THE UNIVERSITY OF CHICAGO

SMOOTH MODELS FOR CERTAIN FIBERED PARTIALLY HYPERBOLIC SYSTEMS

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Dedicated to my parents, Tom and Kari Doucette,

and to my grandmother, Nancy Doucette.

TABLE OF CONTENTS

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ABSTRACT

This thesis studies the existence of smooth models for fibered partially hyperbolic systems. Fibered partially hyperbolic systems are partially hyperbolic diffeomorphisms that have an integrable center bundle, tangent to a continuous invariant fibration by invariant submanifolds.

We prove that under certain restrictions on the fiber and/or bundle, any fibered partially hyperbolic system over a nilmanifold is leaf conjugate to a smooth model that is isometric on the fibers and descends to a hyperbolic nilmanifold automorphism on the base.

CHAPTER 1

INTRODUCTION

Dynamics is the study of the behavior of maps $f : M \to M$ under iteration. Throughout the 70s and 80s, hyperbolic behavior (and in particular Anosov diffeomorphisms) were an extremely active area of study. A diffeomorphism $f : M \to M$ is said to be *Anosov* if the tangent bundle admits a Df-invariant splitting $TM = E^s \oplus E^u$ into a uniformly contracting and a uniformly expanding subbundle.

Anosov diffeomorphisms are remarkable for the wide range of chaotic behaviors they exhibit [13], as well as for their extraordinary degree of stability and rigidity. For example, Anosov diffeomorphisms are structurally stable, and Franks and Manning classified Anosov diffeomorphisms of tori and nilmanifolds by proving that any Anosov diffeomorphism of a torus or nilmanifold is topologically conjugate to a linear model [28], [48].

On the other hand, the existence of an Anosov diffeomorphism on a manifold M imposes strong restrictions on the topology of M. Up to finite covers, all known Anosov diffeomorphisms lie on nilmanifolds, which are manifolds of the form N/Γ, where N is a simply-connected nilpotent Lie group and $N < \Gamma$ is a uniform lattice. In fact, it conjectured that (up to finite covers) only nilmanifolds can support Anosov diffeomorphisms[1].

Despite the limitations on the settings in which Anosov diffeomorphisms can exist, the properties of Anosov diffeomorphisms are extremely interesting. This raises the natural questions of whether hyperbolicity can be weakened in a way that preserves many of these properties while allowing for a much wider range of behaviors. This leads us to the notion of *partial hyperbolicity*.

A diffeomorphism $f : M \to M$ of a closed Riemannian manifold is said to be *partially hyperbolic* if there is a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$, constants

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[1] Note that if true, this conjecture would imply that Franks and Manning's classification result gives a complete classification of Anosov diffeomorphisms.

$0 < \lambda < \hat{\gamma} < 1 < \gamma < \mu$, and constant $C > 0$ such that for all $x \in M$ and all unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$, we have that for all $n \in \mathbb{N}$,

$$\|d_x f^n v^s\| \leq C\lambda^n, \qquad \frac{1}{C}\hat{\gamma}^n \leq \|d_x f^n v^c\| \leq C\gamma^n, \qquad \frac{1}{C}\mu^n \leq \|d_x f^n v^s\|.$$

In other words, Df is uniformly contracting in the direction of the stable bundle E^s, is uniformly expanding in the direction of the unstable bundle E^u, and is less contracting and/or expanding in the direction of the center bundle E^c than it is of the other two bundles. When the constants $\lambda, \hat{\gamma}, \gamma$, and μ are independent of the point $x \in M$, we say that f is *absolutely partially hyperbolic.* When $\lambda, \hat{\gamma}, \gamma$, and μ depend on x, we say that f is *pointwise partially hyperbolic.*

Partially hyperbolic diffeomorphisms were first introduced by Brin and Pesin in their study of frame flows [14] and by Hirsch, Pugh, and Shub in their study of normally hyperbolic foliations [42]. Partially hyperbolic diffeomorphisms are a generalization of *Anosov* diffeomorphisms, and are important because they combine the chaotic behavior of Anosov diffeomorphisms in the directions of the stable and unstable bundles E^s and E^u, respectively, with a more flexible range of behaviors in the direction of the center bundle E^c.

Though partially hyperbolic diffeomorphisms form a much larger and richer class than Anosov diffeomorphisms, partial hyperbolicity, like hyperbolicity, still persists under C^1 perturbations. Partially hyperbolic diffeomorphisms form a much richer class than Anosov diffeomorphisms, but still exhibit many of the same remarkable properties that distinguish Anosov diffeomorphisms (although sometimes under additional assumptions). For more discussion, see Section 2.1.

This thesis focuses on a class of partially hyperbolic systems called *fibered partially hyperbolic diffeomorphisms.* These are partially hyperbolic diffeomorphisms that have an integrable center bundle E^c, tangent to a continuous invariant fibration by compact submanifolds

ifolds[2]. The fibration here allows us to directly observe the interplay between the distinct behavior in the stable/unstable directions and in the transverse center direction.

Fibered partially hyperbolic systems form a rich class of dynamical systems. Beyond the relatively simple examples of skew products on trivial bundles, fibered systems appear as automorphisms of nilmanifolds and play a role in the construction of exotic partially hyperbolic systems (e.g. [31]) and in several rigidity contexts [3], [23], [54]. They also have featured in the proofs of several classification results for partially hyperbolic diffeomorphisms.

A natural notion of equivalence for partially hyperbolic dynamical systems is leaf conjugacy. Two partially hyperbolic diffeomorphisms $f : M \to M$ and $g : M' \to M'$ are said to be *leaf conjugate* if there exists a homeomorphism $h : M \to M'$, a f-invariant foliation W^c_f tangent to the center bundle of f, and a g-invariant foliation W^c_g tangent to the center direction of g such that h maps center leaves of f to center leaves of g (i.e. $h(W^c_f(x)) = W^c_g(h(x))$ and $h(f(W^c_f(x))) = g(h(W^c_f(x))))$.

Classification of partially hyperbolic diffeomorphisms up to leaf-conjugacy is an important problem in the study of partially hyperbolic dynamics because it allows us to understand the behavior of more general partially hyperbolic diffeomorphisms using simpler, easier to study models.

The classification of partially hyperbolic diffeomorphisms up to leaf conjugacy (and also of fibered partially hyperbolic systems) is almost completely open in dimensions greater than three. There are a number of results in dimension three. For example, Hammerlindl and Potrie showed that any partially hyperbolic diffeomorphism on a three-dimensional nilmanifold or torus is leaf conjugate to the “linear" model in its homotopy class [34], [35]. They also provided a partial classification up to leaf conjugacy of partially hyperbolic diffeomorphisms on 3-manifolds whose fundamental group is solvable and has exponential growth [36]. All of these results rely heavily on the topology of three-manifolds and tools from [15], [16], which

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[2] For a more precise definition, see Section 2.2.

only apply in dimension three. As a result, the methods used in dimension three do not generalize well (if at all) to higher dimension.

While there are a lot of results about classification of partially hyperbolic diffeomorphisms in dimension three, almost nothing is known in higher dimension. The results that exist typically require very strong assumptions. For example, Hammerlindl proved that a partially hyperbolic diffeomorphism of the torus $f : \mathbb{T}^d \to \mathbb{T}^d$ ($d > 3$) is leaf conjugate to the linear automorphism of $\mathbb{T}^d$ in its homotopy class under the assumption that the center foliation for f is one-dimensional and the stable and unstable foliations for f are quasi-isometric [34]. Sandfeldt has shown that under similar assumptions of one-dimensional center and quasi-isometry of stable and unstable foliations, a partially hyperbolic diffeomorphism of a nilmanifold modeled on the $(2n + 1)$-dimensional Heisenberg group is leaf conjugate to a nilmanifold automorphism [57].

Even though these results are about the more general class of partially hyperbolic diffeomorphism, fibered partially hyperbolic systems do play a role in their proofs– in constructing the leaf conjugacy, both results show that the original partially hyperbolic systems are fibered. This shows one way in which classification results for fibered partially hyperbolic systems could be of use in understanding the more general class of partially hyperbolic systems.

One thing that makes the study of fibered partially hyperbolic systems challenging is that the fibration of such a system is typically only continuous, and its fibers are C^1 in general. While fibered partially hyperbolic systems are not generally smooth, when they are, the induced map on the base of the fibration, which represents the stable and unstable directions, is Anosov. This allows us to use tools and results about Anosov diffeomorphisms to study smooth fibered partially hyperbolic systems. It also suggests a natural subclass of fibered partially hyperbolic systems to consider.

As noted earlier, Anosov diffeomorphisms are classified on nilmanifolds, but a classification on arbitrary manifolds does not exist. While all known examples of Anosov diffeomorphisms exist on nilmanifolds, it is unknown if nilmanifolds are the only manifolds that can support Anosov diffeomorphisms. In this thesis, we consider fibered partially hyperbolic systems over nilmanifolds. This allows for use and modification of existing results about Anosov diffeomorphisms on nilmanifolds. It also allows us to construct examples of fibered partially hyperbolic systems using existing examples of Anosov diffeomorphisms on nilmanifolds.

1.1 Statements of Main Results

The main result of this thesis establishes the existence of smooth models for certain fibered partially hyperbolic systems. It shows that under certain conditions, every fibered partially hyperbolic system over a nilmanifold is leaf conjugate to a smooth model that is isometric on the fibers and descends to a hyperbolic nilmanifold automorphism on the base.

Theorem A ([25, Theorem A]). *Let* $f : M \to M$ *be a fibered partially hyperbolic system with quotient a nilmanifold* B *and* C^1 *fibers* F *(where* F *is a closed manifold). Suppose that the structure group of the* F*-bundle* M *is* $G \subset \mathrm{Diff}^1(F)$ *and that there exists a Riemannian metric on* F *and a subgroup* I *of* $\mathrm{Isom}(F) \cap G$ *such that the inclusion* $I \hookrightarrow G$ *is a homotopy equivalence.*

Then f *is leaf conjugate to a* C^∞ *fibered partially hyperbolic system* $g : \widehat{M} \to \widehat{M}$ *such that*

1. *The projection of the leaf conjugacy to* B *is a map homotopic to the identity;*
2. *the* F*-bundles* M *and* $\widehat{M}$ *are isomorphic*[3]*;*
3. *the structure group of* $\widehat{M}$ *is* $\mathrm{Isom}(F)$*; and*
4. *the projection of* g *to* B *is a hyperbolic nilmanifold automorphism.*

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[3] This is implicit from the definition of leaf conjugacy, but we state it explicitly for clarity.

Remark 1.1.1.

- The base manifold B in Theorem A a priori might not have a smooth structure. In our results, we assume B is a *topological nilmanifold*, meaning that B is homeomorphic to a nilmanifold. This case can easily be reduced to the case where B is a nilmanifold by replacing the projection map for the bundle with the projection map composed with the homeomorphism. Thus, our results easily extend to a topological nilmanifold B.

- 3. implies that there exists a smooth Riemannian metric on $\widehat{M}$ adapted to g such that g is isometric on fibers. This will be clear from the construction of g in the proof of Theorem A.

- The fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ may act differently on the fibers than the original fibered partially hyperbolic system $f : M \to M$; that is, if $h : M \to \widehat{M}$ is the leaf conjugacy from Theorem A, then $h \circ f$ and $g \circ h$ may not be homotopic.

The main assumption in Theorem A is that the structure group of the F-bundle contain a homotopy equivalent subgroup of $\mathrm{Isom}(F)$. We discuss the necessity of this assumption to our proof in Remark 3.3.3. Finding circumstances where this assumption applies (and thus we can apply Theorem A) comes down to studying the relationship between $\mathrm{Diff}^1(F)$, $\mathrm{Isom}(F)$, and their subgroups.[4] Namely, for which manifolds F and which subgroups G of $\mathrm{Diff}^\infty(F)$ is there a subgroup $H \subset \mathrm{Isom}(F) \cap G$ such that the inclusion $H \hookrightarrow G$ is a homotopy equivalence?

Note that even without the assumption that the structure group of the F-bundle contain a homotopy equivalent subgroup of $\mathrm{Isom}(F)$, our argument gives that the initial fibered partially hyperbolic system $f : M \to M$ is leaf conjugate to an extension over a hyperbolic nilmanifold automorphism. For further discussion and details, see Remark 3.3.3. Notably, in the case where the F-bundle from the fibered partially hyperbolic system $f : M \to M$

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[4] In the following discussion we often replace $\mathrm{Diff}^1(F)$ with $\mathrm{Diff}^\infty(F)$, which we can do since $\mathrm{Diff}^\infty(F) \hookrightarrow \mathrm{Diff}^1(F)$ is a homotopy equivalence (Proposition 2.3.7).

is trivial (i.e. $M = B \times F$), we can construct this extension over a hyperbolic nilmanifold automorphism to be partially hyperbolic.

Proposition B. *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and C^1 fibers F (where F is a closed manifold). Suppose that the F-bundle M is trivial (i.e. that the F-bundle M is isomorphic to $B \times F$).*

Then f is leaf conjugate to a C^∞ fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ such that 1., 2, and 4. from Theorem A hold.

The proof of Proposition B is given at the end of Section 3.3.

The following two corollaries come from answering the above question about subgroups $H \subset \mathrm{Isom}(F) \cap G$ for specific F and G. They are by no means the only such corollaries. (For example, the conclusion of Corollary D holds for any F such that $\mathrm{Diff}^1_0(F)$ is contractible.)

Corollary C. *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and C^1 fibers F, where F is*

1. *a n-sphere S^n for $n = 1, 2, 3$, or*
2. *a hyperbolic 3-manifold*

Then f is leaf conjugate to a C^∞ fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$, which induces a hyperbolic nilmanifold automorphism on the base.

Corollary D. *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and C^1 fibers F and with structure group $\mathrm{Diff}^1_0(F)$. Suppose that F is*

1. *the two or three torus, $\mathbb{T}^2$, $\mathbb{T}^3$,*
2. *a hyperbolic surface, or*
3. *a Haken 3-manifold*

Then f is leaf conjugate to a C^{∞} fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$, which induces a hyperbolic nilmanifold automorphism on the base.

Remark 1.1.2. As noted before, these are by no means the only cases where Theorem A can be applied to get results analogous to the results of Corollaries C and D. Notably, the conclusion of Corollary C holds for any fiber F such that the inclusion $\operatorname{Isom}(F) \hookrightarrow \operatorname{Diff}(F)$ is a homotopy equivalence. For the sake of conciseness, we haven't listed in Corollary C more of the known examples of closed manifolds F for which $\operatorname{Isom}(F) \hookrightarrow \operatorname{Diff}(F)$ is a homotopy equivalence. Other examples include lens spaces, prism and quaternionic manifolds, tetrahedral manifolds, octahedral manifolds, and icosahedral manifolds [4].

Theorem A builds on previous work by Hirsch-Pugh-Shub and by Hammerlindl and Potrie. Hirsch, Pugh, and Shub [42] proved that perturbations of fibered partially hyperbolic systems are fibered partially hyperbolic systems, and that the perturbed system is leaf conjugate to the original system. Hammerlindl [34] proved that a partially hyperbolic diffeomorphism of $\mathbb{T}^3$ is leaf conjugate to a linear automorphism of $\mathbb{T}^3$. Hammerlindl and Potrie [35] proved an analogous result for partially hyperbolic diffeomorphisms of 3-dimensional nilmanifolds.

Corollaries C and D give specific examples of cases where we can apply Theorem A in dimensions 1, 2, and 3, and Proposition B gives an analogue of Theorem A in the case of a trivial bundle. Note that Corollary C shows that the hypothesis of Theorem A holds for all fibered partially hyperbolic systems with one dimensional fiber. Corollaries C and D also show that the conclusion of Theorem A holds for surface bundles, although in the cases of surfaces other than S^2, an added assumption on the structure group of the bundle is needed. The question remains of whether, and in what conditions for other types of fibers in nontrivial bundles.

Remark 1.1.3. If for the fibered partially hyperbolic system $f : M \to M$, $\dim E^s = 1$ (or $\dim E^u = 1$), then the base space B will always be a nilmanifold [7]. This allows us to replace the assumption that the quotient of the fibered partially hyperbolic system $f : M \to M$ is a nilmanifold in Theorem A or in Proposition B with the assumption that $\dim E^s = 1$ (or $\dim E^u = 1$).

One ingredient in the proof of Theorem A, which is of independent interest, is the following generalization of the works of Franks-Manning and of Hiraide to Anosov homeomorphisms of nilmanifolds.

Theorem E ([61, Theorem 2(1)]). *An Anosov homeomorphism of a nilmanifold is topologically conjugate to a hyperbolic nilmanifold automorphism via a conjugacy that is homotopic to the identity.*

This theorem was originally proved by Sumi in [61]. A proof of Theorem E, which follows the same structure as Sumi and Hiraide's proofs in [61] and [41], is provided in Section 2.4 for the sake of completeness.

Anosov homeomorphisms are generalizations of Anosov diffeomorphisms. Many of the important properties of Anosov diffeomorphisms come directly from the fact that Anosov diffeomorphisms are expansive and have the shadowing property. A homeomorphism $f : X \to X$ of a metric space is *expansive* if there exists a constant $c > 0$ such that for all $x, y \in X$, if $d(f^n(x), f^n(y)) < c$ for all $n \in \mathbb{Z}$ then $x = y$. Such a constant c is called an *expansive constant* for f.

The shadowing property says that we can approximate *pseudo-orbits* by actual orbits. More formally, a sequence of points $\{x_i\}_{i \in \mathbb{Z}} \subset X$ is called a δ*-pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A point $z \in X$ is said to ε*-shadow* a sequence of points, $\{x_i\}_{i \in \mathbb{Z}}$, if $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo-orbit is ε-shadowed by a point in X. An expansive homeomorphism with the shadowing property is known as an *Anosov homeomorphism.*

In contrast to some other weakenings of Anosov diffeomorphisms (e.g. hyperbolic homeomorphisms [47, Section IV.9]), Anosov homeomorphisms are not assumed to have invariant foliations.

In Section 2.4, we examine some of the similarities between Anosov homeomorphisms and Anosov diffeomorphisms and give a proof Theorem E.

Theorem E is useful in the proof of our main theorem because given a fibered partially hyperbolic diffeomorphism $f : M \to M$ with associated bundle $\pi : M \to B$, the map induced by f on B is an Anosov homeomorphism. (This follows from a result of Bohnet and Bonatti [8], as we will explain in Section 3.3.)

After proving Theorem E in Section 2.4, we spend the rest of the paper proving Theorem A.

1.2 Sketch of Proof of Theorem A

The proof of Theorem A is split into four parts. Due to the fact that much of the proof will take place in the quotient leaf space B, we denote the partially hyperbolic diffeomorphism on M by $\hat{f} : M \to M$. We denote the homeomorphism that $\hat{f}$ descends to on B by $f : B \to B$.

The strategy of the proof is to first construct,

- a conjugacy $h : B \to B$ between $f : B \to B$ and a hyperbolic nilmanifold automorphism $A : B \to B$, and
- a smooth F-bundle $\widehat{M}$ over B that is isomorphic to the original F-bundle M.

Then lift

- the conjugacy $h : B \to B$ to a homeomorphism $\hat{h} : \widehat{M} \to \widehat{M}$, and
- the hyperbolic nilmanifold automorphism $A : B \to B$ to a partially hyperbolic diffeomorphism $g : \widehat{M} \to \widehat{M}$.

The construction of the conjugacy $h : B \to B$ and the hyperbolic nilmanifold automorphism $A : B \to B$ relies almost entirely on Theorem E. The construction of the smooth bundle $\widehat{M}$ relies on tools developed in Section 2.3. Lifting the conjugacy $h : B \to B$ to a homomorphism $\hat{h} : \widehat{M} \to \widehat{M}$ takes place in Section 3.1, and finally lifting A to a partially hyperbolic diffeomorphism g relies on tools developed in Section 3.2. The entire proof of Theorem A is given in Section 3.3.

CHAPTER 2

BACKGROUND AND PRELIMINARIES

2.1 History

As noted earlier, partial hyperbolicity was originally introduced in the 1970s in the work of Brin an Pesin [14] and of Hirsch, Pugh, and Shub [42]. The definition of partial hyperbolicity is a weakening of hyperbolicity and is in part motivated by the attempt to find a wider class of diffeomorphisms that retain some of the chaotic and rigidity properties of Anosov diffeomorphisms.

In the 1970s, Brin and Pesin studied the dynamics of frame flows over closed negatively curved manifolds. They showed that the time t map of the frame flow of a closed, negatively curved manifold was partially hyperbolic. In addition, they showed that for many closed, negatively curved manifolds, the frame flow is ergodic (and mixing) [14].

Skew products are another important example (or really class of examples) of partially hyperbolic diffeomorphisms. In their study of normally hyperbolic foliations, Hirsch, Pugh, and Shub gave examples of non-Anosov, partially hyperbolic diffeomorphisms that are robustly transitive, and in fact are robustly mixing [26]. Transitivity is one of the chaotic properties exhibited by Anosov diffeomorphisms, and so transitive (and especially robustly transitive) partially hyperbolic diffeomorphisms are of great interest. Another important example of robustly transitive partially hyperbolic diffeomorphisms are Derived from Anosov (DA) diffeomorphisms, which were first exhibited by Mañé [9]. A third important category of examples of partially hyperbolic diffeomorphism are time-1 maps of Anosov flows and their perturbations. This category includes the first example of a stably ergodic, non-Anosov partially hyperbolic diffeomorphism, which was given by Grayson, Pugh and Shub [33]. In the 1990s, Bonatti and Díaz constructed the first example of a robustly transitive partially hyperbolic diffeomorphism homotopic to the identity by perturbing the time one map of an

Anosov flow [9].

The study of robustly transitive partially hyperbolic diffeomorphisms also lead to a number of conjectures and breakthroughs about C^1 generic dynamics. For example, Crovisier proved the weak Palis conjecture[1] by reducing to a classification of partially hyperbolic dynamics with 1-dimensional center [21].

Additionally, the above examples have inspired a large number of conjectures and results. For example, they lead to Pugh and Shub to conjecture that set of stably ergodic diffeomorphisms is open and dense in the set of C^2 volume-preserving partially hyperbolic diffeomorphisms [55]. For more discussion of this and related conjectures, see [17].

A number of the above examples are fibered partially hyperbolic systems. These examples have also motivated a number of conjectures about the classification of partially hyperbolic diffeomorphisms. In 2001, Pujals conjectured that (up to isotopy within partially hyperbolic systems), transitive partially hyperbolic diffeomorphisms in dimension 3 could be divided into three classes: time one maps of Anosov flows, fibered partially hyperbolic systems over $\mathbb{T}^2$, and Anosov diffeomorphisms of $\mathbb{T}^3$ [12]. Rodriguez Hertz, Rodriguez Hertz, and Ures also conjectured that this trichotomy also classified the behavior of dynamically coherent[2] partially hyperbolic diffeomorphisms in dimension 3 up to leaf conjugacy[19].

While both of these conjectures have been proven false in the general setting [10], [11], they have inspired a rich theory of partially hyperbolic diffeomorphisms in dimension three. They have been shown to hold in a number of specific cases:

- On $\mathbb{T}^3$, any dynamically coherent, pointwise partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$, leaf-conjugate to a partially hyperbolic toral automorphism (namely, f's linearization) [35]. If pointwise partial hyperbolicity is replaced with absolute partial

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[1] The weak Palis conjecture says that any diffeomorphism can be C^1 approximated by either a Morse-Smale diffeomorphism or a diffeomorphism that has a transverse homoclinic intersection.

[2] A partially hyperbolic diffeomorphism $f : M \to M$ is said to be *dynamically coherent* if it admits f-invariant foliations $\mathscr{F}^{cu}$ and $\mathscr{F}^{cs}$ that are tangent to the bundles $E^c \oplus E^u$ and $E^s \oplus E^c$ respectively.

hyperbolicity, the assumption that the diffeomorphism be dynamically coherent can be removed [34].

- When M is a nilmanifold that is not $\mathbb{T}^3$, any pointwise partially hyperbolic diffeomorphism $f : M \to M$ is dynamically coherent and is leaf conjugate to a partially hyperbolic nilmanifold automorphism (namely f's linearization) [35]. In both this case and the case where $M = \mathbb{T}^3$, this shows that $f : M \to M$ is a fibered partially hyperbolic system.

- When $\pi_1(M)$ is virtually solvable, but not virtually nilpotent, any dynamically coherent pointwise partially hyperbolic diffeomorphism $f : M \to M$ is leaf conjugate to the time-one map of a suspension Anosov flow [36].

- When M is a closed Seifert fibered 3-manifold, any pointwise partially hyperbolic diffeomorphism $f : M \to M$ that is isotopic to the identity is dynamically coherent, and a finite iterate of f is leaf-conjugate to the time-one map of an Anosov flow [5], [6].

- Finally, when M is hyperbolic, if $f : M \to M$ is a dynamically coherent, pointwise partially hyperbolic diffeomorphism, then a finite iterate of it is leaf-conjugate to the time-one map of an Anosov flow [5], [6].

In this thesis, our study is not limited to dimension 3, however the first interesting examples appear there, which we discuss in the next section.

2.2 Fibered Partially Hyperbolic Systems

We begin with an example. The three-dimensional Heisenberg group is given by

$$\mathrm{Heis} := \left\{ A_{(x,y,z)} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\}$$

with matrix multiplication as the group operation. The group Heis is a nilpotent Lie group whose center consists of the matrices $A_{(0,0,z)}$, with $z \in \mathbb{R}$. Any automorphism of Heis must preserve this center.

Nilmanifolds are quotients of nilpotent Lie groups by discrete subgroups. For the Heisenberg group, a compact quotient can be obtained as follows. Let

$$\Gamma = \{(x, y, z) \in H : \; x, y, 2z \in \mathbb{Z}\},$$

where we use (x, y, z) to denote the matrix $A_{(x,y,z)}$. The quotient Heis/Γ is a compact nilmanifold, an example of a *Heisenberg nilmanifold.* It is a fiber bundle over the 2-torus $\mathbb{T}^2$ with fiber the circle $\mathbb{T}$, where the fibers lie in the "z-direction," tangent to the center of the Lie algebra of Heis.

Any automorphism of Heis that preserves Γ descends to a diffeomorphism of Heis/Γ: since the automorphism preserves the center of Heis, the quotient diffeomorphism preserves the bundle structure. The interesting quotient diffeomorphisms are examples of *fibered partially hyperbolic diffeomorphisms.* An example is the map $f_0 : \mathrm{Heis}/\Gamma \to \mathrm{Heis}/\Gamma$ given by

$$f_0(x, y, z) = \left(2x + y, x + y, z + x^2 + \frac{y^2}{2} + xy\right).$$

Since f_0 preserves the smooth fibration $\mathbb{T} \hookrightarrow \mathrm{Heis}/\Gamma \twoheadrightarrow \mathbb{T}^2$, it induces a diffeomorphism of the base $\mathbb{T}^2$, in this case the hyperbolic linear automorphism $(x,y) \mapsto (2x+y, x+y)$.

This example is partially hyperbolic, meaning that the tangent bundle TN to $N = \operatorname{Heis}/\Gamma$ splits as a df_0-invariant direct sum $TN = E^s \oplus E^c \oplus E^u$ such that for all $x \in N$ and all unit vectors $v^s \in E^s(x),\ v^c \in E^c(x)$, and $v^u \in E^u(x)$, we have that

$$\|d_x f(v^s)\| < \|d_x f(v^c)\| < \|d_x f(v^u)\| \quad \text{and} \quad \|d_x f(v^s)\| < 1 < \|d_x f(v^u)\|.$$

That is, the center direction is dominated by the stable and unstable directions.

Hammerlindl and Potrie proved that if $f\colon N \to N$ is partially hyperbolic and homotopic to f_0, then f is *leaf conjugate* to f_0. In this example, f_0 is a *smooth model*, and any partially hyperbolic diffeomorphism homotopic to it is leaf conjugate to it.

In this paper, we consider the class of *fibered partially hyperbolic diffeomorphisms*. These are partially hyperbolic diffeomorphisms, that, like f_0 above, have an integrable center bundle E^c, tangent to an invariant fibration by compact submanifolds. More precisely $f\colon M \to M$, where M is a closed Riemannian manifold, is a *fibered partially hyperbolic diffeomorphism*[3] with (C^k, $k \geq 1$) fiber X if there exists an f-invariant continuous fiber bundle $\pi : M \to B$ (for some manifold B) with C^k fibers modeled on X, which are tangent to E^c and such that the k-jets along fibers are continuous in M.[4] We say that a fibered partially hyperbolic system (f, π, M) is C^k if the fiber bundle $\pi : M \to B$ is C^k. Note the distinction between a C^k fibered partially hyperbolic system and a fibered partially hyperbolic system with C^k fibers: in a C^k fibered partially hyperbolic system, we require that the bundle $\pi : M \to B$ is C^k, whereas in a fibered partially hyperbolic system with C^k fibers, the bundle $\pi : M \to B$ is merely required to be continuous.

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[3] This is also known as a *fibered partially hyperbolic system with C^k fibers*.

[4] In other words, M is a continuous X-bundle with structure group $\operatorname{Diff}^k(X)$.

2.3 Preliminaries about Fiber Bundles

The goal of this section is to provide several results and definitions about fiber bundles that are necessary for the proof of Theorem A. We begin with some basic definitions.

Definition 2.3.1. Let F and B be topological spaces. A (continuous) *F-bundle* over B with *structure group* G is a space E and a continuous surjective map $\pi : E \to B$ that admits an atlas of *locally trivializing charts*, i.e. such that B admits an open cover $\{U_i\}$ such that for each i, there exists a homeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ such that the diagram in Figure 2.1 commutes. The F-bundle $\pi : E \to B$ is said to be C^k if the map $\pi : E \to B$ is C^k and the maps ϕ_i are C^k diffeomorphisms[5]. We say a F-bundle is smooth if it is C^∞.

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[d, "\pi"] & U_i \times F \arrow[dl, "\mathrm{proj}_1"] \\
U_i &
\end{tikzcd}$$

Figure 2.1: $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ is a local trivialization of π if the diagram commutes. The map $\mathrm{proj}_1 : U_i \times F \to U_i$ is projection onto the first factor.

If G is a topological group that acts on F on the left by homomorphisms, we say that the F-bundle $\pi : E \to B$ has *structure group G* if there are continuous functions $\{\tau_{ij} : U_i \cap U_j \to G\}$ such that for each i, j, we can write $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ as $(x, y) \mapsto (x, \tau_{ij}(x) \cdot y)$. These functions $\{\tau_{ij}\}$ are called *transition functions* for the bundle. If G acts on F by C^k diffeomorphisms, we say that a F-bundle with structure group G has C^k fibers[6].

A *principal G-bundle* is a G-bundle with structure group G, where the structure group G acts on the fiber G by left multiplication.

The structure of a fiber bundle is given by its transition functions, which give how the

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[5] Clearly this requires the spaces F, B, and E to all be C^k manifolds.

[6] Note that a C^k bundle must have at least C^k fibers.

locally trivializing charts for the bundle are glued together. The fact that all the charts must 'agree' on the overlaps is expressed via the *cocyle condition.* A set of functions $\{\tau_{ij} : U_i \cap U_j \to G\}$ satisfy the *cocycle condition* if $\tau_{ij}(x)\tau_{jk}(x) = \tau_{ik}(x)$ holds for all i, j, k and for all $x \in U_i \cap U_j \cap U_k$. It's immediate from the definition of the transition functions (Definition 2.3.1) that the transition functions of a bundle will satisfy the cocycle condition.

All the data of a fiber bundle is contained in the transition functions. More precisely,

Lemma 2.3.2 (Fiber bundle construction theorem)**.** *Let F, B be C^k manifolds ($0 \leq k \leq \infty$), and let G be a topological group with the structure of a C^k manifold that has a C^k left action on F. Given an open cover $\{U_i\}_{i \in A}$ of B and a set of C^k functions $\tau_{ij} : U_i \cap U_j \to G$ such that the cocycle condition, $\tau_{ij}(x)\tau_{jk}(x) = \tau_{ik}(x)$ holds for all $x \in U_i \cap U_j \cap U_k$. Then there exists a C^k F-bundle $\pi : E \to B$ with transition functions τ_{ij}.*

Remark 2.3.3. Lemma 2.3.2 allows us to associate any F-bundle with structure group G with a principal G-bundle that has the same transition functions as the original F-bundle (and vice versa). This is extremely useful because there are many tools and theorems that only apply to principal bundles. We use this trick repeatedly in the proof of Theorem A. Essentially, we use Lemma 2.3.2 to 'convert' our F-bundle to a principal bundle, show whatever we want about that principal bundle, and then use Lemma 2.3.2 to 'convert' our result about the principal bundle to a result about our original F-bundle.

Lemma 2.3.2 is a standard result about fiber bundles, but we provide the proof here for completeness.

Proof. Define the space $E = \bigsqcup_{i \in A} U_i \times F / \sim$, where we define the equivalence relation $\sim$ on $\bigsqcup_{i \in A} U_i \times F = \{(i, x, y); i \in A, x \in U_i, y \in F\}$ by $(j, x, y) \sim (i, x, \tau_{ij}(x) \cdot y)$ for $x \in U_i \cap U_j$. Note that the cocycle condition guarantees that $\sim$ is an equivalence relation. We also define a continuous surjective map $\pi : E \to B$ by $\pi([i, x, h]) = x$.

First, we show that $\pi : E \to B$ is a continuous F-bundle with transition functions τ_{ij}. To do this, we give a set of locally trivializing charts for $\pi : E \to B$. For each

$i \in A$, a homeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ by $\phi_i^{-1}(x, y) = [i, x, y]$ for $x \in U_i, h \in G$. These are locally trivializing charts since $\pi = \mathrm{proj}_1 \circ \phi_i : \pi^{-1}(U_i) \to U_i \times F$. The transition functions corresponding to this set of local trivializations are $\{\tau_{ij}\}$, which we see by computing $\phi_i \circ \phi_j^{-1}(x, y) = \phi_i([j, x, y]) = \phi_i([i, x, \tau_{ij}(x) \cdot y]) = (x, \tau_{ij}(x) \cdot y)$.

We now just need to show that $\pi : E \to B$ is a C^k F-bundle. To do this, we need to show that E is a C^k manifold and that the local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ are C^k diffeomorphisms. (Note this implies that π is C^k (and is, in fact, a submersion if $k \geq 1$) since $\pi = \mathrm{proj}_1 \circ \phi_i$ on x_i.) To see that E is a C^k manifold, we define charts on E. By composing ϕ_i with C^k charts for B and F, we get charts for E. Note that these charts are C^k because the transition maps between different charts (ignoring composition with charts for B and F) are $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$ given by $(x, y) \mapsto (x, \tau_{ij}(x) \cdot y)$ which are C^k since $\tau_{ij} : U_i \cap U_j \to G$ and the action of G on F are both C^k. Our definition of the C^k structure on E immediately implies that the ϕ_i are C^k diffeomorphisms. □

This means that any set of functions $\{\tau_{ij} : U_i \cap U_j \to G\}$ that satisfy the *cocycle condition* (i.e. that $\tau_{ij}(x)\tau_{jk}(x) = \tau_{ik}(x)$ holds for all $x \in U_i \cap U_j \cap U_k$) are transition functions for a fiber bundle. Additionally, the smoothness of a bundle is defined by the smoothness of the transition functions.

Transition functions also classify fiber bundles up to isomorphism. Two F-bundles, $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are *isomorphic* as (continuous[7]) F-bundles over B if there is a homeomorphism $\alpha : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ \alpha$.

Two fiber bundles are isomorphic if they have *cohomologous* transition functions. Two sets of transition functions (i.e. two sets of functions that satisfy the cocycle condition) $\{\tau_{ij} : U_i \cap U_j \to G\}$ and $\{\tau'_{ij} : U_i \cap U_j \to G\}$ are said to be *cohomologous* if there exist continuous functions $t_i : U_i \to G$ such that $\tau'_{ij}(x) = t_i(x)^{-1}\tau_{ij}(x)t_j(x)$ for all $x \in U_i \cap U_j$.

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[7] If $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ are C^k bundles, and the map $\alpha : E_1 \to E_2$ is a C^k diffeomorphism, then we say that E_1 and E_2 are *isomorphic as C^k bundles.* However, in this thesis, when we refer to two bundles being isomorphic, we mean as continuous bundles unless otherwise stated.

Corollary 2.3.4. *Let F, B be topological spaces and G a topological group that has a continuous left action on F. Suppose $\pi : E \to B$ and $\tilde{\pi} : \tilde{E} \to B$ are continuous F-bundles with structure group G. Let $\{(U_i, \phi_i : \pi^{-1}(U_i) \to U_i \times F)\}$ and $\{(U_i, \widetilde{\phi}_i : \widetilde{\pi}^{-1}(U_i) \to U_i \times F)\}$ be locally trivializing charts for E and $\tilde{E}$ respectively, and let $\{\tau_{ij} : U_i \cap U_j \to G\}$ and $\{\widetilde{\tau}_{ij} : U_i \cap U_j \to G\}$ be the associated transition functions for $\{(U_i, \phi_i)\}$ and $\{(U_i, \tilde{\phi}_i)\}$ respectively. Suppose there exist continuous functions $t_i : U_i \to G$ such that $\tilde{\tau}_{ij}(x) = t_i(x)^{-1}\tau_{ij}(x)t_j(x)$ for all $x \in U_i \cap U_j$. Then, E and $\tilde{E}$ are isomorphic as F-bundles with structure group G. If, in addition, the left G action on F is faithful, then the converse holds.*

For a proof of Corollary 2.3.4, see [44, Chapter 5.2].

In the proof of Theorem A, we change the structure group of the fiber bundles we are working with. We now make the notion of changing structure group precise.

Given a continuous homomorphism $\alpha : H \to G$ between two topological groups and a principal H-bundle $q : Q \to B$, we can construct a principal G-bundle from Q and α in the following way. Consider the space

$$Q \times_{\alpha,H} G := Q \times G/\sim, \quad \text{where } (x \cdot h, g) \sim (x, \alpha(h)g), \text{ for } h \in H$$

Note that $Q \times_{\alpha,H} G$ has a free right G-action given by $[x, g] \cdot g' = [x, gg']$. This makes $Q \times_{\alpha,H} G$ a principal G-bundle. Note that the projection map for this bundle $Q \times_{\alpha,H} G \to B$ is given by $[x, g] \mapsto q(x)$.

We say that the principal H-bundle $q : Q \to B$ induces the principal G-bundle $p : P \to B$ if $P \cong Q \times_{\alpha,H} G$. Additionally, we say that a principal G-bundle $p : P \to B$ admits *a reduction of structure group from G to H* if there exists a principal H-bundle $q : Q \to B$ that induces P.

Now, we note the relationship between the transition functions between a bundle and a bundle it induces. Suppose that $t_{ij} : U_i \cap U_j \to H$ are transition functions for the principal H-bundle $q : Q \to B$. The transition functions for the induced bundle $Q \times_{\alpha,H} G$ are given by $\alpha \circ t_{ij} : U_i \cap U_j \to G$.

Next, we give conditions under which a principal bundle admits a reduction of structure group.

Lemma 2.3.5. *Suppose that $\alpha : H \to G$ is a homomorphism that is also a homotopy equivalence. Then any principal G-bundle admits a reduction of structure group from G to H.*

For a proof of Lemma 2.3.5, see [44, Chapter 6]. The proof relies on the theory of *classifying spaces*. Before defining classifying spaces, we recall the definition of the *pullback bundle* induced by bundle and a map of the base.

Let $\pi : E \to B$ be a F-bundle, and let $f : B' \to B$ be a continuous map. The *pullback bundle* $f^*\pi : f^*E \to B'$ is the F-bundle given by the space

$$f^*E = \{(b', z) \in B' \times E : f(b') = \pi(z)\} \subset B' \times E$$

and the projection $f^*\pi = \mathrm{proj}_1 : f^*E \to B$ given by projection onto the first component. If $\tau_{ij} : U_i \cap U_j \to G$ are transition functions for $\pi : E \to B$, then the transition functions for the pullback bundle $f^*\pi : f^*E \to B'$ are given by $\tau_{ij} \circ f : f^{-1}(U_i) \cap f^{-1}(U_j) \to G$.

Now we can define classifying spaces. Let G be a topological group. A principal G-bundle $\pi : EG \to BG$ is a *universal* principal G-bundle if for all CW-complexes X, the map from the set of homotopy classes of maps $X \to BG$ to the set of isomorphism classes of principal G-bundles over X, given by the map $f \mapsto f^*EG$ is a bijection. The base space of a universal principal G-bundle is known as a *classifying space* for G.

Remark 2.3.6. Note that if $H \subset G$ and the homomorphism $\alpha : H \to G$ is inclusion, then the reduction of structure group from G to H from Lemma 2.3.5 has transition functions that are cohomologous to the transition functions of the original principal G bundle.

The following proposition combined with Lemma 2.3.5 shows that any principal $\mathrm{Diff}^1(F)$

bundle admits a reduction of structure group from $\mathrm{Diff}^1(F)$ to $\mathrm{Diff}^\infty(F)$.

Proposition 2.3.7. *If M is a closed, smooth manifold then the inclusion $\mathrm{Diff}^\infty(M) \hookrightarrow \mathrm{Diff}^1(M)$ is a homotopy equivalence.*

Proof. For $0 \leq k \leq \infty$, $\mathrm{Diff}^k(M)$ is an infinite-dimensional separable Fréchet space [32, Section I.4.3]. Since all infinite-dimensional separable Fréchet spaces are homeomorphic to the Hilbert space ℓ^2 [1], we see that $\mathrm{Diff}^k(M)$ is homeomorphic to the Hilbert space ℓ^2. Since ℓ^2 has the homotopy type of a CW-complex [60], we get that $\mathrm{Diff}^k(M)$ is homotopy equivalent to a CW-complex [39]. Thus by Whitehead's Theorem, to show that the inclusion $\iota : \mathrm{Diff}^\infty(M) \hookrightarrow \mathrm{Diff}^1(M)$ is a homotopy equivalence, it is sufficient to show that the induced map on homotopy groups $\iota_* : \pi_*(\mathrm{Diff}^\infty(M)) \to \pi_*(\mathrm{Diff}^1(M))$ is an isomorphism.

To do this, we first recall that for any map in $\varphi \in C^1(M, M)$, we can find a smooth map that is arbitrarily close to and homotopic to φ and that the choice of this map depends continuously on our original map ([46, Theorems 6.21 and 6.28]). This gives us a continuous map $\Phi : \mathrm{Diff}^1(M) \to C^\infty(M, M)$ such that for any $\varphi \in \mathrm{Diff}^1(M)$, $\Phi(\varphi) \simeq \varphi$ and $d(\varphi, \Phi(\varphi)) < \varepsilon$. Since $\mathrm{Diff}^\infty(M) \subset C^\infty(M, M)$ is open (by the inverse function theorem), by choosing ε small enough, we get that $\Phi(\mathrm{Diff}^1(M)) \subset \mathrm{Diff}^\infty(M)$, so we can write $\Phi : \mathrm{Diff}^1(M) \to \mathrm{Diff}^\infty(M)$.

Now, we show that the map induced by $\iota : \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^1(M)$ on homotopy groups is an isomorphism. The map ι_* is surjective because any map $\phi : S^n \to \mathrm{Diff}^1(M)$ is homotopic to the map $\Phi \circ \phi : S^n \to \mathrm{Diff}^\infty(M)$. To see that ι_* is injective, we take a map $\phi : S^n \to \mathrm{Diff}^1(M)$ that is null-homotopic. Let $h_t : S^n \to \mathrm{Diff}^1(M)$ be a null-homotopy for ϕ. Then the map $\Phi \circ \phi : S^n \to \mathrm{Diff}^\infty(M)$ is homotopic to ϕ in $\mathrm{Diff}^1(M)$, and is null-homotopic in $\pi_n(M)$ via the homotopy $\Phi \circ h_t : S^n \to \mathrm{Diff}^\infty(M)$. □

2.4 Anosov Homeomorphisms

The goal of this section is to provide a proof of Theorem E. This result was initially proved by Sumi [61], and we provide a proof which follows the same structure as that of Sumi's for the sake of completeness. Theorem E extends the following result of Franks and Manning to Anosov homeomorphisms of nilmanifolds.

Theorem 2.4.1 ([28],[49]). *An Anosov diffeomorphism of a nilmanifold is topologically conjugate to a hyperbolic nilmanifold automorphism.*

Theorem E generalizes the following result of Hiraide from tori to nilmanifolds.

Theorem 2.4.2 ([41]). *An Anosov homeomorphism of a torus is topologically conjugate to a hyperbolic toral automorphism.*

The proof of Theorem E follows the same basic structure as Sumi's proof, and also the same structure as Hiraide's proof with some modifications to account for being on a nilmanifold instead of a torus.

2.4.1 Notation and Preliminaries for the Proof of Theorem E

This section recalls several properties of Anosov homeomorphisms that will be necessary to the proof of Theorem E.

We assume in the following that M is a connected, closed n-dimensional Riemannian manifold. Let d be the distance function on M induced by the Riemannian metric.

Generalized foliations for Anosov homeomorphisms

In this section, we describe an analogue of the stable manifold theorem for Anosov homeomorphisms that is due to Hiraide [41]. Let $f : M \to M$ be a homeomorphism. For each

$x \in M$, we define the *stable set* (resp. *unstable set*) of f at x as

$$W^s(x) := \left\{ y \in X : d(f^n(x), f^n(y)) \overset{n\to\infty}{\longrightarrow} 0 \right\}$$
$$\left(\text{resp. } W^u(x) := \left\{ y \in X : d(f^{-n}(x), f^{-n}(y)) \overset{n\to\infty}{\longrightarrow} 0 \right\}\right)$$

The collection of stable (resp. unstable) sets for f, which we'll denote by $\mathscr{F}^s_f$ (resp. $\mathscr{F}^u_f$), gives a f-invariant decomposition of M. The stable manifold theorem states that when f is an Anosov diffeomorphism, these collections form foliations. When f is an Anosov homeomorphism, we get the following analogue.

Theorem 2.4.3 ([41, Proposition A])**.** *If $f : M \to M$ is an Anosov homeomorphism of the closed manifold M, then the collections*

$$\mathscr{F}^\sigma_f = \{W^\sigma(x) : \ x \in M\}, \quad \sigma \in \{s, u\}$$

are transverse generalized foliations of M.

Remark 2.4.4. When f is the projection of a fibered partially hyperbolic diffeomorphism $\hat{f}$, the transverse generalized foliations W^s and W^u are, in fact, foliations. They are the projections of the foliations $\widehat{W^s}$ and $\widehat{W^u}$ for $\hat{f}$.

Generalized foliations are a generalization of foliations given by weakening the condition that the leaves be manifolds. More precisely, a collection $\mathscr{F}$ of subsets of M is a *generalized foliation* of M if the following properties hold:

1. $\mathscr{F}$ is a partition of M.

2. Each $L \in \mathscr{F}$ (called a *leaf*) is path-connected.

3. For each $x \in M$, there exist

    - nontrivial, connected subsets $D_x, K_x \subset M$ with $D_x \cap K_x = \{x\}$,

- a connected, open neighborhood $N_x \subset M$ of x,
- a homeomorphism $\phi_x : D_x \times K_x \to N_x$ (called *local coordinates around* x)

such that

(a) $\phi_x(x, x) = x$,

(b) $\phi_x(y, x) = y \ \forall y \in D_x$ and $\phi_x(x, z) = z \ \forall z \in K_x$,

(c) For any $L \in \mathscr{F}$, there is at most a countable set $B \subset K_x$ such that $N_x \cap L = \phi_x(D_x \times B)$.

Two generalized foliations $\mathscr{F}$ and $\mathscr{F}'$ on M are *transverse* if, for each $x \in M$, there exist

- nontrivial, connected subsets $D_x, D'_x \subset M$ with $D_x \cap D'_x = \{x\}$,
- a connected, open neighborhood N_x of x (called *a coordinate domain*),
- a homeomorphism $\phi_x : D_x \times D'_x \to N_x$ (called a *canonical coordinate chart (around x)*),

such that

(a) $\phi_x(x, x) = x$,

(b) $\phi_x(y, x) = y, \ \forall y \in D_x$, and $\phi_x(x, z) = z \ \forall z \in D'_x$,

(c) for any $L \in \mathscr{F}$, there is at most a countable set $B' \subset D'_x$ such that $N_x \cap L = \phi_x(D_x \times B')$,

(d) for any $L' \in \mathscr{F}'$, there is at most a countable set $B \subset D_x$ such that $N_x \cap L' = \phi_x(B \times D'_x)$.

Note that the sole difference between the definitions of a foliation and of a generalized foliation is we don't require the sets D_x and K_x to be manifolds in the definition of a generalized foliation. (If D_x and K_x are manifolds for all $x \in M$, then a generalized foliation $\mathscr{F}$ is, in fact, a topological foliation of M.) While the sets D_x and K_x may fail to be manifolds, the fact that their product, $D_x \times K_x$, is a manifold significantly restricts the ways in which D_x and K_x can fail to be manifolds. In other words, D_x and K_x (and therefore the leaves of $\mathscr{F}$), while not necessarily manifolds themselves, will behave like manifolds in many ways. In fact, the leaves of a generalized foliation are homology manifolds (also known as generalized manifolds). An homology manifold is a topological space that looks like a manifold under homology. This is stated precisely in the following proposition.

Proposition 2.4.5 ([41, Lemma 4.2]). *Let $\mathscr{F}$ be a generalized foliation on a connected manifold without boundary. There exists $0 < p < \dim(M)$ such that any leaf $L \in \mathscr{F}$ and $x \in L$, the relative homology groups, $H_*(L, L \setminus \{x\})$, are given by*

$$H_i(L, L \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \textit{if } i = p \\ 0, & \textit{if } i \neq p \end{cases}.$$

This proposition allows us to define a notion of dimension for generalized foliation. If $\mathscr{F}$ is a generalized foliation of M, then the integer p from Proposition 2.4.5 is called the *dimension* of f. Proposition 2.4.5 also allows us to define orientability for generalized foliations. A p-dimensional generalized foliation is said to be *orientable* if there is a 'locally consistent' choice of generators for the groups $H_p(L, L \setminus \{x\})$, $L \in \mathscr{F}$ and $x \in L$. For more details see [41, Section 4].

Lifts of stable and unstable sets

Much of Franks' and Manning's proofs of Theorem 2.4.1 take place using maps lifted to the universal cover. These arguments exploit the facts that Anosov diffeomorphisms lift to Anosov diffeomorphisms whose stable and unstable sets are lifts of the original stable and unstable manifolds. We'll now give versions of these facts for Anosov homeomorphisms, which will be used in our proof of Theorem E.

We begin with the following set-up. Let M be a closed Riemannian manifold and let $p : \tilde{M} \to M$ be a smooth covering map for M. By lifting the Riemannian metric on M, we see that $\tilde{M}$ is a complete Riemannian manifold.

We can now generalize the previous results about lifts of Anosov diffeomorphisms to Anosov homeomorphisms. These generalizations are due to Hiraide. For details on them and their proofs, see [41, Section 3]. Let $f : M \to M$ be an Anosov homeomorphism. The map f lifts to a homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$. Just as Anosov diffeomorphisms lift to Anosov diffeomorphisms, we observe that an Anosov homeomorphism lifts to an Anosov homeomorphism.

Next, we'll discuss the relationship between the stable and unstable sets of f and $\tilde{f}$. For $\tilde{x} \in \tilde{M}$ and $\varepsilon > 0$, we let $\tilde{W}^s_\varepsilon(\tilde{x})$ and $\tilde{W}^u_\varepsilon(\tilde{x})$ be the local stable and unstable sets of $\tilde{f}$ at $\tilde{x}$.

$$\begin{aligned}
\tilde{W}^s_\varepsilon(\tilde{x}) &= \left\{ y \in \tilde{M} : \ d\left(\tilde{f}^n\left(\tilde{x}\right), \tilde{f}^n\left(y\right)\right) \le \varepsilon, \ \forall n \ge 0 \right\}, \\
\tilde{W}^u_\varepsilon(\tilde{x}) &= \left\{ y \in \tilde{M} : \ d\left(\tilde{f}^{-n}\left(\tilde{x}\right), \tilde{f}^{-n}\left(y\right)\right) \le \varepsilon, \ \forall n \ge 0 \right\}.
\end{aligned}$$

Just as for an Anosov diffeomorphism, the stable and unstable sets for $\tilde{f}$ project down to the stable and unstable sets for f. In fact, locally this projection is an isometry. In addition, the collection of stable (resp.) unstable sets of $\tilde{f}$ forms a generalized foliation, denoted $\mathscr{F}^s_{\tilde{f}}$ (resp. $\mathscr{F}^u_{\tilde{f}}$), and that the stable and unstable generalized foliations for $\tilde{f}$ are transverse.

Indices of fixed points

Let $f : M \to M$ be an Anosov diffeomorphism. The index of f at any fixed point x, denoted $\mathrm{Ind}_x(f)$, will be either ±1 since $d_x f : T_x M \to T_x M$ is hyperbolic. The sign of $\mathrm{Ind}_x(f)$ will depend on the orientation of the stable and unstable subspaces, E^s_x and E^u_x, at x. Thus, if the unstable bundle, E^u, of f is orientable (which implies that the unstable foliation for f is orientable), we can make the fixed point index globally constant, i.e. for all $x, x' \in \mathrm{Fix}(f)$,

$\mathrm{Ind}_x(f) = \mathrm{Ind}_{x'}(f)$. This along with the Lefschetz fixed point theorem tells us that the absolute value of the Lefschetz number of f, denoted $L(f)$, is equal to the number of fixed points of f. This fact is relied upon in the proof of Theorem 2.4.1.

The purpose of this section is to give the following similar result about the fixed point index of an Anosov homeomorphism, which will allow us to use the Lefschetz number to count fixed points. Note that we can define the fixed point index for a fixed point of an Anosov homeomorphism because all fixed points of an Anosov homeomorphism are isolated by expansivity.

Proposition 2.4.6 ([41], Proposition B)**.** *Let $f : M \to M$ be an Anosov homeomorphism of the closed manifold M. If the generalized unstable foliation $\mathscr{F}_f^u$ is orientable, then for sufficiently large m, all the fixed points of f^m have the same index, which is either* 1 *or* -1.

Note that the assumption in this proposition (i.e. that the generalized unstable foliation be orientable) is analogous to the assumption we made in the Anosov case. For the definition of orientability for a generalized foliation, see [41]. The proof of Theorem 2.4.6 can be found in [41, Section 5].

The spectral decomposition

The spectral decomposition is a useful tool for decomposing the non-wandering set of an Anosov diffeomorphism into smaller invariant sets. Recall that given a homeomorphism, $f : M \to M$, a point $x \in M$ is called *nonwandering* if for any neighborhood U of x, $\exists n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. The nonwandering set of f, denoted $\Omega(f)$ is the set of nonwandering points of f. The spectral decomposition admits the following generalization to Anosov homeomorphisms.

Theorem 2.4.7 (Spectral Decomposition, [2])**.** *Let $f : M \to M$ be an Anosov homeomorphism of a compact manifold M. Then, there exist closed, pairwise disjoint sets $X_1, ..., X_k$ and a permutation $\sigma \in S_k$ such that* *(a)* $\Omega(f) = \bigcap_{i=1}^{k} X_i$,

(b) $f(X_i) = X_{\sigma(i)}$, *and*

(c) if for $a > 0$, $\sigma^a(i) = i$, *then* $f^a|_{X_i}$ *is topologically mixing.*

Recall that a continuous map $f : M \to M$ is *topologically mixing* if for any open sets $U, V \subset M$, there exists an integer N such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

2.4.2 Proof of Theorem E

The proof of Theorem E follows the same structure as the proofs of the main result of [41] with a couple of modifications to account for being on a nilmanifold instead of a torus. Before giving the details of the proof, we provide a brief synopsis of the proof and note where it differs from Hiraide's. The argument has three main parts.

- Constructing the hyperbolic nilmanifold automorphism $A : M \to M$. This differs from Hiraide's argument [41] in the same ways that Manning's argument [49] differs from Franks's [28]. The construction of the nilmanifold automorphism $A : M \to M$ is the same as Manning's construction in [49]. The proof that A is hyperbolic follows Hiraide's argument using the same technique that Manning uses in [48] and [49] to get a formula for the Lefschitz number of A in terms of the eigenvalues of A.

- Constructing a semiconjugacy $h : M \to M$ between A and f. Since M is a $K(\pi, 1)$, this construction is the same as that on the torus.

- Proving that the semiconjugacy $h : M \to M$ is actually a conjugacy. This follows the same argument given in Hiraide, with the main modification in Lemma 2.4.13 to construct a homotopy between $\tilde{f}$ and $\tilde{A}$ that doesn't introduce fixed points outside a compact set.

Now, we give the details of the proof. Let $f : M \to M$ be an Anosov homeomorphism of the nilmanifold $M = N/\Gamma$. We begin by finding a candidate for the hyperbolic nilmanifold automorphism in Theorem E. We'll do this following the same procedure as Franks [28], Manning [49], and Hiraide [41]; we'll find a 'linear' model of f, which we'll then show is hyperbolic. Our linear model of f will be a nilmanifold automorphism A that is homotopic to f. The construction of this linear model is identical to that given in [49]. To construct A, we'll show that the induced action of f on $\pi_1(M, e\Gamma)$ can be lifted to an automorphism of Γ. We'll then extend this automorphism to all of N to get our linear model.

Let $f_* : \pi_1(M, e\Gamma) \to \pi_1(M, f(e\Gamma))$ be the homomorphism that f induces on the fundamental group of M. We can view $\pi_1(M, e\Gamma)$ and $\pi_1(M, f(e\Gamma))$ as subgroups of N. To do this, we first identify $\pi_1(M, e)$ with Γ (via the endpoints of the lifts of the loops in the fundamental group). Recall that changing basepoint in the fundamental group is the same as conjugating by some path in M. So in the universal cover of M (i.e. N), the identification that takes $\pi_1(M, e\Gamma)$ to Γ will take $\pi_1(M, f(e\Gamma))$ to $x^{-1}\Gamma x$ for some $x \in N$. By lifting to N, we can view f_* as a homomorphism $\Gamma \to x^{-1}\Gamma x$.

Since we want a homomorphism $\Gamma \to \Gamma$, we compose f_* with conjugation by x^{-1}, which gives us our automorphism of Γ. To summarize, we've shown that we can lift $f_* : \pi_1(M, e\Gamma) \to \pi_1(M, f(e\Gamma))$ to an automorphism of Γ, which is defined up to an inner automorphism of N. We can uniquely extend $f_* : \Gamma \to \Gamma$ to an automorphism $\tilde{A} : N \to N$ [56, Corollary 1 of Theorem 2.11]. Since $\tilde{A}$ preserves Γ, it descends to a nilmanifold automorphism, $A : M \to M$. Note that f is homotopic to A since they induce conjugate maps on $\pi_1(M)$ and M is a $K(\pi, 1)$.

We now claim that the linearization A is hyperbolic, which follows immediately from the following proposition.

Proposition 2.4.8. *Let $f : M \to M$ be an Anosov homeomorphism of a nilmanifold $M = N/\Gamma$. If $A : M \to M$ is a nilmanifold automorphism that is homotopic to f, then A is* *hyperbolic.*

Proof. This proof is a combination of the techniques of Manning [49, Theorem A] and Hiraide [41, Proposition 6.2]. By passing to a double cover of M, it suffices to consider the case where the unstable generalized foliation of f, $\mathscr{F}_f^u$, is orientable. The goal of this proof is to show that A is hyperbolic. More formally, we need to show that D_eA has no eigenvalues of absolute value one. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of D_eA (counted with multiplicity).

The first step in this proof is to relate the number of m-periodic points of f, for $m \in \mathbb{N}$, to the eigenvalues of D_eA. We do this using the Lefschetz fixed point theorem. First, recall that since f^m and A^m are homotopic, their Lefschetz numbers are the same, i.e. $L(f^m) = L(A^m)$. Since $\mathscr{F}_f^u$ is orientable, the Lefschitz fixed point theorem and Proposition 2.4.6 imply that the number of fixed points of f^m, denoted $N(f^m)$, is given by $N(f^m) = |L(f^m)|$ for sufficiently large m. Now, recall from [48] that we can also write $L(f^m) = L(A^m) = \prod_{i=1}^n (1 - \lambda_i^m)$. We've therefore shown that, for sufficiently large m, the number of fixed points of f^m is given by

$$P_m(f) = N(f^m) = \prod_{i=1}^{n} |1 - \lambda_i^m| \, . \tag{2.1}$$

This equation cannot hold if A is not hyperbolic by arguments given in the proof of [41, Proposition 6.2]. □

Recall that when we defined the 'linearization' A of an Anosov homeomorphism $f : M \to M$ of a nilmanifold, we only were able to define A up to an inner automorphism of N because we didn't know whether f had any fixed points. We are now equipped to show that f does indeed have fixed points.

Corollary 2.4.9. *An Anosov homeomorphism of a nilmanifold has at least one fixed point.*

Proof. This follows immediately from the Lefschetz fixed point theorem and Proposition 2.4.8. □

Note that by conjugating the Anosov homeomorphism $f : M \to M$ by a translation, we can assume without loss of generality that f fixes the point $e\Gamma \in M$. We let A be the hyperbolic 'linearization' of f described above.

The goal of the rest of the proof is to construct a conjugacy between A and f. To do this we first construct a semiconjugacy, $h : M \to M$, between A and f.

Proposition 2.4.10. *Let $M = N/\Gamma$ be a nilmanifold, and let $f : M \to M$ be a homeomorphism that fixes the point $e\Gamma \in M$. If f is freely homotopic to a hyperbolic nilmanifold automorphism $A : M \to M$, then there exists a continuous map $h : M \to M$ (freely) homotopic to the identity such that $A \circ h = h \circ f$ and $h(e\Gamma) = e\Gamma$. Furthermore, the map h is the unique map freely homotopic to the identity fixing $e\Gamma$.*

Proof. Since f and A are freely homotopic, M is a $K(\pi, 1)$, and A is hyperbolic, there exists a homomorphism $(h_0)_* : \pi_1(M, e\Gamma) \to \pi_1(M, e\Gamma)$, that is induced by a base point preserving map $h_0 : M \to M$ that is freely homotopic to the identity, such that $A_* \circ (h_0)_* = (h_0)_* \circ f_*$. Under these conditions, [29, Theorem 2.2] states that there exists a unique continuous base point preserving map, $h : M \to M$, that is homotopic to h_0, such that $A \circ h = h \circ f$. □

We complete the proof of Theorem E by proving that the semiconjugacy, $h : M \to M$, from Proposition 2.4.10 is actually a conjugacy. To do this, we just need to show that h is a homeomorphism.

Proposition 2.4.11. *$h : M \to M$ is a homeomorphism.*

Proof. The main step in this argument is to show that h is a local homeomorphism. This combined with the fact that h is surjective (because h is homotopic to the identity and is a proper map) will imply that $h : (M, e\Gamma) \to (M, e\Gamma)$ is a covering map. Then, since h is homotopic to the identity, the covering spaces $h : (M, e\Gamma) \to (M, e\Gamma)$ and $\mathrm{id} : (M, e\Gamma) \to (M, e\Gamma)$ are isomorphic, i.e. that there is a homeomorphism $g : M \to M$ such that $h = \mathrm{id} \circ g$.

This will complete the argument that h is a homeomorphism, and thus gives a conjugacy between A and f.

Thus, all that remains is to show that h is a local homeomorphism. We do this by showing that its lift $\tilde{h} : (N, e) \to (N, e)$ [8] is a local homeomorphism. Recall that Brower's theorem on invariance of domain states that a locally injective continuous map between two manifolds without boundary of the same dimension is a local homeomorphism. Thus, we'll be done if we can show that $\tilde{h}$ is locally injective. In fact, we'll show that $\tilde{h}$ is injective.

First, we note that f lifts to an Anosov homeomorphism $\tilde{f} : (N, e) \to (N, e)$. We recall from Section 2.4.1 that the stable and unstable sets for $\tilde{f}$, denoted $\mathscr{F}^s_{\tilde{f}}$ and $\mathscr{F}^u_{\tilde{f}}$, are transverse generalized foliations on N. The first step in the argument that $\tilde{h}$ is injective is to show that it suffices to prove injectivity of $\tilde{h}$ on stable and unstable leaves of $\tilde{f}$. This follows from the fact that the stable and unstable generalized foliations for $\tilde{f}$ establish a global product structure for N, i.e.

Proposition 2.4.12. *For any points $x, y \in N$ the stable leaf through x and the unstable leaf through y intersect at exactly one point, i.e. the set $\tilde{W}^s(x) \cap \tilde{W}^u(y)$ contains exactly one point.*

Before going through the proof of Proposition 2.4.12, we show how this proposition implies that injectivity of $\tilde{h}$ follows from injectivity on stable and unstable leaves. This argument follows that in [41, p.387-388]. Take $x, y \in N$ such that $\tilde{h}(x) = \tilde{h}(y)$. By Proposition 2.4.12, we can define a point $z := \tilde{W}^s(x) \cap \tilde{W}^u(y)$ to be the intersection of the stable leaf through x and the unstable leaf through y. If we show that $\tilde{h}(x) = \tilde{h}(y) = \tilde{h}(z)$, then injectivity of $\tilde{h}$ will follow from injectivity of the stable and unstable leaves. Thus, it suffices to show that $\tilde{h}(y) = \tilde{h}(z)$.

To prove $\tilde{h}(y) = \tilde{h}(z)$, recall that since $\tilde{A}$ is a hyperbolic automorphism of N, for arbitrary

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[8] When we take this lift, we lift the point $e\Gamma \in M$ to the point $e \in N$. In the rest of this section, we'll be lifting $e\Gamma \in M$ to $e \in N$ unless otherwise noted.

$M_1 > 0$, the map $\tilde{A}$ is expansive with expansive constant M_1. Thus, to show that $\tilde{h}(y) = \tilde{h}(z)$, it suffices to show that there exists a constant $M_1 > 0$ such that for all $n \in \mathbb{Z}$,

$$d\left(\tilde{A}^n \circ \tilde{h}(z), \tilde{A}^n \circ \tilde{h}(y)\right) \leq M_1. \tag{2.2}$$

To see this, first recall that since $\tilde{A} \circ \tilde{h} = \tilde{h} \circ \tilde{f}$ and $\tilde{h}(x) = \tilde{h}(y)$, we have that for all $n \in \mathbb{Z}$,

$$d\left(\tilde{A}^n \circ \tilde{h}(z), \tilde{A}^n \circ \tilde{h}(y)\right) = d\left(\tilde{h} \circ \tilde{f}^n(z), \tilde{h} \circ \tilde{f}^n(y)\right) = d\left(\tilde{h} \circ \tilde{f}^n(z), \tilde{h} \circ \tilde{f}^n(x)\right).$$

In light of these two equations, to prove 2.2, it's sufficient to prove that there exists a constant $M_1 > 0$ such that for all $n \geq 0$, the following two inequalities hold.

$$d\left(\tilde{h} \circ \tilde{f}^{-n}(z), \tilde{h} \circ \tilde{f}^{-n}(y)\right) \leq M_1$$
$$d\left(\tilde{h} \circ \tilde{f}^{n}(z), \tilde{h} \circ \tilde{f}^{n}(x)\right) \leq M_1$$

These inequalities follow immediately from the following two observations,

- Since h is homotopic to the identity, the map $\tilde{h}$ is a bounded distance away from the identity, i.e. there exists a constant $M_0 > 0$ such that $\forall w \in N$, $d\left(\tilde{h}(w), w\right) \leq M_0$.
- The facts that $z \in \tilde{W}^s(x)$ and $z \in \tilde{W}^u(y)$ imply that there exists a constant $C > 0$ such that for all $n \geq 0$,

$$d\left(\tilde{f}^n(x), \tilde{f}^n(z)\right) \leq C \quad \text{and} \quad d\left(\tilde{f}^{-n}(y), \tilde{f}^{-n}(z)\right) \leq C$$

□

Now, all that remains is to prove that the stable and unstable generalized foliations give a global product structure on N, i.e. Proposition 2.4.12. The proof of this follows the proof of

[41, Lemma 6.8], with a single minor change to account for the fact that M is a nilmanifold instead of a torus. We therefore give the general steps in Hiraide's argument and note where modifications need to be made. The argument proceeds in four steps/lemmas.

Lemma 2.4.13. *Let $f : M \to M$ be an Anosov homeomorphism of the nilmanifold $M = N/\Gamma$. Let $\tilde{f} : N \to N$ be a lift of f to N, and let $\tilde{A} : N \to N$ be a hyperbolic automorphism of N. If the C^0-distance between $\tilde{A}$ and $\tilde{f}$ is bounded, then $\tilde{f}$ has exactly one fixed point.*

Proof. This proof is a slight modification of the proof of [41, Lemma 6.5]. The main ingredients in this proof are the Lefschetz fixed point theorem and the homotopy invariance of the Lefschetz number. Since we're working in a space that isn't compact, we need to be careful when using Lefschetz numbers.[9] Since $\tilde{A}$ is a hyperbolic automorphism, its Lefschetz number is $L(\tilde{A}) = \pm 1$. The Lefschetz number of $\tilde{f}$ is defined because $\tilde{f}$ is a bounded distance away from $\tilde{A}$.

Now, we argue that $\tilde{f}$ has at least one fixed point. Since N is contractible, we can construct a homotopy between $\tilde{f}$ and $\tilde{A}$ that does not introduce fixed points outside of a compact set. Thus, $L(\tilde{f}) = L(\tilde{A}) = \pm 1$, which implies that $\tilde{f}$ has at least one fixed point.

The fact that $\tilde{f}$ has at most one fixed point follows from arguments in [41, Lemma 6.7]. □

We now prove that the non-wandering set of f is the whole nilmanifold.

Lemma 2.4.14. *The nonwandering set of an Anosov homeomorphism $f : M \to M$ of a nilmanifold $M = N/\Gamma$ is the entire nilmanifold, i.e. $\Omega(f) = M$.*

Proof. This follows from the same argument as [41, Proposition 6.6]. □

We begin to show that the stable and unstable generalized foliations of $\tilde{f}$ give a global product structure on N.

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[9] Recall that the Lefschetz number of a map $g : X \to X$ is only defined if the set of fixed points $\mathrm{Fix}(g)$ is compact. Two maps have the same Lefschetz number if they are homotopic via a map that does not introduce fixed points out of a compact set. [24]

Lemma 2.4.15. *For $x, y \in N$, the stable manifold of $\tilde{f}$ at x, $\tilde{W}^s(x)$, and the unstable manifold of $\tilde{f}$ at y, $\tilde{W}^u(y)$, intersect at at most one point.*

Proof. This follows from the previous two lemmas along with the spectral decomposition. The details are exactly the same as those in [41, Lemma 6.7]. $\square$

Now to complete the proof of Proposition 2.4.12 we just need to show that $\tilde{W}^s(x)$ and $\tilde{W}^u(y)$ actually intersect for each $x, y \in N$. This follows by gluing together local product neighborhoods given by $\mathscr{F}^s_f$ and $\mathscr{F}^u_f$ using the arguments in [28, Lemma 1.6].

Now, all that remains in the proof of Theorem E is to show that $\tilde{h}$ is injective on the stable and unstable leaves of $\tilde{f}$, which proceeds exactly as in [41].

CHAPTER 3

SMOOTH MODELS

3.1 Lifting the conjugacy on the leaf space

Lemma 3.1.1. *Let $F, M, \tilde{M}, B$ be closed Riemannian manifolds, and let $\pi : M \to B$ and $\widetilde{\pi} : \tilde{M} \to B$ be continuous isomorphic F-bundles. Let $h : B \to B$ be a homeomorphism that is homotopic to the identity. Then, $h : B \to B$ lifts to a homeomorphism $\tilde{h} : M \to \tilde{M}$.*

Proof. First, we note that M and h^*M are isomorphic bundles. This is because we by assumption M and $\tilde{M}$ are isomorphic, and since $h \sim \mathrm{id}$, the bundle $h^*\tilde{M}$ is isomorphic to the bundle $\mathrm{id}^*\tilde{M} = \tilde{M}$. Let $\phi : \tilde{M} \to h^*\tilde{M}$ be a bundle isomorphism, i.e. a homeomorphism such that the diagram in Figure 3.1a commutes. From the definition of $h^*\tilde{M}$, we have that the commutative diagram in Figure 3.1b commutes.[1] Combining these diagrams gives us the commutative diagram from Figure 3.1c. So the continuous map

$$\tilde{h} := \mathrm{proj}_2 \circ \phi : M \to \tilde{M} \tag{3.1}$$

is a lift of h. Since $\tilde{h}$ is a continuous injection, by invariance of domain, $\tilde{h} : M \to \tilde{M}$ is a homeomorphism.

$$\begin{tikzcd}
M \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & h^*\tilde{M} \arrow[dl, "h^*\widetilde{\pi}"] \\
& B &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
h^*\tilde{M} \arrow[r, "\mathrm{proj}_2"] \arrow[d, "h^*\widetilde{\pi}"] & \tilde{M} \arrow[d, "\widetilde{\pi}"] \\
B \arrow[r, "h"] & B
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[dr, "\pi"'] & h^*\tilde{M} \arrow[r, "\mathrm{proj}_2"] \arrow[d, "h^*\widetilde{\pi}"] & \tilde{M} \arrow[d, "\widetilde{\pi}"] \\
& B \arrow[r, "h"] & B
\end{tikzcd}$$

(c)

Figure 3.1

□

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[1] Note that $\mathrm{proj}_2 : h^*\tilde{M} \to \tilde{M}$ is the projection onto the second coordinate from the definition of the pullback bundle $h^*\tilde{M} = \{(b, x) \in B \times M : h(b) = \tilde{\pi}(x)\}$.

3.2 Lifting the Anosov automorphism on the leaf space to a partially hyperbolic system

Lemma 3.2.1. *Let F, E_0, E_1, B be closed Riemannian manifolds. Assume that $p_0 : E_0 \to B$ and $p_1 : E_1 \to B$ are C^k F-bundles with structure group H, where H is a finite-dimensional Lie group with smooth universal bundle, and that the left action of H on F is C^k, and that H acts on F by isometries. Suppose that $\theta : E_0 \to E_1$ is a (continuous) isomorphism of E_0 and E_1 as F-bundles with structure group H over B. Then, there is a C^k isomorphism $\alpha : E_0 \to E_1$ that is an isometry on fibers.*

Proof. We begin by constructing principal H-bundles with the same transition data as E_0 and E_1 using Lemma 2.3.2. We'll call these $q_0 : Q_0 \to B$ and $q_1 : Q_1 \to B$. Let $f_0 : B \to BH$ and $f_1 : B \to BH$ be classifying maps for Q_0 and Q_1. Note that since E_0 and E_1 (and therefore Q_0 and Q_1) are C^k bundles, the maps f_0 and f_1 are C^k.

Since E_0 and E_1 are isomorphic as continuous bundles, we get that there is a homotopy $f : B \times [0, 1] \to BH$ from f_0 to f_1. Since $f|_{B\times\{0,1\}}$ is C^k, then by the Whitney Approximation Theorem (See [46] Theorem 6.26), f_t is homotopic to a C^k map $\overline{f} : B \times [0, 1] \to BH$ relative to $B \times \{0, 1\}$. So, we have a C^k homotopy $\overline{f} : B \times [0, 1] \to BH$ from $\overline{f_0} = f_0$ to $\overline{f_1} = f_1$. If the classifying maps of two C^k principal bundles bundles are homotopic via a C^k homotopy, then the bundles are isomorphic as C^k bundles [44, Chapter 4.9][2]. Thus, we get that the pullback bundles f_0^*EH and f_1^*EH are isomorphic as C^k principal H-bundles over B. Since f_0 and f_1 are the classifying maps for Q_0 and Q_1 respectively, this means that Q_0 and Q_1 are isomorphic as C^k principal H-bundles over B. Since Q_0 and Q_1 have the same transition functions as E_0 and E_1, we get that E_0 and E_1 are isomorphic as C^k bundles with structure group H.

From the definition of a C^k isomorphism of F-bundles with structure group H, we see

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[2] The argument in [44, Chapter 4.9] is only given for continuous bundles, but works for C^k bundles

that this means that there is a C^k isomorphism $\alpha : E_0 \to E_1$, such that, if $(U_{0,i}, \phi_{0,i})$ and $\{(U_{1,j}, \phi_{1,j})\}$ are trivializing atlases for E_0 and E_1 respectively, then there exists functions $d_{ij} : U_{0,i} \cap V_{1,j} \to H$ such that for $x \in U_{0,i} \cap U_{1,j}$ and $y \in F$, we get that $\phi_{1,j} \circ \alpha \circ \phi_{0,i}^{-1}(x, y) = (x, d_{ij}(x) \cdot y)$. Since H acts on F by isometries, we get that α is an isometry on fibers. □

Corollary 3.2.2. *Let F, M, B be closed Riemannian manifolds. Assume that $\pi : M \to B$ is a smooth F-bundle with structure group H, where H is a finite-dimensional Lie group with smooth universal bundle, and that the left action of H on F is smooth, and that H acts on F by isometries. Suppose $A : B \to B$ is a smooth Anosov diffeomorphism, and suppose that A lifts to a homeomorphism $\widehat{A} : M \to M$. Then A lifts to a C^∞ diffeomorphism $g : M \to M$ that is an isometry on fibers of $\pi : M \to M$.*

Proof. Since A lifts to a homeomorphism $\widehat{A} : M \to M$, we can construct a (continuous) isomorphism $\theta : M \to A^*M$ of M and A^*M as F-bundles with structure group H given by $\theta(z) = \left(\pi(z), \widehat{A}(z)\right)$ (See Figure 3.2a). By Lemma 3.2.1, there is a C^∞ isomorphism $\alpha : M \to A^*M$ that is an isometry on fibers. We can then use $\alpha : M \to A^*M$ to define a C^∞ diffeomorphism $g : M \to M$ by $g(z) = \mathrm{proj}_2 \circ \alpha(z)$ (See Figure 3.2b). Since α is an isometry on fibers, we get that g is an isometry on fibers.

$$\begin{tikzcd}
M \arrow[rrd, bend left, "\widehat{A}"] \arrow[rd, "\theta"] \arrow[rdd, bend right, "\pi"'] & & \\
& A^*M \arrow[r, "\mathrm{proj}_2"] \arrow[d, "A^*\pi"] & M \arrow[d, "\pi"] \\
& B \arrow[r, "A"] & B
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
M \arrow[rrd, bend left, "g"] \arrow[rd, "\alpha"] \arrow[rdd, bend right, "\pi"'] & & \\
& A^*M \arrow[r, "\mathrm{proj}_2"] \arrow[d, "A^*\pi"] & M \arrow[d, "\pi"] \\
& B \arrow[r, "A"] & B
\end{tikzcd}$$

(b)

Figure 3.2

□

Proposition 3.2.3. *Let F, M, and B be closed Riemannian manifolds. Assume that $\pi : M \to B$ is a smooth F-bundle and that $f : B \to B$ is an Anosov diffeomorphism. If*

$g : M \to M$ is a diffeomorphism that is a lift of f and such that g is an isometry on fibers of $\pi : M \to B$, then $g : M \to M$ is partially hyperbolic.

Proof. We begin by constructing a Riemannian metric on M, with respect to which g is partially hyperbolic. This construction has three ingredients:

- A smooth family $\langle \cdot, \cdot \rangle^F_x$ of Riemannian metrics on the fibers $\pi^{-1}(x)$ such that $Dg : T\pi^{-1}(x) \to T\pi^{-1}(f(x))$ is an isometry for all $x \in B$. (Such a family exists because g is an isometry on fibers of $\pi : M \to B$.)
- A Riemannian metric $\langle \cdot, \cdot \rangle^B$ on B that is adapted to the Anosov diffeomorphism $f : B \to B$.
- An Ehresmann connection H on M, i.e. H is a smooth subbundle of the tangent bundle TM such that for all $p \in M$, $T_pM = H_p \oplus \ker(D_p\pi)$. Note that from the definition of an Ehresmann connection, we know that $D_p\pi|_{H_p} : H_p \subset T_pM \to T_{\pi(p)}B$ is an isomorphism and the map $p \mapsto H_p$ is smooth.

We define a Riemannian metric $\langle \cdot, \cdot \rangle$ on M by letting for all $p \in M$,

- $\langle v, v' \rangle = \big\langle D_p\pi(v), D_p\pi(v') \big\rangle^B$ for $v, v' \in H_p$,
- $\langle v, v' \rangle = \langle v, v' \rangle^F_{\pi(p)}$ for $v, v' \in \ker(D_p\pi)$, and
- $H_p \perp \ker(D_p\pi)$.

Now, we need to show that g is partially hyperbolic with respect to the metric $\langle \cdot, \cdot \rangle$. To do this, we construct a dominated splitting $TM = E^s \oplus E^c \oplus E^u$ such that g is uniformly contracting on E^s and uniformly expanding on E^u. We begin by letting $E^c = \ker(D\pi)$.

Next, we construct the unstable bundle E^u using a graph transform argument. We begin by lifting the unstable bundle $E^u_f \subset TB$ for the Anosov diffeomorphism $f : B \to B$ to the bundle $\hat{E}^u \subset TM$ given by $\hat{E}^u_p := H_p \cap D_p\pi^{-1}(E^u_f(\pi(p)))$ for $p \in M$. Note that

$D_p\pi^{-1}(E^u_p(\pi(p))) = \hat{E}^u_p \oplus \ker(D_p\pi)$, and that Dg preserves $\hat{E}^u \oplus E^c$ because Df preserves E^u_f and g covers f (so $D_{\pi(p)}f \circ D_p\pi = D_{g(p)}\pi \circ D_p g$).

Let

$$\Sigma = \Big\{\sigma : \hat{E}^u \to E^c :\ \sigma \text{ is fiber preserving over } id, \text{ and } \sigma_p : \hat{E}^u(p) \to E^c(p) \text{ is linear } \forall p \in M\Big\}$$

We put the norm $\|\cdot\|_\Sigma$ on Σ given by

$$\|\sigma\|_\Sigma = \sup_{p\in M} \|\sigma_p\|$$

where $\|\sigma_p\|$ is the operator norm. Note that this norm makes Σ a Banach space.

Now, we want to define a map $\Gamma : \Sigma \to \Sigma$, called the linear graph transform covering g, so that $D_p g(\text{graph}(\sigma_p)) = \text{graph}(\Gamma(\sigma_p))$. We now give $\Gamma : \Sigma \to \Sigma$ explicitly. Since by assumption, Dg preserves $E^c = \ker(D\pi)$ (and Dg preserves $\hat{E}^u \oplus E^c$), we can write for each $p \in M$,

$$D_p g = \begin{pmatrix} A_p & 0 \\ C_p & K_p \end{pmatrix} : \hat{E}^u(p) \oplus E^c(p) \to \hat{E}^u(g(p)) \oplus E^c(g(p)),$$

where

$$A_p : \hat{E}^u(p) \to \hat{E}^u(g(p)), \qquad C_p : \hat{E}^u(p) \to E^c(g(p)), \qquad K_p : E^c(p) \to E^c(g(p))$$

are all linear. Also note that since $D_p g$ is invertible, both A_p and K_p are invertible.

Note that we can write a point in the graph of σ_p as $(v, \sigma_p v) \in \text{graph}(\sigma_p) \subset \hat{E}^u(p) \oplus E^c(p)$. Applying $D_p g$ to this point gives us

$$D_p g(v, \sigma_p v) = \begin{pmatrix} A_p & 0 \\ C_p & K_p \end{pmatrix} \begin{pmatrix} v \\ \sigma_p v \end{pmatrix} = \begin{pmatrix} A_p v \\ C_p v + K_p \sigma_p v \end{pmatrix}$$

So, we can write

$$\begin{aligned} D_p g(\text{graph}(\sigma_p)) &= \left\{ \begin{pmatrix} A_p v \\ C_p v + K_p \sigma_p v \end{pmatrix} : \ v \in \hat{E}^u(p) \right\} \\ &= \left\{ \begin{pmatrix} w \\ \big(C_p + K_p \sigma_p\big) \circ A_p^{-1} w \end{pmatrix} : \ w \in \hat{E}^u(g(p)) \right\} \end{aligned}$$

by reparametrizing. Thus, the requirement that $D_p g(\text{graph}(\sigma_p)) = \text{graph}(\Gamma(\sigma_p))$ is equivalent to saying that

$$\begin{pmatrix} w \\ \big(C_p + K_p \sigma_p\big) \circ A_p^{-1} w \end{pmatrix} = \begin{pmatrix} w \\ \Gamma \sigma_p w \end{pmatrix}$$

for all $w \in \hat{E}^u(g(p))$. This gives us an equation for Γ in terms of C, K, and A:

$$\Gamma \sigma_p = \big(C_p + K_p \sigma_p\big) \circ A_p^{-1} : \hat{E}^u(g(p)) \to E^c(g(p))$$

for all $\sigma \in \Sigma$, $p \in M$, and $v \in \hat{E}^u(p)$. Omitting base points, we get that

$$\Gamma \sigma = (C + K\sigma) \circ A^{-1}$$

Now, our goal is to find an invariant section for Γ (the graph of which we will then show is the unstable bundle E^u for g). To do this, it suffices to show that $\Gamma : \Sigma \to \Sigma$ is a contraction.

Take $\sigma, \sigma' \in \Sigma$. For $p \in M$, we have

$$\begin{aligned}
\|\Gamma\sigma_p - \Gamma\sigma'_p\| &= \left\| \left(C_p + K_p\sigma_p\right) \circ A_p^{-1} - \left(C_p + K_p\sigma'_p\right) \circ A_p^{-1} \right\| \\
&= \left\| \left(C_p + K_p\sigma_p - C_p - K_p\sigma'_p\right) \circ A_p^{-1} \right\| \\
&= \left\| \left(K_p\sigma_p - K_p\sigma'_p\right) \circ A_p^{-1} \right\| \\
&= \left\| K_p \circ \left(\sigma_p - \sigma'_p\right) \circ A_p^{-1} \right\| \\
&\leq \|K_p\| \|\sigma_p - \sigma'_p\| \|A_p^{-1}\| \qquad (3.2)
\end{aligned}$$

We now bound $\|K_p\|$ and $\|A_p^{-1}\|$. Since g is an isometry on fibers of $\pi : M \to B$ and $E^c = \ker(D\pi)$, we get have that $D_pg|_{E^c(p)} = K_p : E^c(p) \to E^c(g(p))$ is an isometry. Thus, $\|K_p\| = 1$. Now, we bound $\|A_p^{-1}\|$ by relating the norm of A to the norm of Df on E^u_f. Since E^u_f is the unstable bundle for the Anosov diffeomorphism $f : B \to B$ and the norm $\| \cdot \|^B$ is adapted to f, we know that there exists a constant $\lambda > 1$ such that for all $w \in E^u_f$, $\|Df(w)\|^B \geq \lambda \|w\|^B$. Take $v \in \hat{E}^u(p)$. Since $D_p\pi(v) \in E^u_f(\pi(p))$, we therefore have that

$$\|D_{\pi(p)}f(D_p\pi(v))\|^B \geq \lambda \|D_p(v)\|^B.$$

Since g covers f, we see that $D_{\pi(p)}f(D_p\pi(v)) = D_{g(p)}\pi(D_pg(v))$ This along with the fact that $E^c = \ker(D\pi)$ and $C_p(v) \in E^c(g(p))$ gives that

$$D_{\pi(p)}f(D_p\pi(v)) = D_{g(p)}\pi(D_pg(v)) = D_{g(p)}\pi(A_p(v) + C_p(v)) = D_{g(p)}\pi(A_p(v))$$

Thus,

$$\lambda \|D_p(v)\|^B \leq \|D_{\pi(p)}f(D_p\pi(v))\|^B = \|D_{g(p)}\pi(A_p(v))\|^B$$

Finally, note that since $v, A_p(v) \in \hat{E}^u(P) \subset H_p$, by our definition of the norm on M, we get that

$$\|D_p\pi(v)\|^B = \|v\| \qquad \text{and} \qquad \|D_{g(p)}\pi(A_p(v))\|^B = \|A_p(v)\|.$$

We've therefore shown that $\lambda\|v\| \leq \|A_p(v)\|$, which implies that $\|A_p^{-1}\| \leq \lambda^{-1}$.

Combining our estimates for the norms of $\|K_p\|$ and $\|A_p^{-1}\|$ with (3.2) gives that

$$\|\Gamma\sigma_p - \Gamma\sigma_p'\| \leq \lambda^{-1}\|\sigma_p - \sigma_p'\|.$$

We have therefore shown that Γ is a contraction map. Then, by the contraction mapping principle, we get a Γ-invariant section $\sigma^u \in \Sigma$. We now define a bundle $E^u \subset TM$ by letting $E^u(p) := \text{graph}(\sigma_p^u)$. Note that E^u is Dg invariant since σ^u is Γ invariant and $D_pg(\text{graph}(\sigma_p)) = \text{graph}(\Gamma(\sigma_p)) = \text{graph}(\sigma_{g(p)})$.

The construction of the bundle E^s is analogous.

We now have a Dg-invariant splitting, $TM = E^s \oplus E^c \oplus E^u$. We now need to show that this splitting is partially hyperbolic. To do this, we construct a new metric $\langle \cdot, \cdot \rangle'$ on M by letting for all $p \in M$,

- $\langle v, v' \rangle' = \langle D_p\pi(v), D_p\pi(v') \rangle^B$ for $v, v' \in E^s$,
- $\langle v, v' \rangle' = \langle D_p\pi(v), D_p\pi(v') \rangle^B$ for $v, v' \in E^u$,
- $\langle v, v' \rangle' = \langle v, v' \rangle_{\pi(p)}^F$ for $v, v' \in E^c$, and
- E^s, E^c, and E^u be pairwise orthogonal.

From our construction of E^s and E^u, we get that Dg is uniformly expanding on E^u and uniformly contracting on E^c with respect to this new metric. Finally, the splitting is dominated because Dg restricted to E^c is an isometry. □

3.3 Proof of Theorem A

First, we recall our setup. Let $\hat{f} : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B, C^1 fibers F (where F is a closed manifold), and structure group $G \subset \mathrm{Diff}^1(F)$. Suppose that there exists a Riemannian metric on F and a subgroup $I \subset \mathrm{Isom}(F) \cap G$ such that the inclusion $I \hookrightarrow G$ is a homotopy equivalence.

The diffeomorphism $\hat{f} : M \to M$ descends to a homeomorphism $f : B \to B$. Our first step is to construct a conjugacy $h : B \to B$ between f and a hyperbolic nilmanifold automorphism $A : B \to B$. This will follow immediately from Theorem E if we can show that $f : B \to B$ is an Anosov homeomorphism. To see why the homeomorphism $f : B \to B$ is Anosov, we first observe that $\hat{f}$ admits an invariant center foliation $\mathscr{F}^c$ whose leaves are the level sets of π and that leaves of $\mathscr{F}^c$ are compact and have trivial holonomy. [3] Thus, by the following result of Bohnet and Bonatti, $f : B \to B$ is an *Anosov homeomorphism.*

Lemma 3.3.1 ([8, Theorem 2, Proposition 4.20])**.** *If $f : M \to M$ is a partially hyperbolic diffeomorphism with an invariant center foliation $\mathscr{F}^c$ with compact leaves and without holonomy, then the homeomorphism $F : M/\mathscr{F}^c \to M/\mathscr{F}^c$ induced by f on the quotient is an Anosov homeomorphism.*

Now, we can apply Theorem E to get that there exists a hyperbolic nilmanifold automorphism $A : B \to B$ and a homeomorphism $h : B \to B$ that is homotopic to the identity such that $A \circ h = h \circ \hat{f}$.

The next step in the proof is to construct a smooth F-bundle $\widehat{\pi} : \widehat{M} \to B$ that is isomorphic to the original bundle $\pi : M \to B$ and such that the structure group of $\widehat{\pi} : \widehat{M} \to B$ is $\mathrm{Isom}(F)$. To do this, we first construct a principal G bundle $p : P \to B$ with the same transition functions as $\pi : M \to B$. Since the inclusion of $I \hookrightarrow G$ is a homotopy inclusion,

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[3] For the definition of holonomy, see [18, Chapter 2]. The fact that the leaves of $\mathscr{F}^c$ have trivial holonomy follows immediately from the definition of a fibered partially hyperbolic system and the definition of holonomy.

by Lemma 2.3.5 there exists a continuous principal I-bundle $q' : Q' \to B$ that has transition functions cohomologous to those of $p : P \to B$. Since $I \subset \text{Isom}(F)$, we can construct (Lemma 2.3.2) a continuous principal $\text{Isom}(F)$-bundle $q : Q \to B$ with the same transition data as $q' : Q' \to B$.

Now, we find a smooth principal $\text{Isom}(F)$ bundle $\hat{q} : \hat{Q} \to B$ that is isomorphic to $q : Q \to B$. This follows immediately from the following lemma along with the fact that $\text{Isom}(F)$ is a locally Euclidean Lie group [52].

Lemma 3.3.2 ([53])**.** *Let K be a Lie group modeled on a locally convex space. Every principal K bundle over a closed manifold is isomorphic to a smooth principal K bundle.*

Now, we use the fiber bundle construction theorem (Lemma 2.3.2) to construct a smooth F-bundle $\widehat{\pi} : \widehat{M} \to B$ with the same transition functions as $\hat{q} : \hat{Q} \to B$. Since $\hat{q} : \hat{Q} \to B$ has transition functions that are cohomologous to those of the original bundle $\pi : M \to B$, we get that the bundle $\widehat{\pi} : \widehat{M} \to B$ is isomorphic to the original bundle $\pi : M \to B$.

Next, we lift the conjugacy $h : B \to B$ to a homeomorphism $\hat{h} : M \to \widehat{M}$. This follows immediately from Lemma 3.1.1.

Finally, we lift the hyperbolic nilmanifold automorphism $A : B \to B$ to a partially hyperbolic diffeomorphism $g : \widehat{M} \to \widehat{M}$. This follows immediately from Corollary 3.2.2 and Proposition 3.2.3. To see why we can apply Corollary 3.2.2 here, we first note that the structure group of $\widehat{\pi} : \widehat{M} \to B$ is $\text{Isom}(F)$, which is a finite dimensional compact Lie group [52]. This implies that $\text{Isom}(F)$ has a smooth universal bundle [53, Lemma I.12]. This completes the proof of Theorem A.

Remark 3.3.3. The fact that the F-bundle $\widehat{M}$ has structure group $\text{Isom}(F)$ is solely used to guarantee that the lift $g : \widehat{M} \to \widehat{M}$ of $A : B \to B$ is partially hyperbolic. Without this fact, the arguments given would allow us to lift $A : B \to B$ to a diffeomorphism, but we would not be able to guarantee that the lift would be a partially hyperbolic diffeomorphism.

This is the only reason that the assumption that that there exists a Riemannian metric on

F and a subgroup $I \subset \mathrm{Isom}(F) \cap G$ such that the inclusion $I \hookrightarrow G$ is a homotopy equivalence is necessary in the proof. Without this assumption, we would be able to get the conjugacy $h : B \to B$ between f and A, and we would be able to construct a smooth F-bundle $\widehat{M}$ over B that is isomorphic to the original F-bundle M.[4] However, the structure group of $\widehat{M}$ would only be $\mathrm{Diff}^{\infty}(F)$, not $\mathrm{Isom}(F)$.

If the structure group of $\widehat{M}$ was $\mathrm{Diff}^{\infty}(F)$, we would still be able to lift $h : B \to B$ to a homeomorphism $\widehat{h} : M \to \widehat{M}$ using Lemma 3.1.1. We would also be able to lift $A : B \to B$ to a diffeomorphism $g : \widehat{M} \to \widehat{M}$ using analogous arguments to the ones we use in Lemma 3.2.1 [5], Corollary 3.2.2, and the proof of Theorem A. However, since the lift g is not an isometry on fibers, the arguments in Proposition 3.2.3 won't apply to show that g partially hyperbolic.

Finding a way to lift $A : B \to B$ to a partially hyperbolic diffeomorphism $g : \widehat{M} \to \widehat{M}$ without requiring that the structure group of $\widehat{M}$ be $\mathrm{Isom}(F)$ or be trivial is a question for further research.

Proposition B is an example of a case where we can overcome the difficulty lifting $A : B \to B$ to a partially hyperbolic diffeomorphism that is discussed in the above remark.

Proof of Proposition B. The setup of Proposition B is that we are given a fibered partially hyperbolic system $\hat{f} : M \to M$ with quotient a nilmanifold B and C^1 fibers F. We assume that the F-bundle M is trivial. Note that this means that the structure group of the bundle $\pi : M \to B$ is the trivial group. We can then proceed with an analogous argument to the one given in the proof of Theorem A to get the conjugacy $h : B \to B$ between the Anosov homeomorphism $f : B \to B$ induced by $\hat{f} : M \to M$ and a hyperbolic nilmanifold

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[4] To construct $\widehat{M}$, we would first use Lemma 2.3.5 and Proposition 2.3.7 to get a F-bundle with structure group $\mathrm{Diff}^{\infty}(F)$ that is isomorphic to M. We then would apply Lemma 3.3.2 with $K = \mathrm{Diff}^{\infty}(F)$, which would give us $\widehat{M}$.

[5] The arguments given in Lemma 3.2.1 apply when the structure group is $\mathrm{Diff}^{\infty}(F)$ (rather than a finite dimensional Lie group that acts on F by isometries) to get a C^{∞} isomorphism between two continuously isomorphic smooth bundles (with structure group $\mathrm{Diff}^{\infty}(F)$). This uses the fact that $\mathrm{Diff}^{\infty}(F)$ has a smooth universal bundle [45, Theorem 44.24].

automorphism $A : B \to B$ and to get a smooth bundle $\widehat{\pi} : \widehat{M} \to B$ with trivial structure group that is isomorphic to the original bundle $\pi : M \to B$. This means that identifying the smooth F-bundle $\widehat{\pi} : \widehat{M} \to B$ with the smooth bundle $\mathrm{proj}_1 : B \times F \to B$ (that is projection onto the first coordinate), we can smoothly lift $A : B \to B$ to the fibered partially hyperbolic diffeomorphism $g : \widehat{M} \cong B \times F \to \widehat{M} \cong B \times F$ given by $g : (x, y) \mapsto (Ax, y)$ for $(x, y) \in B \times F$. $\square$

3.4 Corollaries of the Theorem A

We now explain how Corollaries C and D follow from Theorem A.

To prove Corollary C, we apply Theorem A with $G = \mathrm{Diff}^1(F)$ and with $I = \mathrm{Isom}(F)$. To do this, we just need to show that the inclusion $\mathrm{Isom}(F) \hookrightarrow \mathrm{Diff}^1(F)$ is a homotopy equivalence when $F = S^n$ for $n = 1, 2, 3$ and for F a hyperbolic 3-manifold. In fact showing that the inclusion $\mathrm{Isom}(F) \hookrightarrow \mathrm{Diff}^1(F)$ is a homotopy equivalence is equivalent to showing that the inclusion $\mathrm{Isom}(F) \hookrightarrow \mathrm{Diff}^\infty(F)$ is a homotopy equivalence by Proposition 2.3.7.

1. When $F = S^1$ it is a standard fact that $\mathrm{Diff}^\infty(S^1)$ deformation-retracts to $O(2) = \mathrm{Isom}(S^1)$. When $F = S^2$, Smale [58] proved that the inclusion $\mathrm{Isom}(S^2) \hookrightarrow \mathrm{Diff}^\infty(S^2)$ is a homotopy equivalence. Hatcher [38] proved this for S^3.

2. When F is a hyperbolic 3-manifold, Gabai [30] proved that the inclusion $\mathrm{Isom}(F) \hookrightarrow \mathrm{Diff}^\infty(F)$ is a homotopy equivalence.

Remark 3.4.1. When $n \geq 4$, the inclusion $\mathrm{Isom}(S^n) \hookrightarrow \mathrm{Diff}(S^n)$ is not a homotopy equivalence. This was proved for $n = 4$ in [63]. To see that $\mathrm{Isom}(S^n) \hookrightarrow \mathrm{Diff}(S^n)$ is not a homotopy equivalence for $n \geq 5$, first note that this statement is equivalent to the statement that $\mathrm{Diff}(D^n \text{ rel } \partial D^n)$ is contractible [38, Appendix].

One way to see that $\mathrm{Diff}(D^n \text{ rel } \partial D^n)$ is not contractible for many n is to use the fact that $\pi_0(\mathrm{Diff}(D^n \text{ rel } \partial D^n)) \cong \Theta_{n+1}$ for $n \geq 5$, where Θ_{n+1} is the group of exotic $(n + 1)$-

spheres [39] [59] [20]. For example, this along with the fact that there exist exotic 7-spheres [50] implies that $\mathrm{Isom}(S^6) \hookrightarrow \mathrm{Diff}(S^6)$.

To prove that $\mathrm{Diff}(D^n \text{ rel } \partial D^n)$ is not contractible for $n = 5$, we use the fact that the map $\pi_1(\mathrm{Diff}(D^n \text{ rel } \partial D^n)) \to \pi_0(\mathrm{Diff}(D^{n+1} \text{ rel } \partial D^{n+1}))$ is surjective for $n \geq 5$ [20]. Thus, since $\pi_0(\mathrm{Diff}(D^6 \text{ rel } \partial D^6)) \neq 0$, we get that $\pi_1(\mathrm{Diff}(D^5 \text{ rel } \partial D^5)) \neq 0$, so $\mathrm{Diff}(D^5 \text{ rel } \partial D^5)$ is not contractible.

For $n \geq 7$, the fact that $\mathrm{Diff}(D^n \text{ rel } \partial D^n)$ is not contractible is proved in [22].

This means that we cannot apply Theorem A as above to get an analogous version of Corollary C for S^n, $n \geq 4$.

Now, we prove Corollary D. In Corollary D, the structure group of M is $G = \mathrm{Diff}^1_0(F)$. We prove each case of Corollary D separately using the same strategy: we apply Theorem A by finding a subgroup $I \subset \mathrm{Isom}(F) \cap \mathrm{Diff}^1_0(F)$ such that the inclusion $I \hookrightarrow \mathrm{Diff}^1_0(F)$ is a homotopy equivalence.

1. When $F = \mathbb{T}^n$ for $n = 2, 3$, we choose $I = \mathbb{T}^n$, where $\mathbb{T}^n$ acts on itself by Euclidean isometry via translation. When $F = \mathbb{T}^2$, the inclusion $\mathbb{T}^2 \hookrightarrow \mathrm{Diff}^\infty_0(\mathbb{T}^2)$ is a homotopy equivalence [51, Section 4.1.5]. When $F = \mathbb{T}^3$, $\mathbb{T}^3 \hookrightarrow \mathrm{Diff}^\infty_0(\mathbb{T}^3)$ is a homotopy equivalence [40] [62].

2. When F is a hyperbolic surface, then $\mathrm{Diff}^+_0(F)$ is contractible [27, Thereom 1.14], which means that the hypothesis of Theorem A holds for I the trivial subgroup.

3. When F is a Haken manifold we consider three cases:

    - When F is not a Seifert manifold with coherently orientable fibers, then the components of $\mathrm{Diff}(F)$ are contractible [37], [40], [43, Section 1.3], which means that the hypothesis of Theorem A holds when I is the trivial subgroup.
    - When F is a Seifert manifold with coherently oriented fibers that is not $\mathbb{T}^3$, we take $I = S^1$, where S^1 acts on F by rotating circle fibers of the Seifert fiber

bundle structure. This satisfies the hypothesis of Theorem A because the inclusion $S^1 \hookrightarrow \mathrm{Diff}_0(F)$ is a homotopy equivalence [37], [40], [43, Section 1.3].

- When $F = \mathbb{T}^3$, we dealt with this case in (1).

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