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THE HITCHIN FIBRATION FOR SYMMETRIC PAIRS

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TABLE OF CONTENTS

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ABSTRACT

This thesis consists of two chapters: The first and primary component is dedicated to the Hitchin morphism for symmetric spaces, which is joint work with B. Morrissey. We introduce and describe the "regular quotient" and explain some basic consequences for Higgs bundles. We include an invariant theoretic approach to spectral covers in this setting for the space $\mathrm{GL}_{2n} / \mathrm{GL}_n \times \mathrm{GL}_n$. We also include some work towards an enhanced Grothendieck-Springer style correspondence in for quasi-split pairs, which classifies certain parabolics of H rather than Borels of G. We study the component groups of such covers, including their importance in describing regular centralizers.

In the second chapter, we recount some joint work with B.C. Ngô on companion matrices for clssical groups and G_2. We use the companion matrix construction for GL_n to build canonical sections of the Chevalley map $[\mathfrak{g}/G] \to \mathfrak{g}/\!\!/G$ for classical groups G as well as the group G_2. To do so, we construct canonical tensors on the associated spectral covers. As an application, we make explicit lattice descriptions of affine Springer fibers and Hitchin fibers for classical groups and G_2.

CHAPTER 1

THE HITCHIN FIBRATION FOR SYMMETRIC SPACES

1.1 Introduction

Let G be a reductive group over an algebraically closed field k. Fix a smooth, projective curve C over k and a line bundle D on C whose degree is greater than twice the genus. To such data, one can associate the moduli stack of G Higgs bundles

$$\mathcal{M}_G = \mathrm{Maps}(C, \mathfrak{g}_D/G)$$

where $\mathfrak{g}_D = \mathfrak{g} \otimes D$ is the twisted bundle of Lie algebras and $\mathfrak{g}_D/G$ is the stack quotient. When D is the canonical bundle of C and $k = \mathbb{C}$, this space was introduced by Corlette and Simpson to classify reductive representations of the fundamental group $\pi_1(C)$ in $G(\mathbb{C})$ [9, 46]. In [20], Hitchin introduced a beautiful fibration

$$h_G \colon \mathcal{M}_G \to \mathcal{A}_G$$

where $\mathcal{A}_G$ is an affine space of half the dimension of $\mathcal{M}_G$, which is a global analogue of the characteristic polynomial map, and whose generic fiber is an abelian variety. Since its introduction, the Hitchin fibration has found applications across a wide range of mathematics. Among its remarkable properties: Over $\mathbb{C}$, it is a Lagrangian torus fibration with a known mirror dual (in the sense of SYZ mirror symmetry); it exhibits aesthetic duality phenomina which relate to Geometric Langlands; and it provides a geometric framework for the theory of endoscopy leading to Ngô's proof of the Fundamental Lemma [35].

In this paper, we study a generalized Hitchin fibration associated to a symmetric pair. In particular, we let

$$\theta \colon G \to G$$

be an algebraic involution of G, and let $K = (G^\theta)^\circ$ be the connected component of the fixed points of θ on G. Fix a smooth, closed subgroup H of G such that

$$K \subset H \subset N_G(K).$$

The involution θ induces a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $(+1)$ and (-1) eigenspaces, respectively, of θ on $\mathfrak{g}$. To the G-variety $X = G/H$ we associate a moduli space of relative Higgs bundles

$$\mathcal{M}_X = \mathrm{Maps}(C, \mathfrak{p}_D/H)$$

For $k = \mathbb{C}$ and D the canonical bundle, Gracía-Prada, Gothen, and Riera introduced these relative Higgs bundles to study reductive representations of $\pi_1(C)$ in real forms of G [15]. One still has a relative Hitchin fibration

$$h_X \colon \mathcal{M}_X \to \mathcal{A}_X$$

with $\mathcal{A}_X = \mathrm{Maps}(X, (\mathfrak{p}/\!\!/H)_D)$ the affine space classifying maps to the twisted GIT quotient $\mathfrak{p}/\!\!/H := \mathrm{Spec}\, k[\mathfrak{p}]^H$. The geometry of fibers of these Hitchin systems were studied extensively using spectral covers in [41, 43, 42, 23] among others, and a theory of cameral covers was initiated in [36, 16].

The generic fibers of the Hitchin fibration for symmetric spaces involve two novel geometric behaviors: First, there may be exceptional components. For example, in Schapostnik's thesis work [41] for the symmetric space $X = \mathrm{GL}_{2n}/(\mathrm{GL}_n \times \mathrm{GL}_n)$, fibers are generically identified with a disjoint union of 2^ℓ copies of the Picard stack classifying line bundles on a spectral curve for an explicit ℓ. Worse, the connected components may still fail to be abelian varieties. For example, Hitchin and Schapostnik give a symmetric pair for which the fibers of h_X over a generic a are identified with the space of rank two vector bundles on a spectral curve [23].

In this work, we describe in greater detail the geometry of the Hitchin fibration for symmetric spaces, providing invariant theoretic explanations for the anomalous behavior of the fibers and giving descriptions of the fibration h_X in families. Our results will restrict to the regular locus. That is, we let $\mathfrak{p}^{\mathrm{r}eg}$ denote the set of $x \in \mathfrak{p}$ whose centralizer in H is of minimal dimension. We will restrict to the sub-locus $\mathcal{M}_X^{\mathrm{r}eg}$ of $\mathcal{M}_X$ consisting of maps from C valued in the open substack $\mathfrak{p}_D^{\mathrm{r}eg}/H$. We will make use of new tools provided by the work [33].[1] Following *loc. cit.*, the basic structure theorem can be expressed as follows.

Theorem 1.1.1. *There is a factorization of h_X into*

$$\mathcal{M}_X \xrightarrow{h^{/\!/}} \mathcal{A}_X^{/\!/} \xrightarrow{\phi} \mathcal{A}_X$$

such that, over a dense open locus $\mathcal{A}_X^{\diamondsuit}$ of $\mathcal{A}_X$, the map ϕ is étale and the space $\mathcal{M}_X$ is isomorphic to a space of torsors for a prescribed band over $\mathcal{A}_X^{/\!/}$.

The above theorem is a reflection of some new constructions in invariant theory. Namely, consider the Chevalley style morphism

$$\chi\colon \mathfrak{p}^{\mathrm{r}eg}/H \to \mathfrak{p}/\!\!/H$$

from the stack quotient of H acting on $\mathfrak{p}$ to the corresponding GIT quotient. This map fails to be a gerbe; indeed, there may be multiple regular, nilpotent H orbits in $\mathfrak{p}$. In [16], they suggest studying an intermediate quotient obtained by rigidifying the stack $\mathfrak{p}^{\mathrm{r}eg}/H$ by

1. As [33] is not yet public, care will be taken to ensure this thesis is self-contained. The reader will note, therefore, that Subsection 1.3.1 exposits work on which the author was not a collaborator.

inertia. This intermediate quotient, which we will denote by $\mathfrak{p}^{reg} /\!\!/\!\!/ H$ and call the *regular quotient*, is a coarse moduli space for regular H-orbits in $\mathfrak{p}$. The map χ factors

$$\mathfrak{p}^{reg}/H \to \mathfrak{p}^{reg} /\!\!/\!\!/ H \to \mathfrak{p}/\!\!/H$$

where the first map is a gerbe and the second is a non-separated cover. Motivted by the study of generalized Hitchin systems, [33] gives a construction of such quotients in large generality, but it should be noted that this construction in general returns a DM stack.

Our main result is a completely explicit description of the regular quotient.

Theorem 1.1.2. *Let (G, θ, H) be a symmetric pair with G simple.*

1. *If (G, θ, H) is not the split form on $G = \mathrm{SO}_{4n}$, then there is a Zariski closed subset $Z \subset \mathfrak{c}$ such that*
$$\mathfrak{p}^{\mathrm{reg}} /\!\!/\!\!/ H \simeq \mathfrak{c} \coprod_{Z} \mathfrak{c}$$
consists of two copies of $\mathfrak{c}$ glued along Z. Moreover, this closed subset Z can be computed explicitly via an inductive procedure.

2. *The split form on $G = \mathrm{SO}_{4n}$ can be explicitly described as well, but involves gluing patterns for four sheets. See Example 1.3.48.*

3. *The case of general G can be reduced to the simple case via a compatibility with z-extensions.*

The gerbe

$$\mathfrak{p}^{\mathrm{reg}}/H \to \mathfrak{p}^{\mathrm{reg}} /\!\!/\!\!/ H$$

is banded by the group scheme $I^{\mathrm{reg}} \to \mathfrak{p}^{\mathrm{reg}}$ whose fiber over $x \subset \mathfrak{p}^{\mathrm{reg}}$ is the centralizer of x in H

$$I_x^{\mathrm{reg}} = \{h \in H \colon \mathrm{Ad}(h) \cdot x = x\}.$$

Any question of the behaviour of the map $h^{/\!/}$ must, therefore, be inextricably linked to the question of describing the regular centralizer group scheme I^{reg}. For technical reasons, we restrict our attention in this work to quasi-split symmetric pairs, which is to say, those for which I^{reg} is abelian. For quasisplit symmetric pairs, I^{reg} descends to a smooth, commutative group scheme J over the GIT quotient, which we will denote by $\mathfrak{c} := \mathfrak{p}/\!/H$.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. The abelian algebra $\mathfrak{a}$ comes equipped with a natural root system (the "restricted root system" of the symmetric space), and we denote the corresponding Weyl group by $W_{\mathfrak{a}}$. Let $C = C_H(\mathfrak{a})$. Following the perspective taken in [10] and [35], we seek a flat cover

$$\widetilde{\mathfrak{c}} \to \mathfrak{c}$$

with a group $W_{\widetilde{\mathfrak{c}}}$ acting on $\widetilde{\mathfrak{c}}$ and C over $\mathfrak{c}$ such that J is an open subgroup scheme of the Weil restriction

$$J^1_{\widetilde{\mathfrak{c}}} := \mathrm{Res}^{\widetilde{\mathfrak{c}}}_{\mathfrak{c}}(C \times \widetilde{\mathfrak{c}})^{W_{\widetilde{\mathfrak{c}}}}$$

Attempts to describe J have been made in [16, 29] using the flat cover

$$\mathfrak{a} \to \mathfrak{c}.$$

arising from the Chevalley style isomorphism $\mathfrak{p}/\!/H \simeq \mathfrak{a}/\!/W_{\mathfrak{a}}$. However, this is not sufficient to describe regular centralizers in general: In particular, there are natural examples where the action of $W_{\mathfrak{a}}$ on the centralizer group scheme $C = C_H(\mathfrak{a})$ is trivial while J is not. We describe an enhanced cameral cover

$$\widehat{\mathfrak{c}} \to \mathfrak{c}$$

which is a *reducible*, ramified $W_{\widehat{\mathfrak{c}}}$ cover. We do so in two steps. First, we build a map $\widetilde{\mathfrak{c}} \to \mathfrak{c}$ using a generalized Grothendieck-Springer map as follows

Let $M = C_H(C^\circ)$ be the Levi subgroup of H determined by possibly non-maximal torus

C. Fix parabolic Q with Levi factor M. We define $\widetilde{\mathfrak{p}}^{reg}$ to be the reduced incidence variety parametrizing pairs

$$(x, P) \in \mathfrak{p}^{\mathrm{reg}} \times H/Q$$

such that $C_{\mathfrak{h}}(C_{\mathfrak{h}}(x)) \subset \mathrm{Lie}(P)$.

The following is verified with some explicit computations.

Theorem 1.1.3. *Assume that (G, θ, H) is a quasisplit symmetric pair with no simple factors giving the non-split, quasisplit form on E_6. Then, there exists a cover $\widetilde{\mathfrak{c}} \to \mathfrak{c}$ fitting into a Cartesian diagram*

$$\begin{tikzcd}
\widetilde{\mathfrak{p}}^{reg} \arrow[r] \arrow[d] & \widetilde{\mathfrak{c}} \arrow[d] \\
\mathfrak{p}^{reg} \arrow[r] & \mathfrak{c}
\end{tikzcd}$$

The space $\widetilde{\mathfrak{c}}$ is reducible, with each irreducible component isomorphic to a cover $\widetilde{\mathfrak{c}}_0 \to \mathfrak{c}$ which is a subcover of $\mathfrak{a} \to \mathfrak{c}$.

The cover $\widetilde{\mathfrak{c}}$ is sufficient for describing connected regular centralizers, but not those with disconnected regular centralizers. For this we inflate the components by setting

$$\widehat{\mathfrak{c}} = \widetilde{\mathfrak{c}} \times_{\widetilde{\mathfrak{c}}_0} \mathfrak{a}.$$

We summarize the main result.

Theorem 1.1.4. *Let (G, θ, H) be a quasisplit symmetric pair with the quasisplit form on E_6 not appearing as a simple factor. Let $C = C_H(\mathfrak{a})$ be the centralizer subgroup of a maximal abelian subalgebra $\mathfrak{a} \in \mathfrak{p}$. Then, there is a group homomorphism*

$$J \to \mathrm{Res}_{\mathfrak{c}}^{\widehat{\mathfrak{c}}}(C \times \widehat{\mathfrak{c}})^{W_{\widehat{\mathfrak{c}}}}$$

which is an open embedding. Here, $W_{\widehat{\mathfrak{c}}}$ acts diagonally.

If G is simply connected, then the map above is an isomorphism.

We remark that in the above, the restriction on E_6 is likely not necessary. The argument relies on some explicit computations, which are not feasible for the form E_6 (see Lemma 1.3.65).

As a consequence, we can describe the morphism $h^{/\!\!/}$ of Theorem 1.1.1.

Corollary 1.1.5. *Let $\widetilde{\mathcal{A}}_X$ be the base change of the diagram*

$$\begin{tikzcd} \widetilde{\mathcal{A}}_X \arrow[r] \arrow[d] & \widetilde{\mathfrak{c}}_D \arrow[d] \\ \mathcal{A}_X \arrow[r, "ev"] & \mathfrak{c}_D \end{tikzcd}$$

and $\widetilde{\mathcal{A}}^{/\!\!/}_X$, $\widetilde{\mathcal{M}}_X$ the respective base changes to $\widetilde{\mathcal{A}}_X$. Then, there is an open dense subset $\widetilde{\mathcal{A}}^{\diamondsuit}_X$ such that over $\widetilde{\mathcal{A}}^{\diamondsuit}_X$, the map $\widetilde{\mathcal{M}}_X \to \widetilde{\mathcal{A}}^{/\!\!/}_X$ is a torsor for the action of the abelian group scheme $\mathrm{Bun}_C(\widetilde{\mathcal{A}}^{/\!\!/}_X)$ where by Bun_C we mean the bundles for the constant group scheme C.

Finally, we make these results more concrete in the case of a particularly nice form $X = \mathrm{GL}_{2n}/(\mathrm{GL}_n \times \mathrm{GL}_n)$. This was one of the original forms studied by Schapostnik [41], and is the subject of some of our in-progress work on a version of relative Dolbeault Geometric Langlands. The following is an invariant theoretic version of Schapostnik's results on spectral covers.

Theorem 1.1.6. *Consider the symmetric pair on GL_{2n} with involution*

$$\theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$$

and $H = \mathrm{GL}_n \times \mathrm{GL}_n$ embedded block diagonally. Then, there is a spectral cover

$$\mathfrak{s} \to \mathfrak{c}$$

which is a degree n, irreducible cover such that

$$J \simeq \mathrm{Res}_{\mathfrak{c}}^{\mathfrak{s}}(\mathbb{G}_m)$$

Let $\overline{C} \to C \times \mathcal{A}_X^{\sslash}$ *be the pullpack of the spectral cover* $\mathfrak{s}$*. With notation as in 1.1.1, we have a (noncanonical) identification*

$$\mathcal{M}_X^{\mathrm{reg}} \simeq \mathrm{Pic}(\overline{C}/\mathcal{A}_X^{\sslash}).$$

In fact, in this example we can also be explicit about the cover $\mathcal{A}_X^{\sslash} \to \mathcal{A}_X$ over a large locus. Indeed, we have

Theorem 1.1.7. *We remain in the case of the particular symmetric pair* $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$ *above. Let* $D^{ns} = D \coprod_{D^\times} D$ *denote the line bundle* D *with a doubled zero section, i.e.* D^{ns} *is a bundle on* C *with fiber an affine line with doubled origin. The map* $\mathcal{A}_X^{\sslash} \to \mathcal{A}_X$ *is identified with the map of sections*

$$\Gamma(C, D^{ns}) \to \Gamma(C, D).$$

In particular, let $\mathcal{A}_X^{\diamondsuit}$ *denote those sections which meet the zero section of* D *transversely. Then, there is a decomposition into components*

$$\mathcal{A}_X^{\sslash,\diamondsuit} = \coprod_{i=0}^{n} \mathcal{A}_i^{\sslash}$$

and there is a Cartesian diagram

$$\begin{tikzcd}
\mathcal{A}_i^{\sslash} \arrow[r] \arrow[d] & (C^i \setminus \Delta)/S_i \times (C^{d-i} \setminus \Delta)/S_{d-i} \arrow[d] \\
\mathcal{A}_X^{\diamondsuit} \arrow[r] & (C^d \setminus \Delta)/S_d
\end{tikzcd}$$

where Δ denotes the pairwise diagonal and $d = \deg(D)$.

1.2 Background on Symmetric Pairs

In this section, we review the main results on symmetric pairs that will be used in the study of Hitchin systems appearing in the rest of this paper.

1.2.1 Involutions and the Restricted Root System

Let G be a reductive group over an algebraically closed field k, and $\mathfrak{g}$ its Lie algebra. We assume throughout that $p = \mathrm{char}(k)$ is good for G. Namely, if we let Δ be a basis for the root system Φ of G and if we express the longest element of Φ relative to Δ as $\check{\alpha} = \sum_{\beta \in \Delta} m_\beta \beta$, then p is good for G if and only if $p > m_\beta$ for all $\beta \in \Delta$.

Let $\theta \colon G \to G$ be an algebraic involutive group automorphism. Let $K = (G^\theta)^\circ$ denote the neutral component of the θ fixed points in G. The involution θ induces a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ denote the $(+1)$ and (-1) eigenspaces of θ, respectively. In particular, $\mathfrak{k}$ is the Lie algebra of K.

A complicating fact in the theory of symmetric pairs is that, although all maximal tori of G are conjugate, conjugation does not respect the action of the involution θ on a θ-stable torus. It is essential, therefore, to specify the action of θ on a maximal torus of G when studying root systems. We will mainly restrict our attention to θ-stable tori of G for which the θ fixed subtorus is minimal. We introduce this notion now.

Definition 1.2.1. A θ-split torus A of G is a torus of G such that $\theta(a) = a^{-1}$ for all $a \in A$.

Let A be a fixed θ-split torus of G which is maximal among such tori. The Lie algbera $\mathfrak{a} = \mathrm{L}ie(A)$ is a maximal abelian subalgebra of $\mathfrak{p}$. These subalgebras are important enough to warrant their own name.

Definition 1.2.2. A *θ-Cartan* of $\mathfrak{p}$ is a maximal, abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$.

We will frequently fix a maximal split torus A with Cartan $\mathfrak{a} \subset \mathfrak{p}$ in the sequel. This choice is justified by the following standard proposition.

Proposition 1.2.3. *([48], Lem 26.15) All maximal θ-split tori A are K conjugate. Likewise, all θ-Cartans of $\mathfrak{p}$ are K conjugate.*

Definition 1.2.4. The rank r_θ of an involution θ is the dimension

$$r_\theta = \dim(\mathfrak{a})$$

In general r_θ is less than than the rank of G. When A is a maximal torus of G, we call the involution θ *split*. Every group G has a unique split involution.

We will call a maximal torus T of G *maximally split* if T contains a maximal θ-split torus A.

We now introduce a root system associated to the symmetric space G/K. Fix a maximally split torus T containing a maximal θ-split torus $A \subset T$. Let Φ be the set of roots of G with respect to T, viewed as functions on $\mathfrak{t} = \mathrm{Lie}(T)$.

Definition 1.2.5. The set of *restricted roots* is

$$\Phi_r := \{\alpha|_{\mathfrak{a}} \in \mathfrak{a}^* \colon \alpha \in \Phi,\ \alpha|_{\mathfrak{a}} \neq 1\}.$$

We denote by $r\colon \Phi \to \Phi_r \cup \{0\}$ the restriction map taking $\alpha \mapsto \alpha|_{\mathfrak{a}}$.

It is shown in Lemma 26.16 of [48] that Φ_r forms a (possibly nonreduced) root system. Let $W_{\mathfrak{a}}$, referred to as the *little Weyl group*, be the Weyl group associated to this root system. We can alternatively describe $W_{\mathfrak{a}}$ as a quotient.

Proposition 1.2.6. *([48], Prop. 26.19) There is an isomorphism* $W_{\mathfrak{a}} \simeq N_G(\mathfrak{a})/C_G(\mathfrak{a}) \simeq N_H(\mathfrak{a})/C_H(\mathfrak{a})$*. In particular, the latter acts as a reflection group on* $\mathfrak{a}$*.*

We note that the Weyl group $W_{\mathfrak{a}}$ associated to Φ_r is the same as the Weyl group of the reduced root system Φ_r^{red}.

It is useful to note that that the root system Φ_r^{red} on $\mathfrak{a}$ can be seen as the root system associated to a reductive algebraic group. In fact, such a group is given by the Gaitsgory-Nadler group, introduced for symmetric varieties in [34] and generalized in [14] and [25]. This dual group plays an important role in Langlands duality phenomena, for example as in [40]. For convenience, we state the existence of this dual group below, as we will use its existence in the sequel.

Proposition 1.2.7. *([40], Theorem 3.3.1 and [26], Thoerem 6.7) To any spherical variety, there exists a subgroup* $G_X^\vee \subset G^\vee$ *called the* dual group of X *with maximal torus the canonical torus* A_X^**, canonical up to conjugation by the canonical torus* $T^\vee \subset G^\vee$*.*

Fix a Killing form identifying $\mathfrak{g} \simeq \mathfrak{g}^*$*. In the special case of a symmetric variety* $X = H\backslash G$*, we can take* A_X^* *such that the killing form identifies* $Lie(A_X^*) \subset \mathfrak{g}^*$ *with* $\mathfrak{a} \subset \mathfrak{g}$*.*

1.2.2 Symmetric Pairs and the GIT Quotient

We now introduce the notion of *symmetric pair*, introducing the data of a subgroup H.

Definition 1.2.8. A *symmetric pair* is the data of a triple (G, θ, H) where

$$\theta \colon G \to G$$

is an algebraic involutive group homomorphism on G and H is a smooth, closed subgroup $H \subset G$ such that

$$K \subset H \subset N_G(K).$$

The adjoint action of G on $\mathfrak{g}$ restricts to an action of H on the (-1) eigenspace $\mathfrak{p} \subset \mathfrak{g}$. The following result helps characterize such subgroups H.

Proposition 1.2.9. *Choose a maximal θ split torus A.*

(a) The normalizer is given explicitly by

$$N_G(K) = \{g \in G \colon g\theta(g^{-1}) \in Z(G)\}$$

(b) We have $N_G(K) = F^* \cdot K$ where $F^* = \{a \in A \colon a^2 \in Z(G)\}$. Note that F^* depends on the choice of A. Furthermore, $(F^*)^\circ = Z_-$ is the connected component of the subgroup of $Z(G)$ on which θ acts by inversion, i.e. $\theta(z) = z^{-1}$.

(c) There is a short exact sequence

$$1 \to G^\theta \to N_G(K) \xrightarrow{\tau} (F^*)^2 \to 1$$

where $\tau(g) = g\theta(g^{-1})$ and $(F^*)^2 = \{a^2 \colon a \in F^*\}$.

(d) The group $G^\theta/K = \pi_0(G^\theta)$ is a discrete group.

(e) For any symmetric pair (G, θ, H), the identity component H° is reductive.

Proof. Part (a) follows from the proof of Lemma 1.1 of [44]. Part (b) is Lemma 8.1 in [38]. Part (c) follows immediately from (b). Part (d) is clear as G^θ is finite type. Part (e) is directly from Lemma 8.1 of [38]. □

We have a Chevalley-style result on the GIT quotient $\mathfrak{p}/\!/H := \operatorname{Spec} k[\mathfrak{p}]^H$.

Theorem 1.2.10. *([31], Theorem 4.9 and Corollary 4.10) Fix a θ-Cartan $\mathfrak{a} \subset \mathfrak{p}$. The natural inclusion map $\mathfrak{a} \to \mathfrak{p}$ induces a isomorphisms $\mathfrak{a}/\!/W_{\mathfrak{a}} \simeq \mathfrak{p}/\!/K \simeq \mathfrak{p}/\!/N_G(K)$. In particular, for any closed $K \subset H \subset N_G(K)$, we have $\mathfrak{a}/\!/W_{\mathfrak{a}} \simeq \mathfrak{p}/\!/H$. Note that, by construction, that this map is $\mathbb{G}_m$-equivariant under the homothety actions on both $\mathfrak{a}$ and $\mathfrak{p}$.*

Corollary 1.2.11. *Let $G_X^\vee$ be the dual group of Proposition 1.2.7. There is a natural $\mathbb{G}_m$-equivariant identification of the GIT quotients $\mathfrak{p}/\!\!/K \simeq \mathfrak{g}_X^\vee/\!\!/G_X^\vee$.*

Proof. The lefthand side is isomorphic to $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ by Theorem 1.2.10 while the righthand side is isomorphic to $\mathfrak{a}_X^*/\!\!/W_{G_X}$ where $\mathfrak{a}_X^* \subset \mathfrak{g}_X^\vee$ is a Cartan of G_X. By Proposition 1.2.7, we can choose a Killing form on $\mathfrak{g}$ that identifies $\mathfrak{a}_X^* \subset \mathfrak{g}_X^\vee \subset \mathfrak{g}^\vee$ and $\mathfrak{a} \subset \mathfrak{g}$. Since G_X by definition has its root system the dual of Φ_r^{red}, there is a canonical isomorphism $W_{G_X} \simeq W_{\mathfrak{a}}$. The identification above now follows. □

The invariant theory of this GIT quotient is well studied. We will make use of the following fact.

Lemma 1.2.12. *([31], Lemma 4.11) We can write $k[\mathfrak{a}]^{W_{\mathfrak{a}}} = k[f_1, \ldots, f_r]$ for $r = \dim(\mathfrak{a})$ algebraically independent homogeneous polynomials $f_1, \ldots, f_r$ or degrees $m_1, \ldots, m_r$, respectively, which we will call the* exponents of the root system Φ_r. *Moreover, the sum of these exponents can be computed as*

$$\sum_i m_i = r + \frac{\#\Phi_r^{\mathrm{red}}}{2}.$$

Proof. Follows from taking degree of the left hand and right hand side of the equality in [31], Lemma 4.11, noting that the length of the longest element in $W_{\mathfrak{a}}$ is given by the number of positive roots in the reduced root system. □

1.2.3 Regularity and the Quasi-Split Condition

In [35], the notion of regularity was leveraged to study properties of the Hitchin fibration. In so doing, the regular centralizer group scheme was cemented as a central object in the geometry of Hitchin type systems. In [16], this point of view was explored, and certain cameral covers were introduced with the goal of understanding regular centralizers. In this section, we review these constructions. We will improve on descriptions of regular centralizers in Section 1.3.4.

Definition 1.2.13. We denote by $I = I_H \subset \mathfrak{p} \times H$ the group scheme of centralizers over $\mathfrak{p}$, i.e.

$$I = \{(x,h)\colon h \cdot x = x\}.$$

An element $x \in \mathfrak{p}$ is called *regular* if $\dim(I_x)$ is the minimal possible[2]. Let $\mathfrak{p}^{\mathrm{reg}} \subset \mathfrak{p}$ denote the open subscheme of regular elements in $\mathfrak{p}$.

Remark 1.2.14. By Proposition 1.2.9, H an extension of K by a group $F_H \cdot Z_H$ where F_H is finite and $Z_H \subset Z$ is a subgroup of the center. Hence, the notion of regularity does not depend on the choice of subgroup H–only on the involution θ.

Note that $x \in \mathfrak{p}^{\mathrm{reg}}$ is not in general regular with respect to the action of G on $\mathfrak{g}$. These two notions of regularity agree with the symmetric pair is *quasi-split*, a condition we will return to shortly. In spite of this, regularity for elements of $\mathfrak{p}$ can still be detected by the following dimension criteria with respect to its centralizer in G.

Proposition 1.2.15. *([31], Lemma 4.3) For any $x \in \mathfrak{p}$, x is regular if and only if* $\dim C_G(x) = \dim \mathfrak{a} + \dim C_K(A)$.

We call an element $x \in \mathfrak{p}$ *semisimple* if x is semisimple in $\mathfrak{g}$. That this definition is correct is motivated in part by the following two results on Jordan decompositions in $\mathfrak{p}$.

Lemma 1.2.16. *([31], Lemma 2.1) Any $x \in \mathfrak{p}$ admits a decomposition $x = s+n$ for $s, n \in \mathfrak{p}$, s semisimple, n nilpotent, and $n \in Z_G(s)$.*

Proposition 1.2.17. *Let $x \in \mathfrak{p}$ have decomposition $x = s + n$ as in Lemma 1.2.16, and put $L = Z_G(s)^0$, $\mathfrak{l} = \mathrm{Lie}(L)$, and $\mathfrak{p}_L = \mathfrak{l} \cap \mathfrak{p}$. Then L is θ-stable and x is regular in $\mathfrak{p}$ if and only if n is regular as an element of $\mathfrak{p}_L$.*

Proof. We follow an identical argument to the proof of Proposition 9.12 in [38]. We assume without loss of generality that $s \in \mathfrak{a}$ so that $A \subset L$. By Proposition 1.2.15, x is regular if

2. We note that this is not quite the same as the definition of [33]. However under the assumptions we make on the characteristic of our field this is equivalent to the definition in *loc cit.*

and only if $\dim Z_K(x) = \dim Z_K(A)$ and n is regular in L if and only if $\dim Z_{K\cap L}(n) = \dim Z_{K\cap L}(A)$. We have $Z_K(x)^0 = (Z_K(s) \cap Z_K(n))^0 = Z_{K\cap L}(n)^0$, so that $\dim Z_K(x) = \dim Z_{K\cap L}(n)$. Since $x \in A$, $Z_K(A) = Z_{K\cap L}(x)$, so the result follows. $\square$

As in the Lie algebra $\mathfrak{g}$, the regular, semisimple locus is dense and easy to understand.

Lemma 1.2.18. *Let $\mathfrak{p}^{rs} \subset \mathfrak{p}$ denote the subscheme of regular, semisimple elements in $\mathfrak{p}$.*

(a) Let $x \in \mathfrak{p}$. Then, x is semisimple if and only if x is contained in a Cartan of $\mathfrak{p}$.

(b) The regular, semisimple locus $\mathfrak{p}^{rs}$ is dense in $\mathfrak{p}$.

(c) If $x \in \mathfrak{a}$ is regular, semisimple, then $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$.

Proof. (a) and (b) follow immediately from [31], Corollary 2.10 and Theorem 2.11. (c) follows from [31], Lemma 4.3. $\square$

Let $I^{\mathrm{reg}} = I|_{\mathfrak{p}^{\mathrm{reg}}}$ be the restriction of the centralizer group scheme to the regular locus of $\mathfrak{p}$. In contrast with regular centralizers for the adjoint action of G, the group scheme I^{reg} need not be commutative. A special role in the literature is played by those symmetric pairs for which commutativity holds. This class coincides with the those whose notion of regularity agrees with that of the group G acting on $\mathfrak{g}$. We make the following definition.

Definition 1.2.19. We say a symmetric pair $(\mathfrak{p}, H)$ is *quasi-split* if $\mathfrak{p}^{\mathrm{reg}} \subset \mathfrak{g}^{reg}$; that is, the notion of regularity in $\mathfrak{p}$ under the action of H and $\mathfrak{g}$ under the action of G coincide.

Remark 1.2.20. The quasi-split condition does not depend on the choice of subgroup H, only on the involution θ.

As the author found it difficult to locate a proof in the literature, we include here a proof of several equivalent characterizations for the quasi-split condition.

Proposition 1.2.21. *The following are equivalent:*

1. $(\mathfrak{p}, H)$ *is quasi-split;*

2. $Z_G(A) = T$ *is a maximal (maximally θ-split) torus;*

3. I^{reg} *is a commutative group scheme;*

Proof. The centralizer $Z_G(A)$ includes a maximal θ split torus T; it is abelian if and only if $Z_G(A) = T$. The pair $(\mathfrak{p}, H)$ is quasi-split if and only if for all $x \in \mathfrak{a}^{\mathrm{reg}}$, we have

$$\dim Z_G(x) = r$$

where r is the rank of G, so that all inclusions in $T \subset Z_G(A) \subset Z_G(x)$ are equalities. Hence, (1) and (2) are equivalent.

Assume (1) and (2) hold now. Then if $x \in \mathfrak{p}^{\mathrm{reg}}$, I_x^{reg} is contained in the regular centralizer group scheme of x in G, which is abelian. Hence, (3) holds.

Conversely, if (3) holds, then by Lemma 1.2.18, there exists $x \in \mathfrak{p}^{\mathrm{reg}}$ such that $\mathfrak{z}_{\mathfrak{g}}(A) = \mathfrak{z}_{\mathfrak{g}}(x)$. Then $\dim \mathfrak{z}_{\mathfrak{g}}(A) = \dim \mathfrak{z}_{\mathfrak{g}}(x) = r$ and by Lemma 4.2 of [31], we conclude that $Z_G(A)$ is a maximal torus. □

Proposition 1.2.22. *([29], Lemma 1.6) For θ quasi-split, the little Weyl group $W_{\mathfrak{a}}$ is naturally a subgroup $W_{\mathfrak{a}} \subset W$.*

Let T be a maximal θ-split torus, Φ the root system of G with respect to T, and Φ_r the restricted root system with restriction map

$$r \colon \Phi \to \Phi_r \cup \{0\}$$

Roots of G may, a priori, restrict to zero in $\mathfrak{a} \subset \mathfrak{t}$. For quasi-split forms, this does not happen.

Lemma 1.2.23. *For (G, θ) quasi-split, the set $r^{-1}(0)$ is empty; that is, no root in Φ restricts to zero on $\mathfrak{a}$.*

Proof. Fix θ-stable Cartan $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{a}$ where $\mathfrak{t}_0 \subset \mathfrak{t}_K$ is the $(+1)$ eigenspace of θ on $\mathfrak{t}$. Suppose that $\alpha \in \Phi$ restricts to zero on $\mathfrak{a}$. Let $(x, y) \in \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ be an eigenvector with eigencharacter α. Then, using the compatibility of the bracket on $\mathfrak{g}$ with the Cartan decomposition, we have, for all $t \in \mathfrak{t}_0$ and $a \in \mathfrak{a}$,

$$\alpha(t)(x, y) = \alpha(t + a)(x, y) = \big(\mathrm{a}d(t)(x) + \mathrm{a}d(a)(y), \mathrm{a}d(a)(x) + \mathrm{a}d(t)(y)\big). \tag{1.2.1}$$

In particular, $\mathrm{a}d(a)(y)$ is independent of a, so $y \in \mathfrak{c}_{\mathfrak{p}}(\mathfrak{a})$. But since the form is quasi-split, $\mathfrak{c}_{\mathfrak{p}}(\mathfrak{a}) = \mathfrak{a} \subset \mathfrak{c}_{\mathfrak{p}}(\mathfrak{t}_0)$. [c.f. Levy, Lemma 2.3] Hence, $\mathrm{a}d(t)(y) = 0$ and equation 1.2.1 implies that

$$\mathrm{a}d(a)(x) = \alpha(t)y$$

for all $t \in \mathfrak{t}_0$ and $a \in \mathfrak{a}$. This can only be true if both sides of the expression are uniformly zero, so $\alpha = 0$ is not in Φ, a contradiction. □

We record here an identity that will be important for dimension counts later.

Lemma 1.2.24. *([38], Lemmas 3.1 and 3.2) We have the identity*

$$\dim \mathfrak{k} - \dim \mathfrak{p} = \dim C_K(\mathfrak{a}) - \dim \mathfrak{a}.$$

In particular, if the form is quasi-split, then

$$\dim \mathfrak{k} - \dim \mathfrak{p} = r - 2r_\theta$$

where r is the rank of the group G and $r_\theta = \dim \mathfrak{a}$ is the rank of the involution.

1.2.4 Examples

We state in this section some illustrative examples of symmetric pairs. We will test the main results of this work against this list.

Example 1.2.25. **The Diagonal Case.** Let G_1 be a reductive group, and consider $G = G_1 \times G_1$ with the swapping involution

$$\theta(g,h) = (h,g)$$

Put, $H = K = G_1$ the diagonal copy of G_1 in $G_1 \times G_1$. The Cartan decomposition is given by

$$\mathfrak{g} = \{(x,x)\colon x \in \mathfrak{g}_1\} \oplus \{(x,-x)\colon x \in \mathfrak{g}_1\}.$$

Then, the action of $H = G_1$ on $\mathfrak{p} \simeq \mathfrak{g}_1$ is simply the adjoint representation of G_1, and the restricted root system of $(G_1 \times G_1, G_1)$ is given by the root system for G_1.

Example 1.2.26. **The Case** $\mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$**.** Let $G = \mathrm{GL}_{2n}$ and consider the involution

$$\theta(x) = I_{n,n} x I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

Let $H = K = G^\theta = \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n}$ embedded block diagonally. The Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\}.$$

A Cartan in $\mathfrak{p}$ is rank n, given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

The restricted root system is the type B_n root system computed explicitly as

$$\Phi_r = \{\pm(\delta_j^* \pm \delta_k^*) | j \neq k \text{ and } 1 \leq j, k \leq n\} \cup \{\pm 2\delta_j^* | 1 \leq j \leq n\}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$.

Example 1.2.27. **The Case** $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$**.** Let $G = \mathrm{SO}_{2n}$ with the involution

$$\theta(x) = I_{n,n} x I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

(compare with Example 1.2.26). Let $H = K = \mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$ embedded block diagonally. Note that this is an index 2 subgroup in $G^\theta = S(O_n \times O_n)$. The Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathfrak{so}_n \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} \right\}$$

We fix a Cartan in $\mathfrak{p}$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

The restricted root system is the type D_n root system

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \leq j < k \leq n\}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$ and i is the imaginary unit.

Example 1.2.28. **The Case** $\mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$**,** $m < n$**.** Fix $m < n$, and consider the case of $G = \mathrm{SO}_{2n}$ with the involution

$$\theta(x) = I_{m,2n-m} x I_{m,2n-m} \qquad \text{where } I_{m,2n-m} = \begin{pmatrix} I_m & 0 \\ 0 & -I_{2n-m} \end{pmatrix}$$

Let $H = K = \mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$ be embedded block diagonally. The Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \begin{array}{c} A \in \mathfrak{so}_m, \\ B \in \mathfrak{so}_{2n-m} \end{array} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} : \begin{array}{c} C \text{ is a } 2m \times 2n-m \\ \text{matrix} \end{array} \right\}.$$

We choose

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|c} & \mathbf{0}_{m \times n-m} \quad \delta \quad \mathbf{0}_{m\times n-m} \\ \hline \mathbf{0}_{n-m\times m} & \\ -\delta & \\ \mathbf{0}_{n-m\times m} & \end{array} \right) : \delta \text{ is diagonal } m \times m \right\}.$$

Note that this extends to a Cartan of SO_{2n} given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} : \delta \text{ is diagonal } n \times n \right\},$$

and that the root system for SO_{2n} with respect to $\mathfrak{t}$ is

$$\Phi = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le n \}$$

As $m < n$ the restricted root system is the type B_m root system

$$\Phi_r = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le m \} \cup \{ \pm i\delta_j^* \colon 1 \le j \le m \}.$$

Example 1.2.29. **The Case** $\mathrm{SO}_m \times \mathrm{SO}_{2n-m+1} \subset \mathrm{SO}_{2n+1}$**,** $m \le n$**.** Fix $m \le n$ and consider the case of $G = \mathrm{SO}_{2n+1}$ with the involution

$$\theta(x) = I_{m,2n-m+1} x I_{m,2n-m+1} \quad \text{where } I_{m,2n-m+1} = \begin{pmatrix} I_m & 0 \\ 0 & -I_{2n-m+1} \end{pmatrix}$$

Then, $H = K = \mathrm{SO}_m \times \mathrm{SO}_{2n-m+1} \subset \mathrm{SO}_{2n+1}$ is embedded block diagonally, and the Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \begin{array}{c} A \in \mathfrak{so}_m, \\ B \in \mathfrak{so}_{2n-m+1} \end{array} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} : C \text{ is } 2m \times (2n-m+1) \right\}$$

We choose

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|c} & \mathbf{0}_{m\times(n-m)} \quad \delta \quad \mathbf{0}_{m\times(n-m+1)} \\ \hline \begin{array}{c} \mathbf{0}_{(n-m)\times m} \\ -\delta \\ \mathbf{0}_{(n-m+1)\times m} \end{array} & \end{array} \right) : \delta \text{ is diagonal } m \times m \right\}$$

which sits inside the Cartan of SO_{2n+1}

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \delta & \mathbf{0}_{m\times 1} \\ -\delta & 0 & \\ \mathbf{0}_{1\times m} & & \end{pmatrix} : \delta \text{ is diagonal } n \times n \right\}$$

The root system with respect to the Cartan $\mathfrak{t}$ is

$$\Phi = \{ i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le n \} \cup \{ \pm i\delta_j^* \colon 1 \le j \le n \}$$

In particular, we conclude that the restricted root system is the type B_m root system

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*)\colon 1 \leq j < k \leq m\} \cup \{\pm i\delta_j^*\colon 1 \leq j \leq m\}.$$

Example 1.2.30. **The Case $\mathrm{GL}_n \subset \mathrm{Sp}_{2n}$.** Let $G = \mathrm{Sp}_{2n}$ with the involution

$$\theta(x) = I_{n,n} x I_{n,n} \quad \text{where } I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$$

We take

$$H = K = \left\{ \begin{pmatrix} g & \\ & g^{-t} \end{pmatrix} : g \in \mathrm{GL}_n \right\} \subset \mathrm{Sp}_{2n}$$

We fix a Cartan in $\mathfrak{p}$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\}$$

Note that $\mathfrak{a}$ is also a Cartan of Sp_{2n}, making this a "split" symmetric pair. The restricted root system is thus equal to the root system of Sp_{2n}, namely

$$\Phi_r = \{\pm\delta_j^* \pm \delta_k^*\colon 1 \leq j < k \leq n\} \cup \{\pm 2\delta_j^*\colon 1 \leq j \leq n\}$$

where δ_j^* denotes the dual basis element to the j-th coordinate of δ in $\mathfrak{a}$.

1.2.5 Nilpotent Orbits

In sharp contrast to the case of G acting on $\mathfrak{g}^{reg}$, a symmetric pair may have several distinct H orbits of regular, nilpotent elements. In fact, this will, to a large extend, govern the geometry of the Hitchin fibration for symmetric pairs. In this section, we review results of [27], [44], and [31] on regular nilpotent K-orbits.

Recall that K acts on the nilpotent cone $\mathcal{N}_{\mathfrak{p}} = \mathcal{N} \cap \mathfrak{p}$ of $\mathfrak{p}$. Kostant and Rallis in [27] showed that in characteristic zero, although the nilpotent cone is not necessarily irreducible, it has finite many comonents, each of which contains a unique open orbit for the action of K. Levy extended this result to positive characteristic.

Theorem 1.2.31. *([31], Theorem 5.1) Each irreducible component of $\mathcal{N}_{\mathfrak{p}}$ contains a unique regular K-orbit as an open, dense subset. In particular, irreducible components of $\mathcal{N}_{\mathfrak{p}}$ are in 1-1 correspondence with connected components of $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$.*

Corollary 1.2.32. *The space $\mathcal{N}_{\mathfrak{p}} \backslash \mathcal{N}_{\mathfrak{p}}^{reg}$ is of codimension ≥ 1 in $\mathcal{N}_{\mathfrak{p}}$.*

The number of K-conjugacy classes of regular nilpotents was studied and classified by Sekiguchi over $\mathbb{C}$ [44] and by Levy in positive characteristic [31]. To state the result, we make the following definition.

Definition 1.2.33. An *isogeny of symmetric pairs*

$$(G', \theta', H') \to (G, \theta, H)$$

is an isogeny $G' \to G$ restricting to an isogeny $H' \to H$ such that the following diagram commutes

$$\begin{tikzcd} G' \arrow[r, "\theta'"] \arrow[d] & G' \arrow[d] \\ G \arrow[r, "\theta"] & G \end{tikzcd}$$

We say that two symmetric pairs (G, θ, H) and (G', θ', H') are isogeneous if there exists an isogeny of symmetric pairs between them.

We will make frequent use of the following classification result in computations.

Proposition 1.2.34. *([31], Proposition 6.21) Let G be a simple group, and θ an involution on G. The number of regular nilpotent K orbits (and hence the number of irreducible com-*

ponents of the nilpotent cone) is exactly two if and only if $(G, \theta, H = K)$ is isogenous to one on the following list (listed as pairs (G, K) with involution implied):

$(\mathrm{SL}_{2n}, \mathrm{SO}_{2n})$;
$(\mathrm{SL}_{2n}, S(\mathrm{GL}_n \times \mathrm{GL}_n))$;
$(\mathrm{SO}_{2n+1}, \mathrm{SO}_{2m} \times \mathrm{SO}_{2(n-m)+1})$, $2m < 2(n-m)+1$;
$(\mathrm{Sp}_{2n}, \mathrm{GL}_n)$;
$(\mathrm{SO}_{2n}, \mathrm{SO}_{2m} \times \mathrm{SO}_{2(n-m)})$, $m \neq n/2$;
$(\mathrm{SO}_{4n}, \mathrm{GL}_{2n})$;
$(\mathrm{SO}_{4n+2}, \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1})$;
(G, SL_8), *for G simple of type E_7;*
$(G, G' \times \mathbb{G}_a)$, *for G simple of type E_7 and G' simple of type E_6;*

In addition, the split form $(\mathrm{SO}_{4n}, \mathrm{SO}_{2n} \times \mathrm{SO}_{2n})$ has exactly 4 regular nilpotent orbits. All other symmetric pairs with G simple and $H = K$ have irreducible nilpotent cone in $\mathfrak{p}$, and hence a single regular nilpotent orbit.

Remark 1.2.35. Among the above involutions, only the following are quasi-split:

$(\mathrm{SL}_{2n}, \mathrm{SO}_{2n})$;
$(\mathrm{SL}_{2n}, S(\mathrm{GL}_n \times \mathrm{GL}_n))$;
$(\mathrm{SO}_{2n+1}, \mathrm{SO}_n \times \mathrm{SO}_{n+1})$;
(G, SL_8), for G simple of type E_7;
$(\mathrm{SO}_{4n}, \mathrm{SO}_{2n} \times \mathrm{SO}_{2n})$ (which has 4, not 2, nilpotent orbits)

Remark 1.2.36. Note that the center acts trivially on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$. Hence, for any symmetric pair (G, θ, H), by Proposition 1.2.9 the H-orbits on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ are given by $(\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}/K)/\pi_0(H)$.

If one sets $H = N_G(K)$, then the classification of regular, nilpotent orbits becomes trivial.

Theorem 1.2.37. *([27], Proposition 4; [31], Theorem 5.16) The normalizer group $N_G(K)$ acts transitively on the set of regular nilpotents.*

In particular, $\pi_0(N_G(K))$ acts transitively on $\mathcal{N}_{\mathfrak{p}}^{reg}/K$.

The above classifies nilpotent orbits for involutions on simple G. For our purposes later, we will also need the classification of nilpotent orbits for the diagonal case, which is trivial.

Lemma 1.2.38. *The diagonal case $G_1 \subset G_1 \times G_1$ of Example 1.2.25 has a single nilpotent K orbit on $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$.*

Proof. There is an isomorphism of stacks $\mathfrak{p}/K \simeq \mathfrak{g}_1/G_1$ given by projecting onto the first variable. In particular, this map preserves regularity and induces an isomorphism $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}/K \simeq \mathcal{N}_1^{\mathrm{reg}}/G_1$ where $\mathcal{N}_1$ is the nilpotent cone in $\mathfrak{g}_1$. Since any algebraic group G_1 has a unique regular nilpotent G_1 orbit in $\mathcal{N}_1$, the lemma follows. $\square$

1.2.6 Generalities on Kostant-Rallis Sections

In this subsection, we review the theory of Kostant-Rallis sections, as introduced in [27] and generalized in [31]. We work in the generality of [31]; in particular, in this section, it is essential that $p = char(k)$ is good for G.

In positive characteristic, associated characters replace the $\mathfrak{sl}_2$ triples used in [27]. As this paper will only rely on the existence of sections, we leave the theory of associated characters and their relationship to the more explicit $\mathfrak{sl}_2$ triples to the Appendix in Section 1.6.

Lemma 1.2.39. *([31], Corollary 6.29) Let $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ be a regular nilpotent. Then there exists a slice $e + \mathfrak{v} \subset \mathfrak{p}^{\mathrm{reg}}$ contained in the regular locus of $\mathfrak{p}$ such that the map*

$$e + \mathfrak{v} \to \mathfrak{p}/\!\!/K$$

is an isomorphism whose fiber over $0 \in \mathfrak{p}/\!\!/K$ is e.

Remark 1.2.40. In general, the space $\mathfrak{v} \subset \mathfrak{p}$ is constructed by taking a normal associated character λ to e (see Definition 1.6.6) and then constructing a certain θ stable Lie subalgebra

$\mathfrak{g}^* \subset \mathfrak{g}$ with corresponding Cartan decomposition $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$. The slice is given by taking $\mathfrak{v}$ to be an $\mathrm{A}d(\lambda)$ graded complement to $[\mathfrak{k}^*, e]$ inside of $\mathfrak{p}^*$.

If we suppose that the characteristic of k is either zero or greater than the Coxeter number of G, then the results of Appendix 1.6 gives a bijection between H-conjugacy classes of associated characters and H-conjugacy classes of $\mathfrak{sl}_2$ triples. In this case, we can complete $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$ to a normal $\mathfrak{sl}_2$ triple (e, h, f) (see Definition 1.6.1) uniquely up to $C_K(e)^\circ$ conjugacy, and we can take $e + \mathfrak{v} = e + \mathfrak{c}_{\mathfrak{p}}(f)$ as in [27], Theorem 11.

Fix a Kostant-Rallis section $\mathcal{S} = e + \mathfrak{v}$. For later applications to smoothness, we will need a bit more on the differential of the action map

$$H \times \mathcal{S} \to \mathfrak{p}^{reg}. \tag{1.2.2}$$

We record here the following Lemma.

Lemma 1.2.41. *The differential of the action map* (1.2.2) *at* $(1, e)$ *is surjective.*

Proof. The differential of the above map is identified with the map

$$\mathfrak{h} \oplus \mathfrak{v} \to \mathfrak{p}, \quad (x, y) \mapsto [e, x] + y$$

Since e is regular, the codimension of $[\mathfrak{k}, e]$ in $\mathfrak{p}$ is equal to the dimension of a θ-Cartan, which by Lemma 1.2.39 is exactly the dimension of $\mathfrak{v}$. Moreover, by the construction of $\mathfrak{v}$, it is orthogonal to $[\mathfrak{h}, e]$. Hence by a dimension count, we have $\mathfrak{p} = [\mathfrak{h}, e] + \mathfrak{v}$, and we conclude that the map above is surjective. $\square$

We will study the map $\mathfrak{p} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ produced by Theorem 1.2.10 in some detail; it will provide the underlying structure of the Hitchin fibration for symmetric pairs. In this spirit, we now prove this map's flatness.

Lemma 1.2.42. *The map* $\mathfrak{p} \to \mathfrak{p}/\!\!/K \simeq \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ *is flat, as is the map* $\mathfrak{p}^{reg} \to \mathfrak{p}/\!\!/K$.

Proof. This is a morphism between two smooth schemes. Hence, by miracle flatness, it suffices to show that the fibers are equidimensional. Let $U \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ be the subset of $x \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ whose fiber in $\mathfrak{p}$ is $\dim(K) - r_\theta$ dimensional, with $r_\theta = \dim(\mathfrak{a})$ the rank of the group. Then U contains $0 \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ as Theorem 1.2.31 guarantees $\dim(\mathcal{N}_{\mathfrak{p}}) = \dim(K) - r_\theta$, and U is stable under the action of $\mathbb{G}_m$ as the map is $\mathbb{G}_m$ equivariant. Moreover, U contains the open complement of the image of the root hyperplanes $\cup_\alpha H_\alpha \subset \mathfrak{a}$.Hence, $U = \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ and the map is flat. □

1.3 The regular quotient

1.3.1 Motivation and Generalities

Many facts about the (usual) Hitchin fibration can be abstracted to basic properties in invariant theory. Principally among these is that the morphism

$$\mathfrak{g}^{\mathrm{reg}} \to \mathfrak{g}/\!\!/G$$

is a gerbe banded by the regular centralizer group scheme. For example, this property is the invariant theoretic shadow of why generic fibers of the Hitchin fibration are Picard stacks. However, this is plainly not the case for symmetric pairs, as observed in [16]. We illustrate with an example.

Example 1.3.1. Let $G = \mathrm{SL}_2$ with involution conjugating by the matrix $\mathrm{diag}(1, -1)$. Then, $H = S(\mathbb{G}_m \times \mathbb{G}_m) \simeq \mathbb{G}_m$ acts on $\mathfrak{p} \simeq \mathbf{A}^2$ by the hyperbolic action $x \cdot (a, b) = (xa, x^{-1}b)$. The regular locus is $\mathfrak{p}^{reg} = \mathbf{A}^2 \setminus \{0\}$, and we see that there are two regular orbits lying over the closed orbit $0 \in \mathfrak{p}/\!\!/H$, see Figure 1.

To solve this problem, [16] proposed a rigidification $\mathfrak{p}^{reg} /\!\!\!/\!\!\!/ H$ of the stack $\mathfrak{p}^{reg}/H$ such that there is a factorization

$$\mathfrak{p}^{\mathrm{reg}}/H \to \mathfrak{p}^{\mathrm{reg}} /\!\!\!/\!\!\!/ H \to \mathfrak{p}/\!\!/H$$

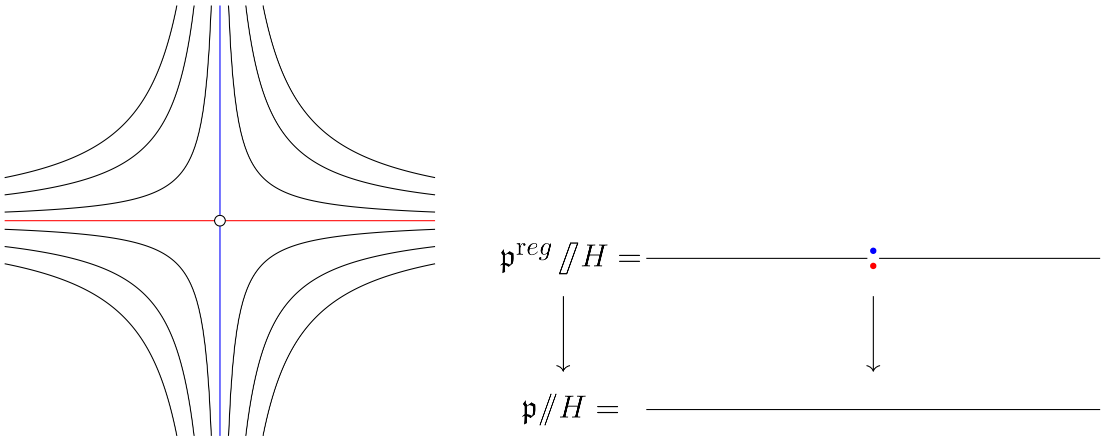

Figure 1.1: (Left) The orbits of $H = S(\mathbb{G}_m \times \mathbb{G}_m)$ acting on $\mathfrak{p}^{reg} \simeq \mathbf{A}^2 \setminus \{0\}$ for the symmetric space $X = \mathrm{SL}_2 / S(\mathbb{G}_m \times \mathbb{G}_m)$. Note the two orbits, drawn in blue and red, whose closure includes the (non-regular) closed orbit $\{0\}$. The regular quotient for this symmetric pair (pictured right) is the affine line with doubled origin.

with the first map being a gerbe and the second a (nonseparated) cover. To illustrate, in Example 1.3.1, $\mathfrak{p} /\!\!/ H \simeq \mathbf{A}^1$ is an affine line while $\mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ is an affine line with doubled origin.

In unpublished work [33] of Ngô and Morrissey, such quotients are introduced in the far greater generality of a reductive group G acting on an affine normal scheme M. Examples of these generalized Hitchin systems include the multiplicative Hitchin system studied most thoroughly in [49], Hitchin systems associated to spherical varieties, and Hitchin systems for higher dimensional varieties in [7]. While it will turn out that $\mathfrak{p}^{reg} /\!\!/ H$ will be a scheme, the resulting quotients $M^{\mathrm{reg}} /\!\!/ G$ are, in general, Deligne-Mumford stacks. For the sake of completeness, we review the general construction of Morrissey and Ngô here. However, the construction is not important for understanding the resulting geometry in our case of interest, and the uninterested reader can safely skip to Section 1.3.2.

Let M be an affine variety acted on by a reductive algebraic group H. Let $I_M \subset M \times H$ be the group scheme over M of stabilizers of the H-action. Following [33] the regular locus is the maximal open subscheme $M^{reg} \subset M$ such that $Lie(I_{M^{reg}}) \to M^{reg}$ is a vector bundle. Theorem 1.3.11 shows that under our assumptions on the characteristic, this definition agrees with our definition of regularity.

Definition 1.3.2 (The Regular Quotient [33]). We define the *regular quotient* $M^{\mathrm{reg}} /\!\!/ H$ to be the stack quotient of the groupoid in algebraic spaces

$$(M^{reg} \times H)/I_M^{\mathrm{reg}} \rightrightarrows M^{reg}$$

where the two right arrows are the projection and action maps.

Assuming smoothness of $I_M^{reg} \to M^{\mathrm{reg}}$ (see Theorem 1.3.11), *loc. cit.* proves that the quotient above exists and the resulting quotient $M^{\mathrm{reg}} /\!\!/ H$ is a Deligne-Mumford stack.

The following result is contained in [33].

Proposition 1.3.3 (Properties of Regular Quotient from [33]). *The regular quotient has the following properties:*

- *If I_M^{reg} is abelian then it descends to a group scheme $J \to M^{reg} /\!\!/ H$.*
- *I_M^{reg} descends to a band in the sense of Giraud [19] $J_{band} \to M^{reg} /\!\!/ H$.*
- *The map $M^{reg}/H \to M^{reg} /\!\!/ H$ is a gerbe banded by J_{band}; when I_M^{reg} is abelian it is a J-gerbe.*
- *If there are compatible $\mathbb{G}_m$ actions on I^{reg} and $\mathfrak{p}^{\mathrm{reg}}$, then there is a canonical $\mathbb{G}_m$ action on $\mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ and the morphism $M^{\mathrm{reg}}/H \to M^{\mathrm{reg}} /\!\!/ H$ is $\mathbb{G}_m$ equivariant.*

The second property is in fact a defining property of the regular quotient:

Proposition 1.3.4 ([33]). *Let V be a scheme such that each fiber of $M^{reg} \to V$ consists of a single G-orbit. Then $V = M^{reg} /\!\!/ G$.*

We will use this property to describe $\mathfrak{p}^{reg} /\!\!/ H$.

1.3.2 The regular quotient and smoothness of stabilizer group schemes via Kostant–Rallis Sections

In this section we describe the regular locus $\mathfrak{p}^{reg}$ as the union of the H-orbits of potentially multiple Kostant–Rallis sections. We use this to deduce smoothness of several of the group schemes considered in the previous section in a way completely analogous to the case of the adjoint action of G on $\mathfrak{g}$ as considered in [18, 39]. We will deduce that the regular quotient can be obtained by gluing together multiple copies of the GIT quotient together. In particular, we show the regular quotient for the action of H on $\mathfrak{p}^{\mathrm{reg}}$ is a (nonseparated) scheme. An explicit description of the gluing will be described in the Section 1.3.3. This is a modification of an argument for the case of the Vinberg monoid found in Proposition 2.12 of [3] and Equation 2.7 and Lemma 2.2.8 of [8]. We will then give a direct argument that I^{reg} descends to the regular quotient.

The key technical input is the following.

Lemma 1.3.5 (Analogue of Lemma 2.2.8 of [8], see also Proposition 2.12 of [3])**.** *Let $U \subset \mathfrak{p}^{reg}$ be stable under the $H \times \mathbb{G}_m$-action. If $U \cap \mathcal{N}_{\mathfrak{p}} = \mathcal{N}_{\mathfrak{p}}^{reg}$ then $U = \mathfrak{p}^{reg}$.*

The following proof is identical to that of [8], we provide it here for completeness.

Proof. We let $F := \mathfrak{p}^{reg} \backslash U$. By assumption this is a $\mathbb{G}_m \times H$ subscheme of $\mathfrak{p}$. Let $\chi|_F$ denote the restriction of $\chi : \mathfrak{p} \to \mathfrak{a}/\!\!/W_a$ to F.

We let $V \subset F$ be the inverse image under $\chi|_F$ of the subset

$$\{x \in \mathfrak{a}/\!\!/W_a \,|\, \dim(\chi_F^{-1}(x)) < \dim(H) - \dim(\mathfrak{a})\}.$$

This is an open subscheme of F by upper semicontinuity. Furthermore, it includes $0 \in \mathfrak{p}$ by Lemma 1.2.32. As V is preserved by $\mathbb{G}_m$ and 0 is in the closure of every $\mathbb{G}_m$-orbit of F we have that $V = F$.

By Lemma 1.2.42 we have that each fiber of χ is of dimension $\dim(\mathfrak{p}) - \dim(\mathfrak{a})$. Suppose that $X \in \mathfrak{p}^{reg} \cap F$, then as F is stable under H, the dimension of the orbit of X (which is inside F) is $\dim(\mathfrak{p}) - \dim(\mathfrak{a})$. This contradicts F being of codimension ≥ 1 in each fiber. $\square$

Remark 1.3.6. Note that Lemma 1.3.5 does note hold for general homogeneous spherical varieties. In particular, for $X = G/H$, it relies on the equidimensionality of $\mathfrak{h}^\perp \to \mathfrak{h}^\perp /\!\!/ H$. This will fail for general homogeneous spherical varieties X.

We now recall that, by Theorem 1.2.31, for each irreducible component $S \in Irr(\mathcal{N}_\mathfrak{p})$ there is a unique regular K-orbit $\mathcal{O}_S$ of $\mathfrak{p}$ in S. Furthermore, by Lemma 1.2.39, there is a (non-unique) Kostant–Rallis section $\kappa_S : \mathfrak{a}/\!\!/W_a \to \mathfrak{p}^{reg}$ such that $\kappa_S(0) \in \mathcal{O}_S$.

Let $\mathcal{I}$ denote the set of H orbits on $\mathcal{N}_\mathfrak{p}^{\text{reg}}$. For each representative pick a Kostant-Rallis section κ_i whose image at 0 is in the regular nilpotent H orbit $i \in \mathcal{I}$. Let $\mathcal{S}_i$ be the image as used to define the Kostant-Rallis section in section 1.2.6. For $i \in \mathcal{I}$, we then have a morphism

$$(H \times \mathcal{S}_i)/I^{\text{reg}} \to \mathfrak{p}^{reg},$$

which is quasifinite and an isomorphism over the regular semisimple locus $\mathfrak{p}^{rs}$. Hence it is birational. As $\mathfrak{p}^{reg}$ is normal, Zariski's main theorem implies that this morphism is an open embedding. Hence we can define $\mathfrak{p}^{\kappa_i,H}$ to be the open subscheme of $\mathfrak{p}^{reg}$ which is the image of this morphism.

Proposition 1.3.7.

$$\mathfrak{p}^{reg} = \bigcup_{i \in \mathcal{I}} \mathfrak{p}^{\kappa_i,H}.$$

Proof. This is an immediate consequence of Lemma 1.3.5 applied to $U = \cup_{i\in\mathcal{I}} \mathfrak{p}^{\kappa_i,H}$. $\square$

Application to smoothness of I^{reg}

We start by proving a Lemma.

Lemma 1.3.8. *We keep notation as in the previous section. If H is smooth and $i \in \mathcal{I}$, then the morphism*

$$H \times \mathcal{S}_i \to \mathfrak{p}^{\kappa_i, H} \tag{1.3.1}$$

is smooth and surjective.

Proof. The morphism (1.3.1) is surjective by definition of $\mathfrak{p}^{\kappa_i,H}$ and is evidently H-equivariant and $\mathbb{G}_m$-equivariant. Moreover, by Lemma 1.2.41, the differential of (1.3.1) at $(1, \kappa_i(0))$ is surjective. The only such neighbourhood is $H \times \mathcal{S}_i$, hence the morphism is smooth. □

Proposition 1.3.9. *For any $i \in \mathcal{I}$, the composition $\chi|_{\mathfrak{p}^{\kappa_i,H}} : \mathfrak{p}^{\kappa_i,H} \hookrightarrow \mathfrak{p} \to \mathfrak{p}/\!\!/H$ is smooth and surjective.*

Proof. This is identical to the proof of Proposition 3.3.3 of [39], namely the composition $H \times \mathcal{S}_i \to \mathfrak{p}^{\kappa_i,H} \to \mathfrak{p}/\!\!/H \cong \mathcal{S}_i$ is identified with the projection to $\mathcal{S}_i$. Hence [2] Tag 02K5 implies $\chi|_{\mathfrak{p}^{\kappa_i,H}}$ is smooth and surjective. □

We denote by $I_{\mathcal{S}_i} := I_H^{reg} \times_{\mathfrak{p}^{reg}} \mathcal{S}_i$ the restriction of I_H^{reg} to $\mathcal{S}_i$.

Proposition 1.3.10. *The map $I_{\mathcal{S}_i} \to \mathcal{S}_i$ is smooth.*

Proof. This proof of Proposition 3.3.5 of [39] carries over to this setting. For completeness we summarize: As schemes over $\mathcal{S}_i$ we have isomorphisms

$$I_{\mathcal{S}_i} \cong \mathcal{S}_i \times_{\mathfrak{p} \times \mathcal{S}_i} (H \times \mathcal{S}_i) \cong \mathcal{S}_i \times_{\mathfrak{p} \times_{\mathfrak{p}/\!\!/H} \mathcal{S}_i} (H \times \mathcal{S}_i) \cong \mathcal{S}_i \times_{\mathfrak{p}} (H \times \mathcal{S}_i).$$

Hence as $H \times \mathcal{S}_i \to \mathfrak{p}^{reg} \hookrightarrow \mathfrak{p}$ is smooth we have that $I_{\mathcal{S}_i} \to \mathcal{S}_i$ is smooth. □

Theorem 1.3.11. *If H is smooth then $I^{reg} \to \mathfrak{p}^{reg}$ is smooth.*

Note that for (G, θ) quasisplit in characteristic 0 this is proved for $I_{G^\theta}^{\mathrm{reg}}$ in [17]. In Theorem 4.7 of [29] this is generalized to the case where the characteristic is $p > 2$ and p is such that

$I_G^{reg} \to \mathfrak{g}^{reg}$ (that is to say the regular centralizers for the adjoint action of G on $\mathfrak{g}$) is smooth (see condition (C3) of [39] for such conditions).

Proof. By Proposition 1.3.7 it is sufficient to show that for each $i \in \mathcal{I}$ (recall this denoted the represented the representatives of the $\pi_0(H)$-orbits in $Irr(\mathcal{N}_{\mathfrak{p}})$) the morphism $I^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \to \mathfrak{p}^{\kappa_i,H}$ is smooth. The proof is now identical to Corollary 3.6 in [39], we provide it only for completeness. The diagram

$$\begin{array}{ccc} H \times I_{\mathcal{S}_i} & \longrightarrow & I^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \\ \downarrow & & \downarrow \\ H \times \mathcal{S}_i & \longrightarrow & \mathfrak{p}^{\kappa_i,H} \end{array}$$

is Cartesian. Hence Lemma 1.3.8 implies that $H \times I_{\mathcal{S}_i} \to I_H^{reg}$ is smooth and surjective. Furthermore the Lemma 1.3.8 and Proposition 1.3.10 tell us that the composition $H \times I_{\mathcal{S}_i} \to \mathfrak{p}^{\kappa_i,H}$ is smooth. Hence $I^{reg}|_{\mathfrak{p}^{\kappa_i,H}} \to \mathfrak{p}^{\kappa_i,H}$ is smooth by Tag 0K25 of [2]. □

Application to the regular quotient

Let $\widetilde{\mathfrak{p}/\!\!/H}$ be the union of $\mathcal{I}$ copies of $\mathfrak{p}/\!\!/H$ where we glue the copies labelled by i and j on the subscheme $U \subset \mathfrak{a}/\!\!/W_a$ where the sections κ_i and κ_j are conjugate. This notation conflicts with the use of tildes in Section 1.3.4, but as it will not appear outside these next two subsections, we hope it will not cause any confusion. Note that *a priori* it is not clear that the gluing is done in a fashion compatible with the $\mathbb{G}_m$-action, and thus it is not clear that $\widetilde{\mathfrak{p}/\!\!/H}$ has a $\mathbb{G}_m$-action coming from the $\mathbb{G}_m$-actions on $\mathfrak{a}/\!\!/W_a$.

Theorem 1.3.12. *Assuming that H is smooth we have a $\mathbb{G}_m$-equivariant isomorphism of schemes*

$$\widetilde{\mathfrak{p}/\!\!/H} \xrightarrow{\cong} \mathfrak{p}^{reg}/\!\!/H,$$

where the $\mathbb{G}_m$-action comes on $\widetilde{\mathfrak{p}/\!\!/H}$ comes from the $\mathbb{G}_m$-action on each copy of $\mathfrak{p}/\!\!/H$. Furthermore this isomorphism commutes with the $\mathbb{G}_m$-equivariant morphisms to $\mathfrak{p}/\!\!/H$.

Remark 1.3.13. Note that the left hand side is definable regardless of whether or not H is smooth while the right hand side is only known to be definable when H is smooth.

Proof. This argument is essentially identical to a similar argument in the Vinberg monoid case found in [33]. The non-equivariant version isomorphism follows immediately from Proposition 1.3.4 and the fact that the map $\mathfrak{p} \to \mathfrak{p}/\!\!/K \cong \mathfrak{a}/\!\!/W_a$ is $\mathbb{G}_m$-equivariant.

These identifications commute with the morphism to $\mathfrak{p}/\!\!/H$, because these are the unique morphisms to $\mathfrak{p}/\!\!/H$ such that the diagrams

$$\begin{tikzcd} \mathfrak{p}^{reg} \arrow[r] \arrow[d] & \mathfrak{p} \arrow[d] \\ \mathfrak{p}^{reg} /\!\!\!/ H \arrow[r] & \mathfrak{p}/\!\!/H \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{p}^{reg} \arrow[r] \arrow[d] & \mathfrak{p} \arrow[d] \\ \widetilde{\mathfrak{p}/\!\!/H} \arrow[r] & \mathfrak{p}/\!\!/H \end{tikzcd}$$

commute.

We hence have a $\mathbb{G}_m$-action on $\widetilde{\mathfrak{p}/\!\!/H}$ via the identification with the regular quotient. Because the morphism $\widetilde{\mathfrak{p}/\!\!/H} \to \mathfrak{p}/\!\!/H$ is $\mathbb{G}_m$-equivariant we hence must have that this $\mathbb{G}_m$ action comes from the $\mathbb{G}_m$-action on each copy of $\mathfrak{p}/\!\!/H$.

The equivariant isomorphism now follows immediately. $\square$

Direct Proof of Gerbe Structure

We now prove some of the results of Proposition 1.3.3 without the use of results from [33]. We also note that the proofs of these results are identical to those of both the Lie algebra case (e.g. [35]) and to those for regular quotients in [33] as such we only include these for completeness.

Proposition 1.3.14. *The maps*

$$\mathfrak{p}^{reg}/H \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}^{reg} /\!\!\!/ H$$

and

$$(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p}^{reg}/\!\!/H)/\mathbb{G}_m$$

are smooth gerbes.

Proof. It is enough to show this for the case of $(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$; the remaining case will follow by pullback to $\widetilde{\mathfrak{p}^{reg}/\!\!/H}$. Consider the pullback off this map along $\mathfrak{p}^{reg} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$. We have a section of the pullback given by the diagonal section

$$\mathfrak{p}^{reg} \to \mathfrak{p}^{reg} \times_{(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m} \mathfrak{p}^{reg} \to \mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m} (\mathfrak{p}^{reg}/H).$$

This gives an identification of $\mathfrak{p}^{reg} \times_{\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m} (\mathfrak{p}^{reg}/H)$ with BI^{reg}, concluding the proof. $\square$

Proposition 1.3.15. *If $(\mathfrak{p}, H)$ is quasisplit then I^{reg} descends to a smooth group scheme*

$$J \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}^{reg}/\!\!/H,$$

and descends further to a smooth group scheme

$$\mathbb{J} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p}^{reg}/\!\!/H)/\mathbb{G}_m.$$

The proof used to define the group scheme of regular centralizers (this can be found in e.g. [35]) for a Lie algebra generalizes immediately to this setting, and indeed to the general setting of regular quotients with I_M^{reg} abelian.

Proposition 1.3.16. *If (G, θ, H) is a symmetric pair and H is smooth, then the map*

$$\mathfrak{p}^{reg}/H \to \widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}^{\mathrm{reg}}/\!\!/H$$

is a J-gerbe. Similarly the map

$$(\mathfrak{p}^{reg}/H)/\mathbb{G}_m \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m \cong (\mathfrak{p}^{\mathrm{reg}} /\!\!/ H)/\mathbb{G}_m$$

is a $\mathbb{J}$*-gerbe.*

Proof. It has already been shown that these spaces are gerbes. We have identifications $\chi^* J \cong I^{reg}$ and $\overline{\chi}^* \mathbb{J} \cong I^{reg}$ for the maps $\chi : \mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$, and $\overline{\chi} : \mathfrak{p}^{reg} \to (\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m$. Therefore, we have that these are J and $\mathbb{J}$ gerbes, respectively. □

Remark 1.3.17. It is important to note that generalizations and refinements of several of the above results are expected. We point out a few of these now.

If $\widetilde{\mathfrak{p}/\!\!/H} \cong \mathfrak{p}/\!\!/H$ (or equivalently there is only one regular nilpotent H-orbit), then we can use a Kostant–Rallis section κ to pull back I^{reg} to get a group scheme $\kappa^* I^{reg}$ on $\widetilde{\mathfrak{p}/\!\!/H}$. If the symmeric pair is quasisplit, but we do not assume that H is smooth, we can still get that $p^*(\kappa^* I_H^{reg}) \cong I_H^{reg}$ (for $p : \mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$).

Secondly, under the same assumption that there is one regular nilpotent H-orbit, we can consider $\kappa^* I^{reg}$ when H is smooth, but the symmetric pair is not quasisplit. Note that we can also use $\mathbb{G}_m^{[2]}$-equivariance (that is to say we consider the usual $\mathbb{G}_m$-action, but we precompose by the squaring map $\mathbb{G}_m \to \mathbb{G}_m$) to get a group scheme on $[(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m^{[2]}]$.

In this case $p^*(\kappa^* I^{reg})$ is I^{reg}. This in particular provides a *group* that $[\mathfrak{p}^{reg}/H \times \mathbb{G}_m^{[2]}] \to [(\widetilde{\mathfrak{p}/\!\!/H})/\mathbb{G}_m^{[2]}]$ is a gerbe for.

As such it is an important question to see whether there are sections of $\mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}/\!\!/H}$ and $[\mathfrak{p}^{reg}/\mathbb{G}_m^{[2]}] \to [\widetilde{\mathfrak{p}/\!\!/H}/\mathbb{G}_m^{[2]}]$ which would allow generalization of these considerations to arbitrary symmetric pairs (with H smooth).

Finally one could ask whether there are then descriptions of the group scheme $\kappa^* I^{reg}$ via Weil restriction. We note that the work of Hitchin–Schaposnik [23] and Branco [5] strongly suggests that in certain examples, one can describe the Weil restriction of SL_2 from a spectral cover.

In the sequel, we will drop the notation $\widetilde{\mathfrak{p}/\!\!/H}$ and only write $\mathfrak{p}^{reg}/\!\!/H$. In particular, as the former is defined when H is not smooth while the latter is not, we will define $\mathfrak{p}^{reg}/\!\!/H$ to be $\widetilde{\mathfrak{p}/\!\!/H}$ when H is not smooth.

1.3.3 Explicit Description of the Regular Quotient

Overview

We now turn to an explicit description of the geometry of the regular quotient $\mathfrak{p}^{reg}/\!\!/H$. We ultimately provide two different descriptions.

The first appears in Theorem 1.3.20, where we describe the regular quotient in terms of certain quotients of component groups. This perspective reduces to computing the differences between certain regular centralizer groups schemes as the group varies.

The second is done by the following multistep procedure:

- Firstly, we reduce to the case of simply connected simple groups and the diagonal case $H = G_1 \overset{\Delta}{\hookrightarrow} G_1 \times G_1$ of Example 1.2.25, using Theorem 1.3.24.

- Secondly, we reduce understanding the orbits above a point in $\mathfrak{a}/\!\!/W_a$ to the case of nilpotent cones of certain Levis, that we call distinguished Levis, in Theorem 1.3.34 and Proposition 1.3.37. This is a Lie algebra version of taking "descendants" described in Section 5.1.1 of [30].

- We use the immediately preceding point to describe the structure for simple, simply connected groups (and the diagonal case, example 1.2.25). Except for the case of $SO(n) \times SO(n) \hookrightarrow SO(2n)$ (considered in Example 1.3.48) this is not complicated due to the fact that there are at most 2 regular H-orbits in the nilpotent cone. The resulting explicit description of the regular quotient is included as Theorem 1.3.38. For such cases, we get a description $\mathfrak{p}^{reg} /\!\!/ H \cong \mathfrak{a}/\!\!/W_a \coprod_U \mathfrak{a}/\!\!/W_a$ for an explicitly described open $U \subset \mathfrak{a}/\!\!/W_a$.

In Section 1.3.3 we explicitly compute the regular quotient in several cases of interest.

Description of the Regular Quotient via Comparison of Regular Centralizers

In this section, we give a first description of the regular quotient using a comparison of regular centralizer group schemes for H and for the full normalizer $N_G(K)$. Throughout this section, we will use subscripts to indicate in which group centralizers are taken; for example, I_H^{reg} will denote the centralizers in the group H.

Let $\mathbf{A} \subset G \times \mathfrak{p}^{rss}$ be the family over $\mathfrak{p}^{\mathrm{rs}}$ whose fiber over $x \in \mathfrak{p}^{rs}$ is the maximal θ-split torus A_x such that $\mathrm{L}ie(A_x) = \mathfrak{z}_{\mathfrak{g}}(x) \cap \mathfrak{p}$. (Note, the existence and uniqueness of such a torus $A \subset G$ is given in [31], Lemma 0.1.)

Moreover, we let $\mathbf{F}^* \subset \mathbf{A}$ denote the family over $\mathfrak{p}^{\mathrm{rs}}$ whose fiber over $x \in \mathfrak{p}^{rs}$ is the subgroup

$$\{a \in A_x \colon a^2 \in Z(G)\} \subset A_x$$

Recall from part (a) of Proposition 1.2.9 that, for a given choice of A, we have $N_G(K) = F^* \cdot K$ where F^* is chosen with respect to A. We use this to determine the structure of $I_{N_G(K)}^{\mathrm{reg}}/I_H^{\mathrm{reg}}$.

Lemma 1.3.18. *Fix a choice of $x \in \mathfrak{p}^{\mathrm{rs}}$ determining A and $F^* \subset A$. Moreover, let Z_- denote the subgroup of the center $Z(G)$ on which θ acts by inversion.*

1. *We have an isomorphism over $\mathfrak{p}^{\mathrm{rs}}$*

$$\left(I_{N_G(K)}^{\mathrm{reg}}/Z_- \cdot I_K^{\mathrm{reg}}\right)\Big|_{\mathfrak{p}^{\mathrm{rs}}} \simeq \coprod_{a \in F^*/Z_- \cdot (F^* \cap K)} \mathfrak{p}^{\mathrm{rs}} \tag{1.3.2}$$

2. The isomorphism (1.3.2) *extends to an isomorphism*

$$I^{\text{reg}}_{N_G(K)}/Z_- \cdot I^{\text{reg}}_K \simeq \coprod_{a \in F^*/Z_- \cdot (F^* \cap K)} U_a \tag{1.3.3}$$

where $U_a \to \mathfrak{p}^{\text{reg}}$ is the inclusion map for an open set $\mathfrak{p}^{\text{rs}} \subset U_a \subset \mathfrak{p}^{\text{reg}}$.

Proof. The inclusion $I^{\text{reg}}_{N_G(K)} \subset N_G(K) \times \mathfrak{p}^{\text{reg}}$ defines a map

$$I^{\text{reg}}_{N_G(K)}/Z_- \cdot I^{\text{reg}}_K \to \big(N_G(K)/Z_- \cdot K\big) \times \mathfrak{p}^{\text{reg}} \tag{1.3.4}$$

with target a constant group scheme with discrete fiber. For any fixed $y \in \mathfrak{p}^{rs}$, let A_y be the fiber of $\mathbf{A}$ at y and $F^*_y \subset A_y$ be the fiber of the group scheme $\mathbf{F}^*$ over y. Then, it is clear from Proposition 1.2.9 that $I^{\text{reg}}_{N_G(K),y} = F^*_y \cdot Z_- \cdot I^{\text{reg}}_{K,y}$. Therefore, the fiber of the map (1.3.4) at y is identified with the identity map

$$F^*_y/Z_- \cdot (F^*_y \cap K) \to F^*_y/Z_- \cdot (F^*_y \cap K)$$

In particular, (1.3.4) is an isomorphism over the regular, semisimple locus, proving part (1).

For (2), we claim that the map (1.3.4) remains an injection over $\mathfrak{p}^{\text{reg}}$. In particular, this amounts to the following claim:

Claim: Let $y \in \mathfrak{p}^{\text{reg}}$. For any $g_1, g_2 \in I^{\text{reg}}_{N_G(K),y}$, if $g_1 = hg_2$ for $h \in K$, then in fact $h \in I^{\text{reg}}_{K,y}$.

Proof of Claim. Since g_1 and g_2 centralize y, we have

$$y = \text{ad}(hg_2) \cdot y = \text{ad}(h)y$$

Hence, $h \in I_{G,y} \cap K = I^{\text{reg}}_{K,y}$. □

It follows that the map (1.3.4) describes the quotient $I^{\text{reg}}_{N_G(K)}/Z_- \cdot I^{\text{reg}}_K$ as the disjoint union of open subsets of $\mathfrak{p}^{reg}$ extending the sheets $\big(N_G(K)/Z_- \cdot K\big) \times \mathfrak{p}^{rs}$. $\square$

Lemma 1.3.19. *The group $N_G(K)$ acts transitively on the fibers of the map $\mathfrak{p}^{\mathrm{reg}} \to \mathfrak{p}/\!\!/K$.*

Proof. Theorem 1.3.12 reduces this to the $N_G(K)$ action on the zero fiber $\mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$. Theorem 1.2.37 proves this case. $\square$

Theorem 1.3.20. *Consider the natural action of the constant group scheme $\underline{N_G(K)} := N_G(K) \times \mathfrak{p}/\!\!/K$ on $\mathfrak{p}^{\mathrm{reg}}$ over $\mathfrak{p}/\!\!/K$.*

1. *A choice of Kostant-Rallis section $\kappa\colon \mathfrak{p}/\!\!/K \to \mathfrak{p}^{\mathrm{reg}}$ gives an identification*

$$\mathfrak{p}^{reg} \simeq \underline{N_G(K)}/\kappa^* I^{\mathrm{reg}}_{N_G(K)},$$

   as schemes over $\mathfrak{p}/\!\!/K$.

2. *The regular quotient $\mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ is identified with the quotient*

$$\mathfrak{p}^{\mathrm{reg}} /\!\!/ H \simeq \frac{\underline{N_G(K)}/\underline{H}}{\kappa^*(I^{\mathrm{reg}}_{N_G(K)}/I^{\mathrm{reg}}_H)} = \frac{\underline{N_G(K)}/\underline{Z_- \cdot H}}{\kappa^*(I^{\mathrm{reg}}_{N_G(K)}/Z_- \cdot I^{\mathrm{reg}}_H)}$$

Proof. By acting on the image of the Kostant–Rallis section κ we gain a surjective morphism $\underline{N_G(K)} \to \mathfrak{p}^{reg}$. This clearly factors through an isomorphism $\underline{N_G(K)}/\kappa^* I^{\mathrm{reg}}_{N_G(K)} \to \mathfrak{p}^{reg}$.

Part (2) then follows by considering the transitive $\underline{N_G(K)}$ action on the right hand side of the description of $\mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ given in Theorem 1.3.12. $\square$

Remark 1.3.21. Since I^{reg}_H and $I^{\mathrm{reg}}_{N_G(K)}$ are affine over $\mathfrak{p}/\!\!/H$, it is tempting to think that Theorem 1.3.20 implies that the map $\mathfrak{p}^{reg} /\!\!/ H \to \mathfrak{p}/\!\!/H$ is affine. However, this is not the case: As the Example 1.3.45 illustrates, nonseparated behavior can occur in codimension greater strictly than 1.

Reduction to the Simple, Simply Connected Case

We begin by reducing to the case of G simple, simply connected. The author is thankful to S. Leslie for his theory of θ-compatible z-extensions (see Section 5 of [30], which dramatically simplifies the exposition.

We begin by recalling the definition and existence of θ-compatible z-extensions. Recall that a z-extension is a surjective homomorphism $\alpha\colon \widetilde{G} \to G$ such that

- $\widetilde{G}$ is connected, reductive over k with derived group simply connected, and
- $\ker(\alpha)$ is a central, split torus in G.

Proposition 1.3.22. *([30] Prop. 5.3) There exists a z-extension $\alpha\colon \widetilde{G} \to G$ together with an involution $\widetilde{\theta}\colon \widetilde{G} \to \widetilde{G}$ such that*

1. *There is a commutative diagram*

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\theta}"] \arrow[d, "\alpha"] & \widetilde{G} \arrow[d, "\alpha"] \\ G \arrow[r, "\theta"] & G \end{tikzcd}$$

2. *Let $\widetilde{Z} = Z(\widetilde{G})$ and $Z = Z(G)$. Let N denote the kernel of the surjection*

$$\widetilde{Z} \to Z$$

   The restriction $\alpha\colon \widetilde{G}^{\widetilde{\theta}} \to G^{\theta}$ is surjective and its restriction

$$\alpha^{-1}(K) \to K$$

   is a z-extension with kernel $N^{\widetilde{\theta}}$.

3. *For every symmetric pair (G, θ, H) on (G, θ), put $\widetilde{H} := \alpha^{-1}(H)$. There is an exact sequence*

$$1 \to N_H \to \widetilde{H} \to H \to 1$$

for $N_H = H \cap N$ connected, and $(\widetilde{G}, \widetilde{\theta}, \widetilde{H})$ is a symmetric pair on $\widetilde{G}$.

4. *Let G^{sc} be the simply connected cover of G with maximal torus $T^{sc} \subset G^{sc}$. Then, one can take $\widetilde{G} = T^{sc} \times^{Z(G^{sc})} G^{sc}$ in the above parts. Moreover, for this choice of $\widetilde{G}$, there is a decomposition*

$$\widetilde{G} = \widetilde{Z} \times \prod_{i=1}^{s} G_i \times \prod_{j=1}^{t} (G'_j \times G'_j) \tag{1.3.5}$$

where $\widetilde{Z} = Z(\widetilde{G}) = T^{sc}$ is the (θ stable) center of $\widetilde{G}$, each G_i and G'_j is simple, and θ restricts to an involution on each of the G_i for $1 \leq i \leq s$ and acts by the swapping involution $\theta(g_1, g_2) = (g_2, g_1)$ on each product $G'_j \times G'_j$ for $1 \leq j \leq t$.

Following [30], we refer to z-extensions satisfying the conditions above as "θ-compatible z-extensions."

Remark 1.3.23. Note that the statement in [30] works rationally, e.g. it is not assumed that k is algebraically closed.

We now give an application of the above to the regular quotient.

Theorem 1.3.24. *Let (G, θ, H) be a symmetric pair. Let $\widetilde{G}$ be as in Equation 1.3.5. For each $1 \leq i \leq s$, let $H_i = G_i \cap \alpha^{-1}(H)$. We denote with subscript i the corresponding daata for the symmetric pair (G_i, θ_i, H_i). Then there is a $\mathbb{G}_m$-equivariant isomorphism*

$$(\mathfrak{p}^{reg} /\!\!/ H) \cong ((Lie(Z(G)^0) \cap \mathfrak{p}) /\!\!/ (\alpha^{-1}(H) \cap Z(G)^0) \times \prod_i \mathfrak{p}_i^{reg} /\!\!/ H_i \times \prod_j \mathfrak{g}'_j /\!\!/ G_j$$

Proof. Since the action of $\widetilde{H}$ on $\mathfrak{p}$ factors through H with finite quotient, the regular locus of these two actions agree. Furthermore, the group scheme of centralizers $\widetilde{I}$ in $\widetilde{H}$ decomposes as a product of centralizers in the subgroups H_i, so $\mathfrak{p}^{reg} = \mathfrak{z}(\mathfrak{g}) \times \prod_i \mathfrak{p}_i^{\mathrm{reg}} \times \prod_j (\mathfrak{g}'_j)^{reg}$. This gives a decomposition of stacks

$$\mathfrak{p}^{reg}/\widetilde{H} \simeq \Big(\mathfrak{z}(\mathfrak{g})/(H \cap Z(G)^0) \times \prod_i \mathfrak{p}_i^{reg}/H_i \times \prod_j (\mathfrak{g}_j')^{reg}/G_j\Big)/N_H.$$

where N_H acts trivially, and moreover,

$$\mathfrak{p}^{reg}/H \simeq \mathfrak{z}(\mathfrak{g})/(H \cap Z(G)^0) \times \prod_i \mathfrak{p}_i^{reg}/H_i \times \prod_j (\mathfrak{g}_j')^{reg}/G_j.$$

The result now follows as the regular quotient respects products and is invariant under trivial actions. □

Reduction to Levi Subgroups

From now on, we assume that G is simple, simply connected. We describe in this section the process of Levi induction needed to compute the gluing loci for the regular quotient. The Levi induction we use is closely related to the degeneration used by S. Leslie in Section 5.1.1 of [30] termed the "descendant" of an element $X \in \mathfrak{a}$. We will first recall this definition and give a root theoretic description of the Lie algebra of this Levi subgroup. Then, we will prove a reduction result, see Proposition 1.3.37, which will prove invaluable for computations.

Definition 1.3.25. Fix a θ-Cartan $\mathfrak{a} \subset \mathfrak{p}$. For an element $X \in \mathfrak{a}$, the descendant of X is the tuple $(G_X^\circ, \theta|_{G_X^\circ}, H \cap G_X^\circ)$ where G_X° is the connected component of the stabilizer of X in G.

Proposition 1.3.26. *For $X \in \mathfrak{a}$, the descendant at X is a symmetric pair.*

Proof. First, note that G_X° is a Levi of G and hence is reductive. Since the adjoint action preserves the Cartan decomposition, G_X° is stable by the action of θ. It is trivial that $H \cap G_X^\circ$ contains the connected component of the fixed point scheme $K_{G_X^\circ} := [(G_X^\circ)^\theta]^\circ$. Moreover, for any $h \in H \cap G_X^\circ$, $\mathrm{Ad}(h)$ by definition lies in $N_G(K) = N_G(G^\theta)$ and so preserves $(G_X^\circ)^\theta$. Hence, $H \cap G_X^\circ$ is contained in the normalizer $N_{G_X^\circ}((G_X^\circ)^\theta) = N_{G_X^\circ}(K_{G_X^\circ})$.

Finally, we note that G_X° is smooth as it is a Levi of G and the characteristic is assumed to be good for G. For H_X, take a character λ whose Lie algebra is the span of X. The subgroup H_X is the fixed points of the image of λ, which is a subgroup of multiplicative type. Hence, by Proposition R.1.1 of [13], H_X is smooth, and therefore any union of its components is smooth. In particular, so is $H \cap G_X^\circ$. $\square$

We now give a root theoretic description of the descendant construction. Fix a maximally θ split torus T of G containing maximal θ-split torus A. For $X \in \mathfrak{a}$, we define the subset of restricted roots $S \subset \Phi_r$ by

$$S = \{\nu \in \Phi_r \colon \nu(X) = 0\}$$

Note that the subset S is constant with respect to the stratification on $\mathfrak{a}$ defined by inclusion in root hyperplanes of $\mathfrak{a}$.

Definition 1.3.27. Let $L = L_X$ be the connected Levi subgroup of G whose Lie algebra is the sum

$$\mathfrak{l} = \mathfrak{l}_X = \mathfrak{t} + \sum_{\beta \in r^{-1}(S)} \mathfrak{g}_\beta$$

where r is the restriction map $r \colon \Phi \to \Phi_r \cup \{0\}$ as in definition 1.2.5. We refer to L as a *distinguished Levi.*

Proposition 1.3.28. *The Levi L defined above is equal to G_X°.*

Proof. The action of $\mathrm{Ad}(X)$ on the root space $\mathfrak{g}_\beta$ for $\beta \in \Phi$ is given by the restricted root $r(\beta)$. In particular, the centralizer $\mathfrak{g}_X$ of X in $\mathfrak{g}$ is the sum of the root spaces $\mathfrak{g}_\beta$ for which $r(\beta) \in \ker(X) = S$. $\square$

We denote $K_L, \mathfrak{p}_L, \mathfrak{a}_L, A_L$, etc. for the corresponding objects in the Levi $L = G_X^\circ$, and $H_L := H \cap L$. Note that $T \subset L$, so that $A_L = A$ and $\mathfrak{a}_L = \mathfrak{a}$. Furthermore,we have

$W_{\mathfrak{a},L} \subset W_{\mathfrak{a}}$ is generated by the reflections across roots in S, and $\Phi_{r,L} = S \subset \Phi_r$. We will relate the structure of the stack $\mathfrak{p}/H$ to the stack $\mathfrak{p}_L/H_L$.

Remark 1.3.29. Note that if one takes $H = K$, it is *not* true in general that $H_L = K_L$. For example, consider the symmetric pair corresponding to the diagonally embedded $\mathrm{SL}_2 \times \mathrm{SL}_2 \subset \mathrm{SL}_4$ (see example 1.2.26). Then, one choice of Levi L of the above form corresponds to the Lie algebra

$$\mathfrak{l} = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : \mathrm{T}r(A) = 0 \right\}.$$

One computes

$$K \cap L = \left\{ \begin{pmatrix} g & \\ & g \end{pmatrix} : \det(g) = \pm 1 \right\}.$$

In particular, $K \cap L$ is disconnected. By definition, $K_L = (L^\theta)^\circ = (K \cap L)^\circ$ is the digaonally embedded copy of SL_2.

We relate the Weyl groups and GIT quotients as follows.

Proposition 1.3.30. *The little Weyl group $W_{\mathfrak{a},L}$ of the Levi L is a subgroup of $W_{\mathfrak{a}}$. Let $D_L \subset \mathfrak{a}$ be the union of hyperplanes $\mathfrak{h}_\alpha$ in $\mathfrak{a}$ such that $\alpha \notin \Phi_{r,L}$ and let $\pi \colon \mathfrak{a} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ be the projection map. The map of GIT quotients*

$$\varphi_L \colon \mathfrak{a}/\!\!/W_{\mathfrak{a},L} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$$

is étale away from $\pi(D_L) \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$.

Proof. Recall that $W_{\mathfrak{a}}$ is generated by reflections given by roots in the restricted root system Φ_r, and similarly for $W_{\mathfrak{a},L}$ with the restricted roots system for the Levi, $\Phi_{r,L}$. [See Richardson, Lemma 4.5.] We claim that $\Phi_{r,L}$ is a subroot system of Φ_r corresponding to roots in L. Indeed, by construction $\mathfrak{a}_L = \mathfrak{a}$, and the root system of L with respect to a maximally

θ-split torus T is a subroot system of G with respect to T. Therefore, restricting to $\mathfrak{a}$ gives a subroot system $\Phi_{\mathfrak{a},L}$ of $\Phi_{\mathfrak{a}}$. It follow that $W_{\mathfrak{a},L} \subset W_{\mathfrak{a}}$ is a subgroup.

Now, consider the map φ_L as above. The projection π factors through φ_L, giving covers π and π_L as below.

$$\begin{tikzcd} \mathfrak{a} \arrow[d, "\pi_L"'] \arrow[dr, "\pi"] & \\ \mathfrak{a}/\!\!/W_{\mathfrak{a},L} \arrow[r, "\varphi_L"'] & \mathfrak{a}/\!\!/W_{\mathfrak{a}} \end{tikzcd}$$

As both sides are quotients of $\mathfrak{a}$, the ramification locus of φ_L is exactly those images $\pi(X) \in \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ such that $X \in \mathfrak{a}^w$ for some $w \in W_{\mathfrak{a}} \setminus W_{\mathfrak{a},L}$. But for any $w \in W_{\mathfrak{a}}$ with minimal presentation $w = s_1 \dots s_n$ for simple reflections s_j, the fixed locus is $\mathfrak{a}^w = \bigcap \mathfrak{h}_j$ where $\mathfrak{h}_j$ is the hyperplane fixed by s_j. In particular, from our earlier description of $W_{\mathfrak{a},L}$, it follows that φ_L is ramified exactly on those $\pi(a)$ such that $a \in \mathfrak{h}_\alpha$ for some $\alpha \in \Phi_r \setminus \Phi_{r,L}$. $\square$

Definition 1.3.31. Let φ_L, π, π_L, and D_L be as above. Let $U_L \subset \mathfrak{a}$ be the complement of D_L in $\mathfrak{a}$, and let V_L denote the image of U_L in $\mathfrak{a}/\!\!/W_{\mathfrak{a},L}$.

Let $i_L : \mathfrak{p}_L \to \mathfrak{p}$ be the inclusion map and let

$$p\colon \mathfrak{p} \to \mathfrak{p}/\!\!/H \quad \text{and} \quad p_L\colon \mathfrak{p}_L \to \mathfrak{p}_L/\!\!/H_L$$

be the projection maps. For any scheme S over $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$, we denote $S|_{U_L} := S \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} U_L$.

Lemma 1.3.32. *The map i_L restricts to a H_L equivariant map*

$$i_L\colon \mathfrak{p}_L^{\mathrm{reg}}|_{U_L} \to \mathfrak{p}^{\mathrm{reg}}|_{U_L}$$

In particular, there is a morphism

$$\chi_L : \mathfrak{p}_L^{reg} /\!\!/ H_L|_{U_L} \to \mathfrak{p}^{reg} /\!\!/ H|_{U_L}.$$

Proof. For any $X \in p_L^{-1}(U_L)$, $(I_G)_X = (I_L)_X$ and hence $(I_{H_L})_X = (I_H)_X \cap (I_L)_X$. Hence the map i_L sends regular elements of $p_L^{-1}(\pi(U_L)) \subset \mathfrak{p}_L$ to regular elements of $\mathfrak{p}$. The result follows. $\square$

Lemma 1.3.33. *We have a canonical isomorphism* $i_L^* I^{\text{reg}}|_{U_L} \simeq I_L^{\text{reg}}|_{U_L}$. *In particular,* $\chi_L^* J|_{V_L} \simeq J_L|_{V_L}$.

Proof. Now, there is a map $J_L|_{U_L} \to \chi_L^* J$ from the inclusion $H_L \subset H$. To show this is an isomorphism on U_L, it suffices to check on fibers. Let $y \in U_L$ have preimage $x \in p_L^{-1}(\pi(U_L))$. Then, we can identify $(J_L)_y = (I_{H_L})_x$ and $(\varphi^* J)_y = (I_H)_x$, where the result now follows. $\square$

Theorem 1.3.34. *The morphism χ_L induces an isomorphism*

$$\mathfrak{p}_L^{reg} /\!\!/ H_L\big|_{U_L} \to \mathfrak{p}^{reg} /\!\!/ H\big|_{U_L}$$

Proof. This follows from the description of the regular quotient in Theorem 1.3.20 combined with the isomorphism of regular centralizers over U_L in Lemma 1.3.33. $\square$

Theorem 1.3.35. *There is a canonical isomorphism of stacks*

$$\psi \colon \mathfrak{p}_L^{\text{reg}}/H_L|_{U_L} \to \mathfrak{p}^{\text{reg}}/H|_{U_L}$$

Proof. Recall that $\mathfrak{p}^{\text{reg}}/H$ is a gerbe over $\mathfrak{p}^{\text{reg}} /\!\!/ H$ banded by J, and similarly, $\mathfrak{p}_L^{\text{reg}}/H_L$ is a gerbe over $\mathfrak{p}_L^{\text{reg}} /\!\!/ H_L$ banded by J_L. Hence, to conclude, it suffices to note that the map $\psi|_{V_L}$ is a map of $J_L \simeq \varphi^* J$ gerbes over $\mathfrak{p}_L^{\text{reg}} /\!\!/ H_L$. $\square$

To conclude, we will reduce to computations of regular nilpotent orbits. To do this we will use Proposition 1.3.37.

Lemma 1.3.36. *Let (G, θ) be a semisimple group with involution θ. Then the intersection* *of all root hyperplanes of the restricted root system is*

$$\bigcap_{\alpha \in \Phi_r} H_\alpha = 0 \in \mathfrak{a}.$$

Proof. For every root hyperplane H_α in $\mathfrak{a}$, let S_α denote the set of all hyperplanes of $\mathfrak{t}$ which restrict to H_α. Let S denote the set of all hyperplanes in $\mathfrak{t}$ which contain $\mathfrak{a}$. Note that for every root hyperplane $H \subset \mathfrak{t}$, $H \cap \mathfrak{a}$ is either a root hyperplane in $\mathfrak{a}$ or is all of $\mathfrak{a}$. Hence, S and S_α as α varies gives a partition of all root hyperplanes of $\mathfrak{t}$. We conclude that

$$\bigcap_{\alpha \in \Phi_r} H_\alpha = \mathfrak{a} \cap \bigcap_{\alpha \in \Phi_r} H_\alpha \subset \left(\bigcap_{H \in S} H \right) \cap \bigcap_{\alpha} \left(\bigcap_{H \in S_\alpha} H \right) = \bigcap_{H \subset \mathfrak{t}:\ \text{root hyperplane}} H = \{0\} \quad \square$$

Proposition 1.3.37. *Let $(\mathfrak{p}, H)$ be a symmetric pair associated with (G, θ) for G a reductive group. Let $Y = \cap_\alpha H_\alpha \subset \mathfrak{a}$ be the intersection of all root hyperplanes in $\mathfrak{a}$. There is then an isomorphism of stacks over $p(Y)$:*

$$\mathfrak{p}/H \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong \mathcal{N}_{\mathfrak{p}}/H \times p(Y)$$

Restricting to regular elements gives:

$$\mathfrak{p}^{reg}/H \times_{\mathfrak{a}/\!\!/W_a} p(Y) \cong \mathcal{N}_{\mathfrak{p}}^{reg}/H \times p(Y)$$

Proof. Let $\widetilde{G} = T^{sc} \times^{Z^{sc}} G^{sc}$ be the θ-compatible z-extension as in part 3 of Proposition 1.3.22 with center $\widetilde{Z} = Z(\widetilde{G})$ and symmetric pair structure $(\widetilde{G}, \widetilde{\theta}, \widetilde{H})$. We will denote by $\widetilde{\mathfrak{p}}$ the (-1) eigenspace of $\widetilde{\theta}$ on $\widetilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$. Likewise, we let $(G^{sc}, \theta^{sc}, H^{sc})$ denote the induced symmetric pair structure on G^{sc} with $\mathfrak{p}^{sc}$ the corresponding (-1) eigenspace in $\mathfrak{g}^{sc}$. Then, from the proof of Theorem 1.3.24, we have

$$\mathfrak{p}/H \simeq ((Lie(Z(G)^0) \cap \mathfrak{p})/\!\!/(\alpha^{-1}(H) \cap Z(G)^0) \times \mathfrak{p}^{sc}/H^{sc}$$

Therefore, by Lemma 1.3.36, we have $p(Y) \simeq ((Lie(Z(G)^0) \cap \mathfrak{p})/\!\!/(\alpha^{-1}(H) \cap Z(G)^0) \times \{0\}$ and the result follows. $\square$

We conclude the description of the enhanced quotient $\mathfrak{p}^{reg} /\!\!/\!\!/ H$ for G simple. Recall from Theorem 1.3.12 that it suffices to describe the gluing on $\widetilde{\mathfrak{p}/\!\!/H}$ explicitly. To describe the gluing on the intersection of some hyperplanes $\cap_i H_i$, we take the Levi L associated to the H_i. Then, by Theorem 1.3.34 and Proposition 1.3.37 the number of sheets of $\widetilde{\mathfrak{p}/\!\!/H}$ over $\cap_i H_i$ is determined by the regular K-orbits of the nilpotent cone for L. This is determined by the list in Proposition 1.2.34. For all simple groups except $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$, there are at most 2 regular nilpotent orbits, so it suffices to describe only the number of sheets in fibers of the map $\mathfrak{p}^{reg} /\!\!/\!\!/ H \to \mathfrak{p}^{reg} /\!\!/ H$. For the $\mathrm{SO}_n \times \mathrm{SO}_n \subset \mathrm{SO}_{2n}$ case, one also needs to compute the gluing pattern of the 4 sheets at the origin as it degenerates. Some results on this case are given in Example 1.3.48.

More formally:

Theorem 1.3.38. *Let $(\mathfrak{p}, H)$ be a symmetric pair corresponding to a simple group G,such that $(K, G) \neq (SO(n) \times SO(n) \subset SO(2n))$. Let $U = \cup_L U_L$, where U_L as in Definition 1.3.31 and L ranges over the subgroups L of the form in Definition 1.3.27, such that $\mathcal{N}^{\mathrm{reg}}_{\mathfrak{p}_L}$ has a single regular H_L-orbit*[3]*. We then have that $\mathfrak{p}^{\mathrm{reg}} /\!\!/\!\!/ H \cong \mathfrak{a}/\!\!/W_a \coprod_U \mathfrak{a}/\!\!/W_a$, and this identification is $\mathbb{G}_m$-equivariant.*

Remark 1.3.39. We note that if $\mathcal{N}_{\mathfrak{p}}$ has one irreducible component by Proposition 1.2.34 we can just directly state that $\mathfrak{p}^{reg} /\!\!/\!\!/ H \cong \mathfrak{a}/\!\!/W_a$.

3. We note that this can be worked out using Proposition 1.2.34, together with, if necessary, computing $\pi_0(H_L)$ and its action on irreducible components of $\mathcal{N}_{\mathfrak{p}_L}$.

Proof of Theorem 1.3.38. This follows immediately from Theorem 1.3.12. □

We note that the remaining case of $(K, G) = (G_1, G_1 \times G_1)$ there is no nonseparrated structure as shown in Example 1.3.42. Finally the following proposition show that the regular semisimple locus is always in the open set U of Theorem 1.3.38

Proposition 1.3.40. *The map $\mathfrak{p}^{reg} /\!\!/\!\!/ H \to \mathfrak{p}^{\mathrm{reg}} /\!\!/ H \simeq \mathfrak{a} /\!\!/ W_{\mathfrak{a}}$ is an isomorphism on the complement of the image of all root hyperplanes in $\mathfrak{a}$.*

Proof. The complement of hyperplanes in $\mathfrak{a} /\!\!/ W_{\mathfrak{a}}$ is the space of semisimple, regular elements $\mathfrak{a}^{rs} \subset \mathfrak{a}$. Let $x \in \mathfrak{p}^{reg}$ lie over the image of $s \in \mathfrak{a}^{rs}$ in $\mathfrak{a}^{rs} /\!\!/ W_{\mathfrak{a}}$. Then by Lemma 1.2.16 we have the Jordan decomposition $x = s + n$. Since s is regular and n is regular nilpotent in its centralizer, by Proposition 1.2.17 we must have $n = 0$. Then, the result follows as there is a unique (closed) orbit of semisimple elements in each fiber of the map $\mathfrak{p}^{\mathrm{reg}} \to \mathfrak{a} /\!\!/ W_{\mathfrak{a}}$. □

Theorem 1.3.41. *Let $(\mathfrak{p}, H)$ be a symmetric pair corresponding to a simple group G,such that $(K, G) \neq (SO(n) \times SO(n), SO(2n))$.*

Then $\mathfrak{p} /\!\!/\!\!/ H \cong \mathfrak{a} /\!\!/ W_a \coprod_U \mathfrak{a} /\!\!/ W_a$ where U is the complement of a closed subvariety which is the union of intersections of root hyperplanes.

Proof. This follows immediately from Theorem 1.3.38 and Proposition 1.3.40. □

Examples

Example 1.3.42. Consider the diagonal case $G_1 \overset{\Delta}{\subset} G_1 \times G_1$ from Example 1.2.25. In this case, we have an isomorphism of stacks

$$\mathfrak{p}/G_1 \to \mathfrak{g}_1/G_1$$

by projecting onto the first factor. The latter quotient is well studied over the regular locus. In particular, it is shown in [10, 35] that the map

$$\mathfrak{g}_1^{\mathrm{reg}}/G_1 \to \mathfrak{g}_1^{\mathrm{reg}} /\!\!/ G_1$$

is a gerbe for the descent of $I_{G_1}^{reg}$ to $\mathfrak{g}_1^{\mathrm{reg}} /\!\!/ G_1$. The regular quotient $\mathfrak{g}_1^{\mathrm{reg}} /\!\!/\!\!/ G_1$ is therefore just the GIT quotient $\mathfrak{g}_1^{\mathrm{reg}} /\!\!/ G_1$. We verify below that this agrees with our inductive construction.

Fix a maximal torus T in G_1 and recall that a Cartan in $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{(X, -X) \colon X \in \mathfrak{t}\},$$

while the restricted root system agrees with the root system on G_1. The distinguished Levi subgroup associated to the sub root system $(\alpha_i, -\alpha_i)$ is of the form $L \simeq L_1 \times L_1$ for L_1 the connected Levi of G_1 with

$$\mathfrak{l}_1 = \mathfrak{t} \oplus \sum_i (\mathfrak{g}_1)_{\alpha_i}$$

The involution θ acts on $L_1 \times L_1$ by swapping factors. But by Lemma 1.2.38, there is a single regular nilpotent H_L orbit for this form. Hence, there is no non-separated structure anywhere on $\mathfrak{p}^{\mathrm{reg}} /\!\!/\!\!/ H$.

Example 1.3.43. We revisit Example 1.2.26. Recall $H = K = \mathrm{GL}_n \times \mathrm{GL}_n \subset \mathrm{GL}_{2n} = G$,

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} : \delta \text{ is diagonal} \right\} \subset \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\} = \mathfrak{p}$$

and

$$\Phi_r = \{\pm(\delta_j^* \pm \delta_k^*) | j \neq k \text{ and } 1 \leq j, k \leq n\} \cup \{\pm 2\delta_j^* | 1 \leq j \leq n\}$$

is a simple root system of type C_n. The little Weyl group has the form $W_{\mathfrak{a}} = \{\pm 1\}^n \ltimes S_n$, with S_n permuting the dual basis δ_j^* and $\{\pm 1\}^n$ acting by changing the sign of the coordinates

δ_j^*.

Note also that if $2\delta_j^* = 2\delta_k^* = 0$, then also $\pm\delta_j^* \pm \delta_k^* = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

$(*)$ For every distinct $1 \leq j < k \leq n$, either $\{\pm 2\delta_j^*, \pm 2\delta_k^*\} \not\subset S$ or else $\{\pm 2\delta_j^*, \pm 2\delta_k^*, \pm\delta_j^* \pm \delta_k^*\} \subset S$.

Now we see that any *simple* subroot system of Φ_r satisfying condition $(*)$ is $W_{\mathfrak{a}}$ conjugate to one of the following:

1. The subroot system $\Phi_r = \{\pm\delta_j^* \pm \delta_k^* \colon 1 \leq j, k \leq n_1\}$ for some $n_1 \leq n$. In this case,

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} * & & * & \\ & \eta' & & \delta' \\ \hline * & & * & \\ & \delta' & & \eta' \end{array} \right) : \delta' \text{ and } \eta' \text{ and diagonal } (n-n_1) \times (n-n_1) \text{ matrices} \right\}$$

so that $L \simeq \mathrm{GL}_{n_1} \times T'$ for a torus T' of rank $2(n-n_1)$.

2. The subroot system $\Phi_r = \{\pm(\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq n_1\}$ for some $n_1 \leq n$. In this case,

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \eta'' & & \delta'' \\ \hline B & & A & \\ & \delta'' & & \eta'' \end{array} \right) : \delta'' \text{ and } \eta'' \text{ are diagonal } (n-n_1) \times (n-n_1) \text{ matrices} \right\}$$

so that $L \simeq (\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \times T''$ for T'' a torus and the involution acting on $\mathrm{GL}_2 \times \mathrm{GL}_2$ by swapping factors. There is a unique nilpotent orbit in this case.

For an arbitrary subroot system $S \subset \Phi_r$ satisfying condition $(*)$, S is a product $S_1 \times \cdots \times S_a \times S_1' \times \cdots \times S_b'$ where S_j is $W_{\mathfrak{a}}$ conjugate to a root system of type (1) and S_j' is $W_{\mathfrak{a}}$

conjugate to a root system of type (2). However, we note that condition $(*)$ immediately implies that $a = 1$. Hence, we reduce to distinguished Levis associated with $S_1 \times S_1' \times \cdots \times S_b'$

For such a root system, we have

$$L \simeq \mathrm{GL}_{2n_1} \times (\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_1}) \times \cdots \times (\mathrm{GL}_{m_b} \times \mathrm{GL}_{m_b}) \times T'$$

for T' a torus of rank $2(n - \sum_j n_j - \sum_k m_k)$. We can see easily that

$$H_L = (\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \times \prod_{k=1}^{b} \mathrm{GL}_{m_k} \times (T')^{\theta}$$

where $(T')^{\theta}$ is a (connected) torus of rank $(n - \sum_j n_j - \sum_k m_k)$. In particular, we see that $|\mathcal{N}_{\mathfrak{p}_L}^{\mathrm{reg}}/H_L| = 2$ by comparing with the list in Proposition 1.2.34 for the first factor of $(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_1}) \subset \mathrm{GL}_{2n_1}$.

We hence conclude with a description of the regular quotient $\mathfrak{p}^{reg} /\!\!/ K$ in this case.

Proposition 1.3.44. *For H, $\mathfrak{p}$, and K as in this example, let $V \subset \mathfrak{a}$ be the subscheme that is the complement of all root hyperplanes for roots of the form $\pm 2\delta_i^*$. Then, we have $\mathfrak{p}^{reg} /\!\!/ K \cong \mathfrak{a}/\!\!/W_a \coprod_U \mathfrak{a}/\!\!/W_a$, where $U := V /\!\!/ W_a \subset \mathfrak{a}/\!\!/W_a$.*

Proof. Follows immediately from the above computations and Theorem 1.3.41. □

Example 1.3.45. Consider the split form $\mathrm{SO}_n \subset \mathrm{SL}_n$. If n is odd, there is only one nilpotent orbit and the regular quotient and GIT quotient agree. We will assume therefore that n is even. We have $\mathfrak{a} = \mathfrak{t}$ is the diagonal Cartan inside $\mathfrak{p} = \mathfrak{sym}_n$ (symmetric $n \times n$ matrices). The restricted root system agrees with the root system for SL_n and so is type A_{n-1}. Any distinguished Levi is $W = W_{\mathfrak{a}}$ conjugate to a block diagonal Levi

$$L = S(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_l}) \subset \mathrm{SL}_n$$

where $n_j \geq 1$ and $\sum_j n_j = n$. It is easy to see that

$$H_L = S(O_{n_1} \times \cdots \times O_{n_l})$$

Recall that the symmetric pair $\mathrm{SO}_{n_j} \subset \mathrm{GL}_{n_j}$ has 2 regular nilpotent orbits in the case where n_j is even and 1 when n_j is odd (see Proposition 1.2.34). Without loss of generality, suppose that $n_1, \ldots, n_a$ are even while $n_{a+1}, \ldots, n_l$ are odd.

If $l > a$ (i.e. some n_j is odd), then the quotient map

$$L = S(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_l}) \to \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a} =: L_1$$

is surjective and carries H_L to the subgroup

$$H_1 := O_{n_1} \times \cdots \times O_{n_a}$$

(L_1, H_1) is a symmetric pair, and one checks that $\pi_0(H_1)$ acts freely on components of $\mathcal{N}^{reg}_{\mathfrak{p}_{L_1}}$. In particular, comparing with list of Proposition 1.2.34, we see that

$$\#(\mathcal{N}^{reg}_{\mathfrak{p}_L}/H_L) = \#(\mathcal{N}^{reg}_{\mathfrak{p}_{L_1}}/H_1) = 2^a/2^a = 1.$$

If $l = a$ (i.e. all indices are even), then $\pi_0(H_L)$ acts freely on the components of $\mathcal{N}^{reg}_{\mathfrak{p}_L}$ and comparing with the list in Proposition 1.2.34 gives

$$\#(\mathcal{N}^{reg}_{\mathfrak{p}_L}/H_L) = 2^a/2^{a-1} = 2.$$

We conclude:

Proposition 1.3.46. *Let n be even, and let $\epsilon_1, \ldots, \epsilon_n$ be coordinates for diagonal matrices, so that $\mathfrak{a} = \mathfrak{t}$ is the locus $\sum_j \epsilon_j = 0$. Let* $\mathrm{Even}Par_n$ *denote the set of even partitions of*

$\{1,\dots,n\}$, i.e. the set of decompositions $\{1,\dots,n\} = S_1 \sqcup \cdots \sqcup S_k$ with each $|S_i|$ even. Put

$$Z_{j_1,\dots,j_l} = \{\epsilon_{j_1} = \epsilon_{j_2} = \cdots = \epsilon_{j_l}\} \subset \mathfrak{a}$$

and $Z_S = \cap_m Z_{S_m}$ for $S \in \mathrm{E}venPar_n$.

Define V to be the complement of

$$\cup_{S \in \mathrm{E}venPar_n} Z_S \subset \mathfrak{a}$$

and let $U = V /\!\!/ W_{\mathfrak{a}} \subset \mathfrak{a}/\!\!/W_{\mathfrak{a}}$. Then in this example, we have $\mathfrak{p}^{\mathrm{reg}} /\!\!/ K \simeq \mathfrak{a}/\!\!/W_{\mathfrak{a}} \coprod_U \mathfrak{a}/\!\!/W_{\mathfrak{a}}$.

Proof. This follows by Theorem 1.3.41 and the computations above. □

Remark 1.3.47. In the statement of Proposition 1.3.46, one can replace $\mathrm{E}venPar_n$ with only those partitions $\{1,\dots,n\} = S_1 \sqcup \cdots \sqcup S_l$ for which $\#S_j = 2$ for all j. The open set U in the Proposition has complement of codimension $n - (n/2)$. Hence this gives an example of a symmetric pair for which the gluing does not occur along the complement of a divisor.

Example 1.3.48. Consider the split form $\mathrm{SO}_{2n} \times \mathrm{SO}_{2n} \subset \mathrm{SO}_{4n}$ of Example 1.2.27. Recall that restricted roots are of the form

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon j \neq k, 1 \leq j,k \leq 2n\}$$

This gives the root system of type D_{2n}, which is simple and simply-laced when $n \geq 2$ and is the product $D_2 = A_1 \times A_1$ when $n = 1$. Recall that for $n \geq 2$ this is the unique family of simple symmetric pairs up to isogeny for which there are 4 regular nilpotent orbits.

For inductive purposes, we will need to describe the case when $n = 1$: When $n = 1$, the root system D_2 is not simple, and the isogeny of Theorem 1.3.24 is the map

$$\xi \colon \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$$

with θ lifting to the involution $\theta(g) = g^{-t}$ on each copy of SL_2, and

$$\xi^{-1}(\mathrm{SO}_2 \times \mathrm{SO}_2) = \mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SL}_2 \times \mathrm{SL}_2 \,.$$

In particular, the regular quotient can be described as a product:

Proposition 1.3.49. *The form* $\mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SO}_4$ *has regular quotient given by the product of regular quotients*

$$\mathfrak{p} /\!\!/ K = \mathfrak{p}_1 /\!\!/ K_1 \times \mathfrak{p}_1 /\!\!/ K_1$$

where $(G_1, \theta_1, H_1) = (\mathrm{SL}_2, g \mapsto g^{-t}, \mathrm{SO}_2)$.

Now, consider the case $\mathrm{SO}_{2n} \times \mathrm{SO}_{2n} \subset \mathrm{SO}_{4n}$ for $n \geq 2$. In this case, the root system is type D_{2n}, is simple, and is simply laced. The little Weyl group is $W_{\mathfrak{a}} = \{\pm 1\}^{n-1} \ltimes S_n$, where S_n acts on the coordinates δ_j^* by permuting the indices j and $\{\pm 1\}^{n-1}$ acts by changing an even number of signs on the δ_j^*. Any *simple* root subsystem of Φ_r is $W_{\mathfrak{a}}$ conjugate to one of the following:

1. $S = \{i(\pm\delta_j^* \pm \delta_k^*)\colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq 2n$. The Levi associated to this root subsystem is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B^t & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : \begin{array}{l} A \in \mathfrak{so}_{m_1}, \text{ and } \delta' \text{ is a diagonal} \\ (2n-m_1) \times (2n-m_1) \text{ matrix} \end{array} \right\}$$

giving $L = \mathrm{SO}_{2m_1} \times T'$ for T' a split torus and θ acting on SO_{2m_1} by conjugation by $\mathrm{d}iag(I_{m_1}, -I_{m_1})$.

2. $S = \{\pm i(\delta_j^* - \delta_k^*)\colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq 2n$. The Levi associated to this S

is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : \begin{array}{c} A \in \mathfrak{so}_{m_1},\ B \in \mathfrak{sym}_{m_1},\ \text{and } \delta' \\ \text{is a diagonal } (2n-m_1)\times(2n-m_1) \text{ matrix} \end{array} \right\}$$

giving $L \simeq \mathrm{GL}_{m_1} \times T'$ for T' a split torus and θ acting on GL_{m_1} by $g \mapsto g^{-t}$.

3. $S = \{\pm i(\delta_j^* + \delta_k^*) \colon 1 \leq j < k \leq m_1\}$ for some *odd* $m_1 < 2n$. The Levi associated to this root subsystem is

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{2n-m_1} & & \delta' \\ \hline -B & & A & \\ & -\delta' & & \mathbf{0}_{2n-m_1} \end{array} \right) : \begin{array}{c} A \in \mathfrak{so}_{m_1},\ B = B^{\ddagger},\ \text{and } \delta' \text{ is} \\ \text{a diagonal } (2n-m_1)\times(2n-m_1) \text{ matrix} \end{array} \right\}$$

where $B^{\ddagger} = \big((-1)^{ij} b_{ji}\big)_{1 \leq i,j \leq m_1}$. This again gives $L \simeq \mathrm{GL}_{m_1} \times T'$.

Any arbitrary subroot system of Φ_r is a product $S = S_1 \times \cdots S_l$ where each S_j is conjugate to one of the three above root subsystems. Of the above, types (2) [when m_1 is even] and (1) can contribute nontrivial regular nilpotent orbits. Suppose that

$$S = \prod_{j=1}^{a} S_j^{(1)} \times \prod_{j=1}^{b} S_j^{(2)} \times \prod_{j=1}^{c} S_j^{(3)}$$

where $S_j^{(k)}$ has rank $m_j^{(k)}$.

Let $H_L^{(2)} = \prod_{j=1}^{b} O_{m_j^{(2)}}$ and $H_L^{(3)} = \prod_{j=1}^{b} O_{m_j^{(3)}}$. Then, we compute

$$H_L = \left(\prod_{j=1}^{a} O_{m_j^{(1)}} \times S\left(\prod_{k=1}^{a} O_{m_k^{(1)}} \times H_L^{(2)} \times H_L^{(3)}\right)\right) \cap \left(S\left(\prod_{j=1}^{a} O_{m_j^{(1)}} \times H_L^{(2)} \times H_L^{(3)}\right) \times \prod_{k=1}^{a} O_{m_k^{(1)}}\right)$$

with the intersection being taken inside

$$\prod_{j=1}^{a} (O_{m_j^{(1)}} \times O_{m_j^{(1)}}) \times H_L^{(2)} \times H_L^{(3)}$$

We recall that $\mathrm{SO}_{m_j} \subset \mathrm{GL}_{m_j}$ has 1 regular, nilpotent K orbit when m_j is odd and 2 when m_j is even, and that $\mathrm{SO}_{m_j} \times \mathrm{SO}_{m_j} \subset \mathrm{SO}_{2m_j}$ has 2 regular nilpotent orbits when m_j is odd and 4 when m_j is even. Define the following invariants:

Let N_1^e be the number of Levis of type (1) with m_j even and N_1^o the number of Levis of type (1) with m_j odd. Let N_2^e and N_2^o be defined similarly for type (2) Levis, and N_3 the number of type (3) Levis. Let ϵ_1 be zero if both $N_1^o = 0$ and $N_1^e \neq 0$ and 1 otherwise, and let ϵ_2 be zero if $\sum_j m_j = 2n$ and $N_2^o = N_3 = 0$ and 1 otherwise. Then, we can count the nilpotent orbits by studying the components of H_L. We find:

$$\#(\mathcal{N}_{\mathfrak{p}_L}^{reg}/H_L) = 2^{2N_1^e + N_1^o + N_2^e} / (2^{N_1^e + \epsilon_1 - 1} \cdot 2^{N_1^e + N_1^o + N_2^e + \epsilon_2 - 1}) = 2^{2 - \epsilon_1 - \epsilon_2}$$

For example, for $\mathrm{SO}_4 \times \mathrm{SO}_4 \subset \mathrm{SO}_8$, we have:

1. There are 4 sheets over the loci:

   (i) Fix an ordering i_j of $\{1, 2, 3, 4\}$. $\{\delta_{i_1}^* = \delta_{i_2}^* = 0,\ \delta_{i_3}^* = \pm\delta_{i_4}^*\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ that are W conjugate to $(\mathrm{SO}_2 \times \mathrm{SO}_2) \times \mathrm{SO}_2 \subset \mathrm{SO}_4 \times \mathrm{GL}_2$.

   (ii) The origin $\{\delta_i^* = 0 \text{ for all } i\}$, i.e. the strata corresponding to the distinguished Levi SO_8.

2. There are 2 sheets over the loci:

   (i) Fix an ordering i_j of $\{1, 2, 3, 4\}$, and fix signs $\epsilon_j \in \{\pm 1\}$. $\{\delta_{i_j}^* = \epsilon_j \delta_{i_{j+1}}^* : j = 1, 2, 3\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ which are W-conjugate to $\mathrm{SO}_4 \subset \mathrm{GL}_4$.

(ii) Fix i_j an ordering of $\{1,2,3,4\}$ and signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. $\{\delta^*_{i_1} = \epsilon_1 \delta^*_{i_2}, \delta^*_{i_3} = \epsilon_2 \delta^*_{i_4}\}$, i.e. strata corresponding to distinguished Levis $H_L \subset L$ which are W conjugate to $S(O_2 \times O_2) \subset \mathrm{GL}_2 \times \mathrm{GL}_2$.

3. 1 sheet over all other strata.

Note that the above list allows us to reduce to *only* Levis conjugate to $(\mathrm{SO}_2 \times \mathrm{SO}_2) \times \mathrm{SO}_2 \subset \mathrm{SO}_4 \times \mathrm{GL}_2$. Note that the regular quotient of this symmetric pair is given by the regular quotient of $\mathrm{SO}_2 \times \mathrm{SO}_2 \subset \mathrm{SO}_4$, whose gluing pattern was studied above.

Example 1.3.50. Consider the more general case of $\mathrm{SO}_m \times \mathrm{SO}_{2n-m} \subset \mathrm{SO}_{2n}$ from Example 1.2.28. Note that this is split for $m = n$ (see previous example for this case with m even) and quasi-split for $m = n-1$. Note furthermore that if m were odd, then there would be a single regular, nilpotent orbit for this pair, and the regular quotient would be equal to the GIT quotient. We will therefore consider only the case where m is even. The restricted root system for this pair is given by

$$\Phi_r = \{i(\pm\delta^*_j \pm \delta^*_k) \colon 1 \leq j < k \leq m\} \cup \{\pm i\delta^*_j \colon 1 \leq j \leq m\}.$$

This is a simple root system of type B_m, and the little Weyl group $W_{\mathfrak{a}} = \{\pm 1\}^m \ltimes S_m$ acts by permutations and sign changes on the δ^*_j, $1 \leq j \leq m$. Note that we again have a condition on subroot systems of Φ_r that can arise as the set of roots vanishing a collection of hyperplanes in $\mathfrak{a}$. Namely,

$(*)$ For every distinct $1 \leq j < k \leq m$, either $\{\pm i\delta^*_j, \pm i\delta^*_k\} \not\subset S$ or else $\{\pm i\delta^*_j, \pm i\delta^*_k, i(\pm\delta^*_j \pm \delta^*_k)\} \subset S$.

A *simple* root subsystem $S \subset \Phi_r$ satisfying condition $(*)$ is $W_{\mathfrak{a}}$ conjugate to one of the following:

1. $\{i(\pm\delta^*_j \pm \delta^*_k) \colon 1 \leq j < k \leq m_1\} \cup \{i(\pm\delta^*_j) \colon 1 \leq j \leq m_1\}$ for some $m_1 \leq m$. This has distinguished Levi given by $L \simeq \mathrm{SO}_{2(m_1+n-m)} \times T'$ where T' is a (not split) torus, and

θ acts on $\mathrm{SO}_{2(m_1+n-m)}$ by conjugation by $\mathrm{diag}(I_{m_1,2n-2m-m_1})$.

2. $\{\pm i(\delta_j^* - \delta_k^*) \colon 1 \le j < k \le m_1\}$ for some $m_1 \le m$. This has associated Levi given by $L = \mathrm{GL}_{m_1} \times T'$ where T' is a (nonsplit) torus and θ acts on GL_{m_1} by the split involution $\theta(g) = g^{-t}$.

An arbitrary subroot system satisfying $(*)$ is a product of at most one Levi of type (1) and an arbitrary number of Levis of type (2). There are two sheets over the strata:

- Fix $m_1, \ldots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\{\delta_j^* = 0 \colon 1 \le j \le m_1\} \cap \bigcap_{l=2}^{r-1} \{\delta_j^* = \delta_k^* \colon m_l + 1 \le j < k \le m_{l+1}\}$.
- Fix $m_1, \ldots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\bigcap_{l=1}^{r-1} \{\delta_j^* = \delta_k^* \colon m_l + 1 \le j < k \le m_{l+1}\}$.

and map $\mathfrak{p} /\!\!/ K \to \mathfrak{p} /\!\!/ K$ is an isomorphism elsewhere.

Example 1.3.51. Consider the case of $\mathrm{SO}_m \times \mathrm{SO}_{2n+1-m} \subset \mathrm{SO}_{2n+1}$, $m \le n$, of Example 1.2.29. Recall that the root system is the simple type B_m root system

$$\Phi_r = \{i(\pm\delta_j^* \pm \delta_k^*) \colon 1 \le j < k \le m\} \cup \{\pm i\delta_j^* \colon 1 \le j \le m\}.$$

The little Weyl group $W_{\mathfrak{a}} = \{\pm 1\}^m \ltimes S_m$ acts on Φ_r by permutation and sign change on the δ_j^*. Note that by Proposition 1.2.34, there is a unique regular, nilpotent K orbit in $\mathfrak{p}$ when m is odd and two when m is even. We therefore restrict to the case when m is even.

Note also that if $i\delta_j^* = i\delta_k^* = 0$, then also $i(\pm\delta_j^* \pm \delta_k^*) = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

$(*)$ For every distinct $1 \le j < k \le n$, either $\{\pm i\delta_j^*, \pm i\delta_k^*\} \not\subset S$ or else $\{\pm i\delta_j^*, \pm i\delta_k^*, i(\pm\delta_j^* \pm \delta_k^*)\} \subset S$.

Any *simple* subroot system of Φ_r is $W_{\mathfrak{a}}$ conjugate to one of the following:

1. $\{i(\pm\delta_j^* \pm \delta_k^*), \pm i\delta_j^* \colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq m$. The associated Levi for this subroot system is $L \simeq \mathrm{SO}_{2(m_1+n-m)+1} \times T'$ for T' a torus and θ acting on $\mathrm{SO}_{2(m_1+n-m)+1}$ by conjugation by the matrix

$$I_{m_1,m_1+2(n-m)+1} = \mathrm{diag}(I_{m_1}, -I_{m_1+2(n-m)+1}).$$

2. $\{i(\pm\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq m_1\}$ for some $m_1 \leq m$. The associated Levi for this subroot system is $L \simeq \mathrm{GL}_{m_1} \times T'$ for T' a torus, and θ acting on GL_{m_1} by $\theta(g) = g^{-t}$.

An arbitrary subroot system of Φ_r is a product of Levis of type (1) and (2) with at most one type (1) Levi appearing. There are 2 sheets precisely over the following strata in $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$:

- Fix $m_1, \ldots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\{\delta_j^* = 0 \colon 1 \leq j \leq m_1\} \cap \bigcap_{l=2}^{r-1} \{\delta_j^* = \delta_k^* \colon m_l + 1 \leq j < k \leq m_{l+1}\}$.
- Fix $m_1, \ldots, m_r$ such that each m_k is even and $\sum_l m_l = 2n$. The $W_{\mathfrak{a}}$ orbit of $\bigcap_{l=1}^{r-1} \{\delta_j^* = \delta_k^* \colon m_l + 1 \leq j < k \leq m_{l+1}\}$.

and the map $\mathfrak{p}/\!\!/\!\!/K \to \mathfrak{p}/\!\!/K$ is an isomorphism elsewhere.

Example 1.3.52. Consider the split form $\mathrm{GL}_n \subset \mathrm{Sp}_{2n}$ of example 1.2.30. The restricted root system agrees with the usual root system, which is type C_n. We use the presentation

$$\Phi_r = \{\pm\delta_j^* \pm \delta_k^* \colon 1 \leq j < k \leq n\} \cup \{\pm 2\delta_j^* \colon 1 \leq j \leq n\}$$

for the root system, where the dual basis δ_j^* is chosen with respect to the coordinate vectors for the Cartan $\mathfrak{a}$ in example 1.2.30. The little Weyl group $W_{\mathfrak{a}} = W = \{\pm 1\}^n \ltimes S_n$ acts on Φ_r by letting S_n act by permuting the δ_j^* and $\{\pm 1\}^n$ act by sign changes on the δ_j^*.

Note also that if $2\delta_j^* = 2\delta_k^* = 0$, then also $\pm\delta_j^* \pm \delta_k^* = 0$. Hence, we need only deal with distinguished Levis associated to subroot systems $S \subset \Phi_r$ satisfying:

$(*)$ For every distinct $1 \leq j < k \leq n$, either $\{\pm 2\delta_j^*, \pm 2\delta_k^*\} \not\subset S$ or else $\{\pm 2\delta_j^*, \pm 2\delta_k^*, \pm\delta_j^* \pm \delta_k^*\} \subset S$.

Now we see that any *simple* subroot system S of Φ_r satisfying condition $(*)$ is $W_{\mathfrak{a}}$ conjugate to one of the following:

1. $\{\pm\delta_j^* \pm \delta_k^* \colon 1 \leq j < k \leq n_1\} \cup \{\pm 2\delta_j \colon 1 \leq j \leq n_1\}$ for some $n_1 \leq n$. The associated Levi has

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} * & & * & \\ & 0 & & \delta' \\ \hline * & & * & \\ & \delta' & & 0 \end{array} \right) : \delta' \text{ is a diagonal } (n-n_1) \times (n-n_1) \text{ matrix} \right\}$$

giving $L \simeq \mathrm{Sp}_{2n_1} \times T'$ for T' a split torus and θ acting on Sp_{2n_1} by conjugation by $\mathrm{d}iag(I_{n_1}, -I_{n_1})$.

2. $\{\pm(\delta_j^* - \delta_k^*) \colon 1 \leq j < k \leq n_1\}$ for some $n_1 \leq n$. The associated Levi has

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} A & & B & \\ & \mathbf{0}_{n-n_1} & & \delta' \\ \hline B & & -A^t & \\ & \delta' & & \mathbf{0}_{n-n_1} \end{array} \right) : \begin{array}{c} A \in \mathfrak{so}_{n_1},\ B \in \mathfrak{sym}_{n_1}, \text{ and } \delta' \text{ is} \\ \text{a diagonal } (n-n_1) \times (n-n_1) \text{ matrix} \end{array} \right\}$$

giving $L \simeq \mathrm{GL}_{n_1} \times T'$ where T' is a split torus and θ acts on GL_{n_1} by the split form $g \mapsto g^{-t}$.

An arbitrary root subsystem of Φ_r satisfying $(*)$ is a product of the above types, with at most one factor of type (1) appearing. In particular, the strata of $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ with two sheets are the following:

1. For any subset $T \subset \{1, \dots, n\}$, $\{\delta_j^* = 0 \colon j \in T\}$. These are the strata corresponding to Levis of type (1).

2. Fix the following data: Even integers n_k such that $\sum_k n_k = n$; an ordering i_j of the numbers $\{1, \dots, n\}$; signs $\epsilon_j \in \{\pm 1\}$. Then, consider the strata

$$\bigcap_k \left\{ \delta_{i_j}^* = \epsilon_j \delta_{i_{j+1}}^* \colon \left(\sum_{l=1}^{k} n_l \right) + 1 \leq j \leq \sum_{l=1}^{k+1} n_l \right\}.$$

This corresponds to Levis which are products of type (2) Levis $L \simeq \prod_k \mathrm{GL}_{n_k}$ where all the n_k are even.

1.3.4 Galois Description of J for Quasi-split Symmetric Pairs

For the results of this section, we restrict to the case of quasisplit symmetric pairs. We will denote $\mathfrak{c} = \mathfrak{p} /\!\!/ H \simeq \mathfrak{a} /\!\!/ W_{\mathfrak{a}}$. In particular, by Proposition 1.2.21, we assume that the regular centralizer group scheme $I^{reg} = I_H^{\mathrm{reg}} \to \mathfrak{p}^{reg}$ is abelian, and hence descends to a smooth, commutative group scheme $J \to \mathfrak{c}$. Our goal in this section is a Galois description of the regular centralizer group scheme J for quasisplit symmetric pairs. More precisely, following the skeleton of Section 2.4 in [35], we seek a flat cover

$$\pi \colon \widetilde{\mathfrak{c}} \to \mathfrak{c},$$

which is with group $W_{\widetilde{\mathfrak{c}}}$ acting on the centralizer group scheme $C = C_H(\mathfrak{a})$, such that J embeds as an open subgroup scheme of the Weyl restriction $\mathrm{Res}_{\mathfrak{c}}^{\widetilde{\mathfrak{c}}}(\widetilde{\mathfrak{c}} \times C_H)^{W_{\widetilde{\mathfrak{c}}}}$. Such a description was the objective of Section 5.1 of [16] and Section 4 of [29] using cameral covers modeled on the flat cover $\mathfrak{a} \to \mathfrak{c}$, which is generically Galois with group $W_{\mathfrak{a}}$. However, this point of view is inadequate for general results on regular centralizers, as illustrated by the following example.

Example 1.3.53. Consider the symmetric pair on $G = \mathrm{SL}_3$ given by the involution θ conjugating by $\begin{pmatrix} 1 & \\ & -I_2 \end{pmatrix}$ with $H = S(\mathbb{G}_m \times \mathrm{GL}_2)$ embedded block diagonally. We choose

$$\mathfrak{a} = \left\{ \begin{pmatrix} & \delta & 0 \\ \delta & & \\ 0 & & \end{pmatrix} : \delta \in k \right\} \simeq \mathbf{A}^1,$$

so that its centralizer is

$$C_H = C_H(\mathfrak{a}) = \left\{ \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} : x^2 y = 1 \right\}$$

One can verify that $W_{\mathfrak{a}} \simeq \{\pm 1\}$ acts trivially on C. Hence, $\mathrm{Res}_{\mathfrak{c}}^{\mathfrak{a}}(C \times \mathfrak{a})^{W_{\mathfrak{a}}} = C \times \mathfrak{c}$ is the constant group scheme. It is trivial to compute the fiber of $\pi_0(J_H)$ at 0 is μ_3 while the fiber of $\mathrm{Res}_{\mathfrak{c}}^{\mathfrak{a}}(C \times \mathfrak{a})^{W_{\mathfrak{a}}} = C \times \mathfrak{c}$ is connected everywhere. Hence, J_H cannot be an open subscheme of $\mathrm{Res}_{\mathfrak{c}}^{\mathfrak{a}}(C \times \mathfrak{a})^{W_{\mathfrak{a}}}$.

To produce a more descriptive cover, we will work with a fundamentally different object: Instead of modeling our cover off a Grothendieck-Springer resolution that classifies certain Borels of the group G, we work with a "parabolic cover" classifying certain parabolics of the subgroup H.

Relations Between Weyl groups

We begin by discussing some various Weyl groups that will emerge in later sections and the relationships between them. Fix a fixed maximal θ-split torus T containing a maximally θ-split torus A. We define $C = C_H(A)$. The connected component C° is a (not necessarily maximal) torus of H. Let $W_C = N_H(C^\circ)/Z_H(C^\circ)$ be the Weyl group of C° in H.

Lemma 1.3.54. *(1) There are inclusions $N_H(A) \subset N_H(C)$ and $C_H(A) \subset C_H(C)$.*

(2) There is a canonical map $\xi\colon W_A \to W_C$.

Proof. Since the form is quasisplit, Proposition 1.2.21 implies that $C^\circ \subset C_H(A) \subset T$. In particular, $C_H(C^\circ)$ is abelian and since $C^\circ \subset C_H(A)$, we have

$$C_H(A) \subset C_H(C_H(A)) \subset C_H(C).$$

Moreover, it is elementary to see that $N_H(A) \subset N_H(C_H(A)^\circ)$. Since conjugation preserves the identity component of $C_H(A)$, we conclude that $N_H(A) \subset N_H(C)$.

The above implies that there is a well-defined map $\xi\colon W_A \to W_C$. □

Remark 1.3.55. We will denote the image of the morphism ξ above by $W_{im} \subset W_C$. In all classical cases, $W_{im} = W_C$; that is, ξ is surjective. The author conjectures that this is true in general, though it has not been checked for exceptional types.

Definition 1.3.56. Let $W_{\mathrm{ker}} \subset W_A$ be the kernel of the map $\xi\colon W_A \to W_C$ constructed above.

Lemma 1.3.57. *The projection map $\pi\colon \mathfrak{a} \to \mathfrak{c}$ as well as its subcover $\mathfrak{a}/\!\!/W_{\mathrm{ker}} \to \mathfrak{c}$ are finite and flat.*

Proof. By the assumption on the characteristic, the characteristic p of the field does not divide $|W_{\mathrm{ker}}|$ and so $\mathfrak{a}/\!\!/W_{\mathrm{ker}}$ is Cohen-Macaulay, see 6.4.6 of [6]. Since $\mathfrak{c}$ is regular, both claims follow by Miracle Flatness. □

Definition and Geometry of Parabolic Covers

We begin with a preliminary lemma relating borels/parabolics of H to those of G.

Lemma 1.3.58. *Any Borel B of H is the intersection $B_G \cap H$ for some θ-stable Borel B_G of G. Moreover, suppose we have a Levi $M \subset H$ of H and suppose that there exists a Levi $M_G \subset G$ such that $M_G \cap H = M$. Then, there exists parabolics $Q \subset H$ with Levi M and $Q_G \subset G$ with Levi M_G such that for any parabolic P of H conjugate to Q, there exists a parabolic P_G of G conjugate to Q_G such that $P = P_G \cap H$.*

Proof. We begin with the statement on Borels, first showing that the intersection any θ-stable Borel of G with H is a Borel of H. Let B_G be a θ-stable Borel of G and put $B = B_G \cap H$. It is clear that B is solvable; we must show that H/B is projective, or equivalently that $H/B \simeq H \cdot B_G \subset G/B_G$ is a closed embedding. We claim that, in fact, $H \cdot B_G = (G/B_G)^\theta$. Let B'_G be another θ-stable Borel of G, and let $g \in G$ be such that $g \cdot B_G = B'_G$. Applying θ gives $\theta(g) \cdot B_G = B'_G$, so that $g^{-1}\theta(g) \in N_G(B_G) = B_G$ and further

$$g^{-1}\theta(g) \in \{b \in B_G \colon \theta(b) = b^{-1}\}.$$

Recall that the image of the morphism $G \to G$ sending $g \mapsto g^{-1}\theta(g)$ is precisely those g in G such that $\theta(g) = g^{-1}$. As B_G is θ-stable, the image of the morphism $B_G \to B_G$ is therefore the intersection

$$\{g \in G \colon \theta(g) = g^{-1}\} \cap B_G = \{b \in B_G \colon \theta(b) = b^{-1}\},$$

and so there exists $b \in B_G$ so that $g^{-1}\theta(g) = b^{-1}\theta(b)$. We have $gb^{-1} \cdot B_G = B'_G$ while also

$$(gb^{-1})^{-1}\theta(gb^{-1}) = b(g^{-1}\theta(g))\theta(b)^{-1} = b(b^{-1}\theta(b))\theta(b)^{-1} = 1.$$

We conclude that B_G and B'_G are G^θ conjugate. Hence, $G^\theta \cdot B_G = (G/B_G)^\theta$ and since the flag variety is invariant under central isogeny, also $H \cdot B_G = (G/B_G)^\theta$. Now, since the fixed locus of an algebraic involution is closed, we conclude that $B_G \cap H$ is Borel.

Since all Borels of H are conjugate under K, and since K conjugacy preserves θ-stability, we conclude that every Borel of H is the intersection of a Borel of G with H.

Now, suppose we have a Levi $M \subset H$ of H and suppose that there exists a Levi $M_G \subset G$ such that $M_G \cap H = M$. Choose any parabolic Q_G of G with Levi factor M_G, and put $Q = Q_G \cap H$. We must show first that Q is a parabolic with Levi M. Let $B_G \subset Q_G$ be a Borel in Q_G and $B = B_G \cap H$ the corresponding Borel of H. Since we have a commutative diagram

$$\begin{tikzcd} H/B \arrow[r] \arrow[d] & H/Q \arrow[d] \\ G/B_G \arrow[r] & G/Q_G \end{tikzcd}$$

with the horizontal arrows being surjective and the left vertical arrow being a closed immersion, it follows that H/Q is a closed subvariety of G/Q_G. In particular, H/Q is projective, and Q is a parabolic in H. That the Levi factor of Q is M follows from intersecting the decomposition $Q_G = M_G \cdot U_G$ where U_G is the unipotent radical of Q_G.

Now, for any parabolic P of H conjugate to Q by $h \in H$, we may take the corresponding conjugate parabolic $h \cdot Q_G$ of G. The result follows. $\square$

We now introduce the central object of study; the parabolic cameral cover.

Definition 1.3.59. Define the Levi subgroup $M := C_H(C^\circ)^\circ$ of H and $M_G := Z_G(C^\circ)^\circ$ denote the analoguous Levi of G.

Let Q and Q_G be parabolics of H and G, respectively, satisfying the conditions of Lemma 1.3.58 for the Levis $M \subset M_G$.

Remark 1.3.60. Note that M, M_G are well defined up to conjugation by H. The parabolics Q and Q_G may involve further choice as parabolics with fixed Levi type are not necessarily conjugate.

Definition 1.3.61. Fix the data of Definition 1.3.59. Then, let

$$\widetilde{\mathfrak{p}}^{reg} = \left\{ (X,P) \colon \begin{array}{l} X \in \mathfrak{p}^{reg},\ \mathrm{L}ie(P) \supset C_{\mathfrak{k}}(C_{\mathfrak{k}}(X)) \text{ and } P \text{ is a} \\ \text{parabolic of } H \text{ which is } H \text{ conjugate to } Q \end{array} \right\}$$

and denote by $\pi_{\mathfrak{p}} \colon \widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ the projection map. We denote by $\tilde{\mathfrak{p}}^{\mathrm{rs}} = \tilde{\mathfrak{p}}^{\mathrm{reg}}|_{\mathfrak{p}^{rs}}$.

Proposition 1.3.62. *Let $W_{H,M} = N_H(M)/M$ denote the relative Weyl group of M. There is a map $\widetilde{\mathfrak{p}}^{\mathrm{rs}} \to \mathfrak{a}/\!\!/W_{\mathrm{ker}} \times_{W_{\mathfrak{a}}/W_{\mathrm{ker}}} W_{H,M} = \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$ fitting into a Cartesian diagram*

$$\begin{tikzcd} \tilde{\mathfrak{p}}^{\mathrm{rs}} \arrow[rr] \arrow[d] & & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\ \mathfrak{p}^{\mathrm{rs}} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a \end{tikzcd}$$

Proof. Let $\widetilde{\mathfrak{a}}^{reg} = \pi_{\mathfrak{p}}^{-1}(\mathfrak{a}^{reg})$ be the closed subscheme of $\widetilde{\mathfrak{p}}^{reg}$ over the subscheme $\mathfrak{a}^{reg} \subset \mathfrak{p}^{\mathrm{rs}}$. Then, we have a projection map $\widetilde{\mathfrak{a}}^{reg} \to \mathfrak{a}^{reg}$.

The relative Weyl group $W_{H,M}$ acts on the set of all parabolics of H with Levi M, and the set of such parabolics conjugate to Q is a $W_{H,M}$ torsor under this action.

The map $\mathfrak{a}^{\mathrm{reg}} \times W_{H,M} \to \widetilde{\mathfrak{a}}^{\mathrm{reg}}$ sending $(X,w) \mapsto (X, w \cdot Q)$ is an isomorphism, and the resulting diagram

$$\begin{tikzcd} \widetilde{\mathfrak{a}}^{reg} \arrow[r, "\cong"] \arrow[d] & \mathfrak{a}^{reg} \times W_{H,M} \arrow[r] & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\ \mathfrak{a}^{reg} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a \end{tikzcd}$$

with the top right arrow given by quotienting by the diagonal action of $W_{\mathfrak{a}}$, is commutative and Cartesian.

Recall from Lemma 1.2.18 that $H \cdot \mathfrak{a}^{reg} = \mathfrak{p}^{\mathrm{rs}}$. The H-orbit $H \cdot (\widetilde{\mathfrak{a}}^{reg}) \to \mathfrak{p}^{rs}$ is therefore a surjective $W_{H,M}$ cover. Since there is an inclusion of $W_{H,M}$ covers

$$H \cdot (\widetilde{\mathfrak{a}}^{reg}) \hookrightarrow \widetilde{\mathfrak{p}}^{\mathrm{r}ss}$$

we conclude that this map is an isomorphism. In particular, since the map $\widetilde{\mathfrak{a}}^{\mathrm{r}eg} \to \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$ is $Z_H(\mathfrak{a}^{\mathrm{r}eg})$ equivariant, it extends to a map

$$p^{\mathrm{rs}} \colon \widetilde{\mathfrak{p}}^{\mathrm{rs}} \to \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}$$

by taking

$$p^{\mathrm{rs}}(h \cdot \gamma) = p^{\mathrm{rs}}(\gamma).$$

This map is H invariant, and hence the diagram

$$\begin{tikzcd}
\tilde{\mathfrak{p}}^{\mathrm{rs}} \arrow[rr] \arrow[d] & & \mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M} \arrow[d] \\
\mathfrak{p}^{\mathrm{rs}} \arrow[r] & \mathfrak{p}/\!\!/H \arrow[r, "\cong"] & \mathfrak{a}/\!\!/W_a
\end{tikzcd}$$

is a Cartesian square. □

We now seek to extend Proposition 1.3.62 over the regular locus. For this, we will need Lemma 1.3.65, which demonstrates the structure of the parabolic cover over the regular nilpotent locus. We begin with some preliminary lemmas.

First, let us set some notation. Fix $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{r}eg}$. We will be interested in the fiber of $\pi_{\mathfrak{p}}$ over e. Let B_G be the unique Borel in G such that $e \in \mathrm{L}ie(B_G)$, and let $T_G \subset B_G$ be the unique maximal torus of B_G. Note that B_G is necessarily θ stable, and hence T_G contains a maximal torus T_H of H. Choose $\mathfrak{a}$ so that $C = C_H(\mathfrak{a}) \subset T_H$, and let S_H denote the set of simple roots of H with respect to T_H and the Borel $B_G \cap H$ (which is Borel by the proof of Lemma 1.3.58). Denote by $V \subset S_H$ the set of simple roots of H which are trivial on C.

We furthermore fix pinnings $\{e^{\mathfrak{g}}_\alpha\}_{\alpha\in\Phi}$ and $\{e^{\mathfrak{k}}_\alpha\}$ of the groups G and H with respect to the choice of tori $T_G \supset T_H$, respectively.

Lemma 1.3.63. *For all simple quasisplit forms except possibly the quasisplit form on E_6, we have:*

1. *The nilpotent*
$$\widetilde{e} = \sum_{\alpha\in S_G} e^{\mathfrak{g}}_\alpha$$
lies in $\mathfrak{p}$.

2. *Consider the nilpotent element*
$$e' := \sum_{\alpha\in S_H\setminus V} e^{\mathfrak{k}}_\alpha \in \mathfrak{h}$$
The nilpotents $\widetilde{e}\in\mathfrak{p}$ and $e'\in\mathfrak{h}$ commute.

Proof. We proceed case-by-case through the classification of quasisplit simple symmetric pairs and some explicit computations. As the definitions of e, e' do not depend on isogeny class or center, we further assume that all pairs are of the form (G,θ,K) for G simple semisimple.

In the case of any split pair (G,θ,K), we have that $C = Z(G)\cap K$, and so $V = S_K$. Hence, $e' = 0$ and the result is trivial.

In the case of $(G,\theta,K) = (\mathrm{SL}_{2n},\theta,S(\mathrm{GL}_n\times\mathrm{GL}_n))$ from example 1.2.26, we have T_K the set of diagonal matrices, and

$$\widetilde{e} = \begin{pmatrix} 0 & I_n \\ N_n & 0 \end{pmatrix} \quad \text{where } N_n = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \text{ is } n\times n$$

On the other hand, $V = \emptyset$ and

$$e' = \begin{pmatrix} N_n & \\ & N_n \end{pmatrix}$$

It is easy to check $[\widetilde{e}, e'] = 0$.

In the case of $(G, \theta, K) = (\mathrm{SL}_{2n+1}, \theta, S(\mathrm{GL}_n \times \mathrm{GL}_{n+1}))$, we have T_K is again the diagonal matrices, and

$$\widetilde{e} = \begin{pmatrix} & \vec{0} & I_n \\ I_n & & \\ \vec{0}^t & & \end{pmatrix} \quad \text{where } \vec{0} \text{ is the } n \times 1 \text{ zero vector.}$$

Then, again $V = \emptyset$ and

$$e' = \begin{pmatrix} N_n & \\ & N_{n+1} \end{pmatrix}$$

It is again an easy check that $[\widetilde{e}, e'] = 0$.

Now consider the case $(G, \theta, K) = (\mathrm{SO}_{2n+2}, \theta, SO_n \times \mathrm{SO}_{n+2})$ of example 1.2.28, with θ given by

$$\theta \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} = \begin{pmatrix} A & -B \\ B^t & D \end{pmatrix}$$

In this case, we take

$$T_G = \left\{ \begin{pmatrix} & a_1 & & & & & \\ -a_1 & & & & & & \\ & & & a_2 & & & \\ & & -a_2 & & & & \\ & & & & \ddots & & \\ & & & & & & a_{n+1} \\ & & & & & -a_{n+1} & \end{pmatrix} : a_j \in k^\times \right\}$$

When n is even, this gives $T_K = T_G$ while when n is odd, T_K is the $n-1$ dimensional subtorus given by $a_{(n+1)/2} = 0$. We divide into cases based on the parity of n: If n is even, we get:

$$\widetilde{e} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \quad \text{where } B = \begin{pmatrix} i & 1 & i & -1 & & & & & & \\ -1 & i & 1 & i & & & & & & \\ & & i & 1 & i & -1 & & & & \\ & & -1 & i & 1 & i & & & & \\ & & & \ddots & & \ddots & & & & \\ & & & & i & 1 & i & -1 & & \\ & & & & -1 & i & 1 & i & & \\ & & & & & & i & 1 & 2i & 0 \\ & & & & & & -1 & i & 2 & 0 \end{pmatrix} \text{is } n \times (n+1)$$

Moreover, V consists of all but two roots of SO_{n+1}, i.e.

$$e' = \begin{pmatrix} \mathbf{0}_n & & & & & \\ & \mathbf{0}_{n-2} & & & & \\ & & & & 0 & 2 \\ & & & & 0 & 2i \\ & & 0 & 0 & & \\ & & -2 & -2i & & \end{pmatrix}$$

So that $[\widetilde{e}, e'] = 0$.

In the case of n odd, we instead get

$$\widetilde{e} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \quad \text{where } B = \begin{pmatrix} 0 & i & 1 & i & -1 & & & & \\ 0 & -1 & i & 1 & i & & & & \\ & & & i & 1 & i & -1 & & \\ & & & -1 & i & 1 & i & & \\ & & & & \ddots & & \ddots & & \\ & & & & & i & 1 & i & -1 \\ & & & & & -1 & i & 1 & i \\ & & & & & & & 2i & 2 \end{pmatrix} \text{is } n \times (n+1)$$

Again, V consists of all but two roots of SO_{n+1}, and we have

$$e' = \begin{pmatrix} \mathbf{0}_n & & & & & & \\ & \mathbf{0}_{n-2} & & & & & \\ & & & & 0 & 2 & 1 \\ & & & & 0 & 2i & i \\ & & 0 & 0 & & & \\ & & -2 & -2i & & & \\ & & -1 & -i & & & \end{pmatrix}$$

We can again verify the commuting property.

The only remaining simple, semisimple quasisplit involution is the quasisplit form on E_6. □

Lemma 1.3.64. *Let $e' = \sum_{\alpha\in S_H\setminus V} e_\alpha^{\mathfrak{k}} \in \mathcal{N}_{\mathfrak{p}}$ for S_H a set of simple roots of H and $V \subset S_H$. Then one can choose an associated cocharacter λ' (See Definition 1.6.3 and Lemma 1.6.5.) such that $C_H(\lambda') = C_H(h')$ for*

$$h' = \sum_{\alpha\in S_H\setminus V} \alpha \in \mathfrak{h}$$

In each H conjugacy class of parabolics P of H, there is a unique parabolic P such that $\mathrm{Lie}(P) \supset C_{\mathfrak{h}}(e')$ and P has Levi factor $C_H(\lambda')$.

Proof. First, we note that h' satisfies the $\mathfrak{sl}_2$ relation

$$[h', e'] = 2e'$$

Similarly, it is elementary to check that the cocharacter $\lambda' = \prod_{\alpha\in S_H\setminus V} \alpha$ is associated to e', and that $C_H(\lambda') = C_H(h')$.

A parabolic P in each conjugacy class of parabolics of H with Levi factor $L = C_H(\lambda')$ is determined by a choice of polarization of the root system generated by $S_H\setminus V$. In particular, we note that for the choice of polarization agreeing with the polarization determining $S_H \subset$

Φ_H, we have $\mathrm{Lie}(P) \supset C_{\mathfrak{h}}(e')$ and this P is unique with this property. $\square$

Lemma 1.3.65. *We keep notation as in Definition 1.3.59. Let G satisfy condition $(*)$. For any regular, nilpotent element $e \in \mathcal{N}_{\mathfrak{p}}^{\mathrm{reg}}$, there exists a parabolic P of H so that P is conjugate to Q and* $\mathrm{Lie}(P) \supset C_{\mathfrak{h}}(C_{\mathfrak{h}}(e))$. *Moreover, there are finitely many parabolics P satisfying these properties.*

Proof. We may assume that G is simple of type not E_6. Moreover, as the fibers of $\pi_{\mathfrak{p}}$ are invariant with respect to the H action, we may reduce to the case $e = \widetilde{e}$ in the notation of Lemma 1.3.63. For such elements, we have $e' \in C_{\mathfrak{h}}(\widetilde{e})$, and so $C_{\mathfrak{h}}(e') \supset C_{\mathfrak{h}}(C_{\mathfrak{h}}(\widetilde{e}))$. Now, by Lemma 1.3.64 applied for H, there is a unique parabolic P of H containing $C_{\mathfrak{h}}(e')$ with the Levi factor $C_{\mathfrak{h}}(\lambda')$. But we can compute

$$C_{\mathfrak{h}}(\lambda') = C_{\mathfrak{h}}\left(\sum_{\alpha \in S_H \setminus V} \alpha\right) = C_{\mathfrak{h}}(C^{\circ}) = M.$$

So the existence is proved.

For the finiteness condition, we note that we can also apply the above argument to show that there exist parabolics P_G of G which are H conjugate to Q_G and which contain $C_{\mathfrak{g}}(C_{\mathfrak{h}}(e))$. By Lemma 1.3.58, each parabolic P of H is obtained by intersecting one of these P_G with H. In particular, the number of parabolics P of H satisfying the conditions of the Lemma is bounded by the number of parabolics P_G of G satisfying the conditions of the Lemma. Since $e \in \mathrm{Lie}(P_G)$ for any such P_G, it follows that each P_G contains the unique Borel B_G whose Lie algebra contains e. As there are finitely many parabolics of fixed type extending a Borel, finiteness follows. $\square$

Theorem 1.3.66. *The map $\widetilde{\mathfrak{p}}^{\mathrm{reg}} \to \mathfrak{p}^{\mathrm{reg}}$ is quasifinite, and there is an action of H on $\widetilde{\mathfrak{p}}^{\mathrm{reg}}$ such that the GIT quotient*

$$\widetilde{\mathfrak{c}} := \widetilde{\mathfrak{p}}^{\mathrm{reg}} /\!\!/ H$$

fits into a Cartesian diagram

$$\begin{tikzcd} \widetilde{\mathfrak{p}}^{reg} \arrow[r] \arrow[d] & \widetilde{\mathfrak{c}} \arrow[d] \\ \mathfrak{p}^{reg} \arrow[r] & \mathfrak{c} \end{tikzcd}$$

and the map $\widetilde{\mathfrak{c}} \to \mathfrak{c}$ *is finite with the components of* $\widetilde{\mathfrak{c}}$ *indexed by* $W_{H,M}/W_C$*, each isomorphic to the finite flat cover* $\mathfrak{a}/\!\!/W_{\ker} \to \mathfrak{c}$*.*

Proof. We first prove the map $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ is quasifinite. Let $x \in \mathfrak{p}^{reg}$ and consider the Jordan decomposition $x = s + n$ as in Lemma 1.2.16. Let L be the distinguished Levi G_s°. Then, there exists a minimal parabolic $P_L \subset L$ such that P_L is conjugate to a fixed parabolic $Q_L \supset C_{H_L}(C)^\circ$ by Corollary 1.3.65. In particular, the Levi factor for P_L has

$$\mathrm{Lie}(M_L) = C_{\mathfrak{h}_L}(C_{\mathfrak{h}_L}(n)) = C_{\mathfrak{h}_L}\big(C_{\mathfrak{h}}(s) \cap C_{\mathfrak{h}}(n)\big) = C_{\mathfrak{h}_L}(C_{\mathfrak{h}}(x)) \subset C_{\mathfrak{h}}(C_{\mathfrak{h}}(x)) = \mathrm{Lie}(M)$$

and since there exist finitely many parabolics P conjugate to Q extending the parabolic P_L we conclude the fiber over x is finite.

Observe also that $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ factors through

$$\widetilde{\mathfrak{p}}^{reg} \subset \mathfrak{p}^{reg} \times H/Q \to \mathfrak{p}^{reg}$$

and so is projective. In particular, we also deduce that the map $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ is finite. In particular, we deduce a unique extension of the morphism

$$(\mathfrak{a}^{reg} \times_{W_{\mathfrak{a}}} W_{H,M}) \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \mathfrak{p}^{rs} \to \widetilde{\mathfrak{p}}^{rs}$$

constructed in Proposition 1.3.62 to a surjective map

$$\xi\colon (\mathfrak{a} \times_{W_{\mathfrak{a}}} W_{H,M}) \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \mathfrak{p}^{reg} \to \widetilde{\mathfrak{p}}^{reg}$$

By Zariski's Main Theorem, this map induces isomorphisms on each irreducible component

$$(\mathfrak{a} \times_{W_{\mathfrak{a}}} W_C) \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \mathfrak{p}^{\mathrm{reg}} = (\mathfrak{a}/\!\!/W_{\mathrm{ker}}) \times_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}} \mathfrak{p}^{\mathrm{reg}} \simeq \widetilde{\mathfrak{p}}^{\mathrm{reg}}_{[w]}$$

In particular, the map ξ constructed above is equivariant with respect the action of the group H, and hence, the GIT quotient $\widetilde{\mathfrak{p}}^{\mathrm{reg}}/\!\!/H$ is a cover of $\mathfrak{c}$. We will denote $\widetilde{\mathfrak{c}} := \widetilde{\mathfrak{p}}^{\mathrm{reg}}/\!\!/H$. The resulting properties of the map

$$\widetilde{\mathfrak{c}} \to \mathfrak{c}$$

are now immediate. □

Galois Description of Regular Centralizers.

For the purposes of this section, we restrict to the setting of quasisplit symmetric pairs with connected regular centralizers. Due to the reliance on Lemma 1.3.65, we exclude the quasisplit form on E_6, though we expect the results of this section should extend to this case.

Let $\mathcal{P}ar \to \widetilde{\mathfrak{p}}^{\mathrm{reg}}$ denote the universal parabolic over $\widetilde{\mathfrak{p}}^{\mathrm{reg}}$ whose fiber over a k-point (x, P) of $\widetilde{\mathfrak{p}}^{\mathrm{reg}}$ is the parabolic P. For each parabolic P, we have a projection $P \to M_P := P/U_P$ to its Levi quotient. Let $\mathcal{M} \to \widetilde{\mathfrak{p}}^{\mathrm{reg}}$ be the universal Levi and

$$\pi\colon \mathcal{P}ar \to \mathcal{M}$$

the projection map.

Lemma 1.3.67. *Let $Z(\mathcal{M})$ denote the subgroup scheme of $\mathcal{M}$ whose fiber over $(x, P) \in \widetilde{\mathfrak{p}}^{\mathrm{reg}}$ consists of the center of the corresponding Levi quotient $Z(M_P)$. Then, $Z(\mathcal{M}) \simeq Z(M) \times \widetilde{\mathfrak{p}}^{\mathrm{reg}}$ is a constant group scheme over $\widetilde{\mathfrak{p}}^{\mathrm{reg}}$.*

Proof. For any two parabolics P, P' which are H conjugate to Q, and any element $h \in H$

conjugating $P' = h \cdot P$, there is an induced map on the quotients

$$h\colon M_P \to M_{P'}$$

and hence also on the corresponding centers. The choice of such an $h \in H$ conjugating P to P' is unique only up to right multiplication by elements of P. However, for any $g \in P$, the induced map

$$g\colon M_P \to M_P$$

is multiplication by an element of M_P and hence the induced map on $Z(M_P)$ is the identity. We conclude that the isomorphism $h\colon Z(M_P) \to Z(M_{P'})$ is canonical, and the lemma follows. $\square$

For every $[w] \in W_{H,M}/W_C$, let $C_{[w]}$ denote the embedding $w \cdot C \subset Z(M)$, where $w \cdot C$ is the conjugation of C by any lift $\dot{w} \in N_H(M)$. Recall the decomposition of $\widetilde{\mathfrak{p}}^{reg}$ into components

$$\widetilde{\mathfrak{p}}^{reg} = \bigcup_{[w] \in W_{H,M}/W_C} \widetilde{\mathfrak{p}}^{\mathrm{reg}}_{[w]}.$$

Proposition 1.3.68. *There is a canonical $W_{H,M}$ equivariant map*

$$\iota\colon \pi_{\mathfrak{p}}^* I^{\mathrm{reg}} \to Z(M) \times \widetilde{\mathfrak{p}}^{reg}$$

which restricts to a map

$$\iota_{[w]}\colon \pi_{\mathfrak{p}}^* I^{\mathrm{reg}}|_{\widetilde{\mathfrak{p}}^{\mathrm{reg}}_{[w]}} \to C_{[w]} \times \widetilde{\mathfrak{p}}^{\mathrm{reg}}_{[w]}$$

for every $[w] \in W_{H,M}/W_C$.

Proof. We describe the map ι first. For this, we need the following claim.

Claim. For every $(x, P) \in \widetilde{\mathfrak{p}}^{\mathrm{reg}}$, there is an inclusion $\pi_{\mathfrak{p}}^* I^{\mathrm{reg}}_x \subset P$.

Proof of Claim. This follows from the definition of $\widetilde{\mathfrak{p}}^{\mathrm{reg}}$ and the assumption that the form be quasi-split. $\square$

For $h \in I_x^{reg}$ and $(x, P) \in \widetilde{\mathfrak{p}}^{reg}$, we consider its image

$$h \mod [\mathrm{Lie}(P), \mathrm{Lie}(P)] \in M_P = \mathcal{M}_{(x,P)}.$$

This defines a map $\iota \colon \pi_{\mathfrak{p}}^* I^{reg} \to \mathcal{M}$. Note that when h is regular semisimple, its image lies in the center $Z(M_P)$ and moreover $h \in Z_{M_P}(h)$, where the latter is identified with $C_{[w]} \subset Z(M)$. Hence, the image of ι lands in $Z(\mathcal{M}) \simeq Z(M) \times \widetilde{\mathfrak{p}}^{reg}$, while the image of its restriction $\iota_{[w]}$ lies in $C_{[w]} \times \widetilde{\mathfrak{p}}_{[w]}^{reg}$. The $W_{H,M}$ equivariance follows from the construction of ι. $\square$

Let $w, w' \in W_{H,M}/W_C$. We can lift identification $\widetilde{\mathfrak{p}}_{[w]}^{reg} \simeq \widetilde{\mathfrak{p}}_{[w']}^{reg}$ to an isomorphism of the constant group schemes $C_{[w]} \times \widetilde{\mathfrak{p}}_{[w]}^{reg} \simeq C_{[w']} \times \widetilde{\mathfrak{p}}_{[w']}^{reg}$ which is well defined up to action by W_C. Therefore, we can descend ι to a morphism

$$\iota_{\mathfrak{c}} \colon J \to \mathrm{Res}_{\mathfrak{c}}^{\widetilde{\mathfrak{c}}}(C \times \widetilde{\mathfrak{c}}^{reg})^{W_{H,M}} \tag{1.3.6}$$

To prove that $\iota_{\mathfrak{c}}$ is an embedding, we will introduce a notion of Levi induction for the parabolic cover. For $x \in \mathfrak{a}$, let $L = G_x^{\circ}$ be the distinguished Levi introduced in section 1.3.3. We use a subscript L to denote corresponding objects for the symmetric pair (L, θ, H_L), e.g. $W_{\mathfrak{a},L}, W_{C,L}$, etc. Note that since $C = C_H(\mathfrak{a})^{\circ} \subset L$, we have $C = C_L$ and so it is immediate that $W_{C,L} \subset W_C$. Moreover, since $W_{\mathfrak{a},L} \subset W_{\mathfrak{a}}$, we get an inclusion of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & W_{\ker,L} \arrow[r] \arrow[d, hook] & W_{\mathfrak{a},L} \arrow[r] \arrow[d, hook] & W_{C,L} \arrow[d, hook] \\ 1 \arrow[r] & W_{\ker} \arrow[r] & W_{\mathfrak{a}} \arrow[r] & W_C \end{tikzcd}$$

Moreover, we have an inclusion of relative Weyl groups given as follows.

Lemma 1.3.69. *For L the distinguished Levi associated to $x \in \mathfrak{a}$ as above, there is a canonical inclusion $W_{H_L,M_L} \subset W_{H,M}$.*

Proof. Let $\mathfrak{z} = \mathfrak{z}(M)$, respectively $\mathfrak{z}_L$ be the Lie algebra of the center of M. We have identities

$$W_{H,M} = N_H(Z(M))/C_H(Z(M)) = N_H(\mathfrak{z})/C_H(\mathfrak{z})$$

and likewise for $W_{H_L,M_L} = N_{H_L}(\mathfrak{z}_L)/C_{H_L}(\mathfrak{z}_L)$. Consider the action of $\mathrm{Ad}(x)$ on $\mathfrak{z}$. There is an eigen-decomposition

$$\mathfrak{z} = \bigoplus_{\lambda} \mathfrak{z}_\lambda$$

where $\mathfrak{z}_\lambda$ is the λ weight space for the action of $\mathrm{Ad}(x)$ and $\mathfrak{z}_0 = \mathfrak{z}_L$. Suppose that $h \in H_L = H \cap G_x^\circ$ normalizes $\mathfrak{z}_L$. Then, for any λ and $y \in \mathfrak{z}_\lambda$,

$$[x, \mathrm{Ad}(h) \cdot y] = [\mathrm{Ad}(h) \cdot x, \mathrm{Ad}(h) \cdot y] = \mathrm{Ad}(h) \cdot [x, y] = \lambda h \cdot y$$

where we used in the first equality that $\mathrm{Ad}(h) \cdot x = x$. Therefore, $\mathrm{Ad}(h)$ preserves $\mathfrak{z}$, and we conclude that $N_{H_L}(\mathfrak{z}_L) \subset N_H(\mathfrak{z})$. The desired inclusion now holds. □

Let $\widehat{\mathfrak{c}} = \widetilde{\mathfrak{c}} \times_{\mathfrak{a}/\!/W_{\mathrm{ker}}} \mathfrak{a}$. The induced map $\widehat{\mathfrak{c}} \to \mathfrak{c}$ is a reducible $W_{\widehat{\mathfrak{c}}} := W_{\mathfrak{a}} \rtimes W_{H,M}$ cover. Let $N(\widehat{\mathfrak{c}})$ be the normalization of $\widehat{\mathfrak{c}}$. $N(\widehat{\mathfrak{c}})$ is a disjoint union

$$N(\widehat{\mathfrak{c}}) \simeq \coprod_{[w] \in W_{H,M}/W_C} \mathfrak{a}$$

Then, for any choice of $[w] \in W_{H,M}/W_C$, we have a natural embedding $N(\widehat{\mathfrak{c}}_L) \hookrightarrow N(\widehat{\mathfrak{c}})$ from Lemma 1.3.69. This map descends to embeddings

$$\widehat{\xi}_{[w]} : \widehat{\mathfrak{c}}_L \to \widehat{\mathfrak{c}} \quad \text{and} \quad \widetilde{\xi}_{[w]} : \widetilde{\mathfrak{c}}_L \to \widetilde{\mathfrak{c}}.$$

Recall the notation $\varphi_L \colon \mathfrak{c}_L \to \mathfrak{c}$. We have a commutative diagram

$$\begin{tikzcd}
N(\widehat{\mathfrak{c}}_L) \arrow[r] \arrow[d, hook] & \widehat{\mathfrak{c}}_L \arrow[r] \arrow[d, hook] & \widetilde{\mathfrak{c}}_L \arrow[r] \arrow[d] & \mathfrak{c}_L \arrow[d] \\
N(\widehat{\mathfrak{c}}) \arrow[r] & \widehat{\mathfrak{c}} \arrow[r] & \widetilde{\mathfrak{c}} \arrow[r] & \mathfrak{c}
\end{tikzcd}$$

Let $\mathfrak{d}_L \subset \mathfrak{c}_L$ and $\mathfrak{d} \subset \mathfrak{c}$ denote the ramification divisor of $\varphi_L \colon \mathfrak{c}_L \to \mathfrak{c}$ and its preimage. (Recall that an explicit description of these divisors pulled back to $\mathfrak{a}$ was given in Section 1.3.3.) Let $U_L \subset \mathfrak{c}_L$ and $U \subset \mathfrak{c}$ be the complement of $\mathfrak{d}_L$ and $\mathfrak{d}$, respectively, and let $\widehat{U}_L$ and $\widehat{U}$ denote the preimages of U_L and U in $\widetilde{\mathfrak{p}}_L^{\mathrm{r}eg}$ and $\widetilde{\mathfrak{p}}^{\mathrm{r}eg}$, respectively.

Proposition 1.3.70. *The map $\iota_{\mathfrak{c}}$ of Proposition 1.3.68 is an inclusion. Moreover, it induces an isomorphism when G is simply connected.*

Proof. Let L be a distinguished Levi of G associated to $x \in \mathfrak{a}$. Over $U_L \subset \mathfrak{c}_L$, the map $\widehat{\xi}_{[w]}$ induces a commutative diagram

$$\begin{tikzcd}
\varphi_L^* \mathrm{Res}_{\mathfrak{c}}^{\widehat{\mathfrak{c}}}(C \times \widehat{\mathfrak{c}})^{W_{\widehat{\mathfrak{c}}}} \arrow[r, hook] & \mathrm{Res}_{\mathfrak{c}_L}^{\widehat{\mathfrak{c}}_L}(C \times \widehat{\mathfrak{c}}_L)^{W_{\widehat{\mathfrak{c}}_L}} \\
\varphi_L^* J \arrow[u, "\iota"] \arrow[r, "\simeq"] & J_L \arrow[u, "\iota_L"']
\end{tikzcd}$$

with the top arrow being an injection as the map $\widehat{\mathfrak{c}}_L \to \widehat{\mathfrak{c}}$ is a $W_{\widehat{\mathfrak{c}}_L}$ equivariant embedding which is an isomorphism onto a union oc components of $\widehat{\mathfrak{c}}$ and the bottom arrow being an isomorphism by Lemma 1.3.33. Hence, ι is an injection on $U \subset \mathfrak{c}$ if and only if ι_L is an injection on U_L.

For the first claim, it suffices to check away from a codimension 2 locus. Hence, it suffices to check for Levis L such that the associated symmetric pair (L, θ, H_L) is of rank 1 in the sense that

$$\dim(\mathfrak{a}) - \dim(\mathrm{L}ie(Z_G)^{\theta=-1}) = 1.$$

Such forms can be classified easily: in particular, the restricted roots systems must be rank 1, and there are only 3 isogeny classes of semisimple symmetric pairs. We may then do a simple computation to verify ι_L is an injection.

Now, note that if ι_L is an isomorphism on U_L, then ι must also be an isomorphism on U. Therefore, to check the second statement, it suffices to check for simply connected symmetric pairs of rank 1. This follows from the computations above. $\square$

It is important that the Galois comparison group scheme $J^1 = \mathrm{Res}_{\mathfrak{c}}^{\widehat{\mathfrak{c}}}(C \times \widehat{\mathfrak{c}})^{W_{\widehat{\mathfrak{c}}}}$ is sufficiently well behaved. For this, we prove the following.

Lemma 1.3.71. *The map $\widehat{\mathfrak{c}} \to \mathfrak{c}$ is Cohen-Macaulay over an open set of $\mathfrak{c}$ whose complement has codimension at least 2.*

In particular, the space J^1 is smooth on an open set whose complement has codimension at least 2.

Proof. By Levi induction, it suffices to check for rank one Levis. If $\widetilde{\mathfrak{c}} \to \mathfrak{c}$ is irreducible, then it agrees with the map $\mathfrak{a}/\!\!/W_{\mathrm{ker}} \to \mathfrak{c}$, which is flat by Lemma 1.3.57. That leaves only the case $(\mathrm{SL}_3, S(\mathbb{G}_m \times \mathrm{GL}_2))$. For this case, let $\mathcal{S} \subset \mathfrak{p}^{reg}$ denote the Kostant-Rallis section

$$\mathcal{S} = \left\{ x_a = \begin{pmatrix} & a & 1 \\ 1 & & \\ a & & \end{pmatrix} : a \in k \right\}$$

Then, after identifying $\mathcal{S} \simeq \mathbf{A}^1$ and $H/B \simeq \mathbf{P}^1$, we may write $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}^{reg}$ as

$$\widetilde{\mathfrak{p}}^{reg} \simeq \{(a, \pm a) \in \mathbf{A}^1 \times \mathbf{P}^1\} \to \mathbf{A}^1$$

In particular, it is a complete intersection and so Cohen-Macaulay. We conclude that $\widetilde{\mathfrak{p}}^{reg} \to \mathfrak{p}$ is flat in this case by Miracle Flatness. $\square$

1.4 General Structure of the Hitchin Fibration

In this section, we introduce the Hitchin morphism and prove the basic structure Theorems 1.4.2 and 1.4.3. Our discussion here will be very limited. Describing the geometry in a more comprehensive way similar to that of [35] and [49] is the subject of ongoing work.

Fix a smooth projective curve C of genus at least 2 and a line bundle D on C of degree $\deg(D) \geq 2g$. Taking $D = K_C$ the canonical bundle will also suffice. Also fix a symmetric pair (G, θ, H), and let $\mathcal{M} = \mathrm{Maps}(C, \mathfrak{p}_D/H)$ denote the stack of maps from the curve C to the twisted stack quotient $\mathfrak{p}_D/H = \mathfrak{p}/H \otimes D$. On k-points, $\mathcal{M}$ classifies pairs

$$\mathcal{M}(k) = \{(\mathcal{T}_H, \sigma) \colon \mathcal{T}_H \text{ is a } H \text{ torsor and } \sigma \in \Gamma(C, \mathcal{T}_H \wedge^{\mathrm{Ad}} \mathfrak{p}_D)\}$$

We have a Hitchin base $\mathcal{A} = \mathrm{Maps}(C, \mathfrak{c}_D)$ classifying maps from C to the twisted GIT quotient $\mathfrak{c}_D = (\mathfrak{p}/\!\!/H)_D$. In particular, by Theorem 1.2.10, there is a $\mathbb{G}_m$ equivariant isomorphism $\mathfrak{c} \simeq \mathbf{A}^r$ where the $\mathbb{G}_m$ action on $\mathbf{A}^r$ is given by exponents $(e_1, \ldots, e_r)$. This induces an identification of affine spaces

$$\mathcal{A} \simeq \oplus_{i=1}^r H^0(C, D^{\otimes e_i})$$

There is a natural Hitchin morphism

$$h \colon \mathcal{M} \to \mathcal{A}$$

induced by the Chevalley map $\mathfrak{p}/H \to \mathfrak{c}$. We restrict our attention to the regular locus in $\mathcal{M}$; namely, we let $\mathcal{M}^{\mathrm{reg}} = \mathrm{Maps}(C, \mathfrak{p}_D^{\mathrm{reg}}/H)$ be the substack of $\mathcal{M}$ classifying maps $C \to \mathfrak{p}_D/H$ which factor through the open substack $\mathfrak{p}_D^{\mathrm{reg}}/H \subset \mathfrak{p}_D/H$. We abuse notation to denote $h \colon \mathcal{M}^{\mathrm{reg}} \to \mathcal{A}$.

To study the geometry of h over the regular locus, we introduce the space

$$\mathcal{A}^{/\!\!/} = \mathrm{Maps}(C, (\mathfrak{p}^{reg} /\!\!/ H)_D).$$

The factorization $\mathfrak{p}^{reg}/H \to \mathfrak{p}^{reg} /\!\!/ H \to \mathfrak{c}$ induces a factorization

$$\mathcal{M}^{\mathrm{reg}} \xrightarrow{h^{/\!\!/}} \mathcal{A}^{/\!\!/} \xrightarrow{\phi} \mathcal{A}.$$

Let $\mathfrak{D} \subset \mathfrak{c}$ be the divisor defined by the function $\prod_{\alpha \in \Phi_r} d\alpha \in k[\mathfrak{a}]$. Following the notation of [35], we will restrict to the following dense open subset of $\mathcal{A}$.

Definition 1.4.1. Let $\mathcal{A}^{\diamondsuit} \subset \mathcal{A}$ denote the locus of maps $C \to \mathfrak{c}_D$ whose image intersects transversely with $\mathfrak{D}$.

We will denote by $\mathcal{A}^{/\!\!/,\diamondsuit}$ denote the base change $\mathcal{A}^{/\!\!/} \times_{\mathcal{A}} \mathcal{A}^{\diamondsuit}$. We prove the first part of the structure theorem now.

Theorem 1.4.2. *The map $\phi\colon \mathcal{A}^{/\!\!/} \to \mathcal{A}$ is étale when restricted to $\mathcal{A}^{\diamondsuit}$.*

Proof. It suffices to prove the étale lifting property. Fix a henselian scheme $\mathbf{D}$. Then, the lifting property is equivalent to proving that, for any diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & (\mathfrak{p}^{reg} /\!\!/ H)_D \arrow[d] \\ C \times \mathbf{D} \arrow[r] & \mathfrak{c}_D \end{tikzcd}$$

there is exists a unique lift $C \times \mathbf{D} \to (\mathfrak{p}^{reg} /\!\!/ H)_D$. This lift can be constructed locally on C: By the assumption that we work over $\mathcal{A}^{/\!\!/}$, the intersection of the preimage of $\mathfrak{D}$ under $C \times \mathbf{D} \to \mathfrak{c}_D$ is a disjoint collection of maps $\mathbf{D} \to \mathfrak{c}_D$. For each such map, there is a unique lift lying in the sheet of $(\mathfrak{p}^{reg} /\!\!/ H)_D$ dictated by the map $C \to (\mathfrak{p}^{reg} /\!\!/ H)_D$. We conclude that the map is étale. $\square$

In addition, we have a general result on the structure of the map $h^{/\!\!/}$. Namely, for $a\colon S \times C \to (\mathfrak{p}^{\mathrm{reg}} /\!\!/ H)_D$, we put $J_a := a^* J$ for J the band on $\mathfrak{p}^{reg} /\!\!/ H$ of Lemma 1.3.3. We may define a scheme over $\mathcal{A}^{/\!\!/}$ whose values over a $\mathcal{A}^{/\!\!/}$ scheme $S \times X \to (\mathfrak{p}^{\mathrm{reg}} /\!\!/ H)_D$ are given by the space of J_a torsors on $S \times C$,

$$\mathcal{P}(S) = (J_a \text{ torsors on } S \times C).$$

Note that although J_a is in general a band and not a group scheme, the notion of a J_a torsor is well defined as the space of torsors of is invariant under inner automorphisms. When (G, θ, H) is quasisplit, $\mathcal{P}$ is a commutative group scheme over $\mathcal{A}$.

Theorem 1.4.3. *Given a section $[\epsilon]\colon \mathcal{A}^{/\!\!/} \to \mathcal{M}^{\mathrm{reg}}$ of $h^{/\!\!/}$, there is an identification*

$$\mathcal{M}^{\mathrm{reg}}|_{\mathcal{A}^{/\!\!/,\diamondsuit}} \simeq \mathcal{P}|_{\mathcal{A}^{/\!\!/,\diamondsuit}}$$

Proof. The proof follows immediately form the fact that $\mathfrak{p}^{\mathrm{reg}}/H \to \mathfrak{p}^{\mathrm{reg}} /\!\!/ H$ is a gerbe banded by J. $\square$

1.5 The Symmetric Pair $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$

In this section, we provide a spectral description of J and the gerbe $\mathfrak{p}^{reg}/K \to \mathfrak{p}^{reg} /\!\!/ K$ for the example of the symmetric pair $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$. This is very closely related to the description of Hitchin fiber in [42].

1.5.1 The Spectral Cover

The map $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \to \mathfrak{t}/\!\!/W$ (here $\mathfrak{t}$ and W are for the group $GL(2n)$) is an embedding by Lemma **??**. Over $\mathfrak{t}/\!\!/W$, there is a natural spectral cover $\bar{\mathfrak{c}}_{\mathrm{GL}(2n)} \to \mathfrak{t}/\!\!/W$ given by $\bar{\mathfrak{c}}_{\mathrm{GL}(2n)} = \mathfrak{t}/\!\!/S_{2n-1}$ for $S_{2n-1} \subset S_{2n} = W$ the index $2n$ subgroup which is the stabilizer of a fixed element in

$\{1, 2, ..., 2n\}$. Explicitly, we have $k[\mathfrak{t}/\!\!/W] = k[a_1, \ldots, a_{2n}]$ where the a_i are the degree i elementary symmetric polynomials in $k[\mathfrak{t}]$, and

$$k[\bar{\mathfrak{c}}_{\mathrm{GL}(2n)}] = k[\mathfrak{t}/\!\!/W][x]/(x^{2n} + a_1 x^{2n-1} + \cdots + a_{2n}).$$

The map $\mathfrak{a}/\!\!/W_a \to \mathfrak{t}/\!\!/W$ corresponds to the map $k[a_1, ..., a_{2n}] \to k[a_2, ..., a_{2n}]$ which sends $a_{2i+1} \mapsto 0$ for each $0 \leq i < n$.

Let $\bar{\mathfrak{c}} = \bar{\mathfrak{c}}_{\mathrm{GL}(2n)} \times_{\mathfrak{t}/\!\!/W} \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ be the restriction of this spectral cover to $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$. Explicitly, we have

$$k[\bar{\mathfrak{c}}] = k[\mathfrak{a}/\!\!/W_{\mathfrak{a}}][x]/(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n})$$

The map $k[\mathfrak{a}/\!\!/W_{\mathfrak{a}}][x] \to k[\mathfrak{a}/\!\!/W_{\mathfrak{a}}][x]/(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n})$ gives a $\mathbb{G}_m$-equivariant embedding $\bar{\mathfrak{c}} \hookrightarrow (\mathfrak{a}/\!\!/W_a) \times \mathbb{A}^1$, where $\mathbb{G}_m$ acts with weight one on $\mathbb{A}^1$.

The involution θ on $GL(2n)$ acts on $\mathfrak{t}$ and also on the quotient $\mathfrak{t}/\!\!/S_{n-1}$. Explicitly, this action takes $x \mapsto -x$ and $a_i \mapsto (-1)^i a_i$. The spectral cover $\bar{\mathfrak{c}} \subset \mathfrak{t}/\!\!/S_{2n-1}$ is preserved by this action, and so we have an involution $i\colon \bar{\mathfrak{c}} \to \bar{\mathfrak{c}}$ defined over $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$ taking $x \mapsto -x$. We denote by

$$p\colon \bar{\mathfrak{c}}/i \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$$

the quotient of the cover $\bar{\mathfrak{c}} \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$. We will refer to the map p as the (generic) spectral cover of $\mathfrak{a}/\!\!/W_{\mathfrak{a}}$. We note that p corresponds to the inclusion

$$k[\mathfrak{a}/\!\!/W_a] \hookrightarrow k[\mathfrak{a}/\!\!/W_a][y]/(y^n + a_2 y^{n-1} + ... + a_n).$$

The map

$$k[\mathfrak{a}/\!\!/W_a][y] \to k[\mathfrak{a}/\!\!/W_a][y]/(y^n + a_2 y^{n-1} + ... + a_n)$$

gives a $\mathbb{G}_m$-equivariant embedding

$$\bar{\mathfrak{c}}/i \hookrightarrow \mathfrak{c} \times \mathbb{A}^1,$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ with weight two.

Finally we want to note that there is a particularly nice description of the set U of Proposition 1.3.44 via the spectral cover.

Proposition 1.5.1. *The set U of proposition 1.3.44 is $k[\mathfrak{a}/\!\!/W_a][a_{2n}^{-1}]$, that is to say it is the complement of the vanishing locus of a_{2n}.*

The interpretation of this in terms of the spectral cover is that the vanishing locus of a_{2n} is precisely the image in $\mathfrak{a}/\!\!/W_a$ of the intersection $((\mathfrak{a}/\!\!/W_a) \times \{0\}) \times_{(\mathfrak{a}/\!\!/W_a)\times\mathbb{A}^1} \bar{\mathfrak{c}}/i$

Proof. This is immediate from the definition of U in proposition 1.3.44. □

Recall that the regular centralizer group scheme $I_K^{reg} \to \mathfrak{p}^{reg}$ descends to a smooth group scheme J on the GIT quotient $\mathfrak{p}^{\mathrm{reg}}/\!\!/(\mathrm{GL}_n \times \mathrm{GL}_n) \simeq \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ since the form is quasi-split. We give a description of J using the spectral cover above.

Proposition 1.5.2. *There is a natural map $J \to \mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)$ where $\mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)$ denotes the Weil restriction of $\mathbb{G}_m$ along the map $p\colon \bar{\mathfrak{c}}/i \to \mathfrak{c}$. This map is an isomorphism.*

Proof. Note that one has the description of regular centralizers of the adjoint action of G on $\mathfrak{g}$ as the Weil restriction of $\mathbb{G}_m$ along the spectral cover $\bar{\mathfrak{c}}_{\mathrm{GL}(2n)} \to \mathfrak{t}/\!\!/W$. In particular, restricting to $\mathfrak{a}/\!\!/W_{\mathfrak{a}} \subset \mathfrak{t}/\!\!/W$, it follows that there is an isomorphism

$$J \xrightarrow{\sim} \mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}}(\mathbb{G}_m)^i$$

of J with the i-invariant locus in $\mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}}(\mathbb{G}_m)^i$. Since

$$\mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}}(\mathbb{G}_m)^i = \mathrm{Res}_{\mathfrak{a}/\!\!/W_{\mathfrak{a}}}^{\bar{\mathfrak{c}}/i}\left(\mathrm{Res}_{\bar{\mathfrak{c}}/i}^{\bar{\mathfrak{c}}}(\mathbb{G}_m)\right)^i$$

it therefore suffices to show that

$$\mathrm{Res}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)^i \simeq \mathbb{G}_m.$$

Note that we have a map

$$\xi\colon \mathbb{G}_m \to \mathrm{Res}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)^i.$$

Namely, an S-point $S \to \mathbb{G}_m \times \bar{\mathfrak{c}}/i$ has image givn by the base change

$$\begin{tikzcd}
S \arrow[r] & \mathbb{G}_m \times \bar{\mathfrak{c}}/i \\
S \times_{\bar{\mathfrak{c}}/i} \bar{\mathfrak{c}} \arrow[r] \arrow[u] & \mathbb{G}_m \times \bar{\mathfrak{c}} \arrow[u]
\end{tikzcd}$$

We claim this map is an isomorphism. Let $\mathcal{D} \subset \bar{\mathfrak{c}}/i$ be the ramification locus of the map $\bar{\mathfrak{c}} \to \bar{\mathfrak{c}}/i$, i.e. the image of the fixed point locus of i in $\bar{\mathfrak{c}}/i$. For $x \in (\bar{\mathfrak{c}}/i) \setminus \mathcal{D}$, we have the stalk

$$\mathrm{Res}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)_x = \mathbb{G}_m \times \mathbb{G}_m$$

with i acting by swapping the two factors. As the preimage of such an x is two points, it is easy to see that the map $\xi_x\colon \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ is the diagonal map. Hence, ξ is an isomorphism away from $\mathcal{D}$. For $x \in \mathcal{D}$, we have the fiber

$$\mathrm{Res}^{\bar{\mathfrak{c}}}_{\bar{\mathfrak{c}}/i}(\mathbb{G}_m)_x = \mathbb{G}_m \times \mathbb{G}_a$$

with i acting by $(y, z) \mapsto (y, -z)$. The map $\xi_x\colon \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_a$ is the inclusion into the first factor, and we conclude that the map ξ is an isomorphism. $\square$

1.5.2 Applications to the Hitchin Fibration for $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$

The regular quotient for the case of $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$ was computed in Example 1.3.43. Now that we have constructed spectral covers, we give an alternate description. This recovers the work of Schapostnik on spectral covers [41].

We will make use of the notation from Section 1.4; in particular, we fix a smooth projective curve C of genus at least 2 and a line bundle D on C of degree at lest $2g$. For any S point $S \times C \to \mathfrak{c}_D$, we define the spectral cover at a to be the base change

$$\begin{tikzcd} \overline{C}_a \arrow[r] \arrow[d] & (\bar{\mathfrak{c}}/i)_D \arrow[d] \\ S \times C \arrow[r] & \mathfrak{c}_D \end{tikzcd}$$

In particular, we will set $\overline{C}$ to be the base change along the evaluation map $\mathcal{A} \times C \to \mathfrak{c}_D$. We can realize $\overline{C}_a$ as a subvariety of the total space $\mathrm{Tot}(D)$ by considering the vanishing locus of the characteristic polynomial equation.

It is immediate from the Weil restriciton description of Proposition 1.5.2 that we have the following description of fibers.

Corollary 1.5.3. *A section* $[\epsilon]\colon \mathcal{A}^{/\!\!/} \to \mathcal{M}^{\mathrm{reg}}$ *induces an isomorphism of schemes over* $\mathcal{A}^{/\!\!/,\diamondsuit}$

$$\mathcal{M}^{\mathrm{reg}}|_{\mathcal{A}^{/\!\!/,\diamondsuit}} \simeq \mathrm{Pic}(\phi^*\overline{C}/\mathcal{A}^{/\!\!/,\diamondsuit}).$$

We note that such a section $[\epsilon]$ is induced by the Kostant-Rallis sections of Section 1.2.6.

We denote by $\mathfrak{D}^{ns} \subset \mathfrak{c}$ the locus over which the map

$$\mathfrak{p}^{reg} /\!\!/ H \to \mathfrak{c}$$

has 2 preimages. In general, we have shown that $\mathfrak{D}^{ns}$ is a Zariski closed subset, but for the example $(\mathrm{GL}_{2n}, \mathrm{GL}_n \times \mathrm{GL}_n)$, $\mathfrak{D}^{ns}$ is a divisor.

Corollary 1.5.4. *For any point $a \in \mathcal{A}(S)$, the image of the map $a\colon S \times C \to \mathfrak{c}_D$ meets the nonseparated divisor $\mathfrak{D}^{ns}$ exactly at the image of the zero section of $\overline{C}_a$ in* $\mathrm{Tot}(D)$.

Proof. As the nonseparated locus was found to be given exactly by the coordinate axes $\delta^* = 0$ in $\mathfrak{a}$, the claim is immediate. $\square$

We now derive the basic geometry of the map $\phi\colon \mathcal{A}^{/\!/,\diamondsuit} \to \mathcal{A}^{\diamondsuit}$. We do so in families, generalizing the earlier work that focused on studying fibers.

Lemma 1.5.5. *Let $d = \deg(D)$. The map $\phi\colon \mathcal{A}^{/\!/,\diamondsuit} \to \mathcal{A}^{\diamondsuit}$ is an étale map of degree d.*

Proof. This follows from Theorem 1.4.2 and the fact that d is the self intersection number of the zero divisor in $\mathrm{Tot}(D)$, see Proposition 9.16 of [11]. $\square$

Let $S^i(C) = (C^i \setminus \Delta)/S_i$ where Δ is the pairwise diagonal in C^i. Note that $S^i(C)$ is an open subscheme of the i-th symmetric power of C. We have an evaluation map

$$\mathcal{A}^{\diamondsuit} \to S^d(C)$$

sending a point a of $\mathcal{A}^{\diamondsuit}$ to the preimage under a of the nonseperated divisor $\mathfrak{D}^{ns}$.

Let Z_1, Z_2 be the two distinct sheets of $\mathfrak{p}^{\mathrm{reg}} /\!/ H$ over $\mathfrak{D}^{ns}$. There is a decomposition

$$\mathcal{A}^{/\!/,\diamondsuit} = \coprod_{i=0}^{d} \mathcal{A}_i^{/\!/,\diamondsuit}$$

where the closed points of $\mathcal{A}_i^{/\!/,\diamondsuit}$ classify maaps $a\colon C \to (\mathfrak{p}^{\mathrm{reg}} /\!/ H)_D$ for which Z_1 has i preimages. Then, we can define

$$\mathcal{A}^{/\!/,\diamondsuit} \to S^i(C) \times S^{d-i}(C)$$

by sending a point a of $\mathcal{A}^{/\!/,\diamondsuit}$ to the preimage under a of Z_1 and Z_2, respectively.

Theorem 1.5.6. *For every $0 \leq i \leq d$, there is a Cartesian diagram*

$$\begin{tikzcd}
\mathcal{A}_i^{\sslash,\diamondsuit} \arrow[r] \arrow[d] & S^i(C) \times S^{d-i}(C) \arrow[d] \\
\mathcal{A}^{\diamondsuit} \arrow[r] & S^d(C)
\end{tikzcd}$$

where both vertical arrows are étale $\binom{d}{i}$ covers.

Proof. It is immediate that the map $\mathcal{A}^{\sslash,\diamondsuit} \to \mathcal{A}^{\diamondsuit}$ is an étale $\binom{d}{i}$ cover and the diagram above commutes. As the right vertical arrow is also étale of degree $\binom{d}{i}$, this implies that the diagram is Cartesian. $\square$

1.6 Appendix to Chapter 1: Sections from $\mathfrak{sl}_2$ Triples

In this appendix, we review the construction of [27] and note an extension of those results to positive characteristic when p is greater than the Coxeter number of G based on the results of [37], [31], [47], and [32]. In particular, we review the theory of normal $\mathfrak{sl}_2$ triples, and derive the Kostant-Rallis section from the construction of the Kostant section. We compare this with the results of [31], reviewed in Section 1.2.6.

Definition 1.6.1. We say an $\mathfrak{sl}_2$ triple (e, h, f) is *normal* if $e, f \in \mathfrak{p}$ and $h \in \mathfrak{k}$. We say that an $\mathfrak{sl}_2$-triple is *principal* if e is regular as an element of $\mathfrak{p}$.

Remark 1.6.2. Note that a principal, normal $\mathfrak{sl}_2$ triple in the sense of Definition 1.6.1 is a principal $\mathfrak{sl}_2$ triple of $\mathfrak{g}$ in the usual sense only in the case of a quasi-split involution.

In the characteristic p case, we will need to pass to associated characters.

Definition 1.6.3. Fix a nilpotent $e \in \mathcal{N}$. An *associated character* of e is a character $\lambda\colon \mathbb{G}_m \to G$ such that $e \in \mathfrak{g}(2; \lambda)$ (where $\mathfrak{g}(k; \lambda)$ is the k-th graded piece of $\mathfrak{g}$ under the grading induced by λ) and there is a Levi subgroup $L \subset G$ such that $\lambda(\mathbb{G}_m) \subset L^{\mathrm{der}}$ and e is distinguished in $\mathrm{L}ie(L)$, i.e. $Z_{L^{\mathrm{der}}}(e)^\circ$ is unipotent.

Lemma 1.6.4. *([37], Prop. 4) Given an associated character λ to a nilpotent e, one can extend e to a unique $\mathfrak{sl}_2$ triple (e, h, f) with $h \in \mathrm{Lie}(\mathrm{i}mage(\lambda))$ and $f \in \mathfrak{g}(-2; \lambda)$.*

We recall the following facts about associated characters and $\mathfrak{sl}_2$ of G up to conjugation.

Lemma 1.6.5. *([32], Prop 18 and [47], Theorem 1.1) Consider the projection*

$$\{(e, \lambda) \colon e \in \mathcal{N} \text{ and } \lambda \text{ is an associated character for } e\}/G \to \mathcal{N}/G$$

where G acts by conjugation on each set.

1. *This map is a bijection in good characteristic.*

2. *The bijection above factors through*

$$\{(e, \lambda) \colon e \in \mathcal{N} \text{ and } \lambda \text{ is an associated character for } e\}/G \to \{\mathfrak{sl}_2\text{-triples}\}/G \to \mathcal{N}/G$$

   where the first map comes from Lemma 1.6.4. The map from G-orbits of $\mathfrak{sl}_2$-triples to $\mathcal{N}/G$ is a bijection if and only if the characteristic of the field is greater than the Coxeter number.

Proof. Part (1) follows from Prop. 18, part 2, of [32]. Part 2 follows from Thoerem 1.1 of [47]. □

We will demand, in addition, that associated characters be compatible with the involution on G in the following sense.

Definition 1.6.6. We say that a character λ is a *normal* associated character with respect to a nilpotent $e \in \mathcal{N}_{\mathfrak{p}}$ of $\mathfrak{p}$ if it is an associated character or e and $\mathrm{I}mage(\lambda) \subset K$.

Lemma 1.6.7. *([31], Cor. 5.4) For any $e \in \mathcal{N}_{\mathfrak{p}}$, there exists a normal associated character λ for e. Moreover, such a character is unique up to conjugation by the connected component of the centralizer $Z_K(e)^\circ$.*

We now deduce the results on $\mathfrak{sl}_2$ triples relevant to our paper.

Lemma 1.6.8. *The map*

$$\{H\text{-}orbits\ of\ normal\ \mathfrak{sl}_2\ triples\} \to \{H\text{-}orbits\ of\ nilpotents\ in\ \mathfrak{p}\}$$

is surjective, i.e. for any $e \in \mathcal{N}_{\mathfrak{p}}$, there exists a normal $\mathfrak{sl}_2$ triple (e, h, f) extending e. Assuming that the characteristic of the field is greater than the Coxeter number, this map is a bijection.

Proof. It suffices to prove this Lemma for $H = K$. In characteristic zero, this follows from [27], Proposition 4.

In characteristic $p > 0$, surjectivity follows from Lemma 1.6.4 and Lemma 1.6.7. Now assume the characteristic is greater than the Coxeter number. Then, we have a sequence of maps

$$\begin{tikzcd} \left\{\begin{matrix} K\text{-orbits of pairs } (e,\lambda) \text{ for} \\ \lambda \text{ associated to } e, \text{ valued in } K \end{matrix}\right\} \arrow[r, "\phi"] \arrow[rr, bend right, "\simeq"'] & \left\{\begin{matrix} K\text{-orbits of normal} \\ \mathfrak{sl}_2\text{-triples} \end{matrix}\right\} \arrow[r, two heads] & \{K\text{-orbits of nilpotents in } \mathfrak{p}\} \end{tikzcd}$$

Since the composite map is an isomorphism, the map ϕ is injective. We claim that it is also surjective. Suppose that a normal $\mathfrak{sl}_2$ triple (e, h, f) is not in the image of ϕ. Then, by Lemma 1.6.7, there is a character λ valued in K associated to e. Moreover, by Lemma 1.6.5, any two associated characters of e are conjugate by an element of $Z_G(e)^\circ$, and there is a unique character λ' associated to e for which (e, h, f) is the corresponding $\mathfrak{sl}_2$ triple. Let $g \in Z_G(e)^\circ$ conjugate λ' and λ, so that g also conjugates (e, h, f) to a normal $\mathfrak{sl}_2$ triple (e, h', f'). Since this g preserves normality of the $\mathfrak{sl}_2$ triple, $\mathrm{L}ie(image\ g \cdot \lambda) \subset \mathfrak{k}$. Since $g \cdot \lambda$ is a one-parameter subgroup, it is connected and hence has image in K. We conclude that $g \cdot \lambda$ is an associated character to e valued in K whose associated $\mathfrak{sl}_2$ triple is (e, h, f). $\square$

Now let $e \in \mathcal{N}_{\mathfrak{p}}^{reg}$ be a regular nilpotent. From a principal, normal $\mathfrak{sl}_2$ triple (e, h, f), one produces a Kostant-Rallis section by considering the slice $e + \mathfrak{c}_{\mathfrak{p}}(f)$.

Theorem 1.6.9. *The map $e + \mathfrak{c}_{\mathfrak{p}}(f) \to \mathfrak{a}/\!\!/W_{\mathfrak{a}}$ is an isomorphism. We will call its inverse a* Kostant-Rallis section *associated to e.*

Moreover, for a given regular nilpotent e in $\mathfrak{p}$, this section is unique up to conjugation by $Z_K(e)^\circ$. In particular, this gives a bijection

$$\{K\text{-orbits of Kostant-Rallis sections}\} \to \{K\text{-orbits of regular nilpotents in } \mathfrak{p}\}.$$

Proof. In characteristic zero, this is the content of [27], Theorem 11.

In characteristic $p > 0$, by [31], Lemma 6.29, it suffices to check that $e+\mathfrak{c}_{\mathfrak{p}}(f)$ is an $\mathrm{A}d(\lambda)$-graded complement of $[\mathfrak{k}, e]$, where $\lambda\colon k^\times \to K$ is an associated character to e. Certainly the slice is $\mathrm{A}d(\lambda)$ graded as e and f are homogeneous with respect to the grading. To show that the slice gives a complement, it suffices to show

$$\mathfrak{p} = \big(e + \mathfrak{c}_{\mathfrak{p}}(f)\big) \oplus [\mathfrak{k}, e]$$

By the proof of Lemma 3.1.3 of [39], we have that

$$\mathfrak{g} = \big(e + \mathfrak{c}_{\mathfrak{g}}(f)\big) \oplus [e, \mathfrak{g}].$$

Intersecting this with $\mathfrak{p}$ and using the fact that $e \in \mathfrak{p}$ gives this result. $\square$

Corollary 1.6.10. *Let $\mathfrak{s} = e + \mathfrak{c}_{\mathfrak{p}}(f)$ be the Kostant-Rallis slice in $\mathfrak{p}$. Then, $\mathfrak{p} = \mathfrak{s} + [e, \mathfrak{p}]$. In particular, if*

$$a\colon H \times \mathfrak{s} \to \mathfrak{p}^{reg}$$

is the action map. Then the differential of a at $0 \in \mathfrak{s}$ is surjective.

Proof. The first claim follows from the proof of Theorem 1.6.9. For the second, we note that the differential is identified with the map

$$\mathfrak{h} \times \mathfrak{s} \to \mathfrak{h}, \quad (x, s) \mapsto [x, e] + s.$$

By the first claim together with the observation that $\mathfrak{h} = \mathfrak{k}$, this is surjective. $\square$

CHAPTER 2

COMPANION MATRIX CONSTRUCTIONS

In this chapter, the author covers some of the joint work with B.C. Ngô on Companion Matrix Constructions. This work is independent of the previous chapter and is related only in the broader heading of Hitchin systems.

2.1 Introduction

Let G be a reductive group over k, and denote by $\mathfrak{g}$ its Lie algebra. The Chevalley map

$$\chi \colon \mathfrak{g} \to \mathfrak{g} /\!\!/ G,$$

where $\mathfrak{g} /\!\!/ G := \operatorname{Spec}(k[\mathfrak{g}]^G)$ denotes the invariant theoretic quotient of $\mathfrak{g}$ by the adjoint action of G, is of fundamental importance in the construction of the Hitchin system [20]. In particular, for $\mathfrak{g} = \mathfrak{gl}_n$, χ sends a matrix to its characteristic polynomial.

In [28], Kostant exhibited a section of the Chevalley map for a general reductive group G under the assumption that the characteristic of k does not divide the order of the Weyl group. Kostant's section was generalized in [4] and [1], including the case of characteristics $p > 2$ for classical groups and the group G_2. As explained in [35], this section can be used to construct sections of the Hitchin fibration and affine Springer fibers. However, Kostant's construction can be counter-intuitive for computations. To illustrate this latter point, consider the case $G = \mathrm{GL}_3(k)$, in which case $\mathfrak{g} /\!\!/ G = \mathbf{A}^3$ is the 3-dimensional affine space. The Kostant section is the map sending

$$(a_1, a_2, a_3) \in \mathfrak{g} /\!\!/ G \quad \mapsto \quad \begin{pmatrix} \frac{a_1}{3} & \frac{a_1^2}{6} + \frac{a_2}{2} & -\frac{4a_1^3}{27} - \frac{a_1 a_2}{3} - a_3 \\ 1 & \frac{a_1}{3} & \frac{a_1^2}{6} + \frac{a_2}{2} \\ 0 & 1 & \frac{a_1}{3} \end{pmatrix} \in \mathfrak{g}$$

If you introduced this problem to an undergraduate student of linear algebra, of course, they would not give you the answer above; they might instead suggest the map:

$$(a_1, a_2, a_3) \in \mathfrak{g}/\!\!/G \quad \mapsto \quad \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} \in \mathfrak{g}$$

sending a characteristic polynomial to its *companion matrix*. The section to the Hitchin map that Hitchin constructed in [20] is not strictly the same as the one of [35] in the sense that he does not rely on the Kostant section but another section that feels more like a generalization of the companion matrix. Instead of the companion matrix, a map $\mathfrak{g}/\!\!/G \to \mathfrak{g}$, we will construct a map $\mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$, where $[\mathfrak{g}/G]$ is the quotient of $\mathfrak{g}$ by the adjoint action of G in the sense of algebraic stack. This section will be called the companion section, which is free of any choice. The present note aims to explicitly construct the companion section for classical groups, including the symplectic and orthogonal groups and G_2. As an application of the companion sections, we will give elementary descriptions of affine Springer fibers and Hitchin fibers for classical groups similar to the description of the Hitchin fibers in the linear case due to Beauville-Narasimhan-Ramanan.

The emphasis of this work is on providing case-by-case explicit formulas for the companion section for classical groups. It is also possible to construct the companion section uniformly. This will be the subject of our subsequent work.

2.2 Tensors defining classical groups

We will recall the standard definition of classical groups as the subgroup of the linear groups fixing certain tensors. This is very well known for symplectic and orthogonal groups but a bit less known for G_2, which in a certain respect could qualify as a classical group as well.

Let V be a $2n$-dimensional vector space over a base field k, V^* its dual vector space.

The linear group $\mathrm{GL}(V)$ acts on the space $\wedge^2 V^*$ of alternating bilinear forms on V with an open orbit. An alternating bilinear form $\mu \in \wedge^2 V^*$ is considered non-degenerate if it lies in this open orbit. This is equivalent to requiring the induced map $\mu : V \to V^*$ to be an isomorphism. The stabilizer of such a non-degenerate alternating bilinear form is a symplectic group G. We note that $\mu \in \wedge^2 V^*$ is non-degenerate if $\wedge^n \mu \in \wedge^{2n} V^*$ is a non-zero vector of the 1-dimensional vector space $\wedge^{2n} V^*$ and as a result, G is contained in the special linear group $\mathrm{SL}(V)$. Then, a G-bundle over a k-scheme S consists of a locally free $\mathcal{O}_S$-module $\mathcal{V}$ of rank $2n$ equipped with an alternating bilinear form $\wedge^2_S \mathcal{V} \to \mathcal{O}_S$ which is non-degenerate fiberwise. Although the embedding of $G = \mathrm{Sp}_{2n}$ into GL_{2n} may differ by conjugation by an element of GL_{2n}, as we are more concerned with G-bundles than G itself, the specific choice of non-degenerate alternating form $\mu \in \wedge^2 V^*$ is immaterial. We will write $G = \mathrm{Sp}_{2n}$.

Let V be a n-dimensional vector space over a base field k, V^* its dual vector space. The linear group $\mathrm{GL}(V)$ acts on the space $S^2 V^*$ of symmetric bilinear forms on V with an open orbit. A symmetric bilinear form $\mu \in S^2 V^*$ is considered non-degenerate if it lies in this open orbit. This is also equivalent to the induced map $\mu : V \to V^*$ being an isomorphism, which in turn is equivalent to the induced map $\wedge^n \mu : \wedge^n V \to \wedge^n V^*$ being an isomorphism of 1-dimensional vector spaces. We note that $\wedge^n V$ and $\wedge^n V^*$ are dual as vector spaces so that for every choice of a basis vector $\omega \in \wedge^n V$, we have a dual basis vector $\omega^* \in \wedge^n V^*$. A basis vector $\omega \in \wedge^n V$ is said to be compatible with μ if the equation $\wedge^n \mu(\omega) = \omega^*$ is satisfied. This equation has exactly two non-zero solutions $\omega \in \wedge^n V$, which differ by a sign. The stabilizer of a non-degenerate symmetric bilinear form $\mu \in S^2 V^*$ is an orthogonal group $O(\mu)$. The stabilizer of a pair (μ, ω) consisting of a non-degenerate symmetric bilinear form $\mu \in S^2 V^*$ and a compatible basis vector $\omega \in \wedge^n V$ is the special orthogonal group $\mathrm{SO}(\mu, \omega)$ which is the neutral component of $O(\mu)$. We note that $\mathrm{SO}(\mu, \omega) = O(\mu) \cap \mathrm{SL}(V)$ so that the special orthogonal group can also be defined as the stabilizer of a pair (μ, ω) as above but without requiring ω being compatible with μ. The stabilizer of any such pair is a special orthogonal group G. A G-bundle over a k-scheme S consists then in a locally free $\mathcal{O}_S$-module $\mathcal{V}$ of rank n equipped a symmetric bilinear form $\wedge^2_S\mathcal{V} \to \mathcal{O}_S$ which is non-degenerate fiberwise. The embedding of $G = \mathrm{SO}_n$ into GL_n depends on the form μ and is well defined only up to conjugation by GL_n. However, as we are more concerned with G-bundles than G itself, choosing a specific non-degenerate symmetric form $\mu \in \wedge^2 V^*$ is immaterial. We will write $G = \mathrm{SO}_n$.

There is a simple tensor definition of G_2 due to Engel [12]. Let V be a 7-dimensional vector space. The linear group $\mathrm{GL}(V)$ acts on the space $\wedge^3 V^*$ of non-degenerate trilinear forms on V with an open orbit. We will follow Hitchin's [21] in formulating the equation defining this open orbit . We will denote the contraction $\wedge^3 V^* \times V \to \wedge^2 V^*$ by μ_v for $\mu \in \wedge^2 V^*$ and $v \in V$. For $v_1, v_2 \in V$ and $\mu \in \wedge^3 V^*$, we then have

$$\mu_{v_1} \wedge \mu_{v_2} \wedge \mu \in \wedge^7 V^*.$$

By choosing a non-zero vector ι of the determinant $\wedge^7 V$, μ gives rise to a symmetric bilinear form $\nu \in S^2 V^*$

$$\nu(v_1, v_2) = \langle \iota, \mu_{v_1} \wedge \mu_{v_2} \wedge \mu \rangle \tag{2.2.1}$$

which is non-degenerate if and only if μ lies in the open orbit of $\wedge^3 V^*$. We will say that μ is a non-degenerate 3-form on V. The stabilizer of a non-degenerate 3-form is a group $G_2 \ltimes \mu_3(k)$ where $\mu_3(k)$ is the group of 3rd roots of unity in k; We obtain the connected component, a group of type G_2, by taking the intersection with $SL(V)$. A G_2-bundle over a k-scheme S is thus a locally free $\mathcal{O}_S$-module $\mathcal{V}$ of rank 7 equipped with an alternating trilinear form $\mu \in \wedge^3 \mathcal{V}^*$ which is non-degenerate fiberwise together with a trivialization of the determinant. Again, a different choice of nondegenerate 3-form μ may give a GL_7 conjugate embedding of G_2 into GL_7. However, such a choice is immaterial for us.

2.3 Spectral cover and the companion matrix

For all groups G discussed previously, including symplectic, special orthogonal, and G_2, G is defined as a subgroup of GL_n fixing certain tensors. We call the inclusion $G \to \mathrm{GL}_n$ the standard representation of G. We also have the induced inclusion of Lie algebras $\mathfrak{g} \to \mathfrak{gl}_n$ compatible with the adjoint actions of G and GL_n. We derive a morphism between invariant theoretic quotients

$$\mathfrak{c} = \mathfrak{g}/\!\!/G \to \mathfrak{gl}_n/\!\!/\,\mathrm{GL}_n = \mathfrak{c}_n$$

which is a closed embedding for symplectic groups, odd special orthogonal groups, and G_2, but not for even orthogonal groups. For GL_n, we have a spectral cover $\mathfrak{s}_n \to \mathfrak{c}_n$, defined in Section 2.3.1, which is a finite flat morphism of degree n so that $\mathcal{O}_{\mathfrak{s}_n}$ is a locally free $\mathcal{O}_{\mathfrak{c}_n}$-module of rank n given with a canonical endomorphism $[x]$ which is the usual companion matrix. The main result of this work can be formulated as follows:

Theorem 2.3.1. *Let G be a symplectic group, odd special orthogonal group, or G_2 group and $G \to \mathrm{GL}_n$ its standard representation. Let $\mathfrak{c} \to \mathfrak{c}_n$ be the induced map of Chevalley quotients which is a closed embedding in these cases. Then the restriction $\mathcal{O}_{\mathfrak{s}_n}$ to $\mathfrak{c}$*

$$\mathcal{V} = \mathcal{O}_{\mathfrak{c}} \otimes_{\mathcal{O}_{\mathfrak{c}_n}} \mathcal{O}_{\mathfrak{s}_n}$$

as locally free $\mathcal{O}_{\mathfrak{c}}$-module affords a canonical tensor defining a G-reduction and the companion matrix for GL_n defines a canonical map $\mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$ which is a section of the natural map $[\mathfrak{g}/G] \to \mathfrak{g}/\!\!/G$. This statement remains valid for even orthogonal groups after replacing $\mathfrak{c} \times_{\mathfrak{c}_n} \mathfrak{s}_n$ by its normalization.

We prove the theorem by a case-by-case analysis. In particular, we will construct the explicit tensors required in each case.

2.3.1 Linear groups

We first recall how the companion matrix is connected to the universal spectral cover in the case GL_n. In this case, the Chevalley quotient $\mathfrak{g}/\!\!/G$ is the n-dimensional affine space $\mathbb{A}^n$ and the map $\chi : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ is given by the characteristic polynomial $\chi(\gamma) = (a_1(\gamma), \ldots, a_n(\gamma))$ where $\gamma \in \mathfrak{g}$ and $a_i(\gamma) = (-1)^i \operatorname{tr}(\wedge^i \gamma)$. In this case we have $\mathfrak{c}_n = \operatorname{Spec}(A_n)$ where $A_n = k[a_1, \ldots, a_n]$. The spectral cover $\mathfrak{s}_n = \operatorname{Spec}(B_n)$ where B_n is the A_n-algebra

$$B_n = A_n[x]/(x^n + a_1 x^{n-1} + \cdots + a_n)$$

which is a free A_n-module of rank n as the images of $1, x, \ldots, x^{n-1}$ form an A_n-basis of B_n. We also note that B_n is a regular k-algebra as it is isomorphic to the polynomial algebra of variables $a_1, \ldots, a_{n-1}, x$. On the other hand, B_n is equipped with an A_n-linear operator $[x] : B_n \to B_n$ given by $b \mapsto bx$. To give a map $\mathfrak{c}_n \to [\mathfrak{gl}_n/\mathrm{GL}_n]$ is equivalent to the data of a rank n vector bundle $\mathcal{E} \to \mathfrak{c}_n$ together with a an $\mathcal{O}_{\mathfrak{c}_n}$-linear endomorphism of $\mathcal{E}$; that is, at the level of modules, a free, rank n A_n module with an A_n linear endomorphism. Hence, B_n with the operator $[x]$ provides us with an A_n-point of $[\mathfrak{gl}_n/\mathrm{GL}_n]$, and we have thus constructed a map $[x] : \mathfrak{c}_n \to [\mathfrak{gl}_n/\mathrm{GL}_n]$ which is a section of $\chi : [\mathfrak{gl}_n/\mathrm{GL}_n] \to \mathfrak{c}_n$. In term of matrices with respect to the A_n-basis of B_n given by $1, x, \ldots, x^{n-1}$, $[x]$ is given by the usual companion matrix

$$x_\bullet = \begin{pmatrix} 0 & & \cdots & -a_n \\ 1 & 0 & \cdots & -a_{n-1} \\ & \cdots & 0 & \\ 0 & & 1 & -a_1 \end{pmatrix} \in \mathfrak{gl}_n(A) \tag{2.3.1}$$

The companion matrix thus gives us a map $x_\bullet : \mathfrak{c} \to \mathfrak{g}$ in the case $G = \mathrm{GL}_n$ taking a point $a = (a_1, \ldots, a_n)$ of $\mathfrak{c}$ to the matrix above. This construction is a section to the characteristic polynomial map. However, it is often more useful to think of $[x]$ as a map $[x] : \mathfrak{c} \to [\mathfrak{g}/G]$ in the case $G = \mathrm{GL}_n$.

Let $\mathfrak{g}$ come equipped with the homothety action of $\mathbb{G}_m$ and $\mathfrak{c}$ with the induced action $t \cdot a_i = t^i a_i$. There is an issue with using the companion matrix to construct a section to the Hitchin map as the companion map $x_\bullet : \mathfrak{c} \to \mathfrak{g}$ is not $\mathbb{G}_\mathrm{m}$-equivariant. We note, however, that the stack-valued map $[x] : \mathfrak{c} \to [\mathfrak{g}/G]$ is almost $\mathbb{G}_\mathrm{m}$-equivariant in the sense that after a base change by the isogeny $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ given by $t \mapsto t^2$, it becomes equivariant because of the identity

$$\mathrm{ad}(\mathrm{diag}(t^{n-1}, t^{n-3}, \ldots, t^{1-n}))(\gamma) = t^{-2} \begin{pmatrix} 0 & 0 & \cdots & -t^{2n} a_n \\ 1 & 0 & \cdots & -t^{2n-2} a_{n-1} \\ & \cdots & & \\ 0 & & 1 & -t^2 a_1 \end{pmatrix}. \tag{2.3.2}$$

This explains why we have a section to the Hitchin map after choosing a square root of the canonical bundle as in [20].

As we intend to use the companion matrix (2.3.1) to construct a canonical section to the Chevalley map $\chi : [\mathfrak{g}/G] \to \mathfrak{c}$ for classical groups, it is useful to further investigate the linear algebraic structure of B_n as an A_n-module. We have a symmetric A_n-bilinear map $\xi : B_n \otimes_{A_n} B_n \to A_n$ given by

$$\xi(b_1 \otimes_{A_n} b_2) = \mathrm{tr}_{B_n/A_n}(b_1 b_2)$$

thus an element $\xi \in S^2_{A_n} B_n^*$. Because this element induces degenerate forms over the ramification locus of B_n over A_n, we need a correction term to get a symmetric bilinear form that is non-degenerate fiberwise. We will describe this correction and the associated nondegenerate form in Lemma 2.3.2.

The pairing ξ defines an A_n-linear map $\mu : B_n \to B_n^*$ where $B_n^* = \mathrm{Hom}_{A_n}(B_n, A_n)$

and $\mu(b_1)(b_2) = \xi(b_1, b_2)$. We note that the A_n-module B_n^* is naturally a B_n-module and $\mu : B_n \to B_n^*$ is B_n linear; thus, it is uniquely determined by the image of $1 \in B_n$ that we will also denote by $\mu \in B_n^*$. We will show that B_n^* is a free B_n-module of rank 1, construct a generator of B_n^* and find an explicit formula for $\mu \in B_n^*$ as a multiple of this generator.

Lemma 2.3.2. *Let us denote by $v_0, \dots, v_{n-1}$ the basis of B_n given by the images of $1, x, \dots,$ x^{n-1} in B_n and $v_0^*, \dots, v_{n-1}^*$ the dual basis of B_n^*. Then $\beta^* = v_{n-1}^*$ is a generator of B_n^* as a B_n-module. Let us denote $f' \in B_n = A_n[x]/(f)$ the image of the derivative*

$$nx^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1} \in A_n[x]$$

of the universal polynomial $f = x^n + a_1x^{n-1} + \cdots + a_n \in A_n[x]$. Then we have $\mu = f'\beta^*$.

Proof. First, the discriminant d of the universal polynomial f, defined as the resultant between f and its derivative, is a nonzero element of the polynomial ring A_n. Indeed, d defines the ramification divisor of the finite flat covering $\mathfrak{s} \to \mathfrak{c}$, which is generically étale for there exist separable polynomials in $k'[x]$ of degree n with coefficients in any infinite field k' containing k. We denote $A' = A_n[d^{-1}]$ the localization of A_n obtained by inverting d, and $B' = B_n \otimes_{A_n} A'$. By construction, f' is an invertible element of B'. The trace map $\mathrm{tr}_{B'/A'} : B' \to A'$ of B' as free A'-module of rank n is now given by the Euler formula (cf. III.6, Lemma 2 in [45])

$$\mathrm{tr}_{B'/A'}\left(\frac{x^k}{f'}\right) = \begin{cases} 0 & \text{if } k < n-1 \\ 1 & \text{if } k = n-1 \end{cases}$$

If $v_0, \dots, v_{n-1}$ denote the basis of B' given by the images of $1, x, \dots, x^{n-1}$ in B_n and $v_0^*, \dots, v_{n-1}^*$ the dual basis of $(B')^*$, then we derive from the Euler formula that the identities

$$\mu(v_i) = f'\left(v_{n-1-i}^* + \sum_{j<i} a'_{i,j} v_{n-1-j}^*\right) \tag{2.3.3}$$

hold in $B_n^* \otimes_{A_n} A'$ for some $a'_{i,j} \in A'$. In particular, we have $\mu(v_0) = f' v_{n-1}^*$. As the localization map $B_n^* \to B_n^* \otimes_{A_n} A'$ is injective, this identity also holds in B_n^*. It follows that $\mu = f' v_{n-1}^*$ as desired. $\square$

As a consequence, we have a canonical nondegenerate bilinear form $\beta^* : B_n \otimes_{A_n} B_n \to A_n$ which is symmetric with respect to which the A_n-linear operator $[x] : B_n \to B_n$ is anti-self-adjoint; that is, for all $v_1, v_2 \in B_n$, we have

$$\beta^*(xv_1, v_2) + \beta^*(v_1, xv_2) = 0.$$

For $G = \mathrm{SL}_n$, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ is the space of traceless matrices. We have $\mathfrak{c} = \mathrm{Spec}(A)$ where $A = k[a_2, \ldots, a_n]$. We note that for $a_1 = 0$, the companion matrix (2.3.1) is traceless and thus gives rise to a A-point on $\mathfrak{sl}_n$. The companion map $\gamma : \mathfrak{c} \to \mathfrak{g}$ induces a map $[\gamma] : \mathfrak{c} \to [\mathfrak{g}/G]$. The latter lays over the point of BG with values in A corresponding to the SL_n-bundle corresponding to rank n vector bundle B equipped with the trivialization of the determinant given by the basis $1, x, \ldots, x^{n-1}$. The formula (2.3.2) shows that the map $[\gamma] : \mathfrak{c} \to [\mathfrak{g}/G]$ is equivariant with respect to the isogeny $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ given by $t \mapsto t^2$ for the diagonal matrix $\mathrm{diag}(t^{n-1}, t^{n-3}, \ldots, t^{1-n})$ belonging to SL_n.

2.3.2 Symplectic groups

In the case $G = \mathrm{Sp}_{2n}$, we have $\mathfrak{c} = \mathrm{Spec}(A)$ with $A = k[a_2, \ldots, a_{2n}]$. The spectral cover $\mathfrak{s} = \mathrm{Spec}(B)$ where

$$B = A[x]/(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n})$$

is a free A-module of rank $2n$, is equipped with an involution $\tau : B \to B$ given $\tau(x) = -x$. The companion matrix (2.3.1) gives a A-linear endomorphism of B as a free A-module. For the companion matrix to produce a section to the Chevalley map $[\mathfrak{g}/G] \to \mathfrak{c}$ in the symplectic case, we need to construct a canonical nondegenerate symplectic form ω on the A-module B

for which γ is anti-self-adjoint in the sense that

$$\omega(\gamma v_1, v_2) + \omega(v_1, \gamma v_2) = 0$$

for all $v_1, v_2 \in B$.

The standard representation $\mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ induces a map on Chevalley bases $\mathfrak{c} \to \mathfrak{c}_{2n} = \mathrm{Spec}(A_{2n})$ where $A_{2n} = k[a_1, \ldots, a_{2n}]$ which identidies $\mathfrak{c}$ with the closed subscheme of $\mathfrak{c}_{2n}$ defined by the ideal generated by $a_1, a_3, \ldots, a_{2n-1}$. We have $B = A \otimes_{A_{2n}} B_{2n}$ where B_{2n} is the finite free A_{2n}-algebra defining the spectral covering of $\mathfrak{c}_{2n}$. If we denote $B^* = \mathrm{Hom}_A(B, A)$ then we have $B^* = A \otimes_{A_{2n}} B^*_{2n}$ where $B^*_{2n} = \mathrm{Hom}_{A_{2n}}(B_{2n}, A_{2n})$. The generator β^*_{2n} of the free B_{2n}-module B^*_{2n} defined in Lemma 2.3.2 then induces a generator β^* of B^* as a free B-module of rank one which can also be viewed as the bilinear form $\beta^* : B \otimes_A B \to A$ given by $b_1 \otimes_A b_2 = \mathrm{tr}_{B/A}(f'^{-1} b_1 b_2)$ after localization.

The bilinear form $\omega : B \otimes_A B \to A$

$$\omega(b_1, b_2) = \beta^*(b_1, \tau(b_2)) = \mathrm{tr}_{B/A}(f'^{-1} b_1 \tau(b_2))$$

with the second identity only making sense after localization of A making f' invertible, is a non-degenerate symplectic form for which $[x]$ is anti-self-adjoint. Indeed, we have

$$\omega(b_1, b_2) = -\omega(b_2, b_1)$$

because $\tau(f') = -f'$ for $f' \in A[x]$ is an odd polynomial as $f \in A[x]$ is an even polynomial. The equation $\omega(xb_1, b_2) + \omega(b_1, xb_2) = 0$ can be derived from $\tau(x) = -x$.

It follows that we have a morphism

$$[x] : \mathfrak{c} \to [\mathfrak{g}/G]$$

which deserves to be called the companion map for the symplectic group. To obtain a companion matrix $x_\bullet : \mathfrak{c} \to \mathfrak{g}$, it is enough to find a trivialization of the G-bundle associated with the non-degenerate symplectic form $\omega : B \otimes_A B \to A$. For most applications, particularly the Hitchin fibration, we only need the section $[x] : \mathfrak{c} \to [\mathfrak{g}/G]$.

2.3.3 Odd special orthogonal groups

In the case $G = \mathrm{SO}_{2n+1}$, we have $\mathfrak{c} = \mathrm{Spec}(A)$ with $A = k[a_2, a_4, \ldots, a_{2n}]$. The spectral cover is defined as $\mathfrak{s} = \mathrm{Spec}(B)$ where $B = A[x]/(f)$ with $f = xf_0$ and $f_0 = x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n}$. B is a free A-module of rank $2n+1$. As in the symplectic case, we will define a symmetric non-degenerate bilinear form $B \otimes_A B \to A$ for which the multiplication by x is anti-self-adjoint.

The standard representation $\mathrm{SO}_{2n+1} \to \mathrm{GL}_{2n+1}$ gives rise to a map $\mathfrak{c} \to \mathfrak{c}_{2n+1} = \mathrm{Spec}(k[a_1, \ldots, a_{2n+1}])$ which is a closed embedding defined by the ideal generated by the functions $a_1, a_3, \ldots, a_{2n+1}$. We have $B = A \otimes_{A_{2n+1}} B_{2n+1}$ where B_{2n+1} is the finite free A_{2n+1}-module of rank $2n+1$ defining the spectral cover in the case GL_{2n+1}. We also have $B^* = A \otimes_{A_{2n+1}} B^*_{2n+1}$ where $B^* = \mathrm{Hom}_A(B, A)$ and $B^*_{2n+1} = \mathrm{Hom}_{A_{2n+1}}(B_{2n+1}, A_{2n+1})$. Following the discussion in the linear case B^*_{2n+1} is a free B_{2n+1} generated by the element $\beta^*_{2n+1} = (f')^{-1}\mu$ where μ is the trace form $\mu(b_1 \otimes_A b_2) = \mathrm{tr}(b_1 b_2)$. It induces a generator β^* of B^* as a B-module. We define the bilinear form $\omega : B \otimes_A B \to A$ by

$$\omega(b_1, b_2) = \beta^*(b_1, \tau(b_2)) = \mathrm{tr}(f'^{-1} b_1 \tau(b_2)). \tag{2.3.4}$$

The bilinear form ω is a nondegenerate bilinear form because β^* is. It is symmetric because $\tau(f') = f'$ as f' is an even polynomial. The equation $\omega(xb_1, b_2) + \omega(b_1, xb_2)$ can be derived from the fact $\tau(x) = -x$.

By choosing a trivialization of the determinant, we obtain a companion map $[x] : \mathfrak{c} \to$

$[\mathfrak{g}/G]$ for $G = \mathrm{SO}_{2n+1}$.

2.3.4 Even special orthogonal groups

The case $G = \mathrm{SO}_{2n}$ is slightly more difficult for the map $\mathfrak{c} \to \mathfrak{c}_{2n}$ induced by the standard representation of SO_{2n} is not a closed embedding. Indeed, we have $\mathfrak{c}_{2n} = \mathrm{Spec}(A_{2n})$ where $A_{2n} = k[a_1, \dots, a_{2n}]$ but $\mathfrak{c} = \mathrm{Spec}(A)$ where $A = k[a_2, \dots, a_{2n-2}, p_n]$ where p_n is the Pfaffian satisfying $p_n^2 = a_{2n}$ does not lie in the image of $A_{2n} \to A$. If B_{2n} is the spectral cover of A_{2n} and $B = A \otimes_{A_{2n}} B_{2n}$ then we have

$$B = A[x]/(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n-2}x^2 + p_n^2).$$

As indicated by Hitchin [20], the true spectral cover for even special orthogonal groups is not B but its blowup $\widetilde{B}$ along the singular locus defined by x. We have

$$\widetilde{B} = A[x, p_{n-1}]/\big(xp_{n-1} - p_n,\ x^{2n-2} + a_2 x^{2n-4} + \cdots + a_{2n-2} + p_{n-1}^2\big)$$

which is a free A-module of rank $2n$ and smooth as a k-algebra. We have an involution τ on B and $\widetilde{B}$ given by $\tau(x) = -x$ and $\tau(p_{n-1}) = -p_{n-1}$.

The dualizing sheaf $\omega_{\widetilde{B}/A}$ is a free rank-one $\widetilde{B}$-module, canonically isomorphic to $\widetilde{B}$ away from the ramification locus. As a $\widetilde{B}$-submodule of $\mathrm{Fr}(\widetilde{B})$ it is generated by the inverse of the different $\mathfrak{D}_{\widetilde{B}/A}$ which is given by the formula

$$\begin{aligned}
\mathfrak{D}_{\widetilde{B}/A} &= \det \begin{pmatrix} -p_{n-1} & f' \\ -x & 2p_{n-1} \end{pmatrix} \\
&= (n-1)x^{2(n-1)} + (n-2)a_2 x^{2(n-2)} + \cdots + a_{2n-2}x^2 + p_{n-1}^2.
\end{aligned}$$

In other words, the bilinear form $\widetilde{B} \otimes_A \widetilde{B} \to A$ given by

$$b_1 \otimes_A b_2 \mapsto \mathrm{tr}_{\widetilde{B}/A}(\mathfrak{D}^{-1}_{\widetilde{B}/A} b_1 b_2)$$

is non-degenerate. As in the symplectic and odd special orthogonal cases, we now consider the symmetric bilinear form

$$\omega(b_1, b_2) = \mathrm{tr}_{\widetilde{B}/A}(\mathfrak{D}^{-1}_{\widetilde{B}/A} b_1 \tau(b_2))$$

Then ω is a non-degenerate symmetric bilinear form because $\tau(\mathfrak{D}_{\widetilde{B}/A}) = \mathfrak{D}_{\widetilde{B}/A}$, and it satisfies

$$\omega(xb_1, b_2) = -\omega(b_1, xb_2).$$

After a choice of trivialization of the determinant of $\widetilde{B}$ as a free A-module of rank n, the multiplication by x gives rise to the companion section $[x] : \mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$ for the odd special orthogonal group $G = \mathrm{SO}_{2n+1}$.

2.3.5 The group G_2

In the case G_2, the invariant quotient is $A = k[e, q]$ with $\deg(e) = 2$ and $\deg(q) = 6$. The spectral cover $\mathfrak{s} = \mathrm{Spec}(B)$ of $\mathfrak{c} = \mathrm{Spec}(A)$ given by

$$B = A[x]/\,(xf_0) \quad \text{for } f_0 = x^6 - ex^4 + \frac{e^2}{4}x^2 + q$$

is a reducible cover of A with two components corresponding to the quotient maps

$$B \to B' = A[x]/(f_0) \quad \text{and} \quad B \to A = A[x]/(x).$$

The cover $\mathfrak{s}' = \operatorname{Spec}(B')$ of A is finite, flat of degree 6, and factors through two subcovers, of degrees 2 and 3, corresponding to the sub-A-algebras

$$A \subset A[y]/\left(y^3 - ey^2 + \frac{e^2}{4}y + q\right) \subset B' \quad \text{where } y = x^2$$

$$A \subset A[z]/\left(z^2 + q\right) \subset B' \quad \text{where } z = x\left(x^2 - \frac{e}{2}\right)$$

Let $\epsilon \in B[q^{-1}]^* := \operatorname{Hom}_{A[q^{-1}]}(B[q^{-1}], A[q^{-1}])$ be dual to f_0; $\delta_i \in B[q^{-1}]^*$ be dual to x^i; and $\eta_i \in B[q^{-1}]^*$ be dual to $x^i z$ for $i = 1, 2, 3$. Let tr_z denote the skew-symmetric bilinear form on B given by

$$\operatorname{tr}_z(g, h) = \operatorname{Tr}_{\operatorname{Frac}(B)/\operatorname{Frac}(A)}\left(\frac{g(x)h(-x)z}{f(x)}\right)$$

We will denote by ρ the 3-form on $B[q^{-1}]$ given by

$$\rho := \delta_1 \wedge \delta_2 \wedge \eta_3 + \delta_1 \wedge \eta_2 \wedge \delta_3 + \eta_1 \wedge \delta_2 \wedge \delta_3 - q \cdot \eta_0 \wedge \eta_1 \wedge \eta_2 + \epsilon \wedge \operatorname{tr}_z \tag{2.3.5}$$

A priori, the 3-form above is valued in $A[q^{-1}]$. The next proposition tells us that it restricts to an element of $\bigwedge^3_A B^*$.

Proposition 2.3.3. *Restricting the 3-form ρ to $B \to B[q^{-1}]$ induces a 3-form $\rho \in \bigwedge^3_A B^*$. In other words, ρ takes values in A when restricted to B.*

Proof. Consider the A-basis of B given by

$$\{1, x^i, x^i z \colon i = 1, 2, 3\}.$$

This differs from the $A[q^{-1}]$-basis

$$\{f_0, x^i, x^i z \colon i = 1, 2, 3\}$$

of $B[q^{-1}]$ only by scaling f_0. As ρ is valued in A on the A-linear span of the latter basis, it suffices to check the contraction $\iota_1\rho$ of ρ along $1 \in B$ is valued in A. We compute

$$\begin{aligned}\iota_1\rho &= \eta_1 \wedge \eta_2 - \frac{e}{2}\eta_2 \wedge \eta_3 + \frac{1}{q}(\mathrm{tr}_z - \delta_1 \wedge \delta_2 + \frac{e}{2}\delta_2 \wedge \delta_3 - i_{x^3z}\mathrm{tr}_z \wedge \epsilon + \frac{e}{2}i_{xz}\mathrm{tr}_z \wedge \epsilon) \\ &= \eta_1 \wedge \eta_2 - \frac{e}{2}\eta_2 \wedge \eta_3 + \left[\frac{1}{q}(\mathrm{tr}_z - \delta_1 \wedge \delta_2 + \frac{e}{2}\delta_2 \wedge \delta_3) - \iota_1\mathrm{tr}_z \wedge \epsilon\right]\end{aligned}$$

Rewriting the latter in terms of a dual basis ξ_i, $i = 0, \ldots, 6$ for the A-basis $\{x^i : i = 0, \ldots, 6\}$ of B, we see that the expression in square brackets above is

$$\iota_1\rho = \epsilon_3 \wedge \epsilon_6 + \epsilon_4 \wedge \epsilon_5 - \frac{3e}{2}\epsilon_5 \wedge \epsilon_6$$

whose image lies in A. □

As the previous proposition illustrates, working with the form ρ requires significantly more computational effort. As such, Propositions 2.3.4 and 2.3.5 will be checked primarily with computer algebra packages. These computations were done in Macaulay2; explicit code for each calculation is referred to in Appendix 2.7.

Proposition 2.3.4. *Let ν be the bilinear form associated to ρ as in equation* (2.2.1) *and let $\omega \in S^2_A B^*$ be the symmetric, nondegenerate form given by the formula* (2.3.4). *Then, $\nu = -2^4 3^2 \omega$.*

Proposition 2.3.5. *The form ρ is compatible with the endomorphism $[x]$, in the sense that*

$$\rho(xb_1, b_2, b_3) + \rho(b_1, xb_2, b_3) + \rho(b_1, b_2, xb_3) = 0.$$

As such, the form ρ together with a trivialization of the determinant gives a map $[x] \colon \mathfrak{c} \to [\mathfrak{g}_2/G_2]$.

2.4 Special components

In the previous section, we gave explicit formulas for the tensors defining the reduction of the vector bundle $\mathcal{O}_{\mathfrak{c}\times_{\mathfrak{c}_n}\mathfrak{s}_n}$ to G so that the companion section for GL_n induces the companion section for classical group G. These explicit formulas may feel like miracles, especially in the G_2 case where a computer algebra system is needed. In this section, we will derive them from the geometry of spectral covers, which makes the construction more conceptual, especially in the G_2 case. In subsequent work, we use this approach to construct the companion section uniformly.

2.4.1 Special form and component associated with a subcover

Let A be a k-algebra, B a finite flat A-algebra of degree n generated by one element $b \in B$, and $A' \subset B$ an A-subalgebra of B such that A' is finite flat of degree m over A generated by one element $a' \in A'$ and B is a finite flat A'-algebra of degree d generated by b. Under these assumptions, we have $B \simeq A[x]/P(x)$ where $P(x)$ is the characteristic polynomial of the A-linear $b : B \to B$ defined as the multiplication by b. Similarly we have $A' \simeq A[x]/(P_1(x))$ where $P_1(x)$ is the the characteristic polynomial of the A-linear operator $a' : A' \to A'$, and $B \simeq A'[x]/P_2(x)$ where $P_2(x)$ is the characteristic polynomial of the A'-linear operator $b : B \to B$.

Assuming that the characteristic of k is greater than d, we want to construct an alternating d-form

$$\omega_{A'} : \wedge_A^d B \to A$$

supported on a special component of $\mathrm{Spec}(S_A^d B)$ isomorphic to $\mathrm{Spec}(A')$. We explain what this means. As far as we know, the concept of non-degeneracy for d-forms is not yet defined for $d \geq 3$ and thus we can prove the it only for $d = 1$ or $d = 2$. However, we expect that the form we construct is non-degenerate for a reasonable definition of this concept. As to the special component, $\bigwedge_A^d B$ is a module over the ring of symmetric tensors $(\bigotimes_A^d B)^{\mathfrak{S}_d}$. We will construct a surjective homomomorphism of A-algebras $(\bigotimes_A^d B)^{\mathfrak{S}_d} \to A'$ which realizes $\operatorname{Spec}(A')$ as an irreducible component of $\operatorname{Spec}((\bigotimes_A^d B)^{\mathfrak{S}_d})$ if B is generically étale over A and A' is a domain.

The homomorphism of A-algebras $(\bigotimes_A^d B)^{\mathfrak{S}_d} \to A'$ is constructed as follows. Let $P_2(x) = x^d + a_1' x^{d-1} + \cdots + a_d'$ be the characteristic polynomial of the A'-linear map $b : B \to B$. Then we have

$$B = A'[x]/(x^d + a_1' x^{d-1} + \cdots + a_d').$$

We consider the polynomial ring $R = k[x_1, \ldots, x_d]$ and the subring S of invariant polynomials under the symmetric group $\mathfrak{S}_d$. We have

$$S = k[x_1, \ldots, x_d]^{\mathfrak{S}_d} = k[\alpha_1, \ldots, \alpha_d]$$

with

$$\alpha_i = (-1)^i \sum_{1 \leq j_1 < \cdots < j_i \leq d} \alpha_{j_1} \ldots \alpha_{j_d}.$$

Since R and S are regular, and R is a finite generated S-module, R is a finite flat S-algebra of degree $d!$. We consider the homomorphism of algebras $S \to A'$ given by $\alpha_i \mapsto a_i'$ and the base change $A' \otimes_S R$ which is a finite flat A'-algebra of degree $d!$ equipped with an action of $\mathfrak{S}_d$. We have $(A' \otimes_S R)^{\mathfrak{S}_d} = A'$. Moreover, for every $i \in \{1, \ldots, d\}$ we have a homomorphism of A'-algebras $B \to R \otimes_S A'$ given by $x \mapsto x_i$ which together give rise to a surjective homomorphism of A'-algebras $\bigotimes_{A'}^d B \to A' \otimes_S R$, which is $\mathfrak{S}_d$-equivariant. We derive a $\mathfrak{S}_d$-equivariant surjective homomorphism of A-algebras

$$\bigotimes_A^d B \to \bigotimes_{A'}^d B \to A' \otimes_S R. \tag{2.4.1}$$

By taking the $\mathfrak{S}_d$-invariant, we obtain the desired homomorphism of algebras

$$S^d_A B = (\bigotimes_A^d B)^{\mathfrak{S}_d} \to A',$$

which is surjective because taking $\mathfrak{S}_d$-invariants is an exact functor under the characteristic assumption.

We will now construct a special d-form on B

$$\omega_{A'} : \bigwedge_A^d B \to A$$

supported on the special component. As above, we have a surjective homomorphism of algebras $\mathfrak{S}_d$-equivariant surjective homomorphisms of A-algebras $\otimes^d_A B \to \otimes^d_{A'} B \to A' \otimes_S R$ which induces a surjective A-linear maps of the alternating parts $\bigwedge^d_A B \to \bigwedge^d_{A'} B \to A' \otimes_S R^{\mathrm{sgn}}$ where R^{sgn} is the direct factor of R as S-module in which $\mathfrak{S}_d$ acts as the sign character. It is known that R^{sgn} is a free S-module generated by $\prod_{1\leq i<j\leq d}(x_i - x_j)$. We thus obtains a surjective A-linear map $\bigwedge^d_A B \to A'$. By composing it with the generator of $\mathrm{Hom}_A(A', A)$ constructed in 2.3.2 we obtain the special d-form $\omega_{A'} : \bigwedge^d_A B \to A$ which is supported by the special by construction.

Let us discuss the non-degeneracy of the special d-form $\omega_{A'} : \bigwedge^d_A B \to A$. For $d = 1$, this follows from Lemma 2.3.2. We can check that it is also non-degenerate everywhere for $d = 2$. For $d \geq 3$, we don't know a general definition of non-degeneracy but it easy to see that the special form $\omega_{A'}$ is everywhere non-zero. In dimension 6 and 7 where the definition of non-degeneracy is available, we will check that the special d-form is everywhere non-degenerate by direct calculation.

2.4.2 Sp_{2n} case

We recall in the case $G = \mathrm{Sp}_{2n}$, we have $\mathfrak{c} = \mathrm{Spec}(A)$ with $A = k[a_2, \dots, a_{2n}]$. The spectral cover $\mathfrak{s} = \mathrm{Spec}(B)$ where

$$B = A[x]/(x^{2n} + a_2 x^{2n-2} + \cdots + a_{2n})$$

is a free A-module of rank $2n$, is equipped with an involution $\tau : B \to B$ given $\tau(x) = -x$. We consider the subalgebra A' of B consisting of elements fixed under τ

$$A' = A[y]/(y^n + a_2 y^{n-1} + \cdots + a_{2n}).$$

We then have $B = A'[x]/(x^2 - y)$.

The construction of the special form and special component in 2.4.1 gives rise to an alternating form

$$\omega_{A'} : \wedge^2_A B \to A$$

supported in the special component $\mathfrak{c}' = \mathrm{Spec}(A')$ of $(\mathfrak{s} \times_{\mathfrak{c}} \mathfrak{s}) /\!\!/ \mathfrak{S}_2$ where $\mathfrak{s} = \mathrm{Spec}(B)$ and $\mathfrak{c} = \mathrm{Spec}(A)$. The homomorphism (2.4.1) $\mathrm{Sym}^2_A(B) \to A'$ can be explicitly computed elements of the form:

$$b \otimes_A 1 + 1 \otimes_A b \mapsto \mathrm{tr}_{B/A'}(b).$$

In particular, $x \otimes_A 1 + 1 \otimes_A x$ be long to the kernel of $\mathrm{Sym}^2_A(B) \to A'$, and in fact on can verify that it is a generator of the kernel. Since $x \otimes_A 1 + 1 \otimes_A x$ annihilates $\omega_{A'}$ we have

$$\omega_{A'}(xb_1, b_2) + \omega_{A'}(b_1, xb_2) = 0$$

for every $b_1, b_2 \in B$. By Lemma 2.3.2, the 2-form $\omega_{A'}$ is everywhere non-degenerate. We can also see by explicit calculation that the form $\omega_{A'}$ is the same as the 2-form we constructed in subsection 2.3.2 by means of the Euler formula.

2.4.3 G_2 case

In the case G_2, the invariant quotient is $A = k[e, q]$ with $\deg(e) = 2$ and $\deg(q) = 6$. The spectral cover $\mathfrak{s} = \operatorname{Spec}(B)$ with

$$B = A[x]/\left(xf_0\right) \quad \text{for } f_0 = x^6 - ex^4 + \frac{e^2}{4}x^2 + q$$

is a reducible cover of A with two components corresponding to the quotient maps

$$B \to B' = A[x]/(f_0) \quad \text{and} \quad B \to A$$

We will define a canonical 3-form on B out of a 3-form and a 2-form on B' associated to subalgebras

$$\begin{aligned} A \subset A' = A[z]/\left(z^2 + q\right) &= k[e, y] \;\subset\; B' = A'[x]/(x^3 - \frac{e}{2}x - z) \\ A \subset A'' = A[y]/\left(y^3 - ey^2 + \frac{e^2}{4}y + q\right) &= k[e, z] \;\subset\; B' = A''[x]/(x^2 - y). \end{aligned}$$

Since both A' and A'' are regular algebras, they are finite flat A-modules of rank 2 and 3, respectively, whereas B are finite flat A'-module and A''-module of rank 3 and 2, respectively. The construction of the special form associated with a subcover gives rise to

$$\omega_{A'} : \wedge_A^3 B' \to A \text{ and } \omega_{A''} : \wedge_A^2 B' \to A$$

supported on the special components $\mathfrak{c}' = \operatorname{Spec}(A')$ and $\mathfrak{c}'' = \operatorname{Spec}(A'')$ of $\mathfrak{s}^{\times_{\mathfrak{c}}^3} /\!\!/ \mathfrak{S}_3$ and $\mathfrak{s}^{\times_{\mathfrak{c}}^2} /\!\!/ \mathfrak{S}_2$, respectively. By arguing as in the symplectic case, we see that $\omega_{A'}$ is annihilated by $x \otimes_A 1 \otimes_A 1 + 1 \otimes_A x \otimes_A 1 + 1 \otimes_A 1 \otimes_A x$ and $\omega_{A'}$ by $x \otimes_A 1 + 1 \otimes_A x$. It follows that as alternating forms, they satisfy the relations:

$$\begin{aligned} \omega_{A'}(xb_1, b_2, b_3) + \omega_{A'}(b_1, xb_2, b_3) + \omega_{A'}(b_1, b_2, xb_3) &= 0 \\ \omega_{A''}(xb_1, b_2) + \omega_{A''}(b_1, xb_2) &= 0 \end{aligned}$$

for all $b_1, b_2, b_3 \in B$.

The form $\omega_{A'}$ agrees with the restriction of the form ρ calculated by Macaulay 2 when restricted to $B' \to B$, with the inclusion given by multiplication by x: Indeed, the restriction of ρ takes value 1 on each of:

$$z \wedge x \wedge x^2, \quad x \wedge zx \wedge x^2, \quad 1 \wedge x \wedge zx^2$$

and $-q$ on $z \wedge zx \wedge zx^2$. This exactly detects the coefficient of z when these wedges are written in terms of the A' basis $1 \wedge x \wedge x^2$ for $\wedge^3_{A'} B'$, which matches $\omega_{A'}$ since the generator of $\mathrm{Hom}_A(A', A)$ as an A'' module detects the coefficient of A'.

We now build a 3-form on B out of the 3-form $\omega_{A'}$ and 2-form $\omega_{A''}$ on B'. Since $B = A[x]/(xf_0)$, $B' = A[x]/(f_0)$ we have exact sequences of free A-modules

$$0 \to A \to B \to B' \to 0 \text{ and } 0 \to B' \to B \to A \to 0$$

where the map $A \to B$ in the first sequence is given by $1 \mapsto f_0$ and the map $B' \to B$ in the second sequence is given by $1 \mapsto x$. It follows an exact sequences

$$0 \to A \oplus B' \to B \to Q \to 0 \text{ and } 0 \to B \to A \oplus B' \to Q \to 0$$

where $Q = A/(q) = B'/(x)$. It follows an exact sequence

$$0 \to \wedge^3_A B^* \to \wedge^3_A (B')^* \oplus \wedge^2_A (B')^* \to \wedge^2 (B')^*/(q) \to 0$$

where the map $\wedge^2_A(B')^* \to \wedge^2(B')^*/(q)$ is the reduction modulo q, and the map $\wedge^3_A(B')^* \to \wedge^2(B')^*/(q)$ is obtained by the composition

$$\wedge^3_A(B')^* \to \wedge^3_A B^* \to \wedge^2_A B^* \to \wedge^2_A(B')^* \to \wedge^2_A(B')^*/(q)$$

where the first map is induced by the projection $B \to B'$, the second is given by contraction with f_0, the third map is induced by the inclusion $B' \to B$ sending $1 \mapsto x$, and the final map is the quotient map. Since $q \wedge^2 (B')^* \simeq \wedge^2(B')^*$ is a free, rank 1 module over the special component of $S^2_A(B')$, there is a unique generator as an A'' module. The 3-form $\omega_{A'}$ and the 2-form $\omega_{A''}$ do not have the same image in $\wedge^2_A(B')^*/(q)$; however, the form $z\omega_{A''}$ is and it gives a generator for the A'' submodule of 2-forms compatible with $\omega_{A'}$. The pair $(\omega_{A'}, z\omega_{A''})$ comes from an element of $\wedge^3_A B^*$ which agrees with the 3-form calculated by Macaulay2.

2.5 Lattice description of affine Springer fibers of classical groups

Let us recall Kazhdan-Lusztig's definition [24] of affine Springer fibers. Let G be a split reductive group defined over a field k and $\mathfrak{g}$ its Lie algebra. Let $F = k((\varpi))$ the field of Laurent formal series and $\mathcal{O} = k[[\varpi]]$ its ring of integers. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. The affine Springer fiber associated with γ is an ind-scheme defined over k whose set of k-points is

$$\mathcal{M}_\gamma(k) = \{g \in G(F)/G(\mathcal{O}) | \mathrm{ad}(g)^{-1}\gamma \in \mathfrak{g}(\mathcal{O})\}.$$

We note that $\mathcal{M}_\gamma$ is non-empty only if the image $a \in \mathfrak{c}(F)$ lies in $\mathfrak{c}(\mathcal{O})$ where $\mathfrak{c} = \mathfrak{g}/\!\!/G$ is the invariant theoretic quotient of $\mathfrak{g}$ by the adjoint action of G. As argued in [35], using the Kostant section, we can define an affine Springer fiber $\mathcal{M}_a$ depending only on a instead of γ, which is isomorphic to $\mathcal{M}_\gamma$.

For $G = \mathrm{GL}_n$, the affine Springer fiber $\mathcal{M}_a$ has a well-known lattice description. In this case, $\mathfrak{c} = \mathbf{A}^n$. If $a = (a_1, \ldots, a_n) \in \mathcal{O}_n$, we form the finite flat $\mathcal{O}$-algebra

$$B_a = \mathcal{O}[x]/(f_a)$$

where $f_a = x^n + a_1 x^{n-1} + \cdots + a_n$ by the base change from the universal spectral cover. As $\gamma \in \mathfrak{g}(F)$ is a regular semisimple element, $B_a \otimes_{\mathcal{O}} F$ is finite and étale over F. We have a well-known lattice description of the affine Springer fiber $\mathcal{M}_a$ in this case.

Theorem 2.5.1. *For $G = \mathrm{GL}_n$ and $a \in \mathfrak{c}^{\mathrm{rs}}(F) \cap \mathfrak{c}(\mathcal{O})$, the set $\mathcal{M}_a(k)$ consists of lattices $\mathcal{V}$ in the n-dimensional vector space $V = B_a \otimes F$ which are also B_a-modules.*

See for example, Section 2 of [50] for an exposition.

For computational purposes, it is desirable to have a lattice description of affine Springer fibers similar to Theorem 2.5.1 for classical groups, which is as simple as in the linear case. This is possible due to the construction of the companion matrix, and in fact, this was our original motivation.

In the cases we have investigated in the paper, i.e., symplectic, special orthogonal, and G_2, we have constructed a finite, flat spectral cover $\mathfrak{s} = \mathrm{Spec}(B)$ of the invariant theoretic quotient $\mathfrak{c} = \mathrm{Spec}(A)$ which is étale over the regular semisimple locus of $\mathfrak{c}$. The degree $d = \deg(B/A)$ is the degree of the standard representation which is $2n$ for Sp_{2n}, $2n+1$ for SO_{2n+1}, $2n$ for SO_{2n} and 7 for G_2. In the case $\mathrm{SO}(2n)$, we must consider the normalization $\tilde{B}$ of B. In each of these cases, we constructed a form ω, which is

- a non-degenerate symplectic form $\omega : B \times B \to A$ satisfying $\omega(xb_1, b_2) + \omega(b_1, xb_2) = 0$ for Sp_{2n}
- a non-degenerate symmetric form $\omega : B \times B \to A$ satisfying $\omega(xb_1, b_2) + \omega(b_1, xb_2) = 0$ for SO_{2n+1}

- a non-degenerate symmetric form $\omega : \tilde{B} \times \tilde{B} \to A$ satisfying $\omega(xb_1, b_2) + \omega(b_1, xb_2) = 0$ for SO_{2n}

- a non-degenerate alternating form $\omega : B \times B \times B \to A$ satisfying

$$\omega(xb_1, b_2, b_3) + \omega(b_1, xb_2, b_3) + \omega(b_1, b_2, xb_3) = 0$$

  for G_2

We also constructed a trivialization of the determinant $\bigwedge_A^d B = A$ in all these cases.

For every $a \in \mathfrak{c}(\mathcal{O}) \cap \mathfrak{c}^{rs}(F)$, we construct a finite flat $\mathcal{O}$-algebra B_a by base change from the spectral cover $\mathfrak{s} \to \mathfrak{c}$. Because $a \in \mathfrak{c}^{rs}(F)$, the generic fiber $V_a = B_a \otimes_{\mathcal{O}} F$ is a finite étale F-algebra of degree d. By pulling back ω, we get a form ω_a which is a non-degenerate alternating F-bilinear form on V_a in the symplectic case, a non-degenerate symmetric F-bilinear form on V_a in the orthogonal case, and a non-degenerate alternating F-trilinear form on V_a in the G_2 case. Moreover, it extends to a non-degenerate form valued in $\mathcal{O}$ on B_a in Sp_{2n}, SO_{2n+1} and G_2 cases and on $\tilde{B}_a$ in the SO_{2n}-case.

Theorem 2.5.2. *The set of k-points of the affine Springer fiber $\mathcal{M}_a$ is the set of $\mathcal{O}$-lattices $\mathcal{V}$ of V_a, which are B_a-modules, such that the restriction of ω_a has value in $\mathcal{O}$ and such that* $\deg(\mathcal{V} : B_a) = 0$ *in* $\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, G_2$ *cases and* $\deg(\mathcal{V} : \tilde{B}_a) = 0$ *in the* SO_{2n} *case.*

The proof of this result follows immediately from the proof of Theorem 2.5.1, as lattices preserved by the nondegenerate form ω_a constructed above are exactly those for which there is a reduction of structure to the classical group G.

2.6 Application to the Hitchin fibration

Let X be a smooth, projective curve over an algebraically closed field k and let G be a reductive group over k with Lie algebra $\mathfrak{g}$. Fix a line bundle L on X such that either

$\deg(L) > 2g-2$ or $L = K$ is the canonical bundle. Denote by $\mathcal{M}$ the moduli stack of Higgs bundles on X, whose k points are given by the set of Higgs bundles

$$\mathcal{M}(k) = \{(E, \phi) \colon E \to X \text{ is a } G \text{ bundle}, \, \phi \in \Gamma(X, \mathrm{ad}(E) \otimes L)\}$$

More succinctly, $\mathcal{M}$ is the mapping stack $\mathcal{M} = \mathrm{Maps}(X, [\mathfrak{g}_L/G])$ where $\mathfrak{g}_L = \mathfrak{g} \wedge^{\mathbb{G}_m} L$ is the twisted bundle of Lie algebras on X.

Recall that under mild hypotheses on the characteristic of k ($\mathrm{char}(k) > 2$ for $G = \mathrm{SO}_n$ and Sp_{2n} and $\mathrm{char}(k) > 3$ for $G = G_2$), the Chevalley isomorphism shows

$$\mathfrak{g}/\!\!/G \simeq \mathfrak{t}/\!\!/W \simeq \mathbf{A}^n$$

is an affine space with $\mathbb{G}_m$ action by weights $d_1, \ldots, d_n$. Let

$$\mathcal{A} = \mathrm{Maps}(X, \mathfrak{g}_L/\!\!/G) \simeq \otimes_{i=1}^{n} \Gamma(X, L^{\otimes d_i})$$

Hitchin, in [20], studied the space $\mathcal{M}$, with appropriate stability conditions imposed, through the fibration that now bears his name:

$$h \colon \mathcal{M} \to \mathcal{A}, \quad (E, \phi) \mapsto \mathrm{char}(\phi)$$

where $\mathrm{char}(\phi)$ is given by composition with the quotient map $[\mathfrak{g}/G] \to \mathfrak{g}/\!\!/G$. Let $\mathcal{M}_a$ denote the fiber of the map h over a point $a \in \mathcal{A}$. In the case that $G = \mathrm{GL}_n$, $d_i = i$ and $\mathrm{char}(\phi) = \sum_i a_i x^i$ is the characteristic polynomial of ϕ, whose coefficients are then sections $a_i \in \Gamma(X, L^{\otimes i})$.

The companion section $[x] : \mathfrak{g}/\!\!/G \to [\mathfrak{g}/G]$ can be used to construct an explicit section to the Hitchin map after extracting a square root of L. This section in many cases is almost the same as the section constructed by Hitchin [20] and [22], but can be different from the section constructed in [35] which is based on the Kostant section. In every case, the Higgs bundle constructed from the companion section will be built out of the structural sheaf of the spectral curve. Note that the following assumes basic $\mathbb{G}_m$ equivariance properties of the relevant forms. For example, in the case of $G = \mathrm{Sp}_{2n}$, we have constructed a canonical alternating form $\omega\colon \wedge^2_A B \to A$ which satisfies $\omega(\lambda\xi) = \lambda^{1-2n}\omega(\xi)$ for any $\lambda \in \mathbb{G}_m$ and $\xi \in \wedge^2_A B$.

In [35], it is shown that over a large open subset of $\mathcal{A}$, there is a close connection, depending on a choice of section, between Hitchin fibers and affine Springer fibers given by the Product Formula. More precisely, let $\mathfrak{D} = \bigcup_\alpha \mathfrak{t}^{s_\alpha}/\!\!/W$ be the divisor consisting of the union of the image of each root hyperplane in $\mathfrak{t}$; in particular, the complement of $\mathfrak{D}$ in $\mathfrak{c}$ is the regular, semisimple locus $\mathfrak{c}^{rs}$. Fix $a \in \mathcal{A}$ such that $a(X) \not\subset \mathfrak{D}$, and let $U \subset X$ be the preimage of $\mathfrak{c}^{rs}$ in X. Given trivialization of the line bundle D on some neighborhood of each point $v \in X \setminus U$, we have a map

$$\prod_{v \in X \setminus U} \mathcal{M}_{x,a} \to \mathcal{M}_a.$$

from the product of affine Springer fibers at the points $x \in X \setminus U$ to the Hitchin fiber, which consists of gluing with the companion section restricted to U. It it induces a universal homeomorphism

$$\prod_{\gamma \in X \setminus U} \mathcal{M}_{\gamma,a} \wedge^{\prod_\gamma \mathcal{P}_\gamma(J_a)} \mathcal{P}_a \to \mathcal{M}_a.$$

The groups $\mathcal{P}_\gamma(J_a)$ and $\mathcal{P}_a$ are discussed in detail in [35]; we will not describe them here. This is proved in [35] under the assumption that $\pi_0(\mathcal{P}_a)$ is finite, and by Bouthier and Cesnavicius in [4] under the only assumption that $a(X) \not\subset \mathfrak{D}$.

As Section 2.5 describes the affine springer fibers $\mathcal{M}_{\gamma,a}$, the product formula above gives an explicit description of Hitchin fibers in the case that $a(X) \not\subset \mathfrak{D}$. Namely, we have the following descriptions for Hitchin fibers under this assumption.

- for $G = \mathrm{GL}_n$, and $a \in \mathcal{A}$ we have a spectral cover $p_a : Y_a \to X$ embedded in the total space $|L|$ of L. We then associate with a the Higgs bundle $E_a = p_{a*}\mathcal{O}_{Y_a}$ and the Higgs fields $\phi : E_a \to E_a \otimes L$ given by the structure of $\mathcal{O}_{Y_a}$ as an $\mathcal{O}_{|L|}$-module.

- for $G = \mathrm{Sp}_{2n}$, and $a \in \mathcal{A}$, we have a spectral cover $p_a : Y_a \to X$ embedded in the total space $|L|$ of L. If $E_a = p_{a*}\mathcal{O}_{Y_a}$ then we have a canonical symplectic form $\wedge^2 E_a \to L^{\otimes(1-2n)}$. If L' is a square root of L then $E'_a = E_a \otimes L'^{\otimes 1-2n}$ will be equipped with a canonical symplectic form with value in $\mathcal{O}_X$ and also equipped with a Higgs fields derived from the the structure of $\mathcal{O}_{Y_a}$ as a $\mathcal{O}_{|L|}$-module.

- for $G = \mathrm{SO}_{2n+1}$, and $a \in \mathcal{A}$, we have a spectral cover $p_a : Y_a \to X$ embedded in the total space $|L|$ of L. If $E_a = p_{a*}\mathcal{O}_{Y_a}$ then we have a canonical non-degenerate symmetric form $S^2 E_a \to L^{\otimes(-2n)}$ so that the vector bundle $E'_a = E_a \otimes L^{\otimes n}$ affords a canonical no-degenerate symmetric form with value in $\mathcal{O}_X$, and also equipped with a Higgs fields derived from the the structure of $\mathcal{O}_{Y_a}$ as a $\mathcal{O}_{|L|}$-module. It also affords a trivialization of the determinant depending on the choice of a square root of L.

- for $G = \mathrm{SO}_{2n}$, and $a \in \mathcal{A}$, we have a spectral cover $p_a : Y_a \to X$ embedded in the total space $|L|$ of L. Using the normalization of the universal spectral cover, we obtain a partial normalization $\tilde{Y}_a$ of Y_a. If $E_a = p_{a*}\mathcal{O}_{\tilde{Y}_a}$ then we have a canonical non-degenerate symmetric form $S^2 E_a \to L^{\otimes(2-2n)}$ so that the vector bundle $E'_a = E_a \otimes L^{\otimes 1-n}$ affords a canonical non-degenerate symmetric form with values in $\mathcal{O}_X$, and also equipped with a Higgs fields derived from the the structure of $\mathcal{O}_{Y_a}$ as a $\mathcal{O}_{|L|}$-module. It also affords a canonical trivialization of the determinant depending on the choice of a square root of L.

- for $G = G_2$, and $a \in \mathcal{A}$, we have a spectral cover $p_a : Y_a \to X$ embedded in the total space $|L|$ of L. If $E_a = p_{a*}\mathcal{O}_{\tilde{Y}_a}$ then we have a canonical non-degenerate 3-form $\wedge^3 E_a \to L^{-9}$ so that the vector bundle $E'_a = E_a \otimes L^{\otimes 3}$ affords a canonical non-degenerate 3-form with value in $\mathcal{O}_X$, and also equipped with a Higgs fields derived from the the structure of $\mathcal{O}_{Y_a}$ as a $\mathcal{O}_{|L|}$-module. It also affords a canonical trivialization of the determinant depending on the choice of a square root of L.

2.7 Appendix to Chapter 3: Computer algebra code and G_2 computations

In this appendix, we give the computer code used to compute the 3-form ρ in Section 2.3.5.

2.7.1 Construction of ρ

To construct ρ, we will use the connection between nondegenerate alternating 3-forms and cross products. Let V be a vector space with a nondegenerate, symmetric bilinear form ν.

Definition 2.7.1. A *cross product* on (V, ν) is a bilinear map

$$c\colon V \otimes V \to V$$

satisfying the following three properties for all $v_1, v_2 \in V$:

1. (Skew symmetry) $c(v_1, v_2) = -c(v_2, v_1)$;
2. (Orthogonality) $\nu(c(v_1, v_2), v_1) = 0$;
3. (Normalization) $\nu\big(c(v_1, v_2), c(v_1, v_2)\big) = \det \begin{pmatrix} \nu(v_1, v_1) & \nu(v_1, v_2) \\ \nu(v_1, v_2) & \nu(v_2, v_2) \end{pmatrix}$

The data of a cross product on (V, ν) is equivalent to the data of a nondegenerate 3-form on V whose associated symmetric bilinear form (see equation (2.2.1)) is a scalar multiple of ν. Indeed, to a cross product c, one associates the 3-form

$$\rho(v_1, v_2, v_3) = \nu\big(c(v_1, v_2), v_3\big) \tag{2.7.1}$$

while for any non-degenerate 3-form ρ, there is a unique cross product c satisfying equation (2.7.1).

Now, consider the free, rank 7 A-module B as in Section 2.3.5 equipped with the symmetric, nondegenerate form ω defined by the formula

$$\omega(g_1, g_2) = \mathrm{tr}_{B/A}\left(\frac{g_1\tau(g_2)}{f'}\right)$$

as in the SO_7 case. Here, $\tau(x) = -x$ is the natural involution on B, and the trace is taken after inverting f' in A. To construct a 3-form on B which is nondegenerate over every k point of A, it suffices to construct a cross product

$$c\colon B \otimes_A B \to B$$

for (B, ω). Moreover, the equation

$$\rho(xg_1, g_2, g_3) + \rho(g_1, xg_2, g_3) + \rho(g_1, g_2, xg_3) = 0$$

is equivalent to the condition

$$c(xg_1, g_2) + c(g_1, xg_2) = xc(g_1, g_2). \tag{2.7.2}$$

To simplify computations further, we note that any form $c\colon B \otimes_A B \to B$ satisfying the conditions of Definition 2.7.1 and equation (2.7.2) can be recovered from its trace:

$$tc\colon B \otimes_A B \to A, \quad (g_1, g_2) \mapsto \mathrm{tr}_{B/A}\big(c(g_1, g_2)\big)$$

Indeed, if we express

$$c(x^i, x^j) = \sum_{l=0}^{6} c_{i,j}^{(l)} x^l$$

then $c_{i,j}^{(6)} = tc(x^i, x^j)$ and

$$tr_{B/A}\big(x^l c(x^i, x^j)\big) = \sum_{r=0}^{l} \binom{l}{r} tc(x^{i+r}, x^{j+l-r})$$

can be expressed in terms of $c_{i,j}^{(m)}$ for $6-l \leq m \leq 6$. This allows us to recover the coefficients $c_{i,j}^{(l)}$ by downward induction on l.

This idea is implemented in the following Macaulay2 code. There is a one-dimensional solution space, which is specialized at a particular point to give the form stated in equation (2.3.5). Note that it is immediate from the computer calculation that the form ρ is valued in B and satisfies the conclusion of Proposition 2.3.5.

```
S=QQ[e,q];
F=frac(S);
R=F[p_(0,0) .. p_(6,6)]; -- ring with p_(i,j)=tc(x^i,x^j),
                                  0\leq i,j\leq 6

 -- The following three commands define tc(x^i,x^j) for i or j between
    7 and 12 using the relation x^7-e*x^5+e^4/4*x^3+q*x=0.
for l from 0 to 5 do [for k from 0 to 6 do p_(k,7+l)=e*p_(k,5+l)-
    (1/4)*e^2*p_(k,3+l)-q*p_(k,1+l)];
for l from 0 to 5 do [for k from 0 to 6 do p_(7+l,k)=e*p_(5+l,k)-
    (1/4)*e^2*p_(3+l,k)-q*p_(1+l,k)];
for l from 0 to 5 do [for k from 7 to 12 do p_(k,7+l)=e*p_(k,5+l)-
    (1/4)*e^2*p_(k,3+l)-q*p_(k,1+l)];

 -- I encodes orthogonality:
I = ideal(flatten for a from 0 to 6 list for k from 0 to 6 list
```

```
    sum(0..k,j->binomial(k,j)*p_(k+j,a+k-j)));


 -- J encodes skew symmetry:
J = ideal( flatten for a from 0 to 6 list for b from 0 to 6 list
    p_(a,b)+p_(b,a) );


 -- The following encodes the normalization condition:
B=R[x]/(x^7-e*x^5+(1/4)*e^2*x^3+q*x);
 -- determinant of norms of x^i,x^j:
f = (i,j) -> coefficient(x^6,(-1)^i*x^(2*i))*coefficient(x^6,(-1)^j*
    x^(2*j))-coefficient(x^6,(-1)^j*x^(i+j))*coefficient(x^6,(-1)^j*
    x^(i+j));
 -- norm of c(x^i,x^j):
g = (i,j) -> coefficient(x^6, (p_(i,j)*(x^6-e*x^4+(1/4)*e^2*x^2+q)+
    sum(0..1,l->binomial(1,l)*p_(i+l,j+1-l))*(x^5-e*x^3+(1/4)*e^2*x)+
    sum(0..2,l->binomial(2,l)*p_(i+l,j+2-l))*(x^4-e*x^2+(1/4)*e^2)+
    sum(0..3,l->binomial(3,l)*p_(i+l,j+3-l))*(x^3-e*x)+sum(0..4,l->
    binomial(4,l)*p_(i+l,j+4-l))*(x^2-e)+sum(0..5,l->binomial(5,l)*
    p_(i+l,j+5-l))*(x)+sum(0..6,l->binomial(6,l)*p_(i+l,j+6-l)))
    *(p_(i,j)*((-x)^6-e*(-x)^4+(1/4)*e^2*(-x)^2+q)+sum(0..1,l->
    binomial(1,l)*p_(i+l,j+1-l))*((-x)^5-e*(-x)^3+(1/4)*e^2*(-x))+
    sum(0..2,l->binomial(2,l)*p_(i+l,j+2-l))*((-x)^4-e*(-x)^2+(1/4)*
    e^2)+sum(0..3,l->binomial(3,l)*p_(i+l,j+3-l))*((-x)^3-e*(-x))+
    sum(0..4,l->binomial(4,l)*p_(i+l,j+4-l))*((-x)^2-e)+sum(0..5,l->
    binomial(5,l)*p_(i+l,j+5-l))*(-x)+sum(0..6,l->binomial(6,l)*
    p_(i+l,j+6-l))) );
```

```
 -- K encodes the normalization condition:
K = ideal(flatten for i from 0 to 6 list for j from 0 to 6 list
    f(i,j)-g(i,j));

Q=R/(I+J+K); -- imposing the relations on our ring of variables
Q2=Q/ideal(p_(6,3)-1,p_(6,4),p_(6,5)-5*e/2); -- specializes to our
    particular form rho

 -- Computation of c from tc:
P=Q2[x]/(x^7-e*x^5+e^2/4*x^3+q);
C=table(for k from 0 to 6 list k, for k from 0 to 6 list k, (i,j) ->
    (p_(i,j)*(x^6-e*x^4+(1/4)*e^2*x^2+q)+sum(0..1,l->binomial(1,l)*
    p_(i+l,j+1-l))*(x^5-e*x^3+(1/4)*e^2*x)+sum(0..2,l->binomial(2,l)*
    p_(i+l,j+2-l))*(x^4-e*x^2+(1/4)*e^2)+sum(0..3,l->binomial(3,l)*
    p_(i+l,j+3-l))*(x^3-e*x)+sum(0..4,l->binomial(4,l)*p_(i+l,j+4-l))*
    (x^2-e)+sum(0..5,l->binomial(5,l)*p_(i+l,j+5-l))*(x)+sum(0..6,l->
    binomial(6,l)*p_(i+l,j+6-l))));
-- This is the matrix for c with respect to the basis x^i, i=0,..,6
netList C -- displays C
```

2.7.2 Nondegeneracy of ρ

Let ρ be the form computed in the previous section, stated explicitly in equation (2.3.5). Note that since we specialized to a particular form in the previous section, it is not yet clear that this form is nondegenerate. For this, we produce the following code in Macaulay2 to explicitly compute the associated bilinear form is as in Proposition 2.3.4. The following uses some basic operations on permutations from the package SpechtModule authored by

Jonathan Niño in Macaulay2.

```
T=permutations {0,1,2,3,4,5,6};

n = (v,w) -> sum(0..7!-1, k-> permutationSign(T_k)*coefficient(x^6,

    v*(C_((T_k)_0))_((T_k)_1))*coefficient(x^6,w*(C_((T_k)_2))_((T_k)_3))

    *coefficient(x^6,(-x)^((T_k)_4)*(C_((T_k)_5))_((T_k)_6))  );

S=table(for k from 0 to 6 list k, for k from 0 to 6 list k, (i,j) ->

    n((-x)^i,(-x)^j);

netList S
```

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