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To my parents

and

in memory of Yichen Yang

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ABSTRACT

This thesis is an amalgam of the two manuscripts [16] and [17] that the author completed during her graduate studies at the University of Chicago. Article [16] was published in *Ergodic Theory and Dynamical Systems* and is reprinted with permission here.

The first part of the thesis contains results obtained in [16], where we study Basmajian-type series identities on holomorphic families of Cantor sets associated to one-dimensional complex dynamical systems. We show that the series is absolutely summable if and only if the Hausdorff dimension of the Cantor set is strictly less than one. Throughout the domain of convergence, these identities can be analytically continued and they exhibit nontrivial monodromy.

The second part of the thesis follows mainly from [17]. In particular, we prove an inequality that must be satisfied by displacement of generators of free Fuchsian groups, which is the two-dimensional version of the $\log(2k-1)$ Theorem for Kleinian groups due to Anderson-Canary-Culler-Shalen [1]. As applications, we obtain quantitative results on the geometry of hyperbolic surfaces such as the two-dimensional Margulis constant and lengths of a pair of based loops, which improves a result of Buser's.

CHAPTER 1

INTRODUCTION

The main body of this thesis, Chapter 3 and 4, is respectively an expanded version of the manuscripts [16] and [17] that the author completed during her graduate studies at the University of Chicago. After giving some preliminaries in Chapter 2, we study Basmajian-type identities on holomorphic families of Cantor sets associated to one-dimensional complex dynamical systems in Chapter 3 and we discuss the displacement problem for free Fuchsian groups in Chapter 4.

1.1 Basmajian-type identities

If $\mathcal{C} \subset [0,1]$ is a Cantor set of zero measure, the Hausdorff dimension of $\mathcal{C}$ is the limit of $-(\log a_n / \log n)^{-1}$, where a_n is the length of the nth biggest component of $[0,1] - \mathcal{C}$. There is no obvious analog of this theorem for an arbitrary Cantor set $\mathcal{C} \subset \mathbb{C}$, but for a *family* of Cantor sets $\mathcal{C}_z \subset \mathbb{C}$ depending holomorphically on a complex parameter z, we can sometimes obtain such a relation.

Note that if the measure of $\mathcal{C} \subset [0,1]$ is zero, then a_n satisfy an identity

$$1 = \sum_{n=1}^{\infty} a_n.$$

For $\mathcal{C}_z \subset \mathbb{C}$ depending on z, we obtain by analytic continuation a formal holomorphic *family* of identities

$$S(z) = \sum_{n=1}^{\infty} a_n(z).$$

Thus it is natural to investigate the conditions under which the right hand side is absolutely summable.

In Chapter 3, we study holomorphic families of Cantor sets $\mathcal{C}_z$ associated to familiar

1-dimensional complex dynamical systems, and for such families we introduce identities of this form (which have a natural geometric interpretation when $\mathcal{C}_z \subset \mathbb{R}$) and show that the right hand sides are absolutely summable if and only if the Hausdorff dimension of $\mathcal{C}_z$ is strictly less than 1.

The identities themselves are of independent interest — even in the case of $\mathcal{C} \subset \mathbb{R}$, where a special case is Basmajian's orthospectrum identity for a hyperbolic surface.

If Σ is a compact hyperbolic surface with geodesic boundary, an *orthogeodesic* $\gamma \subset \Sigma$ is a properly immersed geodesic arc perpendicular to $\partial\Sigma$. Basmajian [2] proved the following identity:

$$\text{length}(\partial\Sigma) = \sum_\gamma 2\log\coth\left(\frac{\text{length}(\gamma)}{2}\right) \tag{1.1}$$

where the sum is taken over all orthogeodesics γ in Σ. The geometric meaning of this identity is that there is a canonical decomposition of $\partial\Sigma$ into a Cantor set (of zero measure), plus a countable collection of complementary intervals, one for each orthogeodesic, whose length depends only on the length of the corresponding orthogeodesic.

For simplicity, one can look at surfaces with a single boundary component. A hyperbolic structure on Σ is the same as a discrete faithful representation $\rho : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{R})$ acting on the upper half-plane model in the usual way. After conjugation, we may assume that $\rho(\partial\Sigma)$ stabilizes the positive imaginary axis ℓ.

Orthogeodesics correspond to double cosets of $\pi_1(\partial\Sigma)$ in $\pi_1(\Sigma)$. For each nontrivial double coset $\pi_1(\partial\Sigma)\alpha\pi_1(\partial\Sigma)$, the hyperbolic geodesic $\alpha(\ell)$ corresponds to another boundary component of $\widetilde{\Sigma}$, and the contribution to Basmajian's identity from this term is $\log(\rho(\alpha)(0)/\rho(\alpha)(\infty))$; hence

$$\text{length}(\partial\Sigma) = \sum_\alpha \log(\rho(\alpha)(0)/\rho(\alpha)(\infty)).$$

If we deform ρ to some nearby representation $\rho_z : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$, and replace each ρ by ρ_z above, we obtain a formula for the *complex length* of $\rho_z(\partial\Sigma)$. This is the desired complexification of Basmajian's identity.

The identity makes sense, and is absolutely convergent, exactly throughout the subset $\mathcal{S}_{<1}$ of Schottky space $\mathcal{S}$ where the Hausdorff dimension of the *limit set* of $\rho_z(\pi_1(\Sigma))$ is less than one. Since log is multivalued, it is important to choose the correct branch at a real representation, and then follow the branch by analytic continuation. The space $\mathcal{S}_{<1}$ is not simply-connected and some terms exhibit monodromy (in the form of integer multiples of $2\pi i$) when they are analytically continued around homotopically nontrivial loops.

Based on Sullivan's dictionary between Kleinian groups and rational maps, we develop a parallel theory for quadratic polynomials $f_c(z) = z^2 + c$. If the complex parameter c is real and is in the interval $(-\infty, -2)$, the Julia set J_c of the quadratic polynomial f_c is a Cantor set of zero measure contained in the real line. Thus there is a natural Basmajian-type identity associated to J_c. If we perturb c off the real line in the complement of the Mandelbrot set $\mathcal{M}$, we obtain a formal family of complexified identities, depending holomorphically on c. Analogous to the case of Basmajian's identity, the complexified identity holds exactly on the subset $(\mathbb{C} \setminus \mathcal{M})_{<1}$ of $\mathbb{C} \setminus \mathcal{M}$ where the Hausdorff dimension of the Julia set is strictly less than one.

$\pi_1(\mathbb{C} \setminus \mathcal{M}) = \mathbb{Z}$ and the generator induces new monodromy terms in the series identity. Our criterion for convergence gives us a method to numerically compute the locus in $\mathbb{C} \setminus \mathcal{M}$ where the Hausdorff dimension of J_c is equal to 1. Experimentally this locus appears to be a topological circle with a "cusp" at -2.

Our main analytic result — the relation between the growth rate of the terms in the right hand side of the identities and the Hausdorff dimension — is proved by using the Thermodynamic Formalism developed by Ruelle, Bowen and others in the 1970's.

The main geometric idea is that the spectrum of the transfer operator is controlled by the geometric contraction rate of sets in a suitable Markov partition, and this in turn can be related to the size of the terms in the identity, precisely because the dynamical systems are conformal.

Now we turn to the relations to L-functions. Dirichlet's unit theorem expresses certain covolumes (of units in an algebraic number field) in terms of special values of L-functions, which have a series decompostion, of which the Riemann zeta function is the simplest example. Basmajian's identity expresses in a similar way a (co)volume as a series, expressed over topological terms. We suggest that the study of our families of Basmajian-type identities is analogous to the idea of studying L-functions expressed in series form.

1.1.1 Statement of results

In section 3.1, we consider an elementary example of our general theory, namely the case of an iterated function system generated by a pair of planar similarities.

In section 3.2, we study complexified Basmajian's identity for Schottky groups and exhibit loops in $\mathcal{S}_{<1}$ with nontrivial monodromy. The main theorem of this section is the following:

Theorem 3.2.8 (Complexified Basmajian's identity). *Suppose $\rho_0 : F_n \to PSL(2, \mathbb{C})$ is a Fuchsian marking corresponding to a hyperbolic surface Σ with geodesic boundaries $a_1, \cdots, a_k$. Let $\alpha_1, \cdots, \alpha_k \in \pi_1 M$ represent the free homotopy classes of $a_1, \cdots, a_k$. If ρ is in the same path component as ρ_0 in $\mathcal{S}_{<1}$, then*

$$\sum_{j=1}^{k} l(\rho(\alpha_j)) = \sum_{p,q=1}^{k} \sum_{w \in \mathcal{L}_{p,q}} \log[\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-] \mod 2\pi i, \tag{1.2}$$

where α_j^+, α_j^- are the attracting and repelling fixed points of $\rho(\alpha_j)$, respectively. Moreover, the series converges absolutely.

Here $\mathcal{L}_{p,q}$ is a set of double coset representatives associated to boundary components a_p and a_q. The key ingredient in the proof of the theorem is the following analytic result, whose proof is deferred until section 3.4.

Theorem 3.2.6. *Given a marked Schottky representation $\rho : F_n \to PSL(2,\mathbb{C})$, the infinite series (1.2) converges absolutely if and only if the Hausdorff dimension of the limit set Λ_Γ of the Schottky group $\Gamma = \rho(F_n)$ is strictly less than one.*

In section 3.3 we study Basmajian-type identities for quadratic polynomials and numerically plot the Hausdorff dimension one locus in $\mathbb{C} \setminus \mathcal{M}$. Our main results are the following:

Theorem 3.3.3. *For complex parameter $c \in (\mathbb{C} \setminus \mathcal{M})_{<1}$, let T_1 and T_2 be the two branches of f_c^{-1} and z_1 be the fixed point of T_1, then the following identity holds*

$$z_1 - (-z_1) = \sum_{w \in \{T_1, T_2\}^*} (-1)^\eta \Big(w(T_1(-z_1)) - w(T_2(-z_1)) \Big), \tag{1.3}$$

where η is the number of T_2's in the word w.

Again, we will state the convergence theorem.

Theorem 3.3.2. *The series in 1.3 is absolutely convergent if and only if the Hausdorff dimension of J_c is strictly less than one.*

The monodromy action of a nontrival loop in $(\mathbb{C} \setminus \mathcal{M})_{<1}$ on J_c is a map ϕ, which, under the symbolic coding, simply exchanges the labels 1 and 2 and hence induces a new identity on J_c:

$$z_2 - (-z_2) = \sum_{w \in \{T_1, T_2\}^*} (-1)^{\eta'} \Big(w(T_2(-z_2)) - w(T_1(-z_2)) \Big) \tag{1.4}$$

where η' is the number of T_1's in the word w.

1.2 Displacement problem for free Fuchsian groups

In [1], Anderson, Canary, Culler and Shalen proved the following remarkable theorem which gives a displacement constraint for the generators of a geometrically finite free Kleinian group.

Theorem 1.2.1 ([1], Theorem 6.1, $\log(2k-1)$ Theorem)**.** *Let $k \geq 2$ be an integer and let Φ be a geometrically finite Kleinian group freely generated by $\{g_1, \cdots, g_k\}$. Let z be any point in $\mathbb{H}^3$ and denote by $d_i = d(z, g_i z)$ the hyperbolic distance between z and $g_i z$. Then we have*

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}. \tag{1.5}$$

In particular, there exists an $i \in \{1, \cdots, k\}$ such that $d_i \geq \log(2k-1)$.

The $\log(2k-1)$ Theorem has many interesting applications in relating quantitative geometry, especially volume estimates, of hyperbolic 3-manifolds and their classical topological invariants. Using topological arguments, one can show that the fundamental group of a hyperbolic 3-manifold contains many free subgroups. The constraints on these free subgroups consequently impose constraints on the geometry of the manifold, which can be used to obtain strong volume estimates (see [1] and [11] for more details).

In Chapter 4, we study the displacement constraint problem in two dimensions. More specifically, we obtain an inequality analogous to (1.5) for free Fuchsian groups.

Theorem 4.3.1. *Let $k \geq 2$ be an integer and let Φ be a free Fuchsian group generated by $\{g_1 \cdots g_k\}$. Let z be any point in $\mathbb{H}^2$ and denote by $d_i = d(z, g_i z)$ the hyperbolic distance between z and $g_i z$. Then we have*

$$\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_i}{2}\right)\right) \leq \frac{\pi}{2}. \tag{1.6}$$

In particular, there exists an $i \in \{1, \cdots, k\}$ *such that*

$$d_i \geq \log \left(\frac{1 + \cos \frac{\pi}{2k}}{1 - \cos \frac{\pi}{2k}} \right).$$

When $k = 2$*, the upper bound is realized by* $\Phi = \Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ *and* $z = \sqrt{-1}$.

Although a free Fuchsian group is also a geometrically finite Kleinian group, our theorem is strictly stronger than the $\log(2k-1)$ Theorem. An easy way to see this is to observe that, when $k = 2$, $\log(3 + 2\sqrt{2})$ is greater than $\log 3$ and it is realized. Rigorous calculations are carried out in Remark 4.3.2.

Theorem 4.3.1 can be applied to study quantitative geometry of hyperbolic surfaces. Recall that, if M is a closed orientable hyperbolic n-manifold, a positive real number ε is called a *Margulis number* for M if for any $z \in \mathbb{H}^n$ and $\alpha, \beta \in \pi_1 M$, $\max\{d(z, \alpha(z)), d(z, \beta(z))\} < \varepsilon$ implies that α and β commute. The Margulis Lemma ([3], Chapter D) states that for every $n \geq 2$ there is a positive constant ε_n called the *Margulis constant* which serves as a Maruglis number for every closed orientable hyperbolic n-manifold. It follows from the Main Theorem that $\log\left(3 + 2\sqrt{2}\right)$ is the 2-dimensional Margulis constant. This is consistent with the number found by Yamada in [24] using the signatures of non-elementary torsion-free 2-generator Fuchsian groups.

Our main theorem also imposes constraints on the lengths of a pair of based loops on a hyperbolic surface. A classical result of Buser's [8] states that if α_1 and α_2 are two intersecting simple closed geodesics on a compact hyperbolic surface, then their lengths ℓ_1 and ℓ_2 must satisfy $\sinh(\ell_1/2) \cdot \sinh(\ell_2/2) \geq 1$. Using the Main Theorem, one can show that the same inequality holds but α_1 and α_2 can be promoted to be a pair of based loops as long as their (free) homotopy classes generate a free group of rank 2.

Now we turn to the proof strategies. We first prove a special case of Theorem 4.3.1 where we assume that the quotient surface Σ_Φ has finite area (Theorem 4.2.1). The proof is based on Culler-Shalen's paradoxical decomposition of the Patterson-Sullivan measure.

If Φ is a free group generated by $\{g_1, \cdots, g_k\}$, then Φ admits a decomposition into a singleton $\{Id\}$ and $2k$ disjoint subsets, each one of which consists of reduced words with a given first letter coming from the symmetric generating set. This decomposition has the property that each of the $2k$ subsets is mapped to the complement of another by left multiplying a generator or the inverse of a generator.

Pick a point z in $\mathbb{H}^2$ and identify the orbit $\Phi \cdot z$ with Φ. Since the Patterson-Sullivan measure μ associated to Φ is constructed as the limit of measures supported on the orbit, it respects the combinatorial structure of the group; that is, there exists a decomposition of μ into $2k$ measures, one for each element in the symmetric generating set, such that each one of the measures is transformed into the complement of another by a generator or its inverse (Proposition 4.1.3).

If the quotient surface Σ_Φ of a Fuchsian group Φ has finite area, then every positive Φ-invariant superharmonic function on $\mathbb{H}^2$ is constant. Then one can show (Proposition 4.1.1) that the Patterson-Sullivan measure μ associated to Φ is in fact the arc length measure on the boundary circle at infinity $\mathbb{S}^1_\infty$. Therefore, we obtain a decomposition of the arc length measure into $2k$ measures. These decomposed measures give an estimate for the displacement of each generator (Lemma 4.2.2), which leads to the inequality involving displacements for all generators.

To prove the theorem when Σ_Φ has infinite area, we prove a simple but seemingly surprising fact (Lemma 4.3.3) that Σ_Φ admits a holomorphic embedding, which is also a homotopy equivalence, into a finite area surface S_∞. To this end, we first show that each funnel end of Σ_Φ is conformal to an annulus. Therefore, Σ_Φ is conformal to the Riemann surface S obtained by gluing an annulus to each boundary geodesic of its convex core $C(\Sigma_\Phi)$. Note that a classical version of this construction is the so-called infinite Nielsen extension due to Bers [5] (see also [15] and [14]). Moreover, S is holomorphically embedded in the finite-area surface S_∞ obtained by gluing a half infinite cylinder $S^1 \times \mathbb{R}^+$ to each boundary geodesic of $C(\Sigma_\Phi)$. The theorem then follows as holomorphic maps are distance non-increasing, the function $\arccos \tanh(x/2)$ is strictly decreasing and the inequality holds on S_∞.

CHAPTER 2

PRELIMINARIES

2.1 Hyperbolic geometry

Let $n \geq 2$ be an integer. The *hyperbolic n-space* is an n-dimensional simply connected Riemannian manifold with constant sectional curvature -1. In this section, we briefly summarize some basic properties of hyperbolic 3-space. For a detailed discussion of hyperbolic spaces, see for example [3] (Chapter A) and [19].

We consider the *upper half space* model $\mathcal{H}$ of hyperbolic 3-space $\mathbb{H}^3$, which is defined as

$$\mathcal{H} = \{(z,t) \mid z \in \mathbb{C}, t > 0\}$$

with the metric $ds^2 = \dfrac{|dz|^2 + dt^2}{t^2}$. There are other models for $\mathbb{H}^3$ and all the models are isometrically diffeomorphic to each other. Chapter A of [3] contains a detailed discussion of those models.

If $P = (z_1, t_1)$ and $Q = (z_2, t_2)$ are two points in $\mathcal{H}$, the hyperbolic distance between them is given by

$$\cosh dist(P,Q) = 1 + \frac{|z_1 - z_2|^2 + |t_1 - t_2|^2}{2t_1 t_2}.$$

In particular, if $z_1 = z_2 = 0$, then $dist(P,Q) = \log |t_1/t_2|$.

The hyperbolic planes in $\mathcal{H}$ are either vertical Euclidean planes or hemispheres with centers in $\mathbb{C}$. Geodesics in $\mathcal{H}$ are vertical Euclidean lines or the great (half) circles on the hemispheres.

The group of orientation preserving isometries of $\mathbb{H}^3$ can be identified with $\mathrm{PSL}(2,\mathbb{C})$. An element of $\mathrm{PSL}(2,\mathbb{C})$ acts on the boundary sphere $\hat{\mathbb{C}} = \partial_\infty \mathbb{H}^3$ by Möbius transformations and the action can be extended in a natural way to the interior of $\mathbb{H}^3$. Any non-trivial element of $\mathrm{PSL}(2,\mathbb{C})$ can have either $0, 1$ or 2 fixed points on $\hat{\mathbb{C}}$, and the element is called *elliptic, parabolic* and *hyperbolic*, respectively.

2.2 Kleinian groups

A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, the group of orientation preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. Kleinian groups are closely related to hyperbolic 3-manifolds as any hyperbolic 3-manifold is the quotient of $\mathbb{H}^3$ by a Kleinian group. In this section, we will only discuss limit sets of Kleinian groups and the Schottky space which we will need in Chapter 3. For a detailed introduction to this vast subject, see for example [19] and [20].

2.2.1 Actions on the boundary sphere and limit sets

Let Γ be a Kleinian group and let $x \in \mathbb{H}^3$. The orbit Γx of x is an infinite discrete subset of $\mathbb{H}^3$. By compactness of $\overline{\mathbb{H}^3}$, it must accumulate on the boundary sphere $\hat{\mathbb{C}}$. We call the set $\Lambda_\Gamma = \overline{\Gamma x} \cap \hat{\mathbb{C}}$ the *limit set* of Γ. Its complement $\Omega_\Gamma = \hat{\mathbb{C}} \setminus \Lambda_\Gamma$ is called the *domain of discontinuity*, which is the maximal open subset of $\hat{\mathbb{C}}$ on which Γ acts properly discontinuously.

If Λ_Γ is finite, then it contains $0, 1$ or 2 points and Γ is said to be *elementary* in this case. If Γ is non-elementary, then Λ_Γ is uncountably infinite and it is the unique closed minimal set for the Γ action on Λ_Γ. The Ahlfors' conjecture which is now a theorem [7] states that Λ_Γ is either the entire Riemann sphere $\hat{\mathbb{C}}$ or it has Lebesgue measure 0. When $\Lambda_\Gamma \neq \hat{\mathbb{C}}$, it is complicated and often exhibits self-similar structures which motivates the study of its Hausdorff dimension.

2.2.2 Marked Schottky space and Schottky groups

Let $F_n = \langle g_1, \cdots, g_n \rangle$ be a free group on n generators. A discrete faithful representation $\rho : F_n \to \mathrm{PSL}(2, \mathbb{C})$ is called a *marked Schottky representation* if there is a subsurface $E \subset \hat{\mathbb{C}}$ (homeomorphic to a sphere with $2n$ holes) with boundary components C_i, C_i' for $i = 1, \cdots, n$ so that $\rho(g_i)(C_i) = C_i'$ and $\rho(g_i)(E) \cap E = C_i'$. The representation is called a *marked Fuchsian representation* or a *Fuchsian marking* if it is conjugate to a representation into $\mathrm{PSL}(2, \mathbb{R})$. Two marked Schottky representations ρ_1 and ρ_2 are *equivalent* if they are conjugate. The space of equivalence classes of marked Schottky representations is called the *marked Schottky space*, denoted by $\mathcal{S}$. A standard reference is [5].

The image of F_n under a Schottky representation is a *Schottky group* Γ. It is well-known that every nontrivial element in Γ is loxodromic and the limit set Λ_Γ is a Cantor set.

$\mathcal{S}$ can be parametrized by the fixed points and the trace squares of the $\rho(g_i)$'s. Denote Fix^- and Fix^+ the repelling and attracting fixed points, respectively. Normalize the representation so that $\mathrm{Fix}^-\rho(g_1) = 0$, $\mathrm{Fix}^+\rho(g_1) = \infty$ and $\mathrm{Fix}^-\rho(g_2) = 1$. Then ρ is uniquely determined by

$$\Big(\mathrm{Fix}^+\rho(g_2), \mathrm{Fix}^-\rho(g_3), \mathrm{Fix}^+\rho(g_3), \cdots, \mathrm{Fix}^+\rho(g_n); \mathrm{tr}^2(\rho(g_1)), \cdots, \mathrm{tr}^2(\rho(g_n))\Big)$$

which gives a parametrization of $\mathcal{S}$ as a subspace of $\hat{\mathbb{C}}^{2n-3} \times \mathbb{C}^{2n}$. It is a standard fact that $\mathcal{S}$ is open and connected. Moreover, $\mathcal{S}_{<1}$, the subspace of Schottky groups whose limit set has Hausdorff dimension strictly less than one, is also open because Hausdorff dimension is an analytic function on the deformation space of Schottky groups (see Corollary 2.4.9).

2.3 Complex dynamics of quadratic polynomials

Complex dynamics studies dynamical systems defined by iterations of rational maps on the Riemann sphere. To serve the purposes of Chapter 3, we restrict ourselves to the simplest case where the map is a quadratic polynomial. In particular, we discuss some basic properties of the dynamics and the parameter space for such maps. Interested readers should refer to [12] for dynamics of quadratic polynomials and [21] for complex dynamics of one variable.

Given a quadratic polynomial $g(z) = \alpha z^2 + \beta z + \gamma$ with $\alpha \neq 0, \beta, \gamma \in \mathbb{C}$, we first observe that $g(z)$ is conjugate by the map $h(z) = \alpha z + \beta/2$ to $f_c = z^2 + c$ where $c = \alpha\gamma + \beta/2 - \beta^2/4$. Therefore, it suffices to restrict our attention to quadratic polynomials of the form $f_c(z) = z^2 + c$ where $c \in \mathbb{C}$.

Let $c \in \mathbb{C}$ and $f_c(z) = z^2 + c$. The *filled Julia set* K_c of f_c is the set of points in the complex plane whose orbits are bounded under iterations of f_c, i.e.

$$K_c \coloneqq \{z \in \mathbb{C} \mid f_c^n(z) \nrightarrow \infty \text{ as } n \to \infty\}$$

and the *Julia set* J_c of f_c is the boundary of K_c.

Example 2.3.1. When $c = 0$, the filled-in Julia set K_0 of $f(z) = z^2$ is the closed unit disk $\mathbb{D}$ and the Julia set $J_0 = \partial\mathbb{D}$ is the unit circle.

The *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$ is a subset of the parameter space consisting of parameters $c \in \mathbb{C}$ such that the orbit of the critical point of f_c, namely 0, stay bounded under iterations of f_c, i.e.

$$\mathcal{M} \coloneqq \{c \in \mathbb{C} \mid f^n(0) \nrightarrow \infty \text{ as } n \to \infty\}.$$

There is a dichotomy of connectedness of the Julia set J_c depending on if c lies in the Mandelbrot set, or in other words, if the critical point escapes to infinity.

Proposition 2.3.2. *J_c is connected if $c \in \mathcal{M}$ and is a Cantor set if $c \notin \mathcal{M}$.*

Note that for higher degree polynomials, the Julia set and its connectedness are more complicated as these polynomials have more than one critical point and some of the critical points escape to infinity while others may stay in the Julia set.

2.4 Thermodynamic formalism

Thermodynamic formalism is a powerful tool to study expanding conformal dynamical systems, especially to estimate Hausdorff dimensions of limit sets. As a generalization of Moran's theorem to nonlinear systems, Bowen [6] established a formula for computing Hausdorff dimension from the *pressure* function, which is also related to the maximum eigenvalue of the *Ruelle transfer operator*.

In this section, we give a brief overview of Thermodynamic Formalism, following [22] (see also [23]). All theorems presented here are classical but we include some simple proofs for the convenience of the reader. In the end, we discuss its applications to Schottky groups and quadratic polynomials.

Throughout this section, $T : \Lambda \to \Lambda$ is a $C^{1+\alpha}$ locally expanding conformal map. The (local) branches of T^{-1} are contractions which form an IFS and $\Lambda \subset \mathbb{R}^d$ is the limit set.

2.4.1 Symbolic coding

$T : \Lambda \to \Lambda$ is said to be a *Markov map* with respect to a *Markov partition* if there exists a finite collection $\mathcal{P}$ of closed subsets $\{P_i\}_{i=1}^m$ such that the following conditions are satisfied:

1. $\Lambda = \bigcup_{i=1}^m P_i$;
2. For each $i = 1, \cdots, m$, P_i is the closure of its interior;
3. For each $i = 1, \cdots, m$, TP_i is a finite union of sets in $\mathcal{P}$.

$\mathcal{P} = \{P_i\}_{i=1}^m$ is called a *Markov partition* for Λ.

Now suppose T is an expanding conformal Markov map and its inverse T^{-1} has n (local) branches. Then the dynamics of the IFS consisting of branches of T^{-1} can be symbolically coded.

Let A be an $n \times n$ matrix such that

$$A_{i,j} = \begin{cases} 1 & \text{if } T(P_i) \supset P_j \\ 0 & \text{otherwise} \end{cases}$$

Since T is expanding, A is *aperiodic*, i.e. there exists an integer $N > 0$ such that all the entries of A^N are positive. For such a matrix A, define (one-sided) *subshift of finite type*

$$\Sigma = \{\underline{i} = (i_j)_{j=0}^{\infty} \mid i_j \in \{1, 2, \cdots, n\}, A_{i_j, i_{j+1}} = 1\}.$$

Endow a metric on Σ

$$d(\underline{x}, \underline{y}) = \frac{1}{2^n}$$

where n is the first place $\underline{x}$ and $\underline{y}$ differ. With the topology induced by d, Σ is a Cantor set.

There is a Hölder continuous projection

$$\pi : \Sigma \to \Lambda$$

such that $\underline{i} = (i_0, i_1, i_2 \cdots)$ gives the sequence of sets $P_{i_0}, P_{i_1}, P_{i_2} \cdots$ visited by the forward orbit of $\pi(\underline{i})$.

The *shift map* $\sigma : \Sigma \to \Sigma$ defined by

$$\sigma(i_0, i_1, i_2, \cdots) = (i_1, i_2, i_3, \cdots)$$

is locally expanding with respect to the metric d. The symbolic dynamical system (Σ, σ) is conjugate to (Λ, T) and is called a *symbolic coding*.

2.4.2 Transfer operator, pressure and Hausdorff dimension

In this subsection, we establish Bowen's formula and deduce that Hausdorff dimension is analytic for an analytic perturbation of T.

Throughout the subsection, we adopt the following notation. For each $\underline{i} = (i_0, i_1, \cdots) \in \Sigma$, write $T_{\underline{i}} = f_{i_n} \circ \cdots \circ f_{i_1}$, where $f_1, \cdots, f_n$ are the local inverse branches of T, and $P_{\underline{i}} = f_{i_n} \circ \cdots \circ f_{i_1} P_{i_0}$. Denote $|P_{\underline{i}}|$ the diameter of the set $P_{\underline{i}}$.

In order to find Hausdorff dimension of Λ, we can cover Λ by (open neighbourhoods of) $P_{\underline{i}}$ with $|P_{\underline{i}}| < \delta$. Then the t-dimensional Hausdorff measure of Λ is estimated by $\sum_{|\underline{i}|=n} |P_{\underline{i}}|^t$, for large n and $0 < t < d$.

Proposition 2.4.1 ([22], Proposition 2.4.1)**.**

1. *There are positive constants B_1 and B_2 such that for any $\underline{i}$ and $x, y \in \Lambda$,*

$$B_1 \leq \frac{|T'_{\underline{i}}(x)|}{|T'_{\underline{i}}(y)|} \leq B_2.$$

2. *There exist constants C_1 and C_2 such that for any $\underline{i}$ and $z \in \Lambda$,*

$$C_1 \leq \frac{|P_{\underline{i}}|}{|T'_{\underline{i}}(z)|} \leq C_2.$$

In particular, there are constants C_1 and C_2 such that for any $n \geq 1$ and $z \in \Lambda$,

$$C_1 \leq \frac{\sum_{|\underline{i}|=n} |P_{\underline{i}}|^t}{\sum_{|\underline{i}|=n} |T'_{\underline{i}}(z)|^t} \leq C_2.$$

Therefore, we see that estimating $\sum_{|\underline{i}|=n} |P_{\underline{i}}|^t$ is equivalent to estimating $\sum_{|\underline{i}|=n} |T'_{\underline{i}}(z)|^t$. However, the latter quantity has a great advantage as it is related to the *transfer operator*, which we define now.

For $\alpha > 0$, let $C^\alpha(\Sigma)$ be the Banach space of real-valued Hölder continuous functions

$f : \Sigma \to \mathbb{R}$, i.e.

$$|f(\underline{x}) - f(\underline{y})| \leq Cd(\underline{x}, \underline{y})^\alpha$$

with Hölder norm given by

$$||f||_\alpha = \sup_{\underline{x}} |f(\underline{x})| + \sup_{\underline{x} \neq \underline{y}} \frac{|f(\underline{x}) - f(\underline{y})|}{d(\underline{x}, \underline{y})^\alpha}$$

Definition 2.4.2 (Transfer operator)**.** For any Hölder continuous potential $\phi \in C^\alpha(\Sigma)$, define the *transfer operator* $\mathcal{L}_\phi : C^\alpha(\Sigma) \to C^\alpha(\Sigma)$ as

$$\big(\mathcal{L}_\phi \psi\big)(\underline{y}) = \sum_{\sigma(\underline{x}) = \underline{y}} e^{\phi(\underline{x})} \psi(\underline{x}).$$

To establish Bowen's formula, we consider the following one-parameter family of Hölder potentials

$$\phi_t = -t \log |T'(\pi(\underline{x}))|, \text{ with } 0 < t < d,$$

and the corresponding transfer operators $\mathcal{L}_t \coloneqq \mathcal{L}_{\phi_t}$. By straightforward computation, $\mathcal{L}_t : C^\alpha(\Lambda) \to C^\alpha(\Lambda)$ is given by

$$\big(\mathcal{L}_t \psi\big)(z) = \sum_{j=1}^{n} |f_j'(z)|^t \psi(f_j(z)).$$

Again, straightfoward computation shows that the n-th iterate of $\mathcal{L}_t$ evaluated at the constant function $\mathbf{1}$ gives the desired quantity, i.e.

$$\big(\mathcal{L}_t^n \mathbf{1}\big)(z) = \sum_{|\underline{i}| = n} |T_{\underline{i}}'(z)|^t, \text{ for any } z \in \Lambda.$$

The following celebrated Ruelle-Perron-Frobenius theorem will allow us to see that the Hausdorff dimension of Λ is in fact related to the spectrum of $\mathcal{L}_t$.

Theorem 2.4.3 (Ruelle-Perron-Frobenius Theorem, [22], Proposition 2.4.2)**.**

1. *$\mathcal{L}_t$ has a simple maximum eigenvalue $\lambda_t > 0$ and there is exponential convergence $|\mathcal{L}_t^n \mathbf{1} - \lambda_{\mathbf{t}}^{\mathbf{n}}| \leq \mathbf{C}\lambda_{\mathbf{t}}^{\mathbf{n}}\theta^{\mathbf{n}}$, for some $C > 1$, $0 < \theta < 1$ and $n \geq 1$.*

2. *There exists a probability measure μ and D_1,$D_2 > 0$ such that for any $n \geq 1$ and $|\underline{i}| = n$ and $x \in \Lambda$*
$$D_1\lambda_t^n \leq \frac{\mu(P_{\underline{i}})}{|T_{\underline{i}}'(x)|^t} \leq D_2\lambda_t^n.$$

3. *The map $\lambda(t) = \lambda_t$ is real analytic and $\lambda'(t) < 0$ for all $t \in \mathbb{R}$.*

Finally, we are in a position to state Bowen's formula for computing Hausdorff dimension.

Definition 2.4.4. For any continuous function $f : \Lambda \to \mathbb{R}$, define *pressure* $P(f)$ to be

$$P(f) = \limsup_{n\to\infty} \frac{1}{n} \log \Big(\sum_{T^n x = x} e^{f(x)+f(Tx)+\cdots+f(T^{n-1}x)} \Big).$$

Theorem 2.4.5 (Bowen [6])**.** *Let $T : \Lambda \to \Lambda$ be a $C^{1+\alpha}$ locally expanding map. Then the Hausdorff dimension t of Λ is the unique solution to $P(-t\log|T'|) = 0$.*

Corollary 2.4.6. $P(-t\log|T'|) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{T^n x = x} \frac{1}{|(T^n)'(x)|^t} \right) = \log \lambda_t.$

Proof. The first equation follows from Definition 2.4.4.

From Proposition 2.4.1, it follows that

$$\sum_{T^n x = x} \frac{1}{|(T^n)'(x)|^t} \asymp \mathcal{L}_t^n \mathbf{1}(\mathbf{x}'), \text{ for any } \mathbf{x}' \in \mathbf{\Lambda},$$

and by Theorem 2.4.3, $\mathcal{L}_t^n \mathbf{1}(\mathbf{x}') \asymp \lambda_{\mathbf{t}}^{\mathbf{n}}$. □

Corollary 2.4.7. *The function $t \mapsto P(-t\log|T'|)$ is strictly monotone decreasing and analytic.*

Proof. It is an immediate consequence of Theorem 2.4.3 (3) and Corollary 2.4.6. □

Remark 2.4.8. $\lambda_s = 1$ when $s =$ Hausdorff dimension of Λ and $\lambda_1 \leq 1$ if and only if $s \leq 1$.

Corollary 2.4.9 (Analyticity of Hausdorff dimension)**.** *Let T_λ be an analytic family of $C^{1+\alpha}$ locally expanding maps. Then the Hausdorff dimension as a function $\lambda \mapsto H.dim(\Lambda_\lambda)$ is real analytic.*

Proof. By Corollary 2.4.7, $f(t,\lambda) = P(-t\log|T'_\lambda|)$ is analytic in t. Also, $\dfrac{\partial f(t,\lambda)}{\partial\lambda} \neq 0$. The corollary then follows from the inverse function theorem. □

2.4.3 Application I: Limit set of Schottky groups

Recall that a Schottky group Γ on n generators is the image $\rho(F_n)$ under a marked Schottky representation $\rho : F_n \to \mathrm{PSL}(2,\mathbb{C})$. By definition, there are n pairs of disjoint Jordan curves C_i, C'_i on the boundary sphere $\hat{\mathbb{C}}$ such that $\rho(g_i)$ takes the exterior of C'_i into the interior of C_i and $\rho(g_i^{-1})$ takes the exterior of C_i into the interior of C'_i. Then the Schottky group action is a conformal IFS on $\hat{\mathbb{C}}$ whose limit set Λ_Γ is contained in the interior of these $2n$ Jordan curves.

The dynamics of Γ has a natural symbolic coding. Set

$$P_i := \text{closed disk bounded by } C_i, \text{ and}$$

$$P_{2n+1-i} := \text{closed disk bounded by } C'_i$$

for each $i = 1, \cdots, n$. Define $T : \Lambda_\Gamma \to \Lambda_\Gamma$ by

$$T(z) = \begin{cases} \rho(g_i^{-1})(z) & \text{if } z \in P_i \\ \rho(g_i)(z) & \text{if } z \in P_{2n+1-i} \end{cases} \quad \text{for } i = 1, \cdots, n.$$

Then T is a Markov map with respect to the Markov partition $\mathcal{P} = \{P_i\}_{i=1}^{2n}$. Furthermore, T is expanding on Λ_Γ (see [6] for details).

By definition,

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and the subshift of finite type is the set

$$\Sigma = \big\{\underline{i} = (i_0, i_1, i_2, \cdots) \mid i_j \in \{g_1, \cdots, g_n, g_1^{-1}, \cdots, g_n^{-1}\}, A_{i_j, i_k} = 1\},$$

which is exactly the set of reduced words in the alphabet $\{g_1^{\pm 1}, \cdots, g_n^{\pm 1}\}$.

2.4.4 Application II: Julia set of quadratic polynomials

For c lying outside of the Mandelbrot set, the quadratic polynomial $f_c(z) = z^2 + c$ is expanding on its Cantor Julia set J_c. In fact, there exists a Markov partition $\{P_1, P_2\}$ with respect to which f_c is a Markov map and satisfies $P_1 \subset f_c(P_1)$, $P_2 \subset f_c(P_1)$, $P_1 \subset f_c(P_2)$, $P_2 \subset f_c(P_2)$.

Then, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

defines the subshift of finite type

$$\Sigma = \big\{\underline{i} = (i_0, i_1, i_2, \cdots) \mid i_j \in \{1, 2\} \text{ for all } j \in \mathbb{N}\big\}.$$

With these symbolic codings, the mechanism of Thermodynamic Formalism applies to Schottky groups and quadratic polynomials, which will be used in the next section to prove the convergence theorems.

CHAPTER 3

BASMAJIAN-TYPE IDENTITIES AND HAUSDORFF DIMENSION OF LIMIT SETS

3.1 Some linear examples

In this section, we give the simplest example to illustrate our main theorem, that of a conformal iterated function system in $\mathbb{C}$ generated by two similarities.

If $T_1, \cdots, T_k : \mathbb{R}^n \to \mathbb{R}^n$ is a finite family of contractions, the *limit set* is the set of accumulation points of the semigroup generated by $\{T_i\}$. These semigroups are usually called iterated function systems or IFS for short.

Dimensions of limit sets of IFSs have been widely studied. Among several notions of dimension, we will concentrate on box dimension and Hausdorff dimension. Conveniently, for the examples we study these two notions of dimension agree. We denote Hausdorff dimension by $\dim_{\mathrm{H}}$ and box dimension by $\dim_{\mathrm{B}}$. For a definition, see [13].

For any compact set $E \subset \mathbb{R}^n$, there is an inequality $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{B}} E$. However, for limit sets of conformal IFSs, there is an important dynamical condition under which the Hausdorff dimension equals the box dimension.

Definition 3.1.1 (Open set condition)**.** The family of maps $T_1, \cdots, T_k : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the *open set condition* if there exists an open set $V \subset \mathbb{R}^n$ such that $T_i(V)$ are contained in V for all $i = 1, \cdots, k$ and $T_i(V) \cap T_j(V) = \emptyset$ for all $i \neq j$.

Proposition 3.1.2 ([22], Proposition 2.1.4)**.** *For a conformal IFS satisfying the open set condition with limit set* Λ*,* $dim_H \Lambda = dim_B \Lambda$*.*

3.1.1 Middle-third Cantor set

The middle-third Cantor set $\mathcal{C}$ can be viewed as a "cut-out set" – a set obtained from the interval $I = [0,1]$ by cutting out a sequence of disjoint intervals $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}) \cdots$ Let a_i denote the length of the ith interval, then

$$a_i = (1/3)^{(\lfloor \log_2 i \rfloor + 1)}, \text{ where } \lfloor x \rfloor \text{ is the floor function.}$$

By construction, it is obvious that the length of the total interval $[0,1]$ equals the infinite sum of lengths of complimentary intervals; namely, we have the following simplest Basmajian-type identity:

$$|[0,1]| = 1 = \sum_{i=1}^{\infty} a_i = \frac{1}{3} + \left(\frac{1}{9} + \frac{1}{9}\right) + \left(\frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27}\right) + \cdots \tag{3.1}$$

On the other hand, $\mathcal{C}$ is also the limit set of the conformal IFS $\{T_1, T_2 : \mathbb{R} \to \mathbb{R}\}$ given by

$$T_1(x) = \frac{x}{3} \text{ and } T_2(x) = \frac{x}{3} + \frac{2}{3}.$$

Note that 0 is the fixed point of T_1, 1 is the fixed point of T_2 and all other points in $\mathcal{C}$ are the images of 0 and 1 under iterations of T_1 and T_2. In terms of T_1, T_2 and their fixed points, identity 3.1 becomes

$$|z_1 - z_2| = \sum_{w \in \{T_1, T_2\}^*} |w(T_2(z_1)) - w(T_1(z_2))| \tag{3.2}$$

where z_1 and z_2 are the fixed points of T_1 and T_2 respectively and the summation is taken over all the words w in the *alphabet* $\{T_1, T_2\}$.

The following theorem due to Falconer shows that the box dimension of a real one-dimensional cut-out set (and in particular $\dim_{\mathrm{H}} \mathcal{C}$) is controlled by the asymptotic sizes of complementary intervals.

Theorem 3.1.3 (Falconer [13]). *Let A be a compact subset of $\mathbb{R}$ and let $\{A_i\}_{i=1}^{\infty}$ be a sequence of disjoint open subintervals of A such that if $a_i = |A_i|$ with $a_1 \geq a_2 \geq a_3 \geq \cdots$, then $|A| = \sum_{i=1}^{\infty} a_i$. Write $E = A \setminus (\bigcup_{i=1}^{\infty} A_i)$. Let*

$$a = -\liminf \frac{\log a_n}{\log n} \text{ and } b = -\limsup \frac{\log a_n}{\log n}.$$

Then

$$\frac{1}{a} \leq \underline{dim}_B E \leq \overline{dim}_B E \leq \frac{1}{b}.$$

Example 3.1.4. For the middle-third Cantor set $\mathcal{C}$,

$$\dim_{\mathrm{H}} \mathcal{C} = \dim_{\mathrm{B}} \mathcal{C} = \log_3 2 = -\left(\lim_{n\to\infty} \frac{\log a_n}{\log n}\right)^{-1}.$$

3.1.2 Semigroups of similarities

Consider the following conformal IFS of two complex similarities $\{f_c, g_c : \mathbb{C} \to \mathbb{C}\}$ given by

$$f_c(z) = cz \quad \text{and} \quad g_c(z) = c(z-1) + 1$$

where $0 < |c| < 1$ is a complex parameter. The limit set Λ_c is either connected or a Cantor set ([10], Lemma 5.2.1).

The middle-third Cantor set corresponds to the case $c = 1/3$. Hence, when Λ_c is a Cantor set, formula 3.2 gives a formal Basmajian-type identity

$$1 = \sum_{n=0}^{\infty} (2c)^n (1 - 2c) \tag{3.3}$$

which is just a geometric series. It is (absolutely) convergent if and only if $|c| < 1/2$.

On the other hand, this conformal IFS satisfies the open set condition, and therefore by Moran's theorem, the Hausdorff dimension of Λ_c is given by

$$\dim_{\mathrm{H}}\Lambda_c = -\log 2/\log|c|,$$

and $\dim_{\mathrm{H}}\Lambda_c < 1$ if and only if $|c| < 1/2$.

Hence, we see that the series in 3.3 is absolutely convergent if and only if $\dim_{\mathrm{H}}\Lambda_c < 1$.

3.2 Schottky Groups: the Complexified Basmajian Identity

Our main goal for this section is to extend Basmajian's identity 1.1 to a suitable subspace of marked Schottky space $\mathcal{S}$ via analytic continuation. There is a "formal" identity for any marked Schottky group Γ. However, the interpretation is problematic unless one deals with the issue of convergence. As in the example of section 2, it turns out that the series is absolutely summable if and only if the limit set Λ_Γ has Hausdorff dimension strictly less than one. Denote by $\mathcal{S}_{<1}$ the space of Schottky groups whose limit set has Hausdorff dimension strictly less than one. Then the main theorem of this section (Theorem 3.2.8) states that this is the maximal domain on which the extended Basmajian's identity holds.

On the other hand, the extended Basmajian's identity may serve as a tool to study the topology of $\mathcal{S}_{<1}$. More specifically, when analytically continued along a loop in $\mathcal{S}_{<1}$, finitely many terms in the series will exhibit monodromy — their imaginary part changes by integer multiples of 2π. We will present examples of such loops.

In the next three subsections, we reformulate both sides of Basmajian's identity in terms of representations.

3.2.1 Complex length

For $A \in \mathrm{PSL}(2,\mathbb{C})$, the *complex length* of A is

$$l(A) = \cosh^{-1}\left(\frac{\mathrm{tr}\ A^2}{2}\right).$$

Note that complex length is only defined up to multiples of $2\pi i$.

Thus, for any $g \in F_n$, the complex length $l(\rho(g))$ is an analytic function of the coordinates on $\mathcal{S}$. The next lemma states that the real part of complex length stays positive when we deform a Fuchsian marking.

Lemma 3.2.1. *Let ρ_0 be a Fuchsian marking with $l(\rho(g))$ real positive for all $g \in F_n$. Then Re$(l(\rho_t(g))) > 0$ for all $g \in F_n$ when analytically continued along a path ρ_t in $\mathcal{S}_{<1}$.*

Proof. If there were some t and some $g \in F_n$ such that $\mathrm{Re}(l(\rho_t(g))) = 0$, then it would contradict the fact that every element in a Schottky group is loxodromic. □

3.2.2 Double cosets, finite state automata and orthogeodesics

In this subsection, we show that orthogeodesics on Σ correspond to certain double cosets of $\pi_1\Sigma$. There is an efficient coset enumeration algorithm which can be implemented for numerical calculations. Looking ahead, this enumeration parallels the encoding of the dynamical systems as a subshift of finite type, a step in the application of the Thermaldynamic Formalism carried out in section 6.

Let $\alpha_1, \cdots, \alpha_k \in \pi_1\Sigma$ represent the free homotopy classes of boundary geodesics $a_1, \cdots, a_k$, respectively. Denote $H_j := \langle\alpha_j\rangle$, the subgroup of $\pi_1\Sigma$ generated by α_j. Let $\mathcal{DC}(\Sigma)$ denote the set of double cosets of the form $H_p w H_q$ where $w \in \pi_1\Sigma$ is not in $H_p \cap H_q$, for $p, q = 1, \cdots, k$. Denote by $[w]_{p,q}$ the class of the double coset $H_p w H_q$.

Proposition 3.2.2. *There is a bijection*

$$\Phi : \mathcal{DC}(\Sigma) \to \{\textit{Orthogeodesics on } \Sigma\}.$$

Proof. Every homotopically nontrivial proper arc has a unique orthogeodesic in its homotopy class (rel. endpoints), which can be seen e.g. by curve shortening. □

Let S be a symmetric generating set of $\pi_1\Sigma$. Choose an ordering on S. This determines a unique reduced lexicographically first (RedLex) representative w of each double coset $H_p w H_q$. Let $\mathscr{L}_{p,q}$ be the set of nontrivial RedLex double coset representatives for fixed p, q. Then the set

$$\mathscr{L} := \coprod_{1 \leq p,q \leq k} \mathscr{L}_{p,q}$$

is naturally in bijection with the set of orthogeodesics on M.

Definition 3.2.3. A *finite state automaton* on a fixed alphabet S is a finite directed graph G with a starting vertex $*$ and a subset of the vertices called the *accept states*, whose oriented edges are labeled by letters of S so that there is at most one outgoing edge with any given label at each vertex. A word is *accepted* by a finite state automaton if there is a path realizing the word which starts with $*$ and ends on an accept state.

For a fixed finite alphabet $\mathcal{A}$, a *language* is a subset of the set of all words on $\mathcal{A}$. A language is *regular* if it consists of exactly the words accepted by some finte state automaton.

Proposition 3.2.4. *$\mathscr{L}$ is a regular language over the alphabet S.*

Proof. The proof follows from the proof of Cannon's theorem ([9], Theorem 3.2.2) and we only give a sketch here. Since $\pi_1\Sigma$ is δ-hyperbolic and H_j's are quasiconvex, the language of all shortest coset representatives is regular. Also, for a given coset, all shortest coset representatives synchronously fellow-travel. Thus, $\mathscr{L}$ is regular. □

The finite state automaton parametrization of a regular language gives rise to a fast coset enumeration algorithm, as demonstrated in the following example.

Example 3.2.5. (Torus with a geodesic boundary) Let Σ be a torus with a single geodesic boundary and let $S = \{a, b, A, B\}$ be a symmetric generating set for $\pi_1\Sigma$, where capital letters denote inverses. Order S by $a > b > A > B$. Then (the free homotopy class of) the boundary geodesic is represented by the commutator $[a, b]$. Let $H := \langle [a, b] \rangle$. By Proposition 3.2.2, the set of orthogeodesics on Σ is parametrized by the set $\mathscr{L}$ of nontrivial RedLex representatives w of HwH.

More precisely, $w \in \mathscr{L}$ if and only if it satisfies the following conditions:

1. w is reduced, i.e. w does not contain aA, Aa, bB or Bb;
2. w does not start with abA or ba;
3. w does not end with BA or bAB.

Figure 3.1 is a finite state automaton parametrizing the regular language $\mathscr{L}$. In this automaton, double-circled-nodes are accept states and red nodes are reject states. Accepted words are obtained by starting at $*$, going along edges and ending at an accept state and the word is the concatenation of edge labels.

3.2.3 Cross ratios

Let $\rho_0 : F_n \to \mathrm{PSL}(2, \mathbb{R})$ be the Fuchsian marking corresponding to a hyperbolic structure on Σ. Suppose $\gamma_{p,q}$ is an orthogeodesic on Σ running from a_p to a_q with RedLex double coset representative w. Then its hyperbolic length length$(\gamma_{p,q})$ satisfies

$$\coth\left(\frac{\mathrm{length}(\gamma_{p,q})}{2}\right) = [\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-] \tag{3.4}$$

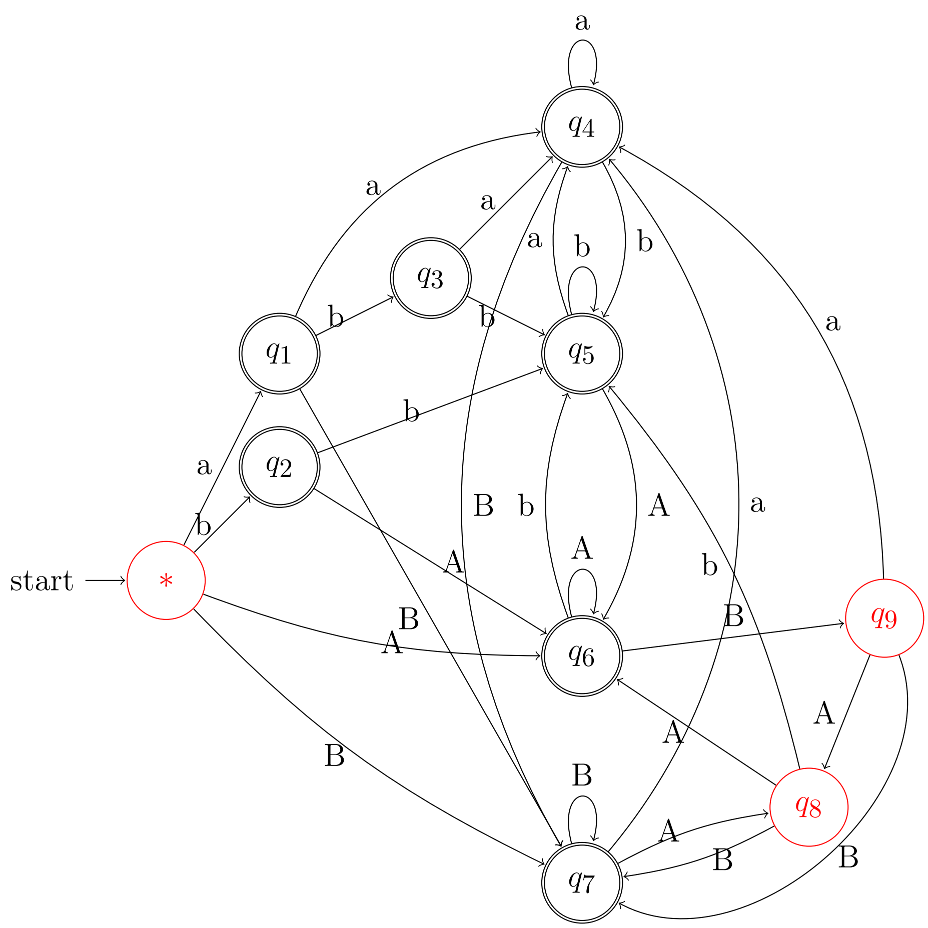

Figure 3.1: A finite state automaton for torus with geodesic boundary

where α_j^+ and α_j^- are respectively the attracting and repelling fixed points of $\rho_0(\alpha_j)$ on S^1_∞, and $[z_1, z_2; z_3, z_4] := \dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is the cross ratio.

Hence, Basmajian's identity 1.1 becomes

$$\sum_{j=1}^{k} l(\rho(\alpha_j)) = \sum_{p,q=1}^{k} \sum_{w \in \mathscr{L}_{p,q}} \log[\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-] \tag{3.5}$$

3.2.4 Complexified Basmajian's identity

When we start deforming a Fuchsian marking, identity 3.5 can be analytically continued only when the series is absolutely convergent.

Theorem 3.2.6. *Given a marked Schottky representation $\rho : F_n \to PSL(2,\mathbb{C})$, the series in 3.5 converges absolutely if and only if the Hausdorff dimension of the limit set Λ_Γ of the Schottky group $\Gamma = \rho(F_n)$ is strictly less than one.*

The proof of the theorem uses the Thermodynamic Formalism and will be presented in section 6, after we introduce the necessary technical tools in section 5.

Proposition 3.2.7. *Let $\rho_t : F_n \to PSL(2,\mathbb{C})$ be a deformation of a Fuchsian marking ρ_0 in $\mathcal{S}_{<1}$. Then the series in 3.5 is uniformly convergent on $[0,1]$.*

Proof. The proposition follows from the proof of Theorem 3.2.6. Since the absolute series is uniformly bounded above by the geometric series $\sum_{k=1}^{\infty} \lambda_1^k$, which is continuous, it is uniformly bounded. □

We are now ready to prove the main theorem of this section.

Theorem 3.2.8 (Complexified Basmajian's identity). *Suppose $\rho_0 : F_n \to PSL(2,\mathbb{C})$ is a Fuchsian marking corresponding to a hyperbolic surface Σ with geodesic boundaries $a_1, \cdots, a_k$. Let $\alpha_1, \cdots, \alpha_k \in \pi_1 M$ represent the free homotopy classes of $a_1, \cdots, a_k$. If ρ is in the same path component as ρ_0 in $\mathcal{S}_{<1}$, then*

$$\sum_{j=1}^{k} l(\rho(\alpha_j)) = \sum_{p,q=1}^{k} \sum_{w \in \mathcal{L}_{p,q}} \log[\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-] \quad mod \ \ 2\pi i \tag{3.6}$$

where α_j^+, α_j^- are the attracting and repelling fixed points of $\rho(\alpha_j)$, respectively. Moreover, the series converges absolutely.

Proof. Let ρ_t in $\mathcal{S}_{<1}$ be an analytic deformation of ρ_0. By Proposition 3.2.7, each side of identity 3.6 is a holomorphic function on $\mathcal{S}_{<1}$ (up to $2\pi i$). In a neighbourhood U of ρ_0, the identity holds on an open subset consisting of Fuchsian markings. Therefore it holds on the entire neighbourhood U. Hence, by analytic continuation, the identity holds for all ρ_t (up to $2\pi i$). □

3.2.5 *Monodromy*

In this subsection, we exhibit two different loops in $\mathcal{S}_{<1}$ along which the identity exhibits nontrivial monodromy. As we analytically continue 3.6 along a loop in $\mathcal{S}_{<1}$ the value of the right hand side might change by multiples of $2\pi i$. We call this the *monodromy* of the loop and observe that it depends only on its homotopy class in $\mathcal{S}_{<1}$. Note that the uniform convergence throughout a compact loop in $\mathcal{S}_{<1}$ implies that only finitely many words can change monodromy.

Let $F_2 = \langle a, b\rangle$ be a free group on two generators with $A = a^{-1}$ and $B = b^{-1}$. Let $L, x \in \mathbb{C}$ be such that $x + 1/x = -L$. Consider the Schottky groups $\Gamma_L := \rho(\langle a, b\rangle) = \langle X_L, Y_L\rangle$ and $\Gamma'_L := \rho'(\langle a, b\rangle) = \langle X_L^2, X_L Y_L^3\rangle$, where

$$X_L = \begin{bmatrix} L & 1 \\ -1 & 0 \end{bmatrix}, Y_L = \begin{bmatrix} 0 & x \\ -\frac{1}{x} & L \end{bmatrix}.$$

Now let $L_t = 5e^{2\pi it}$. Then as t increases from 0 to 1, the trajectory of Γ_{L_t} forms a loop γ in the $\mathrm{PSL}(2,\mathbb{C})-$character variety of F_2 which connects the Fuchsian group Γ_{L_0} to itself. Similarly, the trajectory of Γ'_{L_t} defines another loop γ'.

Numerical calculations suggest that both γ and γ' are in fact in $\mathcal{S}_{<1}$ as the series are absolutely convergent. The homotopy class of these loop in $\mathcal{S}_{<1}$ is distinguished by their monodromy in Table 1.

3.3 Quadratic Polynomials

The main goal of this section is to introduce a Basmajian-type identity on Cantor Julia sets J_c of complex quadratic polynomials $f_c(z) = z^2 + c$ with c lying outside of the Mandelbrot set $\mathcal{M}$. First of all, the series identity is obtained for $c < -2$ as in this case J_c is a cut-out set in $\mathbb{R}$. Similar to the case of Schotkky groups, we state that the series is absolutely

Words	Monodromy along γ	Monodromy along γ'
a	2π	10π
b	2π	6π
A	2π	10π
B	2π	6π
ab	2π	0
AB	2π	0
aB	0	2π
Ab	0	2π
Total change	12π	36π

Table 3.1: Monodromy for the loops γ and γ'

convergent if and only if the Hausdorff dimension of J_c is strictly less than one, which enables us to analytically continue the identity. When we analytically continue the series along a nontrivial loop, the monodromy induces a new Basmajian-type identity on J_c. In addition, using the convergence of the series, we numerically plot the Hausdorff dimension one locus in $\mathbb{C} \setminus \mathcal{M}$.

3.3.1 Basmajian-type identity

Fix $c < -2$. Let

$$T_1 = \sqrt{z - c} \text{ and } T_2 = -\sqrt{z - c}$$

be local branches of f_c^{-1} and let z_1, z_2 be the fixed points of T_1, T_2 respectively. Then the Julia set J_c is obtained as a cut-out set contained in the interval $U = [-z_1, z_1]$ as follows. Set

$$U_0 := U$$

For each $n \geq 1$, let

$$U_n = T_1(U_{n-1}) \cup T_2(U_{n-1}) \text{ and } I_n = U \setminus U_n.$$

Then U_n consists of 2^n intervals whose lengths shrink with n and in the limit $U_\infty = J_c$ is a Cantor set of zero measure.

Lemma 3.3.1. *(Properties of gaps)*

1. $I_1 = [T_2(-z_1), T_1(-z_1)]$ *is the largest gap in* U*.*

2. *Images of* I_1 *under* T_1 *and* T_2 *give rise to all the gaps (i.e.* I_∞*) in* U*.*

3. *The lengths of all the gaps add up to the length of* U*.*

Proof. (1) follows from the monotonicity of T_1 and T_2 and the fact that they are contractions. (2) follows directly from the definitions. Since U is the smallest interval containing J_c and the Cantor set has measure zero, (3) follows. □

By part (3), we have the following identity

$$z_1 - (-z_1) = \sum_{w \in \{T_1, T_2\}^*} (-1)^\eta \Big(w(T_1(-z_1)) - w(T_2(-z_1)) \Big) \tag{3.7}$$

where η is the number of T_2's in the word w.

For a general parameter $c \in \mathbb{C} \setminus \mathcal{M}$, we will show in section 3.4 that

Theorem 3.3.2. *The series in 3.7 is absolutely convergent if and only if the Hausdorff dimension of* J_c *is strictly less than one.*

Again, analyticity of Hausdorff dimension (Corollary 2.4.9) implies that the subspace $(\mathbb{C} \setminus \mathcal{M})_{<1} \coloneqq \{c \notin \mathcal{M} \mid \dim_{\mathrm{H}} J_c < 1\}$ is open.

Hence, by analytic continuation, we obtain

Theorem 3.3.3. *For complex parameter* $c \in (\mathbb{C} \setminus \mathcal{M})_{<1}$*, let* T_1 *and* T_2 *be the two branches of* f_c^{-1} *and* z_1 *be the fixed point of* T_1*, then the following identity holds*

$$z_1 - (-z_1) = \sum_{w \in \{T_1, T_2\}^*} (-1)^\eta \Big(w(T_1(-z_1)) - w(T_2(-z_1)) \Big) \tag{3.8}$$

where η is the number of T_2's in the word w.

3.3.2 Monodromy

In subsection 5.4 we will see that the Julia set J_c can be symbolically coded as the set of strings

$$\Sigma = \left\{ \underline{i} = (i_0, i_1, i_2, \cdots) \mid i_j \in \{1,2\} \text{ for all } j \in \mathbb{N} \right\}.$$

The monodromy group respecting the dynamics can then be identified with $\mathrm{Aut}(\Sigma)$ which is $\mathbb{Z}/2\mathbb{Z}$ ([18]). Hence, the nontrivial monodromy $\phi : J_c \to J_c$ simply exchanges the labels 1 and 2 in the symbolic coding.

This map defines a new identity on J_c, since the original identity would still hold on $\phi(J_c) = J_c$ after being continued along a loop in $(\mathbb{C} \setminus \mathcal{M})_{<1}$ starting and ending at c. Thus,

$$z_2 - (-z_2) = \sum_{w \in \{T_1, T_2\}^*} (-1)^{\eta'} \Big(w(T_2(-z_2)) - w(T_1(-z_2)) \Big) \tag{3.9}$$

where η' is the number of T_1's in the word w.

3.3.3 Hausdorff dimension one locus

Outside of the Mandelbrot set, by analyticity of Hausdorff dimension, the Hausdorff dimension one locus $S(1)$ is a closed analytic set and therefore it has to intersect every ray emanating from the origin. For each fixed ray, we numerically find the points at which the convergence of the series of absolute values changes. Our numerical results as shown in Figure 2 suggest that $S(1)$ is a topological circle connecting -2 to itself with a cusp at -2. Figure 3 is a zoomed in picture near -2. This gives numerical evidence for the following:

Conjecture 3.3.4. $(\mathbb{C} \setminus \mathcal{M})_{>1}$ *is star-shaped centered at* 0.

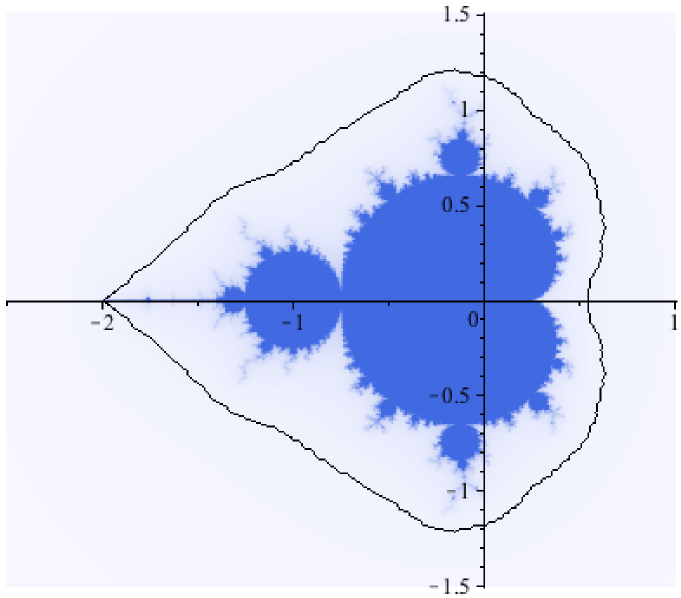

Figure 3.2: Hausdorff dimension one locus in $\mathbb{C} \setminus \mathcal{M}$

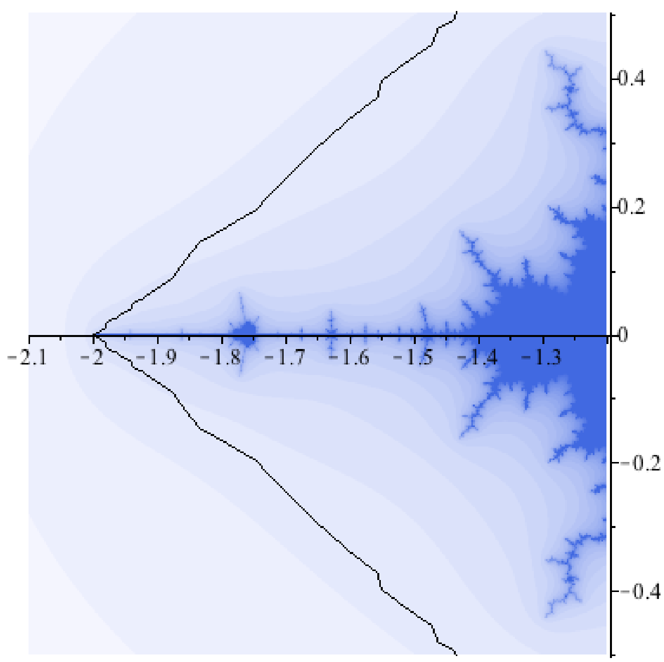

Figure 3.3: Near -2

3.4 Proof of the Convergence Theorems

3.4.1 Quadratic polynomials: proof of Theorem 3.3.2

The proof consists of the following two lemmas.

Lemma 3.4.1. *If the Hausdorff dimension of the Julia set J_c with $c \notin \mathcal{M}$ is less than one, then the series 3.7 converges absolutely.*

Proof. Since the Hausdorff dimension of $J_c < 1$, by Remark 2.4.8, $\lambda_1 < 1$. If a word w is identified with $\underline{i}$ by the symbolic coding, then image $w(I)$ of the largest gap I under w is contained in $P_{\underline{i}}$. Thus, summing up all the words up to length N, we have

$$\sum_{|w|=1}^{N} |w(I)| \leq \sum_{|\underline{i}|=1}^{N} |P_{\underline{i}}| \leq C \sum_{n=1}^{N} \lambda_1^n,$$

for some constant C, since $\sum_{|\underline{i}|=n} |P_{\underline{i}}| \asymp \lambda_1^n$.
Letting $N \to \infty$ gives the desired result. □

Lemma 3.4.2. *There exists a constant C such that for all $n \geq 1$, we have*

$$\sum_{|\underline{i}|=n} |P_{\underline{i}}| - \sum_{|\underline{i}|=n+1} |P_{\underline{i}}| \leq C \sum_{|w|=n} |w(I)|.$$

In particular, if the series is absolutely convergent, then the Hausdorff dimension of the Julia set is less than one.

Proof. Fix $n = |\underline{i}|$. Denote $P_{\underline{i}}^1$ and $P_{\underline{i}}^2$ the two disjoint sets contained in $P_{\underline{i}}$. Using estimates from Proposition 2.4.1,

$$\frac{C_1}{C_2} B_1^n |f_1'| \leq \frac{|P_{\underline{i}}^1|}{|P_{\underline{i}}|} \leq \frac{C_2}{C_1} B_2^n |f_1'|.$$

Since $|f_1'|$ is bounded, there are constants D_1 and D_2 such that

$$D_1 \leq \frac{|P_{\underline{i}}^1|}{|P_{\underline{i}}|} \leq D_2.$$

The same holds for $P_{\underline{i}}^2$. Then,

$$|P_{\underline{i}}| - |P_{\underline{i}}^1| - |P_{\underline{i}}^2| \leq |P_{\underline{i}}|(1 - 2D_1).$$

On the other hand, $\dfrac{|P_{\underline{i}}|}{|w(I)|} \asymp \dfrac{|P_j|}{|f_j(I)|}$ for $j = 1, 2$. Then there are constants D_3 and D_4 such that for any word w identified with $\underline{i}$ via the symbolic coding,

$$D_4 \leq \frac{|P_{\underline{i}}|}{|w(I)|} \leq D_3$$

Therefore, summing up all $\underline{i}$ and w of length n, we have

$$\begin{aligned}\sum_{|\underline{i}|=n} |P_{\underline{i}}| - \sum_{|\underline{i}|=n+1} |P_{\underline{i}}| &\leq (1 - 2D_1) \sum_{|\underline{i}|=n} |P_{\underline{i}}| \\ &\leq (1 - 2D_1) D_3 \sum_{|w|=n} |w(I)|.\end{aligned}$$

The right hand side of the inequality is finite whenever the series is absolute convergent. Thus $\lambda_1^n(1 - \lambda_1) < \infty$. Since n is arbitrary, $\lambda_1 < 1$. □

3.4.2 Schottky groups: proof of Theorem 3.2.6

We begin with a special case which serves as a model for the general case.

Torus with one boundary

Let $\rho_0 : F_2 \to \mathrm{PSL}(2,\mathbb{C})$ be a Fuchsian marking whose underlying surface is a torus with a geodesic boundary a_1. $\pi_1 M = F_2 = \langle a, b\rangle$ with an ordered symmetric generating set $S = \{a > b > A > B\}$, where capital letters denote inverses.

Normalize ρ_0 by conjugation so that $\rho([a,b])$ has ∞ and 0 as attracting and repelling fixed point, respectively.

Then the series at this Fuchsian marking ρ_0 is

$$\sum_{w\in\mathscr{L}} \log[\infty, 0; w\cdot\infty, w\cdot 0] = \sum_{w\in\mathscr{L}} \log\frac{w(0)}{w(\infty)}$$

where $\mathscr{L}$ was specified in Example 3.2.5.

The following three lemmas constitute the proof of Theorem 3.2.6 for this special case.

Lemma 3.4.3. *Take the principal branch of* log. *Then the series* $\sum_{w\in\mathscr{L}} \log\frac{w(0)}{w(\infty)}$ *is absolutely convergent if and only if* $\sum_{w\in\mathscr{L}} w(0) - w(\infty)$ *is.*

Proof. $\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$ for $|z| < 1$. Then, for $|z|$ small, $|\log(1+z)| \asymp |z|$. Therefore, for $|w|$ large,

$$\left|\log\frac{w(0)}{w(\infty)}\right| = \left|\log\left(1 + \frac{w(0)-w(\infty)}{w(\infty)}\right)\right| \asymp \left|\frac{w(0)-w(\infty)}{w(\infty)}\right| \asymp |w(0)-w(\infty)|$$

as the limit set is compact. □

Lemma 3.4.4. *The series*

$$\sum_{w\in\mathscr{L}} w(0) - w(\infty)$$

is absolutely convergent if the Hausdorff dimension of the limit set is less than one.

Proof. Suppose the Hausdorff dimension is less than one. Note that for $|w| = n > 3$, $w(0), w(\infty) \in P_{\underline{i}}$ where first $n-3$ letters of w and $\underline{i}$ are identified, i.e. $|w(0)-w(\infty)| \leq |P_{\underline{i}}|$.

Hence,

$$\sum_{|w|=4}^{N} |w(0) - w(\infty)| \le \sum_{|\underline{i}|=1}^{N} |P_{\underline{i}}| \le C \sum_{k=1}^{N} \lambda_1^k,$$

for some constant C as $\sum_{|\underline{i}|=n} |P_{\underline{i}}| \asymp \lambda_1^n$. The conclusion follows by letting $N \to \infty$. □

Lemma 3.4.5. *If the series*

$$\sum_{w \in \mathscr{L}} w(0) - w(\infty)$$

is absolutely convergent, then the Hausdorff dimension of the limit set is less than one.

Proof. For each n large, if w of length n is identified with $\underline{i}$, denote $P_{\underline{i}}^1$, $P_{\underline{i}}^2$ and $P_{\underline{i}}^3$ the three disjoint sets contained in $P_{\underline{i}}$. Then, exactly the same argument as in the proof of Lemma 3.4.2 shows that there exists a constant C such that for all $\underline{i}$,

$$|P_{\underline{i}}| - |P_{\underline{i}}^1| - |P_{\underline{i}}^2| - |P_{\underline{i}}^3| \le C|w(0) - w(\infty)|.$$

Summing all the words w of length n,

$$\sum_{\underline{i} \sim w} |P_{\underline{i}}| - \sum_{|\underline{i}|=n+1} |P_{\underline{i}}| \le C \sum_{|w|=n} |w(0) - w(\infty)|.$$

Recall that w does not start with abA or ba, for $|\underline{i}| = m$, $4 \cdot 3^{m-3}$ of $4 \cdot 3^{m-1}$ $P_{\underline{i}}$'s are not visited by $w(0)$ or $w(\infty)$. Hence,

$$\sum_{\underline{i} \sim w} |P_{\underline{i}}| = \frac{8}{9} \sum_{|\underline{i}|=n} |P_{\underline{i}}| = \frac{8}{9} \lambda_1^n.$$

For n large, we have

$$\infty > \sum_{|w|=n} |w(0) - w(\infty)| \ge \frac{8}{9} \sum_{|\underline{i}|=n} |P_{\underline{i}}| - \sum_{|\underline{i}|=n+1} |P_{\underline{i}}| = \lambda_1^n (\frac{8}{9} - \lambda_1).$$

Hence, $\lambda_1 < 1$. $\square$

The general case

Lemma 3.4.6. *The series*

$$\sum_{p,q=1}^{k} \sum_{w \in \mathscr{L}_{p,q}} \log[\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-] \tag{3.10}$$

is absolutely convergent if the Hausdorff dimension of the limit set is less than one.

Proof. For each $p, q = 1, \cdots, k$, the absolute convergence of the series

$$\sum_{w \in \mathscr{L}_{p,q}} \log[\alpha_p^+, \alpha_p^-; w \cdot \alpha_q^+, w \cdot \alpha_q^-]$$

is reduced to the torus case (possibly with different starting and ending conditions for RedLex w) and follows from Lemma 3.4.4. $\square$

Lemma 3.4.7. *If the series 3.10 is absolutely convergent, then the Hausdorff dimension of the limit set is less than one.*

Proof. In particular, for $p = q = 1$, the series

$$\sum_{w \in \mathscr{L}_{1,1}} \log[\alpha_1^+, \alpha_1^-; w \cdot \alpha_1^+, w \cdot \alpha_1^-]$$

is absolutely convergent. Then apply the same argument as in the proof of Lemma 3.4.5. $\square$

3.5 Basmajian identity on the modular surface

Let Σ be the modular surface with a single geodesic boundary component. Then $\pi_1(\Sigma) = \mathrm{PSL}(2, \mathbb{Z}) = \langle a, b \mid a^2 = b^3 = 1 \rangle$ and $\pi_1(\partial\Sigma) = \langle ab \rangle$. In this section, we study Basmajian identity on Σ which serves an example to a bigger future research direction where we wish to connect Basmajian-type identities to Number Theory.

Proposition 3.5.1. *If we order the generatoring set as $a > b > B = b^{-1}$, then the shortest lexicographically first double cosets representatives $\pi_1(\partial\Sigma)w\pi_1(\partial\Sigma)$ are exactly the words of the form b or $bab^{\pm 1}ab^{\pm 1}\cdots aB$.*

Proof. Since $a^2 = 1$ and $b^3 = 1$, the reduced words are of the form $ab^{\pm 1}ab^{\pm 1}\cdots$ or $b^{\pm 1}ab^{\pm 1}\cdots$. The word cannot start with ab, aB or Ba and must start with b. Similarly, the word cannot end with ba, Ba or aB and must end with aB. □

For simplicity, let $Q = aB$ and $T = ab$ so that $b = T^{-1}Q$. The regular language of orthogeodesics is

$$\mathscr{L} = \{T^{-1}Q, T^{-1}Qw'Q\}$$

where w' is any word (possibly empty) in the alphabet $\{T, Q\}$.

Now we consider a one-parameter family of representations of $\pi_1(\Sigma) = \mathrm{PSL}(2,\mathbb{Z})$ into $\mathrm{PSL}(2,\mathbb{C})$, sending the generators

$$a = \begin{pmatrix} 1 & -x \\ 1 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} t^2 & -x \\ t^2 & -1 \end{pmatrix},$$

where $x = t^2 - 1 + t^{-2}$. By abuse of notation, we identify elements in $\pi_1(\Sigma)$ with their images. Then the boundary element is mapped to

$$ab = \begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is straightforward to check that S^2 and $(ST)^3$ act projectively as the identity element on $\mathbb{H}^2$.

Remark 3.5.2. Note that $Q = aB$. Projectively,

$$Q = aB = \begin{pmatrix} t^4 & -xt^2 \\ t^2 & -1 \end{pmatrix} = \begin{pmatrix} t^4 & -t^4 + t^2 - 1 \\ t^2 & -1 \end{pmatrix}$$

Basmajian's identity states that

$$\text{length}(\partial\Sigma) = \log t^2 = \sum_{w \in \mathscr{L}} \log \frac{w(0)}{w(\infty)} = \log \frac{t^4 - t^2 + 1}{t^2} + \sum_{w'} \log \frac{Qw'Q(0)}{Qw'Q(\infty)}$$

Note that w' is a word (possibly empty) in $\{T, Q\}$.

Let LHS= $\log t^2 - \log \dfrac{t^4 - t^2 + 1}{t^2}$ and RHS $= \displaystyle\sum_{w'} \log \frac{Qw'Q(0)}{Qw'Q(\infty)}$. We consider series expansions of both sides at $t = \infty$.

$$\begin{aligned} LHS &= \log \frac{t^4}{t^4 - t^2 + 1} \\ &= \frac{1}{t^2} - \frac{1}{2t^4} - \frac{2}{3t^6} - \frac{1}{4t^8} + \frac{1}{5t^{10}} + \frac{1}{3t^{12}} + \frac{1}{7t^{14}} + O\left(\frac{1}{t^{16}}\right) \end{aligned}$$

We observe the following patterns in the series expansion of the LHS.

1. If we let $z = 1/t^2$, then $\dfrac{d}{dz} LHS(z)$ will have integral coefficients.

2. Note the patterns of numerators, denominators and signs. The numerators are periodic in $1, 1, 2$ and the denominators are all the natural numbers. The signs are alternating with 3 negatives and 3 positives except the first term.

Now we expand the first several terms in the RHS series.

$$QQ = \frac{1}{t^2} - \frac{5}{2t^4} + \frac{4}{3t^6} + \frac{3}{4t^8} + \frac{1}{5t^{10}} - \frac{4}{3t^{12}} + \frac{1}{7t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QTQ = \frac{1}{t^4} - \frac{3}{t^6} + \frac{5}{2t^8} + \frac{0}{5t^{10}} + \frac{5}{6t^{12}} - \frac{3}{7t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QQQ = \frac{1}{t^4} - \frac{3}{t^6} + \frac{5}{2t^8} + \frac{0}{5t^{10}} + \frac{5}{6t^{12}} - \frac{3}{7t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QTTQ = \frac{1}{t^6} - \frac{3}{t^8} + \frac{2}{t^{10}} + \frac{3}{2t^{12}} - \frac{1}{t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QTQQ = \frac{1}{t^6} - \frac{4}{t^8} + \frac{6}{t^{10}} - \frac{9}{2t^{12}} + \frac{5}{t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QQTQ = \frac{1}{t^6} - \frac{4}{t^8} + \frac{6}{t^{10}} - \frac{9}{2t^{12}} + \frac{5}{t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

$$QQQQ = \frac{1}{t^6} - \frac{3}{t^8} + \frac{2}{t^{10}} + \frac{3}{2t^{12}} - \frac{1}{t^{14}} + O\left(\frac{1}{t^{16}}\right)$$

We observe the following:

1. $QvQ = QuQ$ where the word u is obtained from the word v by exchanging Q and T.

2. By comparing coefficients, we obtain a countable collection of identities:

$$1 = 1$$

$$-1/2 = -5/2 + 1 + 1$$

$$-2/3 = 4/3 - 3 - 3 + 1 + 1 + 1 + 1$$

$$\dots$$

CHAPTER 4

DISPLACEMENT OF GENERATORS OF FREE FUCHSIAN GROUPS

4.1 Conformal densities and paradoxical decompositions

In this section, we give an expository account of conformal densities and Culler-Shalen's paradoxical decompositions of the Patterson-Sullivan measure. Interested readers are refered to [11] for more details.

4.1.1 Conformal densities

Recall that the Poisson kernel $P : \mathbb{H}^n \times \mathbb{H}^n \times S_\infty^{n-1} \to \mathbb{R}$ is given by

$$P(z, z', \zeta) := (\cosh d(z, z') - \sinh d(z, z') \cos \angle z' z \zeta)^{-1}$$

where d denotes the hyperbolic distance.

Let $n \geq 2$ be an integer and $D \in [0, n-1]$. A *D-conformal density for $\mathbb{H}^n$* is a family $\mathscr{M} = (\mu_z)_{z \in \mathbb{H}^n}$ of finite Borel measures on S_∞^{n-1} such that $d\mu_{z'} = P(z, z', \cdot)^D d\mu_z$ for any $z, z' \in \mathbb{H}^n$. For example, the *area density* $\mathscr{A} = (A_z)_{z \in \mathbb{H}^n}$ is an $(n-1)$-conformal density, where A_z is the normalized area measure on S_∞^{n-1} induced by the round metric centered at z.

Let Γ be a group of isometries of $\mathbb{H}^n$. A conformal density $\mathscr{M} = (\mu_z)_{z \in \mathbb{H}^n}$ is Γ-*invariant* if $\gamma_\infty^* \mu_{\gamma z} = \mu_z$, for all $z \in \mathbb{H}^n$ and $\gamma \in \Gamma$. Clearly, the area density is Γ-invariant.

Given a D-conformal density $\mathscr{M} = (\mu_z)_{z \in \mathbb{H}^n}$, we define a nonnegative function $u_{\mathscr{M}} : \mathbb{H}^n \to [0, 1]$ by $u_{\mathscr{M}}(z) = \mu_z(S_\infty^{n-1})$. In case of the area density $\mathscr{A}$, $u_{\mathscr{A}} \equiv 1$. It can be shown that the function $u_{\mathscr{M}}(z)$ is smooth and satisfies the equation

$$\Delta u_{\mathscr{M}} = -D(n-D-1)u_{\mathscr{M}}$$

where Δ is the (hyperbolic) Laplacian on $\mathbb{H}^n$. In particular, $u_{\mathscr{M}}$ is superharmonic. Moreover, if $\mathscr{M}$ is Γ-invariant for some group Γ of isometries of $\mathbb{H}^n$, then $u_{\mathscr{M}}$ is also Γ-invariant.

The next proposition asserts the uniqueness of a Γ-invariant conformal density for $\mathbb{H}^n$ when all Γ-invariant positive superharmonic functions are constant.

Proposition 4.1.1 ([11], Proposition 3.9)**.** *Let Γ be a non-elementary discrete group of isometries of $\mathbb{H}^n$. Suppose every Γ-invariant positive superharmonic function on $\mathbb{H}^n$ is constant. Then any Γ-invariant conformal density for $\mathbb{H}^n$ is a constant multiple of the area density.*

For the convenience of the reader, we sketch the proof here.

Proof. A Γ-invariant conformal density gives a Γ-invariant positive superharmonic function on $\mathbb{H}^n$ by integrating the Poisson kernel against the measure. Since $u_{\mathscr{M}}(z) \equiv C$, which is a constant, $u_{\mathscr{M}} = u_{C\mathscr{A}}$, where $\mathscr{A}$ is the area density. Moreover, it can be shown that any $(n-1)$-conformal density is determined by the associated harmonic function. Hence, $\mathscr{M} = C\mathscr{A}$. □

4.1.2 Culler-Shalen's paradoxical decompositions

A free group on k generators can be decomposed into $2k+1$ disjoint sets depending on the first letter of the group element. We show in this subsection (Proposition 4.1.3) that there exists a decomposition of the Patterson-Sullivan measure which respects this decomposition of the group. The proof is based on a generalized Patterson's construction of conformal densities for uniformly discrete subsets of $\mathbb{H}^n$.

A subset W of $\mathbb{H}^n$ is *uniformly discrete* if there exists $\delta > 0$ such that $d(z,w) > \delta$ for all $z, w \in W$. In particular, any orbit of a discrete group of isometries of $\mathbb{H}^n$ is uniformly discrete.

Lemma 4.1.2 ([11], Proposition 4.2)**.** *Let W be an infinite uniformly discrete subset of $\mathbb{H}^n$. Let $\mathcal{B}$ be a countable collection of subsets of W which contains W. Then there exists a $D \in [0, n-1]$ and a family $(\mathscr{M}_V)_{V \in \mathcal{B}}$ of D-conformal densities satisfying the following conditions*

1. $\mathscr{M}_W \neq 0$.
2. *For any finite family $(V_i)_{i=1}^m$ of disjoint sets in $\mathcal{B}$ such that $V = \amalg_{i=1}^m V_i \in \mathcal{B}$, we have $\mathscr{M}_V = \sum_{i=1}^m \mathscr{M}_{V_i}$.*
3. *For any $V \in \mathcal{B}$ and any isometry γ of $\mathbb{H}^n$ with $\gamma V \in \mathcal{B}$, we have $\gamma_\infty^*(\mathscr{M}_{\gamma V}) = \mathscr{M}_V$.*
4. *For any $V \in \mathcal{B}$, supp$(\mathscr{M}_V)$ is contained in the limit set of V. In particular, for any finite set $V \in \mathcal{B}$, we have $\mathscr{M}_V = 0$.*

Note that by taking $\mathcal{B} = \{W = \text{any orbit of } \Gamma\}$, we obtain the Patterson-Sullivan density which is Γ-invariant and supported on the limit set of Γ.

Proposition 4.1.3. *Let Γ be a free Fuchsian group with a symmetric generating set $\Psi = \{g_1^\pm \cdots g_k^\pm\}$. Let $z_0 \in \mathbb{H}^2$. Then there exist a number $D \in [0,1]$, a Γ-invariant D-conformal density $\mathscr{M} = (\mu_{z_0})$ for $\mathbb{H}^2$ and a family $(\nu_\psi)_{\psi \in \Psi}$ of Borel measures on S^1_∞ such that*

1. $\mu_{z_0}(S^1_\infty) = 1$,
2. $\mu_{z_0} = \sum_{\psi \in \Psi} \nu_\psi$, *and*
3. *for each $\psi \in \Psi$ we have*

$$\int_{S^1_\infty} P(z_0, \psi^{-1} z_0, \zeta)^D d\nu_{\psi^{-1}}(\zeta) = 1 - \int_{S^1_\infty} d\nu_\psi(\zeta).$$

Proof. Let $W = \Gamma z_0$ be an orbit of Γ. Then

$$W = \{z_0\} \amalg \left(\amalg_{\psi \in \Psi} V_\psi\right)$$

where $V_\psi = \{\gamma z_0 : \gamma \in \Gamma \text{ starts with } \psi\}$. Let $\mathcal{B}$ be a countable collection of subsets of W which has the form $\{z_0\} \amalg (\amalg_{\psi \in \Psi'} V_\psi)$ or $\amalg_{\psi \in \Psi'} V_\psi$ where $\Psi' \subset \Psi$.

Apply Lemma 4.1.2 and set $\mathscr{M} = \mathscr{M}_W$, $\mu_{z_0} = \mu_{W,z_0} \in \mathscr{M}$ and $\nu_\psi = \mu_{V_\psi, z_0} \in \mathscr{M}_{V_\psi}$. Then

$$\mu_{z_0} = \mu_{z_0,z_0} + \sum_{\psi \in \Psi} \mu_{V_\psi, z_0} = 0 + \sum_{\psi \in \Psi} \nu_\psi.$$

Since $\mathscr{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathscr{M}_{W - V_\psi}) = \psi_\infty^*(\mathscr{M} - \mathscr{M}_{V_\psi})$,

$$\mu_{V_{\psi^{-1}}, \psi(z_0)} = \psi_\infty^*(\mu_{z_0} - \nu_\psi).$$

On the other hand, $d\mu_{V_{\psi^{-1}}, \psi(z_0)} = P(z_0, \psi^{-1} z_0, \cdot)^D d\mu_{V_{\psi^{-1}}}$ since $\mathscr{M}_{V_{\psi^{-1}}}$ is a D-conformal density. Hence,

$$\int_{S^1_\infty} P(z_0, \psi^{-1} z_0, \cdot)^D d\nu_{\psi^{-1}} = \int_{S^1_\infty} d(\mu_{z_0} - \nu_\psi) = 1 - \int_{S^1_\infty} d\nu_\psi.$$

□

4.2 The finite co-area case

This section is devoted to proving the main theorem for free Fuchsian groups of rank $k \geq 2$ with finite-area quotient surfaces. The foundation of the proof is the Culler-Shalen's paradoxical decompositions of the Patterson-Sullivan measure developed in the previous section. For each generator, these decomposed measures give rise to a good lower bound for the displacement (Lemma 4.2.2), which allows us to deduce the inequality satisfied by displacement of all the generators.

Theorem 4.2.1. *Let $k \geq 2$ be an integer and let Φ be a finite co-area free Fuchsian group generated by $\{g_1 \cdots g_k\}$. Let z be any point in $\mathbb{H}^2$ and denote by $d_i = d(z, g_i z)$ the hyperbolic distance between z and $g_i z$. Then we have*

$$\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_i}{2}\right)\right) \leq \frac{\pi}{2}. \tag{4.1}$$

In particular, there exists an $i \in \{1, \cdots, k\}$ such that

$$d_i \geq \log\left(\frac{1+\cos\frac{\pi}{2k}}{1-\cos\frac{\pi}{2k}}\right).$$

Proof. According to Proposition 4.1.3, there exist a number $D \in [0,1]$ and Borel measures μ_z, ν_{g_i} and $\nu_{g_i^{-1}}$ satisfying conditions (1)–(3) in the proposition. Also, since Σ_Φ has finite area, it does not admit any non-constant positive superharmonic functions. Hence, by Proposition 4.1.1, the D-conformal density is a constant multiple of the area density which is a 1-conformal density, i.e. $D = 1$.

Let $\alpha_i = \nu_{g_i}(S^1_\infty)$ and $\beta_i = \nu_{g_i^{-1}}(S^1_\infty)$, for $i = 1, \cdots, k$. Without loss of generality, we assume that $\alpha_i \leq \beta_i$ so that $\alpha_i \in (0, 1/2)$ and $\beta_i \in (0, 1)$.

For now, let us assume the following lemma.

Lemma 4.2.2. *Let a and b be numbers such that $0 \leq a \leq 1/2$ and $0 \leq b \leq 1$, let γ be an isometry of $\mathbb{H}^2$ and let z be a point in $\mathbb{H}^2$. Suppose that ν is a measure on S^1_∞ such that*

1. $\nu \leq A_z$
2. $\nu(S^1_\infty) \leq a$, *and*
3. $\displaystyle\int_{S^1_\infty} P(z, \gamma^{-1}z, \cdot)\, d\nu \geq b.$

Then

$$d(z, \gamma z) \geq \log \left(\frac{\tan(b\pi/2)}{\tan(a\pi/2)} \right).$$

Applying the lemma with $a = \alpha_i$ and $b = 1 - \beta_i$, we obtain

$$\begin{aligned} e^{d_i} \geq \frac{\tan(\pi(1-\beta_i)/2)}{\tan(\pi\alpha_i/2)} &= \frac{\cot(\pi\beta_i/2)}{\tan(\pi\alpha_i/2)} \\ &= \frac{\cos(\pi\beta_i/2)\cos(\pi\alpha_i/2)}{\sin(\pi\beta_i/2)\sin(\pi\alpha_i/2)} \\ &= \frac{\cos(\pi p_i)}{\sin(\pi\beta_i/2)\sin(\pi\alpha_i/2)} + 1 \text{ where } p_i = \frac{\alpha_i + \beta_i}{2} \in (0, 1/2). \end{aligned}$$

Note that $\sin(x)\sin(y) \leq \left(\sin\left(\frac{x+y}{2}\right)\right)^2 = \frac{1-\cos(x+y)}{2}$.

Hence,

$$\begin{aligned} e^{d_i} + 1 &\geq \frac{\cos(\pi p_i)}{\sin(\pi\beta_i/2)\sin(\pi\alpha_i/2)} + 2 \\ &\geq \frac{\cos(\pi p_i)}{\sin^2(\pi p_i/2)} + 2 \\ &= \frac{2\cos(\pi p_i)}{1-\cos(\pi p_i)} + 2 \\ &= \frac{2}{1-\cos(\pi p_i)} \end{aligned}$$

Solving for p_i, we obtain

$$\arccos\left(\frac{e^{d_i}-1}{e^{d_i}+1}\right) \leq \pi p_i.$$

Summing over i, we get

$$\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_i}{2}\right)\right) \leq \frac{\pi}{2}.$$

The last part of the theorem is easy to see, for if $d_i < \log\left(\frac{1+\cos\frac{\pi}{2k}}{1-\cos\frac{\pi}{2k}}\right)$ for all $i = 1, \cdots, k$, then $\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_i}{2}\right)\right) > \frac{\pi}{2}$, which is a contradiction. □

Now we turn to the proof of Lemma 4.2.2. The following elementary lemma is needed.

Lemma 4.2.3 ([11], Lemma 5.4)**.** *Let $(X, \mathscr{B})$ be a measure space and let μ and μ_0 be two finite measures such that $0 \leq \mu_0 \leq \mu$. Let C be a Borel set such that $\mu(C) \geq \mu_0(X)$. Let f be a measurable, nonnegative real-valued function on X such that $\inf f(C) \geq \sup f(X - C)$. Then $\int_X f d\mu_0 \leq \int_C f d\mu$.*

Proof of Lemma 4.2.2. Let $h = \text{dist}(z, \gamma z)$ and set $s = \sinh(h)$ and $c = \cosh(h)$. Identify $\overline{\mathbb{H}^2}$ with the unit disk so that z gets mapped to 0 and γz gets mapped to the positive y-axis. The Poisson kernel is given by

$$P(\phi) = P(z, \gamma^{-1} z, \zeta) = (c - s\cos\phi)^{-1}$$

where ϕ is the angle between the vertical axis and the ray from 0 to $\zeta \in S^1_\infty$.

Set $A = A_z$. Since S^1_∞ has the round metric centered at z, the measure $dA = \frac{1}{\pi} d\phi$.

Let $C \subset S^1_\infty$ be the arc defined by the inequality $\phi < \pi a$. Then we have

$$A(C) = \frac{1}{\pi} \int_0^{\pi a} d\phi = a.$$

Note that P is positive and monotone decreasing for $0 \leq \phi \leq \pi$, we have $\inf P(C) \geq \sup P(S^1_\infty - C)$. Then by Lemma 4.2.3,

$$\begin{aligned} b \leq \int_{S^1_\infty} P d\nu \leq \int_C P dA &= \frac{1}{\pi} \int_0^{\pi a} \frac{1}{c - s\cos(\phi)} d\phi \\ &= \frac{2}{\pi} \arctan\left(e^h \tan\left(\frac{\pi a}{2}\right)\right). \end{aligned}$$

Hence,

$$e^h \geq \frac{\tan(\pi b/2)}{\tan(\pi a/2)}.$$

□

4.3 The general case

Our goal for this section is to generalize Theorem 4.2.1 to all free Fuchsian groups. In particular, we prove the Main Theorem stated in Introduction.

Theorem 4.3.1. *Let $k \geq 2$ be an integer and let Φ be a free Fuchsian group generated by $\{g_1 \cdots g_k\}$. Let z be any point in $\mathbb{H}^2$ and denote by $d_i = d(z, g_i z)$ the hyperbolic distance between z and $g_i z$. Then we have*

$$\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_i}{2}\right)\right) \leq \frac{\pi}{2}. \tag{4.2}$$

In particular, there exists an $i \in \{1, \cdots, k\}$ such that

$$d_i \geq \log\left(\frac{1+\cos\frac{\pi}{2k}}{1-\cos\frac{\pi}{2k}}\right).$$

When $k = 2$, the upper bound is realized by $\Phi = \Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ and $z = \sqrt{-1}$.

Remark 4.3.2. Theorem 4.3.1 is strictly stronger than the $\log(2k-1)$ Theorem applied to Fuchsian groups. To see this, let $y_i = \arccos\left(\tanh\left(d_i/2\right)\right), i = 1, \cdots, k$. Then $d_i = \log\left(\frac{1+\cos y_i}{1-\cos y_i}\right)$ and

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2} \text{ if and only if } \sum_{i=1}^{k} (1-\cos y_i) \leq 1.$$

Now assume Theorem 4.3.1, i.e. $\sum_{i=1}^{k} y_i \leq \frac{\pi}{2}$. Then $\sum_{i=1}^{k}(1-\cos y_i)$ attains its maximum value 1 when one of the y_i equals $\pi/2$ and the rest all equal to 0. For another direction, suppose $\sum_{i=1}^{k}(1-\cos y_i) \leq 1$, then letting $y_1 = \cdots = y_k = \arccos(1-1/k)$ gives

$\sum_{i=1}^{k} y_i = k \arccos(1 - 1/k) > \pi/2$ for all $k \geq 2$.

To prove the theorem, we need the following crucial lemma.

Lemma 4.3.3. *Any Riemann surface of finite topological type with negative Euler characteristic admits a holomorphic embedding into a Riemann surface of finite hyperbolic area. Moreover, this embedding is a homotopy equivalence.*

A classical version of this lemma was given by Bers in [5] (see also [15]). Section 4 of [14] contains a more modern discussion. For the convenience of the reader, we give a short proof here.

Proof. Let S be a Riemann surface of finite topological type with negative Euler characteristic. We first show that a funnel end $\mathcal{E}$ of S is conformal to an annulus. This is essentially the infinite Nielsen extension due to Bers [5]. To see this, consider the action of $\pi_1(S)$ on the upper-half space $\mathbb{H}^2$. Up to conjugation, we can assume that the element $\gamma \in \pi_1(S)$ representing the end fixes the positive imaginary axis. Then $\gamma(z) = az$ for some $a > 1$ and $\mathcal{E} = R/\langle\gamma\rangle$, where $R = \{z \in \mathbb{H}^2 \mid Im(z) > 0 \text{ and } Re(z) \geq 0\}$. Consider the map $\phi(z) = e^{-2\pi i \log z/\log a}$. ϕ is holomorphic in R and $\phi(R)$ is the annulus $A = \{1 < |\phi| < e^{\pi^2/\log a}\}$. Since ϕ is γ invariant, $\mathcal{E}$ is confomal to A. Therefore, S is conformal to the surface S' obtained by attaching a finite Euclidean cylinder $S^1 \times (0, t)$ to each boundary geodesic of its convex core $C(S)$.

Consider the Riemann surface S_∞ obtained by attaching a half infinite Euclidean cylinder $S^1 \times \mathbb{R}^+$ to each boundary geodesic of $C(S)$. We claim that S_∞ has finite area. To see this, first note that if γ is a geodesic in the homotopy class of a finite cylinder A, then its hyperbolic length $\ell(\gamma)$ satisfies $\ell(\gamma) < \pi/(\text{modulus of } A)$. Since a half-infinite cylinder contains cylinders of arbitrarily large moduli, the hyperbolic length of the geodesic γ, if it exists, would satisfy $\ell(\gamma) < \varepsilon$ for any $\varepsilon > 0$. Hence, there are no geodesics in the homotopy classes of glued cylinders, i.e. the ends of S_∞ are cusps.

Clearly, S' is embedded in S_∞ and the embedding is a homotopy equivalence. □

Proof of Theorem 4.1. Suppose the quotient surface $\Sigma = \mathbb{H}^2/\Phi$ has infinite area. By Lemma 4.3.3, there is a holomorphic embedding of Σ into a finite area surface Σ_∞. Since holomorphic maps are distance non-increasing with respect to the hyperbolic metric, we have $d_\Sigma(z, g_i(z)) \geq d_{\Sigma_\infty}(z, g_i(z))$, where $z \in \mathbb{H}^2$. Since $\arccos(\tanh(x/2))$ is strictly decreasing, we have

$$\begin{aligned}\sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_\Sigma(z, g_i(z))}{2}\right)\right) &\leq \sum_{i=1}^{k} \arccos\left(\tanh\left(\frac{d_{\Sigma_\infty}(z, g_i(z))}{2}\right)\right) \\ &\leq \frac{\pi}{2}.\end{aligned}$$

The second inequality follows as Σ_∞ has finite area.

Straightforward calculation shows that when $k = 2$ the upper bound is realized by the given group and the point. □

4.4 Quantitative geometry of hyperbolic surfaces

In this section, we discuss applications of Theorem 4.3.1 to quantitative geometry of hyperbolic surfaces.

4.4.1 The two-dimensional Margulis constant

Let M be a closed orientable hyperbolic n-manifold. A positive real number ε is called a *Margulis number* for M if for any $z \in \mathbb{H}^n$ and $\alpha, \beta \in \pi_1 M$, $\max\{d(z, \alpha(z)), d(z, \beta(z))\} < \varepsilon$ implies that α and β commute. The Margulis Lemma ([3], Chapter D) states that for every $n \geq 2$ there is a positive constant which is a Margulis number for every closed orientable hyperbolic n-manifold. The largest such constant ε_n is called the *Margulis constant* for closed orientable hyperbolic n-manifold.

An immediate corollary of Theorem 4.3.1 recovers the following result due to Yamada.

Corollary 4.4.1 (Yamada, [24]). *The two-dimensional Margulis constant* ε_2 *equals* $\log(3+2\sqrt{2})$.

Proof. Let Γ be a discrete group of orientation-preserving isometries of $\mathbb{H}^2$. Then ε is a Margulis number for $\mathbb{H}^2/\Gamma$ if and only if for any non-commuting pair $\alpha, \beta \in \Gamma$ and every point $z \in \mathbb{H}^2$, at least one of α and β moves the point z at least ε distance away from itself. Moreover, the group generated by α and β is free of rank 2. Theorem 4.3.1 (with $k = 2$) gives the desired Margulis number. Furthermore, this number is realized and therefore it is the Margulis constant. $\square$

4.4.2 Lengths of a pair of closed loops

In this subsection, we discuss the consequences of Theorem 4.3.1 on lengths of a pair of based loops on a hyperbolic surface.

Corollary 4.4.2. *If* α_1 *and* α_2 *are two loops on a hyperbolic surface* Σ *based at a point* p *such that they define non-commuting elements in the fundamental group* $\pi_1(\Sigma, p)$*, then*

$$\sinh(\ell_1/2) \cdot \sinh(\ell_2/2) \geq 1$$

where ℓ_i *denotes the hyperbolic length of* $\alpha_i, i = 1, 2$.

Proof. Let z be a lift of the base point p in $\mathbb{H}^2$. Since $[\alpha_1], [\alpha_2] \in \pi_1\Sigma$ generate a free subgroup of rank 2 and $d(z, [\alpha_i]z) = \ell_i$, by Theorem 4.3.1, $\sum_{i=1}^{2} \arccos(\tanh(\ell_i/2)) \leq \pi/2$.

Applying cosine function on both sides of the inequality, we get

$$\tanh(\ell_1/2) \cdot \tanh(\ell_2/2) - \frac{1}{\cosh(\ell_1/2) \cdot \cosh(\ell_2/2)} \geq 0$$

and the desired inequality follows. $\square$

The above corollary improves a result of Buser's [8] which gives the same inequality but requires α_1 and α_2 to be two intersecting simple closed geodesics.

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