THE UNIVERSITY OF CHICAGO

REGULARITY, FREE BOUNDARY ANALYSIS, AND ASYMPTOTIC BEHAVIOR OF FIRST-ORDER MEAN FIELD GAMES

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To my mother

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ABSTRACT

This thesis details recent results concerning the regularity, well-posedness, and long-time behavior of the first-order mean field games system with a local coupling.

First, we prove that when the coupling is unbounded from below (the so-called blow-up assumption) and the density is positive, the system has classical solutions. Our starting point is a transformation due to P.-L. Lions, which gives rise to an elliptic partial differential equation with oblique boundary conditions.

Next, we investigate the extent to which these assumptions can be weakened in the one-dimensional setting. We first prove that the blow-up assumption can be removed, while still obtaining classical solutions, whose long-time behavior can then be fully characterized. Additionally, we show that, for interior times, the solution is still smooth if the positivity assumption of the density is weakened to an appropriate almost-everywhere positivity condition.

Finally, we develop the more challenging setting where the density may vanish on a set of positive measure, which involves studying not only the regularity of the (weak) solutions but also of the emerging free boundary. Our results show that the solution is smooth in regions where the density is strictly positive, and that the density itself is globally continuous. Additionally, the speed of propagation is determined by the behavior of the cost function near small values of the density. When the coupling is entropic, we demonstrate that the support of the density propagates with infinite speed. On the other hand, for a power-type coupling, we establish finite speed of propagation, leading to the formation of a free boundary. We prove that under a natural non-degeneracy assumption, the free boundary is strictly convex and enjoys $C^{1,1}$ regularity. We also establish sharp estimates on the speed of support propagation and the rate of long time decay for the density. Moreover, the density and the gradient of the value function are both shown to be Hölder continuous up to the free boundary. The methods are based on the analysis of a new elliptic equation satisfied by the

Lagrangian flow.

CHAPTER 1

INTRODUCTION

This thesis is concerned with the analysis of the qualitative behavior of solutions to the following forward-backward system of PDE, which arises in the theory of large deviations, optimal transport, and mean field games (MFG), among other places:

$$\begin{cases} -u_t + H(x, Du) = f(m) & (t,x) \in (0,T) \times \Omega, \\ m_t - \operatorname{div}(mD_pH(x, D_xu)) = 0 & (t,x) \in (0,T) \times \Omega, \\ m(0,x) = m_0(x), \ u(T,x) = g(m(T,x)) & x \in \Omega, \end{cases} \tag{1.0.1}$$

where Ω is typically either a subset of $\mathbb{R}^d$ or the d–dimensional torus $\mathbb{T}^d$, $m_0 : \Omega \to [0,\infty)$ is a probability density on Ω, the Hamiltonian $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is strictly convex in the second variable, and $f, g : [0,\infty) \to \mathbb{R}$ are assumed to be increasing.

From a modeling perspective, the solution (u, m) can be interpreted as the Nash equilibrium of a differential game with infinitely many players. Each agent is trying to minimize a cost by controlling her own dynamical state. The function $m(t,x)$ may be interpreted as the density of players at time t and position x, whereas $u(t,x)$ is the optimal cost for a generic player at time t, at the position x, that is,

$$u(t,x) = \inf_{\gamma \in W^{1,\infty}([t,T];\Omega),\, \gamma(t)=x} \int_t^T \big(L(x, \gamma'(s)) + f(m(s,\gamma(s)))\big)\, ds + g(m(T,\gamma(T))),$$

where the Lagrangian $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is the convex conjugate of the Hamiltonian H. In particular, the monotonicity of f and g means that the agents are incentivized to avoid congested regions.

A natural first question to ask about (1.0.1) is whether it is well-posed, and, if yes, to determine the regularity of the solution. It is well-known that standalone Hamilton–

Jacobi equations, even with smooth data, do not have, in general, classical solutions. The system (1.0.1) is, of course, more complicated, because the right hand side $f(m(t,x))$ of the Hamilton–Jacobi equation is not a priori known to have any regularity. Previous results on the well-posedness of (1.0.1) yielded very little regularity. In particular, u was not known to be differentiable, and there was nothing about the continuity of m (see, for instance, [7, 8, 9, 47]).

It turns out that, under adequate assumptions, (1.0.1) has a unique smooth solution (u,m). The first such result was proved in [44, Thm. 1.1], and is the content of Chapter 2. The starting point used to obtain classical solutions is the following key observation of Lions, which he presented in one of his lectures during his course at Collège de France [38]. One may eliminate the function $m = f^{-1}(-u_t + H(x,Du))$ using the first equation of (1.0.1), resulting in a degenerate elliptic equation in the space-time variables with fully non-linear boundary conditions, that is,

$$\begin{cases} -\mathrm{tr}(\mathcal{A}(x,\mathcal{D}u)\mathcal{D}^2u) + a(x,\mathcal{D}u) = 0 & (t,x) \in (0,T)\times\Omega, \\ \mathcal{B}(t,x,\mathcal{D}u) = 0 & (t,x) \in \partial((0,T)\times\Omega), \end{cases} \tag{1.0.2}$$

where $\mathcal{D}u = (Du, u_t)$ denotes the space-time gradient of u, and $\mathcal{D}^2u$ denotes its space-time Hessian matrix. Setting $\chi(\cdot) = f^{-1}(\cdot)f'(f^{-1}(\cdot))$, the matrix $\mathcal{A}$ and the function $\mathcal{B}$ are given by

$$\mathcal{A}(x,s,p) =(D_pH,-1)\otimes(D_pH,-1) + \chi(-s+H(x,p))\begin{pmatrix} D^2_{pp}H(x,p) & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.0.3}$$

$$\mathcal{B}(x,0,z,s,p) =-s + H(x,p) - f(x,m_0(x)), \tag{1.0.4}$$

$$\mathcal{B}(x,T,z,s,p) =-g(x,f^{-1}(-s+H(x,p))) + z. \tag{1.0.5}$$

If $\partial\Omega \neq \emptyset$, then typically the problem is also supplemented by Neumann boundary conditions in the set $(0,T) \times \partial\Omega$.

In view of the convexity of H and the monotonicity of f, this is an elliptic equation. However, the eigenvalues of A degenerate precisely when $\chi = mf'(m) = 0$. This equivalent reformulation strongly suggests that (1.0.1) has smooth solutions, at least inside the region $\{m > 0\}$.

From these observations, two natural questions arise:

Question 1. If the initial density m_0 is smooth and strictly positive, does (1.0.1) have a smooth solution (u, m) such that $m > 0$ everywhere?

Question 2. If the initial density m_0 is not everywhere positive, is the solution (u, m) to (1.0.1) still smooth in the set $\{m > 0\}$? Moreover, what is the shape and regularity of the free boundary $\partial\{m > 0\}$?

The main objective of this thesis is to answer these questions. Question 1 is addressed in Chapters 2 and 3, whereas Question 2 is the main focus of Chapter 4.

CHAPTER 2

REGULARITY IN ARBITRARY DIMENSIONS UNDER BLOW-UP ASSUMPTION

2.1 Introduction

The goal of this chapter is to establish, under adequate assumptions, the existence of smooth solutions to the system:

$$\begin{cases} -u_t + H(x, D_x u) = f(x, m(x,t)) & (x,t) \in Q_T := \mathbb{T}^d \times (0,T), \\ m_t - \operatorname{div}(m D_p H(x, D_x u)) = 0 & (x,t) \in Q_T, \\ m(x,0) = m_0(x),\ u(x,T) = g(x, m(x,T)) & x \in \mathbb{T}^d, \end{cases} \tag{MFG}$$

where $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a strictly convex Hamiltonian of quadratic growth, $f, g : \mathbb{T}^d \times [0,\infty) \to [-\infty, \infty)$ are strictly increasing in their second variable m, f has polynomial growth in m, and m_0 is a strictly positive probability density.

We will impose the blow-up assumption, namely

$$\lim_{m \to 0^+} f(m,x) = -\infty. \tag{2.1.1}$$

From the modeling perspective, this assumption can be understood as imposing a very strong incentive for the players to occupy empty regions.

The main result is stated as follows. We refer to Section 2.2 for the exact assumptions (M), (H), (F), and (G).

Theorem 2.1.1. *Let $0 < \alpha < 1$, and assume that (M), (H), (F), (G), and* (FB) *hold. Then there exists a unique classical solution $(u,m) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$ to (MFG).*

The content and structure of this chapter are described as follows. Section 2.2 explains the general setting and assumptions that will be used, followed by the statements of the preliminary results from the classical literature on quasilinear elliptic equations and oblique derivative problems that will be used to prove existence of classical solutions. In Section 2.3, we obtain all the necessary a priori estimates. The main results, which deal with the system in full generality, are summarized in Theorem 2.3.9. Subsection 2.3.1 contains the L^∞–bounds on the solution u, as well as two-sided bounds for the terminal density $m(\cdot, T)$, obtained through maximum principle methods. In Subsection 2.3.2, the gradient bound is obtained by means of an elaborate application of the Bernstein method, which is the most technical part of this chapter. Finally, Section 2.4 contains the proof of the existence of classical solution, namely the proof of Theorem 2.1.1. It is first explained how a classical result from the theory of oblique derivative problems, due to G.M. Lieberman [37], immediately yields an a priori Hölder estimate for Du up to the boundary in terms of the L^∞–bounds on u and Du. Existence is then proved through an application of the non-linear method of continuity, the classical Schauder estimates for the linear oblique derivative problem, and a variant of a convergence theorem of R. Fiorenza [22, 23, 36].

The main result of this chapter, Theorem 2.1.1, was shown by the author in [44, Thm. 1.1]. This result was also extended in the recent work of Porretta [48], to treat both Neumann boundary conditions and problems in the whole space, and the mean field planning problem, which is discussed in the next chapter. In [45], the author also showed existence of classical solutions for the so-called extended MFG, a generalization of (MFG) introduced by Lions and Souganidis [40], having a fully general continuity equation, and a non-separated Hamiltonian, namely $H = H(x, p, m)$, with arbitrary superlinear growth. In particular, classical solutions were obtained for first order MFG with congestion.

Notation

Let $n, k \in \mathbb{N}$. Given $x, y \in \mathbb{R}^n$, x and y will always be understood to be row vectors, and their scalar product xy^T will be denoted by $x \cdot y$. For any bounded set Ω, with $\Omega \subset Q_T$, $\Omega \subset \mathbb{T}^d$, or $\Omega \subset [0, T]$, and $0 \leq \alpha < 1$, $C^{k,\alpha}(\Omega)$, refers to the space of k times differentiable real-valued functions with α–Hölder continuous k^{th} order derivatives, and, for $u \in C^{0,\alpha}(\Omega)$, the Hölder semi-norm of u will be denoted by $[u]_{\alpha,\Omega}$. For functions $\Phi(x, t, z, p, s) \in C^0(Q_T \times \mathbb{R} \times \mathbb{R}^{d+1})$, where typically $(x, t, z, p, s) = (x, t, u(x, t), D_x u, u_t)$, the conventions $\bar{x} \equiv (x, t)$ and $q \equiv (p, s)$ will always be in place. The notation Du, $D\Phi$ will always refer to the full gradient in all variables, so that, for instance $Du = D_{\bar{x}} u = (D_x u, u_t)$, and $D\Phi = (D_{\bar{x}}\Phi, \Phi_z, D_q\Phi)$. For $(x, t) \in \partial Q_T$, $\nu(x, t) = \pm(0, 0, \ldots, 1)$ denotes the outward pointing unit normal vector. We write $C = C(K_1, K_2, \ldots, K_M)$ for a positive constant C depending monotonically on the non-negative quantities $K_1, \ldots, K_M$. We also define, for $K > 0$, and any set V, $V_K = \{(y, z, q) \in V \times \mathbb{R} \times \mathbb{R}^{d+1} : |z| + |q| \leq K\}$. We write $C^k(\overline{Q_T})^*$ for the dual space of $C^k(\overline{Q_T})$.

2.2 Assumptions and general setting

2.2.1 The MFG system as an elliptic problem

We now present the general elliptic formulation of the MFG system. As explained in Chapter 1, it is an equivalent problem satisfied by u, whenever the pair $(u, m) = (u, f^{-1}(\cdot, -u_t + H(\cdot, D_x u)) \in C^2(\overline{Q_T}) \times C^1(\overline{Q_T})$ is a classical solution to (MFG). It is obtained after eliminating m from the system, and it consists of a quasilinear elliptic equation with a non-linear oblique boundary condition,

$$\begin{cases} Qu = -\mathrm{Tr}(A(x, Du)D^2u) + b(x, Du) = 0 & \text{in } Q_T, \\ Nu = B(x, t, u, Du) = 0 & \text{on } \partial Q_T, \end{cases} \tag{Q0}$$

where, for all $(x,t,z,p,s) \in \overline{Q_T} \times \mathbb{R} \times \mathbb{R}^{d+1}$,

$$A(x,p,s) = (D_pH,-1) \otimes (D_pH,-1) + \chi(x,-s+H(x,p)) \begin{pmatrix} D^2_{pp}H(x,p) & 0 \\ 0 & 0 \end{pmatrix}, \tag{Q1}$$

$$\begin{aligned} b(x,p,s) = &-D_xH(x,p) \cdot D_pH(x,p) + D_xf(x,f^{-1}(x,-s+H(x,p))) \cdot D_pH(x,p) \\ &- \chi(x,-s+H(x,p))\mathrm{Tr}(D^2_{xp}H(x,p)), \end{aligned} \tag{Q2}$$

$$\begin{aligned} B(x,0,z,p,s) = -s + H(x,p) - f(x,m_0(x)),\ B(x,T,z,p,s) \\ = -g(x,f^{-1}(x,-s+H(x,p))) + z, \end{aligned} \tag{B1}$$

with the function $\chi(x,w)$ being defined by

$$\chi(x,w) = f^{-1}(x,w)f_m(x,f^{-1}(x,w)).$$

We remark that the matrix A is clearly non-negative, and since $\det(A) = \chi^d \det D^2_{pp}H$, the condition for degeneracy is $\chi = mf_m = 0$. For future use, we set

$$h(x,w) = \sqrt{\chi(x,w)}.$$

2.2.2 Assumptions

We now state the main assumptions (M), (H), (F), (G) that will be in place throughout this chapter. The differentiability assumptions on the data can naturally be weakened through standard approximation arguments, but in the interest of clarity such matters will not be considered at this stage. In the next chapter, however, it is shown how $C^{3,\alpha} \times C^{2,\alpha}$ solutions may be obtained when m_0 is just assumed to be in $C^{2,\alpha}(\mathbb{T}^d)$. Throughout the assumptions, the quantities $C_0 > 0$ and $0 \le \tau < 1$ are fixed constants.

(M) (Assumptions on m_0) The initial density m_0 satisfies

$$m_0 \in C^4(\mathbb{T}^d),\ m_0 > 0,\ \text{and} \int_{\mathbb{T}^d} m_0 = 1. \tag{M1}$$

(H) (Assumptions on H) The functions $H,\ D_pH,\ D^2_{pp}H$ are four times continuously differentiable, and the following quadratic growth and uniform convexity conditions hold:

$$\frac{1}{C_0} I \le D^2_{pp}H(x,p) \le C_0 I, \tag{H1}$$

$$D_pH(x,p)\cdot p \ge 2H(x,p) - C_0, \tag{H2}$$

$$|D^3_{ppp}H(x,p)| \le C_0(1+|p|)^{-1}, \tag{H3}$$

for all $(x,p) \in \mathbb{T}^d \times \mathbb{R}^d$. The space oscillation of H is at most subquadratic in p, namely

$$|D^3_{xxp}H| \le C(1+|p|)^{\tau},\ |D^3_{xpp}H| \le C_0(1+|p|)^{\tau-1}. \tag{HX}$$

(F) (Assumptions on f) The continuous function $f : \mathbb{T}^d \times [0,\infty) \to [-\infty,\infty)$ is four times continuously differentiable on $\mathbb{T}^d \times (0,\infty)$ and strictly increasing in the second variable, with $f_m > 0$. f satisfies the blow-up assumption

$$\lim_{m\to 0^+} f(m,x) = -\infty. \tag{FB}$$

Furthermore, f grows polynomially as $m \to \infty$, in the sense that its growth is at least of degree zero, namely

$$\liminf_{x \in \mathbb{T}^d \, m \to \infty} m f_m(x,m) > 0, \tag{F1}$$

and its derivative f_m satisfies a polynomial bound $|m f_{mm}| \leq C_0 f_m$, which can be equivalently expressed in terms of $\chi(x,w)$ as

$$|\chi_w| \leq C_0. \tag{F2}$$

The space derivative of f satisfies the same polynomial bound,

$$|m(D_x f)_m| \leq C_0 |D_x f|, \tag{FX1}$$

as well as the control

$$|D_x f|, |D^2_{xx} f| \leq C_0 (1 + |f|^{\tau/2} + |m f_m|^{(1+\tau)/2}). \tag{FX2}$$

(G) (Assumptions on g) The continuous function $g : \mathbb{T}^d \times [0,\infty) \to [-\infty, \infty)$ is four times continuously differentiable on $\mathbb{T}^d \times (0,\infty)$ and strictly increasing in the second variable, with $g_m > 0$. The control required for its space oscillation is that, for each $x \in \mathbb{T}^d$,

$$\lim_{m\to\infty} g(x,m) = \sup_{\mathbb{T}^d \times [0,\infty)} g, \text{ and } g(x,0) = \inf_{\mathbb{T}^d \times [0,\infty)} g, \tag{GX}$$

A few comments should be made about the assumptions on the spatial oscillation. First, we remark that the subquadratic growth assumption (HX) can be interpreted as requiring that the purely quadratic part of H is independent of x. Condition (FX2), on the other hand, can be interpreted as being dual to (HX). Indeed, heuristically, since f is assumed to have polynomial growth, $m f_m \approx f$, and $f = -u_t + H \approx |p|^2$, so both conditions impose the same polynomial growth bound in the variable $|p|$. We consider now the assumption (GX) on the

x–oscillation of g. When g is bounded, the first (resp. second) condition in (GX) corresponds to a purely qualitative control on $|D_x g|$ that becomes stricter as $m \to \infty$ (resp. $m \to 0^+$). From the modeling point of view, it can be interpreted as the requirement that extremely crowded regions (resp. nearly empty regions) have roughly the same terminal value for the players.

Remark 2.2.1. For simplicity of the presentation, we observe that, up to increasing the value of C_0, the following inequalities are trivial consequences of (H1), (HX), (M), and (F), and they will be used freely when pertinent.

$$\frac{1}{C_0}|p|^2 - C_0 \leq H(x,p) \leq C_0|p|^2 + C_0, |D_p H(x,p)| \leq C_0(1+|p|), \tag{2.2.1}$$

$$|D_x H(x,p)| \leq C_0(1+|p|^{1+\tau}),\ |D^2_{xx} H(x,p)| \leq C_0(1+|p|^{1+\tau}), \tag{2.2.2}$$

$$|D^2_{xp} H(x,p)| \leq C_0(1+|p|)^{\tau}, \tag{2.2.3}$$

$$||\chi(\cdot,0)||_{C^0(\mathbb{T}^d)} + ||m_0||_{C^1(\mathbb{T}^d)} + ||f||_{C^2(\mathbb{T}^d \times [\min m_0, \max m_0])} \leq C_0. \tag{2.2.4}$$

2.2.3 Preliminary results

This subsection includes the classical results that will be required in Section 2.4 to obtain the higher regularity from a priori C^1 bounds. In this subsection only, it will not be assumed that the problem (Q0) is explicitly given by (Q1), (Q2), and (B1), but instead (Q,N) will be a general pair of an elliptic quasilinear operator and a fully non-linear boundary operator. In particular, A and b will not necessarily be assumed to be independent of t and u. The first Theorem is the classical interior Hölder gradient estimate for quasilinear equations, due to O. Ladyzhenskaya and N. Uraltseva [37, Lemma 2.1].

Theorem 2.2.2. *Let $u \in C^2(Q_T)$ satisfy $Qu = 0$ in Q_T, with $A(x,t,z,q) \in C^1(Q_T \times \mathbb{R} \times \mathbb{R}^{d+1})$, $b(x,t,z,q) \in C^0(Q_T \times \mathbb{R} \times \mathbb{R}^{d+1})$. Suppose that $||u||_{C^1(Q_T)} \leq K$, and that the constants λ_K, μ_K satisfy, in $Q_{T,K}$,*

$$A \geq \lambda_K I \quad \text{and} \quad \mu_K \geq |A| + |DA| + |b|. \tag{2.2.5}$$

Then, for any $V \subset\subset Q_T$, there exist constants $C = C(K, \mu_K/\lambda_K, \mathrm{dist}(V, \partial Q_T)^{-1})$ and $\gamma = \gamma(K, \mu_K/\lambda_K)$, such that

$$[Du]_{\gamma,V} \leq C.$$

Next is the following local boundary Hölder estimate for the gradient in oblique problems, due to Lieberman [37, Lemma 2.3]. In Theorem 2.2.3, the following definitions are in place:

$$\begin{aligned} B = \{(x,t) \in \mathbb{R}^{d+1} : |(x,t)| < 1\},\ B^+ = \{(x,t) \in B : t > 0\}, \\ B^0 = \{(x,t) \in B : t = 0\},\ B' = \{(x,t) \in B^+ : |(x,t)| < \frac{1}{3}\}. \end{aligned}$$

Theorem 2.2.3. *Let $u \in C^2(B^+ \cup B^0)$ solve $Qu = 0$ in B^+, $Nu = 0$ on B^0, with $A(x,t,z,q) \in C^1(\overline{B^+} \times \mathbb{R} \times \mathbb{R}^{d+1})$, $b(x,t,z,q) \in C^0(\overline{B^+} \times \mathbb{R} \times \mathbb{R}^{d+1})$, $B(x,t,z,q) \in C^1(B^0 \times \mathbb{R} \times \mathbb{R}^{d+1})$, $D_q B(x,t,z,q) \in C^1(B^0 \times \mathbb{R} \times \mathbb{R}^{d+1})$. Assume furthermore that (2.2.5) holds in $\overline{B^+_K}$, as well as, on B^0_K,*

$$\begin{aligned} &\lambda_K \leq -B_s, \text{ and} \\ &\mu_K \geq |D_q B| + |D_z B| + |D_{\bar{x}} B| + |D^2_{qq} B| + |D^2_{qz} B| + |D^2_{q\bar{x}} B| \end{aligned} \tag{2.2.6}$$

Then there exist constants C and γ depending only on K and μ_K/λ_K such that, if $\|u\|_{C^1(B^+ \cup B^0)} \leq K$, then

$$[Du]_{\gamma,B'} \leq C.$$

For the next theorem, which is the basic Schauder estimate for linear oblique problems [25, Theorem 6.30], we recall that $\nu(x,t) = \pm(0,0,\ldots,1)$ denotes the outward pointing normal vector at $(x,t) \in \partial Q_T$.

Theorem 2.2.4. *Assume that $u \in C^2(\overline{Q_T})$ solves the linear problem*

$$-\mathrm{Tr}(\tilde{A}(x,t)D^2u) = \eta_1(x,t) \text{ in } Q_T, \quad \tilde{B}(x,t)\cdot Du = \eta_2(x,t) \text{ on } \partial Q_T,$$

where

$$\tilde{A^{ij}}, \eta_1 \in C^{0,\alpha}(\overline{Q_T}), \quad \tilde{B}, \eta_2 \in C^{1,\alpha}(\partial Q_T), \quad \tilde{A} \geq \lambda I, \text{ and } \tilde{B}\cdot\nu \geq \lambda_0.$$

Then there exists $C = C(\frac{1}{\lambda}, \frac{1}{\lambda_0}, ||\tilde{A}^{ij}||_{C^{0,\alpha}(\overline{Q_T})}, ||\tilde{B}||_{C^{1,\alpha}(\partial Q_T)})$ such that

$$||u||_{C^{2,\alpha}(\overline{Q_T})} \leq C(||u||_{C^0(\overline{Q_T})} + ||\eta_1||_{C^{0,\alpha}(\overline{Q_T})} + ||\eta_2||_{C^{1,\alpha}(\partial Q_T)}).$$

The last result of this subsection is a variant of a convergence theorem of Fiorenza, which is a basic tool for using the method of continuity without the need of a priori second derivative estimates [36, Lemma 2, Corollary 1].

Theorem 2.2.5. *Let $0 < \alpha, \gamma < 1$. For each $n \in \mathbb{N}$, let $u_n \in C^{2,\alpha}(\overline{Q_T})$ be a sequence of solutions to the quasilinear problems $Q_n u = 0$, $N_n u = 0$, where, for $C, K, \gamma, \lambda, \lambda_0$ independent of n,*

$$Q_n u = -\mathrm{Tr}(A_n(x,t,u,Du)D^2u) + b_n(x,t,u,Du), \quad N_n u = B_n(x,t,u,Du),$$

$$||A_n||_{C^1(\overline{Q}_{T,K})} + ||b_n||_{C^1(\overline{Q}_{T,K})} + ||B_n||_{C^2(\overline{Q}_{T,K})} + ||D_q B_n||_{C^2(\overline{Q}_{T,K})} \leq C,$$

$$A_n \geq \lambda I \text{ in } \overline{Q}_{T,K}, \quad \text{and } D_q B_n \cdot \nu \geq \lambda_0 \text{ in } \partial \overline{Q}_{T,K},$$

$$||u_n||_{C^{1+\gamma}(\overline{Q_T})} \leq K,$$

with $u_n \to u$ uniformly, and $(A_n, b_n, B_n) \to (A, b, B)$ uniformly on $\overline{Q}_{T,K}$. Then $u_n \to u$ in $C^{2,\alpha}(\overline{Q_T})$, and u solves (Q0).

2.3 A priori estimates

In this section, we establish a priori estimates for the solution and the gradient, in the case where (MFG) is strictly elliptic. To account for the fact that the functions f and g depend on the space variable, we will make extensive use of the continuous, strictly increasing functions $f_0, g_0, f_1, g_1 : (0, \infty) \to \mathbb{R}$ defined by

$$f_0(m) = \min_{\mathbb{T}^d} f(\cdot, m), \quad g_0(m) = \min_{\mathbb{T}^d} g(\cdot, m), \quad f_1(m) = \max_{\mathbb{T}^d} f(\cdot, m), \quad g_1(m) = \max_{\mathbb{T}^d} g(\cdot, m).$$

2.3.1 Estimates for the solution and the terminal density

We first obtain a priori bounds for the C^0 norm of the solution u. As a corollary, positive, two-sided bounds for the terminal density are established.

Lemma 2.3.1. *For any solution $(u, m) \in C^2(\overline{Q_T}) \times C^1(\overline{Q_T})$ of (MFG), and every $(x, t) \in \overline{Q_T}$, there exists a constant $C = C(C_0)$ such that*

$$g_0 f_1^{-1}(-C) - C(e^{CT} - e^{Ct}) \leq u(x,t) \leq g_1 f_0^{-1}(C) + C(e^{CT} - e^{Ct}). \tag{2.3.1}$$

Proof. The goal here is to modify u into a function that necessarily achieves its maximum at $\{t = T\}$, which is the region of the boundary where, by the strict monotonicity of g, the boundary condition of (Q0) provides information about u. This requires some estimates for the terms in (Q2). By (2.2.3) and (F2),

$$|\chi(x,f)\mathrm{Tr}(D^2_{xp}H(x,D_xu))| \le C(1+|f|)(1+|D_xu|^\tau). \tag{2.3.2}$$

Moreover, by (FX2),

$$|D_xf(x,m(x,t))\cdot D_pH(x,D_xu)| \le C(1+|f|^{(1+\tau)/2})(1+|D_xu|). \tag{2.3.3}$$

Now, given u, define the linear, uniformly elliptic operator Q_u by

$$Q_uv = -\mathrm{Tr}(A(x,Du)D^2v).$$

Notice that $Q_uu = -b(x,Du)$. Let $\zeta \in C^2([0,T])$ be a function to be chosen later, and define

$$v = u + \zeta(t),$$

so that $v_t = u_t + \zeta'(t)$ and $D_xv = D_xu$. This yields, by (2.2.1), (2.2.2), (2.3.2), and (2.3.3),

$$\begin{aligned} Q_uv = -\zeta''(t) + D_xH(x,D_xv)\cdot D_pH(x,D_xv) - D_xf(x,m)\cdot D_pH(x,D_xv) \\ + \chi\mathrm{Tr}(D^2_{xp}H(x,D_xv)) \le -\zeta''(t) + C(1+|D_xv|^{1+\tau})(1+|D_xv|) \\ + C(1+|-v_t+H(x,D_xv)+\zeta'(t)|^{(1+\tau)/2})(1+|D_xv|) \\ + C(1+|-v_t+H(x,D_xv)+\zeta'(t)|)(1+|D_xv|^\tau)) \\ \le -\zeta''(t) + C(1+|D_xv|^3+|v_t|^2) + C(1+|D_xv|)|\zeta'(t)|, \end{aligned}$$

where the constant C increases in each line. Now, set $C_1 = 2C$ and fix C_1, still allowing C to increase at each step. We choose $\zeta(t) = \frac{k}{2C_1}(e^{2C_1t} - e^{2C_1T})$, where $k > 0$ is a parameter. Then,

$$\zeta'(t) = ke^{2C_1t},\ \zeta''(t) = 2C_1|\zeta'(t)|,$$

and, consequently, at any interior maximum point (x,t) of v,

$$0 \leq Q_u v \leq -\zeta''(t) + C_1(1+|\zeta'(t)|) \leq -C_1\zeta'(t) + C_1 = -C_1 k e^{2C_1 t} + C_1 \leq -C_1 k + C_1,$$

which can only hold if $k \leq 1$. Thus, if one chooses $k > 1$, v necessarily achieves its maximum value when $t = 0$ or $t = T$. If this happens at a point (x,t) where $t = 0$, then $u_t + \zeta' = v_t \leq 0$, $D_x u = D_x v = 0$. Therefore,

$$-||H(\cdot,0)||_{C^0(\mathbb{T}^d)} \leq -v_t + H(x,0) = -u_t + H(x, D_x u) - \zeta'(0) = f(x, m_0(x,t)) - \zeta'(0),$$

implying that

$$k = \zeta'(0) \leq f(x, m_0(x,t)) + ||H(\cdot,0)||_{C^0(\mathbb{T}^d)}.$$

Hence, taking $k > \max_{x\in\mathbb{T}^d} f(x, m_0(x)) + ||H(\cdot,0)||_{C^0(\mathbb{T}^d)}$, it follows that v attains its maximum value at $t = T$. At this point, $u_t + \zeta'(t) = v_t \geq 0$, $D_x u = D_x v = 0$, and, as before,

$$||H(x,0)||_{C^0} \geq -v_t + H(x,0) = f(x, m(x,T)) - \zeta'(T),$$

which gives

$$f_0(m(x,T)) \leq f(x, m(x,T)) \leq \zeta'(T) + ||H(\cdot,0)||_{C^0(\mathbb{T}^d)} \leq k e^{2C_1 T} + ||H(\cdot,0)||_{C^0(\mathbb{T}^d)} \leq C.$$

Thus, since $u(x,T) = v(x,T)$, taking into account the surjectivity of $f(x,\cdot)$,

$$\max v = v(x,T) = g(x, m(x,T)) \leq g(x, f_0^{-1}(C)) \leq g_1(f_0^{-1}(C)).$$

Finally, for arbitrary $(x,t) \in Q_T$,

$$u(x,t) = v(x,t) - \zeta(t) \leq g_1(f_0^{-1}(C)) + C(e^{CT} - e^{Ct}).$$

The lower estimate follows from a completely symmetrical argument. □

Corollary 2.3.2. *Let C be the constant from Lemma 2.3.1. Then, for every $x \in \mathbb{T}^d$,*

$$g_1^{-1} g_0 f_1^{-1}(-C) \leq m(x,T) \leq g_0^{-1} g_1 f_0^{-1}(C), \tag{2.3.4}$$

Proof. From the first inequality in (2.3.1), for each $x \in \mathbb{T}^d$,

$$g_0 f_1^{-1}(-C) \leq g(x, m(x,T)),$$

and thus, by definition of g_1,

$$g_0 f_1^{-1}(-C) \leq g_1(m(x,T)). \tag{2.3.5}$$

Observe that the application of g_1^{-1} on both sides of (2.3.5) is possible because, by (GX), the functions g_0 and g_1 have the same range. This yields the first inequality in (2.3.4). The second inequality is obtained through the same reasoning. □

Remark 2.3.3. A minor modification of the proof of Lemma 2.3.1 shows that, when H, f, and g are independent of x, the following sharper estimates hold:

$$g(\min m_0) + (f(\min m_0) - H(0))(T-t) \leq u(x,t) \leq g(\max m_0) + (f(\max m_0) - H(0))(T-t),$$

$$\min m_0 \leq m(x,T) \leq \max m_0.$$

2.3.2 Estimates for the space-time gradient

Given the operator Q from (Q0), we recall that its linearization at $u \in C^2(\overline{Q_T})$ is the linear, uniformly elliptic operator

$$L_u(v) = -\mathrm{Tr}(A(x, Du)D^2v) - D_q\mathrm{Tr}(A(x, Du)D^2u) \cdot Dv + D_q b(x, Du) \cdot Dv. \qquad (2.3.6)$$

The gradient estimate will be obtained through Bernstein's method. Specifically, we will bound $||Du||_{C^0(\overline{Q_T})}$ by evaluating the linearization $L_u(v)$ at appropriately chosen functions $v(x, t) = \Phi(x, t, u, Du)$, where $\Phi(x, t, z, q)$ is convex in q, exploiting the fact that, roughly speaking, convex functions of the gradient are expected to be subsolutions. For this purpose, we first obtain an explicit form for the terms in (2.3.6), as well as a general expression for the linearization applied to such functions v.

Lemma 2.3.4. *Let $\Phi(x, t, p, s) \in C^2(\overline{Q_T} \times \mathbb{R}^{d+1})$, assume that $u \in C^3(\overline{Q_T})$ solves (Q0), and set $v(x, t) = \Phi(x, t, Du(x, t))$. Then, for each $q = (p, s) \in \mathbb{R}^{d+1}$, and for each $\overline{x} = (x, t) \in Q_T$,*

$$\begin{aligned} - D_q\mathrm{Tr}(AD^2u) \cdot q = 2(-D_pHD^2_{xx}u + D_xu_t)D^2_{pp}H \cdot p - \chi D_p(\mathrm{Tr}(D^2_{pp}HD^2_{xx}u)) \cdot p \\ + \chi_w\mathrm{Tr}(D^2_{pp}HD^2_{xx}u)(s - D_pH \cdot p), \quad (2.3.7) \end{aligned}$$

$$\begin{aligned} D_qb(x, Du) \cdot q = -(D_pHD^2_{xp}H) \cdot p - (D_xHD^2_{pp}H) \cdot p + (D_xfD^2_{pp}H) \cdot p \\ + \frac{1}{f_m}D_xf_m \cdot D_pH(-s + D_pH \cdot p) - \chi D_p\mathrm{Tr}(D^2_{xp}H) \cdot p + \chi_w\mathrm{Tr}(D^2_{xp}H)(s - D_pH \cdot p), \\ (2.3.8) \end{aligned}$$

$$L_u v = -\mathrm{Tr}(D^2_{qq}\Phi D^2 u A D^2 u) - \mathrm{Tr}(A D^2_{\bar{x}\bar{x}}\Phi) - 2\mathrm{Tr}(A D^2 u D^2_{\bar{x}q}\Phi) - D_p\Phi \cdot D_x b + D_{\bar{x}}\Phi \cdot D_q b$$
$$+ \sum_{i=1}^{d} \mathrm{Tr}(A_{x_i} D^2 u)\Phi_{p_i} - D_{\bar{x}}\Phi \cdot D_q \mathrm{Tr}(A D^2 u). \quad (2.3.9)$$

Proof. Using (Q1),

$$\begin{aligned}
-(D_q \mathrm{Tr}(AD^2u)) \cdot q =& - D_q(\mathrm{Tr}((D_pH \otimes D_pH + \chi D^2_{pp}H)D^2_{xx}u) - 2D_pH \cdot D_x u_t) \cdot q \\
=& - 2(D_pH D^2_{xx}u D^2_{pp}H) \cdot p - \chi D_p \mathrm{Tr}(D^2_{pp}H D^2_{xx}u) \cdot p \\
& - \chi_w \mathrm{Tr}(D^2_{pp}H D^2_{xx}u)(-s + D_pH \cdot p) + 2(D^2_{pp}H D_x u_t) \cdot p \\
=& 2(-D_pH D^2_{xx}u + D_x u_t)D^2_{pp}H \cdot p - \chi D_p(\mathrm{Tr}(D^2_{pp}H D^2_{xx}u)) \cdot p \\
& + \chi_w \mathrm{Tr}(D^2_{pp}H D^2_{xx}u)(s - D_pH \cdot p),
\end{aligned}$$

which shows (2.3.7). Equation (2.3.8) is an immediate consequence of (Q2). From the definition of v, it follows that

$$Dv = D_{\bar{x}}\Phi + D_q\Phi D^2u,$$

and

$$D^2v = D^2_{\bar{x}\bar{x}}\Phi + (D^2u D^2_{\bar{x}q}\Phi + D^2_{q\bar{x}}\Phi D^2u) + D^2u D^2_{qq}\Phi D^2u + D_q\Phi D^3u.$$

Thus, differentiating the equation $Qu = 0$ and taking the inner product with $D_q\Phi$ yields

$$\begin{aligned}
0 =& D_q\Phi \cdot D_{\bar{x}}(-\mathrm{Tr}(A(x, Du(x,t))D^2u(x,t)) + b(x, Du(x,t))) \\
=& - \mathrm{Tr}(AD_q\Phi D^3u) - \sum_{i=1}^{d+1} \mathrm{Tr}(A_{\bar{x}_i}D^2u)\Phi_{q_i} - D_q\Phi D^2u \cdot D_q\mathrm{Tr}(AD^2u) + D_q\Phi D^2u \cdot D_q b \\
& + D_q\Phi \cdot D_{\bar{x}} b \\
=& - \mathrm{Tr}(A(D^2v - (D^2_{\bar{x}\bar{x}}\Phi + (D^2u D^2_{\bar{x}q}\Phi + D^2_{q\bar{x}}\Phi)D^2u + D^2u D^2_{qq}\Phi D^2u)) \\
& - D_q\mathrm{Tr}(AD^2u) \cdot (Dv - D_{\bar{x}}\Phi) + D_q b \cdot (Dv - D_{\bar{x}}\Phi) - \sum_{i=1}^{d+1} \mathrm{Tr}(A_{\bar{x}_i}D^2u)\Phi_{q_i} + D_q\Phi D_{\bar{x}} b.
\end{aligned}$$

Using the fact that A and b are independent of t, as well as (2.3.6), we obtain

$$\begin{aligned}
0 &= L_u v + \mathrm{Tr}(A(D^2_{\bar{x}\bar{x}}\Phi + (D^2 u D^2_{\bar{x}q}\Phi + D^2_{q\bar{x}}\Phi D^2 u) + D^2 u D^2_{qq}\Phi D^2 u)) + D_q \mathrm{Tr}(AD^2 u)\cdot D_{\bar{x}}\Phi \\
&\quad - D_q b \cdot D_{\bar{x}}\Phi - \sum_{i=1}^{d} \mathrm{Tr}(A_{x_i} D^2 u)\Phi_{p_i} + D_p\Phi \cdot D_x b \\
&= L_u v + \mathrm{Tr}(D^2_{qq}\Phi D^2 u A D^2 u) + \mathrm{Tr}(AD^2_{\bar{x}\bar{x}}\Phi) + 2\mathrm{Tr}(AD^2 u D^2_{\bar{x}q}\Phi) + D_p\Phi\cdot D_x b \\
&\quad - D_q b \cdot D_{\bar{x}}\Phi - \sum_{i=1}^{d} \mathrm{Tr}(A_{x_i} D^2 u)\Phi_{p_i} + D_{\bar{x}}\Phi \cdot D_q \mathrm{Tr}(AD^2 u),
\end{aligned}$$

which proves (2.3.9). $\square$

Corollary 2.3.5. *Let $(u, m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (MFG), and set*

$$\eta_0 = \min(\min_{\mathbb{T}^d} m_0, \min_{\mathbb{T}^d} m(\cdot, T)),\ \eta_1 = \max(\max_{\mathbb{T}^d} m_0, \max_{\mathbb{T}^d} m(\cdot, T)).$$

Then

$$L_u(u_t) = 0, \quad \text{and} \quad -C_0 - f_1(\eta_1) \le u_t \le ||H(\cdot, D_x u)||_{C^0(\overline{Q_T})} - f_0(\eta_0). \tag{2.3.10}$$

Proof. Letting $\Phi(x, t, p, s) = s$ in Lemma 2.3.4, since $D_{\bar{x}}\Phi$, $D_p\Phi$, $D^2\Phi \equiv 0$, it follows that

$$L_u(u_t) = L_u(\Phi(x, t, Du)) = 0.$$

Hence, the maximum and minimum values of u_t are attained in ∂Q_T. (2.3.10) then follows immediately from (2.2.1) and the HJ equation in (MFG). $\square$

By Corollary 2.3.2, this result reduces the problem to estimating $||D_x u||_{C^0}$, but it is also a key ingredient for obtaining that bound, particularly due to the fact that the term $||H(\cdot, D_x u)||_{C^0(\overline{Q_T})}$ has coefficient 1 in (2.3.10). We now begin to simplify the quantity (2.3.9) for the specific Φ that will be used in the proof of the gradient estimate, bounding one of the dominant signed terms by a simpler expression, using matrix algebra.

Lemma 2.3.6. *For each* $(x,t,p,s) \in \overline{Q_T} \times \mathbb{R}^{d+1}$*, set* $\widetilde{H}(x,t,p,s) = H(x,p)$*, and define the matrix* $\tilde{I} = (\delta^{ij}(1-\delta^{i,d+1}))_{i,j=1}^{d+1}$*. Then, for every* $u \in C^2(Q_T)$*,*

$$\begin{aligned}\mathrm{Tr}(D^2_{qq}\widetilde{H}(x,t,Du)D^2uA(x,Du)D^2u) \geq \frac{3}{4C_0}|-D_xu_t + D_pH(x,D_xu)D^2_{xx}u|^2 \\ + \frac{1}{4C_0}\mathrm{Tr}(\tilde{I}D^2uAD^2u) + \frac{3\chi}{4C_0^2}|D^2_{xx}u|^2.\end{aligned} \tag{2.3.11}$$

Proof. By (H1),

$$D^2_{qq}\widetilde{H} \geq \frac{1}{C_0}\tilde{I},$$

thus, since the matrix D^2uAD^2u is non-negative, multiplying both sides of the inequality by this matrix and taking the trace of both sides gives

$$\mathrm{Tr}(D^2_{pp}\widetilde{H}D^2uAD^2u) \geq \frac{1}{C_0}\mathrm{Tr}(\tilde{I}D^2uAD^2u). \tag{2.3.12}$$

Now, by (Q1) and (H1),

$$\begin{aligned}\mathrm{Tr}(\tilde{I}D^2uAD^2u) &= \sum_{k=1}^{d} Du_{x_k}A\cdot Du_{x_k} = \sum_{k=1}^{d}|(D_pH,-1)\cdot Du_{x_k}|^2 + \chi D_xu_{x_k}D^2_{pp}H\cdot D_xu_{x_k} \\ &\geq \sum_{k=1}^{d}|D_pH\cdot D_xu_{x_k} - u_{tx_k}|^2 + \frac{\chi}{C_0}|D_xu_{x_k}|^2 = |D_pHD^2_{xx}u - D_xu_t|^2 + \frac{\chi}{C_0}|D^2_{xx}u|^2.\end{aligned} \tag{2.3.13}$$

Combining (2.3.12) and (2.3.13) yields, as desired,

$$\begin{aligned}\mathrm{Tr}(D^2_{pp}\widetilde{H}D^2uAD^2u) &= \frac{3}{4}\left(\mathrm{Tr}(D^2_{pp}\widetilde{H}D^2uAD^2u)) + \frac{1}{4}(\mathrm{Tr}(D^2_{pp}\widetilde{H}D^2uAD^2u)\right) \\ &\geq \frac{3}{4C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 + \frac{3\chi}{4C_0^2}|D^2_{xx}u|^2 + \frac{1}{4C_0}\mathrm{Tr}(\tilde{I}D^2uAD^2u).\end{aligned}$$

$\square$

The next Lemma continues to simplify the linearizations. Since one of the dominant signed terms will later be shown to be of the order of $|D_xu|^4$, the goal will be to bound everything else by $(4-\epsilon)^{\text{th}}$ powers of $|D_xu|$, $(2-\epsilon)^{\text{th}}$ powers of u_t (dealing with these through Corollary 2.3.5), and second derivative terms that can be dealt with using the other dominant term (2.3.11). The usage of $\Phi(x,t,D_xu,u_t) = H(x,D_xu)$, as opposed to a more standard choice such as $|D_xu|^2$ or $|Du|^2$, is crucial in the next two results, in order to produce structural cancellation of terms that can not be otherwise estimated, as well as to be able to use (2.3.10) without gaining any constant factors in the process.

Lemma 2.3.7. *Let $u \in C^3(\overline{Q_T})$ be a solution to (Q0), and let $c, c' \in \mathbb{R}$. Define*

$$\widetilde{u} = u + c(T-t) + c',\ v_1 = \frac{\tilde{u}^2}{2},\ v_2 = H(\cdot, D_xu).$$

Then, for each $(x,t) \in Q_T$, there exists $C(x,t) > 0$, with

$$C(x,t) = C\left(C_0, ||\tilde{u}||_{C^0(Q_T)}, \frac{1}{\chi(x,f(x,m(x,t)))}, |h_w(x,f(x,m(x,t)))|, c\right),$$

such that

$$L_u(v_1) \leq -\frac{1}{2}|-u_t + D_pH(x,D_xu)D_xu|^2 - \frac{1}{C_0}\chi|D_xu|^2 + C(x,t)(1+|D_xu|^{2+\tau} + \chi|D^2_{xx}u|^2$$
$$|f|^\tau|D_xu|^2 + |f|(1+|D_xu|^\tau) + |f|^{(1+\tau)/2}|D_xu| + \chi + |-D_xu_t + D^2_{xx}uD_pH|^2). \quad (2.3.14)$$

and

$$L_u(v_2) \leq \frac{-1}{2C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 - \frac{\chi}{2C_0^2}|D^2_{xx}u|^2 + C(x,t)(1+|D_xu|^{3+\tau} + \chi(1+|D_xu|^{1+\tau})$$
$$+\chi^{(1+\tau)/2}|D_xu|^2 + |f|(1+|D_xu|^{1+\tau}) + |D_xu|^2|f|^{(1+\tau)/2}). \quad (2.3.15)$$

Proof. Throughout this proof, the number $C = C(x,t)$ may increase at each step, with its size depending on (x,t) only monotonically through $\frac{1}{\chi}$ and $|h_w|$. For this reason, there is no loss of generality in assuming

$$\frac{1}{\chi} + |h_w| \leq C. \tag{2.3.16}$$

Observe first that, since $D^2\tilde{u} = D^2 u$, one has $-\mathrm{Tr}(A(x,Du)D^2\tilde{u}) = -b(x,Du)$. Therefore,

$$-\mathrm{Tr}(A(x,Du)D^2 v_1) = -\tilde{u}b(x,Du) - D\tilde{u}A \cdot D\tilde{u},$$
$$-D_q\mathrm{Tr}(A(x,Du)D^2u) \cdot Dv_1 + D_q b(x,Du) \cdot Dv_1 = -D_q\mathrm{Tr}(AD^2u) \cdot \tilde{u}D\tilde{u} + D_q b \cdot \tilde{u}D\tilde{u}.$$

Consequently, by (Q1) and (H3),

$$\begin{aligned} L_u(v_1) =& -\tilde{u}b - D\tilde{u}A \cdot D\tilde{u} + \tilde{u}(-D_q\mathrm{Tr}(AD^2u) \cdot D\tilde{u} + D_q b \cdot D\tilde{u}) \\ =& -\tilde{u}b - D\tilde{u}(D_pH, -1) \otimes (D_pH, -1) \cdot D\tilde{u} - \chi D_x\tilde{u}D^2_{pp}H \cdot D_x\tilde{u} \\ &+ \tilde{u}(-D_q\mathrm{Tr}(AD^2u) \cdot D\tilde{u} + D_q b \cdot D\tilde{u}) \\ \leq& -|-\tilde{u}_t + D_pH \cdot D_xu|^2 - \frac{1}{C_0}\chi|D_x\tilde{u}|^2 + \tilde{u}(-b - D_q\mathrm{Tr}(ADu) \cdot D\tilde{u} + D_q b \cdot D\tilde{u}) \\ =& -|-\tilde{u}_t + D_pH \cdot D_xu|^2 - \frac{1}{C_0}\chi|D_x\tilde{u}|^2 + \tilde{u}(J_1 + J_2 + J_3). \end{aligned} \tag{2.3.17}$$

The next task will be to estimate the terms J_i. In view of (Q2), (2.2.1), (2.2.2), (2.2.3), (F2), and (FX2),

$$\begin{aligned} |J_1| = |b(x,Du)| &= |-D_xH \cdot D_pH + D_xf \cdot D_pH - \chi\mathrm{Tr}(D^2_{xp}H)| \\ &\leq C(1 + |D_xu|^{2+\tau} + |f|^{(1+\tau)/2}(1 + |D_xu|) + |f|(1 + |D_xu|^\tau)). \end{aligned} \tag{2.3.18}$$

As for J_2, (2.2.1), (H3), (2.3.7), and (2.3.16) imply that

$$\begin{aligned}|J_2| = |-D_q\mathrm{Tr}(AD^2u)\cdot D\tilde{u}| = |2(-D_pHD^2_{xx}u + D_xu_t)D^2_{pp}H\cdot D_xu \\ -\chi D_p(\mathrm{Tr}(D^2_{pp}HD^2_{xx}u))\cdot D_xu + 2hh_w\mathrm{Tr}(D^2_{pp}HD^2_{xx}u)(\tilde{u}_t - D_pH\cdot D_xu)| \\ \le C(1 + |-D_xu_t + D_pHD^2_{xx}u|^2 + |D_xu|^2 + \chi + \chi|D^2_{xx}u|^2) \\ + \frac{1}{8(||\tilde{u}||+1)}|-\tilde{u}_t + D_pH\cdot D_xu|^2. \quad (2.3.19)\end{aligned}$$

By assumptions (HX), (F2), (FX1), and (FX2), together with (2.2.2), (2.3.8) and (2.3.16),

$$\begin{aligned}|J_3| = |D_qb(x,Du)\cdot D\tilde{u}| = |-(D_pHD^2_{xp}H)\cdot D_xu - (D_xHD^2_{pp}H)\cdot D_xu \\ + (D_xfD^2_{pp}H)\cdot D_xu + \left(\frac{1}{f_m}D_xf_m\cdot D_pH\right)(-\tilde{u}_t + D_pH\cdot D_xu) - \chi D_p(\mathrm{Tr}(D^2_{xp}H))\cdot D_xu \\ + \chi_w\mathrm{Tr}(D^2_{xp}H)(\tilde{u}_t - D_pH\cdot D_xu)| \le C(1 + |D_xu|^{2+\tau} + (1+|f|^{(1+\tau)/2})|D_xu| \\ + (1+\frac{1}{\chi})(1+|f|^{\tau/2})(1+|D_xu|)|-\tilde{u}_t + D_pH\cdot D_xu| + (1+|f|)(1+|D_xu|^{\tau}) \\ + (1+|D_xu|^{\tau})|-\tilde{u}_t + D_pH\cdot D_xu|) \le C(1 + |D_xu|^{2+\tau} + |f|^{(1+\tau)/2}|D_xu| \\ + |f|^{\tau}|D_xu|^2 + |f||D_xu|^{\tau}) + \frac{1}{8(||\tilde{u}||+1)}|-\tilde{u}_t + D_pH\cdot D_xu|^2. \quad (2.3.20)\end{aligned}$$

Finally, using (2.3.18), (2.3.19), and (2.3.20) in (2.3.17) yields

$$\begin{aligned}L_u(v_1) \le -\frac{3}{4}|-\tilde{u}_t + D_pHD_xu|^2 - \frac{1}{C_0}\chi|D_xu|^2 + C(1 + |D_xu|^{2+\tau} + \chi|D^2_{xx}u|^2 + |f|^{\tau}|D_xu|^2 \\ + |f|(1+|D_xu|^{\tau}) + |f|^{(1+\tau)/2}|D_xu| + \chi + |-D_xu_t + D^2_{xx}uD_pH|^2),\end{aligned}$$

which proves (2.3.14).

Next is the proof of (2.3.15). In view of Lemma 2.3.4, recalling that $\tilde{H}(x,t,Du) :=$

$H(x, Du)$,

$$\begin{aligned}
L_u v_2 =& L_u(\tilde{H}(x,t,Du)) \\
=& -\operatorname{Tr}(D^2_{qq}\tilde{H}D^2uAD^2u) - \operatorname{Tr}(AD^2_{\bar{x}\bar{x}}\tilde{H}) - 2\operatorname{Tr}(AD^2uD^2_{\bar{x}q}\tilde{H}) - D_p\tilde{H}\cdot D_xb \\
&+ D_{\bar{x}}\tilde{H}\cdot D_qb + \sum_{i=1}^d \operatorname{Tr}(A_{x_i}D^2u)\tilde{H}_{p_i} - D_{\bar{x}}\tilde{H}\cdot D_q\operatorname{Tr}(AD^2u),
\end{aligned}$$

and Lemma 2.3.6 then implies

$$\begin{aligned}
L_u v_2 \leq& \frac{-3}{4C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 - \frac{3\chi}{4C_0^2}|D^2_{xx}u|^2 - \frac{1}{4C_0}\operatorname{Tr}(\tilde{I}D^2uAD^2u) \\
&- \operatorname{Tr}(AD^2_{\bar{x}\bar{x}}\tilde{H}) - 2\operatorname{Tr}(AD^2uD^2_{\bar{x}q}\tilde{H}) - D_pH\cdot D_xb + D_xH\cdot D_pb \\
&+ \sum_{i=1}^d \operatorname{Tr}(A_{x_i}D^2u)H_{p_i} - D_xH\cdot D_p\operatorname{Tr}(AD^2u). \\
=& \frac{-3}{4C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 - \frac{3\chi}{4C_0^2}|D^2_{xx}u|^2 - \frac{1}{4C_0}\operatorname{Tr}(\tilde{I}D^2uAD^2u) \\
&+ K_1 + K_2 + K_3 + K_4 + K_5 + K_6.
\end{aligned} \tag{2.3.21}$$

As before, we proceed to estimate the K_i. Starting with K_1, we observe that, by (Q1), (H1), (2.2.1), and (2.2.3),

$$|K_1| = |\operatorname{Tr}(AD^2_{\bar{x}\bar{x}}\tilde{H})| \leq C(1+|D_xu|^{1+\tau})|A| \leq C(1+|D_xu|^{1+\tau})(1+|D_xu|^2+\chi). \tag{2.3.22}$$

Similarly, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
|K_2| = |2\operatorname{Tr}(AD^2u(\tilde{I}D^2_{\bar{x}q}\tilde{H}))| = 2|\operatorname{Tr}(D^2_{\bar{x}q}\tilde{H}A(\tilde{I}D^2u)^T)| \leq \frac{1}{4C_0}\operatorname{Tr}((\tilde{I}D^2u)A(\tilde{I}D^2u)^T) \\
+ C(\operatorname{Tr}(D^2_{\bar{x}q}\tilde{H}A(D^2_{\bar{x}q}\tilde{H})^T) \leq \frac{1}{4C_0}\operatorname{Tr}(\tilde{I}D^2uAD^2u) + C|D^2_{xp}H|^2(1+|D_xu|^2+\chi) \\
\leq \frac{1}{4C_0}\operatorname{Tr}(\tilde{I}D^2uAD^2u) + C(1+|D_xu|^{2(1+\tau)} + \chi(1+|D_xu|^{2\tau})).
\end{aligned} \tag{2.3.23}$$

Next, we will estimate $|K_3 + K_4|$. Differentiating the equation (Q2) with respect to x, we obtain

$$D_x b(x,p,s) = -D^2_{xx}H \cdot D_pH - D_xH D^2_{xp}H + D_pH D^2_{xx}f + \frac{1}{f_m} D_pH (D_x f_m \otimes (-D_x f + D_x H))$$
$$+ D_x f D^2_{xp}H + \left(D_x f - m D_x f_m + \frac{m f_{mm}}{f_m} D_x f \right) \mathrm{Tr}(D^2_{xp}H) - \chi_w D_x H \mathrm{Tr}(D^2_{xp}H)$$
$$- \chi D_x \mathrm{Tr}(D^2_{xp}H). \quad (2.3.24)$$

Consequently, (2.2.1), (2.2.2), (2.2.3), (F2), (FX1), (FX2), and (2.3.16) yield

$$|-D_x b(x,p,s) + \frac{1}{f_m} D_pH D_x f_m \otimes D_x H| \leq C(1 + |p|^{2+\tau} + (1+|p|)|f|^{(1+\tau)/2} + |f|(1+|p|^\tau)),$$

so that, setting $z_1 = \frac{1}{f_m}(D_pH \cdot D_x f_m)(D_x H \cdot D_p H)$,

$$|K_3 + z_1| = |-D_x b(x, Du) \cdot D_p H + z_1| \leq C(1 + |D_x u|^{2+\tau} + (1+|D_x u|)|f|^{(1+\tau)/2}$$
$$+|f|(1+|D_x u|^\tau))(1+|D_x u|) \leq C(1+|D_x u|^{3+\tau} + (1+|D_x u|^2)|f|^{(1+\tau)/2} + |f|(1+|D_x u|^{1+\tau})).$$
$$(2.3.25)$$

On the other hand, by (2.3.8),

$$|K_4 - z_1| = |D_p b \cdot D_x H - z_1| = |-(D_pH D^2_{xp}H) \cdot D_x H - (D_x H D^2_{pp}H) \cdot D_x H$$
$$+ (D_x f D^2_{pp}H) \cdot D_x H + z_1 - \chi D_p \mathrm{Tr}(D^2_{xp}H) \cdot D_x H + \chi_w \mathrm{Tr}(D^2_{xp}H)(D_pH \cdot D_x H) - z_1|.$$

The terms z_1 and $-z_1$ then cancel out, and therefore (HX), (2.2.2), (2.2.3), and (F2) yield

$$|K_4 - z_1| \leq C(1 + |D_x u|^{2+2\tau} + |f|^{(1+\tau)/2}(1 + |D_x u|^{1+\tau}) + |f||D_x u|^{2\tau}). \quad (2.3.26)$$

The inequalities (2.3.25) and (2.3.26) thus imply

$$|K_3+K_4| \leq C(1+|D_xu|^{3+\tau}+(1+|D_xu|^2)|f|^{(1+\tau)/2}+|f|(1+|D_xu|^{1+\tau})). \qquad (2.3.27)$$

The terms K_5 and K_6 will also be treated jointly. Let $(x,p,s) \in \mathbb{T}^d \times \mathbb{R}^{d+1}$, and set $w=-s+H(x,p)$. It follows from (Q1), (FX1), (HX), and (F2) that

$$\begin{aligned} A_{x_i}(x,p,s) = (D_pH_{x_i},0) \otimes (D_pH,-1) + (D_pH,-1) \otimes (D_pH_{x_i},0) + \chi_w H_{x_i} D^2_{qq}\tilde{H} \\ + O(1+|w|^{\tau/2}+\chi^{(1+\tau)/2}+\chi(1+|p|)^{-1+\tau})\tilde{I}. \end{aligned}$$

Therefore, using (2.3.7), and writing $z_2 = \chi_w \mathrm{Tr}(D^2_{pp}HD^2_{xx}u)(D_pH \cdot D_xH)$,

$$\begin{aligned} |K_5+K_6| = |\sum_{i=1}^{d} \mathrm{Tr}(A_{x_i}D^2u)H_{p_i} - D_p\mathrm{Tr}(AD^2u)\cdot D_xH| \leq |2(D_pHD^2_{xx}u - D_xu_t)D^2_{px}H\cdot D_pH + z_2 \\ + 2(-D_pHD^2_{xx}u + D_xu_t)D^2_{pp}H \cdot D_xH - \chi D_p(\mathrm{Tr}(D^2_{pp}HD^2_{xx}u)) \cdot D_xH - z_2| + C(1+|f|^{\tau/2} \\ + \chi^{(1+\tau)/2} + \chi(1+|D_xu|)^{-1+\tau})|D^2_{xx}u|(1+|D_xu|). \end{aligned}$$

Once more, cancellation occurs and, consequently, (2.3.16), (H3), (2.2.1), (2.2.2), and (2.2.3) imply that

$$\begin{aligned} |K_5+K_6| \leq \frac{1}{4C_0}|D_pHD^2_{xx}u - D_xu_t|^2 + \frac{\chi}{4C_0^2}|D^2_{xx}u|^2 + C(1+|D_xu|^{2+2\tau} \\ + (1+|D_xu|^2)(1+|f|^{\tau}+\chi^{(1+\tau)/2}) + \chi(1+|D_xu|^{2\tau})). \qquad (2.3.28) \end{aligned}$$

Using (2.3.22), (2.3.23), (2.3.27), and (2.3.28) in (2.3.21) yields (2.3.15), completing the proof. □

We can now obtain the a priori gradient bound in terms of bounds for the solution u and the terminal density $m(\cdot,T)$, which were obtained in the previous subsection.

Lemma 2.3.8. *Let $(u,m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (MFG). For $K > 0$, set*

$$\beta_K = ||f||_{C^1(\mathbb{T}^d \times [\frac{1}{K},K])} + ||D_x g||_{C^1(\mathbb{T}^d \times [\frac{1}{K},K])} + \\ \left\|\frac{1}{\chi}\right\|_{C^0(\mathbb{T}^d \times [-K,\infty))} + ||h_w||_{C^0(\mathbb{T}^d \times [-K,\infty))}. \tag{2.3.29}$$

There exist constants C, C_1 with

$$C = C(C_1, \beta_{C_1}),$$

$$C_1 = C_1\left(C_0, T, \frac{1}{T}, \frac{1}{1-\tau}, ||u||_{C^0(\overline{Q_T})}, \max m(T), \frac{1}{\min m(T)}, f_0(\min_{\mathbb{T}^d} m(T))^-\right),$$

such that

$$||Du||_{C^0(\overline{Q_T})} \leq C.$$

Proof. As was mentioned, the proof will proceed through Bernstein's method. By Corollary 2.3.5, it is sufficient to bound the space gradient. Since the estimate will be up to the boundary, as in [38], we linearize the HJ equation that holds at the extremal times:

$$T_u v = -v_t + D_p H(x, Du) D_x v.$$

We now normalize u to have a prescribed sign at the initial and terminal times. That is, we set

$$\widetilde{u} = u + ||u||_{C^0(\overline{Q_T})} + 1 - \frac{2(||u||_{C^0(\overline{Q_T})} + 1)}{T}(T-t),$$

so that

$$|\tilde{u}| \leq C, \quad \widetilde{u}(\cdot, 0) \leq -1,\ \widetilde{u}(\cdot, T) \geq 1, \tag{2.3.30}$$

and define

$$v(x,t) = H(x, D_x u) + \frac{c_1}{2}\widetilde{u}^2,$$

where $0 < c_1 \leq 1$ is a constant to be chosen later. Let $(x_0, t_0) \in \overline{Q_T}$ be a point where v

achieves its maximum value. We will distinguish three cases:

Case 1. $t_0 = T$. Then $D_x v = 0$, $v_t \geq 0$. Therefore, (2.3.30), (H1) (H2), (2.2.2), and the HJ equation in (MFG), together with the fact that $m(\cdot, T) = g^{-1}(\cdot, u(\cdot, T))$, yield

$$
\begin{aligned}
0 \geq T_u v =& T_u(H(x_0, D_x u)) + c_1 \tilde{u}(-\widetilde{u}_t + D_p H(x_0, D_x u) \cdot D_x \widetilde{u}) \\
=& D_x(f(x_0, m(x_0, T))) \cdot D_p H(x_0, D_x u) + c_1 \tilde{u}(-u_t + D_p H(x_0, D_x u) \cdot D_x u - C) \\
\geq& \left(D_x f - \frac{f_m}{g_m} D_x g + \frac{f_m}{g_m} D_x u \right) \cdot D_p H + c_1 \tilde{u}(-u_t + 2H - C) \\
\geq& - C \left(1 + \frac{f_m}{g_m} \right) (1 + |D_x u|) + \frac{f_m}{g_m} D_p H \cdot D_x u + c_1 \tilde{u}(f + H - C) \\
\geq& - C \left(1 + \frac{f_m}{g_m} \right) (1 + |D_x u|) + 2 \left(c_1 \tilde{u} + \frac{f_m}{g_m} \right) H.
\end{aligned}
$$

Thus, by (2.2.1),

$$
|H(x_0, Du(x_0, t_0))| \leq C.
$$

Case 2. $t_0 = 0$. Similarly, we obtain $D_x v = 0$, $v_t \leq 0$, and, since $\tilde{u}(\cdot, 0) \leq -1$,

$$
\begin{aligned}
0 \leq T_u v = & \, D_x(f(x_0, m_0(x_0))) \cdot D_p H + c_1 \tilde{u}(-u_t + D_p H \cdot D_x u - C) \\
& \leq C(1 + |D_x u|) + c_1 \tilde{u}(f(x_0, m_0) + H - C) \leq C(1 + |D_x u|) + C + c_1 \tilde{u} H.
\end{aligned}
$$

This implies $-c_1 \tilde{u}(H(x_0, D_x u)) \leq C(1 + |D_x u|)$, and so, we conclude once more that

$$
|H(x_0, Du(x_0, t_0))| \leq C.
$$

Case 3. $0 < t_0 < T$. Then $Dv = 0$, $D^2 v \leq 0$, which yields

$$
0 \leq L_u v.
$$

In order to make use of Lemma 2.3.7, it is necessary to eliminate the (x_0, t_0) dependence of the "constant" $C(x_0, t_0)$ from the Lemma, which amounts to establishing an a priori upper bound on the quantities $1/\chi$ and $|h_w|$ at the point (x_0, t_0). By (F1) and (F2), $1/\chi$ and $|h_w| = |\chi_w/2\sqrt{\chi}|$ are both bounded above as $w \to \infty$, so it is enough to establish a lower bound for $w = f(x_0, m(x_0, t_0))$. By Corollary 2.3.5, there exists a point $(x_1, t_1) \in \partial Q_T$ where u_t achieves its maximum value. Then, since (x_0, t_0) is a maximum point for v, and the initial and terminal densities are both bounded below a priori,

$$\begin{aligned} f(x_0, m(x_0, t_0)) &= -u_t(x_0, t_0) + H(x_0, D_x u(x_0, t_0)) \geq -u_t(x_1, t_1) + H(x_1, D_x u(x_1, t_1)) \\ &\quad - \frac{c_1}{2}||\tilde{u}||^2_{C^0(\overline{Q_T})} = f(x_1, m(x_1, t_1)) - \frac{c_1}{2}||\tilde{u}||^2_{C^0(\overline{Q_T})} \geq f_0(m(x_1, t_1)) - C \geq -C. \end{aligned}$$

This estimate, together with (H2) and (2.2.1), allows us to identify the dominant power of $|D_x u|$ in the linearization,

$$| - u_t + D_p H \cdot D_x u|^2 \geq (f + H)^2 - C \geq \frac{1}{2C_0^2}|D_x u|^4 - C. \tag{2.3.31}$$

Now, because of the form of the estimate in Lemma 2.3.7, it is also necessary to be able to compare powers of $|f|$ with powers of $|D_x u|$. By Corollary 2.3.5 and (2.2.1),

$$\begin{aligned} f(x_0, m(x_0, t_0)) &\leq -u_t(x_0, t_0) + H(x_0, D_x u(x_0, t_0)) \\ &\leq C + 2H(x_0, D_x u(x_0, t_0)) \leq C(1 + |D_x u|^2). \end{aligned} \tag{2.3.32}$$

With these preliminaries done, we now apply Lemma 2.3.7, obtaining

$$
\begin{aligned}
0 \le L_u(v) = L_u\tilde{H} + c_1 L_u(\frac{\tilde{u}^2}{2}) \le \frac{-1}{2C_0}|-D_x u_t + D_pHD^2_{xx}u|^2 - \frac{\chi}{2C_0^2}|D^2_{xx}u|^2 + C(1+|D_xu|^{3+\tau} \\
+ \chi(1+|D_xu|^{1+\tau}) + \chi^{(1+\tau)/2}|D_xu|^2 + |f|(1+|D_xu|^{1+\tau}) + |D_xu|^2|f|^{(1+\tau)/2}) \\
-\frac{c_1}{2}|-u_t + D_pH\cdot D_xu|^2 - \frac{c_1}{C_0}\chi|D_xu|^2 + Cc_1(\chi|D^2_{xx}u|^2 + |-D_xu_t + D^2_{xx}uD_pH|^2).
\end{aligned}
\tag{2.3.33}
$$

Applying (2.3.31) and (2.3.32) yields

$$
\begin{aligned}
0 \le& -\frac{c_1}{4C_0^2}|D_xu|^4 - \frac{c_1}{C_0}\chi|D_xu|^2 - \frac{1}{2C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 - \frac{\chi}{2C_0^2}|D^2_{xx}u|^2 \\
&+C(1+|D_xu|^{3+\tau}+\chi(1+|D_xu|^{1+\tau})+\chi^{(1+\tau)/2}|D_xu|^2)+Cc_1(\chi|D^2_{xx}u|^2+|-D_xu_t+D^2_{xx}uD_pH|^2) \\
\le& -\frac{c_1}{4C_0^2}|D_xu|^4 - \frac{c_1}{C_0}\chi|D_xu|^2 - \frac{1}{2C_0}|-D_xu_t + D_pHD^2_{xx}u|^2 - \frac{\chi}{2C_0^2}|D^2_{xx}u|^2 + C(1+|D_xu|^{3+\tau} \\
&+\chi(1+|D_xu|^{1+\tau})) + (\frac{C}{c_1} + \frac{c_1}{2C_0}\chi)|D_xu|^2 + Cc_1(\chi|D^2_{xx}u|^2 + |-D_xu_t + D^2_{xx}uD_pH|^2).
\end{aligned}
$$

Now, fix c_1 satisfying $c_1 < \frac{1}{4C_0C(1+C_0)}$, where C is as in the previous line. This gives

$$
0 \le -\frac{c_1}{4C_0^2}|D_xu|^4 - \frac{1}{2C_0}\chi(c_1|D_xu|^2 - 2CC_0(1+|D_xu|^{1+\tau})) + C(1+|D_xu|^{3+\tau}) + \frac{C}{c_1}|D_xu|^2,
$$

which may be rearranged as

$$
\frac{c_1}{4C_0^2}|D_xu|^4 + \frac{1}{2C_0}\chi(c_1|D_xu|^2 - 2CC_0|D_xu|^{1+\tau} - 2CC_0) \le C(1+|D_xu|^{3+\tau}) + \frac{C}{c_1}|D_xu|^2.
$$

This finally implies that

$$
c_1|D_xu|^2 - 2CC_0|D_xu|^{1+\tau} - 2CC_0 \le 0 \quad \text{or} \quad \frac{c_1}{4C_0^2}|D_xu|^4 \le C(1+|D_xu|^{3+\tau}) + \frac{C}{c_1}|D_xu|^2,
$$

either of which yields

$$|H(x_0, D_x u(x_0, t_0))| \le C.$$

$\square$

We now summarize all of the a priori bounds obtained in this section.

Theorem 2.3.9. *Let* $(u, m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ *be a solution to (MFG), and let* β *be defined by (2.3.29). Then there exist constants* L, L_1, K, K_1*, with*

$$L = \left(L_1, |g_1 f_0^{-1}(L_1)|, |g_0 f_1^{-1}(-L_1)|, g_0^{-1} g_1 f_0^{-1}(L_1), \frac{1}{g_1^{-1} g_0 f_1^{-1}(-L_1))} \right), \quad L_1 = L_1(C_0, T),$$

$$K = (K_1, \beta_{K_1}), \quad K_1 = K_1 \left(L, \frac{1}{T}, \frac{1}{1-\tau}, f_0 \left(\frac{1}{L} \right)^- \right),$$

such that

$$||u||_{C^0(\overline{Q_T})} + ||m(T)||_{C^0(\mathbb{T}^d)} + \left\| \frac{1}{m(T)} \right\|_{C^0(\mathbb{T}^d)} \le L \quad \textit{and} \quad ||Du||_{C^0(\overline{Q_T})} \le K.$$

Proof. This result follows simply by the successive application of Lemma 2.3.1, Corollary 2.3.2, and Lemma 2.3.8. $\square$

The following variation of Theorem 2.3.9 shows that, in the standard case where $H(x,p) = H(p) - V(x)$ and $f(x,m) = f(m)$, the condition (F2) that requires f to grow at most polynomially may be significantly relaxed.

Theorem 2.3.10. *The conclusion of Theorem 2.3.9 still holds if condition (F2) is replaced by:*

$$D^2_{xp} H, D^2_{xm} f \equiv 0, \ \textit{and} \limsup_{x \in \mathbb{T}^d, w \to \infty} |h_w(x,w)| < \infty. \tag{HFX*}$$

Proof. We simply address all of the instances in which condition (F2) has been used so far. In the proofs of Lemma 2.3.1, Corollary 2.3.2, and Lemma 2.3.7, (F2) was exclusively used to estimate either space derivatives $D_x f$, $D_x H$, or terms that involve mixed derivatives $D^2_{xm} f$, $D^2_{xp} H$. With (HFX*) in place, such terms are, respectively, either bounded in C^1 norm or trivially zero. Condition (F2) was also used in the proof of Lemma 2.3.8 in order to obtain a bound for $|h_w|$ as $w \to \infty$, but this bound exists here by assumption. □

We note that the condition that (HFX*) imposes on h may be equivalently rewritten, in terms of f, as

$$\limsup_{x \in \mathbb{T}^d, m \to \infty} \frac{1}{m f_m} \left| m \frac{f_{mm}}{f_m} + 1 \right|^2 < \infty.$$

This condition, in particular, allows for f to be combinations of powers m^α,$-m^{-\beta}$, exponentials e^m,$-e^{1/m}$, and such typical examples, as long as one has the required blow-up near $m = 0$ and as $m \to \infty$.

2.4 Classical solutions

To obtain classical solutions, it is necessary to have Hölder estimates for the gradient of the solution in terms of the C^1 norm. The following Lemma, which is merely a restatement of Theorem 2.2.2 and Theorem 2.2.3 in the context of the MFG system, provides such an estimate.

Lemma 2.4.1. *Let $(u, m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (Q0), and set $K = ||u||_{C^1(\overline{Q_T})}$. Let μ_K, $\lambda_K > 0$ be such that (2.2.5) holds in $\mathbb{T}^d_K$, and the conditions (2.2.6) and*

$$\lambda_K \leq D_q B \cdot \nu \tag{2.4.1}$$

hold in $\partial Q_{T,K}$. There exist constants $C > 0,\ 0 < \gamma < 1$, with

$$C = C\left(K, \frac{\mu_K}{\lambda_K}\right),\ \gamma = \gamma\left(K, \frac{\mu_K}{\lambda_K}\right),$$

such that

$$[Du]_{\gamma,\overline{Q_T}} \leq C.$$

Proof. The only thing to remark is that in order to apply Theorem 2.2.3, it is necessary to verify that λ_K can be chosen to satisfy (2.4.1), or, in other words, that N is indeed an oblique boundary operator. This follows directly from (B1), since

$$D_q B(x,0,z,q) \cdot \nu(x,0) = -B_s(x,0,z,q) = 1 > 0,$$
$$D_q B(x,T,z,q) \cdot \nu(x,T) = B_s(x,T,z,q) = g_m f_w^{-1} = \frac{g_m}{f_m} > 0.$$

Therefore, the result follows by applying Theorem 2.2.2 and Theorem 2.2.3 locally, and extracting a finite subcover of $\overline{Q_T}$. The use of Theorem 2.2.3 is particularly straighforward since the boundary of Q_T is already flat. □

The strategy to prove existence will be to use the nonlinear method of continuity, by constructing an explicit homotopy $(Q^\theta, N^\theta)_{\theta\in[0,1]}$ between (Q0) and an elliptic problem that comes from a much simpler MFG system, and trivially has a smooth solution. For each $\theta \in [0,1]$ and each $(x,p,m) \in \mathbb{T}^d \times \mathbb{R}^d \times (0,\infty)$, define

$$H^\theta(x,p) = \theta H(x,p) + (1-\theta)(\frac{1}{2}|p|^2 + f(x,1)),$$

$$g^\theta(x,m) = \theta g(x,m) + (1-\theta)m, \quad m_0^\theta(x) = \theta m_0(x) + (1-\theta),$$

and consider the family of MFG systems

$$\begin{cases} -u_t + H^\theta(\cdot, D_x u) = f(\cdot, m), \;\; u(\cdot, T) = g^\theta(\cdot, m(\cdot, T)) \\ m_t - \operatorname{div}(m D_p H^\theta(\cdot, D_x u)) = 0, \;\; m(\cdot, 0) = m_0^\theta(\cdot) \end{cases} \tag{MFG$_\theta$}$$

We observe that, when $\theta = 0$, the unique solution is $(u, m) \equiv (1, 1)$. Let $(Q^\theta u \;, N^\theta u)$ be the operators for the corresponding elliptic problem associated to (MFG$_\theta$), and let A^θ, b^θ, and B^θ be their coefficients. The following straightforward Lemma is a version of Theorem 2.3.9, tailored to the family (MFG$_\theta$), that also includes the Hölder estimates of Lemma 2.4.1, and provides a priori bounds that hold uniformly in θ.

Lemma 2.4.2. *For each $\theta \in [0, 1]$, let $(u^\theta, m^\theta) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$ be a solution to (MFG$_\theta$). Then there exist constants $C > 0$ and $0 < \gamma < 1$, independent of θ, such that*

$$||u^\theta||_{C^{1,\gamma}(\overline{Q_T})} \le C.$$

Proof. The strategy here is to apply Theorem 2.3.9 and Lemma 2.4.1 to the corresponding MFG system (MFG$_\theta$) that arises from the new data $H^\theta, g^\theta, m_0^\theta$, and proving that those results lead to bounds that are uniform in θ. Let β be defined by (2.3.29), and, for each $\theta \in [0, 1]$, let $C_{0,\theta}$ and $0 \le \tau^\theta < 1$ be any two constants large enough that the inequalities (H1), (H2), (H3), (HX), (FX1), (FX2), (2.2.1), (2.2.2), (2.2.3), and (2.2.4) all hold when H, g, m_0 are replaced by H^θ, g^θ, m_0^θ. Theorem 2.3.9 then yields constants $L_\theta, L_{1,\theta}, K_\theta, K_{1,\theta}$, with

$$L_\theta = \left(L_{1,\theta}, |g_1^\theta f_0^{-1}(L_{1,\theta})|, |g_0^\theta f_1^{-1}(-L_{1,\theta})|, (g_0^\theta)^{-1} g_1^\theta f_0^{-1}(L_{1,\theta}), \frac{1}{(g_1^\theta)^{-1} g_0^\theta f_1^{-1}(-L_{1,\theta}))} \right),$$

$$L_{1,\theta} = M(C_{0,\theta}, T),$$

$$K_\theta = K_\theta(K_{1,\theta}, \beta_{K_{1,\theta}}),\ K_{1,\theta} = K_{1,\theta}\left(L_\theta, \frac{1}{T}, \frac{1}{1-\tau^\theta}, f_0\left(\frac{1}{L_\theta}\right)^-\right),$$

such that

$$||u^\theta||_{C^0(\overline{Q_T})} + ||m^\theta(T)||_{C^0(\mathbb{T}^d)} + \left\|\frac{1}{m^\theta(T)}\right\|_{C^0(\mathbb{T}^d)} \leq L_\theta,\ ||Du^\theta||_{C^0(\overline{Q_T})} \leq K_\theta.$$

The goal is now to show that L_θ, K_θ may be chosen independently of θ. First we prove that this is true for $C_{0,\theta}$ and τ^θ. Conditions (FX1) and (FX2) trivially hold for the same C_0 and the new H^θ, g^θ, m_0^θ, because the functions H, g, m_0 do not appear in those inequalities. Since the map $H^0(p,x) = \frac{1}{2}|p|^2 + f(x,1)$ satisfies $D_pH^0 \equiv p$, it also satisfies (H1), (H3), (HX), and (2.2.3), with C_0 being replaced by a universal constant. Thus, since H^θ is a convex combination of H^0 and H, these inequalities still hold for H^θ, when C_0 is replaced by a convex combination of C_0 and a universal constant. By the same reasoning, conditions (H2), (2.2.1), and (2.2.2) hold for H^θ after replacing C_0 with a convex combination of C_0 and a constant depending only on C_0 and $||f(\cdot,1)||_{C^2(\mathbb{T}^d)} \leq C_0$. Only condition (2.2.4) is left to consider, namely

$$||\chi(\cdot,0)||_{C^0(\mathbb{T}^d)} + ||m_0^\theta||_{C^1(\mathbb{T}^d)} + ||f||_{C^2(\mathbb{T}^d\times[\min m_0^\theta, \max m_0^\theta])} \leq C_{0,\theta}. \tag{2.4.2}$$

The first term is already independent of θ, whereas, noticing that $\min m_0 \leq 1 \leq \max m_0$ and $|D_x m_0^\theta| = \theta |D_x m_0|$,

$$||m_0^\theta||_{C^1(\mathbb{T}^d)} + ||f||_{C^2(\mathbb{T}^d\times[\min m_0^\theta, \max m_0^\theta])} \leq ||m_0||_{C^1(\mathbb{T}^d)} + ||f||_{C^2(\mathbb{T}^d\times[\min m_0, \max m_0])} \leq C_0.$$

Thus, one may select

$$C_{0,\theta} = C_{0,\theta}(C_0),\ \tau^\theta = \tau, \tag{2.4.3}$$

and consequently

$$L_{1,\theta} = L_{1,\theta}(C_{0,\theta}, T) = L_{1,\theta}(C_0, T) := L_1.$$

Now, by definition,

$$g_0^\theta(m) = \theta g_0 + (1-\theta)m, \quad g_1^\theta(m) = \theta g_1 + (1-\theta)m. \tag{2.4.4}$$

Therefore,

$$|g_0^\theta f_1^{-1}(-L_1)| \leq \max(|g_0 f_1^{-1}(-L_1)|, f_1^{-1}(-L_1))) \leq \max(|g_0 f_1^{-1}(-L_1)|, f_0^{-1}(L_1)), \tag{2.4.5}$$

and similarly,

$$|g_1^\theta f_0^{-1}(L_1)| = |\theta g_1 f_0^{-1}(L_1) + (1-\theta) f_0^{-1}(L_1)| \leq \max(|g_1 f_0^{-1}(L_1)|, f_0^{-1}(L_1)). \tag{2.4.6}$$

On the other hand, the following inequalities hold:

$$(g_0^\theta)^{-1} g_1^\theta \leq g_0^{-1} g_1, \quad g_1^{-1} g_0 \leq (g_1^\theta)^{-1} g_0^\theta. \tag{2.4.7}$$

Indeed, by (2.4.4),

$$g_0^\theta g_0^{-1} g_1 = \theta g_0 g_0^{-1} g_1 + (1-\theta) g_0^{-1} g_1 \geq \theta g_1 + (1-\theta) g_0^{-1} g_0 = g_1^\theta,$$

which shows the first inequality in (2.4.7), with the second one following in the same fashion. Now, (2.4.7) yields

$$(g_0^\theta)^{-1} g_1^\theta f_0^{-1}(L_1) \leq g_0^{-1} g_1 f_0^{-1}(L_1), \quad \frac{1}{(g_1^\theta)^{-1} g_0^\theta f_1^{-1}(-L_1)} \leq \frac{1}{g_1^{-1} g_0 f_1^{-1}(-L_1)}. \tag{2.4.8}$$

Thus, (2.4.5), (2.4.6), and (2.4.8) yield

$$L_\theta = L_\theta\left(M, |g_1 f_0^{-1}(L_1)|, |g_0 f_1^{-1}(L_1)|, g_0^{-1} g_1 f_0^{-1}(L_1), \frac{1}{g_1^{-1} g_0 f_1^{-1}(-L_1)}, f_0^{-1}(L_1)\right) := L,$$

and

$$K_{1,\theta} = K_{1,\theta}\left(L, \frac{1}{T}, \frac{1}{1-\tau}, f_0\left(\frac{1}{L}\right)^-\right) := K_1,\ K_\theta = K_\theta(K_1, \beta_{K_1}) := K.$$

Next, we obtain the gradient Hölder estimate with the help of Lemma 2.4.1. We remark that the operator (Q^θ, N^θ) is clearly elliptic and oblique, because it comes from (MFG$_\theta$). Moreover, since A^θ, b^θ, and B^θ and their derivatives are, respectively, continuous functions of (x, t, z, p, s, θ) on the compact sets $\overline{Q}_{T,K} \times [0,1]$ and $\partial Q_{T,K} \times [0,1]$, it follows that there exist constants $\mu_{L+K} > 0$, $\lambda_{L+K} > 0$, independent of θ, satisfying (2.2.5) in $(\mathbb{T}^d)_{L+K}$, and (2.2.6), (2.4.1) in $\partial Q_{T,L+K}$, when the operators (Q, N) are replaced by (Q^θ, N^θ). Lemma 2.4.1 then yields constants $C > 0$, $0 < \gamma < 1$, independent of θ, such that

$$[Du^\theta]_{\gamma,\overline{Q_T}} \leq C.$$

□

With the help of this uniform estimate, the main theorem for the strictly elliptic problem may now be proved.

Proof of Theorem 2.1.1. The uniqueness part of the statement is an immediate consequence of the standard Lasry-Lions monotonicity method, and will be omitted. We define the Banach spaces

$$E = C^{3,\alpha}(\overline{Q_T}),\ F = C^{1,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\partial Q_T),$$

and the continuously differentiable operator $S : E \times [0,1] \to F$ by

$$S(u,\theta) = (Q^\theta u, N^\theta u),\ (u,\theta) \in E \times [0,1].$$

The partial Fréchet derivative of S with respect to the variable u at the point (u,θ) is the corresponding linearization, for fixed θ, of the differential operator (Q^θ, N^θ), namely $(L^1_{(u,\theta)}, L^2_{(u,\theta)})$, where

$$L^1_{(u,\theta)}(w) = -\operatorname{Tr}(A^\theta(x,Du)D^2 w) - D_q \operatorname{Tr}(A^\theta(x,Du)D^2 u) \cdot Dw + D_q b^\theta(x,Du) \cdot Dw,$$
$$L^2_{(u,\theta)}(w) = \begin{cases} -w_t + D_p H^\theta(x, D_x u) \cdot D_x w & \text{if } t = 0, \\ \frac{g_m^\theta}{f_m}(w_t - D_p H^\theta \cdot D_x w) + w & \text{if } t = T. \end{cases}$$

For fixed $(u,\theta) \in E \times [0,1]$, the linear operator $L^1_{(u,\theta)}$ is uniformly elliptic and the linear boundary operator $L^2_{(u,\theta)}$ is oblique. Moreover, the homogeneous problem

$$(L^1_{(u,\theta)} w, L^2_{(u,\theta)} w) = (0,0)$$

has the form

$$-\operatorname{Tr}(\tilde{A}(x,t)D^2 w) + \tilde{b}(x,t) \cdot Dw = 0 \text{ in } Q_T, \quad \tilde{B}(x,t) \cdot Dw + \tilde{c}(x,t)w = 0 \text{ on } \partial Q_T,$$

where $\tilde{B} \cdot \nu > 0$, $\tilde{c} \geq 0$ and $\tilde{c} \not\equiv 0$, which implies that it has only the trivial solution in $C^{3,\alpha}(\overline{Q_T})$. Hence, by the standard Fredholm alternative for linear oblique problems (see [25]), the operator $(L^1_{(u,\theta)}, L^2_{(u,\theta)})$ is invertible in $C^{3,\alpha}(\overline{Q_T})$. The infinite-dimensional implicit function theorem then implies that the set

$$D = \{\theta \in [0,1] : \text{the equation } S(u,\theta) = (0,0) \text{ has a unique solution } u \in C^{3,\alpha}(\overline{Q_T})\}$$

is open in $[0, 1]$.

The next step is to show that D is also closed. Let $\{\theta_n\} \subset D$ be a sequence such that $\theta_n \to \theta \in [0, 1]$, and let $\{u_n\} \subset E$ be the corresponding sequence of solutions to $S(u_n, \theta_n) = (0, 0)$. By Lemma 2.4.2, there exist numbers $C > 0,\ 0 < \gamma < 1$, independent of n, such that

$$||u_n||_{C^{1,\gamma}(\overline{Q_T})} \leq C.$$

The Arzelà-Ascoli Theorem implies that, up to a subsequence, there exists $u \in C^{1,\gamma}(\overline{Q_T})$ such that $u_n \to u$ in $C^1(\overline{Q_T})$. By Theorem 2.2.5, it follows that $u \in C^{2,\alpha}(\overline{Q_T})$, $u_n \to u$ in $C^{2,\alpha}(\overline{Q_T})$, and $S(u, \theta) = 0$. In particular, the u_n are uniformly bounded in $C^{2,\alpha}(\overline{Q_T})$. Now, given $i \in \{1, \ldots, d\}$, differentiating the equation $(Q^{\theta_n}(u_n), N^{\theta_n}(u_n)) = (0, 0)$ yields, for $w = D_{x_i}u_n$,

$$\begin{aligned} L^1_{(u_n,\theta_n)}w =& \operatorname{Tr}(A^{\theta_n}_{x_i}(x, Du_n)D^2u_n) - b^{\theta_n}_{x_i}(x, Du_n), \\ L^2_{(u_n,\theta_n)}w =& \begin{cases} D_{x_i}(f(x, m^{\theta_n}_0(x))) - H^{\theta_n}_{x_i} & \text{if } t = 0, \\ \frac{g^{\theta_n}_m}{f_m}(H^{\theta_n}_{x_i} - f_{x_i}) + g^{\theta_n}_{x_i} & \text{if } t = T. \end{cases} \end{aligned}$$

Therefore, by Theorem 2.2.4, there exists $C > 0$, independent of n, such that

$$||w||_{C^{2,\alpha}(\overline{Q_T})} \leq C,$$

implying that D_xu_n is bounded in $C^{2,\alpha}(\overline{Q_T})$. In particular, $u_n|_{\partial Q_T}$ is bounded in $C^{3,\alpha}(\partial Q_T)$, and by the standard Schauder theory for the Dirichlet problem, u_n is therefore bounded in $C^{3,\alpha}(\overline{Q_T})$. Consequently, $u \in C^{3,\alpha}(\overline{Q_T})$ and $\theta \in D$, proving that D is closed. Since $0 \in D$, it follows that $D = [0, 1]$, which completes the proof. □

The next theorem is the corresponding variant of Theorem 2.1.1 for the case of a fast-growing f, that follows from the estimates in Theorem 2.3.10.

Theorem 2.4.3. *If condition (F2) is replaced by (HFX*), the conclusion of Theorem 2.1.1 holds.*

Proof. All of the results in this section follow in this case by simply replacing the use of Theorem 2.3.9 by Theorem 2.3.10. □

CHAPTER 3

REGULARITY AND LONG-TIME BEHAVIOR IN ONE DIMENSION WITHOUT BLOW-UP ASSUMPTION

3.1 Introduction

In this chapter, we will investigate the extent to which, in the case of one spatial dimension, the blow-up assumption (2.1.1) can be removed while still obtaining smooth solutions. We will also make the first step towards weakening the strict positivity assumption of the initial density, foreshadowing the results of the following chapter on compactly supported solutions in the whole space. Additionally, we show how to treat more general, non-separated Hamiltonians of super-linear growth, as well as an alternative boundary condition at the terminal time known as the planning problem. Finally, we will fully characterize the long time behavior of the solutions. Specifically, we will study the problem

$$\begin{cases} -u_t(x,t) + H(u_x(x,t), m(x,t)) = 0 & (x,t) \in Q_T = \mathbb{T} \times (0,T), \\ m_t(x,t) - (m(x,t)H_p(u_x(x,t), m(x,t)))_x = 0 & (x,t) \in Q_T, \\ m(x,0) = m_0(x),\ u(x,T) = g(m(x,T)) & x \in \mathbb{T}, \end{cases} \tag{MFG}$$

as well as to the so-called planning problem with a prescribed terminal density,

$$\begin{cases} -u_t(x,t) + H(u_x(x,t), m(x,t)) = 0 & (x,t) \in Q_T, \\ m_t(x,t) - (m(x,t)H_p(u_x(x,t), m(x,t)))_x = 0 & (x,t) \in Q_T, \\ m(x,0) = m_0(x),\ m(x,T) = m_T(x) & x \in \mathbb{T}, \end{cases} \tag{MFGP}$$

where $\mathbb{T}$ denotes the 1-dimensional torus, $-H(p,m) : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ and $g(m) : (0,\infty) \to \mathbb{R}$ are strictly increasing in m, H has super-linear growth in p, and $m_0, m_T : \mathbb{T} \to [0,+\infty)$ are probability densities.

The first main result will be the following theorem, which shows that, in the one-dimensional problem, the blow-up assumption (2.1.1) can be completely removed. We refer to Section 3.2 for assumptions (M), (H) (G), (E), (W), and (L), and to the notation subsection for the definition of the function spaces mentioned below.

Theorem 3.1.1. *Let* $0 < \alpha < 1$*, and assume that* (M), (H), (G)*, and* (E) *hold. Then the following statements hold:*

(i) There exists a classical solution $(u, m) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$ *to* (MFGP). *The function* m *is unique, and* u *is unique up to a constant.*

(ii) There exists a unique classical solution $(u, m) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$ *to* (MFG).

The next result of in this chapter establishes interior smoothness of the solutions when, besides removing the blow-up assumption (FB), one also weakens the lower bound assumptions for given densities m_0 and m_T, replacing the latter with the integrability conditions

$$\frac{1}{m_0^\kappa} \in L^1(\mathbb{T}), \ \ \frac{1}{m_T^\kappa} \in L^1(\mathbb{T}) \text{ for some } \kappa > 0. \tag{3.1.1}$$

We observe that, in particular, (3.1.1) allows the initial density to vanish in a set of measure zero. In spite of this fact, the result also shows that m becomes strictly positive instantly after the initial time. Moreover, in the case of (MFG), the density remains bounded below, and the solution remains smooth up to and including $t = T$. We refer to Section 3.6 for the definition of a weak solution.

Theorem 3.1.2. *Let* $0 < \alpha < 1$*, and assume that* (W), (H) (G)*, and* (E) *hold. Then the following statements hold:*

(i) There exists a weak solution

$$(u, m) \in (BV(Q_T) \cap L^\infty(Q_T)) \times (C([0, T], H^{-1}(\mathbb{T})) \cap L^\infty_+(Q_T))$$

to (MFGP). *Moreover,* $(u,m) \in C^{3,\alpha}_{\mathrm{loc}}(Q_T) \times C^{2,\alpha}_{\mathrm{loc}}(Q_T)$ *and* $m > 0$ *in* $(0,T)$. *The function* m *is unique, and* u *is unique up to a constant.*

(ii) Assume, further, that the function H *satisfies, for each* $(p,m) \in \mathbb{R} \times (0,\infty)$,

$$H_p(p,m)p \geq 0. \tag{3.1.2}$$

Then there exists a unique weak solution

$$(u,m) \in (BV(Q_T) \cap L^\infty(Q_T)) \times (C([0,T], H^{-1}(\mathbb{T})) \cap L^\infty_+(Q_T))$$

to (MFG). *Moreover,* $(u,m) \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{T} \times (0,T]) \times C^{2,\alpha}_{\mathrm{loc}}(\mathbb{T} \times (0,T])$, *and* $m > 0$ *in* $(0,T]$.

We will show that the solutions are smooth, and we will establish convergence, as $T \to \infty$, to the solution of the infinite time horizon MFG system,

$$\begin{cases} -v_t(x,t) + \lambda + H(v_x(x,t), \mu(x,t)) = 0 & (x,t) \in \mathbb{T} \times (0,\infty), \\ \mu_t(x,t) - (\mu(x,t) H_p(v_x(x,t), \mu(x,t)))_x = 0 & (x,t) \in \mathbb{T} \times (0,\infty), \\ \mu(x,0) = m_0(x) & x \in \mathbb{T}, \end{cases} \tag{MFGL}$$

where $\lambda = -H(0,1)$.

Concerning the long time behavior of (3.1.1), it was shown by P. Cardaliaguet and P.J. Graber in [8, Thm 5.1] that the rescaled solution $(x,s) \mapsto u(x,sT)/T$ converges, in a certain space $L^p(\mathbb{T} \times (\delta,1))$, to the map $\lambda(1-s)$, while the rescaled density $(x,s) \mapsto m(x,sT)$ converges in $L^p(\mathbb{T} \times (0,1))$ to the invariant measure $\mu \equiv 1$. Our third result shows that, when the marginals are strictly positive, a much stronger statement holds. That is, the solutions satisfy the turnpike property with an exponential rate of convergence, and the limit as $T \to \infty$ of the pair $(u(t) - \lambda(T-t), m(t))$ can be fully characterized as the solution to (MFGL). We emphasize that this is a convergence result *at the original time scale* (cf. [14, Thm 2.6, Thm. 5.1], [16, Thm 4.1, Thm. 5.3]).

Theorem 3.1.3. *Assume that* (M), (H), (G), (E), *and* (L), *hold, and let* $T > 1$. *Assume that* (u^T, m^T) *is either the solution to* (MFG), *or the solution to* (MFGP) *that satisfies* $\int_{\mathbb{T}} v^T(\cdot, \frac{T}{2}) = 0$, *where*

$$v^T(x,t) := u^T(x,t) - \lambda(T-t).$$

Then the following holds:

(i) There exist constants $C, \omega > 0$, *independent of* T, *such that*

$$\|m^T(t) - 1\|_{L^\infty(\mathbb{T})} + \|u_x^T(t)\|_{L^\infty(\mathbb{T})} \leq C(e^{-\omega t} + e^{-\omega(T-t)}), \quad t \in [0,T].$$

Moreover, if (u^T, m^T) *solves* (MFG), *and* (3.1.2) *holds, we have*

$$\|m^T(t) - 1\|_{L^\infty(\mathbb{T})} + \|u_x^T(t)\|_{L^\infty(\mathbb{T})} \leq Ce^{-\omega t}, \quad t \in [0,T].$$

(ii) There exist functions (v, μ) *such that, for each* $T_0 > 0$,

$$v^T \to v \text{ in } C^{3,\alpha}(\mathbb{T} \times [0,T_0]) \text{ as } T \to \infty,$$

and

$$m^T \to \mu \text{ in } C^{2,\alpha}(\mathbb{T} \times [0,T_0]) \text{ as } T \to \infty.$$

Moreover, one has

$$\lim_{t\to\infty} v(\cdot,t) = c, \quad \lim_{t\to\infty} \mu(\cdot,t) = 1 \text{ uniformly in } \mathbb{T}, \tag{3.1.3}$$

where

$$c = \begin{cases} g(1) & \textit{if } (u^T, m^T) \textit{ solves } (\text{MFG}), \\ 0 & \textit{if } (u^T, m^T) \textit{ solves } (\text{MFGP}). \end{cases}$$

Finally, (v, μ) *is the unique classical solution to* (MFGL) *satisfying* (3.1.3) *and*

$$v \in W^{1,\infty}(\mathbb{T} \times (0,\infty)), \ \ \mu^{-1} \in L^{\infty}(\mathbb{T} \times (0,\infty)), \\ \mu - 1 \in L^{1}(\mathbb{T} \times (0,\infty)) \cap L^{\infty}(\mathbb{T} \times (0,\infty)). \quad (3.1.4)$$

In particular, since the Hamiltonian $H(p, m)$ is non-separated, our results yield well-posedness and regularity of MFG systems with congestion, such as

$$\begin{cases} -u_t + \frac{|u_x|^2}{2(m+c_0)^{\alpha}} = f(m) & \text{in } Q_T, \\ m_t - \left(\frac{m}{(m+c_0)^{\alpha}} u_x\right)_x = 0 & \text{in } Q_T, \end{cases} \quad (3.1.5)$$

where $0 < \alpha < 2$, $c_0 \geq 0$, and $f' > 0$. In the same way as in the previous chapter, the problems (MFG) and (MFGP) can be transformed into a single quasilinear elliptic equation in u after eliminating the variable m. Indeed, if one defines H^{-1} by

$$m = H^{-1}(p, H(p, m)),$$

then $m = H^{-1}(u_x, u_t)$ and the problem becomes

$$\begin{cases} Qu := -\text{Tr}(A(Du)D^2 u) = 0 & \text{in } Q_T, \\ Nu := B(x, t, u, Du) = 0 & \text{on } \partial Q_T, \end{cases} \quad (\text{Q})$$

where $Du = (u_x, u_t)$ and, for $(x, z, p, s) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$A(p, s) = \left(H_p + \frac{1}{2}mH_{mp}, -1\right) \otimes \left(H_p + \frac{1}{2}mH_{mp}, -1\right) - \begin{pmatrix} \frac{1}{4}m^2H_{mp}^2 + mH_mH_{pp} & 0 \\ 0 & 0 \end{pmatrix}, \tag{Q1}$$

$$B(x, 0, z, p, s) = -\, s + H(p, m_0(x)), \tag{B1}$$

and

$$B(x, T, z, p, s) = \begin{cases} s - H(p, g^{-1}(z)) & \text{in the case of (MFG)} \\ s - H(p, m_T(x)) & \text{in the case of (MFGP).} \end{cases} \tag{B2}$$

The condition for ellipticity, that is, for the matrix A to be positive, is

$$-4mH_mH_{pp} > m^2H_{mp}^2, \tag{3.1.6}$$

which is also the well-known condition for uniqueness to (MFG) that follows from the Lasry-Lions monotonicity method (see, for instance, Lions, Souganidis [40]). We remark from (3.1.6) that, in particular, the strict positivity of the density is crucial for the regularizing properties of the system. The lower bounds on m obtained in Corollary 3.3.2 and Proposition 3.6.3 both heavily rely on the one-dimensionality assumption, and this is the main obstacle to generalizing the results of this chapter to higher dimensions. Indeed, in dimensions $d > 1$, it remains an open question whether the existence of smooth solutions to local first order MFG systems can still be established if one removes or significantly weakens the blow-up assumption (FB), or if m_0 is not assumed to be bounded away from 0.

Section 3.2 contains all the assumptions that will be in place about the Hamiltonian H, as well as the initial and terminal data. In Section 3.3, we establish an integral displacement convexity formula (see Proposition 3.3.1) that will allow us to bound the density m in terms of its initial and terminal values. Section 3.4 contains the necessary a priori estimates that are needed to prove the existence of classical solutions. In particular, we obtain, in Section 3.4.1, estimates for an ϵ–approximation of (MFGP) via standard MFG systems with a terminal condition of the type $u(\cdot, T) = g(\cdot, m(\cdot, T))$, which we require to prove existence for (MFGP). Finally, we provide a counterexample to existence of solutions to (MFG) when the terminal cost function g is also allowed to depend on the space variable (see Proposition 3.4.5). In Sections 3.5, 3.6, and 3.7, we prove the main results of the chapter, Theorems 3.1.1, 3.1.2, and 3.1.3, respectively.

This chapter is entirely based on the author's joint work [43] with N. Mimikos.

Notation

Let $d, k \in \mathbb{N}$. For $T > 0$, we denote by $Q_T := \mathbb{T} \times (0, T)$, $\overline{Q_T} := \mathbb{T} \times [0, T]$ and $\partial Q_T := \mathbb{T} \times \{0, T\}$. For $\alpha \in (0, 1]$ and $\Omega \subset \mathbb{R}^d$ we denote by $C^{k,a}(\Omega)$, the standard space of k times differentiable scalar functions with α-Hölder continuous k^{th} order derivatives, with the usual norm. Furthermore, we denote by $C^{k,\alpha}_{\text{loc}}(\Omega)$ the functions u that belong to $C^{k,\alpha}(K)$, for all compact sets $K \subset \Omega$. For functions $u : \mathbb{T} \times [0, T] \to \mathbb{R}$, we denote by $\operatorname{osc} u := \max\limits_{(x,t) \in \mathbb{T} \times [0,T]} u(x, t) - \min\limits_{(x,t) \in \mathbb{T} \times [0,T]} u(x, t)$, $Du(x, t) := (u_x(x, t), u_t(x, t))$. We denote by $H^{-1}(\mathbb{T})$ the dual space of the Sobolev space $H^1(\mathbb{T})$, and by $C^{0,\alpha}([0, T]; H^{-1}(\mathbb{T}^d))$ the space of $H^{-1}(\mathbb{T}^d)$–valued functions that are α–Hölder continuous. We write $C = C(K_1, K_2, \ldots, K_M)$ for a positive constant C depending monotonically on the non-negative quantities $K_1, \ldots, K_M$. $\text{BV}(Q_T)$ denotes the space of functions of bounded variation, and $L^\infty_+(Q_T)$ consists of the functions $m \in L^\infty(Q_T)$ such that $m \geq 0$ a.e. in Q_T.

3.2 Assumptions

In what follows, C_0 and γ, α are positive constants, with $\gamma > 1$, and $0 < \alpha < 1$. Moreover, $\overline{C} : (0, \infty) \to [1, \infty)$ is a continuous, strictly positive function. We note that $\overline{C} = \overline{C}(m)$

should be interpreted simply as a positive bound that may blow up both as $m \downarrow 0$ and as $m \uparrow \infty$. Except when explicitly stated, assumptions (M), (H), (G), and (E) will be in place throughout this chapter.

(M) (Assumptions on m_0 and m_T for classical solutions) The given functions m_0 and m_T satisfy

$$m_0, m_T \in C^{2,\alpha}(\mathbb{T}),\ m_0, m_T > 0,\ \text{and}\ \int_{\mathbb{T}} m_0 = \int_{\mathbb{T}} m_T = 1. \tag{M1}$$

(H) (Assumptions on H) The functions H, H_p, and H_{pp} are in $C^4(\mathbb{R}\times(0,\infty))$, and $H_m < 0$. Moreover, for $(p,m) \in \mathbb{R} \times (0,\infty)$,

$$\frac{1}{C_0}(1+|p|)^{\gamma-2} \leq H_{pp} \leq \overline{C}(m)(1+|p|)^{\gamma-2}, \tag{H1}$$

$$pH_p \geq (1+\frac{1}{C_0})H - \overline{C}(m), \tag{H2}$$

$$|H_{ppp}| \leq \overline{C}(m)(1+|p|)^{\gamma-3}, \tag{H3}$$

$$|H_m| \leq \overline{C}(m)(1+|p|)^{\gamma}, \tag{HM1}$$

$$m|H_{mm}| \leq -\overline{C}(m)H_m, \quad m|p||H_{mmp}| \leq -\overline{C}(m)H_m, \tag{HM2}$$

$$|H_{mpp}| \leq \overline{C}(m)(1+|p|)^{\gamma-2} \tag{HM3}$$

(G) (Assumptions on g) The function $g : (0,\infty) \to \mathbb{R}$ is four times continuously differentiable and satisfies, for all $m > 0$,

$$g'(m) > 0. \tag{G1}$$

(E) (Ellipticity of the system) The function H satisfies, for $m > 0$, the condition

$$-4mH_m H_{pp} \geq \left(1 + \frac{1}{C_0}\right) m^2 H_{mp}^2. \tag{E1}$$

(W) (Assumptions on m_0, m_T, H, and g for weak solutions) The functions m_0 and m_T satisfy, for some $\kappa > 0$,

$$m_0,\, m_T \in L^\infty(\mathbb{T}),\; m_0,\, m_T \geq 0,\; \int_{\mathbb{T}} m_0 = \int_{\mathbb{T}} m_T = 1,\; \text{and } \frac{1}{m_0^\kappa}, \frac{1}{m_T^\kappa} \in L^1(\mathbb{T}), \tag{MW}$$

H satisfies, for some constant $s \in (-\kappa - 1, \kappa - 1)$, and for $(p, m) \in \mathbb{R} \times (0, \frac{1}{C_0})$,

$$-H_m(0, m) \leq C_0 m^s,\; -H_m(p, m) \geq \frac{1}{C_0} m^s, \tag{HW}$$

and g satisfies

$$\lim_{m \to 0^+} g(m) > -\infty. \tag{GW}$$

(L) (Assumption on H for the long time behavior) The function H satisfies, for $m > 0$,

$$-4mH_m H_{pp} \geq \frac{1}{\overline{C}(m)}. \tag{HL}$$

Remark 3.2.1. We will impose assumptions *(W)* and *(L)* only in the sections discussing weak solutions and long time behavior, respectively.

Assumption *(W)* significantly weakens the positivity assumption on m but, in exchange, requires a more precise control on the behavior of H and g near small densities.

On the other hand, in the context of our result on the long time behavior of strictly positive classical solutions, no such control near small (or large) densities is needed. However, a different issue arises here: the gradient bounds used throughout the rest of the chapter may degenerate as $T \to \infty$. Indeed, with *(E)* in place, we could rephrase assumption *(L)* as the requirement that the eigenvalues of the elliptic operator (Q1) remain bounded below as $|p| \to \infty$, locally uniformly in $m \in (0,\infty)$[1]. This will allow us to obtain gradient bounds that are uniform in T (see Lemma 3.7.1, where this assumption is used). For example, for the case of a separated Hamiltonian $H(p,m) \equiv H(p) - f(m)$, *(L)* simply reduces to the assumption that H is uniformly convex, which follows automatically from (H1) for $\gamma \geq 2$.

3.3 Displacement convexity and estimates on the density

To obtain estimates for the density at interior times, we will prove an integral formula which, in particular, implies that the quantity

$$\int_{\mathbb{T}} h(m(x,\cdot))dx$$

is a convex function in $[0,T]$ whenever h is convex, provided that (3.1.6) holds.

Theorem 3.3.1. *Let $(u,m) \in C^2(\overline{Q}_T) \times C^1(\overline{Q}_T)$ be a classical solution to*

$$\begin{cases} -u_t + H(u_x,m) = 0, & \text{in } Q_T \\ m_t - (mH_p(u_x,m))_x = 0, & \text{in } Q_T \\ m(\cdot,0) = m_0, & \text{in } \mathbb{T}, \end{cases} \tag{3.3.1}$$

and let $h \in W^{2,\infty}(\mathbb{R})$. Then

$$\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx = \int_{\mathbb{T}} h''(m)\Big(m_t - m_x(H_p + \frac{m}{2}H_{pm})\Big)^2 dx \\ - \int_{\mathbb{T}} h''(m)(m_x)^2\Big(\frac{m^2}{4}H_{pm}^2 + mH_{pp}H_m\Big)dx. \tag{3.3.2}$$

1. The factor $4m$ in (HL) is, of course, inconsequential, because it is a positive function of m. It is only included to emphasize the comparison with (E).

Moreover, there exists $C = C(C_0)$ such that, if $h'' > 0$,

$$\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx \geq \frac{1}{C}\int_{\mathbb{T}} h''(m)(-mH_m H_{pp} m_x^2 + m^2 H_{pp}^2 u_{xx}^2)dx. \tag{3.3.3}$$

Proof. Let $\tilde{h}:\mathbb{R}\to\mathbb{R}$, be a smooth function. Since m satisfies the continuity equation, the following holds for each $t\in[0,T]$:

$$\int_{\mathbb{T}}\Big(m_t(x,t)-(m(x,t)H_p(u_x,m(x,t)))_x\Big)\Big(\partial_t\tilde{h}(m(x,t))-(\tilde{h}(m(x,t))H_p(u_x,m(x,t)))_x\Big)dx = 0. \tag{3.3.4}$$

Expanding equation (3.3.4), we obtain

$$\begin{aligned}
0 = &\int_{\mathbb{T}}(m_t - m_x(H_p + mH_{pm}) - mH_{pp}u_{xx})(\tilde{h}'(m)m_t - m_x(\tilde{h}'(m)H_p + \tilde{h}(m)H_{pm}) - \tilde{h}(m)H_{pp}u_{xx})dx\\
&= \int_{\mathbb{T}}\tilde{h}'(m)(m_t)^2 - m_t m_x\Big[2\tilde{h}'(m)H_p + \Big(\tilde{h}'(m)m + \tilde{h}(m)\Big)H_{pm}\Big]\\
&\quad + m_x H_{pp}u_{xx}\Big[H_p\Big(\tilde{h}'(m)m + \tilde{h}(m)\Big) + 2\tilde{h}(m)mH_{pm}\Big]\\
&\quad + m_x^2\Big[\Big(H_p + mH_{pm}\Big)\Big(\tilde{h}'(m)H_p + \tilde{h}(m)H_{pm}\Big)\Big]\\
&\quad - m_t H_{pp}u_{xx}\Big[\tilde{h}'(m)m + \tilde{h}(m)\Big]\\
&\quad + \tilde{h}(m)m\Big(H_{pp}u_{xx}\Big)^2 dx = A_1 - A_2 + A_3 + A_4 - A_5 + A_6.
\end{aligned}$$

We split term A_3 as follows

$$A_3 = \int_{\mathbb{T}} m_x H_{pp}H_p u_{xx}\Big(\tilde{h}'(m)m + \tilde{h}(m)\Big)dx + 2\int_{\mathbb{T}}\tilde{h}(m)m_x mH_{pm}H_{pp}u_{xx}dx = A_{3.1} + A_{3.2}.$$

From the continuity equation, we have that

$$mH_{pp}u_{xx} = m_t - m_x(H_p + mH_{pm}).$$

Hence, terms $A_{3.2}$ and A_6 can be written as

$$A_{3.2} = 2\int_{\mathbb{T}} m_t m_x H_{pm}\tilde{h}(m)dx - 2\int_{\mathbb{T}} (m_x)^2 H_{pm}\Big(\tilde{h}(m)H_p + m\tilde{h}(m)H_{pm}\Big)dx = A_{3.2.1} - A_{3.2.2}$$

$$A_6 = \int_{\mathbb{T}} \frac{\tilde{h}(m)}{m}\Big[m_t - m_x\Big(H_p + mH_{pm}\Big)\Big]^2 dx$$

$$= \int_{\mathbb{T}} \frac{\tilde{h}(m)}{m}(m_t)^2 - 2\frac{\tilde{h}(m)}{m}m_t m_x\Big(H_p + mH_{pm}\Big) + \frac{\tilde{h}(m)}{m}(m_x)^2\Big(H_p + mH_{pm}\Big)^2 dx = A_{6.1} - A_{6.2} + A_{6.3}.$$

From the Hamilton-Jacobi (HJ for short) equation, we have that

$$H_p u_{xx} = u_{xt} - H_m m_x.$$

Therefore, $A_{3.1}$ may be written as

$$A_{3.1} = \int_{\mathbb{T}} m_x H_{pp} u_{xt}\Big(\tilde{h}'(m)m + \tilde{h}(m)\Big)dx - \int_{\mathbb{T}} (m_x)^2 H_{pp} H_m\Big(\tilde{h}'(m)m + \tilde{h}(m)\Big)dx = A_{3.1.1} - A_{3.1.2}$$

We now begin by grouping together terms A_5, and $A_{3.1.1}$, which yields, for $L(m) = \tilde{h}(m)m,\ L'(m) = \tilde{h}(m) + m\tilde{h}'(m)$,

$$\begin{aligned} A_{3.1.1} - A_5 &= \int_{\mathbb{T}} m_x\Big(\tilde{h}(m) + m\tilde{h}'(m)\Big)H_{pp}u_{xt} - \Big(\tilde{h}(m) + m\tilde{h}'(m)\Big)m_t H_{pp} u_{xx} dx \\ &= \int_{\mathbb{T}} -\partial_t(L(m))(H_p)_x + L'(m)m_t H_{pm} m_x + (L(m))_x \partial_t(H_p) - L'(m)m_x m_t H_{pm} dx \\ &= \int_{\mathbb{T}} \partial_t((L(m))_x)H_p + (L(m))_x\partial_t(H_p)dx = \frac{d}{dt}\int_{\mathbb{T}} (L(m))_x H_p dx. \end{aligned}$$

Next, we group together all the terms with $m_t m_x$ factor, namely A_2, $A_{3.2.1}$, and $A_{6.2}$, which yields

$$-A_2 + A_{3.2.1} - A_{6.2} = -\int_{\mathbb{T}} 2m_t m_x\left(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m}\right)\left(H_p + \frac{m}{2}H_{pm}\right)dx.$$

Collecting the terms involving $(m_t)^2$, namely terms A_1 and $A_{6.1}$, we obtain

$$A_1 + A_{6.1} = \int_{\mathbb{T}} (m_t)^2 \left(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} \right) dx.$$

Finally, we group together the terms involving m_x^2, namely A_4, $A_{3.2.2}$, $A_{6.3}$, and $A_{3.1.2}$:

$$\begin{aligned} A_4 - A_{3.2.2} + A_{6.3} - A_{3.1.2} = \\ \int_{\mathbb{T}} (m_x)^2 \Big[\Big(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} \Big) \Big(H_p + \frac{m}{2} H_{pm} \Big)^2 \Big] dx \\ - \int_{\mathbb{T}} (m_x)^2 \Big[\Big(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} \Big) \Big(\frac{m^2}{4} H_{pm}^2 + m H_{pp} H_m \Big) \Big] dx. \end{aligned}$$

Thus, putting everything together, we obtain

$$\begin{aligned} -\frac{d}{dt} \int_{\mathbb{T}} (L(m))_x H_p dx = \int_{\mathbb{T}} \Big(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} \Big) \Big(m_t - m_x \Big(H_p + \frac{m}{2} H_{pm} \Big) \Big)^2 dx \\ - \int_{\mathbb{T}} m_x^2 \left(\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} \right) \Big(\frac{m^2}{4} H_{pm}^2 + m H_{pp} H_m \Big) dx. \end{aligned} \tag{3.3.5}$$

Next, notice that for a smooth function $h : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{d}{dt} \int_{\mathbb{T}} h(m) dx = \int_{\mathbb{T}} (h(m))_x H_p + m h'(m) (H_p)_x dx = \int_{\mathbb{T}} (h(m) - h'(m) m)_x H_p dx.$$

Thus, if we require that

$$-L(m) = h(m) - h'(m) m,$$

we obtain

$$-\frac{d}{dt} \int_{\mathbb{T}} (L(m))_x H_p dx = \frac{d^2}{dt^2} \int_{\mathbb{T}} h(m) dx.$$

The relation between $h, \tilde{h}$ is

$$m \tilde{h}(m) = h'(m) m - h(m),$$

therefore

$$\tilde{h}(m) = -\frac{h(m)}{m} + h'(m),$$

and, thus,

$$\tilde{h}'(m) + \frac{\tilde{h}(m)}{m} = -\frac{h'(m)}{m} + \frac{h(m)}{m^2} + h''(m) - \frac{h(m)}{m^2} + \frac{h'(m)}{m} = h''(m),$$

from which (3.3.2) follows.

Now, setting $r = 1 - \frac{1}{1+C_0^{-1}}$, we have

$$-\frac{m^2}{4}H_{pm}^2 - mH_mH_{pp} = -\frac{m^2}{4}H_{pm}^2 - (1-r)mH_mH_{pp} - rmH_mH_{pp},$$

and so, applying (E), and multiplying by $h''(m)m_x^2$, (3.3.2) yields

$$\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx \geq \int_{\mathbb{T}} -rh''(m)mH_mH_{pp}m_x^2. \tag{3.3.6}$$

On the other hand, we infer from (E) that

$$\begin{aligned}
&\left(m_t - m_x(H_p + \frac{m}{2}H_{pm})\right)^2 - m_x^2\left(\frac{m^2}{4}H_{pm}^2 + mH_mH_{pp}\right)\\
&\geq \left(m_t - m_xH_p - \frac{m_xm}{2}H_{pm}\right)^2 + \frac{1}{C_0}\left(\frac{m_xm}{2}H_{pm}\right)^2 = (m_t - m_xH_p)^2 - 2(m_t - m_xH_p)\frac{m_xm}{2}H_{pm}\\
&+(1-r)^{-1}\left(\frac{m_xm}{2}H_{pm}\right)^2 = r(m_t - m_xH_p)^2 + \left((1-r)^{\frac{1}{2}}(m_t - m_xH_p) - (1-r)^{-\frac{1}{2}}\frac{m_xm}{2}H_{pm}\right)^2\\
&\geq r(m_t - m_xH_p)^2 = rm^2H_{pp}^2u_{xx}^2.
\end{aligned} \tag{3.3.7}$$

where the last equality follows from the equation of m. As before, multiplying by $h''(m)$ then yields

$$\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx \geq \int_{\mathbb{T}} rh''(m)m^2H_{pp}^2u_{xx}^2. \tag{3.3.8}$$

Combining (3.3.6) and (3.3.8), we conclude that (3.3.3) holds. □

It now follows readily that the density of the solution is bounded above and below in terms of the initial and terminal densities.

Corollary 3.3.2. *Let* $(u, m) \in C^2(\overline{Q}_T) \times C^1(\overline{Q}_T)$ *be a classical solution to* (MFG) *or* (MFGP). *Then, if* $c_1 := \min(\min m_0, \min m(\cdot, T))$*,* $C_1 = \max(\max m_0, \max m(\cdot, T))$*, one has*

$$c_1 \leq m(x,t) \leq C_1, \text{ for all } (x,t) \in \overline{Q}_T. \tag{3.3.9}$$

Proof. The proof follows directly from Proposition 3.3.1 above. Indeed, note that, in view of (E), for any convex function h, the map

$$C(t) := \int_{\mathbb{T}} h(m(x,t))dx$$

is convex, and thus

$$C(t) \leq \max(C(0), C(T)), \text{ for all } t \in [0, T].$$

Hence, setting $h_p(m) = m^p$ and letting $p \to -\infty$ yields the result for the lower bound, whereas letting $p \to +\infty$ yields the upper bound. $\square$

Remark 3.3.3. For dimensions $d > 1$, formula (3.3.2) is no longer true. If one repeats the same argument, the issue will arise at the term $A_{6.2}$. However, in the case of a separated Hamiltonian, i.e. $H(p,m) \equiv H(p) - f(m)$, one still obtains the weaker formula

$$\begin{aligned}\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx = \int_{\mathbb{T}}((h''(m)m^2 - h'(m)m + h(m))(\mathit{Tr}(D^2_{pp}HD^2_{xx}u))^2 \\ + (h'(m)m - h(m))\,\mathit{Tr}((D^2_{pp}HD^2_{xx}u)^2) + h''(m)mf'(m)|Dm|^2)dx. \end{aligned} \tag{3.3.10}$$

In this higher-dimensional setting, it is no longer true that the left hand side is convex whenever h is convex. In particular, the statement is false for negative powers of m, but true for positive powers. Thus, from the proof of Corollary 3.3.2 we see that the upper bound on

m still holds (see [27]).

3.4 Estimates on the solution and the terminal density

In this section we obtain the necessary a priori $L^\infty-$bounds on u, Du, and $m(\cdot, T)$ for solutions to both (MFG) and (MFGP). Combined with the results of the previous section, this will yield global upper and lower bounds on the density. In order to treat the setting of Theorem 3.1.2, where the density may vanish at $\{0, T\}$, we also obtain L^∞-bounds on u that do not depend on the quantities $(\min m_0)^{-1}$, $(\min m_T)^{-1}$.

Theorem 3.4.1. *Let* $(u, m) \in C^2(\overline{Q}_T) \times C^1(\overline{Q}_T)$ *be a classical solution to* (MFG)*, and let* $c_1 = \min m_0, C_1 = \max m_0$*. Then, for each* $(x, t) \in \overline{Q}_T$*,*

$$c_1 \leq m(x, T) \leq C_1, \tag{3.4.1}$$

$$H(0, c_1)(t - T) + g(c_1) \leq u(x, t) \leq H(0, C_1)(t - T) + g(C_1), \tag{3.4.2}$$

and

$$-\int_t^T H(0, \min_{\mathbb{T}}(m(\cdot, s))ds + g(c_1) \leq u(x, t) \leq -\int_t^T H(0, \max_{\mathbb{T}}(m(\cdot, s))ds + g(C_1). \tag{3.4.3}$$

Proof. We will only show the lower bounds, since the argument for the upper bounds is completely symmetrical. Since $H_m < 0$, we may fix $\delta > 0$ and $\epsilon > 0$, such that

$$H(0, c_1) - H(0, c_1 - \delta) < -\epsilon T. \tag{3.4.4}$$

We define

$$w^{\epsilon,\delta}(t) := H(0, c_1 - \delta)(t - T) + \frac{\epsilon}{2}(t - T)^2 + g(c_1 - \delta),$$

and note that

$$w_{xx} = 0, w_{x,t} = 0, w_{tt} = \epsilon.$$

The function $v^{\epsilon,\delta}(x,t) := u(x,t) - w^{\epsilon,\delta}(t)$ has a minimum at some $(x_0, t_0) \in \overline{Q}_T$. If we first assume that $t_0 \in (0,T)$, then it follows that

$$D^2 u - D^2 w^{\epsilon,\delta} \geq 0,$$

which, in view of (Q), implies

$$0 = -\mathrm{Tr}(AD^2 u) \leq -\mathrm{Tr}(AD^2 w^{\epsilon,\delta}) = -\epsilon < 0,$$

a contradiction. On the other hand, assume that $t_0 = 0$. Then,

$$u_t(x_0, 0) \geq w_t^{\epsilon,\delta}(x_0, 0),\ u_x(x_0, 0) = w_x^{\epsilon,\delta}(0) = 0,$$

and thus, using the monotonicity of H and (3.4.4),

$$0 = -u_t(x_0,t_0) + H(0, m_0(x_0)) \leq -w_t^{\epsilon,\delta}(0) + H(0, m_0(x_0)) = -H(0, c_0 - \delta) + H(0, m_0(x_0)) + \epsilon T$$

$$\leq -H(0, c_1 - \delta) + H(0, c_1) + \epsilon T < 0,$$

which is again a contradiction. Hence, the minimum must be achieved at $t_0 = T$. At that point, we have

$$u_t(x_0, T) \leq w_t^{\epsilon,\delta}(T),\ u_x(x_0, T) = w_x^{\epsilon,\delta}(T) = 0.$$

Consequently, from (G1) and the monotonicity of H, we have

$$u(x_0,T) = g(H^{-1}(0, u_t(x_0,T))) \geq g(H^{-1}(0, w_t^{\epsilon,\delta}(T))) = g(H^{-1}(0, H(0, c_1 - \delta)))$$

$$= g(c_1 - \delta) = w^{\epsilon,\delta}(T).$$

We have thus shown that

$$u(x,t) \geq w^{\epsilon,\delta}(t), \text{ for all } (x,t) \in \overline{Q}_T.$$

Letting $\epsilon \to 0$, and then $\delta \to 0$, yields the lower bound in (3.4.2). In particular, for $t = T$, we have

$$g(m(x,T)) \geq g(c_1) \text{ for all } x \text{ in } \mathbb{T},$$

which proves the lower bound in (3.4.1), in view of (G1). Now, we define

$$w(t) = -\int_t^T H(0, c(s))ds + g(c_1),$$

where $c(s) := \min_{\mathbb{T}}\{m(\cdot, s)\}$ is the running minimum of the density. We observe that the function $v(x,t) = u(x,t) - w(t)$ satisfies $v_t = u_t - H(0, c(t))$, $v_x = u_x$. Thus, for any $\epsilon > 0$, at any extremum point of $v - \epsilon t$, the monotonicity of H implies that $v_t = H(0,m) - H(0, c(t)) - \epsilon < 0$. Letting $\epsilon \to 0$ thus implies that v achieves its minimum at $t = T$. Therefore, using (3.4.1), we obtain

$$u(x,t) - w(t) \geq \min_{\mathbb{T}} g(m(\cdot, T)) - g(c_1) \geq 0,$$

and this is precisely the lower bound in (3.4.3). □

Now, for solutions to (MFGP), we do not need to estimate the terminal density, as it is part of the given data. Concerning u, since the solution is only unique up to a constant, we may only bound the oscillation of u, and this is done in the following proposition.

Theorem 3.4.2. *Let $(u, m) \in C^2(\overline{Q}_T) \times C^1(\overline{Q}_T)$ solve (3.3.1). There exists a constant*

$C > 0$, *with*

$$C = C\left(C_0, \int_0^T |H(0, \min_{\mathbb{T}} m(\cdot, s))| ds, \overline{C}(\max_{\overline{Q}_T} m)\right),$$

such that

$$\operatorname{osc}_{\overline{Q}_T} u \leq C\left(T + T^{-\frac{1}{\gamma-1}} + \int_0^T |H(0, \min_{\mathbb{T}} m(\cdot, s)| ds\right).$$

Proof. We define the functions c and w, for $t \in [0, T]$, by

$$c(t) = \min_{\mathbb{T}} m(\cdot, t), \;\; w(t) = -\int_t^T H(0, c(s)) ds.$$

Arguing as in the proof of (3.4.3), we obtain

$$\max_{\overline{Q}_T}(u - w) = \max_{\mathbb{T}}\left(u(\cdot, 0) - w(0)\right), \; \min_{\overline{Q}_T}(u - w) = \min_{\mathbb{T}}\left(u(\cdot, T) - w(T)\right). \tag{3.4.5}$$

Now, in view of (H1) and Proposition 3.4.1, $0 = -u_t + H(u_x, m) \geq -u_t + \frac{1}{C}|u_x|^\gamma - C$. Next, we define γ' by $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. By the Hopf-Lax formula, the function

$$v(x, t) = \min_{y \in \mathbb{R}}\Big(\left(\frac{C}{\gamma}\right)^{\frac{\gamma'}{\gamma}} (T - t)\frac{|x - y|^{\gamma'}}{\gamma'(T - t)^{\gamma'}} + C(T - t) + u(y, T)\Big)$$

then solves, in $\overline{Q}_T$,

$$-v_t(x, t) + \frac{1}{C}|v_x|^\gamma - C = 0, \;\; v(\cdot, T) = u(\cdot, T),$$

and, thus, by the comparison principle,

$$u \leq v.$$

On the other hand, up to increasing the constant C,

$$v(x,0) \leq \frac{C}{T^{\gamma'-1}} + CT + \min_{\mathbb{T}} u(\cdot,T),$$

and so

$$\max_{\mathbb{T}} u(\cdot,0) \leq \max_{\mathbb{T}} v(\cdot,0) \leq \frac{C}{T^{\gamma'-1}} + CT + \min_{\mathbb{T}} u(\cdot,T).$$

In view of (3.4.5), we obtain

$$\operatorname{osc}_{\overline{Q}_T}(u-w) \leq \frac{C}{T^{\gamma'-1}} + CT + w(T) - w(0),$$

and, thus,

$$\operatorname{osc}_{\overline{Q}_T} u \leq \frac{C}{T^{\gamma'-1}} + CT + 2\cdot \operatorname{osc}_{\overline{Q}_T} w \leq \frac{C}{T^{\gamma'-1}} + CT + 2\int_0^T |H(0,c(s))|ds.$$

□

We finally obtain a priori estimates on the gradient of u, while simultaneously treating the case of (MFG) and (MFGP). The proof closely follows that of Lemma 2.3.8, but allows for weaker assumptions due to the $d=1$ setting (see (3.4.9)). In fact, in the special case of a separated Hamiltonian, this proof can be seen to yield a gradient bound that is independent of $\min(m)$.

Theorem 3.4.3. *Let* $(u,m) \in C^3(\overline{Q}_T) \times C^2(\overline{Q}_T)$ *be a classical solution to* (MFG) *or* (MFGP)*. There exists a constant* $C>0$*, with*

$$C = C\Big(C_0, T, T^{-1}, \operatorname{osc}\, u, \gamma, \|m\|_{L^\infty(\overline{Q}_T)}, \|m^{-1}\|_{L^\infty(\overline{Q}_T)},$$
$$\|(m_0)_x\|_{L^\infty(\mathbb{T})}, \|(m_T)_x\|_{L^\infty(\mathbb{T})}, \|\overline{C}\|_{L^\infty[\min m, \max m]}\Big)$$

such that

$$\|Du\|_{L^\infty(\overline{Q}_T)} \le C.$$

Proof. Since $u_t = H(u_x, m)$, and m is bounded above and below, we infer from (H1) and (H2) that it is enough to show that

$$\|u_x\|_{L^\infty(\overline{Q}_T)} \le CT^2.$$

We let

$$\tilde{u} = u - \min u + 1 - \frac{(\text{osc }u + 2)}{T}(T - t),$$

and note that the function $\tilde{u}$ has been constructed to satisfy

$$|\tilde{u}| \le 1 + \text{osc }u, \quad \tilde{u}(\cdot, 0) \le -1, \ \tilde{u}(\cdot, T) \ge 1.$$

Define

$$v(x,t) = \frac{1}{2}u_x^2 + \frac{k}{2}\tilde{u}^2,$$

where $k = \|u_x\|_{Q_T}^{\frac{3}{2}}$. Let $(x_0, t_0) \in \overline{Q}_T$ be a point where v achieves its maximum value. With no loss of generality, we may assume that $p = u_x(x_0, t_0)$ satisfies

$$|p| \ge 1, \ |p|^2 \ge \frac{1}{2}\|u_x\|^2. \tag{3.4.6}$$

We remark here that throughout the proof, the constant C is subject to increase from line to line.

Case 1: $t_0 = T$. For this case we consider the linearization of the HJ equation,

$$T_u v = -v_t + H_p(u_x, m)v_x.$$

Since $v_x = 0$ and $v_t \geq 0$,

$$
\begin{aligned}
0 \geq T_u v = T_u \left(\frac{1}{2}|u_x|^2\right) + k\tilde{u}(-\tilde{u}_t + H_p u_x) \\
= -H_m u_x m_x + k\tilde{u}(-u_t + H_p p - C) \geq -H_m u_x m_x + k\tilde{u}(\frac{1}{C_0}H) - Ck\tilde{u} \\
\geq -H_m u_x m_x + k\tilde{u}\frac{1}{C_0}\left(\frac{1}{\overline{C}(m)}|p|^\gamma - \overline{C}(m)\right) - C|p|^{\frac{3}{2}} \geq -H_m u_x m_x + \frac{1}{C}|p|^{\gamma+\frac{3}{2}} - C|p|^{\frac{3}{2}}.
\end{aligned}
\tag{3.4.7}
$$

If (u, m) solves (MFG), then

$$-H_m u_x m_x = -\frac{H_m}{g'}|p|^2 > 0.$$

On the other hand, if (u, m) solves (MFGP), then

$$| - H_m u_x m_x| \leq C\|(m_T)_x\|_\infty |p|^{\gamma+1}. \tag{3.4.8}$$

In either case, (3.4.7) then implies

$$|p| \leq C.$$

Case 2: $t_0 = 0$. Regardless of whether (u, m) solves (MFG) or (MFGP), this case is dealt with in the same way as was done for $t_0 = T$ when (u, m) solved (MFGP), because, in view of (HM2), we then have the bound

$$| - H_m u_x m_x| \leq C\|(m_0)_x\|_\infty |p|^{\gamma+1}.$$

Case 3: $0 < t_0 < T$. We first observe that, since $v_x = 0$, we have

$$u_x u_{xx} = -k\tilde{u}u_x,$$

and, thus,

$$|u_{xx}| \leq Ck. \tag{3.4.9}$$

We consider the linearization of (Q), namely

$$L_u(w) = -\mathrm{Tr}(A(Du)D^2w) - D_q\mathrm{Tr}(A(Du)D^2u) \cdot Dw.$$

Through direct computation, using (Q1), one obtains

$$\begin{aligned} L_u\left(\frac{1}{2}u_x^2\right) &= -\left|-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right|^2 + \frac{1}{4}m^2H_{mp}^2u_{xx}^2 - mH_mH_{pp}u_{xx}^2 \\ &\leq -\left|-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right|^2, \end{aligned} \tag{3.4.10}$$

where (E1) was used in the last inequality. Similarly,

$$L_u\left(k\frac{1}{2}\tilde{u}^2\right) = -k\left|-\tilde{u}_t + (H_p + \frac{1}{2}mH_{mp})u_x\right|^2 + k\frac{1}{4}m^2H_{mp}^2u_x^2 - kmH_mH_{pp}u_x^2 + E_1 + E_2 + E_3 + E_4, \tag{3.4.11}$$

where

$$E_1 = 2\left(-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right)\left(H_{pp} + \frac{1}{2}mH_{mpp}\right)k\tilde{u}u_x,$$

$$E_2 = \left(\frac{1}{2}H_{mp}H_{mpp} + mH_{mp}H_{pp} + mH_mH_{ppp}\right)u_{xx}k\tilde{u}u_x,$$

$$E_3 = \left(-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right)\frac{2}{H_m}\left(H_{pm} + \frac{1}{2}\left(mH_{mmp} + H_{mp}\right)\right)k\tilde{u}(-\tilde{u}_t + H_pu_x)$$

$$\begin{aligned} E_4 = \frac{1}{H_m}\Bigg(&\frac{1}{2}(mH_{mp}^2 + m^2H_{mp}H_{mmp}) \\ &+ mH_{mm}H_{pp} + mH_mH_{mpp} + H_mH_{pp}\Bigg)u_{xx}k\tilde{u}(-\tilde{u}_t + H_pu_x). \end{aligned}$$

Now we estimate each of the E_i. By Young's inequality, we obtain

$$|E_1| \leq \frac{1}{4}\left|-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right|^2 + C\left|H_{pp} + \frac{1}{2}mH_{mpp}\right|^2 k^2 u_x^2 \tilde{u}^2.$$

As a result of (H1) and (HM3), we thus obtain

$$|E_1| \leq \frac{1}{4}\left|-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)\right|^2 + C|p|^{2\gamma+1}. \tag{3.4.12}$$

Next, to estimate $|E_2|$, we use (3.4.9), (H1) (H3), (HM1), (HM3) and (E1) to obtain

$$|E_2| \leq C|p|^{2\gamma+1}. \tag{3.4.13}$$

For E_3, we have

$$|E_3| \leq \frac{1}{4}\left|-u_{xt} + \left(H_p + \frac{1}{2}mH_{mp}\right)u_{xx}\right|^2 + \frac{Ck^2}{H_m^2}\left(H_{pm}^2 + m^2H_{mmp}^2 + H_{mp}^2\right)|-\tilde{u}_t + H_p u_x|^2. \tag{3.4.14}$$

Now, recalling that $u_t = H(p, m)$, we infer from (H1), (H2), and (3.4.6) that

$$\frac{1}{C}|p|^\gamma \leq |-\tilde{u}_t + H_p u_x| \leq C|p|^\gamma. \tag{3.4.15}$$

Therefore, in view of (H1), (HM1), (HM2), and (E1), as well as the HJ equation, we obtain

$$|E_3| \leq \frac{1}{4}\left|-u_{xt} + (H_p + \frac{1}{2}mH_{mp})u_{xx}\right|^2 + C|p|^{2\gamma+1}. \tag{3.4.16}$$

Finally, for E_4, we observe that (3.4.9), (H1), (HM2), (HM3), (E1), and (3.4.15) yield

$$|E_4| \leq C|p|^{2\gamma+1}. \tag{3.4.17}$$

Now, (E) implies that

$$\left|-\tilde{u}_t + (H_p + \frac{1}{2}mH_{mp})u_x\right|^2 - \frac{1}{4}m^2H_{mp}^2p^2 + mH_mH_{pp}p^2$$
$$\geq \left|-\tilde{u}_t + (H_p + \frac{1}{2}mH_{mp})u_x\right|^2 + \frac{1}{4C_0}m^2H_{mp}^2p^2 = \left|-\tilde{u}_t + \left(H_pu_x + \frac{1}{2}mH_{mp}\right)p\right|^2$$
$$+ \frac{1}{C_0}\left(\frac{1}{2}mH_{mp}p\right)^2 \geq \frac{1}{2}\left|-\tilde{u}_t + \left(H_p + \frac{1}{2}mH_{mp}\right)u_x\right|^2 + \frac{1}{C}|-\tilde{u}_t + H_pu_x|^2. \quad (3.4.18)$$

So, as a result of (3.4.11) and (3.4.15) we get

$$L_u\left(k\frac{1}{2}\tilde{u}^2\right) \leq -\frac{1}{2}k\left|-\tilde{u}_t + \left(H_p + \frac{1}{2}mH_{mp}\right)u_x\right|^2 - \frac{1}{C}|p|^{2\gamma+\frac{3}{2}} + E_1 + E_2 + E_3 + E_4. \quad (3.4.19)$$

Now, since (x_0, t_0) is an interior maximum point of v, we have $L_u(v) \geq 0$. Thus, combining (3.4.12), (3.4.13), (3.4.16), (3.4.17), (3.4.10) and (3.4.19), we conclude

$$0 \leq -\frac{1}{C}|p|^{2\gamma+\frac{3}{2}} + C|p|^{2\gamma+1},$$

which implies

$$|p| \leq C.$$

□

3.4.1 Estimates for MFG with ϵ–penalized terminal condition

In order to obtain classical solutions to (MFGP), it will be necessary to use a natural approximation method, which was previously used in [48] to obtain weak solutions to the second-order planning problem. The solution will be obtained as the limit of solutions to standard MFG systems with a penalized terminal condition. Specifically, we will need to prove estimates for solutions (u^ϵ, m^ϵ) to

$$\begin{cases} -u_t^\epsilon + H(u_x^\epsilon, m^\epsilon) = 0 \text{ in } Q_T, \\ m_t^\epsilon - (m^\epsilon H_p(u_x^\epsilon, m^\epsilon))_x = 0 \text{ in } Q_T, \\ m^\epsilon(x,0) = m_0(x),\ \epsilon u^\epsilon(x,T) = m^\epsilon(x,T) - m_T(x) \text{ on } \partial Q_T. \end{cases} \tag{MFG$_\epsilon$}$$

As long as u^ϵ is bounded in $L^\infty(Q_T)$, the limit is expected to solve (MFGP). This estimate is obtained in the following lemma. While treating this system, we will temporarily assume that $H(0,0)$ is finite. This assumption will be removed in the proof of Theorem 3.1.1.

Lemma 3.4.4. *For $\epsilon > 0$, let $(u^\epsilon, m^\epsilon) \in C^2(\overline{Q_T}) \times C^1(\overline{Q_T})$ be a classical solution to system (MFG$_\epsilon$), and set $c_1 = \min\{\min_{\mathbb{T}} m_0, \min_{\mathbb{T}} m_T\}$, $C_1 = \max\{\max_{\mathbb{T}} m_0, \max_{\mathbb{T}} m_T\}$. Assume that $H(0,0) < \infty$. Then there exists a constant $C > 0$, independent of ϵ, such that*

$$\|u^\epsilon\|_{L^\infty(\overline{Q_T})} \le C. \tag{3.4.20}$$

Furthermore, for all $\epsilon < \frac{1}{C}$, we have

$$\frac{c_1}{2} \le m^\epsilon(x,t) \le 2C_1 \text{ for all } (x,t) \in \overline{Q_T}, \tag{3.4.21}$$

and

$$\|m^\epsilon(T,\cdot) - m_T(\cdot)\|_\infty \le \epsilon C. \tag{3.4.22}$$

Proof. As a result of Proposition 3.4.2, since $H(0, \min_{\overline{Q}_T} m^\epsilon) \le H(0,0)$, there exists

$$C = C(C_0, T, |H(0,0)|, |H(0, \max_{\overline{Q}_T} m^\epsilon)|, \overline{C}(\max_{\overline{Q}_T} m^\epsilon))$$

such that

$$\operatorname{osc}_{\overline{Q}_T}(u^\epsilon) \le C.$$

To make this bound on the oscillation independent of ϵ, we must obtain upper bounds on the density m^ϵ. Note that, from Corollary 3.3.2, it is enough to bound $m^\epsilon(T,\cdot)$ from above. To this end, let $M_0 := \max_{\mathbb{T}} m_0$ and, for $\delta > 0$, define

$$v^\delta(x,t) = u^\epsilon(x,t) + H(0, M_0 + \delta)(T - t).$$

Since $D^2 v^\delta = D^2 u^\epsilon$, we have that v^δ also solves the elliptic equation (Q) in Q_T. Therefore, the maximum of v^δ, must occur at $t = 0$ or $t = T$. If the maximum occurred at $t = 0$, then at that point

$$u_t^\epsilon - H(0, M_0{+}\delta) = v_t^\delta \leq 0,\ v_x^\delta = u_x^\epsilon = 0,$$

and, hence,

$$0 \geq u_t^\epsilon - H(0, M_0 + \delta) = H(0, m_0) - H(0, M_0 + \delta),$$

which is a contradiction because $H_m < 0$. Therefore, for every $\delta > 0$, the maximum occurs at $t = T$, and, letting $\delta \to 0$, we see that the same is true for $\delta = 0$. The maximum value of $v(x,t) := u^\epsilon(x,t) + H(0, M_0)(T - t)$ equals the maximum of $u^\epsilon(x,T)$, since $v(x,T) = u^\epsilon(x,T)$. Letting $x_0 \in \mathbb{T}$ be a point at which this maximum occurs, it follows that $v_t(x_0, T) \geq 0$, and therefore

$$H(0, m^\epsilon(x_0, T)) \geq H(0, M_0),$$

which implies that

$$m^\epsilon(x_0, T) \leq M_0.$$

But, since

$$\epsilon u^\epsilon(x,T) = m^\epsilon(x,T) - m_T(x),$$

we obtain, for each $x \in \mathbb{T}$,

$$\epsilon u^\epsilon(x,T) \leq \epsilon u^\epsilon(x_0,T) = (m^\epsilon(x_0,T) - m_T(x_0)) \leq (M_0 - m_T(x_0)),$$

and, consequently,

$$m^\epsilon(x,T) = \epsilon u^\epsilon(x,T) + m_T(x) \leq M_0 + m_T(x) - m_T(x_0) \leq M_0 + \mathrm{osc}_{\mathbb{T}}(m_T).$$

We have thus shown that the bound on the oscillation of u^ϵ does not depend on ϵ. Furthermore, since

$$\epsilon u^\epsilon(x,T) = m^\epsilon(x,T) - m_T(x),$$

and $m^\epsilon(T,\cdot), m_T(\cdot)$ are both probability densities, we have $\int_{\mathbb{T}} u^\epsilon(\cdot,T) = 0$, so there must exist some $x^\epsilon \in \mathbb{T}$ such that

$$u^\epsilon(x^\epsilon, T) = 0.$$

This implies that, for any $(x,t) \in \overline{Q}_T$,

$$-\mathrm{osc}_{\overline{Q}_T}(u^\epsilon) \leq u^\epsilon(x,t) - u^\epsilon(x^\epsilon,T) \leq \mathrm{osc}_{\overline{Q}_T}(u^\epsilon),$$

which shows (3.4.20). To prove (3.4.21), we require C to be large enough to satisfy $\frac{1}{C}\|u^\epsilon\|_\infty < \frac{1}{2}c_1$. Then for all $\epsilon < \frac{1}{C}$, we have

$$m^\epsilon(x,T) = m_T(x) + \epsilon u^\epsilon(x,T) \geq m_T(x) - \frac{1}{2}c_1 \geq \frac{1}{2}c_1.$$

The upper bound for $m^\epsilon(x,T)$ is obtained similarly. We now conclude by Corollary 3.3.2, since the maxima and minima of m^ϵ both occur at $t = 0, t = T$. Finally, (3.4.22) follows immediately from the terminal condition in (MFG$_\epsilon$) and (3.4.20). □

While the usefulness of (MFG$_\epsilon$) will mainly be as a tool to obtain existence for (MFGP), it can also be used to provide an interesting counterexample. Indeed, one should note that (MFG$_\epsilon$) is not itself a planning problem, but rather a special case of a standard MFG system, which would fit in the framework of (MFG) if the terminal cost function g were allowed to depend on x, as was the case in the previous chapter. The following proposition illustrates the fact that, when such assumptions do not hold, the solution may fail to exist.

Theorem 3.4.5. *Assume that $H(0,0) < \infty$, and that the condition $m_T > 0$ in* (M1) *does not hold, and $m_T(x_0) < 0$ for some $x_0 \in \mathbb{T}$. Then there exists $C > 0$ such that, for all $0 < \epsilon < \frac{1}{C}$, there exists no classical solution to (MFG$_\epsilon$).*

Proof. We assume, by contradiction, that there exists a decreasing sequence $\epsilon_n > 0$, with $\lim_{n\to\infty} \epsilon_n = 0$, such that, for each positive integer n, there exists a solution (u^n, m^n) to system (MFG$_{\epsilon_n}$). Since $H(0,0) < \infty$, the proof of Lemma 3.4.4 shows that, for some constant $C > 0$ independent of $n \in \mathbb{N}$, we have $\|u^n\|_\infty \leq C$. However, this implies that

$$\|m^n(T,\cdot) - m_T(\cdot)\|_\infty \leq C\epsilon_n,$$

while $m^n(x_0,T) \geq 0 > m_T(x_0)$, which is a contradiction. □

We finish our estimates for the ϵ–penalized problem with an analogue of Proposition 3.4.3.

Lemma 3.4.6. *For $\epsilon > 0$, let $(u^\epsilon, m^\epsilon) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$ be a classical solution to system (MFG$_\epsilon$), and assume that $H(0,0) < \infty$. Let c_1 and C_1 be as in Corollary 3.3.2. There exists a constant $C > 0$, independent of ϵ, such that, for $\epsilon < \frac{1}{C}$,*

$$\|Du^\epsilon\|_\infty \leq C.$$

Proof. We first observe that, by Corollary 3.3.2 and Lemma 3.4.4, $\|m^\epsilon\|_{\overline{Q}_T}$ and $\|(m^\epsilon)^{-1}\|_{\overline{Q}_T}$ are bounded a priori in terms of C_1 and c_1^{-1}. The proof of Proposition 3.4.3 may thus be repeated here, with Lemma 3.4.4 replacing the use of Proposition 3.4.2, with one exception.

Namely, the term $-H_m u_x^\epsilon m_x^\epsilon$ in (3.4.7) should be estimated as

$$-H_m u_x^\epsilon m_x^\epsilon = -\epsilon H_m (u_x^\epsilon)^2 - H_m u^\epsilon (m_T)_x \geq -H_m u^\epsilon (m_T)_x,$$

which, in view of (3.4.8), yields the gradient bound in the case $t_0 = T$. The rest of the argument follows unchanged. $\square$

3.5 Existence of classical solutions

In the previous sections, a priori $L^\infty-$bounds were obtained for u, Du, m, and m^{-1}. This is already sufficient to obtain classical solutions to (MFG), following the arguments of Section 2.4. The existence of solutions to (MFGP), on the other hand, is a more delicate issue, because the Neumann type boundary condition that appears in the linearization makes the latter non–invertible. Namely, the linearization of (Q) is

$$\begin{cases} L_u(w) = f & \text{in } Q_T, \\ (-1, H_p(u_x, m)) \cdot Dw = g_1(x) & \text{at } t = 0, \\ (1, -H_p(u_x, m)) \cdot Dw = g_2(x) & \text{at } t = T, \end{cases}$$

which is an oblique boundary value problem that is only solvable for certain functions f, g_1, g_2 satisfying a compatibility condition that itself depends on u. This failure of invertibility precludes the direct use of the implicit function theorem and thus of the method of continuity, which means a different approach is needed. Indeed, we will obtain the solution as the limit as $\epsilon \to 0$ of the solution to the ϵ–penalized problem (MFG$_\epsilon$). We begin by noting, in the following lemma, that for ϵ small enough, the solutions to (MFG$_\epsilon$) are a priori uniformly bounded in $C^{1,\beta}(\overline{Q}_T)$, for some $0 < \beta < 1$, and that the system thus has a classical solution.

Lemma 3.5.1. *Let C be as in Lemma 3.4.4. For all $0 < \epsilon < \frac{1}{C}$, (MFG$_\epsilon$) has a unique smooth solution $(u^\epsilon, m^\epsilon) \in C^{3,\alpha}(\overline{Q_T}) \times C^{2,\alpha}(\overline{Q_T})$. Moreover, there exist constants $K > 0$, $0 < \beta < 1$, independent of ϵ, such that*

$$\|u^\epsilon\|_{C^{1,\beta}} \leq K. \tag{3.5.1}$$

Proof. The a priori C^1-bounds on u^ϵ, as well as $L^\infty-$bounds on m^ϵ and $(m^\epsilon)^{-1}$ (and thus on the ellipticity constants of the system), were all established in Lemmas 3.4.4 and 3.4.6. The Hölder estimate for the gradient then follows in the same way as in Lemma 2.4.1. Indeed, it suffices to verify that, for $(x, t, z, p, s) \in \mathbb{T} \times \{0, T\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the boundary condition

$$B^\epsilon(x, 0, z, p, s) = -s + H(p, m_0(x)), \;\; B^\epsilon(x, T, z, p, s) = s - H(p, \epsilon z + m_T(x)),$$

is oblique. For this purpose, we let $\nu(x, t)$ denote the outward unit normal vector at $(x, t) \in \partial Q_T$. Then we have

$$D_{(p,s)}B^\epsilon(x, 0, z, p, s) \cdot \nu(x, 0) = -B^\epsilon_s(x, 0, z, p, s) = 1 > 0,$$

$$D_{(p,s)}B^\epsilon(x, T, z, p, s) \cdot \nu(x, T) = -B^\epsilon_s(x, T, z, p, s) = 1 > 0$$

and thus the a priori estimate (3.5.1) follows. The proof of existence is then the same as in that of Theorem 2.1.1, through the method of continuity. □

We now have enough information on the ϵ–penalized problem to prove our first theorem.

Proof of Theorem 3.1.1. We initially assume that $m_0, m_T \in C^\infty(\mathbb{T})$. The proof of part (ii), corresponding to (MFG), is identical to the one carried out in Section 2.4. We simply note that the condition $\lim\limits_{m \to 0^+} H(p, m) = +\infty$ in that proof was only used to guarantee the existence of a positive lower bound for the density, which in turn makes the equation (Q) uniformly elliptic. In our case, the lower bound is a consequence of Corollary 3.3.2 and Proposition 3.4.1.

Now, for the case of (MFGP), we remark first that uniqueness of u, up to a constant, follows by the standard Lasry-Lions monotonicity method. To establish existence, we consider first the approximate system (MFG_ϵ), under the assumption $H(0,0) < \infty$. We assume that $\epsilon > 0$ is small enough for Lemma 3.5.1 to guarantee the existence of solutions (u^ϵ, m^ϵ). Letting $0 < \beta < 1$ be as in Lemma 3.5.1, we also have (3.5.1), for some constant $K > 0$ independent of ϵ. We infer that there exist a subsequence $\{u_n\}_n \subset \{u^\epsilon\}_\epsilon$, and $u \in C^{1,\alpha}(\overline{Q_T})$, such that $u_n \to u$ uniformly. Furthermore, in view of Lemma 3.4.4, there exists $C > 0$, independent of ϵ, such that

$$\frac{1}{C} \leq m^\epsilon(x,t) \leq C \text{ for all } (x,t) \in \overline{Q_T}.$$

We let (A, B) and (A_n, B_n), be the quasilinear operators and boundary conditions corresponding, respectively, to u and u_n. Then one has

$$(A_n, B_n) \to (A, B) \text{ locally uniformly,}$$

$$D_q B_n \cdot \nu = 1.$$

Hence, by Fiorenza's convergence theorem for elliptic equations with oblique boundary conditions (see Theorem 2.2.5), we obtain $u_n \to u$ in $C^{2,\alpha}(\overline{Q}_T)$, and u solves (Q), with the boundary condition corresponding to (MFGP). The $C^{3,\alpha}$ regularity (and, in fact, uniform convergence in $C^{3,\alpha}$) then follows readily from the standard Schauder estimates for linear oblique problems, as in Theorem 2.1.1.

The last step will be to remove the assumption that $m_0 \in C^\infty(\mathbb{T})$ and, for (MFGP), the assumptions that $m_T \in C^\infty(\mathbb{T})$ and $H(0,0) < \infty$. We will explain the argument for (MFGP), with the treatment of (MFG) being completely analogous. Consider, for $\delta > 0$, the modified Hamiltonians $H^\delta(p,m) := H(p, m+\delta)$, which satisfy (H) and (E), uniformly in δ, as well as $H^\delta(0,0) < \infty$, and a sequence of C^∞ densities (m_0^δ, m_T^δ), uniformly bounded in $C^{2,\alpha}$ and bounded away from 0, converging uniformly to (m_0, m_T). Let (u^δ, m^δ) be the corresponding solutions to

$$\begin{cases} -u_t^\delta + H^\delta(u_x^\delta, m^\delta) = 0 & \text{in } Q_T, \\ \int_0^T \int_{\mathbb{T}} u^\delta = 0, & \\ m_t^\delta - (m^\delta H_p^\delta(u_x^\delta, m^\delta))_x = 0 & \text{in } Q_T, \\ m^\delta(\cdot, 0) = m_0^\delta, \ m^\delta(\cdot, T) = m_T^\delta & \text{on } \mathbb{T}. \end{cases} \tag{3.5.2}$$

Propositions 3.4.3 and 3.4.2, and Corollary 3.3.2, yield uniform C^1-bounds on u^δ, and thus, as in the proof of Lemma 3.5.1, uniform $C^{1,\beta}$ bounds for some $0 < \beta < 1$. We may thus conclude by letting $\delta \to 0$ and applying Fiorenza's convergence result as above. $\square$

3.6 Regularity of weak solutions

We now study the existence and regularity of solutions to (MFG) and (MFGP) under the weaker assumption that, for some $\kappa > 0$

$$\int_{\mathbb{T}} \frac{1}{m_0^\kappa(x)} dx < \infty, \ \int_{\mathbb{T}} \frac{1}{m_T^\kappa(x)} dx < \infty.$$

We note that, in particular, the above conditions allow for the densities to vanish at a set of measure zero. This, in general, creates significant issues, because (Q) is no longer uniformly elliptic. The key estimate that will allow us to prove smoothness in this setting is an interior lower bound on the density which depends only on t^{-1}, $\|m_0^{-\kappa}\|_1$ (and $(T-t)^{-1}, \|m_T^{-\kappa}\|_1$, in the case of (MFGP)). Indeed, this yields uniform ellipticity of (Q) away from $t = 0$ and $t = T$.

We begin by giving the standard definition of a weak solution (see, for instance, [8, 44, 50]).

Definition 3.6.1 (Definition of weak solution)**.** A pair $(u, m) \in \mathrm{BV}(Q_T) \times L^\infty_+(Q_T)$ is called a weak solution to (MFG) (respectively (MFGP)) if the following conditions hold:

(i) $u_x \in L^2(Q_T), u \in L^\infty(Q_T)$, $m \in C^0([0,T]; H^{-1}(\mathbb{T}))$.

(ii) u satisfies the HJ inequality

$$-u_t + H(u_x, m) \leq 0 \quad \text{in } Q_T,$$

in the distributional sense.

(iii) m satisfies the continuity equation

$$m_t - (mH_p(u_x, m))_x = 0 \text{ in } Q_T, \tag{3.6.1}$$

in the distributional sense.

(iv) We have $m(\cdot, T) \in L^\infty(\mathbb{T})$. Moreover, $m(\cdot, 0) = m_0$ in $H^{-1}(\mathbb{T})$ and $u(\cdot, T) = g(m(\cdot, T))$ in the sense of traces (respectively, $m(\cdot, T) = m_T$ in $H^{-1}(\mathbb{T})$).

(v) The following identity holds:

$$\int\int_{Q_T} m(x,t)(H(u_x, m) - H_p(u_x, m)u_x)dxdt = \int_{\mathbb{T}} (m(x,T)u(x,T) - m_0(x)u(x,0))dx.$$

The following lemma will be needed to show that, for solutions to (MFG), our interior regularity results may be extended up to time $t = T$.

Lemma 3.6.2. *Let* (u, m) *be a smooth solution to* (MFG) *under the assumptions of Theorem 3.1.1 and assume that* (3.1.2) *holds. Then, for every convex function* $h \in C^2(0, \infty)$, *the map*

$$t \to \int_{\mathbb{T}} h(m(x,t))dx$$

is decreasing. Moreover, there exists a constant $C = C(C_0, \|g'\|^{-(\gamma-1)}_{L^\infty([\min m_0, \max m_0])})$ *such that*

$$\frac{d}{dt}\int_{\mathbb{T}} h(m(x,T))dx + \frac{1}{C}\int_{\mathbb{T}} h''(m(x,T))|m_x(x,T)|^\gamma \leq 0.$$

Proof. In view of Proposition 3.3.1, we have that

$$\frac{d^2}{dt^2}\int_{\mathbb{T}} h(m(x,t))dx \geq 0,$$

and, thus, the function

$$d(t) := \frac{d}{dt}\int_{\mathbb{T}} h(m(x,t))dx$$

is increasing. We then infer that the monotonicity will follow if we show that

$$d(T) \leq 0.$$

Since $u(\cdot, T) = g(m(\cdot, T))$, and m satisfies the continuity equation, we have

$$\begin{aligned} d(T) = \int_{\mathbb{T}} h'(m(x,T))m_t(x,T)dx = \int_{\mathbb{T}} h'(m)(mH_p(u_x, m))_x dx \\ = -\int_{\mathbb{T}} h''(m)m_x H_p(m_x g'(m), m). \end{aligned}$$

Now, as a result of (3.1.2) and (H1),

$$H_p(m_x g'(m), m)(m_x g'(m)) \geq \frac{1}{C}|m_x g'(m)|^\gamma,$$

and, therefore,

$$d(T) \leq -\frac{1}{C} \int_{\mathbb{T}} h''(m)|m_x|^\gamma.$$

□

We are now ready to obtain the interior lower bounds on m. Our method of proof relies on the displacement convexity formula (3.3.2), and uses similar techniques to [50, Prop. 5.2].

Theorem 3.6.3. *Let* (u, m) *be a smooth solution to* (MFG) *or* (MFGP)*, under the same assumptions as in Theorem 3.1.1. Assume, furthermore, that* (HW) *holds and, in the case of* (MFG)*, assume that* (3.1.2) *holds. Let*

$$\beta = \frac{2}{\kappa - s - 1},$$

and let $\delta > 0$*. Then, there exist a constant* $C = C(C_0 \| m_0^{-\kappa} \|_{L^1}, \| m_T^{-\kappa} \|_{L^1}, \delta^{-1})$ *such that*

$$m(x,t) \geq \frac{1}{C} \left(\frac{1}{t^{\beta+\delta}} + \frac{1}{(T-t)^{\beta+\delta}} \right)^{-1}. \tag{3.6.2}$$

Furthermore, in the case of (MFG)*, one has*

$$m(x,t) \geq \frac{1}{C} t^{\beta+\delta}. \tag{3.6.3}$$

Proof. Using the displacement convexity formula (3.3.2) for $h(m) = \frac{1}{m^\kappa}$, we have, for each $t \in [0, T]$,

$$\int_{\mathbb{T}} \frac{1}{m^\kappa(x,t)} dx \leq \max \left(\int_{\mathbb{T}} \frac{1}{m_0^\kappa(x)} dx, \int_{\mathbb{T}} \frac{1}{m^\kappa(x,T)} dx \right). \tag{3.6.4}$$

Combined with Lemma 3.6.2 (for the case of (MFG) where $m(\cdot, T)$ is not prescribed), this yields

$$\sup_{t \in [0,T]} \| m^{-\kappa}(t) \|_1 \leq C. \tag{3.6.5}$$

Next, for any $p > 1$, we define the function

$$\phi(t) := \int_{\mathbb{T}} m^{-p\kappa}(t)dx.$$

Using Proposition 3.3.1 with $h(m) = m^{-p\kappa}$, as a result of (E), we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\int_{\mathbb{T}} \frac{m^{-p\kappa}(t)}{p\kappa(p\kappa+1)}dx &\geq -\frac{1}{C}\int_{\mathbb{T}} m^{-p\kappa-1}H_{pp}H_m(m_x)^2dx \geq \int_{\mathbb{T}} \frac{1}{C}m^{-p\kappa-1+s}(m_x)^2dx \\ &\geq \frac{1}{C\frac{-p\kappa+s+1}{2})^2}\int_{\mathbb{T}} \left(\left(m^{\frac{-p\kappa+s+1}{2}}\right)_x\right)^2 dx.\end{aligned}$$

As a result, letting

$$\begin{aligned} C_p &:= \frac{C(p\kappa - s - 1)^2}{4p\kappa(p\kappa+1)}, \\ \lambda &:= \frac{-p\kappa+s+1}{2}, \end{aligned} \tag{3.6.6}$$

we have shown that

$$C_p\phi''(t) \geq \int_{\mathbb{T}} (m^\lambda)_x^2 dx. \tag{3.6.7}$$

From (W), and the fact that $p > 1$, we see that $\lambda < 0$. For each $t \in [0, T]$, since $m(\cdot, t)$ is a probability measure, there exists a point x_0^t such that $m(x_0^t, t) = 1$. By the fundamental theorem of calculus,

$$\left\| m^\lambda(t) - 1 \right\|_\infty^2 = \left\| m^\lambda(t) - m(x_0^t, t)^\lambda \right\|_\infty^2 \leq C\int_{\mathbb{T}} (m^\lambda)_x^2 dx, \tag{3.6.8}$$

and therefore

$$\left\| \frac{1}{m} \right\|_\infty^{2|\lambda|} \leq C\Big(\int_{\mathbb{T}} (m^\lambda)_x^2 dx + 1\Big). \tag{3.6.9}$$

Now, using (3.6.5), we obtain

$$\phi = \int_{\mathbb{T}} \frac{1}{m^{\kappa p}} \leq \int_{\mathbb{T}} \frac{1}{m^\kappa}\left\| \frac{1}{m} \right\|_\infty^{\kappa(p-1)} \leq C\left\| \frac{1}{m} \right\|_\infty^{\kappa(p-1)},$$

and, consequently,

$$C^{-r}\phi^r \le \left\|\frac{1}{m}\right\|_\infty^{2|\lambda|}, \tag{3.6.10}$$

where $r := \frac{2|\lambda|}{\kappa(p-1)}$. From condition (W), we see that $r > 1$. Combining (3.6.7), (3.6.9), and (3.6.10), we obtain

$$C_p\Big(\phi''(t) + 1\Big) - C^{-r}\phi(t)^r \ge 0,$$

that is, for some constant $C = C(p)$,

$$-\phi''(t) + \frac{1}{C}\phi^r \le C. \tag{3.6.11}$$

A straightforward computation then shows that the functions

$$\psi_1(t) = A_p t^{-p\kappa\beta} + K_p,$$

$$\psi_2(t) = A_p(T-t)^{-p\kappa\beta} + K_p,$$

$$\psi(t) = \psi_1(t) + \psi_2(t),$$

are supersolutions of (3.6.11) for large enough A_p, K_p. Therefore, we have

$$\int_{\mathbb{T}} m^{-p\kappa}(t) \le A_p(t^{-p\kappa\beta} + (T-t)^{-p\kappa\beta}) + 2K_p. \tag{3.6.12}$$

Now, going back to (3.6.7) and (3.6.9), we may write

$$\left\|\frac{1}{m}\right\|_\infty^{2|\lambda|}(t) \le C\left(\frac{d^2}{dt^2}\int_{\mathbb{T}} m^{-p\kappa} + 1\right). \tag{3.6.13}$$

In view of (3.3.2), for $q > 0$, the map

$$t \mapsto \int_{\mathbb{T}} m^{-q}(t) \tag{3.6.14}$$

is convex in $[0,T]$. Thus, fixing $t_0 \in (0, \frac{T}{2}]$, we infer that, for each $t \in [t_0, T-t_0]$,

$$\left(\int_{\mathbb{T}} m^{-2|\lambda|q}(t)\right)^{\frac{1}{q}} \leq \frac{2}{t_0} \max\left(\int_{\frac{t_0}{2}}^{t_0}\left(\int_{\mathbb{T}} m^{-2|\lambda|q}\right)^{\frac{1}{q}}, \int_{T-t_0}^{T-\frac{t_0}{2}}\left(\int_{\mathbb{T}} m^{-2|\lambda|q}\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{2}{t_0}\int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}}\left(\int_{\mathbb{T}} m^{-2|\lambda|q}\right)^{\frac{1}{q}}.$$

Letting $q \to \infty$, we obtain

$$\left\|m^{-1}\right\|_{L^{\infty}(\mathbb{T}\times[t_0,T-t_0])}^{2|\lambda|} \leq \frac{2}{t_0}\int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}}\left\|m^{-1}(t)\right\|_{\infty}^{2|\lambda|} dt. \tag{3.6.15}$$

Now, letting $\zeta \in C^{\infty}(Q_T)$ be a test function, supported in $[\frac{t_0}{4}, T-\frac{t_0}{4}]$, such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $[\frac{t_0}{2}, T-\frac{t_0}{2}]$, and $\int_0^T |\zeta''(t)|dt \leq \frac{C}{t_0}$, we see that (3.6.15) implies

$$\left\|m^{-1}\right\|_{L^{\infty}(\mathbb{T}\times[t_0,T-t_0])}^{2|\lambda|} \leq \frac{2}{t_0}\int_0^T\left\|m^{-1}\right\|_{\infty}^{2|\lambda|}(t)\zeta(t)dt. \tag{3.6.16}$$

Hence, recalling (3.6.13) and integrating by parts twice, we infer from (3.6.12) that

$$\left\|m^{-1}\right\|_{L^{\infty}(\mathbb{T}\times[t_0,T-t_0])}^{2|\lambda|} \leq \frac{C}{t_0}\left(\int_0^T\int_{\mathbb{T}}(m^{-p\kappa}\zeta'') + CT\right) \leq C\left(\frac{1}{t_0^{2+p\kappa\beta}} + \frac{1}{t_0}\right),$$

which yields

$$\left\|m^{-1}\right\|_{L^{\infty}(\mathbb{T}\times[t_0,T-t_0])} \leq C\left(\frac{1}{t_0^{\frac{2+p\kappa\beta}{2|\lambda|}}} + \frac{1}{t_0^{\frac{1}{2|\lambda|}}}\right).$$

Now, recalling (3.6.6), we see that

$$\lim_{p\to\infty}\frac{1}{2|\lambda|} = 0 \text{ and } \lim_{p\to\infty}\frac{2+p\kappa\beta}{2|\lambda|} = \beta. \tag{3.6.17}$$

Thus, we may fix p chosen large enough that $\frac{2+\kappa\beta}{2|\lambda|} < \beta + \delta$, and, as a result of (3.6.17),

$$\left\|m^{-1}\right\|_{L^\infty(\mathbb{T}\times[t_0,T-t_0])} \leq C\frac{1}{t_0^{\beta+\delta}}.$$

This implies (3.6.2). Now, for the case of (MFG), we simply observe that, from Lemma 3.6.2, the map (3.6.14) is non-increasing on $[0,T]$, and, thus, (3.6.15) may be strengthened to

$$\left\|m^{-1}\right\|^{2|\lambda|}_{L^\infty(\mathbb{T}\times[t_0,T])} \leq \frac{2}{t_0}\int_{\frac{t_0}{2}}^{T} \left\|m^{-1}\right\|^{2|\lambda|}_{\infty}(t)dt.$$

□

The following lemma is a basic computation exploiting (E1), and will be used in the proof of Theorem 3.1.2 to estimate the terms arising from the Lasry-Lions monotonicity method.

Lemma 3.6.4. *There exists a constant* $C = C(C_0) > 0$ *such that, given* $-\infty < p_0 < p_1 < \infty$ *and* $0 < m_0 < m_1 < \infty$*, we have*

$$\begin{aligned}\big(m_1H_p(p_1,m_1) - m_0H_p(p_0,m_0)\big)(p_1-p_0) - (H(p_1,m_1) - H(p_0,m_0))(m_1-m_0)\\ \geq \frac{m_1+m_0}{C}(p_1-p_0)^2 + \frac{k}{C}(m_1-m_0)^2,\end{aligned} \quad (3.6.18)$$

where $k = \min_{[p_0,p_1]\times[m_0,m_1]}(-H_m(p,m))$*. Moreover, if* H *satisfies* (HW)*, then*

$$\begin{aligned}\big(m_1H_p(p_1,m_1) - m_0H_p(p_0,m_0)\big)(p_1-p_0) - (H(p_1,m_1) - H(p_0,m_0))(m_1-m_0)\\ \geq \frac{m_1+m_0}{C}(p_1-p_0)^2 + \frac{1}{C(s+1)}(m_1^{s+1} - m_0^{s+1})(m_1-m_0).\end{aligned} \quad (3.6.19)$$

Proof. Following the technique carried out in [40], for $z \in [0,1]$, we define

$$\Delta p = p_1 - p_0 \ \Delta m = m_1 - m_0,\ p_z = p_0 + z\Delta p,\ m_s = m_0 + z\Delta m.$$

We then let

$$\phi(z) = (m_z H_p(p_z, m_z) - m_0 H_p(p_0, m_0))\Delta p - (H(p_z, m_z) - H(p_0, m_0))\Delta m,$$

and differentiation yields

$$\phi'(z) = m_z H_{pp}(\Delta p)^2 + m_z H_{mp}\Delta m \Delta p - H_m(\Delta m)^2.$$

Now, in view of (E1), we have, for some constant $C > 0$,

$$-H_m \geq \frac{1}{4H_{pp}} m_z H_{mp}^2 (1 + \frac{1}{C}) - \frac{1}{C} H_m.$$

Therefore,

$$\phi'(z) \geq m_z \left(\frac{1}{\sqrt{1+\frac{1}{C}}} \sqrt{H_{pp}} \Delta p + \frac{\sqrt{1+\frac{1}{C}}}{2\sqrt{H_{pp}}} H_{mp} \Delta m \right)^2 + m_z H_{pp}(\Delta p)^2 (1 - \frac{1}{1+\frac{1}{C}}) - \frac{1}{C} H_m (\Delta m)^2. \quad (3.6.20)$$

If (W) holds, then, up to increasing the constant $C > 0$, as well as using (H1) and (HW), we obtain

$$\phi'(z) \geq \frac{1}{C}(m_z(\Delta p)^2 + m_z^s(\Delta m)^2),$$

and integrating over $[0, 1]$ then yields (3.6.19). The proof of (3.6.18) follows from (3.6.20) in the same way. □

Before proving Theorem 3.1.2, we remind the reader that assumption (M) will not be in place, and will be instead replaced by (W).

Proof of Theorem 3.1.2. For $\epsilon \in (0, 1)$, let m_0^ϵ, m_T^ϵ be smooth, positive densities such that, for $\theta \in \{0, T\}$,

$$m_\theta^\epsilon \to m_\theta \text{ a.e. in } \mathbb{T}, \ \|m_\theta^\epsilon\|_\infty \le C \text{ and } \|(m_\theta^\epsilon)^{-\kappa}\|_1 \le C,$$

where $C > 0$ is a constant independent of ϵ. Let $(u^{\epsilon,1}, m^{\epsilon,1})$ be a smooth solution to (MFGP) obtained from taking m_0^ϵ and m_T^ϵ, respectively, as the initial and terminal densities. Similarly, let $(u^{\epsilon,2}, m^{\epsilon,2})$ be the smooth solution to (MFG) corresponding to the initial density m_0^ϵ. The existence and regularity of such solutions is guaranteed by Theorem 3.1.1. We may further choose the $u^{\epsilon,1}$ to be normalized so that $\int_{\mathbb{T}} u^{\epsilon,1}(T) = 0$.

As in the proof of Proposition 3.6.3, we obtain, for some $C > 0$ independent of ϵ and for $i \in \{1, 2\}$,

$$\|(m^{\epsilon,i})^{-\kappa}\|_1 \le C. \tag{3.6.21}$$

On the other hand, Corollary 3.3.2 and Proposition 3.4.1 yield

$$\|m^{\epsilon,i}\|_\infty \le C, \tag{3.6.22}$$

and (3.6.22), (HW) and Proposition 3.6.3 imply that

$$\int_0^T |H(0, \min_{\mathbb{T}} m^{\epsilon,i}(s)| ds \le C. \tag{3.6.23}$$

Thus, as a result of (GW), Proposition 3.4.1, and Proposition 3.4.2,

$$\|u^{\epsilon,i}\|_\infty \le C. \tag{3.6.24}$$

We will first observe that, up to a subsequence, there is convergence to a weak solution. Indeed, given $0 < \epsilon, \epsilon' < 1$, applying the Lasry-Lions monotonicity method to the corresponding systems yields, for $i \in \{1, 2\}$,

$$\int_{\mathbb{T}}(u^{\epsilon,i}(T)-u^{\epsilon',i}(T))(m^{\epsilon,i}(T)-m^{\epsilon',i}(T))-\int_{\mathbb{T}}(u^{\epsilon,i}(0)-u^{\epsilon',i}(0))(m^{\epsilon,i}(0)-m^{\epsilon',i}(0))$$
$$+\int\int_{Q_T}\left(m^{\epsilon,i}H_p(u_x^{\epsilon,i},m^{\epsilon,i})-m^{\epsilon',i}H_p(u_x^{\epsilon',i},m^{\epsilon',i})\right)(u_x^{\epsilon,i}-u_x^{\epsilon',i})$$
$$-\left(H(u_x^{\epsilon,i},m^{\epsilon,i})-H(u_x^{\epsilon',i},m^{\epsilon',i})\right)(m^{\epsilon,i}-m^{\epsilon',i})=0. \quad (3.6.25)$$

Lemma 3.6.4 therefore yields

$$\int_{\mathbb{T}}(u^{\epsilon,i}(T)-u^{\epsilon',i}(T))(m^{\epsilon,i}(T)-m^{\epsilon',i}(T))-\int_{\mathbb{T}}(u^{\epsilon,i}(0)-u^{\epsilon',i}(0))(m^{\epsilon,i}(0)-m^{\epsilon',i}(0))$$
$$+\int\int_{Q_T}\left(\frac{m^{\epsilon,i}+m^{\epsilon',i}}{C}(u_x^{\epsilon,i}-u_x^{\epsilon',i})^2+\frac{1}{C(s+1)}((m^{\epsilon,i})^{s+1}-(m^{\epsilon',i})^{s+1})(m^{\epsilon,i}-m^{\epsilon',i})\right)\leq 0.$$
$$(3.6.26)$$

Proceeding as in [44, Thm. 1.2], it readily follows that, for $i\in\{1,2\}$, as $\epsilon\to 0$, $(u^{\epsilon,i},m^{\epsilon,i})$ converges to a weak solution (u^i,m^i).

It remains to show the interior regularity. For $\delta>0$, we define

$$I_{1,\delta}=[\delta,T-\delta],\ I_{2,\delta}=[\delta,T].$$

By Proposition 3.6.3, there exists $C=C(\delta^{-1})$ such that, for $t_i\in I_{i,\delta/4}$,

$$m^{\epsilon,i}(\cdot,t_i)\geq\frac{1}{C}. \quad (3.6.27)$$

We must first obtain a priori gradient bounds for $u^{\epsilon,i}$ on $I_{i,\delta/2}$. Setting

$$\phi_1(t)=(t-\delta/4)^{-2/(\gamma-1)}+(T-\delta/4-t)^{-2/(\gamma-1)}\ \phi_2(t)=(t-\delta/4)^{-2/(\gamma-1)},$$

we go through the steps of Proposition 3.4.3, replacing the function v by

$$v_i(x,t) = \frac{1}{2}(u_x^{\epsilon,i})^2 + \frac{1}{2}(\tilde{u}^{\epsilon,i})^2 - K\phi_i(t),$$

where $K > 0$, $\tilde{u}^{\epsilon,i}$ is defined as in Proposition 3.4.3. We consider the maximum point (x_0, t_0) of v_i in $\mathbb{T} \times I_{i,\delta/4}$. In the case of (MFGP), namely $i = 1$, this maximum must be attained in the interior of I_i, since ϕ_i is unbounded near the endpoints. When $i = 2$, the maximum may be attained at $t = T$, and the proof that $|p| \le C$ in this case follows through unchanged from Case 1 of Proposition 3.4.3. If the maximum is achieved at an interior time, the steps of Proposition 3.4.3 yield that if $v_i(x_0, t_0)$ is large enough, then

$$0 \le -|p|^{2\gamma} + |p|^{2\gamma-2} - K(-\phi_i'' + \frac{1}{C}K^\gamma \phi_i^\gamma - C\phi_i).$$

Similarly to Proposition 3.6.3, we see that, if K is chosen large enough, ϕ_i must be a supersolution to

$$-\phi_i'' + \frac{1}{C}K^\gamma \phi_i^\gamma - C\phi_i = 0,$$

which then implies $p \le C$, and thus $|u_x^{\epsilon,i}|$ is bounded on $I_{i,\delta/2}$. In view of (3.6.27) and (3.6.22), $|u_t^{\epsilon,i}| = |H(u_x^{\epsilon,i}, m^{\epsilon,i})|$ is also bounded on $I_{i,\delta/2}$. That is, we have

$$\|u^{\epsilon,i}\|_{C^1(\mathbb{T}\times I_{i,\delta/2})} \le C. \tag{3.6.28}$$

The interior $C^{1,\alpha}$-estimates for quasilinear elliptic equations (see [25, Chapter 13, Thm. 13.6]), followed by the interior Schauder estimates (see [31, Chapter, 2, (1.12)]) then yield, for some $C = C(\delta^{-1})$, and for $i \in \{1, 2\}$,

$$\|u^{\epsilon,i}\|_{C^{3+\alpha}(\mathbb{T}\times I_{1,\delta})} \le C. \tag{3.6.29}$$

For $i = 1$, by virtue of the Arzelà–Ascoli theorem, we may finish the proof by simply letting $\epsilon \to 0$. On the other hand, for $i = 2$ (that is, the case of (MFG)), we require estimates up to the terminal time T. We first observe that (3.6.27), (3.6.22), and (3.6.29) imply that $u^{\epsilon,2}$ solves, in $I_{2,\delta} \times \mathbb{T}$, a system of the form (MFG), where the initial density $m^{\epsilon,2}(\cdot,\delta)$ is bounded below by a positive constant, and bounded above in $C^{2,\alpha}(\mathbb{T})$. Moreover, as in Lemma 3.5.1, (3.6.28) implies that $u^{\epsilon,2}$ is bounded in $C^{1,\beta}$ for some $0 < \beta < 1$. We may now conclude through the same convergence argument as in the proof of Theorem 3.1.1. $\square$

Finally, by requiring some further regularity on the marginals, we establish additional Sobolev regularity for the weak solutions.

Theorem 3.6.5. *Let m_0, m_T satisfy $(m_0)_{xx}, (m_T)_{xx} \in L^1(\mathbb{T})$. Let (u,m) be a weak solution to* (MFG) *or* (MFGP) *under the assumptions of Theorem 3.1.2. Then, for some constant $C > 0$ we have:*

- *In the case of* (MFG)*,*

$$\int_{\mathbb{T}} g'(m(x,T))|m_x(x,T)|^2 + \int_0^T \int_{\mathbb{T}} mH_{pp}(u_{xx})^2 + m^s(m_x)^2 dxdt \le C, \tag{3.6.30}$$

  where $C = C(\|u\|_\infty, \|(m_0)_{xx}\|_1, C_0)$.

- *In the case of* (MFGP)*,*

$$\int_0^T \int_{\mathbb{T}} mH_{pp}(u_{xx})^2 + m^s(m_x)^2 dxdt \le C, \tag{3.6.31}$$

  where $C = C(\|u\|_\infty, \|(m_0)_{xx}\|_1, \|(m_T)_{xx}\|_1, C_0)$.

Proof. We will show the result in the case where (u,m) is smooth, since the general case follows by considering the approximations employed in the proof of Theorem 3.1.2. Differentiating with respect to x the (MFG) or (MFGP), we obtain

$$\begin{cases} -u_{xt} + H_p(u_x, m)u_{xx} + H_m(u_x, m)m_x = 0 \text{ in } Q_T, \\ m_{xt} - (m_x H_p(u_x, m) + mH_{pp}(u_x, m)u_{xx} + mH_{pm}(u_x, m)m_x)_x = 0 \text{ in } Q_T. \end{cases} \tag{3.6.32}$$

Testing against u_x in the equation for m_x above we obtain

$$\int_{\mathbb{T}} m_x(T)u_x(T) - \int_{\mathbb{T}} m_x(0)u_x(0) + \int_0^T \int_{\mathbb{T}} (m_x(-u_{xt} + u_{xx}H_p(u_x, m)) \\ + mu_{xx}^2 H_{pp}(u_x, m) + mu_{xx}H_{pm}(u_x, m)m_x) = 0, \tag{3.6.33}$$

and, therefore,

$$\int_{\mathbb{T}} m_x(T)u_x(T) + \int_0^T \int_{\mathbb{T}} mu_{xx}^2 H_{pp} - H_m(m_x)^2 \\ = -\int_{\mathbb{T}} u(0)(m_0)_{xx}dx - \int_0^T \int_{\mathbb{T}} mu_{xx}H_{pm}m_x. \tag{3.6.34}$$

Now, observe that

$$\left| \int_{\mathbb{T}} u(0)(m_0)_{xx}dx \right| \leq \|u\|_\infty \|(m_0)_{xx}\|_1, \qquad \left| \int_{\mathbb{T}} u(T)(m_T)_{xx}dx \right| \leq \|u\|_\infty \|(m_T)_{xx}\|_1. \tag{3.6.35}$$

Additionally, as a result of (E1), we infer that, for $\delta \in (0, 1)$,

$$\left| mu_{xx}H_{pm}m_x \right| \leq (1-\delta)mu_{xx}^2 H_{pp} + \frac{1}{4(1-\delta)H_{pp}} m|H_{pm}|^2 (m_x)^2 \\ \leq (1-\delta)mu_{xx}^2 H_{pp} - \frac{1}{(1-\delta)(1+\frac{1}{C_0})} H_m(m_x)^2. \tag{3.6.36}$$

We choose $\delta > 0$ small enough so that

$$\frac{1}{(1-\delta)(1+\frac{1}{C_0})} < 1.$$

Using (3.6.35) and (3.6.36) in (3.6.34), we obtain the following. In the case of (MFG), we have

$$\int_{\mathbb{T}} g'(m(T))(m_x(T))^2 dx + \int_0^T \int_{\mathbb{T}} m H_{pp}(u_{xx})^2 dx - H_m(m_x)^2 dx \leq C$$

while in the case of (MFGP), we have

$$\int_0^T \int_{\mathbb{T}} m H_{pp}(u_{xx})^2 dx - H_m(m_x)^2 dx \leq C.$$

We conclude by using the fact that H satisfies (HW). $\square$

3.7 Long time behavior and the infinite horizon problem

In this section, we will characterize the behavior, as $T \to \infty$, of solutions to (MFG) and (MFGP). First, we establish the turnpike property with an exponential rate of convergence. This property shows that, for large values of T, the players spend most of their time close to the equilibrium $m \equiv 1$.

Lemma 3.7.1. *Let* (u, m) *be a solution to* (MFG) *or* (MFGP)*, let* $T > 1$*, and set*

$$c_1 = \min(\min m_0, \min m_T),\ C_1 = \max(\max m_0, \max(m_T)).$$

Then there exist constants $C, \omega > 0$*, with*

$$C = C(C_0, C_1, c_1^{-1}, \|\overline{C}\|_{L^\infty([c_1,C_1])}, \|(m_0)_x\|_\infty, \|(m_T)_x\|_\infty, \|(g')^{-(\gamma-1)}\|_{L^\infty([\min m_0, \max m_0])})$$

and

$$\omega^{-1} = \omega^{-1}(C_0, c_1^{-1}, C_1, \|\overline{C}\|_{L^\infty([c_1,C_1])}),$$

such that

$$\|m(t)-1\|_{L^\infty(\mathbb{T})} + \|u_x(t)\|_{L^\infty(\mathbb{T})} \leq C(e^{-\omega t} + e^{-\omega(T-t)}), \ \ t \in [0,T]. \tag{3.7.1}$$

If (u,m) *solves* (MFG), *and* (3.1.2) *holds, we have*

$$\|m(t)-1\|_{L^\infty(\mathbb{T})} + \|u_x(t)\|_{L^\infty(\mathbb{T})} \leq Ce^{-\omega t}, \ \ t \in [0,T]. \tag{3.7.2}$$

Proof. As in previous arguments, we recall that the constant C may increase at each step. For each $k \in \mathbb{N}$, Proposition 3.3.1 yields

$$\frac{d^2}{dt^2}\int_{\mathbb{T}}(m-1)^{2k}dx \geq 0, \tag{3.7.3}$$

and, as a result of (L) and Corollary 3.3.2,

$$\frac{d^2}{dt^2}\int_{\mathbb{T}}(m-1)^2dx \geq \int_{\mathbb{T}} -2mH_mH_{pp}m_x^2dx \geq \frac{1}{C}\int_{\mathbb{T}}|(m-1)_x|^2\,dx.$$

Since $\int_{\mathbb{T}} m(\cdot,t) \equiv 1$, arguing in the same way as in (3.6.8), we obtain

$$\frac{d^2}{dt^2}\int_{\mathbb{T}}(m-1)^2dx \geq \frac{1}{C}\Big\|m-1\Big\|_\infty^2.$$

Therefore, setting

$$\phi(t) := \int_{\mathbb{T}}(m(t)-1)^2dx,$$

we have

$$-\phi'' + \frac{1}{C}\phi \leq 0. \tag{3.7.4}$$

Moreover, if (u, m) solves (MFG) and (3.1.2) holds, up to increasing the value of C, Lemma 3.6.2 implies that

$$\phi'(T) \leq -\frac{1}{\sqrt{C}}\phi(T). \tag{3.7.5}$$

We now fix the choice $\omega = \frac{1}{2\sqrt{C}}$ (the value of C may still increase in subsequent steps, but the value of ω will not). The comparison principle applied to (3.7.4) then implies that, for each $t \in [0, T]$,

$$\phi(t) \leq \phi(0)e^{-2\omega t} + \phi(T)e^{-2\omega(T-t)} \leq C(e^{-2\omega t} + e^{-2\omega(T-t)}). \tag{3.7.6}$$

Similarly, if (u, m) solves (MFG) and (3.1.2), then (3.7.4), coupled with the Robin boundary condition (3.7.5), readily implies that

$$\phi(t) \leq \phi(0)e^{-2\omega t} \leq Ce^{-2\omega t}. \tag{3.7.7}$$

By using the same convexity arguments as in (3.6.16), in view of (3.7.3), we have

$$\|m(t)-1\|_\infty^2 \leq C\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \|m(s)-1\|_\infty(s)^2 ds \leq C\int_{t-1}^{t+1}\int_{\mathbb{T}}(m-1)^2 = C\int_{t-1}^{t+1}\phi(s)ds. \tag{3.7.8}$$

We now turn our attention to estimating u_x. Fixing $t \in [1, T-1]$, as a result of (H1), Proposition 3.3.1, and Corollary 3.3.2, we obtain, for $s \in [t-1, t+1]$,

$$\frac{1}{C}\int_{\mathbb{T}} u_{xx}^2(s) \leq \frac{d^2}{ds^2}\int_{\mathbb{T}}(m(s)-1)^2.$$

Thus, testing against a bump function $\zeta \geq 0$, which is supported on $[t-1, t+1]$, and identically equals 1 on $[t-\frac{1}{2}, t+\frac{1}{2}]$, we get

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\int_{\mathbb{T}} u_{xx}^2 \leq C\int_{t-1}^{t+1}\int_{\mathbb{T}}(m-1)^2\zeta'' \leq C\int_{t-1}^{t+1}\phi(s)ds. \tag{3.7.9}$$

Differentiating (Q) with respect to x, one sees that $v = u_x$ solves a linear elliptic equation of the form

$$-\mathrm{Tr}(A(x,t)D^2v) + b(x,t) \cdot Dv = 0.$$

Thus, v satisfies the maximum and minimum principles on compact subsets of $\overline{Q}_T$. Applying this observation to $\mathbb{T} \times [t-s, t+s]$, for $s \in (0, \frac{1}{2})$, as well as the fact that, for every $t \in [0,T]$, $\{x \in \mathbb{T} : u_x(x,t) = 0\} \neq \emptyset$, we have

$$\mathrm{osc}_{\mathbb{T}} v(t) \leq \mathrm{osc}_{\mathbb{T}} v(t+s) + \mathrm{osc}_{\mathbb{T}} v(t-s) \leq \int_{\mathbb{T}} |u_{xx}(t+s)| + \int_{\mathbb{T}} |u_{xx}(t-s)|.$$

Integrating in s then yields

$$\mathrm{osc}_{\mathbb{T}} u_x(t) \leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \int_{\mathbb{T}} |u_{xx}|,$$

and, thus, as a result of (3.7.9) and the Cauchy-Schwarz inequality,

$$\|u_x(t)\|_{\infty}^2 \leq C \int_{t-1}^{t+1} \phi(s)ds. \tag{3.7.10}$$

Now, adding (3.7.8) and (3.7.10), followed by (3.7.6), we obtain (3.7.1) for $t \in [1, T-1]$. Similarly, when (u,m) solves (MFG) and (3.1.2) holds, (3.7.7) yields (3.7.2) for $t \in [1, T-1]$. We observe that, for $t \in [0,T] \backslash [1, T-1]$, the bounds on $\|m(t) - 1\|_{\infty}$ given by (3.7.1) and (3.7.2) hold trivially, up to increasing the value of C. Let us see that the same is true for the bounds on $\|u_x(t)\|_{\infty}$ on the interval $[0,1]$. Indeed, we may simply follow the proof of Proposition 3.4.3, applied to the MFG system on the domain $\mathbb{T} \times [0,1]$, with the only change being on Case 1 of that proof, that is, when the maximum value is attained at $t = 1$. For this case, we may simply use the fact that, as a result of (3.7.1) holding for $t = 1$, $|u_x(\cdot, 1)|$ is bounded. Thus, if we take $T = 1$ in Proposition 3.4.2, this yields a bound on $\|u_x\|_{\mathbb{T}\times[0,1]}$ that depends only on C_0, $\|m\|_{L^\infty(\overline{Q}_T)}$, $\|m^{-1}\|_{L^\infty(\overline{Q}_T)}$, $\|(m_0)_x\|_{\infty}$, and $\|\overline{C}\|_{L^\infty([\min m, \max m])}$. A similar argument may be followed on $\mathbb{T} \times [T-1, T]$, which completes the proof. $\square$

Having established the turnpike property, we now follow the program developed in [16] to study the long time behavior. In order to characterize the limit, as $T \to \infty$, of the functions $(u(t) - \lambda(T-t), m(t))$, we first show a uniqueness result for (MFGL).

Lemma 3.7.2. *Assume that (L) holds. Then, up to adding a constant to v, there exists at most one classical solution (v, μ) to* (MFGL) *satisfying* (3.1.4).

Proof. Assume that $(v^1, \mu^1), (v^2, \mu^2)$ are solutions to (MFGL) satisfying (3.1.4). Since $\mu^1 - 1, \mu^2 - 1 \in L^1(\mathbb{T} \times (0, \infty))$, there exists a sequence $T_k \to \infty$ such that

$$\lim_{k\to\infty} \int_{\mathbb{T}} \Big(|\mu^1(\cdot, T_k) - 1| + |\mu^2(\cdot, T_k) - 1| \Big) = 0.$$

Performing the standard Lasry-Lions computation for v^1, v^2 on Q_{T_k}, using Lemma 3.6.4, and noting that

$$\mu^i, (\mu^i)^{-1}, v^i_x, \in L^\infty(\mathbb{T} \times (0, \infty)), \quad i \in \{1, 2\},$$

we obtain

$$\begin{aligned} \frac{1}{C}\left(\int_0^{T_k} \int_{\mathbb{T}} |v^1_x - v^2_x|^2 + |\mu^1 - \mu^2|^2 \right) &\leq \int_{\mathbb{T}} -(v^1(T_k) - v^2(T_k))(\mu^1(T_k) - \mu^2(T_k)) \\ &= \int_{\mathbb{T}} -(v^1(T_k) - v^2(T_k))((\mu^1(T_k) - 1) - (\mu^2(T_k) - 1)). \end{aligned} \tag{3.7.11}$$

Now, since $v^1, v^2 \in L^\infty(\mathbb{T} \times (0, \infty))$, the right hand side converges to 0 as $k \to \infty$. Therefore,

$$\int_0^\infty \int_{\mathbb{T}} |v^1_x - v^2_x|^2 + |\mu^1 - \mu^2|^2 = 0.$$

This implies that $\mu^1 = \mu^2$ and $v^1_x = v^2_x$. From the HJ equations, $v^1_t = v^2_t$, which concludes the proof. □

In the following lemma, we obtain uniform estimates for the solution that are independent of T.

Lemma 3.7.3. *Let* (u^T, m^T) *be a solution to* (MFG) *or* (MFGP) *for* $T > 0$*, and let* $\omega > 0$ *be the constant from Lemma 3.7.1. Set* $v^T = u^T - \lambda(T-t)$*. Then there exists a constant* $C > 0$*, independent of* T*, such that:*

- *If* (3.1.2) *holds and* (u^T, m^T) *solves* (MFG)*, then*

$$|v^T(t) - g(1)| \leq Ce^{-\omega t} \text{ for all } t \in [0, T]. \tag{3.7.12}$$

- *If* (u^T, m^T) *solves* (MFGP)*, and*

$$\int_{\mathbb{T}} v^T\left(\frac{1}{2}T\right) dx = 0, \tag{3.7.13}$$

then we have

$$\|v^T\|_{L^\infty(Q_T)} \leq C \tag{3.7.14}$$

and

$$\|v^T(t)\|_\infty \leq Ce^{-\omega t} \text{ for all } t \in \left[0, \frac{T}{2}\right]. \tag{3.7.15}$$

Proof. First we note that in both (MFG) and (MFGP), as a result of Lemma 3.7.1, the function $v_x^T = u_x^T$ is bounded uniformly, independently of T, and, by Corollary 3.3.2, so are $m^T, (m^T)^{-1}$. Therefore, since H is smooth, and thus locally Lipschitz, we have, for some constant $C > 0$ independent of $T > 0$,

$$|v_t^T| \leq C(|v_x^T| + |m^T - 1|). \tag{3.7.16}$$

Assume first that (u^T, m^T) solves (MFG) and (3.1.2) holds. Integrating the HJ equation in $[t, T]$ and using (3.7.16) along with (3.7.2) in Lemma 3.7.1 we obtain

$$|v^T(t) - v^T(T)| \leq C\int_t^T e^{-\omega s} ds.$$

Furthermore, using the fact that

$$v^T(T) = u^T(T) = g(m^T(T)),$$

and

$$|m^T(T) - 1| \leq Ce^{-\omega T},$$

by increasing the constant C if necessary, we obtain

$$|v^T(t) - g(1)| \leq C(e^{-\omega T} + e^{-\omega t}) \leq 2Ce^{-\omega t},$$

which proves (3.7.12). Next, we assume that (u^T, m^T) solves (MFGP) and (3.7.13) holds. Letting $t < \frac{T}{2}$, and integrating the HJ equation in $[t, \frac{T}{2}]$, we obtain from (3.7.16) and (3.7.1) that

$$\Big|\int_{\mathbb{T}} v^T(\cdot,t)\Big| \leq C\int_t^{\frac{T}{2}} e^{-\omega s} + e^{-\omega(T-s)}ds \leq \frac{2C}{\omega}\Big(e^{-\omega t} + e^{-\omega\frac{T}{2}}\Big) \leq \frac{4C}{\omega}e^{-\omega t}. \tag{3.7.17}$$

Similarly, for $t \geq \frac{T}{2}$ integrating the HJ equation in $[\frac{T}{2}, t]$ yields

$$\Big|\int_{\mathbb{T}} v^T(\cdot,t)dx\Big| \leq C. \tag{3.7.18}$$

Now, for every $t \in [0,T]$, there exists a point $x_t \in \mathbb{T}$ such that $v^T(x_t,t) = \int_{\mathbb{T}} v^T(\cdot,t)$. Therefore,

$$|v^T(x,t)| \leq \mathrm{osc}_{\mathbb{T}} v^T(t) + \Big|\int_{\mathbb{T}} v^T(\cdot,t)\Big|.$$

As a result, in view of (3.7.1), the estimates (3.7.18) and (3.7.17) yield, respectively, (3.7.14) and (3.7.15). □

We are now ready to prove our last result.

Proof of Theorem 3.1.3. We set

$$v^T = u^T - \lambda(T-t),$$

and show that v^T is convergent as $T \to \infty$.

In view of Lemmas 3.7.1 and 3.7.3, as well as (3.7.16), we see that $\|v^T\|_{W^{1,\infty}(Q_T)}$ and $\|m^T\|_\infty$ are bounded, independently of T. We may therefore apply the Arzelà–Ascoli theorem to conclude that, up to extracting a subsequence, there exist $v \in W^{1,\infty}(\mathbb{T} \times [0,\infty))$ and $\mu \in L^\infty(\mathbb{T} \times [0,\infty))$ such that

$$v^T \to v \text{ locally uniformly in } \mathbb{T} \times [0,\infty),$$

and

$$m^T \rightharpoonup \mu \text{ weakly–* in } L^\infty(\mathbb{T} \times (0,\infty)).$$

We now fix $T_0 \in (1,\infty)$, and assume that $T > T_0 + 1$. Then (v^T, m^T) solves the system

$$\begin{cases} -v_t^T + \lambda + H(v_x^T, m^T) = 0 & \text{in } Q_{T_0}, \\ m_t^T - (m^T H_p(v_x^T, m^T))_x = 0 & \text{in } Q_{T_0}, \\ m^T(\cdot, 0) = m_0. \end{cases} \tag{3.7.19}$$

Moreover, as a result of the interior $C^{1,\alpha}$ estimates for quasilinear elliptic equations, and the interior Schauder estimates for linear equations, $m^T(\cdot, T_0)$ is uniformly bounded in $C^{2,\alpha+\epsilon}$, where $\epsilon > 0$ is chosen such that $\alpha + \epsilon < 1$. Therefore, as in the proof of Theorem 3.1.1, we conclude that, as $T \to \infty$,

$$(v^T, m^T) \to (v, \mu) \text{ in } C^{3,\alpha}(\mathbb{T} \times [0, T_0]) \times C^{2,\alpha}(\mathbb{T} \times [0, T_0]). \tag{3.7.20}$$

In particular, this implies that $(v,\mu) \in C^{3,\alpha}_{\text{loc}}(\mathbb{T}\times[0,\infty)) \times C^{2,\alpha}_{\text{loc}}(\mathbb{T}\times[0,\infty))$, and that (v,μ) solves (MFGL). Letting $T\to\infty$ in (3.7.1) yields

$$\|\mu(t)-1\|_\infty + \|v_x(t)\|_\infty \le Ce^{-\omega t}, \tag{3.7.21}$$

which shows that $\mu - 1 \in L^1(\mathbb{T}\times(0,\infty))$. Moreover, since $\|(m^T)^{-1}\|_\infty$ is bounded, we conclude that (3.1.4) holds.

Now, since a subsequence was extracted, we must verify that the limit is uniquely determined. In view of Lemma 3.7.2, μ is uniquely determined, and v is uniquely determined up to a constant. In the case of (MFG) we see from (3.7.12) that

$$\lim_{t\to\infty} \|v(t) - g(1)\|_\infty = 0.$$

On the other hand, in the case of (MFGP), letting $T\to\infty$ followed by $t\to\infty$ in (3.7.15), we obtain

$$\lim_{t\to\infty} \|v(t)\|_\infty = 0.$$

$\square$

CHAPTER 4

SUPPORT PROPAGATION AND FREE BOUNDARY ANALYSIS FOR COMPACTLY SUPPORTED SOLUTIONS

In this chapter, we will analyze the problem in the whole space:

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x,t) \in \mathbb{R} \times (0,T), \\ m_t - (mu_x)_x = 0 & (x,t) \in \mathbb{R} \times (0,T), \end{cases} \tag{4.0.1}$$

where f is an increasing function. As in the previous chapter, we will consider two different conditions at the terminal time. The first is the problem with a prescribed terminal cost,

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x,t) \in \mathbb{R} \times (0,T) \\ m_t - (mu_x)_x = 0 & (x,t) \in \mathbb{R} \times (0,T) \\ m(x,0) = m_0(x),\ u(x,T) = g(m(x,T)), & x \in \mathbb{R}, \end{cases} \tag{MFG}$$

and the second is the planning problem, namely

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x,t) \in \mathbb{R} \times (0,T) \\ m_t - (mu_x)_x = 0 & (x,t) \in \mathbb{R} \times (0,T) \\ m(x,0) = m_0(x),\ m(x,T) = m_T(x), & x \in \mathbb{R}, \end{cases} \tag{MFGP}$$

for some prescribed mass distributions m_0, m_T. So far, in arbitrary dimensions, it was shown in Chapter 2 that solutions are smooth under the blow-up assumption (2.1.1), provided that the marginals m_0, m_T are (strictly) positive, say for positive measures on a compact domain (e.g. on the flat torus) or for Gaussian-like measures on the whole space. It was also established in Chapter 3 that, for the one-dimensional case, assumption (2.1.1) may be removed, thus requiring only the positivity of the marginals.

We will now turn to the more challenging case of compactly supported marginals in the whole space. Regularizing effects of the type $L^1 \to L^\infty$ had been proven to hold (see [34, 48]), but the propagation of the support of the solution, and even basic matters of regularity such as the continuity of the density, had largely remained open issues. This was the state of the art until the joint work [13] of P. Cardaliaguet, A. Porretta, and the author, upon which this chapter is based.

We will address here both the case of finite and infinite speed of propagation of the support of initial measures. Roughly speaking, those two cases correspond to two model choices for the coupling function, namely $f(m) = m^\theta$ for some $\theta > 0$, or $f(m) = \log(m)$. In the latter case, there is infinite speed of propagation, and the solution starting with compact support becomes instantaneously positive and smooth. By contrast, when $f(m) = m^\theta$, we observe finite speed of propagation, and the solution evolves with compact support. This leads to new interesting questions concerning the study of the free boundary $\partial\{m(t) > 0\}$, which is the main focus of this chapter.

By way of analogy, which is also natural from the optimal transport viewpoint, for $f = \log(m)$ the evolution of m is reminiscent of a nondegenerate diffusion, such as the heat equation. On the other hand, the case of a power nonlinearity f resembles the behavior of degenerate slow diffusions such as the flow through a porous medium (see [54]). This analogy becomes more compelling as in fact, when $f(m) = m^\theta$, we exhibit a family of self-similar solutions which evolves from a Dirac mass into a compactly supported measure. These solutions are given by the formula

$$m(x,t) = t^{-\alpha}\left(R - \frac{\alpha(1-\alpha)}{2}\left(\frac{x}{t^\alpha}\right)^2\right)_+^{1/\theta}, \quad \alpha = \frac{2}{2+\theta}, \tag{4.0.2}$$

which is strongly reminiscent of the famous Barenblatt solution for the porous medium equation [3]. The behavior exhibited by this class of compactly supported solutions serves as a prototype for our analysis of problems (MFG) and (MFGP). In order to describe our main results on the propagation of the support and the characterization of the free boundary, we assume henceforth that

$$f(m) = m^{\theta}\,, \quad \theta > 0\,, \tag{4.0.3}$$

and that the initial measure m_0 is a continuous, compactly supported, probability density, with a bump-like shape:

$$\{m_0 > 0\} = (a_0, b_0) \quad \text{and} \quad \frac{1}{C_0}\mathrm{dist}(x, \{a_0, b_0\})^{\alpha_0} \le m_0(x) \le C_0 \mathrm{dist}(x, \{a_0, b_0\})^{\alpha_0}, \tag{4.0.4}$$

for some $\alpha_0, C_0 > 0$. In order to keep our main statement in a simpler form, we will assume here a consistent condition on the terminal density, in case of problem (MFGP),

$$\{m_T > 0\} = (a_1, b_1), \quad \text{and} \quad \frac{1}{C_1}\mathrm{dist}(x, \{a_1, b_1\})^{\alpha_0} \le m_T(x) \le C_1 \mathrm{dist}(x, \{a_1, b_1\})^{\alpha_0}. \tag{4.0.5}$$

However, more general situations will be considered later, allowing for the behavior of m_T at the boundary of its support to differ from the behavior of m_0. Similarly, for problem (MFG), we will require here, for simplicity, consistency between f and the terminal cost coupling g, namely

$$g(s) = c_T\, s^{\theta}\,, \qquad \text{for some } c_T \ge 0. \tag{4.0.6}$$

We may now state the main result, which proves that the unique solution of (MFG) or (MFGP) has a compactly supported density and the free boundary $\partial\{m(t) > 0\}$ consists of two Lipschitz curves, which are $C^{1,1}$ under a suitable non-degeneracy assumption at the initial time. Those curves can be characterized in terms of the flow of optimal trajectories for the agents' optimization problem.

In fact, we will show that u is smooth inside the support of m, and the characteristic flow

$$\begin{cases} \dot{\gamma}(x,\cdot) = -u_x(\gamma(x,\cdot),\cdot) \\ \gamma(x,0) = x \end{cases}$$

is well defined starting from x in the support of m_0. Finally, we also show that the left and right free boundary curves are, respectively, convex and concave, and in problem (MFG) the support spreads outward in time.

Theorem 4.0.1. *Let f be given by (4.0.3), and let $0 < \overline{\alpha} < 1$. Assume that $m_0 : \mathbb{R} \to [0,\infty)$ satisfies (4.0.4), $m_0^\theta \in C^{1,\overline{\alpha}}(a_0,b_0)$, and m_0^θ is semi–convex. In case of problem (MFGP), assume also that $m_T^\theta \in C^{1,\overline{\alpha}}(a_0,b_0)$ satisfies (4.0.5). Let (u,m) be the solution to (MFGP), or to (MFG) with g satisfying (4.0.6). Then $(u,m) \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T]) \cap \{m > 0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T]) \cap \{m>0\})$, and the following holds:*

1. *There exist two functions $\gamma_L < \gamma_R \in W^{1,\infty}(0,T)$, such that*

$$\{m > 0\} = \{(x,t) \in \mathbb{R} \times [0,T] : \gamma_L(t) < x < \gamma_R(t)\}. \tag{4.0.7}$$

Moreover, the flow γ of optimal trajectories is well defined on $(a_0,b_0)\times[0,T]$, we have

$$\gamma \in W^{1,\infty}((a_0,b_0)\times(0,T)) \cap C^{2,\overline{\alpha}}_{\text{loc}}((a_0,b_0)\times[0,T]), \ \ \gamma_x > 0, \ \ \gamma_L(t) = \gamma(a_0,t),\ \gamma_R(t) = \gamma(b_0,t),$$

and γ is a classical solution in $(a_0,b_0)\times(0,T)$ to the elliptic equation

$$\gamma_{tt} + \frac{\theta m_0^\theta}{(\gamma_x)^{2+\theta}}\gamma_{xx} = \frac{(m_0^\theta)_x}{(\gamma_x)^{1+\theta}}. \tag{4.0.8}$$

2. *If we assume further the concavity condition*

$$(m_0^\theta)_{xx} \le 0 \ \textit{in}\ \{x \in (a_0,b_0) : \operatorname{dist}(x,\{a_0,b_0\}) < \delta\} \ \textit{for some}\ \delta > 0, \tag{4.0.9}$$

then we have $\gamma_L, \gamma_R \in W^{2,\infty}(0,T)$*, and there exists* $K > 0$ *such that, for a.e.* $t \in [0,T]$*,*

$$\frac{1}{K} \leq \ddot{\gamma}_L(t) \leq K, \quad and \quad -K \leq \ \ddot{\gamma}_R(t) \leq -\frac{1}{K},$$

where K *depends on* $T, C_0, \theta, \delta^{-1}, \|((m_0^\theta)_{xx})^-\|_\infty$*,* $|(m_0^\theta)_x(a_0^+)|$*,*$|(m_0^\theta)_x(b_0^-)|$ *(and additionally on* c_T *for problem* (MFG)*, and on* C_1 *for problem* (MFGP)*).*

Moreover, when (u,m) *solves* (MFG)*, we have, for* $t \in [0,T]$*,*

$$-K(c_T+(T-t)) \leq \dot{\gamma}_L(t) \leq -\frac{1}{K}(c_T+(T-t)), \quad and \quad \frac{1}{K}(c_T+(T-t)) \leq \dot{\gamma}_R(t) \leq K(c_T+(T-t)).$$

In relation to the main text, Theorem 4.0.1 is a combination of Theorem 4.3.3 (for the existence of the solution (u,m) and its regularity in $\{m > 0\}$), Theorem 4.3.10 (for the description of the free boundary) and Theorem 4.3.14 (for the regularity and convexity of the free boundary).

Remark 4.0.2. We now discuss the nondegeneracy conditions required on m_0^θ at the boundary of its support. First, we note that the $C^{1,\overline{\alpha}}(a_0, b_0)$ (and therefore $W^{1,\infty}(\mathbb{R})$) condition on m_0^θ, together with (4.0.4), implies that $\alpha_0 \geq \frac{1}{\theta}$ in (4.0.4). In turn, the concavity assumption (4.0.9) further restricts the behavior of m_0^θ, forcing $\alpha_0 = \frac{1}{\theta}$ in (4.0.4). However, this condition of a linear, nondegenerate behavior of m_0^θ is natural (a case in point being the self-similar solution itself), and should be compared with standard nondegeneracy conditions on the initial data in other free boundary problems (e.g. in the study of the moving free boundary for the porous medium equation, see [54]).

We also wish to highlight that, when dealing with problem (MFGP), some asymmetry can be observed when requiring some conditions (e.g. concavity-type assumptions) on m_0 but not on m_T. This kind of asymmetry arises because we are referring to the forward flow $\gamma(x,t)$ in our statement. Of course, similar results will hold when reversing the time flow and exchanging the roles of m_0, m_T.

Remark 4.0.3. We stress that the first part of Theorem 4.0.1 remains true under more general conditions than (4.0.5) (respectively, (4.0.6)). We refer the reader to Theorem 4.3.10, which allows for the behavior of m_T at the boundary of its support to be different from the behavior of m_0 (respectively, in case of problem (MFG), for the function $g(s)$ to be a different power than f).

The result in the second part of Theorem 4.0.1 corresponds exactly to the picture described by the self-similar solution (4.0.2). Indeed, our next result shows that the free boundary propagates with strictly convex (resp. concave) behavior at the left (resp. right) free boundary curve. In fact, if we strengthen the concavity assumption on m_0^θ, we show that the free boundary evolves with the optimal speed given by the self-similar solution. Moreover, the long time decay of the density occurs with the same rate, as exhibited by (4.0.2).

Theorem 4.0.4. *Under the assumptions of Theorem 4.0.1, let* (u, m) *be the unique solution to* (MFG) *or* (MFGP), *and let* γ *be the associated flow of optimal trajectories. Assume in addition that* $-K \leq (m_0^\theta)_{xx} \leq -\frac{1}{K}$ *in* (a_0, b_0) *for some* $K > 0$, *and, in case of problem* (MFG), *assume that* $c_T = \kappa_1 T$ *in* (4.0.6). *If we define*

$$\alpha = \frac{2}{2+\theta} \; ; \qquad \mathscr{d}(t) = \begin{cases} t & \textit{if } u \textit{ solves } (\text{MFG}), \\ \operatorname{dist}(t, \{0, T\}) & \textit{if } u \textit{ solves } (\text{MFGP}), \end{cases}$$

then there exists a constant $C > 0$ *such that for every* $(x, t) \in [a_0, b_0] \times [0, T]$,

$$\frac{1}{C}(1 + \mathscr{d}(t)^\alpha) \leq |\operatorname{supp}(m(\cdot, t))| \leq C(1 + \mathscr{d}(t)^\alpha), \quad |\gamma(x, t)| \leq C(1 + \mathscr{d}(t)^\alpha), \tag{4.0.10}$$

$$\frac{1}{C}\frac{m_0(x)}{(1 + \mathscr{d}(t)^\alpha)} \leq m(\gamma(x, t), t) \leq C\frac{m_0(x)}{(1 + \mathscr{d}(t)^\alpha)}, \tag{4.0.11}$$

where

$$C = C\left(C_0, \kappa_1, \kappa_1^{-1}, |a_0|, |b_0|, K\right) \tag{4.0.12}$$

in case of problem (MFG)*, and*

$$C = C\left(C_0, C_1, |a_0|, |b_0|, |a_1|, |b_1|, K\right) \tag{4.0.13}$$

in case of (MFGP)*.*

Theorem 4.0.4 is nothing but Theorem 4.3.15 below. Let us stress that a crucial role in the proof of the above results is played by the equation satisfied by the flow of optimal curves γ, namely (4.0.8). In particular, the Lipschitz regularity of γ is obtained by a maximum principle argument applied to γ_x, which is derived from (4.0.8). We obtain further insight by studying the equation of m in Lagrangian coordinates. Indeed, the function $v = f(m(\gamma(x,t),t))$ satisfies the (degenerate) elliptic equation

$$-\left(\gamma_x(\theta v)^{-1} v_t\right)_t - \left(\gamma_x^{-1} v_x\right)_x = 0, \tag{4.0.14}$$

where one can prove, assuming (4.0.9), that the positive quantity γ_x is bounded below and above. This elucidates the key distinction between the present problem and slow diffusions of porous medium type; equation (4.0.14) is diffusive (rather than parabolic) in the time variable. Relying on this equation, we establish the regularity of m up to the free boundary. Namely, we prove that m is Hölder continuous, through an application of the intrinsic scaling regularity method (see [19, 20, 52]). In turn, we show that Du is Hölder continuous as well. We can summarize these regularity results, contained in Theorem 4.3.21 and 4.3.23 respectively, as follows:

Theorem 4.0.5. *Under the assumptions of Theorem 4.0.4, we have* $f(m) \in C^{\beta}_{\mathrm{loc}}(\mathbb{R}\times(0,T))$ *and* $u \in C^{1,\frac{\beta}{2}}_{\mathrm{loc}}(\mathbb{R}\times(0,T))$ *for some* $\beta \in (0,1)$*.*

Finally, our last result shows that the solutions of (4.0.1) exhibit a different behavior when

$$f(m) = \log(m)\,. \tag{4.0.15}$$

In contrast with the case of a power nonlinearity, the unbounded payoff as $m \downarrow 0$ given by(4.0.15) implies that the support of the density propagates with infinite speed. This behavior is observable in both problem (MFG) (with $g(s) = c_T \log(s)$, $c_T \geq 0$) and in the planning problem (MFGP).

More specifically, under the assumption of m_0 being continuous with compact support (and similarly for m_T in the case of (MFGP)), we establish the existence of classical solutions (u, m) with $m > 0$ in $(0, T)$.

Compared to Theorem 4.0.1, the positivity of solutions on the whole space now makes it much more delicate to use the flow of optimal curves $\gamma(x, t)$, which are no longer confined in a bounded set. This difficulty leads us to require an extra symmetry and monotonicity assumption, namely that m_0 is even and nonincreasing in $(0, \infty)$ (and the same for the terminal density m_T). We are able to take advantage of this assumption by showing that the solution $m(\cdot, t)$ preserves this property for all $t \in (0, T)$, which is in itself a non-trivial feature of the MFG system (see Lemma 4.4.7). However, we no longer require any special behavior of m_0 when vanishing at the boundary of its support, avoiding conditions such as (4.0.4), (4.0.5). In fact, the support now propagates instantly, regardless of the flatness of m_0.

Theorem 4.0.6. *Let f be given by* (4.0.15)*, let $\overline{\alpha} \in (0, 1)$, and assume that m_0 is a continuous, compactly supported, density on $\mathbb{R}$, which is $C^{1,\overline{\alpha}}_{\text{loc}}$ in the set $\{m_0 > 0\}$, even and nonincreasing on $[0, \infty)$.*

1. *If $g(s) = c_T \log(s)$, for some $c_T \geq 0$, then there exists a unique classical solution $(u, m) \in C^2(\mathbb{R} \times (0, T]) \times C^1(\mathbb{R} \times (0, T])$ of* (MFG) *such that m is continuous and*

bounded on $\mathbb{R} \times [0,T]$*, positive on* $\mathbb{R} \times (0,T)$ *with* $m(0) = m_0$*, and* $|x|^2 m(t) \in L^1(\mathbb{R})\,, \frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$*, for every* $t \in (0,T)$*.*

2. *If* $m_T \in C_c(\mathbb{R})$ *is even and nonincreasing on* $[0,\infty)$*, then there exists a unique (up to addition of a constant to* u*) classical solution* $(u,m) \in C^2(\mathbb{R}\times(0,T))\times C^1(\mathbb{R}\times(0,T))$ *of* (MFGP) *such that* m *is continuous and bounded on* $\mathbb{R} \times [0,T]$*, positive on* $\mathbb{R} \times (0,T)$ *with* $m(0) = m_0, m(T) = m_T$*, and* $|x|^2 m(t) \in L^1(\mathbb{R})\,, \frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$*, for every* $t \in (0,T)$*.*

Theorem 4.0.6 is Theorem 4.4.9 below. As mentioned before, the symmetry and monotonicity assumption on m_0, which is required in Theorem 4.0.6, allows us to overcome certain difficulties in the obtention of classical positive solutions in the whole space. These difficulties disappear in compact domains, as is seen in Theorem 4.4.2, where we prove the existence of classical periodic solutions, with $m > 0$ in $(0,T)$, under the only condition that m_0 (and m_T) are continuous and compactly supported.

This chapter is organized as follows. In Section 4.1 we exhibit the class of self-similar solutions which will serve as a prototype for our main results. Section 4.2 presents the key features of smooth, periodic solutions with a positive density: structural properties, displacement convexity, Lipschitz estimates, and a modulus of continuity for the density. Section 4.3, dealing with compactly supported solutions, is the heart of the chapter: starting with the existence of solutions (Subsection 4.3.1), it culminates with the regularity, geometric properties, and long time behavior of the free boundary (Subsection 4.3.2), and the Hölder regularity of m and Du (Subsection 4.3.3). Section 4.4 is devoted to the entropic coupling ($f = \log$) and the infinite speed of propagation. Appendix 4.A contains the computations for the self-similar solutions.

4.1 Self-similar solutions

We now exhibit a family of compactly supported, self-similar solutions of the system

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = m^\theta & (x,t) \in \mathbb{R} \times (0,\infty), \\ m_t - (mu_x)_x = 0 & (x,t) \in \mathbb{R} \times (0,\infty), \\ \int_{\mathbb{R}} m(t)dx = 1 & t \in (0,\infty), \end{cases} \tag{4.1.1}$$

where $\theta > 0$. By a solution of (4.1.1) we mean here that u is Lipschitz continuous and m continuous and nonnegative, the first equation being understood in the sense of viscosity solutions, while the second equation is satisfied in the sense of distributions. The solution is described in the following result.

Theorem 4.1.1. *For $\theta > 0$, let us set*

$$\alpha = \frac{2}{2+\theta}$$

and let R be the unique positive number such that $\int_{\mathbb{R}} \left(R - \frac{1}{2}(\alpha - \alpha^2)y^2\right)_+^{1/\theta} dy = 1$. A solution of (4.1.1) is given by (u,m), with

$$m(x,t) = t^{-\alpha}\phi(x/t^\alpha)\,, \qquad \textit{where} \quad \phi(y) = \left(R - \frac{1}{2}(\alpha(1-\alpha))y^2\right)_+^{1/\theta}$$

and u defined as follows:

(i) either $\theta = 2$ and

$$u(x,t) = \begin{cases} -\frac{1}{4t}x^2 - R\log t & \textit{if } \Delta \leq 0 \\ -\frac{2R|x|}{|x|-\sqrt{\Delta}} - 2R\log(\frac{|x|-\sqrt{\Delta}}{\sqrt{8R}}) & \textit{if } \Delta > 0 \end{cases} \tag{4.1.2}$$

where $\Delta = x^2 - 8Rt$,

(ii) or $\theta \neq 2$ *and*

$$u(x,t) = \begin{cases} -\alpha\frac{x^2}{2t} - R\frac{1}{2\alpha-1}t^{2\alpha-1} & \text{if } \Delta \leq 0 \\ \frac{-R\alpha}{(1-\alpha)(2\alpha-1)}S^{2\alpha-1} - \frac{\alpha R}{1-\alpha}S^{2\alpha-2}(t-S) & \text{if } \Delta > 0, \end{cases} \tag{4.1.3}$$

where $\Delta = |x| - \sqrt{\frac{2R}{\alpha(1-\alpha)}}t^{\alpha}$ *and the function* $S = S(x,t)$ *is defined implicitly by the equation:*

$$S\sqrt{\frac{2R}{\alpha(1-\alpha)}} - |x|S^{1-\alpha} + \sqrt{\frac{2R\alpha}{1-\alpha}}(t-S) = 0. \tag{4.1.4}$$

The explicit construction of u can be understood by distinguishing two regions. First, one shows that, on the support of m, u must be given by

$$u(x,t) = -\alpha\frac{x^2}{2t} + c(t), \quad \text{with} \quad c'(t) = -Rt^{-2\theta/(2+\theta)}. \tag{4.1.5}$$

Outside the support of m, the values of u are extended along the optimal curves, which are straight lines in the set $\{m = 0\}$. This leads to formula (4.1.2) if $\theta = 2$, or to formula (4.1.3) if $\theta \neq 2$. Note that $\Delta \leq 0$ corresponds with the support of m. Finally, we point out that u is defined up to an additive constant. The proof of the statements made in Proposition 4.1.1 will be presented in Appendix 4.A.

We note the following relevant facts about this solution, which will serve as a model for our later assumptions and results:

- At time $t = 0$, the measure corresponds to a Dirac mass at $x = 0$. For positive times $t > 0$, in general, the density m is merely Hölder continuous.

- For each $t > 0$, the function $f(m)$ is always Lipschitz (away from $t = 0$). Moreover, $f(m(\cdot,t))$ is strictly concave within the support, and, in particular, $f(m(\cdot,t))_x$ is non-zero at the endpoints. A weaker, local version of these conditions will serve as our non-degeneracy assumption on the initial distribution $f(m_0)$ (see (4.3.56) and (4.3.57)), in order to prove our main regularity result for the free boundary (Theorem 4.3.14), and the Hölder continuity of m (Theorem 4.3.21). Moreover, the full strict concavity assumption on $f(m_0)$ will yield our result on the optimal speed of support propagation and long time decay of m (Theorem 4.3.15).

- The value function u is smooth on the support of m but u_{xx} blows-up at the interface (see Remark 4.A.4), at least when approaching from outside the support. In fact, it is shown in Proposition 4.A.3 that $u \in C^{1,s}$ for a certain $0 < s < 1$. Accordingly, our general results will show that the $C^{1,s}$ regularity exhibited by this explicit solution in fact holds for arbitrary solutions of the MFG system, at least under the aforementioned non-degeneracy assumption (Theorem 4.3.23).

4.2 Structure and a priori estimates in the periodic setting

4.2.1 Structural properties of the MFG system

The results of this chapter are systematically obtained by establishing a priori estimates on a regularized system which has a smooth solution. In this subsection, we will explain the structural properties of the system (4.0.1), deriving the fundamental identities used throughout this chapter. Here we assume that $f' > 0$ on $(0, \infty)$, and (u, m) is a classical solution to (4.0.1), with m being positive, and u_x has at most a linear growth. Note that we only assume $m(\cdot, 0) = m_0$ to be smooth and positive (no condition on the mass), so that the results of this part are valid for equations with periodic boundary conditions as well as in the whole space.

We begin with the elliptic equation satisfied by u, first derived by Lions in [38] (see also [44, 48]). It is obtained by simply eliminating $m = f^{-1}(-u_t + u_x^2/2)$ from the system, thanks to the fact that $f' > 0$.

Lemma 4.2.1. *Let* (u, m) *be a classical solution to* (4.0.1)*. The map* u *satisfies the quasi-linear elliptic equation*

$$-u_{tt} + 2u_x u_{xt} - (u_x^2 + mf'(m))u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, T) \tag{4.2.1}$$

with $m = f^{-1}(-u_t + u_x^2/2)$*.*

We observe that (4.2.1) is a degenerate elliptic equation for u; the uniform ellipticity being lost when m vanishes. The study of (4.2.1) is the starting point of the regularity theory developed in [38, 44, 48] under conditions ensuring a positive control from below on m. In that case, equation (4.2.1) turns out to be equivalent to the system (4.0.1), at least for classical solutions. As it is customary for quasilinear problems, a key role is played by gradient estimates, which are obtained through the maximum principle. We will recall this approach in Subsection 4.2.2. For this purpose, it is convenient to introduce the linear second order operator

$$Q(v) := -v_{tt} + 2u_x v_{xt} - (u_x^2 + mf'(m))v_{xx}, \tag{4.2.2}$$

and accordingly, the linearized operator generated from (4.2.1):

$$L(v) = Q(v) + 2(u_{xt} - u_x u_{xx})v_x - \left(\frac{mf''(m)}{f'(m)} + 1\right) u_{xx}(-v_t + u_x v_x). \tag{4.2.3}$$

For the rest of this section, the solutions will be tacitly assumed to be sufficiently smooth to justify the computations below.

Lemma 4.2.2. *Let* u *be a classical solution to* (4.2.1)*, and* Q, L *be defined by* (4.2.2) *and* (4.2.3)*. Then we have*

$$L(u_t) = 0\,, \quad L(u_x) = 0$$

and the function $w := f(m)$ satisfies

$$Q(w) - w_x^2 + m(mf''(m) + 2f'(m))u_{xx}^2 = 0. \tag{4.2.4}$$

Proof. The equations satisfied by u_t, u_x are obtained by differentiation of (4.2.1). From $L(u_x) = 0$ we also obtain, by the chain rule,

$$L\left(\frac{1}{2}u_x^2\right) = -(u_{xt} - u_x u_{xx})^2 - mf'(m)u_{xx}^2. \tag{4.2.5}$$

Hence,

$$L(f(m)) = L(-u_t + \frac{1}{2}u_x^2) = -(u_{xt} - u_x u_{xx})^2 - mf'(m)u_{xx}^2 = -f(m)_x^2 - mf'(m)u_{xx}^2. \tag{4.2.6}$$

Now, using (4.2.3), we also have

$$\begin{aligned} L(f(m)) =& Q(f(m)) - 2f(m)_x^2 - \left(\frac{mf''(m)}{f'(m)} + 1\right) u_{xx}(-f(m)_t + u_x f(m)_x) \\ =& Q(f(m)) - 2f(m)_x^2 + \left(\frac{mf''(m)}{f'(m)} + 1\right) u_{xx}(mf'(m)u_{xx}), \end{aligned} \tag{4.2.7}$$

where we used the equation of m in the last step. Putting together (4.2.6) and (4.2.7) yields (4.2.4). $\square$

As our goal is to understand what happens when m vanishes or gets close to 0, we need to introduce more geometric quantities related to the first order system (4.0.1). The first of these is the family of optimal trajectories associated to the HJ equation satisfied by u: we define $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}$ as the solution to

$$\gamma_t(x, t) = -u_x(\gamma(x, t), t) \quad \text{in } \mathbb{R} \times [0, T], \qquad \gamma(x, 0) = x \qquad \text{in } \mathbb{R}. \tag{4.2.8}$$

By the standard theory of Hamilton-Jacobi equations, it is known that $\gamma(x, \cdot)$ is the minimizer of the problem

$$\inf_{\beta\in H^1,\ \beta(t)=x} \int_t^T \frac{1}{2}|\dot{\beta}|^2 + f(m(\beta,s))\ ds + u(\beta(T),T).$$

The fundamental properties of γ are given next.

Lemma 4.2.3. *Let (u,m) be a classical solution to (4.0.1). One has*

$$\gamma_{tt}(x,t) = f'(m(\gamma(x,t),t))m_x(\gamma(x,t),t), \qquad \forall (x,t)\in\mathbb{R}\times(0,T) \qquad \textit{(Euler equation)}, \tag{4.2.9}$$

and

$$\gamma_x(x,t) = \frac{m_0(x)}{m(\gamma(x,t),t)} \qquad \forall (x,t)\in\mathbb{R}\times(0,T) \qquad \textit{(conservation of mass)}, \tag{4.2.10}$$

or, equivalently,

$$\int_{\gamma(x_1,t)}^{\gamma(x_2,t)} m(x,t)dx = \int_{x_1}^{x_2} m_0(x)dx \qquad \forall x_1,x_2\in\mathbb{R},\ t\in[0,T]. \tag{4.2.11}$$

Moreover γ solves the quasilinear elliptic equation

$$-\frac{m_0 f'(m_0/\gamma_x)}{(\gamma_x)^3}\gamma_{xx} - \gamma_{tt} = -\frac{(m_0)_x f'(m_0/\gamma_x)}{(\gamma_x)^2} \qquad \textit{in } \mathbb{R}\times(0,T). \tag{4.2.12}$$

Before proving the lemma, it will be convenient to associate to (u,m) the solution M to the transport equation

$$M_t - u_x M_x = 0 \qquad \text{in } \mathbb{R}\times[0,T], \qquad M(x,0) = -\int_0^x m_0(y)dy \qquad \text{in } \mathbb{R}. \tag{4.2.13}$$

Notice that (4.2.8) defines γ as the curve of characteristics associated to this transport equation.

Lemma 4.2.4. *Let (u,m) be a classical solution to (4.0.1). One has $M_x = -m < 0$ and M*

satisfies the quasilinear elliptic equation

$$-\frac{M_t^2}{M_x^2}M_{xx} + 2\frac{M_t}{M_x}M_{xt} - M_{tt} + M_x f'(-M_x))M_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0,T). \tag{4.2.14}$$

Proof. Differentiating (4.2.13) in space, we see that $\mu := -M_x$ satisfies $\mu_t - (u_x\mu)_x = 0$, with initial condition $\mu(0,x) = m_0$: this is exactly the equation satisfied by m, so that $M_x = -\mu = -m < 0$. On the other hand, by definition $u_x = M_t/M_x$. Taking the derivative w.r.t. x of the equation for u (4.2.1), we obtain (4.2.14). □

Proof of Lemma 4.2.3. As γ solves (4.2.8), we have

$$\gamma_{tt} = -u_{xx}(\gamma,t)(-u_x(\gamma,t)) - u_{xt}(\gamma,t) = f'(m(\gamma,t))m_x(\gamma,t),$$

where the second equality comes from the derivation in space of the HJ equation. This is (4.2.9). As $M_x = -m$, (4.2.10) comes from the derivative in space of the transport equality $M(\gamma(x,t),t) = M_0(x)$. Then (4.2.11) follows by the integration in space of (4.2.10).

By (4.2.9) and then (4.2.10), $m_x(\gamma,t) = \gamma_{tt}/f'(m(\gamma,t)) = \gamma_{tt}/f'(m_0/\gamma_x)$. On the other hand, taking the derivative in space of (4.2.10) gives (using again (4.2.10) and the expression above for $m_x(\gamma,t)$):

$$\gamma_{xx} = \frac{(m_0)_x}{m(\gamma,t)} - \frac{m_0 m_x(\gamma,t)\gamma_x}{(m(\gamma,t))^2} = \frac{(m_0)_x\gamma_x}{m_0} - \frac{\gamma_x^3}{f'(m_0/\gamma_x)m_0}\gamma_{tt}.$$

This is (4.2.12). □

We finally compute the equation satisfied by $v(x,t) = f(m(\gamma(x,t),t))$. The map v is the r.h.s. of the HJ equation viewed from the lens of the optimal trajectories.

Lemma 4.2.5. *Let* (u,m) *be a classical solution to* (4.0.1)*. The map* $v(x,t) = f(m(\gamma(x,t),t))$

satisfies the quasilinear elliptic equation in divergence form:

$$-\left(\frac{v_x}{\gamma_x}\right)_x - \left(\frac{\gamma_x^2}{m_0 f'(m_0/\gamma_x)} v_t\right)_t = 0. \tag{4.2.15}$$

If $f(m) = m^\theta$, this equation simplifies into

$$(i)\ -\left(\frac{v_x}{\gamma_x}\right)_x - \left(\frac{\gamma_x}{\theta v} v_t\right)_t = 0, \qquad or \qquad (ii)\ -v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} + v_x \frac{\theta v}{\gamma_x^3}\gamma_{xx} + \frac{\theta+1}{\theta} v^{-1} v_t^2 = 0, \tag{4.2.16}$$

while if $f(m) = \log(m)$, it becomes

$$-\left(\frac{v_x}{\gamma_x}\right)_x - (\gamma_x v_t)_t = 0. \tag{4.2.17}$$

Proof. Observe that (4.2.9) may be written as

$$\gamma_{tt} = \frac{v_x}{\gamma_x}. \tag{4.2.18}$$

We now compute the time derivative of $v(x,t) = f(m_0(x))/\gamma_x(x,t))$ to find

$$v_t = f'(m_0/\gamma_x)\left(-\frac{m_0 \gamma_{xt}}{\gamma_x^2}\right). \tag{4.2.19}$$

Thus, putting together (4.2.18) and (4.2.19) we get

$$\left(\frac{v_x}{\gamma_x}\right)_x = \gamma_{xtt} = -\left(\frac{v_t \gamma_x^2}{m_0 f'(m_0/\gamma_x)}\right)_t,$$

which gives (4.2.15), and thus (4.2.16)-(i) and (4.2.17). Finally, if $f(m) = m^\theta$, developing (4.2.16)-(i) and using (4.2.19)—which implies that $\gamma_{xt} = -(\gamma_x v_t)/(\theta v)$—leads to (4.2.16)-(ii). □

4.2.2 Displacement convexity estimates on m, Lipschitz estimates on u and existence result

Before studying the problem with compactly supported marginals, we will begin obtaining some estimates for the simpler periodic setting with strictly positive marginals. We are mostly interested in a priori estimates that are independent of $\min m_0$ and $\min m_T$. By approximation with positive densities $m_0^\varepsilon, m_T^\varepsilon$, these estimates will hold for the case in which m_0 and m_T are compactly supported.

Throughout the remainder of this section, $R \geq 1$ will denote a fixed constant, and we will analyze the MFG system on the one-dimensional torus of length R, denoted by $R\mathbb{T}$. Functions defined on $R\mathbb{T}$ are meant to be R-periodic functions on $\mathbb{R}$. We will consider the problem

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x,t) \in R\mathbb{T} \times (0,T), \\ m_t - (mu_x)_x = 0 & (x,t) \in R\mathbb{T} \times (0,T), \end{cases} \tag{4.2.20}$$

complemented either with the initial-terminal conditions

$$m(x,0) = m_0(x), \quad u(x,T) = g(m(x,T)), \quad x \in R\mathbb{T}, \tag{4.2.21}$$

or with the prescribed marginal conditions of the planning problem:

$$m(x,0) = m_0(x), \quad m(x,T) = m_T(x), \quad x \in R\mathbb{T}. \tag{4.2.22}$$

The functions $f, g : (0,\infty) \to \mathbb{R}$ are assumed to satisfy $f', g' > 0$, with $f, g \in C^2(0,\infty)$ and

$$\limsup_{m\to 0^+} mf'(m) < \infty, \ \limsup_{m\to 0^+} \frac{m|f''(m)|}{f'(m)} < \infty. \tag{4.2.23}$$

The initial (and terminal) data $m_0, m_T : R\mathbb{T} \to (0,\infty)$ are understood to be C^1 functions satisfying

$$\int_{R\mathbb{T}} m_0 = \int_{R\mathbb{T}} m_T, \quad m_0, m_T > 0. \tag{4.2.24}$$

Moreover, we also assume that

$$f(m_0), f(m_T) \in C^{1,\overline{\alpha}}(R\mathbb{T}). \tag{4.2.25}$$

The fact that the system (4.2.20) has smooth solutions when the marginals are strictly positive and the data are sufficiently smooth is already known (see [44, Theorem 1.1]). In fact, we will see later (Theorem 4.2.11) that, under the present assumptions, (4.2.20)–(4.2.21) and (4.2.20)–(4.2.22) both admit a classical solution $(u,m) \in C^{2,\overline{\alpha}}(R\mathbb{T}\times(0,T))\times C^{1,\overline{\alpha}}(R\mathbb{T}\times(0,T))$.

We begin by recalling the so-called displacement convexity formula (see [27, 43]), as well as an identity which will later be useful to obtain energy estimates on the density.

Theorem 4.2.6. *Assume that $f \in C^1(0,\infty)$, and let (u,m) be a classical solution to* (4.2.20)*. Then we have*

$$mf'(m)u_{xx}^2 + f(m)_x^2 = (f(m)_t u_x)_x - (f(m)_x u_x)_t. \tag{4.2.26}$$

Moreover, if $h : (0,\infty) \to \mathbb{R}$ is twice differentiable, then

$$\frac{d^2}{dt^2}\int_{R\mathbb{T}} h(m) = \int_{R\mathbb{T}} mh''(m)(mu_{xx}^2 + f'(m)m_x^2). \tag{4.2.27}$$

Proof. We start by multiplying the continuity equation by u_{xx}, which yields

$$mu_{xx}^2 + m_x u_x u_{xx} - m_t u_{xx} = 0.$$

As a result, differentiating the HJ equation for the term $u_x u_{xx}$ we obtain

$$mu_{xx}^2 + m_x(u_{xt} + f'(m)m_x) - m_t u_{xx} = 0, \tag{4.2.28}$$

which, after multiplying by $f'(m)$, yields

$$mf'(m)u_{xx}^2 + f'(m)^2 m_x^2 = (f(m)_t u_x)_x - (f(m)_x u_x)_t.$$

This proves (4.2.26). We note that (4.2.27) is merely special case of [43, Proposition 3.1], but we give a proof for the reader's convenience. Multiplying both sides of (4.2.28) by $mh''(m)$, we obtain

$$mh''(m)(mu_{xx}^2 + f'(m)m_x^2) = h''(m)mm_t u_{xx} - h''(m)m_x u_{xt} = H(m)_t u_{xx} - H(m)_x u_{xt},$$

where $H(m) = mh'(m) - h(m)$. This may be rewritten as

$$mh''(m)(mu_{xx}^2 + f'(m)m_x^2) = (H(m)_t u_x)_x - (H(m)_x u_x)_t \tag{4.2.29}$$

Now, from the continuity equation,

$$h(m)_t = h(m)_x u_x + mh'(m)u_{xx} = (h(m) - mh'(m))_x u_x + (mh'(m)u_x)_x = -H(m)_x u_x + (mh'(m)u_x)_x.$$

Therefore,

$$(H(m)_x u_x)_t = -h(m)_{tt} + (mh'(m)u_x)_{tx}.$$

Substituting in (4.2.29), we obtain

$$\begin{aligned} mh''(m)(mu_{xx}^2 + f'(m)m_x^2) &= h(m)_{tt} + (H(m)u_x)_x - (mh'(m)u_x)_{tx} \\ &= h(m)_{tt} - (h(m)_t u_x)_x - (mh'(m)u_{xt})_x = h(m)_{tt} - (h'(m)(mu_x)_t)_x, \end{aligned}$$

and (4.2.27) then follows by integrating both sides of this equation in space. $\square$

We now note that the density attains its extremum values at the extremal times.

Corollary 4.2.7. *Assume that $f \in C^1(0,\infty)$, and let (u,m) be a classical solution to (4.2.20). Then*

$$\|m\|_\infty \le \max(\|m(\cdot,0)\|_\infty, \|m(\cdot,T)\|_\infty),$$
$$\left\|m^{-1}\right\|_\infty \le \max\left(\left\|m(\cdot,0)^{-1}\right\|_\infty, \left\|m(\cdot,T)^{-1}\right\|_\infty\right).$$

Proof. One first observes that, for any convex function $h : (0,\infty) \to \mathbb{R}$, it follows from (4.2.27) that $\int_{R\mathbb{T}} h(m)$ is convex in time, which yields

$$\int_{R\mathbb{T}} h(m) \le \max\left(\int_{R\mathbb{T}} h(m(\cdot,0)), \int_{R\mathbb{T}} h(m(\cdot,T))\right).$$

The upper bounds on m and m^{-1}, then follow by taking $h(m) = m^p$ and letting $p \to \pm\infty$, respectively. ☐

Remark 4.2.8. These a priori estimates were proved in [27]. They could have also been derived without the displacement convexity formula, as an immediate consequence of Lemma 4.2.12 below.

We will now review the Lipschitz estimates which can be established on u following the approach suggested by P.-L. Lions in [38], and developed later in more generality in [44, 48]. The following L^∞ bounds on u and $m(\cdot,T)$ are well-known consequences of the maximum principle and the Hopf-Lax formula (see [43, Propositions 4.1 and 4.2]).

Theorem 4.2.9. *Assume that $f, g \in C^1(0,\infty)$, $f' > 0, g' \ge 0$, and $m_0, m_T \in C(\mathbb{T})$. If (u,m) is a classical solution to* (4.2.20)–(4.2.21)*, then we have, for $(x,t) \in R\mathbb{T} \times [0,T]$,*

$$\min m_0 \le m(x,T) \le \max m_0,$$

$$f(\min m_0)(T-t) + g(\min m_0) \le u(x,t) \le f(\max m_0)(T-t) + g(\max m_0).$$

Moreover, there exists a constant $C = C(\|f(m_0)\|_\infty, \|f(m_T)\|_\infty)$ *such that, if* (u,m) *solves* (4.2.20)–(4.2.22)*, then*

$$\operatorname{osc}(u) \leq C(T + R^2T^{-1}).$$

We may now obtain a gradient estimate which, crucially, is independent of $\min m$. Hereafter, we denote by Du the vector formed by the space and time first derivatives, that is,

$$Du := (u_x, u_t)\,.$$

Moreover, we denote $\kappa_0 > 0$ to be a constant such that

$$mf'(m), \frac{m|f''(m)|}{f'(m)} \leq \kappa_0 \qquad \forall\ m \in (0, 2\max(\|m_0\|_\infty, \|m_T\|_\infty)]\,. \tag{4.2.30}$$

Such a constant exists by virtue of (4.2.23). It is understood that the term with $\|m_T\|_\infty$ is treated as zero, in case of conditions (4.2.21).

Theorem 4.2.10. *Assume that* (4.2.23), (4.2.24)*, and* (4.2.25) *hold true, and let* (u,m) *be a classical solution to* (4.2.20)–(4.2.21) *or* (4.2.20)–(4.2.22). *There exists a constant* C *such that*

$$\|Du\|_\infty \leq C.$$

where

$$C = \begin{cases} C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|g(m_0)\|_\infty) & \text{if } (u,m) \text{ solves } (4.2.20)\text{–}(4.2.21)\ , \\ C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}) & \text{if } (u,m) \text{ solves } (4.2.20)\text{–}(4.2.22), \end{cases}$$

Proof. By rescaling, it is enough to consider the case $R = 1$. By approximation[1], we may

1. See the proof of Theorem 4.2.11 for the details of such an approximation argument.

also assume that $u \in C^3(\mathbb{T} \times [0,T])$. We begin by noting that, using either Corollary 4.2.7 or Proposition 4.2.9, we have that $\|m\|_\infty$ is controlled by $\|m_0\|_\infty$, and by $\|m_T\|_\infty$ in case of (4.2.22). Hence we will use (4.2.30) for $m(x,t)$ below. We let $v(x,t) = \frac{1}{2}u_x^2 + \frac{1}{2T}\tilde{u}^2$, where $\tilde{u}$ is defined as

$$\tilde{u} = u - \min u + T - \frac{(\operatorname{osc}(u) + 2T)}{T}(T-t),$$

so that $\tilde{u}(x,0) = u - \max u - T \le -T$, $\tilde{u}(x,T) = u - \min u + T \ge T$, and $\|\tilde{u}\|_\infty \le (T + \operatorname{osc}(u))$. Let (x_0, t_0) be a point in which v achieves its maximum value. We note that, by the HJ equation, it is enough to bound v. If $t_0 = 0$, then we have

$$-v_t + u_x v_x = u_x f(m_0)_x + \frac{1}{T}\tilde{u}\left(f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\operatorname{osc}(u) + 2T)\right).$$

Now, we have $v_x = 0$ and $v_t \le 0$, and recall that $\tilde{u}(x,0) \le -T$; so either $f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\operatorname{osc}(u) + 2T) \le 0$, in which case there is nothing to prove, or we deduce

$$0 \le u_x f(m_0)_x - (f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\operatorname{osc}(u) + 2T)).$$

This means

$$\frac{1}{2}u_x^2 \le u_x f(m_0)_x - f(m_0) + \frac{1}{T}(\operatorname{osc}(u) + 2T),$$

and yields the required estimate. The case in which $t_0 = T$ is similar, so we assume now that $0 < t_0 < T$. We begin by noting that, since $v_x = 0$,

$$u_x u_{xx} = -\frac{1}{T}u_x \tilde{u},$$

and, thus,

$$|u_{xx}| \le \frac{C}{T}(T + \operatorname{osc}(u)). \tag{4.2.31}$$

Now, recall the definitions of the linear operators Q, L in (4.2.2), (4.2.3). Using (4.2.5) we have

$$L\left(\frac{1}{2}u_x^2\right) \le -(u_{xt}-u_xu_{xx})^2\,. \tag{4.2.32}$$

On the other hand, since $Q(\tilde{u})=0$, we have

$$Q\left(\frac{1}{2}\tilde{u}^2\right) = -(-\tilde{u}_t+u_x^2)^2 - mf'(m)\tilde{u}_x^2 \le -(-\tilde{u}_t+u_x^2)^2 = -\left(f(m)+\frac{1}{2}u_x^2-\frac{1}{T}(\operatorname{osc}(u)+2T)\right)^2\,.$$

Observe that, by Corollary 4.2.7 and Proposition 4.2.9, $|f(m)|$ is bounded. Therefore, if $f(m)+\frac{1}{4}u_x^2 \le \frac{1}{T}(\operatorname{osc}(u)+2T)$, there is nothing to prove. Else,

$$Q\left(\frac{1}{2}\tilde{u}^2\right) \le -\frac{1}{32}u_x^4 - \frac{1}{2}(-\tilde{u}_t+u_x^2)^2,$$

and, thus, by definition of L, we obtain, using (4.2.30) and (4.2.31),

$$\begin{aligned} L\left(\frac{1}{2T}\tilde{u}^2\right) &\le -\frac{1}{32T}u_x^4 - \frac{1}{2T}(-\tilde{u}_t+u_x\tilde{u}_x)^2 + \frac{2}{T}(u_{xt}-u_xu_{xx})u_x\tilde{u} \\ &\quad -\frac{1}{T}\left(\frac{mf''(m)}{f'(m)}+1\right)u_{xx}(-\tilde{u}_t+u_x\tilde{u}_x)\tilde{u} \le -\frac{1}{32T}u_x^4 + \frac{1}{2T}\left(\kappa_0+1\right)^2(u_{xx})^2\tilde{u}^2 \\ &\quad + \frac{4}{T^2}u_x^2\tilde{u}^2 + (u_{xt}-u_xu_{xx})^2 \le -\frac{1}{64T}u_x^4 + \frac{C}{T^3}(T+\operatorname{osc}(u))^4 + (u_{xt}-u_xu_{xx})^2\,. \end{aligned}$$

Putting toghether the above inequality with (4.2.32) we get

$$L(v) \le -\frac{1}{64T}u_x^4 + \frac{C}{T^3}(T+\operatorname{osc}(u))^4.$$

Now, since (x_0,t_0) is a maximum point for v, we have $L(v)\ge 0$, which yields

$$u_x^4 \le \frac{C}{T^2}(T+\operatorname{osc}(u))^4.$$

Recalling that $\operatorname{osc}(u)$ is estimated from Proposition 4.2.9, we conclude the estimate. □

We now show that, under the present assumptions, (4.2.20)–(4.2.21) and (4.2.20)–(4.2.22) may be solved classically.

Theorem 4.2.11. *Assume that conditions* (4.2.23), (4.2.24), *and* (4.2.25) *hold true. Then the systems* (4.2.20)–(4.2.21) *and* (4.2.20)–(4.2.22) *have a classical solution* $(u, m) \in C^{2,\overline{\alpha}}(R\mathbb{T}\times [0, T]) \times C^{1,\overline{\alpha}}(R\mathbb{T} \times [0, T])$, *with* m *being unique. In the case of (4.2.21),* u *is unique, and in the case of (4.2.22),* u *is unique up to a constant.*

Proof. The uniqueness is a standard result, proved through duality. We will do the proof of existence for (4.2.20)–(4.2.22); the alternative case of (4.2.20)–(4.2.21) requires only minor modifications. We may approximate f and the marginals with $f^\varepsilon \in C^4(0, \infty)$, $m_0^\varepsilon, m_T^\varepsilon \in C^4(R\mathbb{T})$, such that $f^\varepsilon(m_0^\varepsilon), f^\varepsilon(m_T^\varepsilon)$ is uniformly bounded in $C^{1,\overline{\alpha}}(R\mathbb{T})$. Indeed, we may simply take

$$f^\varepsilon = f * \eta_\varepsilon, \quad m_0^\varepsilon = (f^\varepsilon)^{-1}(f^\varepsilon(m_0) * \eta_\varepsilon + c_{0,\varepsilon}), \quad m_T^\varepsilon = (f^\varepsilon)^{-1}(f^\varepsilon(m_T) * \eta_\varepsilon + c_{T,\varepsilon}), \tag{4.2.33}$$

where η^ε is the standard mollifier, and the non-negative numbers $c_{0,\varepsilon}, c_{T,\varepsilon}$ are adequately chosen such that $c_{0,\varepsilon} c_{T,\varepsilon} = 0$, $\int_{R\mathbb{T}} m_0^\varepsilon = \int_{R\mathbb{T}} m_T^\varepsilon$ and $\lim_{\varepsilon\to 0} c_{0,\varepsilon} = \lim_{\varepsilon\to 0} c_{T,\varepsilon} = 0$. With these regularized data, (4.2.20)–(4.2.22) has a unique classical solution $(u^\varepsilon, m^\varepsilon) \in C^3(R\mathbb{T} \times [0, T]) \times C^2(R\mathbb{T} \times [0, T])$ satisfying $\int_{R\mathbb{T}} \int_0^T u^\varepsilon = 0$ (see [43, Thm 1.1]). Moreover, in view of Propositions 4.2.9 and 4.2.10, the solution is bounded in $C^1 \times C^0$, uniformly in ε. The result will then follow by letting $\varepsilon \to 0$ and applying a version of Fiorenza's convergence result, 2.2.5 (see also [25, Lem. 17.29], [36, Lem. 2, Cor. 1], [22, 23]). For completeness, we sketch the details for this argument, which amounts to a proof of Fiorenza's result. From Lemma 4.2.1, the functions u^ε solve the oblique quasilinear elliptic problem

$$\begin{cases} -\mathrm{Tr}(A(u^\varepsilon_x, u^\varepsilon_t)D^2u^\varepsilon) = 0 & (x,t) \in R\mathbb{T} \times [0,T], \\ -u^\varepsilon_t(x,0) + \frac{1}{2}(u^\varepsilon_x)^2(x,0) = f(m^\varepsilon_0(x)) & x \in R\mathbb{T}, \\ -u^\varepsilon_t(x,T) + \frac{1}{2}(u^\varepsilon_x)^2(x,T) = f(m^\varepsilon_T(x)) & x \in R\mathbb{T}, \end{cases}$$

where

$$A(u^\varepsilon_x, u^\varepsilon_t) = \begin{pmatrix} (u^\varepsilon_x)^2 + m^\varepsilon f'(m^\varepsilon) & -u^\varepsilon_x \\ -u^\varepsilon_x & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (u^\varepsilon_x)^2 + f^{-1}(-u^\varepsilon_t + \frac{1}{2}(u^\varepsilon_x)^2)f'(f^{-1}(-u^\varepsilon_t + \frac{1}{2}(u^\varepsilon_x)^2)) & -u^\varepsilon_x \\ -u^\varepsilon_x & 1 \end{pmatrix}.$$

Since $\|u^\varepsilon\|_{C^1}$, $\|m^\varepsilon\|_\infty$ $\|(m^\varepsilon)^{-1}\|_\infty$ are uniformly bounded, this equation is uniformly elliptic, uniformly in ε. Thus, as a result of Lieberman's $C^{1,s}$ estimate for oblique problems (see [37, Lem. 2.3]), there exists $0 < s < 1$ and a constant $C > 0$ such that

$$\|u^\varepsilon\|_{C^{1,s}} \leq C.$$

Therefore if $0 < \varepsilon' < 1$, the difference $v = u^\varepsilon - u^{\varepsilon'}$ solves

$$\begin{cases} -\mathrm{Tr}(A^\varepsilon D^2 v) = \mathrm{Tr}((A^\varepsilon - A^{\varepsilon'})D^2u^{\varepsilon'}) & (x,t) \in R\mathbb{T} \times [0,T], \\ -v_t(x,0) + \frac{1}{2}(u^\varepsilon_x(x,0) + u^{\varepsilon'}_x(x,0))v_x(x,0) = f(m^\varepsilon_0(x)) - f(m^{\varepsilon'}_0(x)) & x \in R\mathbb{T}, \\ -v_t(x,T) + \frac{1}{2}(u^\varepsilon_x(x,T) + u^{\varepsilon'}_x(x,T))v_x(x,0) = f(m^\varepsilon_T(x)) - f(m^{\varepsilon'}_T(x)) & x \in R\mathbb{T}. \end{cases}$$

Let $\alpha' = \min(\overline{\alpha}, s)$. The standard Schauder estimates for linear oblique problems (see, for instance, [36, Lem. 1]) imply that

$$\|v\|_{C^{2,\alpha'}} \le C(\|v\|_\infty + \|u^{\varepsilon'}\|_{C^{2,\alpha'}}\|v\|_{C^1} + \|u^{\varepsilon'}\|_{C^2}\|v^\varepsilon\|_{C^{1,\alpha'}} + \|f(m_0^\varepsilon) - f(m_0^{\varepsilon'})\|_{C^{1,\alpha'}} \\ + \|f(m_T^\varepsilon) - f(m_T^{\varepsilon'})\|_{C^{1,\alpha'}} + \|v\|_{C^1}\|u^\varepsilon + u^{\varepsilon'}\|_{C^{2,\alpha'}}).$$

Recalling the interpolation inequality for Hölder spaces, $\|\cdot\|_{C^2} \le \delta\|\cdot\|_{C^{2,\overline{\alpha}}} + C_\delta\|\cdot\|_{C^0}$, we deduce that

$$\|u^\varepsilon - u^{\varepsilon'}\|_{C^{2,\alpha'}} \le C(o(1)(1+\|u^\varepsilon\|_{C^{2,\alpha'}}+\|u^{\varepsilon'}\|_{C^{2,\alpha'}})+\|u^{\varepsilon'}\|_{C^2}+1) \le \frac{1}{4}(\|u^\varepsilon\|_{C^{2,\alpha'}}+\|u^{\varepsilon'}\|_{C^{2,\alpha'}}))+C$$

as $\varepsilon, \varepsilon' \to 0$. Fixing a small ε', and letting $\varepsilon \to 0$, we see that $\|u^\varepsilon\|_{C^{2,\alpha'}}$ must be bounded. Now repeating the same argument but taking $\alpha' = \overline{\alpha}$, we see that $\|u^\varepsilon\|_{C^{2,\alpha}}$ must be bounded as well, which means u^ε converges to a solution $u \in C^{2,\overline{\alpha}}(R\mathbb{T} \times [0,T])$. In turn, the $C^{1,\overline{\alpha}}$ regularity of m follows from the HJ equation and the C^2 regularity of f. □

4.2.3 Continuity of the density

In this section, we will prove that the function $f(m)$ satisfies a uniform modulus of continuity, independent of $\min m$. This estimate is a crucial step in treating the setting of compactly supported solutions, because it will allow us to prove that the density is globally continuous, despite the lack of a positive lower bound.

Lemma 4.2.12. *Under the assumptions of Theorem 4.2.11, let (u,m) be a classical solution to* (4.2.20). *Then the function $v = f(m)$ satisfies the maximum principle and the minimum principle on each compact subset of $R\mathbb{T} \times [0,T]$.*

Proof. By approximation, we may assume that u and m are smooth. Hence $v = f(m)$

satisfies (4.2.4). Now, from the continuity equation, we have

$$mf'(m)u_{xx} = f(m)_t - f(m)_x u_x = v_t - v_x u_x,$$

and, therefore, substituting in (4.2.4) yields

$$Q(v) - v_x^2 + m(mf''(m) + 2f'(m))(mf'(m))^{-2}(v_t - v_x u_x)^2 = 0. \tag{4.2.34}$$

We notice that Q is a purely second order linear elliptic operator, and the remaining terms of (4.2.34) can be written as first–order terms in v. Thus, v satisfies an elliptic equation with no zero–order terms. This implies that v satisfies the maximum and the minimum principle on every compact subset of $R\mathbb{T} \times [0, T]$, as wanted. □

We now compute an energy estimate for the function v. Recall that κ_0 is given by (4.2.30).

Theorem 4.2.13. *Under the assumptions of Theorem 4.2.11, let* (u, m) *be a classical solution to* (4.2.20)–(4.2.21) *or* (4.2.20)–(4.2.22), *and let* $v = f(m)$. *Then there exists* C *such that*

$$\int_0^T \int_{R\mathbb{T}} |Dv|^2 \le C$$

where

$$C = \begin{cases} C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|g(m_0)\|_\infty) & \textit{if } (u, m) \textit{ solves } (4.2.20)\text{–}(4.2.21)\ , \\ C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}) & \textit{if } (u, m) \textit{ solves } (4.2.20)\text{–}(4.2.22), \end{cases}$$

Proof. Integrating (4.2.26) in space-time yields

$$\int_0^T \int_{R\mathbb{T}} mf'(m)u_{xx}^2 + f(m)_x^2 = \int_{R\mathbb{T}} f(m)_x u_x(0) - f(m)_x u_x(T) \le C - \int_{R\mathbb{T}} f(m)_x u_x(T),$$

where we used the gradient bound from Proposition 4.2.10. If (u, m) solves (4.2.22), then we use the bound on u_x and $f(m_T)_x$. If (u, m) solves (4.2.20), we have $-f(m)_x u_x(T) = -f'(m)g'(m)^{-1}u_x(T)^2 < 0$; so, in any case, we obtain

$$\int_0^T \int_{R\mathbb{T}} mf'(m)u_{xx}^2 + f(m)_x^2 \leq C\,. \tag{4.2.35}$$

The bound on $f(m)_t$ simply follows from the continuity equation:

$$f(m)_t^2 = (f(m)_x u_x + mf'(m)u_{xx})^2 \leq 2(f(m)_x^2 u_x^2 + (mf'(m))^2 u_{xx}^2) \leq C(f(m)_x^2 + mf'(m)u_{xx}^2),$$

where the bound on u_x and (4.2.30) were used in the last inequality. Integrating and using (4.2.35), we get the L^2 bound for $f(m)_t$. Finally, we have proved that

$$\int_0^T \int_{R\mathbb{T}} f(m)_t^2 + f(m)_x^2 \leq C,$$

where C depends on the same quantities as the bound of u_x in Proposition 4.2.10. $\square$

The interior continuity now follows from a classical computation, originally attributed to H. Lebesgue [35], which implies that a sufficient condition for a $W^{1,2}$ function in two variables to be continuous, is for it to satisfy the maximum and minimum principle.

Theorem 4.2.14 (Interior modulus of continuity)**.** *Under the assumptions of Theorem 4.2.11, let (u, m) be a classical solution to (4.2.20). Then the function $v = f(m)$ has the following logarithmic modulus of continuity, valid for all concentric balls B_{r_1} and B_{r_2}, $r_1 \leq r_2$ contained in Q_T.*

$$(\operatorname{osc}_{B_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq \pi \iint_{B_{r_2}} |Dv|^2.$$

Proof. Let $r \in [r_1, r_2]$, and let $\theta_1, \theta_2 \in [0, 2\pi]$. Then, using polar coordinates with origin at the center of the balls B_{r_i},

$$v(r,\theta_2) - v(r,\theta_1) = \int_{\theta_1}^{\theta_2} \frac{\partial v}{\partial \theta} d\theta.$$

Thus, integrating over a half circle and using the Cauchy-Schwarz inequality,

$$\mathrm{osc}_{\partial B_r}(v) \le \sqrt{\pi}\sqrt{\int_0^{2\pi} \left(\frac{\partial v}{\partial \theta}\right)^2 d\theta}.$$

Now, in view of Lemma 4.2.12, v satisfies the maximum and minimum principle, so

$$\mathrm{osc}_{B_{r_1}}(v) \le \mathrm{osc}_{\partial B_r}(v).$$

and, thus,

$$(\mathrm{osc}_{B_{r_1}}(v))^2 \le \pi \int_0^{2\pi} \left(\frac{\partial v}{\partial \theta}\right)^2 d\theta.$$

On the other hand, we have

$$|Dv|^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2 \ge \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2,$$

which then implies that

$$\frac{1}{r}(\mathrm{osc}_{B_{r_1}}(v))^2 \le \pi \int_0^{2\pi} r|Dv|^2 d\theta.$$

Integrating in r from r_1 to r_2 yields the result. □

A slight variant of this argument, by integrating over semi–disks instead, yields the continuity estimate up to the boundary.

Theorem 4.2.15 (Boundary modulus of continuity). *Under the assumptions of Theorem 4.2.11, let $x_0 \in R\mathbb{T}$ and D_{r_1} and D_{r_2} be upper semi–disks of radii $r_1 < r_2$, centered at*

$(x_0, 0)$. *Then we have*

$$(\operatorname{osc}_{D_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq 2\pi \int\int_{D_{r_2}} |Dv|^2 + 4(r_2^2 - r_1^2)\|v_x(0)\|_\infty^2.$$

Similarly, if D_{r_1} *and* D_{r_2} *are lower semi–disks centered at* (x_0, T)*, one has*

$$(\operatorname{osc}_{D_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq 2\pi \int\int_{D_{r_2}} |Dv|^2 + 4(r_2^2 - r_1^2)\|v_x(T)\|_\infty^2.$$

4.3 Finite speed of propagation and compactly supported solutions

This section, which is the main core of the chapter, is dedicated to the study of solutions to problems (MFG) and (MFGP) with *compactly supported density.*

We will first obtain a preliminary existence (and uniqueness) result, under fairly general conditions on the coupling functions f, g, assuming the initial density m_0 (and possibly the terminal density m_T) to be compactly supported. By a suitable choice of approximation of m_0, m_T, we will show the property of finite speed of propagation of the support, and the existence of a unique continuous solution with compactly supported density m. This will allow us to build a rigorous framework for the study of the free boundary $\partial\{(x,t) : m(x,t) > 0\}$, carried out in Subsection 4.3.2, and will be tightly connected to the analysis of the flow of optimal trajectories associated to the optimization problem. The study of the regularity and geometric properties of the free boundary, as well as of its spreading speed, will be analyzed for the model case of $f(m) = m^\theta$, $\theta > 0$. We conclude by establishing space-time Hölder regularity of the pair (m, Du) up to the free boundary.

4.3.1 Well-posedness results

Throughout this subsection, we will assume that

$$f, g \in C^2(0,\infty) \cap C[0,\infty),\ f' > 0,\ g' \geq 0, \text{ and (4.2.23) holds.} \tag{4.3.1}$$

For simplicity, and with no loss of generality, we will also normalize f so that

$$f(0) = 0. \tag{4.3.2}$$

As for the initial density (and possibly terminal density, in case of (MFGP)), we assume that m_0, $m_T \in C_c(\mathbb{R})$ are compactly supported, non-negative functions, such that

$$\int_{\mathbb{R}} m_0 = \int_{\mathbb{R}} m_T = 1, \tag{4.3.3}$$

and

$$m_0, m_T \text{ vanish, respectively, outside the intervals } [a_0, b_0],[a_1, b_1]. \tag{4.3.4}$$

Rather than (4.2.25), we will instead assume that, for some $\overline{\alpha} \in (0,1)$,

$$f(m_0) \in C^{1,\overline{\alpha}}(\{m_0 > 0\}), f(m_T) \in C^{1,\overline{\alpha}}(\{m_T > 0\}). \tag{4.3.5}$$

We remark that, in particular, this assumption implies that $f(m_0), f(m_T) \in W^{1,\infty}(\mathbb{R})$, but it allows for the possibility that $f(m_0)_x$ or $f(m_T)_x$ might be discontinuous at the boundary of the support, when considered as functions in $\mathbb{R}$.

Our first goal will be to show the well-posedness of systems (MFG) and (MFGP). Since we know, from the model case of Section 4.1 that, in general, the solution is not classical, we must work with a preliminary notion of generalized solution. We recall that $C_b(\mathbb{R} \times [0,T])$ denotes the space of bounded continuous functions.

Definition 4.3.1. We say that $(u, m) \in W^{1,\infty}(\mathbb{R} \times (0, T)) \times C_b(\mathbb{R} \times [0, T])$ is a solution to (MFG) (respectively, (MFGP)), if

(i) u is a viscosity solution to the HJ equation

$$-u_t + \frac{1}{2}u_x^2 = f(m) \quad (x, t) \in \mathbb{R} \times (0, T),$$

(ii) m satisfies the continuity equation

$$m_t - (mu_x)_x = 0 \quad (x, t) \in \mathbb{R} \times (0, T),$$

in the distributional sense, with $m(\cdot, 0) = m_0$.

(iii) We have $u(\cdot, T) = g(m(\cdot, T))$ (respectively, $m(\cdot, T) = m_T$).

Remark 4.3.2. Different notions of weak solutions could have been used, alternatively to Definition 4.3.1. In particular, distributional subsolutions of the HJ equation have been frequently used for MFG problems with a local coupling, both in case of final pay-off and in case of planning conditions (see e.g. [7], [12], [28], [47], and the survey [15]). That approach is tightly related to the concept of relaxed minima of variational problems, and avoids, for instance, any requirement of continuity and boundedness of m and u_x. Of course a similar approach would also apply to the present problems. However, since our primary goal in this chapter is the analysis of the free boundary for compactly supported solutions, it is more natural to work from the beginning with the stronger notion of continuous solutions. It is also natural, in that context, to make use of the standard framework of viscosity solutions for HJ equations.

Our goal in this subsection will be to prove the following well-posedness result. In what follows, $C_c(\mathbb{R})$ denotes the space of continuous, compactly supported functions on $\mathbb{R}$. Notice that, in particular, the result shows that the unique solution is such that m has compact support.

Theorem 4.3.3. *Assume that f, g satisfy* (4.3.1), $m_0, m_T \in C_c(\mathbb{R})$ *satisfy* (4.3.3)–(4.3.5), *and κ_0 is as in* (4.2.30). *Then the following holds.*

1. *There exists a unique solution (u, m) to* (MFG). *Moreover, $(u, f(m)) \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T))\cap\{m > 0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T))\cap\{m>0\})$. There exists a constant*

$$C = C(T, T^{-1}, \kappa_0, \|f(m_0)\|_{W^{1,\infty}}, |\text{supp}(m_0)|, \|g(m_0)\|_\infty)$$

such that

$$\|u\|_\infty \le \|f(m_0)\|_\infty T + \|g(m_0)\|_\infty; \qquad \|m\|_\infty \le \|m_0\|_\infty; \qquad \text{supp}(m) \subset [-C, C]\times[0,T],$$

$$\|Du\|_{L^\infty} \le C, \tag{4.3.6}$$

$$\int_0^T \int_{\mathbb{R}} |D(f(m))|^2 \le C, \tag{4.3.7}$$

and, for $(x,t), (\overline{x}, \overline{t}) \in \mathbb{R}\times[0,T]$,

$$|f(m(x,t)) - f(m(\overline{x},\overline{t}))| \le \frac{C}{\sqrt{\log(|x-\overline{x}|^2 + |t-\overline{t}|^2)_-}}. \tag{4.3.8}$$

Finally, if $g' > 0$ or $g \equiv 0$, then $(u, f(m)) \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m>0\})\times C^{1,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m > 0\})$.

2. *There exists a solution (u, m) to* (MFGP). *The function m is unique, u is unique up to a constant on each connected component of $\{m > 0\}$, $(u, f(m)) \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m > 0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m > 0\})$. Moreover, there exist constants $K, C > 0$,*

with

$$K = K(T, T^{-1}, \kappa_0, \|f(m_0)\|_\infty, \|f(m_T)\|_\infty, |\mathrm{supp}(m_0)|, |\mathrm{supp}(m_T)|, \mathrm{dist}(\mathrm{supp}(m_0), \mathrm{supp}(m_T))),$$

$$C = C(K, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}),$$

such that

$$\mathrm{osc}(u) \leq K; \qquad \|m\|_\infty \leq \max(\|m_0\|_\infty, \|m_T\|_\infty); \qquad \mathrm{supp}(m) \subset [-C, C] \times [0, T],$$

and (4.3.6)*,* (4.3.7)*,* (4.3.8) *hold.*

Remark 4.3.4. We note that, due to lack of uniqueness for u in $\{m = 0\}$ for the case of (MFGP), the explicit a priori estimate (4.3.6) for the solutions is limited to the support of m. However, the proof of Theorem 4.3.3 will show that there exists a solution to (MFGP) satisfying the global estimate

$$\|u\|_{W^{1,\infty}(\mathbb{R})} \leq C,$$

where C depends on the data as described above.

From periodic to Neumann boundary conditions

As anticipated in Section 4.2, we intend to use the estimates for the periodic setting in order to build a solution to (MFG) and (MFGP). However, it will be convenient to switch from periodic to Neumann boundary conditions. Indeed, we will see later that this simplifies the analysis of the optimal trajectories, since they do not "wrap around" the domain as they do in the periodic case. The following result shows that we may switch to this point of view while preserving all of the estimates for the periodic setting.

Theorem 4.3.5. *Assume that f, g satisfy* (4.3.1)*, and let $0 < \overline{\alpha} < 1$. Assume that the positive functions m_0^ε, $m_T^\varepsilon \in C(\mathbb{R})$ are such that $m_0^\varepsilon, m_T^\varepsilon \equiv \varepsilon$ outside of the interval $[-r, r]$,*

$\int_{-r}^{r} m_0^\varepsilon = \int_{-r}^{r} m_T^\varepsilon$, and $f(m_0^\varepsilon), f(m_T^\varepsilon) \in C^{1,\overline{\alpha}}(\mathbb{R})$. Assume that $g' > 0$, and that (4.2.23) holds, and let $R > r$. Consider the system

$$\begin{cases} -u_t^\varepsilon + \frac{1}{2}(u_x^\varepsilon)^2 = f(m^\varepsilon) & (x,t) \in [-R,R] \times (0,T) \\ m_t^\varepsilon - (m^\varepsilon u_x^\varepsilon)_x = 0 & (x,t) \in [-R,R] \times (0,T) \\ m^\varepsilon(x,0) = m_0^\varepsilon(x), & x \in [-R,R] \\ u_x^\varepsilon(-R,t) = u_x^\varepsilon(R,t) = 0 & t \in [0,T], \end{cases} \tag{4.3.9}$$

where either $u^\varepsilon(\cdot,T) = g(m^\varepsilon(\cdot,T))$ or $m^\varepsilon(\cdot,T) = m_T^\varepsilon$.

(i) There exists a unique solution $(u^\varepsilon, m^\varepsilon) \in C^{2,\overline{\alpha}}([-R,R]\times[0,T]) \times C^{1,\overline{\alpha}}([-R,R]\times[0,T])$ to (4.3.9) satisfying $u^\varepsilon(\cdot,T) = g(m^\varepsilon(\cdot,T))$. Moreover, there exists a constant $C = C(R,T,T^{-1},\kappa_0,\|f(m_0^\varepsilon)\|_{W^{1,\infty}},\|g(m_0^\varepsilon)\|_\infty)$, such that

$$\|u^\varepsilon\|_\infty \le \|f(m_0^\varepsilon)\|_\infty T + \|g(m_0^\varepsilon)\|_\infty; \qquad \min m_0^\varepsilon \le m^\varepsilon \le \max m_0^\varepsilon, \tag{4.3.10}$$

$$\|Du^\varepsilon\|_\infty \le C; \qquad \int_0^T \int_{-R}^R |D(f(m^\varepsilon))|^2 \le C, \tag{4.3.11}$$

and, for each $(x,t),(\overline{x},\overline{t}) \in [-R,R]\times[0,T]$,

$$|f(m^\varepsilon(x,t)) - f(m^\varepsilon(\overline{x},\overline{t}))| \le C \left(\frac{\int_0^T \int_{-R}^R |D(f(m^\varepsilon))|^2}{\log(|x-\overline{x}|^2 + |t-\overline{t}|^2)_-} \right)^{\frac{1}{2}}. \tag{4.3.12}$$

(ii) Up to an additive constant for u^ε, there exists a unique solution $(u^\varepsilon, m^\varepsilon) \in C^{2,\overline{\alpha}}([-R,R]\times[0,T]) \times C^{1,\overline{\alpha}}([-R,R]\times[0,T])$ to (4.3.9) satisfying $m^\varepsilon(\cdot,T) = m_T^\varepsilon$. Moreover, there exist constants $K, C > 0$, with

$$K = K(R,T,T^{-1},\kappa_0,\|f(m_0^\varepsilon)\|_\infty,\|f(m_T^\varepsilon)\|_\infty)\,,\; C = C(K,\|f(m_0^\varepsilon)\|_{W^{1,\infty}},\|f(m_T^\varepsilon)\|_{W^{1,\infty}})$$

such that

$$\operatorname{osc}(u^\varepsilon) \le K, \quad \min(\min m_0^\varepsilon, \min m_T^\varepsilon) \le m^\varepsilon \le \max(\max m_0^\varepsilon, \max m_T^\varepsilon), \tag{4.3.13}$$

and (4.3.11), (4.3.12) hold.

Proof. For concreteness, we focus on the planning case, $m^\varepsilon(\cdot, T) = m_T^\varepsilon$. For each $x \in [0, 2R]$, define $\tilde{m}_0(x) = m_0^\varepsilon(x - R)$, and $\tilde{m}_T(x) = m_T^\varepsilon(x - R)$. Then these functions can be naturally extended to even, periodic functions $\tilde{m}_0, \tilde{m}_T \in 4R\mathbb{T}$. With these marginals, the solution $(\tilde{u}, \tilde{m})$ to (4.2.20)–(4.2.22), given by Theorem 4.2.11, is even, as well as symmetric with respect to $x = \pm 2R$. In particular, we have

$$\tilde{u}_x(0, t) = \tilde{u}_x(\pm 2R, t) \equiv 0.$$

As a result, the functions $(u^\varepsilon, m^\varepsilon)$ defined by

$$u^\varepsilon(x, t) = \tilde{u}(x + R, t),\ m^\varepsilon(x, t) = \tilde{m}(x + R, t)$$

are classical solutions to (4.3.9). The estimates on $(u^\varepsilon, m^\varepsilon)$ readily follow by applying Corollary 4.2.7 and Propositions 4.2.9, 4.2.10, 4.2.13, 4.2.14, and 4.2.15 to the function $(\tilde{u}, \tilde{m})$. A similar discussion yields the result for the final cost problem, $u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T))$. $\square$

An estimate on the flow of optimal trajectories

Given a solution $(u^\varepsilon, m^\varepsilon)$ to (4.3.9), we may define the flow of optimal trajectories

$$\gamma^\varepsilon : [-R, R] \times [0, T] \to \mathbb{R}$$

according to (4.2.8). We remark that, when $x = \pm R$, the solution is the constant curve $\gamma^\varepsilon(x,\cdot) \equiv x$. Additionally, since $u^\varepsilon \in C^{2,\overline{\alpha}}([-R,R]\times[0,T])$, we have $\gamma^\varepsilon \in C^2([-R,R]\times[0,T])$, and $\gamma^\varepsilon_x > 0$. We begin by showing that the trajectories starting in the support of m_0 remain in a bounded set, independently of R.

Theorem 4.3.6 (Finite propagation of the support)**.** *Under the assumptions of Proposition 4.3.5, let $R \geq 1$, $0 < \varepsilon < 1$, and let $(u^\varepsilon, m^\varepsilon)$ be a solution to (4.3.9), and let γ^ε be the flow of trajectories associated to $(u^\varepsilon, m^\varepsilon)$. Then there exists a constant $\overline{r} = \overline{r}(r, T, T^{-1}, \kappa_0, \|f(m_0^\varepsilon)\|_{W^{1,\infty}}, \|f(m_T^\varepsilon)\|_{W^{1,\infty}}, \|g(m_0^\varepsilon)\|_\infty)$, such that*

$$\|\gamma^\varepsilon\|_{[-r,r]\times[0,T]} \leq \overline{r}.$$

Proof. For simplicity, we write $\gamma^\varepsilon = \gamma$. We will treat separately the planning problem and the final cost problem. First, assume that $m^\varepsilon(\cdot,T) = m_T^\varepsilon$. In view of Lemma 4.2.3, and the facts that $\gamma(-R,\cdot) \equiv -R$, and $m_0 \equiv 0$ outside of $[-r,r]$, we have

$$\int_{-R}^{y} m_T^\varepsilon = \int_{-R}^{-r} m_0^\varepsilon = (R-r)\varepsilon,$$

where $y = \gamma(-r,T)$. Now, the left hand side is (strictly) increasing in y, and

$$\int_{-R}^{-r} m_T^\varepsilon = (R-r)\varepsilon,$$

thus $\gamma(-r,T) = y = -r$. Similarly $\gamma(r,T) = r$. Since $\gamma_x > 0$, this implies that $\gamma([-r,r],T) \subset [-r,r]$. Now, given $x \in [-r,r]$, we recall that

$$\gamma(x,\cdot) = \underset{\beta\in H^1(0,T),\beta(0)=x}{\arg\min} \int_0^T \frac{1}{2}|\dot{\beta}|^2 + f(m^\varepsilon(\beta(t),t))dt + u^\varepsilon(\beta(T),T).$$

Therefore, defining $\beta : [0,T] \to [-r,r]$ to be the straight line segment connecting x and

$\gamma(x,T)$, we have

$$\int_0^T \left(\frac{1}{2}|\gamma_t(x,t)|^2 + f(m^\varepsilon(\gamma(x,t),t))\right) dt + u^\varepsilon(\gamma(x,T),T)$$
$$\leq \int_0^T \left(\frac{1}{2}|\dot{\beta}|^2 + f(m^\varepsilon(\beta(x,t),t))\right) dt + u^\varepsilon(\gamma(x,T),T), \quad (4.3.14)$$

that is,

$$\int_0^T \frac{1}{2}|\gamma_t(x,t)|^2 dt \leq T\left(2\frac{r^2}{T^2} + f(\max m^\varepsilon) - f(\min m^\varepsilon)\right) \leq C.$$

In particular, given $t \in [0,T]$, we have

$$|\gamma(x,t)| \leq r + |\gamma(x,t) - \gamma(x,0)| \leq r + \sqrt{2t}\sqrt{\int_0^T \frac{1}{2}|\gamma_t|^2} \leq r + \sqrt{C} \leq C.$$

This proves the result for the planning case. Next, we assume that $u^\varepsilon(\cdot,T) = g(m^\varepsilon(\cdot,T))$. Observe that, in this case, while we do not know that $\gamma([-r,r],T) \subset [-r,r]$, we instead observe that u^ε is bounded independently of R, due to Proposition 4.3.5. Therefore, we have

$$\int_0^T \frac{1}{2}|\gamma_t(x,t)|^2 dt = u^\varepsilon(x,0) - u^\varepsilon(\gamma(x,T),T) - \int_0^T f(m^\varepsilon(\gamma(x,t),t)dt \leq C.$$

□

Compatible approximations and existence of solutions

In this section, we will apply Proposition 4.3.5 to prove the well-posedness result, Theorem 4.3.3. For that purpose, we will now build suitable $C^{1,\overline{\alpha}}(\mathbb{R})$ approximations $m_0^\varepsilon > 0$ through a mild regularization procedure. For (MFGP), we must also build m_T^ε, while requiring that the two are suitably compatible, in the sense that they have the same mass. Furthermore, since these approximations will be needed later to prove finer results about the free boundary, we will ensure that they are also compatible in the sense that $[a_0, b_0]$ is mapped bijectively onto $[a_1, b_1]$ by the flow $\gamma^\varepsilon(\cdot, T)$.

Definition 4.3.7. Given $r > 0$, we say that $(m_0^\varepsilon, m_T^\varepsilon)$ is a r–compatible approximation of (m_0, m_T) if $(m_0^\varepsilon, m_T^\varepsilon)$ satisfies the assumptions of Proposition 4.3.5, $m_0^\varepsilon \to m_0$, $m_T^\varepsilon \to m_T$ uniformly, and, for every $R > r$,

$$\int_{-R}^{a_0} m_0^\varepsilon = \int_{-R}^{a_1} m_T^\varepsilon \text{ and } \int_{b_0}^{R} m_0^\varepsilon = \int_{b_0}^{R} m_T^\varepsilon. \tag{4.3.15}$$

Lemma 4.3.8 (Construction of r–compatible approximations)**.** *Under the assumption of Proposition 4.3.5, let $r > 0$ be such that*

$$[a_0 - 2, b_0 + 2], [a_1 - 2, b_1 + 2] \subset [-r, r]. \tag{4.3.16}$$

There exists a pair of vector-valued functions $\eta_0, \eta_T \in C_c^\infty(\mathbb{R}, [0, \infty)^3)$ such that, for every $\varepsilon \in (0, 1)$, there exist constant vectors $c_{0,\varepsilon}, c_{T,\varepsilon} \in [0, \infty)^3$ such that the pair $(m_0^\varepsilon, m_T^\varepsilon)$ defined by

$$m_0^\varepsilon = f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon} \cdot \eta_0),\ m_T^\varepsilon = f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2} + c_{T,\varepsilon} \cdot \eta_T) \tag{4.3.17}$$

is a r-compatible approximation of (m_0, m_T). Moreover, we have

$$\lim_{\varepsilon \to 0^+} c_{0,\varepsilon} = \lim_{\varepsilon \to 0^+} c_{T,\varepsilon} = 0, \tag{4.3.18}$$

$$\eta_0 \equiv 0 \text{ in } \left[a_0 - 1, \frac{1}{3}(2a_0 + b_0)\right] \cup \left[\frac{1}{3}(a_0 + 2b_0), b_0 + 1\right],$$
$$\eta_T \equiv 0 \text{ in } \left[a_1 - 1, \frac{1}{3}(2a_1 + b_1)\right] \cup \left[\frac{1}{3}(a_1 + 2b_1), b_1 + 1\right], \tag{4.3.19}$$

and there exists a constant $C = C(|\mathrm{supp}(m_0)|, |\mathrm{supp}(m_T)|, \mathrm{dist}(\mathrm{supp}(m_0), \mathrm{supp}(m_T)))$ *such that*

$$\|f(m_j^\varepsilon)\|_{W^{1,\infty}(\mathbb{R})} \le C\|f(m_j)\|_{W^{1,\infty}(\mathbb{R})} + f(\varepsilon) + |c_{0,\varepsilon}| \cdot \|\eta_j\|_{C^1(\mathbb{R},\mathbb{R}^3)},\ j \in \{0,T\}, \tag{4.3.20}$$

$$[f(m_j^\varepsilon)_x]_{C^{\overline{\alpha}}(\mathbb{R})} \le \frac{C}{f(\varepsilon)}(1 + [f(m_j)_x]_{C^{\overline{\alpha}}(\{m_j>0\})}) + |c_{0,\varepsilon}|[(\eta_j)_x]_{C^{\overline{\alpha}}(\mathbb{R},\mathbb{R}^3)},\ j \in \{0,T\} \tag{4.3.21}$$

$$[f(m_j^\varepsilon)_x]_{C^{\overline{\alpha}}(\{m_j\ge\delta\})} \le \frac{C}{f(\delta)}(1 + [f(m_j)_x]_{C^{\overline{\alpha}}(\{m_j>0\})}) + \frac{C}{f(\delta)}|c_{0,\varepsilon}|[(\eta_j)_x]_{C^{\overline{\alpha}}(\mathbb{R},\mathbb{R}^3)},\ j \in \{0,T\} \tag{4.3.22}$$

Finally if $j \in \{0,T\}$ *and* $f(m_j)$ *is semi–convex, then*

$$\|f(m_j^\varepsilon)_{xx}^-\|_{L^\infty(\mathbb{R})} \le \|f(m_j)_{xx}^-\|_{L^\infty(\mathbb{R})} + |c_{0,\varepsilon}| \cdot \|(\eta_j)_{xx}^-\|_{L^\infty(\mathbb{R})}. \tag{4.3.23}$$

Proof. We let $\eta_{0,1}, \eta_{0,2}, \eta_{0,3} \in C_c^\infty(\mathbb{R})$ be non–zero, non–negative bump functions supported, respectively, on

$$[a_0 - 2, a_0 - 1],\ \left[\frac{1}{3}(2a_0 + b_0), \frac{1}{3}(a_0 + 2b_0)\right],\ \text{and } [b_0 + 1, b_0 + 2].$$

Similarly, we take $\eta_{T,1}, \eta_{T,2}, \eta_{T,3} \in C_c^\infty(\mathbb{R})$ to be supported, respectively, on

$$[a_1 - 2, a_1 - 1],\ \left[\frac{1}{3}(2a_1 + b_1), \frac{1}{3}(a_1 + 2b_1)\right],\ \text{and } [b_1 + 1, b_1 + 2].$$

With this, we can define

$$\eta_0 = (\eta_{0,1}, \eta_{0,2}, \eta_{0,3}),\ \ \eta_T = (\eta_{T,1}, \eta_{T,2}, \eta_{T,3}).$$

We observe first that (4.3.19) holds by construction. Next, note that (4.3.16) guarantees that $m_0^\varepsilon, m_T^\varepsilon \equiv \varepsilon$ outside of $[-r, r]$. For this reason, (4.3.15) will hold for all $R > r$ as long as it holds for $R = r$. Now, we have

$$\int_{-r}^{a_0} m_0^\varepsilon = \int_{-r}^{a_0} f^{-1}(f(\varepsilon) + c_{0,\varepsilon}^1 \eta_{0,1}), \text{ and } \int_{-r}^{a_1} m_T^\varepsilon = \int_{-r}^{a_1} f^{-1}(f(\varepsilon) + c_{T,\varepsilon}^1 \eta_{T,1}).$$

If, say, $a_0 \leq a_1$, then, taking $c_{T,\varepsilon}^1 = 0$, there exists a unique choice of $c_{0,\varepsilon}^1 \geq 0$ that ensures

$$\int_{-r}^{a_0} m_0^\varepsilon = \int_{-r}^{a_1} m_T^\varepsilon.$$

Similarly, if $a_1 < a_0$, we may take $c_{0,\varepsilon}^1 = 0$ and choose an adequate $c_{T,\varepsilon}^1 > 0$. By the same reasoning, there exists a choice of $c_{0,\varepsilon}^3, c_{T,\varepsilon}^3$ such that

$$\int_{b_0}^{r} m_0^\varepsilon = \int_{b_0}^{r} m_T^\varepsilon.$$

We must also ensure that $m_0^\varepsilon, m_T^\varepsilon$ have equal mass. For this purpose, we observe that

$$\int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_0}^{b_0} f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon}^2 \eta_{0,2}), \int_{a_1}^{b_1} m_T^\varepsilon = \int_{a_1}^{b_1} f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2} + c_{T,\varepsilon}^2 \eta_{T,2}).$$

And, as before, depending on which of the quantities

$$\int_{a_0}^{b_0} f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2}), \quad \int_{a_1}^{b_1} f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2}) \tag{4.3.24}$$

is larger, we may choose one of the numbers $c_{0,\varepsilon}^2, c_{T,\varepsilon}^2$ to be zero, which leaves a unique way to choose the remaining one in such a way that

$$\int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_1}^{b_1} m_T^\varepsilon.$$

This, together with (4.3.15), guarantees that the two functions have the same integral over $[-R, R]$. Now, (4.3.18) is a straightforward consequence of (4.3.3), (4.3.4), and the fact that the vectors $c_{0,\varepsilon}, c_{T,\varepsilon}$ were chosen so that

$$c^i_{0,\varepsilon} c^i_{T,\varepsilon} = 0 \text{ for } i \in \{1,2,3\}.$$

It remains to show the Lipschitz, $C^{1,\overline{\alpha}}$, and semi-convexity estimates on $f(m^\varepsilon_0), f(m^\varepsilon_T)$. For concreteness, we will only show these for $f(m^\varepsilon_0)$, since the arguments for $f(m^\varepsilon_T)$ are identical. We have

$$f(m^\varepsilon_0) = \sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon} \cdot \eta_0\,; \qquad f(m^\varepsilon_0)_x = \frac{f(m_0) f(m_0)_x}{\sqrt{f(m_0)^2 + f(\varepsilon)^2}} + c_{0,\varepsilon} \cdot (\eta_0)_x,$$

which readily yields (4.3.20). Now, given $x, y \in [-r, r]$, we may write

$$\begin{aligned}
&f(m^\varepsilon_0)_x(x) - f(m^\varepsilon_0)_x(y) = \\
&c_{0,\varepsilon} \cdot ((\eta_0)_x(x) - (\eta_0)_x(y)) + \frac{(f(m_0)(x) - f(m_0)(y)) f(m_0)_x(x)}{\sqrt{f(m_0)^2(x) + f(\varepsilon)^2}} + f(m_0)(y) f(m_0)_x(y) G_1(x,y) \\
&\qquad = c_{0,\varepsilon} \cdot ((\eta_0)_x(x) - (\eta_0)_x(y)) + \frac{(f(m_0)(x) - f(m_0)(y)) f(m_0)_x(x)}{\sqrt{f(m_0)^2(x) + f(\varepsilon)^2}} \\
&\qquad\quad + f(m_0)(y) \frac{f(m_0)_x(x) - f(m_0)_x(y)}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}} + f(m_0)(y) f(m_0)_x(y) G_2(x,y),
\end{aligned} \tag{4.3.25}$$

where

$$G_1(x,y) = \frac{1}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}} - \frac{1}{\sqrt{f(m_0)(y)^2 + f(\varepsilon)^2}}, \tag{4.3.26}$$

$$G_2(x,y) = \frac{(f(m_0)(y) - f(m_0)(x))(f(m_0)(y) + f(m_0)(x))}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}\sqrt{f(m_0)(y)^2 + f(\varepsilon)^2}(\sqrt{f(m_0)(y)^2 + f(\varepsilon)^2} + \sqrt{f(m_0)(x)^2 + f(\varepsilon)^2})}. \tag{4.3.27}$$

Thus, we obtain

$$
\begin{aligned}
|f(m_0^\varepsilon)_x(x) - f(m_0^\varepsilon)_x(y)| \le |c_{0,\varepsilon}|[\eta_0]_{\overline{\alpha}}|x-y|^{\overline{\alpha}} \\
+ \frac{C}{f(\varepsilon)}\|f(m_0)_x\|_\infty^2|x-y|^{\overline{\alpha}} + \frac{1}{f(\varepsilon)}\|f(m_0)\|_\infty[f(m_0)_x]_{C^{\overline{\alpha}}(\{m_0>0\})}|x-y|^{\overline{\alpha}} \\
+ \frac{C}{f(\varepsilon)}\|f(m_0)_x\|_\infty^2|x-y|^{\overline{\alpha}},
\end{aligned}
$$

which yields (4.3.21). Similarly, if $x, y \in \{m_0 \ge \delta\}$, we obtain (4.3.22). Finally, if m_0 is smooth, then

$$
\begin{aligned}
f(m_0^\varepsilon)_{xx} &= \frac{f(m_0)f(m_0)_{xx}}{\sqrt{f(m_0)^2+f(\varepsilon)^2}} + \frac{f(m_0)_x^2}{\sqrt{f(m_0)^2+f(\varepsilon)^2}} - \frac{f(m_0)^2f(m_0)_x^2}{(f(m_0)^2+f(\varepsilon)^2)^{\frac{3}{2}}} + c_{0,\varepsilon}\cdot(\eta_0)_{xx} \\
&\ge -\|f(m_0)_{xx}^-\|_\infty + \frac{f(m_0)_x^2 f(\varepsilon)^2}{(f(m_0)^2+f(\varepsilon)^2)^{\frac{3}{2}}} - |c_{0,\varepsilon}|\|(\eta_0)_{xx}\|_\infty \\
&\ge -\|f(m_0)_{xx}^-\|_\infty - |c_{0,\varepsilon}|\|(\eta_0)_{xx}\|_\infty,
\end{aligned}
$$

which shows (4.3.23). For a non-smooth m_0, the result then follows by standard approximation. $\square$

Having constructed the necessary approximations, we are now ready to prove the well-posedness theorem.

Proof of Theorem 4.3.3. We begin with the case of (MFGP). We first prove existence of a solution satisfying the estimates. For this purpose, we choose the r-compatible approximations $(m_0^\varepsilon, m_T^\varepsilon)$ as defined by (4.3.17), where $r > 0$ is fixed and chosen large enough to satisfy (4.3.16), and take $R > 2\overline{r}$, where $\overline{r}$ is the constant of Proposition 4.3.6. In view of (4.3.20) from Lemma 4.3.8, the constant C of Proposition 4.3.5 may be chosen independently of ε. In particular, from (4.3.13), we may choose the solutions $(u^\varepsilon, m^\varepsilon)$ of Proposition 4.3.5 in such a way that $\|u^\varepsilon\|_\infty \le C$. Therefore, due to the bounds (4.3.11) and (4.3.12), the family $\{(u^\varepsilon, m^\varepsilon)\}_{\varepsilon\in(0,1)}$ is bounded and equicontinuous, and we may extract a subsequence and conclude that $(u^\varepsilon, m^\varepsilon) \to (u, m) \in W^{1,\infty}([-R, R] \times [0, T]) \times C([-R, R] \times [0, T])$, with (u, m) satisfying the required estimates.

Moreover, if $(x, t) \in \{m > 0\}$, by the equicontinuity of $\{m^\varepsilon\}_{\varepsilon\in(0,1)}$, there exists an open set V containing (x, t) such that $f'(m^\varepsilon)m^\varepsilon > 0$, so that u^ε satisfies a uniformly elliptic equation in V (recall (4.2.1)). As a result, if $0 < t < T$, the standard interior $C^{1,\overline{\alpha}}$ estimates followed by the Schauder estimates (see [25, Thm. 13.6] and [25, Thm. 6.2], respectively) imply that $(u, m) \in C^{2,\overline{\alpha}} \times C^{1,\overline{\alpha}}$ in a neighborhood of (x, t). On the other hand, if $t \in \{0, T\}$, by proceeding as in the proof of Theorem 4.2.11, through the $C^{1,\overline{\alpha}}$ estimates for oblique problems, followed by Fiorenza's convergence argument (as detailed in the proof of Theorem 4.2.11), we see that $(u, m) \in C^{2,\overline{\alpha}} \times C^{1,\overline{\alpha}}$ in a neighborhood of (x, t). In particular, this shows that u solves the HJ equation in the pointwise sense at (x, t).

Now, we claim that $m \in C_c(\mathbb{R} \times [0, T])$, with $\operatorname{supp}(m) \subset [-\overline{r}, \overline{r}] \times [0, T]$. Indeed, this follows from the fact that, for each $t \in [0, T]$, $m(\cdot, t)$ has mass 1 and, by Lemma 4.2.3 and Proposition 4.3.6,

$$\int_{-\overline{r}}^{\overline{r}} m(\cdot, t) = \lim_{\varepsilon\downarrow 0} \int_{-\overline{r}}^{\overline{r}} m^\varepsilon(\cdot, t) \geq \lim_{\varepsilon\downarrow 0} \int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_0}^{b_0} m_0 = 1.$$

On the other hand, we can extend u to $W^{1,\infty}(\mathbb{R} \times (0, T))$ in the following way:

$$u(x, t) = \begin{cases} u(-R, t) & \text{if } x < -R \\ u(R, t) & \text{if } x > R. \end{cases}$$

Notice that, since $u_x^\varepsilon(\pm R, \cdot) \equiv 0$,

$$-u_t^\varepsilon(\pm R, t) = f(m^\varepsilon(\pm R, t)) \to f(0) \text{ uniformly in } t,$$

and, thus,

$$u^{\varepsilon}(\pm R, t) \to f(0)(T-t) + u(\pm R, T) \text{ in } C^{1,1}([0,T]).$$

In particular, we see that, for $|x| > R$,

$$-u_t + \frac{1}{2}u_x^2 = f(0) = f(m).$$

It is then straightforward to verify, by using the basic stability property of viscosity solutions under uniform convergence (see [17]), that this extension (u, m) is a solution to (MFGP), in the sense of Definition 4.3.1, which satisfies all the necessary estimates.

Now, to prove uniqueness, we assume that $(\tilde{u}, \tilde{m})$ is another solution. Since the function $\tilde{u}$ is a viscosity solution to the HJ equation, and it is almost everywhere differentiable, it must also satisfy the equation pointwise almost everywhere. Therefore, it also solves the equation in the distributional sense. Noting that the pairs $(u, m), (\tilde{u}, \tilde{m}) \in W^{1,\infty}(\mathbb{R} \times (0,T)) \times C_b(\mathbb{R} \times [0,T])$ are sufficiently regular to serve as test functions, we may apply the standard Lasry-Lions computation to the pair (u, m) and $(\tilde{u}, \tilde{m})$, obtaining

$$\int_0^T \int_{\mathbb{R}} \frac{1}{2}(m + \tilde{m})|u_x - \tilde{u}_x|^2 + (m - \tilde{m})(f(m) - f(\tilde{m})) = 0.$$

Therefore, since the left hand side is non-negative, we conclude that $m = \tilde{m}$, and $u_x = \tilde{u}_x$ in $\{m > 0\}$. Finally, since m is continuous, $\{m > 0\}$ is an open set, and thus u and $\tilde{u}$ at most differ by a constant on each connected component of $\{m > 0\}$. For the case of (MFG), the proof is completely analogous, noting that in applying Proposition 4.3.5, we take $g^{\varepsilon}(m) = g(m) + \varepsilon m$, to satisfy the strict monotonicity assumption. We note that, unless $g' > 0$, the Lieberman and oblique Schauder estimates may not be applied at $t = T$, hence the weaker conclusion $(u, f(m)) \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T)) \cap \{m > 0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T)) \cap \{m > 0\})$. However, in the special case $g \equiv 0$, u satisfies a smooth Dirichlet boundary condition at $t = T$, and thus we may apply the Ladyzhenskaya-Uralt'seva and Schauder estimates for the

Dirichlet problem (see [25, Thm 13.7, Thm 6.19]) to the limiting function u to still obtain the $C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m>0\})$ regularity. Note that global uniqueness follows because, since m is unique, the terminal value $u(\cdot,T)=g(m(\cdot,T))$ is uniquely determined. $\square$

4.3.2 Analysis of the free boundary

Having established the existence and uniqueness of solutions, with compactly supported density, we now study the set $\partial(\{m>0\})$, where (u,m) is the solution to (MFG) or (MFGP). Henceforth, we restrict our analysis to the case that, for some given constant $\theta>0$, we have

$$f(m)=m^{\theta}\,,\qquad \theta>0\,. \tag{4.3.28}$$

We will focus on the setting in which the initial distribution m_0 (and possibly m_T, in case of problem (MFGP)) are each supported exactly on an interval. That is, we will assume that

$$\{m_0>0\}=(a_0,b_0)\text{ and }\{m_T>0\}=(a_1,b_1), \tag{4.3.29}$$

for some finite a_0,b_0,a_1,b_1. Moreover, we will assume that m_0 and m_T decay like powers near the endpoints of these intervals. In other words, there exist numbers $\alpha_0,\alpha_1>0$ such that

$$\frac{1}{C_0}\text{dist}(x,\{a_0,b_0\})^{\alpha_0}\le m_0(x)\le C_0\text{dist}(x,\{a_0,b_0\})^{\alpha_0},\qquad x\in[a_0,b_0], \tag{4.3.30}$$

$$\frac{1}{C_1}\text{dist}(x,\{a_1,b_1\})^{\alpha_1}\le m_T(x)\le C_1\text{dist}(x,\{a_1,b_1\})^{\alpha_1},\qquad x\in[a_1,b_1]. \tag{4.3.31}$$

For concreteness, we will also assume here that

$$\alpha_0\ge\alpha_1, \tag{4.3.32}$$

but we remark that the opposite case can be studied, with analogous results, by simply considering the time-reversed functions $(-u(x, T-t), m(x, T-t))$, which solve (MFGP), but with the roles of m_0 and m_T reversed.

Correspondingly, in case of problem (MFG), we assume that the coupling function g at final time satisfies, for some given constants $\theta_1 > 0$, $c_1 \geq 0$, with $\theta_1 \geq \theta$,

$$g(m) = c_1 T m^{\theta_1}. \tag{4.3.33}$$

Remark 4.3.9. The factor of T in the definition of g is made explicit in order to adequately state the sharp long time behavior result, Theorem 4.3.15. This is the natural scaling for the final pay-off which is consistent with the behavior of the self-similar solution, see Section 2. However we also note that our assumptions include, in particular, the case of terminal condition $u(\cdot, T) = 0$.

Characterization of the free boundary through the equation satisfied by the flow

In this subsection, we properly establish the existence of the free boundary curves together with their basic characterization. Our main tool will be the elliptic equation satisfied by the flow γ of optimal characteristics (see (4.2.12)). We recall from Section 4.1 that $\alpha \in (0,1)$ is defined by

$$\alpha = \frac{2}{2+\theta}.$$

Theorem 4.3.10 (Characterization of the free boundary and the flow)**.** *Let f be given by* (4.3.28)*. Assume that* (4.3.3)–(4.3.5) *and* (4.3.29)–(4.3.33) *hold, and that $f(m_0)$ is semi–convex. Let (u, m) be a solution to* (MFG) *or* (MFGP)*. Then there exist two functions*

$\gamma_L < \gamma_R \in W^{1,\infty}(0,T)$, *such that*

$$\{m > 0\} = \{(x,t) \in \mathbb{R} \times [0,T] : \gamma_L(t) < x < \gamma_R(t)\}. \tag{4.3.34}$$

Moreover, the flow γ *of optimal trajectories is well defined on* $(a_0, b_0) \times [0,T]$, *we have*

$$\gamma \in W^{1,\infty}((a_0,b_0)\times(0,T))\cap C^{2,\overline{\alpha}}_{\text{loc}}((a_0,b_0)\times[0,T]),\ \ \gamma_x > 0,\ \ \gamma_L(t) = \gamma(a_0,t),\ \gamma_R(t) = \gamma(b_0,t),$$

and γ *is a classical solution in* $(a_0,b_0) \times (0,T)$ *to the elliptic equation*

$$\gamma_{tt} + \frac{\theta f(m_0)}{(\gamma_x)^{2+\theta}}\gamma_{xx} = \frac{f(m_0)_x}{(\gamma_x)^{1+\theta}}. \tag{4.3.35}$$

Moreover, there exists a constant

$$C = \begin{cases} C\left(T, T^{-1}, C_0, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_0)^-_{xx}\|_\infty, \theta, \frac{\theta_1}{\theta}, c_1\right) & \text{if } (u,m) \text{ solves (MFG)},\\ C\left(T, T^{-1}, C_0, C_1, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}, \|f(m_0)^-_{xx}\|_\infty, \theta\right) & \text{if } (u,m) \text{ solves (MFGP)},\end{cases}$$

such that

$$\|\gamma\|_{W^{1,\infty}([a_0,b_0]\times[0,T])} \leq C, \tag{4.3.36}$$

and, for each $(x,t) \in (a_0,b_0) \times [0,T]$,

$$\frac{1}{C} m_0(x) \leq m(\gamma(x,t),t). \tag{4.3.37}$$

Proof. Throughout this proof, as usual, the constant $C > 0$ may increase at each step. We will first treat the case in which (u,m) solves (MFGP). As in the proof of Theorem 4.3.3, we let $(u^\varepsilon, m^\varepsilon)$ be the solution to (4.3.9) given by Proposition 4.3.5, corresponding to the choices of $m_0^\varepsilon, m_T^\varepsilon$ given by (4.3.17). We may normalize the solution to satisfy $\int_{R\mathbb{T}} u^\varepsilon(\cdot, T) = 0$. We also fix R large enough that $(-R,R)$ contains both $[a_0-2,b_0+2]$ and $[a_1-2,b_1+2]$. With this choice, we see from (4.3.17) and (4.3.19) that

$$\int_{-R}^{a_0} m_0^\varepsilon = \int_{-R}^{a_1} m_T^\varepsilon, \text{ and } \int_{b_0}^{R} m_0^\varepsilon = \int_{b_1}^{R} m_T^\varepsilon, \tag{4.3.38}$$

which guarantees that $\gamma^\varepsilon(a_0,T)=a_1$, and $\gamma^\varepsilon(b_0,T)=b_1$, or, equivalently, that $\gamma^\varepsilon(\cdot,T)$ is a bijection between supp m_0 and supp m_T. We are now interested in a Lipschitz bound for γ^ε. Recall that $\gamma_t^\varepsilon = -u_x^\varepsilon(\gamma^\varepsilon(x,t),t)$, so $\|\gamma_t^\varepsilon\|_{L^\infty((a_0,b_0)\times[0,T])}$ is bounded due to Proposition 4.3.5 (we emphasize that our choice of R is already fixed). Moreover, we have $\gamma^\varepsilon \in C^2([-R,R]\times[0,T])$ and, by Lemma 4.2.3 we know that it satisfies (4.2.12); for the case $f(m)=m^\theta$, this equation may be rewritten as

$$\gamma_{tt}^\varepsilon + \frac{\theta f(m_0^\varepsilon)}{(\gamma_x^\varepsilon)^{2+\theta}}\gamma_{xx}^\varepsilon = \frac{f(m_0^\varepsilon)_x}{(\gamma_x^\varepsilon)^{1+\theta}}. \tag{4.3.39}$$

For $K>0$, we set

$$v=\gamma_x^\varepsilon, \quad w=v-Kt^\alpha, \quad \text{and} \quad k_1 = \|f(m_0)_{xx}^-\|_\infty.$$

We now want to show that, if K is chosen sufficiently large,

$$\max_{[a_0,b_0]\times[0,T]} w \le \max_{\partial([a_0,b_0]\times[0,T])} w. \tag{4.3.40}$$

For this purpose we assume first that $m_0^\varepsilon, m_T^\varepsilon$ are smooth. Then γ^ε is smooth as well, and differentiating (4.3.39) with respect to x, we get, for some function $b(x,t)$, and sufficiently small $\varepsilon>0$,

$$v_{tt} + \frac{\theta f(m_0^\varepsilon)}{v^{2+\theta}}v_{xx} + b(x,t)v_x = \frac{f(m_0^\varepsilon)_{xx}}{v^{1+\theta}} \ge \frac{-2k_1}{v^{1+\theta}}, \tag{4.3.41}$$

where equation (4.3.23) from Lemma 4.3.8 was used in the last inequality. Recalling the definition of w, this yields

$$w_{tt} + \frac{\theta f(m_0^\varepsilon)}{v^{2+\theta}} w_{xx} + b(x,t) w_x \geq \frac{-2k_1}{v^{1+\theta}} + K\alpha(1-\alpha)t^{\alpha-2}. \tag{4.3.42}$$

Let (x_0, t_0) be an interior maximum point of w. Then the right hand side of (4.3.42) must be non-positive, which may be rewritten as

$$v(x_0, t_0) \leq \left(\frac{2k_1}{K\alpha(1-\alpha)}\right)^{\frac{1}{1+\theta}} t_0^{\frac{2-\alpha}{1+\theta}} = \left(\frac{2k_1}{K\alpha(1-\alpha)}\right)^{\frac{1}{1+\theta}} t_0^{\alpha}. \tag{4.3.43}$$

Since $\max_{[a_0,b_0]\times[0,T]} w \geq 0$, we have $v(x_0,t_0) \geq Kt_0^\alpha$. Hence inequality (4.3.43) is impossible if we choose K sufficiently large, namely if $K > \left(\frac{2k_1}{\alpha(1-\alpha)}\right)^{\frac{1}{2+\theta}}$. This shows (4.3.40). To remove the assumption that $m_0^\varepsilon, m_T^\varepsilon$ are sufficiently smooth to perform the above computations, we approximate them first with convolutions, as was done in (4.2.33) (but without any need to regularize f), and we simply note that (4.3.40) is stable under such an approximation.

In view of (4.3.40), it now suffices to bound w at the boundary points. When $t = 0$, we have $w = v \equiv 1$, and when $x = a_0$ or $x = b_0$, $m_0^\varepsilon(x) = \varepsilon = \min m^\varepsilon$, so that, recalling (4.2.10),

$$w(x,t) \leq v(x,t) = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x,t),t)} \leq 1.$$

Therefore, we are only left with estimating $v(x,T)$, for $x \in (a_0, b_0)$. We will assume that $x \in (a_0, \frac{1}{2}(a_0+b_0))$, since the converse case is completely symmetric. We first observe that the explicit form of the approximations (4.3.17), where $f(s) = s^\theta$, together with (4.3.30) and (4.3.31), imply, for some constants c_0, c_1:

$$\frac{1}{c_0}(\mathrm{dist}(x,\{a_0,b_0\})^{\alpha_0} + \varepsilon) \leq m_0^\varepsilon(x) \leq c_0(\mathrm{dist}(x,\{a_0,b_0\})^{\alpha_0} + \varepsilon), \qquad x \in (a_0,b_0), \tag{4.3.44}$$

and

$$\frac{1}{c_1}(\mathrm{dist}(x,\{a_1,b_1\})^{\alpha_1}+\varepsilon) \le m_T^\varepsilon(x) \le c_1(\mathrm{dist}(x,\{a_1,b_1\})^{\alpha_1}+\varepsilon), \qquad x\in(a_1,b_1). \quad (4.3.45)$$

Let us set $\overline{x} = \gamma^\varepsilon(x,T)$; since we have

$$\int_{a_0}^{x} m_0^\varepsilon = \int_{a_1}^{\overline{x}} m_T^\varepsilon \quad (4.3.46)$$

we deduce from (4.3.44)–(4.3.45) that $x\to a_0$ if and only if $\overline{x}\to a_1$. Hence we can assume that $\overline{x} \le \frac{a_1+b_1}{2}$ and we estimate

$$\frac{1}{c_0}((x-a_0)^{\alpha_0+1}+\varepsilon(x-a_0)) \le \int_{a_0}^{x} m_0^\varepsilon = \int_{a_1}^{\overline{x}} m_T^\varepsilon \le c_1((\overline{x}-a_1)^{\alpha_1+1}+\varepsilon(\overline{x}-a_1)).$$

In particular, this implies that

$$\frac{1}{c_0}(x-a_0)^{\alpha_0+1} \le c_1(\overline{x}-a_1)^{\alpha_1+1} \text{ or } \frac{1}{c_0}\varepsilon(x-a_0) \le c_1\varepsilon(\overline{x}-a_1),$$

and since $\alpha_0 \ge \alpha_1$ we deduce, for some constant C, that

$$(x-a_0)^{\alpha_0} \le C\,(\overline{x}-a_1)^{\alpha_1}\,.$$

As a result, using (4.3.44)–(4.3.45), we have, for ε small enough,

$$v(x,T) = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x,T),T)} \le \frac{c_0(x-a_0)^{\alpha_0}+\varepsilon}{\frac{1}{c_1}(\overline{x}-a_1)^{\alpha_1}+\varepsilon} \le C.$$

We thus conclude that $w \le C$, hence

$$\gamma_x^\varepsilon(x,t) \le C(1+t^\alpha). \quad (4.3.47)$$

This finally establishes that

$$\|\gamma^\varepsilon\|_{W^{1,\infty}([a_0,b_0]\times[0,T])} \leq C, \tag{4.3.48}$$

and, in particular,

$$\frac{1}{C} m_0^\varepsilon(x) \leq m^\varepsilon(\gamma^\varepsilon(x,t),t). \tag{4.3.49}$$

Now, recalling the proof of Theorem 4.3.3, we see that m^ε converges uniformly to the unique weak solution m of (MFGP). Moreover, $\gamma^\varepsilon \to \gamma$ uniformly in $[a_0,b_0]\times[0,T]$, and (4.3.36), (4.3.37) follow. Now, assume that $x_0 < \gamma(a_0,t_0)$. We have, for ε small enough, $x_0 < \gamma^\varepsilon(a_0,t_0)$, so Lemma 4.2.3 yields

$$\int_{-R}^{x_0} m^\varepsilon \leq \int_{-R}^{a_0} m_0^\varepsilon, \implies \int_{-R}^{x_0} m \leq \int_{-R}^{a_0} m_0 = 0,$$

and, thus, $m(x_0,t_0) = 0$. Similarly, $m(x_0,t_0) = 0$ whenever $x_0 > \gamma(b_0,t_0)$. On the other hand, if $\gamma(a_0,t_0) < x_0 < \gamma(b_0,t_0)$, we have, by continuity, $\gamma(a_0,t_0) = \gamma(c_0,t_0)$ for some $a_0 < c_0 < b_0$. As a result, (4.3.37) yields

$$m(x_0,t_0) = m(\gamma(c_0,t_0),t_0) \geq \frac{1}{C} m_0(c_0) > 0,$$

and this proves (4.3.34). We now recall from the proof of Theorem 4.3.3 that $(u^\varepsilon, f(m^\varepsilon))$ is bounded in $C^{2,\overline{\alpha}}_{\text{loc}}(([-R,R]\times[0,T])\cap\{m>0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}(([-R,R]\times[0,T])\cap\{m>0\})$, independently of ε. Thus, for $(x,t)\in(a_0,b_0)\times[0,T]$, letting $\varepsilon\to 0$ in the relations

$$\gamma_t^\varepsilon = -u_x^\varepsilon(\gamma^\varepsilon(x,t),t), \;\; \gamma_x^\varepsilon = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x,t),t)}$$

shows that $\gamma^\varepsilon \to \gamma$ in $C^2_{\rm loc}((a_0, b_0) \times [0, T])$,

$$\gamma_t = -u_x(\gamma(x,t),t), \;\; \gamma_x = \frac{m_0(x)}{m(\gamma(x,t),t)}, \tag{4.3.50}$$

and, in particular, $\gamma \in C^{2,\overline{\alpha}}_{\rm loc}((a_0,b_0)\times[0,T])$. Finally, letting $\varepsilon \to 0$ in (4.3.39) yields (4.3.35).

We now explain the necessary changes in the above proof to deal with the case that (u, m) solves (MFG). We initially assume that $c_1 > 0$. The first modification lies in the proof of (4.3.47), since our previous argument to estimate $\gamma^\varepsilon_x(\cdot, T)$ does not apply if $m^\varepsilon(\cdot, T) \neq m^\varepsilon_T$. Assume then that the function w achieves its maximum value at a point (x_0, T), with $a_0 < x_0 < b_0$. We first observe that

$$u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T)),$$

so that

$$-\gamma^\varepsilon_t(\cdot, T) = u^\varepsilon_x(\gamma^\varepsilon(\cdot, T), T) = g'(\gamma^\varepsilon(\cdot, T))m^\varepsilon_x(\gamma^\varepsilon(\cdot, T), T)\,.$$

Since $m^\varepsilon_x = \frac{1}{f'}f(m^\varepsilon)_x = \frac{1}{f'}\gamma^\varepsilon_{tt}$, we get

$$-\gamma^\varepsilon_t(\cdot, T) = \frac{g'}{f'}\gamma^\varepsilon_{tt}(\cdot, T) = Tc_1\frac{\theta_1}{\theta}(m^\varepsilon)^{\theta_1-\theta}\gamma^\varepsilon_{tt}.$$

Differentiating this relation with respect to x once more yields

$$T\,c_1\frac{\theta_1}{\theta}(m^\varepsilon)^{\theta_1-\theta}v_{tt} = -v_t - T\,c_1\frac{\theta_1}{\theta}(\theta_1-\theta)(m^\varepsilon)^{\theta_1-\theta-1}(m^\varepsilon_x)^2\gamma_x f'(m^\varepsilon) \leq -v_t.$$

Now, evaluating this at x_0, and using the fact that $w_t \geq 0$ at (x_0, T), we get

$$Tc_1\frac{\theta_1}{\theta}(m^\varepsilon)^{\theta_1-\theta}v_{tt} \leq -K\alpha T^{\alpha-1}.$$

On the other hand, using $w_x = v_x = 0$ and $w_{xx} = v_{xx} \leq 0$, (4.3.41) yields $v_{tt} \geq -\frac{2k_1}{v^{1+\theta}}$.

Hence we get

$$-\frac{2k_1}{v^{1+\theta}}\frac{\theta_1}{\theta}(m^{\varepsilon})^{\theta_1-\theta} \le -K\frac{\alpha T^{\alpha-2}}{c_1}\,.$$

Using the uniform bound on m (see (4.3.10)) we obtain, for ε sufficiently small,

$$v \le \left(\frac{2k_1 c_1\theta_1}{\alpha\theta}\|m_0\|_{\infty}^{\theta_1-\theta}\right)^{\frac{1}{1+\theta}}\frac{T^{\frac{2-\alpha}{1+\theta}}}{K^{\frac{1}{1+\theta}}} := C\frac{T^{\frac{2-\alpha}{1+\theta}}}{K^{\frac{1}{1+\theta}}} = C\frac{T^{\alpha}}{K^{\frac{1}{1+\theta}}}. \tag{4.3.51}$$

If we choose $K > C^{\frac{1+\theta}{2+\theta}}$, then the right hand side of (4.3.51) is bounded above by KT^{α}, which yields $w(x,T) \le 0$, completing the proof of (4.3.47). We also observe that $g = c_1 T m^{\theta}$ satisfies $g' > 0$, which, as seen in the proof of Theorem 4.3.3, is still sufficient to obtain uniform bounds of $(u^{\varepsilon}, f(m^{\varepsilon}))$ in

$$C^{2,\overline{\alpha}}_{\text{loc}}(([-R,R]\times[0,T])\cap\{m>0\}) \times C^{1,\overline{\alpha}}_{\text{loc}}(([-R,R]\times[0,T])\cap\{m>0\}). \tag{4.3.52}$$

When $c_1 = 0$, we can repeat the present proof for $g^{\varepsilon}(m) = \varepsilon m^{\theta}$. The functions $(u^{\varepsilon}, f(m^{\varepsilon}))$ may still be estimated in $C^{2,\overline{\alpha}}_{\text{loc}} \times C^{1,\overline{\alpha}}_{\text{loc}}$ away from $t = T$, and we conclude as before, except that we are only able to obtain that $\gamma \in C^{2,\overline{\alpha}}_{\text{loc}}((a_0,b_0)\times[0,T))$. However, since we know from Theorem 4.3.3 that $u \in C^{2,\overline{\alpha}}_{\text{loc}}((\mathbb{R}\times[0,T])\cap\{m>0\})$, the regularity of γ up to $t = T$ follows from (4.3.50). $\square$

We now obtain the optimal upper bound for the time evolution of the quantity γ_x, which is attained by the self-similar solutions of Section 4.1.

Corollary 4.3.11 (Upper bound on γ_x)**.** *Under the assumptions of Theorem 4.3.10, let (u,m) be a solution to* (MFG) *or* (MFGP)*, let γ be the flow of optimal trajectories for (u,m), and define*

$$\mathscr{d}(t) = \begin{cases} t & \textit{if } u \textit{ solves } \text{(MFG)}, \\ \operatorname{dist}(t,\{0,T\}) & \textit{if } u \textit{ solves } \text{(MFGP)}. \end{cases} \tag{4.3.53}$$

Then there exists a constant $C > 0$, with

$$C = \begin{cases} C\left(C_0, \|f(m_0)^-_{xx}\|_{L^\infty(\{m_0>0\})}, \frac{\theta_1}{\theta}, c_1\right) & \text{if } u \text{ solves (MFG)}, \\ C\left(C_0, C_1, \|f(m_0)^-_{xx}\|_{L^\infty(\{m_0>0\})}\right) & \text{if } u \text{ solves (MFGP)}, \end{cases} \tag{4.3.54}$$

independent of T, such that

$$\gamma_x(x,t) \leq C(1 + \ell(t)^\alpha).$$

Proof. In the proof of Theorem 4.3.10, we showed (4.3.47). In fact, by simply following the proof, one readily sees that the constant C in (4.3.47) may be chosen independently of T, and depending only on the quantities specified in (4.3.54). This observation applies for both (MFG) and (MFGP). Thus, for (MFG) there is nothing left to prove. As for (MFGP), repeating exactly the same argument for the function $w = v - K(T-t)^\alpha$ yields

$$\gamma_x(x,t) \leq C(1 + (T-t)^\alpha).$$

Thus, combining this with (4.3.47), we conclude that, for (u,m) solving (MFGP)

$$\gamma_x(x,t) \leq C(1 + \min(t^\alpha, (T-t)^\alpha)) = C(1 + \ell(t)^\alpha).$$

□

$C^{1,1}$ regularity, strict convexity, strict monotonicity, and long time behavior

In this subsection we obtain, under adequate compatibility and non-degeneracy assumptions on the data, uniform $W^{2,\infty}(0,T)$ estimates for the free boundary. Additionally, we obtain strict convexity and strict concavity for the left and right free boundary curves, respectively, and prove that for the terminal cost problem, (MFG), the boundary is spreading outward. Finally, we quantify the exact rate of propagation of the support and the exact rate of decay in time for the density, which are the ones exhibited by the self-similar solutions of Proposition 4.1.1.

To obtain these extra properties on the free boundary, we strengthen the assumptions of the previous subsection. In particular, we will require the following compatibility condition between terminal and initial data, namely that

$$\alpha_0 = \alpha_1,\ \theta_1 = \theta. \tag{4.3.55}$$

where $\alpha_0, \alpha_1, \theta_1$ are defined in (4.3.30)–(4.3.33). This kind of assumption will guarantee that the function γ_x is well-behaved at $t = T$. We will also strengthen the nondegeneracy assumption on $f(m_0)$ by requiring

$$f(m_0)_{xx} \leq 0 \text{ in } \{x \in (a_0, b_0) : \operatorname{dist}(x, \{a_0, b_0\}) < \delta\} \text{ for some } \delta > 0\,. \tag{4.3.56}$$

We observe that, since $f(m_0)$ is Lipschitz, (4.3.56) necessarily implies that $\alpha_0 = \frac{1}{\theta}$ in (4.3.30), and

$$f(m_0)_x(b_0^-) < 0 < f(m_0)_x(a_0^+). \tag{4.3.57}$$

We begin by obtaining a uniform lower bound on $\gamma_x(\cdot, T)$ for solutions to (MFGP).

Lemma 4.3.12. *Under the assumptions of Theorem 4.3.10, let (u, m) be a solution to* (MFGP)*, and assume that $\alpha_0 = \alpha_1$. Then there exists a constant $C = C(C_0, C_1)$, such that, for $x \in (a_0, b_0)$,*

$$\gamma_x(x, T) \geq \frac{1}{C}.$$

Proof. With no loss of generality, we assume that $x \in (a_0, \frac{1}{2}(a_0 + b_0))$, and that x is close enough to a_0 to guarantee that $\gamma(x, T) \in (a_1, \frac{1}{2}(a_1 + b_1))$. We have, by conservation of mass,

$$\int_{a_0}^{x} m_0 = \int_{a_1}^{\gamma(x,T)} m_T(\gamma(\cdot, T)).$$

Thus, in view of (4.3.30) and (4.3.31),

$$C_0(x-a_0)^{\alpha_0+1} \geq \frac{1}{C_1}(\gamma(x,T)-a_1)^{\alpha_0+1} \implies \frac{(\gamma(x,T)-a_1)^{\alpha_0}}{(x-a_0)^{\alpha_0}} \leq (C_0C_1)^{\frac{\alpha_0}{\alpha_0+1}}.$$

This, combined with (4.3.30) and (4.3.31) once more, implies that

$$\gamma_x(x,T) = \frac{m_0(x)}{m_T(\gamma(x,T),T)} \geq (C_0C_1)^{-1}\frac{(x-a_0)^{\alpha_0}}{(\gamma(x,T)-a_1)^{\alpha_0}} \geq (C_0C_1)^{-(1+\frac{\alpha_0}{\alpha_0+1})}.$$

□

Next we obtain a global lower bound on γ_x. We also show that when (4.3.56) is strengthened to be both strict and global, as occurs in the self-similar solution, this bound can be improved to yield the optimal rate at which γ_x may grow in time, complementing the upper bound of Corollary 4.3.11.

Theorem 4.3.13. *(Lower bounds on γ_x) Under the assumptions of Theorem 4.3.10, let (u,m) be a solution to* (MFG) *or* (MFGP). *Assume that* (4.3.55) *and* (4.3.56) *hold. Then we have:*

(i) There exists a constant $C>0$ such that, for $(x,t)\in(a_0,b_0)\times[0,T]$, $m(\gamma(x,t),t)\leq Cm_0(x)$, that is,

$$\gamma_x(x,t) \geq \frac{1}{C}, \tag{4.3.58}$$

and

$$C = \begin{cases} C(C_0,\delta^{-1}) & \text{if } u \text{ solves (MFG)},\\ C(C_0,C_1,\delta^{-1}) & \text{if } u \text{ solves (MFGP)}. \end{cases} \tag{4.3.59}$$

(ii) Assume, in addition, that $f(m_0)_{xx}\leq -\frac{1}{K}<0$ in (a_0,b_0), and that $c_1>0$ in (4.3.33), *and let $\mathscr{I}$ be defined as in* (4.3.53). *Then there exists a constant $C>0$ such that the*

sharp estimate

$$\gamma_x(x,t) \geq \frac{1}{C}(1 + \text{đ}(t)^{\alpha}) \tag{4.3.60}$$

holds, and

$$C = \begin{cases} C\left(C_0, c_1^{-1}\right) & \textit{if } u \textit{ solves } (\text{MFG}), \\ C\left(C_0, C_1, K\right) & \textit{if } u \textit{ solves } (\text{MFGP}). \end{cases} \tag{4.3.61}$$

Proof. We first treat the case in which (u,m) solves (MFGP). Observe that, since $f \in C^{\infty}(0,\infty)$, the interior Schauder estimates applied to (4.2.1) imply that the solution (u,m) is C^{∞} in the set $\{m > 0\} \cap \{0 < t < T\}$, and so by Lemma 4.2.5 the function

$$v(x,t) = f(m(\gamma(x,t),t))$$

solves, for $(x,t) \in [a_0,b_0] \times [0,T]$,

$$-v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} + v_x \frac{\theta v}{\gamma_x^3} \gamma_{xx} + \frac{\theta+1}{\theta} v^{-1} v_t^2 = 0. \tag{4.3.62}$$

Since $v = \frac{f(m_0)}{\gamma_x^{\theta}}$ (by (4.2.10)), using (4.3.35) and $\gamma_{tt} = -\frac{v_x}{\gamma_x}$ (by (4.2.9)) we deduce

$$\begin{aligned} -v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} &= -\frac{v_x}{\gamma_x^2}\left(\frac{\theta f(m_0)\gamma_{xx}}{\gamma_x^{1+\theta}}\right) - \frac{\theta+1}{\theta} v^{-1} v_t^2 = -\frac{v_x}{\gamma_x^2}\left(\frac{f(m_0)_x}{\gamma_x^{\theta}} - \gamma_x\, \gamma_{tt}\right) - \frac{\theta+1}{\theta} v^{-1} v_t^2 \\ &\leq -\frac{v_x}{\gamma_x^2}\left(\frac{f(m_0)_x}{\gamma_x^{\theta}} - v_x\right). \end{aligned} \tag{4.3.63}$$

Now, given $0 < \varepsilon < 1$, for each $(x,t) \in [a_0,b_0] \times [0,T]$, we let

$$w(x,t) = v(x,t) - Cf(m_0)(x) - \varepsilon t,$$

where $C \geq 1$ is a constant large enough to guarantee that $w(x,T) \leq 0$, and $w \leq 0$ on the set $\{(x,t) : \operatorname{dist}(x, \{a_0, b_0\}) \geq \delta\}$. Such a constant exists because of Lemma 4.3.12, and because, by (4.3.30), $m_0 \geq \frac{1}{C_0}\delta^{\alpha_0}$ on $\{x : \operatorname{dist}(x, \{a_0, b_0\}) \geq \delta\}$. Observe that, as a result of (4.3.56) and (4.3.57), we must have

$$f(m_0)_x \neq 0 \text{ on } \{x : 0 < \operatorname{dist}(x, \{a_0, b_0\}) < \delta\}. \tag{4.3.64}$$

Let (x_0, t_0) be an interior maximum of w on the set

$$S := \{w > -\varepsilon t\} \cap \{\operatorname{dist}(x, \{a_0, b_0\}) < \delta\}.$$

We note that, in S, $v(x,t) > Cf(m_0)$. Moreover, at (x_0, t_0), $v_x = Cf(m_0)_x$ and $D^2 w \leq 0$. Therefore, in view of (4.3.56), (4.3.63), and (4.3.64), we infer that

$$0 \leq -w_{tt} - \frac{\theta v}{\gamma_x^2} w_{xx} \leq \frac{-1}{\gamma_x^2} C \frac{f(m_0)_x^2}{f(m_0)} (v - Cf(m_0)) + \frac{C\theta v}{\gamma_x^2} f(m_0)_{xx} < 0. \tag{4.3.65}$$

This is a contradiction, so the maximum of w in S is achieved at ∂S. At such a point, we have either $w = -\varepsilon t$, $t = 0$, $t = T$, or $\operatorname{dist}(x, \{a_0, b_0\}) = \delta$. By our choice of C, in each of these cases we have $w \leq 0$. Therefore, $w \leq 0$ on all of $[a_0, b_0] \times [0, T]$, that is,

$$f(m(\gamma(x,t), t)) \leq Cf(m_0(x)) + \varepsilon t,$$

and the result follows by letting $\varepsilon \to 0$.

Now, to modify this for the terminal cost problem $u(\cdot, T) = c_1 T f(m(\cdot, T))$ (recall that (4.3.55) holds), the only issue we must address is that we do not know a priori that $w \leq 0$ at $t = T$. It is therefore enough to prove that w cannot achieve a maximum in S at some (x_0, T), where $x_0 \in (a_0, b_0)$. Assume otherwise. On one hand, we have, by (4.2.8) and the continuity equation,

$$v_t(x,T) = f'(m)(m_t + m_x\gamma_t)(\gamma(x,T),T) = f'(m)(m_t - m_x u_x)(\gamma(x,T))$$
$$= f'(m)mu_{xx}(\gamma(x,T),T) = \theta v(x,T)u_{xx}(\gamma(x,T),T). \quad (4.3.66)$$

On the other hand, by the definition of v and the chain rule,

$$f(m)_{xx}(\gamma(x,T),T) = \left(\frac{v_x}{\gamma_x}\right)_x (x,T)\frac{1}{\gamma_x(x,T)}. \quad (4.3.67)$$

Differentiating the terminal condition implies that $u_{xx}(\cdot,T) = c_1 T f(m)_{xx}(\cdot,T)$, so that, in view of (4.3.66) and (4.3.67), we obtain

$$v_t = \theta v u_{xx}(\gamma(x_0,T),T) = \theta c_1 T v f(m)_{xx}(\gamma(x_0,T),T) = \theta c_1 T v \left(\frac{1}{\gamma_x^2}v_{xx} - v_x\frac{1}{\gamma_x^3}\gamma_{xx}\right). \quad (4.3.68)$$

Now, notice that $w_{xx}(x_0,T) = v_{xx} - Cf(m_0)_{xx} \leq 0$ and $f(m_0)_{xx}(x_0) \leq 0$, so $v_{xx} \leq 0$. Hence, in view of (4.3.35) and (4.2.9), we obtain

$$v_t \leq -\theta c_1 T v_x \frac{v}{\gamma_x^3}\gamma_{xx} = -c_1 T v_x \left(\frac{f(m_0)_x}{\gamma_x^{2+\theta}} - \frac{v_x}{\gamma_x^2}\right). \quad (4.3.69)$$

As a result, since $w_t \geq 0$ and $w_x = 0$, that is, $v_t \geq \varepsilon$ and $v_x = Cf(m_0)_x$, recalling (4.3.64) and the definition of S, we get

$$\varepsilon \leq v_t \leq -c_1 T C\frac{1}{\gamma_x^2}\frac{f(m_0)_x^2}{f(m_0)}(v - Cf(m_0)) \leq 0.$$

This is a contradiction, which proves (4.3.58).

To prove part (ii), we now assume that $f(m_0)_{xx} \leq -\frac{1}{K}$, where $K > 0$, and again we focus first on the case where (u,m) solves (MFGP). We repeat the above argument, but with a different choice for the function w, namely

$$w(x,t) = v(x,t) - \zeta(t) f(m_0)(x), \quad \text{where } \zeta(t) = C\left(\frac{1}{t^{\alpha\theta}} + \frac{1}{(T-t)^{\alpha\theta}}.\right) \tag{4.3.70}$$

Since $f(m_0)$ is now globally concave, we may also redefine S to be simply

$$S := \{w > 0\}.$$

Then, instead of (4.3.65), we obtain

$$\begin{aligned} 0 \le -w_{tt} - \frac{\theta v}{\gamma_x^2} w_{xx} &\le \frac{-1}{\gamma_x^2}\zeta(t_0)\frac{f(m_0)_x^2}{f(m_0)} w + \zeta''(t_0) f(m_0) + \frac{\zeta(t_0)\theta v}{\gamma_x^2} f(m_0)_{xx} \\ &\le f(m_0)^{-\frac{2}{\theta}}\left(f(m_0)^{1+\frac{2}{\theta}}\zeta''(t_0) + \zeta(t_0)\theta f(m_0)_{xx} v^{1+\frac{2}{\theta}}\right) \end{aligned} \tag{4.3.71}$$

where we used that $\gamma_x = \left(\frac{f(m_0)}{v}\right)^{\frac{1}{\theta}}$ due to (4.2.10). Since $f(m_0)_{xx} \le -\frac{1}{K}$, and $v > \zeta(t) f(m_0)$ because $w > 0$, we estimate the right hand side of (4.3.71) obtaining

$$0 \le -f(m_0)\left(-\zeta''(t_0) + \theta\frac{1}{K}\zeta(t_0)^{2+\frac{2}{\theta}}\right),$$

It is straightforward to check that, choosing C sufficiently large, independently of T, ζ satisfies

$$-\zeta''(t) + \theta\frac{1}{K}\zeta(t)^{2+\frac{2}{\theta}} > 0.$$

Therefore, with this choice of ζ, we obtain a contradiction. Moreover, since $w(\cdot,0) \equiv w(\cdot,T) \equiv -\infty$, we conclude that the set S must be empty. That is,

$$v(x,t) \le \zeta(t) f(m_0)(x), \quad (x,t) \in (a_0,b_0)\times(0,T),$$

which, combined with (4.3.58), readily implies (4.3.60). Finally, we prove (4.3.60) for the case in which (u, m) solves (MFG). We again define w according to (4.3.70), but this time ζ is defined by

$$\zeta(t) = \frac{C}{t^{\alpha\theta}}, \quad t \in (0, T).$$

We may now follow the same proof as for (MFGP), with the only issue being that we no longer have $w(\cdot, T) \equiv -\infty$, and, thus, we must consider the case of a maximum point (x_0, T) of w. We begin by noticing that, since $w_{xx} = v_{xx} - \zeta(T) f(m_0)_{xx} \le 0$, $w_x = 0$, and $w_t \ge 0$, (4.3.68) implies

$$\zeta'(T) f(m_0) \le v_t \le -\frac{\theta c_1 T v}{\gamma_x^2}\frac{1}{K}\zeta(T) - c_1 T C \frac{1}{\gamma_x^2}\frac{f(m_0)_x^2}{f(m_0)} w \le -\frac{\theta c_1 T v}{\gamma_x^2}\frac{1}{K}\zeta(T).$$

This may be rearranged as

$$v(x_0, T) \le \left(\frac{\alpha K}{c_1}\right)^{\frac{\theta}{2+\theta}} T^{-\alpha\theta} f(m_0)(x_0) := C_2 T^{-\alpha\theta} f(m_0)(x_0).$$

Therefore, if we choose $C > C_2$, we conclude that $v(x_0, T) \le \zeta(T) f(m_0)(x_0)$, as wanted. $\square$

We can now establish our main regularity result for the free boundary.

Theorem 4.3.14 (Regularity and convexity of the free boundary)**.** *Under the assumptions of Theorem 4.3.10, let (u, m) be a solution to* (MFG) *or* (MFGP)*. Assume that (4.3.55) and (4.3.56) hold. Let $\gamma_L = \gamma(a_0, \cdot)$, $\gamma_R = \gamma(b_0, \cdot)$ be, respectively, the left and right free boundary curves. Then $\gamma_L, \gamma_R \in W^{2,\infty}(0, T)$, and there exist constants K_1, K_2, with*

$$K_1 = K_1(C_0, \|\gamma_x\|_\infty, |f(m_0)_x(a_0^+)|^{-1}, |f(m_0)_x(b_0^-)|^{-1}), \tag{4.3.72}$$

and

$$K_2 = K_2(C_0, \|\gamma_x^{-1}\|_\infty, |f(m_0)_x(a_0^+)|, |f(m_0)_x(b_0^-)|) \tag{4.3.73}$$

such that, for a.e. $t \in [0,T]$,

$$\frac{1}{K_1} \le \ddot{\gamma}_L(t) \le K_2, \quad \text{and} \quad -K_2 \le \ddot{\gamma}_R(t) \le -\frac{1}{K_1}. \tag{4.3.74}$$

Moreover, when (u,m) solves (MFG)*, we have, for $t \in [0,T]$,*

$$-K_2(c_1T + (T-t)) \le \dot{\gamma}_L(t) \le -\frac{1}{K_1}(c_1T + (T-t)), \tag{4.3.75}$$

and

$$\frac{1}{K_1}(c_1T + (T-t)) \le \dot{\gamma}_R(t) \le K_2(c_1T + (T-t)). \tag{4.3.76}$$

Proof. By symmetry, it is enough to show the estimates for γ_L. Let $t \in (0,T)$, and let $h > 0$ be such that $(t-h, t+h) \subset (0,T)$. We begin by noting that (4.3.35) may be written as

$$\gamma_{tt} = \frac{1}{\gamma_x}\left(\frac{f(m_0)}{\gamma_x^\theta}\right)_x.$$

We therefore have, for $(x_0, \tau) \in (a_0, \frac{a_0+b_0}{2}) \times [0,T]$,

$$\int_{a_0}^{x_0} \gamma_{tt}(x,\tau)dx = \int_{a_0}^{x_0} \frac{1}{\gamma_x}\left(\frac{f(m_0)}{\gamma_x^\theta}\right)_x dx = \frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) - \int_{a_0}^{x_0}\left(\frac{1}{\gamma_x}\right)_x \frac{f(m_0)}{\gamma_x^\theta}dx, \tag{4.3.77}$$

where in the last step we integrated by parts and used the fact that γ_x is bounded below and $f(m_0)(a_0) = 0$. Using the identity

$$\left(\frac{1}{\gamma_x}\right)_x \frac{f(m_0)}{\gamma_x^\theta} = \frac{1}{\theta}\left(\frac{1}{\gamma_x}\left(\frac{f(m_0)}{\gamma_x^\theta}\right)_x - \frac{f(m_0)_x}{\gamma_x^{1+\theta}}\right),$$

we infer from (4.3.77) that

$$\int_{a_0}^{x_0} \gamma_{tt}(x,\tau)dx = \left(1-\frac{1}{\theta}\right)\frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) + \frac{1}{\theta}\int_{a_0}^{x_0}\left(\frac{1}{\gamma_x}\right)_x \frac{f(m_0)}{\gamma_x^{\theta}}dx + \frac{1}{\theta}\int_{a_0}^{x_0}\frac{f(m_0)_x}{\gamma_x^{1+\theta}}dx.$$

Multiplying this equality by θ and adding to (4.3.77) yields

$$\int_{a_0}^{x_0} \gamma_{tt}(x,\tau)dx = \frac{\theta}{\theta+1}\frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) + \frac{1}{\theta+1}\int_{a_0}^{x_0}\frac{f(m_0)_x}{\gamma_x^{1+\theta}}dx.$$

Recalling that $f(m_0)$ is Lipschitz, with $f(m_0)(a_0) = 0$, and that γ_x is bounded above and below, we conclude that, for x_0 sufficiently close to a_0,

$$\frac{1}{K_1}(x_0 - a_0) \leq \int_{a_0}^{x_0} \gamma_{tt}(x,\tau)dx \leq K_2(x_0 - a_0). \tag{4.3.78}$$

Next, observe that, for $x \in (a_0, x_0)$,

$$\gamma(x,t+h) + \gamma(x,t-h) - 2\gamma(x,t) = \int_0^h \int_{t-s}^{t+s} \gamma_{tt}(x,\tau)d\tau ds,$$

and, therefore, integrating both sides,

$$\int_{a_0}^{x_0} \gamma(x,t+h) + \gamma(x,t-h) - 2\gamma(x,t)dx = \int_0^h \int_{t-s}^{t+s} \int_{a_0}^{x_0} \gamma_{tt}(x,\tau)dxd\tau ds.$$

Using (4.3.78), we see that this yields

$$\frac{1}{K_1}(x_0 - a_0)h^2 \leq \int_{a_0}^{x_0} \gamma(x,t+h) + \gamma(x,t-h) - 2\gamma(x,t)dx \leq K_2(x_0 - a_0)h^2,$$

so, dividing by $(x_0 - a_0)$ and letting $x_0 \to a_0^+$, we obtain

$$\frac{1}{K_1}h^2 \leq \gamma_L(t+h) + \gamma_L(t-h) - 2\gamma_L(t) \leq K_2 h^2,$$

which yields (4.3.74).

Now assume that (u, m) solves (MFG). Recall that, since $u(\cdot, T) = c_1 T f(m(\cdot, T))$, we have

$$\gamma_t(\cdot, T) = -c_1 T \gamma_{tt}(\cdot, T). \tag{4.3.79}$$

We observe that, by Taylor's theorem, for $x \in (a_0, b_0)$ and small $h > 0$, we have

$$\gamma(x, T-h) = \gamma(x, T) - h\gamma_t(x, T) + \int_{T-h}^{T} (s - (T-h))\gamma_{tt}(x, s)ds.$$

Thus, integrating from a_0 to x_0 and using (4.3.79), we obtain

$$\int_{a_0}^{x_0} \gamma(x, T-h) - \gamma(x, T)dx = hc_1 T \int_{a_0}^{x_0} \gamma_{tt}(x, T)dx + \int_{T-h}^{T} \int_{a_0}^{x_0} (s - (T-h))\gamma_{tt}(x, s)dsdx,$$

so we infer from (4.3.78) that, for x_0 sufficiently close to a_0,

$$\frac{c_1 T}{K_1} h(x_0 - a_0) + \frac{1}{2C} h^2 (x_0 - a_0) \leq \int_{a_0}^{x_0} \gamma(x, T-h) - \gamma(x, T)dx \leq K_2 c_1 T h(x_0 - a_0) + \frac{1}{2} C h^2 (x_0 - a_0).$$

Dividing by $(x_0 - a_0)$ and letting $x_0 \to a_0^+$, we see that

$$\frac{c_1 T}{K_1} h + \frac{1}{2C} h^2 \leq \gamma_L(T-h) - \gamma_L(T) \leq K_2 c_1 T h + \frac{1}{2} C h^2.$$

Finally, dividing by h and letting $h \to 0^+$ yields

$$-K_2 c_1 T \leq \dot{\gamma}_L(T) \leq \frac{-c_1 T}{K_1}.$$

Thus, in view of (4.3.74), and noting that $\dot{\gamma}_L(t) = \dot{\gamma}_L(T) - \int_t^T \ddot{\gamma}_L(s)ds$, we obtain (4.3.75).

□

Finally, we show that the support grows with algebraic rate $\alpha = \frac{2}{2+\theta}$, and the density decays to 0 with algebraic rate $-\alpha$, as is expected from the model case of Section 4.1.

Theorem 4.3.15 (Optimal rate of propagation and long time decay)**.** *Under the assumptions of Theorem 4.3.10, let (u,m) be a solution to* (MFG) *or* (MFGP)*, let γ be the associated flow of optimal trajectories, and let $\mathscr{d} : [0,T] \to [0,\infty)$ be defined by* (4.3.53)*. Assume also that $-K \le f(m_0)_{xx} \le -\frac{1}{K}$ in (a_0,b_0) for some $K>0$. Then there exists a constant $C>0$, with*

$$C = \begin{cases} C\left(C_0, c_1, c_1^{-1}, |a_0|, |b_0|, K\right) & \textit{if } u \textit{ solves (MFG)}, \\ C\left(C_0, C_1, |a_0|, |b_0|, |a_1|, |b_1|, K\right) & \textit{if } u \textit{ solves (MFGP)}, \end{cases} \tag{4.3.80}$$

such that, for every $(x,t) \in [a_0,b_0] \times [0,T]$,

$$\frac{1}{C}(1+\mathscr{d}(t)^\alpha) \le |\mathrm{supp}(m(\cdot,t))| \le C(1+\mathscr{d}(t)^\alpha), \quad |\gamma(x,t)| \le C(1+\mathscr{d}(t)^\alpha), \tag{4.3.81}$$

$$\frac{1}{C}\frac{m_0(x)}{(1+\mathscr{d}(t)^\alpha)} \le m(\gamma(x,t),t) \le C\frac{m_0(x)}{(1+\mathscr{d}(t)^\alpha)}. \tag{4.3.82}$$

Proof. Recalling (4.2.10), we observe that (4.3.82) is simply obtained from combining the upper bound on γ_x from Corollary 4.3.11 and the lower bound on γ_x from Proposition 4.3.13,

$$\frac{1}{C}(1+\mathscr{d}(t)^\alpha) \le \gamma_x(x,t) \le C(1+\mathscr{d}(t)^\alpha). \tag{4.3.83}$$

Now, integrating (4.3.83) between a_0 and b_0 immediately yields the first inequality of (4.3.81). Moreover, for the second inequality of (4.3.81), it suffices to show that, for each $t_0 \in [0,T]$, there exists at least one $x_0 \in [a_0,b_0]$ such that $|\gamma(x_0,t_0)| \le C$. When (u,m) solves (MFG), this follows from the fact that, by Theorem 4.3.14, $\mathrm{supp}(m(\cdot,t))$ is expanding, and, thus, in particular, $\gamma(x_0,t_0) = a_0$ for some $x_0 \in [a_0,b_0]$. On the other hand, if (u,m) solves (MFGP), Theorem 4.3.14 implies that $\mathrm{supp}(m)$ is a convex set. Therefore, we may choose $\gamma(x_0,t_0)$ to be the first coordinate of the intersection between $\mathbb{R} \times \{t_0\}$ and the line segment $\{(1-s)(a_0,0)+s(a_1,T) : 0 \le s \le 1\}$.

□

4.3.3 Regularity of the solution up to the free boundary

Intrinsic scaling and Hölder continuity of m

In this subsection, we show how the bounds on γ_x and the intrinsic scaling of the problem allow us to improve the logarithmic modulus of continuity for m to a Hölder one. Throughout the section, we continue to assume that f is given by (4.3.28) , and we will assume the conditions of Theorem 4.3.14, namely (4.3.29)–(4.3.33) together with (4.3.55) and (4.3.56). We will focus on obtaining Hölder regularity estimates for the function

$$v(x,t) := f(m(\gamma(x,t),t)), \qquad (x,t) \in [a_0,b_0] \times [0,T]. \tag{4.3.84}$$

This is equivalent to obtaining Hölder estimates for $m(x,t)$, in view of (4.3.36) and (4.3.58).

Our first result is a simple corollary of the bounds on γ_x, stated in the form of a Harnack inequality (cf. [20, Thm 11.1]).

Theorem 4.3.16 (Harnack inequality)**.** *Let the assumptions of Theorem 4.3.14 be in place, and let (u,m) be a solution to* (MFG) *or* (MFGP)*, and let v be defined by* (4.3.84)*. There exists a constant*

$$C = C(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, \theta)$$

such that the following alternative holds. Let $x_0 \in (a_0,b_0)$, and let $\rho > 0$ be such that $(x_0-\rho, x_0+\rho) \subset (a_0,b_0)$. Then either $\rho \geq \frac{1}{2}\mathrm{dist}(x_0,\{a_0,b_0\})$ and $\sup_{(x_0-\rho,x_0+\rho)\times[0,T]} v \leq C\rho$, or $\rho < \frac{1}{2}\mathrm{dist}(x_0,\{a_0,b_0\})$ and

$$\sup_{(x_0-\rho,x_0+\rho)\times[0,T]} v \leq C \inf_{(x_0-\rho,x_0+\rho)\times[0,T]} v. \tag{4.3.85}$$

Proof. For simplicity, we normalize $a_0 = 0$, and by symmetry we may assume that $x_0 < \frac{b_0}{2}$.

Then, in view of (4.3.30) (where, we recall, $\alpha_0 = \frac{1}{\theta}$ under the current assumptions), and the fact that $\gamma_x = m_0 v^{-\frac{1}{\theta}}$ is bounded above and below, we have, for some constant C, and every $(x,t) \in (x_0 - \rho, x_0 + \rho) \times [0,T]$,

$$\frac{1}{C}x \leq v(x,t) \leq Cx. \tag{4.3.86}$$

Case 1: $\rho < \frac{1}{2}x_0$. Then (4.3.86) implies

$$\sup_{(x_0-\rho,x_0+\rho)\times[0,T]} v \leq C\left(\frac{3x_0}{2}\right) = 3C^2\left(\frac{1}{C}\left(\frac{1}{2}x_0\right)\right) \leq 3C^2 \inf_{(x_0-\rho,x_0+\rho)\times[0,T]} v.$$

Case 2: $\frac{1}{2}x_0 \leq \rho$**.** Then, in view of (4.3.86)

$$\sup_{(x_0-\rho,x_0+\rho)\times[0,T]} v \leq C(x_0+\rho) \leq 3C\rho.$$

□

We note that, unlike in the linear theory, the above Harnack inequality is not yet sufficient to obtain Hölder regularity, because the result does not hold for, say, translations of the solution. Instead, we will proceed by obtaining analogues of the Caccioppoli inequality and De Giorgi type lemmas, adapted to the scaling of the equation satisfied by v (recall (4.2.16)), which is much more diffusive in time than in space near the free boundary.

Lemma 4.3.17 (Intrinsic Caccioppoli inequality)**.** *Let the assumptions of Theorem 4.3.14 hold, and let (u,m) be a solution to* (MFG) *or* (MFGP)*. There exists a universal constant $C > 0$ such that, for each $\zeta \in C_c^\infty((a_0,b_0)\times(0,T))$, and each $k \geq 0$, we have*

$$\int_{a_0}^{b_0}\int_0^T \zeta^2((\psi^\pm(v))_x^2 + (\theta v)^{-1}(\psi^\pm(v))_t^2)dtdx \leq C\int_{a_0}^{b_0}\int_0^T \psi^\pm(v)^2(\zeta_x^2 + (\theta v)^{-1}\zeta_t^2)dtdx, \tag{4.3.87}$$

where $\psi^\pm(v) = (v-k)^\pm$.

Proof. In view of Lemma 4.2.5, and recalling that v is smooth in $(a_0, b_0) \times (0, T)$, we know that v satisfies

$$-\left(\gamma_x^{-1} v_x\right)_x - \left(\gamma_x (\theta v)^{-1} v_t\right)_t = 0. \tag{4.3.88}$$

Testing this equation against the function $\psi(v)^{\pm}\zeta^2$ yields

$$\int_{a_0}^{b_0}\int_0^T \zeta^2(\gamma_x^{-1}v_x^2 + \gamma_x(\theta v)^{-1}v_t^2)(\psi^{\pm})'(v) = -\int_{a_0}^{b_0}\int_0^T 2\zeta\psi^{\pm}(v)(\gamma_x^{-1}v_x\zeta_x + \gamma_x(\theta v)^{-1}v_t\zeta_t). \tag{4.3.89}$$

We may estimate the right hand side as follows:

$$\begin{aligned}\left|\int_{a_0}^{b_0}\int_0^T 2\zeta\psi^{\pm}(v)(\gamma_x^{-1}v_x\zeta_x + \gamma_x(\theta v)^{-1}v_t\zeta_t)\right| &\le \frac{1}{2}\int_{a_0}^{b_0}\int_0^T \zeta^2(\gamma_x^{-1}v_x^2 + \gamma_x(\theta v)^{-1}v_t^2) \\ &\quad + 2\int_{a_0}^{b_0}\int_0^T \psi^{\pm}(v)^2(\gamma_x^{-1}\zeta_x^2 + \gamma_x(\theta v)^{-1}\zeta_t^2). \end{aligned}\tag{4.3.90}$$

We also notice that $(\psi^{\pm})'(v) = \pm\chi_{\pm(v-k)\ge 0}$, so that, as a result of (4.3.89) and (4.3.90),

$$\int_{a_0}^{b_0}\int_0^T \zeta^2(\gamma_x^{-1}(\psi^{\pm}(v))_x^2 + \gamma_x(\theta v)^{-1}(\psi^{\pm}(v))_t^2) \le 4\int_{a_0}^{b_0}\int_0^T \psi^{\pm}(v)^2(\gamma_x^{-1}\zeta_x^2 + \gamma_x(\theta v)^{-1}\zeta_t^2).$$

The result now follows from the fact that γ_x is bounded above and below by positive constants.

□

Our De Giorgi type lemma will be proved for the following special domains, adapted to the scaling of the equation (4.3.88).

Definition 4.3.18. Given $(x_0, t_0) \in (a_0, b_0) \times (0, T)$ and $\rho > 0$, we define the intrinsic rectangle $R_\rho(x_0, t_0)$ of radius ρ centered at (x_0, t_0) by

$$R_\rho(x_0, t_0) := (x_0 - \rho, x_0 + \rho) \times (t_0 - (\theta v(x_0, t_0))^{-\frac{1}{2}}\rho, t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}}\rho).$$

Lemma 4.3.19 (De Giorgi type lemma on intrinsic rectangles)**.** *Let the assumptions of Theorem 4.3.14 hold, and let* (u, m) *be a solution to* (MFG) *or* (MFGP)*. There exists a positive constant* ν*, with*

$$\nu^{-1} = \nu^{-1}(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty),$$

such that the following holds. Let $(x_0, t_0) \in (a_0, b_0) \times (0, T)$*,* $0 < r < \frac{1}{4}$*,*

$$\mu^- = \min_{R_{4\rho}(x_0,t_0)} v, \qquad \mu^+ = \max_{R_{4\rho}(x_0,t_0)} v, \qquad \omega = \mu^+ - \mu^- = \operatorname{osc}_{R_{4\rho}(x_0,t_0)}(v),$$

and

$$\rho \leq \frac{1}{8} \min\left(\operatorname{dist}(x_0, \{a_0, b_0\}), (\theta v(x_0, t_0))^{\frac{1}{2}} \operatorname{dist}(t_0, \{0, T\})\right). \tag{4.3.91}$$

Then

$|\{\pm(v - (\mu^\pm \mp 2r\omega)) > 0\} \cap R_{2\rho}(x_0, t_0)| \leq \nu |R_{2\rho}(x_0, t_0)|$ *implies that* $\pm v \leq \pm(\mu^\pm \mp r\omega)$ *in* $R_\rho(x_0, t_0)$*.*

Proof. We begin by noting that, in view of (4.3.91) and Proposition 4.3.16, $R_{4\rho}(x_0, t_0) \subset (a_0, b_0) \times (0, T)$ and, for some constant $k_1 > 0$,

$$\max_{R_{4\rho}(x_0,t_0)} v \leq k_1 \min_{R_{4\rho}(x_0,t_0)} v. \tag{4.3.92}$$

We now define, for $n \geq 0$,

$$r_n = r + \frac{1}{2^n} r, \qquad \rho_n = \rho + \frac{1}{2^n} \rho, \qquad k_n^\pm = \mu^\pm \mp r_n \omega.$$

and we choose non-negative functions $\zeta_n \in C_c^\infty((a_0, b_0) \times [0, T])$ such that $\zeta_n \equiv 1$ on $R_{\rho_{n+1}}(x_0, t_0)$, $\zeta_n \equiv 0$ outside of $R_{\rho_n}(x_0, t_0)$, and

$$|(\zeta_n)_x| \leq \frac{C 2^n}{\rho}, \qquad |(\zeta_n)_t| \leq \frac{C 2^n (\theta v(x_0, t_0))^{\frac{1}{2}}}{\rho}.$$

As a result of Lemma 4.3.17, (4.3.87) holds when taking $\zeta = \zeta_n$ and

$$\psi(v) = \psi_n^{\pm}(v) := (v - k_n^{\pm})_{\pm}.$$

On $(x,t) \in Q_{2\rho} := (x_0 - 2\rho, x_0 + 2\rho) \times (t_0 - 2\rho, t_0 + 2\rho)$, we now define the rescaled functions $w_n, \overline{\zeta_n}$: by

$$w_n^{\pm}(x,t) = \psi_n^{\pm}(v(x, t_0 + (\theta v(x_0,t_0))^{-\frac{1}{2}}(t - t_0))), \qquad \overline{\zeta}_n(x,t) = \zeta_n(x, t_0 + (\theta v(x_0,t_0))^{-\frac{1}{2}}(t - t_0)).$$

We see that (4.3.87) may be written as

$$\int_{Q_{2\rho}} \overline{\zeta}_n^2((w_n^{\pm})_x^2 + v(x_0,t_0)v^{-1}(w_n^{\pm})_t^2) \leq C \int_{Q_{2\rho}} (w_n^{\pm})^2((\overline{\zeta}_n)_x^2 + v(x_0,t_0)v^{-1}(\overline{\zeta}_n)_t^2).$$

In view of (4.3.92), up to increasing the constant C, we thus obtain

$$\int_{Q_{2\rho}} \overline{\zeta}_n^2 |Dw_n^{\pm}|^2 \leq C \int_{Q_{2\rho}} |D\overline{\zeta}_n|^2 (w_n^{\pm})^2 \leq \frac{C4^n}{\rho^2} \int_{Q_{2\rho}} (w_n^{\pm})^2. \tag{4.3.93}$$

This is now the usual (rather than intrinsic) Caccioppoli inequality, and thus by the standard De Giorgi iteration argument (e.g. see [53, Lem. 5]), writing $w_\infty^{\pm} = \lim_{n\to\infty} w_n^{\pm}$, we see that there exists $\nu > 0$ such that

$$|\{w_0^{\pm} > 0\} \cap Q_{2\rho}| \leq \nu |Q_{2\rho}| \text{ implies that } w_\infty^{\pm} = 0 \text{ in } Q_\rho. \tag{4.3.94}$$

Now, writing

$$\psi_\infty^{\pm}(v) = (v - (\mu^{\pm} \mp r\omega))_{\pm},$$

we have $w_\infty^{\pm} = \psi_\infty^{\pm}(v(x, t_0 + (\theta v(x_0,t_0))^{-\frac{1}{2}}(t - t_0)))$. Scaling back the time variable we have

$|Q_{2\rho}| = (\theta v(x_0, t_0))^{\frac{1}{2}} |R_{2\rho}(x_0, t_0)|$ and

$$|\{w_0^{\pm} > 0\} \cap Q_{2\rho}| = (\theta v(x_0, t_0))^{\frac{1}{2}} |\{\psi_0^{\pm}(v) > 0\} \cap R_{2\rho}(x_0, t_0)|.$$

Thus, we conclude the proof by noticing that (4.3.94) may be equivalently written as:

$$|\{\psi_0^{\pm}(v) > 0\} \cap R_{2\rho}(x_0, t_0)| \leq \nu |R_{2\rho}(x_0, t_0)| \text{ implies that } \psi_{\infty}^{\pm}(v) = 0 \text{ in } R_\rho(x_0, t_0).$$

□

Corollary 4.3.20 (Reduction of oscillation)**.** *Let the assumptions of Theorem 4.3.14 be in place. Assume that the intrinsic rectangle $R_\rho(x_0, t_0)$ satisfies (4.3.91). There exists a constant $0 < \sigma < 1$, independent of the choice of x_0, t_0 and ρ, such that*

$$\operatorname{osc}_{R_\rho(x_0,t_0)} v \leq \sigma \operatorname{osc}_{R_{4\rho}(x_0,t_0)} v. \tag{4.3.95}$$

Proof. The proof of this corollary, included for the reader's convenience, will be a standard application of the classical arguments that yield interior Hölder continuity for functions that satisfy the Caccioppoli inequality (originally due to E. De Giorgi [18]. See also, for instance, [53]). Let ν, r, μ^+, μ^-, and ω be as in Lemma 4.3.19. To simplify notation, we write $R_\rho := R_\rho(x_0, t_0)$.

We begin by defining $z_0 : [-2, 2] \times [-2, 2] \to [-1, 1]$ as

$$z_0(x, t) := 2\omega^{-1} \left(v(x_0 + \rho(x - x_0), t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}} \rho(t - t_0)) - \mu^- \right) - 1.$$

Notice that, since $16 = |[-2, 2] \times [-2, 2]|$, we must have

$$|z_0 \leq 0| \geq 8 \text{ or } |z_0 \geq 0| \geq 8. \tag{4.3.96}$$

We assume the former, and remark that the proof in the alternative case is completely analogous. We define, for $n \geq 1$, $z_n : [-2,2] \times [-2,2] \to (-\infty, 1]$ by

$$z_n(x,t) := \frac{1}{4r^n}(z_0 - (1 - 4r^n)). \tag{4.3.97}$$

Notice that z_n is nonincreasing, and we have

$$z_n \geq 0 \Longleftrightarrow v \geq \mu^+ - 2r^n\omega,$$

and, thus, by a change of variables,

$$\begin{aligned} \fint_{[-2,2]\times[-2,2]} |Dz_n^+|^2 &= \fint_{R_{2\rho}\cap\{v\geq\mu^+-2r^n\omega\}} \frac{\rho^2}{4r^{2n}\omega^2}(v_x^2 + (\theta v(x_0,t_0))^{-1}v_t^2) \\ &= \frac{\rho^2}{4r^{2n}\omega^2} \fint_{R_{2\rho}} (\psi(v)_x^2 + (\theta v(x_0,t_0))^{-1}\psi(v)_t^2), \end{aligned} \tag{4.3.98}$$

where $\psi(v) = (v - \mu^+ + 2r^n\omega)^+$. By Proposition 4.3.16, we thus infer that

$$\int_{[-2,2]\times[-2,2]} |Dz_n^+|^2 \leq \frac{C\rho^2}{r^{2n}\omega^2} \fint_{R_{2\rho}} (\psi(v)_x^2 + (\theta v)^{-1}\psi(v)_t^2) \tag{4.3.99}$$

where, as usual, C is a generic constant that could be increased line by line. On the other hand, Proposition 4.3.16 and Lemma 4.3.17 imply that

$$\int_{R_{2\rho}} (\psi(v)_x^2 + (\theta v)^{-1}\psi(v)_t^2) \leq \frac{C}{\rho^2} \int_{R_{4\rho}} \psi(v)^2 \leq C\omega^2 r^{2n}(\theta v(x_0,t_0))^{-1/2}. \tag{4.3.100}$$

Thus, we deduce from the last two inequalities that

$$\int_{[-2,2]\times[-2,2]} |Dz_n^+|^2 \leq C, \tag{4.3.101}$$

where C is independent of n. Now, assume that, for some $\delta > 0$ and some $n \geq 1$, we have

$$|z_{n-1} \leq 0| \geq 8 + (n-1)\delta. \tag{4.3.102}$$

We will show that, as long as

$$|z_n > 0| \geq \nu, \tag{4.3.103}$$

then (4.3.102) must also hold when n is replaced by $n+1$. Observe that, since we have

$$z_n = \frac{z_{n-1} - (1-r)}{r}, \tag{4.3.104}$$

then (4.3.103) may be rewritten as

$$|z_{n-1} > (1-r)| \geq \nu. \tag{4.3.105}$$

On the other hand, since the first alternative in (4.3.96) was assumed to hold, we have

$$|z_{n-1} \leq 0| \geq |z_0 \leq 0| \geq 8. \tag{4.3.106}$$

We recall that $r < \frac{1}{4}$. A straightforward compactness argument[2] then shows that, in view of (4.3.101), (4.3.105), and (4.3.106), if $\delta > 0$ is chosen sufficiently small, depending only on ν^{-1} and C, then

$$|0 < z_{n-1} < 1 - r| \geq \delta.$$

Thus, we deduce from (4.3.102) and (4.3.104) that

$$|z_n \leq 0| \geq |z_{n-1} \leq 0| + |0 < z_{n-1} < 1 - r| \geq 8 + n\delta.$$

2. The explicit form of this estimate is known as the De Giorgi isoperimetric inequality (see [53, Lem. 10]).

Since the left hand side is bounded above by 16, this recursive process must fail after finitely many steps, and, therefore, there exists $n \geq 1$, depending only on ν^{-1} and C, such that (4.3.103) does not hold. Rewritten in terms of v, this means that

$$|\{v > \mu^+ - 2r^n\omega\} \cap R_{2\rho}| < \nu|R_{2\rho}|.$$

Thus, by Lemma 4.3.19, $v \leq \mu^+ - r^n\omega$ in R_ρ. In particular,

$$\text{osc}_{R_\rho}(v) \leq \mu^+ - r^n\omega - \mu^- = (1 - r^n)\omega = (1 - r^n)\text{osc}_{R_{4\rho}}(v).$$

□

We may now prove our main regularity result for the density.

Theorem 4.3.21 (Hölder continuity of the density up to the free boundary)**.** *Let the assumptions of Theorem 4.3.14 be in place, and let (u, m) be a solution to* (MFG) *or* (MFGP)*. Let $0 < \delta_0 < \min\left(1, \frac{1}{4}T\right)$, and let σ be the constant of Corollary 4.3.20. There exist constants $C > 0$, $0 < s < 1$, with*

$$C = C(\delta_0^{-1}, C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, \theta, (1-\sigma)^{-1}), \quad s^{-1} = s^{-1}(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, (1-\sigma)^{-1}),$$

such that

$$\|f(m(\gamma(\cdot, \cdot), \cdot))\|_{C^s([a_0,b_0]\times[\delta_0,T-\delta_0])} \leq C.$$

In particular, m is Hölder continuous on $\mathbb{R} \times [\delta_0, T - \delta_0]$.

Proof. Let $(x_0, t_0), (x_1, t_1) \in (a_0, b_0) \times [\delta_0, T - \delta_0]$, where $x_1 < x_0$. As usual, we will consider intrinsic rectangles centered at (x_0, t_0), so we abbreviate $R_\rho := R_\rho(x_0, t_0)$. As in the proof of Proposition 4.3.16, we may assume that $a_0 = 0$ and $x_0 < \frac{1}{2}\min(b_0, 1)$, and write, for some constant $k_1 > 1$, and each $(x,t) \in R_{4\rho}$,

$$\frac{1}{k_1} x \leq v(x,t) \leq k_1 x. \tag{4.3.107}$$

Therefore, $x_0 \leq \sqrt{x_0} \leq \sqrt{k_1 v(x_0, t_0)}$. Hence, letting

$$\rho_0 = \frac{1}{8\sqrt{k_1}(1+\sqrt{\theta})} \delta_0 x_0,$$

we see that, in particular, ρ_0 satisfies (4.3.91). Setting now

$$a = \max(|x_1 - x_0|, \sqrt{\theta v(x_0, t_0)} |t_1 - t_0|)$$

we distinguish two alternative cases.

Case 1. $(x_1, t_1) \in R_{4\rho_0}$. Equivalently, we have $a < 4\rho_0$. Let $n \geq 0$ be the unique integer such that

$$\frac{1}{4^n} \rho_0 \leq a < \frac{1}{4^{n-1}} \rho_0.$$

Iterating Corollary 4.3.20, we see that, in view of (4.3.107),

$$\operatorname{osc}_{R_{4^{-(n-1)}\rho_0}}(v) \leq \sigma^n \operatorname{osc}_{R_{4\rho_0}}(v) \leq \sigma^n k_1 (x_0 + 4\rho_0) \leq C \sigma^n \rho_0 \delta_0^{-1}.$$

Moreover, by increasing the value of σ if necessary, we may assume that $\sigma > \frac{1}{4}$, so that $s = -\log \sigma (\log 4)^{-1}$ satisfies $0 < s < 1$. Thus, observing that $(x_1, t_1) \in R_{4^{-(n-1)}\rho_0}$ and $n \geq -\log \left(\rho_0^{-1} a\right) (\log 4)^{-1}$, we have

$$|v(x_1, t_1) - v(x_0, t_0)| \leq C (\rho_0^{-1} a)^s \rho_0 \delta_0^{-1} \leq C \rho_0^{1-s} \delta_0^{-1} (|x_1 - x_0|^s + (\theta v(x_0, t_0))^{\frac{s}{2}} |t_1 - t_0|^s).$$

Case 2. $(x_1, t_1) \notin R_{4\rho_0}$. Then $a \geq 4\rho_0$, and so, since $x_1 < x_0$, appealing to the lower bound on $\gamma_x = \left(f(m_0(x)) v(x,t)^{-1}\right)^{\frac{1}{\theta}}$, and allowing the constant C to increase at each step, we have

$$|v(x_1,t_1)-v(x_0,t_0)| \le C(f(m_0)(x_1)+f(m_0)(x_0)) \le 2CC_0^\theta x_0 \le C\delta_0^{-1}\rho_0$$
$$\le C\delta_0^{-1}a \le C\delta_0^{-1}(|x_1-x_0|+\sqrt{\theta v(x_0,t_0)}|t_1-t_0|).$$

Finally, to see that m itself is also Hölder continuous, we simply observe that, by (4.3.36) and Proposition 4.3.13, the inverse of the map $(x,t)\mapsto \Gamma(x,t) := (\gamma(x,t),t)$ is Lipschitz. Therefore, since $f^{-1} : [0,\infty) \to [0,\infty)$ is Hölder continuous on bounded sets,

$$m = f^{-1}\circ v \circ \Gamma^{-1} : \operatorname{supp}(m)\cap(\mathbb{R}\times[\delta_0, T-\delta_0]) \to \mathbb{R}$$

is the composition of Hölder continuous functions. □

Remark 4.3.22. The result of Theorem 4.3.21 may be improved to obtain Hölder continuity up to the initial time $t=0$, by working with one-sided (in time) analogues of the intrinsic rectangles R_ρ. Moreover, in the case of (MFGP), the Hölder regularity may be established up to $t=T$ as well, by simply imposing on m_T the same assumptions as those of m_0, and applying the continuity result at $t=0$ to the reflected functions $(-u(x,T-t), m(x,T-t))$.

Hölder continuity of Du

We now address the regularity of u in view of the Hölder regularity of m. For (MFGP), since the solution is not unique outside of $\{m>0\}$, we choose to work with the specific u that was constructed in the proof of Theorem 4.3.3. With this choice, for both (MFG) and (MFGP), we observe that u is the unique BUC$(\mathbb{R}\times[0,T])$ viscosity solution to the HJ equation with terminal condition $u(\cdot,T)$. This implies that u satisfies the representation formula

$$u(x,t) = \inf_{\gamma\in H^1((t,T)),\ \gamma(t)=x} \int_t^T \frac{1}{2}|\dot\gamma(s)|^2 + f(m(\gamma(s),s))\, ds + u(\gamma(T),T).$$

We also note that, from the proof of Theorem 4.3.3, the function $-u(x, T-t)$ is also the unique viscosity solution to the HJ equation with terminal condition $-u(\cdot, 0)$, and, therefore, also satisfies the representation formula.

Theorem 4.3.23. *Assume that $f(m)$ is in C^{β}_{loc}. Then the map u is $C^{1,\frac{\beta}{2}}_{\mathrm{loc}}$ in $\mathbb{R} \times (0, T)$.*

Proof. In view of the proof of Theorem 4.3.3, we know that u_x is globally bounded. Let us first check that u is locally semiconcave with a semiconcavity modulus of the form $\omega(r) = Cr^{\beta/2}$. The argument is known and goes back to [6]. Fix $(x, t) \in \mathbb{R} \times I$, where I is a fixed compact sub-interval of $(0, T)$. Let γ be optimal for $u(x, t)$ and $0 < |h|, h' < \tau$ small. We set

$$\gamma_{\pm}(s) := \gamma(\theta_1^{\pm} s + \theta_2^{\pm}) + \theta_3^{\pm}(s - (t+\tau)) \qquad \forall s \in [t \pm h', t+\tau],$$

where $\theta_1^{\pm} = \frac{\tau}{\tau - \pm h'}$, $\theta_2^{\pm} = t - \theta_1^{\pm}(t \pm h')$, $\theta_3^{\pm} = -\frac{\pm h}{\tau - \pm h'}$. Note that

$$\theta_1^{\pm}(t \pm h') + \theta_2^{\pm} = t,\ \theta_1^{\pm}(t+\tau) + \theta_2^{\pm} = t+\tau,\ \gamma_{\pm}(t \pm h') = x \pm h,\ \gamma_{\pm}(t+\tau) = \gamma(t+\tau),$$

so that

$$\begin{aligned}
&u(x+h, t+h') + u(x-h, t-h') - 2u(x,t) \\
&\le \int_{t+h'}^{t+\tau} \frac{1}{2}|\dot\gamma_+|^2 + f(m(\gamma_+, s))ds + \int_{t-h'}^{t+\tau} \frac{1}{2}|\dot\gamma_-|^2 + f(m(\gamma_-, s))ds - 2\int_t^{t+\tau} \frac{1}{2}|\dot\gamma|^2 + f(m(\gamma, s))ds \\
&\le \int_t^{t+\tau} \frac{\theta_1^+}{2}|\dot\gamma + \theta_3^+/\theta_1^+|^2 + \frac{1}{\theta_1^+} f\left(m\left(\gamma(s) + \theta_3^+\left((s-\theta_2^+)/\theta_1^+ - (t+\tau)\right), (s-\theta_2^+)/\theta_1^+\right)\right) ds \\
&\quad + \int_t^{t+\tau} \frac{\theta_1^-}{2}|\dot\gamma + \theta_3^-/\theta_1^-|^2 + \frac{1}{\theta_1^-} f(m(\gamma(s) + \theta_3^-((s-\theta_2^-)/\theta_1^- - (t+\tau)), (s-\theta_2^-)/\theta_1^-))ds \\
&\quad - 2\int_t^{t+\tau} \frac{1}{2}|\dot\gamma|^2 + f(m(\gamma(s), s))ds.
\end{aligned}$$

Using the Hölder regularity of $f(m)$ and the fact that u is bounded we find

$$
\begin{aligned}
&u(x+h,t+h')+u(x-h,t-h')-2u(x,t)\\
&\le \int_t^{t+\tau} \frac{1}{2}(\theta_1^+ + \theta_1^- - 2)|\dot\gamma|^2 + \dot\gamma(\theta_3^+ + \theta_3^-) + \frac{1}{2}((\theta_3^+)^2/\theta_1^+ + (\theta_3^-)^2/\theta_1^-)ds\\
&\quad + \int_t^{t+\tau} (1/\theta_1^+ + 1/\theta_1^- - 2) f(m(\gamma(s),s))ds\\
&\quad + C_I \int_t^{t+\tau} \frac{1}{\theta_1^+}(|\theta_3^+((s-\theta_2^+)/\theta_1^+ - (t+\tau))|^\beta + |((s-\theta_2^+)/\theta_1^+) - s|^\beta)ds\\
&\quad + C_I \int_t^{t+\tau} \frac{1}{\theta_1^-}(|\theta_3^-((s-\theta_2^-)/\theta_1^- - (t+\tau))|^\beta + |((s-\theta_2^-)/\theta_1^-) - s|^\beta)ds\\
&\le C\left(\frac{\tau(h')^2}{\tau^2-(h')^2} + \frac{\tau(h)^2}{\tau^2-(h')^2} + C_I(|h|^\beta\tau + |h'|^\beta\tau)\right)
\end{aligned}
$$

where $C = C(\|u_x\|_\infty)$, since $|\dot\gamma| = |u_x|$. We choose $\tau = (|h| + h')^\delta$ with $\delta = 1 - \beta/2$, which leads to

$$u(x+h,t+h')+u(x-h,t-h')-2u(x,t) \le C(|h|+h')^{1+\frac{\beta}{2}}.$$

This inequality implies the local semiconcavity of u with modulus $\omega(r) = Cr^{\beta/2}$ [5, Thm. 2.1.10].

We now set $w(x,t) = -u(x,T-t)$. Then, as above, w is locally semiconcave with a modulus $\omega(r) = Cr^{\beta/2}$ since the semiconcavity property does not rely on the regularity of the terminal value. Hence u is semiconcave and semiconvex with modulus ω. Following [5, Thm. 3.3.7], the derivatives of u are therefore locally $\beta/2-$Hölder continuous. □

4.4 Infinite speed of propagation: the case of entropic coupling

This section is devoted to the special case of so-called entropic coupling, namely when $f(m) = \log m$. We will show that, in this case, the evolution of m has the property of infinite speed of propagation; marginals with compact support evolve into positive, smooth densities.

4.4.1 The periodic case

As before, we start by considering periodic solutions defined on the torus $R\mathbb{T}$, for arbitrarily large R:

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = \log(m) & (x,t) \in R\mathbb{T} \times (0,T), \\ m_t - (mu_x)_x = 0 & (x,t) \in R\mathbb{T} \times (0,T)\,, \end{cases} \tag{4.4.1}$$

complemented either with prescribed marginals

$$m(x,0) = m_0(x),\ m(x,T) = m_T(x)\,,\ x \in R\mathbb{T}. \tag{4.4.2}$$

or with final pay-off condition

$$u(x,T) = g(m(x,T))\,,\ x \in R\mathbb{T}. \tag{4.4.3}$$

We define solutions which are smooth in $(0,T)$, with traces at $t=0, t=T$ in the space of measures. For this purpose, we denote by $\mathcal{P}(R\mathbb{T})$ the set of Borel probability measure on $R\mathbb{T}$, endowed with the weak$-*$ convergence.

Definition 4.4.1. We say that (u,m) is a (classical) solution of (4.4.1) if $(u,m) \in C^2(R\mathbb{T} \times (0,T)) \times C^1(R\mathbb{T} \times (0,T))$, $m > 0$ in $R\mathbb{T} \times (0,T)$ and the equations are satisfied in a classical sense. For $m_0, m_T \in \mathcal{P}(R\mathbb{T})$, we say that (4.4.2) is satisfied if $m \in C([0,T];\mathcal{P}(R\mathbb{T}))$ and $m(0) = m_0, m(T) = m_T$. Respectively, we say that (4.4.3) is satisfied, if $m \in C([0,T];\mathcal{P}(R\mathbb{T}))$, $m(T) \in L^1(R\mathbb{T})$ and $\lim_{t\to T^-} u(x,t) = g(m(x,T))$ for every $x \in R\mathbb{T}$.

The main result of this subsection is the following theorem.

Theorem 4.4.2. *Assume that $m_0, m_T \in C_c(R\mathbb{T})$. Then the following holds:*

1. *There exists a unique (up to addition of a constant to u) solution (u,m) of (4.4.1)–(4.4.2) such that $m \in L^\infty(R\mathbb{T} \times (0,T))$. In addition, $u,m \in C^\infty(R\mathbb{T} \times (0,T))$.*

2. *Assume that* $g(s) = c_T \log(s)$*, for some* $c_T \geq 0$*. Then there exists a unique solution* (u, m) *of* (4.4.1)–(4.4.3) *such that* $m \in L^\infty(R\mathbb{T}\times(0,T))$*. In addition,* $u, m \in C^\infty(R\mathbb{T}\times(0,T])$ *and* $m > 0$ *in* $(0,T]$.

We recall (see Theorem 4.2.11) that, if $m_0, m_T \in C^{1,\overline{\alpha}}(R\mathbb{T})$ and are strictly positive, then the problems (4.4.1)–(4.4.2) and (4.4.1)–(4.4.3) admit a unique classical solution (u, m) (up to addition of a constant to u, in case of planning conditions (4.4.2)). Therefore, for the proof of Theorem 4.4.2, we will proceed by approximating m_0, m_T with strictly positive smooth measures.

As a first step, we derive estimates which are independent of lower bounds of m_0, m_T as well as independent of R. We denote by $W_2(\mu, \nu)$ the 2–Wasserstein distance between measures $\mu, \nu \in \mathcal{P}(R\mathbb{T})$, and, for a single measure $\mu \in L^1(R\mathbb{T})$, we denote the entropy of μ by $\mathcal{E}(\mu) = \int_{R\mathbb{T}} \mu \log \mu \, dx$ and by $M_2(\mu) = \int_{R\mathbb{T}} |x|^2 \, d\mu$ its second order moment.

Theorem 4.4.3. *Let* (u, m) *be a solution of* (4.4.1)*, and assume that* u, m *are continuous in* $R\mathbb{T} \times [0, T]$.

1. *If* (u, m) *solves* (4.4.1)–(4.4.2)*, there exists a constant* K*, only depending on* T, $\mathcal{E}(m_0)$, $\mathcal{E}(m_T), M_2(m_0), M_2(m_T)$ *(and independent of* R*) such that*

$$\int_0^T \int_{R\mathbb{T}} m\, |u_x|^2 \, dxdt + \int_0^T \int_{R\mathbb{T}} |m \log(m)| \, dxdt \leq K \,, \tag{4.4.4}$$

$$\sup_{t\in(0,T)} \int_{R\mathbb{T}} m(t)|x|^2 \, dx \leq K, \tag{4.4.5}$$

and

$$\sup_{t\in(0,T)} \int_{R\mathbb{T}} m(t)u_x^2 \, dx + \sup_{t\in(0,T)} \left| \int_{R\mathbb{T}} m(t) \, \log m(t) \right| \leq K \,. \tag{4.4.6}$$

As a consequence, it also holds that

$$W_2(m(t), m(s)) \leq K \, |t-s|^{\frac{1}{2}} \qquad \forall t, s \in (0, T) \,. \tag{4.4.7}$$

2. If (u,m) *solves* (4.4.1)–(4.4.3) *(with* $g(m) = c_T \log(m)$*), then the estimates* (4.4.4), (4.4.5), (4.4.7) *hold, for some* K *only depending on* $T, \mathcal{E}(m_0), M_2(m_0)$*. If* $c_T > 0$*, then* (4.4.6) *holds true as well, with* K *depending also on* $(c_T)^{-1}$*.*

Proof. We first consider the case of problem (4.4.1)–(4.4.2). Using the equation of m, we compute

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{R\mathbb{T}} m\,|x|^2\,dx &= \frac{1}{2}\int_{R\mathbb{T}} m_t\,|x|^2\,dx = -\int_{R\mathbb{T}} x\,m\,u_x \\ &\leq \frac{1}{2}\int_{R\mathbb{T}} m\,|x|^2\,dx + \frac{1}{2}\int_{R\mathbb{T}} m\,|u_x|^2\,dx\,.\end{aligned}$$

Hence, Gronwall's lemma implies that, for some constant depending only on T,

$$\int_{R\mathbb{T}} m(t)\,|x|^2\,dx \leq C_T\left(\int_0^T\int_{R\mathbb{T}} m\,|u_x|^2\,dx + \int_{R\mathbb{T}} m_0\,|x|^2\,dx\right). \tag{4.4.8}$$

On the other hand, we know that (m, u_x) is the optimizer of the functional

$$\mathcal{B}(m,v) := \int_0^T\int_{R\mathbb{T}} \frac{1}{2}\,|v|^2 dm + \int_0^T\int_{R\mathbb{T}} m\log(m)\,dxdt\,, \quad \text{subject to } \begin{cases} m_t - (vm)_x = 0 \text{ in } R\mathbb{T}\times(0,T) \\ m(0) = m_0\,, m(T) = m_T\,. \end{cases}$$

This means that

$$\int_0^T\int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T\int_{R\mathbb{T}} m\log(m)\,dxdt \leq \int_0^T\int_{R\mathbb{T}} \frac{1}{2}\,|v|^2 d\mu + \int_0^T\int_{R\mathbb{T}} \mu\log(\mu)\,dxdt$$

for any (μ, v) such that $\mu_t = (\mu v)_x$, with $\mu(0) = m_0, \mu(T) = m_T$. Let us take μ as the Wasserstein geodesic connecting m_0, m_T. By McCann's classical displacement convexity result for Wasserstein geodesics [41], we know that $\int_{R\mathbb{T}} \mu\log(\mu)\,dx$ is convex, hence

$$\int_{R\mathbb{T}} \mu\log(\mu)\,dx \leq \max\left(\mathcal{E}(m_0)\,, \mathcal{E}(m_T)\right)\,.$$

We deduce that

$$\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} m\log(m)\,dxdt \le C_T\,W_2(m_0, m_T) + \max\left(\mathcal{E}(m_0)\,, \mathcal{E}(m_T)\right)\,.$$

This yields, for some constant C independent of R,

$$\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} (m\log(m))_+\,dxdt \le C + \int_0^T \int_{R\mathbb{T}} (m\log m)_- dxdt\,. \tag{4.4.9}$$

Here and below, the constants will be independent of R, although they may depend on T, $\mathcal{E}(m_0)$, $\mathcal{E}(m_T)$, $M_2(m_0)$, $M_2(m_T)$. Since $(s\log s)_- \le c\sqrt{s}$ we have

$$\begin{aligned}\int_0^T \int_{R\mathbb{T}} (m\log m)_- dxdt &\le c\int_0^T \int_{R\mathbb{T}} \sqrt{m}\,dxdt \\ &\le C + \varepsilon \int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt + C_\varepsilon \int_0^T \int_{R\mathbb{T}} \frac{1}{1+|x|^2}\,dxdt\end{aligned} \tag{4.4.10}$$

by Young's inequality. Then (4.4.9) yields

$$\begin{aligned}\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} (m\log(m))_+\,dxdt \le C + \varepsilon \int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt \\ + C_\varepsilon \int_0^T \int_{R\mathbb{T}} \frac{1}{1+|x|^2}\,dxdt\,.\end{aligned} \tag{4.4.11}$$

From (4.4.8), after integration we deduce

$$\begin{aligned}\int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt &\le T\,C_T \int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + C_T\,M_2(m_0) \\ &\le \varepsilon\,T\,C_T \int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt + C(\varepsilon, T, m_0)\,.\end{aligned}$$

Choosing ε suitably small, we get an estimate for $\int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt$. In turn, from (4.4.10) and (4.4.11), we deduce (4.4.4). Now, the right-hand side in (4.4.8) is controlled, and we get the estimate (4.4.5). Moreover, from the equation satisfied by m and the estimate (4.4.4), we immediately deduce (4.4.7).

We are left with the pointwise estimate of $\int_{R\mathbb{T}} mu_x^2\,dx$. For this purpose, we observe that

$$\frac{d}{dt}\left[\int_{R\mathbb{T}} mu_x^2\,dx - \int_{R\mathbb{T}} m\log m\,dx\right] = 0. \tag{4.4.12}$$

Therefore,

$$\int_{R\mathbb{T}} mu_x^2\,dx - \int_{R\mathbb{T}} m\log m\,dx = \frac{1}{T}\int_0^T \left\{\int_{R\mathbb{T}} mu_x^2\,dx - \int_{R\mathbb{T}} m\log m\,dx\right\}dt\,.$$

We deduce that

$$\left|\int_{R\mathbb{T}} mu_x^2\,dx - \int_{R\mathbb{T}} m\log m\,dx\right| \le \frac{1}{T}\left\{\int_0^T\int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T\int_{R\mathbb{T}} |m\log(m)|\,dxdt\right\} \le C\,. \tag{4.4.13}$$

Since, by the displacement convexity formula (4.2.27), we have

$$\int_{R\mathbb{T}} m\log m\,dx \le \max\left(\mathcal{E}(m_0)\,,\mathcal{E}(m_T)\right),$$

we conclude by (4.4.13) that $\left|\int_{R\mathbb{T}} m\log m\,dx\right|$ is bounded, so $\int_{R\mathbb{T}} mu_x^2\,dx$ is bounded above, uniformly in t. This yields (4.4.6).

In case of problem (4.4.1)–(4.4.3), the only difference is in the first estimate, (4.4.4). By the optimality condition, using that $g(s) = c_T\log(s)$, we know that

$$\int_{R\mathbb{T}} u(x,0)\,m_0\,dx \le \frac{1}{2}\int_0^T\int_{R\mathbb{T}} \mu|v|^2\,dxds + \int_0^T\int_{R\mathbb{T}} \mu\,\log\mu\,dxds + c_T\int_{R\mathbb{T}} \mu_T\log(\mu_T)\,dx$$

for any curve $\mu(t)$ such that $\mu(0) = m_0$ and $\mu(T) = \mu_T$, with $\mu_t = (\mu\,v)_x$.

It is enough to choose some μ_T with finite entropy to obtain a global estimate of the right-hand side. Since we also have, by duality,

$$\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} m\log(m)\,dxdt + c_T \int_{R\mathbb{T}} m(T)\log(m(T))dx = \int_{R\mathbb{T}} u(x,0)\,m_0\,dx$$

we get immediately (4.4.9) if $c_T = 0$. If $c_T > 0$, we estimate the term at $t = T$ in a similar way as in (4.4.10), (4.4.11), and we end up with

$$\begin{aligned} &\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} |m\log(m)|\,dxdt + c_T \int_{R\mathbb{T}} |m(T)\log(m(T))|\,dx \\ &\qquad\qquad \le C + \varepsilon \int_0^T \int_{R\mathbb{T}} m\,(|x|^2+1)dxdt + \varepsilon \int_{R\mathbb{T}} m(T)(|x|^2+1)dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad + C_\varepsilon(1+T)\int_{R\mathbb{T}} \frac{1}{1+|x|^2}\,dxdt\,. \end{aligned}$$

Using (4.4.8) and choosing ε suitably small, the estimates (4.4.4), (4.4.5) and (4.4.7) follow as before. Notice that, if $c_T > 0$, we also estimate $\mathcal{E}(m(T))$ and then we are back to the previous case, obtaining (4.4.6) as well. □

Now we show a local bound on the value function u, which is independent of the period R.

Theorem 4.4.4. *Let (u, m) be a solution of (4.4.1), and assume that u, m are continuous in $R\mathbb{T} \times [0, T]$.*

1. *If (u, m) solves (4.4.1)–(4.4.2), there exists a constant $C > 0$, depending on $\|m_0\|_\infty, \|m_T\|_\infty$, $M_2(m_0), M_2(m_T)$, but independent of R, such that, if we normalize u such that $\int u(T)m_T = 0$, then we have*

$$-\frac{C}{t}(1+|x|^2) \le u(x,t) \le \frac{C}{T-t}(1+|x|^2) \qquad \forall t \in (0,T)\,, x \in \mathbb{R}\,. \tag{4.4.14}$$

2. *If (u, m) solves (4.4.1)–(4.4.3) (with $g(m) = c_T \log(m)$), then the estimate (4.4.14)*

holds for some C only depending on $\|m_0\|_\infty, M_2(m_0), c_T$.

Proof. First we consider the case of (4.4.1)–(4.4.2), and we adapt a similar proof given in [48, Lemma 4.2] for bounded domains.

We observe that, from the standard duality between u, m we have

$$\int_0^T \int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T \int_{R\mathbb{T}} m\log(m)\,dxdt = \int_{R\mathbb{T}} u(x,0)m_0 - \int_{R\mathbb{T}} u(x,T)m_T$$

which implies, using Proposition 4.4.3 and $\int u(T)m_T = 0$, that

$$\int_{R\mathbb{T}} u(x,0)m_0 \geq -K\,. \tag{4.4.15}$$

Notice that the constant depends on m_0, m_T through the entropy, by Proposition 4.4.3; however, for bounded functions, $\mathcal{E}(m)$ is itself estimated in terms of $M_2(m)$ and $\|m\|_\infty$.

Now, let us consider the Wasserstein geodesic $\mu(\cdot)$ which connects, in time $(0,t)$, m_0 with any measure λ, supposed to be compactly supported in $(-R/2, R/2)$. This means that $\mu(t) = \lambda$, $\mu(0) = m_0$ and $\mu_s = (\mu v)$ in $(0,t)$, for some v such that $\int_0^t \int_{R\mathbb{T}} |v|^2 d\mu < \infty$. By duality with the equation satisfied by u, we have

$$\begin{aligned}
-\int_{R\mathbb{T}} u(x,t)d\lambda + \int_{R\mathbb{T}} u(x,0)\,m_0\,dx + \frac{1}{2}\int_0^t \int_{R\mathbb{T}} |u_x|^2\,\mu\,dxds = \int_0^t \int_{R\mathbb{T}} \mu\,\log m\,dxds \\
+ \int_0^t \int_{R\mathbb{T}} \mu\,v\,u_x\,dxds \leq \frac{1}{2}\int_t^T \int_{R\mathbb{T}} |u_x|^2\,\mu\,dxds + \frac{1}{2}\int_0^t \int_{R\mathbb{T}} \mu|v|^2\,dxds + T\log(\|m\|_\infty \vee 1)
\end{aligned}$$

which yields

$$\begin{aligned}
\int_{R\mathbb{T}} u(x,t)d\lambda &\geq \int_{R\mathbb{T}} u(x,0)\,m_0\,dx - \frac{1}{2}\int_0^t \int_{R\mathbb{T}} \mu|v|^2\,dxds - T\log(\|m\|_\infty \vee 1) \\
&\geq -K - \frac{c}{t} W_2(\lambda, m_0)^2 - T\log(\|m\|_\infty \vee 1)\,,
\end{aligned}$$

where we used (4.4.15) and the scaling of Wasserstein geodesic. If we let λ converge (in the weak-$*$ topology) towards a Dirac mass δ_{x_0}, we get

$$u(x_0,t) \geq -K - \frac{c}{t}\int_{\mathbb{R}} |x_0-y|^2 dm_0(y) - T\log(\|m\|_\infty \vee 1)\,.$$

We recall that, by Corollary 4.2.7, $\|m(t)\|_\infty$ is controlled by the initial-terminal values. Hence, there exists a constant C, depending on $\|m_0\|_\infty, \|m_T\|_\infty, T$ and $M_2(m_0)$, $M_2(m_T)$, such that

$$u(x_0,t) \geq -\frac{C}{t}(1+|x_0|^2)\,.$$

A similar argument (instead of (4.4.15) we simply use that $\int_{R\mathbb{T}} u(x,T)m_T = 0$) shows the upper bound $u(x_0,t) \leq \frac{C}{T-t}(1+|x_0|^2)$ and concludes the proof of (4.4.14).

In the case of (4.4.3), the duality equality takes the form

$$\int_0^T\int_{R\mathbb{T}} m\,|u_x|^2\,dxdt + \int_0^T\int_{R\mathbb{T}} m\log(m)\,dxdt = \int_{R\mathbb{T}} u(x,0)m_0 - c_T\int_{R\mathbb{T}} m(T)\log(m(T))dx$$

and (4.4.15) follows again from Proposition 4.4.3. We also recall that the L^∞ bound on m is given by Proposition 4.2.9. Then we obtain as before the lower estimate of u. For the upper estimate, we just observe that $\int_{R\mathbb{T}} u(T)m(T)dx = c_T\int_{R\mathbb{T}} m(T)\log(m(T))dx$ and this is bounded above (uniformly with respect to R) if either $c_T = 0$ or $c_T > 0$ (from Proposition 4.4.3). Hence we repeat the argument above using any geodesic connecting $m(T)$ with a Dirac mass. □

In the next step we aim at showing that, if (u,m) is a solution of (4.4.1), then $m(t)$ becomes positive for $t>0$ even if starting from a compactly supported initial measure. For this purpose, we use in a key way the displacement convexity estimates. However, we warn the reader that, while the estimates of Proposition 4.4.3 and 4.4.4 were all independent of the period R, this will not be the case for the following bounds on $\log(m)$.

Lemma 4.4.5. *Let (u,m) be a solution to (4.4.1), where $m_0, m_T > 0$. Set $K = \max(\|m_0\|_\infty,$*

$\|m_T\|_\infty$). *Then, for each integer* $p \geq 1$,

$$\frac{d^2}{dt^2}\int_{R\mathbb{T}} \left|\log\left(\frac{m}{K}\right)\right|^p \geq 0. \tag{4.4.16}$$

Moreover, there exists a constant $C > 0$, *depending on* $\|m_0\|_\infty, \|m_T\|_\infty$, *such that, for each* $t \in [0, T]$,

$$\left\|\log\left(\frac{m(t)}{K}\right)\right\|_\infty^2 \leq R\,\frac{d^2}{dt^2}\int_{R\mathbb{T}} \left|\log\left(\frac{m(t)}{K}\right)\right| + C \tag{4.4.17}$$

Proof. Letting $h(m) = \log(\frac{m}{K})^p$, we obtain

$$h'(m) = p\log\left(\frac{m}{K}\right)^{p-1}\frac{1}{m}, \;\; h''(m) = p\left(p - 1 - \log\left(\frac{m}{K}\right)\right)\log\left(\frac{m}{K}\right)^{p-2}\frac{1}{m^2}.$$

We observe that, by Corollary 4.2.7 and by definition of K, we have $\frac{m}{K} \leq 1$. Hence, in the range of m, h is positive and convex when p is even, and negative and concave when p is odd. The displacement convexity formula (4.2.27) yields

$$\frac{d^2}{dt^2}\int_{R\mathbb{T}} h(m) = \int_{R\mathbb{T}} h''(m)(m^2u_{xx}^2 + m_x^2),$$

which, in particular, shows (4.4.16), and, setting $p = 1$, we obtain

$$\int_{R\mathbb{T}} \left(\log\left(\frac{m}{K}\right)_x\right)^2 \leq \frac{d^2}{dt^2}\int_{R\mathbb{T}} \left|\log\left(\frac{m}{K}\right)\right|.$$

On the other hand, by the fundamental theorem of calculus, and the fact that m is a density, we have

$$\left\|\log\left(\frac{m(t)}{K}\right) - \log\left(\frac{1}{K}\right)\right\|_\infty^2 \leq R\int_{R\mathbb{T}} \left(\log\left(\frac{m}{K}\right)_x\right)^2,$$

and (4.4.17) follows. $\square$

We can now state the (local in time) uniform bound from below, which is independent of $\|m_0^{-1}\|_\infty, \|m_T^{-1}\|_\infty$.

Theorem 4.4.6. *Let (u,m) be a solution to (4.4.1), where $m_0, m_T > 0$. There exists a constant $C_R > 0$, only depending on $\|m_0\|_\infty, \|m_T\|_\infty$ and R, such that, for each $t \in (0,T)$,*

$$\|\log m(t)\|_\infty \leq C_R\left(\frac{1}{t^2} + \frac{1}{(T-t)^2}\right).$$

Proof. Let $t_0 \in [0, \frac{T}{2}]$. In view of (4.4.16), we have, for integers $p \geq 1$ and $s \in (0, \frac{t_0}{2})$,

$$\begin{aligned}
&\left(\frac{1}{(T-2t_0)}\int_{t_0}^{T-t_0}\int_{R\mathbb{T}}\left|\log\left(\frac{m}{K}\right)\right|^{2p}\right)^{\frac{1}{p}} \\
&\qquad\leq \left(\int_{R\mathbb{T}}\left|\log\left(\frac{m(t_0-s)}{K}\right)\right|^{2p} + \int_{R\mathbb{T}}\left|\log\left(\frac{m(T-t_0+s)}{K}\right)\right|^{2p}\right)^{\frac{1}{p}} \\
&\qquad\quad\leq \left(\int_{R\mathbb{T}}\left|\log\left(\frac{m(t_0-s)}{K}\right)\right|^{2p}\right)^{\frac{1}{p}} + \left(\int_{R\mathbb{T}}\left|\log\left(\frac{m(T-t_0+s)}{K}\right)\right|^{2p}\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, integrating in s, we infer that

$$\left(\int_{t_0}^{T-t_0}\int_{R\mathbb{T}}\left|\log\left(\frac{m}{K}\right)\right|^{2p}\right)^{\frac{1}{p}} \leq \frac{2(T-2t_0)^{\frac{1}{p}}}{t_0}\int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}}\left(\int_{R\mathbb{T}}\left|\log\left(\frac{m}{K}\right)\right|^{2p}\right)^{\frac{1}{p}},$$

and letting $p \to \infty$ yields

$$\left\|\log\left(\frac{m}{K}\right)\right\|^2_{L^\infty(R\mathbb{T}\times[t_0,T-t_0])} \leq \frac{2}{t_0}\int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}}\left\|\log\left(\frac{m(t)}{K}\right)\right\|^2_\infty dt.$$

Now, integrating (4.4.17) against a test function ζ supported in $[\frac{t_0}{3}, T-\frac{t_0}{3}]$, satisfying $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $[\frac{t_0}{2}, T-\frac{t_0}{2}]$, and $|\zeta''| \leq \frac{C}{t_0^2}$, we get

$$\left\|\log\left(\frac{m}{K}\right)\right\|^2_{L^\infty(R\mathbb{T}\times[t_0,T-t_0])} \leq \frac{C\,R}{t_0^3}\int_{\frac{t_0}{3}}^{T-\frac{t_0}{3}}\int_{R\mathbb{T}}\left|\log\left(\frac{m}{K}\right)\right| dt + \frac{C}{t_0}.$$

Finally, by Proposition 4.4.4, u is bounded by $\frac{C(1+R^2)}{t_0}$ on $[t_0, T-t_0]$, and, hence, by the HJ equation, we estimate

$$\int_{\frac{t_0}{3}}^{T-\frac{t_0}{3}} \int_{R\mathbb{T}} \left|\log\left(\frac{m}{K}\right)\right| \leq \frac{CR^3}{t_0}\,.$$

This yields

$$\left\|\log\left(\frac{m}{K}\right)\right\|_{L^\infty(R\mathbb{T}\times[t_0,T-t_0])} \leq \frac{C(1+R^2)}{t_0^2},$$

which implies the result. □

Finally, we have all the ingredients for the proof of Theorem 4.4.2.

Proof of Theorem 4.4.2. We start with the case of problem (4.4.1)–(4.4.2). Let $m_0^\varepsilon, m_T^\varepsilon$ be two sequences of functions such that $m_0^\varepsilon, m_T^\varepsilon \in C^{1,\overline{\alpha}}(R\mathbb{T})$, $m_0^\varepsilon, m_T^\varepsilon > 0$ in $R\mathbb{T} \times [0,T]$ and $m_0^\varepsilon, m_T^\varepsilon$ converge uniformly to m_0, m_T respectively. Such an approximation can be readily built by convolution, for instance. By Theorem 4.2.11, there exists a smooth positive solution $(u^\varepsilon, m^\varepsilon)$ of (4.4.1), where we normalize u^ε such that

$$\int_{R\mathbb{T}} u^\varepsilon(T)\, m_T^\varepsilon dx = 0\,.$$

By Corollary 4.2.7, we know that m^ε is uniformly bounded. Then, from Proposition 4.4.4, we deduce that u^ε is locally bounded in $(0,T)$. It also follows from Proposition 4.4.6 that $\log(m^\varepsilon)$ is locally bounded in $(0,T)$ (i.e. m^ε is locally uniformly bounded below). In turn, this implies that u_x is locally bounded in $(0,T)$; one can use for example [48, Thm 6.5] which shows[3] that $\frac{|u_x^\varepsilon|^2}{4} + \log(m^\varepsilon) \leq C_\delta$ for every $x \in R\mathbb{T}, t \in (a+\delta, b-\delta)$, where C_δ only depends on δ and the bound on u^ε in (a,b). Since u^ε satisfies

$$-u_{tt} + 2u_x u_{xt} - (u_x^2+1)u_{xx} = 0\,, \tag{4.4.18}$$

3. The proof in [48] is given for Neumann boundary conditions, but applies identically to periodic solutions

once $(u^\varepsilon)_x$ is locally bounded then the above equation becomes a quasilinear equation which has bounded uniformly elliptic coefficients in any compact subset of $R\mathbb{T} \times (0,T)$. By a standard bootstrap regularity from Schauder's estimates, we deduce that u^ε is locally bounded in $C^{k,\overline{\alpha}}$ (for every $k \in N, \overline{\alpha} \in (0,1)$). Hence, u^ε converges in $R\mathbb{T} \times (0,T)$ towards a function u which is C^∞. Since $m^\varepsilon = \exp(-(u^\varepsilon)_t + |(u^\varepsilon)_x|^2/2)$, we also have m^ε converging to some $m \in C^\infty(R\mathbb{T} \times (0,T))$. But the global estimates also imply that $m \in L^\infty(Q_T)$. In addition, by (4.4.7), we also have that $m^\varepsilon(t)$ is equi-continuous in the Wasserstein space of measures, hence it uniformly converges in $[0,T]$. We deduce that $m \in C^0([0,T];\mathcal{P}(R\mathbb{T}))$ and $m(0) = m_0, m(T) = m_T$. This concludes the proof that (u,m) is a solution of (4.4.1), which is classical inside $(0,T)$.

In case of problem (4.4.1)–(4.4.3), the proof is similar, except that we only approximate m_0. The L^∞ bound on m^ε follows from Proposition 4.2.9, then we argue as before to deduce that m^ε is locally uniformly bounded below, and $u^\varepsilon, m^\varepsilon$ are locally bounded in $C^{k,\overline{\alpha}}$. Applying Propositions 4.2.9 and 4.2.10 to problem (4.4.1)–(4.4.3) for $t \in (T/2,T)$, we conclude that $u_\varepsilon, Du^\varepsilon$ are uniformly bounded up to $t = T$, and m^ε is bounded below up to $t = T$. By regularity of equation (4.4.18) up to the boundary $t = T$, we conclude that u, m are smooth up to $t = T$ and $u(T) = c_T \log(m(T))$.

For the uniqueness of solutions, we use some argument which was already developed for much weaker notions of solutions ([7], [15], [47]). Let (u,m) be a solution which is classical inside, as in Definition 4.4.1, and such that $m \in L^\infty(R\mathbb{T} \times (0,T))$. First we notice that, from (4.4.12), the bound on m implies that $mu_x^2 \in L^1(R\mathbb{T} \times (0,T))$. Next we observe that u satisfies $-u_t + \frac{1}{2}u_x^2 \leq \log(\|m\|_\infty)$, so (up to a time translation) we can assume that $u(\cdot,x)$ is nondecreasing. this implies that $u(t)$ admits one-sided traces at $t = 0$, $t = T$, namely two measurable functions (not necessarily finite) defined as $u(x,0^+) = \lim_{t\downarrow 0} u(x,t) \in \mathbb{R} \cup \{-\infty\}$, $u(x,T^-) = \lim_{t\uparrow 0} u(x,t) \in \mathbb{R} \cup \{+\infty\}$. We first show that $u(0^+) \in L^1(dm_0)$; indeed, since

(u, m) is smooth in $R\mathbb{T} \times (0, T)$, we know that

$$\int_{R\mathbb{T}} u(t_0)m(t_0) - \int_{R\mathbb{T}} u(t_1)m(t_1) = \int_{t_0}^{t_1} \int_{R\mathbb{T}} m u_x^2 + \int_{t_0}^{t_1} \int_{R\mathbb{T}} m \log(m), \qquad \forall\, 0 < t_0 < t_1 < T\,. \tag{4.4.19}$$

Choosing, for instance, $t_1 = \frac{T}{2}$, it follows that $\int_{R\mathbb{T}} u(t_0)m(t_0)$ is bounded below, for every arbitrarily small t_0. Since u is nondecreasing, we deduce $\int_{R\mathbb{T}} u(s)m(t_0) \geq -C$ for any $s > t_0$; letting $t_0 \to 0$ (and using that $m \in C([0, T]; \mathcal{P}(R\mathbb{T})))$ yields $\int_{R\mathbb{T}} u(s)m_0 \geq -C$. Then by monotone convergence we deduce, as $s \to 0^+$, that $\int_{R\mathbb{T}} u(0^+)m_0 \geq -C$. Since the opposite inequality is clear by monotonicity and Proposition 4.4.4, we find that $u(0^+) \in L^1(dm_0)$. Similarly we reason to show that $u(T^-) \in L^1(dm_T)$ (in case of problem (4.4.2)) or that $m(T)\log(m(T)) \in L^1(R\mathbb{T})$ (in case of (4.4.3) with $c_H > 0$).

Now, with a truncation argument, from the equality (4.4.19) we will show that (u, m) satisfies

$$\int_{R\mathbb{T}} u(0^+) dm_0 - \int_{R\mathbb{T}} u(T^-) dm(T) = \int_0^T \int_{R\mathbb{T}} m u_x^2 + \int_0^T \int_{R\mathbb{T}} m \log(m)\,. \tag{4.4.20}$$

Indeed, we first replace u by truncations $u_k := \min(k, \max(u, -k))$, we multiply the HJ equation by m and we integrate in (t_0, t_1); next we can let $t_0 \to 0^+, t_1 \to T^-$ (using the weak-* convergence of m and the strong L^1 convergence of u_k at $t = 0, t = T$). Then we finally let $k \to \infty$ (thanks to $u(0^+) \in L^1(dm_0), u(T^-) \in L^1(dm(T))$) and we obtain (4.4.20).

In a similar way, one can prove that, for any couple of solutions (u, m), $(\tilde{u}, \tilde{m})$, it holds

$$\int_{R\mathbb{T}} u(0^+) dm_0 - \int_{R\mathbb{T}} u(T^-) d\tilde{m}(T) \leq \int_0^T \int_{R\mathbb{T}} \tilde{m} \left(\tilde{u}_x\, u_x - \frac{1}{2} u_x^2 \right) + \int_0^T \int_{R\mathbb{T}} \tilde{m} \log(m)\,. \tag{4.4.21}$$

The proof of (4.4.21) can be done, as before, replacing first u with its truncation u_k and integrating in (t_0, t_1):

$$\int_{R\mathbb{T}} u_k(t_0)\tilde{m}(t_0) - \int_{R\mathbb{T}} u_k(t_1)\tilde{m}(t_1) \le \int_{t_0}^{t_1}\int_{R\mathbb{T}} \tilde{m}\left((u_k)_x\tilde{u}_x - \frac{1}{2}(u_k)_x^2\right) + \int_{t_0}^{t_1}\int_{R\mathbb{T}} \tilde{m}\log(m)\, 1_{\{|u|<k\}}\ .$$

The right-hand side integrand is easily dominated from above. Once more, we can let first $t_0 \to 0, t_1 \to T$ and then $k \to \infty$, in order to get (4.4.21).

From (4.4.20), (4.4.21), the uniqueness follows as in the classical Lasry-Lions monotonicity argument. For problem (4.4.1)–(4.4.2), we take two solutions normalized such that $\int_{R\mathbb{T}} u(T^-)dm_T = \int_{R\mathbb{T}} \tilde{u}(T^-)dm_T = 0$. Using (4.4.20), (4.4.21) for both couples we obtain that

$$\begin{aligned}&\int_0^T\int_{R\mathbb{T}} \tilde{m}\left(\frac{1}{2}u_x^2 - \frac{1}{2}\tilde{u}_x^2 - (u_x - \tilde{u}_x)\,\tilde{u}_x\right) + \int_0^T\int_{R\mathbb{T}} m\left(\frac{1}{2}\tilde{u}_x^2 - \frac{1}{2}u_x^2 - (\tilde{u}_x - u_x)\,u_x\right)\\ &+ \int_0^T\int_{R\mathbb{T}} (m\log(m) - \tilde{m}\log(\tilde{m}))(m - \tilde{m}) \le 0,\end{aligned}$$

which implies $m = \tilde{m}$ and $u_x = \tilde{u}_x$. From the HJ equation we deduce that $u - \tilde{u} = C$, and this concludes the proof. For the problem (4.4.1)–(4.4.3), we proceed similarly and we get uniqueness using the coupled condition $u(T) = g(m(T))$. $\square$

4.4.2 Preservation of monotonicity of the solutions

In this subsection, we show that the MFG system preserves a certain monotonicity property. As the phenomenon does not depend on the specific form of the coupling functions f and g, we suppose here that f and g are smooth and nondecreasing on $(0, \infty)$. We work in the periodic setting and assume the structure condition:

$$\begin{gathered}\text{the densities } m_0, m_T : R\mathbb{T} \to \mathbb{R} \text{ are even,}\\ \text{nonincreasing on } [0, R/2] \text{ and nondecreasing on } [-R/2, 0].\end{gathered} \tag{4.4.22}$$

Lemma 4.4.7. *Assume that* $(u, m) \in C^2(R\mathbb{T} \times [0, T]) \times C^1((R\mathbb{T} \times [0, T]))$ *is the unique classical solution to* (4.2.20)–(4.2.21) *or* (4.2.20)–(4.2.22)*, such that* m *is positive on* $R\mathbb{T} \times [0, T]$ *and* (4.4.22) *holds. Then*

$$\forall t \in [0,T],\ m(\cdot,t) : R\mathbb{T} \to \mathbb{R} \text{ is even, nonincreasing on } [0, R/2] \text{ and nondecreasing on } [-R/2, 0]. \tag{4.4.23}$$

In addition any optimal trajectory $\gamma(x, \cdot)$ *starting from* $x \in [0, R/2]$ *is concave in time. Finally, for the MFG problem* (4.2.20)–(4.2.21)*,* $\gamma(x, \cdot)$ *is nondecreasing in time for any* $x \in [0, R/2]$ *and* u_x *is nonpositive in* $[0, R/2] \times [0, T]$.

Of course, par approximation, this preservation of the structure also holds in the whole space: in the next subsection, we shall use it in the case $f(m) = \log(m)$ to build classical solutions in the whole space. In the case $f(m) = m^\theta$, it shows that, if m_0, m_T are even on $\mathbb{R}$ and nonincreasing on $[0, +\infty)$, then any trajectory starting from $x \in [0, R/2]$ is concave in time: compare with Theorem 4.3.14.

Proof. We do the proof in the MFG case, i.e., when (u, m) solves (4.2.20)–(4.2.21), with the proof for the planning problem (4.2.20)–(4.2.22) being similar and simpler. By the symmetry assumption and the uniqueness of the solution, $m(\cdot, t), u(\cdot, t)$ are even for any $t \in [0, T]$. Thus $m_x(0, t) = u_x(0, t) = 0$. Let us set

$$M(x,t) = 1/2 - \int_0^x m(y,t)dy, \qquad (x,t) \in [0, R/2] \times [0,T].$$

We first note that M is a classical solution to

$$-\mathrm{Tr}\left(\begin{pmatrix} \frac{M_t^2}{M_x^2} & -\frac{M_t}{M_x} \\ -\frac{M_t}{M_x} & 1 \end{pmatrix} D^2_{x,t}M\right) + M_x f'(-M_x)M_{xx} = 0 \qquad \text{in } [0, R/2] \times (0,T) \tag{4.4.24}$$

with boundary condition, for $(x,t) \in [0,R/2] \times [0,T]$,

$$M(0,t) = 1/2,\ M(R/2,t) = 0,\ M(x,0) = M_0(x), \qquad \text{where } M_0(x) = 1/2 - \int_0^x m_0(y)dy, \tag{4.4.25}$$

and

$$M_t(x,T) + M_x(x,T)g'(-M_x(x,T))M_{xx}(x,T) = 0. \tag{4.4.26}$$

The elliptic equation (4.4.24) was proved in Lemma 4.2.4 (where we also explained that $u_x = M_t/M_x$). The boundary conditions (4.4.25) at $x = 0$ and $t = 0$ hold by definition. For $x = R/2$, it comes from the fact that m is a probability measure and from the symmetry. The boundary condition (4.4.26) at $t = T$ comes from the boundary condition for u, which implies that

$$(M_t/M_x)(x,T) = u_x(x,T) = (g'(m)m_x)(x,T) = g'(-M_x(x,T))(-M_{xx}(x,T)).$$

The main part of the proof consists in showing that $x \to M(x,t)$ is convex on $[0,R/2]$ for any $t \in [0,T]$. For this we consider the map

$$\tilde{M}(x,t) = \inf_{\substack{\lambda y + (1-\lambda)z = x, \\ y,z \in [0,R/2],\ \lambda \in [0,1]}} \lambda M(y,t) + (1-\lambda)M(z,t).$$

Note that $\tilde{M} \leq M$ and that $\tilde{M}$ is continuous and satisfies the boundary condition (4.4.25) by our assumption on m_0. We now prove that $\tilde{M}$ is a viscosity supersolution to the elliptic equation (4.4.24) and satisfies the boundary condition (4.4.26) in the viscosity sense.

Assume that ϕ is a test function touching $\tilde{M}$ from below at $(x_0,t_0) \in (0,R/2) \times (0,T]$.

If $t_0 < T$, we have to check that $\phi_x(x_0, t_0) \neq 0$ and that

$$-\mathrm{Tr}\left(\begin{pmatrix} \frac{\phi_t^2}{\phi_x^2} & -\frac{\phi_t}{\phi_x} \\ -\frac{\phi_t}{\phi_x} & 1 \end{pmatrix} D^2_{x,t}\phi\right) + \phi_x f'(-\phi_x)\phi_{xx} \geq 0 \qquad \text{at } (x_0, t_0). \tag{4.4.27}$$

If $t_0 = T$, we have to prove that

$$\phi_t(x_0, T) + \phi_x(x_0, T) g'(-\phi_x(x_0, T))\phi_{xx}(x_0, T) \geq 0. \tag{4.4.28}$$

Note that, if $\tilde{M}(x_0, t_0) = M(x_0, t_0)$, these inequalities hold because M satisfies (4.4.24) and (4.4.26). Thus, we assume from now on that $\tilde{M}(x_0, t_0) < M(x_0, t_0)$. Let $y_0 < x_0 < z_0$ and $\lambda_0 \in (0, 1)$ be optimal in the definition of $\tilde{M}(x_0, t_0)$.

In this step we assume that $t_0 < T$. By optimality of (y_0, z_0, λ_0), and the fact that, by symmetry, $M_{xx}(0, t_0) = m_x(0, t_0) = M_{xx}(R/2, t_0) = m_x(R/2, t_0) = 0$, we have that

$$\begin{gathered} \phi_x(x_0, t_0) = \tfrac{M(z_0,t_0) - M(y_0,t_0)}{z_0 - y_0} < 0 \\ \phi_x(x_0, t_0) = M_x(y_0, t_0) \text{ or } y_0 = 0, \quad \phi_x(x_0, t_0) = M_x(z_0, t_0) \text{ or } z_0 = R/2, \\ \phi_{xx}(x_0, t_0) \leq 0,\ M_{xx}(y_0, t_0) \geq 0,\ M_{xx}(z_0, t_0) \geq 0. \end{gathered} \tag{4.4.29}$$

Fix $\theta, \theta_1, \theta_2 \in \mathbb{R}$ such that $\lambda_0\theta_1 + (1 - \lambda_0)\theta_2 = \theta$, with $\theta_1 = 0$ if $y_0 = 0$ and $\theta_2 = 0$ if $z_0 = R/2$. For h and s small, we have

$$\phi(x_0 + h\theta, t_0 + s) \leq \tilde{M}(x_0 + h, t_0 + s) \leq \lambda_0 M(y_0 + \theta_1 h, t_0 + s) + (1 - \lambda_0) M(z_0 + \theta_2 h, t_0 + s),$$

with an equality at $h = s = 0$. This implies that

$$\phi_t(x_0, t_0) = \lambda_0 M_t(y_0, t_0) + (1 - \lambda_0) M_t(z_0, t_0), \tag{4.4.30}$$

and

$$\begin{pmatrix} \theta^2\phi_{xx} & \theta\phi_{xt} \\ \theta\phi_{xt} & \phi_{tt} \end{pmatrix}(x_0,t_0) \le \lambda_0 \begin{pmatrix} \theta_1^2 M_{xx} & \theta_1 M_{xt} \\ \theta_2 M_{xt} & M_{tt} \end{pmatrix}(y_0,t_0) + (1-\lambda_0) \begin{pmatrix} \theta_2^2 M_{xx} & \theta_2 M_x \\ \theta_2 M_x & M_{tt} \end{pmatrix}(z_0,t_0).$$

Multiplying the previous inequality by $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \ge 0$ and taking the trace gives

$$\begin{aligned} &\Big(\theta^2\phi_{xx} - 2\theta\phi_{xt} + \phi_{tt}\Big)(x_0,t_0) \\ &\qquad \le \lambda_0\Big(\theta_1^2 M_{xx} - 2\theta_1 M_{xt} + M_{tt}\Big)(y_0,t_0) + (1-\lambda_0)\Big(\theta_2^2 M_{xx} - 2\theta_2 M_{xt} + M_{tt}\Big)(z_0,t_0). \end{aligned} \tag{4.4.31}$$

Let us choose $\theta = \phi_t(x_0,t_0)/\phi_x(x_0,t_0)$, $\theta_1 = M_t(y_0,t_0)/M_x(y_0,t_0)$ and $\theta_2 = M_t(z_0,t_0))/M_x(z_0,t_0)$: this choice is licit because, if $y_0 = 0$, the boundary conditions (4.4.25) imply that $M_t(0,t_0) = 0$, and thus $\theta_1 = 0$. We obtain in the same way that $\theta_2 = 0$ if $z_0 = R/2$. Then (4.4.29) and (4.4.30) imply that $\lambda_0\theta_1 + (1-\lambda_0)\theta_2 = \theta$ holds. With this choice of θ, θ_1 and θ_2, (4.4.31) becomes

$$\begin{aligned} &\Big(\frac{\phi_t^2}{\phi_x^2}\phi_{xx} - 2\frac{\phi_t}{\phi_x}\phi_{xt} + \phi_{tt}\Big)(x_0,t_0) \\ &\qquad \le \lambda_0\Big(\frac{M_t^2}{M_x^2}M_{xx} - 2\frac{M_t^2}{M_x^2}M_{xt} + M_{tt}\Big)(y_0,t_0) + (1-\lambda_0)\Big(\frac{M_t^2}{M_x^2}M_{xx} - 2\frac{M_t^2}{M_x^2}M_{xt} + M_{tt}\Big)(z_0,t_0) \\ &\qquad = \lambda_0(M_x f'(-M_x)M_{xx})(y_0,t_0) + (1-\lambda_0)(M_x f'(-M_x)M_{xx})(z_0,t_0), \end{aligned}$$

where we used the equation satisfied by M for the last equality. Recalling that $M_x < 0$, that $f' \ge 0$ and that $M_{xx}(y_0,t_0) \ge 0$ and $M_{xx}(z_0,t_0) \ge 0$ while $\phi_{xx}(x_0,t_0) \le 0$ (by (4.4.29)) gives (4.4.27).

We now assume that $t_0 = T$ and check the boundary condition (4.4.28). To fix the ideas, we assume that $y_0 > 0$ and $z_0 < R/2$, the other cases being similar. Note that (4.4.29) still holds in this case. Moreover, as, for any $s \leq 0$ small we have

$$\phi(x_0, T+s) \leq \tilde{M}(x_0, T+s) \leq \lambda_0 M(y_0, T+s) + (1-\lambda_0) M(z_0, T+s),$$

we get

$$\phi_t(x_0, T) \geq \lambda_0 M_t(y_0, T) + (1-\lambda_0) M_t(z_0, T).$$

Thus, as $\phi_x(x_0, T) < 0$, $g' \geq 0$ and $\phi_{xx}(x_0, T) \leq 0$ while $M_{xx}(y_0, T) \geq 0$ and $M_{xx}(z_0, T) \geq 0$,

$$\begin{aligned}
\phi_t(x_0, T) + \phi_x(x_0, T) g'(-\phi_x(x_0, T)) \phi_{xx}(x_0, T) &\geq \lambda_0 M_t(y_0, T) + (1-\lambda_0) M_t(z_0, T) \\
&\geq \lambda_0 (M_t(y_0, T) + M_x(y_0, T) g'(-M_x(y_0, T)) M_{xx}(y_0, T)) \\
&\quad + (1-\lambda_0)(M_t(z_0, T) + M_x(z_0, T) g'(-M_x(z_0, T)) M_{xx}(z_0, T)) \;=\; 0,
\end{aligned}$$

which is (4.4.28).

Conclusion. We have proved that $\tilde{M}$ is a viscosity supersolution to the elliptic equation satisfied by M (including the boundary conditions). Using the regularity of M, we can choose $\lambda > 0$ (large) and $\varepsilon > 0$ (arbitrarily small) such that the map $\hat{M}_{\varepsilon,\lambda}(x,t) = M(x,t) - 2\varepsilon + \varepsilon \exp\{-\lambda t\}$ is a classical strict subsolution of this equation (including the boundary conditions). This implies that $\hat{M}_{\varepsilon,\lambda} \leq \tilde{M}$. Then, letting $\varepsilon \to 0$, we get $M \leq \tilde{M}$. As by construction, $\tilde{M} \leq M$, we conclude that $M = \tilde{M}$; hence M is convex with respect to the x variable, and then $m = -M_x$ is nonincreasing with respect to the x variable on $[0, R/2] \times [0, T]$. Fix now $x \in [0, R/2]$ and let $\gamma(x, \cdot)$ be optimal solution starting from $(x, 0)$, i.e. the solution to

$$\gamma_t = -u_x(\gamma, t), \qquad \gamma(x, 0) = x.$$

By symmetry and periodicity, $u_x(0,t) = u_x(R/2, t) = 0$ for $t \in [0, T]$. Therefore $\gamma(x,t) \in$

$[0, R/2]$ for $(x,t) \in [0, R/2] \times [0,T]$ and, in view of the Euler equation (4.2.9) and the monotonicity of m, γ is concave in t. Differentiating the terminal condition (4.2.21) with respect to the space variable implies that

$$\gamma_t(x,T) = -g'(m(\gamma(x,T),T))m_x(\gamma(x,T),T) \geq 0.$$

As $\gamma(x,\cdot)$ is concave, we infer that $\gamma_t(x,\cdot)$ is nonnegative on $[0,T]$. As $\gamma_t = -u_x(\gamma)$ and $x \mapsto \gamma(x,t)$ is onto from $[0, R/2]$ to itself, this implies that u_x is nonpositive. □

4.4.3 Solutions in the whole space

We now work in the whole space, returning to the entropic coupling function $f(m) = \log(m)$ (and $g(m) = c_T \log(m(T))$). Our main result is the existence of a classical solution to (MFG) or to (MFGP) under the structure condition (4.4.22). We adapt Definition 4.4.1 to the case that x belongs to the whole space. In what follows, $\mathcal{P}_2(\mathbb{R})$ will denote the set of Borel probability measures on $\mathbb{R}$ with a finite second order moment, equipped with the 2–Wasserstein distance.

Definition 4.4.8. We say that (u,m) is a (classical) solution to (MFG) if $(u,m) \in C^2(\mathbb{R} \times (0,T)) \times C^1(\mathbb{R} \times (0,T))$, with $m > 0$ in $\mathbb{R} \times (0,T]$, if $m \in C([0,T];\mathcal{P}_2(\mathbb{R}))$ with $m(0) = m_0$, if the equations are satisfied in the classical sense for $t \in (0,T)$ and, finally, if $m(T) \in L^1(\mathbb{R})$ and $\lim_{t\to T^-} u(x,t) = g(m(x,T))$ for every $x \in \mathbb{R}$.

Similarly, we say that (u,m) is a (classical) solution to (MFGP) if $(u,m) \in C^2(\mathbb{R} \times (0,T)) \times C^1(\mathbb{R} \times (0,T))$, with $m > 0$ in $\mathbb{R} \times (0,T)$, if $m \in C([0,T];\mathcal{P}_2(\mathbb{R}))$ with $m(0) = m_0$ and $m(T) = m_T$, and if the equations are satisfied in the classical sense for $t \in (0,T)$.

Let us notice that, whenever (4.4.22) holds, in view of the preservation of symmetry property of Lemma 4.4.7, the solutions to the MFG system with periodic and Neumann boundary conditions coincide. For this reason, we will not require the analysis of Subsection

4.3.1 in this case.

Theorem 4.4.9. *Assume that* $f(m) = \log(m)$*.*

1. *Assume that* m_0, m_T *are continuous, compactly supported densities on* $\mathbb{R}$*, even, nonincreasing on* $[0,\infty)$*, with* $m_0 \in C^{1,\overline{\alpha}}_{\text{loc}}(\{m_0 > 0\})$*. Then there exists a unique (up to addition of a constant to* u*) solution* $(u,m) \in C^2(\mathbb{R} \times (0,T)) \times C^1(\mathbb{R} \times (0,T))$ *of* (MFGP) *such that* m *is continuous and bounded on* $\mathbb{R} \times [0,T]$ *and* $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$*, for every* $t \in (0,T)$*.*

2. *Assume that* m_0 *is a continuous, compactly supported density on* $\mathbb{R}$*, even, nonincreasing on* $[0,\infty)$*, with with* $m_0 \in C^{1,\overline{\alpha}}_{\text{loc}}(\{m_0 > 0\})$*, and* $g(s) = c_T \log(s)$*, for some* $c_T \geq 0$*. Then there exists a unique solution* $(u,m) \in C^2(\mathbb{R} \times (0,T]) \times C^1(\mathbb{R} \times (0,T])$ *of* (MFG) *such that* m *is continuous and bounded on* $\mathbb{R} \times [0,T)$ *and* $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$*, for every* $t \in (0,T)$*.*

Let us start with a priori estimates for positive periodic solutions:

Lemma 4.4.10. *Suppose that* (4.4.22) *holds and that* $(u,m) \in C^2(R\mathbb{T} \times [0,T]) \times C^1(R\mathbb{T} \times [0,T])$ *is a classical solution to* (4.4.1)–(4.4.2) *or to* (4.4.1)–(4.4.3) *on* $R\mathbb{T} \times (0,T)$ *with* m *positive in* $R\mathbb{T} \times [0,T]$*. Let* $\gamma : R\mathbb{T} \times [0,T] \to R\mathbb{T}$ *be the associated flow of optimal trajectories.*

1. *(Global estimates) There exists* $C_0 > 0$ *depending only on* T*,* $\mathcal{E}(m_0)$*,* $\mathcal{E}(m_T)$*,* $M_2(m_0)$ *(and* $M_2(m_T)$ *for the planning problem), such that:*

$$\sup_{t \in [0,T]} \int_{R\mathbb{T}} x^2 m(x,t)dx \leq C_0, \qquad W_2(m(t), m(s)) \leq C_0 |t-s|^{1/2} \qquad \forall s,t \in [0,T]. \tag{4.4.32}$$

and

$$-\frac{C_0}{t}(x^2+1) \leq u(x,t) \leq \frac{C_0}{(T-t)}(x^2+1). \tag{4.4.33}$$

Moreover, for any $(x,t) \in (-R/2, R/2) \times [0,T]$,

$$m(x,t) \leq \begin{cases} \max\{\|m_0\|_\infty, \|m_T\|_\infty\} & \textit{if } (u,m) \textit{ satisfies } (4.4.1)\text{–}(4.4.2) \\ \|m_0\|_\infty & \textit{if } (u,m) \textit{ satisfies } (4.4.1)\text{–}(4.4.3) \end{cases} \tag{4.4.34}$$

and

$$(8C_0)^{-1/2}\left(\int_{|x|}^{R/2} m_0(y)dy\right)^{3/2} \leq m(\gamma(x,t),t). \tag{4.4.35}$$

2. *(Interior estimates) Fix any* $\delta \in (0, (1/2) \wedge (T/4))$ *and* $a \in (1, (R/2-1))$, *with* $R > 4$. *For any* $\eta \in (0, R/2)$ *and* $\theta \in (\eta, R/2)$ *such that*

$$\delta^{3/2}(8C_0)^{-1/2}\left|\log\left(\int_{\theta-\eta}^{R/2} m_0(y)dy\right)\right|^{1/2} > 2a, \tag{4.4.36}$$

one has

$$\min_{t\in[\delta,T-\delta]} \gamma(\theta-\eta,t) > a \tag{4.4.37}$$

and

$$\|m, 1/m\|_{C^{2,\overline{\alpha}}((-a,a)\times(\delta,T-\delta))} \leq C(\eta^{-1}, \delta^{-1}, \|m_0\|_\infty, \|m_T\|_\infty, K_\theta, C_0),$$

where

$$K_\theta := \|m_0^{-1}\|_{L^\infty((-\theta,\theta))} + \|m_0\|_{C^{1,\overline{\alpha}}((-\theta,\theta))} + \left(\int_\theta^{R/2} m_0(y)dy\right)^{-1}. \tag{4.4.38}$$

Proof. Estimates (4.4.32) and (4.4.33) are given in Proposition 4.4.3. By Lemma 4.4.7 the solution (u,m) satisfies (4.4.23), and $x \to \gamma(x,t)$ is increasing on $[0, R/2]$, with $\gamma(0,t) = 0$ and $\gamma(R/2,t) = R/2$ for any $t \in [0,T]$.

Step 1: Bounds on the density. The upper bounds on m in (4.4.34) hold by Proposition 4.2.9. Let us now prove the lower bound (4.4.35). Let $x \in [0, R/2)$ and $k = (2C_0)^{1/2}(\int_x^{R/2} m_0(y)dy)^{-1/2}$. We first assume that $k < R/2$. Then, by (4.4.23) (for the second inequality) and (4.4.32) and the choice of k (for the last one), we have

$$\begin{aligned}\int_x^{R/2} m_0(y)dy = \int_{\gamma(x,t)}^{R/2} m(y,t)dy &\leq \int_{\gamma(x,t)\wedge k}^{k} m(y,t)dy + \int_k^{R/2} m(y,t)dy \\ &\leq m(k\wedge\gamma(x,t),t)(k - k\wedge\gamma(x,t)) + k^{-2}\int_k^{R/2} x^2 m(y,t)dy \\ &\leq m(k\wedge\gamma(x,t),t)(k - k\wedge\gamma(x,t)) + \frac{1}{2}\int_x^{R/2} m_0(y)dy.\end{aligned}$$

This implies that $\gamma(x,t) < k$ and (4.4.35) in this case. Next we suppose that $k \geq R/2$. Then the same computation shows that

$$m(\gamma(x,t),t) \geq (2/R)\int_x^{R/2} m_0(y)dy \geq (2C_0)^{-1/2}\left(\int_x^{R/2} m_0(y)dy\right)^{3/2},$$

where the last inequality holds because $k \geq R/2$. Thus we have proved (4.4.35) for $x \in [0, R/2)$. The result for negative x holds by symmetry.

Step 2: Elliptic estimates. We now prove $C^{2,\overline{\alpha}}$ estimates for γ_x and $w = \log(m(\gamma))$. Recalling from (4.2.10) that $\gamma_x(x,t) = m_0(x)/m(\gamma(x,t),t)$, we note that (4.4.34) and (4.4.35) imply that γ_x is locally bounded above and below:

$$\frac{m_0(x)}{\max\{\|m_0\|_\infty, \|m_T\|_\infty\}} \leq \gamma_x(x,t) \leq \frac{(8C_0)^{1/2} m_0(x)}{\left(\int_{|x|}^{R/2} m_0(y)dy\right)^{3/2}} \qquad \forall (x,t) \in (-R/2, R/2)\times[0,T]. \tag{4.4.39}$$

Let $w(x,t) = \log(m(\gamma(x,t),t))$. Then w solves the elliptic equation in divergence form (see (4.2.17)):

$$-(\gamma_x w_t)_t - \left(\frac{1}{\gamma_x} w_x\right)_x = 0 \quad \text{in } (-R/2, R/2)\times(0,T). \tag{4.4.40}$$

Fix $\eta \in (0, R/2)$ and $\delta \in (0, T/4)$. Let $\theta \in (\eta, R/2)$ and K_θ be defined by (4.4.38). As, by (4.4.34), (4.4.35) and (4.4.39),

$$|w| + |\gamma_x| + |1/\gamma_x| \le C(K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0) \qquad \text{on } [-\theta, \theta] \times [0, T],$$

we infer by elliptic regularity that

$$\|w\|_{C^{0,\overline{\alpha}}([-\theta+\eta/3, \theta-\eta/3]\times[\delta/3, T-\delta/3])} \le C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0).$$

Recalling that $\gamma_x = m_0/m(\gamma) = m_0 e^{-w}$, this implies that

$$\|\gamma_x, 1/\gamma_x\|_{C^{0,\overline{\alpha}}([-\theta+\eta/3, \theta-\eta/3]\times[\delta/3, T-\delta/3])} \le C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0).$$

On the other hand (see (4.2.12) with $f(s) = \log s$), γ solves the elliptic equation

$$\gamma_{tt} + \frac{\gamma_{xx}}{\gamma_x^2} = \frac{1}{\gamma_x}\frac{(m_0)_x}{m_0} \quad \text{on } (-R/2, R/2) \times (0, T), \qquad \gamma(x, 0) = x \text{ on } [-R/2, R/2].$$

Using the Schauder estimates, we therefore have

$$\|\gamma\|_{C^{2,\overline{\alpha}}([-\theta+\eta/2, \theta-\eta/2]\times[\delta/2, T-\delta/2])} \le C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0). \tag{4.4.41}$$

Returning to (4.4.40), we obtain, again by the Schauder estimates,

$$\|w\|_{C^{2,\overline{\alpha}}([-\theta+\eta, \theta-\eta]\times[\delta, T-\delta])} \le C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0). \tag{4.4.42}$$

Step 3: Lower bound on γ. We claim that, for any $\delta \in (0,(1/2)\wedge(T/4))$ and any $x \in [0,R/2)$,

$$\min_{t\in[\delta,T-\delta]} \gamma(x,t) \geq (1-\delta)\left((R/2)\wedge\left(\delta^{3/2}(C_0)^{-1/2}\left|\log\left(\int_x^{R/2} m_0(y)dy\right)\right|^{1/2}\right)-1\right). \tag{4.4.43}$$

Proof of (4.4.43)*:* Fix $x \in [0,R/2)$, $t \in [\delta, T-\delta]$ and set

$$a = \max_{s\in[t-\delta^2,t+\delta^2]} \gamma(x,s) + 1.$$

Since we want a lower bound for a, we can assume that $a \leq R/2$. Then, recalling (4.2.11) and the fact that $m(\cdot,t)$ is nonincreasing, we have for any $s \in [t-\delta^2, t+\delta^2]$,

$$\int_x^{R/2} m_0(y)dy = \int_{\gamma(x,s)}^{R/2} m(y,s)dy \geq \int_{\gamma(x,s)}^{a} m(y,s)dy \geq (a-\gamma(x,s))m(a,s) \geq m(a,s).$$

Using the previous inequality together with the HJ equation, and integrating in $(t-\delta^2, t+\delta^2)$, we get

$$\begin{aligned}
2\delta^2 \log\left(\int_x^{R/2} m_0(y)dy\right) &\geq \int_{t-\delta^2}^{t+\delta^2} \log(m(a,s))ds \geq \int_{t-\delta^2}^{t+\delta^2} -u_t(a,s)ds \\
&\geq -\left(\frac{C_0}{t-\delta^2}+\frac{C_0}{T-t-\delta^2}\right)(a^2+1) \geq -8a^2\, C_0\delta^{-1}
\end{aligned}$$

where we used (4.4.33) and the fact that $a \geq 1$ and $\delta/2 \leq t-\delta^2 \leq t+\delta^2 \leq T-\delta/2$. Thus, up to increasing the value of C_0, we obtain

$$a \geq \delta^{3/2} C_0^{-1/2}\left|\log\left(\int_x^{R/2} m_0(y)dy\right)\right|^{1/2}.$$

This proves that

$$\max_{s\in[t-\delta^2,t+\delta^2]}\gamma(x,s) \geq (R/2)\wedge\left(\delta^{3/2}(C_0)^{-1/2}\left|\log\left(\int_x^{R/2} m_0(y)dy\right)\right|^{1/2}\right)-1.$$

As $\gamma(x,\cdot)$ is nonnegative and concave, letting s_0 be a maximum point of $\gamma(x,\cdot)$ in $[t-\delta^2,t+\delta^2]$,

$$\begin{aligned}\gamma(x,t) \geq \left(\frac{t}{s_0}\mathbf{1}_{s_0\in[t,t+\delta^2]}+\frac{T-t}{T-s_0}\mathbf{1}_{s_0\in[t-\delta^2,t]}\right)\gamma(s_0)&\\ \geq \min\left\{\frac{t}{t+\delta^2},\frac{T-t}{T-t+\delta^2}\right\}\gamma(s_0) &\geq (1-\delta)\gamma(s_0).\end{aligned}$$

Using our estimate on $\gamma(x,s_0)=\max_{[t-\delta^2,t+\delta^2]}\gamma$ gives (4.4.43).

Step 4: Interior estimate of m. Fix $\delta\in(0,(1/2)\wedge T/4)$ and $a\in(1,(R/2-1))$, with $R>4$. Now assume that $\eta\in(0,R/2)$ and $\theta\in(\eta,R/2)$ are such that (4.4.36) holds. Then

$$(1-\delta)\left((R/2)\wedge\left(\delta^{3/2}(C_0)^{-1/2}\left|\log\left(\int_{\theta-\eta}^{R/2} m_0(y)dy\right)\right|^{1/2}\right)-1\right)>a+1. \tag{4.4.44}$$

We claim that

$$\|m,m^{-1}\|_{C^{2,\overline{\alpha}}((-a,a)\times(\delta,T-\delta))}\leq C(\eta^{-1},\delta^{-1},K_\theta,\|m_0\|_\infty,\|m_T\|_\infty,C_0).$$

Indeed, let $\gamma^{-1}(\cdot,t):(-R/2,R/2)\to(-R/2,R/2)$ denote the inverse in space of $\gamma(\cdot,t)$. We first claim that $\gamma^{-1}([-a,a]\times[\delta,T-\delta])\subset(-\theta+\eta,\theta-\eta)\times[\delta,T-\delta]$. Indeed, otherwise, there exists $t\in[\delta,T-\delta]$ and $x\in[-a,a]$ such that $(x,t)\notin\gamma((-\theta+\eta,\theta-\eta)\times[\delta,T-\delta])$. This means that $x\notin\gamma((-\theta+\eta,\theta-\eta),t)$. As $\gamma(\cdot,t)$ is continuous, vanishes at $x=0$ and is increasing, this implies that $a\geq|x|\geq\gamma(\theta-\eta,t)$, which, by (4.4.43), contradicts (4.4.44). In particular, (4.4.37) holds. Recalling (4.4.41), γ^{-1} is bounded in $C^{2,\overline{\alpha}}$ in $[-a,a]\times[\delta,T-\delta]$

and then (4.4.42) implies the $C^{2,\overline{\alpha}}$ bound on $\log(m) = w \circ \gamma^{-1}$. This implies that m and $1/m$ are bounded in $C^{2,\overline{\alpha}}$ in $[-a, a] \times [\delta, T - \delta]$. $\square$

Proof of Theorem 4.4.9. (i) We prove the existence of a classical solution. We start with the planning problem (MFGP) and explain at the end of the proof the necessary changes for the (MFG) problem.

Let $[-b_0, b_0]$ (resp. $[-b_T, b_T]$) be the support of m_0 (resp. of m_T). For $R \geq \bar{R} := \max\{b_0, b_T\}$, let $\tilde{m}_0^R$ and $\tilde{m}_T^R$ be the continuous periodic map of period R such that $\tilde{m}_0^R = m_0$ and $\tilde{m}_T^R = m_T$ on $(-R/2, R/2)$. We let $m_0^{R,\varepsilon} = \tilde{m}_0^R * \eta_\varepsilon$ and $m_T^{R,\varepsilon} = \tilde{m}_T^R * \eta_\varepsilon$ where η^ε is a standard mollifier, smooth, even and positive on $\mathbb{R}$. Then, for $R \geq \bar{R}$ and $\varepsilon > 0$, $m_0^{R,\varepsilon}$ and $m_T^{R,\varepsilon}$ are smooth, positive and satisfy (4.4.22). Let $(u^{R,\varepsilon}, m^{R,\varepsilon})$ be the classical solution to (4.2.20)–(4.2.22) given by Theorem 4.2.11. By Lemma 4.4.7, $m^{R,\varepsilon}$ satisfies (4.4.23).

We now prove an interior estimate for $m^{R,\varepsilon}$. Fix $a > 1$ and $\delta \in (0, (1/2) \wedge (T/4))$. Choose $R_0 > 4$ such that $(R_0/2 - 1) \geq a + 1$. Fix also $r_0 \in (0, b^0)$ large enough such that

$$\delta^{3/2} (C_0)^{-1/2} \left| \log \left(\int_{r_0}^{b_0} m_0(y) dy \right) \right|^{1/2} > 2a + 1.$$

This is possible as $\int_{r_0}^{b_0} m_0(y) dy \to 0$ as $r_0 \to (b^0)^-$. We then set $\theta = (r_0 + b_0)/2$ and $\eta = (b_0 - r_0)/2$. We finally choose $\varepsilon_0 \in (0, \eta/4)$ such that, for $\varepsilon \in (0, \varepsilon_0)$,

$$\delta^{3/2} (8C_0)^{-1/2} \left| \log \left(\int_{r_0}^{b_0} m_0^{R,\varepsilon}(y) dy \right) \right|^{1/2} > 2a \quad \text{and} \quad \int_\theta^{R/2} m_0^{R,\varepsilon}(y) dy \geq \frac{1}{2} \int_\theta^{R/2} m_0(y) dy.$$

Then, for any $R \geq R_0$, $\varepsilon \in (0, \varepsilon_0)$, condition (4.4.36) holds for m_0^ε. By Lemma 4.4.10,

$$\begin{aligned} \|m^{R,\varepsilon}, 1/m^{R,\varepsilon}\|_{C^{2,\overline{\alpha}}((-a,a)\times(\delta,T-\delta))} &\leq C(\eta^{-1}, \delta^{-1}, K_\theta^\varepsilon, \|\tilde{m}_0^R\|_\infty, , \|\tilde{m}_T^R\|_\infty, C_0^\varepsilon) \\ &= C(\eta^{-1}, \delta^{-1}, K_\theta^\varepsilon, \|m_0\|_\infty, \|m_T\|_\infty, C_0^\varepsilon), \quad (4.4.45) \end{aligned}$$

where C_0^ε depends only on T, $\mathcal{E}(\tilde{m}_0^R)$, $\mathcal{E}(\tilde{m}_T^R)$, $M_2(\tilde{m}_0^R)$, $M_2(\tilde{m}_T^R)$, and thus only on T, $\mathcal{E}(m_0)$, $\mathcal{E}(m_T)$, $M_2(m_0)$, $M_2(m_T)$, and

$$
\begin{aligned}
K_\theta^\varepsilon &= \|(\tilde{m}_0^{R,\varepsilon})^{-1}\|_{L^\infty((-\theta,\theta))} + \|\tilde{m}_0^{R,\varepsilon}\|_{C^{1,\overline{\alpha}}((-\theta,\theta))} + \left(\int_\theta^{R/2} \tilde{m}_0^{R,\varepsilon}(y)dy\right)^{-1} \\
&\leq \|m_0^{-1}\|_{L^\infty((-\theta-\varepsilon_0,\theta+\varepsilon_0))} + \|m_0\|_{C^{1,\overline{\alpha}}((-\theta-\varepsilon_0,\theta+\varepsilon_0))} + 2\left(\int_{b_0-\eta/2}^{b_0} m_0(y)dy\right)^{-1}, \quad (4.4.46)
\end{aligned}
$$

which is finite since $\theta + \varepsilon_0 = b_0 - \eta$. This shows that

$$
\|m^{R,\varepsilon}, 1/m^{R,\varepsilon}\|_{C^{2,\overline{\alpha}}((-a,a)\times(\delta,T-\delta))} \leq C(a, \delta^{-1}, T, m_0, m_T).
$$

Since $\log(m^{R,\varepsilon})$ is uniformly Lipschitz continuous in $(-a,a)\times(\delta, T-\delta)$ and since the map $u^{R,\varepsilon}$ is a locally uniformly bounded solution of a HJ equation with r.h.s. $\log(m^{R,\varepsilon})$, $u^{R,\varepsilon}$ is uniformly Lipschitz continuous in $(-a/2,a/2)\times(2\delta, T-2\delta)$. By (4.2.1) (with $f=\log s$), we know that $u^{R,\varepsilon}$ satisfies the elliptic equation

$$
-u_{tt} + 2u_x u_{xt} - (u_x^2+1)u_{xx} = 0\,. \tag{4.4.47}
$$

Hence, by elliptic regularity, we obtain

$$
\|u^{R,\varepsilon}\|_{C^{2,\overline{\alpha}}((-a/2,a/2)\times(2\delta,T-2\delta))} \leq C(a, \delta^{-1}, m_0, m_T, C_0).
$$

We can now use the estimates above and the first part of Lemma 4.4.10 to find (a subsequence) of $(u^{R,\varepsilon}, m^{R,\varepsilon})$ which converges, as $\varepsilon \to 0^+$ and then $R\to\infty$, to a pair (u,m) which is a $C^{2,\overline{\alpha}}$ solution of the MFG system (4.0.1) in $\mathbb{R}\times(0,T)$ with $f(m)=\log(m)$ and such that:

$$
m \in C^0([0,T], \mathcal{P}_2(\mathbb{R})),\ m(0)=m_0,\ m(T)=m_T,.
$$

Moreover, by construction, $m(\cdot,t)$ is even and $x \to m(x,t)$ is nonincreasing on $[0,+\infty)$ for any $t \in [0,T]$.

Let us finally check the continuity of m at $t=0$ (the case $t=T$ being symmetric). Let $t_n \to 0^+$. Then the maps $m(\cdot,t_n)$ are nonincreasing on $[0,\infty)$ and converge weakly-* (as measures) to m_0 which has a continuous density: this limit is therefore locally uniform.

We now consider the (MFG) problem. We regularize $m_0^{R,\varepsilon}$ as above and let $(u^{R,\varepsilon}, m^{R,\varepsilon})$ be the classical solution to (4.4.1)–(4.4.3). Let $\gamma^{R,\varepsilon}$ be the associated flow of optimal trajectories. Let us recall that, under our structure condition, $\gamma^{R,\varepsilon}(x,\cdot)$ is concave and nondecreasing in time for any $x \in [0,R/2]$.

Fix $a>1$ and $\delta \in (0,(1/2)\wedge(T/4))$. We can choose R_0, θ, η and $\varepsilon_0>0$ as in the first part of the proof such that, for $R \geq R_0$ and $\varepsilon \in (0,\varepsilon_0)$,

$$\int_\theta^{R/2} m_0^{R,\varepsilon}(y)dy \geq \frac{1}{2}\int_\theta^{R/2} m_0(y)dy > 0$$

and

$$\|m^{R,\varepsilon}, 1/m^{R,\varepsilon}\|_{C^{2,\overline{\alpha}}((-a,a)\times(\delta,T-\delta))} \leq C(a,\delta^{-1},T,m_0,m_T).$$

With this choice we also have (estimate (4.4.43) in Lemma 4.4.10)

$$\min_{t\in[\delta,T-\delta]} \gamma(\theta-\eta,t) > a.$$

Recall also that, by (4.4.34) and (4.4.35), for any $(x,t) \in (-R/2,R/2)\times[0,T]$,

$$(8C_0)^{-1/2}\left(\int_{|x|}^{R/2} m_0^{R,\varepsilon}(y)dy\right)^{3/2} \leq m^{R,\varepsilon}(\gamma(x,t),t) \leq \|m_0^{R,\varepsilon}\|_\infty \leq \|m_0\|_\infty.$$

Hence for any $(y,t) \in [0,a]\times[\delta,T]$, there exists $x \in [0,\theta-\eta]$ with $\gamma^{R,\varepsilon}(x,t) = y$ and therefore

$$\|m_0\|_\infty \geq m^{R,\varepsilon}(y,t) = m^{R,\varepsilon}(\gamma(x,t),t) \geq \frac{1}{2}(8C_0)^{-1/2}\left(\int_\theta^{R/2} m_0(y)dy\right)^{3/2},$$

which proves that $m^{R,\varepsilon}$ is bounded above and below in $[-a,a]\times[\delta,T]$. Next we show a Lipschitz bound for $u^{R,\varepsilon}$ in $[0,a]\times[\delta,T]$. For any $(y,t) \in [0,a]\times[\delta,T]$, there exists $x \in [0,y]$ such that $\gamma^{R,\varepsilon}(x,t) = y$. Recall that $\gamma_t^{R,\varepsilon}(x,t) = -u_x(y,t)$. On the other hand, by concavity of $\gamma^{R,\varepsilon}(x,\cdot)$,

$$0 \leq \gamma^{R,\varepsilon}(x,s) \leq y + \gamma_t^{R,\varepsilon}(x,t)(s-t) \qquad \forall s \in [0,T].$$

Thus (choosing $s=0$ and using that $y \leq a$ and $t \geq \delta$)

$$0 \leq \gamma_t^{R,\varepsilon}(x,t) = -u_x^{R,\varepsilon}(y,t) \leq y/t \leq a/\delta.$$

By symmetry, this proves that

$$\|u_x^{R,\varepsilon}\|_{L^\infty([-a,a]\times[\delta,T])} \leq a/\delta.$$

Let us finally check a local bound for $u^{R,\varepsilon}$. We already have a bound below (Lemma 4.4.10). As $u^{R,\varepsilon}(\cdot,t)$ is nonincreasing on $[0,R/2]$ and $\gamma(0,\cdot)=0$, we have, for $(x,t)\in[0,a]\times[\delta,T]$:

$$\begin{aligned} u^{R,\varepsilon}(x,t) \leq u^{R,\varepsilon}(0,t) &= \int_t^T \frac{1}{2}|\gamma_s^{R,\varepsilon}(0,s)|^2 + \ln(m^{R,\varepsilon}(\gamma^{R,\varepsilon}(0,s)))\, ds + c_T \ln(m^{R,\varepsilon}(\gamma^{R,\varepsilon}(0,T),T)) \\ &\leq C\|m^{R,\varepsilon}, 1/m^{R,\varepsilon}\|_{L^\infty(\{0\}\times[\delta,T])} \leq C(a,\delta^{-1}). \end{aligned}$$

We have proved positive upper and lower bounds for $m^{R,\varepsilon}$ and Lipschitz bounds for $u^{R,\varepsilon}$ independent of R,ε on $[-a,a]\times[\delta,T]$. By the elliptic equation (4.4.47) satisfied by $u^{R,\varepsilon}$, we can infer that

$$\|u^{R,\varepsilon}\|_{C^{2,\overline{\alpha}}([-a/2,a/2]\times[2\delta,T]} \leq C(a,\delta^{-1}).$$

We can then conclude as before.

(ii) The uniqueness of solutions is proved with the same kind of argument used in Theorem 4.4.2. First of all, we observe that, since $m(t) \in \mathcal{P}_2(\mathbb{R})$ and $u(t)/(1+|x|^2) \in L^\infty(\mathbb{R})$, then we have $u(t)m(t) \in L^1(\mathbb{R})$ for any $t \in (0,T)$. This implies that a similar equality as (4.4.19) holds, namely

$$\int_{\mathbb{R}} u(t_0)m(t_0) - \int_{\mathbb{R}} u(t_1)m(t_1) = \int_{t_0}^{t_1}\int_{\mathbb{R}} m u_x^2 + \int_{t_0}^{t_1}\int_{\mathbb{R}} m\log(m) \qquad \forall\, 0 < t_0 < t_1 < T\,. \tag{4.4.48}$$

As in the proof of Theorem 4.4.2, we deduce from (4.4.48) (and the time-monotonicity of u) that $u(0^+) \in L^1(dm_0)$ and $u(T^-) \in L^1(dm(T))$. Then, using a truncation argument for u and the continuity of m, inequality (4.4.48) is extended up to $t_0 = 0$ and $t_1 = T$, obtaining the equivalent of (4.4.20) but integrated for $x \in \mathbb{R}$. Notice that the same truncation argument for u as in Theorem 4.4.2 works here, because m has finite moments (uniformly in time), so actually $\int_{\mathbb{R}} u_k(t)m(t)$ ends up being continuous in $[0,T]$ for fixed k. In a similar way, we obtain the equivalent of (4.4.21) for $x \in \mathbb{R}$, and we conclude from the Lasry-Lions monotonicity argument as in the compact case.

$\square$

APPENDIX

4.A Construction of the self-similar solutions

The goal of this appendix is to prove the statements made in Proposition 4.1.1 regarding the construction of the self-similar solution, as well as to provide a precise analysis of the regularity of the value function u. We begin by showing that, inside the support of m, the system is solved in the classical sense.

Lemma 4.A.1. *Let u, m be defined as in Proposition 4.1.1. Then m is a weak solution of the continuity equation, u is C^1 in the support of m, and u satisfies, in the classical sense,*

$$-u_t + \frac{1}{2}|u_x|^2 = m^\theta \quad in \;\{m > 0\}.$$

Proof. We set $\gamma^{\pm}(t) = \pm(2R/(\alpha - \alpha^2))^{1/2} t^\alpha$, and we note that

$$\{m(t,\cdot) > 0\} = \{(x,t),\; \gamma^-(t) < x < \gamma^+(t)\}.$$

Here we have, according to (4.1.5) and if $m(x,t) > 0$, :

$$-u_t + \frac{1}{2}|u_x|^2 = -\alpha\frac{x^2}{2t^2} + Rt^{-2\theta/(2+\theta)} + \frac{1}{2}\alpha^2\frac{x^2}{t^2} = -(\alpha - \alpha^2)\frac{x^2}{2t^2} + Rt^{-2\theta/(2+\theta)}$$

while

$$f(m(x,t)) = t^{-\alpha\theta}\left(R - \frac{1}{2}(\alpha - \alpha^2)(\frac{x}{t^\alpha})^2\right) = t^{-\alpha\theta}R - \frac{1}{2}(\alpha - \alpha^2)\frac{x^2}{t^{\alpha\theta+2\alpha}}\,.$$

By the definition of α, we have $\alpha\theta = 2\theta/(2+\theta)$, and $\alpha\theta + 2\alpha = 2$. Thus we conclude that

$$-u_t + \frac{1}{2}u_x^2 = f(m(x,t)) \qquad \text{in } \{m > 0\}.$$

On the other hand, for any test function $\varphi \in C_c^\infty(\mathbb{R} \times (0,T))$, we have

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(x,t)m(x,t)dx = \frac{d}{dt}\int_{\gamma^-(t)}^{\gamma^+(t)} \varphi(x,t)t^{-\alpha}\phi(x/t^\alpha)dx$$
$$= \int_{\gamma^-(t)}^{\gamma^+(t)} (\varphi_t(x,t)t^{-\alpha}\phi(x/t^\alpha) + \varphi(x,t)(-\alpha t^{-\alpha-1})(\phi(x/t^\alpha) + xt^{-\alpha}\phi'(x/t^\alpha)))dx$$

where

$$\int_{\gamma^-(t)}^{\gamma^+(t)} \varphi(x,t)x\phi'(x/t^\alpha)dx = -t^\alpha \int_{\bar{\gamma}^-(t)}^{\bar{\gamma}^+(t)} (\varphi_x(x,t)x + \varphi(x,t))\phi(x/t^\alpha)dx.$$

So

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(x,t)m(x,t)dx = \int_{\gamma^-(t)}^{\gamma^+(t)} (\varphi_t(x,t)t^{-\alpha}\phi(x/t^\alpha) - \varphi_x(x,t)(-\alpha t^{-\alpha-1})x\phi(x/t^\alpha)))dx$$
$$= \int_{\mathbb{R}} (\varphi_t(x,t) - \varphi_x(x,t)u_x(x,t))m(x,t)dx.$$

This shows that m solves the continuity equation. $\square$

We now extend the definition of u outside the support of m, and analyze its behavior near the interface. We recall that the free boundary is the set $\{\Delta = 0\}$, where $\Delta = |x| - \sqrt{\frac{2R}{\alpha(1-\alpha)}}t^\alpha$.

Lemma 4.A.2. *For each $(x,t) \in \{\Delta > 0\}$, the equation (4.1.4) has a unique positive solution $S \in (0,t)$, and S is a smooth function of (x,t). Furthermore, the function $S = S(x,t)$ extends continuously to $\{\Delta \geq 0\}$, $S(x,t) = t$ on $\Delta = 0$, and one has the estimates*

$$S(x,t) \geq c_0 \left(\frac{t}{|x| + \Delta}\right)^{\frac{1}{1-\alpha}}, \tag{4.A.1}$$

$$|t - S(x,t)| \leq C_0 t^{1-\frac{\alpha}{2}}\Delta^{\frac{1}{2}}. \tag{4.A.2}$$

where $c_0 = \left(\frac{2R\alpha}{1-\alpha}\right)^{\frac{1}{2(1-\alpha)}}$ *and* $C_0 = \left(\frac{2}{R\alpha(1-\alpha)}\right)^{\frac{1}{4}}$.

Proof. Let us set $C_R := \sqrt{\frac{2R}{\alpha(1-\alpha)}}$; hence the interface $\Delta = 0$ is the curve $|x| = C_R t^{\alpha}$. We define

$$F(x,t,s) = -|x|s^{1-\alpha} + C_R(\alpha t + (1-\alpha)s). \tag{4.A.3}$$

Then, we have

$$\frac{\partial F}{\partial s}(x,t,s) = -(1-\alpha)s^{-\alpha}\left(|x| - C_R s^{\alpha}\right). \tag{4.A.4}$$

Notice that, since $\Delta(x,t) > 0$, $\frac{\partial F}{\partial s}(x,t,s) < 0$ for $s \in [0,t]$. Moreover, $F(x,t,0) = \alpha C_R t > 0$, and $F(x,t,t) = -t^{1-\alpha}\Delta < 0$. Hence there exists a unique $S \in (0,t)$ with $F(x,t,S) = 0$. Now, since $\frac{\partial F}{\partial s}(x,t,S) < 0$, the implicit function theorem guarantees that the function S is smooth in $\{\Delta > 0\}$. We now show a lower bound on S, by taking $s = \left(\frac{\alpha C_R t}{|x|+\Delta}\right)^{\frac{1}{1-\alpha}}$. First we note that, since $\alpha < 1$, we have

$$s < \left(\frac{C_R t}{|x|}\right)^{\frac{1}{1-\alpha}} \leq t \qquad \text{in } \{\Delta > 0\}.$$

Hence $s \in (0,t)$. Moreover, we have

$$F(x,t,s) = -\frac{|x|}{|x| + \Delta}\alpha C_R t + C_R(\alpha t + (1-\alpha)s) \geq 0\,.$$

Consequently, since $\frac{\partial F}{\partial s} < 0$ on $(0,t)$, and $F(x,t,S) = 0$, we have $S \geq s$, that is,

$$S(x,t) \geq \left(\frac{\alpha C_R t}{|x| + \Delta}\right)^{\frac{1}{1-\alpha}}.$$

This is (4.A.1). Finally, we are now concerned with the continuous extension to $\Delta = 0$. First of all, we rewrite (4.1.4) as

$$\alpha C_R(t-S) = S^{1-\alpha}(|x| - C_R S^\alpha) \tag{4.A.5}$$

which yields

$$\alpha C_R(t-S)\left(1 - \frac{1}{\alpha} S^{1-\alpha} \frac{t^\alpha - S^\alpha}{t-S}\right) = S^{1-\alpha}\Delta. \tag{4.A.6}$$

We can then use the elementary inequality

$$\frac{(1-\alpha)}{2}(1-w) \leq \left(1 - \frac{1}{\alpha} w^{1-\alpha} \frac{1-w^\alpha}{1-w}\right),$$

valid for all real numbers $w \in (0,1)$. Using this inequality with $w = \frac{S}{t}$, we deduce from (4.A.6)

$$\frac{\alpha C_R(1-\alpha)}{2t}(t-S)^2 \leq S^{1-\alpha}\Delta \leq t^{1-\alpha}\Delta,$$

which reduces to

$$|t-S| \leq C_0 t^{1-\frac{\alpha}{2}} \Delta^{\frac{1}{2}}$$

by setting $C_0 = \left(\frac{2}{R\alpha(1-\alpha)}\right)^{\frac{1}{4}}$. From this estimate, one sees that if $(x_n, t_n) \in \{\Delta > 0\}$ is such that $(x_n, t_n) \to (x_0, t_0) \in \{\Delta = 0\}$, then $|S(x,t) - t| \to 0$. $\square$

We can now establish the Hölder regularity of Du.

Theorem 4.A.3. *There exists $0 < s < 1$ such that the function u (defined in (4.1.2) or (4.1.3)) is smooth away from $\Delta = 0$, and is uniformly $C^{1,s}$ on compact subsets of $\mathbb{R} \times (0,T)$. Moreover, it is a classical solution of*

$$-u_t + \frac{1}{2}u_x^2 = m^\theta \qquad \text{in } \mathbb{R} \times (0,T).$$

Proof. Given $(x,t) \in \{\Delta > 0\}$, $u(x,t)$ has been defined through the method of characteristics; outside the support of m, the characteristics are straight lines that join (x,t) to a unique point $(\bar{x}, S)$ belonging to the curve $\{\Delta = 0\}$. This means that

$$u(x,t) = u(\bar{x}, S) + \frac{1}{2}(t-S)|\lambda|^2 , \qquad \begin{cases} x = \lambda(t-S) + \bar{x} \,, \qquad \lambda = -u_x(\bar{x}, S) = \frac{\alpha \bar{x}}{S} \\ \bar{x} = C_R S^{\alpha} \end{cases}$$

which leads to the value $S = S(x,t)$ defined by (4.A.5) and, correspondingly to the formulas (4.1.2) or (4.1.3). By construction, relying on the method of characteristics, it follows that u satisfies $-u_t + \frac{1}{2}u_x^2 = 0$ in $\{\Delta \geq 0\}$, and u is actually a viscosity solution in the whole $\mathbb{R} \times (0,T)$.

Let us now look at the regularity of u. Since S is smooth away from $\Delta = 0$, so is u. More precisely, recalling that S is given by the equation

$$\alpha C_R t = S^{1-\alpha}|x| - (1-\alpha) C_R S \tag{4.A.7}$$

we have, by implicit differentiation,

$$S_x = \frac{-\mathrm{sgn}(x) S}{(1-\alpha)(|x| - C_R S^{\alpha})} \,. \tag{4.A.8}$$

Suppose that $\theta \neq 2$, so u is given by (4.1.3), which simplifies into

$$u = -\frac{2\alpha R}{2\alpha - 1} S^{2\alpha - 1} - \frac{\alpha R}{1-\alpha} t \, S^{2\alpha - 2} \,.$$

Therefore,

$$u_x = -2\alpha R \left(1 - \frac{t}{S}\right) S^{2\alpha - 2} S_x,$$

and from (4.A.8) and (4.A.5) we see that this simplifies to:

$$u_x = -\frac{2R}{C_R(1-\alpha)} S^{\alpha-1}\mathrm{sgn}(x) \tag{4.A.9}$$

By Proposition 4.A.2, we deduce that $u \in C^1(\mathbb{R}\times(0,T))$; in fact, we can see that $u \in C^{1,s}$, if we prove a uniform C^s bound for S on compact subsets near $\Delta = 0$. To this purpose, let K be a compact subset of $\Delta > 0$, and let $(x,t),(\overline{x},\overline{t})$ be two points in K. We write $\overline{S} = S(\overline{x},\overline{t})$. With no loss of generality, we may assume that $S \geq \overline{S}$. By formula (4.A.7), we have

$$S^{1-\alpha}|x| - \bar{S}^{1-\alpha}|\bar{x}| - (1-\alpha)C_R(S-\bar{S}) = C_R(t-\bar{t})$$

which yields, by using $S^{1-\alpha} \geq \bar{S}^{1-\alpha} + (1-\alpha)S^{-\alpha}(S-\bar{S})$

$$\frac{(1-\alpha)}{S^\alpha}\left(|x| - C_R S^\alpha\right)(S-\bar{S}) \leq C_R|t-\bar{t}| + \bar{S}^{1-\alpha}|x-\bar{x}|\,.$$

By definition of Δ, and Proposition 4.A.2, we deduce

$$|S-\bar{S}| \leq K_{t,\bar{t}}\,\frac{\left(|x-\overline{x}| + |t-\overline{t}|\right)}{\Delta}\,.$$

Suppose now that $\Delta \geq \left(|x-\overline{x}| + |t-\overline{t}|\right)^\sigma$, for some $\sigma \in (0,1)$. Then we get

$$|S-\bar{S}| \leq K_{t,\bar{t}}\left(|x-\overline{x}| + |t-\overline{t}|\right)^{1-\sigma}\,.$$

On the other hand, if $\Delta < \left(|x-\overline{x}| + |t-\overline{t}|\right)^\sigma$, then we also have

$$\bar{\Delta} < \left(|x-\overline{x}| + |t-\overline{t}|\right)^\sigma + |\Delta - \bar{\Delta}| \leq C_{t,\bar{t}}\left(|x-\overline{x}| + |t-\overline{t}|\right)^\sigma$$

because Δ is a Lipschitz function of (x,t) far from $t=0$ (it is also globally α-Hölder, as well). Hence we estimate, using (4.A.2)

$$|S-\bar{S}| \le |S-t|+|t-\bar{t}|+|\bar{t}-\bar{S}| \le C_0 \max(t,\bar{t})^{1-\frac{\alpha}{2}} \left(\Delta^{\frac{1}{2}} + \bar{\Delta}^{\frac{1}{2}}\right)+|t-\bar{t}| \le \tilde{C}_{t,\bar{t}} \left(|x-\overline{x}| + |t-\bar{t}|\right)^{\frac{\sigma}{2}} .$$

This concludes with the Hölder bound of $S(x,t)$, and therefore with the $C^{1,s}$ regularity of u. Finally, for the case $\theta = 2$, we argue in the same way by using formula (4.1.2); notice that u is explicit in this case, because (4.1.4) is a quadratic equation in $\sqrt{S}$.

$\square$

Remark 4.A.4. We observe that the solution u found above is not $W^{2,\infty}$. In fact, by differentiating once more (4.A.9) one gets

$$u_{xx} = -\sqrt{\frac{2R\alpha}{1-\alpha}} \frac{S^{\alpha-1}}{(|x| - \sqrt{\frac{2R}{\alpha(1-\alpha)}} S^{\alpha})} < 0 \,.$$

Now, as $\Delta \to 0$, $S \to t$, and thus the denominator $|x| - \sqrt{\frac{2R}{\alpha(1-\alpha)}} S^{\alpha} \to 0$. Hence u_{xx} is unbounded.

The same holds for the case $\theta = 2$, since we have

$$u_x = -\frac{4R}{x-\sqrt{\Delta}}, u_{xx} = -\frac{4R}{\sqrt{\Delta}(x-\sqrt{\Delta})} \,.$$

So u_{xx} is again unbounded as $\Delta \to 0$.

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