THE UNIVERSITY OF CHICAGO

PARTIAL DIFFERENTIAL EQUATIONS ARISING FROM TOPOLOGICAL INSULATORS

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Stay healthy and stay Happy!

ABSTRACT

In this thesis, we develop a scattering theory for the asymmetric transport observed at interfaces separating two-dimensional topological insulators. Starting from the spectral decomposition of an unperturbed confining Hamiltonian, we present a limiting absorption principle and construct a generalized eigenfunction expansion for perturbed systems. We then relate the interface conductivity, a current observable quantifying the transport asymmetry, to the scattering matrix associated to the generalized eigenfunctions. In particular, we show that the interface conductivity is concretely expressed as a difference of transmission coefficients and is stable against perturbations. We apply the theory to systems of perturbed Dirac equations with asymptotically linear domain wall.

In the presence of random perturbations in the Hamiltonians, the limiting behavior of the scattering matrix entries as the thickness L of the random medium increases gives rise to a second order diffusion operator by the diffusion approximation theory. We call such diffusion operator a mixed type generalized Kimura diffusion operator. We model the operator and provide the degenerate Hölder space-type estimates for model operators. With the analysis of perturbation term we establish the existence of solutions. We also give proofs of the existence and regularity of the global heat kernel.

We also concern the long-time asymptotics of this degenerate diffusion operators with mixed linear and quadratic degeneracies. In one space dimension, we characterize all possible invariant measures for such a class of operators and in all cases show exponential convergence of the Green's kernel to such invariant measures. We generalize the results to a class of two-dimensional operators including those used in the analysis of topological insulators. Several numerical simulations illustrate our theoretical findings.

CHAPTER 1

INTRODUCTION

1.1 Background and motivation

A characteristic feature of two-dimensional topological insulators is the topologically protected asymmetric transport observed at one-dimensional interfaces separating two insulating bulks. If H is a Hamiltonian describing transport in the two-dimensional system, the asymmetry along the edge is modeled by the following edge conductivity

$$\sigma_I[H] = \mathrm{Tr}\, i[H, P]\varphi'(H) \tag{1.1}$$

where $P = P(x) \in \mathfrak{S}[0, 1]$ and $\varphi(E) \in \mathfrak{S}[0, 1, E_-, E_+]$ are smooth switch functions. Here, $\mathfrak{S}[a, b, c, d]$ is the set of bounded (measurable) functions on $\mathbb{R}$ equal to a for $x < c$ and equal to b for $x > d$ while $\mathfrak{S}[a, b]$ is their union over (finite) $c < d$.

In such a setting, an interface current is flowing in the direction of the $x-$axis while wavefields are concentrated in the vicinity of $y = 0$. We have $2\pi\sigma_I \in \mathbb{Z}$ an integer describing excitations primarily moving e.g. from left to right when $2\pi\sigma_I > 0$.

When $H = H_0$ is an unperturbed operator invariant with respect to spatial translations in x, then plane waves may be identified as generalized eigenfunctions of H_0. In

We consider $H = H_0 + Q$ with H_0 an operator with a well-known spectral decomposition and Q be a short-range perturbation, which for us will be an operator of point-wise multiplication by $Q(x, y)$, a function that decays sufficiently rapidly to 0 as $|x| \to \infty$. The topological protection of the asymmetric transport states that $\sigma_I[H_0] = \sigma_I[H_0 + Q]$ for a large class of perturbations Q.

The main objective is to devise a scattering theory for H. More precisely, for an energy $E \in \mathbb{R}$ within the bulk band gap, we wish to show the existence of generalized plane waves solution of $H\psi_m = E\psi_m$ and construct a scattering matrix S from such functions ψ_m. Our final objective is then to show $2\pi\sigma_I \in \mathbb{Z}$ is directly related to the coefficients of the scattering matrix S.

Dirac operator with domain walls We consider the Dirac operator

$$H = H_0 + Q, \qquad H_0 = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3 \tag{1.2}$$

where $D_x = -i\partial_x$ and $D_y = -i\partial_y$, where $\sigma_{1,2,3}$ are the standard Pauli matrices, and where $m(y)$ is a domain wall that we will take the form $m(y) - y$ equal to a bounded function to simplify the presentation. Here, Q is the operator of multiplication by $Q(x, y)$, which takes values in 2×2 Hermitian matrices. With such perturbation, we proved that H satisfies the following hypothesis.

Hypothesis 1.1.1 ([H1])**.** *(o) We assume that H_0 is a self-adjoint (elliptic) differential operator with domain $H^p \otimes \mathbb{C}^q$ and resolvent operator $R_0(i) = (H_0 - i)^{-1}$ bounded from $\mathcal{H}$ to $H^p \otimes \mathbb{C}^q$.*

(i) For each $\xi \in \mathbb{R}$, $\hat{H}_0(\xi)$ has a compact resolvent and hence purely discrete spectrum. We assume the existence of generalized eigenfunctions in H^p_{-s} for $s > \frac{1}{2}$, solutions

$$\psi_j(x, y; \xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi x}\phi_j(y; \xi), \tag{1.3}$$

of the eigenvalue problem $(H_0 - E_j(\xi))\psi_j = 0$ with $(\phi_j)_j$ an orthonormal basis of $L^2(\mathbb{R}_y) \otimes \mathbb{C}^q$, i.e., $(\phi_j, \phi_k)_{L^2(\mathbb{R}_y)\otimes\mathbb{C}^q} = \delta_{jk}$. Here, $j \in J$ with $J \simeq \mathbb{N}$.

(ii) We assume that the branches of absolutely continuous spectrum $j \to E_j(\xi)$ are smooth and satisfy $|E_j(\xi)| \to \infty$ as $|\xi| \to \infty$ with $\xi \to (1 + |E_j(\xi)|^2)^{-1}$ integrable for $j \in J$. We assume that for any interval $[a, b]$, only a finite number of branches $\xi \to E_j(\xi)$ cross $[a, b]$.

For H_0 an elliptic operator of order p, which is the framework we are interested in, standard ellipticity results show that $\psi_j(x, y; \xi)$ defined in (2.3) is an element in H^p_{-s} for

$s > \frac{1}{2}$. We have by assumption the spectral decomposition

$$H_0 = \sum_j \int_{\mathbb{R}} E_j(\xi)\Pi_j(\xi)d\xi, \qquad \Pi_j(\xi) = \psi_j(\cdot;\xi) \otimes \psi_j(\cdot;\xi), \tag{1.4}$$

where $\Pi_j(\xi)$ are rank-one projectors. Associated to the above decomposition is the following resolution of identity. Let $\Xi = (j, \xi) \in J \times \mathbb{R}$. We define for $f \in L^2(\mathbb{R}^2) \otimes \mathbb{C}^q$ the (unperturbed) Fourier transform:

$$\hat{f}(\Xi) = (\mathcal{F}f)(\Xi) := \int_{\mathbb{R}^2} \overline{\psi_j(x, y, \xi)} \cdot f(x, y)dxdy = (f, \psi_j(\cdot, \xi)), \tag{1.5}$$

where $(f, g) = \int_{\mathbb{R}^2} f(x, y) \cdot \bar{g}(x, y)dxdy$ is the inner product on $\mathcal{H}$, with inverse Fourier transform:

$$f(x, y) = (\mathcal{F}^{-1}\hat{f})(x, y) := \sum_j \int_{\mathbb{R}} \hat{f}(\Xi)\psi_j(x, y, \xi)d\xi. \tag{1.6}$$

The Fourier transform is an isometry from $\mathcal{H} = L^2(\mathbb{R}^2, dxdy) \otimes \mathbb{C}^q$ to $L^2(J \times \mathbb{R}, d\Xi; \mathbb{C})$, with $d\Xi$ the Cartesian product of the counting measure on J and the Lebesgue measure on $\mathbb{R}$.

(iii) The spectral elements $E_j(\xi)$ and $\Pi_j(\xi)$ are assumed to be smooth in ξ with a finite number of critical values. Define

$$Z = \big\{E \in \mathbb{R};\ E = E_j(\xi) \text{ for some } (j, \xi) \in J \times \mathbb{R} \text{ and } \partial_\xi E_j(\xi) = 0\big\}. \tag{1.7}$$

We assume the set Z of critical values to be finite in each bounded interval $[E_-, E_+]$.

(iv) To set up a scattering theory, we finally assume the following completeness property: for any $E \in \mathbb{R}\backslash Z$, then any solution $\psi \in H^p_{-s}$ of $(H_0 - E)\psi = 0$ is a linear combination of the generalized eigenfunctions $\psi_j(x, y; \xi)$ for values of ξ such that $E_j(\xi) = E$. We label $\psi_m(x, y) = \psi_j(x, y; \xi_m)$ for $1 \le m \le \mathrm{M}(E)$ the corresponding solutions at E fixed. Up to

(obvious) relabeling, we thus have

$$\psi_m(x, y; E) = \frac{1}{\sqrt{2\pi}} e^{i\xi_m(E)x} \phi_m(y; \xi_m(E)), \qquad 1 \le m \le \mathrm{M}(E). \tag{1.8}$$

We then show that for each $\Xi = (j, \xi)$, there exist modified generalized eigenfunctions $\psi_j^Q \in H_{-s}^p$ solution of $H\psi_j^Q = E_j(\xi)\psi_j^Q$. For a fixed $\mathbb{R} \ni E \notin Z$, we still denote by ψ_m^Q the solution of the problem $(H - E)\psi_m^Q = 0$ with $\psi_m^Q = \psi_j^Q(\cdot, \xi_m)$ with $E_j(\xi_m) = E$ and $1 \le m \le \mathrm{M}(E)$. Moreover, we will justify the following assumption on the spectral decomposition

$$\text{[H2]} \qquad H = \sum_n \lambda_n \Pi_n + \sum_j \int_{\mathbb{R}} E_j(\xi) \Pi_j^Q(\xi) d\xi, \quad \Pi_j^Q(\xi) = \psi_j^Q(\cdot; \xi) \otimes \psi_j^Q(\cdot; \xi). \tag{1.9}$$

Here, the sum over n corresponds to discrete (locally finite) spectrum with eigenvalues λ_n and (rank-one) projectors $\Pi_n = \psi_n \otimes \psi_n$ for eigenfunctions $\psi_n \in \mathcal{H}$. The above decomposition thus imposes that the branches of absolutely continuous spectrum for H and H_0 be the same. This is consistent with the standard result that two operators H_1 and $H_2 = H_1 + V$ with V a trace-class perturbation have unitarily equivalent absolutely continuous spectrum

A final assumption on the generalized eigenfunctions is that for $|x|$ large, then ψ_m^Q is approximately given by a linear combination of the unperturbed solutions ψ_n for $1 \le n \le \mathrm{M}(E)$. More precisely, define the currents

$$J_m = J_m(E) = \partial_\xi E_m(\xi_m) \neq 0 \tag{1.10}$$

which do not vanish for $E \notin Z$ not being a critical value of the branches of absolutely continuous spectrum. We then assume that for $1 \le m \le \mathrm{M}(E)$,

$$\text{[H3]} \qquad \psi_m^Q(x, y) \approx \sum_{1 \le n \le \mathrm{M}(E)} \alpha_{mn}^{\pm} \psi_n(x, y) = \sum_{1 \le n \le \mathrm{M}(E)} \alpha_{mn}^{\pm} \frac{1}{\sqrt{2\pi}} e^{i\xi_n x} \phi_n(y), \tag{1.11}$$

where $a \approx b$ means that the difference $a - b$ converges to 0 uniformly (in x as a square-integrable function in $y \in \mathbb{R}$) as $x \to \pm\infty$ and where $\alpha^{\pm}$ are the corresponding coefficients in these two limits.

With H satisfying the hypothesis above, we let S be the $(n_+ + n_-) \times (n_+ + n_-)$ scattering matrix

$$S = \begin{pmatrix} T_+ & R_- \\ R_+ & T_- \end{pmatrix}.$$

We deduce from the spectral theorem and the spectral decomposition that

Proposition 1.1.1. *Let $E \in \mathbb{R} \backslash Z$ and ψ_m^Q the associated perturbed generalized eigenfunctions. Then:*

$$\sum_m \left| \frac{\partial \xi_m}{\partial E} \right| (\psi_m^Q, 2\pi i [H, P] \psi_m^Q) = n_+ - n_-. \tag{1.12}$$

As a corollary of the preceding proposition, we obtain the final main result of this section:

Theorem 1.1.2. *The conductivity may be recast as*

$$2\pi\sigma_I = \text{tr } T_+^* T_+ - \text{tr } T_-^* T_- = n_+ - n_-.$$

Scattering theory and diffusion approximation For a given energy level E, a finite number of the edge modes are propagating while the rest are evanescent. In the presence of random fluctuations $\tilde{V}$ coupling the propagating modes (see Hypothesis 4.1 in section 4), the amplitudes of said modes satisfy a closed system of equations (in the y variable). Edge transport is then characterized by a scattering matrix composed of reflection and transmission coefficients. Conductance in such systems is then physically proportional to the trace of the transmission matrix. In the topologically trivial setting, Anderson localization shows that such a conductance decays exponentially as the thickness of the slab of random perturbations increases.

In

Example 1.1.1 (. *The diffusion approximation of the* 3×3 *system gives rise to a second order diffusion operator on the triangle* $T = \{(x, y) : 0 \leq x, y, x + y \leq 1\}$*:*

$$L = \gamma_{12} \left[xy(\partial_x - \partial_y)^2 + (y - x)(\partial_x - \partial_y) \right]$$
$$+\gamma_{23} \left[y(x\partial_x + (y-1)\partial_y)^2 + (y-1)(x\partial_x + (y-1)\partial_y) \right]$$
$$+\gamma_{13} \left[x((x-1)\partial_x + y\partial_y)^2 + (x-1)((x-1)\partial_x + y\partial_y) \right].$$

Let L be a mixed type generalized Kimura diffusion operator, defined on P, a two-dimensional compact manifold with corners, which means that L is a second-order locally elliptic operator in the interior P° with appropriate degeneracy conditions at boundary points. Specifically, in the vicinity of a boundary component, the coefficients of the normal part of the second-order term vanish to order either one or two.

Theorem 1.1.3. *For* $0 < \gamma < 1$*, if the data* $f \in C^{k,2+\gamma}(P), g \in C^{k,\gamma}(P \times [0, T])$*, then the inhomogeneous problem*

$$(\partial_t - L)w = g \text{ in } P \times [0, T] \qquad \text{with} \quad w(0, x, y) = f$$

has a unique solution $w \in C^{k,2+\gamma}(P \times [0, T])$*.*

Based on series expansion of fundamental solutions of model operators and the above result on the heat equation, our second main result is the existence and regularity results of a heat kernel for L. We let P^{reg} denote the union of $\mathring{P}$ and regular edge points and regular corner points (see Definition 3.1.1).

Theorem 1.1.4. *The global heat kernel* $H_t(d_1, d_2, l_1, l_2) \in C^\infty(P^{reg} \times \mathring{P} \times (0, \infty))$ *of the full operator* L *exists and for* $f \in C^0(P)$*, then*

$$v_f := \int_P H_t(d_1, d_2, l_1, l_2) f(l_1, l_2) dl_1 dl_2$$

is the solution of $(\partial_t - L)v_f = 0$ *with* $v_f(0,\cdot,\cdot) = f$.

Associated to the operator L is a C^0 semigroup $\mathcal{Q}_t = e^{tL}$ solution operator of the Cauchy problem

$$\partial_t u = Lu$$

with initial conditions $u(x,0) = f(x)$ at $t = 0$. Our second objective is to analyze the long time behavior of e^{tL}, and in particular convergence to appropriately defined invariant measures. It turns out that the number of possible invariant measures and their type (absolutely continuous with respect to one-dimensional or two-dimensional Lebesgue measures or not) strongly depend on the structure of the coefficients at the boundary. We thus distinguish the boundary into two types: tangent or transverse. For one dimensional case, we have the following results in different boundary types.

Theorem 1.1.5. *If there is only one tangent endpoint* p*, then for any non-quadratic point* x*, the transition probability* $p_t(x,\cdot)$ *converges exponentially to* $\delta(p)$ *in Wasserstein distance.*

If there are two tangent endpoints, then there exists S_0*, satisfying* $S_0(0) = 0$*,* $S_0(1) = 1$ *and* $LS_0 = 0$*, such that for any probability measure* v*,* $\mathcal{Q}_t^* v$ *converges exponentially to* $\delta(0)\int_0^1 v(1-S_0) + \delta(1)\int_0^1 vS_0$ *in Wasserstein distance.*

Theorem 1.1.6. *The invariant measure* μ *in* (4.53) *satisfies a Poincaré inequality. For* $f \in L^2(\mu)$*,*

$$||\mathcal{Q}_t f - \int f\mu||_{L^2(\mu)} \leq e^{-\frac{t}{2C_{LP}}}||f - \int f\mu||_{L^2(\mu)}. \tag{1.13}$$

For any probability measure $v = h\mu$ *with* $h \in L^2(\mu)$*,*

$$||\mathcal{Q}_t^* v - \mu||_{TV} \leq e^{-\frac{t}{2C_{LP}}}||h - 1||_{L^2(\mu)}.$$

For L on a 2 dimensional compact manifold with corners P with only one tangent edge

H, we proved that the Wasserstein distance between the transition probability $p_t(p,\cdot)$ and the invariant measure supported on the tangent edge converges exponentially.

Theorem 1.1.7. *Fix a point p that is not on any quadratic transverse edge. Then the Wasserstein distance between the transition probability $p_t(p,\cdot)$ and the invariant measure supported on the tangent edge converges exponentially:*

$$W(p_t(p,\cdot),\mu_0(x)\delta_0(y)) \le Me^{-\frac{a}{2}t}, t > 0. \tag{1.14}$$

CHAPTER 2

SCATTERING THEORY OF TOPOLOGICALLY PROTECTED EDGE TRANSPORT

2.1 Introduction

A characteristic feature of two-dimensional topological insulators is the topologically protected asymmetric transport observed at one-dimensional interfaces separating two insulating bulks. Applications may be found in many areas in condensed matter physics, photonics, and geophysical sciences

If H is a Hamiltonian describing transport in the two-dimensional system, the asymmetry along the edge is modeled by the following edge conductivity

$$\sigma_I[H] = \operatorname{Tr} i[H, P]\varphi'(H) \tag{2.1}$$

where $P = P(x) \in \mathfrak{S}[0, 1]$ and $\varphi(E) \in \mathfrak{S}[0, 1, E_-, E_+]$ are smooth switch functions. Here, $\mathfrak{S}[a, b, c, d]$ is the set of bounded (measurable) functions on $\mathbb{R}$ equal to a for $x < c$ and equal to b for $x > d$ while $\mathfrak{S}[a, b]$ is their union over (finite) $c < d$. The operator $i[H, P]$ may be interpreted as a current operator while $0 \le \varphi'(E)$ is a density of states. Thus σ_I models the expected value of the current operator for excitations in the system with density $\varphi'(E)$ supported in an energy interval $[E_-, E_+]$ where propagation into the bulk is suppressed. In this paper, we consider the setting where no energy $E \in \mathbb{R}$ is allowed to propagate in the bulk, so that $[E_-, E_+]$ is an arbitrary (bounded) interval in $\mathbb{R}$. The interface conductivity has been used in a variety of contexts

In such a setting, an interface current is flowing in the direction of the $x-$axis while wavefields are concentrated in the vicinity of $y = 0$. We have $2\pi\sigma_I \in \mathbb{Z}$ an integer describing excitations primarily moving e.g. from left to right when $2\pi\sigma_I > 0$. We may then envision the following scattering experiment. When $H = H_0$ is an unperturbed operator invariant with respect to spatial translations in x, then plane waves may be identified as generalized eigenfunctions of H_0. We will show how σ_I may be expressed in terms of such eigenfunctions following

Section 2.2 presents our main framework. Under the assumption that generalized eigenfunctions associated to the problem $H\psi = E\psi$ exist and satisfy a priori constraints, we show that a unitary scattering matrix may be defined and that the edge conductivity (2.1) can indeed be computed from the scattering coefficients. This justifies computations performed in

The rest of the chapter is devoted to providing sufficient conditions for the theory of section 2.2 to apply. This is done in section 2.3 by appealing to the spectral theorem to obtain an appropriate decomposition of H, and in particular to a limiting absorption principle to obtain a detailed description of the absolutely continuous spectrum and point spectrum (and lack of singular continuous spectrum) of H. The construction of generalized eigenfunctions for the perturbed system is given in section 2.4. Finally, section 2.5 verifies all required hypotheses for slight generalizations of the systems of Dirac operators analyzed numerically in

References on scattering theory, the limiting absorption principle, and generalized eigenfunction expansions that are relevant to the current work include

2.2 Current conservation and edge conductivity

This section proposes a framework to relate the asymmetric transport modeled by the edge conductivity to spectral information on the Hamiltonian describing the system. Section 2.2.1 summarizes our main assumptions while section 2.2.2 introduces a current correlation and defines a notion of current conservation and section 2.2.3 finally relates the edge conductivity to a scattering matrix associated to the Hamiltonian.

2.2.1 Assumptions on spectral decomposition of Hamiltonian

Let $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^q$ the space of vector-valued functions with square-integrable entries defined on the Euclidean plane with coordinates (x, y). We define the following functional spaces.

Definition 2.2.1. *For $s \in \mathbb{R}$, we define L^2_s the weighted Hilbert space of all complex valued functions $u(x, y)$ defined in $\mathbb{R}^2$ such that $\langle x\rangle^s u(x, y) \in L^2(\mathbb{R}^2)$ with the norm*

$$\|u\|_{L^2_s} = \Big(\int_{\mathbb{R}^2} \langle x\rangle^{2s} |u(x, y)|^2 dxdy \Big)^{\frac{1}{2}}.$$

Let $\beta \geq 0$. For $p \in \mathbb{N}_*$ and $s \in \mathbb{R}$, H^p_s denote the Hilbert space of L^2_s functions with distribution derivatives in L^2_s up to p-th order, with norm given by

$$\|u\|_{H^p_s} = \Big(\int_{\mathbb{R}^2} \Big[\langle x\rangle^{2s} \langle y\rangle^{2\beta} |u|^2 + \langle x\rangle^{2s} \sum_{|\alpha|=p} |D^\alpha u|^2 \Big] dxdy \Big)^{\frac{1}{2}}. \tag{2.2}$$

Above, $\langle x\rangle := \sqrt{1 + x^2}$. We denote $H^p = H^p_0$. We also denote by H^p_s (L^2_s) the space of vector-valued functions $H^p_s \otimes \mathbb{C}^q$ ($L^2_s \otimes \mathbb{C}^q$). The value of β will be equal to p in our applications.

We start with a self-adjoint operator H_0 from $\mathfrak{D}(H_0) = H^p$ to $\mathcal{H}$ that is invariant with respect to translations in x. We then have $H_0 = \mathcal{F}^{-1}_{\xi\to x} \hat{H}_0(\xi) \mathcal{F}_{x\to\xi}$ with $\mathcal{F}_{x\to\xi}$ Fourier transform in the first variable x and $\mathbb{R} \ni \xi \mapsto \hat{H}_0(\xi)$ a family of self-adjoint operators on $L^2(\mathbb{R}) \otimes \mathbb{C}^q$.

Hypothesis 2.2.1 ([H1])**.** *(o) We assume that H_0 is a self-adjoint (elliptic) differential operator with domain $H^p \otimes \mathbb{C}^q$ and resolvent operator $R_0(i) = (H_0 - i)^{-1}$ bounded from $\mathcal{H}$ to $H^p \otimes \mathbb{C}^q$.*

(i) For each $\xi \in \mathbb{R}$, $\hat{H}_0(\xi)$ has a compact resolvent and hence purely discrete spectrum. We

assume the existence of generalized eigenfunctions in H^p_{-s} for $s > \frac{1}{2}$, solutions

$$\psi_j(x, y; \xi) = \frac{1}{\sqrt{2\pi}} e^{i\xi x} \phi_j(y; \xi), \tag{2.3}$$

of the eigenvalue problem $(H_0 - E_j(\xi))\psi_j = 0$ with $(\phi_j)_j$ an orthonormal basis of $L^2(\mathbb{R}_y) \otimes \mathbb{C}^q$, i.e., $(\phi_j, \phi_k)_{L^2(\mathbb{R}_y)\otimes\mathbb{C}^q} = \delta_{jk}$. Here, $j \in J$ with $J \simeq \mathbb{N}$.

(ii) We assume that the branches of absolutely continuous spectrum $j \to E_j(\xi)$ are smooth and satisfy $|E_j(\xi)| \to \infty$ as $|\xi| \to \infty$ with $\xi \to (1 + |E_j(\xi)|^2)^{-1}$ integrable for $j \in J$. We assume that for any interval $[a, b]$, only a finite number of branches $\xi \to E_j(\xi)$ cross $[a, b]$.

For H_0 an elliptic operator of order p, which is the framework we are interested in, standard ellipticity results show that $\psi_j(x, y; \xi)$ defined in (2.3) is an element in H^p_{-s} for $s > \frac{1}{2}$. We have by assumption the spectral decomposition

$$H_0 = \sum_j \int_{\mathbb{R}} E_j(\xi) \Pi_j(\xi) d\xi, \qquad \Pi_j(\xi) = \psi_j(\cdot; \xi) \otimes \psi_j(\cdot; \xi), \tag{2.4}$$

where $\Pi_j(\xi)$ are rank-one projectors. Associated to the above decomposition is the following resolution of identity. Let $\Xi = (j, \xi) \in J \times \mathbb{R}$. We define for $f \in L^2(\mathbb{R}^2) \otimes \mathbb{C}^q$ the (unperturbed) Fourier transform:

$$\hat{f}(\Xi) = (\mathcal{F}f)(\Xi) := \int_{\mathbb{R}^2} \overline{\psi_j(x, y, \xi)} \cdot f(x, y) dx dy = (f, \psi_j(\cdot, \xi)), \tag{2.5}$$

where $(f, g) = \int_{\mathbb{R}^2} f(x, y) \cdot \bar{g}(x, y) dx dy$ is the inner product on $\mathcal{H}$, with inverse Fourier transform:

$$f(x, y) = (\mathcal{F}^{-1}\hat{f})(x, y) := \sum_j \int_{\mathbb{R}} \hat{f}(\Xi) \psi_j(x, y, \xi) d\xi. \tag{2.6}$$

The Fourier transform is an isometry from $\mathcal{H} = L^2(\mathbb{R}^2, dxdy) \otimes \mathbb{C}^q$ to $L^2(J \times \mathbb{R}, d\Xi; \mathbb{C})$, with $d\Xi$ the Cartesian product of the counting measure on J and the Lebesgue measure on $\mathbb{R}$.

(iii) The spectral elements $E_j(\xi)$ and $\Pi_j(\xi)$ are assumed to be smooth in ξ with a finite

number of critical values. Define

$$Z = \Big\{E \in \mathbb{R};\ E = E_j(\xi) \textit{ for some } (j,\xi) \in J \times \mathbb{R} \textit{ and } \partial_\xi E_j(\xi) = 0\Big\}. \qquad (2.7)$$

We assume the set Z of critical values to be finite in each bounded interval $[E_-, E_+]$.

(iv) To set up a scattering theory, we finally assume the following completeness property: for any $E \in \mathbb{R}\backslash Z$, then any solution $\psi \in H^p_{-s}$ of $(H_0 - E)\psi = 0$ is a linear combination of the generalized eigenfunctions $\psi_j(x,y;\xi)$ for values of ξ such that $E_j(\xi) = E$. We label $\psi_m(x,y) = \psi_j(x,y;\xi_m)$ for $1 \le m \le \mathrm{M}(E)$ the corresponding solutions at E fixed. Up to (obvious) relabeling, we thus have

$$\psi_m(x,y;E) = \frac{1}{\sqrt{2\pi}} e^{i\xi_m(E)x}\phi_m(y;\xi_m(E)), \qquad 1 \le m \le \mathrm{M}(E). \qquad (2.8)$$

The main unperturbed operator of interest in this paper is the massive Dirac Hamiltonian

$$H_0 = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3, \qquad \hat{H}_0(\xi) = \xi\sigma_1 + D_y\sigma_2 + m(y)\sigma_3, \qquad (2.9)$$

with $\sigma_{1,2,3}$ standard Pauli matrices and $D_a = -i\partial_a$ for $a \in \{x,y\}$ and $m(y)$ a domain wall, which for concreteness, equals y up to a bounded perturbation. Then, $p = \beta = 1$ and $q = 2$. That the spectral decomposition (2.4) and all assumptions in Hypothesis [H1] applies to H_0 will be revisited in section **??**; see also

A second natural application of the theory developed here is for the Klein-Gordon operator

$$H_0 = D_x^2 + \mathfrak{a}^*\mathfrak{a}, \qquad \mathfrak{a} = \partial_y + m(y), \quad \mathfrak{a}^* = -\partial_y + m(y) \qquad (2.10)$$

with then $p = \beta = 2$ and $q = 1$. This operator is topologically trivial in the sense that $\sigma_I[H_0 + Q] = 0$ for Q short range

The quantization of the interface conductivity σ_I for Dirac, Klein Gordon, and more general elliptic operators with unbounded domain walls is carried out in

The main objective of this paper is to analyze transport properties for a perturbed operator $H = H_0 + Q$ where Q is a short-range operator. We consider the case where Q is an operator of multiplication by $Q(x, y)$ with the $q \times q$-valued (measurable) function $Q(x, y)$ such that $\langle x\rangle^{1+\varepsilon}|Q| \leq C$ for some $\varepsilon > 0$. Here, $\langle x\rangle = \sqrt{1 + x^2}$. We thus assume Q sufficiently rapidly decaying (only) in the x variable.

We then show that for each $\Xi = (j, \xi)$, there exist modified generalized eigenfunctions $\psi_j^Q \in H^p_{-s}$ solution of $H\psi_j^Q = E_j(\xi)\psi_j^Q$. For a fixed $\mathbb{R} \ni E \notin Z$, we still denote by ψ_m^Q the solution of the problem $(H - E)\psi_m^Q = 0$ with $\psi_m^Q = \psi_j^Q(\cdot, \xi_m)$ with $E_j(\xi_m) = E$ and $1 \leq m \leq \mathrm{M}(E)$. Moreover, we will justify the following assumption on the spectral decomposition

$$[\mathrm{H2}] \qquad H = \sum_n \lambda_n \Pi_n + \sum_j \int_{\mathbb{R}} E_j(\xi)\Pi_j^Q(\xi)d\xi, \quad \Pi_j^Q(\xi) = \psi_j^Q(\cdot;\xi) \otimes \psi_j^Q(\cdot;\xi). \tag{2.11}$$

Here, the sum over n corresponds to discrete (locally finite) spectrum with eigenvalues λ_n and (rank-one) projectors $\Pi_n = \psi_n \otimes \psi_n$ for eigenfunctions $\psi_n \in \mathcal{H}$. The above decomposition thus imposes that the branches of absolutely continuous spectrum for H and H_0 be the same. This is consistent with the standard result that two operators H_1 and $H_2 = H_1 + V$ with V a trace-class perturbation have unitarily equivalent absolutely continuous spectrum

A final assumption on the generalized eigenfunctions is that for $|x|$ large, then ψ_m^Q is approximately given by a linear combination of the unperturbed solutions ψ_n for $1 \leq n \leq \mathrm{M}(E)$. More precisely, define the currents

$$J_m = J_m(E) = \partial_\xi E_m(\xi_m) \neq 0 \tag{2.12}$$

which do not vanish for $E \notin Z$ not being a critical value of the branches of absolutely continuous spectrum. We then assume that for $1 \le m \le \mathrm{M}(E)$,

$$\text{[H3]} \qquad \psi_m^Q(x,y) \approx \sum_{1\le n\le \mathrm{M}(E)} \alpha_{mn}^{\pm}\psi_n(x,y) = \sum_{1\le n\le \mathrm{M}(E)} \alpha_{mn}^{\pm}\frac{1}{\sqrt{2\pi}}e^{i\xi_n x}\phi_n(y), \qquad (2.13)$$

where $a \approx b$ means that the difference $a-b$ converges to 0 uniformly (in x as a square-integrable function in $y \in \mathbb{R}$) as $x \to \pm\infty$ and where $\alpha^{\pm}$ are the corresponding coefficients in these two limits.

2.2.2 Current correlations

Let H be a differential self-adjoint operator of order p as described in the preceding section and so that [H2] and [H3] hold, which we assume for the rest of section 2.2. Let ψ_m and ψ_n two generalized eigenfunctions in H^p_{-s} for $s > \frac{1}{2}$, solutions of

$$H\psi_m = E_m\psi_m, \quad H\psi_n = E_n\psi_n$$

with E_n and E_m in $\mathbb{R}$. Define the current correlation

$$J_{mn}(x_0) = (\psi_n, 2\pi i[H, P(\cdot - x_0)]\psi_m) \qquad (2.14)$$

Here, $(\cdot,\cdot)$ denotes the inner product on $\mathcal{H}$. While $\psi_n \notin \mathcal{H}$, the above integral is well-defined since $[H, P(\cdot - x_0)]$ is a differential operator with coefficients that vanish for $x - x_0$ outside of a compact set and hence mapping H^p_{-s} to L^2_t for any $t \in \mathbb{R}$. We recall that P is a switch function in $\mathfrak{S}[0,1]$. On that compact set in the $x-$variable, both $H\psi_m$ and ψ_n are square integrable.

Lemma 2.2.1. *When $E_m = E_n$, we have the current conservation*

$$J'_{mn}(x_0) = 0 \quad \textit{for all} \quad x_0 \in \mathbb{R}.$$

Proof. Using that $P'(\cdot - x_0)$ (unlike $P(\cdot - x_0)$) is compactly supported, we obtain that

$$-J'_{mn}(x_0) = (\psi_n, 2\pi i[H, P'(\cdot - x_0)]\psi_m) = (\psi_n, 2\pi i(HP'(\cdot - x_0) - P'(\cdot - x_0)H)\psi_m)$$
$$= (H\psi_n, 2\pi i P'(\cdot - x_0)\psi_m) - (\psi_n, 2\pi i P'(\cdot - x_0)H\psi_m) = (E_n - E_m)(\psi_n, 2\pi i P'(\cdot - x_0)\psi_m),$$

which vanishes. □

Consider $H = H_0 + Q$ with Q rapidly decaying at infinity as described in the preceding section. For a fixed energy $E \in \mathbb{R}$, the unperturbed solutions of $(H_0 - E)\psi = 0$ in H^p_{-s} are given by $\psi_m(x, y)$ for $1 \le m \le \mathrm{M}(E)$ in (2.8) while the corresponding perturbed solutions are given by $\psi^Q_m(x, y)$. The number of propagating modes $\mathrm{M}(E)$ equals $n_+ + n_-$ where $n_\pm$ corresponds to the number of currents $\pm J_m > 0$ associated to each unperturbed plane wave and defined in (2.12). From assumption (2.13), we deduce that

$$(\psi^Q_m, 2\pi i[H, P(\cdot - x_0)]\psi^Q_n) \approx \sum_{1\le p,q\le \mathrm{M}(E)} \alpha^\pm_{mp}\bar{\alpha}^\pm_{nq}(e^{i\xi_p x}\phi_p, i[H, P(\cdot - x_0)]e^{i\xi_q x}\phi_q) \quad (2.15)$$

where $\approx$ here is the same sense as above but now as $x_0 \to \pm\infty$. All we need in the sequel is in fact that (2.15) holds rather than the more constraining (2.13). All terms in (2.15) are again clearly defined since $[H, P(\cdot - x_0)]$ is compactly supported in the $x-$vicinity of x_0. We wish to estimate the above right-hand side.

Lemma 2.2.2. *Let P be a switch function in $\mathfrak{S}(0, 1)$. Then we have*

$$(e^{i\xi_m x}\phi_m, i[H, P]e^{i\xi_n x}\phi_n) = \delta_{mn}\partial_\xi E_n(\xi_n) = \delta_{mn}J_n. \quad (2.16)$$

Proof. For an operator A, we denote by $A(x, x')$ its Schwartz kernel. Let us first assume

$\xi_m \neq \xi_n$. Then

$$(i[H,P]e^{i\xi_m x}\phi_m, e^{i\xi_n x}\phi_n) = \int e^{-i\xi_n x}\phi_n^*(y)(i[H_0,P])(x,x',y,y')e^{i\xi_m x'}\phi_m(y')dxdydx'dy'$$
$$= \int e^{-i\xi_n z}\phi_n^*(y)iH_0(z,y,y')\big((P(x')-P(x'+z))e^{i(\xi_m-\xi_n)x'}\big)\phi_m(y')dzdx'dydy'$$
$$=\hat{P}(\xi_n-\xi_m)\int \phi_n^*(y)(e^{-i\xi_n z}-e^{-i\xi_m z})iH_0(z,y,y')\phi_m(y')dzdydy'$$
$$=\hat{P}(\xi_n-\xi_m)\int \phi_n^*(y)(\hat{H}_0(\xi_n,y,y')-\hat{H}_0(\xi_m,y,y'))\phi_m(y')dydy'$$
$$=\hat{P}(\xi_n-\xi_m)(E_n(\xi_n)-E_m(\xi_m))(\phi_n,\phi_m)=0$$

since $E = E_n(\xi_n) = E_m(\xi_m)$ while $\hat{P}$, the Fourier transform of $P - \frac{1}{2}$, is bounded for $\xi_n \neq \xi_m$ (decomposing P as a Heaviside function plus an integrable function while $\hat{P}$ would equal $(\xi_m - \xi_n)^{-1}$ for P the Heaviside function). Note that we may not have (and do not have in practice) $(\phi_n, \phi_m) = 0$ for $n \neq m$ since $\xi_n \neq \xi_m$ for a fixed value of E while the eigenfunctions ϕ_n are orthogonal for different values of E_m at a fixed value of ξ.

When $\xi_n = \xi_m$, we find instead

$$(i[H,P]e^{i\xi_m x}\phi_m, e^{i\xi_m x}\phi_n) = \int e^{-i\xi_m x}\phi_n^*(y)(i[H_0,P])(x,x',y,y')e^{i\xi_m x'}\phi_m(y')dxdydx'dy'$$
$$= \int e^{-i\xi_m z}\phi_n^*(y)iH_0(z,y,y')\big(P(x')-P(x'+z)\big)\phi_m(y')dzdx'dydy'$$
$$= \int \phi_n^*(y)e^{-i\xi_m z}(-z)iH_0(z,y,y')\phi_m(y')dzdydy'$$
$$= \int \phi_n^*(y)\partial_\xi \hat{H}_0(\xi_m,y,y')\phi_m(y')dydy' = (\phi_n, \partial_\xi \hat{H}_0(\xi_m)\phi_m).$$

The modes $\phi_m(\xi)$ satisfy

$$\hat{H}_0(\xi)\phi_m(\xi) = E_m(\xi)\phi_m(\xi).$$

Since the spectral branches $\xi \to E_m(\xi)$ are assumed sufficiently smooth, this yields

$$\partial_\xi \hat{H}_0\phi_m + \hat{H}_0\partial_\xi\phi_m = \partial_\xi E_m\phi_m + E_m\partial_\xi\phi_m$$

from which we deduce

$$(\phi_n, \partial_\xi \hat{H}_0 \phi_m) = \partial_\xi E_m (\phi_n, \phi_m)$$

for any ϕ_n such that $(\hat{H}_0 - E_m)\phi_n = 0$. If $n \neq m$ while $E_m(\xi_m) = E_n(\xi_m)$, then we may choose the eigenvectors ϕ_n and ϕ_m as orthogonal so that $(\phi_n, \partial_\xi \hat{H}_0 \phi_m) = 0$ then (we have that $\partial_\xi E_m \neq 0$ since $E \notin Z$ is not at a critical value of the energy branches).

As a result, when $\xi_n = \xi_m$ we have

$$(e^{i\xi_m x}\phi_n, i[H,P]e^{i\xi_m x}\phi_m) = \delta_{mn}(\phi_m, \partial_\xi \hat{H}_0(\xi_m)\phi_m) = \delta_{mn}\partial_\xi E_m.$$

We used the normalization $\|\phi_m\|^2 = 1$. This concludes the derivation. □

We thus conclude from (2.13) and the above lemma that in the limits $x_0 \to \pm\infty$,

$$(\psi_m^Q, 2\pi i[H, P(\cdot - x_0)]\psi_n^Q) \approx \sum_p J_p \alpha^\pm_{mp} \bar{\alpha}^\pm_{np}. \tag{2.17}$$

2.2.3 Scattering matrix and edge conductivity

We next define the refection and transmission coefficients $R^\pm_{mn}$ and $T^\pm_{mn}$ as

$$\alpha^+_{mn} = \sqrt{\frac{|J_m|}{|J_n|}} T^+_{mn} \quad \text{when} \quad J_m > 0 \text{ and } J_n > 0 \tag{2.18}$$

$$\alpha^-_{mn} = \sqrt{\frac{|J_m|}{|J_n|}} T^-_{mn} \quad \text{when} \quad J_m < 0 \text{ and } J_n < 0 \tag{2.19}$$

$$\alpha^+_{mn} = \sqrt{\frac{|J_m|}{|J_n|}} R^-_{mn} \quad \text{when} \quad J_m < 0 \text{ and } J_n > 0 \tag{2.20}$$

$$\alpha^-_{mn} = \sqrt{\frac{|J_m|}{|J_n|}} R^+_{mn} \quad \text{when} \quad J_m > 0 \text{ and } J_n < 0, \tag{2.21}$$

while we also have $\alpha^-_{mm} = 1$ when $J_m > 0$ and $\alpha^+_{mm} = 1$ when $J_m < 0$. All other coefficients $\alpha^\pm_{ij}$ then vanish. We then have the following result:

Lemma 2.2.3. *The* $(n_+ + n_-) \times (n_+ + n_-)$ *scattering matrix*

$$S = \begin{pmatrix} T_+ & R_- \\ R_+ & T_- \end{pmatrix}$$

is unitary. Here T_+ *is the* $n_+ \times n_+$ *matrix with coefficients* T^+_{mn}*, etc.*

Proof. From (2.17) evaluated at $x_0 \to \pm\infty$ and the current conservation in Lemma 2.2.1 stating that both limits are equal, we deduce that when $J_m > 0$ and $J_n > 0$, then

$$\sum_{J_p>0} \bar{T}^+_{mp} T^+_{np} = \delta_{mn} - \sum_{J_p<0} \bar{R}^+_{mp} R^+_{np}.$$

This shows the orthonormality of the first n_+ columns of S. Considering the other cases $\pm J_m > 0$ and $\pm J_n > 0$ provides the other orthonormality constraints and concludes the proof. □

Lemma 2.2.4. *Let* S *be the above scattering matrix. Then*

$$\text{tr } T^*_+ T_+ - \text{tr } T^*_- T_- = n_+ - n_-. \tag{2.22}$$

Proof. From unitarity of the scattering matrix, we get

$$S^* S = \begin{pmatrix} T^*_+ T_+ + R^*_+ R_+ & T^*_+ R_- + R^*_+ T_- \\ R^*_- T_+ + T^*_- R_+ & R^*_- R_- + T^*_- T_- \end{pmatrix}$$

$$= SS^* = \begin{pmatrix} T_+ T^*_+ + R_- R^*_- & T_+ R^*_+ + R_- T^*_- \\ R_+ T^*_+ + T_- R^*_- & R_+ R^*_+ + T_- T^*_- - \end{pmatrix} = I.$$

Looking at the diagonal terms, we obtain

$$\text{tr } T^*_+ T_+ + R^*_+ R_+ = \text{tr } T_+ T^*_+ + R_- R^*_- = n_+, \quad \text{tr } R^*_- R_- + T^*_- T_- = \text{tr } R_+ R^*_+ + T_- T^*_- = n_-.$$

By cyclicity of the trace or explicit computation of the norm, we deduce that $\mathrm{tr}R_+^*R_+ = \mathrm{tr}R_-R_-^* = \mathrm{tr}R_-^*R_-$ so that $\mathrm{tr}T_+^*T_+ - \mathrm{tr}T_-^*T_- = n_+ - n_-$. This may be written using only one-sided measurements as $n_+ - n_- = \mathrm{tr}\ (T_+^*T_+ + R_-^*R_-) - n_- = n_+ - \mathrm{tr}\ (T_-^*T_- + R_+^*R_+)$. □

We deduce from the spectral theorem and the decomposition (2.11) the following result.

Proposition 2.2.1. *Let $E \in \mathbb{R}\backslash Z$ and ψ_m^Q the associated perturbed generalized eigenfunctions. Then:*

$$\sum_m \Big|\frac{\partial \xi_m}{\partial E}\Big|(\psi_m^Q, 2\pi i[H,P]\psi_m^Q) = n_+ - n_-. \tag{2.23}$$

Proof. We have from the above calculations and sending $x_0 \to +\infty$

$$\begin{aligned}
&\sum_m \Big|\frac{\partial \xi_m}{\partial E}\Big|(\psi_m^Q, 2\pi i[H,P]\psi_m^Q) = \sum_m \Big|\frac{\partial \xi_m}{\partial E}\Big| \sum_n |\alpha_{mn}^+|^2 J_n \\
&= \sum_{J_m>0} \frac{1}{|J_m|} \sum_{J_n>0} |T_{mn}^+|^2 |J_m| + \sum_{J_m<0} \frac{1}{|J_m|}\Big(J_m + \sum_{J_n>0} |R_{mn}^+|^2 |J_m|\Big) \\
&= \sum_{J_m>0, J_n>0} |T_{mn}^+|^2 - n_- + \sum_{J_m<0, J_n>0} |R_{mn}^-|^2 = n_+ - n_-.
\end{aligned}$$

We use here that there are n_+ modes with $J_m > 0$ and n_- modes with $J_m < 0$. □

As a corollary of the preceding proposition, we obtain the final main result of this section:

Theorem 2.2.2. *The conductivity may be recast as*

$$2\pi\sigma_I = \mathrm{tr}\ T_+^*T_+ - \mathrm{tr}\ T_-^*T_- = n_+ - n_-.$$

Proof. From (2.11), we have

$$\varphi'(H) = \sum_n \varphi'(\lambda_n)\Pi_n + \sum_j \int_{\mathbb{R}} \varphi'(E_j(\xi))\Pi_j^Q(\xi)d\xi$$

The operators $i[H,P]\Pi_n$ are trace-class with vanishing trace since

$$\mathrm{Tr}[H,P]\Pi_n = (\phi_n,[H,P]\phi_n) = (\phi_n, HP\phi_n) - (\phi_n, PH\phi_n) = \lambda_n(\phi_n, P\phi_n) - \lambda_n(\phi_n, P\phi_n) = 0.$$

Since the sum over n is finite, it does not contribute to $\sigma_I[H]$. Thus, from the definition of the rank-one projectors Π_j^Q, and identifying $\psi_j^Q(\xi)$ with $\psi_m^Q(E)$ when $E_j(\xi_m) = E$, we find

$$\begin{aligned} 2\pi\sigma_I[H] &= \sum_j \int_{\mathbb{R}} \varphi'(E_j(\xi))(\psi_j^Q(\xi), 2\pi i[H,P]\psi_j^Q(\xi))d\xi \\ &= \sum_m \int_{\mathbb{R}} \Big|\frac{\partial \xi_m}{\partial E}\Big| \varphi'(E)(\psi_m^Q(E), 2\pi i[H,P]\psi_m^Q(E))dE \\ &= \int_{\mathbb{R}} \varphi'(E)(n_+ - n_-)dE = n_+ - n_- = \mathrm{tr}\ T_+^* T_+ - \mathrm{tr}\ T_-^* T_-. \end{aligned}$$

We used $\varphi \in \mathfrak{S}[0,1]$ and Lemma 2.2.4 to conclude. □

2.3 Spectral analysis and limiting absorption principle

This section analyzes spectral properties of H_0 and $H = H_0 + Q$ for operators satisfying the following estimates. We assume that H_0 is a self-adjoint differential operator as described in [H1](o)-(iv) above. The resolvent operator $R_0(z) = (H_0 - z)^{-1}$ then displays different behaviors as z approaches the real-axis with positive or negative imaginary part.

Definition 2.3.1. *For $a < b$, we define*

$$J_+(a,b) = \{\lambda \in \mathbb{C} \mid a < \mathrm{Re}\,\lambda < b,\ 0 < \mathrm{Im}\,\lambda < 1\},$$

$$J_-(a,b) = \{\lambda \in \mathbb{C} \mid a < \mathrm{Re}\,\lambda < b,\ -1 < \mathrm{Im}\,\lambda < 0\},$$

$$J(a,b) = J_+(a,b) \cup J_-(a,b).$$

Our objective is to prove results on the spectrum of H that will allow us to verify hypothesis [H2] in section 2.4. We recall that the spaces L^2_s and H^p_s are introduced in Definition 2.2.1 and that Z is the set of critical values of branches of spectrum of H_0 defined in (2.7).

Remark 2.3.1. *For $s > 0$, the injection $H_s^p \hookrightarrow L^2$ is compact*

We make the following assumptions on the short-range perturbation Q and assume the following a priori estimates.

Hypothesis 2.3.1. *We assume that $Q(x, y)$ is a $q \times q$ Hermitian matrix valued function. Moreover, $|Q(x, y)|$ is bounded (measurable) and for some $h > 1$ and $C = C(h) > 0$,*

$$|Q(x,y)| \leq C\langle x\rangle^{-h}, \qquad (x,y) \in \mathbb{R}^2. \tag{2.24}$$

We recall that $\langle x\rangle = \sqrt{1+x^2}$.

Hypothesis 2.3.2. *Let $\mathbb{R} \ni a < b \in \mathbb{R}$. We assume the following a priori estimates.*

1. *Let $s > \frac{1}{2}$. There is a constant $C = C(s, a, b) > 0$ such that*

$$\|u\|_{H^p_{-s}} \leq C\|(H_0 - \lambda)u\|_{L^2_s}, \tag{2.25}$$

   for all complex numbers $\lambda \in J(a, b)$ and $u \in H_s^p$.

2. *Let $s > 0$, $\epsilon > 0$, and $(a, b) \cap Z = \emptyset$. There is a constant $C = C(s, a, b) > 0$ such that*

$$\|u\||_{H^p_{s-1-\epsilon}} \leq C\|(H_0 - \lambda)u\|_{L^2_s}, \tag{2.26}$$

   for all real numbers $\lambda \in (a, b)$ and $u \in H^p$.

3. *Let $[a, b] \cap Z = \emptyset$ and $[a, b]$ not containing any eigenvalue of H. Then for $h > s > \frac{1}{2}$, there exists a constant $C = C(s, a, b)$ such that*

$$\|u\|_{H^p_{-s}} \leq C\|(H - \lambda)u\|_{L^2_s}, \tag{2.27}$$

   for all $u \in H_s^p$ and $\lambda \in J(a, b)$.

The above hypotheses provide sufficient a priori estimates on $H = H_0 + Q$ to characterize some of its spectral properties. We start by showing that with the above hypotheses, then H has at most discrete point spectrum.

Proposition 2.3.1. *The point spectrum of H is discrete. (In particular, every eigenvalue has finite multiplicity.) The only possible limiting points of families of eigenvalues are in $Z \cup \{\pm\infty\}$.*

Proof. We will show that if $u \in H^p$ is an eigen-function corresponding to λ, i.e., $Hu = \lambda u,\ a < \lambda < b$, where a, b satisfies condition 2 in Hypothesis 2.3.2, then $u \in H^p_s$ for some $s > 0$ and

$$\|u\|_{H^p_s} \leq C\|u\|_{L^2}. \tag{2.28}$$

with some constant C independent of λ.

The proposition is a direct corollary. Indeed, suppose $\{u_n\}$ is a set of eigenfunctions with norm 1 in H^p_s. By (2.28), $\|u_n\|_{L^2}$ is bounded below by a positive constant. Also by Remark 2.3.1, the injection map from H^p_s into L^2 is compact. As $\{u_n\}$ is orthogonal and bounded both below and above in L^2, it must be in a finite set. This implies that H has finite eigenvalues in $[a, b]$ and the multiplicity of each eigenvalue is also finite.

To prove (2.28), we first show that it holds true for $s = 0$. Indeed, since $Hu = \lambda u$,

$$(H_0 - i)u = (\lambda - i)u + g \in L^2, \quad g = -Qu = (H_0 - \lambda)u.$$

Thus $u = R_0(i)[(\lambda - i)u + g]$ with $R_0(i)$ mapping from $\mathcal{H}$ to H^p by ellipticity assumption [H1(o)] and with Q bounded implies that (2.28) holds with $s = 0$. Now, the operator of multiplication by $Q(x, y)$ is a continuous operator from L^2 (and hence H^p) into $L^2_h(\mathbb{R}^2)$ by hypothesis 2.3.1. This implies

$$\|g\|_{L^2_h} \leq C_1 \|u\|_{H^p} \leq C_2 \|u\|_{L^2},$$

with some constant C_1, C_2 independent of λ on a compact interval. For $g \in L^2_h$, we apply condition 2 in Hypothesis 2.3.2, to deduce that $u \in H^p_{h-1-\epsilon}$ for any $\epsilon > 0$. Choosing $0 < \epsilon < h - 1$, we proved (2.28). □

The estimates we obtained in the preceding section naturally yield the following corollary, one of the main results of this section.

Theorem 2.3.3 (Principle of limiting absorption). *Let $a, b \in \mathbb{R}$ such that $[a, b] \cap Z = \emptyset$ and $[a, b]$ does not contain any eigenvalue of H. For $\frac{1}{2} < s < h$, $f \in L^2_s$ and $\operatorname{Im} z \neq 0$, define*

$$u_z(f) = (H - z)^{-1} f.$$

Then for $\lambda \in (a, b)$, there exists $u^{\pm}(\lambda, f)$ such that

$$u_z(f) \to u^{\pm}(\lambda, f) \text{ in } H^p_{-s}$$

as $z \to \lambda \pm 0i$, respectively. Moreover, $u^{\pm}(\lambda, f)$ are solutions of the equation

$$(H - \lambda) u(x) = f(x)$$

and they are continuous functions of λ in the topology of H^p_{-s}.

Proof. We modify standard arguments developed in

From condition 3 in Hypothesis 2.3.2, $u_{z_n}(f)$ is bounded in H^p_{-s}. We can then select (using the same method as described in the proof of Theorem 2.5.2 below) a subsequence $\{u_{z'_n}\}$ from $\{u_{z_n}\}$ which converges locally in $H^p_0(|x| \leq R)$ for any $R > 0$ to some function u_0 in H^p_{-s} and u_0 satisfies that $(H - \lambda) u_0 = 0$.

Next we wish to show that

$$u_{z_n} \to u_0 \text{ in } H^p_{-s}.$$

Assuming the contrary, we select a subsequence $\{u_{z''_n}\}$ such that

$$\|u_{z''_n} - u_0\|_{H^p_{-s}} \geq \delta > 0. \tag{2.29}$$

We then choose (by the method described in the proof of Theorem 2.5.2 below) a subsequence of $\{u_{z''_n}\}$ which we still denote by $\{u_{z''_n}\}$ such that $u_{z''_n} \to u_1$ for some function u_1 in H^p_{-s}. Defining

$$v_n = u_{z'_n} - u_{z''_n}, \quad v_0 = u_0 - u_1,$$

then $v_n \to u_0 - u_1$ in H^p_{-s}, and $(H - \lambda)v_0 = 0$. As in Theorem 2.5.2, we show that $v_0 \in L^2$, which implies that $v_0 = 0$ since $v_0 \in L^2$. This contradicts (2.29).

We now show that the limit u_0 is independent of the choice of the $\{z_n\}$ converging to $\lambda + 0i$. Take another sequence $v_n \to \lambda + 0i$. Then there exists $u_2 \in H^p_{-s}$ such that $u_{v_n} \to u_2$ in H^p_{-s}. We shall show $u_0 = u_2$. To prove this, define

$$w_n = u_{z_n} - u_{v_n}, \quad w_0 = u_0 - u_2.$$

Then $w_n \to w_0$ in H^p_{-s} and $(H - \lambda)w_0 = 0$ which implies $w_0 = 0$. Thus, $u_+(\lambda, f) = u_0$ is well defined.

Finally we show that $u_+(\lambda, f)$ is continuous in λ. In view of the fact that the resolvent $u_z(f)$ is a continuous function of $z \in J_+(a, b)$, it suffices to show that $u_{\lambda+i\eta}$ converges to $u_\lambda(f)$ in H^p_{-s} uniformly respect to $\lambda \in J_+(a, b)$. If we assumed the contrary, there would be a $\delta_1 > 0$ and a sequence of $\eta_n \to 0+$ such that

$$\|u_{\lambda_n+i\eta_n}(f) - u_{\lambda_n}(f)\|_{H^p_{-s}} \geq \delta_1 > 0.$$

On the other hand, for each n we could also select η'_n such that $\eta'_n \to 0+$ and

$$\|u_{\lambda_n+i\eta'_n}(f) - u_{\lambda_n}(f)\|_{H^p_{-s}} \leq \frac{\delta_1}{2} > 0.$$

This implies that

$$\|u_{\lambda_n+i\eta_n}(f) - u_{\lambda_n+i\eta'_n}(f)\|_{H^p_{-s}} \geq \frac{\delta_1}{2} > 0. \tag{2.30}$$

Because $[a, b]$ is compact, we can select a convergent subsequence $\{\lambda'_n\}$ of $\{\lambda_n\}$ such that $\lambda'_n \to \lambda_0$. This contradicts (2.30) because both $u_{\lambda_n+i\eta_n}, u_{\lambda_n+i\eta'_n}$ would converge to $u_{\lambda_0}(f)$. We thus proved that $u_+(\lambda, f)$ is continuous in λ. □

Our final result is the following:

Theorem 2.3.4. *H does not have singular continuous spectrum.*

Proof. Let $E(\lambda)$ be the right-continuous resolution of the identity associated with the self-adjoint operator H. It suffices to show that $(E(\lambda)f, f)$ for $f \in L^2_s$ is continuous when $\lambda \in \mathbb{R}$ is not an eigenvalue.

First we assume that

$$k_n < \alpha < \beta < k_{n+1}$$

for some $n \in \mathbb{Z}$, and $[\alpha, \beta]$ does not contain any eigenvalue, where $\{k_n\}$ label elements in the union of Z defined in (2.7) and the discrete spectrum of H.

For arbitrary a, b with $\alpha < a < b < \beta$, we have the following relation using, e.g.,

2.4 Generalized eigenfunction expansion

In this section, we justify the expansion (2.11) and the corresponding hypothesis [H2] of section 2.2. We thus need to construct generalized eigenfunctions $\psi_j^Q(\cdot;\xi)$ for $\Xi = (j,\xi) \in J \times \mathbb{R}$.

We know that the point spectrum $(\lambda_n)_n$ of H is discrete. Define $Z_H = Z \cup \{(\lambda_n)_n\}$. Let $E \in (a,b)$ with $(a,b) \cap Z_H = \emptyset$. We know from Theorem 2.3.3 in the preceding section that $R(z) = (H-z)^{-1}$ is well defined on $\mathcal{H}$ and bounded uniformly for $z \in J(a,b)$. We may thus define the bounded operators:

$$R^{\pm}(\lambda) = (H - (\lambda \pm i0))^{-1}.$$

For $\Xi = (j,\xi) \in J \times \mathbb{R}$ as defined in section 2.2, we defined $\psi_j(x,y;\xi)$ in (2.3). For $z \in J(a,b)$, it is convenient to introduce the function $A_\Xi(x,y;z)$ defined as

$$A_\Xi(z) = (I - R(z)Q)\psi_j(\xi). \tag{2.31}$$

Associated is the following linear form defined for $f \in L^2_s$ with $s > \frac{1}{2}$:

$$A^*_\Xi f(z) = (f, A_\Xi(z)) := \int_{\mathbb{R}^2} f(x,y) \cdot \bar{A}_\Xi(x,y;z)dxdy. \tag{2.32}$$

We now define the perturbed generalized eigenfunctions

$$\psi_j^{\pm}(\xi) = A_\Xi(E_j(\xi) \pm i0) = (I - R(E_j(\xi) \pm i0)Q)\psi_j(\xi). \tag{2.33}$$

For concreteness, we define $\psi_j^Q(x,y;\xi) = \psi_j^+(x,y;\xi)$, the *outgoing* generalized eigenfunctions, while $\psi_j^-(x,y;\xi)$ corresponds to *incoming* generalized eigenfunctions.

Thanks to Theorem 2.3.3, we observe that $\psi_j^{\pm}(\xi) \in H^p_{-s}(\mathbb{R}^2)$. For each $E \in (a,b)$, there is a finite number of wavenumbers ξ_m such that $E = E_m(\xi_m)$. We denote by $\psi_m^{\pm}(E)$ the corresponding generalized eigenfunctions parametrized by (m, E) rather than (j, ξ).

Let $f \in L^2_s$ and $j \in J$ with $\xi \in \Xi_j$ so that $E_j(\xi) \in (E_-, E_+)\backslash Z_H$. We define the generalized Fourier transform(s) by

$$\tilde{f}^{\pm}(\Xi) = A^*_{\Xi}(E_j(\xi) \pm 0i)f, \tag{2.34}$$

and for any $\lambda_n \in (E_-, E_+)$ an eigenvalue of $H\phi_n = \lambda_n \phi_n$,

$$\tilde{f}_n = (f, \phi_n). \tag{2.35}$$

We then obtain the following results justifying the expansion [H2] in (2.11) in section 2.2. We first verify the following estimate on the generalized Fourier transform.

Lemma 2.4.1. *For $\frac{h}{2} > s > \frac{1}{2}$, there exists a constant $C = C(s, a, b)$ such that*

$$|A^*_{\Xi}(z)f| \leq C\|f\|_{L^2_s} \tag{2.36}$$

for all $\Xi \in J \times \mathbb{R}$, $z \in J(a, b)$ and $f \in L^2_s$.

Proof. By construction, we have

$$A^*_{\Xi}(z)f = (f, A_{\Xi}(z))_{L^2} = (f, (I - R(z)Q)\psi_j(\xi))_{L^2}. \tag{2.37}$$

The multiplication operator Q is a continuous map from L^2_{-s} to L^2_{h-s} and by use of Hypothesis 2.27, we get

$$\begin{aligned} |(f, (I - R(z)Q)\psi_j(\xi))_{L^2}| &\leq \|f\|_{L^2_s} \cdot \|(I - R(z)Q)\psi_j(\xi))\|_{L^2_{-s}} \\ &\leq C\|f\|_{L^2_s} \cdot \|\psi_j(\xi)\|_{L^2_{-s}} \leq C\left(\int_{\mathbb{R}} (1 + |x|)^{-2s} dx\right)^{\frac{1}{2}} \|f\|_{L^2_s} \end{aligned}$$

and hence the result. □

We next state the following properties of the resolvent operator:

Proposition 2.4.1. *For* $s > \frac{1}{2}$, $\operatorname{Im} z \neq 0$ *and* $f \in L^2_s$, *we have*

$$(H - z)A_\Xi(z) = (E_j(\xi) - z)\psi_j(\xi), \tag{2.38}$$

$$\widehat{R(z)f}(\Xi) = \frac{A^*_\Xi(\bar{z})f}{E_j(\xi) - \bar{z}}, \tag{2.39}$$

where $\hat{f}(\Xi) = (f, \psi_j(\xi))$ *is the unperturbed Fourier transform defined in* (2.5).

Proof. We have by construction

$$A_\Xi(z) = (I - R(z)Q)\psi_j(\xi)$$

so that

$$(H - z)A_\Xi(z) = (H_0 - z)\psi_\Xi = (E_j(\xi) - z)\psi_j(\xi).$$

This is (2.38). Thus, from the definition of the (unperturbed) Fourier transform (2.5),

$$\Big(f, \frac{(H - z)A_\Xi(z)}{E_j(\xi) - z}\Big) = \hat{f}(\Xi).$$

From this, we deduce

$$\widehat{R(\bar{z})f}(\Xi) = \Big(R(\bar{z})f, \frac{(H - z)A_\Xi(z)}{E_j(\xi) - z}\Big) = \Big(f, \frac{A_\Xi(z)}{E_j(\xi) - z}\Big) = \frac{A^*_\Xi(z)f}{E_j(\xi) - \bar{z}}.$$

□

We now prove the following eigenfunction expansion result. By proposition 2.3.1, we know that H has discrete point spectrum and each eigenvalue has a finite multiplicity. Let $\{\varphi_n\}$ be the countable set of orthonormalized eigenfunctions of H.

Theorem 2.4.1 (Eigenfunction Expansion Theorem)**.** *For* $f \in L^2_s \subset \mathcal{H}$, *we define* $\tilde{f}$ *as either one of the generalized Fourier transforms* $\tilde{f}^{\pm}$ *in* (2.34)*. Then we have the following*

Parseval relation:

$$\|f\|_{L^2} = \sum_{n=0}^{\infty} |(f, \varphi_n)|^2 + \int_{\mathbb{R}} \sum_{j \in J} |\tilde{f}(\Xi)|^2 d\xi. \tag{2.40}$$

Proof. For fixed ξ, let

$$J_{(\alpha,\beta)}(\xi) = \{j \in J |\ E_j(\xi) \in (\alpha, \beta)\}.$$

Let $[\alpha, \beta]$ contain no eigenvalue of H. We wish to show that

$$((E(\beta) - E(\alpha))f, f) = \int_{\mathbb{R}} \sum_{j \in J_{(\alpha,\beta)}(\xi)} |\tilde{f}(\Xi)|^2 d\xi, \tag{2.41}$$

and for an eigenvalue λ, that

$$((E(\lambda) - E(\lambda-))f, f) = \sum_{i=1}^{k} |(f, \varphi_{\lambda,i})|^2, \tag{2.42}$$

where $\{\varphi_{\lambda,i}\}$ are the orthonormalized eigenfunctions of H associated with the eigenvalue λ.

The second assertion (2.42) reduces to the well-known Parseval equality of Fourier series. To prove the first assertion, we first assume that

$$k_n < \alpha < \beta < k_{n+1}$$

for some $n \in \mathbb{Z}$ where $\{k_n\}$ label elements in Z_H, the union of Z defined in (2.7) with the collection of eigenvalues of H. We make use of

$$((E(\beta) - E(\alpha))f, f) = \frac{1}{2\pi i} \lim_{\eta \downarrow 0} \int_{\alpha}^{\beta} (R(\mu + i\eta)f - R(\mu - i\eta)f, f) d\mu.$$

Since $(E(\lambda)f, f)$ is absolutely continuous with respect to λ, we have using the resolvent equation $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$ and (2.39), that

$$\begin{aligned}
((E(\beta) - E(\alpha))f, f) &= \frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_\alpha^\beta 2i\eta(R(\mu - i\eta)R(\mu + i\eta)f, f)d\mu \\
=&\frac{1}{\pi}\lim_{\eta\downarrow 0}\int_\alpha^\beta \eta\|R(\mu + i\eta)f\|^2 d\mu = \frac{1}{\pi}\lim_{\eta\downarrow 0}\int_\alpha^\beta \eta\|R(\widehat{\mu + i\eta})f\|^2 d\mu \\
=&\frac{1}{\pi}\lim_{\eta\downarrow 0}\int_\alpha^\beta \int_{\mathbb{R}} \sum_j \eta \left|\frac{A_\Xi^*(\mu - i\eta)}{E_j(\xi) - \mu + i\eta} f\right|^2 d\xi d\mu,
\end{aligned} \tag{2.43}$$

using the Parseval identity for $\mathcal{F}$ in (2.5) (i.e., $\mathcal{F}$ is an isometry) and (2.39) for the last line. To analyze the above integral, we recall the result (see

Lemma 2.4.1 implies that $|A_\Xi^*(\mu - i\eta)f|^2 = |\tilde{f}(\Xi)|^2$ is continuous in Ξ and uniformly bounded on $[\alpha, \beta] \times [0, \eta_0]$. Proposition **??** then yields

$$\begin{aligned}
&\lim_{\eta\to 0}\frac{1}{\pi}\int_\alpha^\beta \eta\left|\frac{A_\Xi^*(\mu - i\eta)}{E_j(\xi) - \mu - i\eta} f\right|^2 d\mu \\
&= \lim_{\eta\to 0}\frac{1}{\pi}\int_\alpha^\beta \frac{\eta}{|E_j(\xi) - \mu|^2 + \eta^2}|A_\Xi^*(\mu - i\eta)f|^2 d\mu \ = \begin{cases} |\tilde{f}(\Xi)|^2, & j \in J_{(\alpha,\beta)}(\xi) \\ \frac{1}{2}|\tilde{f}(\Xi)|^2 & E_j(\xi) = \alpha \text{ or } \beta \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.44}$$

Moreover, this integral is uniformly bounded for $\Xi \in \mathbb{R} \times J$ and $(\mu, \eta) \in (\alpha, \beta) \times (0, 1)$.

We split J into two parts:

$$J_1 = \{j \in J |\ d(E_j(\xi), [\alpha, \beta]) \geq 1,\ \forall \xi \in \mathbb{R}\}, \quad J_0 := J \setminus J_1.$$

Here, $d(x, X)$ is Euclidean distance between a point $x \in \mathbb{R}$ and an interval $X \subset \mathbb{R}$. By hypothesis 2.2.1, J_0 is finite. Let $j \in J_0$. Using (2.44) for ξ such that $E_j(\xi) \in [\alpha - 1, \beta + 1]$, and $|E_j(\xi) - \mu + i\eta|^2 \geq C(1 + |E_j(\xi)|^2)$ otherwise, we deduce that there is a constant

$C = C(\alpha,\beta)$ independent of ξ, μ, η such that

$$\sum_{j\in J_0} \int_\alpha^\beta \eta \left| \frac{A_\Xi^*(\mu - i\eta)}{E_j(\xi) - \mu + i\eta} f \right|^2 d\mu \ \le \ \frac{C\|f\|_{L_s^2}}{1 + |E_j(\xi)|^2}. \tag{2.45}$$

For $j \in J_1$, we have by construction that $|E_j(\xi) - \mu - i\eta| \geq 1$. Notice that

$$\begin{aligned} \sum_{j\in J} |A_\Xi^*(z) f|^2 &= \sum_{j\in J} |(f, A_\Xi(z))_{L^2}|^2 = \sum_{j\in J} |(f, (I - R(z)Q)\psi_j(\xi))_{L^2}|^2 \\ &= \sum_{m\in M} |((I - R(\bar z)Q)f, \psi_j(\xi))_{L^2}|^2 = \|(I - \widehat{R(z)}Q)f\ (\xi, y)\|^2_{L^2(y)}. \end{aligned}$$

Since the operator $f \mapsto R(z)Qf$ from L_s^2 to $L^2_{-(s+h)}$ is uniformly bounded for $z \in J(\alpha,\beta)$, we deduce that

$$\sum_{j\in J_1} \int_\alpha^\beta \eta \left| A_\Xi^*(\mu - i\eta) f \right|^2 d\mu$$

is also uniformly bounded for $\xi \in \mathbb{R}$ and $(\mu,\eta) \in (\alpha,\beta)\times(0,1)$. Thus we conclude that there exists a constant $C = C(\alpha,\beta)$ independent of ξ, μ, η such that

$$\sum_{j\in J_1} \int_\alpha^\beta \eta \left| \frac{A_\Xi^*(\mu - i\eta)}{E_j(\xi) - \mu + i\eta} f \right|^2 d\mu \le \frac{C\|f\|_{L_s^2}}{1 + |E_j(\xi)|^2}. \tag{2.46}$$

The estimates in (2.45) and (2.46) are integrable in ξ by Hypothesis 2.2.1. Thus, by the dominated convergence theorem, we deduce from (2.43) and (2.44) that

$$\begin{aligned} ((E(\beta) - E(\alpha))f, f) &= \frac{1}{\pi} \int_{\mathbb{R}} \lim_{\eta\downarrow 0} \int_\alpha^\beta \sum_{j\in J} \eta \left| \frac{A_\Xi^*(\mu - i\eta)}{E_j(\xi) - \mu - i\eta} f \right|^2 d\mu d\xi \\ &= \int_{\mathbb{R}} \sum_{j\in J(\alpha,\beta)(\xi)} |\tilde f(\Xi)|^2 d\xi. \end{aligned} \tag{2.47}$$

This proves the first assertion (2.41) when

$$k_n < \alpha < \beta < k_{n+1}$$

for some $n \in \mathbb{Z}$, where $k_n < k_{n+1}$ are successive numbers in the union of Z defined in (2.7) with the collection of eigenvalues of H. By monotone convergence, we deduce that

$$((E(k_{n+1}-) - E(k_n+))f, f) = \int_{\mathbb{R}} \sum_{j \in J_{(k_n, k_{n+1})}(\xi)} |\tilde{f}(\Xi)|^2 d\xi.$$

This handles the absolutely continuous part of the spectrum of H. It remains to address the discrete set of points k_n, which as in Theorem 2.3.4, carries only point spectrum. This is taken care of by the first term on the right-hand side in (2.40) as in (2.42). This concludes the proof of the theorem. □

2.5 Application to Dirac operators with domain walls

The theory developed in section 2.2 requires that we prove the hypotheses [H1](o-iv), and in particular the spectral decompositions in (2.4) and (2.11). While such a verification can undoubtedly be performed for a large class of problems, some of the steps developed below to analyze the spectrum of H are intricate computationally and hence restricted to systems of Dirac operators similar to those considered in

We thus consider the Dirac operator

$$H = H_0 + Q, \qquad H_0 = D_x\sigma_1 + D_y\sigma_2 + m(y)\sigma_3 \tag{2.48}$$

where $D_x = -i\partial_x$ and $D_y = -i\partial_y$, where $\sigma_{1,2,3}$ are the standard Pauli matrices, and where $m(y)$ is a domain wall that we will take the form $m(y) - y$ equal to a bounded function to simplify the presentation. Here, Q is the operator of multiplication by $Q(x, y)$, which takes values in 2×2 Hermitian matrices.

It is convenient to recast H as UHU^* with U a unitary matrix so that UHU^*, still called

H, takes the form $H = H_0 + Q$, where

$$H_0 = D_x\sigma_3 - D_y\sigma_2 + m(y)\sigma_1 = \begin{pmatrix} -i\partial_x & \mathfrak{a} \\ \mathfrak{a}^* & i\partial_x \end{pmatrix}, \qquad \mathfrak{a} := \partial_y + m(y). \tag{2.49}$$

We recognize in $\mathfrak{a}$ the annihilation operator of the quantum harmonic oscillator when $m(y) = y$. We assume that the range of m is infinite, and for concreteness that:

Hypothesis 2.5.1. *We assume $m(y) - y$ is a bounded function.*

Under this hypothesis, the operators H and H_0 are unbounded elliptic self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with domains of definition $\mathfrak{D}(H) = \mathfrak{D}(H_0)$ the subspace of functions $(\psi_1,\psi_2)^t \in L^2(\mathbb{R}^2;\mathbb{C}^2)$ such that $\nabla\psi_j \in L^2(\mathbb{R}^2;\mathbb{C}^2)$ and $y\psi_j \in L^2(\mathbb{R}^2)$. This was denoted by the space H^1 with $\beta = 1$ and $s = 0$ in (2.2). We assume here that Q decays sufficiently rapidly in x as described in Hypothesis 2.3.1 for the above result to hold

We first observe that since the range of $m(y)$ is unbounded, standard results on Sturm Liouville operators show that $\mathfrak{a}^*\mathfrak{a}$ and $\mathfrak{a}\mathfrak{a}^*$ have a compact resolvent and hence discrete spectrum with simple eigenvalues. It is also straightforward to observe that $\mathfrak{a}$ admits a kernel in $L^2(\mathbb{R})$ of dimension one. The positive eigenvalues of $\mathfrak{a}^*\mathfrak{a}$ and $\mathfrak{a}\mathfrak{a}^*$ are the same and we thus get the existence of simple eigenvalues $\rho_0 = 0 < \rho_n < \rho_{n+1}$ for $n \geq 1$ with ρ_n/n converging to 2 as $n \to \infty$. Moreover, we have the existence of two $L^2(\mathbb{R}; dy)-$orthonormal bases $(\nu_n)_{n\geq 0}$ and $(\mu_n)_{n\geq 1}$ such that

$$\mathfrak{a}^*\mathfrak{a}\nu_n = \rho_n\nu_n, \quad n \geq 0; \qquad \mathfrak{a}\mathfrak{a}^*\mu_n = \rho_n\mu_n, \quad n \geq 1. \tag{2.50}$$

When $m(y) = y$, then ν_n and μ_n are both Hermite functions associated to the quantum harmonic oscillator. We also verify that (for $\|\cdot\|$ the standard $L^2(\mathbb{R}; dy)$ norm)

$$\|f\| + \|yf\| + \|D_y f\| \leq C\|\mathfrak{a}^* f\|, \qquad \|yf\| + \|D_y f\| \leq C(\|\mathfrak{a} f\| + \|f\|). \tag{2.51}$$

The asymmetry between the above two results stems from the fact that $\mathfrak{a}^*$ has trivial L^2-kernel while $\mathfrak{a}\nu_0 = 0$.

Define $\Pi_n^\nu = \nu_n \otimes \nu_n$ for $n \geq 0$ and $\Pi_n^\mu = \mu_n \otimes \mu_n$ for $n \geq 1$ (with $\Pi_0^\mu = 0$ to simplify notation). Then we observe that we have the spectral decomposition:

$$H_0^2 - \lambda^2 = \mathcal{F}_{\xi\to x}^{-1} \sum_{n\geq 0} \int_{\mathbb{R}} d\xi\ (\xi^2 + \rho_n - \lambda^2) \begin{pmatrix} \Pi_n^\nu & 0 \\ 0 & \Pi_n^\mu \end{pmatrix} \mathcal{F}_{x\to\xi}. \tag{2.52}$$

This provides the following explicit expression for the resolvent

$$R_0(\lambda) = (H_0 - \lambda)^{-1} = (H_0 + \lambda)(H_0^2 - \lambda^2)^{-1} \tag{2.53}$$

$$= (H_0 + \lambda)\mathcal{F}_{\xi\to x}^{-1} \sum_{n\geq 0} \int_{\mathbb{R}} d\xi\ (\xi^2 + \rho_n - \lambda^2)^{-1} \begin{pmatrix} \Pi_n^\nu & 0 \\ 0 & \Pi_n^\mu \end{pmatrix} \mathcal{F}_{x\to\xi}. \tag{2.54}$$

Estimates on $R_0(\lambda)$ may thus be obtained by applying $H_0+\lambda$ to the resolvent of the operator H_0^2. The above construction also shows that the eigenvalues of $\hat{H}_0^2(\xi)$ are given explicitly by $E_n^2(\xi) = \xi^2 + \rho_n$ for $n \geq 0$. We now show that $\pm E_n(\xi)$ are indeed eigenvalues of $\hat{H}_0(\xi)$. To simplify notation, we introduce the set M of indices $m = (\pm 1, n)$ for $n \geq 1$ and $0 \equiv (-1, 0)$ for $n = 0$. We then define the eigenvectors $\phi_m(\xi)$ for $m = (\pm, n) \in M$ as

$$\phi_m = \begin{pmatrix} \varphi_m \\ \psi_m \end{pmatrix}, \qquad E_m = \pm(\xi^2 + \rho_n)^{\frac{1}{2}}$$

with $\phi_0 = (\nu_0, 0)^t$ independent of ξ for $m = 0$ and for $n \geq 1$,

$$\phi_m(\xi) = c_n \begin{pmatrix} (E_m(\xi) + \xi)\nu_n \\ \rho_n \mu_n \end{pmatrix}, \qquad c_n^{-2} = 2E_m(\xi)(E_m(\xi) + \xi) > 0. \tag{2.55}$$

We verify that c_m is indeed defined and independent of $\pm$ and hence labeled c_n. Since the functions ν_n and μ_n form an orthonormal basis, we easily deduce that the functions ϕ_m also form an orthonormal basis of $L^2(\mathbb{R};\mathbb{C}^2)$.

From this completeness result, we deduce the spectral decomposition

$$H_0 = \mathcal{F}^{-1}_{\xi\to x} \sum_m \int_{\mathbb{R}} d\xi \; E_m(\xi)\,\Pi^\phi_m(\xi)\; \mathcal{F}_{x\to\xi}, \qquad \Pi^\phi_m(\xi) = \phi_m(\xi)\otimes\phi_m(\xi). \tag{2.56}$$

This provides an explicit expression for (2.4) in hypothesis [H1] with ψ_m defined by (2.3).

We finally turn to the construction of the generalized eigenfunctions (2.8) at a fixed energy level $E \in \mathbb{R}\backslash Z_D$, where for the Dirac operator, we deduce from the explicit expression in (2.55) that the set Z of critical values defined in (2.7) is given explicitly by

$$Z_D = \Big\{ \pm\sqrt{\rho_n};\ n\in\mathbb{N} \Big\}. \tag{2.57}$$

When $m(y) = y$ is a linear domain wall, then we verify that $\rho_n = 2n$. By hypothesis 2.5.1, we deduce from

We construct a basis of $L^2(\mathbb{R};\mathbb{C}^2)$ called $\phi_m(E)$ with a slight abuse of notation. The objective is to construct a basis of solutions of $(\hat{H}_0(\xi_m) - E)\phi_m(E) = 0$. The values ξ_m are defined explicitly by

$$\xi_m = \epsilon_m (E^2 - \rho_n)^{\frac12} \tag{2.58}$$

where $m = (\epsilon_m, n)$ and $(-1)^{\frac12} = i$. Thus ξ_m is real-valued for n sufficiently small and purely imaginary when $E^2 - \rho_n < 0$. Since $E \notin Z_D$, $E^2 - \rho_n \neq 0$. We then define

$$\phi_m(E) = c_n \begin{pmatrix} \sqrt{\rho_n}\mu_n \\ (E - \xi_m(E))\nu_n \end{pmatrix}, \qquad c_n^{-2} = \rho_n + |E - \xi_m|^2. \tag{2.59}$$

The functions $m \to \phi_m(y;E)$ form a basis of $L^2(\mathbb{R};\mathbb{C}^2)$ but no longer an orthonormal one. However, the orthonormalization of this basis is a bounded operator with bounded inverse as we now show.

When $m = (n, \epsilon_m)$ while $q = (p, \epsilon_q)$, we verify that $(\phi_m, \phi_q) = 0$ when $n \neq p$. This is a direct consequence of the orthogonality of the families ν_n and μ_n. However, for $q = m' := (m, -\epsilon_m)$, then we have

$$(\phi_m, \phi_{m'}) = \frac{\rho_n + (E - \epsilon_m)\overline{(E + \epsilon_m)}}{\sqrt{\rho_n + |E - \xi_m|^2} \cdot \sqrt{\rho_n + |E + \xi_m|^2}} = \frac{1 + \frac{\overline{E+\xi_m}}{E+\xi_m}}{2 + 2\frac{|E|^2+|\xi_m|^2}{E^2-\xi_m^2}}.$$

As a consequence, $|(\phi_m, \phi_{m'})| < \frac{1}{2}$ as we verify. The functions (ϕ_m) are thus linearly independent and form a basis of $L^2(\mathbb{R}; \mathbb{C}^2)$ by completeness of the families ν_n and μ_n. The procedure of orthonormalization of the (normalized) basis elements ϕ_m is therefore a operator of norm bounded by 2.

2.5.1 Estimates for unperturbed operator

We now verify that the assumptions made in Hypothesis 2.3.2 hold for the unperturbed Dirac operator.

Proposition 2.5.1. *Estimate (1.) in Hypothesis 2.3.2 holds for the Dirac operator.*

Proposition 2.5.2. *Estimate (2.) in Hypothesis 2.3.2 holds for the Dirac operator.*

The first proposition is useful for $s > \frac{1}{2}$ close to $\frac{1}{2}$ while the second estimate is useful for $s - 1 - \varepsilon > 0$.

Proposition 2.5.1. We introduce

$$u = (H_0 + \lambda)v, \quad v = (H_0^2 - \lambda^2)^{-1}(H_0 - \lambda)u.$$

We want to show that

$$\|v\|_{H^2_{-s}} \leq C\|(H_0 - \lambda)u\|_{L^2_s} = \|(H_0^2 - \lambda^2)v\|_{L^2_s}. \tag{2.60}$$

Using (3.5.2), we find

$$H_0^2 - \lambda^2 = \begin{pmatrix} D_x^2 + \mathfrak{a}^*\mathfrak{a} - \lambda^2 & 0 \\ 0 & D_x^2 + \mathfrak{a}\mathfrak{a}^* - \lambda^2 \end{pmatrix}.$$

Let $\{\nu_n\}$, $\{\mu_n\}$ be the (orthonormal basis of) eigenfunctions of $\mathfrak{a}^*\mathfrak{a}, \mathfrak{a}\mathfrak{a}^*$,

$$\mathfrak{a}^*\mathfrak{a}\nu_n = \rho_n\nu_n, \quad \mathfrak{a}\mathfrak{a}^*\mu_n = \rho_n\mu_n.$$

We also have $\mathfrak{a}\nu_0 = 0$. Consider

$$(D_x^2 + \mathfrak{a}^*\mathfrak{a} - \lambda^2)v_1 = g_1$$

with the expansion

$$v_1 = \sum_{n\geq 0} v_{1n}(x)\nu_n(y)$$

so that

$$(D_x^2 + \rho_n - \lambda^2)v_{1n} = g_{1n}$$

with obvious notation.

We then use

We then sum over n using the orthonormality of the families ν_n and μ_n to get

$$\|D_x^2 v_1\|_{L^2_{-s}} + \|v_1\|_{L^2_{-s}} \leq C\|(H_0^2 - \lambda^2)v_1\|_{L^2_s}.$$

Now we use the relation

$$\mathfrak{a}^*\mathfrak{a}v_1 = (H_0^2 - \lambda^2)v_1 + \lambda^2 v_1 - D_x^2 v_1$$

and the above inequality to deduce from (2.51) bounds for $D_y^2 v_1$ as well $y^2 v_1$ in weighted L^2

space. This implies that (2.60) holds for v_1. We perform the same calculation for

$$v_2 = \sum_{n\geq 1} v_{2n}(x)\mu_n(y).$$

Thus, (2.60) holds. It remains to apply $(H_0 + \lambda)$ to v, count derivatives and powers of y, and obtain

$$\|u\|_{H^1_{-s}} \leq C\|(H_0 - \lambda)u\|_{L^2_s}. \tag{2.61}$$

This concludes the derivation. □

While the proof of Proposition 2.5.1 was based on the spectral decomposition of $\mathfrak{a}^*\mathfrak{a}$ leading to that of $H_0^2 - \lambda^2$, we now base the proof of Proposition 2.5.2 when $\lambda \equiv E$ is real-valued on the direct plane wave expansion of $H_0 - \lambda$ using the eigen-elements $\phi_m(E)$ in (2.59).

We first need the following result on the operator D:

Lemma 2.5.1. *Let $u \in H^1(\mathbb{R})$, and $s \geq 0$.*

When $\lambda \in i\mathbb{R}$ and $\epsilon > 0$, there is $C = C(s, \epsilon)$ such that

$$\|u\|_{H^1_{s-1-\epsilon}} \leq C(1 + |\lambda|)\Big\|\Big(\frac{d}{dx} - \lambda\Big)u\Big\|_{L^2_s}. \tag{2.62}$$

When $\lambda \in \mathbb{R}$, there is $C = C(s)$ such that

$$\|u\|_{L^2_s} \leq \frac{C}{|\lambda|(|\lambda| \wedge 1)^s}\Big\|\Big(\frac{d}{dx} - \lambda\Big)u\Big\|_{L^2_s}, \quad \Big\|\frac{du}{dx}\Big\|_{L^2_s} \leq \frac{C(1 + |\lambda|)}{|\lambda|(|\lambda| \wedge 1)^s}\Big\|\Big(\frac{d}{dx} - \lambda\Big)u\Big\|_{L^2_s}. \tag{2.63}$$

Proof. We start with $\lambda \in \mathbb{R}$. Since $u \in H^1$, we have $\lim\limits_{|x|\to\infty} u(x) = 0$ and u may be expressed in two ways:

$$u(x) = \int_{-\infty}^{x} f(t)e^{\lambda(x-t)}dt = \int_{\infty}^{x} f(t)e^{\lambda(x-t)}dt, \qquad f(x) = \Big(\frac{d}{dx} - \lambda\Big)u(x).$$

When $x \leq 0$,

$$|u(x)|^2 \leq \int_{-\infty}^{x} |f(t)|^2(1+t^2)^s dt \cdot \int_{\infty}^{x} (1+t^2)^{-s} dt \leq C\|f\|_{L^2_s}^2 \cdot (1+x^2)^{1-2s}.$$

A similar estimate holds for $x \geq 0$. Multiplying with $(1+x^2)^{s-1-\epsilon}$ with $\epsilon > 0$, we have $u \in L^2_t$ and

$$\|u\|_{L^2_t} \leq C\|f\|_{L^2_s}, \quad t = s-1-\epsilon.$$

Together with $u' = f + \lambda u$, we obtained the inequality (2.62).

Consider now the case $\lambda \in \mathbb{R}$. Denote $\partial_x \equiv \frac{d}{dx}$. The second inequality in (2.63) is a consequence of the first one and the fact that $\partial_x u = (\partial_x - \lambda)u + \lambda u$. We may assume $\lambda \geq 0$ without loss of generality as the case $\lambda < 0$ then holds after $x \to -x$. Let us define $f = (\partial_x - \lambda)u$. Let $\varepsilon > 0$ and define $w_\varepsilon(x) = \langle \varepsilon x \rangle^s$. We find for $s \geq 0$ that

$$\left\| \frac{w'_\varepsilon}{w_\varepsilon} \right\|_\infty \leq C\varepsilon.$$

The result in L^2 (with norm $\|\cdot\|$) for $\beta = 0$ holds. Indeed in the Fourier domain,

$$\hat{u}(\xi) = \frac{1}{i\xi - \lambda}\hat{f}, \qquad \|u\| \leq \frac{1}{\lambda}\|f\|,$$

by the Parseval equality. For $s > 0$ we have

$$(\partial_x - \lambda)(w_\varepsilon u) = w_\varepsilon f - \frac{w'_\varepsilon}{w_\varepsilon} w_\varepsilon u.$$

Thus

$$\|w_\varepsilon u\| \leq \frac{1}{\lambda}(\|w_\varepsilon f\| + C\varepsilon \|w_\varepsilon u\|).$$

Choosing ε so that $C\varepsilon = \frac{1}{2}(\lambda \wedge 1)$, we deduce that

$$\|w_\varepsilon u\| \leq \frac{C}{\lambda}\|w_\varepsilon f\|.$$

When $\lambda \gtrsim 1$, we choose $\varepsilon \sim 1$ so that $w_\varepsilon \sim \langle x\rangle^s$ and the result is clear. When $\lambda \lesssim 1$, we use that

$$\lambda^s\langle x\rangle^s \sim \varepsilon^s\langle x\rangle^s \leq w_\varepsilon(x) \leq \langle x\rangle^s.$$

This shows that for $\lambda \lesssim 1$, $\|\langle x\rangle^s u\| \leq C\lambda^{-1-s}\|\langle x^s\rangle f\|$. □

Proposition 2.5.2. We use the basis $\phi_m(y;E)$ at a fixed $a < \lambda = E < b$ to decompose any smooth function (x, y) as

$$u(x,y) = \sum_{m\in M} u_m(x)\phi_m(y;E).$$

We showed that ϕ_m formed a basis equivalent to an orthonormal one in the sense that at each fixed x,

$$\|u(x,y)\|^2_{L^2_y} \approx \sum_m |u_m(x)|^2. \tag{2.64}$$

Here $a \approx b$ when for some constant $C > 0$ we have $C^{-1}a \leq b \leq Ca$. We also know that

$$(\hat{H}_0(\xi_m) - \lambda)\phi_m = 0.$$

Thus,

$$(H_0 - \lambda)u_m\phi_m = (D_x - \xi_m)u_m(\varphi_m, 0)^t - (D_x + \xi_m)(0, \psi_m)^t.$$

Hence using the above

$$\|(H_0 - \lambda)u\|^2_{L^2_s} \approx \sum_m \|(D_x - \xi_m)u_m\|^2_{L^2_s} + \|(D_x + \xi_m)u_m\|^2_{L^2_s}.$$

Recall that $\xi_m = \varepsilon_m(\lambda^2 - \lambda_n)^{\frac{1}{2}}$.

When $\xi_m \in \mathbb{R}$, we use (2.62) to deduce that for $s-1-\epsilon \geq 0$,

$$\|u_m\|_{H^1_{s-1-\epsilon}} \leq C\|(D_x - \xi_m)u_m\|_{L^2_s}.$$

When $\xi_m \in i\mathbb{R}$, we use (2.62) to deduce that

$$\|u_m\|_{H^1_s} \leq \frac{C(1+|\xi_m|)}{|\xi_m|(|\xi_m| \wedge 1)^s}\|(D_x - \xi_m)\|_{L^2_s}.$$

Since $a < \lambda < b$ and $(a,b) \cap Z_D = \emptyset$, we deduce from the definition of ξ_m that $|\xi_m(\lambda)|$ is bounded below uniformly in that interval. Note, however, that $|\xi_m(\lambda)|$ tends to 0 as λ approaches Z_D.

Summing these equalities over m and using (2.64), we deduce that

$$\|u\|^2_{L^2_{s-1-\epsilon}} + \|D_x u\|^2_{L^2_{s-1-\epsilon}} \leq C\|(H_0 - \lambda)u\|^2_{L^2_s}$$

for $s \geq 0$. The system $(H_0 - \lambda)u = f$ may be recast as

$$\mathfrak{a}^* u_2 = f_1 - (D_x - \lambda)u_1, \quad \mathfrak{a} u_1 = f_2 + (D_x + \lambda)u_2$$

From the properties of $\mathfrak{a}$ and $\mathfrak{a}^*$, this implies that

$$\|y u_j\|_{L^2_{s-1-\epsilon}} + \|D_y u_j\|_{L^2_{s-1-\epsilon}} \leq C\|f\|_{L^2_s} + \|u\|_{L^2_{s-1-\epsilon}} \leq C\|f\|_{L^2_s}.$$

This concludes the derivation of (2.26). □

2.5.2 Perturbed Dirac operator

We now derive estimates for the resolvent of the perturbed Dirac operator $H = H_0 + Q$.

Theorem 2.5.2. *Condition (3.) in Hypothesis 2.3.2 holds for the perturbed Dirac operator* *when Q satisfies the assumptions in Hypothesis 2.3.1.*

Before proving the theorem, we state the following intermediate result:

Lemma 2.5.2. *For $a < b$, $h > s > \frac{1}{2}$ and $\lambda \in J(a,b)$, and for $R > 0$ sufficient large, there exists a constant $C = C(s,a,b,R)$ such that*

$$\|u\|^2_{H^1_{-s}} \leq C\Big(\|(H-\lambda)u\|^2_{L^2_s} + \int_{|x|\leq R} |u(x)|^2 dx\Big) \tag{2.65}$$

for all $u \in H^1_s$ and $\lambda \in J(a,b)$.

Proof. From Proposition 2.5.1 and (2.25), we have for $s > \frac{1}{2}$ and $C_1 > 0$ that

$$\|u\|_{H^1_{-s}} \leq C_1\|(H_0-\lambda)u\|_{L^2_s} \leq C_1(\|(H-\lambda)u\|_{L^2_s} + \|Qu\|_{L^2_s}). \tag{2.66}$$

It remains to estimate $\|Qu\|_{L^2_s}$. Fix $R > 0$. We first estimate $\|Qu\|_{L^2_s(\{|x|\leq R\})}$. Since $|Q(x,y)|$ is bounded,

$$\|Qu\|_{L^2_s(\{|x|\leq R\})} \leq C\|u\|_{L^2(\{|x|\leq R\})}.$$

We next estimate $\|Qu\|_{L^2_s}(\{|x| \geq R\})$. By (2.24),

$$\|Qu\|^2_{L^2_s(\{|x|>R\})} \leq (1+R)^{2s-2h} \iint_{\{|x|\leq R\}} |u|^2|1+x^2|^{-s} dxdy \leq (1+R)^{2s-2h}\|u\|^2_{L^2_{-s}}. \tag{2.67}$$

Choosing R large enough so that $(1+R)^{2s-2h} < \frac{1}{2C_1}$, combined with (2.66) and (2.67), we obtain

$$\frac{1}{2}\|u\|^2_{H^1_{-s}} \leq 2C_1\Big(\|(H-\lambda)u\|^2_{L^2_s} + C_1(R)\int_{|x|\leq R} |u(x,y)|^2 dxdy\Big).$$

This proves (2.65). □

We are now ready to prove the main theorem.

Theorem 2.5.2. By Lemma 2.5.2, it suffices to show that for a fixed $R > 0$,

$$\|u\|_{L^2(|x|\leq R)} \leq C\|(H-\lambda)u\|_{L^2_s}, \tag{2.68}$$

with some constant C. Assuming the contrary, there is a sequence $\{u_n\}$ of H^1_s and a sequence $\{\lambda_n\}$ of $J_+(a,b)$ such that

$$\int_{|x|\leq R} |u_n(x,y)|^2 dxdy = 1,\ (H-\lambda_n)u_n \to 0 \text{ in } L^2_s. \tag{2.69}$$

Define $f_n = (H-\lambda_n)u_n$ and $g_n = f_n - Qu_n$. We may assume that $\lambda_n \to \lambda_0$ where $\lambda_0 \in [a,b]$ is a non-eigenvalue real number. Indeed, Im $\lambda_0 > \eta_0 > 0$ would imply that for n large enough, we would have $\|u_n\|_{L^2} \leq \eta_0^{-2}\|(H-\lambda_n)u_n\|_{L^2}$, which contradicts (2.69). From Lemma 2.5.2, there exists a constant C_1 such that

$$\|u_n\|^2_{H^1_{-s}} \leq C_1 \left(\|(H-\lambda_n)u_n\|_{L^2_s} + \|u_n\|_{L^2(|x|\leq R)}\right). \tag{2.70}$$

Thus $\{u_n\}$ is bounded in H^1_{-s}. So by Rellich's theorem we can select a subsequence of $\{u_n\}$ (which we still denote by $\{u_n\}$) which converges in $L^2_{\{|x|\leq R\}}$ for any $R > 0$. We denote this limit as u_0. Since Q is bounded, Qu_n also converges to Qu_0 in $L^2_{\{|x|\leq R\}}$. This implies that $Qu_n \to Qu_0$ in L^2_s by (2.67) and the fact that $\{u_n\}$ is bounded in H^1_{-s}. Thus $g_n \to g_0$ in L^2_s, and

$$(H_0 - \lambda_n)u_n = f_n - Qu_n = g_n. \tag{2.71}$$

By standard ellipticity results for the Dirac operator H_0, we have that

$$\|u\|_{H^1(|x|\leq R)} \leq C(R)\left(\|u\|_{L^2(|x|\leq R+1)} + \|H_0 u\|_{L^2(|x|\leq R+1)}\right). \tag{2.72}$$

Then by (2.71), (2.72) and $g_n \to g_0$ in L^2_s, we have

$$u_n \to u_0 \text{ in } H^1(|x| \le R). \tag{2.73}$$

Applying Lemma 2.5.2, we have $u_n \to u_0$ in $H^1_0(|x| \le R)$ so that

$$\int_{|x|\le R} |u_0(x,y)|^2 dxdy = 1, \quad (H - \lambda_0)u_0 = 0. \tag{2.74}$$

If we can show $u_0 \in L^2$, then $u_0 = 0$ since λ_0 is not an eigenvalue. By (2.71),

$$(H_0 - \lambda_0)u_n = g_n + (\lambda_n - \lambda_0)u_n. \tag{2.75}$$

As constructed in section **??**, $\{\phi_m(\lambda_0)\}$ which we abbreviate as ϕ_m below form a basis of $L^2(\mathbb{R}, \mathbb{C}^2)$. For $u \in H^1_s$ with $(H_0 - \lambda_0)u = f$, we use the decomposition

$$u = \sum_{m\in M}^{\infty} u_m(x)\phi_m(y).$$

We claim that

$$\Big(\frac{d}{dx} + i\xi_m\Big)u_m(x) = \begin{cases} -i\frac{\lambda}{\xi_m} f_m(x) + \frac{i(\xi_{m'}+\lambda)}{\xi_{m'}} f_{m'} & m \neq 0 \\ -if_m & m = 0. \end{cases} \tag{2.76}$$

It is enough to prove the result for $u \in C_c^\infty$ then pass to the limit since C_c^∞ is dense in H^1_s. By recalling $(H_0 + \lambda)(H_0 - \lambda_0)$ in (3.5.2) and that $(\hat{H}_0(\xi_m) - \lambda_0)\phi_m = 0$, it follows that

$$(H_0 + \lambda)(H_0 - \lambda)u = \sum_m (-u''_m(x) - \xi_m^2)\phi_m,$$

$$(H_0 + \lambda)f = (-if'_m(x) - \xi_m f_m(x))\sigma_3\phi_m.$$

A linear decomposition gives that,

$$\sigma_3\phi_m = -\frac{\lambda}{\xi_m}\phi_m + \frac{\xi_m+\lambda}{\xi_m}\phi_{m'},\ m\neq 0;\ \sigma_3\phi_m = -\phi_m,\ m=0 \tag{2.77}$$

where $\{m, m'\} = (\pm, n)$, so that $\xi_{m'} = -\xi_m$. Since $u_m, f_m, f_{m'}$ all have limit 0 at $\pm\infty$, we deduce (2.76).

Applying (2.76) to (2.75), for $m = 0$, gives

$$I_{n,0} := (u_{n,0}, g_{n,0}) = -\xi_0\|u_{n,0}\|_{L^2} - \overline{\lambda_n - \lambda_0}\|u_{n,0}\|_{L^2}.$$

For $m \neq 0$, we write, for $m \neq 0$,

$$\frac{d}{dx}\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix} = \begin{pmatrix} -i\xi_m & 0 \\ 0 & i\xi_m \end{pmatrix}\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix} + A\begin{pmatrix} g_{n,m} \\ g_{n,m'} \end{pmatrix} + (\lambda_n - \lambda_0)A\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}$$

where $A = \begin{pmatrix} -\frac{i\lambda_0}{\xi_m} & \frac{i(-\xi_m+\lambda_0)}{-\xi_m i} \\ \frac{i(\xi_m+\lambda_0)}{\xi_m i} & \frac{i\lambda_0}{\xi_m} \end{pmatrix}$. Each entry of A is bounded because ξ_m is bounded away from 0 and $|\xi_m|$ is unbounded as n grows. Thus

$$I_{n,m} := \left(\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}, \sigma_3 A\begin{pmatrix} g_{n,m} \\ g_{n,m'} \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}, \frac{d}{dx}\sigma_3\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix} + \begin{pmatrix} i\xi_m & 0 \\ 0 & i\xi_m \end{pmatrix}\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix} - (\lambda_n - \lambda_0)\sigma_3 A\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}\right)$$

$$= i\xi_m\Big(\|u_{n,m}\|_{L^2}^2 + \|u_{n,m'}\|_{L^2}^2\Big) - \overline{\lambda_n - \lambda_0}\left(\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}, \sigma_3 A\begin{pmatrix} u_{n,m} \\ u_{n,m'} \end{pmatrix}\right).$$

Since $\lambda_n \to \lambda_0$ and A is bounded as ξ_m is away from 0, so for n sufficiently large, there exists a positive constant $U > 0$ such that for all n, m,

$$\operatorname{Im} I_{n,m} - \operatorname{Re} I_{n,m} \geq U \left(\|u_{n,m}\|_{L^2}^2 + \|u_{n,m'}\|_{L^2}^2 \right). \tag{2.78}$$

Summing $I_{n,m}$ over m,

$$\sum_{m \in M,\ \epsilon_m > 0} I_{n,m} \xrightarrow{n \to \infty} \left(u_0, g_0^{\text{twist}} \right) \tag{2.79}$$

where $\begin{pmatrix} g_{n,m}^{\text{twist}} \\ g_{n,m'}^{\text{twist}} \end{pmatrix} = \sigma_3 A \begin{pmatrix} g_{n,m} \\ g_{n,m'} \end{pmatrix}$, since $g_n^{\text{twist}} \in L_s^2$ and $g_n^{\text{twist}} \to g_0^{\text{twist}}$ in L_s^2. Thus, by (2.78), (2.79),

$$U\|u_n\|_{L^2}^2 \leq \operatorname{Im} \sum_{m \in M,\ \epsilon_m > 0} I_{n,m} - \operatorname{Re} \sum_{m \in M,\ \epsilon_m > 0} I_{n,m} \xrightarrow{n \to \infty} \operatorname{Im} \left(u_0, g_0^{\text{twist}} \right) - \operatorname{Re} \left(u_0, g_0^{\text{twist}} \right). \tag{2.80}$$

We deduce that $\{u_n\}$ is bounded in L^2, which implies that $u_0 \in L^2$ and hence $u_0 = 0$. This concludes the proof of the theorem. □

2.5.3 Scattering matrix and conductivity

The objective of this section is to prove [H3] for the Dirac operator. Fix an energy $E \in \mathbb{R} \backslash Z_H$. We decompose the generalized eigenfunction $\psi_m^Q(x, y; E)$ in the basis of $\phi_m = \phi_m(y; \xi_m(E))$ as

$$\psi_m^Q(x, y; E) = \sum_{q \in M} A_q(x) \phi_q(y) = \sum_{q \in M} B_q(x) e^{i \xi_q x} \phi_q(y). \tag{2.81}$$

The decomposition consists of finitely many propagating modes and countably many evanescent modes. We wish to show that in the limit $x \to \pm\infty$, ψ_m^Q is well approximated as a linear combination of propagating modes.

We define $M(E) \subset M$ as the subset of propagating modes at a fixed energy E. The cardinality of $M(E)$ was denoted by $\mathrm{M}(E)$ in Hypothesis [H1](iv).

Proposition 2.5.3. *The generalized eigenfunctions satisfy the following approximate decomposition:*

$$\psi_m^Q(x,y) \approx \sum_{q\in M(E)} \alpha_{mq}^{\pm} e^{i\xi_q x}\phi_q(y), \tag{2.82}$$

with respect to norm $|||u||| = \max_{x\in\mathbb{R}} \|u(x,\cdot)\|_{L^2(\cdot)}$*; see* (2.86) *below for a more precise statement.*

Proof. $\psi_m^Q(x,y;E)$ satisfies that $(H_0 - E)\,\psi_m^Q = -Q\psi_m^Q$. We decompose $-Q\psi_m^Q$ in the basis $\phi_m(y;\xi_m(E))$ as

$$-Q\psi_m^Q = \sum_{q\in M} a_q(x)\phi_q(y), \qquad a_q(x) = (-Q\psi_m^Q,\ \phi_q(y))_y.$$

Since $\psi_m^Q \in H^1_{-s}$ for $s > \frac{1}{2}$ and $|Q(x,y)| = O(|x|^{-h})$ for some $h > 1$, then $Q\psi_m^Q \in L_t^2$ for some $t > \frac{1}{2}$ and hence $a_q(x) \in L_t^2$.

Let $p = (-\epsilon_q, n)$ be the conjugate of $q = (\epsilon_q, n)$. Then it holds that

$$(H_0 - E)\left[A_q(x)\phi_q(y) + A_p(x)\phi_p(y)\right] = a_q(x)\phi_q(y) + a_p(x)\phi_p(y). \tag{2.83}$$

The proof is then based on a direct computation. From (2.83) and by use of $(\hat{H}_0(\xi_m) - E)\phi_m(E) = 0$ and (2.77), we have

$$\begin{aligned}\left[-iA_q'(x) - \xi_q A_q(x)\right]\sigma_3\phi_q + \left[-iA_p'(x) - \xi_p A_p(x)\right]\sigma_3\phi_p &= a_q(x)\phi_q(y) + a_p(x)\phi_p(y)\\ = a_q(x)\left[\frac{E}{\xi_q}\sigma_3\phi_q + (1 - \frac{E}{\xi_q})\sigma_3\phi_p\right] &+ a_p(x)\left[(1 + \frac{E}{\xi_q})\sigma_3\phi_q - \frac{E}{\xi_q}\sigma_3\phi_p\right].\end{aligned}$$

Thus

$$-iA_q'(x) - \xi_q A_q(x) = \frac{E}{\xi_q} a_q(x) + \left(1 + \frac{E}{\xi_q}\right) a_p(x),$$

which implies that

$$B_q'(x) = ie^{-i\xi_q x} \left[\frac{E}{\xi_q} a_q(x) + \left(1 + \frac{E}{\xi_q}\right) a_p(x)\right].$$

When $\xi_q \in \mathbb{R}$, since $a_q, a_p \in L_s^2$ for some $s > \frac{1}{2}$, then $a_q, a_p \in L^1$, and hence $B_q(x)$ converges to two constants as $x \to \pm\infty$, i.e., $\lim_{x \to \pm\infty} B_q(x) = \alpha_{mq}^{\pm}$. Moreover,

$$|B_q(x) - \alpha_{mq}^{\pm}| = O(|x|^{\frac{1}{2}-t}) \text{ as } x \to \pm\infty. \tag{2.84}$$

When $\xi_q \in i\mathbb{R}$, then by applying Lemma 2.5.1, it follows that there exists a constant independent of q such that $\|A_q(x)\|_{H_t^1} \leq C\|a_q\|_{L_t^2} + C\|a_p\|_{L_t^2}$ so $\lim_{|x|\to\infty} A_q(x) = 0$. Moreover, $(1+|x|)^t A_q(x) \in H^1$, and thus by Sobolev's inequality, there exists a constant C_1 such that

$$(1 + |x|)^t |A_q(x)| \leq C_1 \|a_q\|_{L_t^2}.$$

Together with the fact that ϕ_q has L^2 norm 1 for all q, we derive that

$$\Big\| \sum_{q \in M(E)} A_q(x)\phi_q(y) \Big\|_{L_y^2} \leq \frac{C_2}{(1+|x|)^t} \sum_{\xi_q \in i\mathbb{R}} \|a_q\|_{L_s^2}. \tag{2.85}$$

Therefore, by (4.30) and (4.31), we have that as $x \to \pm\infty$,

$$\Big\| \psi_m^Q(x,y) - \sum_{m \in M(E)} \alpha_{mq}^{\pm} \phi_q(y) \Big\|_{L_y^2} = O(|x|^{\frac{1}{2}-t}) \text{ as } x \to \pm\infty. \tag{2.86}$$

This concludes the proof of the proposition. □

This concludes our analysis of the edge conductivity for the Dirac operator. This establishes for this model the main result of this paper, namely Theorem 2.2.2, relating the two natural notions associated to asymmetric transport: edge conductivity and the difference of transmissions in a scattering experiment.

CHAPTER 3

A MIXED TYPE GENERALIZED KIMURA OPERATOR

3.1 Introduction

Let L be a mixed type generalized Kimura diffusion operator, acting on functions defined on a 2-dimensional manifolds of corner P.

A paracompact Hausdorff topological space P is a 2-dimensional manifold with corners if for every $p \in P$, there is a neighborhood $\mathcal{U}_p$ and a homeomorphism ψ_p from $\mathcal{U}_p$ to a neighborhood of 0 in $\mathbb{R}_+^l \times \mathbb{R}^{2-l}$ for some $l \in \{0, 1, 2\}$, with $\psi_p(p) = 0$ and the overlap maps are diffeomorphisms. (Recall that a mapping between two relatively open sets in $\mathbb{R}^n \times \mathbb{R}^{N-n}$ is a diffeomorphism if it is the restriction of a diffeomorphism between two absolute open sets in $\mathbb{R}^N$.) Specifically, if $\psi_p : \mathcal{U}_p \to \mathcal{V}_p$ is the homeomorphism, then for $p \neq q$:

$$\psi_p \circ \psi_q^{-1} : \psi_q \left(\mathcal{U}_q \cap \mathcal{U}_p\right) \longrightarrow \psi_p \left(\mathcal{U}_q \cap \mathcal{U}_p\right)$$

is a diffeomorphism. If such a map ψ_p exists, we say that the point p is an interior point if $l = 0$, an edge point if $l = 1$, a corner if $l = 2$. The codimension l is well defined after imposing smoothness structures. It is due to the fact that the wedges $\{(r, \theta) : r \geq 0, 0 \leq \theta \leq A\}$ with various angles (acute angle, π, obtuse angle, 2π) are different diffeomorphism classes. The definition of manifold with corners excludes the wedge with obtuse angles, hence a non-convex polyhedron appears as the simplest counterexample.

Definition 3.1.1. *Let P be a two-dimensional compact manifold with corners. A second order operator L defined on P is called a generalized Kimura diffusion operator of second kind if it satisfies the following set of conditions:*

1. *L is elliptic in the interior of P.*

2. *If q is an edge point, then there are local coordinates (x, y) so that in the neighborhood*

$$\mathcal{U} = \{0 \leq x < 1,\ |y| < 1\}$$

the operator takes one of the following two forms:

$$L = ax\partial_x^2 + bx\partial_{xy} + c\partial_y^2 + d\partial_x + e\partial_y \tag{2a}$$

$$L = ax^2\partial_x^2 + bx\partial_{xy} + c\partial_y^2 + dx\partial_x + e\partial_y. \tag{2b}$$

We assume that all coefficients $a(x, y), b(x, y), c(x, y), d(x, y), e(x, y)$ lie in $C^\infty(\mathcal{U})$. We call q a regular edge point, infinity edge point, respectively.

3. *If q is a corner, then there are local coordinates (x, y) so that in the neighborhood*

$$\mathcal{U} = \{0 \leq x < 1,\ 0 \leq y < 1\}$$

the operator takes one of the following three forms:

$$L = ax\partial_x^2 + bxy\partial_{xy} + cy\partial_y^2 + d\partial_x + e\partial_y \tag{2c}$$

$$L = ax^2\partial_x^2 + bxy\partial_{xy} + cy\partial_y^2 + dx\partial_x + e\partial_y \tag{2d}$$

$$L = ax^2\partial_x^2 + bxy\partial_{xy} + cy^2\partial_y^2 + dx\partial_x + ey\partial_y. \tag{2e}$$

We assume that all coefficients $a(x, y), b(x, y), c(x, y), d(x, y), e(x, y)$ lie in $C^\infty(\mathcal{U})$. We call q a regular regular corner, mixed corner, infinity corner, respectively.

4. *The vector field is inward pointing at edge points of type (2a) and corners of type (2c), vertical at edge points of (2b), vertical up at corners of type (2d).*

5. *$a(x, y),\ c(x, y)$ are strictly positive on P.*

When the coefficients of the normal part of the second order term vanish exactly to order one along all the boundary components, L is the *generalized Kimura operators* as introduced by C. Epstein and R. Mazzeo in

Let E be a regular edge of P, i.e., so that all the points on E are of type 2a. We say L is tangent to E if at any point p on E, the vector field perpendicular to the edge vanishes at p, i.e., $d(p) = 0$, and L is transverse to E if there exists a $c_E > 0$ such that $d(p) > c_E$.

The results in this chapter are derived under the following assumption:

Assumption 3.1.1. *L is either tangent or transverse to any regular edge.*

In this case we say p is a *tangent point* if it lies on one tangent edge, otherwise we say it is a *transverse point*.

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Based on series expansion of fundamental solutions of model operators and the above result on the heat equation, our second main result is the existence and regularity results of a heat kernel for L. We let P^{reg} denote the union of $\mathring{P}$ and regular edge points and regular corner points (see Definition 3.1.1).

Theorem 3.1.1. *The global heat kernel $H_t(d_1, d_2, l_1, l_2) \in C^\infty(P^{reg} \times \mathring{P} \times (0, \infty))$ of the full operator L exists and for $f \in C^0(P)$, then*

$$v_f := \int_P H_t(d_1, d_2, l_1, l_2) f(l_1, l_2) dl_1 dl_2$$

is the solution of $(\partial_t - L)v_f = 0$ with $v_f(0, \cdot, \cdot) = f$.

This heat kernel is smooth in (d_1, d_2) when $(d_1, d_2) \in P^{reg}$. In other words, this includes source contributions at (d_1, d_2) any point on the regular part of bP. When (d_1, d_2) is on the infinity edge, we may prove that $H_t(d_1, d_2, l_1, l_2)$ is the product of a delta function on the infinity edge and a one-dimensional heat kernel along that edge; we do not present the details here. The diffusion coefficients vanishing to second order in the normal direction essentially imply that the infinity edge as the name indicates is indeed at infinity in the following sense:

any diffusion starting from $\mathring{P}$ would never reach the infinity edge, while a diffusion starting from infinity edge would never enter $\mathring{P}$.

Comparison with previous research. There is a rich literature addressing the fundamental solution of the Kimura operator. In

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Outline of this paper. The plan of this chapter is as follows. In section **??** we introduce the degenerate Hölder space associated with L. The operator L is modeled at different boundary points by the model operator L_M acting on the model spaces. In section 3.3 we derive the explicit fundamental solutions of model operators and make a careful analysis of the solution operator in the degenerate Hölder spaces. After this, in section 3.4 we prove the existence of solutions of the equation

$$(\partial_t - L)u = g \text{ in } P \times (0, T],\ u(0, p) = f(p)$$

with data f, g in the degenerate Hölder spaces. In section 3.5 we establish the existence and regularity results of the heat kernel of L.

Notation 1. *In the following we denote*

- P*: a compact two-dimensional manifold with corners*
- E_{reg}*: the union of regular edge points*
- E_∞*: the union of infinity edge points*
- C_{reg}*: the set of regular corners*
- C_{mix}*: the set of mixed corners*
- C_∞*: the set of infinity corners*
- P^{reg}*:* $\mathring{P} \cup E_{reg} \cup C_{reg}$

3.2 Hölder Weighted Space

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Definition 3.2.1. *We define the metric spaces*

1. S_{c_reg}: $(\mathbb{R}^2_+, d)$ *equipped with the norm*

$$d((x_1,x_2),(x_1',x_2')) = 2|\sqrt{x_1} - \sqrt{x_1'}| + 2|\sqrt{x_2} - \sqrt{x_2'}|.$$

2. S_{c_mix}: $(\mathbb{R}^2_+, d)$ *equipped with the norm*

$$d((x,y),(x',y')) = 2|\sqrt{x} - \sqrt{x'}| + |lny - lny'|.$$

3. S_{c_∞}: $(\mathbb{R}^2_+, d)$ *equipped with the norm*

$$d((y_1,y_2),(y_1',y_2')) = |lny_1 - lny_1'| + |lny_2 - lny_2'|.$$

4. S_{e_reg}: $(\mathbb{R}_+ \times \mathbb{R}, d)$ *equipped with the norm*

$$d((x,y),(x',y')) = 2|\sqrt{x} - \sqrt{x'}| + |y - y'|.$$

5. S_{e_∞}: $(\mathbb{R} \times \mathbb{R}_+, d)$ *equipped with the norm*

$$d((y_1,y_2),(y_1',y_2')) = |y_1 - y_1'| + |lny_2 - lny_2'|.$$

We use the (x,y) notation above: L is Kimura in the direction of x, and is elliptic or quadratic in the direction of y. To unify the notation, we make the following convention of first tangential derivative of y:

$$D_y u = \begin{cases} \partial_y u & L \text{ is elliptic in } y, i.e. 3 \\ y\partial_y u & L \text{ is quadratic in } y, i.e. 1,2,4 \end{cases}.$$

Definition 3.2.2. *Let S represents one of the metric spaces above. We denote*

1. *$\dot{C}^k(S)$: the closure in $C^k(S)$ of compactly supported smooth functions*
2. *$\dot{C}^{k,2}(S)$: the closure in $C^k(S)$ of compactly supported smooth functions with respect to the norm:*

$$||f||_{k,2} := ||f||_{C^{k-1}} + \sup_{|\alpha|+|\beta|=k} ||(\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta} f||_2,$$

$$||f||_2 := ||f||_\infty + ||(\partial_x f, \mathbf{D_y} f)||_\infty + \sum_{|\alpha|+|\beta|=2} ||(\sqrt{\mathbf{x}}\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta} f||_\infty.$$

For $0 < \gamma < 1$, set the norm

$$[f]_\gamma := \sup_{\mathbf{x} \neq \mathbf{x}'} \frac{|f(\mathbf{x}) - f(\mathbf{x}')|}{d(\mathbf{x}, \mathbf{x}')^\gamma},$$

$$[f]_{2+\gamma} := [f]_\gamma + [(\partial_{\mathbf{x}} f, \mathbf{D_y} f)]_\gamma + \sum_{|\alpha|+|\beta|=2} [(\sqrt{\mathbf{x}}\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta} f]_\gamma.$$

The space $C^{k,\gamma}(S)$, $C^{k,2+\gamma}(S)$ are the subspace of $\dot{C}^k(S)$, $\dot{C}^{k,2}(S)$ consisting of functions f for which the norm

$$||f||_{k,\gamma} := ||f||_{C^k} + \sup_{|\alpha|+|\beta|=k} [\partial_{\mathbf{x}}^{\alpha} \mathbf{D}_{\mathbf{y}}^{\beta} f]_\gamma,$$

$$||f||_{k,2+\gamma} := ||f||_{C^k} + \sup_{|\alpha|+|\beta|=k} [\partial_{\mathbf{x}}^{\alpha} \mathbf{D}_{\mathbf{y}}^{\beta} f]_{2+\gamma}.$$

are finite, respectively.

Similarly we define the parabolic Hölder spaces.

Definition 3.2.3. *We denote*

1. $\dot{C}^{k,\frac{k}{2}}(S\times[0,T])$*: the closure in* $C^{k,\frac{k}{2}}(S\times[0,T])$ *of compactly supported smooth functions*

2. $\dot{C}^{k+2,\frac{k}{2}+1}(S\times[0,T])$*: the closure in* $C^{k,\frac{k}{2}}(S\times[0,T])$ *of compactly supported smooth functions with respect to the norm:*

$$||f||_{k+2,\frac{k}{2}+1} := ||f||_{C^{k,\frac{k}{2}}} + \sup_{|\alpha|+|\beta|+2|j|=k} ||(\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta}\partial_t^j f||_{2,1},$$

$$||f||_{2,1} := ||f||_{\infty} + ||(\partial_t f, \partial_{\mathbf{x}} f, \mathbf{D_y} f)||_{\infty} + \sup_{|\alpha|+|\beta|=2} |||(\sqrt{\mathbf{x}}\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta} f||_{\infty}.$$

For $0<\gamma<1$*, set the norm*

$$[f]_{\gamma} := \sup_{(t,\mathbf{x})\neq(s,\mathbf{x}')} \frac{|f(t,\mathbf{x})-f(s,\mathbf{x}')|}{\left(d(\mathbf{x},\mathbf{x}')+\sqrt{|t-s|}\right)^{\gamma}},$$

$$[f]_{2+\gamma} := [f]_{\gamma} + [(\partial_t f, \partial_{\mathbf{x}} f, \mathbf{D_y} f)]_{\gamma} + \sum_{|\alpha|+|\beta|=2} [(\sqrt{\mathbf{x}}\partial_{\mathbf{x}})^{\alpha}(\mathbf{D_y})^{\beta} f]_{\gamma}.$$

The space $C^{k,\gamma}(S\times[0,T])$*,* $C^{k,2+\gamma}(S\times[0,T])$ *are the subspaces of* $\dot{C}^{k,\frac{k}{2}}(S\times[0,T])$*,* $\dot{C}^{k+2,\frac{k}{2}+1}(S\times[0,T])$ *consisting of functions* f *for which the norm*

$$||f||_{k,\gamma} := ||f||_{C^{k,\frac{k}{2}}} + \sup_{|\alpha|+|\beta|+2j=k} [\partial_{\mathbf{x}}^{\alpha}\mathbf{D}_{\mathbf{y}}^{\beta}\partial_t^j f]_{\gamma},$$

$$||f||_{k,2+\gamma} := ||f||_{C^{k,\frac{k}{2}}} + \sup_{|\alpha|+|\beta|+2j=k} [\partial_{\mathbf{x}}^{\alpha}\mathbf{D}_{\mathbf{y}}^{\beta}\partial_t^j f]_{2+\gamma}$$

are finite, respectively.

3.3 Model Operator

3.3.1 Fundamental Solutions

1. Analysis of L_M **at** C_{reg} (

2. Analysis of L_M at E_{reg} If the boundary point $q \in E_{reg}$, then there is a neighborhood U_q of q and smooth local coordinates (x, y) centered at q, in terms of which L takes the form

$$L = x\partial_x^2 + a(x,y)\partial_y^2 + b(x,y)\partial_{xy} + d(x,y)\partial_x + e(x,y)\partial_y \tag{3.1}$$

where $a(0,0) = 1$. We introduce the model operator in the space S_{e_reg}:

$$L_M = x\partial_x^2 + \partial_y^2 + d\partial_x.$$

The solution kernel of L_M is

$$K_t(x, y, x_1, y_1) = p_t^d(x, x_1)k_t^e(y, y_1)$$

where $k_t^e(y, y_1) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(y-y_1)^2}{4t}}$ is the heat kernel.

3. Analysis of L_M at C_{mix} If the boundary point $q \in C_{mix}$, then there is a neighborhood U_q of q and smooth local coordinates (x, y) centered at q, in terms of which L takes the form

$$L = a(x,y)x\partial_x^2 + b(x,y)y^2\partial_y^2 + c(x,y)xy\partial_{xy} + d(x,y)\partial_x + e(x,y)y\partial_y \tag{3.2}$$

where $a(0,0) = 1$. We introduce the model operator in the space S_{c_mix}:

$$L_M = x\partial_x^2 + b\partial_y^2 + d\partial_x + by\partial_y$$

where $b = b(0,0)$. The solution kernel of L_M is

$$K_t(x, y, x_1, y_1) = p_t^d(x, x_1)k_{bt}^{e'}(y, y_1)$$

where $k_{bt}^{e'}(y, y_1) = \frac{1}{\sqrt{4\pi bt}}e^{-\frac{(\ln y - \ln y_1)^2}{4bt}}\frac{1}{y_1}$.

4. Analysis of L_M at E_∞ If the boundary point $q \in E_\infty$, then there is a neighborhood U_q of q and smooth local coordinates (x, y) centered at q, in terms of which L takes the form

$$L = a(y_1, y_2)\partial_{y_1}^2 + b(y_1, y_2)y_2^2\partial_{y_2}^2 + c(y_1, y_2)y_1y_2\partial_{y_1y_2} + d(y_1, y_2)\partial_{y_1} + e(y_1, y_2)y_2\partial_{y_2} \quad (3.3)$$

where $a(0, 0) = 1$. We introduce the model operator in the space S_{e_∞}:

$$L_M = \partial_{y_1}^2 + by_2^2\partial_{y_2}^2 + by_2\partial_{y_2}$$

where $b = b(0, 0)$. The solution kernel of L_M is

$$K_t(y_1, y_2, y_1', y_2') = k_t^e(y_1, y_1')k_{bt}^{e'}(y_2, y_2').$$

5. Analysis of L_M at C_∞ If the boundary point q is in C_∞, then there is a neighborhood U_q of q and smooth local coordinates (x, y) centered at q, in terms of which L takes the form

$$L = a(y_1, y_2)y_1^2\partial_{y_1}^2 + b(y_1, y_2)y_2^2\partial_{y_2}^2 + c(y_1, y_2)y_1y_2\partial_{y_1y_2} + d(y_1, y_2)y_1\partial_{y_1} + e(y_1, y_2)y_2\partial_{y_2}. \quad (3.4)$$

We introduce the model operator in the space S_{c_∞}:

$$L_M = ay_1^2\partial_{y_1}^2 + by_2^2\partial_{y_2}^2 + ay_1\partial_{y_1} + by_2\partial_{y_2}$$

where $a = a(0, 0), b = b(0, 0)$. The solution kernel of L_M is

$$K_t(y_1, y_2, y_1', y_2') = k_{at}^{e'}(y_1, y_1')k_{bt}^{e'}(y_2, y_2').$$

Definition 3.3.1. *The coordinates and forms of L introduced above are called* ***local adapted coordinates*** *and* ***local normal forms centered at*** *q.*

3.3.2 Hölder Estimates

Let S be one of the spaces in $C_{reg}, C_{mix}, C_{\infty}, E_{reg}, E_{\infty}$.

Proposition 3.3.1. *Let $k \in \mathbb{N}$, $R > 0$, and $0 < \gamma < 1$, assume that $f \in C^{k,\gamma}(S)$ and f is supported in $B_R^+(\boldsymbol{0})$. The solution v to*

$$(\partial_t - L_M)v(t,x,y) = 0,\ v(0,x,y) = f \tag{3.5}$$

belongs to $C^{k,\gamma}(S \times [0,T])$, and there is a constant $C_{k,\gamma,R}$ so that

$$||v||_{k,\gamma} \leq C_{k,\gamma,R}||f||_{k,\gamma}.$$

If $f \in \mathcal{C}^{k,2+\gamma}(S)$, then

$$||v||_{k,2+\gamma} \leq C_{k,\gamma,R}||f||_{k,2+\gamma}.$$

Proof. At C_{reg} and E_{reg}, these are known results in

Let $f \in C^{k,\gamma}(\mathbb{R}_+ \times \mathbb{R})$, we first decompose f as

$$f = f_1 + f_2 = f_1 + f(x,-\infty)\chi(y)$$

where $\chi \in \mathcal{C}_c^{\infty}([-\infty,\infty)),\ \chi(-\infty) = 1$ such that $||f_2||'_{k,\gamma} \leq 2||f||'_{k,\gamma}$. Then $f_1 \in C^{k,\gamma}(\mathbb{R}_+ \times \mathbb{R})$ vanishes at infinity, for which we can apply

This leaves the Cauchy problem with initial data f_2. We begin by writing

$$K_t f_2(x,y) = \int_S K_t(x,y,x_1,y_1)f(x_1,0)\chi(y_1)dx_1dy_1$$
$$= \left(\int_0^{\infty} p_t^d(x,x_1)f(x_1,0))dx_1\right)\left(\int_{-\infty}^{\infty} k_t^e(y,y_1)\chi(y_1)dy_1\right) =: I_1(t,x) \cdot I_2(t,y).$$

Since $f(x,0) \in C^{k,\gamma}(\mathbb{R}_+), \chi(y) \in C^{k,\gamma}(\mathbb{R})$, then $I_1(t,x) \in C^{k,\gamma}(\mathbb{R}_+ \times [0,T]), I_2(t,y) \in$

$C^{k,\gamma}(\mathbb{R} \times [0,T])$ are solutions to one-dimensional Cauchy problem respectively. Moreover, since

$$\lim_{y \to -\infty} I_2(t,y) = \chi(-\infty) = 1,$$

$K_t f_2$ can be continuous extended to $y = -\infty$ by $K_t f_2(x, -\infty) = I_1(t,x)$. Moreover for $\alpha + \beta + 2j = k, t > 0$, $\partial_y(\partial_x^\alpha \partial_y^\beta \partial_t^j) K_t f_2, \partial_{yy}(\partial_x^\alpha \partial_y^\beta \partial_t^j) K_t f_2$ can also be continuously extended to $y = -\infty$ by 0.

So $K_t f_2 \in C^{k,\gamma}(\mathbb{R}_+ \times \mathbb{R} \times [0,T])$ is the solution to the Cauchy problem with initial data f_2 such that

$$||K_t f_2||'_{k,\gamma} \le C||f_2||'_{k,\gamma} \le 3C||f||'_{k,\gamma}.$$

In all, let $v = v_1 + K_t f_2$, then v satisfies (3.5) up to the quadratic boundary $y = 0$ and

$$||v||_{k,\gamma} \le 6C||f||_{k,\gamma}.$$

Particularly along the edge $y = 0$, $v(t,x,0) = I_1(t,x)$ and satisfies

$$(\partial_t - x\partial_x^2 - d\partial_x)v(t,x,0) = 0 \text{ with } v(0,x,0) = f(x,0).$$

If $f \in C^{k,2+\gamma}(\mathbb{R}_+ \times \mathbb{R})$, this shall be established similarly. For the remaining case when $S = E_{reg}, E_\infty$, the proofs are essentially the same as above and we don't give details here. $\square$

Proposition 3.3.2. *Let $k \in \mathbb{N}$, $R > 0$, and $0 < \gamma < 1$. Assume that $g \in C^{k,\gamma}(S \times [0,T])$ and g is supported in $B_R^+(0) \times [0,T]$. The solution u to*

$$(\partial_t - L_M)u(t,x,y) = g(t,x,y), \; u(0,x,y) = 0 \tag{3.6}$$

belongs to $C^{k,2+\gamma}(S \times [0,T])$*, and there is a constant* $C_{k,\gamma,R}$ *so that*

$$||u||_{k,2+\gamma} \leq C_{k,\gamma,R}(1+T)||g||_{k,\gamma}.$$

The tangential first derivatives satisfy a stronger estimate: there is a constant C *so that if* $T \leq 1$*, then*

$$||D_y u||_{\gamma,T} \leq CT^{\frac{\gamma}{2}}||u||_{\gamma,T}. \tag{3.7}$$

Proof. At C_{reg} and E_{reg}, these are known results in

Let $g \in C^{k,\gamma}(S \times [0,T])$. We first decompose g as

$$g = g_1 + g_2 = g_1 + g(t,x,0)\chi(y)$$

where $\chi \in \mathcal{C}_c^{\infty}([-\infty,\infty)),\ \chi(-\infty) = 1$ such that $||g_2||'_{k,\gamma} \leq 2||g||'_{k,\gamma}$. Then $g_1 \in C^{k,\gamma}(\mathbb{R}_+ \times \mathbb{R} \times [0,T])$ vanishes at infinity, for which we can apply

This leaves the inhomogeneous problem with g_2. We begin by writing

$$A_t g_2 = \int_0^t \int_S K_{t-s}(x,y,x_1,y_1)g(s,x_1,0)\chi(y_1)dx_1dy_1ds$$
$$= \int_0^t \left(\int_0^{\infty} p^d(t-s,x,x_1)g(s,x_1,0))dx_1\right)\left(\int_{-\infty}^{\infty} k_{t-s}^e(y,y_1)\chi(y_1)dy_1\right)ds$$
$$= \int_0^t I_1(t,s,x)\cdot I_2(t,s,y)ds.$$

If $g(t,x,0) \in C^{k,\gamma}(\mathbb{R}_+ \times [0,T]), \chi(y) \in C^{k,\gamma}(\mathbb{R} \times [0,T])$, then $\int_0^t I_1(t,s,x)ds \in C^{k,2+\gamma}(\mathbb{R}_+ \times [0,T])$, $\int_0^t I_2(t,s,y)ds \in C^{k,2+\gamma}(\mathbb{R} \times [0,T])$ are solutions to one-dimensional inhomogeneous problem respectively. So

$$(\partial_t - L_M)A_t g_2 = g(t,x,0)\chi(y) + \int_0^t (\partial_t - L_M)I_1(t,s,x)\cdot I_2(t,s,y)ds = g(t,x,0)\chi(y).$$

and

$$|\partial_y A_t g_2| \leq \sqrt{T}||g_2||_\infty. \tag{3.8}$$

In the following we first assume that $k = 0$, $A_t g_2, \partial_x A_t g_2, x\partial_x^2 A_t g_2$ can be continuously extended to $y = -\infty$ since $\lim_{y\to-\infty} I_2(t,s,y) = \chi(-\infty) = 1$. Next we verify the hölder continuity condition.

$$|x\partial_x^2 A_t g_2 - x'\partial_x^2 A_t g_2| \leq \int_0^t |x\partial_x^2 I_1(t,s,x) - x'\partial_x^2 I_1(t,s,x)| \cdot I_2(t,s,y)ds$$
$$\leq \int_0^t |x\partial_x^2 I_1(t,s,x) - x'\partial_x^2 I_1(t,s,x)|ds \leq C(1+T)|\sqrt{x} - \sqrt{x'}|^\gamma ||g(t,x,0)||_{k,\gamma}.$$

Similarly

$$|\partial_y A_t g_2 - \partial_y' A_t g_2| \leq \int_0^t |\partial_y I_2(t,s,y) - \partial_y' I_2(t,s,y)| \cdot I_1(t,s,y)ds \tag{3.9}$$
$$\leq \int_0^t |\partial_y I_2(t,s,y) - \partial_y' I_2(t,s,y)|ds \leq CT^{\frac{\gamma}{2}}|y-y'|^\gamma ||\chi||_{k,\gamma}. \tag{3.10}$$

When $k > 0$, for $\alpha + \beta + 2j = k$, using the formula

$$(\partial_x^\alpha \partial_y^\beta \partial_t^j)A_t g_2 = \int_0^t \int_{\mathbb{R}_+\times\mathbb{R}} p_{t-s}^{d+\alpha}(x,x_1)k_s^e(y,y_1)(\partial_{x_1}^\alpha \partial_{y_1}^\beta L_{d+\alpha,M}^j)g_2(x_1,y_1)dx_1dy_1ds$$
$$+\sum_{r=0}^{j-1} \partial_t^{j-r-1}\partial_x^\alpha \partial_y^\beta L_{d+\alpha,M}^r g_2,$$

so $A_t g_2, \partial_x(\partial_x^\alpha \partial_y^\beta \partial_t^j)A_t g_2, x\partial_x^2(\partial_x^\alpha \partial_y^\beta \partial_t^j)A_t g_2$ can be continuously extended to $y = -\infty$ and hölder continuity conditions hold.

Above all $A_t g_2 \in C^{k,2+\gamma}(\mathbb{R}_+ \times \mathbb{R} \times [0,T])$ is the solution to the inhomogeneous problem with g_2 such that

$$||A_t g_2||'_{k,2+\gamma} \leq C(1+T)||g_2||'_{k,\gamma} \leq 3C(1+T)||g||'_{k,\gamma}.$$

Let $u = u_1 + A_t g_2$, then u solves the inhomogeneous problem ((3.6)) and $||u||_{2+\gamma} \leq 6C(1 + T)||g||_{k,\gamma}$. (3.7) follows from (3.8), (3.9). Particularly along the edge $y = 0$, $v(t, x, 0) = \int_0^t I_1(t, s, x)ds$ and satisfies

$$(\partial_t - x\partial_x^2 - d\partial_x)v(t, x, 0) = g(x, 0) \text{ with } v(0, x, 0) = 0.$$

This concludes the proof for $S = C_{mix}$. Taken together, we obtain the conclusion. The remaining case $S = E_{reg}, E_\infty$ shall be treated similarly. □

3.4 Existence of Solution

We now return to our principal goal, namely the analysis of L defined on P. The estimates proved in the previous sections allow us to prove existence of a unique solution to the inhomogeneous problem

$$(\partial_t - L)w = g \ in \ P \times [0, T] \tag{3.11}$$

$$\text{with } w(0, x, y) = f. \tag{3.12}$$

Definition 3.4.1. *Let $\mathfrak{M} = \{(W_j, \phi_j) : j = 1, \cdots, K\}$ be a cover of bP by normal coordinate charts, $W_0 \subset\subset$ int P, covering $P \setminus \cup_{j=1}^K W_j$ and let $\{\varphi_j : j = 0, \cdots, K\}$ be a partition of unity subordinate to this cover. A function $f \in C^{k,\gamma}(P)$ provided $(\varphi_j f) \circ \phi_j \in \mathcal{C}^{k,\gamma}(W_j)$ for each j. We define a global norm on $\mathcal{C}^{k,\gamma}(P)$ by setting*

$$||f||_{k,\gamma} = \sum_{j=0}^{K} ||(\varphi_j f) \circ \phi_j||_{k,\gamma}^{W_j}.$$

Theorem 3.4.1. *For $0 < \gamma < 1$, if the data $f \in C^{k,2+\gamma}(P), g \in C^{k,\gamma}(P \times [0, T])$, then equation (3.11) has a unique solution $w \in C^{k,2+\gamma}(P \times [0, T])$.*

Uniqueness of the solution can be obtained from the following maximum principle.

Proposition 3.4.1. *(Maximum Principle) Let $u \in C^{k,2+\gamma}(P)$ be a subsolution of $\partial_t u \leq Lu$ on $P \times [0,T]$, then*

$$\max_{P\times[0,T]} u(t,x,y) = \max_{P} u(0,x,y). \tag{3.13}$$

Proof. We show that if $u(0,x,y) \leq 0$, then $u(t,x,y) \leq 0$ for $0 < t \leq T$. The standard argument (see

For a point q on infinity edge E_∞, by definition, there exist local coordinates (x,y) centered at q and a neighborhood $\mathcal{U} = \{0 \leq x < 1, |y| < 1\}$ such that L has the form

$$L = ax^2\partial_x^2 + bx\partial_{xy} + c\partial_y^2 + V.$$

After a coordinate map $z = -\ln x$, L transforms to a uniform elliptic operator L' on $D = [0,\infty) \times [0,1]$. Let $v(t,x,z) = u(t,x,e^{-z})$. Then $(\partial_t - L')v \leq 0$ on D. Because $u \in C^0(P \times [0,T])$, v is bounded on $[0,\infty) \times [0,1]$, so there exists $A, a > 0$ such that $v \leq Ae^{a(x^2+z^2)}$.

We use the fundamental solution of the parabolic equation constructed in

For $\epsilon_1, \epsilon_2 > 0$, set

$$v_\tau(t,x,z) = v(t,x,z) - \epsilon_1 U_\tau(t,x,z) + \epsilon_2 \frac{1}{1+t},$$

then v_τ is a strict subsolution of $(\partial_t - L')v_\tau < 0$. We choose τ so that $\frac{G}{4\tau} > a$. Let

$$D_R = (0,\tau'] \times (D \cap bB_R(0)),$$

$B_R(0)$ is the ball of radius of R around the origin in $\mathbb{R}^2$ and $0 < \tau' < \tau$. Then with R sufficiently large, v_τ is negative on D_R. Let $\epsilon_1, \epsilon_2 \to 0$, we see $v(t,x,z) \leq 0$ on the infinity, therefore $u(t,q) \leq 0$ for $0 < t \leq \tau'$. Therefore we proved that

$$\max_{P\times[0,T]} u(t,x,y) \leq \max_{P} u(0,x,y).$$

$\square$

3.4.1 Parametrix construction

Let P denote the two-dimension compact manifold with corners, $C_r = \{C_{r,i}\}$ the set of regular corners, $C_m = \{C_{m,i}\}$ the set of mix corners, $C_\infty = \{C_{\infty,i}\}$ the set of infinity corners, $E_r = \{E_{r,i}\}$ the set of regular edges, $E_\infty = \{E_{\infty,i}\}$ the infinity edges, then

$$bP = C_r \bigcup C_m \bigcup C_\infty \bigcup E_r \bigcup E_\infty.$$

First we fix a function $\varphi_U \in C^\infty(P)$ that is equal to 1 in a neighborhood of bP. Let U be a neighborhood of bP such that $bU \cap intP$ is a smooth hypersurface in P, and $\bar{U} \subset\subset \varphi_U^{-1}(1)$. The subset $P_U = P \cap U^c$ is a smooth compact manifold with boundary and $L \mid_{P_U}$ is a non-degenerate elliptic operator.

$$P = U \bigsqcup P_U, \bar{U} \subset\subset \varphi_U^{-1}(1)$$

We can double P_U across its boundary to obtain $\widetilde{P_U}$, which is a compact manifold without boundary. And the operator L can be extended to a classical elliptic operator $\widetilde{L}$ on $\widetilde{P_U}$.

The classical theory of non-degenerate parabolic equations on compact manifolds without boundary, applies to construct an exact solution operator

$$u_i = \widetilde{Q^t}[(1-\varphi_U)g]$$

to the inhomogeneous equation:

$$(\partial_t - \tilde{L})u_i = (1 - \varphi_U)g \text{ in } \widetilde{P_U} \times [0, T]$$
$$\text{with } u_i(0, p) = 0, \ p \in \widetilde{P_U}.$$

This operator defines bounded maps from $C^{k,\gamma}(\widetilde{P_U} \times [0, T]) \to C^{k,2+\gamma}(\widetilde{P_U} \times [0, T])$ for any $0 < \gamma < 1, k \in \mathbb{N}$. We set interior parametrix

$$\widehat{Q_i^t} = \psi \widetilde{Q^t}[(1 - \varphi_U)g] \tag{3.14}$$

where we choose $\psi \in C_c^\infty(P_U)$ so that $\psi \equiv 1$ on a neighborhood of the support of $(1 - \varphi_U)$.

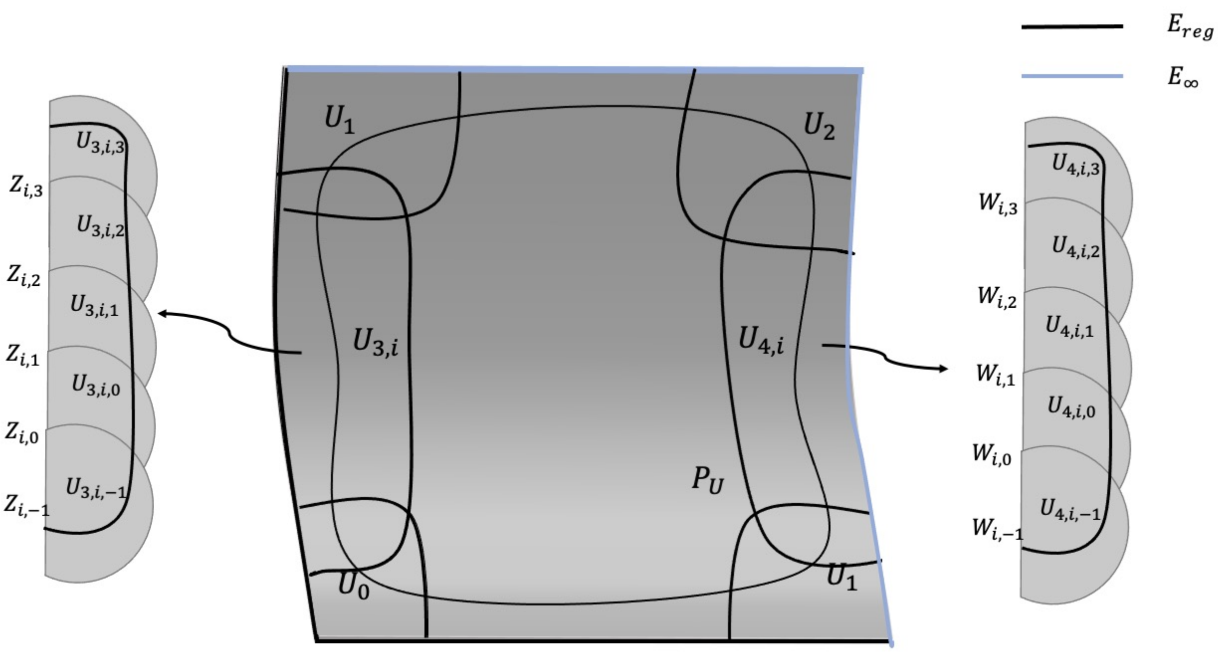

Figure 3.1: Covering in the proof

To build the boundary parametrix, we construct the ϵ-grid as follows. Let

$$\mathfrak{U} = \{U_{0,i}, U_{1,i}, U_{2,i}, U_{3,i}, U_{4,i}\}$$

be an NCC covering of bP, where $U_{0,i}, U_{1,i}, U_{2,i}$ are 5ϵ-neighborhoods of i-th regular corner, i-th mix corner, i-th infinity corner, respectively. By shrinking the neighborhoods we can assume that $U_{0,i}, U_{1,i}, U_{2,i}$ are disjoint.

All of the charts in $U_{3,i}, U_{4,i}$ have coordinates lying in $\mathbb{R}_+ \times \mathbb{R}$. We let $Z_{i,j}, W_{i,j}$ be the points in $U_{3,i}, U_{4,i}$ with coordinates

$$\{Z_{i,j} = (0, \epsilon j) : j \in \mathbb{Z}\}, \{W_{i,j} = (0, \epsilon j) : j \in \mathbb{Z}\}$$

respectively. And we let $U_{3,i,j}$, $U_{4,i,j}$ to be the 5ϵ-neighborhood of $Z_{i,j}, W_{i,j}$. To simplify notation we do not keep track of the dependence on ϵ. We illustrate the covering in Fig 3.1.

Let (x, y) denote normal cubic coordinates in one of these neighborhoods, $U_{n,i,j}$ (if $n = 0, 1, 2$, assume $j = 0$). In these coordinates the operator L takes the normal form $L_{i,j}$. We let $L_{n,i,j,M}$ be the corresponding model operator and let $A^t_{n,i,j}$ denote the solution operator for the model problem

$$(\partial_t - L_{n,i,j,M})u = g, u(x, 0) = 0.$$

For $n = 3, 4$, let $\chi_n \equiv 1$ in $(0, 2)\times(-2, 2)$ and vanishing outside $(0, 3)\times(-3, 3)$, $\phi_n \equiv 1$ in $(0, 4)\times(-4, 4)$ and vanishing outside $(0, 5)\times(-5, 5)$, and $\chi_n, \phi_n \in C^\infty_c(\mathbb{R}_+\times\mathbb{R})$, respectively. We define

$$\widetilde{\chi_{n,i,j}}(x, y) = \chi_n(\frac{x}{\epsilon^2}, \frac{y - \epsilon j}{\epsilon}), \phi_{n,i,j}(x, y) = \phi_i(\frac{x}{\epsilon^2}, \frac{y - \epsilon j}{\epsilon}).$$

For $n = 0, 1, 2$, we let

$$\widetilde{\chi_{n,i,j}}, \widetilde{\phi_{n,i,j}} = \begin{cases} \chi_n(\frac{x_1}{\epsilon^2}, \frac{x_2}{\epsilon^2}), \phi_n(\frac{x_1}{\epsilon^2}, \frac{x_2}{\epsilon^2}) & n = 0 \\ \chi_n(\frac{x}{\epsilon}, \frac{y}{\epsilon^2}), \phi_n(\frac{x}{\epsilon}, \frac{y}{\epsilon^2}) & n = 1 \\ \chi_n(\frac{y_1}{\epsilon}, \frac{y_2}{\epsilon}), \phi_n(\frac{y_1}{\epsilon}, \frac{y_2}{\epsilon}) & n = 2 \end{cases}$$

where $\chi_n \equiv 1$ in $(0, 2) \times (0, 2)$ vanish outside $(0, 3) \times (0, 3)$, $\phi_n \equiv 1$ in $(0, 4) \times (0, 4)$ vanish outside $(0,5)\times(0,5)$, and $\chi_n, \phi_n \in C_c^\infty(\mathbb{R}_+^2)$, respectively.

By abuse of notation, we still denote $\widetilde{\chi_{n,i,j}}, \phi_{n,i,j}$ to indicate their pullback to P. It is clear that there exists $S>0$, independent of ϵ, such that for $r \in P$,

$$\chi(r) = \sum_{i=0}^{4} \sum_{i,j} \widetilde{\chi_{n,i,j}}(r) \leq S.$$

It is clear that $\chi_\epsilon(r) \geq 1$ for $r \in bP$. We can arrange to have $\varphi_U = 1$ on the set $\chi_\epsilon \geq \frac{1}{2}$ and *Supp* $\varphi_U \subset \chi_\epsilon^{-1}([\frac{1}{16}, S])$. Based on that, $\mathfrak{U}' = \{U_{n,i,j}\}$ together with P_U is a partition of P.

To get a partition of unity of a neighborhood of bP subordinate to $U_{i,j}$, we replace the functions $\{\widetilde{\chi_{n,i,j}}\}$ with

$$\chi_{n,i,j} = \varphi_U \cdot \left(\frac{\widetilde{\chi_{n,i,j}}}{\sum_{n=0}^{4} \sum_{i,j} \widetilde{\chi_{n,i,j}}} \right).$$

For each $\epsilon > 0$, we define a boundary parametrix by setting

$$\widehat{Q_{b,\epsilon}^t} = \sum_{n=0}^{4} \sum_{i,j} \phi_{n,i,j} A_{n,i,j}^t \chi_{n,i,j}. \tag{3.15}$$

In total we set

$$\widehat{Q^t} g = \widehat{Q_{b,\epsilon}^t} g + \widehat{Q_i^t} g. \tag{3.16}$$

3.4.2 Perturbation Estimate

Next we are going to analyze the perturbation term,

$$
\begin{aligned}
(\partial_t - L)\widehat{Q^t_{b,\epsilon}}g &= (\partial_t - L)\left(\sum_{n=0}^{4}\sum_{i,j}\phi_{n,i,j}A^t_{n,i,j}\chi_{n,i,j}g\right) \\
= \left(\sum_{i=0}^{4}\sum_{i,j}\chi_{n,i,j}g\right) &+ \left(\sum_{i=0}^{4}\sum_{i,j}\phi_{n,i,j}(L_{n,i,j,M} - L)A^t_{n,i,j}[\chi_{n,i,j}g]\right) \\
&+ \left(\sum_{i=0}^{4}\sum_{i,j}[\phi_{n,i,j}, L]A^t_{n,i,j}[\chi_{n,i,j}g]\right) \\
&= \varphi_U g + E^0_\epsilon(g) + E^1_\epsilon(g)
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_U &= \sum_{n=0}^{4}\sum_{i,j}\chi_{n,i,j}, \\
E^0_\epsilon(g) &= \sum_{n=0}^{4}\sum_{i,j}\phi_{n,i,j}(L_{n,i,j,M} - L)A^t_{n,i,j}[\chi_{n,i,j}g] \\
E^1_\epsilon(g) &= \sum_{n=0}^{4}\sum_{i,j}[\phi_{n,i,j}, L]A^t_{n,i,j}[\chi_{n,i,j}g].
\end{aligned}
$$

And

$$
(\partial_t - L)\widehat{Q^t_i}g = (1-\varphi_U)g + [\psi, L]\widehat{Q^t_i}[(1-\varphi)g] = (1-\varphi_U)g + E^\infty_i(g).
$$

Taken together,

$$
(\partial_t - L)\widehat{Q^t}g = g + E^0_\epsilon(g) + E^1_\epsilon(g) + E^\infty_i(g). \tag{3.17}
$$

As $\phi \equiv 1$ on the support χ, the norm of E^1_ϵ is bounded and by $Ce^{-\frac{M}{T}}$ as $T \to 0$ for some $M > 0$ by

Then we estimate E^0_ϵ.

n=0

In $U_{0,i}$, under local adapted coordinates, the operator L takes the form

$$L = x\partial_{xx} + y\partial_{yy} + xyb(x,y)\partial_{xy} + d(x,y)\partial_x + e(x,y)\partial_y.$$

$$E^0_{0,\epsilon} = \phi_\epsilon(L_M - L)A^t[\chi_\epsilon g] = -\phi_\epsilon(xyb(x,y)\partial_{xy} + \tilde{b}\partial_x + \tilde{e}\partial_y)A^t[\chi g].$$

Since ϕ_ϵ is supported in the set where $x, y \leq 5\epsilon^2$, we have

$$||E^0_{0,\epsilon}||_\infty \leq C\epsilon^2||\chi_\epsilon g||_{0,\gamma} \leq C\epsilon^{2-\gamma}||g||_{0,\gamma}. \tag{3.18}$$

We used

$$[fgh]_\gamma \leq ||fg||_\infty[h]_\gamma + ||fh||_\infty[g]_\gamma + ||gh||_\infty[f]_\gamma$$

and $[\phi_\epsilon]_\gamma \leq C\epsilon^{-\gamma}$, so

$$[E^0_{0,\epsilon}]_\gamma \leq C\epsilon^{2-\gamma}||\chi_\epsilon g||_\gamma \leq C\epsilon^{2-2\gamma}||g||_{0,\gamma} \tag{3.19}$$

where the constant C is independent of ϵ. Combining (3.18) and (3.19), we conclude

$$||E^0_{0,\epsilon}||_{0,\gamma} \leq C\epsilon^{2-2\gamma}||g||_{0,\gamma}. \tag{3.20}$$

n=3

In a neighborhood U_1 of an infinity edge point, under local adapted coordinates, the operator L takes the form

$$L = x\partial_{xx} + xyb(x,y)\partial_{xy} + c(x,y)y^2\partial_{yy} + d(x,y)\partial_x + e(x,y)y\partial_y$$

and the model operator:

$$L_M = x\partial_{xx} + y^2\partial_{yy} + d\partial_x + y\partial_y.$$

We split

$$L - L_M = L^r + e(x,y)y\partial_y, \tag{3.21}$$

$$L^r = xyb(x,y)\partial_{xy} + \tilde{c}(x,y)y^2\partial_{yy} + \tilde{d}(x,y)\partial_x. \tag{3.22}$$

There are two types of errors:

$$E^{0,r}_{1,\epsilon}g = -\phi_\epsilon L^r A^t[\chi_\epsilon g],\ E^{0,t}_{1,\epsilon}g = -\phi_\epsilon e(x,y)y\partial_y A^t[\chi_\epsilon g].$$

Then by Proposition (3.3.2) and $[\chi_\epsilon]_\gamma = O(\epsilon^{-\gamma})$ and since

$$|\chi_\epsilon(y) - \chi_\epsilon(y')| \le C\epsilon^{-\gamma}|y-y'|^\gamma \le C\epsilon^{-\gamma}|\text{ln}y - \text{ln}y'|^\gamma,$$

we have

$$[E^{0,t}_{1,\epsilon}]_\gamma \le C\epsilon^{-\gamma}T^{\frac{\gamma}{2}}||\chi_\epsilon g||_{0,\gamma} + CT^{\frac{\gamma}{2}}||\chi_\epsilon g||_{0,\gamma} \le C\epsilon^{-2\gamma}T^{\frac{\gamma}{2}}||g||_{0,\gamma},$$
$$||E^{0,t}_{1,\epsilon}||_\infty \le CT^{\frac{\gamma}{2}}||\chi_\epsilon g||_\infty \le CT^{\frac{\gamma}{2}}\epsilon^{-\gamma}||g||_\infty.$$

Taken together

$$||E^{0,t}_{1,\epsilon}||_{0,\gamma} \le C\epsilon^{-2\gamma}T^{\frac{\gamma}{2}}||g||_{0,\gamma}. \tag{3.23}$$

Next we estimate the remaining term $E^{0,r}_{1,\epsilon}g$. Since ϕ_ϵ is supported on $x \le 5\epsilon^2, y \le 5\epsilon$, the vanishing properties of the coefficients of L^r implies that the L^∞-term is bounded by

$$||E^{0,r}_{1,\epsilon}(g)||_\infty \leq C\epsilon||\chi_\epsilon g||_{0,\gamma} \leq C\epsilon^{1-\gamma}||g||_{0,\gamma}.$$

To estimate the Hölder semi-norm we need to consider terms

$$||\phi_\epsilon xyb(x,y)||_\infty \left[\partial_{xy}A^t[\chi_\epsilon g]\right]_\gamma, ||\phi_\epsilon\tilde{c}(x,y)||_\infty y^2\partial_{yy}\left[A^t[\chi_\epsilon g]\right]_\gamma,$$
$$||\phi_\epsilon\tilde{d}(x,y)||_\infty\partial_x\left[A^t[\chi_\epsilon g]\right]_\gamma$$

and

$$\left[\phi_\epsilon xyb(x,y)\right]_\gamma ||\partial_{xy}A^t[\chi_\epsilon g]||_\infty, \left[\phi_\epsilon\tilde{c}(x,y)\right]_\gamma ||y^2\partial_{yy}A^t[\chi_\epsilon g]||_\infty,$$
$$\left[\phi_\epsilon\tilde{d}(x,y)\right]_\gamma ||\partial_x A^t[\chi_\epsilon g]||_\infty.$$

The first three terms are bounded by

$$C\epsilon||\chi_\epsilon g||_{0,\gamma} \leq C\epsilon^{1-\gamma}||g||_{0,\gamma}.$$

For any $0 < \gamma' \leq \gamma < 1$, the second three terms are bounded by

$$C\epsilon^{1-\gamma}||\chi_\epsilon g||_{0,\gamma'} \leq C\epsilon^{1-\gamma-\gamma'}||g||_{0,\gamma}.$$

We therefore fix a $0 < \gamma' \leq \gamma$ so that $\gamma + \gamma' < 1$, then

$$||E^{0,r}_{1,\epsilon}g||_{0,\gamma} \leq C\epsilon^{1-\gamma-\gamma'}||g||_{0,\gamma}. \tag{3.24}$$

n=1,2,4

These cases are estimated as above. We still split $L - L_M$ into two parts L^r and the first tangential parts like (3.21). The vanishing properties of coefficients of L^r and the estimates of first tangential parts give the same results as

$$||E_{1,\epsilon}^{0,t}||_{0,\gamma} \leq C\epsilon^{-2\gamma}T^{\frac{\gamma}{2}}||g||_{0,\gamma},$$
$$||E_{1,\epsilon}^{0,r}g||_{0,\gamma} \leq C\epsilon^{1-\gamma-\gamma'}||g||_{0,\gamma}.$$

Proof of Theorem 3.4.1

Case when k=0. We first prove the case when $k = 0$. We write the solution $w = u + v$, where v solves the homogeneous Cauchy problem with initial data $v(0, x, y) = f(x, y)$ and u solves the inhomogeneous problem with $u(0, x, y) = 0$.

Recall the perturbation term (3.17),

$$(\partial_t - L)\widehat{Q^t}g = g + E_\epsilon^0(g) + E_\epsilon^1(g) + E_i^\infty(g).$$

The support of the kernel of $E_i^\infty g$ has a positive distance from the diagonal and therefore this is a compact operator in the metric topology of $C^{0,\gamma}(P \times [0, T])$, tending to zero as $T \to 0$. Using the estimations in the previous subsections,

$$||E_\epsilon^0(g) + E_\epsilon^1(g)||_{0,\gamma,T} \leq C\left[\epsilon^{2-2\gamma} + \epsilon^{-2\gamma}T^{\frac{\gamma}{2}} + \epsilon^{1-\gamma-\gamma'}\right] \cdot ||g||_{0,\gamma,T}.$$

Fix a $\delta > 0$, we choose ϵ so that $\epsilon^{2-2\gamma} + \epsilon^{1-\gamma-\gamma'} = \delta$. If we choose T_0 sufficiently small, then the operator $E^t = E_\epsilon^{0,t} + E_\epsilon^{1,t} + E_i^\infty$ has norm strictly less than 1, and therefore the operator $Id + E^t$ is invertible as a map from $C^{k,\gamma}(P \times [0, T_0])$ to itself. Thus the operator

$$Q^t = \widehat{Q^t}\left(Id + E^t\right)^{-1}$$

is the right inverse to $(\partial_t - L)$ up to time T_0 and is a bounded map

$$Q^t : C^{0,\gamma}(P \times [0, T_0]) \to C^{0,2+\gamma}(P \times [0, T_0]).$$

After constructing $\widehat{Q^t}$, the solution operator for the inhomogeneous problem, we build a similar boundary parametrix for the homogeneous Cauchy problem, which we then glue to the exact solution operator for P_U. For each n let $\widehat{Q^t_{n,i,j}}$ be the solution operator for the homogeneous Cauchy problem defined by the model operator in $U_{i,j}$. And let $\widehat{Q^t_{00}}$ be the exact solution operator for the Cauchy problem $(\partial_t - L)u = 0$ on W_0 with Dirichlet data on $bW_0 \times [0, \infty)$. For each $\epsilon > 0$, we define a parametrix by setting

$$\widehat{Q^t_0} = \sum_{i=0}^{4} \sum_{i,j} \phi_{n,i,j,\epsilon} \widehat{Q^t_{n,i,j}} \chi_{n,i,j,\epsilon} + \psi \widehat{Q^t_{00}} (1 - \varphi_U). \tag{3.25}$$

Then

$$(\partial_t - L)\widehat{Q^t_0} = E^t f : C^{0,2+\gamma}(P) \to C^{0,\gamma}(P \times [0, T_0])$$

is a bounded map and a slightly stronger statement is $\lim_{t \to 0} ||E^t|| = 0$.

Finally we set $Q^t_0 f = \widehat{Q^t_0} f - \widehat{Q}^t E^t f$, then

$$Q^t_0 : C^{0,2+\gamma}(P) \to C^{0,2+\gamma}(P \times [0, T_0])$$

is bounded and so is the solution operator of the Cauchy problem.

Higher order regularity. We want to establish the convergence of $(Id + E^t)^{-1}$ in the operator norm defined by $C^{k,\gamma}(P \times [0, T_0])$. We follow the proof in

We have proved the existence of solutions to the inhomogeneous problem and Cauchy problem in $[0, T_0]$, where T_0 does not depend on the initial data. To show the solution exists for all $t > 0$, we can apply this proof again with initial data $f(0, \cdot) = v(T_0, \cdot), g(t, \cdot) =$

$g(t+T_0,\cdot)$. This extends the solution to $[0,2T_0]$. We can repeat this process k times until $kT_0 \geq t$.

3.5 The Heat Kernel

Now we are prepared to construct the heat kernel of the solution operator:

$$(\partial_t - L)v_f = 0 \text{ with } v_f(0,\cdot,\cdot) = f. \tag{3.26}$$

We want to show the following result:

Theorem 3.5.1. *The global heat kernel $H_t(d_1,d_2,l_1,l_2) \in C^\infty(P^{reg} \times \mathring{P} \times (0,\infty))$ of the full operator L exists and for $f \in C^0(P)$, then*

$$\int_P H_t(d_1,d_2,l_1,l_2)f(l_1,l_2)dl_1dl_2$$

is the solution of $(\partial_t - L)v_f = 0$ with $v_f(0,\cdot,\cdot) = f$.

When $(d_1,d_2) \in \mathring{P}$, the existence of the heat kernel follows from a standard construction for elliptic operators and the theory developed in the preceding sections. We may indeed construct a heat kernel $\tilde{H}_t(d_1,d_2,l_1,l_2)$ for (l_1,l_2) in an open neighborhood of (d_1,d_2) and obtain by standard elliptic regularity that $H_t - \tilde{H}_t$ solves (3.11) with a smooth right-hand side. A nontrivial aspect of the above result is that we also construct a heat kernel when (d_1,d_2) is on the regular part of bP, where elliptic regularity is not applied. Instead we use fundamental solutions of model operators and use a series expansion to construct and analyze the heat kernel.

3.5.1 Overview of main ideas

We continue working under the similar $\epsilon-$grid covering constructed in Section 3.4.1. Instead of requiring L having normal forms in each $U_{n,i,j}$, we require L takes the form in (3.1.1) but without mixed second-order derivative term. This can be achieved by taking ϵ small enough and through coordinate change.

To construct the global heat kernel of the full operator L, we first construct the local heat kernel $\{q_{n,i}^t\}$ in $U_{n,i}$, which satisfies that for $\forall q \in \mathring{U}_{n,i}$,

$$(\partial_t - L)q_{n,i}^t(\cdot, q) = 0, \quad \lim_{t\to 0} q_{ij}^t(\cdot, q) = \delta_q(\cdot). \tag{3.27}$$

Then we patch them together to construct the global kernel parametrix.

Construction of Local heat kernel: Fix q in the interior of $U_{n,i}$, we fix a smooth compactly supported function $h(x,y) \in C_c^\infty(U_{n,i})$ and $h(x,y) \equiv 1$ on Supp $\phi_{n,i}$. Letting

$$\widetilde{L} = L_{q,M} + h(x,y)(L - L_{q,M}) \tag{3.28}$$

where $L_{q,M}$ is the model operator we choose corresponding to q. Then $\widetilde{L}$ equals to L on Supp $\phi_{i,j}$, so the problem transform to construct the Green function q_t of the solution operator in this neighborhood:

$$(\partial_t - \widetilde{L})q_t(\cdot, q) = 0 \text{ with } \lim_{t\to 0^+} q_t(q, \cdot) = \delta(q). \tag{3.29}$$

The idea is to approximate $q_t(\cdot, q)$ by the kernel of the model operator L_M modeled at q and control the perturbation term to be sufficiently smooth.

Let K_t be the fundamental solution of $L_{q,M}$, $\mathbf{K}_t$ be the solution operator of $L_{q,M}$, $\mathbf{B}_t = (\widetilde{L} - L_{q,M})\mathbf{K}_t$, an observation is that for $N \geq 1$,

$$(\partial_t - \widetilde{L})\left(q_t(\cdot, q) - K_t(t, \cdot, q) - \sum_{i=1}^{N-1} \mathbf{K}_\mathbf{t}^i \mathbf{B}\delta\right) = \mathbf{B}^N\delta \tag{3.30}$$

where $\mathbf{B}\delta = (\widetilde{L} - L_M)K_t$.

3.5.2 Existence of Local Heat Kernel

Case $q \in U_0$.

We first consider the case when q is in a neighborhood U_0 of a regular corner. Under the local adapted coordinates (**??**), L takes the form

$$L = a(x_1,x_2)x_1\partial_{x_1}^2 + b(x_1,x_2)x_2\partial_{x_2}^2 + d(x_1,x_2)\partial_{x_1} + e(x_1,x_2)\partial_{x_2}.$$

Fix a point $q = (x_2, y_2)$ in the interior of U_0. We introduce the model operator $L_{q,M}$ in S_{c_reg}:

$$L_{q,M} = a(q)x_1\partial_{x_1}^2 + b(q)x_2\partial_{x_2}^2 + d(q)\partial_{x_1} + e(q)\partial_{x_2}. \tag{3.31}$$

Then the kernel formula $K_t(x,y,x_1,y_1)$ is the product of two one-dimensional kernel formula:

$$K_t(x,y,x_2,y_2) = p_{a(q)t}^{d(q)/a(q)}(x,x_2)p_{b(q)t}^{e(q)/b(q)}(y,y_2). \tag{3.32}$$

Proposition 3.5.1. *Denote* $d(t,x,y) = \frac{(\sqrt{x}-\sqrt{x_2})^2}{2a(q)t} + \frac{(\sqrt{y}-\sqrt{y_2})^2}{2b(q)t}$. *Assume that for* $(x,y) \in \mathbb{R}_+^2$, $j \le \frac{3}{2}$

$$|g(t,x,y)| \le \frac{1}{t^j}e^{-d(t,x,y)}$$

and for some $0 < \gamma < 1$,

$$|g(t,x,y) - g(t,x',y)| \le \frac{1}{t^{j+\frac{\gamma}{2}}}|\sqrt{x} - \sqrt{x'}|^\gamma \left(e^{-d(t,x,y)} + e^{-d(t,x',y)}\right) \tag{3.33}$$

$$|g(t,x,y) - g(t,x,y')| \le \frac{1}{t^{j+\frac{\gamma}{2}}}|\sqrt{y} - \sqrt{y'}|^\gamma \left(e^{-d(t,x,y)} + e^{-d(t,x',y)}\right). \tag{3.34}$$

Then there exists a constant $C > 0$ such that

$$|\boldsymbol{B}g(t,x,y)| \le C\frac{\sqrt{t}}{t^j}e^{-\frac{d(t,x,y)}{2}}$$

and

$$|\boldsymbol{B}g(t,x,y) - \boldsymbol{B}g(t,x',y)| \le \frac{C}{t^{j+\frac{\gamma-1}{2}}}|\sqrt{x}-\sqrt{x'}|^\gamma \left(e^{-\frac{d(t,x,y)}{2}} + e^{-\frac{d(t,x',y)}{2}}\right)$$
$$|\boldsymbol{B}g(t,x,y) - \boldsymbol{B}g(t,x,y')| \le \frac{C}{t^{j+\frac{\gamma-1}{2}}}|\sqrt{y}-\sqrt{y'}|^\gamma \left(e^{-\frac{d(t,x,y)}{2}} + e^{-\frac{d(t,x',y)}{2}}\right).$$

Proof. By replacing $a(q)t, b(q)t$ with t in the kernel formula (3.32), we might as well assume that $a(q) = b(q) = 1$. Through a Taylor expansion of the coefficient of L at q, we see

$$L - L_{q,M} = \Theta((x-x_2) + (y-y_2))(L_x + L_y) \tag{3.35}$$

where $L_x = x\partial_x^2 + d(q)\partial_x, L_y = y\partial_y^2 + e(q)\partial_y$. In the following proof we will use a number of kernel estimates, which we present and prove in the Appendix.

First we assume that $d(q) \ge \frac{1}{2}$ or $d(q) = 0$, $e(q) \ge \frac{1}{2}$ or $e(q) = 0$. We use the estimate of $p_t^d(x,y)$ (5.1.1) to see

$$p_{t-s}^d(x,x_1) \le \frac{C}{\sqrt{t-s}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2(t-s)}}\frac{1}{\sqrt{x_1}},$$

so we deduce that:

$$|\mathbf{K}_t g| \le Ce^{-\frac{(\sqrt{x}-\sqrt{x_2})^2+(\sqrt{y}-\sqrt{y_2})^2}{2t}}.$$

Next we turn to estimate $\mathbf{B}g$. It suffices to consider L_x so that

$$\widetilde{L} - L_M = \Theta((x-x_2) + (y-y_2))L_x. \tag{3.36}$$

Since the $t-$regularity of $(\widetilde{L} - L_M)K_{t-s}$ is $(t-s)^{-2}$ which is not integrable in s, using the fact that

$$\int_0^\infty x\partial_x^2 p_t^d(x,y)dy = 0,$$

we split $\mathbf{B}g$ into two parts

$$\mathbf{B}g = (\tilde{L} - L_M)\int_{\frac{t}{2}}^t \int_{\mathbb{R}_+^2} K_{t-s}(x,y,x_1,y_1)(g(s,x_1,y_1) - g(s,x,y_1))dx_1dy_1ds \tag{3.37}$$

$$+(\tilde{L} - L_M)\int_0^{\frac{t}{2}} \int_{\mathbb{R}_+^2} K_{t-s}(x,y,x_1,y_1)g(s,x_1,y_1)dx_1dy_1ds. \tag{3.38}$$

The first term is bounded by

$$(\tilde{L} - L_M)\int_{\frac{t}{2}}^t \int_{\mathbb{R}_+^2} K_{t-s}(x,y,x_1,y_1)\frac{1}{s^{j+\frac{\gamma}{2}}}|\sqrt{x} - \sqrt{x_1}|^\gamma$$
$$\cdot\left(e^{-\frac{(\sqrt{x_1}-\sqrt{x_2})^2}{2s}} + e^{-\frac{(\sqrt{x}-\sqrt{x_2})^2}{2s}}\right)e^{-\frac{(\sqrt{y_1}-\sqrt{y_2})^2}{2s}}dx_1dy_1ds.$$

We use the estimates of derivatives (5.1.3), (5.28) to see

$$|\partial_x p_{t-s}^d(x,x_1)|, |x\partial_x^2 p_{t-s}^d(x,x_1)| \leq \frac{C}{(t-s)^{\frac{3}{2}}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2+(y^s-y_1^s)^2}{2(t-s)}}\frac{1}{\sqrt{x_1}}. \tag{3.39}$$

By the integral

$$\int_{-\infty}^\infty \frac{1}{\sqrt{t-s}}e^{-\frac{(x-x_1)^2}{t-s}}\frac{1}{\sqrt{s}}e^{-\frac{(x_1-x_2)^2}{s}}dx_1 = \frac{1}{\sqrt{t}}e^{-\frac{(x-x_2)^2}{t}} \tag{3.40}$$

and the contribution of coefficients of $\tilde{L} - L_M$:

$$|x - x'| \cdot e^{-\frac{(x-x')^2}{t}} \leq C\sqrt{t}e^{-\frac{(x-x')^2}{2t}}, \tag{3.41}$$

we can deduce that (3.37) is integrable and

$$|\mathbf{B}g| \leq C\frac{1}{\sqrt{t}}e^{-\frac{(\sqrt{x}-\sqrt{x_2})^2+(\sqrt{y}-\sqrt{y_2})^2}{2t}}.$$

Finally we turn to estimate $[\cdot]_\gamma$ part of $\mathbf{B}g$. We consider two cases. In the first case when $|\sqrt{x}-\sqrt{x'}|\geq\sqrt{t}$,

$$|\mathbf{B}g(t,x,y)-\mathbf{B}g(t,x',y)|\leq|\mathbf{B}g(t,x,y)+\mathbf{B}g(t,x',y)|\cdot\left(\frac{|\sqrt{x}-\sqrt{x'}|}{t}\right)^\gamma.$$

In the second case when $|\sqrt{x}-\sqrt{x'}|\leq\sqrt{t}$,

$$|\sqrt{x}-\sqrt{x_2}|\leq|\sqrt{x'}-\sqrt{x_2}|+\sqrt{t},$$

so that multiplying this term with $(e^{-\frac{(\sqrt{x}-\sqrt{x_2})^2}{t}}+e^{-\frac{(\sqrt{x'}-\sqrt{x_2})^2}{t}})$,

$$|\sqrt{x}-\sqrt{x_2}|\cdot(e^{-\frac{(\sqrt{x}-\sqrt{x_2})^2}{t}}+e^{-\frac{(\sqrt{x'}-\sqrt{x_2})^2}{t}})\leq C\sqrt{t}(e^{-\frac{(\sqrt{x}-\sqrt{x_2})^2}{2t}}+e^{-\frac{(\sqrt{x'}-\sqrt{x_2})^2}{2t}}).\quad(3.42)$$

1. If $x'<\frac{x}{3}$, it suffices to show that

$$|x\partial_x^2\mathbf{K}_tg(t,x,y)|\leq C\frac{x^{\frac{\gamma}{2}}}{t^{j+\frac{\gamma}{2}}}e^{-\frac{d(x,y)}{2t}}.$$

This is obtained similarly to $\mathbf{B}g$ in (3.37): we replace the estimate (3.39) with

$$x\partial_x^2p(t-s,x,x_1)dx_1\leq\frac{Cx^{\frac{\gamma}{2}}}{(t-s)^{\frac{3+\gamma}{2}}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2+(y^s-y_1^s)^2}{2(t-s)}}dx_1^s.$$

Then

$$\begin{aligned}&|(x-x_2)x\partial_x^2\mathbf{K}_tg(t,x,y)-(x'-x_2)x'\partial_x^2\mathbf{K}_tg(t,x',y)|\\&\leq|(x-x_2)-(x'-x_2)|x\partial_x^2\mathbf{K}_tg(t,x,y)\\&+|(x'-x_2)|\cdot|x\partial_x^2\mathbf{K}_tg(t,x,y)-x'\partial_x^2\mathbf{K}_tg(t,x',y)|.\end{aligned}$$

The first term is bounded by $C\frac{|\sqrt{x}-\sqrt{x'}|^{\gamma}}{t^{j+\frac{\gamma-1}{2}}}e^{-\frac{d(x,y)}{2t}}$. The estimate below

$$\begin{aligned}|x\partial_x^2\mathbf{K}_t g(t,x,y) - x'\partial_x^2\mathbf{K}_t g(t,x',y)| &\le C\frac{x^{\frac{\gamma}{2}}}{t^{j+\frac{\gamma}{2}}}\left(e^{-\frac{d(x,y)}{2t}} + e^{-\frac{d(x',y)}{2t}}\right)\\ &\le C\frac{|\sqrt{x}-\sqrt{x'}|^{\gamma}}{t^{j+\frac{\gamma}{2}}}\left(e^{-\frac{d(x,y)}{2t}} + e^{-\frac{d(x',y)}{2t}}\right)\end{aligned}$$

and (3.42) show that the second term is bounded by $C\frac{|\sqrt{x}-\sqrt{x'}|^{\gamma}}{t^{j+\frac{\gamma-1}{2}}}e^{-\frac{d(x,y)}{2t}}$.

2. We assume that $\frac{x}{3} < x' < x$,

   Since we have established the Hölder continuity of the first derivative, it suffices to establish the Hölder continuity when $\tilde{L} - L_M = (x - x_2)(x\partial_x^2 + d\partial_x)$. We split $\mathbf{B}^2 g$ into two parts

$$\mathbf{B}g(t,x,y) - \mathbf{B}g(t,x',y) = \tag{3.43}$$

$$(\tilde{L} - L_M)\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}_+^2}\left[K_{t-s}(x,y,x_1,y_1) - K_{t-s}(x',y,x_1,y_1)\right]g(s,x_1,y_1)dx_1dy_1ds \tag{3.44}$$

$$+(\tilde{L} - L_M)\int_0^{\frac{t}{2}}\int_{\mathbb{R}_+^2}\left[K_{t-s}(x,y,x_1,y_1) - K_{t-s}(x',y,x_1,y_1)\right]g(s,x_1,y_1)dx_1dy_1ds \tag{3.45}$$

$$= A + B. \tag{3.46}$$

With $J = [\alpha, \beta]$, $\sqrt{\alpha} = \frac{3\sqrt{x'} - \sqrt{x}}{2}$, $\sqrt{\beta} = \frac{3\sqrt{x} - \sqrt{x'}}{2}$, we have

$$A = \int_{\frac{t}{2}}^{t} [\int_J (\tilde{L} - L_M) K_{t-s}(x, x_1, y, y_1)(g(s, x_1, y_1) - g(s, x, y_1)) dx_1 dy_1 - \tag{3.47}$$

$$\int_J (\tilde{L} - L_M) K_{t-s}(x', x_1, y, y_1)(g(s, x_1, y_1) - g(s, x', y_1)) dx_1 dy_1 - \tag{3.48}$$

$$\int_{J^c} (\tilde{L} - L_M) K_{t-s}(x, x_1, y, y_1)(g(s, x, y_1) - g(s, x', y_1)) dx_1 dy_1 + \tag{3.49}$$

$$\int_{J^c} ((\tilde{L} - L_M) K_{t-s}(x, x_1, y, y_1) \tag{3.50}$$

$$-(\tilde{L} - L_M) K_{t-s}(x', x_1, y, y_1))(g(s, x_1, y_1) - g(s, x', y_1)) dx_1 dy_1] ds \tag{3.51}$$

$$= I_1 + I_2 + I_3 + I_4. \tag{3.52}$$

We first estimate I_3. Based on the observation that for $t > 0$,

$$\partial_t p_d(t, x, x_1) = L_{d,x} p_d(t, x, x_1) = L^*_{d,x_1} p_d(t, x, x_1),$$

the operator $L^t_x = \partial_{x_1}(\partial_{x_1} x_1 - d)$, so we can perform x_1-integral to obtain that

$$I_3 \leq \int_{\frac{t}{2}}^{t} [(\partial_{x_1} x_1 - d) K_{t-s}(x, \alpha, y, y_1) - (\partial_{x_1} x_1 - d) K_{t-s}(x, \beta, y, y_1)]$$
$$(g(s, x, y_1) - g(s, x', y_1)) dx_1 dy_1 ds.$$

We use the Hölder estimate of g (3.33) and

If $0 < d(q) \leq \frac{1}{2}$ or $0 < e(q) \leq \frac{1}{2}$, all the arguments are essentially similar to those above except that we need to replace the integral (3.40) with the estimate: for $0 < d < 1$,

$$\int_{-\infty}^{\infty} \frac{1}{(t-s)^d} e^{-\frac{(\sqrt{x} - \sqrt{x_1})^2}{t-s}} \frac{1}{x_1^{1-d}} \frac{1}{\sqrt{s}} e^{-\frac{(\sqrt{x_1} - \sqrt{x_2})^2}{s}} dx_1 \tag{3.53}$$

$$\leq C_p \frac{1}{(t-s)^d \sqrt{s}} \left(\sqrt{\frac{(t-s)s}{t}} \right)^{2d} e^{-\frac{(\sqrt{x} - \sqrt{x_2})^2}{t}}. \tag{3.54}$$

Notice that the degree of time t on the right hand side above is independent of d, so in total the regularity of time t in this case is same as that when $d > \frac{1}{2}$. □

Then we can show the existence of local heat kernel and analyze their behavior as $t \to 0$.

Proposition 3.5.2. *There exists a local heat kernel satisfying* (3.27) *in* U_0. *Moreover, for any* $f \in C_c(U_0)$,

$$(\partial_t - L) \int_{U_0} q_t(x, y, x_1, y_1) f(x_1, y_1) dx_1 dy_1 = 0, \tag{3.55}$$

and

1. *If* $(0, 0)$ *is a transverse point,*

$$\lim_{t \to 0} \int_{U_0} q_t(0, 0, x_1, y_1) f(x_1, y_1) dx_1 dy_1 = f(0, 0). \tag{3.56}$$

2. *If* $(0, 0)$ *is a tangent point,*

$$\lim_{t \to 0} \int_{U_0} q_t(0, 0, x_1, y_1) f(x_1, y_1) dx_1 dy_1 = 0. \tag{3.57}$$

Proof. First we show that $\mathbf{B}\delta = (\tilde{L} - L_M)K_t$ satisfies the assumption in Proposition 3.5.1. We begin by writing

$$K_t(x, y, x_1, y_1) = p_t^{d(x_1,y_1)}(x, x_1) p_t^{e(x_1,y_1)}(y, y_1).$$

1. If $d(0, 0) > 0$**:** by shrinking the neighborhood U_0, we might as well assume that $d(x, y) \in [d_1, d_2]$ for some $0 < d_1 < d_2$. Then by the estimates in (5.1.1), (5.1.3), (5.28),

$$|\partial_x p_{d(x_1,y_1)}(t, x, x_1)|, |x\partial_x^2 p_{d(x_1,y_1)}(t, x, x_1)| \leq \frac{C}{t^{\frac{3}{2}}} e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}} (x_1^{d_1-1} + \frac{1}{\sqrt{x_1}}),$$
$$|p_{d(x_1,y_1)}(t, x, x_1)| \leq \frac{C}{\sqrt{t}} e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}} (x_1^{d_1-1} + \frac{1}{\sqrt{x_1}}).$$

2. If $d(0,0)=0$**:** so $d(0,y)=0$ by Assumption 3.1.1. Again by shrinking U_0 we may assume that $d(x,y)\in[0,\frac{1}{2}]$. A more precise estimate is

$$p_{d(x,y)}(t,x,y)\le\begin{cases}\frac{1}{t^{d(x,y)}}e^{-\frac{x+y}{t}}y^{d(x,y)-1}\psi_{d(x,y)}(\frac{xy}{t^2}) & \text{when } \frac{xy}{t^2}<1\\ \frac{1}{\sqrt{yt}}e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} & \text{when } \frac{xy}{t^2}>1\end{cases}.$$

Using the expansion of ψ_d at 0, for $0\le z\le 1$,

$$\psi_d(z)=\psi_d(0)+\psi_d'(c)z$$

for some $c\in[0,1]$. ψ_d' is uniform bounded on $d\in(0,\frac{1}{2}]\times[0,1]$. Since $\psi_d(0)=\frac{1}{\Gamma(d)}$, $\frac{1}{\Gamma(0)}=0$ and $\frac{\partial}{\partial d}\frac{1}{\Gamma(d)}$ is uniformly bounded on $(0,\frac{1}{2}]$, where $\Gamma(\cdot)$ is the usual Gamma function, we have

$$\psi_{d(x,y)}(\frac{xy}{t^2})\le\frac{1}{\Gamma(d(x,y))}+C\frac{xy}{t^2}\le C(y+\frac{xy}{t^2}).$$

In all

$$p_{d(x,y)}(t,x,x_1)\le C\frac{1}{\sqrt{x_1t}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}}.$$

Similarly we can show that the derivatives of $p_{d(x,y)}(x,x_1)$ are bounded by

$$|\partial_x p_{d(x_1,y_1)}(t,x,x_1)|,|x\partial_x^2 p_{d(x_1,y_1)}(t,x,x_1)|\le\frac{C}{t^{\frac{3}{2}}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}}\frac{1}{\sqrt{x_1}}.$$

Above all, there exists $d_1,d_2>0$ such that

$$|\partial_x p_{d(x_1,y_1)}(t,x,x_1)|,|x\partial_x^2 p_{d(x_1,y_1)}(t,x,x_1)|\le\frac{C}{t^{\frac{3}{2}}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}}x_1^{d_1-1} \tag{3.58}$$

$$|p_{d(x_1,y_1)}(t,x,x_1)|\le\frac{C}{\sqrt{t}}e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{2t}}x_1^{d_1-1}, \tag{3.59}$$

and

$$|\partial_y p_{e(x_1,y_1)}(t,y,y_1)|, |y\partial_y^2 p_{e(x_1,y_1)}(t,y,y_1)| \leq \frac{C}{t^{\frac{3}{2}}} e^{-\frac{(\sqrt{y}-\sqrt{y_1})^2}{2t}} y_1^{d_2-1} \tag{3.60}$$

$$|p_{e(x_1,y_1)}(t,y,y_1)| \leq \frac{C}{\sqrt{t}} e^{-\frac{(\sqrt{y}-\sqrt{y_1})^2}{2t}} y_1^{d_2-1}. \tag{3.61}$$

Finally because

$$L - L_M = \Theta((x-x_2)+(y-y_2))(x\partial_x^2 + \partial_x + y\partial_y^2 + \partial_y),$$

combined with the estimates above, we conclude that

$$|(L-L_M)K_t| \leq \frac{C}{t^{\frac{3}{2}}} e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2+(\sqrt{y}-\sqrt{y_1})^2}{2t}} x_1^{d_1-1} y_1^{d_2-1}, \tag{3.62}$$

so we can iterate Proposition 3.5.1 until $\mathbf{B}^5\delta \in \mathcal{C}^{0,\gamma}(U_0 \times [0,T])$, for which by the results in Theorem 3.4.1, there exists the solution $Q^t\mathbf{B}^5\delta \in \mathcal{C}^{0,2+\gamma}(P \times [0,T])$. Recall that

$$(\partial_t - \widetilde{L})\left(q_t(\cdot,q) - K_t(t,\cdot,q) - \sum_{i=1}^{N-1} \mathbf{K_t}\mathbf{B}^i\delta\right) = \mathbf{B}^N\delta, \tag{3.63}$$

hence there exists a local heat kernel $q_t(x,y,x_2,y_2)$ in U_0, which has the expansion

$$q_t(x,y,x_2,y_2) = K_t(x,y,x_2,y_2) + \sum_{i=1}^{4} \mathbf{K_t}\mathbf{B}^i\delta + Q^t\mathbf{B}^5\delta. \tag{3.64}$$

Moreover, the derivatives of it

$$\partial_x q_t(x,y,x_1,y_1), \partial_y q_t(x,y,x_1,y_1), x\partial_x^2 q_t(x,y,x_1,y_1), y\partial_y^2 q_t(x,y,x_1,y_1)$$

are also bounded by the right hand side of (3.62).

Therefore, on $[t_0,t_1]$, the integral of these derivatives converges uniformly on $t \in [t_0,t_1], (x,y) \in$

U_0. Hence we can exchange derivatives with integration to obtain that

$$(\partial_t - L)\int_{U_0} q_t(x,y,x_1,y_1)f(x_1,y_1)dx_1dy_1 = \int_{U_0}(\partial_t - L)q_t(x,y,x_1,y_1)f(x_1,y_1)dx_1dy_1 = 0. \tag{3.65}$$

Finally we turn to prove (3.5.2). Based on the expansion (3.64), it suffices to show that (3.5.2) holds for the fundamental solution

$$K_t(0,0,x,y) = p_t^{d(x,y)}(0,x)\cdot p_t^{e(x,y)}(0,y). \tag{3.66}$$

1. If $(0,0)$ **is a transverse point,** we assume that $d(x,y), e(x,y) \in [d_1, d_2]$ for some $0 < d_1 < d_2$. We denote by

$$K_t'(0,0,x,y) = p_t^{d(0,0)}(0,x)\cdot p_t^{e(0,0)}(0,y). \tag{3.67}$$

Since $\lim_{t\to 0} K_t'(0,0,x,y) = \delta(0,0)$, it is suffices to show that

$$\lim_{t\to 0}(K_t(0,0,x,y) - K_t'(0,0,x,y)) = 0. \tag{3.68}$$

We have

$$\begin{aligned} &|K_t(0,0,x,y) - K_t'(0,0,x,y)| \\ \le\ &|(p_t^{d(x,y)}(0,x) - p_t^{d(0,0)}(0,x)||p_t^{e(x,y)}(0,y)| + |(p_t^{e(x,y)}(0,y) - p_t^{e(0,0)}(0,y)||p_t^{d(0,0)}(0,x)| \\ =:\ &I + II. \end{aligned}$$

By the mean value theorem, there exists $c \in [d,e]$ such that

$$|p^d(t,0,y) - p^e(t,0,y)| = |\frac{\partial}{\partial_d}p_c(t,0,y)|\cdot|d-e|.$$

Since

$$|\frac{\partial}{\partial_d}p_c(t,0,y)| = \frac{1}{y}e^{-\frac{y}{t}}\left(\frac{y}{t}\right)^c |\frac{1}{\Gamma(c)}\ln\frac{y}{t} - \frac{\Gamma'(c)}{\Gamma^2(c)}| \leq \frac{C}{y}e^{-\frac{y}{t}}\left(\frac{y}{t}\right)^c\left[ln\frac{y}{t}+1\right]$$
$$\leq \frac{C}{y}e^{-\frac{y}{t}}(\left(\frac{y}{t}\right)^{d_1} + \left(\frac{y}{t}\right)^{d_2})\left[ln\frac{y}{t}+1\right],$$

hence

$$|p_{d_2(x,y)}(t,0,y) - p_{d_2(0,0)}(t,0,y)| \leq C(x+y)\frac{1}{y}e^{-\frac{y}{t}}(\left(\frac{y}{t}\right)^{d_1} + \left(\frac{y}{t}\right)^{d_2})\left[ln\frac{y}{t}+1\right]$$

by the Taylor expansion of d_2 at (0,0). It is not hard to show that

$$\lim_{t\to 0}\int_0^\infty xp_{d_1(0,0)}(t,0,x)dx = 0$$
$$\lim_{t\to 0}\int_0^\infty y\frac{1}{y}e^{-\frac{y}{t}}(\left(\frac{y}{t}\right)^M + \left(\frac{y}{t}\right)^N)\left[ln\frac{y}{t}+1\right]dy = 0.$$

Therefore we have

$$\lim_{t\to 0}\int_{U_0} II\cdot f(x_1,y_1)dx_1dy_1 = 0.$$

Also we can obtain that

$$\lim_{t\to 0}\int_{U_0} I\cdot f(x_1,y_1)dx_1dy_1 = 0,$$

which leads to (3.68).

2. If $(0,0)$ **is a tangent point,** we assume that $e(0,0)=0$. Then $e(x,0)=0$ by the assumption (3.1.1). We also assume that $e(x,y)\in[0,M]$. We have

$$p_t^{e(x,y)}(0,y) = \frac{1}{t^{e(x,y)-1}}y^{e(x,y)}e^{-\frac{y}{t}}\frac{1}{\Gamma(e(x,y))} \leq C\frac{1}{t^{e(x,y)-1}}y^{e(x,y)+1}e^{-\frac{y}{t}}$$
$$\leq Ct(p_t^1(0,y) + p_t^{M+1}(0,y)).$$

If $d(0,0)=0$, then it is also bounded by $Ct(p_t^1(0,y)+p_t^{M+1}(0,y))$, so

$$\lim_{t\to 0}\ K_t(0,0,x,y)=0$$

in the sense of distribution. If $d(0,0)>0$, by the estimation in the previous case,

$$\lim_{t\to 0}\ (p_t^{d(x,y)}(0,x)-p_t^{d(0,0)}(0,x))=0,$$

combined with

$$\lim_{t\to 0}\ p_t^{d(0,0)}(0,x)p_t^{e(x,y)}(0,y)=0,$$

we obtain that

$$\lim_{t\to 0} K_t(0,0,x,y)=0 \tag{3.69}$$

in the sense of distribution. □

Other cases

1. q in a neighborhood of a regular edge point p. When $q=(x_2,y_2)\in U_3$ is in a neighborhood of a regular edge point, under local adapted coordinates, L takes the form

$$L=a(x,y)x\partial_x^2+b(x,y)\partial_{yy}+d(x,y)\partial_x+e(x,y)\partial_y.$$

Replacing $d(t,x,y)$ in Proposition 3.5.1 with

$$d(t,x,y)=\frac{(\sqrt{x}-\sqrt{x_2})^2}{2a(q)t}+\frac{(y-y_2)^2}{4b(q)t},$$

we can similarly prove the existence of the local heat kernel through its series expansion

$$q_t(x,y,x_2,y_2) = K_t(x,y,x_2,y_2) + \sum_{i=1}^{4} \mathbf{K_t B}^i \delta + Q^t \mathbf{B}^5 \delta$$

where $K_t(x,y,x_2,y_2) = p_{a(q)t}^{d(q)/a(q)}(x,x_2) k_{b(q)t}^{e}(y,y_2)$. By comparing $K_t(p,q)$ with fundamental solution of model operator with coefficients frozen at p, we can derive the result like in 3.68, 3.69. That being said, if p is a transverse point, then $\lim_{t\to 0} q_t(p,\cdot) = \delta(p)$, if p is a tangent point, then $\lim_{t\to 0} q_t(p,\cdot) = 0$.

2. q in a neighborhood of a mixed corner/infinity edge point p. When $q = (x_2,y_2) \in U_1$ is in a neighborhood of a regular edge point, under local adapted coordinates 3.2, L takes the form

$$L = a(x,y)x\partial_x^2 + b(x,y)y^2\partial_{yy} + d(x,y)\partial_x + e(x,y)y\partial_y.$$

We can similarly prove the existence of the local heat kernel through its series expansion

$$q_t(x,y,x_2,y_2) = K_t(x,y,x_2,y_2) + \sum_{i=1}^{4} \mathbf{K_t B}^i \delta + Q^t \mathbf{B}^5 \delta.$$

where $K_t(x,y,x_2,y_2) = p_{a(q)t}^{d(q)/a(q)}(x,x_2)\frac{1}{\sqrt{4\pi t}} exp\left[-\frac{(\ln y - \ln y_2)^2}{4b(x_2,y_2)t}\right]\frac{1}{y_2}$. Particularly when $y = 0$,

$$q_t(x,0,x_2,y_2) = p_t(x,x_2)\delta_0(y_2),$$

where $p_t(x,x_2)$ is a 1-dimensional heat kernel of Kimura operator on $y = 0$. This implies that the infinity edge is isolated from $\mathring{P}$, the diffusion starting from the E_∞ stay on it.

3. when q in a neighborhood of an infinity corner p. When $q = (x_2, y_2) \in U_3$ is in a neighborhood of a regular edge point, under local adapted coordinates, L takes the form

$$L = a(x,y)x^2\partial_x^2 + b(x,y)y^2\partial_y^2 + d(x,y)x\partial_x + e(x,y)y\partial_y.$$

Again we can similarly prove the existence of the local heat kernel through its series expansion. $q_t(x, y, x_2, y_2)$ takes the form

$$p_t(x,y,x_2,y_2)exp\left[-\frac{(\ln x - \ln x_2)^2}{4a(x_2,y_2)t} - \frac{(\ln y - \ln y_2)^2}{4b(x_2,y_2)t}\right]\frac{1}{x_2 y_2},$$

where $p_t(x, y, x_2, y_2)$ is bounded for $t > 0$. When $x = 0$ or $y = 0$,

$$q_t(0,y,x_2,y_2) = \delta_0(x_2) \cdot p_t(y,y_2)$$
$$q_t(x,0,x_2,y_2) = p_t(x,x_2) \cdot \delta_0(y_2)$$

where p_t is the 1-dimensional heat kernel on $y = 0$. In particular

$$q_t(0,0,x_2,y_2) = \delta_{(0,0)}(x_2,y_2),$$

which means $(0, 0)$ is an isolated point.

In all for $q \in \mathring{P}$, if $p \in P^{reg}$, the heat kernel $q_t(p, q)$ is well defined and continuous at p. In other cases, $q_t(p, q) = 0$. Specifically $q_t(p, \cdot)$ degenerates to a 1-dimensional along the quadratic edge when $p \in E_\infty \cup C_{mix}$, and $q_t(p, \cdot)$ is the delta function at p when $p \in C_\infty$. On the other hand for $p \in P^{reg}$, we investigated the limit behavior of $q_t(p, \cdot)$ when $t \to 0$. It tends to $\delta(p)$ if p is a transverse point and tends to 0 in the sense distribution if p is a tangent point. We summarize this in the following table:

p		p_0
C_{reg}	transverse	δ_p
	tangent	0
C_{mix}		0
C_∞		0
E_{reg}	transverse	δ_p
	tangent	0
E_∞		0

Table 3.1: Limit of $q_t(p,\cdot)$ as $t \to 0$

Proposition 3.5.3. *Fix $\alpha > 0$ and $k, l \in \mathbb{N}$, there exists constants $C, c > 0$ depending on α, k, l such that if $|\sqrt{x} - \sqrt{x_1}| \geq \alpha > 0$, $|y - y_1| \geq \alpha > 0$ and $0 \leq x, x_1, y, y_1 \leq L$, then*

$$|(\partial_x)^k p_d(t, x, x_1)| \leq Ce^{-c/t} x_1^{d-1}, \qquad |(\partial_y)^l k_t^e(y, y_1)| \leq Ce^{-c/t}.$$

Proof. For the first estimate, let $\lambda = \frac{x}{t}, w = \frac{x_1}{t}$. When $k = 0$,

$$p_d(t, x, y) y^{1-d} = \frac{1}{t^d} e^{-(\lambda+w)} \psi_d(\lambda w).$$

1. If $\lambda w \leq 1$, then this term is bounded by

$$\frac{C}{t^d} e^{-(\lambda+w)} \leq \frac{C}{t^d} e^{-(\sqrt{\lambda}-\sqrt{w})^2} = \frac{C}{t^d} e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{t}}.$$

2. If $\lambda w \geq 1$, using the asymptotic expansion $\psi_d(z) \sim \frac{z^{\frac{1}{4}-\frac{d}{2}} e^{2\sqrt{z}}}{\sqrt{4\pi}}$,

$$\frac{1}{t^d} e^{-(\lambda+w)} \psi_d(\lambda w) \sim \frac{1}{\sqrt{4\pi} t^d} e^{-(\sqrt{\lambda}-\sqrt{w})^2} (\lambda w)^{\frac{1}{4}-\frac{d}{2}}.$$

The right hand side is bounded by $\frac{1}{\sqrt{4\pi} t^d} e^{-(\sqrt{\lambda}-\sqrt{w})^2}$ if $d \geq \frac{1}{2}$, and bounded by

$\frac{L^{\frac{1}{2}-d}}{\sqrt{4\pi t}} e^{-(\sqrt{\lambda}-\sqrt{w})^2}$ if $0 < d \leq \frac{1}{2}$. Overall,

$$p_d(t,x,y) \leq \frac{C}{t^{d+\frac{1}{2}}} e^{-\frac{(\sqrt{x}-\sqrt{x_1})^2}{t}} \frac{1}{y^{d-1}}.$$

The estimates for higher order derivatives are proved similarly.

The second estimate is not hard to get since we can repeatedly use that for $\forall b \geq 0$,

$$\frac{1}{t^b} e^{-\frac{(y-y_1)^2}{4ct}} \leq \frac{C_b}{t^b} \frac{t^b}{|y-y_1|^{2b}} \leq C_{b,\alpha}.$$

□

3.5.3 Proof of Theorem 3.5.1

So far we have constructed the local heat kernel $q_{ij}^t(\cdot, q)$ for $\forall q \in \mathring{P}$. By the classical elliptic theory, there exists the Dirichlet heat kernel q_t^U in P_U. We patch them together by defining the global kernel parametrix:

$$q_t(d_1,d_2,l_1,l_2) = \sum_{i=0}^{4} \sum_{j} \phi_{i,j,\epsilon}(d_1,d_2)\, q_t^{ij}\left((\psi_{ij,\epsilon}(d_1), \psi_{ij,\epsilon}(d_2), \psi_{ij,\epsilon}(l_1), \psi_{ij,\epsilon}(l_2)\right) \tag{3.70}$$

$$\cdot \chi_{ij,\epsilon}(l_1,l_2) \cdot |\det \psi_{ij}(l_1,l_2)| \tag{3.71}$$

$$+\psi q_t^U(d_1,d_2,l_1,l_2)(1-\varphi_U). \tag{3.72}$$

Now set

$$e_t(d_1,d_2.l_1,l_2) = (\partial_t - L) q_t(d_1,d_2,l_1,l_2).$$

Again

$$e_t(d_1,d_2,l_1,l_2) = \left(\sum_{i=0}^{3}\sum_{j}\phi_{i,j,\epsilon}(\widetilde{L^{ij}} - L^{ij})q^t_{i,j}\chi_{i,j,\epsilon}\right) + \left(\sum_{i=0}^{3}\sum_{j}[\phi_{i,j,\epsilon}, L]q^t_{i,j}\chi_{i,j,\epsilon}\right)$$
$$+[\psi, L]q^U_t(1-\varphi_U).$$

By construction, $e_t(d_1,d_2,l_1,l_2)$ is supported on $P_U \times P \times [0,\infty)$. To be more precise, the support of $e_t(d_1,d_2,l_1,l_2)$ is in an off-diagonal region: $d\left((d_1,d_2),(l_1,l_2)\right) > \alpha$ for some α, hence for $\forall T>0$, in $[0,T]$ it is bounded and by $Ce^{-\frac{C_\alpha}{t}}$ for some $C_\alpha > 0$ using the estimate in Proposition 3.5.3. We let $A^t e_t$ be the solution to the inhomogeneous problem

$$(\partial_t - L)A^t e_t(\cdot,l_1,l_2) = e_t(\cdot,l_1,l_2) \text{ in } P\times[0,T] \text{ with } A^0(\cdot,l_1,l_2)=0,$$

Thus the global heat kernel is given by

$$H_t\left(d_1,d_2,l_1,l_2\right) = q_t\left(d_1,d_2,l_1,l_2\right) - A^t e_t\left(d_1,d_2,l_1,l_2\right).$$

Next we investigate the regularity of the heat kernel H_t. If L is Kimura operator, i.e., all the edges of P are of Kimura type, as studied in

For $\forall t_0 \in (0,T]$, fix $q\in \mathring{P}$, the estimation in Proposition 3.5.1 shows that the local heat kernel $q_{t_0}(\cdot,q)$ are in local $C^{0,\gamma}$ spaces, and the perturbation term $A^t e_t(\cdot,q)$ is also in $C^{0,\gamma}(P)$. Thus $H_{t_0}(\cdot,q)\in C^{0,\gamma}(P)$. We apply the regularity statement above to the Cauchy problem with initial condition $H_{t_0}(\cdot,q)$, giving that $H_t(\cdot,q)\in C^\infty(P\times(t_0,T])$. Letting $t_0\to 0, T\to\infty$, then we have $(\cdot,t)\mapsto H_t(\cdot,q)\in C^\infty(P\times(0,\infty))$ for $\forall q\in\mathring{P}$ in the Kimura case.

When L has a mixed type of boundary conditions, fix $q\in\mathring{P}$, we choose a neighborhood U_Q of all the quadratic edge with $q\notin U_Q$ and $\chi\in C^\infty(U_Q^c)$ so that $\chi\equiv 1$ away from U_Q, $\chi(q)=1$. Let K be a Kimura operator on P so that the transverse/tangent boundary conditions align with L. We define the new operator

$$\tilde{L} = \chi L + (1 - \chi)K,$$

such defined $\tilde{L}$ is a Kimura operator, $\tilde{L} = L$ away from U_Q, particularly $\tilde{L}(q) = L(q)$. Denote the heat kernel of $\tilde{L}$ by $\tilde{H}_t$, we have

$$(\partial_t - L)(H_t - \tilde{H}_t)(\cdot, q) = (L - \tilde{L})\tilde{H}_t(\cdot, q) \tag{3.73}$$

$$\lim_{t \to 0}(H_t - \tilde{H}_t)(\cdot, q) = 0. \tag{3.74}$$

We have shown that $\tilde{H}_t \in C^\infty(P \times (0, \infty))$. Since $L = \tilde{L}$ at q, the support of $(L - \tilde{L})\tilde{H}_t(\cdot, q)$ is away from q, so $(t, \cdot) \mapsto (L - \tilde{L})\tilde{H}_t(\cdot, q)$ is smooth and its high order derivatives are bounded by $e^{-\frac{C}{t}}$ for some $C > 0$. Moreover since $q \in \mathring{P}$, when constructing the parametrix 3.70 of global heat kernel $H_t(\cdot, q), \tilde{H}_t(\cdot, q)$, we can choose the same Dirichlet heat kernel in a vicinity of q. Then the two remaining perturbation terms $A^t e_t(\cdot, q), A^t \tilde{e}_t(\cdot, q)$ are both in $C^{k,\gamma}(P \times [0, T])$ for $k \in \mathbb{N}$. This roughly shows that $(H_t - \tilde{H}_t)(\cdot, q) \in C^{k,\gamma}(P \times [0, T])$. Therefore by Theorem 3.5.1 $H_t - \tilde{H}_t \in C^\infty(P \times [0, T])$. The argument above works for $\forall T > 0$, so indeed we showed that $(\cdot, t) \mapsto H_t(\cdot, q) \in C^\infty(P \times (0, \infty))$.

Finally for the regularity of the forward variable, for $p \in P^{reg}$, $H_t(p, \cdot)$ is a solution to the Kolmogorov forward equation $(\partial_t - L^*)H_t(p, \cdot) = 0$. Therefore by standard hypoellicity results for parabolic operators (

CHAPTER 4

LONG-TIME BEHAVIOR OF A MIXED KIND OF KIMURA DIFFUSION OPERATOR

4.1 Introduction

This chapter analyzes the long time behavior of diffusion processes with infinitesimal generator given by a mixed type Kimura operator L on one-dimensional and two-dimensional manifolds with corners.

Let L be a degenerate second-order differential operator. For an edge point p, when written in an adapted system of local coordinates on $\mathbb{R}_+ \times \mathbb{R}$, L takes the form:

$$L = a(x,y)x^m \partial_{xx} + b(x,y)x^{m-1}\partial_{xy} + c(x,y)\partial_{yy} + d(x,y)x^{m-1}\partial_x + e(x,y)\partial_y, \tag{4.1}$$

where we assume that a, b, c, d, e are smooth functions and that $a(x,y) > 0$ and $c(x,y) > 0$. Also $m \in \{1, 2\}$. When $m = 1$, the edge $x = 0$ is of Kimura type in that the coefficients vanish linearly towards it. When $m = 2$, the edge $x = 0$ is of quadratic type as the coefficients now vanish quadratically.

For a corner p, as an intersection point of two edges, L has the following normal form:

$$L = a(x,y)x^m \partial_{xx} + b(x,y)x^{m-1}y^{n-1}\partial_{xy} + c(x,y)y^n\partial_{yy} + d(x,y)x^{m-1}\partial_x + e(x,y)y^{n-1}\partial_y, \tag{4.2}$$

when written in an adapted system of local coordinates on $\mathbb{R}^2_+$, where $a(x,y) > 0$, $c(x,y) > 0$, and $m, n \in \{1, 2\}$.

Associated to the operator L is a C^0 semigroup $\mathcal{Q}_t = e^{tL}$ solution operator of the Cauchy problem

$$\partial_t u = Lu$$

with initial conditions $u(x, 0) = f(x)$ at $t = 0$. The operator e^{tL} and some of its properties are presented in detail in

It turns out that the number of possible invariant measures and their type (absolutely continuous with respect to one-dimensional or two-dimensional Lebesgue measures or not) strongly depend on the structure of the coefficients (a, b, c, d, e). We thus distinguish the different boundary types that influence the long time asymptotics of transition probabilities.

Definition 4.1.1. *A Kimura edge E is called a tangent (Kimura) edge when $d(0, y) = 0$ and a transverse (Kimura) edge when $d(0, y) > 0$.*

A quadratic edge E is called a tangent (quadratic) edge when $\frac{d(0,y)}{a(0,y)} < 1$, a transverse (quadratic) edge when $\frac{d(0,y)}{a(0,y)} > 1$, and a neutral (quadratic) edge when $\frac{d(0,y)}{a(0,y)} = 1$.

We assume that:

Assumption 4.1.1. *Every edge is either tangent, transverse, or neutral.*

Note that we do not consider the setting with $d(0, y) < 0$ on a Kimura edge. In such a situation, diffusive particles pushed by the drift term $d(0, y) < 0$ have a positive probability of escaping the domain P. We would then need to augment the diffusion operator with appropriate boundary conditions.

The long-time analysis in two-dimensions for a class of operators including a specific example of interest in the field of topological insulators

To describe all invariant measure on the interval $[0, 1]$ in one dimensional, consider

$$L = a(x)x^{m(0)}(1-x)^{m(1)}\frac{d^2}{dx^2} + b(x)x^{m(0)-1}(1-x)^{m(1)-1}\frac{d}{dx}, \tag{4.3}$$

with $a(x), b(x) \in C^\infty([0, 1])$.

For $i = 0, 1$, we say that $x = i$ is of Kimura type when $m(i) = 1$ and of quadratic type when $m(i) = 2$. When $x = i$ is of Kimura type, we assume that the vector field $b(x)\frac{d}{dx}$ is inward pointing at $x = i$. For brevity, we use $\widetilde{a}$ and $\widetilde{b}$ to denote

$$\widetilde{a}(x) = a(x)x^{m(0)}(1-x)^{m(1)}, \quad \widetilde{b}(x) = b(x)x^{m(0)-1}(1-x)^{m(1)-1}. \tag{4.4}$$

Definition 4.1.2. *When $x = 0$ (1 resp.) is a Kimura endpoint, we say it is a tangent point if $b(0) = 0$ ($b(1) = 0$ resp.) and a transverse point if $b(0) > 0$ ($b(1) < 0$ resp.).*

When $x = 0$ (1 resp.) is a quadratic endpoint, we say it is a tangent point if $\frac{b(0)}{a(0)} < 1$ ($\frac{b(1)}{a(1)} > -1$ resp.), a transverse point if $\frac{b(0)}{a(0)} > 1$ ($\frac{b(1)}{a(1)} < -1$ resp.), and a neutral point if $\frac{b(0)}{a(0)} = 1$ ($\frac{b(1)}{a(1)} = -1$ resp.).

The quadratic endpoint and the tangent Kimura endpoint are sticky boundary points in the sense that the Dirac measure supported on them is an invariant measure. When both endpoints are transverse, there is another invariant measure μ with full support on the whole interval. By computing the index of L on an appropriate Hölder space, we characterize the kernel space of $\overline{L}^*$ composed of invariant measures for the diffusion L.

In both cases, starting from a point in P, the corresponding transition probability of the diffusion converges to the invariant measure at an exponential rate. In cases with at least one tangent boundary point, we consider a functional space of functions that vanish at the tangent boundary points and show that L has a spectral gap on such a space. In the absence of tangent boundary points, we prove that the invariant measure μ satisfies an appropriate Poincaré inequality so that L also admits a spectral gap in $L^2(\mu)$. Our main convergence results for $\mathcal{Q}_t = e^{tL}$, whose properties are described in Theorem 4.2.1, are summarized in Theorems 4.4.1 and 4.4.2 below.

In two space dimensions, we do not consider all possible invariant measures as a function of the nature of the drift terms d and e in the vicinity of edges or corners. Instead, we restrict ourselves to the following case:

Assumption 4.1.2. *For L on a 2 dimensional compact manifold with corners P, there is exactly one tangent edge H, and when restricted to H, $L|_H$ is transverse to both boundary* *points.*

This case involves exactly one tangent edge with two transverse boundary points so that, applying results from the one-dimensional case, we find that L has a unique invariant measure μ fully supported on (the one-dimensional edge) H. Starting from any point p not on the quadratic edge, we show in Theorem 4.5.2 that the transition probability converges to μ at an exponential rate in the Wasserstein distance sense. The main tool used in the convergence is the construction of a Lyapunov function in Theorem 4.5.1.

The setting of P a triangle with two transverse Kimura edges while the third edge is quadratic with transverse endpoints as described in Assumption 4.1.2 finds applications in the analysis of the asymmetric transport observed at an edge separating topological insulators

There is a large literature on the analysis of the long-time behavior of e^{tL} when L is non-degenerate and when L is of Kimura type. In the latter case, L is the generalized Kimura operator studied in

However, in the presence of quadratic edge/point, L_γ is not Fredholm. The reason is that near such quadratic edges or points, the operator may be modeled by an elliptic (non-degenerate) operator on an infinite domain (with thus continuous spectrum in the vicinity of the origin). We thus need another approach that builds on the following previous works. In

An outline of the rest of this paper is as follows. The semigroup e^{tL} is analyzed in section 4.2 in the one-dimensional case. The space of invariant measures associated with a one-dimensional diffusion L, which depends on the structure of the drift term at the two boundary points, is constructed in section 4.3; see Table 4.1 for a summary. The exponential convergence of the kernel of e^{tL} (the Green's function) to an appropriate invariant measure over long times is demonstrated in section 4.4.

The operator e^{tL} in the two-dimensional setting is analyzed in

4.2 The C^0 Semigroup in one-space dimension

Let L be the one-dimensional mixed-type Kimura operator on $[0,1]$ given in (4.3). Let $\mathcal{Q}_t = e^{tL}$ be the solution operator of the Cauchy problem for the generator L and denote by $q_t(x,y)$ its kernel. Its main properties are summarized in the following result:

Theorem 4.2.1. *The operator $\mathcal{Q}_t$ defines a positivity preserving semigroup on $C^0([0,1])$. For $f \in C^0([0,1])$, the function $u(x,t) = \mathcal{Q}_t f(x)$ solves the Cauchy Problem for L with initial condition $f(x)$ in the sense that*

$$\lim_{t\to 0+} ||\mathcal{Q}_t f - f||_{\mathcal{C}^0} = 0. \tag{4.5}$$

Proof. If both endpoints are of Kimura type, L is the 1D Kimura operator. By

For the remaining case, we might as well assume that $x=0$ is a Kimura endpoint and $x=1$ a quadratic endpoint. We intend to build the global solution out of local solution near the boundary. Let $([0,1-\eta],\phi_0),([\eta,1],\phi_1)$ for some $0<\eta<\frac{1}{4}$ small be the coordinate charts so that pulling back L to these coordinate charts gives two local operators

$$L_0 = x\partial_x^2 + b_0\partial_x + xc(x)\partial_x, \quad x \in [0,\phi_0(1-\eta)),$$
$$L_1 = \partial_z^2 + d(z)\partial_z, \quad z \in (\phi_1(\eta),\infty).$$

We extend these two local operators to the whole sample space

$$\widetilde{L}_0 = x\partial_x^2 + b_0\partial_x + xc(x)\varphi_0(x)\partial_x, \quad \widetilde{L}_1 = \partial_z^2 + d(z)\varphi_1(z)\partial_z$$

where $\varphi_0(x)$ is a smooth cutoff function so that

$$\varphi_0(x) = \begin{cases} 1 & \text{for } x \in [0,\phi_0(1-2\eta)] \\ 0 & \text{for } x > \phi_0(1-\eta), \end{cases} \qquad \varphi_1(z) = \begin{cases} 1 & \text{for } z \in [\phi_1(2\eta),\infty) \\ 0 & \text{for } z < \phi_1(\eta). \end{cases}$$

Let $\widetilde{Q}_t^0, \widetilde{Q}_t^1$ be the solution operators of $\widetilde{L}_0, \widetilde{L}_1$ and denote their kernels by $\widetilde{q}_t^0, \widetilde{q}_t^1$ respectively. Define smooth cutoff functions $0 \leq \chi, \psi_0, \psi_1 \leq 1$ so that

$$\text{supp}\psi_0 \subset [0, 1-2\eta],\ \text{supp}\psi_1 \subset [2\eta, 1],\ \psi_0|_{\text{supp}\chi} \equiv 1,\ \psi_1|_{\text{supp}(1-\chi)} \equiv 1. \tag{4.6}$$

Given $f \in C^0([0,1])$ and $g \in C^0([0,1] \times [0,T])$, set the homogeneous and inhomogeneous solution operator as

$$\widetilde{Q}_t f = \psi_0 \widetilde{Q}_t^0[\chi f] + \psi_1 \widetilde{Q}_t^1[(1-\chi)f],$$
$$A_t g = \int_0^t \widetilde{Q}_{t-s} g(s) ds.$$

Then

$$(\partial_t - L)\widetilde{Q}_t f = E_t^0 f := [\psi_0, L]\widetilde{Q}_t^0[\chi f] + [\psi_1, L]\widetilde{Q}_t^1[(1-\chi)f],$$
$$(\partial_t - L)A_t g = (Id - E_t)g := g - [\psi_0, L]A_t^0[\chi g] - [\psi_1, L]A_t^1[(1-\chi)g].$$

Our choice of χ, ψ_0, ψ_1 (4.6) ensures that $\text{dist}(\text{supp}[\psi_0, L], \text{supp}\chi) > 0$, $\text{dist}(\text{supp}[\psi_1, L], \text{supp}(1-\chi)) > 0$, which ensures that E_t^0, E_t are bounded operators with operator norms bounded by $O(e^{-\frac{c}{t}})$ for some constant $c > 0$ as $t \to 0^+$. Hence for $T > 0$ small enough, there exists an inverse $(Id - E_t)^{-1}$, which can be expressed as a convergent Neumann series in the operator norm topology of $C^0([0,1] \times [0,T])$. Finally we can express the solution operator by

$$\mathcal{Q}_t f = \widetilde{Q}_t f - A_t (Id - E_t)^{-1} E_t^0 f.$$

Since both $\widetilde{Q}_t^0, \widetilde{Q}_t^1$ are strongly continuous, then so is $\widetilde{Q}_t$:

$$\lim_{t \to 0} ||\widetilde{Q}_t f - f||_{C^0} = 0.$$

As $(Id - E_t)^{-1}E_t^0$ is a bounded map from $C^0([0,1]$ to $C^0([0,1]\times[0,T])$, we have

$$||A_t(Id - E_t)^{-1}E_t^0||_{C^0([0,1])\to(C^0([0,1]),t)} = o(t).$$

Therefore (4.5) holds. Let $\widetilde{q}_t, h_t$ be the heat kernels of $\widetilde{Q}_t, (Id - E_t)^{-1}E_t^0$. We express the heat kernel of $\mathcal{Q}_t$ as

$$q_t(x,y) = \widetilde{q}_t(x,y) - \int_0^t \int_0^1 \widetilde{q}_{t-s}(x,z)h_s(z,y)dzds.$$

□

4.3 Invariant Measures in dimension one

In this section, we aim to find all invariant measures of L in spatial dimension one. For convenience of computation, in this section we first choose a global coordinate ϕ so that $(W_0,\phi),(W_1,\phi)$ is a cover of $[0,1/3],[2/3,1]$ under which L takes the following normal form:

1. In $([0,1/3],\phi)$, L_0 has two possible forms:

$$L_0 = x\partial_x^2 + b(x)\partial_x, \text{ if 0 is a Kimura endpoint}$$
$$L_0 = \partial_z^2 + b(z)\partial_z, \text{ if 0 is a quadratic endpoint}$$

2. In $([2/3,1],\phi)$, L_1 has two possible forms:

$$L_1 = (1-x)\partial_x^2 + b(x)\partial_x, \text{ if 1 is a Kimura endpoint}$$
$$L_1 = \partial_z^2 + b(z)\partial_z, \text{ if 1 is a quadratic endpoint}$$

where we use b to denote the first-order term in all cases.

Notation 2. *We call the global coordinate ϕ on $[0,1]$* heat coordinates *if L_0, L_1 have forms* *above under ϕ.*

Let

$$b_\pm = \lim_{x\to 1,0} b(x) \qquad \text{or} \qquad b_\pm = \lim_{z\to\pm\infty} b(z). \tag{4.7}$$

A straightforward derivation shows that, if in the original coordinate, $L_0 = x^2\partial_x^2 + (b_- + 1)x\partial_x$, then after turning to heat coordinates $x = e^z$, L_0 takes the form $\partial_z^2 + b_-\partial_z$.

4.3.1 Functional settings and index associated to L

Our functional setting involves local Hölder spaces, which differ from the usual Hölder space (with $|\cdot|_{k+\gamma}$ to denote its norm) near the boundaries and are variations of those used in

$f \in C^1(U)$ belongs to $C^{1+\gamma}(U)$ if the functions $\partial_z f, f$ can be continuously extend to $z = -\infty$ and local $C^{1+\gamma}$ norm is finite:

$$||f||_{1+\gamma,U} = |f|_{1+\gamma,U} + ||\partial_z f||_{\gamma,U}; \tag{4.8}$$

$f \in C^2(U)$ belongs to $C^{2+\gamma}(U)$ if the functions $\partial_z^2 f, \partial_z f, f$ extend continuously to $z = -\infty$ and the local $C^{2+\gamma}$ norm is finite:

$$||f||_{2+\gamma,U} = |f|_{2+\gamma,U} + ||\partial_z^2 f||_{\gamma,U}. \tag{4.9}$$

For Kimura type boundaries, with $U = (0, c]$,

1. *$f \in C^0(U)$ belongs to $\mathcal{D}^\gamma(U)$ if the function xf can be continuously extend to $x = 0$, and the local C^γ norm is finite:*

$$||f||_{\gamma,U} = |xf|_{\gamma,U}; \tag{4.10}$$

2. *$f \in C^0(\overline{U}) \cap C^1(U)$ belongs to $C^{1+\gamma}(U)$ if the function $x\partial_x f$ can be continuously extend to $x = 0$ and vanish, and the local $\mathcal{C}^{1+\gamma}$ norm is finite:*

$$||f||_{1+\gamma,U} = |f|_{\gamma,U} + |x\partial_x f|_{\gamma,U}; \tag{4.11}$$

3. *$f \in C^1(\overline{U}) \cap C^2(U)$ belongs to $C^{2+\gamma}(U)$ if the function $x\partial_x^2$ can be continuously extend to $x = 0$ and vanish, and the local $\mathcal{C}^{2+\gamma}$ norm is finite:*

$$||f||_{2+\gamma,U} = |f|_{\gamma,U} + |\partial_x f|_{\gamma,U} + |x\partial_x^2 f|_{\gamma,U}. \tag{4.12}$$

Remark 4.3.1. *At neutral quadratic endpoint, interchanging the two integral signs, we have*

$$\int_{z_1}^{z_2} \int_{-\infty}^{z} f(s) s ds dz = \int_{-\infty}^{z_2} (z_2 - s) f(s) ds,$$

so $\int_{-\infty}^{z_2} s f(s) ds < \infty$.

We now build global norms on spaces of functions on $[0, 1]$ out of the above local norms.

Definition 4.3.1. *Let $W_2 \subset\subset (0, 1)$ covering $[0, 1] \setminus (W_0 \cup W_1)$ and $\varphi_0, \varphi_1, \varphi_2$ be a partition of unity subordinate to this cover. A function $f \in C^{2+\gamma}([0, 1])$ if $(\varphi_i f) \circ \phi \in C^{2+\gamma}(W_i)$ for each i and the global norm is*

$$||f||_{2+\gamma} = \sum_i ||(\varphi_i f) \circ \phi||_{2+\gamma,W_i}.$$

Motivated by (4.19) below, we define $f \in C^{1+\gamma}([0, 1])$ if $\phi_i' \cdot (\varphi_i f) \circ \phi \in C^{1+\gamma}(W_i)$ for each i and the global norm is

$$||f||_{1+\gamma} = \sum_i ||\phi_i'(\varphi_i f) \circ \phi||_{1+\gamma,W_i}.$$

Finally $f \in \mathcal{D}^\gamma([0,1])$ *if* $(\phi_i')^2 \cdot (\varphi_i f) \circ \phi \in C^\gamma(W_i)$ *for each* i *and the global norm is*

$$||f||_\gamma = \sum_i ||(\phi_i')^2 (\varphi_i f) \circ \phi||_{\gamma, W_i}.$$

Different choices of coverings give rise to equivalent norms. Endowed with these norms, we now show that the domains and target spaces of M are all Banach spaces.

Proposition 4.3.1. *For* $0 < \gamma < 1$*, all the spaces defined in Definition 4.3.1 are all Banach spaces.*

Proof. Take $C^{2+\gamma}([0,1])$ as an example. Let

$$i : C^{2+\gamma}([0,1]) \to C^{2+\gamma}(W_0) \times C^{2+\gamma}(W_1) \times C^{2+\gamma}(W_2)$$

be the inclusion by mapping f to $((\varphi_0 f) \circ \phi, (\varphi_1 f) \circ \phi, (\varphi_2 f) \circ \phi)$. This inclusion is closed since $\varphi_0, \varphi_1, \varphi_2$ is a partition of unity. Thus we need to show that each local $(0, \gamma)$ space defined at the beginning of this subsection is Banach.

The cases when $U = (0, c]$ are verified in

For $C^{1+\gamma}(U), C^{2+\gamma}(U)$, the above proof applies to show that there exists a limit $u \in C^{1+\gamma}(U), C^{2+\gamma}(U)$ respectively, and $\partial_z f, \partial_{zz} f \in \mathcal{D}^\gamma(U)$, so that these two spaces are also Banach spaces. $\square$

$$L_\gamma : C^{2+\gamma}([0,1]) \longrightarrow \alpha \cdot \mathcal{D}^\gamma([0,1]), \tag{4.13}$$

where $\alpha(x) = x^{m(0)}(1-x)^{m(1)}$ with $m(0) = 1$ if $x = 0$ is Kimura and $m(0) = 0$ otherwise, while $m(1)$ is defined similarly. We first state the main theorem of this section and leave the proof to Section 3.3.

Theorem 4.3.1. *L_γ is a Fredholm operator and its index is*

$$ind\ (L_\gamma) = \kappa^+ + \kappa^-$$

where κ^+, κ^- are the number of positive b_+ and negative b_-, respectively, where $b_\pm$ is defined in (4.7).

4.3.2 Null Space of $\overline{L}^*$

Let $\overline{L}$ denote the $C^0([0,1])$-graph closure of L with domain $C^{2+\gamma}([0,1])$. Having obtained the index of L_γ, we are now able to find the null space of the adjoint operator $\overline{L}^*$. We first use the maximum principle below to find the kernel space of L_γ.

Lemma 4.3.1 (Maximum Principle)**.** *Suppose that $w \in C^{2+\gamma}([0,1])$ is a subsolution of L, $Lw \geq 0$ in a neighborhood, U of a transverse boundary point p. If w attains a local maximum at p, then w is a constant on U.*

Proof. The case when L is of generalized Kimura type is studied in

By subtracting $v(-\infty)$ from v, we may assume that $v(-\infty) = 0$. Integrating $(\partial_z^2 + b\partial_z)v$ we see that $\partial_z v + bv \geq 0$ in a neighborhood U of $-\infty$. Thus $\partial_z v \geq -bv \geq 0$ in U. As w is not a constant, we can expand this neighbourhood until some z_0 such that $\partial_z v > 0$. Then $v(z_0) = \int_{-\infty}^{z_0} \partial_z v dz > 0$, which contradicts the fact that $-\infty$ is a local maximum. □

Theorem 4.3.2. *dim ker $L_\gamma = 2$ if and only if L_γ is tangent to both endpoints; otherwise dim ker $L_\gamma = 1$.*

Proof. The kernel of L_γ is in the linear space of $\{1, S(x)\}$, where $S(x)$ is the scale function of the process. For

$$L = \widetilde{a}(x)\partial_{xx} + \widetilde{b}(x)\partial_x, \tag{4.14}$$

the scale function is defined as

$$S(x) = C\int^{x} \exp\left[-\int_{\frac{1}{2}}^{\eta} \frac{\widetilde{b}(\xi)}{\widetilde{a}(\xi)} d\xi\right] d\eta \tag{4.15}$$

and has derivatives

$$S'(x) = \exp\left[-\int^{x} \frac{\widetilde{b}(\xi)}{\widetilde{a}(\xi)} d\xi\right], \qquad S''(x) = -S'(x)\frac{\widetilde{b}(x)}{\widetilde{a}(x)}.$$

Denote $\frac{\widetilde{b}(x)}{\widetilde{a}(x)} = \frac{c(x)}{x(1-x)}$ and $c_0 = c(0)$, $c_1 = c(1)$. The integrand $\exp\left[-\int_{\frac{1}{2}}^{\eta} \frac{\widetilde{b}(\xi)}{\widetilde{a}(\xi)} d\xi\right] \sim \eta^{-c_0}, (1-\eta)^{c_1}$ as η approaches $0+, 1-$ respectively, so S is integrable when $c_0 < 1, c_1 > -1$.

For Kimura point $x = 0$, if $0 < c_0 < 1$, then $xS''(x) \sim x^{-c_0}$ as $x \to 0$, which is not finite and thus not in $\mathcal{D}^{\gamma}$. If $c_0 = 0$, S'' is smooth at $x = 0$. For a quadratic point $x = 0$, if $c_0 < 1$, turning into heat coordinates $z = lnx$, $S(z) \sim e^{(1-c_0)z}$, it is in local $C^{2+\gamma}$ space. We conclude that S is in $C^{2+\gamma}([0,1])$ if and only if L is tangent to both endpoints. □

Definition 4.3.2. *We define*

$$bP_{ter}(L) = \{quadratic\ endpoint, P\}$$

if L is transverse to both endpoints, otherwise we define

$$bP_{ter}(L) = \{quadratic\ endpoint,\ Kimura\ tangent\ endpoint\}.$$

Here we use the notation bP_{ter} from

We first explain that to each element of $bP_{\text{ter}}([0,1])$ there is an element of the nullspace of $\overline{L}^*$. For any quadratic endpoint or Kimura tangent endpoint p, p is a sticky boundary point so $\delta(p)$ is in ker $\overline{L}^*$. For $w \in C^{2+\gamma}([0,1])$, we have

$$\langle L_{\gamma} w, \delta(p)\rangle = 0,$$

that is to say $\delta(p) \in \text{Ker } L_\gamma^*$. This equality still holds for $w \in \text{Dom}(\overline{L})$, where $\overline{L}$ is the $\mathcal{C}^0$ graph closure of L_γ. Hence $\delta(p) \in \text{Dom}(\overline{L}^*)$, and $\overline{L}^*\delta(p) = 0$. If L is transverse to both endpoints, we can explicitly construct μ, which is supported on the whole interval, as follows:

Construction of μ We first reduce L to the standard form

$$L_z = \frac{1}{2}\partial_{zz} - \nabla U(z)\partial_z = -\frac{1}{2}\partial_z^*\partial_z$$

with $\partial_z^* = \partial_z + 2\nabla U(z)$ on a probability space (X, μ). It is known that L_z has an invariant measure μ

$$\mu(dz) = \frac{e^{-2U(z)}}{Z}dz$$

where Z is a normalizing constant. We now check that μ is a probability measure in the different boundary cases.

1. **Two transverse quadratic boundary.** For $L = x^2(1-x)^2\partial_{xx} + x(1-x)b(x)\partial_x$, we first do a coordinate change:

$$z = \frac{1}{\sqrt{2}}\ln\frac{x}{1-x}, \quad x = \frac{e^{\sqrt{2}z}}{1+e^{\sqrt{2}z}},$$

so that

$$L_z = \frac{1}{2}\partial_{zz} - \nabla U(z)\partial_z, \; z \in (-\infty, \infty)$$

with $\nabla U = -\frac{1}{\sqrt{2}}\left(b(x) + 2x - 1\right)$. Then

$$e^{-2U(z)} = \begin{cases} \Theta(e^{\sqrt{2}(b(0)-1)z}), \; z \to -\infty \\ \Theta(e^{\sqrt{2}(b(1)+1)z}), \; z \to +\infty \end{cases}.$$

Thus, $Z < \infty$ exactly when $b(0) > 1, b(1) < -1$, i.e. both quadratic endpoints are transverse.

2. **One transverse Kimura and one quadratic boundary.** For $L = x^2(1-x)\partial_{xx} + xb(x)\partial_x$, we do a coordinate change by letting

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{2}x\sqrt{1-x}}$$

so that

$$L_z = \frac{1}{2}\partial_{zz} - \nabla U(z)\partial_z,\ z \in (-\infty, 0]$$

with $\nabla U = -\frac{1}{\sqrt{2}}(\frac{3x-2+2b(x)}{2\sqrt{1-x}})$. Then

$$e^{-2U(z)} = \begin{cases} \Theta(e^{\sqrt{2}(b(0)-1)z}),\ z \to -\infty \\ \Theta(z^{-1-2b(1)}),\ z \to 0 \end{cases}.$$

Thus, $Z < \infty$ exactly when $b(0) > 1, b(1) < 0$, i.e. both endpoints are transverse.

3. **Two transverse Kimura boundary.** For $L = x(1-x)\partial_{xx} + b(x)\partial_x$, we do a coordinate change

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{2x(1-x)}}$$

so that

$$L_z = \frac{1}{2}\partial_{zz} - \nabla U(z)\partial_z,\ z \in [0, \frac{\pi}{\sqrt{2}}]$$

with $\nabla U = -\frac{x-1/2+b(x)}{\sqrt{2x(1-x)}}$. Then

$$e^{-2U(z)} = \begin{cases} \Theta(z^{2b(0)-1}), \ z \to 0 \\ \Theta(z^{-1-2b(1)}), \ z \to \frac{\pi}{\sqrt{2}} \end{cases} . \tag{4.16}$$

Again, $Z < \infty$ exactly when $b(0) > 0, b(1) < 0$, i.e. both endpoints are transverse.

Proposition 4.3.2.

$$dim\ ker\ \overline{L}^* = |bP_{ter}(L)|.$$

Proof. We know that dim ker $L_\gamma = 2$ if L is tangent to both endpoints and otherwise dim ker $L_\gamma = 1$. This is equivalent to:

$$\text{dim ker } L_\gamma = \max\{1, |\text{tangent points}|\}.$$

By Theorem 4.3.1, $\text{ind}(L_\gamma) = \kappa^+ + \kappa^- = |\text{tangent quadratic endpoints}|$, so dim ker $L_\gamma^* = |bP_{ter}(L_\gamma)|$, where

$$bP_{ter}(L_\gamma) = \begin{cases} P & L \text{ is transverse/neutral to both endpoints,} \\ \text{Kimura tangent endpoint} & \text{otherwise.} \end{cases}$$

Write

$$C^0([0,1])^* = C_0^0([0,1])^* \oplus A^*$$

where $A^* = \{\delta(p)|\ p \text{ quadratic}\}$. $A^* \subset \text{ker } \overline{L}^*$. Since $\widetilde{a} \cdot \mathcal{D}^\gamma([0,1])$ is dense in the subspace $C_0^0([0,1])$, every $\mu \in$ ker $L_\gamma{}^*$ can be uniquely extended to a measure in ker $\overline{L}^* \cap C_0^0([0,1])^*$, so there is an inclusion map

$$i : \ker \overline{L}^* \cap C_0^0([0,1])^* \longrightarrow \ker L_\gamma^*.$$

And codim$(i) = 1$ if there are one or more neutral point(s) and no tangent point, while codim$(i) = 0$. This is equivalent to

$$\text{codim}(i) = |bP_{\text{ter}}(L_\gamma)| - |bP_{ter}(L)| + |\text{quadratic endpoint}|.$$

In conclusion

$$\dim \ker \overline{L}^* = \dim \ker L_\gamma^* - \text{codim}(i) + |\text{quadratic point}| = |bP_{ter}(L)|.$$

□

The following table summarizes the invariant measures found in the ten different cases of interest:

	K Trans	K Tan	Q Trans	Q Tan **
Kimura Transverse (K Trans)	μ^*	δ_1	μ, δ_1	δ_1
Kimura Tangent (K Tan)	δ_0	δ_0, δ_1	δ_0	δ_0, δ_1
Quadratic Transverse (Q Trans)	μ, δ_0	δ_0, δ_1	μ, δ_0, δ_1	δ_0, δ_1
Quadratic Tangent (Q Tan)**	δ_0	δ_0, δ_1	δ_1	δ_0, δ_1

* μ refers to an invariant measure supported on (0,1)
** include neutral case

Table 4.1: Invariant measures for all cases of boundary types at $x = 0$ (rows) and at $x = 1$ (columns).

4.3.3 Proof of Theorem 4.3.1

For convenience of computation, we fix the spatial domain as the interval $[0, 3]$ instead of $[0, 1]$ whenever convenient in this subsection.

Outline of proof

It remains to characterize the range of L_γ. First, we turn the second-order operator L_γ to a first-order system M in (4.18), with an index equal to the index of L_γ (see Lemma 4.3.2 below). Next we continuously deform M to another first-order system $\widetilde{M}$ with constant coefficients in the vicinity of the two endpoints. Such a deformation does not change the index (see Proposition 4.3.3 below), so the original problem now is equivalent to the easier linear system $\widetilde{M}$. These constructions are presented in subsection 3.3.2.

The problem $\widetilde{M}u = f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, given $u(1) \in \mathbb{R}^2$, has a uniquely defined solution u. In order for u to belong to the domain space, we observe that αf_1 has to vanish at tangent Kimura endpoints and $u(1), f$ have to be related by

$$E_- u(1) = I_1 f, \quad E_+ \Lambda u(1) = -I_2 f; \tag{4.17}$$

see the definition of $I_1, I_2, E_\pm$, and Λ in Section 3.4. Conversely if αf_1 vanishes at tangent Kimura endpoints and if we can find $u(1)$ satisfying the relation (4.17), then the uniquely defined u is the solution of $\widetilde{M}u = f$ (see Lemma 4.3.3, Lemma 4.3.4). Thus f in the range of $\widetilde{M}$ is equivalent to the existence of $u(1) \in \mathbb{R}^2$ satisfying (4.17). This construction is presented in subsection 3.3.3.

We next show in Lemma 4.3.6 that the above constraint is equivalent to

$$\Lambda I_1 f + I_2 f \in \ker(E_+) + \Lambda \cdot \ker(E_-).$$

We thus define the linear map

$$\Phi : C^{1+\gamma}([0,3]) \times \mathcal{D}^\gamma([0,3]) \longrightarrow \frac{\mathbb{R}^2}{\ker(E_+) + \Lambda \ker(E_-)},$$

by assigning f to the coset $[\Lambda I_1 f + I_2 f]$. Clearly the range of $\widetilde{M}$ lies in the kernel of

Φ. A computation in Lemma 4.3.7 shows that Φ is surjective and $\dim \ker(\Phi)/R(\widetilde{M}) =$ |kimura tangent points|. We now have all the ingredients to compute the codimension of $\widetilde{M}$. The final results and proofs are given in subsection 3.3.4.

Reduction to a first order system

To simplify notation, we consider L_γ defined on the interval $[0,3]$. We rewrite the system $L_\gamma u = f$ as a first-order system M:

$$M : C^{2+\gamma}([0,3]) \times C^{1+\gamma}([0,3]) \longrightarrow C^{1+\gamma}([0,3]) \times \mathcal{D}^{\gamma}([0,3]). \tag{4.18}$$

associated to the expression

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u_0' \\ u_1' \end{pmatrix} + A(L_\gamma)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0' - u_1 \\ u_1' + \frac{b}{a}u_1 \end{pmatrix}, \qquad A(L_\gamma) := \begin{pmatrix} 0 & -1 \\ 0 & \frac{b}{a} \end{pmatrix}.$$

where a, b are the coefficients of second-order and first-order term of L under heat coordinate ϕ, i.e. $L_\phi = a(x)\partial_{xx} + b(x)\partial_x$.

For a smooth coordinate change $z = z(x)$, we let L_z be the operator corresponding to L under coordinate change and define $A(L_z)$ and M_z as above. If $M\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$, then

$$M_z \begin{pmatrix} u_0[x(z)] \\ u_1[x(z)]x'(z) \end{pmatrix} = \begin{pmatrix} f_0[x(z)]x'(z) \\ f_1[x(z)](x'(z))^2 \end{pmatrix}. \tag{4.19}$$

In the local charts $([0,1],\phi^{-1}), ([2,3],\phi^{-1})$, $A(t) = \begin{pmatrix} 0 & -1 \\ 0 & b(z) \end{pmatrix}$ when 0 is a quadratic point with heat coordinate, while $A(t) = \begin{pmatrix} 0 & -1 \\ 0 & \frac{b(x)}{x} \end{pmatrix}$ when 0 is a Kimura point.

Lemma 4.3.2. *Assume that M is Fredholm, then L_γ is also Fredholm and*

$$ind(L_\gamma) = ind(M). \tag{4.20}$$

Proof. There is an isomorphism between $\ker(M) \longrightarrow \ker(L_\gamma)$ by mapping (u, u') to u, so the dimensions of the kernel spaces are the same.

If $(f_0, f_1) \in R(M)$ and $M \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$, then $L_\gamma u_0 = \widetilde{a} f_0' + \widetilde{b} f_0 + \widetilde{a} f_1$. Based on this observation we define the map

$$\pi : C^{1+\gamma}([0,3]) \times \mathcal{D}^\gamma([0,3]) \longrightarrow \alpha \cdot \mathcal{D}^\gamma([0,3])$$
$$(f_0, f_1) \mapsto \widetilde{a} f_0' + \widetilde{b} f_0 + \widetilde{a} f_1.$$

We claim that

$$\text{codim ran}(\pi) = \text{codim ran}(L_\gamma) - \text{codim ran}(M). \tag{4.21}$$

We first show that

$$f \in \text{ran}(M) \Leftrightarrow \pi f \in \text{ran}(L_\gamma). \tag{4.22}$$

The inclusion part is straightforward. For the converse part, if $\pi(f_0, f_1) = L_\gamma u \in \text{ran}(L_\gamma)$, we verify that $M \begin{pmatrix} u \\ u' - f_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in \text{ran}(M)$. Next we show that

$$\pi(\text{ran}(M)) = \text{ran}(L_\gamma). \tag{4.23}$$

Again the inclusion part is straightforward. For the converse part, if $g = L_\gamma u \in \text{ran}(L_\gamma)$, then $\pi(0, \frac{g}{\widetilde{a}}) = g \in \text{ran}(L_\gamma)$. Note that by (4.22), this implies that $(0, \frac{g}{\widetilde{a}}) \in \text{ran}(M)$. (4.22) and (4.23) imply that the quotient map defined by

$$\pi' : [C^{1+\gamma}([0,3]) \times \mathcal{D}^{\gamma}([0,3])]/\text{ran}(M) \longrightarrow \alpha \cdot \mathcal{D}^{\gamma}([0,3])/\text{ran}(L_{\gamma})$$

is injective. So codim ran(π') = codim ran(L_γ) − codim ran(M). At the same time, $\pi(\text{ran}(M)) = \text{ran}(L_\gamma)$ indicates codim ran(π') = codim ran(π) so that (4.21) holds. Clearly, π is surjective since $\pi(0, \frac{h}{a}) = h$. Combined with (4.21), this proves (4.20). □

Index of the model operator $\widetilde{M}$ Alongside M, we consider the operator $\widetilde{M}$ associated with a continuous function $A(t)$ such that

$$A(t) = A_- \text{ for } t \leq 1, A(t) = A_+ \text{ for } t \geq 2$$

where $A_- = \begin{pmatrix} 0 & -1 \\ 0 & b_- \end{pmatrix}$ if 0 is a quadratic point, $A_- = \begin{pmatrix} 0 & -1 \\ 0 & \frac{b_-}{x} \end{pmatrix}$ if 0 is a kimura point, and A_+ is similarly defined. Since M and $\widetilde{M}$ can be reduced to each other by a continuous deformation in the class of Fredholm operators, we have

Proposition 4.3.3. *The operator M is Fredholm if and only if the operator $\widetilde{M}$ is Fredholm and*

$$ind(M) = ind(\widetilde{M}). \tag{4.24}$$

Proof. We need to show that $M - \widetilde{M}$ is a compact map since then by

Given a bounded sequence u_k in $C^{2+\gamma}([0,3]) \times C^{1+\gamma}([0,3])$, we need to show that there exists a convergent subsequence v_k of $(M - \widetilde{M})u_k$. v_k convergent in $C^{1+\gamma}([0,3]) \times \mathcal{D}^{\gamma}([0,3])$ is equivalent to $v_k|_{[0,1]}, v_k|_{[1,2]}, v_k|_{[2,3]}$ convergent in the local spaces, respectively. Thus it suffices to show that $(M - \widetilde{M})|_{[0,1]}, (M - \widetilde{M})|_{[2,3]}$ are compact.

To check both Kimura and quadratic cases, we assume that L_0 is Kimura type and L_1

is of quadratic type.

At $x = 0$, given a bounded sequence (u_n, q_n) in $C^{2+\gamma}([0,1]) \times C^{1+\gamma}([0,1])$, we have

$$(M - \widetilde{M})(u_n, q_n) = (0, \frac{b(x) - b_-}{x} q_n).$$

Since $\frac{b(x)-b_-}{x}$ is smooth and bounded, it remains to show by definition of $\mathcal{D}^\gamma([0,1])$ that xq_n has a convergent subsequence in $C^\gamma([0,1])$.

Since $\{q_n, xq_n'\}$ are bounded in $C^\gamma([0,1])$, then $\{xq_n\}$ are uniformly bounded and uniformly equicontinuous. Therefore by the Arzela-Ascoli theorem, there exists a convergent subsequence v_n of xq_n in $C^0([0,1])$. For the $[\cdot]_\gamma$ part, notice that for $g_n(y) = v_n(x)$ under the coordinate change $y = x^\gamma$,

$$[v_n]_\gamma = O(||g_n'||_\infty). \tag{4.25}$$

Since $g_n' = \frac{1}{\gamma} v_n' x^{1-\gamma}$ and $\{v_n'\}$ are uniformly bounded in $C^\gamma([0,1])$, $\{g_n'\}$ are uniformly bounded and uniformly equicontinuous. Thus, there exists a subsequence of g_n' convergent in $C^0([0,1])$. By (4.25), the corresponding subsequence of v_n converges in $C^\gamma([0,1])$. Therefore $(M - M')|_{[0,1]}$ is a compact operator.

At $x = 3$, we prove the result using heat coordinates in $[-\infty, 0]$. Given a bounded sequence (u_n, q_n) in $C^{2+\gamma}((-\infty, 0]) \times C^{1+\gamma}((-\infty, 0])$, write

$$(M - \widetilde{M})(u_n, q_n) = (0, (b(z) - b_+)q_n).$$

Since $\{q_n'\}$ is bounded in $C^\gamma(T)$for any compact set T, there exists a convergent subsequence of $\{q_n\}$ in $C^\gamma(T)$. We choose a convergent subsequence $\{q_1^k\}_{k\geq 1}$ on $[-1, 0]$, and $\{q_2^k\}_{k\geq 1}$ a convergent subsequence of $\{q_1^k\}$ on $[-2, 0]$. Iteratively, we choose $\{q_n^k\}_{k\geq 1}$ to be a convergent subsequence of $\{q_{n-1}^k\}$ on $[-n, 0]$.

We show below that the diagonal sequence $\{(b(z) - b_+)\, q_k^k\}_{k\geq 1}$ converges in $\mathcal{D}^\gamma((-\infty, 0])$.

Since $||q_k^k||_\gamma$ is bounded and $b(z)-b_-$ tends to 0 as z tends to $-\infty$, there exist $N>0, K>0$ such that

$$\max\Big\{|(b(z)-b_-)q_k^k|, \quad [(b(z)-b_-)q_k^k]_\gamma,$$
$$\sup_{z_1\le z_2\le -N}\left|\int_{z_1}^{z_2}(b(z)-b_-)q_k^k dz\right|\Big\} \le \epsilon/4, \text{ on } (-\infty,-N]$$
$$\{q_k^k\}_{k\ge K} \text{ converge in } C^\gamma([-N,0]).$$

So we can pick N' such that $||(b(z)-b_-)q_m^m-(b(z)-b_-)q_n^n||_{\gamma,[-N,0]} < \frac{\epsilon}{2}$ when $m,n>N'$. Combining the above, we have $||(b(z)-b_-)q_m^m-(b(z)-b_-)q_n^n||_{\gamma,(-\infty,0]} < \epsilon$ when $m,n > N'$. We proved that the diagonal sequence $\{(b(z)-b_+)\,q_k^k\}_{k\ge 1}$ converged in $\mathcal{D}^\gamma((-\infty,0])$. Therefore $(M-M')|_{[2,3]}$ is a compact operator. It is not hard to show that $(M-M')|_{[1,2]}$ is also a compact operator so that $M-\widetilde{M}$ is compact. □

Representation of solutions of the modeling operator

The results in the previous subsection show that $\text{ind}(L_\gamma)=\text{ind}(\widetilde{M})$ if $\widetilde{M}$ is Fredholm. Now we turn to proving that $\widetilde{M}$ is Fredholm and computing the index of $\widetilde{M}$. Recall that

$$\widetilde{M}: C^{2+\gamma}([0,3])\times C^{1+\gamma}([0,3]) \longrightarrow C^{1+\gamma}([0,3])\times \mathcal{D}^\gamma([0,3]) \tag{4.26}$$

is associated to the expression

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u_0' \\ u_1' \end{pmatrix} + A(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad A(t)=A_- \text{ for } t\le 1, \quad A(t)=A_+ \text{ for } t\ge 2.$$

Let $f\in C^{1+\gamma}([0,3])\times\mathcal{D}^\gamma([0,3])$ be given, and suppose $u\in C^{2+\gamma}([0,3])\times C^{1+\gamma}([0,3])$

solves

$$\frac{du}{dt} + Au = f.$$

Let $Y = Y(t)$ denote the fundamental matrix associated to $\widetilde{M}$, defined in different intervals by

$$\frac{dY}{dt} + AY(t) = 0,\ Y(1) = I, \text{on } [0,2)$$
$$\frac{dY}{dt} + AY(t) = 0,\ Y(2+) = I, \text{on } (2,3].$$

Then $\Lambda := Y(2-)$ is an invertible matrix and

$$u(2) = \Lambda u(1) + \Lambda \int_1^2 Y(s)^{-1} f(s) ds.$$

Given $u(1)$, we can uniquely determine $u(t)$ on $[0,3]$:

$$u(t) = Y(t)u(1) - \int_t^1 Y(t)Y^{-1}(s)f(s)ds,\ t \leq 1 \tag{4.27}$$

$$u(t) = Y(t)\left(\Lambda u(1) + \Lambda \int_1^2 Y(s)^{-1} f(s) ds\right) + \int_2^t Y(t)Y^{-1}(s)f(s)ds,\ t \geq 2. \tag{4.28}$$

We analyze the above solutions for different boundary points. For quadratic endpoints, we directly assume t is in heat coordinates.

1. Quadratic endpoints with $b_\pm \neq 0$, $A_\pm = \begin{pmatrix} 0 & -1 \\ 0 & b_\pm \end{pmatrix}$ constant matrix. Then

$$Y(t) = \begin{cases} e^{-(t-1)A_-}, & t \in (-\infty, 1] \\ e^{-(t-2)A_+}, & t \in [2, +\infty). \end{cases} \tag{4.29}$$

$A_\pm$ has two different eigenvalues $0, b_\pm$. We let E_- be the spectral projector associated to the set of all positive eigenvalues of A_-, and E_+ be the spectral projector associated to the set of all negative eigenvalues of A_+. Apply the operator E_-, E_+ on $(4.27), (4.28)$, and let t go to $-\infty, \infty$ to get:

$$E_-u(1) = \int_{-\infty}^{1} e^{(s-1)A_-}E_-f(s)ds, \tag{4.30}$$

$$E_+\Lambda u(1) = -E_+\Lambda\int_1^2 Y(s)^{-1}f(s)ds - \int_2^\infty E_+e^{(s-2)A_+}f(s)ds. \tag{4.31}$$

Using this relation, we recast $u(t)$ for $t < 1$ as:

$$u(t) = e^{-(t-1)A_-}(I - E_-)u(1) - \int_t^1 e^{-(t-s)A_-}(I - E_-)f(s)ds + \int_{-\infty}^t e^{-(t-s)A_-}E_-f(s)ds \tag{4.32}$$

and for $t > 2$ as:

$$u(t) = e^{-(t-1)A_+}(I - E_+)\Lambda\left[u(1) + \int_1^2 Y(s)^{-1}f(s)ds\right] \tag{4.33}$$
$$+ \int_2^t e^{-(t-s)A_+}(I - E_+)f(s)ds - \int_t^\infty e^{-(t-s)A_+}E_+f(s)ds.$$

2. Kimura transverse point In this case

$$Y(t) = \begin{pmatrix} 1 & F(t^{-b_-}) - F(1) \\ 0 & t^{-b_-} \end{pmatrix}, \ t \in [0,1],$$
$$Y(t) = \begin{pmatrix} 1 & F((\frac{t}{2})^{-b_+}) - F(1) \\ 0 & (\frac{t}{2})^{-b_+}. \end{pmatrix}, \ t \in [2,3],$$

where F denotes the antiderivative. We let $E_\pm = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Multiplying by $E_-Y(t)^{-1}$ both sides of (4.27), (4.28), and letting $t \to 0, t \to 3$, respectively, we obtain

$$E_-u(1) = \int_0^1 E_-Y(s)^{-1}f(s)ds, \tag{4.34}$$

$$E_+\Lambda u(1) = -E_+\Lambda \int_1^2 Y(s)^{-1}f(s)d - \int_2^3 E_+Y^{-1}(s)f(s)ds. \tag{4.35}$$

Using this relation we recast $u(t)$ for $t < 1$ as:

$$u(t) = Y(t)(I - E_-)u(1) + Y(t)\int_0^t E_-Y^{-1}(s)f(s)ds - Y(t)\int_t^1 (I - E_-)Y^{-1}(s)f(s)ds \tag{4.36}$$

and for $t > 2$ as:

$$u(t) = Y(t)(I - E_+)\Lambda \left(u(1) + \int_1^2 Y(s)^{-1}f(s)ds\right) \tag{4.37}$$
$$+ \int_2^t Y(t)(I - E_+)Y^{-1}(s)f(s)ds - \int_t^3 Y(t)E_+Y^{-1}(s)f(s)ds.$$

3. Quadratic endpoints with $b_\pm = 0$ In this case

$$Y(t) = \begin{pmatrix} 1 & t-1 \\ 0 & 1 \end{pmatrix}, \; t \in [0,1], \qquad Y(t) = \begin{pmatrix} 1 & t-2 \\ 0 & 1. \end{pmatrix}, \; t \in [2,3].$$

We let $E_\pm = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Multiply by e^t on both sides of (4.27), (4.28) and let $t \to \pm\infty$. We again obtain (4.34), (4.35) with the integral lower/upper points $0, 3$ replaced by $-\infty, \infty$. So we can re-express $u(t)$ as (4.36), (4.37) again with integral lower/upper points $0, 3$ replaced with $-\infty, \infty$.

4. Kimura tangent point In this case, $A_\pm = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. We observe that if $f = (f_0, f_1) = \widetilde{M}(u_0, u_1) \in \operatorname{ran}(\widetilde{M})$, then $f_1 = u_1'$, so that $s(s-3)f_1(s)$ vanishes at Kimura tangent endpoints by definition of $C^{1+\gamma}([0,3])$. We next let $E_\pm = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Obviously (4.34), (4.35) are true then. So we can also recast $u(t)$ as (4.36), (4.37).

Summarizing these boundary cases, we make the following definition.

Definition 4.3.3. *We define*

$$I_1 f = \int_l^1 E_- Y(s)^{-1} f(s) ds,$$
$$I_2 f = E_+ \Lambda \int_1^2 Y(s)^{-1} f(s) ds + \int_2^r E_+ Y^{-1}(s) f(s) ds,$$

with $l, r = -\infty, \infty$ at a quadratic endpoint and $l, r = 0, 3$ at a Kimura endpoint.

In all cases, when f satisfying the corresponding boundary conditions is in the range of $\widetilde{M}$, then $I_1 f, I_2 f$ are related by

$$E_- u(1) = I_1 f, \quad E_+ \Lambda u(1) = -I_2 f. \tag{4.38}$$

For f satisfying the relation (4.38), we fix a constant $u(1)$ as in (4.38), and then define $u(t)$ on the whole domain by (4.36) and (4.37) (or (4.32) and (4.33) for quadratic endpoints). We now show that such a u is indeed the solution.

Lemma 4.3.3. *For $U = (-\infty, c]$ or $U = [c, \infty)$, if $f = (f_0, f_1) \in C^{1+\gamma}(U) \times \mathcal{D}^\gamma(U)$ together with some $u(1) \in \mathbb{R}^2$ satisfy (4.38), then $u(t)$ defined by (4.32),(4.33) is in $C^{2+\gamma}(U) \times C^{1+\gamma}(U)$ and solves*

$$\frac{du}{dt} + Au = f.$$

Proof. When $b_\pm \neq 0$, $A_\pm$ can be diagonalized as

$$A_\pm = \begin{pmatrix} 1 & -1 \\ 0 & b_\pm \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_\pm \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & b_\pm \end{pmatrix}^{-1} = CB_\pm C^{-1}.$$

By changing a basis $u \mapsto C^{-1}u$, we may assume that $A_\pm$ has the diagonal form $B_\pm$ and $E_\pm$ is the corresponding projector of $A_\pm$. For convenience we only prove (4.32) as (4.33) is obtained similarly.

For the solution $u = (u_0, u_1)$, it is straightforward to see that $u_0 \in C^2((-\infty, c])$, $\lim_{z\to-\infty} \partial_z u = \lim_{z\to\ -\infty} \partial_z^2 u = 0$ and satisfies $\partial_z^2 u + b_- \partial_z u = f_1' + f_2 + bf_1$. Hence the classical result of elliptic operator in Hölder spaces gives that $u_0 \in C^{2+\gamma}((-\infty, c])$. Then $u_1 = u_0' - f_1 \in C^{1+\gamma}((-\infty, c])$.

When $b_\pm = 0$, we explicitly express $u(t)$ for $t < 1$ as follows:

$$u_0(t) = u_0(1) - u_1(1) + tu_1(t) - \int_t^1 [f_0(s) - sf_1(s)]ds, \tag{4.39}$$

$$u_1(t) = u_1(1) - \int_t^1 f_1(s)ds. \tag{4.40}$$

Since f satisfies (4.30), by Remark 4.3.1, u_0, u_1 are integrable up to $-\infty$. $u_1'(t) = f_1(t) \in C^\gamma((-\infty, c])$ so $u_1 \in C^{1+\gamma}((-\infty, c])$. And

$$u_0'(t) = u_1(1) + f_0(t) - \int_t^0 f_1(s)ds, \quad u_0''(t) = f_0'(t) + f_1(t).$$

$u_0'' \in C^\gamma((-\infty, c])$ and u_0' is integrable up to $-\infty$, so $u_0' \in C^{1+\gamma}((-\infty, c])$. Thus, $u \in C^{2+\gamma}((-\infty, c]) \times C^{1+\gamma}((-\infty, c])$ and solves $\frac{du}{dt} + Au = f$. □

Lemma 4.3.4. *For $U = [0, 1]$ or $U = [2, 3]$, if $f = (f_0, f_1) \in C^{1+\gamma}(U) \times \mathcal{D}^\gamma(U)$ together with some $u(1) \in \mathbb{R}^2$ satisfies (4.38) and $s(s-3)f_1(s)$ vanishes at Kimura tangent endpoints,* *then* $u(t)$ *defined by* (4.36),(4.37) *is in* $C^{2+\gamma}(U)\times C^{1+\gamma}(U)$ *and solves*

$$\frac{du}{dt}+Au=f.$$

Proof. At a transverse Kimura point, we write $u(t)$ for $t<1$ as

$$u(t)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}u(1)+\int_0^t\begin{pmatrix}0 & s^{b_-}G(t)\\ 0 & t^{-b_-}s^{b_-}\end{pmatrix}f(s)ds-\int_t^1\begin{pmatrix}1 & s^{b_-}G(s)\\ 0 & 0\end{pmatrix}f(s)ds, \tag{4.41}$$

where $G(t)=F(t^{-b_-})-F(1)$. By direct computation we have

$$u_1'(t)=f_1(t)-at^{-(b_-+1)}\int_0^t s^{b_-}f_1(s)ds,$$
$$u_0'(t)=t^{-b_-}\int_0^t s^{b_-}f_1(s)ds+f_0(t),u_0''(t)=f_1(t)-b_-t^{-(b_-+1)}\int_0^t s^{b_-}f_1(s)ds+f_0'(t).$$

Thus $u_0'',u_1'\in\mathcal{D}^\gamma([0,1])$ and by definition $u\in C^{2+\gamma}(U)\times C^{1+\gamma}(U)$ and solves $\frac{du}{dt}+Au=f$.

At a tangent Kimura point, for $t\in[0,1]$, we write $u(t)$ for $t<1$ as:

$$u_0(t)=u_0(1)+(t-1)u_1(1)-\int_t^1[f_0(s)+(t-s)f_1(s)]ds, \tag{4.42}$$
$$u_1(t)=u_1(1)-\int_t^1 f_1(s)ds. \tag{4.43}$$

u_1 is integrable up to $t=0$ iff $\lim_{s\to 0}sf_1(s)=0$. In such a case, $u_1'(t)=f_1(t)\in\mathcal{D}^\gamma([0,1])$ and so $u_1\in C^{1+\gamma}([0,1])$. Since

$$u_0'(t)=u_1(1)+f_0(t)-\int_t^1 f_1(s)ds,\ u_0''(t)=f_0'(t)+f_1(t),$$

$u\in C^{2+\gamma}(U)\times C^{1+\gamma}(U)$ and solves $\frac{du}{dt}+Au=f$. For $t\in[2,3]$, the proof is essentially the same. □

Proof of Theorem 4.3.1

Lemma 4.3.5. *dim ker*$(\widetilde{M})$ = *dim* [*ker*$(E_-) \cap \Lambda^{-1}$*ker*(E_+)].

Proof. If $\widetilde{M}u = 0$, then $\Lambda u(1) \in \ker(E_+)$ and similarly $u(1) \in \ker(E_-)$. On the other hand, we set $u(1) \in \ker(E_-) \cap \Lambda^{-1}\ker(E_+)$, and define u at the quadratic boundary point by

$$\text{for } t \leq 1 : u(t) = e^{-(t-1)A_-}u(1), \quad \text{for } t \geq 2 : u(t) = e^{-(t-2)A_+}\Lambda u(1)$$

and at the Kimura endpoint by

$$\text{for } t \leq 1 : u(t) = \Phi(t)u(1), \quad \text{for } t \geq 2 : u(t) = \Phi(t)\Lambda u(1)$$

and $u(t)$ is uniquely characterized on $[1,2]$ by $u(1)$. Then, $u \in C^{2+\gamma}([0,3] \times C^{1+\gamma}([0,3])$ and solves $\widetilde{M}u = 0$. When such u exists, it is uniquely determined by $u(1)$. We thus proved that the map

$$N : \ker(\widetilde{M}) \longrightarrow \ker(E_-) \cap \Lambda^{-1}\ker(E_+)$$

by assigning u to $u(1)$ is a bijection. Therefore $\dim \ker(\widetilde{M}) = \dim\,[\ker(E_-) \cap \Lambda^{-1}\ker(E_+)]$. □

Lemma 4.3.6. *The condition* (4.38) *is equivalent to*

$$\Lambda I_1 f + I_2 f \in \mathit{ker}(E_+) + \Lambda \cdot \mathit{ker}(E_-). \tag{4.44}$$

Proof. Suppose (4.38) is satisfied, then for some $v_1, v_2 \in \mathbb{R}^2$ we have

$$u(1) = I_1 f + (I - E_-)v_2, \quad \Lambda u(1) = -I_2(f) + (I - E_+)v_1.$$

Multiplying the first equality by Λ and subtracting the second equality, we obtain (4.44).

Conversely, suppose that (4.44) is true. Then, for some $v_1, v_2 \in \mathbb{R}^2$,

$$I_2 f + \Lambda I_1 f = -\Lambda(I - E_-)v_2 - (I - E_+)v_1.$$

Define $u(1) = I_1 f + (I - E_-)v_2$. Using that $E_\pm^2 = E_\pm$, $E_\pm(I - E_\pm) = 0$, we verify that (4.38) is satisfied. $\square$

We define the linear map

$$\Phi : C^{1+\gamma}([0,3]) \times \mathcal{D}^\gamma([0,3]) \longrightarrow \frac{\mathbb{R}^2}{\ker(E_+) + \Lambda \cdot \ker(E_-)}.$$

by assigning f to the coset $[\Lambda I_1 f + I_2 f]$.

$\mathrm{ran}(\widetilde{M}) \subset \ker(\Phi)$ is already known and moreover

$$\mathrm{ran}(\widetilde{M}) \subset \ker(\Phi) \cap \{(f_1, f_2) : s(s-3)f_2(s) \text{ vanishes at Kimura tangent points.}\}$$

Indeed we show below that they are equal and thus obtain

Lemma 4.3.7. *We have*

1. *dim ker*$(\Phi)/$*ran*$(\widetilde{M}) = |$*Kimura tangent points*$|$,
2. Φ *is surjective.*

Proof. We show that

$$\ker(\Phi) \cap \{(f_1, f_2) : s(s-3)f_2(s) \text{ vanishes at Kimura tangent points}\} = \mathrm{ran}(\widetilde{M}). \quad (4.45)$$

First we assume that there is no Kimura tangent endpoint. Given $f \in \ker(\Phi)$, then for some $u(1) \in \mathbb{R}^2$, $E_+\Lambda u(1) = -I_1 f$, $E_- u(1) = I_2 f$. Define u by (4.36), (4.37). Then by Lemmas 4.3.3, 4.3.4, $u \in C^{2+\gamma}([0,3]) \times C^{1+\gamma}([0,3])$ solves $\widetilde{M}u = f$, so that $f \in \mathrm{ran}(\widetilde{M})$, and hence $\mathrm{ran}(\widetilde{M}) = \ker(\Phi)$.

If there is at least one Kimura tangent point, then $\ker(E_+) + \Lambda\ker(E_-) = \mathbb{R}^2$, so $\ker(\Phi)$ is the whole space $C^{1+\gamma}([0,3]) \times \mathcal{D}^\gamma([0,3])$. For f in the left hand space of (4.45), we define u by (4.36), (4.37). Again by Lemmas 4.3.3, 4.3.4, such u is the solution of $\widetilde{M}u = f$, and hence we proved (4.45) in this case.

Next we prove that Φ is surjective. If there is at least one Kimura tangent endpoint, then $\ker(E_+) + \Lambda\ker(E_-) = \mathbb{R}^2$, so the target space of Φ is $\{0\}$ and Φ is surjective.

Now we assume that there is no Kimura tangent point. Pick $x \in \mathbb{R}^2$. We need to find $f \in C^{1+\gamma}([0,3]) \times \mathcal{D}^\gamma([0,3])$ such that

$$\Lambda I_1 f + I_2 f - x \in \ker(E_+) + \Lambda \cdot \ker(E_-).$$

Recall that

$$I_1 f = E_+\Lambda \int_{-1}^{1} Y(s)^{-1} f(s)ds + \int_1^\infty e^{(s-1)A_+} E_+ f(s)ds,$$
$$I_2 f = \int_{-\infty}^{-1} e^{(s+1)A_-} E_- f(s)ds.$$

We first take

$$f(t) = 0 \ \forall\ t < 1, \qquad f(t) = -gA_+ x \ \forall t > 1.$$

We could choose an appropriate function $g \in L^1([1,\infty) \cap \mathcal{C}_b^0(\mathbb{R})$ so that

$$I_1(f) + \Lambda I_2 f - x = -(I - E_+)x$$

and therefore $\Phi(f) = [x]$. However f would not be continuous. Thus modifying this idea we set

$$f_k(t) = 0 \text{ for } t \leq 1,$$
$$f_k(t) = k(1-t)gA_+x \text{ for } 1 \leq t \leq 1 + \frac{1}{k},$$
$$f_k(t) = -gA_+x \text{ for } t \geq 1 + \frac{1}{k}.$$

We compute

$$I_1(f) + \Lambda I_2 f - x = -(I - E_+)x + x_k$$

where $x_k \to 0$ as $k \to \infty$. Since the projection map is continuous we have $\Phi(f_k) \to [x]$ and since the image of Φ is closed we conclude that $[x] \in \text{ran}(\Phi)$. We construct f similarly in all other cases. □

From Lemma 3.7 we have

$$\begin{aligned} \text{codim ran}(\widetilde{M}) &= \dim \ker(\Phi)/\text{ran}(\widetilde{M}) + \dim \text{ran } \Phi \\ &= |\text{kimura tangent endpoints}| + 2 - \dim(\ker(E_+) + \Lambda \ker(E_-)). \end{aligned}$$

so that

$$\begin{aligned} \text{ind}(L_\gamma) =& \text{ind}(M) = \text{ind}(\widetilde{M}) \\ =& \dim[\ker(E_-) \cap \Lambda^{-1}\ker(E_+)] - [2 - \dim(\ker(E_+) \\ & + \Lambda \ker(E_-))] - |\text{kimura tangent endpoints}| \\ =& \dim \ker(E_+) + \dim \ker(E_-) - 2 - |\text{Kimura tangent endpoints}|. \end{aligned}$$

From the definitions of $E_{\pm}$ we conclude that

$$\dim \ker(E_+) + \dim \ker(E_-) = 1 + 1 + |\text{tangent endpoints}|,$$

and hence

$$\begin{aligned}\text{ind}(L_\gamma) &= 1 + 1 + |\text{tangent endpoints}| - 2 - |\text{tangent kimura endpoints}| \\ &= |\text{tangent quadratic endpoints}| \qquad = \kappa^+ + \kappa^-.\end{aligned}$$

4.4 Exponential convergence to invariant measures

In this section we study the convergence rate to the invariant measure in two different cases. First, when there is at least one tangent boundary, then the invariant measure is Dirac measure(s) at the tangent boundary(ies) and quadratic endpoints (if any). To estimate the convergence rate we use Lyapunov functions constructed in Section 4.4.1. These Lyapunov functions display different asymptotic behaviors at different boundary endpoints, and turn out to lie in the function spaces $C(\alpha, \beta)$ (see Definition 4.4.1). We show that the growth bound of the generator of $\mathcal{Q}_t$ on $C(\alpha, \beta)$ is not larger than the Lyapunov function rate (4.48), and thus negative. From this, we conclude that the transition probability converges exponentially to the corresponding invariant measure in a Wasserstein distance. The main results are summarized in Theorem 4.4.1.

In the second case where both boundary points are transverse, the invariant measures include a unique invariant measure μ supported on the whole domain $[0, 1]$ and Dirac measures at quadratic endpoints (if any). Motivated by underdamped Langevin dynamics, we wish to prove and then apply a Poincaré inequality in $L^2(\mu)$ to show exponential convergence. Using tools developed in

4.4.1 Case of one/two tangent boundary points

Function Space $C(\alpha, \beta)$

When there is at least one tangent boundary point, we consider a function space isometrically isomorphic to $C^0([0,1])$:

Definition 4.4.1. *For $\alpha, \beta \in \mathbb{R}$, we define the function space*

$$C(\alpha, \beta) := x^{\alpha}(1-x)^{\beta} C^0([0,1])$$

equipped with norm

$$||f||_{\alpha,\beta} := \left|\left| \frac{f}{x^{\alpha}(1-x)^{\beta}} \right|\right|_{C^0}.$$

We will specify α, β in different cases later (see Sec.4.4.1 for more detail) but list their ranges as:

$$\alpha, \beta = \begin{cases} \in (0,1) & \text{Kimura/quadratic tangent} \\ 0 & \text{Kimura transverse} \\ < 0 & \text{quadratic transverse.} \end{cases}$$

Remark 4.4.1. *The solution operator $\mathcal{Q}_t$ constructed in Theorem 4.2.1 is also the solution operator of the Cauchy problem on $C(\alpha, \beta)$. To see this, the two local operators $\widetilde{Q}_t^0, \widetilde{Q}_t^1$ naturally act on $C(\alpha, \beta)$ and so does the perturbation operator.*

We let A be the generator of $\mathcal{Q}_t$ on $C(\alpha, \beta)$. We use the notation $f(x_0) \sim 0$ in $C(\alpha, \beta)$ if $\frac{f}{x^{\alpha}(1-x)^{\beta}}(x_0) = 0$ and $f \succ 0 (\succeq 0)$ in $C(\alpha, \beta)$ if $\frac{f}{x^{\alpha}(1-x)^{\beta}}$ is strictly positive (nonnegative).

Lemma 4.4.1 (Positive Minimum Principle)**.** *For $\lambda \in \mathbb{R}$, the operator $B := A - \lambda$ satisfies* *the positive minimum principle on $C(\alpha, \beta)$, i.e.*

$$\textit{for every } 0 \preceq f \in D(A) \textit{ and } x \in [0,1],\ f(x) \sim 0 \textit{ implies } (Bf)(x) \succeq 0.$$

Proof. We say $f \in C^0([0,1])$ is in $D^2([0,1])$ if f is twice differentiable up to Kimura endpoint and if at quadratic endpoint if any, say $x = 0$, then $x\partial_x f, x^2\partial_x^2 f$ has a continuous limit 0 at $x = 0$. Consider the space

$$D([0,1]) := x^{\alpha^*}(1-x)^{\beta^*} g,\ g \in D^2([0,1]),$$

where $\alpha^* = \alpha, \beta^* = \beta$ if x=0,1 is quadratic and 0 if Kimura.

Since $C^2([0,1])$ is dense in $C^0([0,1]$, then $D^2([0,1]) \supset C^2([0,1])$ is also dense in $C^0([0,1]$, and hence $D([0,1])$ is dense in $C^{\alpha,\beta}([0,1])$. And for $t > 0$, $\mathcal{Q}_t(D([0,1])) \subset D([0,1])$, so by

Assume $f \in D([0,1])$ and $f(x_0) = 0$. If x_0 is an interior point, then $\partial_x f(x_0) = 0, \partial_x^2 f(x_0) \geq 0$, so $Bf(x_0) \geq 0$. If x_0 is Kimura tangent, since f is twice differentiable at x_0, $Bf(x_0) = 0$. If x_0 is Kimura transverse, in this case, $\partial_x f(x_0) \geq 0, x\partial_x^2 f(x_0) = 0$, and so $Bf(x_0) \geq 0$.

If x_0 is a quadratic endpoint, locally near 0, $f = x^\alpha g$ for $g \in D^2([0,1])$. As a function $g(x) \geq 0, g(0) = 0$ where $x\partial_x g, x^2\partial_x^2 g$ have a continuous limit 0 at $x = 0$, so $\lim_{x\to 0} \frac{Af}{x^\alpha} = 0$. Thus, $(Bf)(x_0) \sim 0$.

□

To analyze the semigroup Q_t generated by A, we first recall the **spectral bound** s and the **growth bound** w_1 for A defined as

$$s(A) := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \tag{4.46}$$

$$w_1 := \inf_{w\in\mathbb{R}}\{||Q_t x|| \leq Me^{wt}||x||_{D(A)}, \forall x \in D(A), t \geq 0 \text{ for suitable } M\} \tag{4.47}$$

and let

$$\lambda_0 := \inf_{\lambda \in \mathbb{R}} \{Af \leq \lambda f,\ 0 \prec f\}. \tag{4.48}$$

$C(\alpha, \beta)$ equipped the norm $|| \cdot ||_{\alpha,\beta}$ makes it a Banach lattice as defined in

Given $\lambda > \lambda_0$, by definition of λ_0, there exists $0 < u \in C(\alpha, \beta)$ such that $Au \leq \lambda u$. We define a strict half norm P_u on $C(\alpha, \beta)$:

$$P_u(f) = \sup_{x \in [0,1]} \frac{f^+(x)}{u(x)}, \qquad f^+ := \max(f, 0).$$

Since $u > 0$, P_u is well defined on $C(\alpha, \beta)$. P_u gives rise to a norm

$$||f||_p := P_u(f) + P_u(-f)$$

which is equivalent to the norm on $C(\alpha, \beta)$.

Let $B = \overline{L} - \lambda$, then $Bu \leq 0$. We now show B is P_u-dissipative (

Fix f. If $f \leq 0$, define $\phi_f := 0$. If f is positive at least at one point, denote by x_0 a point such that $P_u(f) = \frac{f^+(x_0)}{u(x_0)}$. Now consider $\phi_f \in C(\alpha, \beta)'$ such that

$$(g, \phi_f) = \frac{g(x_0)}{u(x_0)}.$$

Clearly such ϕ_f satisfies the second condition. Next we check the first condition. Let $f \in D(B)$, if $P_u(f) = 0$, then $f \leq 0$, so $(Bf, \phi_f) = (Bf, 0) = 0$. If $P_u(f) > 0$, since $P_u(f) = \frac{f(x_0)}{u(x_0)} \geq \frac{f(x)}{u(x)}$, then

$$P_u(f) \cdot u - f \geq 0,\ (P_u(f) \cdot u - f)(x_0) = 0. \tag{4.49}$$

By the positive minimum principle Lemma 4.4.1,

$$Bf(x_0) \le P_u(f)(Bu)(x_0) \le 0$$

i.e. $(Bf, \phi_f) \le 0$.

Let P_t be the semigroup generated by B. We show that P_t is P_u-contraction, i.e. $P_u(P_t f) \le P_u(f), \forall f \in D(B)$. Let $f \in D(B)$, $t > 0$, then

$$P_u(f) = (f, \phi_f) = (f - tBf + tBf, \phi_f) \le (f - tBf, \phi_f) \le P_u(f - tBf).$$

Thus for $\lambda > w_1$, $f \in C(\alpha, \beta)$, $(\lambda - B)R(\lambda, B)f = f$, so $P_u(\lambda R(\lambda, B)f) \le P_u(f)$, then by the formula $P_t f = \lim_{n\to\infty} \left(\frac{n}{t} \cdot R(\frac{n}{t}, B)\right)^n f$, we have

$$P_u(P_t f) \le P_u(f).$$

Thus the closure $\overline{B}$ generates a P_u-contraction semigroup, hence the closure $\overline{A} = \overline{B} + \lambda$ of A generates a positive semigroup of type $w_1(\bar{A}) \le \lambda$. Hence we showed that $\lambda_0 \ge w_1(A) = s(A)$. □

Estimation of λ_0 and rate of convergence

In this part, we show that $\lambda_0 < 0$, which implies the exponential convergence to δ measures at tangent boundaries. In order to do this, we construct u that satisfies $Lu < \lambda u$ for some $\lambda < 0$. The strategy of construction is to first use the behavior of L in the vicinity of two the boundary points to construct u near the boundaries and then find a connecting interior function.

Boundary Construction *1. Quadratic endpoint.* In the vicinity of quadratic point, $L = x^2\partial_{xx} + (b(x)+1)x\partial_x$ and recall $b_- = b(0)$. Let

$$u = Ax^c, A > 0.$$

Taking $c = \frac{-b_-}{2}$,

$$\frac{Lu}{u} = (c + b(x))c,$$

is negative in a neighborhood of 0.

2.Kimura endpoint. In the vicinity of a Kimura point, $L = x\partial_{xx} + b(x)\partial_x$ with $b = b(0)$. Let

$$u = \begin{cases} Ax^c,\ 0 < c < 1 & b = 0 \\ A(1-x)^c,\ c > 0 & b > 0 \end{cases}.$$

In both cases

$$\frac{Lu}{u} = (c - 1 + b(x))cx^{-1}$$
$$\frac{Lu}{u} = c(1-x)^{-2}[(c-1)x - b(x)(1-x)]$$

are negative in a neighborhood of 0.

Interior Construction Suppose we have constructed u in $[0, x_1]$ and $[x_2, 1]$. We now need to construct an interior function u that satisfies the boundary conditions. Inside $[x_1, x_2]$, suppose L has the form

$$L = \partial_{xx} + b(x)\partial_x,$$

where $b(x)$ is smooth on $[x_1, x_2]$. Let

$$B(x) = \int_{x_1}^{x} b(t)dt, \tag{4.50}$$

then B is smooth on $[x_1, x_2]$. We want Lu strictly negative, that is

$$u_{xx} + b(x)u_x < 0.$$

It is equivalent to $e^{B(x)}u_x$ being a decreasing function, i.e. there exists some positive function f on $[x_1, x_2]$, such that

$$(e^{B(x)}u_x)' = -f. \tag{4.51}$$

Case I: one tangent, one transverse points Without loss of generality, we assume a tangent boundary at $x = 0$. Now by construction of u at both boundaries, $u_x(x_1) > 0$, $u_x(x_2) > 0$. We first modify the constant A in the construction near $x = 1$ such that

$$e^{B(x_2)}u_x(x_2) < e^{B(x_1)}u_x(x_1).$$

Next, by (4.51), $f(x_1) = -e^{B(x_1)}Lu(x_1) > 0$, $f(x_2) = -e^{B(x_2)}Lu(x_2) > 0$ are two positive fixed constants. Then we can set f to be a positive function with $f(x_1)$, $f(x_2)$ fixed and

$$\int_{x_1}^{x_2} f(t)dt = e^{B(x_1)}u_x(x_1) - e^{B(x_2)}u_x(x_2).$$

Such an f guarantees that $e^{B(x)}u_x(x) > e^{B(x_2)}u_x(x_2) > 0$ for $x \in [x_1, x_2]$ which guarantees positiveness of u_x and hence positiveness of u.

Case II: two tangent boundaries By construction of u at both boundaries, $u_x(x_1) > 0$, $u_x(x_2) < 0$. We directly get $e^{B(x_2)}u_x(x_2) < e^{B(x_1)}u_x(x_1)$.

Denoting $F(x) = \int_{x_1}^{x} f(t)dt$, then $F(x)$ is a differentiable increasing function on $[x_1, x_2]$ with $F(x_1) = 0$. Then by considering derivatives of u

$$u_x(x) = e^{-B(x)} \left[e^{B(x_1)} u_x(x_1) - F(x) \right]$$

we know u is first increasing then decreasing on $[x_1, x_2]$, which implies u is lower bounded by $\min(u(x_1), u(x_2))$. Hence we only need to assign a positive f such that

$$\int_{x_1}^{x_2} f(t)dt = e^{B(x_1)} u_x(x_1) - e^{B(x_2)} u_x(x_2).$$

Theorem 4.4.1. *If there is only one tangent endpoint p, then for any non-quadratic point x, the transition probability $p_t(x, \cdot)$ converges exponentially to $\delta(p)$ in Wasserstein distance.*

If there are two tangent endpoints, then there exists S_0, satisfying $S_0(0) = 0$, $S_0(1) = 1$ and $LS_0 = 0$, such that for any probability measure v, $\mathcal{Q}_t^* v$ converges exponentially to $\delta(0) \int_0^1 v(1 - S_0) + \delta(1) \int_0^1 v S_0$ in Wasserstein distance.

Proof. If there is only one tangent endpoint, say $x = 0$, then for any $f \in C^0([0, 1])$ with $\mathrm{Lip}(f) \leq 1$, f can be decomposed as

$$f = f_0 + f(0), f_0 \in C(\alpha, \beta).$$

Then by Proposition **??**,

$$||\mathcal{Q}_t f - f(0)||_{C(\alpha,\beta)} = ||\mathcal{Q}_t f_0||_{C(\alpha,\beta)} \leq M e^{\lambda_0 t} ||f_0||_{C(\alpha,\beta)}$$

with some constant $M > 0$ independent of f. Since $f_0(0) = 0, \mathrm{Lip}(f_0) \leq 1$ and $0 < \alpha < 1, \beta \leq 0$, then $||f_0||_{C(\alpha,\beta)} \leq C_{\alpha,\beta}$ for some constant $C_{\alpha,\beta} > 0$ only dependent on α, β. Hence

$$|\mathcal{Q}_t f(x) - f(0)| \leq M C_{\alpha,\beta} e^{\lambda_0 t} x^{\alpha} (1 - x)^{\beta}. \tag{4.52}$$

Notice that only when $x = 1$ is a quadratic endpoint do we have $\beta < 0$. Hence for any non-quadratic point x, the transition probability $q_t(x, \cdot)$ converges to $\delta(0)$ in the sense of Wasserstein distance at an exponential rate. This convergence is uniform on any compact interval away from quadratic transverse point. If $x = 1$ is a quadratic point, then $q_t(x, \cdot) = \delta(1)$ for $\forall t > 0$.

If both endpoints are tangent, by Theorem 4.3.2, the kernel of L is in the linear space of $\{1, S\}$. Then we can take S_0 as a linear combination of 1 and S, such that $S_0(0) = 0$ and $S_0(1) = 1$. For any $f \in C^0([0,1])$ with $\mathrm{Lip}(f) \leq 1$, we decompose this as

$$f = f_0 + (1 - S_0)f(0) + S_0 f(1), \quad f_0 \in C(\alpha, \beta).$$

Again, by Proposition **??**,

$$||\mathcal{Q}_t f - (1 - S_0)f(0) - S_0 f(1)||_{C(\alpha,\beta)} = ||\mathcal{Q}_t f_0||_{C(\alpha,\beta)} \leq M e^{\lambda_0 t} ||f_0||_{C(\alpha,\beta)}$$

with $M > 0$ independent of f. Since $0 < \alpha, \beta < 1$, $f_0(0) = f_0(1) = 0$ and $\mathrm{Lip}(f_0) \leq 1$, then

$$||\mathcal{Q}_t f - (1 - S_0)f(0) - S_0 f(1)||_0 \leq M_{\alpha,\beta} ||\mathcal{Q}_t f - (1 - S_0)f(0) - S_0 f(1)||_{C(\alpha,\beta)},$$
$$||f_0||_{C(\alpha,\beta)} \leq C_{\alpha,\beta}$$

for some constant $M_{\alpha,\beta}, C_{\alpha,\beta} > 0$ only dependent on α, β. Hence,

$$||\mathcal{Q}_t f - (1 - S_0)f(0) - S_0 f(1)||_0 \leq M C_{\alpha,\beta} M_{\alpha,\beta} e^{\lambda_0 t},$$

so that the transition probability $q_t(x, \cdot)$ converges uniformly on $[0, 1]$ to $(1 - S_0(x))\delta(0) + S_0(x)\delta(1)$ at an exponential rate in Wasserstein distance. Hence starting from any probability measure v, $\mathcal{Q}_t^* v$ converges exponentially to $\delta(0) \int v(1 - S_0) + \delta(1) \int v S_0$ in Wasserstein distance. □

Remark 4.4.2. *Consider as a the first example the case with one tangent boundary point, say $x = 0$, quadratic tangent while $x = 1$ is Kimura transverse. For instance,*

$$L = x^2(1-x)\partial_{xx} + bx(a-x)\partial_x \text{ on [0,1]}, \ ab < 1, \ b(a-1) < 0.$$

Consider $f = x^c$. If we choose $0 < c < 1 - ab$, then

$$\frac{Lf}{f} = \frac{Lx^c}{x^c} = [(1-x)(c-1) + b(a-x)]\, c \le max(c - 1 + ab, b(a-1))c < 0.$$

So

$$\lambda_0 \le \min_{0<c<1-ab} \max(c-1+ab, b(a-1))c < 0.$$

As a second example, consider two tangent boundary points , for instance

$$L = x^2(1-x)\partial_{xx} \text{ on } [0,1].$$

In this case, we should expect $p(t, x, \cdot) \to (1 - S_0)\delta_0 + S_0\delta_1$, where S_0 in this case is x. By setting $f = x^c(1-x),\ 0 < c < 1$, then

$$Lf = (c(c-1) - c(c+1)x)x^c(1-x) \le \lambda f.$$

Here, $\lambda = c(c-1) < 0$. So

$$\lambda_0 \le \min_{0<c<1} c(c-1) = -\frac{1}{4}.$$

4.4.2 Case with two transverse boundaries

Recall that we have constructed the invariant measure μ when both endpoints are transverse in Section 3.2. We first introduce a (W)-Lyapunov-Poincaré inequality that will be used to analyze the long time behavior of the transition probability.

(W)-Lyapunov-Poincaré inequality

Consider $U(z) = \int_0^z b(s)ds$ with $b(s)$ negative when $s \to -\infty$ and positive when $s \to \infty$. Then U grows linearly to $+\infty$ near $z = \pm\infty$. L can be taken as

$$L_z = \frac{1}{2}\partial_{zz} - \nabla U(z)\partial_z = -\frac{1}{2}\partial_z^*\partial_z$$

with $\partial_z^* = \partial_z + 2\nabla U(z)$ on a probability space (X, μ), with μ the invariant measure

$$\mu(dz) = \frac{e^{-2U(z)}}{Z}dz \tag{4.53}$$

where the normalizing constant $Z < \infty$ due to the linear growth of U . The main advantage of switching to $L^2(\mu)$ is L is self adjoint on $L^2(\mu)$. Indeed

$$\int_{\mathbb{R}} (\partial_z\varphi)\phi d\mu = \int_{\mathbb{R}} \varphi(-\partial_z\phi + 2\phi\nabla U(z))d\mu = \int_{\mathbb{R}} \varphi\partial_z^*\phi d\mu$$

so that

$$(\varphi, L\phi)_{L^2(\mu)} = (L\varphi, \phi)_{L^2(\mu)}.$$

For some function $W \in D(L)$ with $W \geq 1$, let $I_W(t) = \int Q_t^2 f W d\mu$. Then

Local Poincaré inequality In the transverse case, we can not find a global Lyapunov function. We have to weaken the condition to define a **local Lyapunov function** V as

$$LV \leq -\alpha V + \beta \mathbf{1}_{\mathcal{C}},\ V \geq 1 \tag{4.54}$$

for some set $\mathcal{C}$.

Definition 4.4.2 (Local Poincaré inequality). *Let Ω be a subset of the whole space X (we will use $\mathcal{C} \subset \Omega$). We say that μ satisfies a local Poincaré inequality on Ω if there exists some constant κ_Ω such that for all nice f with $\int_X f d\mu = 0$,*

$$\int_\Omega f^2 d\mu \leq \kappa_U \int_X |\nabla f|^2 d\mu + \frac{1}{\mu(\Omega)} \left(\int_\Omega f d\mu \right)^2. \tag{4.55}$$

We now show the:

Proposition 4.4.1. *Assume that there is some local Lyapunov function $V \geq 1$ under some set $\mathcal{C}$ such that (4.54) holds, and μ satisfies a local Poincaré inequality on $\Omega \supset \mathcal{C}$ with moreover*

$$\beta\mu(\Omega^c) < \alpha\mu(\Omega). \tag{4.56}$$

Then we can find $\lambda \geq 0$ such that if $W = V + \lambda$, then μ satisfies a (W)-Lyapunov-Poincaré inequality.

Proof. Multiply (4.54) by f^2 and integrate to get

$$\int_X f^2 LV d\mu \leq -\alpha \int_X f^2 V d\mu + \beta \int_C f^2 d\mu.$$

Let $\int_X f d\mu = 0$. The local Poincaré inequality implies

$$\begin{aligned}\int_\Omega f^2 d\mu &\leq \kappa_\Omega \int_X |\nabla f|^2 d\mu + \frac{1}{\mu(\Omega)} \left(\int_\Omega f d\mu \right)^2 \\ &\leq \kappa_\Omega \int_X |\nabla f|^2 d\mu + \frac{1}{\mu(\Omega)} \left(- \int_{\Omega^c} f d\mu \right)^2 \\ &\leq \kappa_\Omega \int_X |\nabla f|^2 d\mu + \frac{\mu(\Omega^c)}{\mu(\Omega)} \left(\int_{\Omega^c} f^2 d\mu \right),\end{aligned}$$

so that

$$\int_X f^2\left(LV+\alpha V\right)d\mu \le \beta\int_C f^2 d\mu \le \beta\int_\Omega f^2 d\mu$$
$$\le \beta\kappa_U\int_X |\nabla f|^2 d\mu + \frac{\beta\mu(\Omega^c)}{\mu(\Omega)}\left(\int_{\Omega^c} f^2 V d\mu\right).$$

We re-organize the inequality as,

$$\alpha\left(1-\frac{\beta\mu(\Omega^c)}{\alpha\mu(\Omega)}\right)\int_X f^2 V d\mu \le \beta\kappa_\Omega\int_X |\nabla f|^2 d\mu - \int_X f^2 LV d\mu$$
$$\le \int_X |\nabla f|^2 (V+\lambda) d\mu - \int_X f^2 L(V+\lambda) d\mu$$
$$= \int_X |\nabla f|^2 (V+\lambda) d\mu - \int_X Lf^2 (V+\lambda) d\mu.$$

Here we take $\lambda = (\beta\kappa_U - 1)_+$ so that $\beta\kappa_\Omega \le V+\lambda$. Since,

$$\int_X f^2 (V+\lambda) d\mu \le (1+\lambda)\int_X f^2 V d\mu,$$

then

$$\alpha\left(1-\frac{\beta\mu(\Omega^c)}{\alpha\mu(\Omega)}\right)\frac{1}{1+\lambda}\int_X f^2(V+\lambda)d\mu \le \int_X \left(|\nabla f|^2(V+\lambda) - Lf^2(V+\lambda)\right)d\mu$$

and $\frac{1}{C_{LP}} = \alpha\left(1-\frac{\beta\mu(\Omega^c)}{\alpha\mu(\Omega)}\right)/(1+\lambda)$. □

Long time behaviors

Let us come back to the operator $L = \frac{1}{2}\Delta - \nabla U\cdot\nabla$ on (X,μ). We first construct the local Lyapunov function V with $\mathcal{C}$ satisfying (4.54).

Lemma 4.4.2. *There exists a local Lyapunov function.*

Proof. Let $X = [x_l, x_r],\ -\infty \leq x_l < x_r \leq \infty$ and

$$V = \begin{cases} e^U & x_l = -\infty \\ 2 - (x(z) - x_l) & x_l > -\infty. \end{cases}$$

We define V similarly in the vicinity of x_r. We check that $LV \leq -\alpha V,\ V > 1$ for some $\alpha > 0$ in a neighborhood x_l:

1. When $x_l = -\infty$, for $V = e^U$,

$$LV = V\Big(\frac{1}{2}\Delta U - \frac{1}{2}|\nabla U|^2\Big).$$

By the transverse condition, $\lim\limits_{z\to-\infty} \nabla U = -b_- < 0$, which implies $\lim\limits_{z\to\infty} \Delta U = 0$. So

$$\lim_{z\to-\infty} \Big(\frac{1}{2}\Delta U - \frac{1}{2}|\nabla U|^2\Big) = -\frac{1}{2}b_-^2 < 0.$$

Since $\mathop{lim}\limits_{z\to-\infty} U(z) = \infty$, for $0 < \alpha < \frac{b_-^2}{2}$, we have $LV \leq -\alpha V,\ V \geq 1$ in a neighborhood of x_l.

2. When $x_l > -\infty$, for $V = 2 - (x(z) - x_l)$,

$$LV = L_x(2 - x) = -b(x).$$

By the transverse condition, $b_- > 0$, so $LV(0) < 0$ and since $V(0) = 2 > 0$, $LV \leq -\alpha V,\ V \geq 1$ for some $\alpha > 0$ in a neighborhood of x_l.

In all cases, $LV \leq -\alpha V,\ V > 1$ for some $\alpha > 0$ in a neighborhood x_l, x_r.

After constructing V in a neighborhood of x_l and x_r, we extend it to a complement $\mathcal{C}$ of these two neighborhoods as a twice differentiable function with $V \geq 1$. Since $\mathcal{C}$ is compact, LV, V are bounded on $\mathcal{C}$, and we can find some $\beta > 0$ large enough that V is a Lyapunov function for $\mathcal{C}$:

$$LV \leq -\alpha V + \beta \mathbf{1}_{\mathcal{C}}.$$

□

Next we verify that μ satisfies a local Poincaré inequality (4.4.2). Indeed by

We now state our main result of the section:

Theorem 4.4.2. *The invariant measure μ in* (4.53) *satisfies a Poincaré inequality. For $f \in L^2(\mu)$,*

$$||\mathcal{Q}_t f - \int f\mu||_{L^2(\mu)} \leq e^{-\frac{t}{2C_{LP}}}||f - \int f\mu||_{L^2(\mu)}. \tag{4.57}$$

For any probability measure $v = h\mu$ with $h \in L^2(\mu)$,

$$||\mathcal{Q}_t^* v - \mu||_{TV} \leq e^{-\frac{t}{2C_{LP}}}||h - 1||_{L^2(\mu)}.$$

Proof. By construction, $\mathcal{C}$ is a compact set in the interior of X. We choose a larger set U with $\mathcal{C} \subset U \subset int(X)$ such that $\beta\mu(U^c) < \alpha\mu(U)$. μ is bounded on U, so μ satisfies a local Poincaré inequality on U. Applying Proposition 4.4.1, μ satisfies a (W)-Lyapunov-Poincaré inequality. Thus by (**??**), for all f such that $\int f^2 W\mu < \infty$ and $\int f\mu = 0$,

$$\int (\mathcal{Q}_t f)^2 W\mu \leq e^{-\frac{t}{C_{LP}}} \int f^2 W\mu.$$

Since $W \in L^1(\mu)$, by

For the second assertion, we use the symmetry property $\mathcal{Q}_t^*(h\mu) = (\mathcal{Q}_t h)\mu$ and (4.57) to derive that

$$||\mathcal{Q}_t^* v - \mu||_{\text{TV}} = ||\mathcal{Q}_t h - 1||_{L^1(\mu)} \leq ||\mathcal{Q}_t h - 1||_{L^2(\mu)} \leq e^{-\frac{t}{2C_{LP}}}||h - 1||_{L^2(\mu)}.$$

□

Remark 4.4.3. *Regarding logarithmic Sobolev inequalities, which are stronger than Poincaré inequalities, we may use a criterion on a ball of radius R (*

4.5 Long time behaviour in two dimension

In this section we analyze the long time behavior of transition probability on a two-dimensional manifold with corners. We refer to

Let P be a paracompact Hausdorff topological space. A chart of $p \in P$ is a pair $(\mathcal{U}_p, \psi_p)$ where $\mathcal{U}_p$ is a neighborhood of p and ψ_p is a homeomorphism with $\psi_p(p) = 0$ from $\mathcal{U}_p$ to a neighborhood of 0 in $\mathbb{R}^l_+ \times \mathbb{R}^{2-l}$ for some $l \in \{0, 1, 2\}$. Two charts $(\mathcal{U}_p, \psi_p), (\mathcal{U}_q, \psi_q)$ are said to be compatible if

$$\psi_p \circ \psi_q^{-1} : \psi_q(\mathcal{U}_p \cap \mathcal{U}_q) \longrightarrow \psi_p(\mathcal{U}_p \cap \mathcal{U}_q)$$

is a diffeomorphism. The codimension l is well defined for $p \in P$. We say that the point p is an interior point if $l = 0$, an edge point if $l = 1$, a corner if $l = 2$. A two dimensional manifold with corners is paracompact Hausdorff topological space equipped with a maximal compatible atlas.

4.5.1 Lyapunov function

We make the following assumption:

Assumption 4.5.1. *For L on a 2 dimensional compact manifold with corners P, there is exactly one tangent edge H, and when restricted to H, $L|_H$ is transverse to both boundary points.*

We say that V is a Lyapunov function if V is strictly positive except on H, $V|_H \equiv 0$ and *for some* $\lambda_0 < 0$,

$$LV \le \lambda_0 V. \tag{4.58}$$

We make the following assumption:

Assumption 4.5.2. *There is a twice differentiable function* $\rho(p)$ *such that*

a. $\rho > 0$ *except on* H, $\rho|_H \equiv 0$;

b. $\nabla\rho \neq 0$ *nowhere vanishing;*

c. *For* p *in an edge* E, *if* $\nabla_E \rho(p) = 0$, *then* $\rho \equiv c$ *in a relative neighborhood of* p *along* E;

d. *There is no local minimum point of* ρ *other than on* H.

The third assumption ensures that ρ *is constant or strictly monotonic along any edge* E. *In the presence of such* ρ, *there exists a stratification of* P *so that:*

1. P *is covered by the layers:*

$$P = \bigcup_{i=0}^{n} P_{[k_i,k_{i+1}]} := \bigcup_{i=0}^{n} \{p \in P : \rho(p) \in [k_i, k_{i+1}]\}, \tag{4.59}$$
$$0 = k_0 < k_1 < \cdots k_n = \max_P \rho.$$

2. *Each layer* $P_{[k_i,k_{i+1}]}$ *is covered by a finite collection of closed sets* U_{i,p_j} , *i.e.*

$$P_{[k_i,k_{i+1}]} = \bigcup_j U_{i,p_j}, \tag{4.60}$$

where p_j *is a point in* U_{i,p_j} *and the range of* ρ *in each* U_{i,p_j} *is* $[k_i, k_{i+1}]$.

We leave the construction of the stratification to Lemma 5.3.2 in the Appendix. We then have the following result:

Theorem 4.5.1. *There exists a Lyapunov function* V *on* P.

Proof. We first analyze the local behavior of ρ, L. Note that for

$$L = A(x,y)\partial_{xx} + B(x,y)\partial_{xy} + C(x,y)\partial_{yy} + D(x,y)\partial_x + E(x,y)\partial_y,$$

under the new coordinates (η, ρ), the ordinary differential ρ-part L_ρ is

$$L_\rho = (A\rho_{xx} + B\rho_{xy} + C\rho_{yy})\partial_{\rho\rho} + (L\rho)\partial_\rho. \tag{4.61}$$

Given any point $p \in P$, we discuss the local property of L at point p subject to the following cases.

1. p is an interior point There exists a closed set including p equipped with coordinates (x, ρ) that is diffeomorhic to the rectangle $R := [0,1] \times [\rho(p), \rho(p) + \epsilon]$. In this case, L_ρ is elliptic of the form

$$L_\rho = a(x,\rho)[\partial_{\rho\rho} + b(x,\rho)\partial_\rho], (x,\rho) \in R, \tag{4.62}$$

where $a(x, \rho) > 0$.

2. p is an edge point Under local adapted coordinates (x, y),

a. if $\rho \equiv \rho(p)$ along the edge $y = 0$, then (x, ρ) are local adapted coordinates. There is a neighborhood of p that is diffeomorphic to the rectangle

$$R := [0,1] \times [\rho(p) - \epsilon, \rho(p)].$$

L_ρ is degenerate at $\rho(p)$ of the form: for $(x, \rho) \in R$,

$$\textbf{Kimura} : L_\rho = a(x,\rho)\left[(\rho(p) - \rho)\partial_\rho^2 + c(x,\rho)\partial_\rho\right] \tag{4.63}$$

$$\textbf{quadratic} : L_\rho = a(x,\rho)\left[(\rho - \rho(p))^2\partial_\rho^2 + d(x,\rho)(\rho(p) - \rho)\partial_\rho\right] \tag{4.64}$$

where $a(x, \rho) > 0$.

b. If the gradient of ρ along the edge $\nabla_x \rho(p) \neq 0$, then (y, ρ) are local adapted coordinates. There is a neighborhood of p that is diffeomorphic to $R := [0, 1] \times [\rho(p), \rho(p) + \epsilon]$ for some $\epsilon > 0$. In this case L_ρ is elliptic of the same form with (4.62) in R.

3. p **is a corner** Under local adapted coordinates (x, y),

a. if along one (e.g. x) edge $\nabla_x \rho(p) = 0$, then $\rho \equiv \rho(p)$ in a relative open neighborhood of p on the x-edge. Since $\nabla_y \rho(p) \neq 0$, then by rescaling x, (x, ρ) are also local adapted coordinates at p; this case is then the same as in (4.63), (4.64). See the first picture in Figure 4.1.

b. If $\nabla_x \rho(p), \nabla_y \rho(p) \neq 0$, then (x, ρ) are not adapted coordinates. In this case, there exists a neighborhood of p that is diffeomorphic to an irregular area inside a rectangle

$$T := \{(x, \rho) : \rho \geq \rho(x)|_{y=0}\} \subset [0, 1] \times [\rho(p) - \epsilon_1, \rho(p) + \epsilon_2].$$

See the second and third pictures in Figure 4.1. L_ρ is then degenerate when $x = 0$ and elliptic when $x \neq 0$.

If p is neither a local maximum nor minimum, T is made up of a rectangle and an irregular area T_1, we can extend the coefficients of L_ρ to the rectangle supplemented by the dashed lines. Then in these two rectangles, L_ρ are both degenerate when $x = 0$ and elliptic when $x \neq 0$.

We now start the construction of V iteratively in each layer. Such V generally depends only on the variable ρ, i.e., $V(p) = V(\rho(p))$, except near the corners.

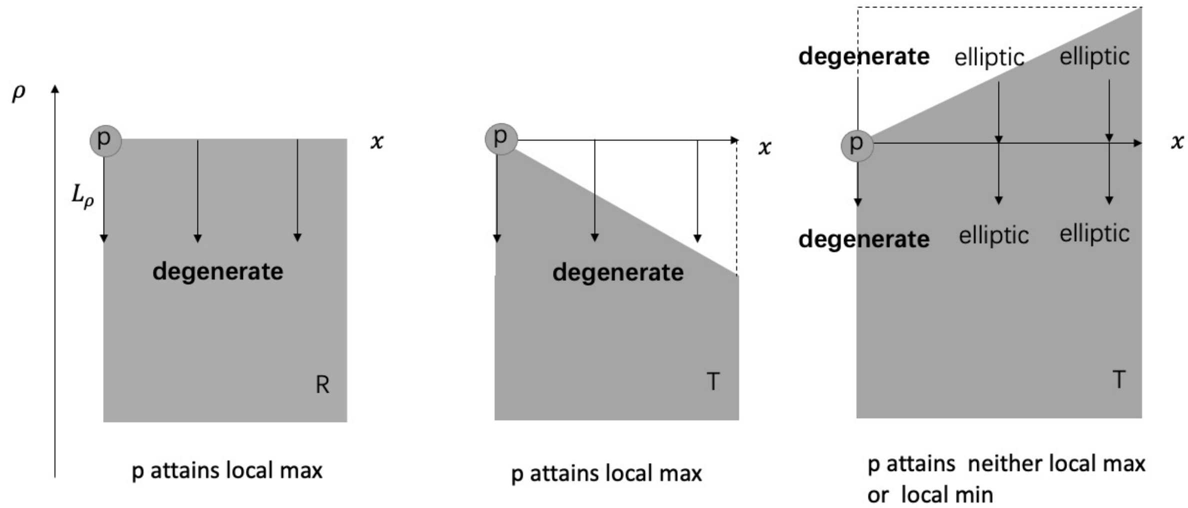

Figure 4.1: L_ρ at a corner

Case $i = 0$ $\quad P_{[0,k_1]}$ is covered by finite rectangles $R := [0,1] \times [0,k_1]$ where L_ρ takes the form

$$\textbf{Kimura} : L_\rho = a(x,\rho)\left(\rho\partial_\rho^2 + b(x,\rho)\partial_\rho\right) \tag{4.65}$$

$$\textbf{quadratic} : L_\rho = a(x,\rho)(\rho^2\partial_\rho^2 + c(x,\rho)\rho\partial_\rho) \tag{4.66}$$

where $a(x,\rho) > 0, b(x,\rho) < 1, c(x,\rho) < 1$ since H is a tangent edge. We take $V = \rho^\nu$, where ν depends on the considered case. In case (4.65), we choose $0 < \nu < \min\limits_{(x,\rho)}(1 - b(x,\rho))$ so that $L_\rho\rho^\nu = a\nu(\nu + b - 1)\rho^{\nu-1} \leq -\lambda\rho^\nu$ for some $\lambda > 0$. In case (4.66), we choose $0 < \nu < \min\limits_{(x,\rho)}(1 - c(x,\rho))$ so that $L_\rho\rho^\nu = a\nu(\nu + c - 1)\rho^\nu \leq -\lambda\rho^\nu$ for some $\lambda > 0$.

Case $i \geq 1$ $\quad P_{[k_i,k_{i+1}]}$ is then covered by four kinds of disjoint collections of open set R as we analyzed before, namely:

I. R does not contain any corner or local maximum point, L_ρ is elliptic for $\forall x$ of the form (4.62);

II. R contains an edge and ρ is constant along this edge, so L_ρ takes the form (4.63), (4.64);

III. R contains a corner that is a strict local maximum point of ρ,

IV. R contains a corner that is neither a local maximum point nor local minimum point of ρ.

At this layer $P_{[k_i,k_{i+1}]}$, R_{II} and R_{III} are both disconnected from the other three types. R_I may only be connected with R_{IV}. There maybe more than one R of type II, but since L_ρ takes a uniform form in these R_{II}, in our construction below we may regard these R_{II} as a uniform R and for convenience of notation we still let $x \in [0,1]$. R_{III} will be treated in this way as well. For R_I, we can merge these R_I as a uniform one as well. If there exists R_{IV}, we construct V separately in R_{IV} so that it may be patched with V in R_I.

Thus, it suffices to construct V in these four cases separately. Since $k_i > 0$ when $i \geq 1$, we assume that $\rho(p) = 1$.

Case I: Fix an arbitrary x_0, we choose $f(\rho)$ so that

$$f'(\rho) < b(x_0, \rho) - b(x, \rho).$$

Starting from $\rho = 1$, we let, $V_\rho = \exp(f(\rho) - \int^\rho b(x,z)dz)$, then $\frac{L_\rho V_\rho}{V_\rho} = [f' - b(x_0, \rho) + b(x, \rho)] < 0$.

Case II: In the Kimura case (4.63), we fix x_0 so that $c(x_0, 1) = \max_{x\in[0,1]} c(x, 1)$. We choose $f(\rho)$ so that

$$f'(\rho) < \frac{c + c(x_0, \rho) - c(x, \rho)}{1 - \rho}, c = -c(x_0, 1) > 0$$

on $[1 - \epsilon, 1]$. Denote $H^x(\rho) = \exp\left[\int^\rho \frac{c(x,z)}{1-z} dz\right] \sim (1 - \rho)^{-c(x,1)}$, and let

$$V_\rho = \frac{(1 - \rho)^c e^{f(\rho)}}{H^{x_0}(\rho)}.$$

Then, $\frac{L_\rho V_\rho}{V_\rho} = [(1-\rho)f' - c - c(x_0, \rho) + c(x, \rho)] < 0$.

In the quadratic case (4.64), we fix x_0 so that $d(x_0, 1) = \max_{x \in [0,1]} d(x, 1)$. We choose $f(\rho)$ so that

$$(1-\rho)f' - (d-1) - d(x_0, \rho) + d(x, \rho) < 0, d-1 = -d(x_0, 1) - 1 > 0$$

on $[1-\epsilon, 1]$. Denote $H^x(\rho) = \exp\left[\int^\rho \frac{d(x,z)}{1-z} dz\right] \sim (1-\rho)^{-d(x,1)}$, let

$$V_\rho = \frac{(1-\rho)^{d-1} e^{f(\rho)}}{H^{x_0}(\rho)}.$$

Then, $\frac{L_\rho V_\rho}{V_\rho} = (1-\rho)[(1-\rho)f' - (d-1) - d(x_0, \rho) + d(x, \rho)] < 0$.

Case III: If the corner p is a strict local maximum point, we can extend the coefficients of L_ρ to the rectangle which is supplemented by the dashed lines so that L_ρ is has the same degree of degeneracy at $\rho = \rho(p)$.

Case IV: In this case, if L has the same degeneracy type towards two edges intersecting at the corner p. Let U be a neighborhood of p inside T and introduce new coordinates (x, z) so that $z = \rho$ in $T \setminus U$, $z_\rho > 0$ and p attains local maximum of z. So in the (x, z)-coordinate, L_z has the same kind of degeneracy as L_ρ in an irregular area T'. We extend the coefficients of L_z to a rectangle so that L_z takes the form (4.63), (4.64). Taking the solutions V to these two forms which we already constructed, we see V is infinity at the corner of two quadratic edges.

If L has different types of degeneracy towards the two edges, then it takes the form

$$L = \left[xa(x,\rho) + (\rho-1)^2\right] \partial_{\rho\rho} + xb(x,\rho)\partial_{xx} + c(x,\rho)\partial_\rho + d(x,y)\partial_x + x(\rho-1)e(x,y)\partial_{x\rho} \tag{4.67}$$

on $[0,1] \times [0,1]$, $\rho \in [0,1]$, where $a, b, c, d > 0$.

When $x = 0$, $L_\rho^0 = (1-\rho)^2\partial_{\rho\rho} + c(x,\rho)\partial_\rho$. Clearly a decreasing linear function V would satisfy $L_\rho^0 V < 0$. But such V has negative derivative, which is not desired. So our strategy is to first construct $f_0(\rho), f_1(\rho)$ at $x = 0, 1$ so that

$$L_\rho^0 f_0, L_\rho^1 f_1 < 0,\ \ f_1'(\rho) > 0.$$

We then patch them together with the desired boundary conditions. We choose two non-negative functions $\chi_0(x), \chi_1(x)$ so that

$$\chi_0(0) > 0, \chi_0 = 0 \text{ in } [1-\epsilon, 1],$$
$$\chi_1(0) = 0, \chi_1 = \chi(1) \text{ in } [1-\epsilon, 1].$$

Suppose that $\max\limits_{(x,\rho)} -\frac{xb\chi_1''+d\chi_1'}{\chi_1} < M$ for some $M > 0$ and let $c_{\min} = \min\limits_{[0,1]\times[0,1]} c(x,\rho)$. We choose another two non-negative functions $h_0(\rho), h_\epsilon(\rho)$ so that

$$h_0(0) = 0, h_0'(0), h_0''(0) = 0, h_0(\rho) = \frac{c_{min}}{2} - M(1-\rho) \text{ on } [1 - \frac{c_{min}}{2M}, 1]$$
$$h_\epsilon \in C_c^2([0, 1 - \frac{c_{min}}{2M} + \epsilon]), h_\epsilon(0) = 0, h_\epsilon'(0) = V'(0), h_\epsilon''(0) = V''(0),$$
$$L_\rho h_\epsilon < 0 \text{ on } [0, 1 - \frac{c_{min}}{2M}].$$

Then for $\rho \in [1 - \frac{c_{min}}{2M}, 1]$,

$$\frac{(L_\rho + xe(\rho-1)\chi_1'\partial_\rho)h_0}{h_0} > -\frac{xb\chi_1'' + d\chi_1'}{\chi_1}. \tag{4.68}$$

Indeed we can rescale χ_1 so that the coefficient of ∂_ρ in $L_\rho + xe(\rho-1)\chi_1'\partial_\rho$ is bigger than $\frac{c_{min}}{2}$, so the left hand side term is no less than M, hence (4.68) is true.

We let

$$V(x,\rho) = c_0 f(x,\rho) h_0(\rho) + h_\epsilon(\rho)$$

where $f(x,\rho) = \chi_0(x) f_0(\rho) + \chi_1(x) f_1(\rho), c_0 > 0$ is a constant.

We want to ensure that $L(fh_0) < 0$ on $\rho \in [1 - \frac{c_{min}}{2M}, 1]$. The coefficient of f_1 in $L(fh_0)$ is

$$(xb\chi_1'' + d\chi_1')h_0 + \chi_1(x) L_\rho h_0 + xe(x,\rho)(\rho - 1)\chi_1'(x) h_0'(\rho) \tag{4.69}$$

through a direct computation in(4.70) below, which is positive by (4.68). So we can subtract a positive constant from f_1 so that $L(fh_0) < 0$ when $\rho \in [1 - \frac{c_{min}}{2M}, 1]$. We compute:

$$\begin{aligned} Lf =& \chi_0(x) L\rho f_0 + \chi_1(x) L_\rho f_1 + xb(x,\rho)\left[\chi_0'' f_0 + \chi_1'' f_1\right] \\ &+ d(x,y)\left[\chi_0' f_0 + \chi_1' f_1\right] + x(\rho - 1) e(x,y)\left[\chi_0' f_0' + \chi_1' f_1'\right] \\ L(f(x,\rho)h_0(\rho)) =& (Lf) \cdot h_0 + f \cdot L_\rho h_0 + h_0'(\rho)\left[xe(x,\rho) f_x + 2(xa(x,\rho) + \rho^2) f_\rho\right] \end{aligned} \tag{4.70}$$

Finally for $c_0 > 0$ small enough, V is positive since $h_\epsilon > 0$ and $L_\rho h_\epsilon < 0$ for $\rho \in [0, 1 - \frac{c_{min}}{2M} + \epsilon]$. We adjust h_ϵ so that $\max_{\rho \in [1 - \frac{c_{min}}{2M}, 1]} L_\rho h_\epsilon < \max_{\rho \in [1 - \frac{c_{min}}{2M}, 1]} - L_\rho(c_0 f h_0)$ and $LV < 0$. Such V is finite by construction, so $\frac{LV}{V} < 0$. V satisfies the boundary conditions at $\rho = 0$ and when $x \in [1 - \epsilon, 1]$, V only depends on ρ, so this V could be extended to a global function in this layer.

By induction, we construct V layer by layer, and such V is positive on $P \setminus H$ while V is ∞ at corners which are the intersection of two quadratic edges and on the quadratic edges where ρ is constant. We have finitely many layers and each layer is covered by finite closed rectangles or irregular areas, and we showed that $\frac{LV}{V} < 0$ in each of this area, so V is a Lyapunov function indeed satisfying $LV \leq \lambda_0 V$ for some $\lambda_0 < 0$. □

Remark 4.5.1. *The Lyapunov function V we constructed above behaves like ρ^c where ρ is* *the distance to the tangent edge and c is some positive constant.*

V gives rise to a norm $||\cdot||_{P_V}$:

$$||f||_{P_V} := \sup_P \frac{f^+}{V} + \sup_P \frac{f^-}{V}.$$

and we define the space $C^V(P)$:

$$C^V(P) := V \cdot C^0(P)$$

equipped with the norm $||\cdot||_{P_V}$. By saying $f \succ 0$ if $||f||_{P_V} > 0$, let

$$\lambda_0 := \inf_{\lambda \in \mathbb{R}} \{Af \leq \lambda f,\ 0 \prec f\}.$$

Clearly, $\lambda_0 < 0$. A similar argument of Proposition **??** *gives rise to the following result:*

Proposition 4.5.1. *There exists some $M > 0$ so that for $f \in C^V(P)$, $t > 0$,*

$$||P_t f||_{P_V} \leq M ||f||_{P_V} e^{\lambda_0 t}.$$

Fix a point $p \in P$ neither on H nor any quadratic edge, we denote $p_t(p, \cdot)$ the transition probability of the diffusion starting from p. A corollary is

Corollary 4.5.1. *For $f \in C^V(P), t > 0$*

$$|P_t f(p)| \leq V(p) ||P_t f||_{P_V} \leq M V(p) ||f||_{P_V} e^{\lambda_0 t}. \tag{4.71}$$

Remark 4.5.2. *Consider the example in*

4.5.2 Exponential convergence to invariant measure

First, there exists a tubular neighborhood of H that is diffeomorphic to $U = [0,1]\times[0,1]$ with H mapped to $[0,1]\times\{0\}$. We fix a non-negative function $h \in C_c(U)$ satisfying $h|_H \equiv 1, h|_{y=1} \equiv 0$. Then we integrate the transition probability $p_t(p,x,y)$ with respect to h in U:

$$q_h(t,x) := \int_0^1 p_t(p,x,y)h(y)dy. \tag{4.72}$$

It can be interpreted as the marginal density with respect to coordinate along H (x) and limited to some neighbourhood of H (specified by h).

We already know that there exists a unique invariant measure μ_0 supported on H. The total variation distance between $q_h(t,x)$ and $\mu_0(x)$ can be bounded by χ^2-divergence or Kullback-Leibler (KL) divergence. To show the exponential decay of the χ^2-divergence or the Kullback-Leibler(KL), convergence requires a Poincaré inequality or a logarithmic Sobolev inequality, respectively, for μ_0. It turns out that if $L|_H$ has a Kimura endpoint, then the χ^2-divergence is not finite. And if $L|_H$ has a quadratic endpoint, then μ_0 does not satisfy the logarithmic Sobolev inequality. So in the proof of the following theorem, we employ an interpolation divergence as constructed in

For $1 < r \le 2$, define

$$\psi(u) = 3 + \left(\frac{5}{2} - \frac{3}{r-1}\right)(u-3) + \frac{9}{r(r-1)}\left[\left(\frac{u}{3}\right)^r - 1\right], u \ge 0. \tag{4.73}$$

Such ψ is convex, $\psi(1) = 0$ and $\psi(u) \sim \frac{9}{r(r-1)3^r}u^r$ at $+\infty$.

Lemma 4.5.1. *On $I\times[0,1]$, L takes the form*

$$L = a(x,y)\partial_{xx} + b(x,y)y^m\partial_{yy} + c(x,y)y\partial_{xy} + d(x,y)\partial_x + e(x,y)^{m-1}\partial_y, \quad m \in \{1,2\}. \tag{4.74}$$

The proofs for the cases H being a quadratic edge or Kimura edge are essentially the same.

So in the following proof, we assume that $m = 2$. In addition, we will prove below that in an appropriate set of variables, we may choose

$$c(x, 1) = 0. \tag{4.75}$$

Regarding $q_h(t, x)$ as the marginal density limited to neighbourhood of H, its total measure approaches 1 at an exponential rate of λ_0. Indeed

$$\begin{aligned}\int_I q_h(t, x)dx = &\int_I \int_0^1 p_t(p, x, y)h(y)dxdy = 1 - \int_I \int_0^1 p_t(p, x, y)[1 - h(y)]dxdy \\ &\geq 1 - Me^{\lambda_0 t}\end{aligned} \tag{4.76}$$

for some $M > 0$. Moreover, it satisfies the backwards equation:

$$\partial_t q(t, x) = \int_0^1 L^* p_t(p, x, y)h(y)dy = L_t^* q_h(t, x) + v(t, x) \tag{4.77}$$

where by integrating by parts

$$\begin{aligned}L_t^* q(t, x) =&\partial_{xx} \int_0^1 a(x, y)p_t(p, x, y)h(y)dy - \partial_x \int_0^1 c(x, y)yp_t(p, x, y)h'(y)dy \\ &- \partial_x \int_0^1 d(x, y)p_t(p, x, y)h(y)dy \\ =:&\partial_{xx}(B(t, x)q_h(t, x)) - \partial_x(A(t, x)q_h(t, x))\end{aligned}$$

$$v(t, x) = \int_0^1 b(x, y)y^2 p_t(p, x, y)h''(y)dy + \int_0^1 e(x, y)yp_t(p, x, y))h'(y)dy. \tag{4.78}$$

We make a few remarks on these terms. First, $||v(t, x)||_1$ and $||[A(t, x) - d(x, 0)]q_h(t, x)||_1$ decay exponentially at the rate λ_0. Second, $c(x, 1) = 0$ ensures that $\frac{c(x,y)yh'(y)}{h(y)}$ is bounded, *so that by choosing U sufficient small we may assume that*

$$\left|\partial_x[B(t,x)-a(x,0)]\right|+\left|\frac{\mu_0'}{\mu_0}(B(t,x)-a(x,0))\right|+\left|(A(t,x)-d(x,0))\right|\leq\epsilon. \tag{4.79}$$

*We are now ready to estimate $||q_h(t,x)-\mu_0(x)||_{TV}$. Let $f(t,x)=\frac{q_h(t,x)}{\mu_0(x)}$. Then by Lemma **??**,*

$$||q_h(t,x)-\mu_0(x)||_{TV}\leq\frac{C_\psi}{P(t)}\sqrt{\int\psi(f(t,x))\mu}+Me^{-\lambda_0 t}, \tag{4.80}$$

where $\mathbb{P}(t)=||q_h(t,x)||_1$. We differentiate the term

$$\begin{aligned}\frac{d}{dt}(\psi(f(t,x)),\mu_0)=&(\psi'(f),\partial_t q_h(t,x))\\ =&(L_0\psi'(f),f\mu_0)+((L_t-L_0)\psi'(f),f\mu_0)+(\psi'(f),v(t,x))\\ =:&-\int_I\psi''(f)(\partial_x f)^2\mu_0+II+III.\end{aligned}$$

We will show below that

$$II\leq2\epsilon\int_I\psi''(f)(\partial_x f)^2\mu_0+\epsilon\int_I\psi(f)\mu_0dx+Me^{\lambda_0 t} \tag{4.81}$$

$$III\leq\epsilon\int_I\psi(f)\mu_0dx+Me^{(1-\epsilon)\lambda_0 t}. \tag{4.82}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}(\psi(f(t,x)),\mu_0)\leq&-(1-3\epsilon)\int_I\psi''(f)(\partial_x f)^2\mu_0+2\epsilon\int_I\psi(f)\mu_0+Me^{\lambda_0 t}\\ \leq&-\left[\frac{1-3\epsilon}{C_\psi}-2\epsilon\right]\int_I\psi(f)\mu_0+Me^{\lambda_0 t}.\end{aligned}$$

As a consequence,

$$\int_I \psi(f(t,x))\mu_0 \le Me^{-at} \tag{4.83}$$

where $a = \min(\frac{1-2\epsilon}{C_\psi} - 2\epsilon, -\lambda_0)$. *Finally combined with* (4.80)*, we obtain that*

$$||q_h(t,x) - \mu_0(x)||_{TV} \le Me^{-\frac{a}{2}t},$$

for some constant $M > 0$. *This only leaves the proof of* (4.81)*,* (4.82)*, and* (4.75)*. We leave the proof of* (4.82) *to Lemma 5.3.1 in the Appendix.*

Proof of (4.81) *To see* (4.81)*, we integrate by parts to see*

$$II = -(\psi''(f)(\partial_x f)^2, (B(t,x) - a(x,0))\mu_0) - (C(t,x)\partial_x \psi'(f), f\mu_0) =: II_1 + II_2$$

where

$$C(t,x) = \partial_x[B(t,x) - a(x,0)] + \frac{\mu_0'}{\mu_0}(B(t,x) - a(x,0)) - (A(t,x) - d(x,0)).$$

It is easy to see that

$$II_1 \le \epsilon \int_I \psi''(f)(\partial_x f)^2 \mu_0 \tag{4.84}$$

since by (4.79) $|B(t,x) - a(x,0)| \le \epsilon$. *We use Hölder and Cauchy-Schwarz inequalities to bound*

$$|II_2| \le (C(t,x)(\psi''(f))^2(\partial_x f)^2 |f|^{2-p}, \mu_0) + (C(t,x)|f|^p, \mu_0) =: II_{21} + II_{22}.$$

Since $\psi''(f)|f|^{2-p}$ *is bounded by some constant independent of* ϵ*, by rescaling* $\epsilon' = \frac{\epsilon}{F}$ *in* (4.79) *we obtain*

$$II_{21} \le \epsilon \int_I \psi''(f)(\partial_x f)^2 \mu_0. \tag{4.85}$$

We split II_{22} *into two parts*

$$\int_{f\le 1} C(t,x)|f|^p \mu_0 + \int_{f\ge 1} C(t,x)|f|^p \mu_0.$$

The first term decreases exponentially as $e^{\lambda_0 t}$ *using* $|f|^p \le f$ *and that* $(C(t,x), q_h(t,x))$ *decreases exponentially. The second term is bounded by* $\int \psi(f)\mu_0$ *multiplied by some constant. To see this, when* $0 \le f \le 3$*, because of the convexity of* ψ*,*

$$\psi(f) - \psi(1) - \psi'(1)(f-1) \ge \frac{1}{2} \min_{0\le f\le 3} \psi''(f)(f-1)^2,$$

and $\psi(f) \sim f^p$ *near* ∞*. Thus,*

$$(f-1)^2 \le C(1+f^{2-p})(\psi(f) - \psi(1) - \psi'(1)(f-1)).$$

Hence

$$\int_I \psi(f)\mu_0 \ge \int \frac{(f-1)^2}{1+f^{2-p}}\mu_0 \ge \int_{f\ge 1} \frac{(f-1)^2}{1+f^{2-p}}\mu_0 \ge M \int_{f\ge 1} \psi(f)\mu_0. \tag{4.86}$$

Gathering the above estimates,

$$II \le II_1 + II_{21} + II_{22} \le \epsilon \int_I \psi''(f)(\partial_x f)^2 \mu_0 + \epsilon \int_I \psi(f)\mu_0 + Me^{\lambda_0 t}. \tag{4.87}$$

Proof of (4.75) *If we consider the new coordinate* (t, y)*, then the coefficient of mixed derivative* ∂_{tx} *is*

$$A(x,y) = cyt_x + 2by^2 t_y$$

To make $A(x,1) = 0$*, we choose*

$$t_x(x,1) = 1, \qquad \text{and then} \qquad t_y(x,1) = -\frac{c(x,1)}{2b(x,1)}.$$

Let $f(x,y) = \partial_{xy} t = \partial_{yx} t$*, we choose* $f(x,y) = f(x,1)h(y)$ *and*

$$t_x(x,y) = \int_0^y f(x,z)dz + g(x), \; t_y(x,y) = \int_0^x f(z,y)dz + l(y). \tag{4.88}$$

Plugging (4.88) into t_x, t_y*, we find*

$$\int_0^1 f(x,y)dy + g(x) = 1, \; \int_0^x f(z,1)dz + l(1) = -\frac{c(x,1)}{2b(x,1)},$$

and hence

$$f(x,1) = -\frac{\partial}{\partial x}\frac{c(x,1)}{2b(x,1)}, \; g(x) = 1 - f(x,1)\int_0^1 h(y)dy.$$

Now we only need to make t_x *everywhere positive. Then, the coordinate change* $(x,y) \mapsto (t,y)$ *would be bijective on* $[0,1]^2$*. Now by (4.88),*

$$t_x(x,y) = f(x,1)\int_0^y h(z)dz + 1 - f(x,1)\int_0^1 h(y)dy = 1 - f(x,1)\int_y^1 h(z)dz.$$

We can choose a smooth function $h(z)$ *so that* $|f(x,1)| \cdot |\int_y^1 h(z)dz| < 1$ *for all* $(x,y) \in [0,1]^2$*. Therefore* $t(x,y)$ *exists since* $t_{xy} = t_{yx}$ *and* $A(x,1) = 0$*.*

As a consequence we have the following estimate in Wasserstein distance,

Theorem 4.5.2. *Fix a point* p *that is not on any quadratic transverse edge. Then the Wasserstein distance between the transition probability* $p_t(p, \cdot)$ *and the invariant measure* *supported on the tangent edge converges exponentially:*

$$W(p_t(p,\cdot),\mu_0(x)\delta_0(y)) \le Me^{-\frac{a}{2}t}, t > 0. \tag{4.89}$$

Proof. For any $f \in C^0(P)$ with $\text{Lip}(f) \le 1$,

$$\int_P fp_t - \int_P f\mu(x)\delta_0(y) = \int_{P\setminus U} fp_t + \int_U fp_t - \int_U f\mu(x)\delta_0(y).$$

$\int_{P\setminus U} fp_t$ is bounded by $Me^{-\lambda_0 t}$with constant M independent of f. The second term is

$$\begin{aligned}
&\int_U fp_t(p,x,y)dxdy - \int_U f\mu(x)\delta_0(y) \\
=&\int_I\int_0^1 p_t(p,x,y)[f(x,y) - f(x,0)h(y)]dxdy \\
&+\left[\int_I\int_0^1 p_t(p,x,y)f(x,0)h(y)dxdy - \int_I f(x,0)\mu(x)dx\right].
\end{aligned}$$

It is bounded by $||q_h(t,x)-\mu(x)||_{TV}$. The first term also has exponential decay because $[f(x,y)-f(x,0)h(y)] \in C^V(P)$ with bounded norm independent of f. Hence the Wasserstein distance between between $p_t(p,\cdot)$ and $\mu(x)\delta_0(y)$ decays exponentially with the same rate $\frac{a}{2}$. □

CHAPTER 5

APPENDIX

5.1 Kimura kernel Estimates

We now provide several kernel estimates which are crucial in the construction of the local heat kernel at boundary points. Recall that

$$p_t^d(x,y) = \left(\frac{y}{t}\right)^d e^{-\frac{x+y}{t}} \psi_d(\frac{xy}{t^2})\frac{1}{y},\ d>0,$$
$$p_t^0(x,y) = e^{-\frac{x}{t}}\delta(y) + \left(\frac{x}{t}\right) e^{-\frac{x+y}{t}} \psi_2(\frac{xy}{t^2})\frac{1}{t},\ d=0.$$

Lemma 5.1.1. *There exists a constant $C_d > 0$ uniformly bounded for $d \in [0,B]$ depending on d such that*

$$p_d(t,x,y) \leq \frac{C_d}{\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}},\ d \geq \frac{1}{2} \text{ or } d=0, y\neq 0, \tag{5.1}$$
$$p_d(t,x,y) \leq C_d \max\left(\frac{1}{t^d} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} y^{d-1}, \frac{1}{\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}\right),\ d<\frac{1}{2}. \tag{5.2}$$

Proof. Using the kernel formula,

$$p_t^d(x,y)y = \left(\frac{y}{t}\right)^d e^{-\frac{x+y}{t}} \psi_d(\frac{xy}{t^2}).$$

Let $\lambda = \frac{x}{t}, w = \frac{y}{t}$. First we treat the case when $d \geq \frac{1}{2}$ and want to show that

$$w^{d-\frac{1}{2}} e^{-(\lambda+w)} \psi_d(\lambda w) \leq C e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}. \tag{5.3}$$

1. If $\lambda w \leq 1$, then $\psi_d(\lambda w) \leq C_d$ since ψ_d is smooth and $e^{-(\lambda+w)} \leq e^{-(\sqrt{\lambda}-\sqrt{w})^2}$, so it remains

to show that

$$w^{d-\frac{1}{2}} = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}). \tag{5.4}$$

If $w \leq 1$, $w^{d-\frac{1}{2}} \leq 1 \leq e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}$. If $w > 1$, then $w \leq \lambda$ since $\lambda w \leq 1$, so

$$e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}} \geq e^{\frac{\left(\sqrt{\frac{1}{w}}-\sqrt{w}\right)^2}{2}} = \Omega(w^{d-\frac{1}{2}}).$$

2. If $\lambda w \geq 1$, using the asymptotic expansion $\psi_d(z) \sim \frac{z^{\frac{1}{4}-\frac{d}{2}}e^{2\sqrt{z}}}{\sqrt{4\pi}}$,

$$w^{d-\frac{1}{2}}e^{-(\lambda+w)}\psi_d(\lambda w) = O\left(e^{-(\sqrt{\lambda}-\sqrt{w})^2}(\frac{w}{\lambda})^{\frac{d}{2}-\frac{1}{4}}\right).$$

It remains to show that

$$(\frac{w}{\lambda})^{\frac{d}{2}-\frac{1}{4}} = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}). \tag{5.5}$$

Let $k = \frac{w}{\lambda}$, then $\lambda\sqrt{k} \geq 1$,

$$e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}} = e^{\frac{\lambda(\sqrt{k}-1)^2}{2}} \geq e^{\frac{\frac{1}{\sqrt{k}}(\sqrt{k}-1)^2}{2}} = \Omega(k^{\frac{d}{2}-\frac{1}{4}}). \tag{5.6}$$

In the remaining case when $0 < d \leq \frac{1}{2}$,

1. If $\lambda w \leq 1$, $p_d(t,x,y)y^{1-d}t^d = e^{-(\lambda+w)}\psi_d(\lambda w) \leq \psi_d(1)e^{-(\sqrt{\lambda}-\sqrt{w})^2}$
2. If $\lambda w \geq 1$, again we need to show that (5.3), for which we can apply the same estimate (5.6).

In the last case when $d = 0$, for $y \neq 0$,

$$\sqrt{y}tp_0(t,x,y) = \frac{x\sqrt{y}}{t^{\frac{3}{2}}}e^{-\frac{x+y}{t}}\psi_2(\frac{xy}{t^2}) = \lambda\sqrt{w}e^{-(\lambda+w)}\psi_2(\lambda w).$$

1. If $\lambda w \leq 1$, then the term above is bounded by

$$C\sqrt{\lambda}e^{-(\lambda+w)} = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

2. If $\lambda w \geq 1$, then

$$\lambda\sqrt{w}e^{-(\lambda+w)}\psi_2(\lambda w) \sim \left(\frac{\lambda}{w}\right)^{\frac{1}{4}} e^{-(\sqrt{\lambda}-\sqrt{w})^2} = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

□

Lemma 5.1.2. *There exists a constant $C_d > 0$ uniformly bounded for $d \in [0, B]$ such that*

$$\sqrt{x}\partial_x p_d(t,x,y) \leq \frac{C_d}{t\sqrt{y}}e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}, \quad d \geq \frac{1}{2} \text{ or } d = 0, y \neq 0,$$
$$\sqrt{x}\partial_x p_d(t,x,y) \leq \frac{C_d}{\sqrt{t}} \max\left(\frac{1}{t^d}e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}}y^{d-1}, \frac{1}{\sqrt{yt}}e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}\right), \quad d < \frac{1}{2}.$$

Proof. Let $\lambda = \frac{x}{t}, w = \frac{y}{t}$, then

$$\sqrt{xy}t\partial_x p_d(t,x,y) = \frac{\sqrt{x}}{\sqrt{y}}(\frac{y}{t})^d e^{-\frac{x+y}{t}} \left|\frac{y}{t}\psi_d'(\frac{xy}{t^2}) - \psi_d(\frac{xy}{t^2})\right| = \frac{\sqrt{\lambda}}{\sqrt{w}}w^d e^{-(\lambda+w)}|w\psi_d'(\lambda w) - \psi_d(\lambda w)|. \tag{5.7}$$

We want to show that

$$\frac{\sqrt{\lambda}}{\sqrt{w}}w^d e^{-(\lambda+w)}|w\psi_d'(\lambda w) - \psi_d(\lambda w)| = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

1. If $\lambda w \leq 1$, it suffices to show that

$$\frac{\sqrt{\lambda}}{\sqrt{w}}w^d = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

Let $k = \frac{\lambda}{w}$, then $w \leq \frac{1}{\sqrt{k}}$, fix k,

$$\ln k + 2d\ln w - (\sqrt{k}-1)^2 w \tag{5.8}$$

has maximal value at $min(\frac{1}{\sqrt{k}}, \frac{2d}{(\sqrt{k}-1)^2})$. If $\frac{1}{\sqrt{k}} \leq \frac{2d}{(\sqrt{k}-1)^2}$, then $M \leq k \leq N$ for some $M, N > 0$, so

$$lnk + 2dlnw - (\sqrt{k}-1)^2 w \leq (1-d)lnk - \frac{(\sqrt{k}-1)^2}{\sqrt{k}}$$

is bounded above. If $\frac{2d}{(\sqrt{k}-1)^2} \leq \frac{1}{\sqrt{k}}$, then

$$lnk + 2dlnw - (\sqrt{k}-1)^2 w \leq lnk - 2dln(\sqrt{k}-1)^2 + (2dln2d - 2d).$$

The right-hand side tends to $-\infty$ as $k \to 0$, and tends to $-\infty$ if $d > \frac{1}{2}$ or a finite number if $d = \frac{1}{2}$ as $k \to \infty$, so it is bounded above.

2. If $\lambda w \geq 1$, then

$$\frac{\sqrt{\lambda}}{\sqrt{w}} w^d e^{-(\lambda+w)} |w\psi_d'(\lambda w) - \psi_d(\lambda w)| \sim (\frac{\lambda}{w})^{\frac{1}{4}-\frac{d}{2}} e^{-(\sqrt{\lambda}-\sqrt{w})^2} (\sqrt{w} - \sqrt{\lambda}).$$

In general,

1. If $\lambda w \leq 1$, $p_d(t,x,y) y^{1-d} t^{d+\frac{1}{2}} = \sqrt{\lambda} e^{-(\lambda+w)} |w\psi_d'(\lambda w) - \psi_d(\lambda w)| = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}})$ by (5.4).
2. If $\lambda w \geq 1$, again we need to show that (5.3), for which we can apply the same estimate (5.6)

□

Lemma 5.1.3. *There exists a constant $C_d > 0$ uniformly bounded for $d \in [0, B]$ such that*

$$x\partial_x^2 p_t^d(x,y) \leq \frac{C_d}{t\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}, \ d \geq \frac{1}{2} \ \textit{or} \ d = 0, y \neq 0,$$
$$x\partial_x^2 p_t^d(x,y) \leq \frac{C_d}{t} \max\left(\frac{1}{t^d} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} y^{d-1}, \frac{1}{\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}\right), \ d < \frac{1}{2}.$$

Proof. Let $\lambda = \frac{x}{t}, w = \frac{y}{t}$, then

$$\sqrt{y}ttx\partial_x^2 p_d(t,x,y) = w^{d-\frac{1}{2}}e^{-(\lambda+w)}[(\lambda+w)\psi_d(\lambda w) - 2\lambda w\psi_d'(\lambda w) - dw\psi_d'(\lambda w)]. \quad (5.9)$$

For $d > \frac{1}{2}$, we want to show that

$$w^{d-\frac{1}{2}}e^{-(\lambda+w)}[(\lambda+w)\psi_d(\lambda w) - 2\lambda w\psi_d'(\lambda w) - dw\psi_d(\lambda w)] = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

1. If $\lambda w \leq 1$, it suffices to show that

$$(\lambda + w + 1)w^{d-\frac{1}{2}} = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

2. If $\lambda w \geq 1$, then

$$\begin{aligned} &w^{d-\frac{1}{2}}e^{-(\lambda+w)}[(\lambda+w)\psi_d(\lambda w) - 2\lambda w\psi_d'(w) - dw\psi_d(\lambda w)] \\ &\sim (\frac{\lambda}{w})^{\frac{1}{4}-\frac{d}{2}}e^{-(\sqrt{\lambda}-\sqrt{w})^2}[(\sqrt{w}-\sqrt{\lambda})^2 + O(\frac{w+\lambda}{\sqrt{\lambda w}}+1)]. \end{aligned}$$

Next we treat the case when $d < \frac{1}{2}$.

1. If $\lambda w \leq 1$, $p_d(t,x,y)y^{1-d}t^{d+\frac{1}{2}} = \sqrt{\lambda}e^{-(\lambda+w)}|w\psi_d'(\lambda w) - \psi_d(\lambda w)| = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}})$ by (5.4).
2. If $\lambda w \geq 1$, again we need to show that (5.3), for which we can apply the same estimate (5.6).

For $d = 0$, we want to show that

$$2\lambda e^{-(\lambda+w)}\left[w^{\frac{3}{2}}\psi_2'(\lambda w) - \sqrt{w}\psi_2(\lambda w)\right] + \lambda^2 e^{-(\lambda+w)}\left[w^{\frac{5}{2}}\psi_2''(\lambda w) - 2w^{\frac{3}{2}}\psi_2'(\lambda w) + \sqrt{w}\psi_2(\lambda w)\right] = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

When $\lambda w \leq 1$, the left hand side is $O((\sqrt{\lambda} + \sqrt{w} + \lambda^{\frac{3}{2}})e^{-(\lambda+w)}) = O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}})$. When

$\lambda w > 1$, the left hand side has asymptotic

$$e^{-(\sqrt{\lambda}+\sqrt{w})^2}\left[\frac{1}{(\lambda w)^{\frac{1}{4}}}(\sqrt{w}-\sqrt{\lambda})+\left(\frac{\lambda}{w}(\sqrt{\lambda}-\sqrt{w})^2\right)^{\frac{1}{4}}\right]=O(e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}).$$

□

Lemma 5.1.4. *There exists a constant $C_d > 0$ uniformly bounded for $d \in [0, B]$ such that*

$$\partial_x p_d(t,x,y) \leq \frac{C_d}{t\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}, \quad d \geq \frac{1}{2} \text{ or } d = 0, y \neq 0, \tag{5.10}$$

$$\partial_x p_d(t,x,y) \leq C_d \max\left(\frac{1}{t^{d+1}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} y^{d-1}, \frac{1}{t\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}\right), \quad d < \frac{1}{2}. \tag{5.11}$$

Proof. First compute

$$yt\partial_x p_d(t,x,y) = (\frac{y}{t})^d e^{-\frac{x+y}{t}} \left|\frac{y}{t}\psi'_d(\frac{xy}{t^2}) - \psi_d(\frac{xy}{t^2})\right|.$$

Let $\lambda = \frac{x}{t}, w = \frac{y}{t}$. For the case $d \geq \frac{1}{2}$, we want to show that

$$w^{d-\frac{1}{2}} e^{-(\lambda+w)} \left|w\psi'_d(\lambda w) - \psi_d(\lambda w)\right| \leq C e^{\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}. \tag{5.12}$$

The proof is similar to the proof above, except that we now use the asymptotic expansion

$$\psi_d(z) \sim \frac{z^{\frac{1}{4}-\frac{d}{2}} e^{2\sqrt{z}}}{\sqrt{4\pi}}, \psi'_d(z) \sim \frac{z^{-\frac{1}{4}-\frac{d}{2}} e^{2\sqrt{z}}}{\sqrt{4\pi}}$$

and ψ'_d is also continuous at 0. The case when $0 \leq d \leq \frac{1}{2}$ can be proved similarly. □

Lemma 5.1.5. *For $k \in \mathbb{N}$, there exists a constant $C_d > 0$ depending on k uniformly bounded* *for $d \in [0, B]$ such that*

$$(y\partial_y)^k p_d(t,x,y) \le \frac{C_d}{\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}}, \quad d \ge \frac{1}{2} \text{ or } d=0, y \neq 0, \tag{5.13}$$

$$(y\partial_y)^k p_d(t,x,y) \le C_d \max\left(\frac{1}{t^d} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{t}} y^{d-1}, \frac{1}{\sqrt{yt}} e^{-\frac{(\sqrt{x}-\sqrt{y})^2}{2t}} \right), \quad d < \frac{1}{2}. \tag{5.14}$$

Proof. Let $\lambda = \frac{x}{t}, w = \frac{y}{t}$, then $(w\frac{\partial}{\partial w})^k p = (y\frac{\partial}{\partial y})^k p,$. We first consider the case when $d \ge \frac{1}{2}$.

$$\begin{aligned}
p_t^d(\lambda, w) &= \frac{1}{t} w^{d-1} e^{-(\lambda+w)} \psi_d(\lambda w) =: I_0 \\
w\partial_w p_t^d(\lambda, w) &= \frac{1}{t} w^d e^{-(\lambda+w)} \left[\lambda\psi_d'(\lambda w) - \psi_d(\lambda w)\right] + (d-1)p_t^d(\lambda, w) =: I_{11} + (d-1)I_0 \\
(w\partial_w)^2 p_t^d(\lambda, w) &= \frac{1}{t} w^d e^{-(\lambda+w)} \left[\lambda^2 \psi_d''(\lambda w) - 2\lambda\psi_d'(\lambda w)\right. \\
&\quad + \psi_d(\lambda w)) + (2d-1)I_1 - d(d-1)I_0 \\
&\quad =: I_{21} + (2d-1)I_1 - d(d-1)I_0
\end{aligned}$$

By induction, for $k \in \mathbb{N}$,

$$(w\partial_w)^k p_t^d(x,y) = \frac{1}{t} w^d e^{-(\lambda+w)} \left[\sum_{i=0}^{k} (-1)^{k-i} + \begin{pmatrix} k \\ i \end{pmatrix} \lambda^i \psi_d^{(i)}(\lambda w) \right] + \sum_{j=0}^{k-1} c_j (w\partial_w)^j p_t^d(x,y) \tag{5.15}$$

so we are left to show that

$$I := w^{d-\frac{1}{2}} e^{-(\lambda+w)} \left[\sum_{i=0}^{k} (-1)^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \lambda^i \psi_d^{(i)}(\lambda w) \right] \le C e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}}$$

Use that

$$\frac{d^k}{d_z^k}\psi_d = (\sqrt{z})^{1-d-k} I_{d-1+k}(2\sqrt{z}) = \psi_{d+k}(z), \tag{5.16}$$

$$\psi_d(z) \sim \frac{z^{\frac{1}{4}-\frac{d}{2}} e^{2\sqrt{z}}}{\sqrt{4\pi}}, z \to \infty \tag{5.17}$$

1. If $\lambda w \leq 1$, then $I \leq C w^{d-\frac{1}{2}} \left(1+\lambda+...+\lambda^k\right) e^{-(\lambda+w)} = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}})$ since $d \geq \frac{1}{2}$;

2. If $\lambda w \geq 1$,

$$I \sim \frac{1}{\sqrt{4\pi}} w^{d-\frac{1}{2}} e^{-(\sqrt{\lambda}-\sqrt{w})^2} \left[\sum_{i=0}^{k} (-1)^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} \lambda^i (\lambda w)^{\frac{1}{4}-\frac{d+i}{2}}\right]$$
$$\sim \frac{1}{\sqrt{4\pi}} \left(\frac{\lambda}{w}\right)^{\frac{1}{4}-\frac{d}{2}+\frac{k}{4}} \frac{1}{(\lambda w)^{\frac{k}{4}}} e^{-(\sqrt{\lambda}-\sqrt{w})^2}$$

so $I \leq \frac{1}{\sqrt{4\pi}} \left(\frac{\lambda}{w}\right)^{\frac{1}{4}-\frac{d}{2}+\frac{k}{4}} e^{-(\sqrt{\lambda}-\sqrt{w})^2} = O(e^{-\frac{(\sqrt{\lambda}-\sqrt{w})^2}{2}})$ by (5.5).

The case when $0 \leq d \leq \frac{1}{2}$ can be proved similarly. □

5.2 Heat kernel estimates

Next we give kernel estimates for the heat kernel $k_t^e(x, x_1) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x_1)^2}{4t}}$.

Lemma 5.2.1. *For* $c > 0,\ 0 < s < t$, *there exists a constant* $C > 0$ *such that*

$$\int_{-\infty}^{\infty} |\partial_y k_{t-s}^e(y, y_1) - \partial_y k_{t-s}^e(y', y_1)| \cdot |y - y_1|^\gamma \cdot e^{-\frac{(y_1-y_2)^2}{4cs}} dy_1 \tag{5.18}$$
$$\leq \frac{C}{\sqrt{t-s}} |y-y'|^\gamma \left(e^{-\frac{(y-y_2)^2}{4ct}} + e^{-\frac{(y'-y_2)^2}{4ct}}\right). \tag{5.19}$$

For $x_1 < x_2$, *denote* $\alpha_e = \frac{3x_1-x_2}{2},\ \beta_e = \frac{3x_2-x_1}{2}$, *let* $J = [\alpha_e, \beta_e]$.

Lemma 5.2.2. *For $0 < \gamma < 1$, there is a C such that*

$$\int_0^t \int_J |\partial_x^2 k_{t-s}^e(y, y_1)||y - y_1|^\gamma e^{-\frac{(y_1-y_2)^2}{4s}} dy_1 ds \le C|y - y'|^\gamma e^{-\frac{(y-y_2)^2}{4t}}, \tag{5.20}$$

$$\int_0^t \int_J |\partial_x^2 k_{t-s}^e(y', y_1)||y' - y_1|^\gamma e^{-\frac{(y_1-y_2)^2}{4s}} dy ds \le C|y - y'|^\gamma e^{-\frac{(y'-y_2)^2}{4t}}. \tag{5.21}$$

Lemma 5.2.3. *For $c > 0$, $0 < s < t$, there exists a constant $C > 0$ such that*

$$\int_0^t \int_{J^c} |\partial_y^2 k_{t-s}^e(y, y_1) - \partial_{y'}^2 k_{t-s}^e(y', y_1)| \cdot |y - y_1|^\gamma \cdot e^{-\frac{(y_1-y_2)^2}{4s}} dy_1 ds,$$
$$\le C|y - y'|^\gamma \left(e^{-\frac{(y-y_2)^2}{8t}} + e^{-\frac{(y'-y_2)^2}{8t}} \right).$$

Proof. These are all corollaries of

5.3 Proof in Chapter 4

Lemma 5.3.1. *Fix $p \in P$ that is not on the quadratic edge or tangent Kimura edge. For $q \in H \times [A, 1]$, the transition probability $p_t(p, q)$ has a pointwise upper bound:*

$$p_t(p, q) \le C \cdot exp\left[-\frac{x^2}{4C_2 t} + C_0|x| + C_1 t\right], \tag{5.22}$$

for some constant $C, C_0, C_1, C_2 > 0$. Therefore

$$\left| \int_I \psi'(f) v(t, x) dx \right| \le \epsilon \int_I \psi\mu + \sqrt{t} e^{(1-\epsilon)\lambda_0 t}. \tag{5.23}$$

Proof. We can follow the proof of

Firstly, unlike the setting under the usual Rimmannian measure on P, our proof introduces a weighted measure $d\mu$ on P. The principal symbol of L induces a Riemannian metric dV on P. For each Kimura boundary surface $H_i, 1 \le i \le \eta$, define $\rho_i(p)$ to be the

Riemannian distance from the point $p \in P$ to H_i. Then

$$B_i|_{H_i} := \frac{1}{4} L\rho_i^2|_{H_i}$$

are coordinate-invariant quantities. We also let B_i denote a smooth extension from H_i to P of the coefficients. Then we define the weighted measure $d\mu$ by

$$d\mu(p) := \prod_{i=1}^{\eta} \rho_i(p)^{2B_i-1} dV.$$

In an adapted system of local coordinates of a corner p, L takes the form: $m+n=2$,

$$\begin{aligned} L =& \sum_{i=1}^{m} x_i \partial_{x_i x_i} + \sum_{i,j=1}^{m,n} b(x,y) x_i y_j \partial_{x_i y_j} + \sum_{j=1}^{n} c(x,y) y_j^2 \partial_{y_j y_j} \\ &+ \sum_{i=1}^{m} d(x,y)\partial_{x_i} + \sum_{j=1}^{n} e(x,y) y_j \partial_{y_j}. \end{aligned} \tag{5.24}$$

The weighted measure $d\mu$ is a multiple by a smooth function of

$$d\mu(x,y) = \prod_{i=1}^{m} x_i^{d_i(x,y)-1} dx_i \prod_{j=1}^{n} \frac{1}{y_j} dy_j.$$

For $\alpha \in \mathbb{R}$ and $\phi \in C^\infty(P)$ satisfying $|\nabla\phi| \leq 1$ under the Riemannian metric on P, define $H_t^{\alpha,\phi}(P)$ on $L^2(\mu)$ by

$$H_t^{\alpha,\phi} f(x) = e^{-\alpha\phi(x)} H_t[e^{\alpha\phi} f](x).$$

Instead of the estimates of $||H_t^{\alpha,\phi}||_{2\to 2}$ in

Proof of (??) To see this, compute the derivative:

$$\frac{\partial}{\partial_t} ||H_t^{\alpha,\phi} f||_2^2 = 2(H_t^{\alpha,\phi} f, e^{-\alpha\phi} L e^{\alpha,\phi} H_t^{\alpha,\phi} f)_{L^2(\mu)}.$$

Taking $g = H_t^{\alpha,\phi} f$, to derive (**??**) it is sufficient to have

$$(e^{-\alpha\phi} g, Le^{\alpha\phi} g) \leq [\alpha^2 + C_0|\alpha| + C_1](g,g). \tag{5.25}$$

We associate a bilinear form $Q(u,v)$ to $-(Lu,v)_\mu$ by letting

$$Q(u,v) = (Lu,v)_{L^2(\mu)},$$

which can be written as

$$Q(u,v) = Q_{\text{sym}} + (Tu,v)_{L^2(d\mu)}$$

where $Q_{\text{sym}}(u,v)$ is a symmetric bilinear form and T is a first order vector field on P. We need to estimate it in two cases.

Case I: In a neighborhood that is away from any Kimura edge, then L is uniformly elliptic under $d\mu$ and of the form

$$L = a\partial_x^2 + 2b\partial_{xy} + c\partial_y^2 + V,$$

where V is a vector field. Then

$$Q_{\text{sym}}(u,v) = -\int [au_x v_x + cu_y v_y + b(u_x v_y + u_y v_x)] d\mu.$$

T is a vector field with bounded smooth coefficients. Take $u = e^{\alpha\phi} g, v = e^{-\alpha\phi} g$. Then

$$Q(u,v) = Q(g,g) + \alpha^2 \int (a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2) g^2 d\mu + \alpha(T\phi, g^2)_{L^2(d\mu)}.$$

L is uniformly parabolic so we can bound $||Q(g,g)||$ by $C||g||^2_{L^2(d\mu)}$ for some constant $C > 0$. Using the fact that $|\nabla\phi| \leq 1$, the second term and the third term are bounded by $\alpha^2||g||^2_{L^2(d\mu)}$, $C|\alpha|\cdot||g||^2_{L^2(d\mu)}$ for some constant $C > 0$, respectively. So we obtained (5.25).

Case II: In a neighborhood of the tangent Kimura edge, say suppose in a neighborhood of a corner which is the intersection of two Kimura edges, L is degenerate. Under local coordinates (5.24),

$$L = x\partial_{xx} + y\partial_{yy} + xyb(x,y)\partial_{xy} + d_1(x,y)\partial_x + d_2(x,y)\partial_y.$$

We refer to

Take $u = e^{\alpha\phi}g, v = e^{-\alpha\phi}g$. Then,

$$(Lu,v) = Q(g,g) + \alpha^2\int g^2(x\phi_x^2 + y\phi_y^2 + axy\phi_x\phi_y)d\mu + \alpha(T\phi, g^2)_{L^2(d\mu)}.$$

By

Proof of (5.22) Fix $p \in P$ that is not on a quadratic edge or tangent Kimura edge, $q \in H\times[A,1], r_1, r_2 > 0$ so that $B_{r_1}(p), B_{r_2}(q) \subset P$. Notice that if q is an edge point on a Kimura edge, $B_{r_2}(q)$ is taken to be the semi-ball that lies in P.

Let χ_1, χ_2 be the indicatrix functions of $B_{r_1}(p), B_{r_2}(q)$. Instead of deriving the mean value inequalities from local Sobolev inequality in

Proof of (5.23) Having obtained the pointwise upper bound of $p_t(p,q)$, we can estimate $v(t,x)$.We choose $\phi(q) = \phi(p) - C_2x$ with some constant C so that $|\nabla\phi| \leq 1$. Since $h|_{H\times[0,A]} \equiv 1$, by definition of $v(t,x)$ (4.78), we only need to integrate y in $[A,1]$. We use the pointwise upper bounded above to obtain the following estimate of $v(t,x)$:

$$v(t,x) \leq C\cdot\exp\left[-\frac{x^2}{4C_2t} + C_0|x| + C_1t\right], \tag{5.26}$$

for some constant $C_0, C_1, C_2 > 0$. Next we estimate $\int \psi'(f)v(t,x)dx$. We use Hölder's inequality with $\frac{1}{r}+\frac{1}{s}=1$ to obtain

$$\left|\int \psi'(f)v(t,x)dx\right| \leq ||\psi'(f)\mu^{1-\frac{1}{p}}||_{\frac{p}{p-1}} \cdot ||\mu^{\frac{1}{p}-1}v^{1-\frac{1}{ps}}||_{pr} \cdot ||v^{\frac{1}{ps}}||_{ps}. \tag{5.27}$$

First it is easy to see

$$||v^{\frac{1}{ps}}||_{ps} = O(e^{\frac{\lambda_0}{ps}t}) \tag{5.28}$$

has exponential decay with rate $\frac{\lambda_0}{ps}$. We next estimate the first term. Since $\psi(x) \sim x^p$ as $x \to \infty$ and taking into account that ψ is negative only on a finite interval, we have

$$||\psi'(f)\mu^{1-\frac{1}{p}}||_{\frac{p}{p-1}}^{\frac{p}{p-1}} \leq M_1 \int |\psi|\mu \leq M_1 \int \psi\mu + M_2. \tag{5.29}$$

For the second term, notice that μ^{-1} has asymptotic behavior $O(e^{a|x|})$ for $a > 0$, and $v(t,x)$ decays quadratically in $|x|$ by (5.26). So the second term is intergable while it may have exponential growth with respect to t. Indeed after a direct calculation

$$\begin{aligned}||\mu^{\frac{1}{p}}v^{1-\frac{1}{ps}}||_{pr} &= O\left(\sqrt{t}\cdot\exp\left[4\left(C_0+\frac{a(p-1)}{p-\frac{1}{s}}\right)^2(1-\frac{1}{ps})\right]\right)\\ &= O\left(\sqrt{t}\cdot\exp\left[4\left(C_0+a\right)^2(1-\frac{1}{ps})\right]\right)\end{aligned}$$

since $\frac{1}{s} < 1, p < 2$. Regarding (5.28), for $0 < \epsilon < 1$, we take $\frac{1}{ps} = 1 - \epsilon\frac{\lambda_0}{4(C_0+a)^2+\lambda_0}$ so that

$$||\mu^{\frac{1}{p}-1}v^{1-\frac{1}{ps}}||_{pr} \cdot ||v^{\frac{1}{ps}}||_{ps} = O(e^{(1-\epsilon)\lambda_0}). \tag{5.30}$$

Taken (5.27), (5.29), (5.30) and an interpolation of arithmetic-geometric mean inequality together, we obtain

$$\left|\int \psi'(f)v(t,x)dx\right| \leq \epsilon \int \psi\mu + \sqrt{t}e^{(1-\epsilon)\lambda_0 t}. \tag{5.31}$$

□

Lemma 5.3.2. *There exists a stratification as described in* (4.59)*.*

Proof. We first take a finite covering of H by coordinate neighborhoods of the form R based at points of H. Let k_1 be the minimum of the height (normal direction in the local coordinates) of these rectangles. Then by taking the closure of these rectangles and shrinking the height, $P_{[k_0,k_1]}$ is covered by closed rectangles that satisfy the condition 4.60. We can do the similar job at each level of the corners. Specifically, for a corner p, the level $L_{\rho(p)} := \rho = \rho(p)$ is the disjoint union

$$L_{\rho(p)} = L_{\rho(p),\max} \bigcup L_{\rho(p),i}$$

where $L_{\rho(p),\max}$ are the collection of points that attain the local maximum. $L_{\rho(p),\max}$, $L_{\rho(p),i}$ are both compact and a disjoint union of finite points and the edges out of condition 4.5.2.c. We cover $L_{\rho(p),\max}, L_{\rho(p),i}$ by finite open sets of the form R or T. Again, by taking the closure and shrinking the height of these open sets, there are $\epsilon_1, \epsilon_2 \geq 0$ so that $P_{[\rho(p)-\epsilon_1,\rho(p)+\epsilon_2]}$ is covered by the closed sets that satisfy the condition 4.60. Up to now, the remaining parts of P we have not covered are a disjoint unions of level intervals. Suppose $P_{[a,b]}$ is one of level interval, we first cover $P_{[a,b]}$ by finite open rectangles with two sides lie on two ρ-levels, then we can cut these rectangles so that $P_{[a,b]}$ is covered by closed rectangles that satisfy the condition 4.60. Thus we can cover these level intervals by the desired stratification. □

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