5.6.3)
$$

By the universal property of pullbacks we get a unique map

$$\pi_L : \mathcal{Y} \longrightarrow \mathcal{X}_{(x_s),L} = \mathcal{X}_{(x_s)} \times_S S_L \simeq \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{X},x_s} \otimes_{\mathcal{O}_K} \mathcal{O}_L)$$

We show π_L is the desired simultaneous resolution, i.e. that $\mathcal{Y}$ simultaneously resolves the RDP singularity of the local affine neighborhood $\mathcal{X}_{(x_s),L}$ in $\mathcal{X}_L$. Scheme $\mathcal{Y}$ is smooth over S_L via pullback in (5.6.3) and π_L is proper as the base–change of proper morphism $\pi : \widetilde{\mathcal{S}} \to \mathcal{S} \times_{\mathfrak{h}/\!/W} \mathfrak{h}$ along $S_L \to \mathfrak{h}$. Lastly we need only pass to the special fiber and note that, via proper base–change, the induced map $(\pi_L)_s : \mathcal{Y}_s \to (\mathcal{X}_{(x_s),L})_s = (\mathcal{X}_{(x_s)})_s$ is just the minimal resolution of surfaces $\widetilde{\mathcal{S}}_0 \to \mathcal{S}_0$, where the subscript denotes the fiber over zero, so we are done.

To show that L/K has minimal degree, let M/K be any finite extension so that $\mathcal{X}_M$ has a local model of the singularity over $S_M = \mathrm{Spec}(\mathcal{O}_M)$ admitting a simultaneous resolution. Via diagram (5.6.1) we obtain a non-zero morphism

$$S_M \longrightarrow \widetilde{\mathcal{Z}} \simeq \coprod_i \mathrm{Spec}(L_i) \longrightarrow S_L$$

exhibiting M as an extension of L as well.

Finally, we show (ii). Part (i) together with ([7], Thms. 1 and 2) implies that over base S_L there exists a simultaneous resolution $\widetilde{\pi}_L : \widetilde{\mathcal{X}} \to \mathcal{X}_L$, where $\widetilde{\mathcal{X}}$ is an algebraic space; note $\widetilde{\pi}_L$ is obtained from the 'local model' resolution π_L of (i). Since the fibers of $\widetilde{\mathcal{X}}$ are smooth 2–dimensional, they are schemes themselves, thus in particular $\mathrm{R}\Psi_{\widetilde{\mathcal{X}}}\mathbb{Q}_\ell$ makes sense as a sheaf on $\widetilde{\mathcal{X}}_s$. Since the map of generic fibers $(\widetilde{\pi}_L)_{\eta_L} : \widetilde{\mathcal{X}}_{\eta_L} \to (\mathcal{X}_L)_\eta$ is an isomorphism by construction, proper base-change gives

$$\mathrm{R}\Psi_{\mathcal{X}}\mathbb{Q}_\ell = \mathrm{R}\Psi_{\mathcal{X}_L}((\widetilde{\pi}_L)_{\eta*}\mathbb{Q}_\ell) \simeq \mathrm{R}(\widetilde{\pi}_L)_{s*}\mathrm{R}\Psi_{\widetilde{\mathcal{X}}}\mathbb{Q}_\ell \simeq \mathrm{R}(\widetilde{\pi}_L)_{s*}\mathbb{Q}_\ell$$

as $\widetilde{\mathcal{X}}$ is smooth. Passing to the stalk at singularity $x_s \in \mathcal{X}_s(k)$ yields (5.6.2) since the exceptional divisor is $\widetilde{\pi}_L^{-1}(x_s)$ by definition. $\qquad\square$

We now advance towards defining a W-action on the nontrivial nearby cycles stalk $(\mathrm{R}\Psi_{\mathcal{X}}\mathbb{Q}_\ell)_{x_s} \simeq \mathrm{R}\Gamma(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ mimicking the monodromy Weyl action defined in ([58], §4.2 and §4.3); we will eventually identify this new action with the Springer W-action as it is given in Theorem 5.5.7.

By Lemma 5.5.2 we have that

$$\widetilde{\chi}^{\mathrm{rs}} : \widetilde{\mathfrak{g}}^{\mathrm{rs}} \simeq G/T \times_S \mathfrak{h}^{\mathrm{rs}} \longrightarrow \mathfrak{h}^{\mathrm{rs}}$$

is just the projection to the second factor, whence we can restrict the simultaneous resolution diagram (3.5.3) to obtain a cartesian square

$$\begin{array}{ccc}
\widetilde{\mathfrak{g}}^{\mathrm{rs}} & \xrightarrow{\ \pi^{\mathrm{rs}}\ } & \mathfrak{g}^{\mathrm{rs}} \\
\downarrow{\scriptstyle \widetilde{\chi}^{\mathrm{rs}}} & & \downarrow{\scriptstyle \chi^{\mathrm{rs}}} \\
\mathfrak{h}^{\mathrm{rs}} & \xrightarrow{\ \psi^{\mathrm{rs}}\ } & \mathfrak{h}^{\mathrm{rs}}/\!\!/W
\end{array}$$

where the horizontal maps are Galois W-torsors. The content of the following lemma ensures the square stays cartesian when we restrict the Grothendieck alteration to the Slodowy slice (or, equivalently, when we restrict diagram (4.6.1) to the regular semisimple locus). Below we set $\mathcal{S}^{\mathrm{rs}} = \mathcal{S} \times_g \mathfrak{g}^{\mathrm{rs}}$ and $\widetilde{\mathcal{S}}^{\mathrm{rs}} = \widetilde{\mathcal{S}} \times_{\widetilde{\mathfrak{g}}} \widetilde{\mathfrak{g}}^{\mathrm{rs}}$[3].

Lemma 5.6.3 ([61], Lem. 3). *The following diagram is cartesian.*

$$\begin{array}{ccc}
\widetilde{\mathcal{S}}^{rs} & \xrightarrow{\ \pi^{rs}\ } & \mathcal{S}^{rs} \\
\downarrow{\scriptstyle \widetilde{\chi}^{rs}} & & \downarrow{\scriptstyle \chi^{rs}} \\
\mathfrak{h}^{rs} & \longrightarrow & \mathfrak{h}^{rs}/\!\!/W
\end{array} \qquad (5.6.4)$$

---

3. One needs to ensure these fiber products are nonempty. To this end, one observes that $\mathcal{S}_{\overline{\eta}}$ and $\mathcal{S}_s$ intersect only regular and subregular orbits, which are nonempty ([57], §5.5).

The desired W-action will in a sense follow from the "monodromy" of diagram (5.6.4). We first note a 'homotopical' lemma for ℓ-adic cohomology. It is originally due to Springer ([61], Prop. 1) but Laumon has generalized it to the relative scheme setting ([42], Lemme 5.5), and this is the version we use.

Lemma 5.6.4. *Let $f : X \to S$ be an S-scheme endowed with a $\mathbb{G}_{m,S}$-action contracting X to a section $p : Y \simeq S$ and let $\mathcal{F}$ be a $\mathbb{G}_m$-equivariant sheaf on X. Then $\mathrm{R}f_*\mathcal{F} \simeq p_*\mathcal{F}|_Y$.*

Lemma 5.6.5. *If $\widetilde{\mathcal{S}}_0$ is the fiber of $\widetilde{\chi}$ over the zero $\mathcal{O}_K$-section of $\mathfrak{h}$, then we have canonical isomorphisms*

$$\mathrm{H}^i((\widetilde{\mathcal{S}})_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i((\widetilde{\mathcal{S}}_0)_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell) \qquad (5.6.5)$$

$$\mathrm{H}^i((\widetilde{\mathcal{S}})_s, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \qquad (5.6.6)$$

Proof. Recall that, by Definition 4.5.2, $\mathcal{S}, \widetilde{\mathcal{S}}$ and $\mathfrak{h}$ are equipped with appropriate $\mathbb{G}_{m,S}$-actions so that both $\mathrm{R}\pi_*\mathbb{Q}_\ell$ and $\mathrm{R}\widetilde{\chi}_*\mathbb{Q}_\ell$ are $\mathbb{G}_{m,S}$-equivariant sheaves; the first and last actions contract $\mathcal{S}$ and $\mathfrak{h}$ to x and the origin, respectively. Let $f : \widetilde{\mathcal{S}} \to S$ and $g : \mathcal{S} \to S$ be the structure morphisms over $S = \mathrm{Spec}(\mathcal{O}_K)$, then by the Leray spectral sequence

$$E_2^{pq} = \mathrm{R}^p g_*(\mathrm{R}^q \pi_* \mathbb{Q}_\ell) \implies \mathrm{R}^{p+q} f_* \mathbb{Q}_\ell$$

By Lemma 5.6.4, since the $\mathbb{G}_{m,S}$-action on $\mathcal{S}$ contracts it to the fiberwise nilpotent subregular section $x \simeq S$ (with image $\mathrm{im}(x) \subset \mathcal{S}$), we have $E_2^{pq} = 0$ for $p > 0$ and

$$\mathrm{R}^q f_* \mathbb{Q}_\ell \simeq E_2^{0q} = \mathrm{R}^q \pi_* \mathbb{Q}_\ell|_{\mathrm{im}(x)} \qquad (5.6.7)$$

Let $j : \overline{\eta} \hookrightarrow S$ and $i : s \hookrightarrow S$ be the inclusions of the geometric generic and special points of

84

the trait. By properness of π and base-change, (5.6.7) yields

$$j^* \mathrm{R}^q f_* \mathbb{Q}_\ell \simeq j^*_{x_{\overline{\eta}}} \mathrm{R}^q \pi_* \mathbb{Q}_\ell \simeq \mathrm{H}^q(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell) \tag{5.6.8}$$

$$i^* \mathrm{R}^q f_* \mathbb{Q}_\ell \simeq i^*_{x_s} \mathrm{R}^q \pi_* \mathbb{Q}_\ell \simeq \mathrm{H}^q(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \tag{5.6.9}$$

where $j_{x_{\overline{\eta}}} : x_{\overline{\eta}} \hookrightarrow \mathcal{S}$ and $i_{x_s} : x_s \hookrightarrow \mathcal{S}$ are the inclusions of the generic and special fibers of section x into $\mathcal{S}$. Now the absolute version of the aforementioned Leray spectral sequence $E_2^{pq} = \mathrm{H}^p(\mathcal{S}, \mathrm{R}^q \pi_* \mathbb{Q}_\ell) \implies \mathrm{H}^{p+q}(\widetilde{\mathcal{S}}, \mathbb{Q}_\ell)$ yields isomorphisms

$$\mathrm{H}^q((\widetilde{\mathcal{S}})_{\overline{\eta}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^q(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell), \;\; \mathrm{H}^q((\widetilde{\mathcal{S}})_s, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^q(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \tag{5.6.10}$$

by ([61], Lemma 2), since $(\mathcal{S})_{\overline{\eta}}$ and $(\mathcal{S})_s$ are Slodowy slices over K resp. k in the usual sense ([57], §5.1). Proper base-change maps are canonical, hence all isomorphisms in (5.6.8), (5.6.9) and (5.6.10) are canonical. Applying the same Leray spectral sequence argument to the map $\pi_0 : \widetilde{\mathcal{S}}_0 \longrightarrow \mathcal{S}_0$, which is still $\mathbb{G}_{m,S}$-equivariant (*loc. cit.*, Lemma 2), we obtain after base–change again that $\mathrm{H}^i((\widetilde{\mathcal{S}}_0)_{\overline{\eta}}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^i((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$, so that altogether we get the canonical isomorphisms (5.6.5), (5.6.6) induced from inclusions $\mathcal{B}_x \hookrightarrow \widetilde{\mathcal{S}}_0 \hookrightarrow \widetilde{\mathcal{S}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Proposition 5.6.6. *For a geometric point* $\overline{h} \to \mathfrak{h}^{rs}/\!\!/W$ *we have* $\mathrm{H}^i(\mathcal{S}_{\overline{h}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$.

Proof. Note that $\overline{h}$ necessarily specializes to $0 \in \mathfrak{h}/\!\!/W(k)$. We first investigate the case where the residue field of $\overline{h}$ is k, following [61]: let h be a lift of $\overline{h}$ along W-cover $\psi : \mathfrak{h} \to \mathfrak{h}/\!\!/W$ and let L be the line connecting h and 0 in $\mathfrak{h}$, $\widetilde{\chi}_L : \widetilde{\mathcal{S}}_L \longrightarrow L$ the $\mathbb{G}_m$-equivariant restriction of $\widetilde{\chi}$ to L. Note $\widetilde{\mathcal{S}}_L$ is $\mathbb{G}_m$-stable by the action $\widetilde{\mu}$ defined in Section 4.5 (factoring through $\mathbb{G}_{m,S} \to \mathbb{G}_{m,k}$).

We claim $\mathcal{E} := \mathrm{R}^i \widetilde{\chi}_{L*} \mathbb{Q}_\ell$ is a lisse sheaf. $\mathcal{E}|_{L \setminus \{0\}}$ is locally constant since the map $\mathbb{G}_m \times \widetilde{\mathcal{S}}_h \longrightarrow \widetilde{\mathcal{S}}_L \setminus \widetilde{\mathcal{S}}_0$ given by $(t, s) \mapsto \widetilde{\mu}(t, s)$ is an étale covering. Now by ([62, Tag 03Q7])

the stalk at zero is

$$\mathcal{E}_0 = \mathrm{H}^i(\widetilde{\mathcal{S}}_L \times_L L_{(0)}, \mathbb{Q}_\ell), \quad L_{(0)} = \mathrm{Spec}(\mathcal{O}_{L,0}^{\mathrm{sh}})$$

Applying the argument of Lemma 5.6.5 to $\mathbb{G}_m$-equivariant map $\widetilde{\chi}_L$ (i.e. looking at its associated Leray spectral sequence) we observe $(\mathrm{R}^i\widetilde{\chi}_{L*}\mathbb{Q}_\ell)_0 \xrightarrow{\sim} \mathrm{H}^i((\widetilde{\mathcal{S}})_s, \mathbb{Q}_\ell)$ and hence $\mathcal{E}_0 \simeq \mathrm{H}^i((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell)$ by Lemma 5.6.5, Equation (5.6.6).

As $L_{(0)}$ is a DVR there is a unique generic (geometric) point η_L of $L_{(0)}$ with specialization $\eta_L \to 0$. We then have

$$\mathcal{E}_{\eta_L} \simeq \mathrm{H}^i(\widetilde{\mathcal{S}}_{\eta_L}, \mathcal{F}) \simeq \mathrm{H}^i(\widetilde{\mathcal{S}}_L \times_L L_{(0)}, \mathbb{Q}_\ell)$$

with the first isomorphism due to $\mathcal{E}|_{L\setminus\{0\}}$ being locally constant, and the second isomorphism due to $\widetilde{\chi}_L$ being smooth and ([22], p. 56). Altogether we get that cospecialization $\mathcal{E}_0 \to \mathcal{E}_\eta$ is bijective, so by the criterion for local constancy of sheaves ([62, Tag 0GJ7]) we conclude that $\mathcal{E}$ is lisse. In particular $\mathrm{H}^i(\widetilde{\mathcal{S}}_h, \mathbb{Q}_\ell) \simeq \mathrm{H}^i((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$. We have $\widetilde{\mathcal{S}}_h \simeq \mathcal{S}_{\overline{h}}$ via Lemma 5.6.4, since the Grothendieck alteration induces an isomorphism over the regular semisimple locus, so altogether we get the desired isomorphism.

The case of $\overline{h}$ having zero-characteristic residue field works similarly by using Lemma 5.6.5, which yields $\mathrm{H}^i(\mathcal{S}_{\overline{h}}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_{\overline{\eta}}}, \mathbb{Q}_\ell)$. By Lemma 5.5.6 we are therefore done. $\qquad\square$

We isolate from the proof a Weyl action of interest. Let $\alpha_h : \mathrm{H}^i(\mathcal{S}_{\overline{h}}, \mathbb{Q}_\ell) \to \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ denote the isomorphism of Proposition 5.6.6. By Lemma 5.6.3, $h \to \mathfrak{h}^{\mathrm{rs}}$ is a lift of $\overline{h}$ and so $\mathcal{S}_{\overline{h}} \simeq \widetilde{\mathcal{S}}_h$. In the proof of Proposition 5.6.6 we constructed an isomorphism $\mathrm{H}^i(\widetilde{\mathcal{S}}_h, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ and so in particular $\mathrm{H}^i(\widetilde{\mathcal{S}}_h, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\widetilde{\mathcal{S}}_{w \cdot h}, \mathbb{Q}_\ell)$, where $w \cdot h \to \mathfrak{h}^{\mathrm{rs}}$ is a w–translate of h under the reflection W–action on $\mathfrak{h}^{\mathrm{rs}}$. Thus $\overline{h} = \psi(h) = \psi(w \cdot h)$ and

$\mathrm{H}^i(\widetilde{\mathcal{S}}_{w \cdot h}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{S}_{\overline{h}}, \mathbb{Q}_\ell)$, altogether giving the composite automorphism

$$\mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \xrightarrow{\alpha_h^{-1}} \mathrm{H}^i(\mathcal{S}_{\psi(h)}, \mathbb{Q}_\ell) \simeq \mathrm{H}^i(\mathcal{S}_{\psi(w \cdot h)}, \mathbb{Q}_\ell) \xrightarrow{\alpha_{w \cdot h}} \mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \tag{5.6.11}$$

Corollary 5.6.7. *The* $\mathrm{Gal}(L/K)$*-action on* $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ *is the restricton of the "monodromy"* W*–action coming from the automorphisms* $\alpha_{w \cdot h} \circ \alpha_h^{-1}$ *in* (5.6.11).

Proof. By Lemma 5.6.2 there exists a finite extension L/K so that the singularity of $\mathcal{X}/S$ admits a simultaneous resolution. We therefore have that the monodromy action of $\mathrm{Gal}(\overline{\eta}/\eta)$ factors through finite quotient $\mathrm{Gal}(L/K)$, and we get a map $\mathrm{Spec}(\mathcal{O}_L) \to \mathfrak{h}$ with closed point $s \mapsto 0$ and generic point $\eta_L \mapsto h \in \mathfrak{h}^{\mathrm{rs}}$; let $\overline{h}$ be the geometric point over h (NB. this is different than $\overline{h}$ defined in Proposition 5.6.6; here $\overline{h} \to \mathfrak{h}^{\mathrm{rs}}$).

We also set $\chi' : \mathcal{S}' \to \mathfrak{h}$ to be the base–change of χ along $\psi : \mathfrak{h} \to \mathfrak{h}/\!\!/W$. The singularity $x_s \in \mathcal{X}_s(k)$, viewed as the generic point in the subregular orbit of $\mathcal{S}_0$, is fixed by the W–cover, so that χ' is still a map of henselianized schemes ($\mathcal{S}'$ completed at x_s and $\mathfrak{h}$ completed at 0).

Let $\mathrm{R}\Psi_h^0 \mathbb{Q}_\ell \in \mathrm{D}^b(\mathcal{S}_0')$ be the sliced nearby cycles corresponding to data $(x_s, \overline{h}, \mathrm{sp})$ with $\mathrm{sp} : \overline{h} \to 0$ (Definition 5.3.2). If $i : \mathcal{S}_0' \hookrightarrow \mathcal{S}'$ and $j : \mathcal{S}' \times_{\mathfrak{h}} \mathfrak{h}_{(\overline{h})} \to \mathcal{S}'$ are the respective immersions, with j induced by the specialization map sp, we have

$$\mathrm{R}\Psi_h^0 \mathbb{Q}_\ell = i^* \mathrm{R}j_* \mathbb{Q}_\ell$$

Now the Grothendieck alteration $\pi : \widetilde{\mathcal{S}} \to \mathcal{S}$ has Stein factorization $\widetilde{\mathcal{S}} \to \mathcal{S}' \to \mathcal{S}$; we denote $\widetilde{\mathcal{S}} \to \mathcal{S}'$ also by π. Then π is proper, and so the formation of

$$\mathrm{R}\overleftarrow{\pi}_* : \mathrm{D}^b(\widetilde{\mathcal{S}} \overleftarrow{\times}_{\mathfrak{h}} \mathfrak{h}) \longrightarrow \mathrm{D}^b(\mathcal{S}' \overleftarrow{\times}_{\mathfrak{h}} \mathfrak{h})$$

commutes with base–change on $\mathcal{S}'$ and $\mathfrak{h}$ (see Section 5.3). Note that $\widetilde{\chi} : \widetilde{\mathcal{S}} \to \mathfrak{h}$ is smooth, so $\mathrm{R}\Psi_{\widetilde{\chi}} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell$ ([36], Ex. 1.7(b)) and in particular base–changing to $\mathcal{S}_0' \to \mathcal{S}'$ and $0 \to \mathfrak{h}$

yields (cf. Equation (5.1.2))

$$R\Psi_h^0 \mathbb{Q}_\ell \simeq R(\pi_0)_* \mathbb{Q}_\ell \tag{5.6.12}$$

Here $\pi_0 : \widetilde{\mathcal{S}}_0 \to \mathcal{S}'_0$ is the induced minimal resolution of surface $\mathcal{S}'_0$, and the left-hand side is $R\Psi_h^0(R(\pi_{(h)})_* \mathbb{Q}_\ell) \simeq R\Psi_h^0 \mathbb{Q}_\ell$ since $\pi_{(h)} : \widetilde{\mathcal{S}} \times_{\mathfrak{h}} \mathfrak{h}_{(\overline{h})} \to \mathcal{S}' \times_{\mathfrak{h}} \mathfrak{h}_{(\overline{h})}$ is an isomorphism by construction. Moreover the stalk of $R\Psi_h^0 \mathbb{Q}_\ell$ at x_s is equipped via (5.6.12) with the monodromy W-action specified after the proof of Proposition 5.6.6.

On the other hand, by Proposition (3.4.3) χ' is smooth (hence locally acyclic) outside the *finite* set of isolated singularities of each fiber. Therefore by Theorem 5.3.4 the sliced nearby cycles $R\Psi_h^0 \mathbb{Q}_\ell$ are Ψ-good, i.e. they commute with any base-change on $\mathfrak{h}$. Base-changing to $\mathrm{Spec}(\mathcal{O}_L) \hookrightarrow \mathfrak{h}$ yields the following computation of Milnor tube cohomology (see Remark 5.3.5 and Equation (5.3.4))

$$(R\Psi_h^0 \mathbb{Q}_\ell)_{x_s} \simeq R\Gamma(\mathcal{S}' \times_{\mathfrak{h}} \mathfrak{h}_{(\overline{h})}, \mathbb{Q}_\ell) \xrightarrow{\sim} R\Gamma(\mathcal{S}' \times_{\mathfrak{h}} \overline{h}, \mathbb{Q}_\ell) \simeq (R\Psi_{\mathcal{X}} \mathbb{Q}_\ell)_{x_s} \tag{5.6.13}$$

where the middle restriction isomorphism is a consequence of Ψ-goodness and the last isomorphism follows upon identifying $\mathcal{S}' \times_{\mathfrak{h}} \mathrm{Spec}(\mathcal{O}_L) \simeq \mathcal{X}_{(x_s)}$. The right-hand side of (5.6.13) is naturally equipped with the monodromy $\mathrm{Gal}(L/K)$-action (Section 5.1), so via this restriction we see that the $\mathrm{Gal}(L/K)$–action on $H^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ comes from restricting the monodromy W-action on $H^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$. $\qquad\qquad\square$

Remark 5.6.8. Implicit in Corollary 5.6.7 is the fact that the monodromy W-action on $H^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ comes from the action of $\pi_1^{\text{ét}}(\mathfrak{h}^{\mathrm{rs}} /\!/ W)$ on the stalks $H^2(\widetilde{\mathcal{S}}_{\overline{h}}, \mathbb{Q}_\ell)$ of local system $R^2 \widetilde{\chi}_* \mathbb{Q}_\ell$, as explained in ([61], §5). This retrieves the cohomological version of the monodromy results in ([58], §4.2-4.3). In the complex situation, Slodowy proves a stronger result, i.e. that $\mathcal{B}_{x_s}$ is homotopy-equivalent to a general fiber of $\widetilde{\chi}$ and so we get a W-action on the *homotopy type* of $\mathcal{B}_{x_s}$. By comparison, we obtain a weaker result here because we cannot pass to the differentiable category the way Slodowy does over $\mathbb{C}$.

So far we have defined two W-actions on $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$, the Springer W-action and the monodromy W-action. We next show that they coincide, by adapting the argument of ([33], §1.9) to our ℓ-adic setting. Since $\mathcal{B}_{x_s}$ is defined over k, in the following proposition we may work over k instead of S.

Proposition 5.6.9. *The Springer and monodromy W-actions on $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ coincide.*

Proof. Let $\overline{h} \to 0$ be a specialization from a point in $\mathfrak{h}^{\mathrm{rs}} /\!/ W(k)$ and consider its associated cospecialization map $\mathrm{H}^2((\mathcal{S}_0)_s, \mathrm{R}\pi_*\mathbb{Q}_\ell) \overset{\mathrm{cosp}}{\longrightarrow} \mathrm{H}^2(\mathcal{S}_{\overline{h}}, \mathrm{R}\pi_*\mathbb{Q}_\ell)$. Since $\mathrm{R}\pi_*\mathbb{Q}_\ell|_{\mathcal{S}_{\overline{h}}} \simeq \pi^{\mathrm{rs}}_* \mathbb{Q}_\ell$ and π^{rs} is a finite étale W-cover, we have

$$\mathrm{H}^2(\mathcal{S}_{\overline{h}}, \mathrm{R}\pi_*\mathbb{Q}_\ell) = \mathrm{H}^2(\mathcal{S}_{\overline{h}}, \pi^{\mathrm{rs}}_*\mathbb{Q}_\ell) \simeq \mathrm{H}^2\Big(\coprod_{w \in W} \widetilde{\mathcal{S}}_{w \cdot h}, \mathbb{Q}_\ell \Big) \simeq \mathrm{H}^2((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W]$$

where h is a chosen lift of $\overline{h}$ in $\mathfrak{h}^{\mathrm{rs}}$ yielding all other lifts $w \cdot h$, and the last isomorphism follows from Proposition 5.6.6. The right-hand side is further identified with $\mathrm{H}^i(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W]$ via Lemma 5.6.5, equipped with the monodromy W-action on the left. On the other hand by proper base-change we have

$$\mathrm{H}^2((\mathcal{S}_0)_s, \mathrm{R}\pi_*\mathbb{Q}_\ell) \simeq \mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \simeq \mathrm{H}^2((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell)$$

with the latter isomorphism again due to Lemma 5.6.5, and the right-hand side is equipped with the Springer W-action coming from the left-hand side and the Springer W-action on $\mathrm{R}\pi_*\mathbb{Q}_\ell$. These W-modules fit together into diagram

$$
\begin{array}{ccccc}
\mathrm{H}^2(\mathcal{S}_{\overline{h}}, \mathrm{R}\pi_*\mathbb{Q}_\ell) & \overset{\simeq}{\longrightarrow} & \bigoplus_{w \in W} \mathrm{H}^2(\widetilde{\mathcal{S}}_{w \cdot h}, \mathbb{Q}_\ell) & \overset{\simeq}{\longrightarrow} & \mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W] \\
{\scriptstyle\mathrm{cosp}} \uparrow & & & & {\scriptstyle\simeq} \uparrow \\
\mathrm{H}^2((\mathcal{S}_0)_s, \mathrm{R}\pi_*\mathbb{Q}_\ell) & \overset{\simeq}{\longrightarrow} & \mathrm{H}^2((\widetilde{\mathcal{S}}_0)_s, \mathbb{Q}_\ell) \overset{i}{\longrightarrow} \mathrm{H}^2((\widetilde{\mathcal{S}}_0)_s \times W \cdot h, \mathbb{Q}_\ell) & \simeq & \mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W]
\end{array}
$$

where the bottom-right isomorphism is the Künneth map, the bottom row has $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ equipped with the Springer W-action and the top row has $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$ equipped with the

monodromy W-action. The image of i is by definition $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell[W]^W \simeq \mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$, so the right vertical isomorphism identifies the monodromy and Springer W-actions on $\mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$. $\square$

We conclude this section with the proof of parts (i) and (ii) of the Main Theorem 1.3.1.

***Proof of* Theorem 1.3.1 (i), (ii)**. By assumption $p = \mathrm{char}(k)$ is sufficiently good, so $p \nmid |\mathrm{Gal}(L/K)|$ and hence the monodromy action of $\mathrm{Gal}(\overline{\eta}/\eta)$ is tame, hence generated by the Kummer character t_ℓ. We may therefore identify $\mathrm{Gal}(L/K) = \langle t_\ell \rangle$ with a cyclic subgroup $\langle w \rangle$ of W, and so by Proposition 5.6.9, Theorem 5.5.7 and Corollary 5.2.3 we have that the monodromy action on $\mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is the restriction to $\langle w \rangle$ of the reflection action of W on $\mathfrak{h}^\vee \simeq \mathrm{H}^2(\mathcal{B}_{x_s}, \mathbb{Q}_\ell)$. This retrieves (i), and by construction we achieve good reduction after base-change L/K, yielding (ii) as well. $\square$

Corollary 5.6.10. *Suppose the base $S = \mathrm{Spec}(\mathcal{O}_K)$ of model $\mathcal{X}$ intersects the discriminant divisor $\Delta \subset \mathfrak{h}/\!\!/W$ transversely, then $\mathrm{Gal}(L/K)$ is a multiple of the Coxeter number.*

Proof. As a consequence of Theorem 1.3.1 we have an equivalent description of the action of monodromy in the complex case and the *tame* mixed-characteristic case. We may thus identify the fundamental homogeneous generators of $\mathcal{O}_K[\mathfrak{h}]^W$ with coordinates on $\mathfrak{h}/\!\!/W \simeq \mathrm{Spec}(\mathcal{O}_K[\![t_1, \cdots, t_r]\!])$ so that the highest degree of a fundamental generator is $\mathrm{Cox}(\mathfrak{g})$. Then the line $\mathrm{Spec}(\mathcal{O}_K)$ is transverse to the discriminant if and only if all W-invariant polynomials of degree $< \mathrm{Cox}(\mathfrak{g})$ vanish on $\mathrm{Spec}(\mathcal{O}_K)$ ([58], p. 38). Conversely any Coxeter element $w \in W$ acting as a reflection on $\mathfrak{h} \simeq \mathfrak{h}^\vee$ has a 1-dimensional eigenspace for eigenvalue ζ_N, $N = \mathrm{Cox}(\mathfrak{g})$, which is mapped to a transversal line in $\mathfrak{h}/\!\!/W$ under a degree-$\mathrm{Cox}(\mathfrak{g})$ cover. Since L is a field extension on which $\mathcal{X}/S$ admits a simultaneous resolution, we have that $\mathrm{Cox}(\mathfrak{g}) = \mathrm{ord}(w) \mid \mathrm{Gal}(L/K)$. $\square$

A rephrasing of the above Corollary says that if S transversely intersects the discriminant locus, then the monodromy $\mathrm{Gal}(\overline{\eta}/\eta)$ acts through a *Coxeter* class in W.

5.7 Explicit monodromy actions on degenerations.

This section concerns part (iii) of Theorem 1.3.1. For certain types of singularities we may directly compute (up to conjugacy) the elements acting as monodromy operators on $H^2(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_\ell)$. We assume the situation in Notation 5.6.1 once more, i.e. we work over a complete trait $S = \mathrm{Spec}(\mathcal{O}_K)$ and consider flat proper surfaces $\mathcal{X}/S$ so that $\mathcal{X}_s$ has a singularity of type A_n and $p = \mathrm{char}(k)$ is sufficiently good for the singularity $(p > n + 1)$. After coordinate change we have an affine neighborhood of the singularity (see Example 2.5.5)

$$\mathrm{Spec}\left(\frac{\mathcal{O}_K[\![x, y, z]\!]}{x^2 + z^2 + y^{n+1} + u_n \pi^{a_n} y^{n-1} + \cdots + u_2 \pi^{a_2} y + u_1 \pi^{a_1}}\right) \longrightarrow \mathrm{Spec}(\mathcal{O}_K) \qquad (5.7.1)$$

where $u_i \in \mathcal{O}_K^\times$. In this case $W = S_{n+1}$ and the W-cover $\mathfrak{h} \longrightarrow \mathfrak{h}/\!\!/W$ is given in coordinates by

$$\mathrm{Spec}\left(\frac{\mathcal{O}_K[\![s_1, \cdots, s_{n+1}]\!]}{s_1 + \cdots + s_{n+1}}\right) \longrightarrow \mathrm{Spec}(\mathcal{O}_K[\![t_1, \cdots, t_n]\!]),$$

$$(s_1, \cdots, s_{n+1}) \longmapsto \sigma_i(s_1, \cdots, s_{n+1}), \ i = 2, \cdots, n + 1 \qquad (5.7.2)$$

where σ_i are the symmetric polynomials of i-th degree (see [64], §3 or [15], §2.7). The full monodromy group W acts on $H^2(\mathcal{B}_x, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^n$ via the standard symmetric S_{n+1}-representation. We also know in this case ([64]) that the base-changed miniversal family over $\mathfrak{h}$ is given by

$$V = \{x^2 + z^2 + (y - s_1) \cdots (y - s_{n+1}) = 0\}$$

after suitably multiplying variables s_i by units in $\mathcal{O}_K$. Family V may then be simultaneously resolved by means of the graph of the map

$$V \longrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^1, \quad (x, y, z) \longmapsto \left(\left[x : \prod_{i=1}^{k}(y - s_i)\right]\right)_{k=1}^{n+1} \qquad (5.7.3)$$

We describe here a general principle that will allow us to pass between the mixed-characteristic setting of $\mathrm{Spec}(K)$ and the equal-characteristic setting of $\mathrm{Spec}(k(\!(t)\!))$. The following is a direct application of Abhyankar's lemma ([26], §XIII.5).

Lemma 5.7.1. *Let* $T = \mathrm{Spec}\left(\dfrac{\mathcal{O}_K[\![t, u]\!]}{tu - \pi}\right)$ *and* $\mathrm{Spec}(\mathcal{O}_K), \mathrm{Spec}(k[\![t]\!]) \to T$ *be the fiber maps induced respectively from* $u \mapsto 1, u \mapsto 0$. *Restricting to the open dense scheme* $T\left[\frac{1}{t}\right] \simeq \mathbb{G}_{m, \mathcal{O}_K}$ *and the associated maps* $\mathrm{Spec}(K) \to \mathbb{G}_{m, \mathcal{O}_K} \leftarrow \mathrm{Spec}(k(\!(t)\!))$ *yields an isomorphism of (tame) étale fundamental groups*

$$\pi_1^{\mathrm{tame}}(\mathrm{Spec}(K), \overline{\eta}) \xrightarrow{\ \sim\ } \pi_1^{\text{ét}}(\mathbb{G}_{m, \mathcal{O}_K}, \overline{\eta}) \xleftarrow{\ \sim\ } \pi_1^{\mathrm{tame}}(\mathrm{Spec}(k(\!(t)\!)), s)$$

Under our restrictions on the characteristic, the ramified covers of K that we consider are automatically tame, so in what follows we may view family (5.7.1) as a family over $\mathrm{Spec}(k[\![t]\!])$ with parameter t instead of π for the purposes of ramification theory. We use this fact implicitly in the statements below.

Proposition 5.7.2. *Suppose we have a factorization* $x^2 + z^2 + (y^{r_1} - v_1 \pi^{b_1}) \cdots (y^{r_k} - v_k \pi^{b_k})$ *of the family in (5.7.1) so that* $r_i, b_i \geq 1$, $(r_i, b_i) = 1$ *and* $v_i \in \mathcal{O}_K^\times$. *Then the associated model achieves good reduction after a totally ramified base-change of order* $\mathrm{lcm}(r_1, \cdots, r_k)$ *and the element* $w \in S_{n+1}$ *acting as the monodromy operator has cycle-type* $(r_1, \cdots, r_k)$.

Proof. Let $f(y) = (y^{r_1} - v_1 \pi^{b_1}) \cdots (y^{r_k} - v_k \pi^{b_k}) \in \mathcal{O}_K[y]$. By assumption $\overline{f}(y) = y^{n+1}$ for the mod π polynomial, and by the Newton polygon of f we get slopes $\mu_i = -\frac{b_i}{r_i}$, which may appear with multiplicity (equal to the number of factors of form $y^{r_i} - \pi^{b_i}$). Since $\mathcal{O}_K$ is henselian and the degrees r_i are coprime to b_i we get that each factor is irreducible by Gauss's lemma, hence the degree of its splitting field is divisible by r_i. On the other hand $r_i < p$ and $\mathcal{O}_K$ contains all roots of unity prime to p so the splitting field degree is exactly r_i, yielding an r_i-cycle in S_{n+1}.

Since any element of S_{n+1} can be written as a product of disjoint cycles and disjoint

cycles commute, we observe that $f(y)$ splits after an extension of degree $\mathrm{lcm}(r_1, \cdots, r_k)$, corresponding to an element of cycle-type $(r_1, \cdots, r_k)$ as we may reorder the factors in decreasing order of r_i. The base-changed model then admits a (small) resolution of the form (5.7.3) where the s_i correspond to the roots of $f(y)$; in other words we get a smooth model obtained as a base-change of $\widetilde{\mathcal{S}} \to \mathfrak{h}$ to $\mathrm{Spec}(\mathcal{O}_L)$, where $\mathcal{O}_L/\mathcal{O}_K$ is the unique totally ramified extension of $\mathcal{O}_K$ of degree $\mathrm{lcm}(r_1, \cdots, r_k)$. Conjugacy classes in S_{n+1} bijectively correspond to cycle-types of this form, which proves the last statement. $\qquad\square$

Corollary 5.7.3 (Part (iii) of Theorem 1.3.1). *Any conjugacy class in S_{n+1} may act as the monodromy operator on a surface family degenerating to an A_n-singularity.*

For A_n-singularities, we derive another proof of Corollary 5.6.10 using the explicit equation of the miniversal deformation.

Corollary 5.7.4. *Assume $\mathrm{Spec}(\mathcal{O}_K)$ intersects the discriminant divisor Δ transversely, then for the resulting model with an A_n-type singularity at the special fiber we have that monodromy acts a Coxeter element.*

Proof. We assume again that there is an affine étale neighborhood of the singular point in X of the form $\{x^2 + z^2 + f(y) = 0\}$ for $f(y) = y^{n+1} + u_n \pi^{a_n} y^{n-1} + \cdots + u_2 \pi^{a_2} y + u_1 \pi^{a_1}$ with $a_i \geq 1$ (otherwise the proof of Proposition 5.7.2 shows that we get a singularity of A_m-type for $m < n$ when we reduce mod π). Viewing $\mathrm{Spec}(\mathcal{O}_K)$ as a line inside $\mathfrak{h}/\!\!/W$, it intersects Δ transversely exactly when it is not tangent to the tangent cone of Δ. The tangent cone is $V(\sigma_{n+1})$ for σ_{n+1} the $(n+1)$-th symmetric polynomial generator of S_{n+1}. When the affine étale neighborhood of the singular point in $\mathcal{X}$ is smooth, we satisfy the non-tangency condition.

By the Jacobian criterion we see that the point corresponding to ideal (x, y, z, π) is singular if and only if $a_1 \geq 2$ so we have $a_1 = 1$. In this case $f(y)$ is Eisenstein, hence irreducible in $\mathcal{O}_K$ by Gauss's lemma. The Galois group of its splitting field therefore contains

93

an element of order $n+1$ i.e. an $(n+1)$-cycle, and it is cyclic, hence generated by an $(n+1)$-cycle. We thus get the conjugacy class of Coxeter elements in S_{n+1}. $\qquad\square$

5.8   Higher rational singularities and beyond

The results of this chapter apply to rational singularities of multiplicity 2, but the argument can be adapted to rational singularities of higher multiplicity. In [67], Wahl extends the argument of Brieskorn–Slodowy to general rational singularities by incorporating some results of Artin ([7], §1 and §4).

The idea is as follows: given a rational singularity $\operatorname{Spec}(R)$, where R is a normal 2-dimensional local ring with structure map to $\mathbb{C}$, we have a space of deformations Def_R and a (smooth) space of deformations admitting a simultaneous resolution Res_R along with a finite morphism $\Phi : \operatorname{Res}_R \to \operatorname{Def}_R$ ([7], Thm. 1). In the case of RDPs Φ is surjective (as every deformation of an RDP admits a simultaneous resolution), but for higher multiplicities it is not, and one calls its image the *Artin component* Art_R, an irreducible component of Def_R.

If we take $X = \widetilde{\operatorname{Spec}(R)} \to \operatorname{Spec}(R)$ to be the minimal resolution of the rational singularity, we may interpret Res_R as a space of versal deformations of X. As the exceptional divisor $E \subset X$ contains rational curve components of various multiplicities, one may blow down any (-2)-curves to obtain an intermediate variety $X \to Y \to \operatorname{Spec}(R)$, where Y now contains RDPs corresponding to the blown-down (-2)-curves.

The method of Brieskorn–Slodowy may be adapted in this case to show that $\operatorname{Res}_R = \operatorname{Def}_X \to \operatorname{Def}_Y$ is finite Galois with Galois group $\prod_i W_i$, where each W_i is the Weyl group associated to a (largest) connected (-2)-configuration in the dual graph of divisor E, hence corresponds to each RDP in Y. A cohomological argument due to Lipman ([67], §2) then shows $\operatorname{Def}_Y \hookrightarrow \operatorname{Def}_R$ so that we overall get a finite branched cover $\operatorname{Res}_R \to \operatorname{Art}_R$ with Galois group $\prod_i W_i$, similar to the RDP case $\mathfrak{h} \to \mathfrak{h}/\!\!/ W$.

The arguments of Wahl a priori work over $\mathbb{C}$, but due to their functorial nature and

94

results from this Chapter we can also extend the step involving $\text{Def}_X \to \text{Def}_Y$ to mixed characteristic, and then adapt Lipman's argument to this setting too. Thus we get a complete description of the monodromy of rational singularities in mixed characteristic, parallel to results of Shepherd-Barron ([56]); details are currently being written up in a forthcoming paper.

5.9 An application to K3 surfaces

In [32], Hassett–Tschinkel study invariants of *derived equivalences* of K3 surfaces defined over $\mathbb{C}$ or local fields of equal and mixed characteristic. By a derived equivalence between two K3 surfaces X and Y we mean an equivalence between $\text{D}^b(X) := \text{D}^b(\mathcal{O}_X\text{-Mod})$ and $\text{D}^b(Y)$, the respective bounded derived categories of coherent sheaves on X and Y.

One of their main results ([32], Prop. 31) is that, for derived equivalent K3 surfaces defined over $\mathbb{C}((t))$, the monodromy operator on H^2 acting via a product of Weyl groups is equivalent to both X, Y admitting integral models with an RDP K3 surface reduction. The construction of such a model crucially uses the Torelli theorem and the relation between monodromy and Weyl groups in the ADE case, which up to now was only available in the complex case.

Our main theorem extends this observation when X, Y are also defined over a mixed characteristic complete field, as long as the residue characteristic p is good (i.e. we are in the tame ramification case). The argument of [32] applies in this context as well and we find that the RDP reduction of a K3 surface depends only on a monodromy characterization of its $\text{H}^2_{\text{ét}}$, hence it depends only on the derived equivalence class of X. We summarize our findings as follows.

Corollary 5.9.1. *For a K3 surface defined over a mixed characteristic complete field with suitably good residue characteristic p, admitting a model with RDP reduction is a derived invariant.*

APPENDIX A

RELATIVE PERVERSITY OF NEARBY CYCLES

Throughout we use the following conventions: S is a noetherian scheme of finite type over $\mathbb{Z}$, and $f : X \to S$ is a noetherian S-scheme of finite type.

Such schemes admit a *dimension* function ([2], Exposé XIV) as follows: $\delta_S : |S| \to \mathbb{Z}$ maps $s \mapsto \dim(\overline{\{s\}})$ and via f we obtain a canonical dimension function for X:

$$\delta(x) = \delta_S(f(x)) + \operatorname{trdeg}(k(x)/k(f(x))) \tag{A1}$$

For schemes equipped with dimension functions, the natural perverse t-structure we consider is

$${}^p D^{\leq 0}(X) = \{\mathcal{F} \in D^b(X) \mid \mathcal{H}^i(\mathcal{F}_x) = 0 \quad \text{for } i > -\delta(x), \ x \to X\} \tag{A2}$$

It can be shown that the truncation functors preserve $D^b_c(X)$, that this perverse t-structure is stable under Verdier duality and that any other choice of dimension function δ would yield the same t-structure up to a locally constant shift. In the case of X/k finite-type over a field this recovers the usual notion of *middle* perversity.

When S is a strictly henselian trait and X/S has a dimension function as in (A1), Drinfeld has shown the following:

Proposition A1 ([11], Appendix A). *If* $\mathcal{F} \in D^b_c(X_\eta, \Lambda)$ *is perverse then* $R\Psi\mathcal{F}[-1] \in D^b_c(X_s, \Lambda)$ *is also perverse.*

The discrepancy between this result and perversity of nearby cycles without shift ([11], Prop. 4.4.2) comes from a difference in the choice of dimension function δ; in this case, the t-structure on X_η is shifted to the left by $\operatorname{codim}_{\overline{\{\eta\}}}(s) = 1$.

We now return to the setting of Section 5.3 and consider nearby cycles over base S of dimension ≥ 2. Let $\eta \neq s$ be two points of S so that $s = \overline{s}$ and η specializes to s. We may check the perversity of the sliced nearby cycles (Definition 5.3.2) as follows:

Lemma A2. Let K be the residue field of η in the above setting. Then there exists a discrete valuation ring $\mathcal{O}_K$ with fraction field K and a morphism $\mathrm{Spec}(\mathcal{O}_K) \to S$ mapping generic point $\eta_K \mapsto \eta$ and closed point $s_K \mapsto s$.

Proof. Omitted; see ([30], II.4.4) for a related statement. $\square$

Associated to $\mathrm{R}\Psi_f(\mathcal{F}) \in \mathrm{D}^b(X \overset{\leftarrow}{\times}_S S)$ we have the sliced nearby cycles $\mathrm{R}\Psi_\eta^s(\mathcal{F}) \in \mathrm{D}^b(X_s)$, which is also given by $i^* \mathrm{R}j_* \mathcal{F}|_{X_{(\overline{\eta})}}$, where $i : X_s \hookrightarrow X$ and $j : X_{(\overline{\eta})} = X \times_S S_{(\overline{\eta})} \to X_{(s)}$. We may also consider a variant of the slice construction given by

$$\mathrm{R}\Psi^s : \mathrm{D}^b(X) \to \mathrm{D}^b(X_s \overset{\leftarrow}{\times}_S S) \tag{A3}$$

where $\mathrm{R}\Psi^s(\mathcal{F}) = (i \overset{\leftarrow}{\times}_S \mathrm{id}_S)^* \mathrm{R}\Psi(\mathcal{F})$. This is what Lu–Zheng call "sliced" nearby cycles in [44] (and they refer to our sliced nearby cycles as "shredded" as $\mathrm{R}\Psi_\eta^s$ is a further restriction of $\mathrm{R}\Psi^s$).

In the case where S is local henselian, we have an equivalence of $X_s \overset{\leftarrow}{\times}_S S$ with the *fiber product* of toposes $X \times_{\underset{S}{\leftarrow}} S$ (see [44], Lemma 4.1) and so, following Hansen–Scholze, we call $\mathrm{R}\Psi^s(\mathcal{F})$ relatively perverse if the slice $\mathrm{R}\Psi_\eta^s(\mathcal{F}|_{X_{(\eta)}}) \in \mathrm{D}^b(X_s)$ is perverse for all specializations $\eta \to S_{(\eta)} \to S_{(s)} = S$. We may assume here η is a physical point of S and the construction of $\mathrm{R}\Psi_\eta^s$ makes a choice of a geometric point $\overline{\eta} \to \eta$.

In good cases we may confirm that nearby cycles are relatively perverse in this sense:

Proposition A3. Suppose that pair $(f, \mathbb{Q}_\ell)$ is Ψ-good and $s \to S$ is a geometric point so that $S = S_{(s)}$. Then for any $\eta \in S$ specializing to s we have that $\mathrm{R}\Psi_\eta^s \mathbb{Q}_\ell[-\mathrm{codim}_{\overline{\{\eta\}}}(s)]$ is perverse.

Proof. By Lemma 2 we may construct a morphism $\phi : \mathrm{Spec}(\mathcal{O}_K) \to S$ from a DVR $\mathcal{O}_K$ with fraction field $K = k(\eta)$ such that $\phi(\eta_K) = \eta$ and $\phi(s_K) = s$. By Ψ-goodness, formation of $\mathrm{R}\Psi_\eta^s \mathbb{Q}_\ell$ commutes with base-change ϕ, so if $\phi' : X_{\mathcal{O}_K} \to X$ and $\phi'' : X_{\mathcal{O}_K, s_K} \overset{\leftarrow}{\times}_{\mathcal{O}_K} \overline{\eta}_K \to$

$X_s \overset{\leftarrow}{\times}_S \overline{\eta}$ are the base-change morphisms induced by ϕ we have

$$(\phi'')^* \mathrm{R}\Psi_\eta^s \mathbb{Q}_\ell \simeq \mathrm{R}\Psi_{\eta_K}^{s_K}((\phi')^* \mathbb{Q}_\ell) \simeq \mathrm{R}\Psi\mathbb{Q}_\ell \tag{A4}$$

where the right-hand side denotes the classical nearby cycles for $X_{\mathcal{O}_K} \to \mathrm{Spec}(\mathcal{O}_K)$ as in ([4], Exposé XIII). The latter is a sheaf on $X_{\mathcal{O}_K, s_K} \overset{\leftarrow}{\times}_{\mathcal{O}_K} \overline{\eta}_K \simeq X_{\mathcal{O}_K, s_K} \times BG$, where $G = \mathrm{Gal}(k(\overline{\eta})/k(\eta))$. Thus there is a natural monodromy action of G on $\mathrm{R}\Psi_\eta^s \mathbb{Q}_\ell$ and Drinfeld's argument (Proposition 1) adapts without change to show that $\mathrm{R}\Psi_\eta^s \mathbb{Q}_\ell$ is perverse up to a shift. The shift is determined by the difference between the naive t-structures on X_s and $X_\eta \hookrightarrow X_{(\overline{\eta})}$, which the dimension function shifts to the left by $\mathrm{codim}_{\overline{\{\eta\}}}(s)$. $\qquad\square$

By the hypotheses on S coming from the above proposition, we yield the relative perversity of $\mathrm{R}\Psi^s \mathbb{Q}_\ell$.

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